

“QUANTAL” BEHAVIOR IN CLASSICAL PROBABILITY

K. A. Kirkpatrick

*New Mexico Highlands University
Las Vegas, New Mexico 87701
E-mail: kirkpatrick@physics.nmhu.edu*

Received 9 February 2003; revised 17 March 2003

A number of phenomena generally believed characteristic of quantum mechanics and seen as interpretively problematic—the incompatibility and value-indeterminateness of variables, the non-existence of dispersion-free states, the failure of the standard marginal-probability formula, the failure of the distributive law of disjunction and interference—are exemplified in an emphatically non-quantal system: a deck of playing cards. Thus the appearance in quantum mechanics of incompatibility and these associated phenomena requires neither explanation nor interpretation.

Key words: quantum interpretation, “realism,” compatibility, classical models.

1. INTRODUCTION

If I will show you a probabilistic system that exhibits these phenomena:

- Q1. Observations of the several variables of the system cannot be made simultaneously—the processes for their observation are mutually inconsistent.
- Q2. Variables are *incompatible*: The statistics of the observation of two different variables in succession depend on the order of their observation; joint probability distributions of such incompatible variables do not exist.
- Q3. The system has no dispersion-free states—if, in a particular preparation of the system, one variable is sharp, the variable(s) incompatible with it cannot be.

- Q4. An observation whose result is ignored may affect the statistics of a succeeding, incompatible, observation—an apparent contradiction of the formula for marginal probabilities.
- Q5. Under certain circumstances, a disjunction of several values of a variable may fail to distribute through the conjunction with a succeeding, incompatible, observation—*interference* may occur.

Would you not assume any system such as this to be a quantum-mechanical one? Such an assumption would be most reasonable: Each phenomenon on this list has been considered, by one author or another, to be characteristic of quantum mechanics; each has been considered inexplicable by (and unacceptable from the viewpoint of) classical physics; each has inspired interpretations of quantum mechanics (Copenhagenism, quantum probability, quantum logic, . . . ; for an excellent review, see Ref. 1). But that assumption, however reasonable, would be incorrect—the system is unarguably classical, consisting of playing cards being drawn from a deck under an unusual, but straightforward, scheme.¹

I will argue, in Sec. 7, that the existence of such an example invalidates every call for the interpretation of quantum mechanics, every claim of metaphysical difficulty regarding quantum mechanics, which is based on the statistical phenomena Q2–Q5. If a problem of meaning or understanding were raised by these phenomena, it would not be a problem for the understanding of quantum mechanics *per se* but a problem for the understanding of the category *probabilistic models of sequences of variable-evaluation events in systems having several variables*—one such system being quantum mechanics, and another being our example.

In Sec. 2, I present the general probabilistic concepts and some necessary notation. In Sec. 3, I give formal definitions of Q2–Q5 (Eqs. (4), (7), (9), and (10), respectively), and prove that Eq. (4), the definition of compatibility, is equivalent to the quantum-mechanical definition in terms of commuting operators. In Sec. 4, I present examples of a classical probabilistic system which exhibits each of the properties Q1–Q5; the statistical properties of these systems are summarized in Eq. (11) and the discussion following. In Sec. 5, I clarify how Q4 and Q5 may occur (both in classical probability and in quantum mechanics) without violating basic probability identities: I deal with the problem of the completeness of the values of variables in the derivation of the marginal-probability formula by introducing the concept of *manifestation*, and then use this to solve a problem

¹The cards are selected using ordinary (chaotic) mechanical shuffling; the resulting system is deterministic (although probabilistic) and completely describable by classical mechanics—as far from quantum mechanical as is possible for a physical system to be.

(noted by Margenau [2]) in which the marginal-probability formula seems to fail in quantum mechanics. In Sec. 6, I discuss the matter of value-indeterminate variables, called "nonreality" in quantum mechanics, and show that it is a natural occurrence in nondeterministic (as contrasted with simply chaotically probabilistic) systems. I provide several appendices: Appendix A gives a brief summary of the probability of propositions, Appendix B contains the mathematical analysis of the example system, and Appendix C contains several exercises which illustrate Q1, Q2, and Q4 in a very simple system.

2. THE GENERAL SETTING: NOTATION

In this section, I discuss the theory of sequences of events in a probabilistic system described by more than one variable. This does not require an extension of standard probability theory (which is summarized in Appendix A—merely, to avoid ambiguities, the introduction of several new terms and some notation (which are new only because textbooks do not consider systems of several independent variables).

The system. A probabilistic system having several variables, P , Q , \dots , whose possible values are discrete: $\{p_1, p_2, \dots\}$, $\{q_1, q_2, \dots\}$, \dots . The values of each variable are disjoint and complete (cf. Appendix A). (If it doesn't lead to ambiguity, the proposition that a variable has a particular value will be abbreviated to the value itself: $P = p_j$ will be written simply p_j .)

Manifested events. An *event* is an occurrence at which at least one variable of the system takes on a value randomly; this is brought about by a physical interaction of the system with its exterior. Which variable takes on a value randomly depends on the details of the physical interaction, or *manifestation*; at each event, then, a particular variable is manifested. The dynamics of a probabilistic system deals with a (time) sequence of events.²

The v-state. An event is described by the values which have occurred; I will call this description the value state (the "*v-state*"); the theory of the system yields probabilities for the various branches, or *v-states*, possible for the event.

Preparation and the p-state. A *preparation* is an event process which erases any effect of the system's prior history on the probabilities of succeeding events. A preparation of a system determines the system's probability- or preparation-state (the "*p-state*"), a function which allows the calculation of probabilities of every possible succeeding sequence of events. The *p-state* is to be distinguished from the "state"

²Both quantum mechanics and my card system are probability theories of sequences of events; neither can be treated as a probability theory of values, because in neither can the set of all value propositions be given a Boolean logical structure.

of classical physics (to which, in a sense, the v -state corresponds); it does not describe what is, but only the probabilities of what might be. Because different preparations may result in the same p -state, a p -state is implied by, but does not specify, a preparation. In quantum mechanics, the p -state is equivalent with the statistical operator of the system. Thus

$$\text{Preparation} \Rightarrow p\text{-state} \xleftrightarrow{\text{QM}} \text{statistical operator}. \quad (1)$$

Though the p -state is determined by the preparation, it may then change according to a deterministic dynamics (the Schrödinger equation, for example); it does not, however, change according to the outcomes of events (occurrences). We will write probability expressions with the p -state (say s) as a subscript: $\text{Pr}_s(\cdot)$.

Event sequence notation. An event's ordinal position in a sequence of events will be denoted by a superscript in brackets: The event E followed by the event F is denoted $E^{[1]} \wedge F^{[2]}$. Because this notation is rather awkward, we introduce the following simplifications which will allow us to avoid the use of explicit ordinal superscripts for the most part:

(a) When the terms in the probability expressions are in the "natural" order and no ambiguity arises, the sequence ordinals will be dropped; thus $\text{Pr}_s(p_j)$ always means $\text{Pr}_{s[0]}(p_j^{[1]})$, and $\text{Pr}_s(q_k | p_j)$ always means $\text{Pr}_{s[0]}(q_k^{[2]} | p_j^{[1]})$.

(b) In probability expressions involving a conjunction such as $\text{Pr}_{s[0]}(x^{[1]} \wedge y^{[2]})$, we introduce the symbol $\&$, "and then," which implies the sequential order of conjunction:

$$x \& y \equiv x^{[1]} \wedge y^{[2]}. \quad (2)$$

Then $\text{Pr}_s(x \& y)$ always means $\text{Pr}_{s[0]}(x^{[1]} \wedge y^{[2]})$.

