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DERIVED CATEGORIES, DERIVED EQUIVALENCES AND REPRESENTATION THEORY

THORSTEN HOLM

Definition:
A derived category ... is when you take complexes seriously! (L.L. Scott [Sc]*)*

The aim of this chapter is to give a fairly elementary introduction to the (not very elementary) subject of derived categories and equivalences. Especially, we emphasize the applications of derived equivalences in representation theory of groups and algebras in order to illustrate the importance and usefulness of the concept. We try to keep the necessary prerequisites as low as possible. Ideally, these notes should be accessible for an audience with a good background in general algebra and some basic knowledge in representation theory, category theory and homological algebra. Of course, this means that these notes cannot be a comprehensive treatment. Most theorems have to be stated without proof but in any case we point to the relevant literature and wherever possible we provide examples in order to illustrate the results.

In Section 1 abelian categories are introduced and we give a proof of Morita's theorem. Section 2 deals with triangulated categories; in particular we study the homotopy category of complexes. Section 3 contains the definition of derived categories. In Section 4 we discuss tilting theory and indicate the development towards the important results of J. Rickard. As an illustration we discuss derived equivalences for Brauer tree algebras. In Section 5 we sketch some of the main applications of derived equivalences in representation theory.

1. Abelian Categories

Let R be a ring (always associative and with a unit element). Denote by R-Mod the category of (left-) R-modules and by R-mod the category of finitely generated (left-) R-modules. These module categories are the main object of study in representation theory and they carry a lot of important additional structure.

A category $\mathcal{C}$ is called an *additive category* if

- the morphism sets $Hom_{\mathcal{C}}(X,Y)$ are abelian groups and the compositions
$$Hom_{\mathcal{C}}(Y,Z) \times Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X,Z)$$
are bilinear
- $\mathcal{C}$ has a zero object 0 (i.e. for any object X the sets $Hom_{\mathcal{C}}(X,0)$ and $Hom_{\mathcal{C}}(0,X)$ consist only of one element)
- for any $X, Y \in \mathcal{C}$ there exists a direct sum $X \oplus Y \in \mathcal{C}$.

The basic examples of additive categories we have in mind are the module categories R-Mod and R-mod. But also the full subcategory R-Proj of R-Mod consisting of all projective R-modules, and the category $P_R = R$-proj of all finitely generated projective R-modules provide examples of additive categories.

But R-Mod carries even more structure; before explaining it we need some terminology. Let $\mathcal{A}$ be an additive category and $\phi : A \to B$ a morphism in $\mathcal{A}$. A *kernel of* ϕ is an object K in $\mathcal{A}$ together with a morphism $\iota : K \to A$ such that $\phi \circ \iota = 0$ and with the property that every morphism $\psi : X \to A$ in $\mathcal{A}$ with $\phi \circ \psi = 0$ factors uniquely through ι. Dually, a *cokernel of* ϕ consists of an object C in $\mathcal{A}$ together with a morphism $\pi : B \to C$ such that $\pi \circ \phi = 0$ and such that every morphism $\sigma : B \to Y$ with $\sigma \circ \phi = 0$ factors uniquely through π. Kernels and cokernels need not exist in general. But note that if a kernel or a cokernel exists then it is unique up to unique isomorphism. The *image of* ϕ is defined to be the kernel of the cokernel of ϕ. The *coimage of* ϕ is the cokernel of the kernel of ϕ.

The reader may wish to check that in the case of a module category R-Mod this general definition actually coincides with the well-known notions of kernel, cokernel, image and coimage of a module homomorphism (see Exercise 2.).

Assume we are given an additive category $\mathcal{A}$ in which every morphism ϕ has a kernel and a cokernel (e.g. $\mathcal{A} = R$-Mod). Then there exist canonical morphisms $\bar{\phi} : \mathrm{coim}(\phi) \to \mathrm{im}(\phi)$ (the details of this straightforward construction should be carried out in Exercise 2.).

An additive category $\mathcal{A}$ is called an *abelian category* if

- each morphism ϕ in $\mathcal{A}$ has a kernel and a cokernel
- the canonical morphisms $\bar{\phi} : \mathrm{coim}(\phi) \to \mathrm{im}(\phi)$ are isomorphisms.

For example, R-Mod is an abelian category. In fact, for $\phi : A \to B$ we have $\bar{\phi} : A/\ker(\phi) \xrightarrow{\cong} \mathrm{im}(\phi)$ by the homomorphism theorem. But the subcategory R-Proj is not abelian in general (choose your favourite algebra and see that the kernel or cokernel of a homomorphism between projective modules need not be projective).

In some sense the categories of modules over rings are rather general examples of abelian categories. To make this precise we need some more terminology. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. An *additive functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that each induced map $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ is a homomorphism of abelian groups.

An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *left exact* (*right exact*) if it preserves kernels (cokernels). F is called *exact* if it is both left and right exact. F is called a *full embedding* if F is injective on objects and induces bijections $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ for all objects A, A' in $\mathcal{A}$.

For example, for a ring R and an R-module A the functor $\mathrm{Hom}_R(A, -) : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ is left exact. For any right R-module B the functor $B \otimes_R - : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ is right exact.

A category $\mathcal{A}$ is called *small* if the collection of objects in $\mathcal{A}$ forms a set.

Theorem 1.1 (Mitchell, [Mi])**.** *Let $\mathcal{A}_0$ be a small abelian category. Then there exists a ring R and an exact full embedding $F : \mathcal{A}_0 \to R$-*Mod.

As a consequence, any theorem about modules involving only a finite diagram and notions like exactness, existence or vanishing of morphisms etc. holds in any abelian category $\mathcal{A}$ (in fact, take $\mathcal{A}_0 \subseteq \mathcal{A}$ a small abelian subcategory containing all objects involved).

