# SEVEN LECTURES ON THE UNIVERSAL ALGEBRAIC GEOMETRY

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#### PREFACE

These lectures are devoted to the subject I've got interested in during my work in Hebrew University. They are naturally connected with my previous works on universal algebra and algebraic logic in database theory. On the other hand, an essential influence on them is provided by talks with E.Rips and Z.Sela and discussions on algebraic geometry in a free group.

A great role for me play communications with A.Bokut, I.Dolgachev, B.Kunyavskii, R.Lypianskii, G.Mashevitzky, A.Miasnikov, A.Mikhalev, A.Olshanski, E.Plotkin, V.Remeslennikov, A.Shmelkin, N.Vavilov, E.Winberg, E.Zelmanov and with many other colleagues.

I would like to distinguish specially A.Berzins, who, in particular, completely studied the situation in the classical variety Var - P (see 5.3), and whose key idea gives rise to the general theorems 5 and 5' from the same Section.

Lecture 7 returns me to database and knowledge base theory and relies on the joint work with T.Plotkin.

This text was prepared for lectures in USA and Canada in September-October, 2000 and does not contain, as a rule, proofs of results. On the other hand, it contains a list of problems of different levels of difficulty.

I am very grateful to Hebrew University for the support during preparation of the lectures.

The bibliography is not complete and consists of the works which have direct and indirect relation to the topic of the lectures. Most of the proofs can be found there.

The lectures are mostly prepared by E.Plotkin. I am also very grateful to Mrs. M.Beller for typing.

I would like to add that I thank circumstances for bringing me to the problems I describe here.

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# LECTURE 1

# WHAT IS THE UNIVERSAL ALGEBRAIC GEOMETRY

# Contents

- 1. Generic view
- 2. Classical algebraic geometry from the point of view of universal algebraic geometry
- 3. Some information from universal algebra
- 4. Algebras with a fixed algebra of constants
- 5. Special categories

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#### 1. Generic view.

Universal algebraic geometry is a system of algebraic concepts and problems going back to classical algebraic geometry. It is mainly algebra, not quite geometry. However, this algebra inherits some geometric intuition; preserves a certain geometric characteristic which is not necessarily explicit.

The key idea of universal algebraic geometry is that every variety of algebras  $\Theta$  has its own algebraic geometry. It varies if the variety  $\Theta$  and an algebra  $H \in \Theta$  vary, but the core notions are common. We look at algebras  $H \in \Theta$  from the point of view of algebraic geometry in  $\Theta$ . On the other hand, supposing that an algebra H is given, we consider algebraic varieties and their invariants. This is what is used to study in the classical algebraic geometry.

Universality of geometry means that the variety  $\Theta$  can be an arbitrary fixed variety of algebras. For every algebra  $H \in \Theta$  we consider the category of algebraic sets  $K_{\Theta}(H)$ . This is a geometric invariant of the algebra  $H \in \Theta$ . We are looking for conditions which provide the categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  to be isomorphic or equivalent for different  $H_1$ ,  $H_2$  in  $\Theta$ . This is one of the main problems of the theory. For the whole variety  $\Theta$  its geometrical invariant is presented by the special category  $K_{\Theta}$ . All  $K_{\Theta}(H)$  are the subcategories in  $K_{\Theta}$ . There is a problem of isomorphism or equivalence of the categories  $K_{\Theta_1}$  and  $K_{\Theta_2}$  for the different varieties  $\Theta_1$  and  $\Theta_2$ .

In the above setting, the classical algebraic geometry occupies a distinguished place. It is based on the variety of commutative and associative rings with unity and is connected with the corresponding varieties of algebras over fields. Every such variety over a field P is denoted by Var-P and is called a classical variety over P.

If we replace the variety of associative and commutative algebras by a variety of associative but not necessarily commutative algebras over different P, then we come to non-commutative algebraic geometry.

On the other hand, in group theory there is a strong interest to investigate equations in groups and, especially, in free groups. The corresponding results give rise to the sketch of the special algebraic geometry. The most interesting results are obtained for groups with a given group of constants G. The corresponding variety is denoted by Grp-G.

Something similar is known for semigroups and other algebraic systems.

Now, one can say that universal algebraic geometry stands in the same position with respect to all these special theories as does universal algebra with respect to groups, rings, semigroups, etc.

To every variety of algebras  $\Theta$  its elementary logic (i.e., first order logic) is associated. We consider algebraic logic in  $\Theta$ . The elementary properties of an algebra  $H \in \Theta$  are considered from the point of view of logic in  $\Theta$ . Geometrical and logical properties of algebras in  $\Theta$  are often well interacted. The same is valid for geometrical and logical relations between different algebras in  $\Theta$ .

Besides, we will consider algebraic geometry in first order logic, which generalizes usual equational algebraic geometry. This outlet to logic is motivated geometrically and also by various applications, in particular, in computer science.

# 2. Classical algebraic geometry from the point of view of universal algebraic geometry.

Let us fix an infinite ground field P. This field generates the variety of algebras which was denoted by Var-P, i.e., the variety of all commutative and associative algebras with unity over the given P. The polynomial algebra  $P[X] = P[x_1, \ldots, x_n]$  is the free algebra in Var-P over the set X.

A field L is an extension of the field P if there is an injection  $P \to L$ . Every such field L is also an algebra in Var-P.

To every map  $\mu: X \to L$  there is a P-homomorphism  $\mu: P[X] \to L$  in one-to-one correspondence. Namely, if  $f = f(x_1, \ldots, x_n) \in P[X]$ , then  $f^{\mu} = f(x_1^{\mu}, \ldots, x_n^{\mu})$  is an element in L. The affine space  $L^{(n)} = L^X$  can be viewed as the set of all homomorphisms Hom(P[X], L). A point  $a = (a_1, \ldots, a_n) \in L^{(n)}$  is identified with the homomorphism  $\mu = \mu_a: P[X] \to L$  by the rule  $\mu(x_i) = a_i$ . Then  $f^{\mu} = f(a)$  and the point a is a root of the polynomial f, if  $f \in \text{Ker } \mu$ .

We consider affine algebraic sets over the given L. These sets are defined by the systems of equations with coefficients from P and solutions in the field L. Since the algebraic sets are supposed to be subsets in  $L^{(n)}$ , we can consider them also as subsets in Hom(P[X], L).

Let A be an arbitrary set of points  $\mu: P[X] \to L$ . It corresponds an ideal U in P[X] defined by

$$U = A' = \bigcap_{\mu \in A} \operatorname{Ker} \mu.$$

This is the set of all polynomials f, such that every point  $\mu \in A$  is a root of f. Every such ideal is called an L-closed ideal.

On the other hand, let U be a subset in the algebra P[X]. One can consider U as a system of equations. It corresponds the set of points  $A \subset \text{Hom}(P[X], L)$ , defined by the rule

$$A = U'_L = U' = \{\mu : P[X] \to L \mid U \subset \operatorname{Ker} \mu\}.$$

This means that a point  $\mu$  belongs to A if  $\mu$  is a root of every polynomial from U. Such sets A are called *algebraic sets* or *closed sets*.

Define two closures:

$$A'' = (A')'; \quad U_L'' = U'' = (U_L')'.$$

#### Definition 1.

Two extensions  $L_1$  and  $L_2$  of the ground field P are called *geometrically equivalent*, if for every finite set  $X = \{x_1, \ldots, x_n\}$  and for every  $U \subset P[X]$ , the equality

$$U_{L_1}^{"} = U_{L_2}^{"}$$

takes place.

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#### Theorem 1.

1. If the field P is algebraically closed, then all its extensions are geometrically equivalent.

2. If every two extensions of P are geometrically equivalent, then P is algebraically closed.

The first part of the theorem follows from Hilbert Nullstellensatz, the second one is deduced easily from definitions.

We consider the category of algebraic sets over the given L and fixed P. Denote this category by  $K_P(L)$ . Its objects are algebraic sets A with different X.

To the category  $K_P(L)$  corresponds the dual category  $C_P(L)$ , whose objects are P-algebras of the form P[X]/A'. We later define these categories in an arbitrary variety  $\Theta$ .

If the fields  $L_1$  and  $L_2$  are geometrically equivalent, then the categories  $C_P(L_1)$  and  $C_P(L_2)$  coincide and the categories  $K_P(L_1)$  and  $K_P(L_2)$  are isomorphic.

The category  $K_P(L)$  can be considered as a natural geometric invariant of the P-algebra L. It can also be viewed as a kind of characteristic which measures P-algebraic closeness of the field L. In other words, it characterizes possibilities to solve in L systems of equations with coefficients from P. We say that a field L is P-algebraically closed if, for any finite X and a proper ideal  $U \subset P[X]$ , the set  $U'_L$  is not empty. If the field P is algebraically closed, then each of its extensions L is P-algebraically closed. If  $L_1$  and  $L_2$  are P-algebraically closed, then they are geometrically equivalent. If  $L_1$  is P-algebraically closed, and  $L_1$  and  $L_2$  are geometrically equivalent, then  $L_2$  is P-algebraically closed. The definition above can be viewed as a definition of strict algebraic closeness of the field L over L0. The weak algebraic closeness of the field L1 over L2 means that every polynomial with one variable and coefficients from L2 has roots in L3. In fact, strict and weak closeness coincide. Indeed, algebraic (absolute) closure L2 of the field L3 coincides with the weak L4-closure of the L5. Therefore, L6 is contained in every L5-closed extension of L6. Then, Hilbert's Nullstellensatz works.

In the definitions above the classical Nullstellensatz can be formulated as follows: if the field L is P-algebraically closed then for every ideal  $U \in P[X]$  the equality

$$U_L'' = \sqrt{U}$$

holds.

The pointed above equivalence of definitions plays a crucial role in the algebraic geometry in different varieties Var - P. This equivalence fails if we doing general varieties  $\Theta$  and  $H \in \Theta$ .

Note that we distinguish algebraic sets in affine space and algebraic varieties. It is assumed that an algebraic variety is an algebraic set considered up to isomorphisms in the category  $K_P(L)$ .

The following problem is of special interest.

Let  $L_1$  and  $L_2$  be two extensions of the field P. When are the categories  $K_P(L_1)$  and  $K_P(L_2)$  isomorphic?

Let  $\mu_1: P \to L_1$  and  $\mu_2: P \to L_2$  be injections defining extensions. An isomorphism of extensions means that there is a commutative diagram

$$P \xrightarrow{h_1} L_1$$

$$\downarrow^{\mu}$$

$$L_2$$

where  $\mu$  is an isomorphism of rings.

Two extensions  $h_1: P \to L_1$  and  $h_2: P \to L_2$  are called *semi-isomorphic* if there is a diagram

$$P \xrightarrow{h_1} L_1$$

$$\sigma \downarrow \qquad \qquad \downarrow \mu$$

$$P \xrightarrow{h_1} L_2$$

where  $\mu$  is an isomorphism of rings and  $\sigma$  is an automorphism of the field P.

The semi-isomorphism does not imply the equivalence but if  $L_1$  and  $L_2$  are semi-isomorphic then the categories  $K_P(L_1)$  and  $K_P(L_2)$  are isomorphic.

**Definition 2.** Two extensions  $L_1$  and  $L_2$  are said to be geometrically equivalent up to a semi-isomorphism, if there exists a field L, such that L is semi-isomorphic to  $L_1$  and geometrically equivalent to  $L_2$ .

**Theorem 2.** Two categories  $K_P(L_1)$  and  $K_P(L_2)$  are isomorphic if and only if the extensions  $L_1$  and  $L_2$  are geometrically equivalent up to a semi-isomorphism.

The proof of this theorem is based on consideration of automorphisms of the category of polynomial algebras P[X] with different X. It can be proved that every such an automorphism is a semi-inner automorphism. Definitions of inner and semi-inner automorphisms are natural and will be given in Lecture 3.

In order to get further information on geometric equivalence of fields, we use the idea of a quasi-identity.

A quasi-identity u in the variety Var-P is a formula of the form

$$f_1 \equiv 0 \land f_2 \equiv 0 \land \cdots \land f_n \equiv 0 \Rightarrow f \equiv 0$$

where  $f_i \in P[X], i = 1, \ldots, n$ .

**Theorem 3.** Two extensions  $L_1$  and  $L_2$  of a field P are geometrically equivalent if and only if they satisfy the same quasi-identities.

Thus in this case geometric equivalence coincides with equivalence in logik of quasi-identities in the variety Var - P.

This theorem implies that any extension L of P is geometrically equivalent to each its ultrapower. Every L and every its ultrapower have the same elementary theories. The following problem looks quite natural:

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**Problem 1.** When two geometrically equivalent extensions  $L_1$  and  $L_2$  of a field P have different elementary theories in the logik of Var - P?

There are examples of such kind. Let, for example, P be an algebraically closed field and L be its non-algebraically closed extension. Then L and P are geometrically equivalent, while they are not elementary equivalent. In particular, they are not elementary equivalent in the logic of Var - P.

Indeed, let  $f(x) = \alpha_0 + \alpha_1 x \cdots + \alpha_n x^n$  be a polynomial with the coefficients in L and without roots in L. Take a polynomial over P

$$\varphi(x, y_0, \dots, y_n) = y_0 + y_1 x + \dots + y_n x^n$$

and consider a P-formula

$$\forall y_0 \dots y_n \exists x (\varphi(x, y_0, \dots, y_n) = 0).$$

This formula holds in P and does not hold in L.

In the Problem 1 one has to consider the general situation.

#### 3. Some information from universal algebra.

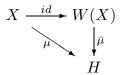
Almost all that which has been determined for the classical case admits natural generalizations.

Let us recall some information from universal algebra. Fix a signature  $\Omega$ , i.e., a set of symbols of operations of the arbitrary arity. Consider the class of all  $\Omega$ -algebras. Every set of identities, satisfied by the operations from  $\Omega$ , specify some subclass in the class of all  $\Omega$ -algebras. Every such subclass  $\Theta$  is called a variety of algebras. Groups, rings, semigroups, associative and Lie rings, associative and Lie algebras over fields, Var-P are all examples of varieties.

Speaking of algebra, we think of universal algebra, i.e., algebra in an arbitrary but fixed variety  $\Theta$ . One can assume that this is a group, or an associative or Lie algebra. Algebra can be viewed as an universal or a concrete one.

In each variety  $\Theta$  there are free algebras W(X), where X is a set. This is a significant feature of varieties which plays the crucial role in the theory under consideration.

The characteristic property of W(X) can be described by the commutativity of the following diagram



Here, H is an arbitrary algebra in  $\Theta$ , id is the identity map,  $\mu$  is an arbitrary map, and  $\overline{\mu}$  is the homomorphism in  $\Theta$  uniquely determined by  $\mu$ .

In other words, this diagram means that the set X freely generates W(X) (or, X is a basis for W), the variables from X take values from H by the rule  $\mu$ , and the homomorphism  $\overline{\mu}$  computes in H the values of arbitrary elements  $w \in W$ .

Elements from W are used to call words, terms, polynomials, or  $\Theta$ -polynomials. They are constructed from the variables from X by the rules of the variety  $\Theta$ . Every  $w \in W$  is uniquely presented as  $w = w(x_1, \ldots, x_n), x_i \in X$ , and uniqueness means that  $w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)$  holds if and only if it is an identity in  $\Theta$ .

Every variety  $\Theta$  can be regarded as category whose morphisms are homomorphisms in  $\Theta$ . To each homomorphism  $\mu: G \to H$  corresponds its kernel Ker  $\mu$ . This is a special binary relation, defined by the rule:  $g_1(\text{Ker }\mu)g_2$  if and only if  $g_1^{\mu} = g_2^{\mu}$ . A relation T in G is called a congruence if  $g_1Tg_1' \wedge \cdots \wedge g_nTg_n'$  implies  $g_1 \ldots g_n \omega Tg_1' \ldots g_n' \omega$  for every n-ary operation  $\omega$  in  $\Omega$ . The kernel  $Ker\mu$  is always a congruence.

For every congruence T in G one can take a factor algebra G/T with the natural homomorphism  $G \to G/T$ . The diagram

$$G \xrightarrow{\mu} H$$

$$\downarrow^{\mu_1}$$

$$G/Ker\mu$$

gives a canonical decomposition of a homomorphism  $\mu$ . The set of all homomorphisms  $G \to H$  is denoted by  $\operatorname{Hom}(G, H)$ .

Every variety of algebras is closed under taking Cartesian products, subalgebras and homomorphic images. In particular, if I is a set and  $H \in \Theta$ , then the Cartesian power  $H^I$  also belongs to  $\Theta$ . If |I| = n, then  $H^I$  is denoted by  $H^{(n)}$ . The algebra  $H^{(n)}$  is viewed as an affine space of points  $(a_1, \ldots, a_n)$ ,  $a_i \in H$ . Let  $X = \{x_1, \ldots, x_n\}$ . Then there is a bijection

$$\alpha_X : \operatorname{Hom}(W(X), H) \to H^{(n)},$$

where for every  $\nu: W \to H$ ,

$$\alpha_X(\nu) = (\nu(x_1), \dots, \nu(x_n)).$$

This bijection induces a structure of algebras in  $\Theta$  on  $\operatorname{Hom}(W(X), H)$ . By definition, if  $\omega \in \Omega$  is an m-ary operation,  $\nu_1, \ldots, \nu_m \in \operatorname{Hom}(W(X), H)$ , then

$$(\nu_1 \dots \nu_m \omega)(x) = \nu_1(x) \dots \nu_m(x)\omega,$$

for every  $x \in X$ .

We define the notion of a commutative algebra H. For a commutative algebra H the equality

$$(\nu_1, \dots \nu_m \omega)(w) = \nu_1(w) \dots \nu_n(w)\omega$$

always holds for every  $w \in W$ .

Let  $w_1, w_2 \in \Omega$  be two operations of arity  $m_1, m_2$ , respectively. Consider the matrix

$$\begin{pmatrix} x_{11} & \dots & x_{1m_1} \\ \dots & \dots & \dots \\ x_{m_21} & \dots & x_{m_2m_1} \end{pmatrix}$$

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Two operations  $w_1$  and  $w_2$  are said to be commutative if

$$((x_{11} \dots x_{1m_1}\omega_1) \dots (x_{m_21} \dots x_{m_2m_1})\omega_1)\omega_2 =$$

$$= ((x_{11} \dots x_{m_21}\omega_2) \dots (x_{1m_1} \dots x_{m_2m_1}\omega_2)\omega_1.$$

An algebra H is called commutative if every two operations, not necessarily different, commute. For nullary operations this means that all of them coincide.

If H is a commutative algebra, then the algebra Hom(W(X), H) is also commutative and it is a subalgebra in the algebra of mappings  $G^{W(X)}$ .

In the general case the set  $\operatorname{Hom}(W(X), H)$ ,  $|X| < \infty$  is considered as an affine space over an algebra H.

Let now  $\Theta^0$  denote the category of all free algebras W(X) in  $\Theta$  with finite X. The category  $\Theta^0$  is a full subcategory in the category  $\Theta$ . Morphisms in  $\Theta^0$  are presented by arbitrary homomorphisms  $s:W(X)\to W(Y)$ .

Let us make remarks on equations. Let us fix a finite set  $X = \{x_1, \ldots, x_n\}$  and consider equations of the form w = w',  $w, w' \in W(X)$ . Every such equation with the given X is considered also as a formula in the logic in  $\Theta$ . In the later case we write  $w \equiv w'$ .

A point  $\nu: W(X) \to H$  is a root of the equation  $w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)$ , if  $w(x_1^{\nu}, \ldots, x_n^{\nu}) = w'(x_1^{\nu}, \ldots, x_n^{\nu})$ . This also means that the pair (w, w') belong to Ker  $\nu$ . We will identify the pair (w, w') and the equation w = w'.

In order to get a reasonable geometry in  $\Theta$  one has to consider the equations with constants. The next subsection deals with constants.

# 4. Algebras with a fixed algebra of constants.

Let  $\Theta$  be an arbitrary variety of algebras, G be a fixed algebra in  $\Theta$ , |G| > 1. Consider the new variety, denoted by  $\Theta^G$ . First we define the category  $\Theta^G$ . The objects in  $\Theta^G$  have the form  $h: G \to H$ , where h is a homomorphism in  $\Theta$ , not necessarily injective. Morphisms in  $\Theta^G$  are presented by the commutative diagrams

$$G \xrightarrow{h} H$$

$$\downarrow^{\mu}$$

$$H'$$

where  $\mu, h, h'$  are homomorphisms in  $\Theta$ .

An algebra H, considered as a G-algebra, is denoted by (H, h). In this way, to elements  $g \in G$  correspond constants, i.e., nullary operations in H. We add them to the signature  $\Omega$ . This signature allows us to consider a variety of G-algebras. Identities of a variety of G-algebras are presented by identities of  $\Theta$  and by the defining relations of the algebra G (see [P11] for details).

A free in  $\Theta^G$  algebra W = W(X) has the form  $G * W_0(X)$ , where  $W_0(X)$  is the free in the  $\Theta$  algebra over X, \* is the free product in  $\Theta$  and the embedding  $i_G : G \to W(X) = G * W_0(X)$  follows from the definition of free product.

A G-algebra (H, h) is called a faithful G-algebra if  $h: G \to H$  is an injection. A free algebra  $(W, i_G)$  is faithful. A G-algebra G with the identical  $G \to G$  is also faithful. All other G-algebras G are isomorphic to this one. All of them are simple, i.e., they do not have faithful subalgebras and congruences. Let (H,h) be a G-algebra, and  $\mu: H \to H'$  is a homomorphism in  $\Theta$ . Then, by  $h' = \mu h$ , H' becomes a G-algebra, and  $\mu$  is a homomorphism of G-algebras. Since one can start from an arbitrary congruence T in H and from the natural congruence H/T, we say that T is called faithful if the G algebra H/T is faithful. A congruence T is faithful if and only if  $g_1^h = g_2^h$  is equivalent to  $g_1 = g_2$ .

Let a homomorphism



be given, and let (H', h') be a faithful G-algebra. Then (H, h) is a faithful G-algebra. If  $T = \text{Ker } \mu$ , then T is a faithful congruence and H/T is also faithful.

Note that the inclusion  $G \in \Theta$  means that all the constants of G-algebra H are covered by the elements from G.

**Examples.** A variety Var-P is a variety of the type  $\Theta^G$ , where  $\Theta$  is a variety of associative and commutative rings with 1, and G is a field P. In this example, elements of the field P are the constants in the P-algebras. They considered as nullary operations, and, simultaneously, using multiplication, we can look at them as unary operations.

G-groups is another example of G-algebras. Here, elements from G also can be viewed as unary operations.

We denote also a free G-algebra W=W(X) by G[X]. Since  $\Theta^G$  is a variety of algebras, all constructions like Cartesian and free products, subalgebras and homomorphisms are naturally defined for  $\Theta^G$ .

Consider, further, the special condition on  $\Theta^G$  denoted by (\*). Namely, we assume that the algebra G generates the whole variety  $\Theta^G$ , i.e., in G there are no non-trivial identities with the coefficients from G. This condition is fulfilled in Var-P if the field P is infinite and in  $\operatorname{Grp} - F = (\operatorname{Grp})^F$ .

Every faithful G-algebra H contains G as a subalgebra. Thus, (\*) implies that every faithful G-algebra H generates the whole variety  $\Theta^G$ , i.e., in  $\Theta^G$  there are no proper subvarieties containing faithful algebras.

In the category  $\Theta^G$  along with morphisms, one can consider also semimorphisms. They have the form

$$G \xrightarrow{h} H$$

$$\sigma \downarrow \qquad \qquad \downarrow \mu$$

$$G \xrightarrow{h'} H'$$

where  $\sigma \in \text{End } G$ . Then, we can consider semi-isomorphic G-algebras.

Another possibility is to vary also the algebra of constants G. This leads to the

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diagram of the form

$$G \xrightarrow{h} H$$

$$\sigma \downarrow \qquad \qquad \downarrow \mu$$

$$G' \xrightarrow{h'} H'$$

with component-wise multiplication.

Let us make a remark on equations. The equations of the form w = w' with  $w, w' \in W(X) = G * W_0(X)$  are equations with constants. Consider systems of such equations T. If T is a congruence, then T has a solution in a faithful G-algebra H if and only if T is a faithful congruence in W = W(X). Thus, a system T has a solution, if T is contained in a faithful congruence in W. Note that, by definition, all faithful congruences are proper.

# 5. Special categories.

Let  $\Theta$  be a variety of algebras and  $H \in \Theta$ . Consider the category of affine spaces  $K^0_{\Theta}(H)$ . Its objects are represented by affine spaces  $\mathrm{Hom}(W(X),H)$ . Morphisms have the form

$$\tilde{s}: \operatorname{Hom}(W(X), H) \to \operatorname{Hom}(W(Y), H),$$

where  $s: W(Y) \to W(X)$  is a morphism in  $\Theta^0$ . Here,  $\tilde{s}$  is defined by  $\tilde{s}(\nu) = \nu s$  for every point  $\nu: W(X) \to H$ .

If an algebra H generates the variety  $\Theta$  then  $K_{\Theta}^{0}(H)$  is dual to the category  $\Theta^{0}$ .

Another category is denoted by  $\operatorname{Pol}_{\Theta}(H)$ . Its objects have the form of affine spaces  $H^{(n)}$ , while morphisms are the special polynomial maps

$$s^{\alpha}:H^{(n)}\to H^{(m)}.$$

If, further,  $s:W(Y)\to W(X)$  is a morphism in  $\Theta$  and  $|Y|=m,\,|X|=n,$  then there is a commutative diagram

$$\operatorname{Hom}(W(X), H) \xrightarrow{\tilde{s}} \operatorname{Hom}(W(Y), H)$$

$$\alpha_X \downarrow \qquad \qquad \downarrow \alpha_Y$$

$$H^{(n)} \xrightarrow{s^{\alpha}} H^{(m)}$$

This yields a canonical duality between  $K_{\Theta}^{0}(H)$  and  $\operatorname{Pol}_{\Theta}$ -H. In this diagram, all maps are algebra homomorphisms, if in  $\Theta$  all operations commute.

Denote  $\operatorname{Bool}_{\Theta}(H)$  to be a special category of Boolean algebras. Its objects  $\operatorname{Bool}(W(X),H)$  are the Boolean algebras of subsets in  $\operatorname{Hom}(W(X),H)$ .

Morphisms

$$s_* : \operatorname{Bool}(W(X), H) \to \operatorname{Bool}(W(Y), H)$$

are defined for every  $s: W(X) \to W(Y)$ . Here,  $\tilde{s}: \operatorname{Hom}(W(Y), H) \to \operatorname{Hom}(W(X), H)$  is given, and for every  $A \subset \operatorname{Hom}(W(X), H)$ ,  $B = s_*A$  is the inverse image of the set A under the map  $\tilde{s}$ .

