HYPERCONTRACTIVITY FOR FREE PRODUCTS

MARIUS JUNGE, CARLOS PALAZUELOS,
JAVIER PARCET, MATHILDE PERRIN AND ÉRIC RICARD

Abstract. In this paper, we obtain optimal time hypercontractivity bounds for the free product extension of the Ornstein-Uhlenbeck semigroup acting on the Clifford algebra. Our approach is based on a central limit theorem for free products of spin matrix algebras with mixed commutation/anticommutation relations. With another use of Speicher's central limit theorem, we may also obtain the same bounds for free products of q-deformed von Neumann algebras interpolating between the fermonic and bosonic frameworks. This generalizes the work of Nelson, Gross, Carlen/Lieb and Biane. Our main application yields hypercontractivity bounds for the free Poisson semigroup acting on the group algebra of the free group $\mathbb{F}_n$, uniformly in the number of generators.

Introduction

The two-point inequality was first proved by Bonami and rediscovered years later by Gross [6, 14]. In the context of harmonic analysis, this inequality was central for Bonami's work on the relation between integrability of a function and the decay properties of its Fourier coefficients. It was also instrumental in Beckner's theorem on the optimal constants for the Hausdorff-Young inequality [2]. On the other hand, motivated by quantum field theory, Gross used it as a key step towards his logarithmic Sobolev inequalities [14]. More recently, the two-point inequality has also produced very important applications in computer science and in both classical and quantum information theory [8, 11, 22, 23]. If $1 < p \leq q < \infty$ and $\alpha, \beta \in \mathbb{C}$, Bonami-Gross inequality can be rephrased for $r = e^{-t}$ as follows

$$\Big(\sum_{\varepsilon = \pm 1} \Big| \frac{(1+\varepsilon r)\alpha + (1-\varepsilon r)\beta}{2^{1+\frac{1}{q}}} \Big|^q \Big)^{\frac{1}{q}} \leq \Big(\frac{|\alpha|^p + |\beta|^p}{2} \Big)^{\frac{1}{p}} \ \Leftrightarrow \ r \leq \sqrt{\frac{p-1}{q-1}}.$$

It can be regarded —from Bonami's viewpoint— as the optimal hypercontractivity bound for the "Poisson semigroup" on the group $\mathbb{Z}_2$, while Gross understood it as the optimal hypercontractivity bound for the Ornstein-Uhlenbeck semigroup on the Clifford algebra with one generator $\mathcal{C}(\mathbb{R})$. Although the two-point inequality can be generalized in both directions, harmonic analysis has developed towards other related norm inequalities in the classical groups —like Λ_p sets in $\mathbb{Z}$— instead of analyzing the hypercontractivity phenomenon over the compact dual of other discrete groups. Namely, to the best of our knowledge only the cartesian products of $\mathbb{Z}_2$ and $\mathbb{Z}$ have been understood so far, see [40]. The first goal of this paper is to replace cartesian products by free products, and thereby obtain hypercontractivity inequalities for the free Poisson semigroups acting on the group von Neumann algebras associated to $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ and $\mathbb{G}_n = \mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$.

Let G denote any of the free products considered above and let $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ stand for the corresponding left regular representation. The group von Neumann algebra $\mathcal{L}(\mathrm{G})$ is the weak operator closure of the linear span of $\lambda(\mathrm{G})$. If e denotes the identity element of G, the algebra $\mathcal{L}(\mathrm{G})$ comes equipped with the standard trace $\tau(f) = \langle \delta_e, f\delta_e \rangle$. Let $L_p(\mathcal{L}(\mathrm{G}), \tau)$ be the L_p space over the noncommutative measure space $(\mathcal{L}(\mathrm{G}), \tau_{\mathrm{G}})$ —the so called noncommutative L_p spaces— with norm $\|f\|_p^p = \tau|f|^p$. We invite the reader to check that $L_p(\mathcal{L}(\mathrm{G}), \tau) = L_p(\mathbb{T})$ for $\mathrm{G} = \mathbb{Z}$ after identifying $\lambda_{\mathbb{Z}}(k)$ with $e^{2\pi i k \cdot}$. In the general case, the absolute value and the power p are obtained from functional calculus for this (unbounded) operator on the Hilbert space $\ell_2(\mathrm{G})$, see [35] for details. If $f = \sum_g \widehat{f}(g)\lambda(g)$, the free Poisson semigroup on G is given by the family of linear maps

$$\mathcal{P}_{\mathrm{G},t} f \, = \, \sum_{g \in \mathrm{G}} e^{-t|g|} \widehat{f}(g) \lambda(g) \quad \text{with} \quad t \in \mathbb{R}_+.$$

In both cases $\mathrm{G} \in \{\mathbb{F}_n, \mathbb{G}_n\}$, $|g|$ refers to the Cayley graph length. In other words, $|g|$ is the number of letters (generators and their inverses) which appear in g when it is written in reduced form. It is known from [17] that $\mathcal{P}_{\mathrm{G}} = (\mathcal{P}_{\mathrm{G},t})_{t \ge 0}$ defines a Markovian semigroup of self-adjoint, completely positive, unital maps on $\mathcal{L}(\mathrm{G})$. In particular, $\mathcal{P}_{\mathrm{G},t}$ defines a contraction on $L_p(\mathcal{L}(\mathrm{G}))$ for every $1 \le p \le \infty$. The hypercontractivity problem for $1 < p \le q < \infty$ consists in determining the optimal time $t_{p,q} > 0$ above which

$$\|\mathcal{P}_{\mathrm{G},t} f\|_q \, \le \, \|f\|_p \qquad \text{for all} \qquad t \ge t_{p,q}.$$

In our first result we provide new hypercontractivity bounds for the free Poisson semigroups on those group von Neumann algebras. If $g_1, g_2, \ldots, g_n$ stand for the free generators of $\mathbb{F}_n$, we will also consider the symmetric subalgebra $\mathcal{A}_{sym}^n$ of $\mathcal{L}(\mathbb{F}_n)$ generated by the self-adjoint operators $\lambda(g_j) + \lambda(g_j)^*$. In other words, we set

$$\mathcal{A}_{sym}^n \, = \, \big\langle \lambda(g_1) + \lambda(g_1)^*, \ldots, \lambda(g_n) + \lambda(g_n)^* \big\rangle''.$$

Theorem A. *If $1 < p \le q < \infty$, we find*:

i) *Optimal time hypercontractivity for $\mathbb{G}_n$*

$$\big\|\mathcal{P}_{\mathbb{G}_n,t} : L_p(\mathcal{L}(\mathbb{G}_n)) \to L_q(\mathcal{L}(\mathbb{G}_n))\big\| \, = \, 1 \ \Leftrightarrow \ t \ge \frac{1}{2} \log \frac{q-1}{p-1}.$$

ii) *Hypercontractivity for $\mathbb{F}_n$ over twice the optimal time*

$$\big\|\mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{L}(\mathbb{F}_n)) \to L_q(\mathcal{L}(\mathbb{F}_n))\big\| \, = \, 1 \quad \textit{if} \quad t \ge \log \frac{q-1}{p-1}.$$

iii) *Optimal time hypercontractivity in the symmetric algebra $\mathcal{A}_{sym}^n$*

$$\big\|\mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{A}_{sym}^n) \to L_q(\mathcal{A}_{sym}^n)\big\| \, = \, 1 \ \Leftrightarrow \ t \ge \frac{1}{2} \log \frac{q-1}{p-1}.$$

Theorem A i) extends Bonami's theorem for $\mathbb{Z}_2^n$ to the free product case with optimal time estimates. According to the applications in complexity theory and quantum information of Bonami's result, it is conceivable that Theorem A could be of independent interest in those areas. These potential applications will be explored in further research. Theorem A ii) gives the first hypercontractivity estimate for the free Poisson semigroup on $\mathbb{F}_n$, where a factor 2 is lost from the expected optimal time. This is related to our probabilistic approach to the problem and a little distortion must be done to make $\mathbb{F}_n$ fit in. Theorem A iii) refines this, providing optimal time estimates in the symmetric algebra $\mathcal{A}_{sym}^n$. We also obtain optimal time $L_p \to L_2$ hypercontractive estimates for linear combinations of words with length less than or equal to one. Apparently, our probabilistic approach in this paper is limited to go beyond the constant 2 in the general case. We managed to push it to $1+\frac{1}{4}\log 2 \sim 1.173$ in the last section. Actually, we have recently found in [20] an alternative combinatorial/numerical method which yields optimal $L_2 \to L_q$ estimates for $q \in 2\mathbb{Z}$ and also reduces the general constant to $\log 3 \sim 1.099$ for $1 < p \leq q < \infty$. The drawback of this method is the numerical part: the larger is the number of generators n, the harder is to implement and test certain pathological terms in a computer. In this respect, Theorem A ii) is complementary since —at the price of a worse constant— we obtain uniform estimates in n.

As we have already mentioned, it is interesting to understand the two-point inequality as the convergence between the *trigonometric point of view* outlined above and the *gaussian point of view*, which was developed along the extensive study of hypercontractivity carried out in the context of quantum mechanics and operator algebras. The study of hypercontractivity in quantum mechanics dates back to the work of Nelson [31] who showed that semiboundedness of certain Hamiltonians H associated to a bosonic system can be obtained from the (hyper)contractivity of the semigroup $e^{-tA_\gamma} : L_2(\mathbb{R}^d, \gamma) \to L_2(\mathbb{R}^d, \gamma)$, where A_γ is the Dirichlet form operator for the Gaussian measure γ on $\mathbb{R}^d$. After some contributions [12, 18, 36] Nelson finally proved in [32] that the previous semigroup is contractive from $L_p(\mathbb{R}^d, \gamma)$ to $L_q(\mathbb{R}^d, \gamma)$ if and only if $e^{-2t} \leq \frac{p-1}{q-1}$; thus obtaining the same optimal time as in the two-point inequality. By that time a new deep connection was shown by Gross in [14], who established the equivalence between the hypercontractivity of the semigroup e^{-tA_μ}, where A_μ is the Dirichlet form operator associated to the measure μ, and the logarithmic Sobolev inequality verified by μ. During the next 30 years hypercontractivity and its equivalent formulation in terms of logarithmic Sobolev inequalities have found applications in many different areas of mathematics like probability theory, statistical mechanics or differential geometry. We refer the survey [16] for an excellent exposition of the topic.

The extension of Nelson's theorem to the fermonic case started with Gross' papers [13, 15]. Namely, he adapted the argument in the bosonic case by considering a suitable Clifford algebra $\mathcal{C}(\mathbb{R}^d)$ on the fermion Fock space and noncommutative L_p spaces on this algebra after Segal [37]. In particular, hypercontractivity makes perfectly sense in this context by considering the corresponding Ornstein-Uhlenbeck semigroup

$$\mathcal{O}_t := e^{-tN_0} : L_2(\mathcal{C}(\mathbb{R}^d), \tau) \to L_2(\mathcal{C}(\mathbb{R}^d), \tau).$$

Here N_0 denotes the fermion number operator, see Section 1 for the construction of the Clifford algebra $\mathcal{C}(\mathbb{R}^d)$ and a precise definition of the Ornstein-Uhlenbeck semigroup on fermion algebras. After some partial results [15, 27, 28], the optimal time hypercontractivity bound in the fermionic case was finally obtained by Carlen and Lieb in [9]

$$\big\| \mathcal{O}_t : L_p(\mathcal{C}(\mathbb{R}^d)) \to L_q(\mathcal{C}(\mathbb{R}^d)) \big\| = 1 \ \Leftrightarrow \ t \geq \frac{1}{2} \log \frac{q-1}{p-1}.$$

The proof deeply relies on the optimal 2-uniform convexity for matrices from [1].

Beyond its own interest in quantum mechanics, these contributions represent the starting point of hypercontractivity in the noncommutative context. This line was continued by Biane [4], who extended Carlen and Lieb's work and obtained optimal time estimates for the q-Gaussian von Neumann algebras Γ_q introduced by Bozejko, Kümmerer and Speicher [7]. These algebras interpolate between the bosonic and fermonic frameworks, corresponding to $q = \pm 1$. The semigroup for $q = 0$ acts diagonally on free semi-circular variables —instead of free generators as in the case of the free Poisson semigroup— in the context of Voiculescu's free probability theory [39]. We also refer to [19, 21, 24, 25, 26] for related results in this line. On the other hand, the usefulness of the two-point inequality in the context of computer science has motivated some other extensions to the noncommutative setting more focused on its applications to quantum computation and quantum information theory. In [3], the authors studied extensions of Bonami's result to matrix-valued functions $f : \mathbb{Z}_2^n \to M_n(\mathbb{C})$, finding optimal estimates for $q = 2$ and showing some applications to coding theory. In [30], the authors introduced quantum boolean functions and obtained hypercontractivity estimates in this context with some consequences in quantum information theory, see also the recent work [29].

The very nice point here is that, although our main motivation to study the Poisson semigroup comes from harmonic analysis, we realized that a natural way to tackle this problem is by means of studying the Ornstein-Uhlenbeck semigroup on certain von Neumann algebras. In particular, a significant portion of Theorem A follows from our main result, which extends Carlen and Lieb's theorem to the case of free product of Clifford algebras. The precise definitions of reduced free products which appear in the statement will be recalled for the non-expert reader in the body of the paper.

