Extremal graph theory

David Conlon

Lecture 1

The basic statement of extremal graph theory is Mantel's theorem, proved in 1907, which states that any graph on n vertices with no triangle contains at most $n^2/4$ edges. This is clearly best possible, as one may partition the set of n vertices into two sets of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and form the complete bipartite graph between them. This graph has no triangles and $\lfloor n^2/4 \rfloor$ edges.

As a warm-up, we will give a number of different proofs of this simple and fundamental theorem.

Theorem 1 (Mantel's theorem) *If a graph G on n vertices contains no triangle then it contains at most $\frac{n^2}{4}$ edges.*

First proof Suppose that G has m edges. Let x and y be two vertices in G which are joined by an edge. If $d(v)$ is the degree of a vertex v, we see that $d(x) + d(y) \leq n$. This is because every vertex in the graph G is connected to at most one of x and y. Note now that

$$\sum_x d^2(x) = \sum_{xy \in E} (d(x) + d(y)) \leq mn.$$

On the other hand, since $\sum_x d(x) = 2m$, the Cauchy-Schwarz inequality implies that

$$\sum_x d^2(x) \geq \frac{\left(\sum_x d(x)\right)^2}{n} \geq \frac{4m^2}{n}.$$

Therefore

$$\frac{4m^2}{n} \leq mn,$$

and the result follows. $\square$

Second proof We proceed by induction on n. For $n = 1$ and $n = 2$, the result is trivial, so assume that we know it to be true for $n - 1$ and let G be a graph on n vertices. Let x and y be two adjacent vertices in G. As above, we know that $d(x)+d(y) \leq n$. The complement H of x and y has $n-2$ vertices and since it contains no triangles must, by induction, have at most $(n-2)^2/4$ edges. Therefore, the total number of edges in G is at most

$$e(H) + d(x) + d(y) - 1 \leq \frac{(n-2)^2}{4} + n - 1 = \frac{n^2}{4},$$

where the -1 comes from the fact that we count the edge between x and y twice. $\square$

Third proof Let A be the largest independent set in the graph G. Since the neighborhood of every vertex x is an independent set, we must have $d(x) \leq |A|$. Let B be the complement of A. Every edge in G must meet a vertex of B. Therefore, the number of edges in G satisfies

$$e(G) \leq \sum_{x \in B} d(x) \leq |A||B| \leq \left(\frac{|A|+|B|}{2}\right)^2 = \frac{n^2}{4}.$$

Suppose that n is even. Then equality holds if and only if $|A| = |B| = n/2$, $d(x) = |A|$ for every $x \in B$ and B has no internal edges. This easily implies that the unique structure with $n^2/4$ edges is a bipartite graph with equal partite sets. For n odd, the number of edges is maximised when $|A| = \lceil n/2 \rceil$ and $|B| = \lfloor n/2 \rfloor$. Again, this yields a unique bipartite structure. $\square$

The last proof tells us that not only is $\lfloor n^2/4 \rfloor$ the maximum number of edges in a triangle-free graph but also that any triangle-free graph with this number of edges is bipartite with partite sets of almost equal size.

The natural generalisation of this theorem to cliques of size r is the following, proved by Paul Turán in 1941.

Theorem 2 (Turán's theorem) *If a graph G on n vertices contains no copy of K_{r+1}, the complete graph on $r+1$ vertices, then it contains at most $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$ edges.*

First proof By induction on n. The theorem is trivially true for $n = 1, 2, \ldots, r$. We will therefore assume that it is true for all values less than n and prove it for n. Let G be a graph on n vertices which contains no K_{r+1} and has the maximum possible number of edges. Then G contains copies of K_r. Otherwise, we could add edges to G, contradicting maximality.

Let A be a clique of size r and let B be its complement. Since B has size $n - r$ and contains no K_{r+1}, there are at most $\left(1 - \frac{1}{r}\right) \frac{(n-r)^2}{2}$ edges in B. Moreover, since every vertex in B can have at most $r - 1$ neighbours in A, the number of edges between A and B is at most $(r-1)(n-r)$. Summing, we see that

$$e(G) = e_A + e_{A,B} + e_B \leq \binom{r}{2} + (r-1)(n-r) + \left(1 - \frac{1}{r}\right) \frac{(n-r)^2}{2} = \left(1 - \frac{1}{r}\right) \frac{n^2}{2},$$

where $e_A, e_{A,B}$ and e_B are the number of edges in A, between A and B and in B respectively. The theorem follows. $\square$

Second proof We again assume that G contains no K_{r+1} and has the maximum possible number of edges. We will begin by proving that if $xy \notin E(G)$ and $yz \notin E(G)$, then $xz \notin E(G)$. This implies that the property of not being connected in G is an equivalence relation. This in turn will imply that the graph must be a complete multipartite graph.

Suppose, for contradiction, that $xy \notin E(G)$ and $yz \notin E(G)$, but $xz \in E(G)$. If $d(y) < d(x)$ then we may construct a new K_{r+1}-free graph G' by deleting y and creating a new copy of the vertex x, say x'. Since any clique in G' can contain at most one of x and x', we see that G' is K_{r+1}-free. Moreover,

$$|E(G')| = |E(G)| - d(y) + d(x) > |E(G)|,$$

contradicting the maximality of G. A similar conclusion holds if $d(y) < d(z)$. We may therefore assume that $d(y) \geq d(x)$ and $d(y) \geq d(z)$. We create a new graph G'' by deleting x and z and creating two extra copies of the vertex y. Again, this has no K_{r+1} and

$$|E(G'')| = |E(G)| - (d(x) + d(z) - 1) + 2d(y) > |E(G)|,$$

so again we have a contradiction.

We now know that the graph is a complete multipartite graph. Clearly, it can have at most r parts. We will show that the number of edges is maximised when all of these parts have sizes which differ by at most one. Indeed, if there were two parts A and B with $|A| > |B| + 1$, we could increase the number of edges in G by moving one vertex from A to B. We would lose $|B|$ edges by doing this, but gain $|A| - 1$. Overall, we would gain $|A| - 1 - |B| \geq 1$. □

This second proof also determines the structure of the extremal graph, that is, it must be r-partite with all parts having size as close as possible. So if $n = mr + q$, we get q sets of size $m + 1$ and $r - q$ sets of size m.

Lecture 2

A perfect matching in a bipartite graph with two sets of equal size is a collection of edges such that every vertex is contained in exactly one of them.

Hall's (marriage) theorem is a necessary and sufficient condition which allows one to decide if a given bipartite graph contains a matching. Suppose that the two parts of the bipartite graph G are A and B. Then Hall's theorem says that if, for every subset U of A, there are at least $|U|$ vertices in B with neighbours in U then G contains a perfect matching. The condition is clearly necessary. To prove that it is sufficient we use the following notation.

For any subset X of a graph G, let $N_G(X)$ be the set of neighbours of X, that is, the set of vertices with a neighbour in X.

Theorem 1 (Hall's theorem) *Let G be a bipartite graph with parts A and B of equal size. If, for every $U \subset A$, $|N_G(U)| \geq |U|$ then G contains a perfect matching.*

Proof Let $|A| = |B| = n$. We will prove the theorem by induction on n. Clearly, the result is true for $n = 1$. We therefore assume that it is true for $n - 1$ and prove it for n.

If $|N_G(U)| \geq |U| + 1$ for every non-empty proper subset U of A, pick an edge $\{a, b\}$ of G and consider the graph $G' = G - \{a, b\}$. Then every non-empty set $U \subset A \backslash \{a\}$ satisfies

$$|N_{G'}(U)| \geq |N_G(U)| - 1 \geq |U|.$$

Therefore, there is a perfect matching between $A\backslash\{a\}$ and $B\backslash\{b\}$. Adding the edge from a to b gives the full matching.

Suppose, on the other hand, that there is some non-empty proper subset U of A for which $|N(U)| = |U|$. Let $V = N(U)$. By induction, since Hall's condition holds for every subset of U, there is a matching between U and V. But Hall's condition also holds between $A\backslash U$ and $B\backslash V$. If this weren't the case, there would be some W in $A\backslash U$ with fewer than $|W|$ neighbours in $B\backslash V$. Then $W \cup U$ would be a subset of A with fewer than $|W \cup U|$ neighbours in B, contradicting our assumption. Therefore, there is a perfect matching between $A\backslash U$ and $B\backslash V$. Putting the two matchings together completes the proof. $\square$

A Hamiltonian cycle in a graph G is a cycle which visits every vertex exactly once and returns to its starting vertex. Dirac's theorem says that if the minimum degree $\delta(G)$ of a graph G is such that $\delta(G) \geq n/2$ then G contains a Hamiltonian cycle. This is sharp since, if one takes a complete bipartite graph with one part of size $\lceil \frac{n}{2} - 1 \rceil$ (and the other the complement of this), then it cannot contain a Hamiltonian cycle. This is simply because one must pass back and forth between the two sets.

Theorem 2 (Dirac's theorem) *If a graph G satisfies $\delta(G) \geq \frac{n}{2}$, then it contains a Hamiltonian cycle.*

Proof First, note that G is connected. If it weren't, the smallest component would have size at most $n/2$ and no vertex in this component could have degree $n/2$ or more.

Let $P = x_0 x_1 \ldots x_k$ be a longest path in G. Since it can't be extended, every neighbour of x_0 and x_k must be contained in P. Since $\delta(G) \geq n/2$, we see that $x_0 x_{i+1}$ is an edge for at least $n/2$ values of i

with $0 \le i \le k-1$. Similarly, $x_i x_k$ is an edge for at least $n/2$ values of i. There are at most $n-1$ values of i with $0 \le i \le k-1$. Therefore, since the total number of edges of the form $x_0 x_{i+1}$ or of the form $x_i x_k$ with $0 \le i \le k-1$ is at least n, there must be some i for which both $x_0 x_{i+1}$ and $x_i x_k$ are edges in G.

We claim that

$$C = x_0 x_{i+1} x_{i+2} \dots x_k x_i x_{i-1} \dots x_0$$

is a Hamiltonian cycle. Suppose not and that there is a set of vertices Y which are not contained in C. Then, since G is connected, there is a vertex x_j and a vertex y in Y such that $x_j y$ is in $E(G)$. But then we may define a path P' starting at y, going to x_j and then around the cycle C which is longer than P. This would contradict our assumption about P. $\square$

A tree T is a connected graph containing no cycles. The Erdős-Sós conjecture states that if a tree T has t edges, then any graph G with average degree t must contain a copy of T. This conjecture has been proven, for sufficiently large graphs G, by Ajtai, Komlós, Simonovits and Szemerédi. Here we prove a weaker version of this conjecture.

Theorem 3 *If a graph G has average degree $2t$, it contains every tree T with t edges.*

Proof We start with a standard reduction, by showing that a graph of average degree $2t$ has a subgraph of minimum degree t. If the number of vertices in G is n, the number of edges in G is at least tn. If there is a vertex of degree less than t, delete it. This will not decrease the average degree. Moreover, the process must end, since any graph with fewer than $2t$ vertices cannot have average degree $2t$.

We now use this condition to embed the vertices of the tree greedily. Suppose we have already embedded j vertices, where $j < t+1$. We will try to embed a new vertex which is connected to some already embedded vertex. By the minimum degree condition, there are at least t possible places to embed this vertex. At most $t-1$ of these are blocked by already embedded vertices, so the embedding may always proceed. $\square$

Lecture 3

For general graphs H, we are interested in the function $ex(n, H)$, defined as follows.

$$ex(n, H) = \max\{e(G) : |G| = n, H \not\subset G\}.$$

Turán's theorem itself tells us that

$$ex(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

We are now going to deal with the general case. We will show that the behaviour of the extremal function $ex(n, H)$ is tied intimately to the chromatic number of the graph H.

Definition 1 *The chromatic number $\chi(H)$ of a graph H is the smallest natural number c such that the vertices of H can be coloured with c colours and no two vertices of the same colour are adjacent.*

The fundamental result which we shall prove, known as the Erdős-Stone-Simonovits theorem, is the following.

Theorem 1 (Erdős-Stone-Simonovits) *For any fixed graph H and any fixed $\epsilon > 0$, there is n_0 such that, for any $n \geq n_0$,*

$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right) \frac{n^2}{2} \leq ex(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) \frac{n^2}{2}.$$

For the complete graph K_{r+1}, the chromatic number is $r+1$, so in this case the Erdős-Stone-Simonovits theorem reduces to an approximate version of Turán's theorem. For bipartite H, it gives $ex(n, H) \leq \epsilon n^2$ for all $\epsilon > 0$. This is an important theme, one we will return to later in the course.

To prove the Erdős-Stone-Simonovits theorem, we will first prove the following lemma, which already contains most of the content.

Lemma 1 *For any natural numbers r and t and any positive ϵ with $\epsilon < 1/r$, there exists an n_0 such that the following holds. Any graph G with $n \geq n_0$ vertices and $\left(1 - \frac{1}{r} + \epsilon\right) \frac{n^2}{2}$ edges contains $r+1$ disjoint sets of vertices $A_1, \ldots, A_{r+1}$ of size t such that the graph between A_i and A_j, for every $1 \leq i < j \leq r+1$, is complete.*

Proof To begin, we find a subgraph G' of G such that every degree is at least $\left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) |V(G')|$. To find such a graph, we remove one vertex at a time. If, in this process, we reach a graph with ℓ vertices and there is some vertex which has fewer than $\left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \ell$ neighbors, we remove it.