(c) All other orders of occurrence in probability expressions will require the explicit use of the ordinal superscripts—e.g., "retrodiction," $\text{Pr}_{s[0]}(x^{[1]} | y^{[2]})$.

Filtered preparation. The conditional probability $\text{Pr}_s(\cdot | x)$ refers to the probability distribution of the subset of the preparations which in the succeeding event satisfied the proposition $X = x$; this distribution is the same as the distribution of a preparation consisting of s followed by the filter which passes only that subset $X = x$:

$$\text{Pr}_s(\cdot | x) = \text{Pr}_{s \& x}(\cdot). \quad (3)$$

"*Observation.*" Throughout this paper I will use the term "observation" for its simplicity and familiarity. But it is too easy to infer from the use of this term the existence of an observer, which connotes human conscious involvement and a concomitant collection of metaphysical difficulties. As a physicist, not a metaphysicist, I always mean by "observation" only the minimal physical interactions necessary to assure the occurrence of an event; presumably, a sufficiently clever human would then be able to observe the value manifested in that event. Further, in probabilistic systems, that which is "observed" is in most cases (think of flipping a coin, drawing a card, passing a spin system through a Stern-Gerlach device) given its value by the very process of "observation" (thus the scare-quotes, which, the point having been made, I henceforth drop).

3. COMPATIBILITY AND INTERFERENCE

3.1. Compatibility; Q2

Here is a formal definition of compatibility (informally stated in Q2; recall the notation introduced in Eq. (2)):

Definition. The two variables P and Q are *compatible* iff

$$\Pr_s(p_j \& q_k) = \Pr_s(q_k \& p_j) \quad (4)$$

for all indices, for every preparation state s .³

The use of the quantum-mechanical term *compatibility* for this classical definition is not arbitrary: The following theorem establishes that this definition is *equivalent* with quantum mechanics' commuting operators definition.

Theorem. Two variables P and Q of a quantum system are compatible, $\Pr_s(p_j \& q_k) = \Pr_s(q_k \& p_j)$ for all j, k , and s iff their corresponding operators \mathbf{P} and \mathbf{Q} satisfy $\mathbf{PQ} = \mathbf{QP}$.

Proof: Expressing Eq. (4), the compatibility of the variables P and Q , in quantal terms (utilizing the usual "sandwich" form for the probability of successive events, with the proposition $R = r_j$ being represented

³Incompatibility does not arise in elementary probability texts, in which the usual elementary examples of sequences are the drawing of balls from a urn or cards from a deck, almost always done either *with* replacement or *without* replacement; in either case the probability is independent of the order of occurrence. However, there are many other replacement schemes (e.g., "replace if red, discard if green") which do not lead to this symmetry.

by the 1-projector $\mathbf{P}[r_j] \equiv |r_j\rangle\langle r_j|$, we have

$$\text{Tr}\{\rho \mathbf{P}[p_j] \mathbf{P}[q_k] \mathbf{P}[p_j]\} = \text{Tr}\{\rho \mathbf{P}[q_k] \mathbf{P}[p_j] \mathbf{P}[q_k]\}. \quad (5)$$

Equation (4) holds for all p-states s , hence Eq. (5) must hold for all statistical operators ρ , so

$$\mathbf{P}[p_j] \mathbf{P}[q_k] \mathbf{P}[p_j] = \mathbf{P}[q_k] \mathbf{P}[p_j] \mathbf{P}[q_k]. \quad (6)$$

Introduce [3] the operator $\mathbf{C} \equiv \mathbf{P}[p_j] \mathbf{P}[q_k] - \mathbf{P}[q_k] \mathbf{P}[p_j]$. By Eq. (6), $\mathbf{C}^\dagger \mathbf{C}$ is the zero operator, hence $\|\mathbf{C}|x\rangle\|^2 = 0$ for all $|x\rangle$, hence \mathbf{C} is the zero operator, i.e., $\mathbf{P}[p_j]$ and $\mathbf{P}[q_k]$ commute; by Eq. (5), this implies the compatibility of P and Q , completing the implicative circle: the compatibility of the variables P and Q is equivalent with the commutativity of the basis projectors $\mathbf{P}[p_j]$ and $\mathbf{P}[q_k]$.

The variables P and Q are represented by the hermitian operators \mathbf{P} and \mathbf{Q} , whose eigenexpansions are $\mathbf{P} = \sum_t p_t \mathbf{P}[p_t]$ and $\mathbf{Q} = \sum_t q_t \mathbf{P}[q_t]$, respectively. As is well-known, the commutativity of $\mathbf{P}[p_j]$ and $\mathbf{P}[q_k]$ is equivalent with the commutativity of \mathbf{P} and \mathbf{Q} , which is thus equivalent with the compatibility of P and Q . \square

3.2. Sharpness; Q3

A variable is said to be *sharp* in a given event if it has no statistical dispersion; in probability terms, $\text{Pr}_s(p_j) \in \{0, 1\}$. The condition Q3 is expressed by

$$\text{Pr}_s(q_k | p_j) \neq 1 \quad (7)$$

(i.e., if a system has been filtered to a sharp value of P , then a succeeding observation of an incompatible variable Q cannot yield a sharp value).

3.3. Marginal Probability; Q4

In the case of the sequences $\{p_j \& q\}$ (i.e., $\{p_j^{[1]} \wedge q^{[2]}\}$), the formula of marginal probability (cf. Appendix A) would seem to imply

$$\sum_t \text{Pr}_s(p_t \& q) = \text{Pr}_{s[0]}(q^{[2]}). \quad (8)$$

This expresses the erroneous assumption (based on the completeness of the values of a variable) that $\text{Pr}_s((\bigvee_j p_j) \& q) = \text{Pr}_s(T \& q)$ is independent of the variable P whose values are disjoined (or summed

over)—that is, that $\text{Pr}_{s[0]}(q^{[2]})$ itself is defined. This is generally not the case (and is Q4 on the list of "quantal" phenomena). We will discuss this further in Sec. 5, where we develop the correct form of the formula for marginal probability of a sequence, Eq. (16).

This error, moreover, is often compounded by ignoring the sequential index, writing

$$\sum_t \text{Pr}_s(p_t \& q) = \text{Pr}_s(q). \quad (9)$$

This seems to imply that $\text{Pr}_{s[0]}(q^{[2]}) = \text{Pr}_{s[0]}(q^{[1]})$; this is generally incorrect, even when $\text{Pr}_{s[0]}(q^{[2]})$ is defined. Although the error is rather obvious, it is exactly the error made by Margenau which led to his questioning the use of classical probability in quantum mechanics; we discuss this further in Sec. 5.5.

3.4. Interference; Q5

In physical examples of classical wave systems (optics, acoustics, ocean waves), the energy is additive for independent waves. As we analyze a wave process, we may arbitrarily divide it into several alternate disjoint wave subprocesses, all having a common endpoint. Interference is the difference (or, qualitatively, the existence of a difference) between the value of the energy at the endpoint of the process and the sum of the values of the energies of each of the several parallel subprocesses at that endpoint. (If this difference is made a function of the endpoint, we refer to an interference *pattern*.) Interference is not itself a directly observable phenomenon; rather, it is an artifact of the analysis of the physical system, and is defined only in relation to the particular analytical decomposition. For example, in the double-slit apparatus there is interference between the left and right slits, but there is no interference between the upper halves of the two slits and their lower halves—though the resulting pattern on the screen is the same in either case.