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# LECTURE 2

# ALGEBRAIC SETS AND ALGEBRAIC VARIETIES

# Contents

- 1. Systems of equations and algebraic sets
- 2. Lattices of algebraic sets
- 3. Categories of algebraic sets
- 4. On the notion of algebraically closed algebras
- 5. Changing the basic variety of algebras
- 6. Zariski topology

LECTURE 2

#### 1. Systems of equations and algebraic sets.

Let  $\Theta$  be a variety of algebras, W=W(X) be the free algebra in  $\Theta$  with  $|X|<\infty$ .

Fix an algebra H from  $\Theta$ . Consider equations of the form  $w \equiv w'$ ,  $w, w' \in W$ . Denote the value of this equation in H by  $\operatorname{Val}_H(w \equiv w')$ . It is defined by

$$Val_H(w \equiv w') = \{ \mu : W \to H \mid w^{\mu} = w'^{\mu} \}.$$

Systems of equations in W are denoted by T. They can be viewed as binary relations in W. Some of these relations can be congruences in the algebra W.

Consider, also, sets of points  $A \subset \text{Hom}(W, H)$ . The Galois correspondence between T and A is as follows:

$$\left\{ \begin{array}{l} T' = A = \{\mu : W \to H \mid T \subset \operatorname{Ker} \mu\} = T'_H \\ A' = T = \bigcap_{\mu \in A} \operatorname{Ker} \mu \end{array} \right.$$

The same can be written in the form,

$$T' = \bigcap_{(w,w') \in T} \operatorname{Val}_H (w \equiv w')$$

$$A' = \{ w \equiv w' \mid A \subset \operatorname{Val}_H (w \equiv w') \}$$

**Definition 1.** Every set A such that A = T' for some T is called a closed or an H-closed set.

These are algebraic sets over the algebra H.

**Definition 2.** Algebraic sets considered up to isomorphisms in the category of algebraic sets  $K_{\Theta}(H)$  are called *algebraic varieties*.

The definition of the category  $K_{\Theta}(H)$  is given in the item 3 of this lecture. If A is an arbitrary set of points, then its closure is defined to be A'' = (A')'.

**Definition 3.** Congruences T in W of the form T = A' are called H-closed congruences.

For an arbitrary system of equations T its closure is defined by  $T''_H = (T'_H)'$ .

**Proposition 1.** A congruence T in W is H-closed if and only if for some set I there is an injection

$$W/T \to H^I.$$

Let, further, T be a congruence in W, and  $\mu_0: W \to W/T$  be a natural homomorphism. Consider the sets Hom(W/T, H) and  $\text{Hom}(W/T, H)\mu_0$ , where

$$\text{Hom}(W/T, H)\mu_0 = \{\nu\mu_0 : W \to H\}, \ \nu \in \text{Hom}(W/T, H).$$

Denote  $\tilde{\mu}_0(\nu) = \nu \mu_0$ .

**Proposition 2.** The following formula

$$T'_H = A = \operatorname{Hom}(W/T, H)\mu_0$$

holds. Moreover,  $\tilde{\mu}_0 : \text{Hom}(W/T, H) \to A$  is a bijection.

Let us consider, separately, the case H is a commutative algebra. Then the sets  $\operatorname{Hom}(W,H)$  and  $\operatorname{Hom}(W/T,H)$  are also commutative algebras. Every algebraic set A is a subalgebra in  $\operatorname{Hom}(W,H)$ . The bijection  $\tilde{\mu}_0: \operatorname{Hom}(W/T,H) \to A$  is an isomorphism of algebras.

Now, let us make some remarks on trivial cases.

A congruence T is called a zero congruence if wTw' means that w and w' coincide in W. Here T is a relation of equality. T is called a unity or a non-proper congruence, if wTw' is fulfilled for every  $w, w' \in W$ . These congruences are denoted T = 0, T = 1, respectively.

We have 0' = Hom(W, H) and, therefore, Hom(W, H) is an algebraic set. As for 1', this is either an empty set, or a zero point in Hom(W, H), sending W to the zero subalgebra in H, if the latter exists. A zero subalgebra consists of one element and is defined by a unique nullary operation.

It is easy to see that if A = Hom(W, H), then T = A' is the congruence of all identities of an algebra H in the free algebra W. This is the minimal closed congruence in W. If an algebra H generates the whole variety  $\Theta$ , then A' = 0. If A is the empty set, then A' = 1. If A consists of one zero point, then A' = 1 as well. If T is the empty set, then assume that T' = Hom(W, H).

For arbitrary algebras H and G, denote

$$(H - \operatorname{Ker})(G) = \bigcap_{\nu: G \to H} \operatorname{Ker} \nu.$$

Let T be a congruence in W and take  $\mu_0: W \to W/T$ . Then

**Proposition 3.** (Hilbert's Nullstellensatz)

$$T_H'' = \mu_0^{-1}((H - \operatorname{Ker} \mu)(W/T)).$$

This proposition can be considered as one of the forms of Hilbert's theorem. There are, also, others.

Let A be an algebraic set. To each element  $w \in W$  corresponds a map

$$\overline{w}:A\to H$$

defined by  $\overline{w}(\nu) = w^{\nu}$  for every  $\nu \in A$ . Such  $\overline{w}$  are called regular maps. One can define the algebra of regular maps. This algebra is isomorphic to an algebra W/A', which is called the *coordinate algebra* of the algebraic set A.

From the Proposition 1 follows that an algebra  $G \in \Theta$  can be presented as a coordinate algebra of an algebraic set A over given algebra  $H \in \Theta$  if and only if G is finitely generated algebra and there is an injection  $G \to H^I$  for some set I.

# 2. Lattices of algebraic sets.

Given a set I, the following formulas hold:

- 1.  $(\cup A_{\alpha})' = \cap A'_{\alpha}$
- 2.  $(\cup T_{\alpha})' = \cap T'_{\alpha}$
- 3.  $\cup T'_{\alpha} \subset (\cap T_{\alpha})'$ 4.  $\cup A'_{\alpha} \subset (\cap A_{\alpha})'$

where  $\alpha \in I$ .

If all  $A_{\alpha}$  and  $T_{\alpha}$  are H-closed sets, then

- 5.  $(\cup T'_{\alpha})'' = (\cap T_{\alpha})'$
- 6.  $(\cup A'_{\alpha})'' = (\cap A_{\alpha})'$ .

Thus, one can state that the intersection of algebraic sets is an algebraic set, and the intersection of closed congruences is again a closed congruence.

It is evident that the union of two closed congruences is not a closed congruence, and the union of two algebraic sets is not necessarily an algebraic set (say, if H is a commutative group it is not an algebraic set).

For given  $H \in \Theta$  and W = W(X),  $|X| < \infty$ , denote by

 $Alv_H(W)$  the set of all algebraic sets in Hom(W, H),

 $Cl_H(W)$  the set of all H-closed congruences in W.

This give rise to the functors

$$Alv_H: \Theta^0 \to Set$$
,

$$Cl_H: \Theta^0 \to Set.$$

The first functor is covariant, while the second one is contravariant. If  $s:W(Y)\to$ W(X) is a morphism in  $\Theta^0$ , then for every  $B \in Alv_H(W(Y))$  the corresponding  $s_*B = A$  is contained in  $Alv_H(W(X))$ .

There is a map

$$s_* : \operatorname{Alv}_H(W(Y)) \to \operatorname{Alv}_H(W(X)).$$

If  $T \in Cl_H(W(X))$  and  $s_*T$  is the inverse image of T in W(Y), then  $s_*T \in$  $Cl_H(W(Y))$ . This gives the map

$$s_*: \operatorname{Cl}_H(W(X)) \to \operatorname{Cl}_H(W(Y)).$$

The sets  $Alv_H(W)$  and  $Cl_H(W)$  constitute the lattices. If A and B are the elements from  $Alv_H(W)$ , then we set

$$A\overline{\cup}B = (A \cup B)''.$$

If  $T_1$  and  $T_2$  belong to  $Cl_H(W)$ , then

$$T_1 \overline{\cup} T_2 = (T_1 \cup T_2)''.$$

**Proposition 4.** Lattices  $Alv_H(W)$  and  $Cl_H(W)$  are dual.

The duality is determined by  $A \to A'$  and the properties 5,6 above.

In the general case, we cannot state that the functors  $Alv_H$  and  $Cl_H$  are coordinated with lattice operations.

**Definition 4.** An algebra H is called geometrically stable, if for every W(X) = W and any  $A, B \in Alv_H(W)$ , the equality

$$A\overline{\cup}B = A \cup B$$

takes place.

This is equivalent to the fact that for closed systems of equations  $T_1$  and  $T_2$  in W, the equality

$$T_1' \cup T_2' = (T_1 \cap T_2)'$$

holds.

If H is a stable algebra, then every lattice  $Alv_H(W)$  is a distributive lattice in  $Bool_{\Theta}(W, H)$ . A dual lattice  $Cl_H(W)$  is also distributive.

In this case, the functors  $Cl_H$  and  $Alv_H$  are the functors to the category of distributive lattices.

**Proposition 5.** If  $s: W(Y) \to W(X)$  is surjective, then the maps

$$s_*: \mathrm{Cl}_H(W(X)) \to \mathrm{Cl}_H(W(Y))$$
 and

$$s^* : \operatorname{Alv}_H(W(Y)) \to \operatorname{Alv}_H(W(X))$$

are homomorphisms of lattices.

Here the map  $s^*$  is conjugated to the map  $s_*$ .

For every algebraic set  $A \in Alv_H(W)$ , denote by L(A) the lattice of all algebraic sets in A. This lattice is considered as a lattice invariant of A.

Another invariant is based on the semigroup  $\operatorname{End}(H)$ . For every  $\delta: H \to H$  we have  $\tilde{\delta}: \operatorname{Hom}(W, H) \to \operatorname{Hom}(W, H)$  by the rule  $\tilde{\delta}(\nu) = \delta \nu$ . Thus, the semigroup  $\operatorname{End}(H)$  acts in the affine space  $\operatorname{Hom}(W, H)$ . It is clear that every algebraic set A in this space is invariant with respect to the action of  $\operatorname{End}(H)$ . Thus, we can look for the structure of A with respect to this action.

#### 3. Categories of algebraic sets.

Denote the category of algebraic sets in  $\Theta$  over the given H by  $K_{\Theta}(H)$ . Its objects are denoted by

where A is an algebraic sets in the affine space Hom(W(X), H).

In order to define morphisms

$$(X,A) \rightarrow (Y,B)$$

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we start from  $s: W(Y) \to W(X)$ . We say that s is admissible with respect to A and B, if  $\tilde{s}(\nu) = \nu s \in B$  for every  $\nu \in A$ . The equivalent condition is  $A \subset s_*B$ . If s is admissible with respect to A and B, it corresponds the map  $[s]: A \to B$ . We write  $[s]: (X,A) \to (Y,B)$ . The multiplication of such morphisms is defined in a natural way and the category  $K_{\Theta}(H)$  is defined.

Let us define now the category  $C_{\Theta}(H)$ . Its objects are algebras from  $\Theta$  of the form W(X)/T, where  $|X| < \infty$ , and T is an H-closed congruence. Morphisms are presented by homomorphisms

$$\sigma: W(Y)/T_2 \to W(X)/T_1.$$

Here,  $C_{\Theta}(H)$  is a full subcategory in the category  $\Theta$ .

Every such  $\sigma$  can be represented as  $\sigma = \overline{s}$ , where  $s : W(Y) \to W(X)$  is an admissible homomorphism of free algebras with respect to congruences  $T_2$  and  $T_1$ . This means that  $wT_2w'$  implies  $w^sT_1w'^s$ .

**Proposition 6.** A homomorphism  $s: W(Y) \to W(X)$  is admissible with respect to A and B if and only if s is admissible with respect to  $B' = T_2$  and  $A' = T_1$ .

Besides, if  $s_1: W(Y) \to W(X)$  is admissible with respect to A and B, and, hence, with respect to  $T_2$  and  $T_1$ , then the equality  $[s_1] = [s]$  holds if and only if  $\overline{s}_1 = \overline{s}$ .

This leads to the following theorem

Theorem 1. The transition

$$(X,A) \to W(X)/A'$$

defines the duality of the categories  $K_{\Theta}(H)$  and  $C_{\Theta}(H)$ .

This fact confirms the correctness of the definitions of  $K_{\Theta}(H)$  and  $C_{\Theta}(H)$ .

Define now the categories  $K_{\Theta}$  and  $C_{\Theta}$ . In these categories an algebra H is not fixed. The objects of  $K_{\Theta}$  have the form

where A is an algebraic set in  $\operatorname{Hom}(W(X), H)$ . The objects in  $C_{\Theta}$  have the form

where T is an H-closed congruence in W(X).

Consider morphisms

$$(X, A, H_1) \to (Y, B, H_2).$$

Take  $s: W(Y) \to W(X)$  and a homomorphism  $\delta: H_1 \to H_2$ .

Consider a commutative diagram

$$W(Y) \xrightarrow{s} W(X)$$

$$\downarrow^{\nu} \qquad \qquad \downarrow^{\nu}$$

$$H_2 \xrightarrow{\delta} H_1$$

For every point  $\nu: W(X) \to H_1$  denote  $(s, \delta)(\nu) = \nu' = \delta \nu s$ .

A pair  $(s, \delta)$  is said to be admissible with respect to A and B, if  $\nu' = (s, \delta)(\nu) \in B$  for every  $\nu \in A$ . Let  $(s, \delta)$  be an admissible pair with respect to A and B. Fix  $\delta$  and consider the map  $[s]_{\delta} : A \to B$ . The pair  $([s]_{\delta}, \delta)$  is considered to be a morphism

$$(X, A, H_1) \rightarrow (Y, B, H_2).$$

Let now

$$([s]_{\delta}, \delta): (X, A, H_1) \rightarrow (Y, B, H_2)$$
 and

$$([s']_{\delta'}, \delta') : (Y, B, H_2) \to (Z, C, H_3)$$

be two morphisms. Then, set

$$([s']_{\delta'}, \delta')([s]_{\delta}, \delta) = ([ss']_{\delta'\delta}, \delta'\delta) : (X, A, H_1) \to (Z, C, H_3).$$

These definitions are correct and the category  $K_{\Theta}$  is defined.

The categories  $K_{\Theta}$  and  $C_{\Theta}$  are dual.

The categories  $K_{\Theta}(H)$  and  $C_{\Theta}(H)$  are subcategories in  $K_{\Theta}$  and  $C_{\Theta}$ , and duality between  $K_{\Theta}$  and  $C_{\Theta}$  induces duality between  $K_{\Theta}(H)$  and  $C_{\Theta}(H)$ .

Let T be a system of equations in W(X).

Define a full subcategory  $K_{\Theta}(T)$  in  $K_{\Theta}$ .

The objects have the form

where X is fixed,  $A = T'_H$ , and H, A are changed. The category  $K_{\Theta}(T)$  characterizes possibilities to solve the system T in different algebras  $H \in \Theta$ .

Consider separately the case when there is an injection  $\delta: G \to H$ . It can be proven that this injection induces in  $K_{\Theta}$  an isomorphism between the category  $K_{\Theta}(G)$  and a subcategory in  $K_{\Theta}(H)$ .

#### 4. On the notion of algebraically closed algebras.

As previously in the classical situation, we consider  $K_{\Theta}(H)$  as a geometrical invariant of the algebra H. This category is viewed also as a measure of algebraic closeness of an algebra H. An algebra H can be not an algebraically closed algebra, but H is algebraically closed with some measure.

Consider the notion of algebraically closed algebras.

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**Definition 5.** An algebra  $H \in \Theta$  is called *algebraically closed* in  $\Theta$ , if for every finite X and every non-unity congruence T in W(X) this T has a non-zero root  $\mu: W(X) \to H$ .

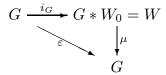
Here,  $\operatorname{Ker} \mu$  is a non-unity congruence and  $T \subset \operatorname{Ker} \mu$ . In particular, if T is a maximal congruence in W = W(X), then  $\operatorname{Ker} \mu = T$  and T is an H-closed congruence. Every maximal congruence T is H-closed.

It is clear that if an algebra H is contained in some other  $H_1 \in \Theta$ , and H is algebraically closed, then  $H_1$  is also algebraically closed.

Consider this definition with respect to varieties of the kind  $\Theta^G$ . An algebra  $H \in \Theta$  is considered as a G-algebra for different G with an injection  $h: G \to H$ . For such (H,h) and different G, the definition of algebraic closeness with respect to different varieties  $\Theta^G$  differ.

In Var-P, an algebra L over P can be P-closed but not absolutely algebraically closed. Here and in the general situation, absolutely algebraically closed means algebraic closeness in the variety  $\Theta^L$ .

In general case all equations and systems of equations have coefficients in H. So, fix  $G \in \Theta$  and consider a homomorphism in  $\Theta^G$ 



For every such  $\mu$  the congruence  $T = \operatorname{Ker} \mu$  is faithful and maximal. The condition of algebraic closeness of G in  $\Theta^G$  means that if T is a maximal faithful congruence in W, then  $T = \operatorname{Ker} \mu$  for some  $\mu$ .

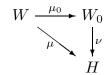
This is well-coordinated with the situation of the field P in Var-P. Note, that there are other approaches to the notion of algebraic closeness.

Consider systems of equalities T of the form  $w \equiv w'$  and inequalities  $w \not\equiv w'$ . Such a system has a solution if there exists a proper congruence  $T_1$  such that all  $w \equiv w'$  from T belong to  $T_1$  and all  $w \not\equiv w'$  from T do not belong to  $T_1$ . Now, one can say that an algebra H is an algebraically closed, if for any compatible system T of equalities and inequalities for every finite X there is a point  $\mu: W(X) \to H$ , which is a root of such T.

Such a point of view on algebraic closeness of an algebra H is closer to algebraic geometry in logic (see lecture 6) than to equational algebraic geometry.

# 5. Changing of the basic variety of algebras.

Let  $\Theta$  be a variety of algebras,  $\Theta_0 \subset \Theta$  be a subvariety, and X be a finite set. Then W = W(X) and  $W_0 = W_0(X)$  are the free algebras in  $\Theta$  and  $\Theta_0$ , respectively. The homomorphism  $\mu_0 : W(X) \to W_0(X)$  with the kernel  $T = \text{Ker } \mu$  consists of identities in W of the variety  $\Theta_0$ . For every such H there is a decomposition



To every point  $\mu: W \to H$  uniquely corresponds  $\nu: W_0 \to H$  with  $\mu = \nu \mu_0 = \tilde{\mu}_0(\nu)$ . We have the bijection  $\mu_0: \text{Hom}(W_0, H) \to \text{Hom}(W, H)$ .

**Proposition 7.** The bijection  $\tilde{\mu}_0$  induces the isomorphism of lattices of the algebraic sets  $\text{Alv}_H(W)$  and  $\text{Alv}_H(W_0)$ ; and the lattices  $\text{Cl}_H(W)$  and  $\text{Cl}_H(W_0)$ .

The same bijections  $\mu_0$  for different finite  $X_0$  is used to prove that

**Theorem 2.** The categories  $K_{\Theta}(H)$  and  $K_{\Theta_0}(H)$  are canonically isomorphic.

For  $\Theta_0 \subset \Theta$  one can consider the categories  $K_{\Theta_0}$  and  $K_{\Theta}$ . It can be proved that the first one is isomorphic to a subcategory of the second.

Let, again,  $H \in \Theta_0 \subset \Theta$ . Then,

**Proposition 8.** If H is algebraically closed in  $\Theta$ , then H is algebraically closed in  $\Theta_0$ .

The opposite statement is not true in general.

One can imagine the situation when a system of equations T in the given W = W(X) in  $\Theta$  contradicts to identities of the algebra H. Then T does not have a common root in H.

One can consider algebraic closeness of the given H with respect to a different finite X. Then the question is whether the absolute algebraic closeness follows from the algebraic closeness with respect to one-element set X. The answer depends on  $\Theta$  and on the structure of algebraic closure of an arbitrary H in  $\Theta$ .

#### 6. Zariski topology.

This is the minimal topology in the space Hom(W, H) in which all algebraic sets are closed.

If the algebra H is stable, then algebraic sets are, precisely, all closed sets. In the general case, the closed sets in Zariski topology are represented by finite unions of algebraic sets and their arbitrary intersections.

There is another approach which we now discuss.

Consider formulas of a more general form called pseudoequalities. They have the form

$$w_1 = w'_1 \vee \cdots \vee w_n \equiv w'_n, \quad w_i, w'_i \in W = W(X).$$

One can consider also pseudoequations. If u is the pseudoequality above, then its value  $Val_H(u)$  in the algebra H is defined by

$$\operatorname{Val}_H(u) = \operatorname{Val}_H(w_1 \equiv w_1') \vee \cdots \vee \operatorname{Val}_H(w_n \equiv w_n').$$

Similar to the usual equalities, for every system T of pseudoequalities, one can define

$$T'_H = A = \bigcap_{u \in T} \operatorname{Val}_H(u).$$

The sets A of such form are called pseudoalgebraic sets.

Every algebraic set is pseudoalgebraic. Conversely, if an algebra H is stable, then every pseudoalgebraic set is algebraic.

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**Proposition 9.** Closed sets in Zariski topology coincide with pseudoalgebraic sets.

One can consider the category of affine spaces  $K_{\Theta}^{0}(H)$  and every category  $K_{\Theta}(H)$  as categories with topological objects. The morphisms defined above turn out to be continuous maps.

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# LECTURE 3

# GEOMETRIC PROPERTIES AND GEOMETRIC RELATIONS OF ALGEBRAS

#### Contents

- 1. Geometric equivalence of algebras
- 2. Geometric equivalence and quasi-identities
- 3. Geometric similarity of algebras
- 4. Similarity and equivalence
- 5. Similarity and semi-isomorphisms
- 6. Geometrically Noetherian algebras
- 7. Geometric stability

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#### 1. Geometric equivalence of algebras.

Given a variety  $\Theta$ .

**Definition 1.** Algebras  $H_1$  and  $H_2$  in  $\Theta$  are geometrically equivalent if for every finite X and every system of equations T in the free algebra W = W(X), the equality

$$T_{H_1}^{"} = T_{H_2}^{"}$$

takes place.

This is equivalent to the fact that always  $Cl_{H_1}(W) = Cl_{H_2}(W)$ , i.e., the functors  $Cl_{H_1}$ , and  $Cl_{H_2}$  coincide.

Also the lattices  $Cl_{H_1}(W)$  and  $Cl_{H_2}(W)$  coincide and the lattices  $Alv_{H_1}(W)$  and  $Alv_{H_2}(W)$  are isomorphic.

**Proposition 1.** Every algebra H is geometrically equivalent to any of its cartesian powers  $H^{I}$ .

This proposition follows from the fact that the algebras  $H_1$  and  $H_2$  are geometrically equivalent if and only if for any W and a congruence T in W the equality

$$(H_1 - \text{Ker})(W/T) = (H_2 - \text{Ker})(W/T)$$

takes place.

**Proposition 2.** Let  $\Theta$  be a variety and  $\Theta_0$  a subvariety in  $\Theta$ . The algebras  $H_1$  and  $H_2$  from  $\Theta_0$  are geometrically equivalent in  $\Theta_0$  if and only if they are equivalent in  $\Theta$ .

Consider one more point of view on the notion of geometric equivalence. Let  $H_1$  and  $H_2$  from  $\Theta$  be given. There exists a canonical functor

$$F = F(H_1, H_2) : K_{\Theta}(H_1) \to K_{\Theta}(H_2).$$

If (X, A) is an object in  $K_{\Theta}(H_1)$ , then

$$F((X,A)) = (X, (A')'_{H_2}).$$

The definition of the action of F on morphisms is more complicated and is omitted.

**Proposition 3.** The algebras  $H_1$  and  $H_2$  are geometrically equivalent if and only if the functors  $F(H_1, H_2)$  and  $F(H_2, H_1)$  are mutually inverse.

In particular, if  $H_1$  and  $H_2$  are geometrically equivalent, then the categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  are isomorphic.

Isomorphism of categories can be established also from the following commutative diagram

$$C_{\Theta}(H_1) \longrightarrow C_{\Theta}(H_2)$$

$$\downarrow \qquad \qquad \downarrow$$

$$K_{\Theta}(H_1) \longrightarrow K_{\Theta}(H_2)$$

Here, the upper arrow is an identity map, and the vertical arrows are dual.

As an example, let us consider the following result which is due to A. Berzins.

**Proposition 4.** Two abelian groups  $H_1$  and  $H_2$  are geometrically equivalent if and only if

- 1. They have the same exponents,
- 2. For every prime p the exponents of Sylow subgroups  $H_{1p}$  and  $H_{2p}$  coincide.

### 2. Geometric equivalence and quasi-identities.

A quasi-identity in  $\Theta$  has the form

$$w_1 \equiv w_1' \wedge \cdots \wedge w_n \equiv w_n' \to w \equiv w'.$$

We can assume that all  $w_i, w'_i, w, w'$  belong to the same W(X) with finite X.

**Theorem 1.** If algebras  $H_1$  and  $H_2$  are geometrically equivalent, then they have the same quasi-identities.

In particular, if algebras  $H_1$  and  $H_2$  are geometrically equivalent, then

$$Var(H_1) = Var(H_2).$$

**Corollary.** If two groups  $G_1$  and  $G_2$  are geometrically equivalent and one of them is torsion-free, then the second one is also torsion-free.

**Problem 2.** Is it true that if  $H_1$  and  $H_2$  have the same quasi-identities then they are geometrically equivalent? Is it true that if  $H_1$  and  $H_2$  are elementary equivalent, then they are geometrically equivalent?

There is a negative answer to both of these questions. Namely, there exists a group H and its ultrapower  $\tilde{H}$  such that H and  $\tilde{H}$  are not geometrically equivalent. Similar fact is true also for G-groups. This very important expected result is obtained in [MR2] and uses ideas of the dissertation of V.A. Gorbunov (1996) [Gor]. For groups there are also beautiful solutions in [GoSh] and [BlG].

For associative and Lie algebras this result is also valid (see [Pl11] for details. For associative algebras it uses the result from [Li]).

On the other hand, recall that in classical situation every extension L of the field P is geometrically equivalent to every its ultrapower.

We will consider also generalized (infinitary) quasi-identities. They have the form

$$\wedge_{\alpha \in I} (w_{\alpha} \equiv w'_{\alpha}) \to w \equiv w'.$$

Here I is an arbitrary set, and all  $w_{\alpha}, w'_{\alpha}, w, w'$  belong to one and the same W(X) with finite X.

**Theorem 2.** Algebras  $H_1$  and  $H_2$  are geometrically equivalent if and only if they have the same generalized quasi-identities.

This theorem is based on the following form of the theorem about zeroes.

Let T be an arbitrary system of equations in W(X) with finite X. Denote by I the set of all indices  $\alpha$  of equations  $w_{\alpha} \equiv w'_{\alpha}$  from T. Then

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**Proposition 5.** Inclusion  $w \equiv w' \in T''_H$  is satisfied if and only if

$$\wedge_{\alpha \in I} (w_{\alpha} \equiv w'_{\alpha}) \to (w \equiv w')$$

is fulfilled in the algebra H.