Suppose that this process terminates when we have reached a graph G' with n' vertices. To show that n' is not too small, consider the total number of edges that have been removed from the graph. When the graph has ℓ vertices, we remove at most $\left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \ell$ edges. Therefore, the total number of edges removed is at most

$$\sum_{\ell=n'+1}^{n} \left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \ell = \left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \frac{(n - n')(n + n' + 1)}{2} \leq \left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \frac{(n^2 - n'^2)}{2} + \frac{(n - n')}{2}.$$

Also, since G' has at most $\frac{n'^2}{2}$ edges, we have

$$|e(G)| \leq \left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \frac{(n^2 - n'^2)}{2} + \frac{(n - n')}{2} + \frac{n'^2}{2} = \left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right) \frac{n^2}{2} + \left(\frac{1}{r} - \frac{\epsilon}{2}\right) \frac{n'^2}{2} + \frac{(n - n')}{2}.$$

But we also have $|e(G)| \geq \left(1 - \frac{1}{r} + \epsilon\right) \frac{n^2}{2}$. Therefore, the process stops once

$$\left(\frac{1}{r} - \frac{\epsilon}{2}\right) \frac{n'^2}{2} - \frac{n'}{2} < \epsilon \frac{n^2}{4} - \frac{n}{2},$$

that is, when $n' \approx \sqrt{\epsilon r} n$. From now on, we will assume that we are working within this large well-behaved subgraph G'.

We will show, by induction on r, that there are $r+1$ sets $A_1, A_2, \ldots, A_{r+1}$ of size t such that every edge between A_i and A_j, with $1 \leq i < j \leq r+1$, is in G'. For $r = 0$, there is nothing to prove.

Given $r > 0$ and $s = \lceil 3t/\epsilon \rceil$, we apply the induction hypothesis to find r disjoint sets $B_1, B_2, \ldots, B_r$ of size s such that the graph between every two disjoint sets is complete. Let $U = V(G') \backslash \{B_1 \cup \cdots \cup B_r\}$ and let W be the set of vertices in U which are adjacent to at least t vertices in each B_i.

We are going to estimate the number of edges missing between U and $B_1 \cup \cdots \cup B_r$. Since every vertex in $U \backslash W$ is adjacent to fewer than t vertices in some B_i, we have that the number of missing edges is at least

$$\tilde{m} \geq |U \backslash W|(s - t) \geq (n' - rs - |W|) \left(1 - \frac{\epsilon}{3}\right) s.$$

On the other hand, every vertex in G' has at most $\left(\frac{1}{r} - \frac{\epsilon}{2}\right) n'$ missing edges. Therefore, counting over all vertices in $B_1 \cup \cdots \cup B_r$, we have

$$\tilde{m} \leq rs \left(\frac{1}{r} - \frac{\epsilon}{2}\right) n' = \left(1 - \frac{r\epsilon}{2}\right) sn'.$$

Therefore,

$$|W| \left(1 - \frac{\epsilon}{3}\right) s \geq (n' - rs) \left(1 - \frac{\epsilon}{3}\right) s - \left(1 - \frac{r\epsilon}{2}\right) sn' = \epsilon \left(\frac{r}{2} - \frac{1}{3}\right) sn' - \left(1 - \frac{\epsilon}{3}\right) rs^2.$$

Since ϵ, r and s are constants, we can make $|W|$ large by making n' large. In particular, we may make $|W|$ such that

$$|W| > \binom{s}{t}^r (t - 1).$$

Every element in W has at least t neighbours in each B_i. There are at most $\binom{s}{t}^r$ ways to choose a t-element subset from each of $B_1 \cup \cdots \cup B_r$. By the pigeonhole principle and the size of $|W|$, there must therefore be some subsets $A_1, \ldots, A_r$ and a set A_{r+1} of size t from W such that every vertex in A_{r+1} is connected to every vertex in $A_1 \cup \cdots \cup A_r$. Since $A_1, \ldots, A_r$ are already joined in the appropriate manner, this completes the proof. $\square$

Note that a careful analysis of the proof shows that one may take $t = c(r, \epsilon) \log n$. It turns out that this is also best possible (see example sheet).

It remains to prove the Erdős-Stone-Simonovits theorem itself.

Proof of Erdős-Stone-Simonovits For the lower bound, we consider the Turán graph given by $r = \chi(H) - 1$ sets of almost equal size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$. This has roughly the required number of vertices and it is clear that every subgraph of this graph has chromatic number at most $\chi(H) - 1$.

For the upper bound, note that if H has chromatic number $\chi(H)$, then, provided t is large enough, it can be embedded in a graph G consisting of $\chi(H)$ sets $A_1, A_2, \ldots, A_{\chi(H)}$ of size t such that the graph between any two disjoint A_i and A_j is complete. We simply embed any given colour class into a different A_i. The theorem now follows from an application of the previous lemma. $\square$

Lecture 4

The main aim of the next two lectures will be to prove the famous regularity lemma of Szemerédi. This was developed by Szemerédi in his work on what is now known as Szemerédi's theorem. This astonishing theorem says that for any $\delta > 0$ and $k \geq 3$ there exists an n_0 such that, for $n \geq n_0$, any subset of $\{1, 2, \ldots, n\}$ with at least δn elements must contain an arithmetic progression of length k. The particular case when $k = 3$ had been proven earlier by Roth and is accordingly known as Roth's theorem.

Our initial purpose in proving the regularity lemma will be to give another proof of the Erdős-Stone-Simonovits theorem. However, its use is widespread throughout extremal graph theory and we will see a number of other applications in the course.

Roughly speaking, Szemerédi's regularity lemma says that no graph is entirely random because every graph is at least somewhat random. More precisely, the regularity lemma says that any graph may be partitioned into a finite number of sets such that most of the bipartite graphs between different sets are random-like. To be absolutely precise, we will need some notation and some definitions.

Let G be a graph and let A and B be subsets of the vertex set. If we let $E(A, B)$ be the set of edges between A and B, the density of edges between A and B is given by

$$d(A, B) = \frac{|E(A, B)|}{|A||B|}.$$

Definition 1 *Let G be a graph and let A and B be two subsets of the vertex set. The pair (A, B) is said to be ϵ-regular if, for every $A' \subset A$ and $B' \subset B$ with $|A'| \geq \epsilon |A|$ and $|B'| \geq \epsilon |B|$,*

$$|d(A', B') - d(A, B)| \leq \epsilon.$$

We say that a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ is ϵ-regular if

$$\sum \frac{|X_i||X_j|}{n^2} \leq \epsilon,$$

where the sum is taken over all pairs (X_i, X_j) which are not ϵ-regular.

That is, a bipartite graph is ϵ-regular if all small induced subgraphs have approximately the same density as the full graph and a partition of the vertex set of a graph G into smaller sets is ϵ-regular if almost every pair forms a bipartite graph which is ϵ-regular. The regularity lemma is now as follows.

Theorem 1 (Szemerédi's regularity lemma) *For every $\epsilon > 0$ there exists an M such that, for every graph G, there is an ϵ-regular partition of the vertex set of G with at most M pieces.*

In order to prove the regularity lemma, we will associate a function, known as the mean square density, with each partition of $V(G)$. We will prove that if a particular partition is not ϵ-regular it may be further partitioned in such a way that the mean square density increases. But, as we shall see, the mean square density is bounded above by 1, so we eventually reach a contradiction.

Definition 2 *Let G be a graph. Given a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ of the vertex set of G, the mean square density of this partition is given by*

$$\sum_{1 \leq i,j \leq k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2.$$

We now observe that since $\sum_{1\le i,j\le k}\frac{|X_i||X_j|}{n^2}=1$ and $0\le d(X_i,X_j)\le 1$, the mean square density also lies between 0 and 1.

Lemma 1 *For every partition of the vertex set of a graph G, the mean square density lies between* 0 *and* 1.

Another important property of mean square density is that it cannot increase under refinement of a partition. That is, we have the following.

Lemma 2 *Let G be a graph with vertex set $V(G)$. If $X_1,X_2,\dots,X_k$ is a partition of $V(G)$ and $Y_1,Y_2,\dots,Y_\ell$ is a refinement of $X_1,X_2,\dots,X_k$, then the mean square density of $Y_1,Y_2,\dots,Y_\ell$ is at least the mean square density of $X_1,X_2,\dots,X_k$.*

Proof Because the Y_i partition is a refinement of the X_i partition, every X_i may be rewritten as a disjoint union $X_{i1}\cup\cdots\cup X_{ia_i}$, where each $X_{ia_i}=Y_j$, for some j. Now, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
d(X_i,X_j)^2 &= \left(\sum_{s,t}\frac{|X_{is}||X_{jt}|}{|X_i||X_j|}d(X_{is},X_{yt})\right)^2\\
&\le \left(\sum_{s,t}\frac{|X_{is}||X_{jt}|}{|X_i||X_j|}\right)\left(\sum_{s,t}\frac{|X_{is}||X_{jt}|}{|X_i||X_j|}d(X_{is},X_{yt})^2\right)\\
&= \sum_{s,t}\frac{|X_{is}||X_{jt}|}{|X_i||X_j|}d(X_{is},X_{yt})^2.
\end{aligned}$$

Therefore,

$$\frac{|X_i||X_j|}{n^2}d(X_i,X_j)^2\le\sum_{s,t}\frac{|X_{is}||X_{jt}|}{n^2}d(X_{is},X_{yt})^2.$$

Adding over all values of i and j implies the lemma. □

An analogous result also holds for bipartite graphs G. That is, if G is a bipartite graph between two sets X and Y, $\cup_i X_i$ and $\cup_i Y_i$ are partitions of X and Y and $\cup_i Z_i$ and $\cup_i W_i$ refine these partitions, then

$$\sum_{i,j}\frac{|X_i||Y_j|}{n^2}d(X_i,Y_j)^2\le\sum_{i,j}\frac{|Z_i||W_j|}{n^2}d(Z_i,W_j)^2.$$

We will now show that if X and Y are two sets of vertices and the graph between them is not-regular then there is a partition of each of X and Y for which the mean square density increases.

Lemma 3 *Let G be a graph and suppose X and Y are subsets of the vertex set $V(G)$. If $d(X,Y)=\alpha$ and the graph between X and Y is not ϵ-regular then there are partitions $X=X_1\cup X_2$ and $Y=Y_1\cup Y_2$ such that*

$$\sum_{1\le i,j\le 2}\frac{|X_i||Y_j|}{|X||Y|}d(X_i,Y_j)^2\ge\alpha^2+\epsilon^4.$$

Proof Since the graph between X and Y is not ϵ-regular, there must be two subsets X_1 and Y_1 of X and Y, respectively, with $|X_1| \geq \epsilon|X|$, $|Y_1| \geq \epsilon|Y|$ and $|d(X_1, Y_1) - \alpha| > \epsilon$. Let $X_2 = X \backslash X_1$, $Y_2 = Y \backslash Y_1$ and $u(X_i, Y_j) = d(X_i, Y_j) - \alpha$. Then

$$\begin{aligned}
\epsilon^4 &\leq \sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} u(X_i, Y_j)^2 \\
&= \sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} d(X_i, Y_j)^2 - 2\alpha \sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} d(X_i, Y_j) + \alpha^2 \sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} \\
&= \sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} d(X_i, Y_j)^2 - \alpha^2.
\end{aligned}$$

Note that the second line holds since

$$\sum_{1 \leq i,j \leq 2} \frac{|X_i||Y_j|}{|X||Y|} d(X_i, Y_j) = d(X, Y) = \alpha.$$

The result therefore follows. $\square$

Lecture 5

To complete the proof of the regularity lemma, we need to prove that if a partition is not ϵ-regular there is a refinement of this partition which has a higher mean square density. This is taken care of in the following lemma.

Lemma 1 *Let G be a graph and let $X_1 \cup X_2 \cup \cdots \cup X_k$ be a partition of the vertices of G which is not ϵ-regular. Then there is a refinement $X_{11} \cup \cdots \cup X_{1a_1} \cup \cdots \cup X_{k1} \cup \cdots \cup X_{ka_k}$ such that every a_i is at most 2^{2k} and the mean square density is at least ϵ^5 larger.*

Proof Let $I = \{(i,j) : (X_i, X_j) \text{ is not } \epsilon\text{-regular}\}$. Let α^2 be the mean square density of G with respect to $X_1 \cup \cdots \cup X_k$.

For each $(i,j) \in I$, the previous lemma gives us partitions $X_i = A_1^{ij} \cup A_2^{ij}$ and $X_j = B_1^{ij} \cup B_2^{ij}$ for which

$$\sum_{1 \le p,q \le 2} \frac{|A_p^{ij}||B_q^{ij}|}{|X_i||X_j|} d(A_p^{ij}, B_q^{ij})^2 \ge d(X_i, X_j)^2 + \epsilon^4.$$

For each i, let $X_{i1} \cup \cdots \cup X_{ia_i}$ be the partition of X_i which refines all partitions which arise from partitioning X_i or X_j into A_is or B_is. Note that this partition has at most 2^{2k} pieces, that is, $a_i \le 2^{2k}$. Moreover, since refining bipartite partitions does not decrease the mean square density, we have

$$\sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_{ip}||X_{jq}|}{|X_i||X_j|} d(X_{ip}, X_{jq})^2 \ge d(X_i, X_j)^2 + \epsilon^4,$$

for all $(i,j) \in I$. Multiplying both sides of the equation by $\frac{|X_i||X_j|}{n^2}$ and summing over all (i,j), we have

$$\begin{aligned}
\sum_{1 \le i,j \le k} \sum_{p=1}^{a_i} \sum_{q=1}^{a_j} \frac{|X_{ip}||X_{jq}|}{n^2} d(X_{ip}, X_{jq})^2 &\ge \sum_{1 \le i,j \le k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2 + \epsilon^4 \sum_{(i,j) \in I} \frac{|X_i||X_j|}{n^2} \\
&\ge \alpha^2 + \epsilon^5.
\end{aligned}$$

The result follows. □

We now have all the ingredients necessary to finish the proof.

Proof of Szemerédi's regularity lemma Start with a trivial partition into one set. If it is ϵ-regular, we are done. Otherwise, there is a partition into at most 4 sets where the mean square density increases by ϵ^5.