Theorem B. *Let $\mathcal{M}_\alpha = \mathcal{C}(\mathbb{R}^{d_\alpha})$ be the Clifford algebra with d_α generators for any $1 \le \alpha \le n$ and construct the corresponding reduced free product von Neumann algebra $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2 * \cdots * \mathcal{M}_n$. If $\mathcal{O}_\alpha = (\mathcal{O}_{\alpha,t})_{t\ge 0}$ denotes the Ornstein-Uhlenbeck semigroup acting on $\mathcal{M}_\alpha$, consider the free product semigroup $\mathcal{O}_\mathcal{M} = (\mathcal{O}_{\mathcal{M},t})_{t\ge0}$ given by $\mathcal{O}_{\mathcal{M},t} = \mathcal{O}_{1,t} * \mathcal{O}_{2,t} * \cdots * \mathcal{O}_{n,t}$. Then, we find for $1 < p \le q < \infty$*

$$\big\| \mathcal{O}_{\mathcal{M},t} : L_p(\mathcal{M}) \to L_q(\mathcal{M}) \big\| = 1 \quad \Leftrightarrow \quad t \ge \frac{1}{2} \log \frac{q-1}{p-1}.$$

It is relevant to point out a crucial difference between our approach and the one followed in [6, 9, 32]. Indeed, in all those cases the key point in the argument is certain basic inequality —like Bonami's two-point inequality or Ball/Carlen/Lieb's convexity inequality for matrices— and the general result follows from an inductive argument due to the tensor product structure of the problem. However, no tensor product structure can be found in our setting (Theorems A and B). In order to face this problem, Biane showed in [4] that certain optimal hypercontractive estimates hold in the case of spin matrix algebras with mixed commutation/anticommutation relations, and then applied Speicher's central limit theorem [38]. In this paper we will extend Biane's and Speicher's results by showing that a wide range of von Neumann algebras can also be approximated by these spin systems. Namely, the proof of Theorem B will show that the same result can be stated in a much more general context. As we shall explain, we may consider the free product of Biane's mixed spins algebras which in turn gives optimal hypercontractivity estimates for the free products of q-deformed algebras with $q_1, q_2, \ldots, q_n \in [-1, 1]$.

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1. **Preliminaries**

In this section we briefly review the definition of the CAR algebra and the Ornstein-Uhlenbeck semigroup acting on it. We also recall the construction of the reduced free product of a family of von Neumann algebras and introduce the free Ornstein-Uhlenbeck semigroup on a reduced free product of Clifford algebras.

1.1. **The Ornstein-Uhlenbeck semigroup.** The standard way to construct a system of d fermion degrees of freedom is by means of the antisymmetric Fock space. Let us consider the d-dimensional real Hilbert space $\mathcal{H}_\mathbb{R} = \mathbb{R}^d$ and its complexification $\mathcal{H}_\mathbb{C} = \mathbb{C}^d$. Define the Fock space

$$\mathcal{F}(\mathcal{H}_\mathbb{R}) \, = \, \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \mathcal{H}_\mathbb{C}^{\otimes m}$$

for some fixed unit vector $\Omega \in \mathcal{H}_\mathbb{C}$ called the vacuum. If S_m denotes the symmetric group of permutations over $\{1, 2, \ldots, m\}$ and $i(\beta)$ the number of inversions of the permutation β, we define the hermitian form $\langle \cdot, \cdot \rangle$ on $\mathcal{F}(\mathcal{H}_\mathbb{R})$ by $\langle \Omega, \Omega \rangle = 1$ and the following identity

$$\big\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \big\rangle \, = \, \delta_{mn} \sum_{\beta \in S_m} (-1)^{i(\beta)} \langle f_1, g_{\beta(1)} \rangle \cdots \langle f_m, g_{\beta(m)} \rangle.$$

It is not difficult to see that the hermitian form $\langle \cdot, \cdot \rangle$ is non-negative. Therefore, if we consider the completion of the quotient by the corresponding kernel, we obtain a Hilbert space that we will call again $\mathcal{F}(\mathcal{H}_\mathbb{R})$. Let us denote by $(e_j)_{j=1}^d$ the canonical basis of $\mathcal{H}_\mathbb{R} = \mathbb{R}^d$. Then, we define the j-th fermion annihilation operator acting on $\mathcal{F}(\mathcal{H}_\mathbb{R})$ by linearity as $c_j(\Omega) = 0$ and

$$c_j(f_1 \otimes \cdots \otimes f_m) \, = \, \sum_{i=1}^{m} (-1)^{i-1} \langle f_i, e_j \rangle \, f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_m.$$

Its adjoint c_j^* is called the j-th fermion creation operator on $\mathcal{F}(\mathcal{H}_\mathbb{R})$. It is determined by $c_j^*(\Omega) = e_j$ and $c_j^*(f_1 \otimes \cdots \otimes f_m) = e_j \otimes f_1 \otimes \cdots \otimes f_m$. It is quite instrumental to observe that $c_i c_j + c_j c_i = 0$ and $c_i c_j^* + c_j^* c_i = \delta_{ij} \mathbf{1}$. The basic free Hamiltonian on $\mathcal{F}(\mathcal{H}_\mathbb{R})$ is the fermion number operator

$$N_0 = \sum_{j=1}^{d} c_j^* c_j.$$

It generates the fermion oscillator semigroup $(\exp(-tN_0))_{t \geq 0}$. Then, one defines the configuration operators $x_j = c_j + c_j^*$ for $1 \leq j \leq d$. Denote by $\mathcal{C}(\mathbb{R}^d)$ the unit algebra generated by them. Note that these operators verify the canonical anti-commutation relations (CAR)

$$x_i x_j + x_j x_i = 2\delta_{ij} \qquad \text{and} \qquad x_j^* = x_j.$$

It is well-known that $\mathcal{C}(\mathbb{R}^d)$ can be concretely represented as a subalgebra of the matrix algebra $\mathbb{M}_{2^d}$ by considering d-chains formed by tensor products of Pauli matrices. The key point for us is that the 2^d distinct monomials in the x_j's define a basis of $\mathcal{C}(\mathbb{R}^d)$ as a vector space. Indeed, given any subset A of $[d] := \{1, 2, \ldots, d\}$ we shall write $x_A = x_{j_1} x_{j_2} \cdots x_{j_s}$ where $(j_1, j_2, \ldots, j_s)$ is an enumeration of A in increasing order. If we also set $x_\emptyset = \mathbf{1}$, it turns out that $\{x_A \mid A \subset [d]\}$ is a linear basis of $\mathcal{C}(\mathbb{R}^d)$. In particular, any $X \in \mathcal{C}(\mathbb{R}^d)$ has the form

$$X \, = \, \alpha_\emptyset \mathbf{1} \, + \sum_{s=1}^d \sum_{1 \le j_1 < \cdots < j_s \le d} \alpha_{j_1, \ldots, j_s} x_{j_1} \cdots x_{j_s}.$$

The vacuum Ω defines a tracial state τ on $\mathcal{C}(\mathbb{R}^d)$ by $\tau(X) = \langle X\Omega, \Omega\rangle$. We denote by $L_p(\mathcal{C}(\mathbb{R}^d), \tau)$ or just $L_p(\mathcal{C}(\mathbb{R}^d))$ the associated non-commutative L_p-space. The map $X \mapsto X\Omega$ defines a continuous embedding of $\mathcal{C}(\mathbb{R}^d)$ into $\mathcal{F}(\mathbb{R}^d)$ which extends to a unitary isomorphism $L_2(\mathcal{C}(\mathbb{R}^d)) \simeq \mathcal{F}(\mathbb{R}^d)$. Then, instead of working on the Fock space $\mathcal{F}(\mathbb{R}^d)$ and with the semigroup $\exp(-tN_0)$, we can equivalently consider $\mathcal{C}(\mathbb{R}^d)$ and the Ornstein-Uhlenbeck semigroup on $\mathcal{C}(\mathbb{R}^d)$ defined by

$$\mathcal{O}_t(X) \, = \, \alpha_\emptyset \mathbf{1} \, + \sum_{s=1}^d e^{-ts} \sum_{1 \le j_1 < \cdots < j_s \le d} \alpha_{j_1, \ldots, j_s} x_{j_1} \cdots x_{j_s}.$$

If $1 < p \le q < \infty$, the main result in [9] yields

$$\big\| \mathcal{O}_t : L_p(C(\mathbb{R}^d)) \to L_q(C(\mathbb{R}^d)) \big\| \, = \, 1 \ \Leftrightarrow \ t \ge \frac{1}{2} \log \frac{q-1}{p-1}.$$

1.2. **Free product of von Neumann algebras.** Let $(A_j, \phi_j)_{j \in J}$ be a family of unital C*-algebras with distinguished states ϕ_j whose GNS constructions $(\pi_j, \mathcal{H}_j, \xi_j)$ with $\mathcal{H}_j = L_2(A_j, \phi_j)$ are faithful. Let us define

$$\mathring{A}_j = \Big\{ a \in A_j \, \big| \, \phi_j(a) = 0 \Big\} \qquad \text{and} \qquad \mathring{\mathcal{H}}_j = \xi_j^\perp$$

so that $A_j = \mathbb{C}\mathbf{1} \oplus \mathring{A}_j$ and $\mathcal{H}_j = \mathbb{C}\xi_j \oplus \mathring{\mathcal{H}}_j$. Note that we have natural maps $i_j = A_j \to \mathcal{H}_j$ such that $\phi_j(a^*b) = \langle i_j(a), i_j(b)\rangle_{\mathcal{H}_j}$ for every $j \in J$. Let us consider the full Fock space associated to the free product

$$\mathcal{F} \, = \, \mathbb{C}\Omega \oplus \bigoplus_{\substack{m \ge 1 \\ j_1 \ne j_2 \ne \cdots \ne j_m}} \mathring{\mathcal{H}}_{j_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{j_m}$$

with inner product

$$\big\langle h_1 \otimes \cdots \otimes h_m, h_1' \otimes \cdots \otimes h_n' \big\rangle \, = \, \delta_{mn} \prod_{j=1}^m \langle h_j, h_j' \rangle.$$

Each algebra A_j acts non-degenerately on $\mathcal{F}$ via the map $\omega_j : A_j \to \mathcal{B}(\mathcal{F})$ in the following manner. Since we can decompose every $z \in A_j$ as $z = \phi_j(z)\mathbf{1} + a$ with $\phi_j(a) = 0$, it suffices to define $\omega_j(a)$. Let $h_1 \otimes \cdots \otimes h_m$ be a generic element in $\mathcal{F}$ with $h_i \in \mathcal{H}_{j_i} \ominus \mathbb{C}\xi_{j_i}$. If $j \ne j_1$, we set

$$\omega_j(a)\big(h_1 \otimes \cdots \otimes h_m\big) \, = \, i_j(a) \otimes h_1 \otimes \cdots \otimes h_m.$$

When $j = j_1$ we add and subtract the mean to obtain

$$\begin{aligned}\omega_j(a)\big(h_1 \otimes \cdots \otimes h_m\big) &= \big\langle \xi_j, \pi_j(a)(h_1)\big\rangle_{\mathcal{H}_j} h_2 \otimes \cdots \otimes h_m \\ &+ \Big(\pi_j(a)(h_1) - \big\langle \xi_j, \pi_j(a)(h_1)\big\rangle_{\mathcal{H}_j} \xi_j\Big) \otimes h_2 \otimes \cdots \otimes h_m.\end{aligned}$$

The faithfulness of the GNS construction of (A_j, ϕ_j) implies that the representation ω_j is faithful for every $j \in J$. Thus, we may find a copy of the algebraic free product

$$A \,=\, \mathbb{C}\Omega \oplus \bigoplus_{\substack{m \geq 1 \\ j_1 \neq j_2 \neq \cdots \neq j_m}} \overset{\circ}{A}_{j_1} \otimes \cdots \otimes \overset{\circ}{A}_{j_m}$$

in $\mathcal{B}(\mathcal{F})$. The reduced free product of the family $(A_j, \phi_j)_{j \in J}$ is the C*-algebra generated by these actions. In other words, the norm closure of A in $\mathcal{B}(\mathcal{F})$. It is denoted by

$$(A, \phi) \,=\, *_{j \in J}(A_j, \phi_j),$$

where the state ϕ on A is given by

$$\phi(\mathbf{1}) \,=\, 1 \qquad \text{and} \qquad \phi(a_1 \otimes \cdots \otimes a_m) = 0$$

for $m \geq 1$ and $a_i \in \overset{\circ}{A}_{j_i}$ with $j_1 \neq j_2 \neq \cdots \neq j_m$. Each A_j is naturally considered as a subalgebra of A and the restriction of ϕ to A_j coincides with ϕ_j. It is helpful to think of the elementary tensors above $a_1 \otimes \cdots \otimes a_m$ as words of length m, where the empty word Ω has length 0. In this sense, a word $a_1 \otimes \cdots \otimes a_m$ can be identified with the product $a_1 a_2 \cdots a_m$ via the formula $a_1 \cdots a_m \Omega = a_1 \otimes \cdots \otimes a_m$.

This construction also holds in the category of von Neumann algebras. Let $(\mathcal{M}_j, \phi_j)_{j \in J}$ be a family of von Neumann algebras with distinguished states ϕ_j whose GNS constructions $(\pi_j, \mathcal{H}_j, \xi_j)$ are faithful. Then, the corresponding reduced free product von Neumann algebra is the weak-$*$ closure of $*_{j \in J}(\mathcal{M}_j, \phi_j)$ in $\mathcal{B}(\mathcal{F})$ which will be denoted by $(\mathcal{M}, \phi) \,=\, \overline{*}_{j \in J}(\mathcal{M}_j, \phi_j)$. As before, the $\mathcal{M}_j$'s are regarded as von Neumann subalgebras of $\mathcal{M}$ and the restriction of ϕ to $\mathcal{M}_j$ coincides with ϕ_j. A more complete explanation of the reduced free product of von Neumann algebras can be found in [39]. Let us now consider a family $(\Lambda_j : \mathcal{M}_j \to \mathcal{M}_j)_{j \in J}$ of normal, completely positive, unital and trace preserving maps. Then, it is known from [5, Theorem 3.8] that there exists a map $\Lambda \,=\, *_{j \in J}\Lambda_j : \mathcal{M} \to \mathcal{M}$ such that $\Lambda(x_1 x_2 \cdots x_m) = \Lambda_{j_1}(x_1) \cdots \Lambda_{j_m}(x_m)$, whenever $x_i \in \mathcal{M}_{j_i}$ is trace 0 and $j_i \neq j_{i+1}$ for $1 \leq i \leq m-1$. This map is called the free product map of the Λ_j's. In particular we may take $\mathcal{M}_j = \mathcal{C}(\mathbb{R}^d)$ for $1 \leq j \leq n$ and $\Lambda_j = \mathcal{O}_{j,t}$, the Ornstein-Uhlenbeck semigroup on $\mathcal{M}_j$ at time t. The resulting free product maps $\mathcal{O}_{\mathcal{M}} = (\mathcal{O}_{\mathcal{M},t})_{t \geq 0}$ with $\mathcal{O}_{\mathcal{M},t} = \mathcal{O}_{1,t} * \mathcal{O}_{2,t} * \cdots * \mathcal{O}_{n,t}$ will be referred to as the *free Ornstein-Uhlenbeck semigroup* on the reduced free product von Neumann algebra $\mathcal{M}$.

