

# Quantum Information Flow: A Computer Science Perspective

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## 1 Introduction

Quantum Information and Computation (QIC) is concerned with the use of quantum-mechanical systems to carry out computational and information-processing tasks [29]. It emerged from the recognition that quantum phenomena such as entanglement—Einstein’s “spooky action at a distance”—should be seen not as a *bug* but as a *feature*. Some first fruits of this were the Deutch-Jozsa, Shor and Grover algorithms, the BB84 and Ekert 91 public key distribution schemes, the quantum teleportation protocol and several variants [29]. However, although the point of view is new, the *methods* have largely remained traditional: the manipulations of complex numbers, vectors and matrices in “computational bases” built from *kets*  $|0\rangle$  and  $|1\rangle$  bear some comparison with the acrobatics with bits and bytes in the early days of computer programming.

### 1.1 The need for high-level methods

Why does this matter? The current tools available for developing quantum algorithms and protocols are *deficient* in two main respects.

Firstly, they are too *low-level*. Quantum algorithms are currently mainly described using the ‘network model’ corresponding to circuits in classical computation. One finds a plethora of ad hoc calculations with ‘bras’ and ‘kets’, normalizing constants, matrices etc. The arguments

for the benefits of a high-level, conceptual approach to designing, programming and reasoning about quantum computational systems are just as compelling as for classical computation. Moreover, there is the whole issue of integrating quantum and classical features, which would surely be mandatory in any realistic system.

At a more fundamental level, the standard mathematical framework for quantum mechanics (which is essentially due to von Neumann [37]) is actually *insufficiently comprehensive* for informatic purposes. In describing a protocol such as teleportation, or any quantum process in which *the outcome of a measurement is used to determine subsequent actions*, the von Neumann formalism leaves feedback of information from the classical or macroscopic level back to the quantum *implicit* and *informal*, and hence not subject to rigorous analysis and proof. As quantum protocols and computations grow more elaborate and complex, this point is likely to prove of increasing importance. For example, the development of secure distributed quantum communication schemes will involve an interplay between classical and quantum components, distributed agents, and all the subtle concepts pertaining to information security. It will be *harder* to specify and reason about quantum information security than classical information security, which is already a major topic of current research.

Furthermore, there are many fundamental issues in QIC which remain very much open. The current low-level methods seem unlikely to provide an adequate basis for addressing them. For example:

- What are the precise structural relationships between superposition, entanglement and mixedness as quantum informatic resources? Or, more generally,
- Which features of quantum mechanics account for differences in computational and informatic power as compared to classical computation?
- How do quantum and classical information interact with each other, and with a spatio-temporal causal structure?
- Which quantum control features (e.g. iteration) are possible and what additional computational power can they provide?
- What is the precise logical status and axiomatics of (No-)Cloning and (No-)Deleting, and more generally, of the quantum mechanical formalism as a whole?

These questions gain additional force from the fact that a variety of different quantum computational architectures and information-processing scenarios are beginning to emerge. While at first it seemed that the notions of Quantum Turing Machine and the quantum circuit model could supply canonical analogues of the classical computational models, recently some very different models for quantum computation have emerged, e.g. Raussendorf and Briegel's *one-way quantum computing* model [31, 33] and *measurement based quantum computing* in general [25], *adiabatic quantum computing* [18], *topological quantum computing* [19] etc. These new models have features which are both theoretically and experimentally of great interest, and the methods developed to date for the circuit model of quantum computation do not carry over straightforwardly to them. In this situation, we can have no confidence that a comprehensive paradigm has yet been found. It is more than likely that we have overlooked many new ways of letting a quantum system compute.

Thus there is a need to design structures and develop methods and tools which apply to *non-standard quantum computational models* where most of the current methods fail, in particular the *one-way quantum computing* model and *measurement based quantum computing*

in general. We must also address the question of how the various models compare — can they be interpreted in each other, and which computational and physical properties are preserved by such interpretations?

## 1.2 Categorical Quantum Mechanics

Our approach is very different to previous work on the Computer Science side of this interdisciplinary area, which has focussed on quantum algorithms and complexity. The focus has rather been on developing *high-level methods* for Quantum Information and Computation (QIC)—languages, logics, calculi, type systems etc.—analogous to those which have proved so essential in classical computing [2]. This has led to nothing less than a recasting of *the foundations of Quantum Mechanics itself*, in the more abstract language of category theory. A key contribution is the paper with Bob Coecke [5], in which we develop an axiomatic presentation of quantum mechanics in the general setting of *strongly compact closed categories*, which is adequate for all the needs of QIC.

Specifically, we show that we can recover the key quantum mechanical notions of *inner-product*, *unitarity*, *full and partial trace*, *Hilbert-Schmidt inner-product* and *map-state duality*, *projection*, *positivity*, *measurement*, and *Born rule* (which provides the quantum *probabilities*), axiomatically at this high level of abstraction and generality. Moreover, we can derive the correctness of protocols such as quantum teleportation, entanglement swapping and logic-gate teleportation [9, 20, 39] in a transparent and very conceptual fashion. Also, while at this level of abstraction there is no underlying field of complex numbers, there *is* still an intrinsic notion of ‘scalar’, and we can still make sense of *dual vs. adjoint* [5, 6], and *global phase and elimination thereof* [11]. Further work in this line has shown how a range of notions including *mixed states*, *completely positive maps*, *Jamiołkowski map-state duality*, *decoherence*, *generalized measurements* and *Naimark’s theorem* can be elegantly incorporated in this framework [35, 12, 13].

Moreover, this formalism has two important additional features. Firstly, it goes *beyond* the standard Hilbert-space formalism, in that it is able to capture classical as well as quantum information flows, and the interaction between them, *within the formalism*. For example, we can capture the idea that the result of a measurement is used to determine a further stage of quantum evolution, as e.g. in the teleportation protocol [9], where a unitary correction must be performed after a measurement; or also in measurement-based quantum computation [31, 32]. Secondly, this categorical axiomatics can be presented in terms of a *diagrammatic calculus* which is extremely intuitive, and potentially can replace low-level computation with matrices by much more conceptual — and automatable — reasoning. Moreover, this diagrammatic calculus can be seen as a proof system for a logic, leading to a radically new perspective on what the right logical formulation for Quantum Mechanics should be. This latter topic is initiated in [7], and developed further in [17].

## 1.3 Outline of the Lecture

The present document is a written-up version of a lecture, rather than a full paper. It emphasizes the basic ideas and intuition, rather than formal details. Much of the mathematics will be cast in a (hopefully appealing) diagrammatic form.