Morita Theory. We briefly recall the now classical Morita theory for rings. For more details the interested reader may consult the relevant textbooks, like [A-F], [C-R] or [Be1]. Let R, S be rings. Considering modules as invariants for rings the basic question is:

*When are R-*Mod *and S-*Mod *equivalent as abelian categories?*

Assume we are given (mutually inverse) equivalences $F : R\text{-Mod} \to S\text{-Mod}$ and $F' : S\text{-Mod} \to R\text{-Mod}$ of abelian categories. Set $P = F'(S) \in R\text{-Mod}$. From the corresponding properties of S we can deduce that the following assertions hold:

(0) P is finitely generated.
(i) P is projective.
(ii) Every R-module is a homomorphic image of a direct sum of copies of P.
(iii) $S \cong \mathrm{End}_R(P)^{op}$.

Every finitely generated R-module satisfying (i) and (ii) is called a *progenerator* for R-Mod.

For example, let R be an artinian ring. By Wedderburn's theorem, $R/\mathrm{rad}(R) \cong \oplus \mathrm{Mat}_{n_i}(\Delta_i)$ (with suitable division rings Δ_i). Moreover, ${}_RR = \oplus n_i P_i$ where the P_i are the projective indecomposable R-modules. Clearly, $P := \oplus m_i P_i$ is a progenerator if and only if $m_i > 0$ for all i. Then for $S := \mathrm{End}_R(P)^{op}$ we have $S/\mathrm{rad}(S) \cong \oplus \mathrm{Mat}_{m_i}(\Delta_i)$, i.e., the simple S-modules are of dimension m_i.

A complete answer to the above question on the equivalence of module categories was obtained by K. Morita in the 1950's.

Theorem 1.2 (Morita)**.** *For rings R, S the following assertions are equivalent.*

(i) *There exists an equivalence of abelian categories R-*Mod$\xrightarrow{\simeq}$ *S-*Mod.

(ii) *There exists a progenerator P for R-*Mod *such that* $S \cong \mathrm{End}_R(P)^{op}$.

(iii) *There exist bimodules ${}_RP_S$ and ${}_SQ_R$ and surjective bimodule homomorphisms* $\phi : P \otimes_S Q \to R$ *and* $\psi : Q \otimes_R P \to S$ *satisfying the identities* $x\psi(y \otimes x') = \phi(x \otimes y)x'$ *and* $y\phi(x \otimes y') = \psi(y \otimes x)y'$ *for all* $x, x' \in P,\ y, y' \in Q$.

(iv) *The functors* $Q \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ *and* $P \otimes_S - : S\text{-Mod} \to R\text{-Mod}$ *provide mutually inverse equivalences of abelian categories between R-*Mod *and S-*Mod *(where P, Q are as in (iii)).*

Proof: (i) $\Rightarrow$ (ii): Set $P = F'(S)$ as above.

(ii) $\Rightarrow$ (iii): $S := \mathrm{End}_R(P)^{op}$ acts on P via $x \cdot f = f(x)$ for $x \in P$, $f \in S$, thus providing an R-S-bimodule ${}_RP_S$. Set $Q = \mathrm{Hom}_R(P, R)$; this becomes an S-R-bimodule in the natural way, i.e., $(f \cdot g)(x) = g(f(x))$ and $(g \cdot r)(x) = g(rx)$ for $f \in S$, $g \in Q$, $r \in R$ and $x \in P$. We define bimodule homomorphisms by setting $\phi : P \otimes_S Q \to R$, $\phi(x \otimes g) = g(x)$ and $\psi : Q \otimes_R P \to S$, $\psi(g \otimes x)(x') = g(x')x$. It is straightforward to check that these maps satisfy the required identities. The crucial point is to show that ϕ and ψ are surjective. Firstly, since P is a progenerator, there exists a surjective homomorphism $f : \oplus_I P \to R$; choose $x = (x_i)_{i \in I}$ such that $f(x) = 1$. Then $x = \sum_{i \in I} \iota_i(x_i)$, where ι_i are the natural embeddings. Hence, $1_R = f(x) = \sum_{i \in I}(f \circ \iota_i)(x_i) = \phi(\sum_{i \in I} x_i \otimes (f \circ \iota_i)) \in \mathrm{im}(\phi)$, and ϕ is surjective. (Note that only finitely many $x_i \neq 0$, i.e., all sums are finite.)

Secondly, the finitely generated module P has the form $P = Rx_1 + \ldots + Rx_n$ and there is a corresponding surjection $f : F := \oplus_{i=1}^n R \to P$. Since P is projective there exists a homomorphism $g : P \to F$ such that $f \circ g = \mathrm{id}_P$. For $i = 1, \ldots, n$ set $g_i = \pi_i \circ g \in \mathrm{Hom}_R(P, R) = Q$, where π_i are the canonical projections. Then any $x \in P$ has the form $x = \sum_{i=1}^n g_i(x)x_i$. It follows $1_S = \mathrm{id}_P = \psi(\sum_{i=1}^n g_i \otimes x_i) \in \mathrm{im}(\psi)$, and also ψ is surjective.

(iii) $\Rightarrow$ (iv): It suffices to show that the maps ϕ and ψ are actually isomorphisms. (In fact, by associativity of the tensor product we then have $M \cong P \otimes_S (Q \otimes_R M)$ and $N \cong Q \otimes_R (P \otimes_S N)$ for all $M \in R$-Mod and $N \in S$-Mod.) We shall show that $\ker(\phi) = 0$, a similar argument works for ψ. Let $\phi(\sum_i x_i \otimes y_i) = 1 \in R$ (use that ϕ is surjective). Suppose $\sum_j z_j \otimes w_j \in \ker(\phi) \subseteq P \otimes_S Q$. Then

$$\begin{aligned}
\textstyle\sum_j z_j \otimes w_j &= \textstyle\sum_j z_j \otimes w_j \phi(\sum_i x_i \otimes y_i) \\
&= \textstyle\sum_{i,j} z_j \otimes w_j \phi(x_i \otimes y_i) \\
&= \textstyle\sum_{i,j} z_j \otimes \psi(w_j \otimes x_i) y_i \\
&= \textstyle\sum_{i,j} z_j \psi(w_j \otimes x_i) \otimes y_i \\
&= \textstyle\sum_{i,j} \phi(z_j \otimes w_j) x_i \otimes y_i \\
&= 0.
\end{aligned}$$

Finally, the direction (iv) $\Rightarrow$ (i) is trivial. □

Definition 1.3. *Rings R, S satisfying the properties of Theorem 1.2 are said to be* Morita equivalent.