If, at stage i, we have a partition into k pieces and this partition is not ϵ-regular, there is a partition into at most $k2^{2k} \le 2^{2^k}$ pieces whose mean square density is at least ϵ^5 greater. Because the mean square density is bounded above by 1, this process must end after at most ϵ^{-5} steps. The number of pieces in the final partition is at most a tower of 2s of height $2\epsilon^{-5}$. □

The tower function $t_i(x)$ is defined by $t_0(x) = x$ and, for $i \ge 0$, $t_{i+1}(x) = 2^{t_i(x)}$. The bound given in the proof above is $t_{2\epsilon^{-5}}(2)$, which is clearly enormous. Surprisingly, as was shown by Gowers, there are graphs where, to get an ϵ-regular partition, one needs roughly that many pieces in the partition.

In this and the next lecture, we will prove a beautiful consequence of the regularity lemma, the triangle removal lemma, and show how one may deduce Roth's theorem from it. The triangle removal lemma says that if a graph contains very few triangles, then one may remove all such triangles by removing very few edges. Though this lemma sounds simple, its proof is surprisingly subtle. We begin with what is known as a counting lemma.

Lemma 2 *Let G be a graph and let X, Y, Z be subsets of the vertex set $V(G)$. Suppose that (X, Y), (Y, Z) and (Z, X) are ϵ-regular and that $d(X, Y) = \alpha$, $d(Y, Z) = \beta$ and $d(Z, X) = \gamma$. Then, provided $\alpha, \beta, \gamma \geq 2\epsilon$, the number of triangles xyz with $x \in X$, $y \in Y$ and $z \in Z$ is at least*

$$(1 - 2\epsilon)(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon)|X||Y||Z|.$$

Proof For every x, let $d_Y(x)$ and $d_Z(x)$ be the number of neighbours of x in Y and Z, respectively. Then the number of $x \in X$ with $d_Y(x) < (\alpha - \epsilon)|Y|$ is at most $\epsilon|X|$. Suppose otherwise. Then there will be a subset X' of X of size at least $\epsilon|X|$ such that the density of edges between X' and Y is less than $\alpha - \epsilon$. But this would contradict regularity. We may similarly show that there are at most $\epsilon|X|$ values of x for which $d_Z(x) < (\gamma - \epsilon)|Z|$. If $d_Y(x) > (\alpha - \epsilon)|Y|$ and $d_Z(x) > (\gamma - \epsilon)|Z|$, the number of edges between $N(x) \cap Y$ and $N(x) \cap Z$, and consequently the number of triangles containing x, is at least

$$(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon)|Y||Z|.$$

Summing over all $x \in X$ gives the result. □

We will complete the proof of the removal lemma in the next lecture.

Lecture 6

We are now ready to prove the triangle removal lemma.

Theorem 1 (Triangle removal lemma) *For every $\epsilon > 0$ there exists $\delta > 0$ such that, for any graph G on n vertices with at most δn^3 triangles, it may be made triangle-free by removing at most ϵn^2 edges.*

Proof Let $X_1 \cup \cdots \cup X_M$ be an $\frac{\epsilon}{4}$-regular partition of the vertices of G. We remove an edge xy from G if

1. $(x, y) \in X_i \times X_j$, where (X_i, X_j) is not an $\frac{\epsilon}{4}$-regular pair;
2. $(x, y) \in X_i \times X_j$, where $d(X_i, X_j) < \frac{\epsilon}{2}$;
3. $x \in X_i$, where $|X_i| \leq \frac{\epsilon}{4M} n$.

The number of edges removed by condition 1 is at most $\sum_{(i,j) \in I} |X_i||X_j| \leq \frac{\epsilon}{4} n^2$. The number removed by condition 2 is clearly at most $\frac{\epsilon}{2} n^2$. Finally, the number removed by condition 3 is at most $Mn \frac{\epsilon}{4M} n = \frac{\epsilon}{4} n^2$. Overall, we have removed at most ϵn^2 edges.

Now, suppose that some triangle remains in the graph, say xyz, where $x \in X_i$, $y \in X_j$ and $z \in X_k$. Then the pairs (X_i, X_j), (X_j, X_k) and (X_k, X_i) are all $\frac{\epsilon}{4}$-regular with density at least $\frac{\epsilon}{2}$. Therefore, since $|X_i|, |X_j|, |X_k| \geq \frac{\epsilon}{4M} n$, we have, by the counting lemma that the number of triangles is at least

$$\left(1 - \frac{\epsilon}{2}\right) \left(\frac{\epsilon}{4}\right)^3 \left(\frac{\epsilon}{4M}\right)^3 n^3.$$

Taking $\delta = \frac{\epsilon^6}{2^{20} M^3}$ yields a contradiction. □

We now use this removal lemma to prove Roth's theorem. We will actually prove the following stronger theorem.

Theorem 2 *Let $\delta > 0$. Then there exists n_0 such that, for $n \geq n_0$, any subset A of $[n]^2$ with at least δn^2 elements must contain a triple of the form $(x, y), (x + d, y), (x, y + d)$ with $d > 0$.*

Proof The set $A + A = \{x + y : x, y \in A\}$ is contained in $[2n]^2$. There must, therefore, be some z which is represented as $x + y$ in at least

$$\frac{(\delta n^2)^2}{(2n)^2} = \frac{\delta^2 n^2}{4}$$

different ways. Pick such a z and let $A' = A \cap (z - A)$ and $\delta' = \frac{\delta^2}{4}$. Then $|A'| \geq \delta' n^2$ and if A' contains a triple of the form $(x, y), (x + d, y), (x, y + d)$ for $d < 0$, then so does $z - A$. Therefore, A will contain such a triple with $d > 0$. We may therefore forget about the constraint that $d > 0$ and simply try to find some non-trivial triple with $d \neq 0$.

Consider the tripartite graph on vertex sets X, Y and Z, where $X = Y = [n]$ and $Z = [2n]$. X will correspond to vertical lines through A, Y to horizontal lines and Z to diagonal lines with constant values of $x + y$. We form a graph G by joining $x \in X$ to $y \in Y$ if and only if $(x, y) \in A$. We also join x and z if $(x, z - x) \in A$ and y and z if $(z - y, y) \in A$.

If there is a triangle xyz in G, then $(x, y), (x, y+(z-x-y)), (x+(z-x-y), y)$ will all be in A and thus we will have the required triple unless $z = x + y$. This means that there are at most $n^2 = \frac{1}{64n}(4n)^3$ triangles in G. By the triangle removal lemma, for n sufficiently large, one may remove $\frac{\delta}{2}n^2$ edges and make the graph triangle-free. But every point in A determines a degenerate triangle. Hence, there are at least δn^2 degenerate triangles, all of which are edge disjoint. We cannot, therefore, remove them all by removing $\frac{\delta}{2}n^2$ edges. This contradiction implies the required result. $\square$

This implies Roth's theorem as follows.

Theorem 3 (Roth) *For all $\delta > 0$ there exists n_0 such that, for $n \geq n_0$, any subset A of $[n]$ with at least δn elements contains an arithmetic progression of length 3.*

Proof Let $B \subset [2n]^2$ be $\{(x, y) : x - y \in A\}$. Then $|B| \geq \delta n^2 = \frac{\delta}{4}(2n)^2$ so we have $(x, y), (x + d, y)$ and $(x, y + d)$ in B. This translates back to tell us that $x - y - d$, $x - y$ and $x - y + d$ are in A, as required. $\square$

To prove Szemerédi's theorem by the same method, one must first generalise the regularity lemma to hypergraphs. This was done by Gowers and, independently, by Nagle, Rödl, Schacht and Skokan. This method also allows you to prove the following more general theorem.

Theorem 4 (Multidimensional Szemerédi) *For any natural number d, any $\delta > 0$ and any subset P of $\mathbb{Z}^d$, there exists an n_0 such that, for any $n \geq n_0$, every subset of $[n]^d$ of density at least δ contains a homothetic copy of P, that is, a set of the form $k.P + \ell$, where $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^d$.*

The theorem proved above corresponds to the case where $d = 2$ and $P = \{(0, 0), (1, 0), (0, 1)\}$. Szemerédi's theorem for length k progressions is the case where $d = 1$ and $P = \{0, 1, 2, \ldots, k - 1\}$.

Lecture 7

We are now ready to give the promised alternative proof of Erdős-Stone-Simonovits. To begin, we will need a counting lemma which generalises that given earlier for triangles.

Lemma 1 *Let $\epsilon > 0$ be a real number. Let G be a graph and suppose that $V_1, V_2, \ldots, V_r$ are subsets of $V(G)$ such that $|V_i| \geq 2\epsilon^{-\Delta} t$ for each $1 \leq i \leq r$ and the graph between V_i and V_j has density $d(V_i, V_j) \geq 2\epsilon$ and is $\frac{1}{2}\epsilon^{\Delta}\Delta^{-1}$-regular for all $1 \leq i < j \leq r$. Then G contains a copy of any graph H on t vertices with chromatic number r and maximum degree Δ.*

Proof Since the chromatic number of H is at most r, we may split $V(H)$ into r independent sets $U_1, \ldots, U_r$. We will give an embedding f of H into G so that $f(U_i) \subset V_i$ for all $1 \leq i \leq r$.

Let the vertices of H be $u_1, \ldots, u_t$. For each $1 \leq h \leq t$, let $L_h = \{u_1, \ldots, u_h\}$. For each $y \in U_j \backslash L_h$, let T_y^h be the set of vertices in V_j which are adjacent to all already embedded neighbours of y. That is, letting $N_h(y) = N(y) \cap L_h$, T_y^h is the set of vertices in V_j adjacent to every element of $f(N_h(y))$. We will find, by induction, an embedding of L_h such that, for each $y \in V(H) \backslash L_h$, $|T_y^h| \geq \epsilon^{|N_h(y)|}|V_j|$.

For $h = 0$ there is nothing to prove. We may therefore assume that L_h has been embedded consistent with the induction hypothesis and attempt to embed $u = u_{h+1} \in U_k$ into an appropriate $v \in T_u^h$. Let Y be the set of neighbours of u which are not yet embedded. We wish to find an element $v \in T_u^h \backslash f(L_h)$ such that, for all $y \in Y$, $|N(v) \cap T_y^h| \geq \epsilon |T_y^h|$. If such a vertex v exists, taking $f(u) = v$ and $T_y^{h+1} = N(v) \cap T_y^h$ will complete the proof.

Let B_y be the set of vertices in T_u^h which are bad for $y \in Y$, that is, such that $|N(v) \cap T_y^h| < \epsilon |T_y^h|$. Note that, by induction, if $y \in U_\ell$, $|T_y^h| \geq \epsilon^{\Delta}|V_\ell|$. Therefore, we must have $|B_y| < \frac{1}{2}\epsilon^{\Delta}\Delta^{-1}|V_k|$, for otherwise the density between B_y and T_y^h would be less than ϵ, contradicting the regularity assumption on G. Hence, since $|V_k| \geq 2\epsilon^{-\Delta} t$,

$$\left| T_u^h \backslash \cup_{y \in Y} B_y \right| > \epsilon^{\Delta}|V_k| - \Delta \frac{1}{2}\epsilon^{\Delta}\Delta^{-1}|V_k| \geq t.$$

Since at most $t-1$ vertices have already been embedded, an appropriate choice for $f(u)$ exists. $\square$

In fact, there are at least $\frac{1}{2}\epsilon^{\Delta}|V_k| - t$ choices for each vertex u. Therefore, if H has d_i vertices in U_i, the lemma tells us that, for $|V_i| \gg 2\epsilon^{-\Delta} t$, we have at least

$$c_H(\epsilon) \prod_{i=1}^{r} |V_i|^{d_r}$$

copies of H, where $c_H(\epsilon)$ is an appropriate constant. Like the triangle counting lemma, we could make the constant $c_H(\epsilon)$ reflect the densities between the various V_i, but I simply wanted to note that the graph G contained a positive proportion of the total number of possible copies of H.

We are now ready to give another proof of the Erdős-Stone-Simonovits theorem. That is, we will show that for any r-chromatic graph H and n sufficiently large, $ex(n, H) \leq \left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$.

Alternative proof of Erdős-Stone-Simonovits Let H be a graph with t vertices, chromatic number r and maximum degree Δ. Suppose that G is a graph on n vertices with at least $\left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$ edges. We will show how to embed H in G. Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_M$ be a $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta}\Delta^{-1}$-regular partition of the vertex set of G. We remove edges as in the triangle-removal lemma, removing xy if

1. $(x, y) \in X_i \times X_j$, where (X_i, X_j) is not $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta}\Delta^{-1}$-regular;

2. $(x, y) \in X_i \times X_j$, where $d(X_i, X_j) < \frac{\epsilon}{4}$;

3. $x \in X_i$, where $|X_i| < \frac{\epsilon}{16M}n$.

The total number of edges removed is at most $\frac{\epsilon}{16}n^2$ from the first condition, since if I is the set of (i, j) corresponding to non-regular pairs (X_i, X_j), we have

$$\sum_{(i,j)\in I} |X_i||X_j| \le \frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta}\Delta^{-1}n^2 \le \frac{\epsilon}{16}n^2.$$

The total number of edges removed by condition 2 is clearly at most $\frac{\epsilon}{4}n^2$ and the total number removed by condition 3 is at most $\frac{\epsilon}{16}n^2$.