If T is a finite set, then with  $T''_H$  the usual quasi-identity is associated. Thus,

**Theorem 2'.** Algebras  $H_1$  and  $H_2$  are geometrically equivalent with respect to finite T if and only if  $H_1$  and  $H_2$  generate the same quasivariety.

Note also

**Theorem 2".** Algebras  $H_1$  and  $H_2$  are geometrically equivalent if and only if

$$LSC(H_1) = LSC(H_2).$$

Here, L, S, and C are the standard closure operators on classes of algebras, used in the characterization of prevarieties.

For any class  $\mathfrak{X}$  the class  $LSC(\mathfrak{X})$  is a locally closed prevariety over  $\mathfrak{X}$  which is contained in the quasivariety, generated by  $\mathfrak{X}$  [PPT].

For every algebra  $H \in \Theta$ , finitely generated algebras in the prevariety SC(H) are the algebras presented as coordinate algebras of algebraic sets over H.

It follows from [MR2] that the class  $LSC(\mathfrak{X})$  is not a quasivariety and, moreover, not an axiomatized class. In this sense, the relation of geometric equivalence of algebras is not an axiomatizable relation. This relation is axiomatizable in terms of generalized quasi-identities [Pl10].

#### 3. Geometric similarity of algebras.

This notion generalizes the notion of geometric equivalence of algebras, and, like the notion of geometric equivalence, is associated with the problem of isomorphism of categories of algebraic sets. It is the main notion of the lecture 5. Geometric equivalence means that the functors  $\operatorname{Cl}_{H_1}$  and  $\operatorname{Cl}_{H_2}$  coincide. Geometric similarity assumes the more complicated connection between  $\operatorname{Cl}_{H_1}$  and  $\operatorname{Cl}_{H_2}$ .

If  $H_1$  and  $H_2$  are geometrically equivalent, then  $Var(H_1) = Var(H_2)$  and the categories  $Var(H_1)^0$  and  $Var(H_2)^0$  are also coincide.

Let us consider the functors

$$\operatorname{Cl}_{H_1}:\operatorname{Var}(H_1)^0\to\operatorname{Set}$$
 and  $\operatorname{Cl}_{H_2}:\operatorname{Var}(H_2)^0\to\operatorname{Set}.$ 

Similarity of algebras means that there is an isomorphism  $\varphi : Var(H_1)^0 \to Var(H_2)^0$  with the commutative diagram

$$\begin{array}{ccc}
\operatorname{Var}(H_1)^0 & \xrightarrow{\varphi} & \operatorname{Var}(H_2)^0 \\
\operatorname{Cl}_{H_1} \searrow & & \swarrow \operatorname{Cl}_{H_2}
\end{array}$$

Commutativity of the diagram indicates an isomorphism (not necessarily equality) of the functors  $Cl_{H_1}$  and  $Cl_{H_2} \varphi$ . This isomorphism  $\alpha = \alpha(\varphi)$  depends on the isomorphism of categories  $\varphi$  and is constructed in a special way.

The notion of geometric similarity appears to be sophisticated, but in "good" cases it reduces to geometric equivalence.

We will use, further, some remarks on congruences in free algebras.

Let the algebra W = W(X) be given. For a congruence T in W, consider a relation  $\rho = \rho(T)$  in the semigroup  $\operatorname{End}(W)$ . Let us set  $\nu \rho \nu', \nu, \nu' \in \operatorname{End}(W)$ , if  $\nu(w)T\nu'(w)$  holds for every  $w \in W$ .

Here,  $\rho$ -equivalence on  $\operatorname{End}(W)$ , and  $\nu\rho\nu'$  implies  $(\nu_1\nu)\rho(\nu_1\nu')$  for every  $\nu_1\in\operatorname{End}(W)$ .

Let now  $\rho$  be an arbitrary congruence on  $\operatorname{End}(W)$ . Define a relation  $T=T(\rho)$  on the algebra W:

$$w_1 T w_2 \Leftrightarrow \exists w \in W, \nu, \nu' \in \text{End}(W)$$

such that  $w^{\nu} = w_1$ ,  $w^{\nu'} = w_2$ , and  $\nu \rho \nu'$ .

**Proposition 6.** If T is a congruence on the algebra W and  $\rho = \rho(T)$ , then  $T(\rho) = T$ .

If T is a fully characteristic congruence on W, then  $\rho = \rho(T)$  is a congruence in the semigroup  $\operatorname{End}(W)$ , and the semigroup  $\operatorname{End}(W)/\rho$  is isomorphic to the semigroup  $\operatorname{End}(W/T)$ .

Now let an isomorphism of categories

$$\varphi: \operatorname{Var}(H_1)^0 \to \operatorname{Var}(H_2)^0$$

be given.

We want to define a function  $\alpha = \alpha(\varphi) : \operatorname{Cl}_{H_1} \to \operatorname{Cl}_{H_2} \varphi$  which yields an isomorphism of functors. A function  $\alpha$  for every W from  $\operatorname{Var}(H_1)^0$  is given by the map  $\alpha_W : \operatorname{Cl}_{H_1}(W) \to \operatorname{Cl}_{H_2} \varphi(W)$ , and for every  $s : W \to W'$ , the commutative diagram

$$\begin{array}{ccc}
\operatorname{Cl}_{H_1}(W') & \xrightarrow{\operatorname{Cl}_{H_1}(s)} & \operatorname{Cl}_{H_1}(W) \\
 \alpha_{W'} \downarrow & \alpha_W \downarrow \\
\operatorname{Cl}_{H_2} \varphi(W') & \xrightarrow{\operatorname{Cl}_{H_2} \varphi(s)} & \operatorname{Cl}_{H_2} \varphi(W)
\end{array}$$

should be fulfilled. This means that  $\alpha = \alpha(\varphi)$  is an isomorphism of functors  $\text{Cl}_{H_1}$  and  $\text{Cl}_{H_2} \varphi$ . Let us construct such an  $\alpha$ .

First for any  $\Theta$  define functions  $\beta$  and  $\gamma$  which for an object W from  $\Theta^0$  give the maps  $\beta_W$  and  $\gamma_W$ .

The function  $\beta_W$  is defined by  $\beta_W(T) = \rho = \rho(T)$ , where T is a congruence in W and  $\rho(T)$  is defined above.

The function  $\gamma_W$  is given by  $\gamma_W(\rho) = T = T(\rho)$ , where  $\rho$  is an equivalence in  $\operatorname{End}(W)$ . We have

$$\gamma_W(\beta_W(T)) = T$$

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if T is a congruence in W.

For every  $T \in \operatorname{Cl}_{H_1}(W)$  define

$$\alpha(\varphi)_W(T) = \gamma_{\varphi(W)}(\varphi(\beta_W(T)).$$

Here, if  $\rho$  is a relation on  $\operatorname{End}(W)$  then  $\varphi(\rho) = \rho^*$  is a relation on  $\varphi(W)$  defined by  $\mu \rho^* \mu'$  if and only if  $\mu = \varphi(\nu)$ ,  $\mu' = \varphi(\nu')$ ,  $\nu \rho \nu'$ . Here  $\alpha(\varphi)_W(T)$  is a relation on the algebra  $\varphi(W)$  which is not necessarily a congruence.

The meaning of the function  $\alpha = \alpha(\varphi)$  is that it represents the action of the isomorphism  $\varphi$  on the congruences of free algebras.

Define now a function  $\tau$ . Let W, W' be two objects in  $\Theta^0$ . Then for any congruence T in W' define  $\tau_{W,W'}(T) = \rho$  to be a relation over the set  $\operatorname{Hom}(W, W')$  defined by

$$s\rho s' \Leftrightarrow w^s T w^{s'}, \ \forall w \in W,$$

where s and s' are homomorphisms  $W \to W'$ . We have  $\tau_{W,W} = \beta_W$ . We say that  $\alpha$  is compatible with  $\tau$  if for every W and W', and a congruence  $T \triangleleft W'$ ,

$$\varphi(\tau_{W,W'}(T)) = \tau_{\varphi(W),\varphi(W')}(\alpha(\varphi)_{W'}(T))$$

holds.

Now we state the main definition.

**Definition 2.** Two algebras  $H_1$  and  $H_2$  are called *geometrically similar* if there exists an isomorphism

$$\varphi: \operatorname{Var}(H_1)^0 \to \operatorname{Var}(H_2)^0$$
,

such that the function  $\alpha = \alpha(\varphi) : \operatorname{Cl}_{H_1} \to \operatorname{Cl}_{H_2} \varphi$  is compatible with the function  $\tau$  and for every object W in  $\operatorname{Var}(H_1)^0$  there is a bijection

$$\alpha(\varphi)_W: \mathrm{Cl}_{H_1}(W) \to \mathrm{Cl}_{H_2}(\varphi(W)).$$

From this definition it follows that a function  $\alpha = \alpha(\varphi)$  defines an isomorphism of functors  $\text{Cl}_{H_1}$  and  $\text{Cl}_{H_2} \varphi$ .

In addition, the same  $\alpha(\varphi)$  defines an isomorphism of lattices

$$\alpha(\varphi)_W : \mathrm{Cl}_{H_1}(W) \to \mathrm{Cl}_{H_2}(\varphi(W)).$$

**Theorem 3.** The algebras  $H_1$  and  $H_2$  are geometrically similar if and only if there exists an isomorphism  $\varphi : \operatorname{Var}(H_1)^0 \to \operatorname{Var}(H_2)^0$ , and an isomorphism of functors  $\alpha = \alpha(\varphi) : \operatorname{Cl}_{H_1} \to \operatorname{Cl}_{H_2} \varphi$  which is compatible with the function  $\tau$ .

Note that

- 1. If  $H_1$  and  $H_2$  are geometrically equivalent, then they are similar. Here  $Var(H_1) = Var(H_2)$  and for  $\varphi$  take the identity map.
- 2. A relation of geometric similarity is symmetric, reflexive and transitive.
- 3. If  $\Theta_0$  is a subvariety in  $\Theta$ , and  $H_1, H_2 \in \Theta_0$ , then  $H_1$  and  $H_2$  are similar in  $\Theta$  if and only if they are similar in  $\Theta_0$ .

#### 4. Similarity and equivalence.

Consider the case when  $Var(H_1) = Var(H_2) = \Theta$ . In this case  $\varphi$  is an automorphism of the category  $\Theta^0$ .

**Definition 3.** An automorphism  $\varphi: C \to C$  of an arbitrary category C is called *inner* if it corresponds to a function s which attaches an isomorphism  $s_A: A \to \varphi(A)$  to every object A from C such that for any  $\nu: A \to B$  there is a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\nu} & B \\
s_A \downarrow & s_B \downarrow \\
\varphi(A) & \xrightarrow{\varphi(\nu)} & \varphi(B)
\end{array}$$

i.e.,

$$\varphi(\nu) = s_B \nu s_A^{-1}.$$

From this definition follows that an automorphism  $\varphi$  is inner if and only if the functor  $\varphi$  is isomorphic to identity functor of the category C.

**Theorem 4.** If the algebras  $H_1$  and  $H_2$  are geometrically similar with respect to the inner automorphism  $\varphi$ , then they are equivalent.

Let us make some remarks on inner automorphisms of categories.

If an inner automorphism  $\varphi$  is realized by a function s, then we write  $\varphi = \hat{s}$ . Let now  $\varphi$  be an arbitrary automorphism and  $\hat{s}$  be an inner automorphism. Take an automorphism  $\varphi \hat{s} \varphi^{-1}$  and consider a function  $\varphi(s)$  defined by the rule

$$\varphi(s)_B = \varphi(s_{\varphi^{-1}B})$$

for any object B in the given category C.

Let us check that

$$\varphi \hat{s} \varphi^{-1} = \widehat{\varphi(s)}.$$

Thus if Aut(C) is a group of all automorphisms of the category C, then there is a normal subgroup Int(C) of all inner automorphisms.

Let now  $\varphi \in \operatorname{Aut}(C)$  and A be an arbitrary object in C. Then  $\varphi$  induces an isomorphism of semigroups  $\operatorname{End}(A)$  and  $\operatorname{End}(\varphi(A))$  and the groups  $\operatorname{Aut}(A)$  and  $\operatorname{Aut}(\varphi(A))$ . If  $\varphi(A) = A$  then  $\varphi$  induces an automorphism of the semigroup  $\operatorname{End}(A)$  and of the group  $\operatorname{Aut}(A)$ . If  $\varphi$  is an inner automorphism, then the corresponding induced automorphism of the semigroup  $\operatorname{End}(A)$  and the group  $\operatorname{Aut}(A)$  is also inner.

**Definition 4.** A variety  $\Theta$  is called *perfect* if every automorphism of the category  $\Theta^0$  is an inner automorphism.

We will consider the examples of perfect  $\Theta$ . We can state that if  $\Theta$  is a perfect variety and  $Var(H_1) = Var(H_2) = \Theta$ , then  $H_1$  and  $H_2$  are similar if and only if they are equivalent.

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Let us define the weakly inner automorphisms  $\varphi: C \to C$ . An automorphism  $\varphi$  is called *weakly inner* if there exists a function s with isomorphisms  $s_A: A \to \varphi(A)$ , such that for every  $\nu: A \to A$  holds

$$\varphi(\nu) = s_A \nu s_A^{-1} : \varphi(A) \to \varphi(A).$$

It can be proven that in Theorem 4 one can proceed from a weakly inner automorphism. Using this fact, one can prove

**Proposition 7.** Two abelian groups  $H_1$  and  $H_2$ , such that each of them generates the whole variety of abelian groups, are similar if and only if they are equivalent.

A reasonable conjecture states that the same is true for  $H_1$  and  $H_2$ , generating the variety of all groups Grp.

The solution of this problem is closely connected with the following problem.

**Problem 3.** Let F = F(X) be the non-commutative free group with finite X. It is known that the group Aut(F) is a perfect group. Is the same true for the semigroup End(F)?

Let  $\varphi$  be an automorphism of the semigroup  $\operatorname{End}(F)$ . This  $\varphi$  induces an automorphism of the group  $\operatorname{Aut}(F)$ . Thus, there exists an element  $\sigma \in \operatorname{Aut}(F)$ , such that  $\varphi(x) = \sigma x \sigma^{-1}$  for every  $x \in \operatorname{Aut}(F)$ . Is it true that the same holds for every  $x \in \operatorname{End}(F)$ ?

# 5. Similarity and semiisomorphisms.

Let  $\Theta$  be an arbitrary variety,  $G \in \Theta$ , and consider the variety  $\Theta^G$ . Assume that for  $G \in \Theta^G$  the condition (\*) (see Lecture 2) is fulfilled. This means that if  $(H_1, h_1)$  and  $(H_2, h_2)$  are two faithful G-algebras, then each of them generates the variety  $\Theta^G$ .

In the category  $\Theta^G$  one can consider semiisomorphisms and semiisomorphic algebras.

**Theorem 5.** If algebras  $(H_1, h_1)$  and  $(H_2, h_2)$  are semiisomorphic, then they are similar.

Thus, if there is a sequence of G-algebras

$$(H_1, h_1) = (H_1^0, h_1^0), (H_1^1, h_1^1), \dots (H_1^n, h_1^n) = (H_2, h_2),$$

such that the neighbors are geometrically equivalent or semiisomorphic, then  $(H_1, h_1)$  and  $(H_2, h_2)$  are similar.

<sup>&</sup>lt;sup>2</sup>This question is solved positively in [For]. From this follows that the conjecture is also true. From the result of Formanek follows that the variety Grp is weekly perfect. Thus, arises the following problem. Is it true that the variety Grp is perfect? The positive answer can be obtained using the technique of Formanek. The same question stands for the variety of all abelian groups.

**Definition 5.** Algebras  $(H_1, h_1)$  and  $(H_2, h_2)$  are called geometrically equivalent up to a semiisomorphism, if there exists (H, h) such that  $(H_1, h_1)$  and (H, h) are semiisomorphic, and (H, h) is equivalent to  $(H_2, h_2)$ .

We will consider semi-inner automorphisms of the category  $(\Theta^G)^0$ .

Let  $\varphi: (\Theta^G)^0 \to (\Theta^G)^0$  be an automorphism of the category of free algebras in  $\Theta^G$ . These algebras have the form

$$W(X) = G * W_0(X) = W = G * W_0.$$

Consider a pair  $(\sigma, s)$ , where  $\sigma$  is an automorphism of the algebra G and  $s_W : W \to \varphi(W)$  is an algebra isomorphism in  $\Theta$  for every W. The commutative diagram

$$G \xrightarrow{i_G} W$$

$$\sigma \downarrow \qquad \qquad s_W \downarrow$$

$$G \xrightarrow{i'_G} \varphi(W)$$

defines a semiisomorphism

$$(\sigma, s_W): (W, i_G) \to (\varphi(W), i_G').$$

It is supposed that with the automorphism  $\varphi$  one can associate such a pair  $(\sigma, s)$  and, besides, for every morphism  $\nu: W \to W'$  there is a commutative diagram

$$W \xrightarrow{\nu} W'$$

$$(\sigma, s_W) \downarrow \qquad (\sigma, s_{W'}) \downarrow$$

$$\varphi(W) \xrightarrow{\varphi(\nu)} \varphi(W')$$

Then,

$$\varphi(\nu) = (\sigma, s_{W'})(1, \nu)(\sigma, s_W)^{-1} = (1, s_{W'}\nu s_W^{-1}) = s_{W'}\nu s_W^{-1},$$

and this is a morphism in  $(\Theta^G)^0$ .

**Definition 6.** An automorphism  $\varphi$  is called *semi-inner*, if there exists a pair  $(\sigma, s)$  with the property above.

We denote  $\varphi = (\widehat{\sigma, s})$ .

It can be proved that semi-inner automorphisms constitute a subgroup in  $\operatorname{Aut}((\Theta^G)^0)$  which contains the normal subgroup of inner automorphisms.

From the definition 6 follows that an automorphism  $\varphi$  is semi-inner if and only if the functor  $\varphi$  is semiisomorphic to the identity functor of the category  $(\Theta^G)^0$ .

The main theorem here is as follows:

**Theorem 6.** If the similarity of algebras  $(H_1, h_1)$  and  $(H_2, h_2)$  is given by a semi-inner automorphism  $\varphi$ , then they are equivalent up to a semi-isomorphism.

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# 6. Geometrically Noetherian algebras.

We define a property of an algebra  $H \in \Theta$ , which always takes place in the classical situation and plays a crucial part.

**Definition 7.** An algebra H in the variety  $\Theta$  is called *geometrically Noetherian* if for every finite set X and every system of equations T in W = W(X), there exists a finite subsystem  $T_0 \subset T$ , such that  $T'_H = T'_{0H}$ , or, in other words,  $T''_H = T''_{0H}$ .

We say that T and  $T_0$  are an equivalent systems of equations.

**Proposition 8.** An algebra H is geometrically Noetherian if and only if for every finite X the lattices  $Cl_H(W(X))$  and  $Alv_H(W(X))$  satisfy the maximal and minimal condition, respectively.

**Proposition 9.** Let  $\Theta_0$  be a subvariety in  $\Theta$  and  $H \in \Theta^0$ . Then H is geometrically Noetherian in  $\Theta$  if and only if it is geometrically Noetherian in  $\Theta_0$ .

**Definition 8.** A variety  $\Theta$  is called *Noetherian* if every finitely generated algebra from  $\Theta$  is Noetherian.

**Examples.** 1. The variety Var - P is noetherian.

- 2. Arbitrary variety of nilpotent groups is noetherian.
- 3. In the variety of arbitrary associative algebras over the field P every noetherian subvariety can be distinguished by special known identities.

**Proposition 10.** If the variety  $\Theta$  is Noetherian, then every algebra H in  $\Theta$  is geometrically Noetherian. In particular, if the variety Var(H) is Noetherian, then the algebra H is geometrically Noetherian.

**Theorem 7.** Let  $H_1$  and  $H_2$  be two geometrically Noetherian algebras. They are geometrically equivalent if they have the same quasi-identities.

**Theorem 8.** Let  $H_1$  and  $H_2$  have the same quasi-identities. Then, if  $H_1$  is geometrically Noetherian, then  $H_2$  is also geometrically Noetherian.

**Corollary.** If an algebra H is geometrically Noetherian, then each of its cartesian power and ultrapower is geometrically Noetherian.

The same is true for arbitrary filtered power.

**Proposition 11.** If algebras  $H_1$  and  $H_2$  are geometrically similar and one of them is geometrically Noetherian, then the second one is also geometrically Noetherian.

It can be easily seen that every subalgebra of a geometrically Noetherian algebra is also geometrically Noetherian, and a finite cartesian product of geometrically Noetherian algebra is also a Noetherian algebra. Every finite algebra is geometrically Noetherian.

Geometrically Noetherian algebras admit a Noetherian Zariski topology in the corresponding affine spaces and the Lasker-Noether Theorem on the decomposition of algebraic varieties.

**Definition 9.** A variety of the type  $\Theta^G$  is called *faithfully Noetherian* if it is Noetherian with respect to faithful congruences.

If the variety  $\Theta^G$  is faithfully Noetherian, then each of its faithful algebras (H,h) is geometrically Noetherian.

There are many important results on geometrically Noetherian groups (see [Br], [Guba], [BMR1], [BMR0] and others. There are also many interesting problems, especially for the case of G in  $\Theta^G$ .

# 7. Geometric stability.

This notion was already defined and will be used and discussed in the next lecture. Let us make now some remarks on G-algebras.

**Proposition 12.** Let the algebras  $(H_1, h_1)$  and  $(H_2, h_2)$  be semiisomorphic. Then,

- 1. If one of them is stable, then the second one is also stable.
- 2. If one of them is geometrically Noetherian, then the second one is also geometrically Noetherian.

Besides that, if  $(H_1, h_1)$  and  $(H_2, h_2)$  are two arbitrary G-algebras, then their product  $(H_1 \times H_2, h_1 \times h_2)$  cannot be a stable algebra [Be]. From this follows that if  $(H_1, h_1)$  and  $(H_2, h_2)$  are equivalent and one of them is stable, then the second one is not necessarily stable.

**Definition 10.** An algebra H in the variety  $\Theta$  is called *geometrically distributive* if every lattice  $\mathbb{C}_H(W)$  is distributive.

If H is stable, then H is geometrically distributive.

**Problem 4.** Is it true that an algebra H is geometrically distributive if and only if it is similar to a stable algebra?

One can consider also geometrically modular algebras.

It can be proven that if an algebra  $G \in \Theta$  admits faithful finite dimensional linearization over a field, then it is geometrically Noetherian in  $\Theta$  and in  $\Theta^G$ . This fact, in particular, relates to finite dimensional, associative and Lie algebras. (For groups, see [BMR], the general case is done by A.Belov [BelP]). The proof uses the natural Zariski topology over the same field.

A.Miasnikov and V.Remeslennikov have noticed the following interesting question. Is it true that a free Lie algebra F(X) with finite X is geometrically noetherian?

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# LECTURE 4

# ALGEBRAIC GEOMETRY IN GROUP-BASED ALGEBRAS.

#### Contents

- 1. Group-based algebras
- 2. Zero divisors and nilpotent elements
- 3. Group-based algebras with an algebra of constants
- 4. Domains
- 5. Algebraic geometry in group-based algebras
- 6. Stability
- 7. Theorems on zeroes

# 1. Group-based algebras.

**Definition 1.** A group-based algebra H is an additive group (not necessarily commutative) with some additional signature  $\Omega$ . For every  $\omega \in \Omega$  of the arity  $n(\omega) = n > 0$  the condition  $0 \cdots 0\omega = 0$  should be fulfilled.

Here 0 is the zero element in the additive group H and on the left side we have n times for 0.

Such group-based algebras are called also  $\Omega$ -groups. They were introduced by P. Higgins [Hi] in 1956, see also [Ku], [Pl7].

A group is an  $\Omega$ -group with empty set  $\Omega$ , in rings the set  $\Omega$  consists of a single multiplication, in modules over a ring R all elements of R belong to the set  $\Omega$ . Groups over rings, considered by R.Lyndon [L2] are also  $\Omega$ -groups.

In arbitrary  $\Omega$ -group G we have the usual commutator  $[a,b]=-a-b+a+b=-a+a^b$ , and  $\omega$ -commutators for  $\omega$  with  $n(\omega)>0$ . By definition we have:

$$[a_1, \cdots, a_n; b_1, \cdots, b_n; \omega] = -a_1 \cdots a_n \omega - b_1 \cdots b_n \omega + (a_1 + b_1) \cdots (a_n + b_n) \omega.$$

From this definition follows that if all  $a_1, \dots, a_n$  are zeros or all  $b_1, \dots, b_n$  are zeros then the  $\omega$ -commutator is a zero.

In rings we have:

$$[a_1, a_2; b_1, b_2; \cdot] = a_1b_2 + b_1a_2.$$

Now about ideals in  $\Omega$ -groups. Let  $\mu: H \to H_1$  be a homomorphism of two  $\Omega$ -groups, U is the coimage of the zero element from  $H_1$ . Then:

- 1) U is closed relative to all  $\omega \in \Omega$  with  $n(\omega) > 0$ .
- 2) U is a normal subgroup in additive group H.
- 3) The  $\omega$ -commutator  $[a_1, \dots, a_n; b_1, \dots, b_n; \omega]$  belongs to U if  $a_1, \dots, a_n \in U$ , and  $b_1, \dots, b_n$  are arbitrary elements of H.

**Definition 2.** An ideal U in H is a set U with the three conditions above.

From the definition follows also that the commutator  $[a_1, \dots, a_n; b_1, \dots, b_n; \omega]$  belongs to U if all  $b_1, \dots, b_n$  in U and  $a_1, \dots, a_n$ - are arbitrary. The ideal in a group is a normal subgroup, the ideal in a ring is a usual ideal. It can be proved that if T is a congruence in an  $\Omega$ -group H, then the class [0] = U for T is an ideal and we have: [a] = a + U for arbitrary  $a \in H$ . So we have the one-to-one correspondence between congruences and ideals. We write now  $\operatorname{Ker} \mu = U$  instead of  $\operatorname{Ker} \mu = T$  and H/U instead of H/T.

Now let A and B be two sets in the  $\Omega$ -group H,  $\{A, B\}$  be an  $\Omega$ -subgroup in H, generated by A and B.

**Definition 3.** The mutual commutant [A, B] is the ideal in  $\{A, B\}$ , generated by all commutators of the kind  $[a, b], a \in A, b \in B$  and all  $[a_1, \dots, a_n; b_1, \dots, b_n; \omega]$ , were  $a_1, \dots, a_n \in A, b_1, \dots, b_n \in B$ .

We have [A, B] = [B, A] and for every two ideals  $U_1$  and  $U_2$  we have:  $[U_1, U_2] \subset U_1 \cap U_2$ . Let now  $U_1$  and  $U_2$  be ideals in the associative ring R. Then:

$$[U_1, U_2] = U_1 U_2 + U_2 U_1.$$

In the Lie case we have:  $[U_1, U_2] = U_1U_2$ , in groups it is usual mutual commutant.

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**Definition 4.** The group-based algebra H is abelian if [H, H] = 0.