Example 1.4. Let R be a ring, $n \in \mathbb{N}$. Then $P := \underbrace{R \oplus \ldots \oplus R}_{n}$ is a progenerator for R-Mod, and $S := \mathrm{End}_R(P)^{op} \cong \mathrm{Mat}_n(R)$. Thus, for any $n \in \mathbb{N}$ the rings R and $\mathrm{Mat}_n(R)$ are Morita equivalent.

Remark 1.5. 1.) For Morita-equivalent rings R, S and any $M \in R$-Mod, an equivalence $F : R\text{-Mod} \to S\text{-Mod}$ induces a bijection between the submodules of M and the submodules of $F(M)$ (which preserves inclusions, sums and intersections). In particular, M is simple (or semisimple, indecomposable, artinian, noetherian,...) if and only if $F(M)$ has this property.

2.) If R and S are Morita-equivalent rings then the centers $Z(R) \cong Z(S)$ are isomorphic as rings ([A-F], 21.10).

Exercises

1. Recall the definition of direct sums in a category. Dualize to get the definition of direct products. Deduce that for a family of objects $\{A_i\}_{i \in I}$ the direct sum $\oplus_I A_i$ and the direct product $\prod_I A_i$ are uniquely defined up to unique isomorphism. Show that for $|I| < \infty$ there exists a canonical isomorphism $\bigoplus_I A_i \xrightarrow{\cong} \prod_I A_i$.
2. Let $\mathcal{A}$ be an additive category, $\phi : A \to B$ a morphism in $\mathcal{A}$. Give the details for the existence of the canonical morphism $\bar{\phi} : \mathrm{coim}(\phi) \to \mathrm{im}(\phi)$. Identify $\ker(\phi)$, $\mathrm{coker}(\phi)$, $\mathrm{im}(\phi)$, $\mathrm{coim}(\phi)$ and $\bar{\phi}$ in R-Mod.
3. Let $\mathcal{A}$ be the category of filtered abelian groups, i.e., the objects of $\mathcal{A}$ are of the form $A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_1 \subseteq A_2 \subseteq \ldots$ are chains of abelian groups; the morphisms $f : A \to B$ in $\mathcal{A}$ are families $(f_n : A_n \to B_n)_{n \in \mathbb{N}}$ of homomorphisms of abelian groups such that $f_n|_{A_{n-1}} = f_{n-1}$ for all n. Then $\mathcal{A}$ is an additive category and each morphism has a kernel and a cokernel. But show that $\mathcal{A}$ is not abelian. (Hint: $A[1] := \cup_{n \in \mathbb{N}} A[1]_n$ where $A[1]_n = A_{n+1}$; the canonical map $A \to A[1]$ is mono and epi, but not invertible.)

2. Triangulated Categories

Given an abelian category $\mathcal{A}$, there are 'natural' situations in which one would like to pass to some quotient category $\bar{\mathcal{A}}$ which may no longer be abelian. Main examples are provided by

- A a selfinjective algebra, $\mathcal{A} = A$-Mod, and $\bar{\mathcal{A}} = A$-$\underline{\mathrm{Mod}}$ the *stable module category* ($\bar{\mathcal{A}}$ has the same objects as $\mathcal{A} = A$-Mod but the morphisms are equivalence classes of homomorphisms where two morphisms are called equivalent if their difference factors through a projective module.)
- $\mathcal{A}$ an abelian category, $\mathbf{C}(\mathcal{A})$ the category of complexes over $\mathcal{A}$, and $\bar{\mathcal{A}} = \mathbf{K}(\mathcal{A})$ the *homotopy category* (to be defined below).

But although not being abelian, there is still an interesting structure on these two sorts of quotient categories, namely they are *triangulated categories*. In this section we shall mainly discuss the homotopy categories and their triangulated structure. For the stable category see section 5 below, and Chapter III, section 5.

Let $\mathcal{C}$ be an additive category. A *complex over* $\mathcal{C}$ is a sequence

$$\ldots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \ldots$$

of objects and morphisms in $\mathcal{C}$ such that $d^n d^{n-1} = 0$ for all n.

Let X, Y be complexes over $\mathcal{C}$. A *homomorphism of complexes* $f : X \to Y$ is a sequence $f = (f^n)_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $d^n f^n = f^{n+1} d^n$ for all n.

This defines a category $\mathbf{C}(\mathcal{C})$, the *category of complexes* over $\mathcal{C}$. If $\mathcal{C}$ is abelian (additive) then also $\mathbf{C}(\mathcal{C})$ is abelian (additive).

A complex $X = (X^n)$ is *bounded (bounded below, bounded above)* if $X^n = 0$ for $|n| \gg 0$ ($n \ll 0$, $n \gg 0$). These bounded complexes form the objects of full subcategories $\mathbf{C}^b(\mathcal{C})$, $\mathbf{C}^+(\mathcal{C})$ and $\mathbf{C}^-(\mathcal{C})$ of $\mathbf{C}(\mathcal{C})$.

Morphisms $f, g : X \to Y$ in $\mathbf{C}(\mathcal{C})$ are called *homotopic*, denoted $f \sim g$, if there exist maps $s^n \in \mathrm{Hom}_{\mathcal{C}}(X^n, Y^{n-1})$ such that $f^n - g^n = d^{n-1}s^n + s^{n+1}d^n$ for all n.

For $X, Y \in \mathbf{C}(\mathcal{C})$ set $\mathrm{Ht}(X,Y) = \{f : X \to Y \mid f \text{ homotopic to zero}\}$. This set is closed under composition (from left or right) with arbitrary morphisms of $\mathbf{C}(\mathcal{C})$, i.e., it is an 'ideal' in $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(X,Y)$.

Definition 2.1. *The* homotopy category $\mathbf{K}(\mathcal{C})$ *has the same objects as* $\mathbf{C}(\mathcal{C})$*, but the morphisms are given by equivalence classes* $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(X,Y)/\mathrm{Ht}(X,Y)$.