We shall sketch some background material on basic Quantum Mechanics, discuss entanglement and teleportation, and show how structural and compositional methods can be used to illuminate quantum information flow, and can be presented in diagrammatic form.

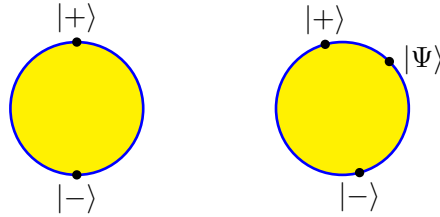
## 2 Introduction to Quantum Mechanics

**Bits and Qubits Bits:**

- have two values 0, 1
- are freely readable and duplicable
- admit arbitrary data transformations

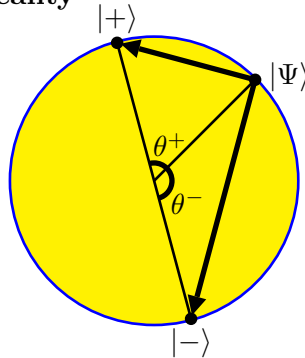
**Qubits:**

- have a ‘sphere’ of values spanned by  $|0\rangle$ ,  $|1\rangle$



- measurements of qubits
  - have two outcomes  $|-\rangle$ ,  $|+\rangle$
  - change the value  $|\psi\rangle$
- admit *unitary transformations*, i.e. antipodes and angles are preserved.

‘Truth makes an angle with reality’



We have *partial constant maps*  $Q \rightarrow Q$  on the sphere  $Q$

$$\mathbf{P}_+ : |\psi\rangle \mapsto |+\rangle \quad \mathbf{P}_- : |\psi\rangle \mapsto |-\rangle$$

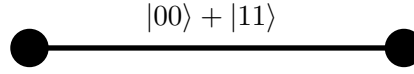
which have chance  $\text{prob}(\theta^\#)$  for  $\# \in \{+, -\}$ .

We know the value *after* the measurement, but not what the qubit was *before* the measurement!

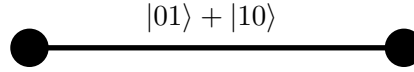
So measurements change the state, and apparently lose information. This seems like bad news! However ...

### 3 Entanglement

**Quantum Entanglement** Bell state:



EPR state:



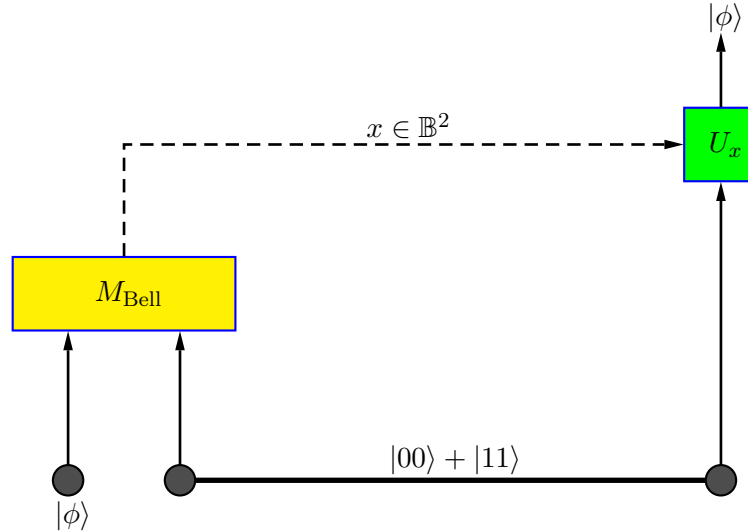
Compound systems are represented by *tensor product*:  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Typical element:

$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

*Superposition* encodes *correlation*. Einstein's ‘spooky action at a distance’. Even if the particles are spatially separated, measuring one has an effect on the state of the other.

Bell’s theorem: QM is *essentially non-local*.

**From ‘paradox’ to ‘feature’: Teleportation**



**Entanglement: where does the magic come from?** A pair of qubits is *not* described by a pair

$$|\psi_1, \psi_2\rangle \in Q \times Q$$

but by a *linear function*

$$|f : Q \rightarrow Q\rangle \in Q \otimes Q$$

This follows from the isomorphism (in finite dimensions)

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \multimap \mathcal{H}_2.$$

**Entangled states as linear maps** Indeed,  $\mathcal{H}_1 \otimes \mathcal{H}_2$  is spanned by

$$\begin{array}{ccc} |11\rangle & \cdots & |1m\rangle \\ \vdots & \ddots & \vdots \\ |n1\rangle & \cdots & |nm\rangle \end{array}$$

hence

$$\sum_{i,j} \alpha_{ij} |ij\rangle \longleftrightarrow \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \longleftrightarrow |i\rangle \mapsto \sum_j \alpha_{ij} |j\rangle$$

Pairs  $|\psi_1, \psi_2\rangle$  are a special case —  $|ij\rangle$  in a well-chosen basis.

This is **Map-State Duality**.

**Some notation for projectors** A projector onto the 1-dimensional subspace spanned by a vector  $|\psi\rangle$  will be written  $\mathbf{P}_\psi$ . It is essentially (up to scalar multiples) a “partial constant map”

$$\mathbf{P}_\psi : |\phi\rangle \mapsto |\psi\rangle.$$

This will correspond e.g. to a branch of a (projective, non-degenerate) measurement, or to a preparation.

We combine this notation with Map-State Duality: we write a projector  $\mathbf{P}_\psi$  on a tensor product space  $\mathcal{H}_1 \otimes \mathcal{H}_2$  as  $\mathbf{P}_f$ , where  $f$  is the linear map  $\mathcal{H}_1 \rightarrow \mathcal{H}_2$  associated to  $\psi$  under Map-State Duality.

**On the trail of structure** The identity map

$$|\text{id} : Q \rightarrow Q\rangle \in Q \otimes Q \quad |11\rangle + \cdots + |nn\rangle \longleftrightarrow |i\rangle \mapsto |i\rangle$$

is the *Bell state*.