2. **The free Ornstein-Uhlenbeck semigroup**

This section is devoted to the proof of Theorem B. Of course, we may and will assume for simplicity that $d_\alpha = d$ for all $1 \leq \alpha \leq n$. The key idea is to describe the free product of fermion algebras and the corresponding Ornstein-Uhlenbeck semigroup as the limit objects of certain spin matrix models and certain semigroups defined on them. In this sense, we will extend Biane's results [4] by showing that these matrix models can be used to describe a wide range of operator algebra frameworks.

Note that the free Ornstein-Uhlenbeck semigroup restricted to a single free copy $\mathcal{M}_\alpha$ coincides with the fermion oscillator semigroup on $\mathcal{M}_\alpha$. In particular, we know from Carlen and Lieb's theorem [9] that the optimal time in Theorem B must be greater than or equal to $\frac{1}{2}\log\frac{q-1}{p-1}$. This proves the necessity, it remains to prove the sufficiency. Given $1 \le \alpha \le n$ and recalling that $[d]$ stands for $\{1,2,\ldots,d\}$, we denote by $(x_i^\alpha)_{i\in[d]}$ the generators of $\mathcal{M}_\alpha = \mathcal{C}(\mathbb{R}^d)$. A reduced word in the free product $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2 * \cdots * \mathcal{M}_n$ is then of the form

$$x \,=\, x_{A_1}^{\alpha_1}\cdots x_{A_\ell}^{\alpha_\ell} \tag{2.1}$$

with $A_j \subset [d]$ and $\alpha_j \neq \alpha_{j+1}$. The case $\ell = 0$ refers to the empty word $\mathbf{1}$. If we set $s_j = |A_j|$ and write $A_j = \{i_{s_1+\cdots+s_{j-1}+1},\ldots,i_{s_1+\cdots+s_{j-1}+s_j}\}$ —labeling the indices in a strictly increasing order— x can be written as follows

$$x \,=\, \overbrace{x_{i_1}^{\alpha_1}\cdots x_{i_{s_1}}^{\alpha_1}}^{x_{A_1}^{\alpha_1}}\,\overbrace{x_{i_{s_1+1}}^{\alpha_2}\cdots x_{i_{s_1+s_2}}^{\alpha_2}}^{x_{A_2}^{\alpha_2}}\cdots\overbrace{x_{i_{s_1+\cdots+s_{\ell-1}+1}}^{\alpha_\ell}\cdots x_{i_{s_1+\cdots+s_\ell}}^{\alpha_\ell}}^{x_{A_\ell}^{\alpha_\ell}}. \tag{2.2}$$

In what follows, we will use the notation $|x| = |A_1| + \cdots + |A_\ell| = s_1 + \cdots + s_\ell$.

2.1. **Spin matrix model.** Given $m \ge 1$, we will describe a spin system with mixed commutation and anticommutation relations which approximates the free product of fermions $\mathcal{M}$ as $m \to \infty$. Let us first recall the construction of a spin algebra in general. In our setting, we will need to consider three indices. This is why we introduce the sets $\Upsilon = [n]\times[d]\times\mathbb{Z}_+$ and $\Upsilon_m = [n]\times[d]\times[m]$ for $m \ge 1$. Let $\varepsilon : \Upsilon\times\Upsilon \to \{-1,1\}$ be any map satisfying

- ε is symmetric: $\varepsilon(x,y) = \varepsilon(y,x)$,
- $\varepsilon \equiv -1$ on the diagonal: $\varepsilon(x,x) = -1$.

Given $m \ge 1$, we will write ε_m to denote the truncation of ε to $\Upsilon_m\times\Upsilon_m$. Consider the complex unital algebra $\mathcal{A}_{\varepsilon_m}$ generated by the elements $(x_i^\alpha(k))_{(\alpha,i,k)\in\Upsilon_m}$ which satisfy the commutation/anticommutation relations

$$x_i^\alpha(k)x_j^\beta(\ell) - \varepsilon\big((\alpha,i,k),(\beta,j,\ell)\big)x_j^\beta(\ell)x_i^\alpha(k) \,=\, 2\delta_{(\alpha,i,k),(\beta,j,\ell)} \tag{2.3}$$

for $(\alpha,i,k),(\beta,j,\ell)\in\Upsilon_m$. We endow $\mathcal{A}_{\varepsilon_n}$ with the antilinear involution such that $x_i^\alpha(k)^* = x_i^\alpha(k)$ for every $(\alpha,i,k)\in\Upsilon_m$. If we equip Υ_m with the lexicographical order, a basis of the linear space $\mathcal{A}_{\varepsilon_m}$ is given by $x_\emptyset^{\varepsilon_m} = \mathbf{1}_{\mathcal{A}_{\varepsilon_m}}$ and the set of reduced words written in increasing order. Namely, elements of the form

$$x_A^{\varepsilon_m} \,=\, x_{i_1}^{\alpha_1}(k_1)\cdots x_{i_s}^{\alpha_s}(k_s),$$

where $A = \{(\alpha_1,i_1,k_1),\ldots,(\alpha_s,i_s,k_s)\}\subset\Upsilon_m$ with $(\alpha_j,i_j,k_j) < (\alpha_{j+1},i_{j+1},k_{j+1})$ for $1 \le j \le s-1$. For any such element we set $|x_A^{\varepsilon_m}| = |A| = s$. Define the tracial state on $\mathcal{A}_{\varepsilon_m}$ given by $\tau_{\varepsilon_m}(x_A^{\varepsilon_m}) = \delta_{\emptyset,A}$ for $A\subset\Upsilon_m$. The given basis turns out to be orthonormal with respect to the inner product $\langle x,y\rangle = \tau_{\varepsilon_m}(x^*y)$. Let $\mathcal{A}_{\varepsilon_m}$ act by left multiplication on the Hilbert space $\mathcal{H}_{\mathcal{A}_{\varepsilon_m}} = (\mathcal{A}_{\varepsilon_m},\langle\cdot,\cdot\rangle)$ to get a faithful $*$-representation of $\mathcal{A}_{\varepsilon_m}$ on $\mathcal{H}_{\mathcal{A}_{\varepsilon_m}}$. We may endow $\mathcal{A}_{\varepsilon_m}$ with the von Neumann algebra structure induced by this representation and denote by $L_p(\mathcal{A}_{\varepsilon_m},\tau_{\varepsilon_m})$ the associated non-commutative L_p-space. At this point, it is natural to define the ε_m*-Ornstein-Uhlenbeck semigroup* on $\mathcal{A}_{\varepsilon_m}$ by

$$\mathcal{S}_{\varepsilon_m,t}(x_A^{\varepsilon_m}) \,=\, e^{-t|x_A^{\varepsilon_m}|} x_A^{\varepsilon_m}. \tag{2.4}$$

Biane extended hypercontractivity for fermions to these spin algebras in [4]

$$\big\|\mathcal{S}_{\varepsilon_m,t} : L_p(\mathcal{A}_{\varepsilon_m}) \to L_q(\mathcal{A}_{\varepsilon_m})\big\| \,=\, 1 \;\Leftrightarrow\; t \,\geq\, \frac{1}{2}\log\frac{q-1}{p-1}, \tag{2.5}$$

whenever $1 < p \leq q < \infty$. We will also use the following direct consequence of Biane's result. Namely, given $1 \leq p < \infty$ and $r \in \mathbb{Z}_+$ we may find constants $C_{p,r} > 0$ such that the following inequality holds uniformly for all $m \geq 1$ and all homogeneous polynomials P of degree r in $|\Upsilon_m|$ noncommutative indeterminates satisfying (2.3) and written in reduced form

$$\Big\|P\big((x_i^\alpha(k))_{(\alpha,i,k)\in\Upsilon_m}\big)\Big\|_{L_p(\mathcal{A}_{\varepsilon_m})} \,\leq\, C_{p,r}\Big\|P\big((x_i^\alpha(k))_{(\alpha,i,k)\in\Upsilon_m}\big)\Big\|_{L_2(\mathcal{A}_{\varepsilon_m})}. \tag{2.6}$$

According to (2.5), it is straightforward to show that we can take $C_{p,r} = (p-1)^{r/2}$.

2.2. **A central limit theorem.** In order to approximate the free product $\mathcal{M}$ of Clifford algebras, we need to choose the commutation/anticommutation relations randomly. More precisely, we consider a probability space (Ω,μ) and a family of independent random variables

$$\varepsilon\big((\alpha,i,k),(\beta,j,\ell)\big) : \Omega \to \{-1,1\} \quad \text{for} \quad (\alpha,i,k) < (\beta,j,\ell)$$

which are distributed as follows

$$\mu\Big(\varepsilon\big((\alpha,i,k),(\beta,j,\ell)\big) = -1\Big) \,=\, \begin{cases} 1 & \text{if } \alpha = \beta, \\ 1/2 & \text{if } \alpha \neq \beta. \end{cases} \tag{2.7}$$

In particular, this means that all the generators $(x_i^\alpha(k))_{i\in[d],k\in[m]}$ anticommute for $\alpha \in [n]$ fixed and all $m \geq 1$. Therefore, the algebra $\mathcal{A}_{\varepsilon_m}^\alpha$ generated by them is isomorphic to $\mathcal{C}(\mathbb{R}^{dm})$. Formally, we have a matrix model for each $\omega \in \Omega$. In this sense, the generators $x_i^\alpha(k)$ and the algebras $\mathcal{A}_{\varepsilon_m}^\alpha$ are also functions of ω. In order to simplify the notation, we will not specify this dependence unless it is necessary for clarity in the exposition. Define also the algebra

$$\tilde{\mathcal{A}}_{\varepsilon_m}^\alpha \,=\, \Big\langle \tilde{x}_i^\alpha(m) \;\big|\; i \in [d] \Big\rangle$$

with generators given by

$$\tilde{x}_i^\alpha(m) \,=\, \frac{1}{\sqrt{m}}\sum_{k=1}^m x_i^\alpha(k).$$

Lemma 2.1. *The von Neumann algebra $\tilde{\mathcal{A}}_{\varepsilon_m}^\alpha$ is canonically isomorphic to $\mathcal{C}(\mathbb{R}^d)$.*

Proof. It suffices to prove that the generators verify the CAR relations. All of them are self-adjoint since the same holds for the x_i^α's. Since α is fixed, our choice (2.7) of the sign function ε and (2.3) give

$$\tilde{x}_i^\alpha(m)\tilde{x}_j^\alpha(m) + \tilde{x}_j^\alpha(m)\tilde{x}_i^\alpha(m) \,=\, \frac{1}{m}\sum_{k=1}^m\sum_{\ell=1}^m x_i^\alpha(k)x_j^\alpha(\ell) + x_j^\alpha(\ell)x_i^\alpha(k) \,=\, 2\delta_{ij}. \quad \square$$

We will denote by $\Pi(s)$ the set of all partitions of $[s] = \{1, 2, \dots, s\}$. Given $\sigma, \pi \in \Pi(s)$, we will write $\sigma \le \pi$ if every block of the partition σ is contained in some block of π. We denote by σ_0 the smallest partition, in which every block is a singleton. Given an s-tuple $\underline{i} = (i_1, \cdots, i_s) \in [N]^s$ for some N, we can define the partition $\sigma(\underline{i})$ associated to $\underline{i}$ by imposing that two elements $j, k \in [s]$ belong to the same block of $\sigma(\underline{i})$ if and only if $i_j = i_k$. We will denote by $\Pi_2(s)$ the set of all pair partitions. That is, partitions $\sigma = \{V_1, \cdots, V_{s/2}\}$ such that $|V_j| = 2$ for every block V_j. In this case, we will write $V_j = \{e_j, z_j\}$ with $e_j < z_j$ so that $e_1 < e_2 < \cdots < e_{s/2}$. For a pair partition $\sigma \in \Pi_2(s)$ we define the set of crossings of σ by

$$I(\sigma) \, = \, \Big\{ (k, \ell) \, \Big| \, 1 \le k, \ell \le s, \; e_k < e_\ell < z_k < z_\ell \Big\}.$$

Moreover, given an s-tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ such that $\sigma \le \sigma(\underline{\alpha})$, we can define the set of crossings of σ with respect to $\underline{\alpha}$ by $I_{\underline{\alpha}}(\sigma) = \{(k, \ell) \in I(\sigma) : \alpha_{e_k} \neq \alpha_{e_\ell}\}$. This notation allows us to describe the moments of reduced words in $\mathcal{M}$ with a simple formula. Indeed, the following lemma arises from [38, Lemma 2] and a simple induction argument like the one used below to prove identity (2.9).

Lemma 2.2. *If $\underline{i} \in [d]^s$ and $\underline{\alpha} \in [n]^s$ we have*

$$\tau\big(x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}\big) \, = \, \delta_{s \in 2\mathbb{Z}} \sum_{\substack{\sigma \in \Pi_2(s) \\ \sigma \le \sigma(\underline{i}), \sigma(\underline{\alpha}) \\ I_{\underline{\alpha}}(\sigma) = \emptyset}} (-1)^{\# I(\sigma)}.$$

We can now prove that the moments of the free product von Neumann algebra $\mathcal{M}$ are the almost everywhere limit of the moments of our matrix model. More explicitly, we find the following central limit type theorem.