Overall, we have removed at most $\frac{3\epsilon}{8}n^2$ edges. Hence, the graph G' that remains after all these edges have been removed has density at least $1 - \frac{1}{r-1} + \frac{\epsilon}{8}$. It must, therefore, contain a copy of K_r. We may suppose that this lies between sets $V_1, \dots, V_r$ (some of which may be equal). Because of our removal process, $|V_j| \ge \frac{\epsilon}{16M}n$, the graph between V_i and V_j has density at least $\frac{\epsilon}{4}$ and is $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta}\Delta^{-1}$-regular. Therefore, if

$$\frac{\epsilon}{16M}n \ge 2\left(\frac{\epsilon}{8}\right)^{-\Delta}t,$$

an application of the previous lemma with $\frac{\epsilon}{8}$ implies that G contains a copy of H. $\square$

Because of the observation made after the previous lemma, we know that, for n large, G not only contains one copy of any given r-chromatic graph H, it must contain $cn^{v(H)}$ copies. This phenomenon, that once one passes the extremal density one gets a very large number of copies rather than one single copy, is known as supersaturation.

Lecture 8

We are now going to begin an in-depth study of the extremal number for bipartite graphs. We have already seen that if H is a bipartite graph $ex(n, H) \leq \epsilon n^2$ for any $\epsilon > 0$. We now prove a much stronger result, due essentially to Kővári, Sós and Turán.

Theorem 1 *For any natural numbers s and t with $s \leq t$, there exists a constant c such that*

$$ex(n, K_{s,t}) \leq cn^{2-\frac{1}{s}}.$$

Proof Suppose that we have a graph G with n vertices, at least $cn^{2-\frac{1}{s}}$ edges and not containing $K_{s,t}$ as a subgraph. Note that the average degree of G is $2cn^{1-\frac{1}{s}}$. We are going to count pairs (v, S) consisting of sets S of size s all elements of which are connected by an edge to v. On the one hand, the number of such pairs is given by

$$\sum_v \binom{d(v)}{s} \geq n\binom{\frac{1}{n}\sum_v d(v)}{s} \geq n\binom{2cn^{1-\frac{1}{s}}}{s} \geq n\frac{c^s n^{s-1}}{s!} = c^s\frac{n^s}{s!},$$

for n sufficiently large. On the other hand, the number of pairs (v, S) is at most

$$(t-1)\binom{n}{s} \leq (t-1)\frac{n^s}{s!},$$

for otherwise there would be some set S of s vertices which have t neighbours in common. Therefore, if we choose c so that $c^s \geq t-1$, we have a contradiction. □

By being a little more careful, we could have obtained the bound

$$ex(n, K_{s,t}) \leq (1+o(1))\frac{1}{2}(t-1)^{\frac{1}{s}}n^{2-\frac{1}{s}}.$$

It is known that this bound is sharp in various specific cases. For example, when $H = K_{1,t}$ and n satisfies some divisibility assumptions, there is a graph on n vertices with $\frac{1}{2}(t-1)n$ edges which does not contain any copies of H - simply take a graph such that every vertex has degree $t-1$.

A more interesting example is $H = K_{2,2}$. The following $K_{2,2}$-free construction, due to Erdős, Rényi and Sós, allows us to show that $ex(n, K_{2,2}) \approx \frac{1}{2}n^{3/2}$.

Construction of $K_{2,2}$-free graph Let p be a prime and consider the graph on $n = p^2 - 1$ vertices whose vertex set is $\mathbb{Z}_p \times \mathbb{Z}_p \backslash \{(0,0)\}$ and where (x, y) is joined to (a, b) if and only if $ax + by = 1$.

For a fixed (x, y), there are exactly p solutions (a, b) to $ax + by = 1$. To see this, we must split into some subcases. If $x = 0$, then there is a unique non-zero solution for b and anything works for a. Similarly, if $y = 0$, a is uniquely determined and b may be anything. If both x and y are non-zero, it is elementary to see that any choice of b gives rise to a unique choice of a, i.e., $a = x^{-1}(1 - by)$.

Therefore, (x, y) has degree at least $p-1$ (one of the solutions could be $(a, b) = (x, y)$, which we ignore) and the graph has at least $\frac{1}{2}n(p-1) \approx \frac{1}{2}n^{3/2}$ edges. Moreover, the graph does not contain a $K_{2,2}$. Suppose otherwise and that $(a, b), (x, y), (a', b'), (x', y')$ is a $K_{2,2}$. Then the set of simultaneous equations $ux + vy = 1$ and $ux' + vy' = 1$ would have two solutions, $(u, v) = (a, b)$ and (a', b'), which is clearly impossible, since any two distinct lines meet in at most one point.

This construction works for $n = p^2 - 1$, but the result that $ex(n, H) \approx \frac{1}{2}n^{3/2}$ follows for all n since we know that, for n large, there exists a prime between $\sqrt{n} - n^{1/3}$ and $\sqrt{n}$ (though this is a very deep result). $\square$

There is also a result of Füredi extending this result. It says that, for each t,

$$ex(n, K_{2,t+1}) \approx \frac{\sqrt{t}}{2}n^{3/2}.$$

There is also a construction, due to Brown, which gives a lower bound $ex(n, K_{3,3}) \geq c'n^{5/3}$. Roughly speaking, take a prime $p \equiv 3 (\mathrm{mod}\ 4)$ and consider the graph on p^3 vertices whose vertex set is $\mathbb{Z}_p^3$, where (x, y, z) is joined to (a, b, c) if and only if $(a-x)^2 + (b-y)^2 + (c-z)^2 = 1$. For any given (x, y, z), there will be on the order of p^2 elements (a, b, c) to which it is connected. There are, therefore, around $c'n^{5/3}$ edges in the graph. Moreover, the unit spheres around the three distinct points $(x, y, z), (x', y', z')$ and (x'', y'', z'') cannot meet in more than two points, so the graph does not contain a $K_{3,3}$. The result follows for all n by a similar argument to above.

Other than the constructions mentioned, there is also an impressive construction of Alon, Kollár, Rónyai and Szabó which shows that if $t \geq (s-1)! + 1$, the upper bound we gave at the start of the lecture is essentially sharp, that is, $ex(n, K_{s,t}) \geq c'n^{2-\frac{1}{s}}$. This includes, though without sharp multiplying constants, all the cases discussed thusfar.

Apart from this, almost the best known lower bound follows from the following random construction.

Theorem 2 *For any $s, t \geq 2$, there exists a constant c′ such that*

$$ex(n, K_{s,t}) \geq c'n^{2-\frac{(s+t-2)}{(st-1)}}.$$

Proof Choose each edge in the graph randomly with probability $p = \frac{1}{2}n^{-\frac{(s+t-2)}{(st-1)}}$. The expected number of copies of $K_{s,t}$ is

$$p^{st}\binom{n}{s}\binom{n}{t} \leq p^{st}n^{s+t}.$$

Phrased differently, if J is the random variable counting copies of $K_{s,t}$, then $\mathbb{E}(J) \leq p^{st}n^{s+t}$. On the other hand, the expected number of edges in the graph is $p\binom{n}{2} \geq \frac{1}{4}pn^2$. Again, if I is the random variable counting the number of edges in the graph, then $\mathbb{E}(I) \geq \frac{1}{4}pn^2$. By linearity of expectation,

$$\mathbb{E}(I - J) = \mathbb{E}(I) - \mathbb{E}(J) \geq \frac{1}{4}pn^2 - p^{st}n^{s+t} \geq \frac{1}{8}pn^2 = \frac{1}{16}n^{2-\frac{(s+t-2)}{(st-1)}}.$$

The final inequality follows since $p^{st}n^{s+t} \leq \frac{1}{8}pn^2$. This in turn follows from

$$p^{st-1}n^{s+t-2} \leq \left(\frac{1}{2}\right)^{st-1} \leq \frac{1}{8}.$$

Therefore, there exists some graph G on n vertices for which $I - J \geq \frac{1}{16}n^{2-\frac{(s+t-2)}{(st-1)}}$. We may therefore remove one edge from each of the $K_{s,t}$, removing all copies of $K_{s,t}$ and still be left with a graph containing $\frac{1}{16}n^{2-\frac{(s+t-2)}{(st-1)}}$ edges. $\square$

Lecture 9

The aim of this lecture is to prove that if a bipartite graph H has one side whose maximum degree is Δ, then $ex(n, H) \leq c(H)n^{2-\frac{1}{\Delta}}$. This clearly generalises the result from the last lecture that $ex(n, K_{s,t}) \leq n^{2-\frac{1}{s}}$ when $s \leq t$.

We will use a very powerful technique known as dependent random choice. Roughly speaking, if one has a bipartite graph G between sets A and B, then dependent random choice allows one to find a large subset of A' where every collection of Δ elements in A' has many joint neighbours in B. This then allows one to embed subgraphs with maximum degree Δ on one side with impunity.

Lemma 1 (Dependent random choice) *Let G be a bipartite graph with vertex sets A and B, each of size n. Suppose that the graph has density α, that is, that there are αn^2 edges. Then, for any natural number $r \geq 1$, there exists a set $A' \subset A$ of size greater than $\frac{1}{2}\alpha^r n$ such that every subset of A' of size r has at least $\frac{1}{2^r}\alpha n^{1/r}$ common neighbours in B.*

Proof For each vertex v, let $d(v)$ be its degree. For randomly chosen vertices $b_1, \cdots, b_r \in B$ (allowing repetitions), let I be the random variable giving the size of the common neighborhood. What is the expectation of I? By convexity of the function x^r, we see that

$$\begin{aligned} \mathbb{E}(I) &= \sum_{v \in A} \mathbb{P}(b_1, \ldots, b_r \in N(v)) = \sum_{v \in A} \left(\frac{d(v)}{n}\right)^r \\ &\geq \frac{n\left(\frac{\sum_{v \in A} d(v)}{n}\right)^r}{n^r} = \frac{n(\alpha n)^r}{n^r} = \alpha^r n. \end{aligned}$$

We will say that an r-tuple is bad if it has fewer than $\gamma^r|B|$ common neighbours. Let J be the random variable counting the number of bad r-tuples in the common neighborhood of $b_1, \ldots, b_r$. Note that any bad r-tuple has at most $\gamma^r|B|$ common neighbours in B. Therefore, the probability that a randomly chosen b_i is contained in this set is at most γ^r. Hence, because we made r random choices of b_i, the probability that such an r-tuple be contained in the intersection of their neighborhoods is at most γ^{r^2}. Therefore, the expected number of bad r-tuples in the common neighborhood of $b_1, \ldots, b_r$ satisfies

$$\mathbb{E}(J) \leq \gamma^{r^2}\binom{|A|}{r} \leq \gamma^{r^2}n^r.$$

By linearity of expectation, we have

$$\mathbb{E}(I - J) \geq \alpha^r n - \gamma^{r^2}n^r.$$

Choose $\gamma = \frac{1}{2}\alpha^{1/r}n^{-(r-1)/r^2}$. Then $\gamma^{r^2}n^r \leq \frac{1}{2}\alpha^r n$ and, therefore, $\mathbb{E}(I - J) \geq \frac{1}{2}\alpha^r n$. Therefore, there exists a set A_0 for which $I - J \geq \frac{1}{2}\alpha^r n$. Hence, we may remove the set of bad r-tuples from A_0 and be left with a set A' of size at least $\frac{1}{2}\alpha^r n$ which has no bad r-tuples. Since there are no bad r-tuples, every set of r elements in A has at least $\gamma^r n = \frac{1}{2^r}\alpha n^{1/r}$ neighbours in common. $\square$

We may now prove the required estimate on $ex(n, H)$, where H has one side with bounded maximum degree Δ.

Theorem 1 *Let H be a graph between two sets U and V such that the degree of every vertex in V is at most Δ. Then there exists a constant c such that*

$$ex(n, H) \leq cn^{2-\frac{1}{\Delta}}.$$

Proof Let G be a graph of size n and suppose that G has at least $cn^{2-\frac{1}{\Delta}}$ edges. Then there is a bipartite subgraph G' of G between two sets A and B containing at least half the edges of G, that is, at least $\frac{c}{2}n^{2-\frac{1}{\Delta}}$. To see this, we simply choose the sets A and B at random, placing a vertex in each of A or B with probability $\frac{1}{2}$. The probability that a given edge lies between the two sets is then just $\frac{1}{2}$. Therefore, the expected number of edges in such a cut is $\frac{1}{2}e(G)$. In particular, there must be some cut for which we do have $\frac{1}{2}e(G)$ edges.

We now have a bipartite graph G' between two sets A and B, each of size at most n, with $\frac{c}{2}n^{2-\frac{1}{\Delta}}$ edges. An application of the dependent random choice lemma with $r = \Delta$ and $\alpha = \frac{c}{2}n^{-1/\Delta}$ tells us that there is a subset A' of A of size at least

$$\frac{1}{2}\alpha^{\Delta}n \geq \frac{1}{2}\left(\frac{c}{2}\right)^{\Delta}$$

such that every Δ-tuple in A' has at least

$$\frac{1}{2^r}\alpha n^{1/\Delta} \geq \frac{c}{2^{r+1}}$$

common neighbours.

Provided $\frac{1}{2}\left(\frac{c}{2}\right)^{\Delta} \geq |U|$ and $\frac{c}{2^{r+1}} \geq |V|$, we have an embedding of H. To see this, let $u_1, \ldots, u_t$ be the vertices of U. Embed them in any fashion into A'. For any given vertex $v \in V$, suppose that $u_{i_1}, \ldots, u_{i_\Delta}$ are its neighbours. Then these vertices have at least $|V|$ common neighbours in B, so even if we have embedded previous elements of V, there is still place to embed v. $\square$

A graph H is said to be d-degenerate if every subgraph contains a vertex of degree at most d. Equivalently, H is d-degenerate if there is an ordering $\{v_1, \ldots, v_t\}$ of the vertices such that any v_j has at most d neighbours v_i with $i < j$.

An old conjecture of Burr and Erdős says that if a bipartite graph H is d-generate, then $ex(n, H) \leq cn^{2-\frac{1}{d}}$. This would be strictly stronger than the result we proved in this lecture. The best result currently known, due to Alon, Krivelevich and Sudakov, is that $ex(n, H) \leq cn^{2-\frac{1}{4d}}$.