This means that the group H is abelian and for every  $\omega \in \Omega$ ,  $n = n(\omega) > 0$  we have:

$$(a_1 + b_1) \cdots (a_n + b_n)\omega =$$
  
=  $a_1 \cdots a_n \omega + b_1 \cdots b_n \omega$ 

for every  $a_1, \dots, a_n$ ;  $b_1, \dots, b_n$ . In rings this means that ab = 0 for every a and b.

**Definition 5.** A group-based algebra H is antiabelian if:

- 1. If U is an abelian ideal in H then U=0.
- 2. If  $U_1$  and  $U_2$  are two nontrivial ideals then the ideal  $U_1 \cap U_2$  is also non-trivial.

## 2. Zero divisors and nilpotent elements.

From now on all algebras are group-based algebras. Let an algebra H be given,  $a \in G$  and (a) be the ideal in H, generated by the element a.

**Definition 6.** A non-zero element  $a \in G$  is called a zero divisor if for some non-zero  $b \in G$  we have

$$[(a), (b)] = 0.$$

Consider this notion for associative and Lie rings and groups. We are studying the case without zero divisors.

**Proposition 1.** A ring R is without zero divisors if and only if for every non-zero a and b there exist  $c_1, c_2, c'_1, c'_2$  such that

$$c_1 a c_2 \cdot c'_1 b c'_2 \neq 0$$
 or  $c'_1 b c'_2 \cdot c_1 a c_2 \neq 0$ .

Corollary. The class of all R without zero divisors is an axiomatizable class.

**Proposition 2.** The Lie ring L is without zero divisors if and only if for every non-zero a and b in L there exist  $m \geq 0$ ,  $n \geq 0$ , and elements  $c_1, \dots, c_m, c'_1, \dots, c'_n \in L$  such that:

$$[[a, c_1, \cdots, c_m]; [b, c'_1, \cdots, c'_n]] \neq 0.$$

Corollary. The class of all L without zero divisors, may not be an axiomatizable class.

**Proposition 3.** The group G is a group without zero divisors iff for every non-unite a and b in G there exists  $c \in G$  such that:

$$[a, b^c] \neq 1,$$

where  $b^c = c^{-1}bc$ .

Corollary. The class of all groups without zero divisors is an axiomatizable class.

Groups without zero divisors were considered in [BMR] and in [BPP] under the name antiabelian. In [BMR] it was proved (in additional conditions) that the free product of two groups without zero divisors is a group without zero divisors as well.

From the corollary above we can conclude that the ultraproduct of groups without zero divisors is also a group without zero divisors.

Now the following general result.

**Proposition 4.** An  $\Omega$ -group H does not have zero divisors iff H is antiabelian.

**Examples.** An algebra H is simple if H has not non-trivial ideals. Every non-abelian and simple algebra is antiabelian and so without zero-divisors. Every free group and every free Lie algebra is antiabelian. Every free associative or free associative and commutative algebras are without zero divisors, and thus such an algebras are antiabelian. It is easy to characterize all antiabelian finite dimensional associative and Lie algebras.

Note that using commutants in  $\Omega$ -groups the nilpotent and solvable  $\Omega$ -groups are naturally defined.

**Definition 7.** An element a in an  $\Omega$ -group H is called strictly nilpotent if the ideal (a) is nilpotent. An element a in an  $\Omega$ -group H is called weakly nilpotent if the ideal (a) is solvable.

It is easy to see that if a is a weakly nilpotent element then in (a) there is a zero divisor. If a is a strictly nilpotent element then a is a zero divisor.

## 3. Group-based algebras with an algebra of constants.

For an arbitrary variety  $\Theta$  and an arbitrary  $G \in \Theta$  we consider a new variety of algebras with constants from G, which is denoted by  $\Theta^G$ . We take for  $\Theta$  a variety of  $\Omega$ -groups and let G be a fixed  $\Omega$ -group from  $\Theta$ . As before, the key point for considering such a variety is the fact that groups and rings in their pure form are not so convenient for constructing an algebraic geometry. For algebraic geometry we need some large set of constants in the signature  $\Omega$ , and we need equations with constants.

For a given  $h: G \to H$  the ideal U in H is the same as the ideal in G-algebra H. For such U we have the embedding  $\bar{h}: G \to H/U$ , induced by h. If h is faithful and  $Im(h) \cap U = 0$ , then  $\bar{h}$  is also faithful.

Let us define now relative ideals and relative zero divisors in respect to the group G.

Now we consider the notion of relative ideal or G-ideal. We connect the notion of ideal with the algebra of constants G.

**Definition 8.** The set  $U \subset H$  in the algebra  $h: G \to H$  is a G-ideal if

- 1) U is closed in respect to all  $\omega \in \Omega$ ,  $n(\omega) > 0$ , and U is a subgroup in the additive group H,
- 2) For every  $a \in U$  and  $g \in G$  we have  $[a, h(g)] \in U$ ,

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3)  $[a_1, \dots, a_n; h(g_1), \dots, h(g_n); \omega] \in U$  if  $\omega \in \Omega, n(\omega) > 0; a_1, \dots, a_n \in U, g_1, \dots, g_n \in G.$ 

If the Conditions 1,2,3, are fulfilled the set U is called invariant in respect to constants from G.

From the conditions 1, 2, 3) we have also:

$$[h(g_1), \cdots, h(g_n); a_1, \cdots, a_n; \omega] \in U.$$

Every ideal at the same time is an G-ideal.

The algebra G can be considered as a G-algebra with the identical  $G \to G$ . In this case the G-ideal in G is the same as ideal.

For every  $a \in H$  by  $(a)^G$  we denote the G-ideal, generated by a.

**Definition 9.** A non-zero  $a \in H$  we call a G-zero divisor if for some  $b \neq 0$  we have

$$[(a)^G, (b)^G] = 0.$$

Every zero divisor in H is also a G-zero divisor. So if H is without G-zero divisors then H is without (absolute) zero divisors. Such H is an antiabelian.

**Proposition 5.** Let  $h: G \to H$  be a G-group. Then H is without G-zero divisors if and only if for every non-unite a and b there exists  $c \in G$  such that

$$[a, b^{h(c)}] \neq 1.$$

We can consider now a set T of equations

$$[x, y^{h(c)}] = 1, \qquad c \in G.$$

Let T' = A be the set of all solutions in the given H. We can say that H is without G-zero divisors iff from  $(a, b) \in A$  follows a = 1 or b = 1. In other words the infinitary formula

$$\underset{c \in G}{\wedge} ([x, y^{h(c)}] = 1) \rightarrow (x = 1) \lor (y = 1).$$

holds in H. The similar can be considered in Lie-L and in Assoc - R.

An algebra  $G \to G$  is without G-zero divisors iff G is antiabelian.

**Examples.** As before, if  $\Theta = Grp$ , then we write  $\Theta^G = Grp - G$ . In the similar Lie case we have Lie-L, and in associative one  $\Theta^G = Assoc - R$ .

#### 4. Domains.

A G-algebra H is called a domain if H is without G-zero divisors. It is trivial and very important that every G-subalgebra in the domain H is also a domain.

Now let  $h_1: G \to H_1$  and  $h_2: G \to H_2$  be two faithful G-algebras. We have in this case an isomorphism  $h_0: Im(h_1) \to Im(h_2)$ . Using this  $h_0$  we can consider the amalgamated free product

$$h: G \to H_1 *_G H_2.$$

That is a free product in  $\Theta(G)$ . In [BMR] it is proven that if  $H_1$  and  $H_2$  from Grp - G are domains then their free product is also a domain.

**Definition 10.** An ideal U in G-algebra H is called prime if H/U is a domain.

For every G-algebra H denote by Spec(H) the set of all prime ideals in H.

If H/U is a domain then this algebra is antiabelian. In this case the ideal U is irreducible, U cannot be non-trivially represented as  $U = U_1 \cap U_2$ .

Let now  $\mu: H \to H_1$  be a homomorphism of G-algebras and  $H_1$  be a domain. Then  $U = \text{Ker } \mu$  is prime.

**Proposition 6.** Let H and  $H_1$  be two semiisomorphic G-algebras. Then  $H_1$  is a domain if and only if H is a domain.

**Proposition 7.** Let  $\mu: H \to H_1$  be a homomorphism of G-algebras,  $a \in H$ . Then we have

$$((a)^G)^{\mu} = (a^{\mu})^G.$$

This Proposition plays an important role in the proofs of Theorems 1 and 2 below.

## 5. Algebraic geometry in group-based algebras.

Let us repeat the basic notions for the case when the variety  $\Theta$  is a variety of  $\Omega$ -groups.

Let X be a finite set and W = W(X) a free in  $\Theta$  algebra over X. For the given  $H \in \Theta$  the set of homomorphisms Hom(W, H) we consider as an affine space.

For every point  $\mu: W \to H$  we have the kernel  $U = Ker\mu$  and  $\mu$  is a solution of some equation  $w \equiv 0$  iff  $w \in Ker\mu$ .

Now let U be a set of "polynomials" in W and A be a set of points in the space Hom(W, H). We establish the following Galois correspondence:

$$\left\{ \begin{array}{l} U' = A = \{\mu | U \subset Ker\mu\} \\ A' = U = \bigcap_{\mu \in A} Ker\mu \end{array} \right.$$

The set A such that A = U' for some U we call an algebraic set over the algebra H. The ideal U of the form U = A' for some A we call an H-closed ideal.

As usual we call algebraic sets also algebraic varieties.

Now let two algebraic varieties A and B in Hom(W,G) be given,  $A' = U_1$  and  $B' = U_2$ . Then:  $(A \cap B) = (U_1 \cup U_2)'$  and so the intersection of two algebraic varieties is also an algebraic variety. But we can not say the same for the union  $A \cup B$ . In general, it is not true that  $A \cup B = (U_1 \cap U_2)'$ . It depends on the choice of the algebra H.

## 6. Stability.

**Definition 11.** The algebra H is said to be *stable* if for every W = W(X) and every two algebraic sets A and B in the space Hom(W, H) the union  $A \cup B$  is also an algebraic set.

If H is stable then in the category  $K_{\Theta}(H)$  every object A can be considered as a topological space. The morphisms in  $K_{\Theta}(H)$  are well coordinated with the topology, they are continuous maps.

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In the classical case all fields and all domains are stable. But if  $\Theta = Grp$  is the variety of all groups and H is abelian then H is not stable.

Now the main question: when the algebra H is stable? And the main idea: the notions to be stable and without zero divisors are close to each other.

Consider now the varieties of the type  $\Theta^G$ .

**Theorem 1.** If a G-algebra H is a domain then H is stable.

Now we consider the following special identities, the so-called CD-identities (commutator distributivity). They are defined for all  $\omega \in \Omega$ ,  $n = n(\omega) > 0$  and have the form:

$$[x_1 + y_1, \cdots, x_n + y_n; z_1, \cdots, z_n; \omega] =$$

$$= [x_1, \cdots, x_n; z_1, \cdots, z_n; \omega] +$$

$$+ [y_1, \cdots, y_n; z_1, \cdots, z_n; \omega].$$

Here all  $x_i, y_i, z_i$  in X.

**Definition 12.** An algebra  $H \in \Theta$  is called a CD-algebra if all CD-identities are hold in H.

**Proposition 8.** If the algebra G is a CD-algebra and G has zero-divisors, then G is not stable in  $\Theta^G$ .

**Theorem 2.** Let G be a CD-algebra in  $\Theta^G$ . Then G is stable if and only if G has no zero divisors.

CD-conditions are fulfilled in groups, associative and Lie algebras. In these cases the theorem was proved by A.Berzins. The result is true also in arbitrary rings.

#### 7. Theorems on zeroes.

We consider the case when the algebra G is algebraically closed in the variety  $\Theta^G$ . Let first the variety  $\Theta$  be an arbitrary variety, i.e., not necessarily the variety of group-based algebras.

If T is a maximal faithful congruence in the algebra  $W = G * W_0$ , then  $T = Ker\mu$  for some point  $\mu$ .

G-algebra G is a simple algebra since it has no faithful congruences.

**Definition 13.** G-algebra H is called semisimple if it is approximated by simple G-algebras. G-algebra H is called locally semisimple, if every finitely generated subalgebra in H is simple.

**Theorem 3.** Let algebra G be algebraically closed in the variety  $\Theta^G$  and let H be a faithful locally semisimple G-algebra. Then for every faithful congruence T in W = W(X) with finite X, the congruence  $T''_H$  is the intersection of all maximal faithful congruences in W which contain T.

Proof By definition,  $T''_H$  is the intersection of all  $Ker\mu$  where  $T \subset Ker\mu$ . For every such  $\mu$  take in H the image  $Im\mu = H_1$ . The algebras  $H_1$  and  $W/Ker\mu$  are faithful and semisimple. Therefore, the congruence  $Ker\mu$  is the intersection of

some maximal faithful congruences. Then  $T''_H$  is an intersection of maximal faithful congruences.

Let now  $T_1$  be a maximal faithful congruence in W containing T. Show that  $T_1$  contains  $T''_H$ . We have  $T_1 = Ker\mu$  for some point  $\mu : W \to G$ . This point  $\mu$  is a homomorphism  $W \to H$ . This means that  $T_1 = T''_{1H}$ . Since  $T \subset T_1$  then  $T''_H \subset T''_{1H}$ . This implies the theorem.

**Corollary.** If  $H_1$  and  $H_2$  are faithful locally simple G-algebras, and G is algebraically closed, then  $H_1$  and  $H_2$  are geometrically equivalent.

Let us come back to the situation of group-based algebras. Define relative nilpotents, i.e., G-nilpotents.

Let (H, h) be a  $G - \Omega$ -group and  $a \in H$ .

**Definition 14.** An element a is called G-nilpotent if in  $\Omega$ -group  $(a)^G$  there is a series of G-invariant ideals

$$0 = U_0 \subset U_1 \cdots \subset U_n = (a)^G$$

with the abelian factors  $U_{i+1}/U_i$ . If, moreover,  $[(a)^G, U_{i+1}] \subset U_i$  then a is said to be strictly G-nilpotent element.

In groups ( $\Omega$  is empty) an element a is a nilpotent element if the group  $(a)^G$  is solvable. If  $(a)^G$  is nilpotent then a is strictly nilpotent.

In associative rings both these notions coincide. Namely, an element  $a \in H$  is a nilpotent if there exists n such that

$$b_0 a b_1 a b_2 a \cdots b_{n-1} a b_n = 0$$

for any  $b_i = h(g_i), g_i \in G$ .

If the ring is commutative then this condition is equivalent to  $a^n = 0$ .

If the algebra H is a domain then it has no non-zero nilpotent elements.

Given G-algebra H denote N(H) the ideal in H generated by all its G-nilpotent elements. This is the N-radical of H. The function N can be iterated and this gives rise to the upper N-radical. Denote it by  $\tilde{N}(H)$ . There are no non-trivial nilpotent elements in  $H/\tilde{N}(H)$ .

If U is an ideal in H then denote by  $\sqrt{U}$  the inverse image in H of the radical  $\tilde{N}(H/U)$ .

It is clear that if a G-homomorphism  $\mu: H \to H'$  is given and  $U \subset Ker\mu$  then  $\sqrt{U}$  is also contained in  $Ker\mu$  in case H' is a domain. This implies

**Proposition 9.** Let U be an ideal in the free G-algebra  $W = G * W_0$ , and let H be a G-domain. Then

$$\sqrt{U} \subset U_H''$$
.

We are looking for conditions which imply the equality in the formula above.

**Definition 15.** A variety is called special, if for every finitely generated G-algebra H its radical  $\tilde{N}(H)$  is an intersection of maximal faithful congruences in H.

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**Theorem 3'.** If an algebra G is algebraically closed and the variety  $\Theta^G$  is special then for every faithful G-domain H the equality

$$\sqrt{U} = U_H''$$
.

holds for every ideal U and every W.

In fact, this theorem, as well as the theorem 3, immediately follows from definitions. The problem is to study the situations when all this is applicable.

Here the ideas of radical, semisimplicity and algebraic closure are naturally intersected in the theorems 3 and 3'.

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#### LECTURE 5

## ISOMORPHISMS OF CATEGORIES OF ALGEBRAIC SETS

#### Contents

- 1. Correct isomorphism
- 2. Isomorphism, similarity, equivalence
- 3. Perfect and semiperfect varieties of algebras
- 4. Other results
- 5. Problems

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#### 1. Correct isomorphism.

Let  $H_1$  and  $H_2$  be two algebras in the variety  $\Theta$ . We are looking for an isomorphism of categories

$$F: K_{\Theta}(H_1) \to K_{\Theta}(H_2).$$

Denote  $\Theta_1 = Var(H_1)$  and  $\Theta_2 = Var(H_2)$ .

The isomorphism F one-to-one corresponds to the isomorphism

$$F: K_{\Theta_1}(H_1) \to K_{\Theta_2}(H_2).$$

which is also denoted by F.

Let  $W^1 = W^1(X)$  and  $W^2 = W^2(X)$  be the free algebras in  $\Theta_1$  and  $\Theta_2$  respectively, X is a finite set.

**Definition 1.** An isomorphism F is called correct isomorphism, if

1.  $F(Hom(W^1(X), H_1) = Hom(W^2(Y), H_2)$  for some Y such that |Y| = |X|. 2. If  $[s]: A \to Hom(W^1(X), H_1)$  is an identity embedding, then

$$F([s]); F(A) \rightarrow Hom(W^2(Y), H_2)$$

is also an identity embedding.

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Categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  are correctly isomorphic if there exists a correct isomorphism of categories  $K_{\Theta_1}(H_1)$  and  $K_{\Theta_2}(H_2)$ .

Let us translate this definition to the language of dual categories. Consider the diagram

$$K_{\Theta_1}(H_1) \xrightarrow{F} K_{\Theta_2}(H_2)$$

$$\downarrow_{F_1} \qquad \qquad \downarrow_{F_2}$$

$$C_{\Theta_1}(H_1) \xrightarrow{\Phi} C_{\Theta_2}(H_2)$$

Here,  $F_1$  and  $F_2$  are dual isomorphisms and  $\Phi$  is an isomorphism of categories induced by F.

We consider the correctness condition in terms of the functor  $\Phi$ . First, let us connect directly the functors  $\Phi$  and F.

Let us take objects (X, A) in  $K_{\Theta_1}(H_1)$  and (Y, B) in  $K_{\Theta_2}(H_2)$ .

Let  $T = A'_{H_1}$  and  $T^* = B'_{H_2}$ ,  $A = T'_{H_1}$ , and  $B = T^*'_{H_2}$ . Then F(X, A) = (Y, B) if and only if  $\Phi(W^1(X)/T) = W^2(Y)/T^*$ .

Consider morphisms. Let  $[s]:(X_1,A_1)\to (X_2,A_2)$  be a morphism in  $K_{\Theta_1}(H_1)$  and

$$\bar{s}: W^1(X_2)/T_2 \to W^1(X_1)/T_1$$

be the corresponding morphism in  $C_{\Theta_1}(H_1)$ .

In the categories  $K_{\Theta_2}(H_2)$  and  $C_{\Theta_2}(H_2)$  we have:

$$[s_1]:(Y_1,B_1)\to (Y_2,B_2)$$

and

$$\bar{s_1}: W^2(Y_2)/T_2^* \to W^2(Y_1)/T_1^*$$
.

Now, note that

$$F[s]: F((X_1, A_1)) \to F((X_2, A_2))$$

is

$$[s_1]:(Y_1,B_1)\to (Y_2,B_2)$$

if and only if

$$\Phi(\bar{s}): \Phi(W^1(X_2)/T_2) \to \Phi(W^1(X_1)/T_1)$$

equals to

$$\bar{s_1}: W^2(Y_2)/T_2^* \to W^2(Y_1)/T_1^*$$
.

**Definition 2.** An isomorphism  $\Phi: C_{\Theta_1}(H_1) \to C_{\Theta_2}(H_2)$  is a correct isomorphism if

- 1.  $\Phi(W^1(X)) = W^2(Y)$  for some Y, |Y| = |X|.
- 2. If  $\mu_X: W^1(X) \to W^1(X)/T_1$  is the natural homomorphism in  $C_{\Theta_1}(H_1)$ , then

$$\Phi(\mu_X): \Phi(W^1(X)) \to \Phi(W^1(X)/T_1)$$

is also a natural homomorphism,

$$\Phi(\mu_X) = \mu_Y : W^2(Y) \to W^2(Y)/T_2.$$

The main result here is as follows.

**Theorem 1.** An isomorphism  $F: K_{\Theta_1}(H_1) \to K_{\Theta_2}(H_2)$  is a correct isomorphism if and only if  $\Phi: C_{\Theta_1}(H_1) \to C_{\Theta_2}(H_2)$  is a correct isomorphism.

## 2. Isomorphism, similarity, equivalence.

In the paper [Pl5] the following theorems are proved

**Theorem 2.** The categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  is correctly isomorphic if and only if the algebras  $H_1$  and  $H_2$  are geometrically similar.

Let now  $H_1$  and  $H_2$  be abelian groups, each of them generates the variety of all abelian groups, and let  $\Theta$  be the variety of all groups.

**Theorem 3.** Categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  are correctly isomorphic if and only if  $H_1$  and  $H_2$  are geometrically equivalent.

Let  $H_1$  and  $H_2$  be groups such that  $Var(H_1) = Var(H_2) = \Theta = Grp$ .

**Conjecture.** Categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  are correctly isomorphic if and only if the groups  $H_1$  and  $H_2$  are geometrically equivalent.

The crucial point here is the problem 3 on Aut(End(F)), where F is free, is stated earlier.<sup>3</sup>

## **Problems**

**Problem 5.** What is the situation in semigroups, i.e.,  $H_1$  and  $H_2$  are commutative semigroups.

**Problem 6.** What is the situation in modules over commutative rings.

**Problem 7**. What is the relation between arbitrary isomorphisms of the categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$  and correct isomorphisms? This problem depends on the choice of  $\Theta$  and  $H \in \Theta$ .

The following two problems are of the very general nature.

**Problem 8.** Consider the question of equivalence of the categories  $K_{\Theta}(H_1)$  and  $K_{\Theta}(H_2)$ .<sup>4</sup>

**Problem 9.** When the categories  $K_{\Theta_1}$  and  $K_{\Theta_2}$  are isomorphic or equivalent. Consider, in particular, the case when  $\Theta_1$  and  $\Theta_2$  are equivalent categories. Consider also the cases  $\Theta_1 = \Theta^{G_1}$ ,  $\Theta_2 = \Theta^{G_2}$  for the different algebras  $G_1$  and  $G_2$  in the given  $\Theta$ .

Note that in [Pl6] and [Pl8] there is an invariant approach (without equations) to the category of algebraic varieties and it is quite natural to proceed here from the idea of equivalence of categories. There are some results in this direction.

## 3. Perfect and semiperfect varieties of algebras.

3.1. In the first lecture the theorem on isomorphism of the categories of type  $K_P(L)$  was formulated. The key role in this theorem played the notion of semiisomorphism of P-algebras. All this was done for the variety Var - P.

<sup>&</sup>lt;sup>3</sup>Since the problem 3 is solved positively (E.Formanek [For]), the conjecture is true.

<sup>&</sup>lt;sup>4</sup>This problem is considered in the paper [Pl11]

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Now we consider the general situation. We take varieties of the type  $\Theta^G = \Theta - G$  and consider the question on isomorphisms of the categories of the type  $K_{\Theta^G}(H)$  where H is a faithful G-algebra. We are going to apply Theorem 2 in the G-algebras case.

Fix a variety  $\Theta$  and  $G \in \Theta$ . Consider the variety  $\Theta^G$  and the category  $(\Theta^G)^0$  of free in  $\Theta^G$  algebras of the form  $W(X) = G * W_0(X)$  with different finite X.

We are interested in automorphisms of this category, and want to find out when all of them are inner or semi-inner.

We assume that the condition (\*) is fulfilled. This means that the algebra G generates the variety  $\Theta^G$ . Results on automorphisms of the category  $(\Theta^G)^0$  we apply to the problem of isomorphism of categories of algebraic varieties with faithful G-algebras H.

In the given conditions the category  $(\Theta^G)^0$  is dual to the category of affine spaces  $K^0_{\Theta^G}(G)$ , while the last one is connected with the category of polynomial maps Pol - G.

Let  $\varphi: (\Theta^G)^0 \to (\Theta^G)^0$  be an automorphism of the category  $(\Theta^G)^0$ . It corresponds the automorphism  $\tau$  of the category  $K^0_{\Theta^G}(G)$ . The automorphisms  $\varphi$  and  $\tau$  are connected by the following rules.

First of all

$$\tau(Hom(W(X),G) = Hom(\varphi(W(X)),G).$$

Let further,  $s:W(Y)\to W(X)$  be a morphism in  $(\Theta^G)^0$ . In the category  $K^0_{\Theta^G}(G)$  it corresponds

$$\tilde{s}:(Hom(W(X),G)\to Hom(W(Y),G)).$$

For every  $\nu: W(X) \to G$  we have  $\tilde{s}(\nu) = \nu s$ . Then

$$\tau(\tilde{s})(\nu) = \nu \varphi(s) = \varphi(\tilde{s})(\nu)$$

for every point  $\nu: \varphi(W(X)) \to G$ ,  $\tau(\tilde{s}) = \varphi(\tilde{s})$ . This gives one-to-one correspondence between  $\varphi$  and  $\tau$ .

3.2. We would like to explain that every automorphism  $\tau$  of the category of affine spaces is, in a sense, a quasi-inner automorphism. Denote the category of affine spaces by  $K_{\Theta^G}^0(G) = K_{\Theta^G}^0$ .

Recall that every point  $\nu:W\to G$  satisfies the commutative diagram

$$G \xrightarrow{i_G} W = G * W_0$$

$$\downarrow^{\nu}$$

$$G$$

Let  $\nu: W \to G$  be a point. Consider the homomorphism defined by

$$W \xrightarrow{\nu} G \xrightarrow{i_G} W$$

Check that  $i_G \nu: W \to W$  is an endomorphism of the algebra W in the variety  $\Theta^G$ . We have to check that there is the commutative diagram

$$G \xrightarrow{i_G} W \bigvee_{i_G \nu} \bigvee_{W}$$

We have

$$(i_G\nu)i_G = i_G(\nu i_G) = i_Gid_G = i_G.$$

Denote  $i_G \cdot \nu = \bar{\nu}$ . For every  $w \in W$  the element  $\bar{\nu}(w) = i_G(\nu(w))$  is a constant in W and, therefore,  $\bar{\nu}$  is called a constant endomorphism. Every endomorphism  $s: W \to W$  leaves constants and, therefore,  $(s\bar{\nu})(w) = \bar{\nu}(w)$ ,  $s\bar{\nu} = \bar{\nu}$ . Endomorphism  $\bar{\nu}$  defines the map

$$\stackrel{\simeq}{\overline{\nu}}$$
:  $Hom(W,G) \to Hom(W,G)$ .