Theorem 2.3. *If $\underline{i} \in [d]^s$ and $\underline{\alpha} \in [n]^s$ we have*

$$\lim_{m \to \infty} \tau_{\varepsilon_m} \Big(\tilde{x}_{i_1}^{\alpha_1}(m)(\omega) \cdots \tilde{x}_{i_s}^{\alpha_s}(m)(\omega) \Big) \, = \, \tau\big(x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}\big) \quad a.e.$$

Proof. We will first prove that the convergence holds in expectation. For $\omega \in \Omega$ fixed, by developing and splitting the sum according to the distribution we obtain

$$\begin{aligned} (2.8) \qquad & \tau_{\varepsilon_m} \Big(\tilde{x}_{i_1}^{\alpha_1}(m)(\omega) \cdots \tilde{x}_{i_s}^{\alpha_s}(m)(\omega) \Big) \\ & = \frac{1}{m^{s/2}} \sum_{\underline{k} \in [m]^s} \tau_{\varepsilon_m} \big(x_{i_1}^{\alpha_1}(k_1)(\omega) \cdots x_{i_s}^{\alpha_m}(k_s)(\omega) \big) \\ & = \frac{1}{m^{s/2}} \sum_{\sigma \in \Pi(s)} \underbrace{\sum_{\substack{\underline{k} \in [m]^s \\ \sigma(\underline{k}) = \sigma}} \tau_{\varepsilon_m} \big(x_{i_1}^{\alpha_1}(k_1)(\omega) \cdots x_{i_s}^{\alpha_s}(k_s)(\omega) \big)}_{\mu_\sigma(\omega)}. \end{aligned}$$

We claim that

$$\lim_{m \to \infty} \frac{1}{m^{s/2}} \mu_\sigma(\omega) \, = \, 0$$

for every $\sigma \in \Pi(s) \backslash \Pi_2(s)$ and all $\omega \in \Omega$. Indeed, the upper bound $\mu_\sigma(\omega) \le m^r$ holds when σ has r blocks since $|\tau_{\varepsilon_m}(x_{i_1}^{\alpha_1}(k_1)(\omega) \cdots x_{i_s}^{\alpha_s}(k_s)(\omega))| \le 1$. Hence, the limit above vanishes for $r < s/2$. It then suffices to show that the same limit vanishes when σ contains a singleton $\{j_0\}$. However, in this case we have

$$\tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega)\big) \;=\; 0$$

whenever $\sigma(\underline{k}) = \sigma$ since the j_0-th term can not be cancelled. This proves our claim. Hence, the only partitions which may contribute in the sum (2.8) are pair partitions $\sigma = \{\{e_1, z_1\}, \ldots, \{e_{\frac{s}{2}}, z_{\frac{s}{2}}\}\}$. In particular, if s is odd we immediately obtain that the trace converges to zero in (2.8). Note that given such a pair partition σ, we must have that $\sigma \leq \sigma(\underline{\alpha})$ and $\sigma \leq \sigma(\underline{i})$. Indeed, if this is not the case we will have $i_{e_j} \neq i_{z_j}$ or $\alpha_{e_j} \neq \alpha_{z_j}$ for some $j = 1, 2, \ldots, s/2$. Now, for every $\underline{k} \in [m]^s$ such that $\sigma(\underline{k}) = \sigma$ we have $k_{e_j} = k_{z_j} \neq k_\ell$ for every $\ell \neq e_j, z_j$. Thus, the only way for the elements

$$x_{i_{e_j}}^{\alpha_{e_j}}(k_{e_j})(\omega) \quad \text{and} \quad x_{i_{z_j}}^{\alpha_{z_j}}(k_{z_j})(\omega)$$

to cancel is to match each other. Thus, we can assume that $(\alpha_{e_j}, i_{e_j}) = (\alpha_{z_j}, i_{z_j})$.

We have seen that the letters of our word should match in pairs. We are now reduced to study the sign which arises from the commutation/anticommutation relations to cancel all elements. Assume that σ has a crossing with respect to $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$. That is, there exists $(k, \ell) \in I(\sigma)$ such that $\alpha_{e_k} \neq \alpha_{e_\ell}$. Then we find that

$$\mathbb{E}_\omega \tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega)\big) \;=\; 0$$

for every $(k_1, \ldots, k_s)$ such that $\sigma(k_1, \ldots, k_s) = \sigma$. Indeed, define the sign function

$$\varepsilon_{(k,\ell)}^{\underline{\alpha}} := \varepsilon\big((\alpha_{e_\ell}, i_{e_\ell}, k_{e_\ell}), (\alpha_{z_k}, i_{z_k}, k_{z_k})\big).$$

If σ has such a crossing, we obtain (among others) this sign only once when canceling the letters associated to $(\alpha_{e_k}, i_{e_k}, k_{e_k})$ and $(\alpha_{z_k}, i_{z_k}, k_{z_k})$ as well as $(\alpha_{e_\ell}, i_{e_\ell}, k_{e_\ell})$ and $(\alpha_{z_\ell}, i_{z_\ell}, k_{z_\ell})$. Furthermore, by independence and since $\mathbb{E}_\omega \varepsilon_{(k,\ell)}^{\underline{\alpha}} = 0$ we get

$$\begin{aligned}
&\mathbb{E}_\omega \tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_1)(\omega)\big) \\
&\qquad = \;\; \pm\,\mathbb{E}_\omega\Big(\prod_{(k,\ell)\in I_{\underline{\alpha}}(\sigma)} \varepsilon_{(k,\ell)}^{\underline{\alpha}}\Big) \;=\; \pm \prod_{(k,\ell)\in I_{\underline{\alpha}}(\sigma)} \mathbb{E}_\omega \varepsilon_{(k,\ell)}^{\underline{\alpha}} = 0,
\end{aligned}$$

where $\pm$ denotes a possible change of signs depending on the crossings of σ. Then, we can also rule out these kind of partitions and we can assume that $\sigma \in \Pi_2(s)$ is such that $\sigma \leq \sigma(\underline{i}), \sigma(\underline{\alpha})$ and $I_{\underline{\alpha}}(\sigma) = \emptyset$. In this case, we do not need to commute two letters (α, i, k) and (β, j, ℓ) with $\alpha \neq \beta$. Hence we will obtain deterministic signs coming from the commutations, which only depend on the number of crossings of σ. More precisely, given $\sigma \in \Pi_2(s)$ satisfying the properties above and $\underline{k} \in [m]^s$ such that $\sigma(\underline{k}) = \sigma$ we have

$$\tau_{\varepsilon_m}\Big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega)\Big) = (-1)^{\# I(\sigma)} \quad \text{for every} \quad \omega. \tag{2.9}$$

Indeed, this can be proved inductively as follows. Using that $I_{\underline{\alpha}}(\sigma) = \emptyset$, there must exists a connected block of consecutive numbers in $[s]$ so that the following properties hold

- The letters in that block are related to a fixed α.
- The product of the letters in that block equals $\pm\mathbf{1}$.
- The block itself is a union of pairs of the partition $\sigma \in \Pi_2(s)$.

If π denotes the restriction of σ to our distinguished block —well defined by the third property— the sign given by the second property equals $(-1)^{\# I(\pi)}$. After canceling this block of letters, we may start again by noticing that $I_{\underline{\beta}}(\sigma \setminus \pi) = \emptyset$ where $\underline{\beta}$ is the restriction of $\underline{\alpha}$ to the complement of our distinguished block. This allows to restart the process. In the end we obtain $(-1)^{\# I(\sigma)}$ as desired. We deduce that

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{E}_\omega \tau_{\varepsilon_m}\Big(\tilde{x}_{i_1}^{\alpha_1}(m)(\omega)\cdots\tilde{x}_{i_s}^{\alpha_s}(m)(\omega)\Big)\\
&\quad = \lim_{m\to\infty} \frac{1}{m^{s/2}} \mathbb{E}_\omega \sum_{\substack{\sigma\in\Pi_2(s)\\ \sigma\le\sigma(\underline{i}),\sigma(\underline{\alpha})\\ I_{\underline{\alpha}}(\sigma)=\emptyset}} \sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma}} (-1)^{\# I(\sigma)} \ = \sum_{\substack{\sigma\in\Pi_2(s)\\ \sigma\le\sigma(\underline{i}),\sigma\le\sigma(\underline{\alpha})\\ I_{\underline{\alpha}}(\sigma)=\emptyset}} (-1)^{\# I(\sigma)}.
\end{aligned}$$

Here we have used that

$$\lim_{m\to\infty} \frac{|\{\underline{k}\in[m]^s : \sigma(\underline{k})=\sigma\}|}{m^{s/2}} = \lim_{m\to\infty} \frac{m(m-1)\cdots(m-\frac{s}{2}+1)}{m^{s/2}} = 1.$$

By Lemma 2.2, this proves convergence in expectation and completes the first part of the proof. It remains to prove almost everywhere convergence in ω. Let us define the random variables

$$X_m(\omega) = \tau_{\varepsilon_m}\Big(\tilde{x}_{i_1}^{\alpha_1}(m)\cdots\tilde{x}_{i_s}^{\alpha_s}(m)\Big).$$

By the dominated convergence theorem, it suffices to show

$$\lim_{m\to\infty} \mu\Big(\big\{\sup_{M\ge m} \big|X_M - \mathbb{E}_\omega[X_M]\big| \ge \alpha\big\}\Big) = 0$$

for every $\alpha>0$. According to Tchebychev's inequality, we find

$$\mu\Big(\big\{\sup_{M\ge m} \big|X_M - \mathbb{E}_\omega[X_M]\big| \ge \alpha\big\}\Big) \le \frac{1}{\alpha^2}\sum_{M=m}^{\infty} V[X_M],$$

where $V[X_M] = \mathbb{E}_\omega[X_M^2] - (\mathbb{E}_\omega[X_M])^2$ denotes the variance of X_M. We will prove the upper bound $V[X_M] \le C(s)/M^2$ for every M, for some contant $C(s)$ depending only on the length s. This will suffice to conclude the argument. To this end we write

$$V[X_M] = \frac{1}{M^s} \sum_{\sigma,\pi\in\Pi(s)} \sum_{\substack{\underline{k}\,:\,\sigma(\underline{k})=\sigma\\ \underline{\ell}\,:\,\sigma(\underline{\ell})=\pi}} D_{\underline{k},\underline{\ell}}, \tag{2.10}$$

where

$$\begin{aligned}
D_{\underline{k},\underline{\ell}} &= \mathbb{E}_\omega\Big[\tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega)\big)\tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(\ell_1)(\omega)\cdots x_{i_s}^{\alpha_s}(\ell_s)(\omega)\big)\Big]\\
&- \mathbb{E}_\omega\Big[\tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega)\big)\Big]\mathbb{E}_\omega\Big[\tau_{\varepsilon_m}\big(x_{i_1}^{\alpha_1}(\ell_1)(\omega)\cdots x_{i_s}^{\alpha_s}(\ell_s)(\omega)\big)\Big]
\end{aligned}$$

for $\underline{k}=(k_1,\ldots,k_s)$ and $\underline{\ell}=(\ell_1,\ldots,\ell_s)$. Now, reasoning as above one can see that whenever σ or π has a singleton, all the corresponding terms in the sum (2.10) are equal to zero. Thus, we may write $\sigma=\{V_1,\ldots,V_{r_\sigma}\}$ and $\pi=\{W_1,\ldots,W_{r_\pi}\}$ with $r_\sigma, r_\pi \le \frac{s}{2}$. If neither σ nor π are pair partitions, we will have $r_\sigma, r_\pi \le \frac{s}{2}-1$ and the part of the sum in (2.10) corresponding to these pairs (σ,π) can be bounded above in absolute value by $C(s)/M^2$ as desired. Then, it remains to control the rest of the terms in (2.10). To this end, we assume that σ is a pair partition. Actually, a cardinality argument as before allows us to conclude that π must be either a pair partition or a partition with all blocks formed by two elements up to a possible four element block. In the following, we will explain how to deal with the case in which π is a pair partition. The other case can be treated exactly in the same way, being actually even easier by cardinality reasons. Let us fix two pair partitions σ and π and let us consider $\underline{k} = (k_1, \ldots, k_s)$ and $\underline{\ell} = (\ell_1, \ldots, \ell_s)$ such that $\sigma(\underline{k}) = \sigma$ and $\sigma(\underline{\ell}) = \pi$. When rearranging the letters in the traces defining $D_{\underline{k},\underline{\ell}}$, the deterministic signs —$\alpha = \beta$ in (2.7)— do not have any effect in the absolute value of $D_{\underline{k},\underline{\ell}}$. On the other hand, the random signs —$\alpha \neq \beta$ in (2.7)— makes the second term of $D_{\underline{k},\underline{\ell}}$ vanish. Thus, $D_{\underline{k},\underline{\ell}} \neq 0$ if and only if $I_{\underline{\alpha}}(\sigma) \neq \emptyset \neq I_{\underline{\alpha}}(\pi)$ and we obtain the same random signs coming from crossings in $I_{\underline{\alpha}}(\sigma)$ and $I_{\underline{\alpha}}(\pi)$. In particular, we should find at least two signs

$$\begin{aligned} &\varepsilon\big((\alpha_p, i_p, k_p), (\alpha_q, i_q, k_q)\big)(\omega) \quad (\alpha_p \neq \alpha_q) \quad \text{from} \quad x_{i_1}^{\alpha_1}(k_1)(\omega) \cdots x_{i_s}^{\alpha_s}(k_s)(\omega), \\ &\varepsilon\big((\alpha_u, i_u, \ell_u), (\alpha_v, i_v, \ell_v)\big)(\omega) \quad (\alpha_u \neq \alpha_v) \quad \text{from} \quad x_{i_1}^{\alpha_1}(\ell_1)(\omega) \cdots x_{i_s}^{\alpha_s}(\ell_s)(\omega). \end{aligned}$$

By independence, this implies that

$$\big\{(\alpha_p, i_p, k_p), (\alpha_q, i_q, k_q)\big\} \, = \, \big\{(\alpha_u, i_u, \ell_u), (\alpha_v, i_v, \ell_v)\big\}.$$

Moreover, since we also need $\sigma \leq \sigma(\underline{\alpha})$ for non-vanishing terms, we can conclude that $k_p \neq k_q$ and $\ell_u \neq \ell_v$. Therefore, the sets $\{k_1, \ldots, k_s\}$ and $\{\ell_1, \ldots, \ell_s\}$ must have four elements (corresponding to two different blocks) in common. This implies that the part of the sum in (2.10) corresponding to pairs (σ, π) of pair partitions is bounded above by

$$C'(s) \frac{M^{s/2} M^{(s-4)/2}}{M^s} \, = \, \frac{C'(s)}{M^2}$$

for a certain constant $C(s)'$ as we wanted. This completes the proof. □

Let x be a word in the reduced free product of Clifford algebras $\mathcal{M}$, which written in reduced form is given by (2.1). In what follows, we will associate to x an element $\tilde{x}(m)$ in $\mathcal{A}_{\varepsilon_m}$ given by

$$\tilde{x}(m) \, = \, \tilde{x}_{A_1}^{\alpha_1}(m) \cdots \tilde{x}_{A_\ell}^{\alpha_\ell}(m). \tag{2.11}$$

If we develop x as in (2.2), then we can write $\tilde{x}(m)$ as

$$\overbrace{\tilde{x}_{i_1}^{\alpha_1}(m) \cdots \tilde{x}_{i_{s_1}}^{\alpha_1}(m)}^{\tilde{x}_{A_1}^{\alpha_1}(m)} \overbrace{\tilde{x}_{i_{s_1+1}}^{\alpha_2}(m) \cdots \tilde{x}_{i_{s_1+s_2}}^{\alpha_2}(m)}^{\tilde{x}_{A_2}^{\alpha_2}} \cdots \overbrace{\tilde{x}_{i_{s_1+\cdots+s_{\ell-1}+1}}^{\alpha_\ell}(m) \cdots \tilde{x}_{i_{s_1+\cdots+s_\ell}}^{\alpha_\ell}(m)}^{\tilde{x}_{A_\ell}^{\alpha_\ell}(m)}.$$

2.3. **Hypercontractivity bounds.** In this subsection we prove Theorem B. The result below can be obtained following verbatim the proof of [4, Lemma 4] just replacing Theorem 7 there by Theorem 2.3 above.