Starting with $\mathbf{C}^b(\mathcal{C})$*,* $\mathbf{C}^+(\mathcal{C})$ *or* $\mathbf{C}^-(\mathcal{C})$ *one similarly obtains homotopy categories* $\mathbf{K}^b(\mathcal{C})$*,* $\mathbf{K}^+(\mathcal{C})$ *and* $\mathbf{K}^-(\mathcal{C})$*.*

Remark 2.2. The homotopy category $\mathbf{K}(\mathcal{C})$ is an additive category, but not abelian in general, even if $\mathcal{C}$ is abelian. (Once we have shown below that $\mathbf{K}(\mathcal{C})$ is a triangulated category, this can be deduced from the general fact that an abelian category which is triangulated has to be semisimple, i.e., every short exact sequence splits [G-M], IV.1.) In particular, the notion of a short exact sequence shall be replaced in $\mathbf{K}(\mathcal{C})$ by a more suitable 'weaker' notion. These are the *(distinguished) triangles* to be defined below.

For any $k \in \mathbb{Z}$ we have the *shift functor* $[k] : \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$ which is defined on objects by $X[k]^n = X^{n+k}$, $d^n_{X[k]} = (-1)^k d_X^{n+k}$ and on morphisms by $f[k] : X[k] \to Y[k]$, $f[k]^n = f^{n+k}$ for all $n \in \mathbb{Z}$. In particular, we have an automorphism $[1] : \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$.

A *triangle in* $\mathbf{K}(\mathcal{C})$ is a sequence $X \to Y \to Z \to X[1]$ of objects and morphisms in $\mathbf{K}(\mathcal{C})$. A *morphism of triangles* is a triple (f,g,h) of morphisms making the following diagram commutative (in the homotopy category)

$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & X[1] \\ f\downarrow & & g\downarrow & & h\downarrow & & \downarrow f[1] \\ X' & \to & Y' & \to & Z' & \to & X'[1] \end{array}$$

Such a morphism is called an *isomorphism of triangles* if f, g and h are isomorphisms.

Definition 2.3. *For* $f \in \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(X,Y)$ *the* mapping cone $M(f) \in \mathbf{C}(\mathcal{C})$ *is defined by*

$$M(f)^n = X^{n+1} \oplus Y^n \quad and \quad d^n_{M(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

We have in $\mathbf{C}(\mathcal{C})$ the canonical morphisms $\alpha(f) : Y \to M(f)$, $\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta(f) : M(f) \to X[1]$, $\beta(f)^n = (\mathrm{id}_{X^{n+1}}, 0)$.

A *standard triangle in* $\mathbf{K}(\mathcal{C})$ is a triangle of the form $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$. A *distinguished triangle in* $\mathbf{K}(\mathcal{C})$ is a triangle which is isomorphic (in $\mathbf{K}(\mathcal{C})$) to a standard triangle.

Definition 2.4. *A* triangulated category *is an additive category* $\mathcal{C}$ *together with an additive automorphism* $T = [1]$ *(the translation functor), and a collection of distinguished triangles such that the following axioms are satisfied:*

(TR0) *Any triangle isomorphic to a distinguished triangle is a distinguished triangle.*

(TR1) *For each object* X, $X \xrightarrow{id} X \to 0 \to X[1]$ *is a distinguished triangle.*

(TR2) *Any morphism* $u : X \to Y$ *can be embedded into a distinguished triangle* $X \xrightarrow{u} Y \to Z \to X[1]$.

(TR3) *If* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is a distinguished triangle then also* $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ *is a distinguished triangle, and vice versa.*

(TR4) *If* $X \xrightarrow{u} Y \to Z \to X[1]$ *and* $X' \xrightarrow{u'} Y' \to Z' \to X'[1]$ *are distinguished triangles then each commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{u'} & Y' \end{array}$$

can be embedded into a morphism of triangles.

(TR5) *(Octahedral axiom) Given distinguished triangles* $X \xrightarrow{u} Y \to Z' \to X[1]$, $Y \xrightarrow{v} Z \to X' \to Y[1]$, $X \xrightarrow{vu} Z \to Y' \to X[1]$ *there exists a distinguished triangle*

$Z' \to Y' \to X' \to Z'[1]$ making the following diagram commutative

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
\| & & \downarrow v & & \downarrow & & \| \\
X & \xrightarrow{vu} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
\downarrow u & & \| & & \downarrow & & \downarrow \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1]
\end{array}$$

Theorem 2.5. *Let $\mathcal{C}$ be an additive category. The homotopy category $\mathbf{K}(\mathcal{C})$ is a triangulated category.*

Sketch of proof: By definition, (TR0) and (TR2) hold.

For (TR3) it suffices to consider a standard triangle $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$. We show that the diagram

$$\begin{array}{ccccccc}
Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] & \xrightarrow{-f[1]} & Y[1] \\
\| & & \| & & & & \| \\
Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\alpha(\alpha(f))} & M(\alpha(f)) & \xrightarrow{\beta(\alpha(f))} & Y[1]
\end{array}$$

can be completed to an isomorphism of triangles. Define $\phi : X[1] \to M(\alpha(f))$ by setting $\phi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}$. Conversely, set $\psi : M(\alpha(f)) \to X[1]$, $\psi^n = (0, \mathrm{id}_{X^{n+1}}, 0)$. We leave it to the reader to check the required properties, i.e., to check that $\psi\phi = \mathrm{id}_{X[1]}$, $\phi\psi \sim \mathrm{id}_{M(\alpha(f))}$, $\alpha(\alpha(f)) \sim \phi\beta(f)$ and $\beta(\alpha(f))\phi = -f[1]$.

For (TR1) consider the zero map $f : 0 \to X$. Its mapping cone is $M(f) = X$, and $\alpha(f)^n = \mathrm{id}_{X^n}$. Thus $0 \to X \xrightarrow{id} X \to 0$ is a standard triangle and then shifting it by (TR3) gives a distinguished triangle $X \xrightarrow{id} X \to 0 \to X[1]$.