Note that for every  $\nu_0: W \to G$  we have  $\widetilde{\overline{\nu}}(\nu_0) = \nu$ . Indeed,  $\widetilde{\overline{\nu}}(\nu_0) = \nu_0 \cdot \overline{\nu} = \nu_0(i_G\nu) = (\nu_0i_G)\nu = id_G \cdot \nu = \nu$ .

Thus, the map  $\widetilde{\overline{\nu}}$  takes an arbitrary  $\nu_0$  to one and the same element  $\nu$ , and, therefore  $\widetilde{\overline{\nu}}$  is a constant map.

Consider an arbitrary  $\sigma: W \to W$ , such that  $s\sigma = \sigma$  for every  $s: W \to W$ . Since one can take for s an endomorphism taking w to a constant,  $\sigma$  takes any w to a constant.

It can be shown that to every such  $\sigma$  one-to-one corresponds  $\nu:W\to G$  such that  $\sigma=\bar{\nu}$ .

Let now  $\varphi: (\Theta^G)^0 \to (\Theta^G)^0$  be an automorphism of the category  $(\Theta^G)^0$  and let  $\tau: K_{\Theta^G}^0 \to K_{\Theta^G}^0$  be the corresponding automorphism in the category of affine spaces.

In particular, the constant

$$\stackrel{\sim}{\overline{\nu}}$$
:  $Hom(W,G) \to Hom(W,G)$ 

is characterized by the condition  $s\bar{\nu} = \bar{\nu}$  for every s. This condition can be rewritten as  $\tilde{\nu}\tilde{s} = \tilde{\nu}$ . Apply  $\tau$  to the equality  $\tilde{\nu}\tilde{s} = \tilde{\nu}$ . We have

$$(\widetilde{\widetilde{\nu}}\widetilde{s})^{\tau} = \widetilde{\widetilde{\nu}}^{\tau} \cdot \widetilde{s}^{\tau} = \widetilde{\widetilde{\nu}}^{\tau} \colon Hom(W^{1},G) \to Hom(W^{1},G)$$

Since  $\tilde{s}^{\tau}$  is an arbitrary,  $\tilde{\overline{\nu}}^{\tau}$  is a constant, i.e.,  $\tilde{\overline{\nu}}^{\tau} = \tilde{\overline{\nu}}_1$ , where  $\nu_1 : \varphi(W) = W^1 \to G$  is a point. Denote  $\nu_1 = \mu(\nu)$ . The map

$$\mu = \mu_W : Hom(W, G) \to Hom(W^1, G) = Hom(\varphi(W), G)$$

is a bijection. For every point  $\nu: W \to G$  there is  $\widetilde{\overline{\nu}}^{\tau} = \underbrace{\overline{\mu_W(\nu)}}$ .

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**Definition 3.** An automorphism  $\tau$  of the category  $K_{\Theta(G)}^0$  is called quasi-inner, if for an arbitrary  $\tilde{s}: Hom(W^2, G) \to Hom(W^1, G)$ , the formula

$$\tilde{s}^{\tau} = \mu_{W^1} \tilde{s} \mu_{W^2}^{-1}.$$

takes place.

**Theorem 4.** (see [Be2] for Var - P) Every automorphism  $\tau$  of the category  $K_{\Theta^G}^0$  is quasi-inner, i.e., for every  $s: W(Y) \to W(X)$  we have

$$\tilde{s}^{\tau} = \mu_{W^1} \tilde{s} \mu_{W^2}^{-1} : Hom(\varphi(W(X), G) \to Hom(\varphi(W(Y), G)).$$

Note that the definitions of inner and semi-inner automorphism in the category  $(\Theta^G))^0$  is well correlated with the definition of quasi-inner automorphism in the category  $K_{\Theta^G}^0$ . Theorem 4 plays a key role in the proofs of the theorems on conditions on an automorphism  $\varphi$  to be inner or semi-inner.

3.3. Let us define now a substitutional automorphism of the category  $(\Theta^G)^0$ . Let, first,  $\varphi$  be a substitution on the objects W = W(X) of the category  $(\Theta^G)^0$ . Suppose that if  $\varphi(W(X)) = W(Y)$  then |X| = |Y|. Let  $X = \{x_1, \dots, x_n\}$ ,  $Y = \{y_1, \dots, y_n\}$ . Define isomorphism  $s_W: W(X) \to \varphi(W(Y))$  by  $s_W(x_i) = y_i, i = 1, \dots, n$ . If, further,  $\nu: W^1 \to W^2$  is a morphism, then set

$$\varphi(\nu) = s_{W^2} \nu s_{W^1}^{-1} \colon \varphi(W^1) \to \varphi(W^2).$$

From the substitution  $\varphi$  we come to the automorphism  $\bar{\varphi}$ . We call  $\bar{\varphi}$  substitutional automorphism.

**Proposition 1.** Every automorphism  $\varphi$  in the category  $(\Theta^G))^0$  can be presented in the form

$$\varphi=\varphi^2\varphi^1$$

where  $\varphi^2$  is a substitutional automorphism depending on  $\varphi$  and  $\varphi^1 = (\varphi^2)^{-1}\varphi$  does not change objects.

Similar decomposition can be considered for any variety, not necessarily of the form  $\Theta^G$ . Such a decomposition of an automorphism of the category  $\Theta^0$  implies, under some conditions, the decomposition of the corresponding relation of similarity for algebras in  $\Theta$ .

3.4. Let us pass to the category of polynomial maps Pol-G and let  $s: W(Y) \to W(X)$ , where  $X = \{x_1, \dots, x_n\}, Y = \{y_1, \dots, y_n\}$ , be given. Consider the commutative diagram (Lecture 1)

$$Hom(W(X), G) \xrightarrow{\tilde{s}} Hom(W(Y), G)$$

$$\downarrow^{\alpha_X} \qquad \qquad \downarrow^{\alpha_Y}$$

$$G^{(n)} \xrightarrow{s^{\alpha}} G^{(m)}$$

We have  $s^{\alpha} = \alpha_{Y}\tilde{s}\alpha_{X}^{-1}$ ,  $\tilde{s} = \alpha_{Y}^{-1}s^{\alpha}\alpha_{X}$  (Lecture 1). Let  $\varphi$  be an automorphism in  $\Theta^{G^{0}}$  and  $\varphi$  does not change objects. The corresponding automorphism  $\tau$  also acts identically on objects. It corresponds an automorphism  $\tau^{\alpha}$  of the category Pol - G which also preserves the objects. Let  $s^{\alpha}: G^{(n)} \to G^{(m)}$  be given. We set  $\tau^{\alpha}(s^{\alpha}) = \alpha_{Y}\tau(\tilde{s})\alpha_{X}^{-1}$ , where  $\tilde{s} = \alpha_{Y}^{-1}s^{\alpha}\alpha_{X}$ . Here  $\tau^{\alpha}$  is an automorphism of the category Pol - G. To automorphism  $\tau$  corresponds a function  $\mu$ , defining  $\tau$  as a quasi-inner automorphism. Here

$$\mu_X = \mu_{W(X)} = \mu_W : Hom(W, G) \to Hom(W, G).$$

This is a bijection and  $W = \varphi(W)$ . It corresponds a bijection

$$\mu_n: G^{(n)} \to G^{(n)}.$$

Take  $a = (a_1, ..., a_n) \in G^{(n)}$ . Let

$$\mu_n(a) = \alpha_X(\mu_X(\alpha_X^{-1}(a))) = \alpha_X \mu_X \alpha_X^{-1}(a).$$

Then,

$$\mu_n = \alpha_X \mu_X \alpha_X^{-1}, \ \mu_X = \alpha_X^{-1} \mu_n \alpha_X.$$

For the given homomorphism  $s: W(Y) \to W(X), |X| = n, |Y| = m$ , we have

$$\tilde{s}$$
:  $Hom(W(X), G) \rightarrow Hom(W(Y), G)$ 

and

$$\tilde{s}^{\tau} = \mu_Y \tilde{s} \mu_X^{-1} = \tilde{s}_1 : Hom(W(X), G) \to Hom(W(Y), G).$$

Here  $\mu_Y \tilde{s} = \tilde{s}_1 \mu_X$ . In the category Pol - G this equality has the form

$$\mu_m s^{\alpha} = s_1^{\alpha} \mu_n.$$

3.5. Let us make some general remarks on the transition  $W \to \varphi(W)$  in arbitrary  $\Theta$ . Here,  $\varphi$  can change objects. Consider  $s: W(X) \to W(X)$ ,  $X = \{x_1, \ldots, x_n\}$  and present it as  $s = (s_1, \ldots, s_n)$ , where all  $s_i$ ,  $i = 1, \ldots, n$  are morphisms  $W(x) \to W(X)$ . Here,  $s_i$  are defined by the condition

$$s_i(x) = s(x_i) = w_i(x_1, \dots, x_n) = w_i.$$

The presentation  $s = (s_1, \ldots, s_n)$  depends on the basis X. We write  $s = (w_1, \ldots, w_n)$ . Consider an automorphism  $\varphi \colon \Theta^0 \to \Theta^0$ . What can be said about the equality

$$\varphi(s) = (\varphi(s_1), \dots, \varphi(s_n))?$$

We will see that application of  $\varphi$  preserves the corresponding presentation, but this is a presentation in some special base, connected with  $\varphi$ .

Consider a system of injections  $(\varepsilon_1, \ldots, \varepsilon_n)$ ,

$$\varepsilon_i:W(x)\to W(X).$$

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**Definition 4.** We say that  $(\varepsilon_1, \ldots, \varepsilon_n)$  freely defines an algebra W, if for arbitrary morphisms  $f_1, \ldots, f_n, f_i : W(x) \to W(X)$ , there exist a unique  $s : W(X) \to W(X)$ , such that  $f_i = s\varepsilon_i, i = 1, 2, \ldots, n$ .

**Proposition 2.** A collection  $(\varepsilon_1, \ldots, \varepsilon_n)$  freely defines an algebra W if and only if the elements  $\varepsilon_1(x), \ldots, \varepsilon_n(x)$  freely generate W.

Consider, further, automorphism  $\varphi$  of the category  $\Theta^0$  with the condition  $\varphi(W(x)) = W(y)$ .

**Proposition 3.** Let the set of morphisms  $(\varepsilon_1, \ldots, \varepsilon_n)$ ,  $\varepsilon_i: W(x) \to W(X)$  freely define  $W = W(X), X = \{x_1, \ldots, x_n\}$ . Then the set  $(\varphi(\varepsilon_1), \ldots, \varphi(\varepsilon_n))$ ,

$$\varphi(\varepsilon_i): \varphi(W(x)) = W(y) \to \varphi(W(X)) = W(Y)$$

freely defines W(Y).

**Definition 5.** A variety  $\Theta$  is called a regular variety, if for any free algebra W = W(X), |X| = n, every other system of free generators of W, also consists of n elements.

Now we can state that if  $W(Y) = \varphi(W(X))$ , then |Y| = |X| if  $\Theta$  is regular. Fix  $(\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_i(x) = x_i$ , take  $(\varphi(\varepsilon_1), \ldots, \varphi(\varepsilon_n))$  and  $y_1' = \varphi(\varepsilon_1)(y), \ldots, y_n' = \varphi(\varepsilon_n)(y), Y' = \{y_1', \ldots, y_n'\}$ .

**Proposition 4.** If an endomorphism  $s: W \to W$  in the basis  $X = \{x_1, \ldots, x_n\}$  has presentation  $s = (s_1, \ldots, s_n)$ , then in the base Y' we have the presentation  $\varphi(s) = (\varphi(s_1), \ldots, \varphi(s_n))$ .

A transition from the base X to another base determines an automorphism used in the proof of the Theorem 5.

3.6. Now we will formulate the Theorem 5.

**Definition 6.** A variety  $\Theta^G$  is called perfect if every automorphism  $\varphi : (\Theta^G)^0 \to (\Theta^G)^0$  is inner. If every  $\varphi$  is semi-inner, then  $\Theta^G$  is called semiperfect.

We consider conditions which provide  $\Theta^G$  to be perfect or semiperfect.

To every decomposition  $\varphi = \varphi^2 \varphi^1$  from the proposition 1 corresponds a decomposition  $\tau = \tau^2 \tau^1$ . The function  $\mu$  corresponds to  $\tau$  and we have a pair  $(\mu, \varphi)$ . There are the corresponding pairs for  $\tau^2$  and  $\tau^1$ . Then  $(\mu, \varphi) = (\mu^2 \mu^1, \varphi^2 \varphi^1)$  and  $(\mu^2 \mu^1)_W = \mu^2_{\varphi^1(W)} \mu^1_W$ . This decomposition rule is used in the proofs of the theorems.

Let an automorphism  $\varphi$  does not change the objects. For every W = W(X) the automorphism  $\varphi$  induces an automorphism of the semigroup EndW.

Take W = W(x),  $X = \{x\}$  and let  $\varphi_0$  be the corresponding induced automorphism of the semigroup End(W).

**Definition 7.** An automorphism  $\psi$  of the semigroup EndW is called semi-inner if  $\psi$  is connected with the diagram

$$G \xrightarrow{i_G} W(x) = G * W_0(x)$$

$$\downarrow s$$

$$G \xrightarrow{i_G} W(x) = G * W_0(x)$$

in such a way that for every endomorphism  $\nu:W(x)\to W(x)$ , the equality  $\varphi(\nu)=s\nu s^{-1}$  holds.

Here  $(\sigma, s)$  is a semiautomorphism of the algebra  $W = G * W_0$ . The same  $\psi$  is called inner if take  $\sigma = 1$ .

**Definition 8.** A semigroup EndW is called perfect if every its automorphism is inner. A semigroup EndW is called semiperfect if every its automorphism is semi-inner.

Let now  $\psi$  be an automorphism of the semigroup EndW where W=W(x). It corresponds an automorphism  $\bar{\psi}: (\Theta^G)^0 \to (\Theta^G)^0$ . This  $\bar{\psi}$  is constructed in such a way that if  $\psi$  inner or semi-inner, then  $\bar{\psi}$  is also inner or semi-inner.

Return to the initial  $\varphi: (\Theta^G)^0 \to (\Theta^G)^0$  and to the corresponding  $\varphi_0$ . Then  $\bar{\varphi}_0$  as well as  $\varphi$  preserves the objects. This  $\bar{\varphi}_0$  also induces  $\varphi_0$  in End(W).

Let  $\varphi = \varphi_1 \bar{\varphi}_0$ ,  $\varphi_1 = \varphi \cdot \bar{\varphi}_0^{-1}$ ,  $\varphi_1$  preserves the objects and, besides, does not change constant endomorphisms of the algebra W(x). Decomposition of  $\varphi$  gives rise to decomposition of  $\tau$ ,  $\tau = \tau_1 \tau_2$ , where  $\tau_2$  corresponds to the automorphism  $\bar{\varphi}_0$ . If now  $\mu$  is a function for  $\tau$ , then  $\mu = \mu^1 \mu^2$ ,  $\mu_W = \mu_W^1 \cdot \mu_W^2$ . Let now  $W_0 = W(x)$ . Then  $\mu_{W_0} = \mu_{W_0}^1 \cdot \mu_{W_0}^2$ . But  $\mu_{W_0} = \mu_{W_0}^2$  since  $\varphi$  and  $\bar{\varphi}_0$ ,  $\tau_1$  and  $\tau_2$  coincide on  $W_0$ . Therefore  $\mu_{W_0}^1 = 1$ . This precisely means that  $\varphi_1$  does not change constant endomorphisms of the algebra W(x). Applying this fact and Proposition 4 it can be proved that  $\varphi_1$  is an inner automorphism.

The main theorem here is the following

**Theorem 5.** If the semigroup EndW(x) is perfect in  $\Theta^G$ , then the variety  $\Theta^G$  is perfect too. If the semigroup EndW(x) is semiperfect then the variety  $\Theta^G$  is semiperfect.

The proof of this theorem uses various reductions, based on transitions from  $\varphi$  to  $\tau$ , from  $\tau$  to  $\tau^{\alpha}$ , and to the corresponding functions  $\mu$  and bijections of the form  $\mu_X$  and  $\mu_n$ .

Using Theorem 2 and the lecture 3, we have also

**Theorem 5'.** Let  $H_1$  and  $H_2$  be faithful G-algebras. Consider the categories  $K_{\Theta^G}(H_1)$  and  $K_{\Theta^G}(H_2)$ . Then,

- 1. If the semigroup EndW(x) is semiperfect then the categories are isomorphic if and only if  $H_1$  and  $H_2$  are geometrically equivalent up to a semiisomorphism.
- 2. If the semigroup EndW(x) is perfect then the isomorphism of categories coincides with geometrical equivalence of  $H_1$  and  $H_2$ .

Automorphisms of categories of free algebras of varieties are studied in the paper [MPP].

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#### 4. Other results.

A.Berzins proved [Be2] that

- 1. If P is an infinite field, and P[x] is algebra of polynomials with one variable x, then the semigroup EndP[x] is semiperfect, i.e. every its automorphism is semi-inner.
- 2. Let F be a free non-commutative group. The semigroup  $End(F*\{x\})$  is semiperfect.

So we have

**Theorem 6.** The variety Var - P is semiperfect. If the field P does not have automorphisms, then Var - P is perfect.

**Theorem 7.** The variety Grp - F is semiperfect.

And, further

- 1. If  $L_1$  and  $L_2$  are two extensions of the field P then the categories  $K_P(L_1)$  and  $K_P(L_2)$  are correctly isomorphic if and only if  $L_1$  and  $L_2$  are geometrically equivalent up to a semiisomorphism.
- 2. If  $H_1$  and  $H_2$  are two faithful F-groups then the categories  $K_{Grp-F}(H_1)$  and  $K_{Grp-F}(H_2)$  are isomorphic if and only if  $H_1$  and  $H_2$  are geometrically equivalent up to a semiisomorphism. It can be shown that in this case  $H_1$  and  $H_2$  are, in fact, equivalent.

#### 5. Problems.

**Problem 10.** What is the situation for Lie F-algebras, where F is a free Lie algebra.

**Problem 11**. What is the situation for associative F-algebras, where F is a free associative algebra or an infinite dimensional over its center skew field [Pl11].

It should be noted that Theorem 5 cannot be applied to associative algebras over a field, since a field P does not generate the whole variety of associative algebras over P and the condition (\*) does not fulfill.

This lecture concludes the part devoted to equational algebraic geometry. We present now some general view on the situation in this part.

First of all note that there are problems which relate to the universal theory. Some of them have been mentioned. However, the principal thing is to consider situations in the various special  $\Theta$  and special  $H \in \Theta$ .

The algebraic geometry in groups is on rise now, see [BMR], [KhM] [MR1], [MR2], [BMRo], [Se] and others. It is quite reasonable to expect the similar breakthrough in Lie algebras and semigroups.

For the case of associative algebras over a field or over on some other algebra of constants it is necessary to clarify how all this is connected with the theory which is used to call non-commutative algebraic geometry. In particular, it would be quite reasonable to compare the notions of noetherian variety of algebras and geometrically noetherian algebras with the notion of noetherian scheme in the non-commutative algebraic geometry.

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One has to distinguish also the cases of noetherian and non-noetherian non-commutative geometry. Algebraic set  $A \subset Hom(W,H)$  is called correct if for every system of equations T in W such that T' = A there exists a finite  $T_0 \subset T$ , such that  $T'_0 = A$ . In the opposite case the set A is called non-correct. A set A is called almost correct if A is non-correct but all its proper algebraic subsets are correct. If an algebra H is not geometrically noetherian, then there exist non-correct algebraic sets over H. Each non-correct algebraic set contains an almost correct subset. This follows from the following observation. Take in the lattice  $Alv_H(W)$  an infinite descending system  $A_\alpha$ ,  $\alpha \in I$ , consisting of non-correct algebraic sets. Denote  $A = \bigcap_{\alpha \in I} A_\alpha$ . Then A is also non-correct algebraic set. Indeed, take  $T_\alpha = A'_\alpha$ ,  $T'_\alpha = A_\alpha$ , and let T be  $\bigcup T_\alpha$ . Then T' = A and A' = T''. Suppose that the set A is correct and  $A = T'_0$  where  $T_0$  is a finite subset in T. Since  $T_0$  is a finite set,  $T_0$  is contained in some  $T_\alpha$ . Therefore,  $T''_0 = T'' \subset T_\alpha$ . This is impossible. Therefore, the set A is non-correct.

There is the following general problem. Which almost correct sets A arise for the given non-geometrically noetherian algebra H? This question , first of all, relates to associative non-commutative algebras over a field.

The next natural object is modules over rings.

Note that there are some results for algebraic geometry for group representations [KP].

There are many problems associated with the solution of equations in specific groups, say in  $GL_2(p)$ , and with the investigation of the corresponding algebraic sets.

Note one more problem.

Let K be a ring with the unity and Mod - K be the category of K-modules which is considered as a variety.

The notion of semiisomorphism makes sense also for this category.

**Problem 12**. For which K can be stated that all automorphisms of the category of free K-modules are semi-inner?<sup>5</sup>.

Is the theory similar to constructed one, possible in this case? The solution of this problem is connected with the solution of the problem of similarity for K-modules.

<sup>&</sup>lt;sup>5</sup>This is true if a ring K is left noetherian [MPP]

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## LECTURE 6

# ALGEBRAIC GEOMETRY IN FIRST ORDER LOGIC

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## Contents

1. Introduction

- 2. Algebraic logic
- 3. Elementary (algebraic) sets
- 4. Galois theory in logic
- 5. Lattices of elementary sets
- 6. Zariski topology
- 7. Geometrical properties

#### 1. Introduction.

#### 1.1. The initial idea

Equational algebraic geometry is a geometry whose algebraic sets are determined by the systems of equations of a special type:  $w \equiv w'$ . These are equalities in logic.

Here we proceed from arbitrary formulas of elementary  $\Theta$ -logic. A formula u is considered as an equation, and systems T of such generalized equations determine generalized algebraic sets. We call them *elementary sets*. The point of view on the Zariski topology, which is a main topology in such a geometry, is correspondingly changed.

For this new algebraic geometry we need a special category of formulas which takes the role that the category  $\Theta^0$  plays in the equational theory.

This category of formulas assumes the transition from logic to algebraic logic. The logic is built accordingly to some variety of algebras  $\Theta$ .

## 1.2. Algebra

Fix a variety  $\Theta$ . Now, keeping in mind applications from Lecture 7, we proceed from the situation when algebras are multy-sorted (not necessarily one-sorted). Fix for  $\Theta$  a set of sorts  $\Gamma$  which is now finite, but in general it may be infinite. Every algebra  $G \in \Theta$  is recorded as  $G = (G_i, i \in \Gamma)$ . Operation in the signature  $\Omega$  is a  $\Gamma$ -sorted one. For every  $\omega \in \Omega$  we have its type  $\tau = \tau(\omega) = (i_1, \ldots, i_n; j), \ i, j \in \Gamma$ . An operation  $\omega$  of the type  $\tau$  is a mapping  $\omega : G_{i_1} \times \cdots \times G_{i_n} \to G_j$ . All operations of the signature  $\Omega$  satisfy some set of identities. This fixes the variety  $\Theta$  of  $\Gamma$ -sorted  $\Omega$ -algebras. Let us switch to homomorphisms in  $\Theta$  and to free algebras. A homomorphism of algebras in  $\Theta$  has the form

$$\mu = (\mu_i, i \in \Gamma) : G = (G_i, i \in \Gamma) \to G' = (G'_i, i \in \Gamma).$$

Here  $\mu_i: G_i \to G_i'$  are mappings of sets, coordinated with operations in  $\Omega$ . A congruence  $Ker\mu = (\text{Ker }\mu_i, i \in \Gamma)$  is the kernel of a homomorphism  $\mu$ .

We consider multisorted sets  $X=(X_i,i\in\Gamma)$  and the corresponding free in  $\Theta$  algebras

$$W = W(X) = (W_i, i \in \Gamma).$$

A set X and a free algebra W can be presented as a free union of all  $X_i$  and all  $W_i$ , respectively.

Every (multisorted) mapping  $\mu: X \to G$  is extended up to a homomorphism  $\mu: W \to G$ . Denote the set of all such  $\mu$  by  $\operatorname{Hom}(W,G)$ . If all  $X_i$  are finite, we treat this set as an affine space. Homomorphisms  $\mu: W \to G$  are points of this space.

For the given  $G = (G_i, i \in \Gamma)$  and  $X = (X_i, i \in \Gamma)$  we can consider the set

$$G^X = (G_i^{X_i}, i \in \Gamma).$$

It is the set of mappings

$$\mu = (\mu_i, i \in \Gamma): X \to G.$$

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Then we have the natural bijection  $\operatorname{Hom}(W,G) \to G^X$ . More information about multisorted algebras can be found in [Pl1].

Now let us pass to the models. Fix some set of symbols of relations  $\Phi$ . Every  $\varphi \in \Phi$  has its type  $\tau = \tau(\varphi) = (i_1, \ldots, i_n)$ . A relation, corresponding to  $\varphi$ , is a subset in the Cartesian product  $G_{i_1} \times \ldots \times G_{i_n}$ . Now,  $\Phi\Theta$  denotes the class of all models  $(G, \Phi, f)$ , where  $G \in \Theta$ , and f is a realization of the set  $\Phi$  in G. As for homomorphisms of models, they are homomorphisms of the corresponding algebras which are coordinated with relations.

#### 1.3 Logic.

We consider logic in the given variety  $\Theta$ . For every finite X it is determined a logical signature

$$L = L_X = \{ \lor, \land, \neg, \exists x, \ x \in X \},\$$

where X is  $\bigcup_{i\in\Gamma} X_i$  for a finite  $\Gamma$ . We consider a set (more precisely, an L-algebra) of formulas  $L\Phi W$  over the free algebra W=W(X). This algebra is an L-algebra of formulas of FOL over the given  $\Theta$  and  $\Phi$  and for the given X.

First we define the atomic formulas. They are equalities of the form  $w \equiv w'$  with  $w, w' \in W$  of the same sort and the formulas  $\varphi(w_1, \ldots, w_n)$ , where  $w_i \in W$  and all  $w_i$  are positioned according to the type  $\tau = \tau(\varphi)$  of the relations  $\varphi$  and to the sorts. The set of all atomic formulas we denote by  $M = M_X$ . Define  $L\Phi W$  as the absolutely free  $L_X$ -algebra over  $M_X$ .

Let us consider another example of an  $L_X$ -algebra.

Given W = W(X) and  $G \in \Theta$ , as before, we denote by  $\operatorname{Bool}(W, G)$  the Boolean algebra  $\operatorname{Sub}(\operatorname{Hom}(W,G))$  of all subsets in  $\operatorname{Hom}(W,G)$ . We define also the action of quantifiers in  $\operatorname{Bool}(W,G)$ . Let A be a subset in  $\operatorname{Hom}(W,G)$  and  $x \in X_i$  be a variable of the sort i. Then  $\mu: W \to G$  belongs to  $\exists xA$  if there exists  $\nu: W \to G$  in A such that  $\mu(y) = \nu(y)$  for every  $y \in X$  of the sort  $j, j \neq i$ , and for every  $y \in X_i$ ,  $y \neq x$ . Thus we get an L-algebra  $\operatorname{Bool}(W,G)$ .