Lemma 2.4. *If $p \geq 1$, we have*

$$\lim_{m \to \infty} \Big\| \sum\nolimits_j \rho_j \tilde{x}_j(m) \Big\|_{L_p(\mathcal{A}_{\varepsilon_m})} \, = \, \Big\| \sum\nolimits_j \rho_j x_j \Big\|_{L_p(\mathcal{M})} \quad a.e.$$

for any finite linear combination $\sum_j \rho_j x_j$ of reduced words in the free product $\mathcal{M}$.

Lemma 2.5. *Given x a reduced word in the free product $\mathcal{M}$, let $\tilde{x}(m)$ be the element in $\mathcal{A}_{\varepsilon_m}$ associated to x as in* (2.11)*. Then, there exists a decomposition $\tilde{x}(m) = \tilde{x}_1(m) + \tilde{x}_2(m)$ with the following properties*

i) $\langle \tilde{x}_1(m), \tilde{x}_2(m) \rangle = 0$ *a.e.*,

ii) $\mathcal{S}_{\varepsilon_n,t}(\tilde{x}_1(m)) = e^{-t|x|}\tilde{x}_1(m)$,

iii) $\lim_{m\to\infty} \|\tilde{x}_1(m)\|_{L_2(\mathcal{A}_{\varepsilon_m})} = 1$ *a.e.*

In particular, we deduce that

$$\lim_{m\to\infty} \|\tilde{x}_2(m)\|_{L_2(\mathcal{A}_{\varepsilon_m})} \;=\; 0 \quad a.e.$$

Proof. If we set $s = |x|$ and σ_0 denotes the singleton partition, define

$$\begin{aligned} \tilde{x}_1(m)(\omega) &= \frac{1}{m^{s/2}} \sum_{\substack{\underline{k}\in[m]^s \\ \sigma(\underline{k})=\sigma_0}} x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega), \\ \tilde{x}_2(m)(\omega) &= \frac{1}{m^{s/2}} \sum_{\sigma\in\Pi(s)\setminus\{\sigma_0\}} \sum_{\substack{\underline{k}\in[m]^s \\ \sigma(\underline{k})=\sigma}} x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega). \end{aligned}$$

Clearly $\tilde{x}(m) = \tilde{x}_1(m) + \tilde{x}_2(m)$ point wise and $\|\tilde{x}(m)\|_{L_2(\mathcal{A}_{\varepsilon_m})} = 1$. Property i) is easily checked. Indeed, consider $\underline{k}, \underline{\ell} \in [m]^s$ with $\sigma(\underline{k}) = \sigma_0$ and $\sigma(\underline{\ell}) \in \Pi(s)\setminus\{\sigma_0\}$. Since the k_i's are all distinct and the ℓ_i's are not we must have

$$\tau_{\varepsilon_m}\Big(x_{i_1}^{\alpha_1}(k_1)(\omega)\cdots x_{i_s}^{\alpha_s}(k_s)(\omega) x_{i_s}^{\alpha_s}(\ell_s)(\omega)\cdots x_{i_1}^{\alpha_1}(\ell_1)(\omega)\Big) \;=\; 0.$$

The second property comes from the definition of the semigroup (2.4) and the fact that for every $\underline{k}$ with $\sigma(\underline{k}) = \sigma_0$, we have no cancellations. Now it remains to show that

$$\lim_{m\to\infty} \frac{1}{m^s} \sum_{\substack{\underline{k},\underline{\ell}\in[m]^s \\ \sigma(\underline{k})=\sigma(\underline{\ell})=\sigma_0}} \tau_{\varepsilon_m}\Big(\underbrace{x_{i_1}^{\alpha_1}(k_1)\cdots x_{i_s}^{\alpha_s}(k_s)}_{x_{\underline{i}}^{\underline{\alpha}}(\underline{k})}\underbrace{x_{i_s}^{\alpha_s}(\ell_s)\cdots x_{i_1}^{\alpha_1}(\ell_1)}_{x_{\underline{i}}^{\underline{\alpha}}(\underline{\ell})^*}\Big) \;=\; 1.$$

Indeed, if $\{k_1,\ldots,k_s\} \neq \{\ell_1,\ldots,\ell_s\}$ the trace clearly vanishes and it suffices to consider the case $\{k_1,\ldots,k_s\} = \{\ell_1,\ldots,\ell_s\}$. Note that, the trace above is different from 0 if and only if $(\alpha_j, i_j, k_j) = (\alpha_{\beta(j)}, i_{\beta(j)}, \ell_{\beta(j)})$ for some permutation $\beta \in S_s$ and every $1 \le j \le s$. If we assume $k_s \neq \ell_s$, we get $(\alpha_j, i_j, k_j) = (\alpha_s, i_s, \ell_s)$ for certain $j < s$. This means that $x_{i_j}^{\alpha_j}(k_j)$ and $x_{i_s}^{\alpha_s}(k_s)$ belong to different α-blocks since the i_j's are pairwise distinct in a fixed α-block. Thus, to cancel these elements we must cross a β-block with $\beta \neq \alpha_s$. Since the k's are all different, the ε-signs corresponding to these commutations appear just once. We can argue in the same way for every $1 \le j \le s$ and conclude that

$$\mathbb{E}_\omega\, \tau_{\varepsilon_m}\Big(x_{i_1}^{\alpha_1}(k_1)\cdots x_{i_s}^{\alpha_s}(k_s)x_{i_s}^{\alpha_s}(\ell_s)\cdots x_{i_1}^{\alpha_1}(\ell_1)\Big) \;=\; 0$$

unless $k_j = \ell_j$ for all $1 \le j \le s$. Therefore

$$\begin{aligned} \lim_{m\to\infty} \mathbb{E}_\omega \|\tilde{x}_1(m)\|_{L_2(\mathcal{A}_{\varepsilon_m})}^2 &= \lim_{m\to\infty} \frac{1}{m^s} \sum_{\substack{\underline{k}\in[m]^s \\ k_i\neq k_j}} 1 \\ &= \frac{m(m-1)\cdots(m-s+1)}{m^s} \;=\; 1. \end{aligned}$$

Finally, arguing as in the proof of Theorem 2.3 we see that the same limit holds for almost every $\omega \in \Omega$. This proves iii). The last assertion follows from i), iii) and the identity $\|\tilde{x}(m)\|_2 = 1$. The proof is complete. $\square$

Lemma 2.6. *If $p \geq 1$, we have*

$$\lim_{m\to\infty} \Big\| \mathcal{S}_{\varepsilon_m,t}\Big(\sum\nolimits_j \rho_j \tilde{x}_j(m)\Big)\Big\|_{L_p(\mathcal{A}_{\varepsilon_m})} = \Big\| \mathcal{O}_{\mathcal{M},t}\Big(\sum\nolimits_j \rho_j x_j\Big)\Big\|_{L_p(\mathcal{M})} \quad a.e.$$

for any finite linear combination $\sum_j \rho_j x_j$ of reduced words in the free product $\mathcal{M}$.

Proof. According to Lemma 2.5, we have

$$\lim_{m\to\infty} \Big\| \Big(\mathcal{S}_{\varepsilon_m,t} - e^{-t|x|}\mathbf{1}_{\mathcal{A}_{\varepsilon_m}}\Big)\big(\tilde{x}(m)\big)\Big\|_{L_2(\mathcal{A}_{\varepsilon_m})} = 0 \quad \text{a.e.}$$

for any reduced word $x \in \mathcal{M}$ and the associated $\tilde{x}(m)$'s $\in \mathcal{A}_{\varepsilon_m}$ given by (2.11). Thus

$$\lim_{m\to\infty} \Big\| \mathcal{S}_{\varepsilon_m,t}\Big(\sum\nolimits_j \rho_j \tilde{x}_j(m)\Big) - \sum\nolimits_j e^{-t|x_j|}\rho_j \tilde{x}_j(m)\Big\|_{L_2(\mathcal{A}_{\varepsilon_m})} = 0 \quad \text{a.e.}$$

Then (2.6) implies that the same limit vanishes in the norm of $L_p(\mathcal{A}_{\varepsilon_m})$. On the other hand, since $\mathcal{O}_{\mathcal{M},t}(x_j) = e^{-t|x_j|}x_j$, the assertion follows from Lemma 2.4. □

Proof of Theorem B. Let $1 < p \leq q < \infty$. By construction, the algebraic free product A is a weak-$*$ dense involutive subalgebra of $\mathcal{M}$. In particular, it is dense in $L_p(\mathcal{M})$ for every $p < \infty$. Given a finite sum $z = \sum_j \rho_j x_j \in A$, consider the corresponding sum $\tilde{z}(m) = \sum_j \rho_j \tilde{x}_j(m) \in \mathcal{A}_{\varepsilon_m}$ following (2.11). Given any $t \geq \frac{1}{2}\log(q-1/p-1)$, we may apply Lemmas 2.4 and 2.6 in conjunction with Biane's theorem (2.5) to conclude

$$\begin{aligned} \|\mathcal{O}_{\mathcal{M},t}(z)\|_{L_q(\mathcal{M})} &= \lim_{m\to\infty} \big\|\mathcal{S}_{\varepsilon_m,t}(\tilde{z}(m))\big\|_{L_q(\mathcal{A}_{\varepsilon_m})} \\ &\leq \lim_{m\to\infty} \|\tilde{z}(m))\|_{L_p(\mathcal{A}_{\varepsilon_m})} = \|z\|_{L_p(\mathcal{M})}. \end{aligned}$$

The necessity of the condition $t \geq \frac{1}{2}\log(q-1/p-1)$ was justified above. □

2.4. **Further comments.** Note that the argument we have used in the proof of Theorem B still works in a more general setting. More precisely, we may replace the fermion algebras $\mathcal{M}_\alpha = \mathcal{C}(\mathbb{R}^d)$ by spin system algebras $\mathcal{A}_\alpha$, where the generators x_i^α satisfy certain commutation and anticommutation relations given by a sign ε^α as follows

$$x_i^\alpha x_j^\alpha - \varepsilon^\alpha(i,j)x_j^\alpha x_i^\alpha = 2\delta_{ij} \quad \text{ for } 1 \leq i,j \leq d.$$

Indeed, we just need to replace (2.7) by

$$\mu\Big(\varepsilon\big((\alpha,i,k),(\beta,j,\ell)\big) = -1\Big) = \begin{cases} \varepsilon^\alpha(i,j) & \text{if } \alpha = \beta, \\ 1/2 & \text{if } \alpha \neq \beta. \end{cases}$$

This yields optimal time hypercontractivity bounds for the Ornstein-Uhlenbeck semigroup on the free product of spin matrix algebras. An additional application of Speicher's central limit theorem allows us to obtain optimal hypercontractivity estimates for the Ornstein-Uhlenbeck semigroup on the free product of q-deformed algebras Γ_q, $-1 \leq q \leq 1$.

Remark 2.7. Slight modifications in (2.7) lead to von Neumann algebras which are still poorly understood. For instance, let us fix a function $f : [1,n]\times[1,n] \to [-1,1]$ which is symmetric and assume that

$$\mu\big(\{\varepsilon((\alpha,i,k),(\beta,j,\ell)) = +1\}\big) = \frac{1+f(\alpha,\beta)}{2}.$$

As usual we will assume that all the random variables $\varepsilon(x,y)$ are independent. Then it is convenient to first calculate expectation of the joint moments of

$$\tilde{x}_i^\alpha(m) = \frac{1}{\sqrt{m}} \sum_{k=1}^m x_i^\alpha(k).$$

Again, only the pair partitions survive and we get

$$\lim_{m\to\infty} \mathbb{E}_\omega \tau_{\varepsilon_m}(\tilde{x}_{i_1}^{\alpha_1}(m)\cdots\tilde{x}_{i_s}^{\alpha_s}(m)) \quad = \quad \sum_{\substack{\sigma\in\Pi_2(s)\\ \sigma\le\sigma(\underline{i}),\sigma(\underline{\alpha})}} \prod_{(k,\ell)\in I(\sigma)} f(\alpha_{e_k},\alpha_{e_\ell}).$$

As above, we will have hypercontractivity with the optimal constant for the limit gaussian systems (they indeed produce a tracial von Neumann algebra). As an illustration, let us consider $n=2$, $q_1,q_2\in[-1,1]$, $f(1,1)=q_1q_2$ and $f(1,2)=f(2,1)=f(2,2)=q_2$. We deduce immediately that

i) The von Neumann subalgebra generated by
$$x_i^1 = \lim_m \tilde{x}_i^1(m),$$
for $i=1,\ldots,d$ is isomorphic to $\Gamma_{q_1q_2}(\mathbb{R}^d)$, generated by d q_1q_2-gaussians.

ii) The von Neumann subalgebra generated by
$$x_i^2 = \lim_m \tilde{x}_i^2(m),$$
for $i=1,\ldots,d$ is isomorphic to $\Gamma_{q_2}(\mathbb{R}^d)$, generated by d q_2-gaussians.

iii) Let $A\subset[s]$ and let $y_i=x_{j_i}^1$ for $i\in A$ (and $\alpha_i=1$) and $y_i=x_{j_i}^2$ ($\alpha_i=2$) otherwise. Let η_0 be the partition of $[s]$ defined by the possible values of (j_i,α_i). Then we get
$$\tau(y_1y_2\cdots y_s) = \sum_{\eta_0\ge\sigma\in\Pi_2(s)} q_1^{\mathrm{inversion}(\sigma|A)} q_2^{\mathrm{inversion}(\sigma)}.$$
Here $\sigma|A$ is the restriction of σ to A where we count only inversions inside A. This construction is considered in [10] for constructing new Brownian motions.