For (TR4) it again suffices to consider standard triangles. By assumption we have a diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] \\
\downarrow u & & \downarrow v & & & & \downarrow u[1] \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{\alpha(f')} & M(f') & \xrightarrow{\beta(f')} & X'[1]
\end{array}$$

in which the left square commutes in $\mathbf{K}(\mathcal{C})$, i.e., there exist $s^n : X^n \to Y'^{n-1}$ such that $v^n f^n - f'^n u^n = s^{n+1} d_X^n + d_{Y'}^{n-1} s^n$. The question is whether this diagram can be completed to a morphism of triangles. We define $w : M(f) \to M(f')$ by setting $w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix}$ and leave it to the reader to verify the details.

The verification of the octahedral axiom (TR5) is rather long and tedious and we refrain from reproducing it here; for a proof see e.g. [K-S], 1.4 $\square$

Exercise: Fill in the details in the proof of Theorem 2.5.

3. Derived Categories

Let $\mathcal{C}$ be an abelian category. There is an obvious embedding of $\mathcal{C}$ into the category of complexes $\mathbf{C}(\mathcal{C})$ given by mapping an object X to the complex $\ldots \to 0 \to X \to 0 \to \ldots$ (concentrated in degree zero). Before discussing another 'natural' embedding of $\mathcal{C}$ into $\mathbf{C}(\mathcal{C})$ we need to introduce the fundamental notion of the cohomology of a complex.

Cohomology. Let $X := \ldots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \ldots$ be a complex in $\mathbf{C}(\mathcal{C})$. Denote by $\iota : \ker(d^n) \to X^n$ the natural embedding. By the universal property of $\ker(d^n)$ there exists a unique morphism $a^{n-1} : X^{n-1} \to \ker(d^n)$ such that $\iota a^{n-1} = d^{n-1}$.

The nth *cohomology of the complex* X is defined to be the object $H^n(X) := \operatorname{coker}(a^{n-1})$ in $\mathcal{C}$. For example, in R-Mod the cohomology is the factor module $H^n(X) = \ker(d^n)/\operatorname{im}(d^{n-1})$.

A complex X is called *exact in degree* n if $H^n(X) = 0$; X is called *exact* if $H^n(X) = 0$ for all $n \in \mathbb{Z}$.

Any morphism $f : X \to Y$ in $\mathbf{C}(\mathcal{C})$ induces maps $H^n(f) : H^n(X) \to H^n(Y)$. For example, in R-Mod the map $H^n(f)$ is given by $x + \operatorname{im}(d_X^{n-1}) \mapsto f(x) + \operatorname{im}(d_Y^{n-1})$.

Hence, we obtain for all $n \in \mathbb{Z}$ the *cohomology functor* $H^n : \mathbf{C}(\mathcal{C}) \to \mathcal{C}$.

Exercise: Show that if morphisms f, g in $\mathbf{C}(\mathcal{C})$ are homotopic, then they induce the same map in cohomology, i.e., $H^n(f) = H^n(g)$.

A category $\mathcal{C}$ has *enough projective objects* if for each object X in $\mathcal{C}$ there exists a projective object P in $\mathcal{C}$ and an epimorphism $P \to X$. In this case any object X has a *projective resolution*, i.e., a complex

$$P^* = \ldots \to P^{-n} \to P^{-n+1} \to \ldots \to P^{-1} \to P^0$$

with all P^i projective objects in $\mathcal{C}$, together with a map $\varepsilon : P^0 \to X$ (augmentation) such that the complex $\ldots \to P^{-1} \to P^0 \xrightarrow{\varepsilon} X \to 0$ is exact.

Dually, $\mathcal{C}$ has *enough injective objects* if for each object X there exists an injective object I and a monomorphism $\iota : X \to I$; in this case any object has an injective resolution $I^* = I^0 \to I^1 \to I^2 \ldots$, i.e., the complex $0 \to X \xrightarrow{\iota} I^*$ is exact.

Let $\mathcal{C}$ be an abelian category with enough projective objects (or enough injective objects). Then we can also define a map $\mathcal{C} \to \mathbf{C}(\mathcal{C})$ by sending X to a projective resolution (or injective resolution) of it. One idea behind derived categories is that *in the derived category an object X should be identified with all its projective and injective resolutions.*

Let X be an object in $\mathcal{C}$, and P^* a projective resolution of X with augmentation map $\varepsilon : P^0 \to X$. Then ε gives rise to a homomorphism of complexes (also denoted by ε)

$$\begin{array}{ccccccc} \dots & \longrightarrow & P^{-1} & \xrightarrow{d^{-1}} & P^0 & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow \varepsilon & & \\ \dots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

In degrees different from zero both complexes have zero cohomology, and clearly the induced map $H^n(\varepsilon)$ is the zero map for $n \neq 0$. In degree zero, ε induces an isomorphism

$$H^0(\varepsilon) : \underbrace{H^0(P^*)}_{coker(d^{-1})} \xrightarrow{\cong} \underbrace{H^0(X)}_{X}.$$

This observation leads to the next definition which is fundamental for the construction of derived categories.

Definition 3.1. *Let $f : X \to Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. Then f is called a* quasi-isomorphism *if f induces isomorphisms in cohomology, i.e., if for all n the map $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism.*

If $f, g : X \to Y$ are homotopic then $H^n(f) = H^n(g)$. So the definition also makes sense for morphisms in the homotopy category $\mathbf{K}(\mathcal{C})$.

Idea for the construction of derived categories: We want quasi-isomorphisms to be invertible (in order to identify an object X with all its resolutions). Therefore, we shall 'localize' the homotopy category $\mathbf{K}(\mathcal{C})$ with respect to the class of quasi-isomorphisms.

Localization of categories. Let $\mathcal{B}$ be a category and S a class of morphisms in $\mathcal{B}$. The aim is to define a localized category $\mathcal{B}[S^{-1}]$ and a localizing functor $L : \mathcal{B} \to \mathcal{B}[S^{-1}]$ with the following properties.

(i) $L(s)$ is an isomorphism in $\mathcal{B}[S^{-1}]$ for any $s \in S$.
(ii) Any functor $F : \mathcal{B} \to \mathcal{B}'$ (for some category $\mathcal{B}'$) transforming the elements of S into isomorphisms in $\mathcal{B}'$ factors uniquely over L.