A measurement of  $Q \otimes Q$  has four outcomes

$$|f_1\rangle, |f_2\rangle, |f_3\rangle, |f_4\rangle \quad (\text{cf. } |00\rangle, |01\rangle, |10\rangle, |11\rangle)$$

and corresponding projectors

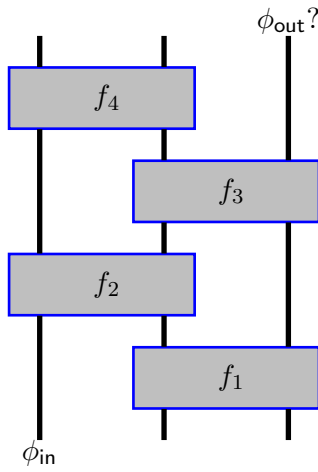
$$\mathbf{P}_f : Q \otimes Q \rightarrow Q \otimes Q :: |g\rangle \mapsto |f\rangle$$

E.g. the Bell state is produced by

$$\mathbf{P}_{\text{id}} : Q \otimes Q \rightarrow Q \otimes Q :: |g\rangle \mapsto |\text{id}\rangle$$

**Key Question:** *Do entangled states qua functions compose (somehow)?*

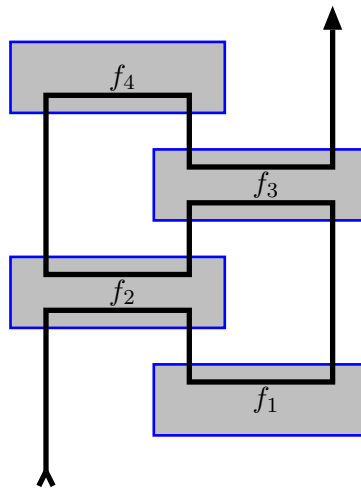
**What is the output?**



$$(\mathbf{P}_{f_4} \otimes 1) \circ (1 \otimes \mathbf{P}_{f_3}) \circ (\mathbf{P}_{f_2} \otimes 1) \circ (1 \otimes \mathbf{P}_{f_1}) : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$$

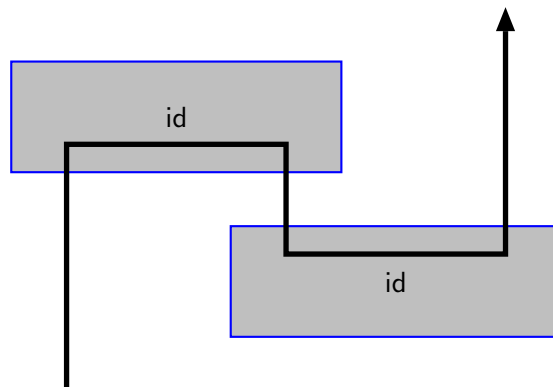
$$\phi_{\text{out}} = f_3 \circ f_4 \circ f_2^\dagger \circ f_3^\dagger \circ f_1 \circ f_2(\phi_{\text{in}})$$

**Follow the line!**



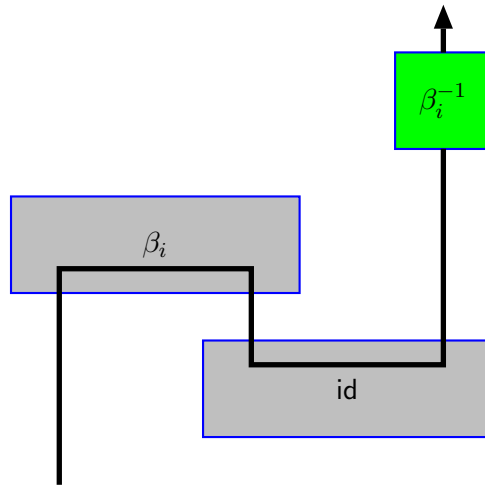
$$f_3 \circ f_4 \circ f_2^\dagger \circ f_3^\dagger \circ f_1 \circ f_2$$

**Teleportation: basic case**



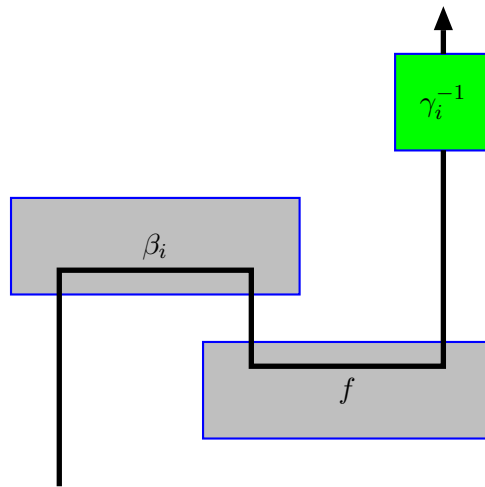
$$\text{id} \circ \text{id} = \text{id}$$

**Teleportation: general case**



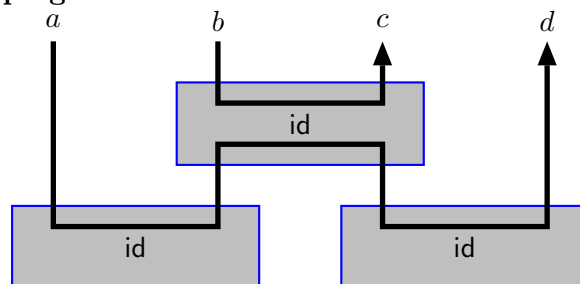
$$\beta_i^{-1} \circ \text{id} \circ \beta_i = \text{id}, \quad 1 \leq i \leq 4$$

**Logic-Gate Teleportation**



$$\gamma_i^{-1} \circ f \circ \beta_i = f, \quad 1 \leq i \leq 4$$

**Entanglement Swapping**

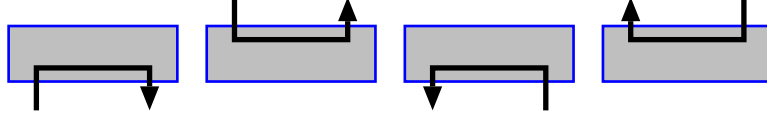




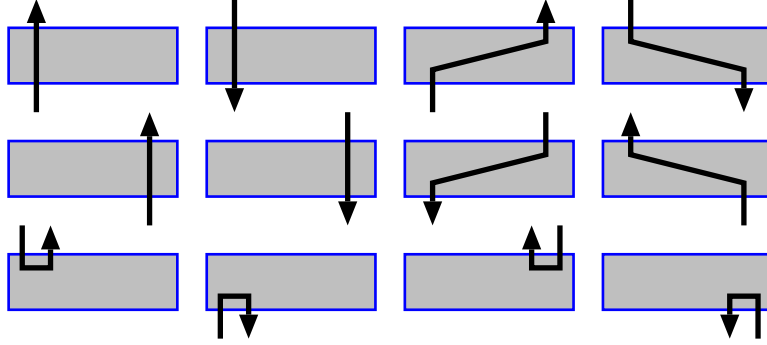
$$\text{id} \circ \text{id} \circ \text{id} = \text{id}$$

## The Logic of Lines

Permitted:

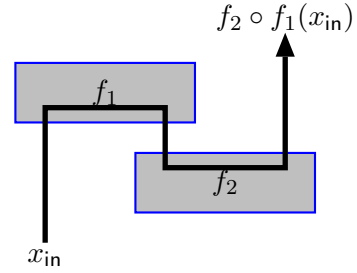


Forbidden:



Sets and Relations have the same structure!