But classical wave interference is not directly applicable to quantum mechanics (nor to any probabilistic system): theories of probabilistic systems predict the probabilities of the occurrence of a value of a variable, not the value itself, so the energy can't be used in the definition. Of course, there are quantum systems so similar to the classical cases, both physically and mathematically (e.g., the atomic Young apparatus) that the idea of interference transferred rather directly, without formal definition; however, for many other situations (e.g., Wigner's recombining Stern-Gerlach apparatus), the analogy is much less direct. Though it seems we all "know it when we see it," still there is need for an explicit definition of a generalization of interference to probabilistic systems, one expressed in terms not restricted to the quantum formalism.

The probability of the disjunction of disjoint subprocesses is additive, thus the wave concept of interference is generalized in a natural way to probabilistic phenomena by giving the the role of the *wave energy at the endpoint* to the *probability of the process*: In a probabilistic system, interference is the difference between the probability of the process and the sum of the probabilities of the subprocesses. Based on this line of thought, we offer the following definition of interference in probabilistic systems:

Definition. Given an event E_P compatible with P for which, for all preparation states s ,

$$\Pr_s(E_P) = \sum_{t \in D} \Pr_s(p_t), \quad (10a)$$

the *interference* of E_P with respect to $\{p_j | j \in D\}$ is

$$I(E_P, \{p_j | j \in D\}, s, q) = \Pr_s(E_P \& q) - \sum_{t \in D} \Pr_s(p_t \& q). \quad (10b)$$

Thus (in the case $D = \{1, 2\}$) E_P appears to be the disjunction $p_1 \vee p_2$ —but this “disjunction” doesn’t distribute: $E_P \wedge q \not\equiv (p_1 \wedge q) \vee (p_2 \wedge q)$. This phenomenon of quantum interference (Q5 on the list of “quantal” properties) was described by Feynman [4] as the “heart” of quantum mechanics, its “only mystery.”⁴ In quantum mechanics, interference arises exactly in the case that E_P appears to be the disjunction $p_1 \vee p_2$, but the apparent alternatives p_1 and p_2 are physically indistinguishable (as in, for example, the atomic double-slit apparatus) and the apparent disjunction fails to distribute.⁵

4. A CLASSICAL SYSTEM EXHIBITING THE PROPERTIES Q1–Q5

Here is an example of an entirely classical probabilistic system which illustrates the ordinary nature of much of “quantum probability”: incompatibility, the non-existence of dispersion-free ensembles, the “failure” of the marginal-probability formula, and interference.

In order to make this system dramatically non-quantal, I construct it using playing cards. About these cards the reader need only

⁴Feynman’s statement was made prior to Bell’s publications, hence the singularity of the mystery; but nothing about Bell’s insights makes interference any *less* mysterious.

⁵This failure of distribution of disjunction is, of course, one basis for the introduction of “quantum logic.” Another (related) reason is described in footnote 8.

know that each carries two marks, the "face" and the "suit" (traditionally with names such as King, Queen, . . . , and Spades, Hearts, . . . , respectively), and that the suits are marked in two colors, red (Hearts and Diamonds) and black (Spades and Clubs). I treat face, suit, and color as system variables; each variable is "observed" (given a value) according to the rules of observation described below.

Interference may arise from a manifestation process which treats several values of a variable identically, as in the case of a degenerate value in quantum mechanics. In this card example, a natural choice for degeneracy is the *color* of the suit. The color of a card may be observed in either of the following ways: We may observe the suit and report the color (the failure to have a value for the suit is a matter of ignorance—which is what we mean by "we ignored the suit"); in such a case, Eq. (10b) vanishes. Alternatively, we may observe the color using a manifestation which processes the red suits identically, but differently from the black suits; in our example this leads to interference (the nonvanishing of Eq. (10b)).

System S A deck of playing cards, having two variables, *Face* and *Suit*, each of which may take on a disjoint set of values: K, Q, J, and S, H, D, respectively. The variable *Color* (a function of *Suit*) takes on the values R (red) and B (black). Duplicate cards are allowed, with the restriction that each *Face* and each *Suit* appear in equal numbers (so their *a priori* probabilities are equal).⁶ The variable under consideration (*Face*, *Suit*, or *Color*) is denoted by P ; its values are $\{p_j\}$.

Observation. To observe a variable P :

1. Shuffle the subdeck.
2. Report p_j , the P -value of the subdeck's top card.
3. Construct a new subdeck consisting of all cards for which $P = p_j$.

E.g., to observe *Face*, shuffle the subdeck and report the *Face*-value of its top card, Q, say, then construct a new subdeck consisting of all the Q's in the deck.

To observe *Color*, shuffle the subdeck; if, say, the top card's *Suit*-value is H (a red card), report the *Color* R, then construct a new subdeck consisting of all the deck's R cards: all its H's and all its D's.

⁶For example, in the case of two-valued variables we might use the deck $\{KS, KS, KH, QS, QH, QH\}$: three of each value.

Preparation. To prepare the system in the state "the value of P is p_j ":

Repeat

1. observe any other, incompatible, variable Q
(ignoring the result);

2. observe P

until $P = p_j$

(This results in a subdeck consisting of all cards for which $P = p_j$.)

The system S exemplifies incompatibility (Q1): The processes for observing *Suit* and *Face* cannot be carried out simultaneously. For example, if the card on the top of the subdeck is QH, then the reporting of *Face* would require the construction of a new subdeck consisting of all the Q's, while the reporting of *Suit* would require the construction of a new subdeck consisting of all the H's. It is impossible to carry out these two constructions simultaneously (unless the deck contains no H's and no Q's other than the QH).⁷

The statistical behavior of S is summarized in Eq. (11), in which I use the following notation: P , Q , X , and Y are system variables; P and Q are different variables, with possible values $\{p_j\}$ and $\{q_k\}$, respectively, while x and y are (not necessarily distinct) values of the (not necessarily different) variables X and Y . (In the example, the variables are *Face* or *Suit*, their values are K, Q, ..., S, H, ...). The equalities in Eq. (11) hold for all value indices; the inequalities hold for at least some values. The system is prepared in the state s .

$$\text{Pr}_{s[0]}(p_k^{[2]} | p_j^{[1]}) = \delta_{jk} \quad (\text{repeatability}); \quad (11a)$$

$$\text{Pr}_s(q_k | p_j) = \text{Pr}_s(p_j | q_k) \quad (\text{reciprocity}); \quad (11b)$$

$$\text{Pr}_s(x \& y) = \text{Pr}_x(y) \text{Pr}_s(x) \quad (\text{Markovian}); \quad (11c)$$

$$\text{Pr}_s(p_k \& p_l) = \text{Pr}_s(p_l \& p_k) \quad (\text{self-compatibility}); \quad (11d)$$

$$\text{Pr}_s(p_k \& q_l) \neq \text{Pr}_s(q_l \& p_k) \quad (\text{incompatibility}); \quad (11e)$$

$$\text{Pr}_s(q_k | p_j) \neq 1 \quad (\text{if } P \text{ is sharp, } Q \text{ is not}); \quad (11f)$$

$$(\exists j \ni \text{Pr}_s(p_j) \notin \{0, 1\}) \implies \quad (\text{marginal probability formula "fails"}); \quad (11g)$$

⁷Of course, one might cheat and look at both values as marked on the card; however, it is impossible to follow both subdeck-construction rules, and thus the cheat would be meaningless, irrelevant to the behavior of the system—as irrelevant as, say, having just determined the z -component of spin, flipping a coin to "determine" the x -component.