The objects in $\mathcal{B}[S^{-1}]$ are defined to be the same as in $\mathcal{B}$. But the morphisms are changed. For each $s \in S$ take a variable x_s and construct an oriented graph Γ as follows. The vertices of Γ are the objects of $\mathcal{B}$. Any morphism in $\mathcal{B}$ gives an arrow in Γ; in addition we take for each $s \in S$ an arrow in the reverse direction, labelled by x_s. Two paths in Γ are called equivalent if they can be obtained from each other by a sequence of the following operations:

- Replace two consecutice arrows by their composition.

- Replace compositions $X \xrightarrow{s} Y \xrightarrow{x_s} X$ (resp. $Y \xrightarrow{x_s} X \xrightarrow{s} Y$) by $X \xrightarrow{id} X$ (resp. $Y \xrightarrow{id} Y$), or vice versa.

Define a *morphism in* $\mathcal{B}[S^{-1}]$ to be an equivalence class of paths in Γ with the same starting and end point. Composition of morphisms in $\mathcal{B}[S^{-1}]$ is induced by concatenation of paths.

The functor $L : \mathcal{B} \to \mathcal{B}[S^{-1}]$ is defined to be the identity on objects, and a morphism $X \to Y$ in $\mathcal{B}$ is mapped by L to its equivalence class in $\mathcal{B}[S^{-1}]$.

Exercise: Check that $\mathcal{B}[S^{-1}]$ and $L : \mathcal{B} \to \mathcal{B}[S^{-1}]$ satsify the required properties (i) and (ii) above.

We apply this general construction to the following situation: $\mathcal{C}$ an abelian category, $\mathbf{K}(\mathcal{C})$ the homotopy category of complexes, S the class of quasi-isomorphisms in $\mathbf{K}(\mathcal{C})$.

Proposition 3.2. *Let $\mathcal{C}$ be an abelian category. There exists a category $\mathbf{D}(\mathcal{C}) := \mathbf{K}(\mathcal{C})[S^{-1}]$ and a functor $L : \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C})$ such that:*

(i) *For any quasi-isomorphism f in $\mathbf{K}(\mathcal{C})$, $L(f)$ is an isomorphism in $\mathbf{D}(\mathcal{C})$.*

(ii) *Any functor $F : \mathbf{K}(\mathcal{C}) \to \mathcal{D}$ ($\mathcal{D}$ any category) transforming quasi-isomorphisms into isomorphisms factors uniquely through $L : \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C})$.*

Definition 3.3. $\mathbf{D}(\mathcal{C})$ *is called the* derived category *of the abelian category $\mathcal{C}$.*

Starting with the homotopy categories $\mathbf{K}^b(\mathcal{C})$, $\mathbf{K}^+(\mathcal{C})$, $\mathbf{K}^-(\mathcal{C})$ one similarly obtains derived categories $\mathbf{D}^b(\mathcal{C})$, $\mathbf{D}^+(\mathcal{C})$ and $\mathbf{D}^-(\mathcal{C})$. $\mathbf{D}^b(\mathcal{C})$ is called the bounded derived category *of $\mathcal{C}$.*

So far, the description of morphisms in the derived category is not very convenient. They are just represented by expressions of the form (writing $x_s = s^{-1}$) $f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \ldots \circ s_k^{-1} \circ f_{k+1}$ where the s_i are quasi-isomorphisms and the f_i are arbitrary morphisms in $\mathbf{K}(\mathcal{C})$. In order to 'compute' with such expressions one needs techniques like 'finding a common denominator' etc. This can be done with a suitable restriction on the class S of morphisms.

Definition 3.4. *Let $\mathcal{B}$ be a category and S a class of morphisms in $\mathcal{B}$. S is called a* multiplicative system *if the following conditions are satisfied:*

(S1) $\mathrm{id}_X \in S$ *for any object X in $\mathcal{B}$.*

(S2) *S is closed under composition.*

(S3) *Given $s \in S$ and a morphism f in $\mathcal{B}$, then there exist an object W and morphisms g, t where $t \in S$ such that*

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ t\downarrow & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array} \quad \textit{resp.} \quad \begin{array}{ccc} W & \xleftarrow{g} & Z \\ t\uparrow & & \uparrow s \\ X & \xleftarrow{f} & Y \end{array}$$

are commutative diagrams.

(S4) *For $f, g \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$ the following conditions are equivalent:*
- *there exists $s \in S$, $s \in \mathrm{Hom}_{\mathcal{B}}(Y, Y')$ for some Y' with $sf = sg$*
- *there exists $t \in S$, $t \in \mathrm{Hom}_{\mathcal{B}}(X', X)$ for some X' with $ft = gt$.*

Let us briefly reflect on the importance of multiplicative systems for the description of morphisms in localized categories. Consider in $(S3)$ the paths $x_s f$ and gx_t. As $ft = sg$ we have $x_s f t x_t = x_s s g x_t$. Using the equivalence relation on paths it follows that $x_s f = gx_t$ in $\mathcal{B}[S^{-1}]$. Hence, condition $(S3)$ yields that each expression $s^{-1} \circ f$ can be replaced by an expression of the form $g \circ t^{-1}$, i.e., in the description of morphisms in $\mathcal{B}[S^{-1}]$, like $f_1 \circ s_1^{-1} \circ \ldots \circ s_k^{-1} \circ f_{k+1}$, we can move all denominators to the right. Similarly, using the other diagram in $(S3)$ we could move all denominators to the left. Thus, if we localize a category $\mathcal{B}$ with respect to a multiplicative system S then any morphism in $\mathcal{B}[S^{-1}]$ can be represented by an expression of the form $f \circ s^{-1}$. We usually write them as *roofs*

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where $s \in S$ and f is a morphism in $\mathcal{B}$ (the equivalence relation on roofs corresponding to the equivalence of paths will be discussed below).

Proposition 3.5. *The quasi-isomorphisms form a multiplicative system in the homotopy category $\mathbf{K}(\mathcal{C})$.*

Rough Idea of Proof: (see e.g. [K-S], 1.6.7 for details)

Study a more general situation, namely localizing a triangulated category $\mathcal{B}$ with respect to a *null system.*

(Step 1) A class N of objects in $\mathcal{B}$ is called a *null system* if it satisfies the following conditions:

(N1) $0 \in N$.