We see that we can combine different q gaussian random variables in one von Neumann algebra with a prescribed interaction behaviour. With this method we recover the construction from [10] of a non-stationary Brownian motion B_t. Indeed one can choose $0=t_0<t_1<\cdots<t_d$ such that B_t is an abstract Brownian motion [10] and the random variables $s_t(j)=B_t-B_{t_j}$ are $q_0\cdots q_j$-Brownian motions. In this construction we needed a q_1-Brownian motion over a q_2-Brownian motion and hence the choice of the product q_1q_2 above. Although it is no longer trivial to determine the number operator, we see that hypercontractivity is compatible with non-stationarity. The algebras generated for arbitrary symmetric f could serve as models for q_1-products over q_2-products, although in general there is no q-product of arbitrary von Neumann algebras.

3. **The free Poisson semigroup**

In this section we prove Theorem A and optimal hypercontractivity for linear combinations of words in $\mathbb{F}_n$ with length lower than or equal to 1. Let us start with a trigonometric identity, which follows from the binomial theorem and the identity $2\cos x = e^{ix} + e^{-ix}$

$$(\cos x)^m \ = \ \frac{1}{2^{m-1}} \sum_{0 \le k \le [\frac{m}{2}]} \binom{m}{k} \frac{\cos((m-2k)x)}{2^{\delta_{m,2k}}}.$$

Let g_j denote one of the generators of $\mathbb{F}_n$. Identifying $\lambda(g_j)$ with $\exp(2\pi i \cdot)$, the von Neumann algebra generated by $\lambda(g_j)$ is $\mathcal{L}(\mathbb{Z})$ and the previous identity can be rephrased as follows for $u_j = \lambda(g_j)$

$$(u_j + u_j^*)^m \ = \sum_{0 \le k \le [\frac{m}{2}]} \binom{m}{k} v_{j,m-2k}, \tag{3.1}$$

with $v_{j,k} = u_j^k + (u_j^*)^k$ for every $k \ge 1$ and $v_0 = \mathbf{1}$. We will also need a similar identity in $\mathbb{G}_{2n}$. Let $z_1, z_2, \ldots, z_{2n}$ denote the canonical generators of $\mathbb{G}_{2n}$, take $x_j = \lambda(z_j)$ for $1 \le j \le 2n$ and consider the operators $a_{j,0} = \mathbf{1}, b_{j,0} = 0$ and

$$a_{j,k} \ = \ \underbrace{x_{2j-1}x_{2j}x_{2j-1}\cdots}_{k} \qquad , \qquad b_{j,k} \ = \ \underbrace{x_{2j}x_{2j-1}x_{2j}\cdots}_{k}. \tag{3.2}$$

If we set $\zeta_j = u_j + u_j^*$ and $\psi_j = x_{2j-1} + x_{2j}$, let us consider the $*$-homomorphism $\Lambda : \mathcal{A}_{sym}^n \to \mathcal{L}(\mathbb{G}_{2n})$ determined by $\Lambda(\zeta_j) = \psi_j$. The result below can be proved by induction summing by parts.

Lemma 3.1. *If $m \ge 0$, we find*

$$(x_{2j-1} + x_{2j})^m \ = \sum_{0 \le k \le [\frac{m}{2}]} \binom{m}{k} (a_{j,m-2k} + b_{j,m-2k}).$$

Moreover, $v_{j,k} \in \langle u_j + u_j^* \rangle$ and we have $\Lambda(v_{j,k}) = a_{j,k} + b_{j,k}$ for every $k \ge 0$.

Proof of Theorem A. As observed in the Introduction, the group von Neumann algebra $\mathcal{L}(\mathbb{Z}_2)$ is $*$-isomorphic to the Clifford algebra $\mathcal{C}(\mathbb{R})$. Moreover, the Poisson and Ornstein-Uhlenbeck semigroups coincide in this case. In particular, the first assertion follows from $\mathcal{L}(\mathbb{G}_n) = \mathcal{L}(\mathbb{Z}_2) * \cdots * \mathcal{L}(\mathbb{Z}_2) \simeq \mathcal{C}(\mathbb{R}) * \cdots * \mathcal{C}(\mathbb{R})$, by applying Theorem B with $d = 1$. To prove the second assertion, we consider the injective group homomorphism determined by

$$\Phi : g_j \in \mathbb{F}_n \ \mapsto \ x_{2j-1}x_{2j} \in \mathbb{G}_{2n}.$$

This map clearly lifts to an isometry $L_p(\mathcal{L}(\mathbb{F}_n)) \to L_p(\mathcal{L}(\mathbb{G}_{2n}))$ for all $p \ge 1$. Moreover, since $|\Phi(g)| = 2|g|$, we see that Φ intertwines the corresponding free Poisson semigroup up to a constant 2. More precisely, $\Phi \circ \mathcal{P}_{\mathbb{F}_n,t} \ = \ \mathcal{P}_{\mathbb{G}_{2n},t/2} \circ \Phi$ for all $t > 0$. Hence, if $1 < p \le q < \infty$ and $f \in L_p(\mathcal{L}(\mathbb{F}_n))$, we obtain from the result just proved that

$$\big\| \mathcal{P}_{\mathbb{F}_n,t} f \big\|_{L_q(\mathcal{L}(\mathbb{F}_n))} = \big\| (\mathcal{P}_{\mathbb{G}_{2n},t/2} \circ \Phi) f \big\|_{L_q(\mathcal{L}(\mathbb{G}_{2n}))} \le \|\Phi f\|_{L_q(\mathcal{L}(\mathbb{G}_{2n})} = \|f\|_{L_q(\mathcal{L}(\mathbb{F}_n))},$$

whenever $t \ge \log(q - 1/p - 1)$. It remains to prove the last assertion iii). The necessity of the condition $t \ge \frac{1}{2}\log(q - 1/p - 1)$ can be justified following Weissler argument in [40, pp 220]. Therefore, we just need to prove sufficiency. According to [33], $\chi_{[-2,2]}(s)/\pi\sqrt{4-s^2}$ is the common distribution of ζ_j and ψ_j. Moreover, since both families of variables are free, the tuples $(\zeta_1, \ldots, \zeta_n)$ and $(\psi_1, \ldots, \psi_n)$ must have the same distribution too. Therefore, for every polynomial P in n non-commutative variables we have

$$\big\|P(\zeta_1,\ldots,\zeta_n)\big\|_{L_p(\mathcal{A}_{sym}^n)} = \big\|P(\psi_1,\ldots,\psi_n)\big\|_{L_p(\mathcal{L}(\mathbb{G}_{2n}))}$$

for every $1 \le p \le \infty$. In particular, the $*$-homomorphism $\Lambda : \mathcal{A}_{sym}^n \to \mathcal{L}(\mathbb{G}_{2n})$ determined by $\Lambda(\zeta_j) = \psi_j$ for every $1 \le j \le n$ extends to an L_p isometry for every $1 \le p \le \infty$. We claim that

$$\Lambda\big(\mathcal{P}_{\mathbb{F}_n,t}(P(\zeta_1,\ldots,\zeta_n))\big) = \mathcal{P}_{\mathbb{G}_{2n},t}\big(P(\psi_1,\ldots,\psi_n)\big)$$

for every polynomial P in n non-commutative variables. It is clear that the last assertion iii) of Theorem A follows from our claim above in conjunction with the first assertion i), already proved. By freeness of the semigroups involved and the fact that Λ is a $*$-homomorphism, it suffices to justify the claim for $P(X_1, X_2, \ldots, X_n) = X_j^m$ with $1 \le j \le n$ and $m \ge 0$. However, this follows directly from Lemma 3.1. $\square$

In the lack of optimal time estimates for $\mathbb{F}_n$ through the probabilistic approach used so far —see [20] for related results— we conclude this paper with optimal hypercontractivity bounds for linear combinations of words with length lower than or equal to 1. We will use two crucial results, the second one is folklore and it follows from the "invariance by rotation" of the CAR algebra generators.

- *The Ball/Carlen/Lieb convexity inequality* [1]
$$\Big(\frac{\mathrm{Tr}|A+B|^p + \mathrm{Tr}|A-B|^p}{2}\Big)^{\frac{2}{p}} \ge \big(\mathrm{Tr}|A|^p\big)^{\frac{2}{p}} + (p-1)\big(\mathrm{Tr}|B|^p\big)^{\frac{2}{p}}$$
for any $1 \le p \le 2$ and any given pair of $m \times m$ matrices A and B.
- *A Khintchine inequality for fermion algebras*
$$\Big\|\sum_{j=1}^d \rho_j x_j\Big\|_p = \Big(\sum_{j=1}^d |\rho_j|^2\Big)^{\frac{1}{2}}$$
whenever $1 \le p < \infty$, $\rho_j \in \mathbb{R}$, $x_j = x_j^*$ and $x_i x_i + x_j x_i = 2\delta_{ij}$.

Theorem 3.2. *Let us denote by $\mathcal{W}_1$ the linear span of all words in $\mathcal{L}(\mathbb{F}_n)$ of length lower than or equal to 1. Then, the following optimal hypercontractivity bounds hold for $1 < p \le 2$, every $t \ge -\frac{1}{2}\log(p-1)$ and all $f \in \mathcal{W}_1$*

$$\|\mathcal{P}_{\mathbb{F}_n,t} f\|_{L_2(\mathcal{L}(\mathbb{F}_n))} \le \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}.$$

Proof. The optimality of our estimate follows once again from Weissler argument in [40, pp 220]. Moreover, it suffices to show the inequality for the extreme case $e^{-t} = \sqrt{p-1}$. The key point in the argument is the use of the $*$-homomorphism $\Phi : \mathcal{L}(\mathbb{F}_n) \to \mathcal{L}(\mathbb{G}_{2n})$ defined in the proof of Theorem A in conjunction with our characterization of $\mathcal{L}(\mathbb{G}_{2n})$ using a spin matrix model. Indeed, we will consider here exactly the same matrix model with $2n$ free copies and just one generator per algebra. More precisely, given $m \ge 1$ we will consider $x^\alpha(k)$ with $1 \le \alpha \le 2n$ and $1 \le k \le m$ verifying the same relations as in (2.7) depending on the corresponding random functions $\varepsilon((\alpha,k),(\beta,\ell))$. We also set

$$\tilde{x}^\alpha(m) = \frac{1}{\sqrt{m}}\sum_{k=1}^m x^\alpha(k)$$

as usual. Note that this model describes —in the sense of Theorem 2.3— the algebra $\mathcal{L}(\mathbb{G}_{2n})$. In fact, according to Lemma 2.4 we know that for every trigonometric polynomial $z = \sum_j \rho_j x_j \in \mathcal{L}(\mathbb{G}_{2n})$ in the span of finite words, we can define the corresponding elements $\tilde{z}(m) = \sum_j \rho_j \tilde{x}_j(m) \in \mathcal{A}_{\varepsilon_m}$ such that

$$\lim_{m\to\infty} \|\tilde{z}(m)\|_{L_p(\mathcal{A}_{\varepsilon_m})} \, = \, \|z\|_{L_p(\mathcal{L}(\mathbb{G}_{2n}))}$$

almost everywhere. Furthermore, by dominated convergence we find

$$\lim_{m\to\infty} \mathbb{E}_\omega \|\tilde{z}(m)\|_{L_p(\mathcal{A}_{\varepsilon_m})} \, = \, \|z\|_{L_p(\mathcal{L}(\mathbb{G}_{2n}))}.$$

We first consider a function $f = a_0\mathbf{1} + a_1\lambda(g_1) + b_1\lambda(g_1)^* + \ldots + a_n\lambda(g_n) + b_n\lambda(g_n)^*$ in $\mathcal{W}_1$ such that $\arg(a_\alpha) = \arg(b_\alpha)$ for all $1 \le \alpha \le n$. By the comments above, we have for every $1 < p < 2$

$$\begin{aligned} \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^2 &= \|\Phi f\|_{L_p(\mathcal{L}(\mathbb{G}_{2n}))}^2 \\ &= \lim_{m\to\infty} \mathbb{E}_\omega \Big\| a_0\mathbf{1} + a_1\tilde{x}^1(m)\tilde{x}^2(m) + b_1\tilde{x}^2(m)\tilde{x}^1(m) \\ &+ \cdots + a_n\tilde{x}^{2n-1}(m)\tilde{x}^{2n}(m) + b_n\tilde{x}^{2n}(m)\tilde{x}^{2n-1}(m)\Big\|_{L_p(\mathcal{A}_{\varepsilon_m})}^2. \end{aligned}$$

Now, we claim that $\|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^2$ is bounded below by

$$\lim_{m\to\infty} \mathbb{E}_\omega \Big(|a_0|^2 + \frac{p-1}{m^2} \sum_{\substack{1\le\alpha\le n \\ 1\le k\le m}} \Big\| \sum_{1\le\ell\le m} \Big(a_\alpha + b_\alpha \varepsilon\big((2\alpha-1,k),(2\alpha,\ell)\big)\Big) x^{2\alpha}(\ell) \Big\|_p^2 \Big).$$