	S	H	D
K	0.1	0.4	0.5
Q	0.4	0.5	0.1
J	0.5	0.1	0.4
$\Pr(\text{Suit}_k^{[1]} \mid \text{Face}_j^{[0]}) = \Pr(\text{Face}_j^{[1]} \mid \text{Suit}_k^{[0]})$			

Table 1. The basic probabilities; note that there are no dispersion-free states (Q3). (Multiplying each table entry by (a multiple of) 10 gives the number of duplicates of that card.)

- (α) $\Pr_s(R) = \Pr_s(H \vee D) \forall s$, but (the interference of R)
 (β) $\Pr_s(R \& q_k) \neq \Pr_s((H \vee D) \& q_k)$ relative to $H \vee D$. (11h)

(These results are derived in Appendix B for any number of variables with any number of values.)

From Eq. (11a) we see that, following the observation of, say, *Face*, the ensemble is sharp in *Face*: the observation is *repeatable*. In quantum mechanics, Eqs. (11b) and (11c) hold for nondegenerate values. (Eq. (11c) may be written equivalently as

$$\Pr_s(y \mid x) = \begin{cases} \Pr_x(y), & \text{if } \Pr_s(x) \neq 0, \\ \text{undefined,} & \text{otherwise;} \end{cases} \quad (11c')$$

filtering a manifestation to a specific result erases any "memory" of the earlier preparation state.)

Equation (11d) shows that, as in quantum mechanics, the variables of this system are compatible with themselves. Eqs. (11e)–(11h) are the archetypal "quantal" effects Q2–Q5.

Tables 1–4 illustrate results for a simple version of the system, involving just two variables, each with three values. Table 3 shows an apparent failure of completeness, and Table 4 shows an apparent failure of the distributive rule; we discuss these "failures" in the succeeding section.

The system \mathcal{S} exhibits the phenomena Q1–Q5 as a result of the laws of probability and an "intelligent" choice of selection rules and cards—it does not simulate them (in the sense that a system constructed using Newtonian mechanics and an "intelligent" choice of parameters and initial conditions *exhibits* the orbital phenomena of the solar system; an orrery, or a combination of observational data and Kepler's Laws, *simulates* those phenomena). Further, \mathcal{S} does not approximate the specific quantitative results of quantum mechanics (which, of

course, also satisfies Q2–Q5); \mathcal{S} is neither a model nor a mechanization of quantum mechanics.

5. MANIFESTATION

5.1. Apparent Problems with Completeness

A naive consideration of the completeness of the values of a variable would suggest that, since $\{K, Q, J\}$ exhausts the possibilities of *Face* (in the decks being considered in \mathcal{S}), it must be that $K \vee Q \vee J = T$. Similarly, it must be that $S \vee H \vee D = T$. And this is correct in many ways; for example, $\Pr_x(K \vee Q \vee J) = \Pr_x(K) + \Pr_x(Q) + \Pr_x(J) = 1$ for any preparation x . However, note that

$$\Pr_Q((K \vee Q \vee J) \& K) = 0, \quad (12a)$$

but

$$\begin{aligned} & \Pr_Q((S \vee H \vee D) \& K) \\ &= \Pr_Q(S \& K) + \Pr_Q(H \& K) + \Pr_Q(D \& K) \\ &= \Pr_S(K) \Pr_Q(S) + \Pr_H(K) \Pr_Q(H) + \Pr_D(K) \Pr_Q(D) \geq 0 \end{aligned} \quad (12b)$$

(using Eqs. (11a)–(11c)). This can vanish only in a specially selected deck such that the K 's and the Q 's share no suit—not the general case. But these expressions would necessarily be equal were both $K \vee Q \vee J$ and $S \vee H \vee D$ equal to T .

Thus $K \vee Q \vee J$ and $S \vee H \vee D$ cannot both be true (hence equal) at the same time. But if $K \vee Q \vee J \neq T$ it must be that *Face* has no value at all. This is exactly the case in our cardgame example: The observations of *Face* and *Suit* are incompatible processes, hence *Face* and *Suit* cannot have values simultaneously.⁸

5.2. Manifestation

But if a variable does not always have a value, there must be a process which causes a variable P to take on one of its values; I call that process the *manifestation* of P and denote it M_P . Each event in this probability system of variable values must be a manifestation. The

⁸The erroneous identification of the two probability-1 disjunctions $\bigvee_j p_j$ and $\bigvee_k q_k$ (of incompatible variables P and Q respectively) has misled us to quantum logic by suggesting that we “patch together” the Boolean proposition lattices of P and Q into a non-modular lattice, by identifying their least elements and identifying their greatest elements.

	S	H	D
K	0.09	0.36	0.45
Q	-0.16	-0.20	-0.04
J	-0.25	-0.05	-0.20
$\Pr(\text{Face}_k^{[1]} \wedge \text{Suit}_l^{[2]} \mid K) - \Pr(\text{Suit}_l^{[1]} \wedge \text{Face}_k^{[2]} \mid K)$			

Table 2. incompatibility (Q2). Note that the two variables are incompatible in every value, and that the incompatibilities are not symmetric.

probability $\Pr_x(S \& K)$, for example, has meaning only in the context of the *manifestation history* $M_{\text{Suit}} \& M_{\text{Face}}$ as the condition of the probability: $\Pr_x(S \& K \mid M_{\text{Suit}} \& M_{\text{Face}})$.

With the understanding that this manifestation history is a necessary part of the probability expression, it is often safe to leave it implicit—but not always: If we make the manifestations in Eq. (12) explicit, we may safely identify the disjunctions as true; for example, we may write Eq. (12b) as

$$\begin{aligned} & \Pr_Q((S \vee H \vee D) \& K \mid M_{\text{Suit}} \& M_{\text{Face}}) \\ &= \Pr_Q(T \& K \mid M_{\text{Suit}} \& M_{\text{Face}}). \end{aligned} \quad (13)$$

The $M_{\text{Face}}^{[2]}$ may be made implicit ("dropped") without harm, but $\Pr_{Q^{[0]}}(K^{[2]})$ would be ambiguous regarding the occurrence at [1], so the $M_{\text{Suit}}^{[1]}$ (and $M_{\text{Face}}^{[1]}$, in Eq. (12a)) must be left explicit. Thus, we may write Eqs. (12a) and (12b) as

$$\Pr_Q(K \mid M_{\text{Face}}) = 0, \quad (14a)$$

$$\Pr_Q(K \mid M_{\text{Suit}}) \geq 0; \quad (14b)$$

that these may differ is no surprise.

5.3. Congruence with the Manifestation History

Let us generalize this. Accounting for the manifestation history and the disjointness of $\{p_j\}$,

$$\sum_t \Pr_s(p_t \& q) = \Pr_s((\bigvee_t p_t) \& q \mid M_P \& M_Q). \quad (15)$$

Conditionalize the RHS probability on the disjunction $\bigvee_t p_t$ (whose probability is 1), and note that, while $M_Q^{[2]}$ may be made implicit without ambiguity, the manifestation $M_P^{[1]}$ must be denoted explicitly, thereby obtaining the correct form of the marginal-probability formula for sequences involving incompatible variables:

$$\sum_t \Pr_s(p_t \& q) = \Pr_s(q \mid M_P). \quad (16)$$

The RHS expresses Q4 explicitly: *Though we have ignored the value of an observed variable, we may not ignore the fact of that variable's observation.*

This discussion leads to the rule

The probability of an event-sequence necessarily includes the manifestation history in the condition of the probability expression (and the event-sequence must be congruent with that manifestation history).⁹

The simultaneous manifestation $M_{Face}^{[1]} \wedge M_{Suit}^{[1]}$ is impossible (as already pointed out, in Sec. 4. An impossible condition leads to an undefined conditional probability (cf. Eq. (A2)); thus $\Pr_x(K \wedge S) \equiv \Pr_x(K^{[1]} \wedge S^{[1]} \mid M_{Face}^{[1]} \wedge M_{Suit}^{[1]})$ is undefined—disallowed logically, not by decree. Generally,

The simultaneous manifestation of the values of several incompatible variables is impossible, hence the probability of the simultaneous conjunction (or disjunction) of their values is meaningless—their joint probability distribution does not exist.