Now let us define a mapping

$$\operatorname{Val}_f^X : M_X \to \operatorname{Bool}(W, G),$$

where f is a model which realizes the set  $\Phi$  in the given G. If  $w \equiv w'$  is an equality of the sort i, then we set:

$$\mu: W \to G \in \operatorname{Val}_f^X(w \equiv w') = \operatorname{Val}^X(w \equiv w')$$

if  $\mu_i(w) = \mu_i(w')$  in G. Here the point  $\mu$  is a solution of the equation  $w \equiv w'$ . If the formula is of the form  $\varphi(w_1, \ldots, w_n)$ , then

$$\mu \in \operatorname{Val}_f^X(\varphi(w_1,\ldots,w_n))$$

if  $\varphi(\mu(w_1), \ldots, \mu(w_n))$  is valid in the model  $(G, \Phi, f)$ . Here  $\mu(w_j) = \mu_{i_j}(w_j)$ ,  $i_j$  is the sort of  $w_j$ . The mapping  $\operatorname{Val}_f^X$  is uniquely extended up to the *L*-homomorphism

$$\operatorname{Val}_f^X : L\Phi W \to \operatorname{Bool}(W, G).$$

Thus, for every formula  $u \in L\Phi W$  we have its value  $\operatorname{Val}_f(u)$  in the model  $(G, \Phi, f)$ , which is an element in  $\operatorname{Bool}(W, G)$ .

Every formula  $u \in L\Phi W$  can be viewed as an equation in the given model. The point  $\mu: W \to G$  is the solution of the "equation" u if  $\mu \in \operatorname{Val}_f(u)$ .

#### 1.4 Geometrical Aspect.

In the L-algebra of formulas  $L\Phi W, W=W(X)$ , we consider its various subsets T, i.e., sets of formulas. We regard T also as a system of equations. On the other hand, we consider subsets A in the affine space  $\operatorname{Hom}(W,G)$ , i.e., elements of the L-algebra  $\operatorname{Bool}(W,G)$ . For each given model  $(G,\Phi,f)$  and for these T and A we establish the following Galois correspondence:

$$T^f = A = \bigcap_{u \in T} \operatorname{Val}_f(u)$$
  
 $A^f = T = \{u | A \subset \operatorname{Val}_f(u)\}.$ 

Here  $A = T^f$  is a locus of all points satisfying the system of equations  $u \in T$ . Every set A of such kind is said to be an *algebraic set* (or closed set, or elementary set), determined for the given model.

The set A can be treated also as a relation between elements of G, derived from equalities and relations of the basic set  $\Phi$ . The relation  $A = T^f$  belongs to the multisorted set

$$G^X = \{G_i^{X_i}, \ i \in \Gamma\}.$$

The set T of the form  $T=A^f$  for some A is an f-closed set. For an arbitrary T we have its closure  $T^{ff}=(T^f)^f$  and for every  $A\subset \operatorname{Hom}(W,G)$  we have the closure  $A^{ff}=(A^f)^f$ .

It is easy to understand that the following rule takes place:

The formula v belongs to the set  $T^{ff}$  if and only if the formula

$$(\bigwedge_{u \in T} u) \to v$$

holds in the model  $(G, \Phi, f)$ .

If the set T is infinite then the corresponding formula is infinitary.

Free in  $\Theta$  algebras W(X) with finite X are the objects of the category, denoted by  $\Theta^0$ . Morphisms of this category  $s: W(X) \to W(Y)$  are arbitrary homomorphisms of algebras. The category  $\Theta^0$  is a full subcategory in the category  $\Theta$ .

Basing on the first order logic in the given  $\Theta$ , we intend to build a category which is similar, in a sense, to the category  $\Theta^0$  in the equational logic. Thus we pass from pure logic to the algebraic logic. The sets of the type  $T=A^f$  look here more attractive.

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#### 2. Algebraic logic

#### 2.1 The main idea.

Algebraic logic deals with algebraic structures, related to various logical structures, i.e., with logical calculi. Boolean algebras relate to classical propositional logic, Heyting algebras relate to non-classical propositional logic, Tarski cylindric algebras and Halmos polyadic algebras relate to FOL.

Every logical calculus assumes a set of formulas of the calculus, axioms of logic and rules of inference. On this basis the syntactical equivalence of formulas, well correlated with their semantical equivalence, is defined. The transition from pure logic to the algebraic logic is based on treating logical formulas up to a certain equivalence, i.e., squeezed formulas. Such transition leads to various special algebraic structures, in particular to the structures mentioned above.

As for logical calculi, usually they are associated with some infinite sets of variables. Denote such a set by  $X^0$ . In our situation it is a multisorted set  $X^0 = (X_i^0, i \in \Gamma)$ . Keeping in mind algebraic geometry in logic, knowledge theory and its geometrical aspect we will use a system of all finite subsets  $X = (X_i, i \in \Gamma)$  of  $X^0$  instead of this infinite universum. This leads to multisorted logic and multisorted algebraic logic. Every formula has a definite type (sort) X. Denote the new set of sorts by  $\Gamma^0$ . It is a set of all finite subsets of the initial set  $X^0$ .

## 2.2 Halmos Categories.

Fix some variety of algebras  $\Theta$ . This means that a finite set of sorts  $\Gamma$ , a signature  $\Omega = \Omega(\Theta)$  connected with  $\Gamma$ , and a system of identities  $Id(\Theta)$  are given.

Define Halmos categories for the given  $\Theta$ .

First, for the given Boolean algebra B we define its existential quantifiers [HMT]. Existential quantifiers are the mappings  $\exists: B \to B$  with the conditions:

- 1)  $\exists 0 = 0,$
- 2)  $a < \exists a,$
- 3)  $\exists (a \land \exists b) = \exists a \land \exists b, \ 0, a, b \in B.$

The universal quantifier  $\forall : B \to B$  is defined dually:

- 1)  $\forall 1 = 1,$
- $2) \quad a > \forall a,$
- 3)  $\forall (a \lor \forall b) = \forall a \lor \forall b$ .

Let B be a Boolean algebra and X a set. We say that B is a quantifier X-algebra if a quantifier  $\exists x : B \to B$  is defined for every  $x \in X$  and for every two elements  $x, y \in X$  the equality  $\exists x \exists y = \exists y \exists x$  holds true.

One may consider also quantifier X-algebras B with equalities over W(X). In such algebras to each pair of elements  $w, w' \in W(X)$  of the same sort it corresponds an element  $w \equiv w' \in B$  satisfying the conditions

- 1)  $w \equiv w$  is the unit in B
- 2)  $(w_1 \equiv w'_1 \wedge \ldots \wedge w_n \equiv w'_n) < (w_1 \ldots w_n \omega \equiv w'_1 \ldots w'_n \omega)$  where  $\omega$  is an operation in  $\Omega$  and everything is coordinated with the type of operation.

Now we will give a general definition of the Halmos category for the given  $\Theta$ , which will be followed by examples.

A Halmos category H for an arbitrary finite  $X = (X_i, i \in \Gamma)$  fixes some quantifier X-algebra H(X) with X-equalities. H(X) is an object in H.

The morphisms in H correspond to morphisms in the category  $\Theta^0$ . Every morphism  $s_*$  in H has the form

$$s_*: H(X) \to H(Y),$$

where  $s: W(X) \to W(Y)$  is a morphism in  $\Theta^0$ . We identify  $s_*$  and s.

We assume that

- 1) The transitions  $W(X) \to H(X)$  and  $s \to s_*$  constitute a (covariant) functor  $\Theta^0 \to H$ .
  - 2) Every  $s_*: H(X) \to H(Y)$  is a Boolean homomorphism.
  - 3) The coordination with the quantifiers is as follows:
    - 3.1)  $s_1 \exists x a = s_2 \exists x a, \quad a \in H(X), \text{ if } s_1 y = s_2 y \text{ for every } y \in X, y \neq x.$
- 3.2)  $s\exists xa=\exists (sx)(sa)$  if  $sx=y\in Y$  and y=sx not in the support of  $sx',\,x'\in X,\,x'\neq x.$ 
  - 4) The following conditions describe coordination with equalities
- 4.1)  $s_*(w \equiv w') = (sw \equiv sw')$  for  $s: W(X) \to W(Y), w, w' \in W(X)$  are of the same sort.
- 4.2)  $s_w^x a \wedge (w \equiv w') < s_{w'}^x a$  for an arbitrary  $a \in H(X), x \in X, w, w'$  of the same sort with x in W(X), and  $s_w^x \colon W(X) \to W(X)$  is defined by the rule:  $s_w^x(x) = w, sy = y, y \in X, y \neq x$ .

So, the definition of Halmos category is given.

#### **2.3** The example $\operatorname{Hal}_{\Theta}(G)$ .

Fix an algebra G in the variety  $\Theta$ . Define the Halmos category  $\operatorname{Hal}_{\Theta}(G)$  for the given G. Take a finite set X and consider the space  $\operatorname{Hom}(W(X),G)$ . We have defined the action of quantifiers  $\exists x$  for all  $x \in X$  in the Boolean algebra  $\operatorname{Bool}(W(X),G)$ . The equality  $w \equiv w'$  in  $\operatorname{Bool}(W(X),G)$  is defined as a diagonal, coinciding with the set of all  $\mu:W(X)\to G$  for which  $w^{\mu}=w'^{\mu}$  holds true. It is easy to check that in this case the algebra  $\operatorname{Bool}(W(X),G)$  turns to be a quantifier X-algebra with equalities. We set

$$\operatorname{Hal}_{\Theta}(G)(X) = \operatorname{Bool}(W(X), G).$$

Let now  $s: W(X) \to W(Y)$  be given in  $\Theta^0$ . We have:

$$\tilde{s}$$
: Hom $(W(Y), G) \to \text{Hom}(W(X), G)$ 

defined by  $\tilde{s}(\nu) = \nu s$  for any arbitrary  $\nu: W(Y) \to G$ .

Now, if A is a subset in  $\operatorname{Hom}(W(X),G)$ , then  $\nu \in s_*A = sA$  if and only if  $\tilde{s}(\nu) = \nu s \in A$ .

We have a mapping:

$$s_*: \operatorname{Bool}(W(X), G) \to \operatorname{Bool}(W(Y), G)$$

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which is a Boolean homomorphism. One can also check that  $s_*$  satisfies the conditions 3–4. Thus, the Halmos category  $\text{Hal}_{\Theta}(G)$  is defined.

Note that to each  $s_*$  it corresponds a conjugate mapping

$$s^* : \operatorname{Bool}(W(Y), G) \to \operatorname{Bool}(W(X), G),$$

where the set  $s^*B$  is the  $\tilde{s}$ -image of the set B for every  $B \subset \text{Hom}(W(Y), G)$ .

Here,  $s^*$  is not a boolean homomorphism, but it preserves sums and zero.

It may be seen [Pl1] that such conjugate mapping can be defined in any Halmos category. Note the obvious relation between the categories  $Hal_{\Theta}(G)$  and  $Bool_{\Theta}(G)$ .

## 2.4 Multisorted Halmos algebras.

Fix some infinite set  $X^0 = (X_i^0, i \in \Gamma)$  and let  $\Gamma^0$  be the set of all finite subsets  $X = (X_i, i \in \Gamma)$  in  $X^0$ . In this section multisorted algebra means  $\Gamma^0$ -sorted. Every such algebra is of the form  $H = (H(X), X \in \Gamma^0)$ .

A few words about the signature of the algebras to be constructed. First, the signature includes  $L_X$  for every X together with equalities ( $w \equiv w', w, w'$  of one sort in W(X)) as nullary operations. This is the signature in H(X). Second, we consider symbols of operations of the type  $s: W(X) \to W(Y)$ . To each such symbol corresponds an unary operation  $s: H(X) \to H(Y)$ . Denote this signature of all  $L_X$ , all equalities, and all  $s: W(X) \to W(Y)$  by  $L_{\Theta}$ . This is the signature of FOL in  $\Theta$  in the multisorted variant.

Consider further the variety of  $\Gamma^0$ -sorted  $L_{\Theta}$ -algebras, denoted by  $\operatorname{Hal}_{\Theta}$ . The identities of this variety exactly copy the definition of Halmos category. We call algebras from  $\operatorname{Hal}_{\Theta}$  multisorted Halmos algebras.

Every such algebra can be considered as a small Halmos category.

#### 2.5 Algebras of formulas.

First consider a multisorted set of atomic formulas  $M = (M(X), X \in \Gamma^0)$ , with  $M(X) = M_X$  defined as above. All  $w \equiv w'$  are viewed as symbols of nullary operations-equalities. The set of symbols of relations  $\Phi$  is fixed.

Denote by  $H_{\Phi\Theta} = (H_{\Phi\Theta}(X), X \in \Gamma^0)$  an absolutely free  $L_{\Theta}$ -algebra over the set M. This is the algebra of formulas of pure FOL in the given  $\Theta$ .

Now denote by  $\tilde{H}_{\Phi\Theta}$  the result of factorization of the algebra  $H_{\Phi\Theta}$  by the identities of the variety  $\operatorname{Hal}_{\Theta}$ . It is a free Halmos algebra over the set of atomic formulas M.

Let us introduce the following defining relations:

$$(*) s_*\varphi(w_1,\ldots,w_n) = \varphi(sw_1,\ldots,sw_n)$$

for all  $s: W(X) \to W(Y)$  and all formulas of the type  $\varphi(w_1, \ldots, w_n)$  in M(X).

In the sequel the principal role will play the Halmos algebra  $\operatorname{Hal}_{\Theta}(\Phi) = \operatorname{Hal}_{\Phi\Theta}$ , defined as a factor algebra of the free algebra  $\tilde{H}_{\Phi\Theta}$  by the relations of the (\*) type. Elements of this algebra are defined as squeezed formulas.

Consider now values of formulas. First of all consider a mapping

$$\operatorname{Val}_f = (\operatorname{Val}_f^X, X \in \Gamma^0) \colon M \to \operatorname{Hal}_{\Theta}(G)$$

for the model  $(G, \Phi, f)$ . Here the mappings  $\operatorname{Val}_f^X : M_X \to \operatorname{Bool}(W(X), G) = \operatorname{Hal}_{\Theta}(G)(X)$  have been defined.

This mapping is uniquely extended up to the homomorphisms

$$\operatorname{Val}_f: H_{\Phi\Theta} \to \operatorname{Hal}_{\Theta}(G),$$
  
 $\operatorname{Val}_f: \tilde{H}_{\Phi\Theta} \to \operatorname{Hal}_{\Theta}(G).$ 

Note that the relations (\*) hold in every algebra  $\operatorname{Hal}_{\Theta}(G)$  and this gives a canonical homomorphism of Halmos algebras

$$Val_f: \operatorname{Hal}_{\Theta}(\Phi) \to \operatorname{Hal}_{\Theta}(G).$$

It determines the value of the formulas  $\operatorname{Val}_f(u)$  (pure and squeezed) in the given model  $(G, \Phi, f)$ .

It is easy to see that the kernel  $Ker(Val_f)$  is precisely the elementary theory of the model  $(G, \Phi, f)$  in the logic of the variety  $\Theta$ .

In fact, elementary theory of the algebra G or the model  $(G, \Phi, f)$  is considered also on the logic of the variety  $\Theta^G$ . This logic is more reach in respect to the given G.

We call two pure formulas u and v of the given type X semantically equivalent, if  $\operatorname{Val}_f(u) = \operatorname{Val}_f(v)$  for every model  $(G, \Phi, f)$ .

The following main theorem takes place:

**Theorem 1.** Two formulas u and v are semantically equivalent if and only if the corresponding squeezed formulas  $\overline{u}$  and  $\overline{v}$  coincide in the algebra  $\operatorname{Hal}_{\Theta}(\Phi)$ .

This theorem explains the role of algebra  $\operatorname{Hal}_{\Theta}(\Phi)$  as a main structure of the multisorted algebraic logic for FOL in the given  $\Theta$ . The same algebra plays the essential part in the algebraic geometry in the FOL in  $\Theta$ . Besides that, the role of the algebras  $\operatorname{Hal}_{\Theta}(G)$  is underlined by the following theorem:

**Theorem 2.** All algebras  $\operatorname{Hal}_{\Theta}(G)$  over different  $G \in \Theta$  generate the variety of Halmos algebras  $\operatorname{Hal}_{\Theta}$ .

Define the notion of the logical kernel of a homomorphism.

Let the homomorphism  $\mu: W(X) \to G$  be given. One can view its kernel Ker  $\mu$  as a system of all formulas  $w \equiv w'$  with w, w' of the same sort in W(X), for which  $\mu \in \operatorname{Val}(w \equiv w')$ .

The logical kernel Log Ker  $\mu$  naturally generates the standard Ker  $\mu$ . We set: the formula  $u \in \operatorname{Hal}_{\Phi\Theta}(X)$  belongs to Log Ker( $\mu$ ) if the point  $\mu$  lies in  $\operatorname{Val}_f(u)$ , i.e.,  $\mu$  is a solution of the "equation" u in the given model  $(G, \Phi, f)$ . It is easy to understand, that for every point  $\mu$  its logical kernel is an ultrafilter of the Boolean algebra  $\operatorname{Hal}_{\Phi\Theta}(X)$ . It is also clear, that the kernel Ker  $\mu$  is the set of all equalities in the logical kernel.

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## 3. Elementary (algebraic) sets

#### 3.1 Preliminary remarks.

In the sequel we call the sets below algebraic sets although it would be more sensible to call them elementary sets.

Algebraic sets are the sets, determined by FOL formulas. We work with squeezed formulas, i.e., formulas of the algebra  $\operatorname{Hal}_{\Theta}(\Phi) = \operatorname{Hal}_{\Phi\Theta}$ .

For the given place X consider sets of formulas T in  $\operatorname{Hal}_{\Phi\Theta}(X)$  and the sets of points A in the space  $\operatorname{Hom}(W(X), G)$ . We establish a Galois correspondence, determined by the given model  $(G, \Phi, f)$ . It looks like

$$T^f = A = \bigcap_{u \in T} \operatorname{Val}_f(u) = \{\mu | T \subset \operatorname{Log} \operatorname{Ker}(\mu)\}$$
$$A^f = T = \{u | A \subset \operatorname{Val}_f(u)\} = \bigcap_{\mu \in A} \operatorname{Log} \operatorname{Ker}(\mu).$$

As above, we call a set A represented as  $A = T^f$  an algebraic set or elementary set for the given model  $(G, \Phi, f)$ .

The set T, represented as  $A^f = T$ , is always a filter of Boolean algebra  $\operatorname{Hal}_{\Phi\Theta}(X)$ , since by the definition it is an intersection of ultrafilters. We call it f-closed filter. If A is an algebraic set then  $T = A^f$  can be considered also as the elementary theory of the given A. One can consider here the Boolean algebra  $\operatorname{Hal}_{\Phi\Theta}(X)/T$  for this T. If  $T^f = A$  and  $A^f = T$ , then the algebra  $\operatorname{Hal}_{\Phi\Theta}(X)/T$  is considered as an invariant of the algebraic set A. This invariant is a coordinate algebra of the set A. It can be represented as an algebra of regular functions determined on A (see [P12]).

Every algebraic set, defined in Section 1, is also an algebraic set according to this new definition. The opposite is not true, because in the new variant additional operations of the type  $s: W(X) \to W(Y)$  are involved in the formulas.

Consider now the relations between Galois correspondence and morphisms of Halmos categories.

For every  $s: W(X) \to W(Y)$  and every A of the type X we considered a set  $B = s_*A$  of the type Y. If B is of the type Y, then  $A = s^*B$  is of the type X. Define the operations  $s_*$  and  $s^*$  on the sets of formulas.

If T is a set of formulas in  $\operatorname{Hal}_{\Phi\Theta}(Y)$ , then  $s_*T$  is a set of formulas in  $\operatorname{Hal}_{\Phi\Theta}(X)$  defined by the rule:

$$u \in s_*T \Leftrightarrow su \in T$$
.

If T is a set of formulas in  $\operatorname{Hal}_{\Phi\Theta}(X)$ , then  $s^*T$  is contained in  $\operatorname{Hal}_{\Phi\Theta}(Y)$  and it is defined by

$$u \in s^*T$$
 if  $u = sv$ ,  $v \in T$ .

The following theorem takes place:

## Theorem 3.

1. If T lies in  $\operatorname{Hal}_{\Phi\Theta}(X)$ , then

$$(s^*T)^f = s_*T^f = sT^f.$$

2. If  $B \subset \text{Hom}(W(Y), G)$ , then

$$(s^*B)^f = s_*B^f.$$

3. If  $A \subset \text{Hom}(W(X), G)$ , then  $s^*A^f \subset (s_*A)^f$ .

It follows from these rules that

- 1. If  $A = T^f$  is an algebraic set, then sA is also an algebraic set.
- 2. If  $T = B^f$  is f-closed, then  $sT = s_*T$  is f-closed.

## **3.2.** Categories $K_{\Phi\Theta}(f)$ and $C_{\Phi\Theta}(f)$ .

Fix a model  $(G, \Phi, f)$  and define a category of algebraic sets  $K_{\Phi\Theta}(f)$  for this model. Objects of this category have the form (X, A), where  $A = T^f$  for some T. X is the place for both A and T.

Let us now define morphisms  $(X,A) \to (Y,B)$ . Proceed from  $s: W(Y) \to W(X)$ . We say that s is admissible for A and B if  $\tilde{s}(\nu) = \nu s \in B$  for any  $\nu \in A$ . It is clear that s is admissible for A and B if  $A \subset sB$ . A mapping  $[s]: A \to B$  corresponds to each s admissible for A and B.

We consider weak and exact categories  $K_{\Phi\Theta}(f)$ . In the first one the morphisms look like  $s:(X,A)\to (Y,B)$ , while in the second one like  $[s]:(X,A)\to (Y,B)$ .

If  $s_1$  is admissible for A and B and  $s_2$  for B and C, then  $A \subset s_1B$ ,  $B \subset s_2C$ ,  $s_1B \subset s_1s_2C$ , and  $s_1s_2$  is admissible for A and C.

Define now a category  $C_{\Phi\Theta}(f)$ . Its objects are Boolean algebras of the type  $\operatorname{Hal}_{\Phi\Theta}(X)/T$ , where  $T=A^f$  for some A.

Consider morphisms

$$\operatorname{Hal}_{\Phi\Theta}(Y)/T_2 \stackrel{\overline{s}}{\longrightarrow} \operatorname{Hal}_{\Phi\Theta}(X)/T_1.$$

We proceed here from  $s: W(Y) \to W(X)$  and pass to the new  $s: \operatorname{Hal}_{\Phi\Theta}(Y) \to \operatorname{Hal}_{\Phi\Theta}(X)$ . Assume that  $su \in T_1$  for every  $u \in T_2$ . The homomorphism s is admissible for  $T_2$  and  $T_1$  in this sense. Define homomorphisms  $\overline{s}$  for such s. This defines morphisms in  $C_{\Phi\Theta}(f)$ .

The next two propositions determine the correspondence between the categories  $K_{\Phi\Theta}(f)$  and  $C_{\Phi\Theta}(f)$ .

**Proposition 1.** Homomorphism  $s: W(Y) \to W(X)$  is admissible for varieties (X, A) and (Y, B) if and only if this s is admissible for  $T_2 = B^f$  and  $T_1 = A^f$ .

**Proposition 2.** If  $s_1, s_2: W(Y) \to W(X)$  are admissible for A and B, then  $[s_1] = [s_2]$  implies  $\overline{s_1} = \overline{s_2}$ .

It follows from these two propositions that the transition

$$(X,A) \to \operatorname{Hal}_{\Phi\Theta}(X)/A^f$$

determines a contravariant functor

$$K_{\Phi\Theta}(f) \to C_{\Phi\Theta}(f)$$

for weak and exact categories  $K_{\Phi\Theta}(f)$ . Duality takes place under some additional conditions. In particular, this is the case when we proceed from the variety  $\Theta^G$  for the given G in  $\Theta$ .

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## **3.3** Categories $K_{\Phi\Theta}$ and $C_{\Phi\Theta}$ .

In the categories  $K_{\Phi\Theta}$  and  $C_{\Phi\Theta}$  the model  $(G, \Phi, f)$  is not fixed. Objects of  $K_{\Phi\Theta}$  have the form (X, A; G, f). Here f is a realization of the set  $\Phi$ , fixed for the category  $K_{\Phi\Theta}$ , in the algebra G, and  $A = T^f$  for some  $T \subset \operatorname{Hal}_{\Phi\Theta}(X)$ .

Define morphisms

$$(X, A; G_1, f_1) \rightarrow (Y, B; G_2, f_2).$$

They act on all components of the objects. Proceed from the commutative diagram

$$W(Y) \xrightarrow{s} W(X)$$

$$\downarrow^{\nu} \qquad \qquad \downarrow^{\nu}$$

$$G_2 \xrightarrow{\delta} G_1$$

Consider a pair  $(s, \delta)$  and write  $(s, \delta)(\nu) = \nu' = \delta \nu s$ .

Let now  $A = T_1^{f_1}$  be of the type X and  $B = T_2^{f_2}$  of the type Y. We say that the pair  $(s, \delta)$  is admissible for A and B if  $(s, \delta)(\nu) \in B$  for every  $\nu \in A$ .

We need some further auxiliary remarks. For every  $\delta: G_1 \to G_2$  and every X we have a mapping

$$\tilde{\delta}: \operatorname{Hom}(W(X), G_1) \to \operatorname{Hom}(W(X), G_2)$$

defined by the rule

$$\tilde{\delta}(\nu) = \delta \nu, \nu \in \text{Hom}(W(X), G_1).$$

Determine  $\delta_*A \subset \operatorname{Hom}(W(X), G_1)$  for every  $A \subset \operatorname{Hom}(W(X), G_2)$ , setting

$$\nu \in \delta_* A$$
 if  $\delta \nu = \tilde{\delta}(\nu) \in A$ .

We change notations to  $\delta_* A = \delta A$ .

If  $A \subset \operatorname{Hom}(W(X), G_1)$ , then  $\delta^*A \subset \operatorname{Hom}(W(X), G_2)$  and  $\nu \in \delta^*A$  if  $\nu = \delta\nu_1, \nu_1 \in A$ .

Now we can say that the pair  $(s, \delta)$  is admissible for A and B if  $\delta^*A \subset sB$ , or, the same,  $A \subset \delta sB = s\delta B$ .

We have morphisms

$$(s,\delta):(X,A;G_1,f_1)\to (Y,B;G_2,f_2)$$

and

$$([s], \delta): (X, A; G_1, f_1) \to (Y, B; G_2, f_2)$$

for the admissible  $(s, \delta)$ . Here  $[s]: A \to B$  is a mapping, induced by the pair  $(s, \delta)$ . We get weak and exact categories  $K_{\Phi\Theta}$ . It can be proven that the pair  $(s, \delta)$  is admissible for A and B if and only if the homomorphism  $s: \operatorname{Hal}_{\Phi\Theta}(Y) \to \operatorname{Hal}_{\Phi\Theta}(X)$  is admissible in respect to  $T_2 = B^{f_2}$  and  $T_1 = (\delta^* A)^{f_2}$ . This leads to a natural definition of the category  $C_{\Phi\Theta}$  with contravariant functor  $K_{\Phi\Theta} \to C_{\Phi\Theta}$ .