(N2) $X \in N$ if and only if $X[1] \in N$.

(N3) If $X \to Y \to Z \to X[1]$ is a distinguished triangle and $X, Y \in N$, then $Z \in N$.

(Step 2) Given a null system N, the class

$$S(N) := \{f : X \to Y \mid \exists \text{ dist. triangle } X \xrightarrow{f} Y \to Z \to X[1] \text{ with } Z \in N\}$$

can be proved to be a multiplicative system.

(Step 3) For $\mathcal{B} = \mathbf{K}(\mathcal{C})$ we get that $N := \{X \in \mathbf{K}(\mathcal{C}) \mid H^n(X) = 0 \ \forall n \in \mathbb{Z}\}$ is a null system, and $S(N)$ is the class of quasi-isomorphisms. □

For a description of the morphisms in $\mathcal{B}[S^{-1}]$ in the language of roofs the notion of equivalence of paths has to be transferred. Let $\mathcal{B}$ be a category and S a multiplicative system. Call two roofs

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \sim \quad \begin{tikzcd} & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

equivalent if there exists a commutative diagram

$$\begin{tikzcd} & & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[drrr, "f" near start] & & X'' \arrow[dlll, "t"' near start] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd}$$

with $r \in S$. Then $\sim$ is an equivalence relation and the equivalence classes of roofs form the morphisms in $\mathcal{B}[S^{-1}]$ (see [G-M], III.2 for details). Moreover, the composition of morphisms is given by

$$\begin{tikzcd} & & X'' \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

where X'', $t' \in S$, f' exist by axiom $(S3)$ of a multiplicative system.

Proposition 3.6. *Let $\mathcal{C}$ be an abelian category. The derived category $\mathbf{D}(\mathcal{C})$ (and also $\mathbf{D}^*(\mathcal{C})$ for $* \in \{b, +, -\}$) is a triangulated category.*

Rough Idea of Proof: (1) Additivity: For adding two morphisms represented by roofs

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

we need a 'common denominator'. By $(S3)$ there exist morphisms s', t' in $\mathbf{K}(\mathcal{C})$ with t' a quasi-isomorphism such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{s'} & X'' \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow{s} & X \end{array}$$

is commutative. Now, s, t and t' are quasi-isomorphisms, so s' is a quasi-isomorphism. We can now define the sum of the two morphisms to be represented by the roof

$$\begin{tikzcd} & Z \arrow[dl, "ts'"'] \arrow[dr, "gs' + ft'"] & \\ X & & Y \end{tikzcd}$$

(2) Triangles: More generally, consider a triangulated category $\mathcal{B}$, a null system N and $L : \mathcal{B} \to \mathcal{B}[S(N)^{-1}]$ the canonical localization functor which is given by the identity on objects and which maps a morphism $f : X \to Y$ to the roof

$$\begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

Call a triangle in $\mathcal{B}[S(N)^{-1}]$ a distinguished triangle if it is isomorphic to the image under L of a distinguished triangle in $\mathcal{B}$. One can show that in this way, $\mathcal{B}[S(N)^{-1}]$ becomes a triangulated category. □

An example of the structure of $\mathbf{D}(\mathcal{C})$. Let $\mathcal{C}$ be an abelian category. Consider the full subcategory $\mathbf{C}_0(\mathcal{C}) = \{X = (X^n, d^n) \in \mathbf{C}(\mathcal{C}) \mid d^n = 0 \ \forall n\}$ of the category of complexes $\mathbf{C}(\mathcal{C})$. Because of the zero differentials there is an isomorphism of categories

$$\mathbf{C}_0(\mathcal{C}) \cong \prod_{n\in\mathbb{Z}} \mathcal{C}$$

given by $X \mapsto (X^n)_{n\in\mathbb{Z}}$. Moreover, there exists the cohomology functor

$$H : \mathbf{K}(\mathcal{C}) \to \mathbf{C}_0(\mathcal{C}) \ , \ \begin{cases} (X^n, d^n) \mapsto (H^n(X), 0) \\ (f : X \to Y) \mapsto (H^n(f) : H^n(X) \to H^n(Y)) \end{cases} .$$

The functor H transforms quasi-isomorphisms in $\mathbf{K}(\mathcal{C})$ into isomorphisms in $\mathbf{C}_0(\mathcal{C})$, hence it factors over the derived category

$$\begin{tikzcd} \mathbf{K}(\mathcal{C}) \arrow[rr, "H"] \arrow[dr, "L"'] & & \mathbf{C}_0(\mathcal{C}) \\ & \mathbf{D}(\mathcal{C}) \arrow[ur, "G"'] & \end{tikzcd}$$

where L is the localization functor and the functor G has to map the morphism represented by the roof

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

to the morphism $(H^n(f) \circ H^n(s)^{-1})_{n\in\mathbb{Z}}$ in $\mathbf{C}_0(\mathcal{C})$.

An abelian category $\mathcal{C}$ is called *semisimple* if each short exact sequence in $\mathcal{C}$ splits (i.e. is isomorphic to an exact sequence of the form $0 \to X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{\pi_2} Y \to 0$).

For example, if R is a semisimple ring then the module category R-mod is a semisimple category.

The following result shows that derived categories of semisimple categories have a particularly nice structure (a proof can be found in [G-M], III.2.4).

Proposition 3.7. *Let $\mathcal{C}$ be a semisimple category. Then $G : \mathbf{D}(\mathcal{C}) \xrightarrow{\simeq} \mathbf{C}_0(\mathcal{C})$ is an equivalence of categories.*

The indecomposable objects in the derived category $\mathbf{D}(\mathcal{C}) \simeq \mathbf{C}_0(\mathcal{C})$ of a semisimple category are precisely the *stalk complexes* $S[n] = \ldots \to 0 \to S \to 0 \to \ldots$ (concentrated in degree n) where $n \in \mathbb{Z}$ and S is a simple object in $\mathcal{C}$.