5.4. Significance of Manifestation

In a probabilistic system, events are physical processes, interactions with the exterior of the system; the nature of the event, in particular the identity of the variable which is randomly affected ("takes on a value") in an event, depends on the details of the physical interaction. We have expressed this as *manifestation*, a necessary component of any discussion of processes in physically realistic probabilistic systems (quantal or other). Manifestation is (due to) the physical interaction of the system under consideration with its exterior; in the absence of

⁹The event-sequences congruent with a given manifestation history form a Boolean event space.

	$K^{[1,2]}$	$Q^{[1,2]}$	$J^{[1,2]}$
$K^{[0]}$	-0.90	0.40	0.50
$Q^{[0]}$	0.40	-0.50	0.10
$J^{[0]}$	0.50	0.10	-0.60
$\Pr ((S \vee H \vee D)^{[1]} \wedge Face_k^{[2]} \mid Face_j^{[0]}) - \Pr (Face_k^{[1]} \mid Face_j^{[0]})$			

Table 3. The effect of ignoring a prior observation of *Suit*—an apparent failure of the marginal-probability formula (Q4).

such interaction, there is no event, no probabilistic branching. Manifestation implements Bohr’s dictum that the entire situation must be taken into account; while Bohr’s requirement is (merely) metaphysical, the requirement that manifestation be taken into account is a logical consequence of this analysis.

Both quantum mechanics and the theory describing *S* are probability theories of *sequences of events*, not of the events themselves. These sequences’ probabilities are defined in terms of a classical Boolean probability space whose elements are the sequences congruent with a given manifestation history. When we carry out all analysis in terms of the sequence-element, no anomalies arise. It may appear that these probabilities fail the Kolmogorov postulates if we forget their contextuality: that congruence. Taking into account the physical situation—the manifestation history—properly restricts the choice of the sequence of events, guaranteeing a Boolean probability space. As we see in these ordinary systems, this contextuality is a respectable, non-subjective property of a probabilistic system with more than one variable.

It is common in probability applications to ignore (“sum out”) the outcome of an event; we’ve seen, however, that in general *we may not ignore the fact that an outcome was ignored*—we must take into account the fact of the manifestation of the ignored variable to avoid numerous anomalies, such as the apparent failure of the formula for marginal probability, the appearance of non-Boolean probability structures, and difficulties such as the “Curious” results of Ref. 5 (which I discuss in Ref. 6).

5.5. Margenau, Marginal Probabilities, and Manifestation

For a quantum mechanical system prepared in the state Ψ , we have

$$\Pr_{\Psi}(q_k) = |\langle q_k | \Psi \rangle|^2. \tag{17a}$$

	$K^{[2]}$	$Q^{[2]}$	$J^{[2]}$
$K^{[0]}$	-0.005	0.020	-0.015
$Q^{[0]}$	0.020	-0.080	0.060
$J^{[0]}$	-0.015	0.060	-0.045
$\Pr(R^{[1]} \wedge Face_k^{[2]} Face_j^{[0]}) - \Pr((H \vee D)^{[1]} \wedge Face_k^{[2]} Face_j^{[0]})$			

Table 4. Numerical results demonstrating interference (Q5)—the apparent failure of the distributive rule $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$.

But also, for the non-degenerate values $\{p_j\}$ (whose occurrence, in quantum mechanics, is Markovian: $\Pr_\Psi(q_k | p_j) = \Pr_{p_j}(q_k)$), we have

$$\begin{aligned} \sum_t \Pr_\Psi(p_t \& q_k) &= \sum_t \Pr_\Psi(q_k | p_t) \Pr_\Psi(p_t) \\ &= \sum_t |\langle q_k | p_t \rangle|^2 |\langle p_t | \Psi \rangle|^2. \end{aligned} \quad (17b)$$

According to the conventionally accepted formula for marginal probability (Eq. (A3)), these should be equal; however, as Margenau [2] pointed out, they are not:

$$\sum_t |\langle q_k | p_t \rangle|^2 |\langle p_t | \Psi \rangle|^2 \neq |\langle q_k | \Psi \rangle|^2. \quad (18)$$

Margenau interpreted this as establishing the failure of classical probability within quantum mechanics.

However, as exemplified in Sec. 4, this “failure” occurs in ordinary probability settings. The marginal-probability formula for sequences of events is correctly given by Eq. (16); Eq. (17b) is to be equated, not with Eq. (17a), but with $\Pr_\Psi(q_k | M_P)$. Now (cf. Eq. (3))

$$\Pr_\Psi(q_k | M_P) = \Pr_{\Psi \& M_P}(q_k) = \text{Tr}\{\rho_P(\Psi) |q_k\rangle \langle q_k|\}, \quad (19)$$

with $\rho_P(\Psi)$, the p-state after preparation in Ψ followed by manifestation of P , given by

$$\rho_P(\Psi) = \sum_t |p_t\rangle \langle p_t| |\Psi\rangle \langle \Psi| |p_t\rangle \langle p_t|; \quad (20)$$

thus

$$\Pr_{\Psi}(q_k | M_P) = \sum_t |\langle q_k | p_t \rangle|^2 |\langle p_t | \Psi \rangle|^2, \quad (21)$$

in accordance with the correct marginal-probability formula Eq. (16). No failure of classical probability within quantum mechanics arises here—only a failure to apply classical probability correctly.

6. INDETERMINATE VALUES

There is a further property which has played a major role in the interpretation of quantum mechanics—that of the "nonreality" of quantum systems, the value-indeterminate nature of variables:

Q6. The variables are value-indeterminate, having no value except as one arises upon observation.

Value-indeterminacy is suggested by Q1–Q3—Bohr included it in his metaphysical principle of *complementarity*. In quantum mechanics, it is suggested even more strongly by results arising directly out of the formalism [7, 8]. Although quite disturbing to classically-trained physicists with a bias toward determinism (that is, all of us), it is not in fact particularly strange—merely by making the system \mathcal{S} of Sec. 4 truly nondeterministic, we obtain a real system (\mathcal{S}_{vi} , below) which is seen by internal analysis to be nonrealistic (value-indeterminate).

First, some definitions:

Definition. *Value-Determinate System.* At every instant there exists, for each variable of the system, a value which would be the result were that variable to be the next observed.

(This does not require that, at such instant, the variable physically have the value that it will, if observed, display.)

Definition. *Deterministic System.* The outcome of a future event is a function of the values of the variables of the system at an earlier time. (This is *future-determinism*, all we need for the following development.) In a deterministic system the outcome of the next observation of a variable is determined by the present state, so the result of that future observation has a value now: *A deterministic system is necessarily value-determinate.*

6.1. Examples of Value-Determinacy and -Indeterminacy

In the system \mathcal{S} , introduced and discussed in Sec. 4, the shuffling of the deck was taken to be deterministic (e.g., mechanical); in that

case, the outcome of the next pick exists now—the variables are value-determinate. However, this determinism is not necessary—the choice of the next card may be made nondeterministically, and this may lead to value-indeterminacy. To implement this nondeterministic choice we may use any of a number of physical systems with random behavior—nuclear decay, Josephson junction tunneling, photons impinging on a beam splitter—to generate truly nondeterministic random numbers to pick the cards. Thus, consider S_{vi} :

System S_{vi} . The system S_{vi} is exactly as S except that the deck is shuffled nondeterministically.