Let us define the categories  $K_{\Phi\Theta}(G)$  and  $C_{\Phi\Theta}(G)$ . Here G is a fixed algebra in  $\Theta$ , while the realizations f of the set  $\Phi$  in G change.

The objects in  $K_{\Phi\Theta}(G)$  have the form

The morphisms

$$(X, A, f_1) \to (Y, B, f_2)$$

are defined according to the general definition of the morphisms in  $K_{\Phi\Theta}$  with identical  $\delta = \varepsilon : G \to G$ .

Objects in  $C_{\Phi\Theta}(G)$  have the form

$$(\operatorname{Hal}_{\Phi\Theta}(X)/T, f)$$
, where  $T = A^f$ 

for some A of the type X.

The transition

$$(X, A; f) \to (\operatorname{Hal}_{\Phi\Theta}(X)/A^f, f)$$

determines the functor  $K_{\Phi\Theta}(G) \to C_{\Phi\Theta}(G)$ . Here  $K_{\Phi\Theta}(G)$  is a subcategory in  $K_{\Phi\Theta}$  and every  $K_{\Phi\Theta}(f)$  is a subcategory in  $K_{\Phi\Theta}(G)$ . The same for C.

#### 4. Galois theory in logic.

Galois theory we are talking about is tied in with the considered algebraic geometry and, besides, it will be used in the next lecture.

## 4.1 Automorphisms.

Let  $\delta: G \to G$  be an automorphism of the algebra  $G, \delta \in AutG$ . Then for every X we have a substitution

$$\tilde{\delta}: \operatorname{Hom}(W(X), G) \to \operatorname{Hom}(W(X), G).$$

This substitution induces an automorphism of the Boolean algebra Bool(W(X), G). For each element A from this Boolean algebra we have  $\delta_*A = \delta A$ .

Actually, it is an automorphism of the quantifier X-algebra with equalities. This leads to the automorphism  $\sigma = \delta_*$  of the Halmos algebra  $\operatorname{Hal}_{\Theta}(G)$ . We have a representation

$$Aut(G) \to Aut(\operatorname{Hal}_{\Theta}(G)).$$

**Theorem 4.** The given representation is an isomorphism of the groups of automorphisms Aut(G) and  $Aut(Hal_{\Theta}(G))$ .

Thus, the group of automorphisms AutG can be considered as a group of automorphisms of the algebra  $\operatorname{Hal}_{\Theta}(G)$ .

Let, further,  $\delta: G_2 \to G_1$  be an isomorphism of algebras in  $\Theta$ . As above, we may take the isomorphism of Halmos algebras

$$\delta_* : \operatorname{Hal}_{\Theta}(G_1) \to \operatorname{Hal}_{\Theta}(G_2).$$

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This isomorphism is well correlated with the homomorphisms of the type  $Val_f$ .

Let the model  $(G_2, \Phi, f)$  be given. The isomorphism  $\delta: G_2 \to G_1$  uniquely determines the model  $(G_1, \Phi, f^{\delta})$ , isomorphic to the initial model.

We have the commutative diagram

$$\operatorname{Hal}_{\Theta}(\Phi)$$

$$\operatorname{Val}_{f^{\delta}} \longrightarrow \operatorname{Hal}_{\Theta}(G_{2})$$

$$\operatorname{Hal}_{\Theta}(G_{1}) \xrightarrow{\delta_{*}} \operatorname{Hal}_{\Theta}(G_{2})$$

The arrow  $\operatorname{Val}_{f^{\delta}}$  is uniquely determined by the other two arrows due to the diagram commutativity.

## 4.2 The main theorems of Galois theory.

Consider the algebra  $\operatorname{Hal}_{\Theta}(G)$  together with its group of automorphisms  $\operatorname{Aut}(G)$ . Let us define the standard Galois correspondence.

Let H be a subset of Aut(G). Then H'=R is a subalgebra of  $Hal_{\Theta}(G)$  (with equalities), consisting of all A with  $\delta A = A$  for every  $\delta \in H$ .

Let R be a subset in  $\operatorname{Hal}_{\Theta}(G)$ . Then R' = H is a subgroup in  $\operatorname{Aut}(G)$  consisting of all  $\delta$  with  $\delta A = A$  for every  $A \in R$ .

Define closures R'' and H''.

**Theorem 5.** If algebra G is finite, then every subalgebra in  $Hal_{\Theta}(G)$  and every subgroup in Aut(G) are closed. There is one-to-one correspondence between them.

Let now two finite algebras,  $G_1$  and  $G_2$ , be given. Take a subalgebra  $R_1$  in  $\operatorname{Hal}_{\Theta}(G_1)$  and a subalgebra  $R_2$  in  $\operatorname{Hal}_{\Theta}(G_2)$ .

**Theorem 6.** If there is an isomorphism  $\gamma: R_1 \to R_2$ , then there exists an isomorphism  $\delta: G_2 \to G_1$ , such that  $\delta_*: \operatorname{Hal}_{\Theta}(G_1) \to \operatorname{Hal}_{\Theta}(G_2)$  induces the given  $\gamma: R_1 \to R_2$ .

Here.

1. 
$$R_2 = \delta_*(R_1),$$
  
2.  $R'_1 = \delta R'_2 \delta^{-1}$ 

2. 
$$R_1' = \delta R_2' \delta^{-1}$$

 $R'_1$  and  $R'_2$  determine  $R_1$  and  $R_2$  with  $R_2 = \delta_*(R_1)$ .

Note that the considered Galois theory goes back to the papers by M.I.Krasner [KR] and it would be reasonable to call it the Galois-Krasner theory.

#### 4.3 Automorphisms and algebraic sets.

Denote a group of automorphisms of the model  $(G, \Phi, f)$  by Aut(f). It is a subgroup of Aut(G).

**Theorem 7.** Every algebraic set  $A = T^f$  is invariant under the action of the group  $\operatorname{Aut}(f)$ . If  $A = T^f$  and  $\delta \in \operatorname{Aut}(f)$ , then  $\delta A = A$ .

This theorem to some extent determines the group structure of algebraic sets. Consider  $\operatorname{Val}_f: \operatorname{Hal}_{\Theta}(\Phi) \to \operatorname{Hal}_{\Theta}(G)$  for the model  $(G, \Phi, f)$  and let again  $R_f$ be the image of this homomorphism.  $R_f$  is a subalgebra in  $\operatorname{Hal}_{\Theta}(G)$  consisting of algebraic sets of the form  $A = \operatorname{Val}_f(u)$ . Here T consists of one element u. We call A a simple algebraic set.

**Theorem 8.**  $R'_f = \operatorname{Aut}(f)$  holds for every model  $(G, \Phi, f)$ . If the algebra G is finite, then  $\operatorname{Aut}(f)' = R_f$ .

In particular, it follows that if the algebra G is finite, then every algebraic set  $A = T^f$  is a simple algebraic set,  $A = \operatorname{Val}_f(u)$  for some  $u \in \operatorname{Hal}_{\Theta} = \operatorname{Hal}_{\Theta}(\Phi)$ .

Note in the addition to Theorem 7 that the set  $A \subset \operatorname{Hom}(W, G)$  is elementary (algebraic) set over a model with the algebra G, if and only if  $\delta A = A$ ,  $\delta \neq 1$  holds for some  $\delta \in \operatorname{Aut}(G)$ . We can select a set  $\Phi$  and its realization f in G by  $\delta$  in such a way that  $\delta \in \operatorname{Aut}(f)$  and, simultaneously, we have  $\delta A = A$  for every  $\delta \in \operatorname{Aut}(f)$ .

## 5. Lattices of elementary sets.

For every model  $(G, \Phi, f)$  and every finite set X denote:  $Alv_f(X)$  is the set of all elementary sets for the model  $(G, \Phi, f)$  in the space Hom(W, G);  $Cl_f(X)$  is the set of all closed sets T in  $Hal_{\Phi\Theta}(X)$ .

We consider both these sets as lattices as well. The intersection operation is defined here as set theory intersection, while the union is defined by the rules:

$$A \nabla B = (A \cup B)^{ff}; A, B \in Alv_f(X)$$

$$T_1 \nabla T_2 = (T_1 \cup T_2)^{ff}; \ T_1, T_2 \in \text{Cl}_f(X).$$

The transitions  $A \to A^f = T$  and  $T \to T^f = A$  give the duality of lattices.

Further, we can specify various levels of logic, all of which are the parts of the algebra  $\operatorname{Hal}_{\Theta}(\Phi)$ . Levels of algebraic geometry are connected with them. In particular, equational geometry is connected with equational logic. The attitude to the pointed lattices changes. Let us popint out some interesting levels.

 $L_0$  – equational logic. Its formulas have the form  $w \equiv w'$ . They are equalities.

 $L_1$  – pseudoequational logic. Its formulas are pseudoequalities of the form  $w_1 \equiv w_1' \vee \cdots \vee w_n = w_n'$ .

 $L_2$  – universal logic over equalities. The formulas have the form  $w_1 \equiv w_1' \lor \cdots \lor w_n \equiv w_n' \lor v_1 \not\equiv v_1' \lor \ldots v_m \not\equiv v_m'$ .

 $L_3$  – we mean here positive logic whose formulas are built without negations.

 $L_4$  – universal logic (without quantifiers).

 $L_5$  – all that is built from atoms without quantifiers and negations (only  $\vee$  and  $\wedge$ ).

 $L_6$  – the whole algebra  $\operatorname{Hal}_{\Theta}(\Phi)$ .

Finally, we say that the logic L is  $\vee$ -closed, if  $u, v \in L$  implies  $u \vee v \in L$ . With each of these logics L a definite level  $\ell$  of geometry is associated. A notion of logical kernel of a homomorphism relates to the level  $\ell$ . If the homomorphism  $\mu: W(X) \to G$  in  $\Theta$  is given, then we set  $\ell - \log \ker(\mu)$  is the set of formulas u of the level  $\ell$  in  $\operatorname{Hal}_{\Theta\Phi}(X)$  for which  $\mu \in \operatorname{Val}_f(u)$ . We assume that the algebra G is included in the model  $(G, \Phi, f)$ .

According to these considerations, we localize Galois connections, all sets of formulas T are on a definite level  $\ell$  in a definite logic L. Let us rewrite these connections

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For every  $A \subset \text{Hom}(W,G)$  we have

$$T = A^{f,\ell} = \bigcap_{\mu \in A} (\ell - \log \ker(\mu)) = \{ u \mid u \in \ell \quad \text{and} \quad A \subset \operatorname{Val}_f(u) \}.$$

If, further T is of the level  $\ell$ , then

$$T^f = A = \bigcap_{u \in T} \operatorname{Val}_f(u) = \{ \mu : W \to G \mid T \subset \ell - \log \ker(\mu) \}.$$

Consider corresponding Galois closures. For the given  $A \subset \operatorname{Hom}(W(X), G)$  we have:  $A^{ff,\ell} = (A^{f,\ell})^f$ . For the given T of the level  $\ell$  we have  $T^{ff,\ell} = (T^f)^{f,\ell}$ . Consider also  $\ell$ , f-closed A and T. We consider the lattices  $\operatorname{Alv}_f(X)$  and  $\operatorname{Cl}_f(X)$  on the level  $\ell$  as well. We denote them  $\operatorname{Alv}_{f,\ell}(X)$  and  $\operatorname{Cl}_{f,\ell}(X)$ , respectively.

#### 6. Zariski topology.

We speak of topology in the affine space  $\operatorname{Hom}(W,G), W = W(X)$ , for the given model  $(G,\Phi,f)$  on the given level of logic  $\ell$ . We suppose here to consider only positive formulas.

For the given level  $\ell$  a Zariski topology in  $\operatorname{Hom}(W,G)$  over the model  $(G,\Phi,f)$  is a topology generated by all  $\ell$ -algebraic sets A in  $\operatorname{Hom}(W,G)$  as closed sets.

**Theorem 9.** If the level  $\ell$  is a  $\vee$ -closed logic, then all closed sets in the Zariski topology are exactly all  $\ell$ -algebraic sets.

In this case the lattice  $\operatorname{Alv}_{f,\ell}(X)$  is a sublattice of the lattice  $\operatorname{Bool}(W,G)$ . It is distributive, as is the lattice  $\operatorname{Cl}_{f,\ell}(X)$ .

Interesting case of Zariski topology is provided by empty set  $\Phi$ . In the special situation when  $\Theta = \text{Var-}R$  and the field R is the field of real numbers, the case when  $\Phi$  is an one-element relation  $\varphi$  which is the order relation is of special interest.

It is natural to treat all the considerations of this lecture in the classical situation of the variety Var-P.

## 7. Geometrical properties.

We work at the beginning in the largest FOL– $L_6$ .

**Definition 1.** The model  $(G, \Phi, f)$  is called *geometrically noetherian* if for every finite set X and every set of formulas T in  $\operatorname{Hal}_{\Phi\Theta}(X)$  in T there is some finite part  $T_0$  with  $T_0^f = T^f$ . This means that  $T_0^{ff} = T^{ff}$ .

**Theorem 10.** The model  $(G, \Phi, f)$  is geometrically noetherian if and only if the minimality condition holds in the lattice  $Alv_f(X)$ . Correspondingly, in the lattice  $Cl_f(X)$  we have the maximality condition.

Now let  $T_0 = \{u_1, \ldots, u_n\}$  and u is  $u_1 \wedge \cdots \wedge u_n$ . Then  $T_0^f = \operatorname{Val}_f(u) = \{u\}^f$ . Thus, we may claim that if the model  $(G, \Phi, f)$  is geometrically noetherian, then every algebraic set over this model is a simple algebraic set.

However, the corresponding element  $u = u_1 \wedge \cdots \wedge u_n$  does not necessarily belong to the initial set T. We call a model  $(G, \Phi, f)$  weak geometrically noetherian

if every algebraic set over it is a simple algebraic set. Weak noetherian model is not necessarily geometrically noetherian.

However, we may claim that every finite model is geometrically noetherian (compare Theorem 8 in Galois theory). We may also claim that any finite cartesian product of geometrically noetherian models is also a geometrically noetherian model. A submodel of a geometrically noetherian model is also geometrically noetherian. The similar properties not true in general in the transition to cartesian powers.

The notion of the model to be geometrically noetherian may be also defined on different special levels  $\ell$ . It is easy to understand that if the model  $(G, \Phi, f)$  is noetherian on the absolute level  $\ell_6$ , then it is noetherian on each other level  $\ell$ .

We may assume that by lowering the level  $\ell$ , one may obtain good properties. For example, for which level  $\ell$  the property of the model  $(G, \Phi, f)$  to be noetherian implies that each of its cartesian powers is noetherian as well. It is valid for  $L_0$  and for the logic of all atomic formulas. It is not clear when else this is true. It is most likely never (see also Theorem 11 below).

Let us pass to the notion of geometrical equivalence of two models. Let two models  $(G_1, \Phi, f_1)$  and  $(G_2, \Phi, f_2)$  with the same  $\Phi$  be given.

**Definition 2.** The models  $(G_1, \Phi, f_1)$  and  $(G_2, \Phi, f_2)$  are geometrically equivalent if  $T^{f_1 f_1} = T^{f_2 f_2}$  holds for every finite X and every T in  $\text{Hal}_{\Phi\Theta}(X)$ .

If the models  $(G_1, \Phi, f_1)$  and  $(G_2, \Phi, f_2)$  are geometrically equivalent, then

- 1. The lattices  $Cl_{f_1}(X)$  and  $Cl_{f_2}(X)$  coincide, while the lattices  $Alv_{f_1}(X)$  and  $Alv_{f_2}(X)$  are isomorphic.
- 2. The categories  $C_{\Phi\Theta}(f_1)$  and  $C_{\Phi\Theta}(f_2)$  coincide, while the categories  $K_{\Phi\Theta}(f_1)$  and  $K_{\Phi\Theta}(f_2)$  are isomorphic.
- 3. These models are elementary equivalent.

It follows from the first claim that if the models are geometrically equivalent and one of them is geometrically noetherian, then the second one is also geometrically noetherian.

**Theorem 11.** Let the model  $(G, \Phi, f)$  be geometrically noetherian. Then each of its ultrapower is also geometrically noetherian, and all these ultrapowers are geometrically equivalent to the initial  $(G, \Phi, f)$ .

All the described notions are naturally connected with the logic of generalized (infinitary) formulas of the kind  $(\wedge_{u \in T}) \to v$ , or, what is the same  $T \to v$ . For the geometrically noetherian models it is sufficient to proceed from the usual finite formulas.

Note that the notion of geometrical equivalence of models can be also considered on different logical levels. Here, if the models are equivalent on the absolute level  $\ell_6$ , then they are equivalent on every other level.

Elementary equivalence of the models does not generally imply their geometrical equivalence. However, we may claim the following

**Theorem 12.** If two models are elementary equivalent and one of them is geometrically noetherian, then the second one is as geometrically noetherian and these

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models are geometrically equivalent.

In concern with the notion of geometrically noetherian model let us return to the notion of the logical kernel of a homomorphism of the form  $\mu:W(X)\to G$ . It is easy to see that if  $\operatorname{Log}_f \operatorname{Ker}(\mu)$  is such a kernel and  $\operatorname{Val}_f(\operatorname{Log}_f \operatorname{Ker}(\mu))$  is its image in the algebra  $R_f(X)$  then this image is a principal ultrafilter if the model  $(G,\Phi,f)$  is geometrically noetherian. This ultrafilter is generated by the algebraic set  $\{\mu\}^{ff}$ .

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## LECTURE 7

# DATABASES AND KNOWLEDGE BASES GEOMETRICAL ASPECT

## Contents

- 1. Introduction
- 2. Categories of elementary knowledge
- 3. Databases and Knowledge bases
- 4. Equivalence of Databases and Knowledge bases
- 5. Conclusion

## 1. Introduction.

We use here the material from lecture 6, mainly, in knowledge theory.

Elementary knowledge is considered to be a first order knowledge, i.e., knowledge that can be represented by the means of the First Order Logic (FOL). The corresponding applied field (field of knowledge) is based on some variety of algebras  $\Theta$ , which is arbitrary but fixed and can be multisorted. This variety  $\Theta$  is considered as a knowledge type, like in database theory, databases of a data type  $\Theta$  are considered. We also fix a set of symbols of relations  $\Phi$ . Finally, the subject of knowledge is a model  $(G, \Phi, f)$ , where G is an algebra in  $\Theta$  and f is a realization of the set  $\Phi$  in G. It is a model in the ordinary mathematical meaning. Similar to the previous lecture, we write f instead of  $(G, \Phi, f)$  for short. Given  $\Phi$ , we denote the corresponding applied field by  $\Phi\Theta$ .

FOL is also oriented on the variety  $\Theta$ .

We assume that every knowledge under consideration is represented by three components:

1) The description of the knowledge.

It is a syntactical part of knowledge, written out in the language of the given logic. The description reflects, what we do want to know.

- 2) The subject of the knowledge which is an object in the given applied field, i.e., an object for which we determine knowledge.
- 3) Content of the knowledge (its semantics).

It is assumed also that there is a finite type of the description of knowledge, which is denoted by X. This X determines the space where the content of knowledge is calculated.

We consider knowledge as a triple of the form (Description, Subject, Content). In the knowledge base a query is the description of the knowledge. The reply to query is content of knowledge. Subject of the knowledge is fixed for the whole knowledge base.

The first two components of knowledge are relatively independent, while the third one is uniquely determined by the previous two components. In the theory under consideration, this third component has a geometrical nature. In some sense it is an algebraic variety in an affine space. If T is a description of knowledge and  $(G, \Phi, f)$  is a subject, then  $T^f$  denotes the content of knowledge. We would like to fill the content with its own structure, algebraic or geometric, and we consider some elements of such structure.

On the one hand, the language is constructed according to primary notions of algebraic geometry. On the other hand, it uses algebraic logic. We want to underline that there are three aspects in our approach to knowledge representation: logical (for knowledge description), algebraic (for the subject of knowledge) and geometric (in the content of knowledge).

The core point of the Lecture is elementary knowledge, i.e., First Order Knowledge. The main goal is to construct a model to represent some non-elementary knowledge about elementary knowledge using universal algebra approach. For the

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solution of this problem we join the methods of algebraic logic and universal algebraic geometry in logic, both defined over an arbitrary variety of algebras  $\Theta$ . Let me stress also that we make emphasis on the geometric nature of knowledge.

We consider categories of elementary knowledge. Language of categories in the knowledge theory is a good way to organize and systematize primary elementary knowledge. Morphisms in the knowledge category give links between knowledge. In particular, one can speak of isomorphic knowledge. The categorical approach also allows us to use ideas of monada and comonada [ML]. It turns out that this leads to some general views on enrichment and computation of knowledge. Enrichment of a structure can be associated with a suitable monada over a category, while the corresponding computation is organized by comonada.

Let us make one more remark. In every well described field of knowledge one can study the category of elementary knowledge, belonging to this field. Consideration of such categories might be of special interest.

### 2. Categories of elementary knowledge.

# **2.1** The category $Know_{\Phi\Theta}(f)$ .

Fix a model (subject of knowledge)  $(G, \Phi, f)$ . Let us define a category of knowledge for this model and denote it by  $\operatorname{Know}_{\Phi\Theta}(f)$ . It is a category for the given subject of knowledge. The objects of the category  $\operatorname{Know}_{\Phi\Theta}(f)$  have the form (X, T, A). Their meaning is knowledge. We do not fix the subject of knowledge, although it occurs here implicitly, since it is fixed once for all. The set X is multisorted in general. It marks the "place" where knowledge is situated. The set X points also the "place of the knowledge" - the space of the knowledge Hom(W(X), G) while the subject of the knowledge  $(G, \Phi, f)$  is given. The set T is the description of the knowledge in the algebra  $Hal_{\Phi\Theta}(X)$ , and  $A = T^f$ ,  $A \subset Hom(W(X), G)$ , is the content of knowledge, depending on T and f. The set  $T^{ff} = A^f$  is the full description of the knowledge (X, T, A) which is a Boolean filter in  $\operatorname{Hal}_{\Phi\Theta}(X)$ .

Now about morphisms  $(X, T_1, A) \to (Y, T_2, B)$ . Take  $s: W(Y) \to W(X)$ . We have also  $s: \operatorname{Hal}_{\Phi\Theta}(Y) \to \operatorname{Hal}_{\Phi\Theta}(X)$ . This is a homomorphism of Boolean algebras. The first s gives also

$$\tilde{s}$$
:  $Hom(W(X), G) \to Hom(W(Y), G)$ .

As above, the first s is admissible for A and B if  $\tilde{s}(\nu) = \nu s \in B$  for every point  $\nu: W(X) \to G$  in A.

As we know, s is admissible for A and B if and only if for every  $u \in B^f$  we have  $su \in A^f$ . This is for the second s, for which we have also a homomorphism  $\overline{s}: \operatorname{Hal}_{\Phi\Theta}(Y)/B^f \to \operatorname{Hal}_{\Phi\Theta}(X)/A^f$ . It is easy to prove that s is admissible for A and B if and only if  $su \in A^f$  holds for every  $u \in T_2$ . We consider such s as a morphism

$$s: (X, T_1, A) \rightarrow (Y, T_2, B)$$

in the weak category  $\operatorname{Know}_{\Phi\Theta}(f)$ . We have  $\tilde{s}(\nu) = \nu s \in B$  if  $\nu \in A$ , and s induces a mapping  $[s]: A \to B$ . Simultaneously arises a mapping  $s: T_2 \to A^f$  and a we have a homomorphism

$$\overline{s}$$
:  $\operatorname{Hal}_{\Phi\Theta}(Y)/B^f \to \operatorname{Hal}_{\Phi\Theta}(X)/A^f$ .

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We have already mentioned that  $\overline{s}_1 = \overline{s}_2$  follows from  $[s_1] = [s_2]$ . Thus, we can consider the morphisms of the form

$$[s]: (X, T_1, A) \to (Y, T_2, B)$$

in the exact category  $Know_{\Phi\Theta}(f)$ .

The canonical functors

$$Know_{\Phi\Theta}(f) \to K_{\Phi\Theta}(f)$$

for weak and exact categories are given by the transition  $(X, T, A) \to (X, A)$ . In this transition we "forget" to fix the description of knowledge.

# 2.2 The category $Know_{\Phi\Theta}$ .

Let us define the category of elementary knowledge for the whole applied field  $\Phi\Theta$ ; the subject of the knowledge  $(G, \Phi, f)$  is not fixed. As earlier, we proceed from the category  $\Phi\Theta$  whose morphisms are homomorphisms in  $\Theta$ . They ignore the relations from  $\Phi$ .

Objects of the category  $Know_{\Phi\Theta}$  are knowledge, represented by

$$(X, T, A; (G, \Phi, f)),$$

and we write (X, T, A; G, f), because  $\Phi$  is fixed for the category. Here X denotes the place of knowledge;  $A = T^f$ , G and f may change.

Consider morphisms:

$$(X, T_1, A; G_1, f_1) \rightarrow (Y, T_2, B; G_2, f_2).$$

We apply the same approach as before with some extensions.

Start from  $s:W(Y)\to W(X)$  and  $\delta:G_1\to G_2$ . These s and  $\delta$  should correlate. Let us explain the correlation condition. Take a set  $A_1=\{\delta\nu,\nu\in A\}=\delta^*A$  and take further  $T_1^\delta=A_1^{f_2}$ . Correlation of s and  $\delta$  means that  $su\in T_1^\delta$  holds for any  $u\in T_2$ . The same for every  $u\in B^{f_2}$ . The last also says that there is a homomorphism

$$\overline{s}$$
:  $\operatorname{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \operatorname{Hal}_{\Phi\Theta}(X)/A_1^{f_2}$ .

The first of the two mappings  $(s, \delta)$ :  $A \to B$  and  $s: T_2 \to T_1^{\delta}$  transforms the content of knowledge, while the second one acts on the description. Here  $T_2$  and  $T_1^{\delta}$  describe knowledge, associated with the same subject  $(G_2, \Phi, f_2)$ .

With fixed  $\delta$  there is also an exact mapping  $([s], \delta) : A \to B$ . We come to weak and exact categories  $Know_{\Phi\Theta}$ . The morphisms of the first one are  $(s, \delta)$  and in the second one they are  $([s], \delta)$  for  $(X, T_1, A; G_2, f_1) \to (Y, T_2, B; G_2, f_2)$ .

The canonical functor  $Know_{\Phi\Theta} \to K_{\Phi\Theta}$  is defined by the transition

$$(X, T, A; G, f) \rightarrow (X, A; G, f).$$

As above, we remove description of knowledge from the information about it.