Lecture 10

In this and the next lecture, we will consider the extremal problem for some of the most obvious examples of bipartite graphs, cycles of even length. For cycles of length 4, we have already seen that $ex(n, C_4) \approx \frac{1}{2}n^{3/2}$.

The main theorem we will prove over the next couple of lectures is the upper bound $ex(n, C_{2k}) \leq cn^{1+1/k}$, due to Bondy and Simonovits. For $k = 2, 3$ and 5, that is, for C_4, C_6 and C_{10}, this is known to be sharp. A quick probabilistic argument, similar to that used earlier for complete bipartite graphs, gives the following general lower bound. We leave the details to the reader.

Theorem 1 *There exists a constant c such that*

$$ex(n, C_{2k}) \geq cn^{1+1/(2k-1)}.$$

There is also an explicit construction, due to Lazebnik, Ustimenko and Woldar, which does better, giving $ex(n, C_{2k}) \geq cn^{1+2/(3k-3)}$. However, this is beyond the reach of the course.

In order to prove the Bondy-Simonovits theorem, $ex(n, C_{2k}) \leq cn^{1+1/k}$, we will need some preliminary lemmas. Both concern cycles with an extra chord.

Lemma 1 *Let H be a cycle with an extra chord. Let (A, B) be a non-trivial partition of $V(H)$, that is, there is some edge crossing the partition. Then, unless H is bipartite between A and B, H contains paths of every length $\ell < |H|$ which begin in A and end in B.*

Proof Label the vertices of H as $0, 1, \ldots, t-1$, where $t = |H|$. Suppose that H does not contain cycles which start in A and end in B for every possible length $\ell < t$. We will focus on a particular class of path, saying that a path is good if it begins in A, ends in B and does not use the chord of H. Let s be the smallest integer such that there is no good path of length s. Then $s > 1$, since there is at least one edge between A and B. If this edge is the chord, it will automatically imply that there is some other edge across the partition. We also have that $s \leq t/2$. This is because, by symmetry, the existence of a good path of length j implies the existence of a good path of length $t - j$.

Let χ be the characteristic function of A. Then, for any j, $\chi(j + s) = \chi(j)$, where addition is taken modulo t. Let $d = \text{hcf}(s, t)$. Then there are p and q such that $ps+qt = d$ and, therefore, $\chi(j) = \chi(j+d)$, for all j. But then there is no good path of length d. Therefore, since s was the smallest number with this property, $d = s$ and s divides t. This also implies that for every i which is not a multiple of s, there will be good paths of length i.

We will now find paths of all remaining lengths is, where $1 \leq i \leq t/s - 1$, by using the chord. Suppose first that the chord joins two vertices at distance r, where $1 < r \leq s$, say 0 and r. We know from above that there are good paths of length $s + r - 1$. In particular, there is some j such that $\chi(j) \neq \chi(j+s+r-1)$. By shifting, we may assume that $-s < j \leq 0$. Therefore, since $j+s+r-1 \geq r$ and $\chi(j) \neq \chi(j + is + r - 1)$, the path $j, j+1, \ldots, 0, r, r+1, \ldots, j+is+r-1$ is a path of length is beginning in A and ending in B. We need to verify that $j + is + r - 1 < t + j$, that is, that it doesn't loop all the way around, but this follows easily for $i \leq t/s - 1$.

We therefore assume that the chord is $0r$, where $s < r < t-s$. Let $-s < j < 0$ and consider the paths $j, j+1, \ldots, 0, r, r-1, \ldots, r-j-s+1$ and $s+j, s+j-1, \ldots, 0, r, r+1, \ldots, r-j-1$, each of length s. If either of them is a path starting in A and ending in B, we may extend it to produce a well-behaved path of length is until the number of unused vertices in the two arcs defined by the chord is less than s in both arcs. At this point, $is + 1 \geq t - 2(s-1)$ and, since s divides t, $is = t - s$, so we already have everything. Similarly, if either of the paths $0, r, r-1, \ldots, r-s+1$ or $0, r, r+1, \ldots, r+s-1$ begin in A and end in B, then H contains well-behaved paths of all lengths less than t.

We may therefore assume that, for $-s < j < 0$,

$$\chi(r-j+1) = \chi(r-j-s+1) = \chi(j) = \chi(s+j) = \chi(r-j-1).$$

The first and third equalities are by shifting. The second and fourth follow because the paths $j, j+1, \ldots, 0, r, r-1, \ldots, r-j-s+1$ and $s+j, s+j-1, \ldots, 0, r, r+1, \ldots, r-j-1$ must each have both endpoints in one of A or B. Similarly, we may assume that $\chi(r+s+1) = \chi(r+s-1)$. This implies that $\chi(i) = \chi(i+2)$ for every vertex i. Therefore, $s = 2$.

We may conclude therefore that t is even and that the vertices of the cycle alternate between A and B. It is easy now to see that if the chord is contained with one of A or B, then the graph contains paths of all length less than t which start in A and end in B. Therefore, the chord goes between A and B and H is bipartite, as required. □

The second lemma we need is a condition for a graph to contain a cycle with an extra chord.

Lemma 2 *Any bipartite graph G with minimum degree $d \geq 3$ contains a cycle of length at least $2d$ with an extra chord.*

Proof Let P be the longest path in G, visiting vertices $x_1, \ldots, x_p$ in that order. x_1 has at least $d \geq 3$ neighbours in G. By the maximality of P, these must all lie in P. Suppose that they are $x_{i_1}, \ldots, x_{i_d}$ with $i_1 < \cdots < i_d$. Every two neighbours of x_1 must be at least 2 apart, since G is bipartite. Therefore, since $i_1 = 2$, we must have $i_d \geq 2d$. The required cycle with chord is formed by taking the path from x_1 to x_{i_d} and adding the edges $x_1 x_{i_2}$ and $x_1 x_{i_d}$. □

We will also need two simple lemmas which we have proved in previous lectures.

Lemma 3 *Every graph G contains a subgraph whose minimum degree is at least half the average degree of G.*

Lemma 4 *Every graph G contains a bipartite subgraph with at least half the edges of G.*

Lecture 11

We now begin the proof proper of the Bondy-Simonovits theorem.

Theorem 1 *For any natural number $k \geq 2$, there exists a constant c such that*

$$ex(n, H) \leq cn^{1+1/k}.$$

Proof Suppose that G is a C_{2k}-free graph on n vertices with at least $cn^{1+1/k}$ edges. Then the average degree of G is at least $2cn^{1/k}$, so there exists some subgraph H for which the minimum degree is at least $cn^{1/k}$.

Fix an arbitrary vertex x of H. Let $i \geq 0$, let V_i be the set of vertices that are at distance i from x with respect to the graph H. In particular, $V_0 = \{x\}$ and $V_1 = N(x)$. Let $v_i = |V_i|$ and let H_i be the bipartite subgraph $H[V_{i-1}, V_i]$ induced by the disjoint sets V_{i-1} and V_i.

Claim 1 *For $1 \leq i \leq k-1$, none of the graphs $H[V_i]$ or H_{i+1} contain a bipartite cycle of length at least $2k$ with a chord.*

Proof Suppose, on the contrary, that there is a bipartite cycle with a chord F, of length at least $2k$, in $H[V_i]$. Let $Y \cup Z$ be the bipartition of $V(F)$.

Let $T \subset H$ be a breadth first-search tree beginning at x. That is, we begin at the root node x. The first layer will consist of the neighbours of x, labelling them as we uncover them. At the jth step, we look at layer $j-1$. For the first vertex in the ordering, we look at its neighbours that have not yet occurred and label them as they occur. Then we do the same in order for every vertex in the $(j-1)$st level. This will give us the jth level with all vertices labelled.

Let y be the vertex which is farthest from x in the tree T and which still dominates the set Y, that is, every vertex in Y is a descendant of y. Clearly, the paths leading from y to Y must branch at y. Pick one such branch (leading to a non-trivial subset of Y), defined by some child z of y and let A be the set of descendents of z which lie in Y. Let $B = (Y \cup Z) \backslash A$. Since $Y \backslash A \neq \emptyset$, B is not an independent set of F.

Let ℓ be the distance between x and y. Then $\ell < i$ and $2k - 2i + 2\ell < 2k \leq v(F)$. By the main lemma from the last lecture, since F is not bipartite with respect to the partition into A and B, we can find a path $P \subset F$ of length $2k - 2i + 2\ell$ which starts in $a \in A$ and ends in $b \in B$. Since the path has even length and the partition into Y and Z is bipartite, b must be in Y. Let P_a and P_b be the unique paths in T that connect y to a and b. They intersect only at y, since a is a descendent of z and b is not. Also, they each have length $i - \ell$. Therefore, the union of the paths P, P_a and P_b forms a C_{2k} in H, which contradicts our assumption.

The proof follows similarly for H_{i+1} if we take $Y = V(F) \cap V_i$. □

We also know that if a bipartite graph has minimum degree $d \geq 3$ then it contains a cycle of length at least $2d$ with an extra chord. We may therefore assume that, for $1 \leq i \leq k-1$, the average degrees $d(H[V_i])$ and $d(H_{i+1})$ of $H[V_i]$ and H_{i+1} satisfy

$$d(H[V_i]) \leq 4k - 4 \text{ and } d(H_{i+1}) \leq 2k - 2.$$

For example, if $H[V_i]$ has average degree greater than $4k-4$, it has a bipartite subgraph with average degree greater than $2k-2$ and, therefore, a bipartite subgraph with minimum degree greater than $k-1$. This would then imply that the graph contained a bipartite cycle of length at least $2k$ with a chord, which would contradict the claim. The bound for $d(H_{i+1})$ follows similarly.

We will now show inductively that, provided n is sufficiently large,

$$\frac{e(H_{i+1})}{v_{i+1}} \leq 2k$$

for every $0 \leq i \leq k-1$. For $i = 0$, this is true, since every edge in V_1 is connected to x by only one edge. Suppose that we want to prove it for some $i > 0$. Then, by induction and the bound on $d(H[V_i])$,

$$e(H_{i+1}) = \sum_{y \in V_i} d_{V_{i+1}}(y) \geq \left(\delta(H) - \frac{4k-4}{2} - 2k\right) v_i \geq \left(cn^{1/k} - 4k + 2\right) v_i \geq \frac{c}{2} n^{1/k} v_i \geq 2kv_i.$$

In particular, $V_{i+1} \neq \emptyset$ and the average degree of vertices of V_i with respect to H_{i+1} is at least $2k$. But since $d(H_{i+1}) \leq 2k-2$, we must have that the average degree of V_{i+1} with respect to H_{i+1} is at most $2k-2$, that is, $e(H_{i+1}) \leq (2k-2)v_{i+1}$, implying the required bound.

Note now that we have

$$\frac{c}{2} n^{1/k} v_i \leq e(H_{i+1}) \leq 2kv_{i+1}.$$

Therefore,

$$\frac{v_{i+1}}{v_i} \geq \frac{c}{4k} n^{1/k}.$$

This implies that

$$v_k \geq \left(\frac{c}{4k}\right)^k n.$$

This is a contradiction if $c \geq 4k$, completing the proof. □

We have shown that $ex(n, C_{2k}) \leq (4k + o(1))n^{1+1/k}$. A slightly more careful rendering of this proof, due to Pikhurko, allows one to show $ex(n, C_{2k}) \leq (k - 1 + o(1))n^{1+1/k}$.

Lecture 12

In this lecture we will consider the extremal number for odd cycles. We already know, by the Erdős-Stone-Simonovits theorem, that $ex(n, C_{2k+1}) \approx \frac{n^2}{4}$. Here we will use the so-called stability approach to prove that, for n sufficiently large, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.

The idea behind the stability approach is to show that a C_{2k+1}-free graph with roughly the maximal number of edges is approximately bipartite. This will be the first lemma below. Then one uses this approximate structural information to prove an exact result. This will be the theorem.

Lemma 1 *For every natural number $k \geq 2$ and $\epsilon > 0$ there exists $\delta > 0$ and a natural number n_0 such that, if G is a C_{2k+1}-free graph on $n \geq n_0$ vertices with at least $\left(\frac{1}{4} - \delta\right) n^2$ edges, then G may be made bipartite by removing at most ϵn^2 edges.*

Proof We will prove the result for $\delta = \frac{\epsilon^2}{100}$ and n sufficiently large. We begin by finding a subgraph G' of G with large minimum degree. We do this by deleting vertices one at a time, forming graphs $G = G_0, G_1, \ldots, G_\ell$, at each stage removing a vertex with degree less than $\frac{1}{2}\left(1 - 4\delta^{1/2}\right)|V(G_\ell)|$, should it exist. By doing this, we delete at most $4\delta^{1/2} n$ vertices. Otherwise, we would have a C_{2k+1}-free graph G' on $n' = \left(1 - 4\delta^{1/2}\right) n$ vertices with at least

$$\begin{aligned}
e(G') &> e(G) - \sum_{i=n'+1}^{n} \frac{1}{2}\left(1 - 4\delta^{1/2}\right) i \\
&\geq \left(\frac{1}{4} - \delta\right) n^2 - \frac{1}{2}\left(1 - 4\delta^{1/2}\right)\left(\binom{n+1}{2} - \binom{n'+1}{2}\right) \\
&\geq \frac{n'^2}{4} + 2\delta^{1/2} n^2 - 4\delta n^2 - \delta n^2 - \frac{1}{2}\left(1 - 4\delta^{1/2}\right)(n - n')n \\
&= \frac{n'^2}{4} + 2\delta^{1/2} n^2 - 5\delta n^2 - 2\delta^{1/2} n^2 + 8\delta n^2 \geq \frac{n'^2}{4}(1 + \delta).
\end{aligned}$$

But, by the Erdős-Stone-Simonovits theorem, for n sufficiently large G' will therefore contain a copy of C_{2k+1}, so we've reached a contradiction. We therefore have a subgraph G' with $n' \geq (1 - 4\delta^{1/2})n$ vertices and minimum degree at least $\frac{1}{2}\left(1 - 4\delta^{1/2}\right) n'$.