The statistics of S_{vi} and of S are identical, satisfying all of Eq. (11). S_{vi} is perfectly real (and easily constructed),¹⁰ but it is not “realistic”—it is clearly a value-*indeterminate* system: if the most recently observed variable were, say, *Face*, then whatever value would appear if *Suit* were to be observed in a moment *does not exist* before that process of observation, and will only be brought into existence by the nondeterministic shuffling of the deck at the time of observation.

The value-indeterminacy of S_{vi} arises strictly from its nondeterministic card choice; the systems S and S_{vi} are otherwise identical. However, nondeterminism does not, in and of itself, imply value-indeterminacy, as we see in S_{vd} :

System S_{vd} . The system S_{vd} is a modification of S , in which the manifestation rule is carried out in the order 2, 3, 1 (i.e., shuffle last).

E.g., observing *Face* following the prior observation of *Suit* = H, the subdeck consists of all the H's. Report the *Face*-value of its top card, Q, say; create a new subdeck consisting of all the Q's, then shuffle it.

S_{vd} is value-determinate (whether the shuffling is deterministic or nondeterministic): both *Face* and *Suit* have values (those of the top card) *prior* to an observation of either—although only one can be observed, the other being disturbed by that observation. (The statistical properties of S_{vd} , including interference, are identical with S and S_{vi} .)

6.2. Discussion: Value Indeterminacy in Quantum Mechanics

These examples show us that “nonrealism” is a straightforward possibility in nondeterministic systems. However, in contrast with Q2–Q5,

¹⁰ S_{vi} is not “classical”; classical physics is the study of a strictly deterministic world. Neither, however, does the presence of a Josephson junction, say, make S_{vi} “quantal”; no aspect of the theory of quantum mechanics need be used in its analysis.

which are statistical, that is, phenomenological, value-indeterminacy is an ontic property with no characteristic empirical consequence. It can be demonstrated only by analysis of the workings of the system: \mathcal{S} and \mathcal{S}_{vi} have identical observable behavior, but one is value-determinate, the other value-indeterminate. Furthermore, while nondeterminism is necessary for value indeterminacy, as we have seen from the example of \mathcal{S}_{vd} it is not sufficient.

We should note here that interference depends neither on value-indeterminacy nor on nondeterminism: it occurs in the value-determinate \mathcal{S}_{vd} as well as in the value-indeterminate \mathcal{S}_{vi} , both nondeterministic, as well as in the deterministic and value-determinate \mathcal{S} .

The problem of indeterminate values—the lack of "reality" of the values of variables—has been a central difficulty for the interpretation of quantum mechanics: Ref. 9, for example, concludes with "In this book we have been mainly concerned with the difficulties encountered by a simple-minded realism of possessed values"; because of indeterminacy, Ref. 10 considers reality to be "veiled"; the consistent-histories interpretation and the various modal interpretations all have as their central purpose the avoidance of value-indeterminate variables, while Copenhagenism goes to the other extreme, metaphysically demanding value-indeterminacy under the philosophical principle of complementarity.

However, in the ordinary, non-quantal system \mathcal{S}_{vi} , the values are indeterminate—the mechanism of this system makes it clear that a newly manifested variable *had no value* prior to its manifestation. Value-indeterminacy is a normal possibility in a nondeterministic system (at a certain point in the manifestation process, a nondeterministic choice brings one variable's value into existence, and, at the same moment, pushes the other variable's value out of existence).¹¹ Thus, value indeterminacy has no "explanation" beyond the ontic fact of nondeterminism: though we have complete understanding of their internal structure and behavior, we gain no explanation of the nonrealism of our classical examples beyond the analysis which demonstrates it.

We may caricature as the view of Heisenberg and of Bohr, respectively, that the incompatibility of variables is *epistemic* (it is merely that we have no technique, even, perhaps, in principle, to observe at one instant the values of incompatible variables) or *ontic* (it is that they have no such values). Each of these alternatives is, in fact, quite possible, as \mathcal{S} and \mathcal{S}_{vi} illustrate. Epistemic and ontic incompatibility are not other than the value-determinacy or -indeterminacy of the incompatible variables.

The mystery of the indeterminate values of quantum mechanics is not to be resolved through (unattainable) detailed knowledge of the

¹¹The classic example of indeterminate value is due to Aristotle: $B =$ "There will be a sea-battle tomorrow." Then $B \vee \sim B$ is true, but, assuming nondeterminism in human affairs, neither B nor $\sim B$ has a truth value today.

system, nor by deep metaphysics; it is not other than the mystery of nondeterminism.

7. CONCLUSION

Each of the phenomena Q1–Q6 has been considered, by one author or another, to be characteristic of quantum mechanics, inexplicable and unacceptable from the viewpoint of classical physics. The problematic nature of these apparently quantal properties—incompatibility, the non-existence of dispersion-free pure states, interference, value-indeterminacy—seems to call for the “interpretation” of quantum mechanics—but, unfortunately, not in any single interpretational direction: The problems raised by Q1–Q3 encouraged Bohr’s complementarity principle and “Copenhagenism”; the failure of the distributive rule of logic (Q4 and Q5) has given rise to various quantum probabilities and logics. The various modal and consistent-histories interpretations arose primarily in order to solve the “problem” of quantal “nonrealism,” Q6.¹²

There are, of course, interpretive difficulties with quantum mechanics other than those associated with Q1–Q6: The issues of “collapse” and the Measurement Problem (alias “Schrödinger’s Cat”), which have generated the decoherence approach; and the impossibility, established by Bell, of reducing quantum mechanics (specifically, its distant correlations) to classical-mechanical kinetic theory (which includes, of course, any scheme based on systems of the sort presented in this paper). I have not dealt with any of these issues in this paper.¹³

But the phenomena Q2–Q5 are, from a probability viewpoint, quite unexceptional, even expected, and certainly comprehensible: they have all been exemplified in an ordinary (“classical”) probabilistic system \mathcal{S} . The appearance of Q2–Q5 in this emphatically non-quantal system establishes decisively that none of these phenomena are quantal.

Both quantum mechanics and \mathcal{S} (absolutely distinct from quantum mechanics) are examples of probabilistic systems having more than one variable; because Q2–Q5 appear in both, we must conclude that

¹²This oversimplification of the development of these interpretive systems is not, I think, misleading in the present context.

¹³However, I might comment that the first of these is greatly clarified by recalling that a the state concept in a probabilistic system is categorically different from that of a deterministic system; the second and third are similarly greatly clarified by the recognition that a probabilistic system, which quantum mechanics certainly is, may well be (though there is no phenomenological test) irreducibly probabilistic, that is, nondeterministic, hence its variables may well be value-indeterminate (Q6)—in which case the usual derivations of the Bell conditions fail, and the positivity requirement in Fine’s derivation [11] loses cogency.

these phenomena are characteristic of (some subset of) probabilistic systems of several variables, but are characteristic neither of S nor of quantum mechanics.¹⁴ Nor are these phenomena in any way *quantal* weirdness: any metaphysical problems these phenomena may present are problems for the entirety of their subset of theories, and require no special interpretations of quantum mechanics for the understanding of their implications (although their appearance in this card game example rather removes the sense there might be "implications" needing "understanding").