Hilbert space  $\rightsquigarrow$  **set**  
 linear map  $\rightsquigarrow$  **relation**  
 tensor product  $\rightsquigarrow$  **cartesian product**



$$(\mathbf{P}_{f_1} \otimes 1) \circ (1 \otimes \mathbf{P}_{f_2})(x_{\text{in}} \otimes \cdots) = \cdots \otimes (f_2 \circ f_1)(x_{\text{in}})$$

Here the corresponding notion of ‘projector’ is

$$\mathbf{P}_R = R \times R = \{((x_1, y_1), (x_2, y_2)) \mid x_1 R y_1 \wedge x_2 R y_2\}$$

## 4 Categorical Quantum Mechanics

**Axiomatization: Key Features**    **Physical View**

- A setting where *compound systems*, and *series and parallel composition* of *typed processes* can be described.
- In this setting, we axiomatize *Bell states* and *teleportation*.

**Algebraic View**

- Symmetric monoidal categories
- Strong compact closure
- Leading to *graphical calculi*.

### Logical View

- Graphical calculi = *proof nets*.
- Simplification of diagrams = *Cut Elimination*.

**Categories** A category  $\mathcal{C}$  has *objects* (types)  $A, B, C, \dots$ , and for each pair of objects  $A, B$  a set of *morphisms*  $\mathcal{C}(A, B)$ . (Notation:  $f : A \rightarrow B$ ). It also has identities  $\text{id}_A : A \rightarrow A$ , and composition  $g \circ f$  when types match:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Categories allow us to deal explicitly with *typed processes*, e.g.

Logic	Programming
Propositions	Data Types
Proofs	Programs

For QM:

Types of system (e.g. qubits  $Q$ )  $\rightsquigarrow$  objects  
Processes (e.g. measurements)  $\rightsquigarrow$  morphisms

**Symmetric Monoidal Categories** We also want an associative connective  $\otimes$  to represent compound systems.

A *symmetric monoidal category* comes equipped with a bifunctor which acts on *both* objects *and* morphisms:

$$A \otimes B \quad f_1 \otimes f_2 : A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2$$

There is also a symmetry operation

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

which satisfies some ‘obvious’ rules, e.g. naturality:

$$\begin{array}{ccc}
A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2} & B_1 \otimes B_2 \\
\sigma_{A_1, A_2} \downarrow & & \downarrow \sigma_{B_1, B_2} \\
A_2 \otimes A_1 & \xrightarrow{f_1 \otimes f_2} & B_2 \otimes B_1
\end{array}$$

**The Logic of Tensor Product** Tensor can express *independent* or *concurrent* actions (mathematically: bifactoriality):

$$\begin{array}{ccc}
 A_1 \otimes A_2 & \xrightarrow{f_1 \otimes \text{id}} & B_1 \otimes A_2 \\
 \text{id} \otimes f_2 \downarrow & & \downarrow \text{id} \otimes f_2 \\
 A_1 \otimes B_2 & \xrightarrow{f_1 \otimes \text{id}} & B_1 \otimes B_2
 \end{array}$$

But tensor is *not* a cartesian product, in the sense that *we cannot reconstruct an ‘element’ of the tensor from its components*.

This turns out to comprise the absence of

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 \text{Cf. } A \vdash A \wedge A & & A_1 \otimes A_2 \xrightarrow{\pi_i} A_i \\
 & & A_1 \wedge A_2 \vdash A_i
 \end{array}$$

What else do we need for reasoning about quantum entanglement?

**Strong Compact Closure** It suffices to require the following structure on a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbf{I}, \sigma)$ :

- A monoidal involutive assignment  $()^*$  on objects.
- An identity on objects, contravariant, involutive, strictly monoidal functor  $()^\dagger$  (so we take the adjoint as primitive).
- An assignment of units  $\eta_A : \mathbf{I} \rightarrow A^* \otimes A$  such that  $\eta_{A^*} = \sigma_{A^*, A} \circ \eta_A$ .

We can then *define*  $\epsilon_A = \eta_A^\dagger \circ \sigma_{A, A^*}$ .

Triangular identities:

$$\begin{aligned}
 A & \xrightarrow{1_A \otimes \eta_A} A \otimes A^* \otimes A \xrightarrow{\epsilon_A \otimes 1_A} A = 1_A \\
 A^* & \xrightarrow{\eta_A \otimes 1_A} A^* \otimes A \otimes A^* \xrightarrow{1_A \otimes \epsilon_A} A^* = 1_{A^*}
 \end{aligned}$$

## Examples

- Sets, relations and cartesian product **(Rel,  $\times$ )**. Here  $\eta_X \subseteq \{*\} \times (X \times X)$  and we have

$$\eta_X = \epsilon_X^c = \{(*, (x, x)) \mid x \in X\}.$$

- Vector spaces over a field  $\mathbb{K}$ , linear maps and tensor product **(FdVec $_{\mathbb{K}}$ ,  $\otimes$ )**. The unit and counit in **(FdVec $_{\mathbb{C}}$ ,  $\otimes$ )** are

$$\begin{aligned}
 \eta_V : \mathbb{C} & \rightarrow V^* \otimes V :: 1 \mapsto \sum_{i=1}^{i=n} \bar{e}_i \otimes e_i \\
 \epsilon_V : V \otimes V^* & \rightarrow \mathbb{C} :: e_j \otimes \bar{e}_i \mapsto \langle \bar{e}_i \mid e_j \rangle
 \end{aligned}$$