Since $ex(n, C_{2k}) = o(n^2)$, we know that for n (and therefore n') sufficiently large, the graph G' will contain a cycle of length $2k$. Let $a_1 a_2 \ldots a_{2k}$ be such a cycle. Note that $N(a_1)$ and $N(a_2)$ cannot intersect, for otherwise there would be a cycle of length $2k+1$. Moreover, each of the two neighborhoods must contain a small number of edges. Indeed, if $N(a_1)$ contained more than $4kn'$ edges, then our result from Lecture 2 on extremal numbers for trees would imply that there was a path of length $2k$ in $N(a_1)$. But then the endpoints could be joined to a_1 to give a cycle of length $2k+1$. Therefore, we have two large disjoint vertex sets $N(a_1)$ and $N(a_2)$, each of size at least $\frac{1}{2}\left(1 - 4\delta^{1/2}\right) n' \geq \frac{1}{2}\left(1 - 8\delta^{1/2}\right) n$ such that each contains at most $4kn'$ edges. We can make the graph bipartite by deleting all the edges within $N(a_1)$ and $N(a_2)$ and all of the edges which have one end in the complement of these two sets. In total, this is at most

$$8kn' + 8\delta^{1/2} n^2$$

edges. Therefore, for n sufficiently large and $\delta = \frac{\epsilon^2}{100}$, we will have deleted at most ϵn^2 edges, which gives the required result. □

Lecture 13

In the last lecture we showed that a C_{2k+1}-free graph with roughly $\frac{n^2}{4}$ edges must be approximately bipartite. We will now refine this structure to prove that the graph must be exactly bipartite for C_{2k+1}-free graphs of maximum size.

Theorem 1 *For n sufficiently large, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.*

Proof Let G be a C_{2k+1}-free graph on n vertices with the maximum number of edges. It will have at least $\lfloor \frac{n^2}{4} \rfloor$ edges. Note that it is sufficient to prove the result in the case where G has minimum degree at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n$. For suppose that we knew the result under this assumption for all $n \geq n_0$. As in the previous lemma, we form a sequence of graphs $G = G_0, G_1, \ldots, G_\ell$. If there is a vertex in G_ℓ of degree less than $\frac{1}{2}(1 - 4\epsilon^{1/2})|V(G_\ell)|$, we remove it, forming $G_{\ell+1}$. This process must stop before we reach a graph G' with $n' = (1 - 4\epsilon^{1/2})n$ vertices. Otherwise, we would have a graph with n' vertices and more than $(1 + \epsilon)\frac{n'^2}{4}$ edges. It would therefore, for n sufficiently large, contain a copy of C_{2k+1}, which would be a contradiction. When we reach the required graph, we will have a graph with $n' > (1 - 4\epsilon^{1/2})n$ vertices, minimum degree at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n'$ and more than $\lfloor \frac{n'^2}{4} \rfloor$ edges, so we will have a contradiction if the removal process begins at all. Hence, we may assume that the minimum degree of G at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n$.

By the previous lemma, we know that G is approximately bipartite between two sets of size roughly $\frac{n}{2}$. Consider a bipartition $V(G) = A \cup B$ such that $e(A) + e(B)$ is minimised. Then $e(A) + e(B) < \epsilon n^2$, where ϵ may be taken to be arbitrarily small provided n is sufficiently large. We may assume that A and B have size $\left(\frac{1}{2} \pm \epsilon^{1/2}\right) n$. Otherwise, $e(G) < |A||B| + \epsilon n^2 < \frac{n^2}{4}$, contradicting the choice of G as having maximum size. Let $d_A(x) = |A \cap N(x)|$ and $d_B(x) = |B \cap N(x)|$ for any vertex x. Note that for any $a \in A$, $d_A(a) \leq d_B(a)$. Otherwise, we could improve the partition by moving a to B. Similarly, $d_B(b) \leq d_A(b)$ for any $b \in B$.

Let $c = 2\epsilon^{1/2}$. We claim that there are no vertices $a \in A$ with $d_A(a) \geq cn$. If $d_A(a) \geq cn$, then also $d_B(a) \geq cn$. Moreover, $A \cap N(a)$ and $B \cap N(a)$ span a bipartite graph with no path of length $2k - 1$ and, therefore, there are at most $4kn$ edges between them. For n sufficiently large, this gives $(cn)^2 - 4kn > e(A) + e(B)$ missing edges between A and B. Therefore, $e(G) < |A||B| \leq \frac{n^2}{4}$, a contradiction. Similarly, there are no vertices $b \in B$ with $d_B(b) \geq cn$.

Now suppose that there is an edge in A, say aa'. Then

$$|N_B(a) \cap N_B(a')| > d(a) - cn + d(a') - cn - |B| > \left(\frac{1}{2} - 9\epsilon^{1/2}\right) n.$$

Let $A' = A \backslash \{a, a'\}$ and $B' = N_B(a) \cap N_B(a')$. There is no path of length $2k - 1$ of the form $b_1 a_1 b_2 a_2 \ldots b_{k-1} a_{k-1} b_k$ between A' and B'. But this implies that there is no path of any type of length $2k$ (remember that since the graph is bipartite a path must alternate sides). But this implies that the number of edges between A' and B' is at most $4kn$. This then implies that the number of edges in the graph is at most

$$e(A', B') + e(A \backslash A', V(G)) + e(V(G), B \backslash B') \leq 4kn + 2n + 10\epsilon^{1/2} n^2,$$

a contradiction for n large. $\square$

More generally, there is a result of Simonovits which shows that if H is a graph with $\chi(H) = t$ and $\chi(H \backslash e) < t$, for some edge e, then $ex(n, H) = ex(n, K_t)$ for n sufficiently large. We say that such graphs are colour-critical. It is easy to verify that odd cycles are colour-critical.

Lecture 14

We will now turn our attention to hypergraphs. An r-uniform hypergraph $\mathcal{G}$ on vertex set V is a collection of subsets of V of size r. The complete r-uniform hypergraph $K_n^{(r)}$ is a hypergraph on n vertices where every r-element subset of the vertex set is an edge. Our concern will be with the following function. Given an r-uniform hypergraph $\mathcal{H}$ and a natural number n, let

$$ex(n, \mathcal{H}) = \max\{e(\mathcal{G}) : |\mathcal{G}| = n, \mathcal{H} \not\subset \mathcal{G}\}.$$

Sometimes it will be convenient to talk about the Turán density, rather than the exact extremal function, for r-uniform hypergraphs $\mathcal{H}$. This is given by

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{ex(n, \mathcal{H})}{\binom{n}{r}}.$$

It is not hard to show that this density is well-defined. For graphs, the Erdős-Stone-Simonovits theorem tells us that if H has chromatic number t, then $\pi(H) = 1 - \frac{1}{t-1}$. For hypergraphs, much less is known. Even in the simple case where $\mathcal{H} = K_4^{(3)}$, we only know that

$$\frac{5}{9} \leq ex(n, K_4^{(3)}) \leq 0.561666.$$

The lower bound is not hard to come by. Take three vertex sets V_1, V_2 and V_3, each of size $n/3$. We let an edge uvw be in $\mathcal{G}$ if $u, v \in V_i$ and $w \in V_{i+1}$, for $i = 1, 2, 3$, or if $u \in V_1$, $v \in V_2$, $w \in V_3$. It is straightforward to check that this contains no $K_4^{(3)}$ and that its density is $5/9$. The upper bound, on the other hand, is much more difficult to obtain, using a computational technique known as flag algebras.

Over the next two lectures we will study the general case $\pi(K_s^r)$, showing that

$$1 - \left(\frac{r-1}{s-1}\right)^{r-1} \leq \pi(K_s^{(r)}) \leq 1 - \binom{s-1}{r-1}^{-1}.$$

Note that for $r = 2$ this just reduces to Turán's theorem.

We will start with the upper bound. It will be convenient in what follows to flip the definition and to take $T(n, s, r)$ to be the minimum number of edges in an r-uniform hypergraph $\mathcal{G}$ on n vertices such that any subset with s vertices contains at least one edge. We also define a density version $t(s, r) = \lim_{n\to\infty} \binom{n}{r}^{-1} T(n, s, r)$. Note that $t(s, r) + \pi(K_s^{(r)}) = 1$. Our main result of this lecture will now be that

$$T(n, s, r) \geq \frac{n-s+1}{n-r+1} \binom{s-1}{r-1}^{-1} \binom{n}{r},$$

a result due to de Caen. That $t(s, r) \geq \binom{s-1}{r-1}^{-1}$ then follows easily.

To begin, we will prove an inequality which relates the number of copies of cliques with various sizes. Given an r-uniform hypergraph $\mathcal{G}$ on n vertices, let N_s be the number of copies of $K_s^{(r)}$ in $\mathcal{G}$.

Lemma 1

$$N_{s+1} \geq \frac{s^2 N_s}{(s-r+1)(s+1)} \left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right),$$

provided $N_{s-1} \neq 0$.

Proof Let P be the number of pairs (S,T), where S and T are sets of size s with $|S\cap T| = s-1$, S spans a copy of $K_s^{(r)}$ and T does not. We will count P in two different ways to get a bound.
On the one hand, for each $i = 1, \ldots, N_{s-1}$, let a_i be the number of copies of $K_s^{(r)}$ which contain the ith copy of $K_{s-1}^{(r)}$. Note that $\sum_{i=1}^{N_{s-1}} a_i = sN_s$. Therefore,

$$P = \sum_{i=1}^{N_{s-1}} a_i(n-s+1-a_i) = (n-s+1)\sum_{i=1}^{N_{s-1}} a_i - \sum_{i=1}^{N_{s-1}} a_i^2 \leq (n-s+1)sN_s - N_{s-1}^{-1}s^2N_s^2,$$

where the inequality follows from Cauchy-Schwarz.
On the other hand, let the copies of $K_s^{(r)}$ be $B_1, \ldots, B_{N_s}$ and let b_i be the number of $K_{s+1}^{(r)}$ containing B_i. For each B_j, there are $n-s-b_j$ ways to choose $x \notin B_j$ such that $B_j \cup \{x\}$ does not span a $K_{s+1}^{(r)}$. For any such x, there must be some $C \subseteq B_j$ of size $r-1$ such that $C \cup \{x\}$ is not an edge. Therefore, for every $y \in B_j \backslash C$, the pair $(B_j, B_j \cup \{x\} \backslash y)$ is counted by P. Hence,

$$P \geq \sum_{j=1}^{N_s} (n-s-b_j)(s-r+1) = (s-r+1)((n-s)N_s - (s+1)N_{s+1}),$$

where we used that $\sum_{j=1}^{N_s} b_j = (s+1)N_{s+1}$. Comparing the upper and lower bounds gives the result. $\square$

For graphs, this is known as the Moon-Moser inequality. The hypergraph case is due to de Caen. From it, we may derive the following lemma.

Lemma 2

$$N_s \geq N_{s-1}\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(e(G) - F(n,s,r)),$$

where $F(n,s,r) = r^{-1}((n-r+1) - \binom{s-1}{r-1}^{-1}(n-s+1))\binom{n}{r-1}$.

Proof We prove the result by induction on s. For $s = r$, we have $N_s = e(G)$. This is easily seen to accord with the inequality.
Suppose, therefore, that the inequality holds for s. We will prove it for $s+1$. By the Moon-Moser inequality and the induction hypothesis,

$$\begin{aligned}
\frac{N_{s+1}}{N_s} &\geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right) \\
&\geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(e(G)-F(n,s,r)) - \frac{(r-1)(n-s)+s}{s^2}\right) \\
&= \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}e(G) - \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}F(n,s,r) - \frac{(r-1)(n-s)+s}{(s-r+1)(s+1)}.
\end{aligned}$$

It remains to show that

$$F(n,s+1,r) \geq F(n,s,r) + \frac{((r-1)(n-s)+s)}{s-r+1}\frac{(s+1)\binom{n}{r-1}}{r^2\binom{s+1}{r}}.$$

A long but relatively straightforward computation allows us to show that equality actually holds. The result follows. □

De Caen's result now follows easily.

Theorem 1
$$T(n,s,r) \geq \frac{n-s+1}{n-r+1}\binom{s-1}{r-1}^{-1}\binom{n}{r}.$$

Proof From the previous lemma and since the $F(n,s,r)$ increase with s, we must have $e(\mathcal{G}) \leq F(n,s,r)$ for any $K_s^{(r)}$-free graph. Therefore,
$$T(n,s,r) \geq r^{-1}\binom{s-1}{r-1}^{-1}(n-s+1)\binom{n}{r-1} = \frac{n-s+1}{n-r+1}\binom{s-1}{r-1}^{-1}\binom{n}{r},$$
as required. □

Lecture 15

In this lecture, we will prove the lower bound

$$\pi(K_s^{(r)}) \geq 1 - \left(\frac{r-1}{s-1}\right)^{r-1}.$$

Equivalently, we shall show that $t(s,r) \leq \left(\frac{r-1}{s-1}\right)^{r-1}$.

The construction which we shall analyse is as follows. Given n vertices, divide them into $s-1$ roughly equal parts $A_1, \ldots, A_{s-1}$. A set B of size r is an edge of $\mathcal{G}$ if and only if there is some j such that $\sum_{i=1}^{k} |B \cap A_{j+i}| \geq k+1$ for each $1 \leq k \leq r-1$ (taking addition modulo $s-1$ in the subscript). Note that for $r=3$ and $s=4$, this is precisely the complement of the construction described in the previous lecture.