No explanation of the appearance of Q2-Q5 in quantum mechanics is necessary beyond noting that quantum mechanics is a probabilistic system which has more than one variable. It may be that analysis of the workings of a system exhibiting Q2-Q5 will shed light on the particulars of the mechanism by which they are expressed (though, in quantum mechanics, seventy-five years of trying have yielded no such prize). That, in fact, not much such light can or need be shed is supported by the theorem of Sec. 3.1: Given the Hilbert-space formalism of quantum mechanics, statistical incompatibility of variables requires neither more nor less than the noncommutativity of the variables' operators: there is no room for further explanation (in the sense that, as mechanical conservation laws are sufficient to explain the center-of-mass motion of colliding billiard balls, no room is left for further explanation of that center-of-mass motion though study of the internal elastic response during the collision.)

Q1 is not a phenomenon, but an ontic property; its empirical results arise probabilistically in Q2-Q5. Q1 is seen to be true for the classical system presented here by analysis of that system; it is *suspected* to be true of quantum mechanics from numerous analyses of the experimental conditions necessary to manifest distinct variables, all of which support this suspicion. (Q1 could be shown to be *true* of quantum mechanics only through an analysis of its internal workings, but—as far as we know or believe—quantum mechanics *has* no internal workings.)

Value-indeterminacy (Q6), the "non-reality" of variables, is (obviously) the norm for the variables of a nondeterministic system: their values leap into existence at each event. In the case of several variables, it is *repeatability* (per von Neumann)—the value-determinacy of an already-observed variable—which makes the value-indeterminacy of the other variables seem "wrong." Non-reality of this kind is a problem only given a belief in an underlying determinism for quantum mechanics. Whatever metaphysical difficulties Q1-Q6—incompatibility,

¹⁴Socrates' mortality is characteristic, not of Socrates, but of a subset of all beings, the *mortal* beings, of which he is a member. Although a study of the details of Socrates' mortality may bear fruit (or hemlock), questions regarding the meaning of his mortality—its "interpretation"—must be directed at, and studied within the context of, *mortal beings*.

nondeterminism and nonrealism—may bring, such difficulties are not particular to quantum mechanics, nor do they call for heroic efforts of quantum interpretation; to the extent that an interpretation of quantum mechanics is based on *these* properties, it is irrelevant.

APPENDIX A: ELEMENTARY PROBABILITY OF PROPOSITIONS

Each event of a probabilistic system is characterized by the set of propositions which take on truth values; there can be no presumption that all propositions regarding the system need take on a truth value at each event.¹⁵ For all propositions which do take on truth values in a given event we have the following:¹⁶

$$\Pr(F) = 0 \leq \Pr(a) \leq 1 = \Pr(T), \quad (\text{A1a})$$

$$\Pr(a \vee b) + \Pr(a \wedge b) = \Pr(a) + \Pr(b), \quad (\text{A1b})$$

with F and T the absurd and trivial propositions, respectively; hence

$$\Pr(\sim a) = 1 - \Pr(a). \quad (\text{A1c})$$

The set $\{a_j\}$ is *disjoint* iff, whenever all $\{a_j\}$ take on values, $a_j \wedge a_{j'} = F$, $j \neq j'$; a disjoint set satisfies $\sum_t \Pr_s(a_t) = \Pr_s(\bigvee_t a_t)$ for all preparations s . The set $\{a_j\}$ is *complete* iff, whenever all $\{a_j\}$ take on values, $\bigvee_t a_t = T$; hence, a disjoint, complete set satisfies $\sum_t \Pr_s(a_t) = 1$ for all preparations s .

The conditional probability (probability conditioned on an occurrent fact), defined by

$$\Pr(b | a) = \begin{cases} \Pr(a \wedge b) / \Pr(a), & \Pr(a) > 0, \\ \text{undefined}, & \text{otherwise,} \end{cases} \quad (\text{A2})$$

is the probability of the truth of the proposition b given that the fact stated by the proposition a occurs. Examples of the condition a would be "the coin was flipped," "the Jokers were removed from the deck," "the first card was a King"; equally well (though not seen in this paper), the condition may be yet to occur: "the probability of drawing a King

¹⁵One might draw a card from a deck, or one might throw the deck into the air; the proposition p = "A card landed on the chair" is true or false in the second event, but has neither meaning nor truth value in the first event.

¹⁶Disjunction ("or") is indicated by \vee ; conjunction ("and"), by \wedge ; negation ("not") by \sim .

given that the card drawn after it is a Spade." The condition is an occurrent fact—the only place, in a probabilistic theory, where "what actually happens" can appear.

Given the disjoint and complete set $\{p_j\}$ and an arbitrary value q , what can be said of $\sum_t \text{Pr}_s(p_t \wedge q)$? Because $\{p_j\}$ is disjoint, $\{p_j \wedge q\}$ is disjoint; hence $\sum_t \text{Pr}_s(p_t \wedge q) = \text{Pr}_s((\bigvee_t p_t) \wedge q)$. Since the set $\{p_j\}$ is complete, $\bigvee_t p_t = T$, thus

$$\sum_t \text{Pr}_s(p_t \wedge q) = \text{Pr}_s(q), \quad (\text{A3})$$

the formula of marginal probability. (The use of the term "marginal" refers to row- and column-sums in the margins of a table of probabilities of $p_j \wedge q_k$.)

APPENDIX B: THE SYSTEM OF SEC. 4

System: a deck of cards marked with three¹⁷ "variables," P , Q , and R (think of *Face*, *Suit*, and, say, *Letter*), each taking on V values denoted respectively p_k , q_l , and r_m . The specific card denoted $(p_k \cdot q_l \cdot r_m)$ appears $N(p_k \cdot q_l \cdot r_m)$ times in the deck. We have (in all permutations) $N(p_k \cdot q_l) = \sum_m N(p_k \cdot q_l \cdot r_m)$ and $N(p_k) = \sum_{lm} N(p_k \cdot q_l \cdot r_m)$. The restriction that each value of each variable has equal *a priori* probability requires $N(p_k) = N(q_l) = N(r_m) \equiv N$; the total number of cards in the deck is then NV , and the fractional occurrence of each card in the deck is

$$f(p_k \cdot q_l \cdot r_m) = n_{klm}/V, \quad (\text{B1})$$

where

$$n_{klm} \equiv N(p_k \cdot q_l \cdot r_m)/N. \quad (\text{B2})$$

Note that all *double* sums of n_{klm} equal 1, and all *triple* sums equal V .

1. Analysis of the System

According to the rules of system \mathcal{S} (and systems \mathcal{S}_{vi} and \mathcal{S}_{vd} , as well), the probability of the occurrence of the specific card $(p_k \cdot q_l \cdot r_m)$ at the top of the subdeck, the system having been prepared in x and the value $P = p_j$ (or $Q = q_k$) having been observed, is

$$\text{Pr}_x((p_k \cdot q_l \cdot r_m) \mid p_j) = \delta_{kj} n_{jlm}, \quad (\text{B3a})$$

$$\text{Pr}_x((p_k \cdot q_l \cdot r_m) \mid q_j) = \delta_{lj} n_{kjm}. \quad (\text{B3b})$$

¹⁷The generalization to more than three variables is obvious, and has no effect on the results, Eqs. (B4), (B6), (B9), and (B11).