**Duality, Names and Conames** For each morphism  $f : A \rightarrow B$  in a compact closed category we can construct a *dual*  $f^* : B^* \rightarrow A^*$ :

$$f^* = B^* \xrightarrow{\eta_A \otimes 1} A^* \otimes A \otimes B^* \xrightarrow{1 \otimes f \otimes 1} A^* \otimes B \otimes B^* \xrightarrow{1 \otimes \epsilon_B} A^*$$

a *name*

$$\lceil f \rceil: \mathbf{I} \rightarrow A^* \otimes B \quad = \quad \mathbf{I} \xrightarrow{\eta} A^* \otimes A \xrightarrow{1 \otimes f} A^* \otimes B$$

and a *coname*

$$\lfloor f \rfloor : A \otimes B^* \rightarrow \mathbf{I} \quad = \quad A \otimes B^* \xrightarrow{f \otimes 1} B \otimes B^* \xrightarrow{\epsilon} \mathbf{I}$$

The assignment  $f \mapsto f^*$  extends  $A \mapsto A^*$  into a contravariant endofunctor with  $A \simeq A^{**}$ . In any compact closed category, we have

$$\mathcal{C}(A \otimes B^*, \mathbf{I}) \simeq \mathcal{C}(A, B) \simeq \mathcal{C}(\mathbf{I}, A^* \otimes B).$$

**Mathematical Foundations of Quantum Mechanics Old Picture** (von Neumann 1932) State space is Hilbert space. Operations are operators on this space — unitaries for ‘undisturbed dynamics’, self-adjoint operators for measurements. There is only one type!

**New Picture** QM can be done in *any strongly compact closed category with biproducts*. Compound systems represented by *abstract tensor*  $A \otimes B$ . Branching on measurement outcomes can now be represented explicitly using *abstract direct sum*  $A \oplus B$ :

$$A \xrightarrow{M} \bigoplus_{i=1}^n A_i \xrightarrow{\bigoplus_{i=1}^n U_i} \bigoplus_{i=1}^n B_i$$

Propagation of the outcome of a measurement performed on one part of a compound system to other parts — “classical communication” — can be expressed using *distributivity*:

$$(A_1 \oplus A_2) \otimes B \cong (A_1 \otimes B) \oplus (A_2 \otimes B).$$

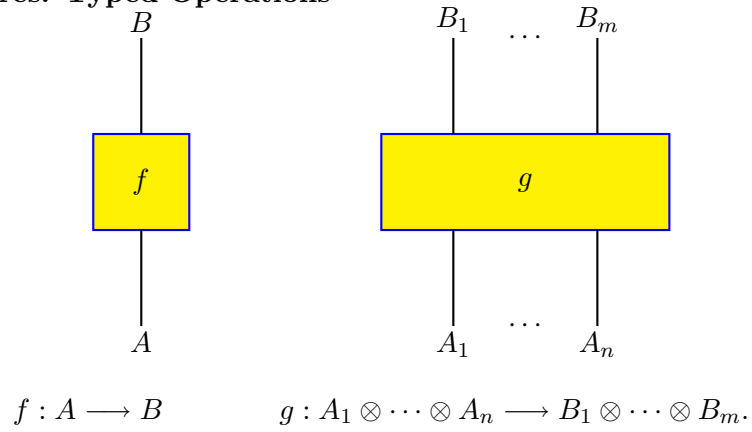
## Teleportation categorically

$$\begin{array}{ccccccc}
Q & \xrightarrow[(1)]{(1 \otimes \eta)} & Q \otimes Q \otimes Q & \xrightarrow[(2)]{\langle \lfloor U_i \rfloor \rangle_{i=1}^4 \otimes 1} & \bigoplus_{i=1}^4 I \otimes Q & \xrightarrow[(3)]{\text{dist}} & \bigoplus_{i=1}^4 Q \\
\parallel & & & & & & \parallel \\
Q & \xrightarrow[\langle 1 \rangle_{i=1}^{i=4}]{} & & & & & \bigoplus_{i=1}^4 Q
\end{array}$$

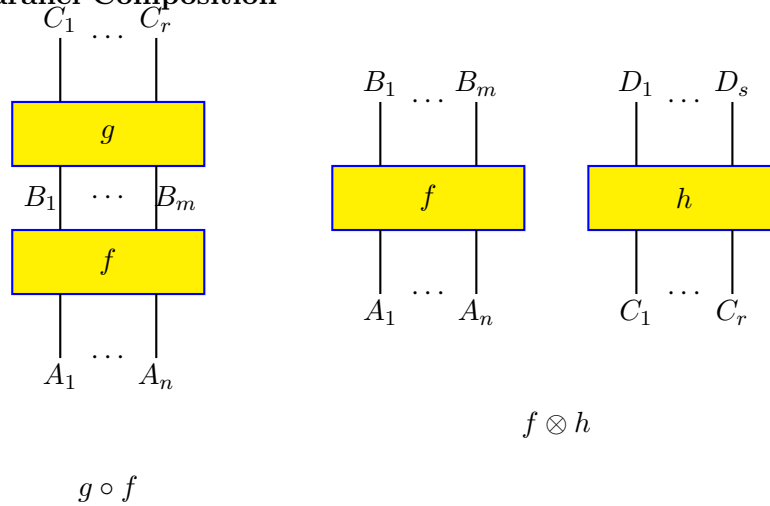
- (1) **Produce EPR pair**
- (2) **Perform measurement in Bell-basis**
- (3) **Propagate classical information**
- (4) **Perform unitary correction.**