If this is true, we can apply Khintchine's inequality for fixed α and k to get

$$\begin{aligned} &\mathbb{E}_\omega \Big\| \sum_{1\le\ell\le m} \Big(a_\alpha + b_\alpha \varepsilon\big((2\alpha-1,k),(2\alpha,\ell)\big)\Big) x^{2\alpha}(\ell) \Big\|_p^2 \\ &= \mathbb{E}_\omega \Big\| \sum_{1\le\ell\le m} \Big(|a_\alpha| + |b_\alpha| \varepsilon\big((2\alpha-1,k),(2\alpha,\ell)\big)\Big) x^{2\alpha}(\ell) \Big\|_p^2 \\ &= \sum_{1\le\ell\le m} \Big(|a_\alpha|^2 + |b_\alpha|^2 + 2|a_\alpha b_\alpha| \mathbb{E}_\omega \varepsilon\big((2\alpha-1,k),(2\alpha,\ell)\big)\Big) \, = \, m(|a_\alpha|^2 + |b_\alpha|^2). \end{aligned}$$

Here, we have used that the ε's are centered for $\alpha \neq \beta$. Therefore, we finally obtain

$$\|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^2 \, \ge \, |a_0|^2 + (p-1) \sum_{\alpha=1}^n (|a_\alpha|^2 + |b_\alpha|^2) \, = \, \|\mathcal{P}_{\mathbb{F}_n,t} f\|_{L_2(\mathcal{L}(\mathbb{F}_n))}^2$$

for $e^{-t} = \sqrt{p-1}$. Therefore, it suffices to prove the claim. To this end, note that

$$\|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^2 \, = \, \lim_{m\to\infty} \mathbb{E}_\omega \big\| A_m + x^1(1) B_m \big\|_{L_p(\mathcal{A}_{\varepsilon_m})}^2,$$

where A_m and B_m are given by

$$\begin{aligned} A_m &= a_0\mathbf{1} + \frac{1}{m} \sum_{\substack{2\le k\le m \\ 1\le\ell\le m}} \Big(a_1 + b_1\varepsilon\big((1,k),(2,\ell)\big)\Big) x^1(k)x^2(\ell) \\ &+ \frac{1}{m} \sum_{1\le k,\ell\le m} \Big[a_2 x^3(k)x^4(\ell) + b_2 x^4(k)x^3(\ell) + \ldots + b_n x^{2n}(k)x^{2n-1}(\ell)\Big] \end{aligned}$$

and $B_m = \frac{1}{m}\sum_{1\le\ell\le m} \big(a_1 + b_1\varepsilon\big((1,1),(2,\ell)\big)\big)x^2(\ell)$. Then, since the spin matrix model is unaffected by the change of sign of one generator and A_m, B_m do not depend on $x^1(1)$, we deduce $\|A_m + x^1(1)B_m\|_p = \|A_m - x^1(1)B_m\|_p$. Therefore, applying Ball/Carlen/Lieb inequality we conclude that

$$\|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^2 \ \geq \ \lim_{m\to\infty} \mathbb{E}_\omega \Big(\|A_m\|_{L_p(\mathcal{A}_{\varepsilon_m})}^2 + (p-1)\|B_m\|_{L_p(\mathcal{A}_{\varepsilon_m})}^2 \Big),$$

where we have used that $\|x^1(1)B_m\|_p = \|B_m\|_p$ for every ω and every p. If we apply the same strategy with $x^1(2), \ldots, x^1(m)$, it is not difficult to obtain the following lower bound

$$\begin{aligned} \|f\|_{L_p(\mathcal{L}(\mathbb{F}_d))}^2 \ & \geq \ \lim_{m\to\infty} \mathbb{E}_\omega \Big\| a_0\mathbf{1} + a_2\tilde{x}^3(m)\tilde{x}^4(m) + b_2\tilde{x}^4(m)\tilde{x}^3(m) \\ & + \ \cdots \ + a_n\tilde{x}^{2n-1}(m)\tilde{x}^{2n}(m) + b_n\tilde{x}^{2n}(m)\tilde{x}^{2n-1}(m) \Big\|_p^2 \\ & + \ \frac{p-1}{m^2} \sum_{1\leq k\leq m} \Big\| \sum_{1\leq \ell\leq m} \Big(a_1 + b_1\varepsilon\big((1,k),(2,\ell)\big)\Big) x^2(\ell) \Big\|_p^2. \end{aligned}$$

Our claim follows iterating this argument on $2 \leq \alpha \leq n$. It remains to consider an arbitrary $f = a_0\mathbf{1} + a_1\lambda(g_1) + b_1\lambda(g_1)^* + \ldots + a_n\lambda(g_n) + b_n\lambda(g_n)^* \in \mathcal{W}_1$. Let us set $(\theta_\alpha, \theta_\alpha') = (\arg(a_\alpha), \arg(b_\alpha))$ and $(\nu_\alpha, \nu_\alpha') = (\frac12(\theta_\alpha + \theta_\alpha'), \frac12(\theta_\alpha - \theta_\alpha'))$ for each $1 \leq \alpha \leq n$. Consider the 1-dimensional representation $\pi : \mathbb{F}_n \to \mathbb{C}$ determined by $\pi(g_\alpha) = \exp(i\nu_\alpha')$ for the α-th generator g_α. According to the L_p-analog of Fell's absorption principle [34], we have from the first part of the proof that

$$\begin{aligned} \|\mathcal{P}_{\mathbb{F}_n,t}f\|_2 \ & \leq \ \Big\| a_0\mathbf{1} + \sum_{\alpha=1}^n |a_\alpha|e^{i\nu_\alpha}\lambda(g_\alpha) + |b_\alpha|e^{i\nu_\alpha}\lambda(g_\alpha)^* \Big\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \\ & = \ \Big\| a_0\mathbf{1} + \sum_{\alpha=1}^n |a_\alpha|e^{i\nu_\alpha}\pi(g_\alpha)\lambda(g_\alpha) + |b_\alpha|e^{i\nu_\alpha}\pi(g_\alpha^{-1})\lambda(g_\alpha)^* \Big\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \\ & = \ \Big\| a_0\mathbf{1} + \sum_{\alpha=1}^n a_\alpha\lambda(g_\alpha) + b_\alpha\lambda(g_\alpha)^* \Big\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \ = \ \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}. \end{aligned}$$

The proof is complete. □

We finish this section with further results on $L_p \to L_2$ estimates for the free Poisson semigroup. The key point here is to use a different model for Haar unitaries. In the sequel, we will denote by $\mathbb{M}_2$ the algebra of 2×2 matrices.

Lemma 3.3. *If $u_j = \lambda(g_j)$ and $x_j = \lambda(z_j)$, the map*

$$u_j \mapsto \left[\begin{array}{cc} 0 & x_{2j-1} \\ x_{2j} & 0 \end{array} \right]$$

determines a trace preserving $*$-homomorphism $\pi : \mathcal{L}(\mathbb{F}_n) \to \mathbb{M}_2\overline{\otimes}\mathcal{L}(\mathbb{G}_{2n})$ such that

$$\pi \circ \mathcal{P}_{\mathbb{F}_n,t} \ = \ \big(Id_{\mathbb{M}_2} \otimes \mathcal{P}_{\mathbb{G}_{2n},t}\big) \circ \pi.$$

Proof. Since $\pi(u_j)$ is a unitary w_j in $\mathbb{M}_2\overline{\otimes}\mathcal{L}(\mathbb{G}_{2n})$ and $\mathbb{F}_n$ is a free group, a unique $*$-homomorphism $\pi : \mathcal{L}(\mathbb{F}_n) \to \mathbb{M}_2\overline{\otimes}\mathcal{L}(\mathbb{G}_{2n})$ is determined by the w_j's. Thus, it suffices to check that π is trace preserving. The fact that $\pi(\lambda(g))$ has trace zero in $\mathbb{M}_2\overline{\otimes}\mathcal{L}(\mathbb{G}_{2n})$ for every $g \neq e$ follows easily from the equalities

$$\pi(u_1)^{2k} = \left[\begin{array}{cc} a_{1,2k} & 0 \\ 0 & b_{1,2k} \end{array} \right] \qquad \pi(u_1)^{2k+1} = \left[\begin{array}{cc} 0 & a_{1,2k+1} \\ b_{1,2k+1} & 0 \end{array} \right]$$

and its analogous formulae for the product of different generators. Here, we have used the notations introduced in (3.2). The second assertion can be checked by simple calculations. The proof is complete. $\square$

Biane's theorem relies on an induction argument [4, Lemma 2] which exploits the Ball-Carlen-Lieb convexity inequality stated before Theorem 3.2. In fact, our proof of Theorem 3.2 follows the same induction argument. We will now consider spin matrix models with operator coefficients. More precisely, given a finite von Neumann algebra $(\mathcal{M},\tau)$, we will look at $\mathcal{M}\overline{\otimes}\mathcal{A}_{\varepsilon_m}$. In particular, following the notation in Section 2 every $x \in \mathcal{M}\overline{\otimes}\mathcal{A}_{\varepsilon_m}$ can be written as $x = \sum_A \rho_A \otimes x_A^{\varepsilon_m}$ where $\rho_A \in \mathcal{M}$ for every $A \subset \Upsilon_m$. Then, the induction argument easily leads to the inequality below provided that $e^{-t} \le \sqrt{p-1}$

$$\|x\|^2_{L_p(\mathcal{M}\overline{\otimes}\mathcal{A}_{\varepsilon_m})} \ge \sum_{A\subset\Upsilon_m} e^{-2t|A|}\|\rho_A\|^2_{L_p(\mathcal{M})}.$$

For our purpose we will consider $\mathcal{M} = \mathbb{M}_2$ with its normalized trace, so that

$$\|a\|_p \ge 2^{\frac{1}{2}-\frac{1}{p}}\|a\|_2$$

for every $a \in \mathbb{M}_2$. Let $x = \sum_A \rho_A \otimes x_A^{\varepsilon_m}$ be as above. Let us also define $\mathcal{U}$ as the (possible empty) set of the subsets A of Υ_m such that ρ_A is a multiple of a unitary. In particular, $\|\rho_A\|_{L_2(\mathcal{M})} = \|\rho_A\|_{L_p(\mathcal{M})}$ for every $A \in \mathcal{U}$. Then, letting $y = \sum_{A\in\mathcal{U}} \rho_A \otimes x_A^{\varepsilon_m}$, the following estimate holds provided $e^{-t} \le \sqrt{p-1}$

$$\|x\|_p^2 \ge \big\|Id_{\mathbb{M}_2}\otimes\mathcal{S}_{\varepsilon_m,t}(y)\big\|_2^2 + 2^{1-\frac{2}{p}}\big\|Id_{\mathbb{M}_2}\otimes\mathcal{S}_{\varepsilon_m,t}(x-y)\big\|_2^2, \tag{3.3}$$

where the right hand side norms are taken in $\mathbb{M}_2\overline{\otimes}\mathcal{A}_{\varepsilon_m}$. Our first application of this alternative approach is that Weissler's theorem [40] can be proved using probability and operator algebra methods.

Proposition 3.4. *If $1 < p \le q < \infty$, we find*

$$\big\|\mathcal{P}_{\mathbb{Z},t} : L_p(\mathcal{L}(\mathbb{Z})) \to L_q(\mathcal{L}(\mathbb{Z}))\big\| \ = \ 1 \ \Leftrightarrow \ t \ge \frac{1}{2}\log\frac{q-1}{p-1}.$$

Proof. We will assume that $q = 2$ since the optimal time for every p, q can be obtained from this case by means of standard arguments involving log-Sobolev inequalities. We follow here the same approximation procedure of Lemmas 2.4 and 2.5 with $n = 2$ and $d = 1$. Consider a reduced word $x = x_{\alpha_1}\cdots x_{\alpha_s}$ in $\mathcal{L}(\mathbb{G}_2)$, so that $\alpha_j \in \{1,2\}$ and $\alpha_j \ne \alpha_{j+1}$. We then form the associated element

$$\tilde{x}(m)(\omega) = \frac{1}{m^{s/2}}\sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma_0}} x^{\alpha_1}(k_1)(\omega)\cdots x^{\alpha_s}(k_s)(\omega) \in \mathcal{A}_{\varepsilon_m}.$$

Note that restricting to $\sigma(\underline{k}) = \sigma_0$ implies that there will be no repetitions of the elements $x^{\alpha_j}(k_j)$, hence no simplifications in $\tilde{x}(m)$. As we showed in the proof of Lemma 2.5, the terms with repetitions do not play any role. On the other hand, Lemma 2.4 easily extends to operator coefficients so that for any $1 \le p \le 2$, every $\rho_j \in \mathbb{M}_2$ and every reduced word $x_j \in \mathcal{L}(\mathbb{G}_2)$, we have

$$\lim_{m\to\infty}\Big\|\sum_j \rho_j\otimes\tilde{x}_j(m)\Big\|_{L_p(\mathbb{M}_2\overline{\otimes}\mathcal{A}_{\varepsilon_m})} = \Big\|\sum_j \rho_j\otimes x_j\Big\|_{L_p(\mathbb{M}_2\overline{\otimes}\mathcal{L}(\mathbb{G}_2))} \quad \text{a.e.} \tag{3.4}$$

Let us denote by $u = \lambda(g_1)$ the canonical generator of $\mathcal{L}(\mathbb{Z})$. By the positivity of $\mathcal{P}_{\mathbb{Z},t}$ and a density argument, it suffices to show that $\|\mathcal{P}_{\mathbb{Z},t} f\|_{L_2(\mathcal{L}(\mathbb{Z}))} \leq \|f\|_{L_p(\mathcal{L}(\mathbb{Z}))}$ for every positive trigonometric polynomial

$$f = \rho_0 \mathbf{1} + \sum_{j=1}^{d} (\rho_j u^j + \overline{\rho}_j u^{*j}).$$