It turns out to be convenient to analyse the construction in rather different terms.

Lemma 1 *A lorry driver needs to follow a certain closed route. There are several petrol stations along the route and the total amount of fuel in these stations is sufficient for the route. Then there is a starting point from which the route can be completed.*

Proof Suppose that the driver had enough fuel for the journey and consider the journey starting from an arbitrary point where the driver still picks up the fuel at every station, even though he doesn't need it. Then the point at which the fuel reserves are lowest during this route can be used as a starting point for another route where the fuel supply never drops below zero. □

We first prove that $\mathcal{G}$ has the fundamental property required to show that $T(n,s,r) \leq e(\mathcal{G})$.

Lemma 2 *Every subset of $\mathcal{G}$ with s vertices contains an edge.*

Proof Consider any set S of size s. We will use the lorry driver model where we travel through all n vertices, going through the A_i in order. Imagine, in the lorry driver model, that every element of S represents a unit of fuel and that it takes $\frac{s}{s-1}$ units of fuel to travel from A_i to A_{i+1}. Then S contains enough fuel for a complete circuit. Hence, by the previous lemma, there is an appropriate starting point to complete the circuit.

Let B be the first r elements of S that are encountered on this circuit. Since $r \geq (r-1)\frac{s}{s-1}$, the lorry can advance distance $r-1$ using just the fuel from B. This implies that B is an edge, as $\lceil k\frac{s}{s-1} \rceil = k+1$ for $1 \leq k \leq r-1$. Thus any set of size s contains an edge, as required. □

All that remains to be done is to estimate the number of edges in $\mathcal{G}$.

Lemma 3

$$e(\mathcal{G}) = (1+o(1))\left(\frac{r-1}{s-1}\right)^{r-1}\binom{n}{r}.$$

Proof We will count ordered edges $x_1 \ldots x_r$. Each edge will be in $A_{j+1} \cup A_{j+2} \cup \cdots \cup A_{j+r-1}$ for some j. We may choose an ordered edge $x_1 \ldots x_r$ by choosing (i) a starting position j, (ii) a choice of

A_{j+i}, $1 \le i \le r-1$, in which to place each x_ℓ and (iii) a vertex for each x_ℓ within each assigned part. There are $s-1$ choices in step (i) and $\left(\frac{n}{s-1}\right)^r + O(n^{r-1})$ choices in step (iii).

In step (ii), there are $(r-1)^r$ ways to assign the parts if we ignore the required inequalities in the intersection sizes (i.e. that there should be enough fuel for the lorry). We claim that given any assignment, there is exactly one cyclic permutation which satisfies the required inequalities. More precisely, if we assign b_i of the x_ℓ to A_{j+i} for $1 \le i \le r-1$, then there is exactly one c with $1 \le c \le r-1$ such that the shifted sequence $b'_i = b_{c+i}$ (addition taken modulo $r-1$) satisfies $\sum_{i=1}^k b'_i \ge k+1$ for each $1 \le k \le r-1$. To see this, consider a lorry that makes a circuit of the A_{j+i}, $1 \le i \le r-1$, where each of the x_ℓ has a unit of fuel, but now it takes one unit of fuel to advance from A_{j+i} to A_{j+i+1}, and the lorry is required to always have one spare unit of fuel. It is clear that a valid starting point for the lorry is equivalent to a shifted sequence satisfying the required inequalities. Imagine that the driver starts with enough fuel to drive around the route and consider the journey starting from an arbitrary point. Then the point at which the fuel reserves are lowest during this route is a starting point for a route where there is always a spare unit of fuel. This is the unique point at which the fuel reserves are lowest and so it gives the unique cyclic permutation satisfying the required inequalities. To see this note that if i_1 and i_2 with $i_1 \le i_2$ both worked, we would have

$$r = \sum_i b_i = \sum_{i=i_1}^{i_2-1} b_i + \sum_{i=i_2}^{i_1-1} b_i \ge (i_2 - i_1 + 1) + ((r-1) - (i_2 - i_1) + 1) = r+1,$$

a contradiction. We deduce that there are $(r-1)^{r-1}$ valid assignments in step (ii).

Putting everything together gives

$$\frac{1}{r!}(s-1)(r-1)^{r-1}(1+o(1))\left(\frac{n}{s-1}\right)^r = (1+o(1))\left(\frac{r-1}{s-1}\right)^{r-1}\binom{n}{r},$$

as required. □

Extremal graph theory - Example Sheet 1

1. Show that if G is a graph with n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, then G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles. Show that, for $n \geq 3$, this result is sharp.

2. Let G be a non-bipartite graph with more than $\frac{1}{4}(n-1)^2 + 1$ edges. Show that G contains a triangle. Show that, for all odd $n \geq 5$, there is a triangle-free non-bipartite graph with $\frac{1}{4}(n-1)^2 + 1$ edges.

3. By considering a random ordering of the vertices of a graph G, show that the size $\alpha(G)$ of the largest independent set in G satisfies
$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$
Deduce Turán's theorem.

4. Let S be a set of diameter 1 in the plane, that is, no two points are at distance more than 1. Show that the number of pairs of points of S whose distance is greater than $\frac{1}{\sqrt{2}}$ is at most $\lfloor \frac{n^2}{3} \rfloor$, where $n = |S|$. Moreover, for $n \geq 2$, show that this is sharp.

5. Show that Hall's theorem does not hold for infinite graphs. That is, find an infinite bipartite graph between sets A and B for which every subset of A has at least the cardinality of A neighbours in B, but there is no way of matching A to a subset of B.

6. Suppose that the edges of the complete graph K_n have been coloured with two colours, red and blue. Show that there are at least $\frac{n(n-1)(n-5)}{24}$ monochromatic triangles.

7. Determine $\lim_{n\to\infty} ex(n,H)/\binom{n}{2}$ for each of the platonic solids.

8. Show that an ϵ-regular partition of a graph G is also an ϵ-regular partition of its complement $\overline{G}$.

9. Suppose $ex(n,H) \leq \rho\binom{n}{2}$ whenever $n \geq n_0$. Show by averaging that, for n large, any graph G on n vertices with more than $(\rho+\epsilon)\binom{n}{2}$ edges contains at least $c(\epsilon)n^{v(H)}$ copies of H.

10. Show that, for any natural number t and any $\delta > 0$ there exists an n_0 such that, for $n \geq n_0$, if G is a bipartite graph between $\{1, 2, \ldots n\}$ and $\{1, 2, \ldots, n\}$ with at least δn^2 edges, there is a complete bipartite graph $K_{t,t}$ between two sets U and V of size t, where U and V are arithmetic progressions of length t with the same common difference.

Example sheet 1 - solutions

1. We will prove the result by induction on n. For $n = 3$, a subgraph with three edges contains one triangle, as expected. Similarly, for $n = 4$, it is easily checked that any subgraph with 5 edges must contain two triangles.

   Assume now that a graph with $n-2$ vertices and at least $\lfloor \frac{(n-2)^2}{4} \rfloor + 1$ edges contains at least $\lfloor \frac{n-2}{2} \rfloor$ triangles. We will prove the required result also holds for n. Suppose that we have a graph G on n vertices with $\lfloor \frac{n^2}{4} \rfloor + 1 = \lfloor \frac{(n-2)^2}{4} \rfloor + 1 + (n-1)$ edges but with fewer than $\lfloor \frac{n}{2} \rfloor$ triangles. Let x and y be two vertices which are joined by an edge but are not contained in a triangle. This is certainly possible, since $3(\lfloor \frac{n}{2} \rfloor - 1) \leq \lfloor \frac{n^2}{4} \rfloor + 1$. Therefore, as usual $d(x) + d(y) \leq n$. Moreover, the neighborhoods $N(x)$ and $N(y)$ of x and y must be disjoint. We now know that the graph $H = G - \{x, y\}$ contains at least $\lfloor \frac{(n-2)^2}{4} \rfloor + 1$ edges. It must therefore contain at least $\lfloor \frac{n-2}{2} \rfloor$ triangles. But the number of edges between $N(x)$ and $N(y)$ is at most $\lfloor \frac{(n-2)^2}{4} \rfloor$. Therefore, one of $N(x)$ and $N(y)$ must contain an edge. This yields one further triangle and proves the result.

   To show that the result is sharp, we just take the bipartite graph between sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ and add one extra edge in the set of size $\lceil \frac{n}{2} \rceil$.

2. We will prove the result by induction on n. For $n = 4$ and $n = 5$, there are no non-bipartite triangle-free graphs with n vertices and 4 or 6 edges respectively.

   Assume now that any non-bipartite graph on $n-2$ vertices with more than $\frac{1}{4}(n-3)^2 + 1$ edges contains a triangle. Let G be a non-bipartite graph on n vertices with more than $\frac{1}{4}(n-1)^2 + 1$ vertices and assume that it contains no triangles. Let xy be an edge in G. Since G is triangle-free, $N(x)$ and $N(y)$ both form independent sets. But the union of the two sets cannot be everything, for otherwise G would have to be bipartite. Therefore $d(x) + d(y) \leq n - 1$. This implies that the number of edges in $H = G - \{x, y\}$ is more than $\frac{1}{4}(n-3)^2 + 1$. If H is not bipartite, then, by induction, H contains a triangle and we are done. Therefore, the graph H must be bipartite.

   If H is bipartite, let A and B be the sets in the partition with $|A| \geq |B|$. Neither x nor y can have neighbours in both A and B. Otherwise, we would have a triangle. Moreover, if x only has neighbours in A and y only has neighbours in B (or vice versa), the graph G is bipartite. Therefore, all of the neighbours of x and y lie in A or B. Since $|A| \geq |B|$, the maximum number of edges occurs when all neighbours of x and y are in A. In this case, we have $|A||B| + |A| + 1$ edges. Since $|B| = n - |A| - 2$, this is maximised by taking $|A| = \lfloor \frac{n-1}{2} \rfloor$.

   To show that this is sharp for odd values of n, take two sets, one of size $\frac{n-1}{2}$ and the other of size $\frac{n-3}{2}$, and place every edge between them. Then take two extra vertices x and y, join them and connect one (and only one) of them to every vertex in the piece of size $\frac{n-1}{2}$, insisting that each of x and y has at least one neighbor in this set. This yields a graph G which is not bipartite (it has a 5-cycle), contains no triangle and has $\frac{1}{4}(n-1)^2 + 1$ edges.

3. Let $<$ be a uniformly chosen ordering of V. Define

   $$I = \{v \in V : \{v, w\} \in E \Rightarrow v < w\}.$$

   Let X_v be the indicator random variable which indicates whether or not $v \in I$. That is, it takes value 1 if $v \in I$ and 0 otherwise. Let $X = \sum_{v \in V} X_v = |I|$. For each v,

   $$\mathbb{E}[X_v] = \mathbb{P}[v \in I] = \frac{1}{d(v) + 1},$$

since $v \in I$ if and only if it is the smallest element among v and its neighbours. Therefore

$$\mathbb{E}[X] = \sum_{v \in V} \frac{1}{d(v)+1}.$$

In particular, there exists some ordering for which $|I| \geq \sum_{v \in V} \frac{1}{d(v)+1}$. But it is easily verified that the set of elements in I form an independent set.

To deduce Turán's theorem, suppose that G is a graph with more than $\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}$ edges. Its complement $\overline{G}$ has fewer than

$$\binom{n}{2} - \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} = \frac{1}{r-1}\frac{n^2}{2} - \frac{n}{2}$$

edges. Now the function $\sum_v \frac{1}{d(v)+1}$ will be minimised when all of the $d(v)$ have size as close as possible. Therefore, taking $d(v) = \frac{1}{r-1}n - 1 - \epsilon$ for each v, we have

$$\alpha(\overline{G}) \geq \sum_v \frac{1}{d(v)+1} \geq \frac{n}{\frac{n}{r-1} - \epsilon} > r - 1.$$

Since an independent set in $\overline{G}$ is a clique in G, this implies Turán's theorem.

4. Let $S = \{x_1, \ldots, x_n\}$. Consider the graph G formed by joining two vertices if the distance between them is greater than $1/\sqrt{2}$. If we can show that G contains no copy of K_4, then Turán's theorem will imply that there are at most $\frac{2}{3}\frac{n^2}{2} = \frac{n^2}{3}$ edges in G, as required.

   To prove that G contains no K_4, we begin by noting that the convex hull of any four points forms either a line, a triangle or a quadrilateral. In any of these cases, there will be three points x_i, x_j and x_k such that the angle $x_i x_j x_k$ is at least 90 degrees.

   Now, consider the triangle formed by x_i, x_j and x_k. If both $d(x_i, x_j)$ and $d(x_j, x_k)$ are greater than $\frac{1}{\sqrt{2}}$, then $d(x_i, x_k)$ will be greater than 1, which contradicts the assumption about the set S. Therefore, at least one of $x_i x_j$ or $x_j x_k$ is not in G, so the graph does not contain a K_4.

   To show that it is sharp, let r be a real number with $0 < r < \left(1 - \frac{1}{\sqrt{2}}\right)/4$ and let $p = \lfloor \frac{n}{3} \rfloor$. Take an equilateral triangle with side length $1 - 2r$ and draw a circle of radius r around each of the vertices. Place $x_1, \ldots, x_p$ in the first circle, $x_{p+1}, \ldots, x_{2p}$ in the second circle and $x_{2p+1}, \ldots, x_n$ in the third circle. We may also insist that x_1 and x_n are distance 1 exactly apart to give the set diameter 1. If x_i and x_j are in different pairs, they are distance greater than $\frac{1}{\sqrt{2}}$ apart and if they are in the same set their distance is smaller than this. Therefore, there are $\lfloor \frac{n^2}{3} \rfloor$ pairs with $d(x_i, x_j) > \frac{1}{\sqrt{2}}$.