## 5 The Graphical Calculus

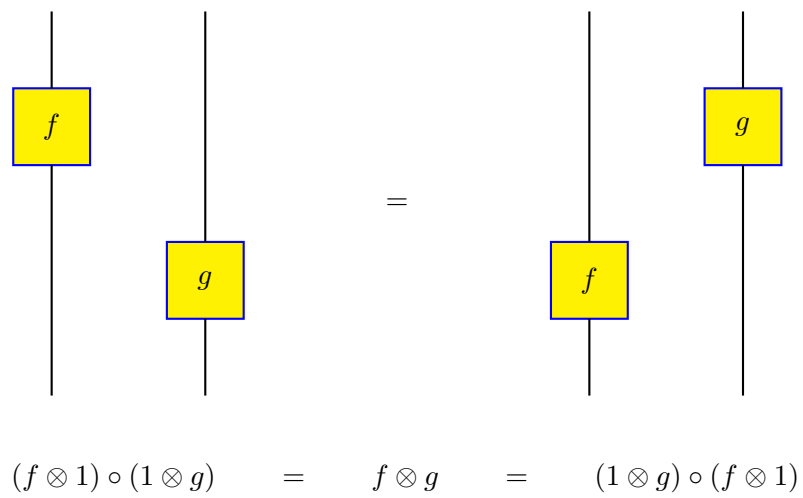
### Boxes and Wires: Typed Operations



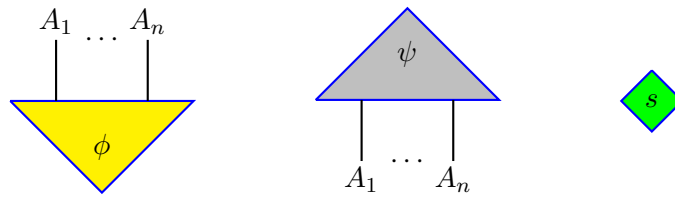
### Series and Parallel Composition



### Geometry absorbs Functoriality, Naturality



## Bras, Kets and Scalars



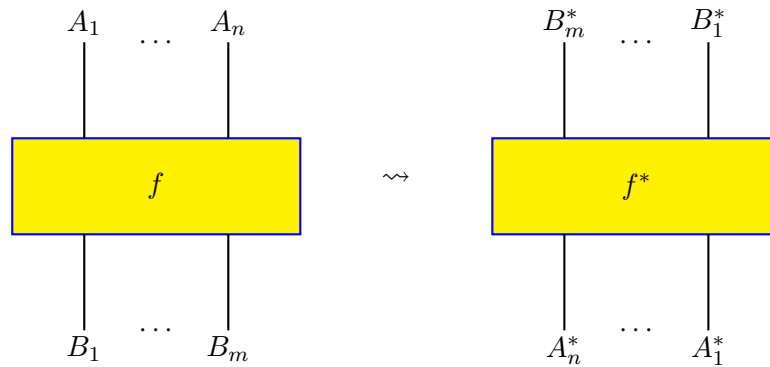
$$\phi : A_1 \otimes \cdots \otimes A_n \longrightarrow \mathbf{I} \quad \psi : \mathbf{I} \longrightarrow A_1 \otimes \cdots \otimes A_n \quad s : \mathbf{I} \longrightarrow \mathbf{I}.$$

Bras: no outputs

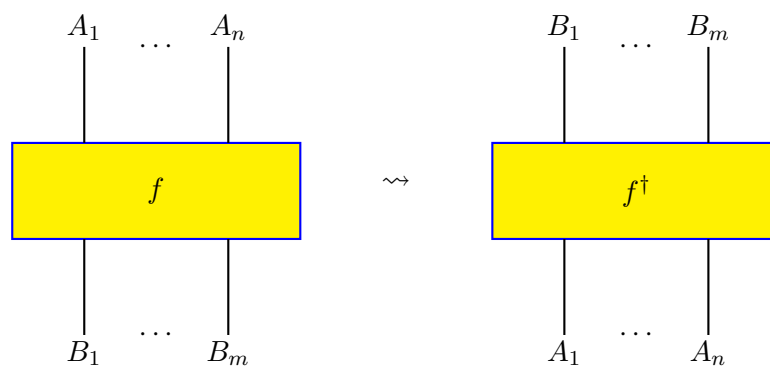
Kets: no inputs

Scalars: no inputs or outputs.

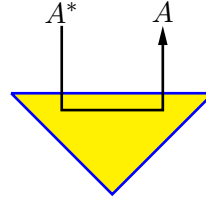
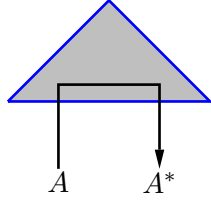
## Dual Arrows — 180° rotation



## Adjoint Arrows — reflection in the $x$ -axis



## Cups and Caps

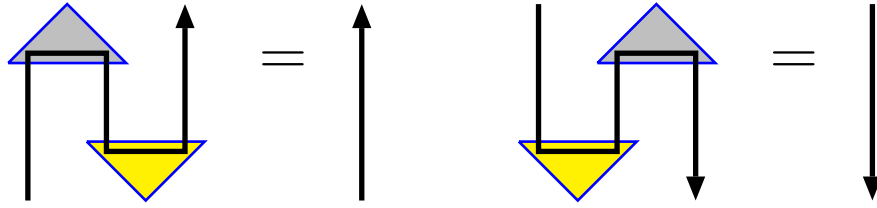


$$\epsilon_A : A \otimes A^* \longrightarrow I$$

$$\eta_A : I \longrightarrow A^* \otimes A.$$

Caps = Bell States; Cups = Bell Tests.

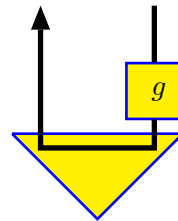
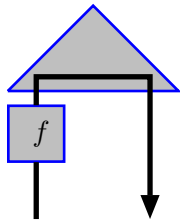
**Graphical Calculus for Information Flow** **Compact Closure:** The basic algebraic laws for units and counits.



$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A$$

$$(1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

**Names and Conames in the Graphical Calculus**



$$\lfloor f \rfloor : A \otimes B^* \rightarrow I$$

$$\lceil f \rceil : I \rightarrow A^* \otimes B$$

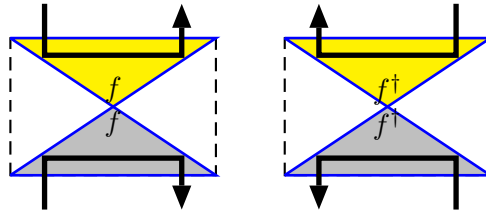
$$\mathcal{C}(A \otimes B^*, I) \simeq \mathcal{C}(A, B) \simeq \mathcal{C}(I, A^* \otimes B).$$

**Bipartite Projectors** Information flow in entangled states can be captured mathematically by the isomorphism

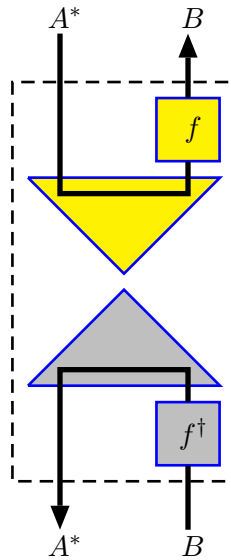
$$\mathbf{Hom}(A, B) \cong A^* \otimes B.$$

This leads to a *decomposition* of bipartite projectors into “names” (preparations) and “conames” (measurements).