# **2.3** Categories $K_{\Phi\Theta}(G)$ and $Know_{\Phi\Theta}(G)$ .

Algebra  $G \in \Theta$  is fixed in the categories  $K_{\Phi\Theta}(G)$  and  $\operatorname{Know}_{\Phi\Theta}(G)$ . A set of symbols of relations  $\Phi$  is fixed as usual, but realizations f of  $\Phi$  in G may change. Thus,  $K_{\Phi\Theta}(G)$  is a subcategory in  $K_{\Phi\Theta}$  and  $\operatorname{Know}_{\Phi\Theta}(G)$  is a subcategory in  $\operatorname{Know}_{\Phi\Theta}$ . Here the corresponding  $\delta: G \to G$  are identical homomorphisms. Objects of the category  $K_{\Phi\Theta}(G)$  are now recorded as (X, A, f), and those of the category  $\operatorname{Know}_{\Phi\Theta}(G)$  as (X, T, A, f). There is a canonical functor  $\operatorname{Know}_{\Phi\Theta}(G) \to K_{\Phi\Theta}(G)$ . As for morphisms

$$(X, A, f_1) \to (Y, B, f_2)$$
 and  $(X, T_1, A, f_1) \to (Y, T_2, B, f_2),$ 

we can note that  $A=A_1, A_1^{f_2}=T_1^{\delta}$  and  $A^{f_2}=T_1^{f_1f_2}$ . Hence, the corresponding admissible  $s:W(Y)\to W(X)$  transfers each  $u\in T_2$  into  $su\in T_1^{f_1f_2}$  and it induces a homomorphism

$$\overline{s}: \operatorname{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \operatorname{Hal}_{\Phi\Theta}(X)/A^{f_2}.$$

Every s gives a mapping  $[s]: A \to B$ .

### 3. Databases and Knowledge bases

### 3.1 Databases.

The proposed model of a database differs from that of [Pl 1]. We want to compare databases and knowledge bases. Geometrical aspect in databases reflects the fact that the reply to the query can be considered as an algebraic variety. It is a simple algebraic variety. In knowledge bases we deal with arbitrary algebraic varieties. But this is not the only difference.

Database is represented as a category.

Let us fix an algebra  $G \in \Theta$  and consider an (admissible) set F of realizations f of the set of symbols of relations  $\Phi$  in G. These f are instances of a database. For every instance  $f \in F$  we have a model  $(G, \Phi, f)$ , all of them forming a system of models, denoted by  $(G, \Phi, F)$ . We call this system a multimodel.

Consider a DB as a category whose objects have the form

$$\operatorname{Val}_f: \operatorname{Hal}_{\Phi\Theta} \to R_f, \quad f \in F.$$

Here, as above,  $R_f$  is a subalgebra in the algebra  $\operatorname{Hal}_{\Theta}(G)$ , coinciding with the image of the homomorphism  $\operatorname{Val}_f$ . If u is a query to a database, then the reply to this query in the instance f is  $u * f = Val_f(u)$ .

The morphisms are homomorphisms  $\gamma:R_{f_1}\to R_{f_2}$  with the commutative diagrams

$$\begin{array}{c}
\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}} R_{f_1} \\
\downarrow^{\gamma} \\
R_{f_2}
\end{array}$$

We call these  $\gamma$  correct homomorphisms. The diagram associates replies to the same query in different instances and for different  $\gamma$ .

Note that all algebras  $R_f$  are simple algebras and hence all  $\gamma: R_{f_1} \to R_{f_2}$  are injective.

Denote the database by  $DB(G, \Phi, F)$ .

# 3.2 Knowledge bases.

We fix again a multimodel  $(G, \Phi, F)$  and consider a knowledge base  $KB(G, \Phi, F)$ . This knowledge base is a category, whose objects are:

$$\operatorname{Val}_f: \operatorname{Hal}_{\Phi\Theta} \to K_{\Phi\Theta}(f), f \in F.$$

The mapping  $\operatorname{Val}_f$  transforms formulas of the algebra  $\operatorname{Hal}_{\Phi\Theta}$  into the objects of the category  $K_{\Phi\Theta}(f)$ , which is a subcategory in  $K_{\Phi\Theta}(G)$ . Denote by  $R_{\Phi\Theta}(f)$  a full subcategory in  $K_{\Phi\Theta}(f)$ , whose objects form a subalgebra  $R_f$  in  $\operatorname{Hal}_{\Theta}(G)$ .

In each object for every description of knowledge T the content of knowledge  $A = T^f$  is calculated. This A is considered as an object of the category  $K_{\Phi\Theta}(f)$  with all its internal and external ties in this category.

Morphisms of the category  $KB(G, \Phi, F)$  are represented as follows:

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}} K_{\Phi\Theta}(f_1)$$

$$\downarrow^{\gamma}$$

$$K_{\Phi\Theta}(f_2)$$

This diagram needs explanation. Here,  $\gamma$  is a functor of categories and commutativity of the diagram is supposed on the level of the objects of categories who are elements of the algebra  $\operatorname{Hal}_{\Theta}(G)$ . This  $\gamma$  induces the diagram

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}} R_{f_1}$$

$$\downarrow^{\gamma}$$

$$R_{f_2}$$

Thus, there is a canonical functor

$$KB(G, \Phi, f) \rightarrow DB(G, \Phi, f).$$

This functor shows what is in common for databases and knowledge bases. A homomorphism of algebras  $R_{f_1} \to R_{f_2}$  corresponds to the functor  $\gamma: K_{\Phi\Theta}(f_1) \to K_{\Phi\Theta}(f_2)$ . Furthermore, we assume that if T is a set of formulas in  $\operatorname{Hal}_{\Phi\Theta}(X)$ , then

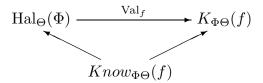
$$\gamma(T^{f_1}) = T^{f_2}.$$

This is a strengthened commutativity of the diagram. It connects knowledge content for the same description in different instances and for different  $\gamma$ .

In the knowledge category  $\operatorname{Know}_{\Phi\Theta}(G)$  we can distinguish a subcategory for the given set of instances F.

Note also the ties between knowledge base and knowledge category.

We consider the following commutative diagram:



for every  $f \in F$ .

The right arrow in the object (X, T, A) "forgets" the component T, while the left one "forgets" the component A. Such diagrams associate the category of knowledge with the knowledge base.

### 4. Equivalence of databases and knowledge bases

# 4.1. Equivalence of databases.

Let two databases with different  $(G_1, \Phi_1, F_1)$  and  $(G_2, \Phi_2, F_2)$  be given. We are interested in informational equivalence of these databases; another approach see in [PT].

Consider pairs  $(\alpha, \gamma)$  where  $\alpha: F_1 \to F_2$  is a mapping of sets and  $\gamma$  is a function, defining a homomorphism  $\gamma_f: R_f \to R_{f^{\alpha}}$  for every  $f \in F_1$ .

The pair  $(\alpha, \gamma)$  is called an *equivalence* of the corresponding databases if  $\alpha$  is a bijection and every  $\gamma_f, f \in F_1$ , is an isomorphism of algebras. Databases are equivalent, if there exists an equivalence  $(\alpha, \gamma)$  between them.

Let us motivate this definition. Take first a kernel  $\operatorname{Ker}(\operatorname{Val}_f)$  for every  $\operatorname{Val}_f$ :  $\operatorname{Hal}_{\Phi\Theta} \to R_f$  and pass to a factor algebra  $Q_f = \operatorname{Hal}_{\Phi\Theta}/\operatorname{Ker}(\operatorname{Val}_f)$ . Let here  $\delta_f : \operatorname{Hal}_{\Phi\Theta} \to Q_f$  be a natural homomorphism. Represent a homomorphism  $\operatorname{Val}_f$  as  $\operatorname{Val}_f = \operatorname{Val}_f^{\circ} \delta_f$  where  $\operatorname{Val}_f^{\circ} : Q_f \to R_f$  is an isomorphism. We have a diagram of isomorphisms for  $f \in F_1$ 

Here  $\beta_f^{\circ} = (\operatorname{Val}_{f^{\alpha}}^{\circ})^{-1} \gamma_f \operatorname{Val}_f^{\circ}$ .

Along with the natural homomorphism  $\delta_f$  we fix also some function of choice  $\delta_f^{-1}: Q_f \to \operatorname{Hal}_{\Phi\Theta}$  which chooses a definite  $u \in \operatorname{Hal}_{\Phi\Theta}$  with  $\delta_f(u) = q$  for every  $q \in Q_f$ .

Consider special functions  $\beta$  and  $\beta'$ . The function  $\beta$  gives a mapping  $\beta_f$ :  $\operatorname{Hal}_{\Phi_1\Theta} \to \operatorname{Hal}_{\Phi_2\Theta}$  for every  $f \in F_1$ . First we define a homomorphism  $\beta_f^1$ :  $\operatorname{Hal}_{\Phi_1\Theta} \to Q_{f^{\alpha}}$  by  $\beta_f^1 = \beta_f^{\circ} \delta_f$ . Now,  $\beta_f = \delta_{f^{\alpha}}^{-1} \beta_f^1 = \delta_{f^{\alpha}}^{-1} \beta_f^{\circ} \delta_f$ . Here  $\beta_f$  is a multisorted mapping of algebras, which is not, obviously, a homomorphism.

The function  $\beta'$  chooses a mapping  $\beta'_f: \operatorname{Hal}_{\Phi_2\Theta} \to \operatorname{Hal}_{\Phi_1\Theta}$  for every  $f^{\alpha} \in F_2$ . It is constructed similarly:

$$\beta_f' = \delta_f^{-1}(\beta_f^{\circ})^{-1}\delta_{f^{\alpha}}.$$

Let now  $u \in \operatorname{Hal}_{\Phi_1\Theta}$  and  $f \in F_1$ . Then

$$(u * f)^{\gamma_f} = \gamma_f(\operatorname{Val}_f(u)) = \gamma_f(\operatorname{Val}_f^{\circ} \delta_f(u)) =$$

$$= \operatorname{Val}_{f^{\alpha}}^{\circ} (\operatorname{Val}_{f^{\alpha}}^{\circ})^{-1} \gamma_f \operatorname{Val}_f^{\circ} \delta_f(u) = \operatorname{Val}_{f^{\alpha}}^{\circ} \beta_f^{\circ} \delta_f(u) =$$

$$= \operatorname{Val}_{f^{\alpha}}^{\circ} \delta_{f^{\alpha}} \delta_{f^{\alpha}}^{-1} \beta_f^{\circ} \delta_f(u) = \operatorname{Val}_{f^{\alpha}} \beta_f(u) = u^{\beta_f} * f^{\alpha}.$$

Analogously, if  $u \in \operatorname{Hal}_{\Phi_2\Theta}$  and  $f_1 = f^{\alpha} \in F_2$ , then

$$(u * f^{\alpha})^{\gamma_f^{-1}} = u^{\beta_f'} * f.$$

Thus, the reply to the query in the first database can be obtained via the second one and vice versa.

Consider separately a case when  $\Phi_1=\Phi_2=\Phi$  and every isomorphism  $\gamma_f$  is correct. The last means that

$$\begin{array}{c}
\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f}} R_{f} \\
\operatorname{Val}_{f^{\alpha}} & \downarrow^{\gamma_{f}} \\
R_{f^{\alpha}}
\end{array}$$

holds for every  $f \in F_1$ . Also, for every query  $u \in \operatorname{Hal}_{\Phi\Theta}$  and every  $f \in F_1$  we have  $(f * u)^{\gamma_f} = u * f^{\alpha}$ .

Take a morphism in the first database:

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}} R_{f_1}$$

$$\downarrow^{\gamma}$$

$$R_{f_2}$$

and construct the morphism in the second database:

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1^{\alpha}}} R_{f_1}^{\alpha}$$

$$\downarrow^{\gamma^{\alpha}}$$

$$R_{f_2^{\alpha}}$$

Here,  $\gamma^{\alpha} = \gamma_{f_2} \gamma \gamma_{f_1}^{-1}$  can be found from the diagram

$$R_{f_1} \xrightarrow{\gamma_{f_1}} R_{f_1^{\alpha}}$$

$$\downarrow^{\gamma^{\alpha}}$$

$$R_{f_2} \xrightarrow{\gamma_{f_2}} R_{f_2^{\alpha}}$$

We need to check that  $\gamma^{\alpha}$  is correct:  $\gamma^{\alpha} \operatorname{Val}_{f_{1}^{\alpha}} = \operatorname{Val}_{f_{2}^{\alpha}}$ . We have:  $\gamma^{\alpha} \operatorname{Val}_{f_{1}^{\alpha}} = \gamma_{f_{2}} \gamma \gamma_{f_{1}}^{-1} \gamma_{f_{1}} \operatorname{Val}_{f_{1}} = \gamma_{f_{2}} \gamma \operatorname{Val}_{f_{1}} = \gamma_{f_{2}} \operatorname{Val}_{f_{2}} = \operatorname{Val}_{f_{2}^{\alpha}}$ . Hence, the transition  $\gamma \to \gamma^{\alpha}$ 

gives the functor which is an isomorphism of databases, and in this specific situation equivalence of databases turns out to be their isomorphism.

Let us return to the general case. The relations with the special functions of choice  $\beta$  and  $\beta'$  for the sets  $F_1$  and  $F_2$  can be now represented as commutative diagrams:

$$\begin{array}{cccc} \operatorname{Hal}_{\Phi_1 \Theta} & \xrightarrow{\operatorname{Val}_f} & R_f & \operatorname{Hal}_{\Phi_1 \Theta} & \xrightarrow{\operatorname{Val}_f} & R_f \\ \beta_f & & & \gamma_f & \beta_f' & \gamma_f^{-1} \\ \operatorname{Hal}_{\Phi_2 \Theta} & \xrightarrow{\operatorname{Val}_{f^{\alpha}}} & R_{f^{\alpha}} & \operatorname{Hal}_{\Phi_2 \Theta} & \xrightarrow{\operatorname{Val}_{f^{\alpha}}} & R_{f^{\alpha}} \end{array}$$

These diagrams along with further remarks mean that equivalence of databases in general can be treated as some semi-isomorphism or skew isomorphism [Pl1]. The remarks are as follows.

Let  $\gamma: R_{f_1} \to R_{f_2}$  be a morphism in the first database. Take  $\gamma^{\alpha} = \gamma_{f_2} \gamma \gamma_{f_1}^{-1}$ . Check that for every  $u \in \operatorname{Hal}_{\Phi_1\Theta}$  it holds

$$\gamma^{\alpha} \operatorname{Val}_{f_1^{\alpha}}(\beta_{f_1}(u)) = \operatorname{Val}_{f_2^{\alpha}}(\beta_{f_2}(u)).$$

Indeed,

$$\gamma^{\alpha} \operatorname{Val}_{f_{1}^{\alpha}}(\beta_{f_{1}}(u)) = \gamma^{\alpha}(\beta_{f_{1}}(u) * f_{1}^{\alpha}) =$$

$$= \gamma^{\alpha} \gamma_{f_{1}}(u * f_{1}) = \gamma_{f_{2}} \gamma \gamma_{f_{1}}^{-1} \gamma_{f_{1}}(u * f_{1}) =$$

$$= \gamma_{f_{2}} \gamma(u * f_{1}) = \gamma_{f_{2}}(u * f_{2}) = \beta_{f_{2}}(u) * f_{2}^{\alpha} =$$

$$= \operatorname{Val}_{f_{2}^{\alpha}}(\beta_{f_{2}}(u)).$$

Thus,  $\gamma^{\alpha}$  is not anymore a morphism in the second database, but some "skew" morphism. The same for the transition from the second database to the first one with the mapping  $\beta'$ .

#### 4.2 Equivalence of knowledge bases.

Again we regard multimodels  $(G_1, \Phi_1, F_1)$  and  $(G_2, \Phi_2, F_2)$  and the related knowledgebases, denoted by  $KB_1$  and  $KB_2$ .

Consider pairs  $(\alpha, \gamma)$  where  $\alpha: F_1 \to F_2$  is a bijection of sets and  $\gamma$  is a function, determining an isomorphism of weak categories:

$$\gamma_f: K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha).$$

We assume that the isomorphism  $\gamma_f$  induces isomorphism of algebras  $\gamma_f: R_f \to R_{f^{\alpha}}$ . It preserves the type X and is coordinated with the inclusion of sets of points in affine spaces.

Consider every such pair  $(\alpha, \gamma)$  as an equivalence of knowledge bases: knowledge bases are equivalent if there is some equivalence  $(\alpha, \gamma)$ .

As it was done for databases, let us pass to motivations and distinguish the case when  $\Phi_1 = \Phi_2 = \Phi$  and isomorphism  $\gamma_f$  is correct. The last means that

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_f} K_{\Phi\Theta}(f)$$

$$Val_{f^{\alpha}} \qquad \qquad \downarrow^{\gamma_f}$$

$$K_{\Phi\Theta}(f^{\alpha})$$

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for every  $f \in F_1$ .

Take an arbitrary set of formulas T in the definite  $\operatorname{Hal}_{\Phi\Theta}(X)$ . Then:

$$T^{f^{\alpha}} = \bigcap_{u \in T} \operatorname{Val}_{f^{\alpha}}(u) = \bigcap_{u \in T} \gamma_f \operatorname{Val}_f(u) =$$
$$= \gamma_f(\bigcap_{u \in T} \operatorname{Val}_f(u)) = \gamma_f T^f.$$

We have used here the correlation of  $\gamma_f$  with inclusions which brings correlation with intersections. Thus, all  $T^{f^{\alpha}}$  and  $T^f$  are well correlated by the isomorphism  $\gamma_f$ .

Check now that the pair  $(\alpha, \gamma)$  gives an isomorphism of categories  $KB_1$  and  $KB_2$ . If  $\operatorname{Val}_f: \operatorname{Hal}_{\Phi\Theta} \to K_{\Phi\Theta}(f)$  is an object in the first category, then the corresponding object of the second category is  $\operatorname{Val}_{f^{\alpha}}: \operatorname{Hal}_{\Phi\Theta} \to K_{\Phi\Theta}(f^{\alpha})$ , and vice versa. Consider a morphism

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}} K_{\Phi\Theta}(f_1)$$

$$\downarrow^{\gamma}$$

$$K_{\Phi\Theta}(f_2)$$

The corresponding diagram is

$$\operatorname{Hal}_{\Phi\Theta} \xrightarrow{\operatorname{Val}_{f_1}^{\alpha}} K_{\Phi\Theta}(f_1^{\alpha})$$

$$\downarrow^{\gamma^{\alpha} = \gamma_{f_2} \gamma \gamma_{f_1}^{-1}}$$

$$K_{\Phi\Theta}(f_2^{\alpha})$$

This second diagram is actually a morphism in the second category.

Thus, in this special situation equivalence of knowledge bases is reduced to their isomorphism. The same fact has been established for the databases.

Now let us make some remarks on the general case. For the given set of formulas T in  $\operatorname{Hal}_{\Phi_1\Theta}(X)$  and the mapping  $\beta_f: \operatorname{Hal}_{\Phi_1\Theta} \to \operatorname{Hal}_{\Phi_2\Theta}$  consider a set  $T^{\beta_f}$  in  $\operatorname{Hal}_{\Phi_2\Theta}(X)$ , defined by

$$T^{\beta_f} = \{ \beta_f(u) | u \in T \}.$$

Check, that  $T^{\beta_f f^{\alpha}} = \gamma_f T^f$  for every  $f \in F_1$ . We have

$$T^{\beta_f f^{\alpha}} = \bigcap_{u \in T} \operatorname{Val}_{f^{\alpha}}(\beta_f(u)) = \bigcap_{u \in T} (\beta_f(u) * f^{\alpha}) =$$
$$= \bigcap_{u \in T} \gamma_f(u * f) = \gamma_f(\bigcap_{u \in T} \operatorname{Val}_f(u)) = \gamma_f T^f.$$

Similarly, if T is a set of formulas in  $\operatorname{Hal}_{\Phi_2\Theta}(X)$ , then

$$T^{\beta_f'f} = \gamma_f^{-1} T^{f^{\alpha}}.$$

The pointed relations mean that equivalence of knowledge bases  $KB_1$  and  $KB_2$  provides the corresponding informational equivalence on the level of the transition from the description of knowledge to its content.

Let now  $\gamma: K_{\Phi_1\Theta}(f_1) \to K_{\Phi_1\Theta}(f_2)$  be a morphism in  $KB_1$ . Take  $\gamma^{\alpha} = \gamma_{f_2} \gamma \gamma_{f_1}^{-1}$ . One can check that  $\gamma^{\alpha} T^{\beta_{f_1} f_1^{\alpha}} = T^{\beta_{f_2} f_2^{\alpha}}$  for every T in  $\operatorname{Hal}_{\Phi_1\Theta}(X)$ . In particular, it means that  $\gamma^{\alpha}: K_{\Phi_2\Theta}(f_1^{\alpha}) \to K_{\Phi_2\Theta}(f_2^{\alpha})$  is not a morphism in  $KB_2$ . Also here we have some "skew" property which needs additional motivation [Pl1].

#### 4.3. Main results.

Two multimodels  $(G_1, \Phi, F_1)$  and  $(G_2, \Phi, F_2)$  with the same  $\Phi$  are called geometrically equivalent if there is a bijection  $\alpha : F_1 \to F_2$  such that for every  $f \in F_1$  models  $(G_1, \Phi, f)$  and  $(G_2, \Phi, f^{\alpha})$  are geometrically equivalent.

**Theorem 1.** If multimodels  $(G_1, \Phi, F_1)$  and  $(G_2, \Phi, F_2)$  are geometrically equivalent, then the knowledge bases  $KB_1$  and  $KB_2$  are isomorphic and, hence, equivalent.

Let us say that two multimodels  $(G_1, \Phi, F_1)$  and  $(G_2, \Phi, F_2)$  are isomorphic if there is a bijection  $\alpha: F_1 \to F_2$  such that for every  $f \in F_1$  the models  $(G_1, \Phi, f)$ and  $(G_2, \Phi, f^{\alpha})$  are isomorphic. If multimodels are isomorphic, then they are geometrically equivalent and the corresponding bases  $KB_1$  and  $KB_2$  are isomorphic. If here the algebras  $G_1$  and  $G_2$  are finite, then the opposite is true as well.

**Theorem 2.** If algebras  $G_1$  and  $G_2$  in  $(G_1, \Phi, F_1)$  and  $(G_2, \Phi, F_2)$  are finite, then the knowledge bases  $KB_1$  and  $KB_2$  are isomorphic if and only if the multimodels are isomorphic.

This theorem gives the algorithm of verification of isomorphism of two finite knowledge bases.

Consider now the question of equivalence of two finite knowledge bases. Here we need additional definitions.

**Definition 1.** Let  $(G_1, \Phi_1, f_1)$  and  $(G_2, \Phi_2, f_2)$  be two models  $(\Phi_1 \text{ and } \Phi_2 \text{ may be different})$ . We call them automorphic equivalent, if

- 1) Algebras  $G_1$  and  $G_2$  are isomorphic
- 2) Groups  $Aut(f_1)$  and  $Aut(f_2)$  are conjugated by some isomorphism of algebras  $G_1$  and  $G_2$ .

In other words, there exists an isomorphism  $\delta: G_2 \to G_1$  such that

$$\operatorname{Aut}(f_2) = \delta^{-1} \operatorname{Aut}(f_1) \delta.$$

**Definition 2.** Two multimodels  $(G_1, \Phi_1, F_1)$  and  $(G_2, \Phi_2, F_2)$  are automorphic equivalent, if for some bijection  $\alpha : F_1 \to F_2$  the models  $(G_1, \Phi_1, f)$  and  $(G_2, \Phi_2, f^{\alpha})$  are automorphic equivalent for every  $f \in F_1$ .

**Proposition 1.** If the multimodels  $(G_1, \Phi_1, F_1)$  and  $(G_2, \Phi_2, F_2)$  are automorphic equivalent then the corresponding knowledge bases  $KB_1$  and  $KB_2$  are equivalent.

Let now the multimodels  $(G_1, \Phi_1, F_1)$  and  $(G_2, \Phi_2, F_2)$  with finite  $G_1, G_2 \in \Theta$  be given;  $KB_1$  and  $KB_2$  are the corresponding knowledge bases.

**Theorem 3.** Knowledge bases  $KB_1$  and  $KB_2$  are equivalent if and only if their multimodels are automorphic equivalent.

The proof of the theorem is based on Galois theory from the previous lecture. It also gives the algorithm of verification of two finite knowledge bases (compare [PT]).

Let us note that it is natural to consider semi-isomorphisms along with the isomorphisms of knowledge bases with the same  $\Phi$ . Semi-isomorphisms are described by the diagrams of the form

$$\begin{array}{ccc}
\operatorname{Hal}_{\Phi\Theta} & \xrightarrow{\operatorname{Val}_f} & K_{\Phi\Theta}(f) \\
\downarrow^{\sigma} & & \downarrow^{\gamma_f} \\
\operatorname{Hal}_{\Phi\Theta} & \xrightarrow{\operatorname{Val}_{f^{\alpha}}} & K_{\Phi\Theta}(f^{\alpha})
\end{array}$$

where  $\sigma$  is an automorphism of the algebra  $\operatorname{Hal}_{\Phi\Theta}$  and  $\gamma_f$  is an isomorphism of the categories. According to such a diagram, we have

$$\gamma_f \operatorname{Val}_f(u) = (u * f)^{\gamma_f} = \operatorname{Val}_{f^{\alpha}}(\sigma u) = \sigma u * f^{\alpha}$$

for every  $u \in \operatorname{Hal}_{\Phi\Theta}$ . Thus, we may replace the mappings  $\beta_f$  and  $\beta_f'$  in the general situation by universal mappings  $\sigma$  and  $\sigma^{-1}$ . In this case all the pictures with "skew" become more visible. We see that semi-isomorphism implies equivalence of knowledge bases, as well as isomorphism does.

We can also consider a category of knowledge bases for fixed  $\Phi$  and  $\Theta$ . These are categories with usual (special) morphisms, and categories with semimorphisms. We can consider monads and comonads in them.

#### 5. Conclusion

The main problem in computation of knowledge is to find the content of knowledge  $A = T^f$  by the given description of knowledge T. Since  $A = \bigcap_{u \in T} \operatorname{Val}_f(u)$ , we need to compute the sets  $\operatorname{Val}_f(u)$  for various  $u \in \operatorname{Hal}_{\Phi\Theta}$ .

Pass from the algebra  $\operatorname{Hal}_{\Phi\Theta}$  to the corresponding free Halmos algebra  $\tilde{H}=\tilde{H}_{\Phi\Theta}$  (see Lecture 6). We treat every u as an element in  $\tilde{H}$ . We consider an algebra  $\tilde{H}$  to be the constructive one. In other words, every element u is well represented by atoms. Thus, the computation of the sets  $\operatorname{Val}_f(u)$  with arbitrary u is reduced to that for atoms. The last is somehow made for constructive models  $(G, \Phi, f)$ . All above can be realized via some comonada in the category of knowledge bases. One may also consider monads in these categories for enrichment of knowledge.

Let us note the following general problem: what is the inference of knowledge in the categories of knowledge under consideration? The problem is to well formalize the corresponding idea. 84

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