5. We take $A = B = \mathbb{N}$. We connect the vertex 1 in A to everything in B and, for $i > 1$, we connect i in A to $i-1$ in B. This then satisfies Hall's condition but contains no matching.

6. The number of monochromatic triangles is at least

$$\frac{1}{2}\left(\sum_v \binom{r_v}{2} + \sum_v \binom{b_v}{2} - \binom{n}{3}\right),$$

where r_v and b_v are the red and blue degrees, respectively, of the vertices v over which we are summing. (To prove this formula, consider, in turn, the contribution of monochromatic and non-monochromatic triangles to the sum.) This is maximised when $r_v = b_v = (n-1)/2$ for all v. A quick calculation then implies that the number of monochromatic triangles is at least $\frac{n-5}{12}\binom{n}{2}$, as required.

7. This clearly reduces to determining the chromatic number of each of the graphs. One may easily verify that $\chi(\text{Tetrahedron}) = 4$, $\chi(\text{Cube}) = 2$, $\chi(\text{Octahedron}) = 3$, $\chi(\text{Dodecahedron}) = 3$ and $\chi(\text{Icosahedron}) = 4$.

8. This follows easily from the definition.

9. By assumption, any set of size n_0 containing more than $\rho\binom{n}{2}$ edges contains a copy of H. For at least $\frac{\epsilon}{2}\binom{n}{n_0}$ choices of a set N of size n_0, we must have that the number of edges in N is at least $\left(\rho + \frac{\epsilon}{2}\right)\binom{n_0}{2}$. If, on the contrary, this wasn't the case, we would have

$$\sum_N e(G[N]) \leq \binom{n}{n_0}\left(\rho + \frac{\epsilon}{2}\right)\binom{n_0}{2} + \frac{\epsilon}{2}\binom{n}{n_0}\binom{n_0}{2} = (\rho + \epsilon)\binom{n}{n_0}\binom{n_0}{2}.$$

On the other hand, we have

$$\sum_N e(G[N]) = \binom{n-2}{n_0-2}e(G) > \binom{n-2}{n_0-2}(\rho+\epsilon)\binom{n}{2} = (\rho+\epsilon)\binom{n}{n_0}\binom{n_0}{2},$$

which would be a contradiction. Now, every set of size n_0 with density $\rho + \frac{\epsilon}{2}$ contains a copy of H. Therefore, the number of copies of H is at least

$$\binom{n-v(H)}{n_0-v(H)}^{-1}\frac{\epsilon}{2}\binom{n}{n_0} = \frac{\epsilon}{2}\binom{n_0}{v(H)}^{-1}\binom{n}{v(H)}.$$

The required result follows with $c(\epsilon) = \frac{\epsilon}{2}\binom{n_0}{v(H)}^{-1}$.

10. Given a bipartite graph G between $\{1, 2, \ldots n\}$ and $\{1, 2, \ldots, n\}$ of density at least δ, we may describe a subset of $[n]^2$ of density at least δ by including (i, j) if and only if there is an edge between i and j. If we now apply the multidimensional version of Szemerédi's theorem with $d = 2$ and $P = \{(i, j) : 0 \leq i, j \leq t-1\}$, we get a subset of the form $\{(u+ki, v+kj) : 0 \leq i, j \leq t-1\}$. This implies the theorem with $U = \{u + ki : 0 \leq i \leq t-1\}$ and $V = \{v + kj : 0 \leq j \leq t-1\}$ being arithmetic progressions of length t with common difference k.

Extremal graph theory - Example Sheet 2

1. Prove that the Turán density of an r-uniform hypergraph $\mathcal{H}$, that is, $\lim_{n\to\infty} ex(n, \mathcal{H})/\binom{n}{r}$, is well-defined.

2. Given a set of n points in the plane, prove that there are at most $cn^{3/2}$ pairs of points which are a unit distance apart.

3. Prove that if H is a bipartite graph with t vertices and m edges, then $ex(n, H) \geq cn^{2-\frac{t-2}{m-1}}$.

4. Show that a bipartite graph of density ϵ between sets A and B contains a complete bipartite subgraph $K_{a,b}$ with a subset of A of size $a = \epsilon|A|$ and a subset of B of size $b = 2^{-|A|}|B|$. Show also that if $|A| = |B| = n$, one may find a complete bipartite subgraph $K_{s,t}$ with $s = c(\epsilon)\log n$ and $t = n^{1/2}$.

5. Suppose that a graph G contains δn^3 triangles. Use the result of the previous question to show that G must also contain a blow-up $K_{t,t,t}$, that is, a graph with three vertex sets of size t such that every two vertices in different vertex sets are connected, where $t = c(\delta)\log n$.

6. Use the result of the previous question with the supersaturation property to prove the particular case of the Erős-Stone-Simonovits theorem when $\chi(H) = 3$.

7. Prove that for any $\epsilon > 0$ there exists a graph H of chromatic number 3 such that $ex(n, H) > \frac{1}{4}n^2 + c_H n^{2-\epsilon}$.

8. Show that a graph H of chromatic number t can satisfy $ex(n, H) = ex(n, K_t)$ only if it is colour-critical, that is, if there is an edge e such that $\chi(H \backslash e) < t$.

9. Let $\mathcal{H} = K^{(3)}_{t,t,t}$ be the complete tripartite 3-uniform hypergraph with three sets each of size t. Prove that $ex(n, \mathcal{H}) \leq cn^{3-\frac{1}{t^2}}$.

10. The cube Q_t is the graph on vertex set $\{0, 1\}^t$ where two vertices are connected if and only if they differ in exactly one coordinate. The Ramsey number of a graph H is the smallest number n such that in any 2-colouring of the complete graph K_n on n vertices there is guaranteed to be a monochromatic copy of H. Use dependent random choice to prove that $r(Q_t) \leq 2^{ct}$.

Example sheet 2 - solutions

1. Consider an r-uniform hypergraph on $n+1$ vertices which contains no copy of $\mathcal{H}$. Then no subset of n vertices contains a copy of $\mathcal{H}$ either. Hence, any subset of size n contains at most $ex(n, \mathcal{H})$ edges. Therefore, by averaging over subsets of size n,
$$ex(n+1, \mathcal{H}) = \frac{1}{\binom{n+1-r}{n-r}} \sum_{|U|=n} e(U) \leq \frac{n+1}{n+1-r} ex(n, \mathcal{H}).$$
Dividing either side by $\binom{n+1}{r}$ yields
$$\frac{ex(n+1, \mathcal{H})}{\binom{n+1}{r}} \leq \frac{ex(n, \mathcal{H})}{\binom{n}{r}}.$$
Therefore, since these ratios are decreasing and bounded below by 0, they must approach a limit.

2. Let X be a set of n points in the plane. Form a graph G by connecting two vertices if and only if they are distance 1 apart. It is easy to check that the graph contains no copy of $K_{2,3}$. Therefore $e(G) \leq ex(n, K_{2,3}) \leq cn^{3/2}$, as required.

3. This is a standard application of the probabilistic method. Consider the random graph $G_{n,p}$ where $p = cn^{-\frac{t-2}{m-1}}$. The expected number of edges is at least $pn^2/8$ and the expected number of copies of H is at most $p^m n^t$. For c sufficiently small, we have
$$p^m n^t \leq \frac{1}{16} pn^2.$$
Therefore, since $\mathbb{E}(\text{edges} - \text{copies of } H) \geq \frac{1}{16}pn^2$, we may remove all copies of H and still be left with a graph which has $\frac{1}{16}pn^2 = \frac{c}{16}n^{2-\frac{t-2}{m-1}}$ edges.

4. By the convexity of the function $f(x) = \binom{x}{a}$ and the fact that the average degree of B is at least $a = \epsilon|A|$, we conclude that the number of pairs (U, v) with U a subset of A of size a and v a vertex in B connected to every element of U is at least
$$\sum_{v \in B} \binom{d(v)}{a} \geq |B| \binom{\frac{1}{|B|}\sum_{v\in B} d(v)}{a} \geq |B|.$$
Since A has at most $2^{|A|}$ subsets, the pigeonhole principle implies that for some $U \subset A$ of size a there are at least $b = 2^{-|A|}|B|$ elements of B which are connected to every element of U. This yields the required copy of $K_{a,b}$.

   For the second part, note that
$$\sum_{v \in B} \binom{d(v)}{s} \geq |B| \binom{\epsilon|A|}{s} \geq |B| \frac{(\epsilon|A|/2)^s}{s!},$$
where the inequalities follow from the convexity of $f(x) = \binom{x}{s}$ and the fact that $s = c(\epsilon)\log n \leq \epsilon|A|/2$. If the graph does not contain $K_{s,t}$ then we know that every subset of A of size s has at most $t-1$ common neighbours. Therefore,
$$|B| \frac{(\epsilon|A|/2)^s}{s!} \leq (t-1)\binom{|A|}{s} < t\frac{|A|^s}{s!} = n^{1/2}\frac{|A|^s}{s!}.$$
But for $c(\epsilon)$ sufficiently small, $(\epsilon/2)^s|B| \geq n^{1/2}$, so this is a contradiction.

5. Partition the set of vertices into three sets at random. This easily yields a partition for which there are at least $c\delta n^3$ triangles with one vertex in each part. We will therefore, without loss of generality, assume that we have three vertex sets V_1, V_2 and V_3 and that there are δn^3 triangles with one vertex in each part. Let E_{23} be the set of edges between V_2 and V_3 which are contained in at least $\frac{\delta}{2}n$ triangles. Note that $|E_{23}| \geq \frac{\delta}{2}n^2$. Otherwise, we would have at most $|E_{23}|n + n^2\frac{\delta}{2}n < \delta n^3$ triangles, contradicting our assumption.

   Using the second part of the previous question, we may now find a complete graph between two sets $W_2 \subset V_2$ and $W_3 \subset V_3$, where $|W_2| = |W_3| = c(\delta)\log n$. Now consider a complete matching M between W_2 and W_3. M will have $c(\delta)\log n$ edges. Consider the bipartite graph between M and the set V_1, where m and v are joined if any only if they form a triangle together. Since every edge in M is in E_{23}, there are at least $|M|\frac{\delta}{2}n = \frac{\delta}{2}|M||V_1|$ edges in the graph. Therefore, applying the first part of the previous question, we can find a subset M' of M of size $\frac{\delta}{2}|M|$ and a subset of W_1 of V_1 of size $2^{-|M'|}n \geq n^{1/2}$, for $c(\delta)$ sufficiently small. Let X_2 and X_3 be the two endpoints of the matching M' and let X_1 be a subset of W_1 of size $|M'|$. The graph between X_1, X_2 and X_3 is the required blow-up of the triangle.

6. By supersaturation, we know that as soon as we have density $\frac{1}{2} + \epsilon$, we have δn^3 triangles. By the previous question, this implies the existence of a large blow-up. This will contain any 3-chromatic graph provided n is sufficiently large. (Note that this question and the last may be generalised to give a full proof of Erdős-Stone-Simonovits.)

7. Let $H = K_{t,t,t}$, that is, there are three vertex sets of size t and any two vertices in different parts are connected. Consider also a graph G consisting of two vertex sets U and V of size $n/2$, where U is empty and V contains a graph L containing no copy of $K_{t,t}$. We know that there exist such graphs with at least $c(n/2)^{2-2/t}$ edges. We will assume that L has this many edges and, therefore, that G has $\frac{1}{4}n^2 + c'n^{2-2/t}$ edges.

   It is elementary to check that G contains no copy of H. Indeed, any copy of H clearly cannot be contained entirely within V. Therefore, there is some vertex in U. But the neighborhood of this vertex in H contains a copy of $K_{t,t}$ and the neighborhood of this vertex in G must lie entirely inside V, so we have a contradiction.

8. Let H be a graph which is not colour-critical, that is, removing any edge still leaves one with a graph of chromatic number t. Consider the Turán graph, which consists of $t-1$ vertex subsets of size as equal as possible and add a single edge e in one of the vertex sets. We will show that this graph, which necessarily contains a copy of K_t, does not contain a copy of H. Clearly, any copy of H must contain the edge e. So for any copy we get two vertices in the same vertex subset and, by construction, every other vertex must lie in distinct vertex subsets. But then the edge e is the only obstruction to making the graph $(t-1)$-chromatic, so deleting the edge e from H will yield a graph of chromatic number $t-1$. This contradicts the definition of colour-critical.

9. Suppose that we have a hypergraph $\mathcal{G}$ with n vertices and cn^{3-1/t^2} edges not containing $K_{t,t,t}$ as a subgraph. Note that the average number of edges containing a 2-edge is $3cn^{1-1/t^2}$. We will count pairs (e, T) consisting of edges e and sets of vertices T, of size t, such that every vertex in T forms an edge with e. The number of such pairs is at least

$$\sum_e \binom{d(e)}{t} \geq \binom{n}{2}\binom{\frac{1}{\binom{n}{2}}\sum_e d(e)}{t} \geq \binom{n}{2}\binom{3cn^{1-1/t^2}}{t} \geq \binom{n}{2}c^t n^{t-1/t}t! = c'\frac{n^{t+2-1/t}}{t!}.$$

Therefore, since there are $\binom{n}{t}$ possible choices for T, there exists some T_1 of size t with common neighborhood a graph E of size at least $c'n^{2-1/t}$. Provided c and hence c' is sufficiently large, we may now apply the result that we know for ordinary graphs to find sets T_2 and T_3 of size t such that there is a complete subgraph of E between them. This completes the proof.

10. I will not write out a full proof of this. Instead, I refer the reader to the survey paper 'Dependent random choice' by Jacob Fox and Benny Sudakov. A complete proof of the required result is contained in Lemma 2.1 and Theorem 3.4, though I heartily recommend the full paper.