In graphical notation:

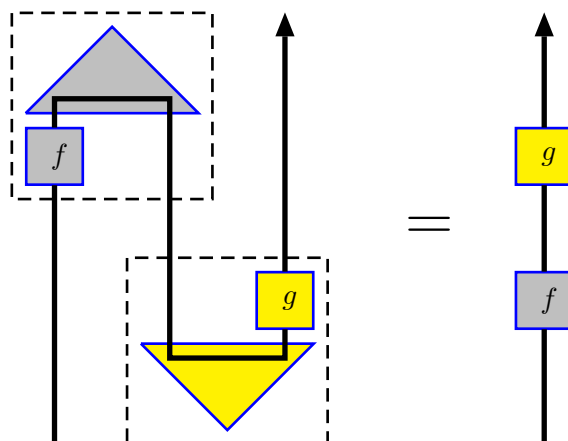


**Projectors Decomposed**

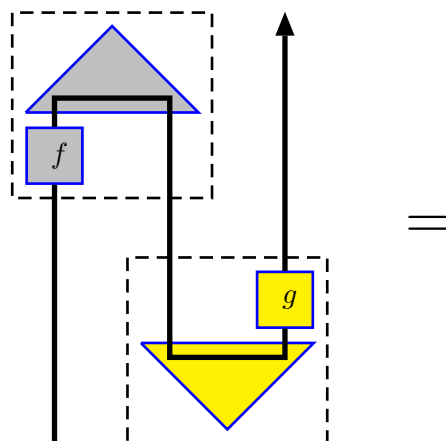


**Compositionality** The key algebraic fact from which teleportation (and many other protocols) can be derived.

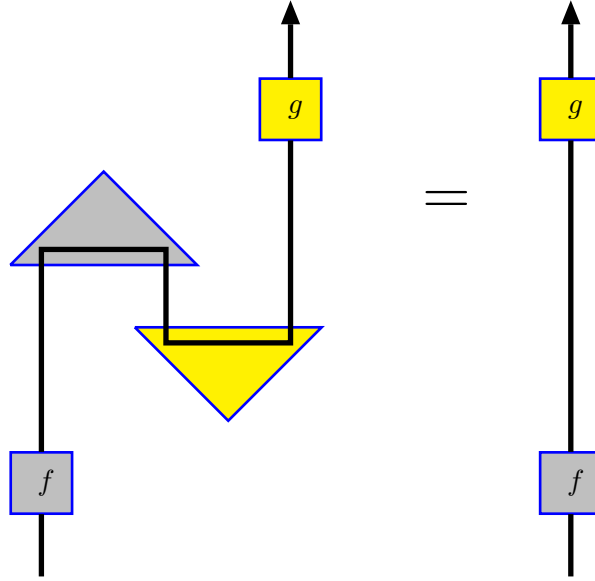




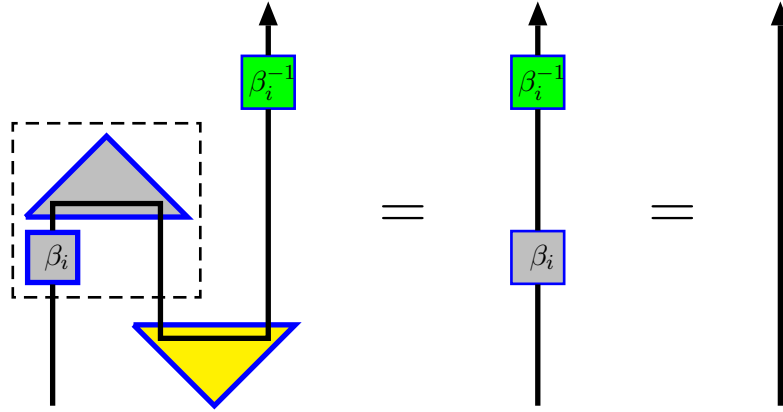
Compositionality ctd



Compositionality ctd



Teleportation diagrammatically



## 6 Conclusions

**Further Developments** The setting is very rich:

- Compound systems, unitary operations, projectors, preparations of entangled states, Dirac bra-ket notation, traces, scalars, the Born rule, Hilbert-Schmidt norm, etc.
- Dictionary of interpretation of QM concepts in the abstract setting

**Some Results**

- Characterization of free strongly compact closed categories. Justification of the diagrammatic calculus.
- CP construction (Selinger). Generalizes the passage from Hilbert spaces and linear maps to completely positive operators on density matrices, hence from the pure state to the mixed state setting for QM.

- (Coecke and Pacquette) Diagrammatic proof of a generalization of Naimark's theorem.

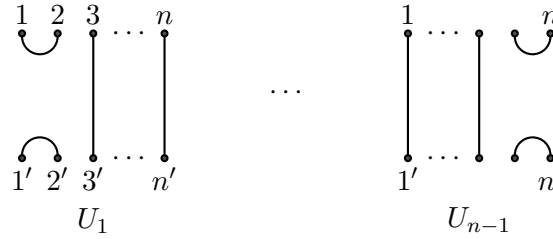
**It's Logic!** The graphical calculus can be seen as a *calculus of proofs* for a certain *logic* — which is highly non-classical, (in particular *resource-sensitive*, so e.g. it builds in ‘No Cloning’), but also very different from the Birkhoff-von Neumann quantum logic.

Simplification of diagrams — ‘straightening out the lines’ — corresponds to *normalization* or *cut-elimination* of proofs.

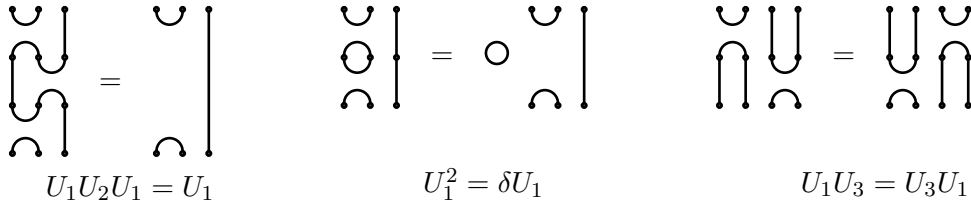
### Some Results

- A strongly normalizing proof net calculus for the logic of strongly compact closed categories with biproducts.
- Full Completeness for this calculus, i.e. it characterizes the free such categories.
- (Duncan) Extension to free constructions over *monoidal categories*.

**Temperley-Lieb Algebra** Generators:



Relations:



**Connections with Knots: the Kauffman bracket**

$$\langle \text{crossing} \rangle = A \langle \text{cup/cap} \rangle + B \langle \text{cup/cap} \rangle$$

The Temperley-Lieb algebra carries a representation of the braid group. The trace of the image of a braid yields a scalar — the Jones polynomial of the corresponding link (with appropriate choices for  $A$  and  $B$  and a normalizing factor to give invariance under the first Reidemeister move). This lifts to yield 3-manifold invariants and 3-D TQFT's.