Summing Eqs. (B3) over l and m , we have

$$\Pr_x(p_k | p_j) = \delta_{kj}, \quad (\text{B4a})$$

$$\Pr_x(p_k | q_j) = N(p_k \cdot q_j)/N. \quad (\text{B4b})$$

Equation (7b) establishes that

$$\Pr_x(p_k | q_j) = \Pr_x(q_j | p_k). \quad (\text{B5})$$

$\Pr_x(y | p_k)$ does not depend on the preparation state x (assuming $\Pr_x(p_k) \neq 0$)—that is, the system's probabilities are Markovian:

$$\Pr_x(p_k \& y) = \Pr(y | p_k) \Pr_x(p_k). \quad (\text{B6})$$

Henceforth in this Appendix we drop the preparation-state subscript from conditional probabilities.

Consider the "marginal probability" summation (where neither of the variables X or Y is the variable P):

$$\begin{aligned} \sum_t \Pr_x(p_t \& y) &= \sum_t \Pr(y | p_t) \Pr_x(p_t) \\ &= \frac{1}{N^2} \sum_t N(p_t \cdot x) N(p_t \cdot y). \end{aligned} \quad (\text{B7})$$

However, $\Pr(y | x)$ is 0 or 1, if x and y are values of the same variable, or $N(x \cdot y)/N$, if they are values of distinct variables X, Y . It is obvious, in the former case, that $\sum_t \Pr_x(p_t \& y) \neq \Pr_x(y)$; numerical examples easily establish the occurrence of this inequality in the latter case.

2. Analysis of the System—Interference

Introduce the additional variable Π with values $\{\pi_j\}$. Π is a function of the variable P , $\Pi = f(P)$; the function is defined by $f(p_1) = \pi_1$, $f(p_2) = \pi_1$, $f(p_j) = \pi_{j-1}$, $j > 2$. Then $N(\pi_1 \cdot q_k \cdot r_l) = N(p_1 \cdot q_k \cdot r_l) + N(p_2 \cdot q_k \cdot r_l) = 2N$, and $N(\pi_j \cdot q_k \cdot r_l) = N(p_{j+1} \cdot q_k \cdot r_l) = N$ for $j > 1$. The rule for the observation of a variable is unchanged. (In the example of Sec. 4, P is *Suit*, p_1 is H, p_2 is D, p_3 is S, Π is *Color*, π_1 is R and π_2 is B.)

The probability of the occurrence of a specific card at the top of the subdeck, the system having been prepared in x and the value $P = p_j$, $Q = q_k$, or $\Pi = \pi_1$ having been observed, is

$$\Pr((\pi_1 \cdot q_l \cdot r_m) | p_j) = (\delta_{1j} + \delta_{2j}) n_{jlm}, \quad (\text{B8a})$$

$$\Pr((\pi_1 \cdot q_l \cdot r_m) | q_j) = \delta_{lj} (n_{1lm} + n_{2lm}), \quad (\text{B8b})$$

$$\Pr((p_j \cdot q_l \cdot r_m) | \pi_1) = (\delta_{j1} n_{1lm} + \delta_{j2} n_{2lm})/2. \quad (\text{B8c})$$

Summing Eqs. (B8a) and (B8c) over l and m , we find

$$\Pr(\pi_1 | p_j) = \Pr(p_1 | p_j) + \Pr(p_2 | p_j) = 2\Pr(p_j | \pi_1); \quad (\text{B9a})$$

summing Eq. (B8b) over l and m , and in Eq. (B8c) exchanging j and l and then summing over l and m , we find

$$\Pr(\pi_1 | q_j) = \Pr(p_1 | q_j) + \Pr(p_2 | q_j) = 2\Pr(q_j | \pi_1). \quad (\text{B9b})$$

In the definition of interference, Eq. (10), we take $E_P = \Pi_1$. Then Eqs. (B9) show that Eq. (10a) is satisfied, and Eq. (10b), the interference, becomes

$$\Pr(\pi_1 \& y | x) - \Pr((p_1 \vee p_2) \& y | x). \quad (\text{B10})$$

Because π_1 completely specifies the p-state, Eq. (B6) generalizes to

$$\Pr(\pi_1 \& y | x) = \Pr(y | \pi_1) \Pr(\pi_1 | x) = \sum_{s,t=1}^2 \Pr(y | p_s) \Pr(p_t | x)$$

(using Eq. (B9)). Thus the interference is given by

$$-\Pr(y | p_1) \Pr(p_2 | x) - \Pr(y | p_2) \Pr(p_1 | x), \quad (\text{B11})$$

which does not vanish in general. (This *classical* interference arises from the off-diagonal terms of a sum, exactly as in the corresponding quantum-mechanical expression.)

APPENDIX C: EXERCISES

These back-of-the-envelope exercises introduce the reader to some of the principles involved in several-variable stochastic systems.

Given a deck of cards, we may choose to observe the value of either *Face* or *Suit*; the rule for doing so is:

1. Draw a card.
- 2a. To observe *Face*: report the card's *Face* value; if *Face* = K, return the card to the deck, otherwise discard it.
- 2b. To observe *Suit*: report the card's *Suit* value; if *Suit* = H, return the card to the deck, otherwise discard it.

The deck is {KS, QH}; hence {K, Q} is a complete set of values of *Face*, and {S, H} is a complete set of values of *Suit*.

Exercise 1. (Q1) Show that it is not always possible to observe both *Face* and *Suit* simultaneously. (Hint: suppose the card drawn is KS.)

Exercise 2. (Q2) Show that observations of *Suit* and *Face* do not commute temporally; for example, show that $\Pr(K^{[1]} \wedge S^{[2]}) = 1/4$, while $\Pr(S^{[1]} \wedge K^{[2]}) = 0$. What does this imply regarding the existence of joint distributions (such as $\Pr(S \wedge K)$)?

Exercise 3. (Q4) Show that Eq. (8) gives an ambiguous value for $\Pr(K^{[2]})$, by showing that $\Pr((K \vee Q)^{[1]} \wedge K^{[2]}) = 3/4$, while $\Pr((S \vee H)^{[1]} \wedge K^{[2]}) = 0$. How can this be reconciled with the apparent fact that $K \vee Q = S \vee H = T$?

(No exercise regarding Q3 is possible: because observations on this system do not repeat, “sharpness” has no meaning.)

REFERENCES

1. M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).
2. H. Margenau, “Measurements in quantum mechanics,” *Ann. Phys.* **23**, 469–485 (1963).
3. J. E. G. Farina, “An elementary approach to quantum probability,” *Am. J. Phys.* **61**(5), 466–468 (1993).
4. R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Vol. III* (Addison-Wesley, Reading, MA, 1965).
5. D. Z. Albert, Y. Aharonov, and S. D’Amato, “Curious new statistical prediction of quantum mechanics,” *Phys. Rev. Lett.* **54**(1), 5–7 (1985).
6. K. A. Kirkpatrick, “Classical three-box ‘paradox’,” *J. Phys. A* **36** (17), 4891–4900 (2003); arXiv: quant-ph/0207124.
7. J. S. Bell, “On the problem of hidden variables in quantum mechanics,” *Rev. Mod. Phys.* **38**(3), 447–452 (1966).
8. S. Kochen and E. P. Specker, “The problem of hidden variables in quantum mechanics,” *J. Math. Mech.* **17**(1), 59–87 (1967).
9. M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Paperbacks, Oxford, 1989).
10. B. d’Espagnat, *Veiled Reality* (Addison-Wesley, Reading, MA, 1995).
11. A. Fine, “Joint distributions, quantum correlations, and commuting observables,” *J. Math. Phys.* **23**(7), 1306–1310 (1982).