To this end, we use the map π from Lemma 3.3 and construct

$$\begin{aligned} x \; = \; \pi(f) \; &= \; \left[\begin{array}{cc} \rho_0 & 0 \\ 0 & \rho_0 \end{array}\right] \otimes \mathbf{1} \\ &+ \; \sum_{\ell \geq 1} \left[\begin{array}{cc} \rho_{2\ell} & 0 \\ 0 & \overline{\rho}_{2\ell} \end{array}\right] \otimes a_{1,2\ell} + \left[\begin{array}{cc} \overline{\rho}_{2\ell} & 0 \\ 0 & \rho_{2\ell} \end{array}\right] \otimes b_{1,2\ell} \\ &+ \; \sum_{\ell \geq 1} \left[\begin{array}{cc} 0 & \rho_{2\ell+1} \\ \overline{\rho}_{2\ell+1} & 0 \end{array}\right] \otimes a_{1,2\ell+1} + \left[\begin{array}{cc} 0 & \overline{\rho}_{2\ell+1} \\ \rho_{2\ell+1} & 0 \end{array}\right] \otimes b_{1,2\ell+1}. \end{aligned}$$

To use our approximation procedure, we consider the element $\tilde{x}(m) \in \mathbb{M}_2 \overline{\otimes} \mathcal{A}_{\varepsilon_m}$ associated to x. We start noting that $\tilde{x}(m)$ is self-adjoint. Now, in order to use (3.3) and make act $Id_{\mathbb{M}_2} \otimes \mathcal{S}_{\varepsilon_m,t}$, we must write $\tilde{x}(m)$ in reduced form. That is, for every $\underline{k} \in [m]^s$ with $\sigma(\underline{k}) = \sigma_0$ and $\underline{\alpha} = (\alpha_1, \cdots, \alpha_s) \in \{1,2\}^s$ with $\alpha_j \neq \alpha_{j+1}$, we want to understand the matrix coefficients $\gamma^{\underline{\alpha}}(\underline{k})$ of $x^{\underline{\alpha}}(\underline{k}) = x^{\alpha_1}(k_1) \cdots x^{\alpha_s}(k_s)$, where the latter is an element in the basis of $\mathcal{A}_{\varepsilon_m}$. In fact, it suffices to show that these matrix coefficients are multiples of unitaries, so that all the subsets A of Υ_m are in $\mathcal{U}$ and we do not loose any constant when applying (3.3). Let us first assume that $s = 2\ell + 1$ is odd. Since by definition there is no simplifications in $\tilde{x}(m)$, the term $x^{\underline{\alpha}}(\underline{k})$ will only appear in the element in $\mathcal{A}_{\varepsilon_m}$ associated to either $a_{1,2\ell+1}$ or $b_{1,2\ell+1}$. By the commutation relations, we see that $x^{\underline{\alpha}}(\underline{k})^* = \pm x^{\underline{\alpha}}(\underline{k})$. Then its matrix coefficient must also satisfy $\gamma^{\underline{\alpha}}(\underline{k})^* = \pm \gamma^{\underline{\alpha}}(\underline{k})$. Moreover, one easily checks that it also has the shape

$$\left[\begin{array}{cc} 0 & \delta \\ \mu & 0 \end{array}\right]$$

from the above formula of x. Hence $\delta = \pm \overline{\mu}$ and $\gamma^{\underline{\alpha}}(\underline{k})$ is a multiple of a unitary (this can also be directly seen from the formula of x). If $s = 2\ell$, the term $x^{\underline{\alpha}}(\underline{k})$ will appear in the elements associated to the two reduced words $a_{1,2\ell}$ and $b_{1,2\ell}$. Since the commutation relations only involve signs, after a moment of thought we can conclude that $\gamma^{\underline{\alpha}}(\underline{k})$ has the shape

$$\left[\begin{array}{cc} \delta & 0 \\ 0 & \overline{\delta} \end{array}\right].$$

Hence, it is a multiple of a unitary. Actually, we also know that δ is either real or purely imaginary. Once we have seen that the matrix coefficients of $\tilde{x}(m)$ written in reduced form are multiples of unitaries, we can conclude the proof as in Theorem B. Indeed, using Lemma 3.3, (3.3) and (3.4), we get

$$\begin{aligned} \|f\|_{L_p(\mathbb{T})} \; &= \; \|x\|_{L_p(\mathbb{M}_2 \overline{\otimes} \mathcal{L}(\mathbb{G}_2))} \\ &= \; \lim_{m \to \infty} \|\tilde{x}(m)\|_{L_p(\mathbb{M}_2 \overline{\otimes} \mathcal{A}_{\varepsilon_m})} \\ &\geq \; \lim_{m \to \infty} \|(Id_{\mathbb{M}_2} \otimes \mathcal{S}_{\varepsilon_m,t}) \tilde{x}(m)\|_{L_2(\mathbb{M}_2 \overline{\otimes} \mathcal{A}_{\varepsilon_m})} \\ &= \; \|(Id_{\mathbb{M}_2} \otimes \mathcal{P}_{\mathbb{G}_2,t})(x)\|_{L_2(\mathbb{M}_2 \overline{\otimes} \mathcal{L}(\mathbb{G}_2))} = \|\mathcal{P}_{\mathbb{Z},t}(f)\|_{L_2(\mathbb{T})}, \end{aligned}$$

where the limits are taken a.e. and $t \geq -\frac{1}{2}\log(p-1)$. The proof is complete. $\square$

A slight modification of the previous argument allows us to improve Theorem A ii) for $q=2$. In fact, by a standard use of log-Sobolev inequalities we may also improve the $L_p \to L_q$ hypercontractivity bound, see Remark 3.7 below.

Theorem 3.5. *If $1 < p \leq 2$, we find*

$$\big\| \mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{L}(\mathbb{F}_n)) \to L_2(\mathcal{L}(\mathbb{F}_n)) \big\| = 1 \quad if \quad t \geq \frac{1}{2}\log\frac{1}{p-1} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{2}\Big)\log 2.$$

Proof. Once again, by positivity and density it suffices to prove the assertion for a positive trigonometric polynomial $f \in \mathcal{L}(\mathbb{F}_n)$. If $\underline{j} = (j_1, \ldots, j_d)$, we will use the notation $|\underline{j}| = d$ and $u_{\underline{j}} = \lambda(g_{\underline{j}})$ with $g_{\underline{j}} = g_{j_1} \cdots g_{j_d}$ a reduced word in $\mathbb{F}_n$, so that

$$f = \sum\nolimits_{\underline{j}} \rho_{\underline{j}} u_{\underline{j}}.$$

Here we use the usual convention that $g_{-k} = g_k^{-1}$. We use again the trace preserving $*$-homomorphism $\pi : \mathcal{L}(\mathbb{F}_n) \to \mathbb{M}_2 \overline{\otimes} \mathcal{L}(\mathbb{G}_{2n})$ coming from Lemma 3.3. This gives the identity

$$\pi(u_{\underline{j}}) = \left[\begin{array}{cc} 0 & x_{2j_1-1} \\ x_{2j_1} & 0 \end{array}\right] \left[\begin{array}{cc} 0 & x_{2j_2-1} \\ x_{2j_2} & 0 \end{array}\right] \cdots \left[\begin{array}{cc} 0 & x_{2j_d-1} \\ x_{2j_d} & 0 \end{array}\right]$$

with the convention that for $j > 0$, $x_{-2j} = x_{2j-1}$ and $x_{-2j-1} = x_{2j}$. If $d = 0$, we set $g_{\underline{j}} = e$ and $\pi(u_{\underline{j}}) = Id_{\mathbb{M}_2}$. Hence with $x = \pi(f)$, summing up according to the length we obtain

$$\begin{aligned} x \ = \ & \left[\begin{array}{cc} \rho_0 & 0 \\ 0 & \rho_0 \end{array}\right] \otimes \mathbf{1} \ + \sum_{\substack{|\underline{j}|=2\ell \\ \ell \geq 1}} \left[\begin{array}{cc} \rho_{\underline{j}} & 0 \\ 0 & \rho_{-\underline{j}} \end{array}\right] \otimes x_{2j_1-1} x_{2j_2} \cdots x_{2j_{2\ell}} \\ & + \sum_{\substack{|\underline{j}|=2\ell+1 \\ \ell \geq 0}} \left[\begin{array}{cc} 0 & \rho_{\underline{j}} \\ \rho_{-\underline{j}} & 0 \end{array}\right] \otimes x_{2j_1-1} x_{2j_2} \cdots x_{2j_{2\ell+1}-1}. \end{aligned} \tag{3.5}$$

We repeat the arguments used in the proof of Proposition 3.4 to approximate x by a spin model $\tilde{x}(m)(\omega)$ with operator coefficients. That is, $x_{\alpha_1} \cdots x_{\alpha_s} \in \mathcal{L}(\mathbb{G}_{2n})$ is associated to

$$\tilde{x}(m)(\omega) = \frac{1}{m^{s/2}} \sum_{\substack{\underline{k} \in [m]^s \\ \sigma(\underline{k}) = \sigma_0}} x^{\alpha_1}(k_1)(\omega) \cdots x^{\alpha_s}(k_s)(\omega) \in \mathcal{A}_{\varepsilon_m}.$$

Note that the contribution to x given by (3.5) of words of length 0 and 1 is

$$\left[\begin{array}{cc} \rho_0 & 0 \\ 0 & \rho_0 \end{array}\right] \otimes 1 + \sum_{j \in \mathbb{Z} \setminus \{0\}} \left[\begin{array}{cc} 0 & \rho_j \\ \rho_{-j} & 0 \end{array}\right] \otimes x_{2j-1}.$$

Since f is self-adjoint, we have $\rho_{-j} = \overline{\rho}_j$ for $j \in \mathbb{Z} \setminus \{0\}$. Hence the matrix coefficients corresponding to the words of length 0 and 1 in the approximation are multiples of unitaries. We will have $\{A \subset \Upsilon_m \, : \, |A| \leq 1\} \subset \mathcal{U}$ with the notations of (3.3), and decompose $f = g + h$, where g is the part of f of degree less than 1 and h is supported by the words of length greater or equal than 2. Observe that g

and h are orthogonal. Let $t = t_0 + t_1$ with $t_0 = -\frac{1}{2}\log(p-1)$. Since h has valuation 2, we have

$$\|\mathcal{P}_{\mathbb{F}_n,t_0+t_1}(h)\|_2 \le e^{-2t_1}\|\mathcal{P}_{\mathbb{F}_n,t_0}(h)\|_2.$$

Thus thanks to (3.3), as in the proof of Proposition 3.4, we get by orthogonality

$$\begin{aligned} \|f\|_p^2 &\ge \|\mathcal{P}_{\mathbb{F}_n,t_0}(g)\|_2^2 + 2^{1-\frac{2}{p}}\|\mathcal{P}_{\mathbb{F}_n,t_0}(h)\|_2^2 \\ &\ge \|\mathcal{P}_{\mathbb{F}_n,t}(g)\|_2^2 + 2^{1-\frac{2}{p}}e^{4t_1}\|\mathcal{P}_{\mathbb{F}_n,t}(h)\|_2^2 \\ &\ge \|\mathcal{P}_{\mathbb{F}_n,t}(g)\|_2^2 + \|\mathcal{P}_{\mathbb{F}_n,t}(h)\|_2^2 = \|\mathcal{P}_{\mathbb{F}_n,t}(f)\|_2^2, \end{aligned}$$

provided that $e^{-4t_1}2^{\frac{2}{p}-1} \le 1 \Leftrightarrow t_1 \ge \frac{1}{2}(\frac{1}{p}-\frac{1}{2})\log 2$. This completes the proof. □

Remark 3.6. Let σ be the involutive $*$-representation on $\mathcal{L}(\mathbb{F}_n)$ exchanging u_j and $u_j^* = u_{-j}$ for all $j \ge 1$. So that if $f = \sum_{\underline{j}} \rho_{\underline{j}} u_{\underline{j}}$, then $\sigma(f) = \sum_{\underline{j}} \rho_{-\underline{j}} u_{\underline{j}}$. Denote by $\mathcal{L}(\mathbb{F}_n)^\sigma$ the fixed point algebra of σ, it clearly contains $\mathcal{A}^n_{sym}$. The above arguments actually prove that $\mathcal{P}_{\mathbb{F}_n,t}$ is hypercontractive on $\mathcal{L}(\mathbb{F}_n)^\sigma$ from L_p to L_2 with optimal time. Indeed, under this symmetric condition for f all the matrix coefficients will be multiples of unitaries. Then using the equivalence between hypercontractivity with optimal time and log-Sobolev inequality, one sees that Theorem A iii) can be extended to $\mathcal{L}(\mathbb{F}_n)^\sigma$. The Gross' argument to deduce general hypercontractive inequalities $L_p \to L_q$ from $L_p \to L_2$ estimates in this setting are recalled in [20].

Remark 3.7. We claim that

$$\frac{1}{2}\log\frac{1}{p-1} + \frac{1}{2}\Big(\frac{1}{p}-\frac{1}{2}\Big)\log 2 \le \frac{\beta}{2}\log\frac{1}{p-1}$$

with $\beta = 1 + \frac{\log(2)}{4}$. This is not difficult to prove by using basic computations. Then in particular Theorem 3.5 proves that we have hypercontractive $L_p \to L_2$ estimates for $t \ge -\frac{\beta}{2}\log(p-1)$. Then Gross' arguments relying on log-Sobolev inequality apply when the time has this shape, and the constant 2 given by Theorem A ii) can be replaced by the better constant $\beta = 1 + \frac{1}{4}\log(2) \sim 1.17$. Hence for any $1 < p \le q < \infty$ we get

$$\big\|\mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{L}(\mathbb{F}_n)) \to L_q(\mathcal{L}(\mathbb{F}_n))\big\| = 1 \quad \text{if} \quad t \ge \frac{\beta}{2}\log\frac{q-1}{p-1}.$$

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Marius Junge
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green St. Urbana, IL 61891. USA
junge@math.uiuc.edu

Carlos Palazuelos
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera 13-15. 28049, Madrid. Spain
carlospalazuelos@icmat.es

Javier Parcet
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera 13-15. 28049, Madrid. Spain
javier.parcet@icmat.es

Mathilde Perrin
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera 13-15. 28049, Madrid. Spain
mathilde.perrin@icmat.es

Éric Ricard
Laboratoire de Mathématiques Nicolas Oresme
Université de Caen Basse-Normandie
14032 Caen Cedex, France
eric.ricard@unicaen.fr