Temperley-Lieb algebras are free (non-symmetric) strongly compact closed categories over one self-dual generator.

We can also apply ideas from TCS, Logic to yield the first direct (no quotients) purely combinatorial description of the Temperley-Lieb algebra.

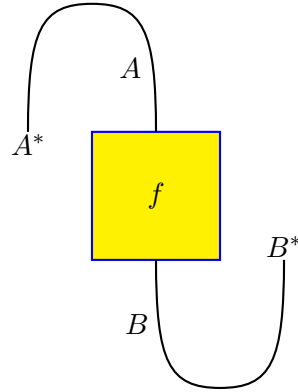
**Some Papers** Papers available from the arXiv or my webpages

<http://web.comlab.ox.ac.uk/oucl/work/samson.abramsky/>

- Abramsky, S. and Coecke, B. (2004) *A categorical semantics of quantum protocols*. Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LiCS'04), IEEE Computer Science Press. (extended version at arXiv: quant-ph/0402130)
- Abramsky, S. and Duncan, R. W. (2006) *Categorical Quantum Logic*. Mathematical Structures in Computer Science, 2006.
- Abramsky, S. (2006) *Temperley-Lieb Algebras: From Knot Theory to Logic and Computation via Quantum Mechanics*. Submitted.
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- S. Abramsky, R. F. Blute and P. Panangaden (1999). Nuclear and trace ideals in tensor \*-categories. *Journal of Pure and Applied Algebra* **143**, 3–47, 1999.

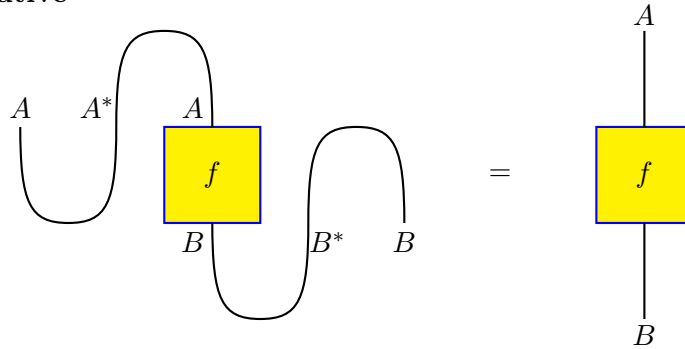
## Appendix: Diagrammatic Proofs

### Definition of Duality



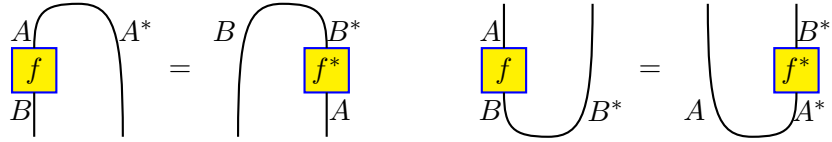
$$f^* = (1 \otimes \epsilon_B) \circ (1 \otimes f \otimes 1) \circ (\eta_A \otimes 1).$$

### Duality is Involutive

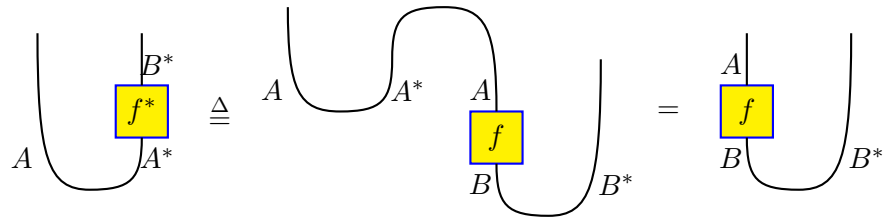


$$f^{**} = f.$$

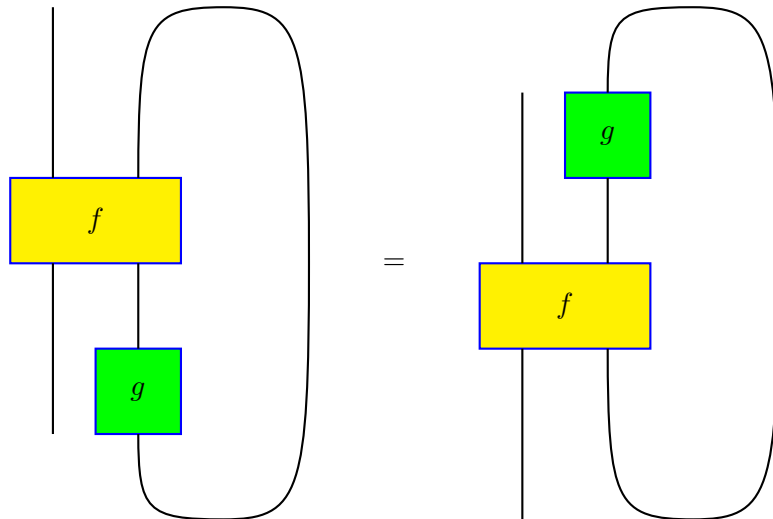
**Moving Boxes round Cups and Caps**



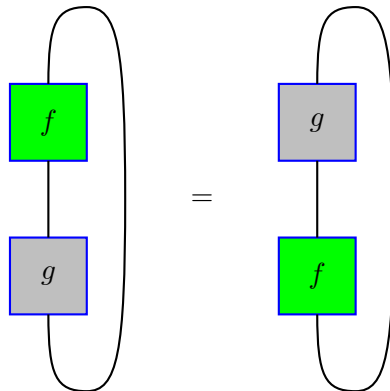
**Diagrammatic Proof**



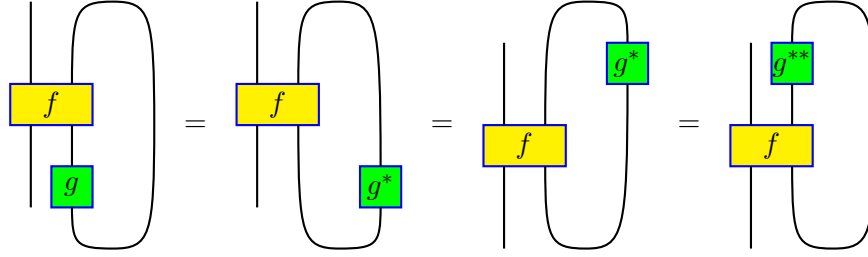
**Feedback Dinaturality**



**Application: Invariance of Trace Under Cyclic Permutations**



## Graphical Proof of Feedback Dinaturality



We use  $g^{**} = g$  to conclude.

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