Let A, B be semisimple algebras over a field K and assume that there is an equivalence of triangulated categories $F : \mathbf{D}^b(A\text{-mod}) \xrightarrow{\simeq} \mathbf{D}^b(B\text{-mod})$. Let $\{S_1, \ldots, S_l\}$ be the set of simple A-modules (up to isomorphism) and similarly, let $\{T_1, \ldots, T_m\}$ be the set of simple B-modules. The equivalence functor F has to map indecomposable objects to indecomposable objects, i.e., there exists a bijection $\sigma : \{1, \ldots, l\} \to \{1, \ldots, m\}$ such that $F(S_i) = T_{\sigma(i)}[n_i]$ for some $n_i \in \mathbb{Z}$. (In fact, as F is a triangle equivalence, it can be checked that different simple A-modules can not be mapped to shifts of the same simple B-module.) In particular, *semisimple algebras with equivalent (bounded) derived module categories have the same number of simple modules.*

We want to apply this to a typical situation in group representation theory. Let G, H be finite groups, p a prime, and $(K, \mathcal{O}, k)$ a (splitting) p-modular system for G and H. Let $A = KGe$ be a p-block algebra of G and $B = KHf$ a p-block algebra of H. To any bounded complex $X \in \mathbf{D}^b(A\text{-mod})$ (and similarly for complexes over B) we can associate its *virtual character* $\sum_{n\in\mathbb{Z}}(-1)^n ch(X^n)$ where $ch(X^n)$ is the character afforded by the module X^n.

Then, an equivalence of triangulated categories $F : \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ induces a map on the virtual characters given by $ch(S_i) \mapsto (-1)^{n_i} ch(T_{\sigma(i)})$. Hence, an equivalence between the derived module categories of two blocks induces a *correspondence with signs* between the (ordinary) irreducible characters of the blocks KGe and KHf.

Example 3.8 (A_5 vs. A_4)**.** Let G be the alternating group A_5. The ordinary character table of A_5 has the form

	id	(..)(..)	(...)	(.....)	(.....)'
1	1	1	1	1	1
χ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3'	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

For a prime p denote by $B_p(G)$ the principal p-block. Let P be a Sylow p-subgroup of G. We shall show that in our example there exists an isomorphism between the groups of virtual characters

$$I_p : \mathbb{Z}Irr(B_p(N_G(P))) \xrightarrow{\cong} \mathbb{Z}Irr(B_p(G))$$

such that

- I_p is an isometry.
- I_p preserves character degrees modulo p.

- I_p preserves the values on p-elements.

Such a character correspondence I_p between blocks is called a *perfect isometry* [Br].

Consider our example in the case $p = 2$. Here, $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, a Klein four group, and $N_{A_5}(P) \cong A_4$. The character table of A_4 has the form

	id	(..)(..)	(...)	(...)'
1	1	1	1	1
α_1	1	1	$-\frac{1+\sqrt{-3}}{2}$	$-\frac{1-\sqrt{-3}}{2}$
α_1'	1	1	$-\frac{1-\sqrt{-3}}{2}$	$-\frac{1+\sqrt{-3}}{2}$
α_3	3	-1	0	0

The principal 2-blocks of A_5 and A_4 consist of the characters $Irr(B_2(A_5)) = \{1, \chi_3, \chi_3', \chi_5\}$ and $Irr(B_2(A_4)) = \{1, \alpha_1, \alpha_1', \alpha_3\}$. Then a perfect isometry is provided by the map

$$I_2 : \mathbb{Z}Irr(B_2(N_{A_5}(P))) \xrightarrow{\cong} \mathbb{Z}Irr(B_2(A_5)) \ , \ \begin{pmatrix} 1 \\ -\alpha_1 \\ -\alpha_1' \\ -\alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \chi_3 \\ \chi_3' \\ \chi_5 \end{pmatrix}.$$

Exercise: Find such perfect isometries between the principal blocks of A_5 and $N_{A_5}(P)$ for the primes $p = 3$ and 5.

By these perfect isometries, the ordinary characters of the principal 2-blocks of A_5 and A_4 are closely connected. Now the natural problem occurs how to 'explain' these similarities. We shall see that the corresponding block algebras are *not* Morita equivalent. To this end, let us have a closer look at the group algebras (where we have to be rather short on the representation theoretic background). Let k be an algebraically closed field of characteristic 2. The group algebra kA_4 has three simple modules, each one-dimensional, say L_0, L_1, L_2. The corresponding indecomposable projective kA_4-modules can be described by their radical layers:

$$\begin{matrix} & L_0 & \\ L_1 & \oplus & L_2 \\ & L_0 & \end{matrix} \qquad \begin{matrix} & L_1 & \\ L_2 & \oplus & L_0 \\ & L_1 & \end{matrix} \qquad \begin{matrix} & L_2 & \\ L_0 & \oplus & L_1 \\ & L_2 & \end{matrix}$$

Now we shall compare this structure with the situation for the alternating group A_5. The principal block B of the group algebra kA_5 (in fact, kA_5 decomposes into the principal block and one other block of defect zero) has three simple modules, the trivial module k of dimension 1 and M_1, M_2 each of dimension 2. The radical series of the indecomposable projective B-modules look like:

$$\begin{array}{ccc} \begin{array}{ccc} & k & \\ M_1 & & M_2 \\ k & \oplus & k \\ M_2 & & M_1 \\ & k & \end{array} & \qquad \begin{array}{c} M_1 \\ k \\ M_2 \\ k \\ M_1 \end{array} & \qquad \begin{array}{c} M_2 \\ k \\ M_1 \\ k \\ M_2 \end{array} \end{array}$$

From the different structures of the projective indecomposable modules it is now apparent that the module categories of the principal 2-block algebras of A_5 and A_4 can not be equivalent, i.e., the principal 2-blocks of A_5 and A_4 are not Morita equivalent. So in order to explain the phenomenon that the characters of the 2-blocks of A_5 and A_4 are nevertheless closely related by a perfect isometry we need weaker notions of equivalence than the classical Morita equivalence. This will be the main topic of the next section where we will also come back to this example.

4. Tilting Theory

Morita's theorem states that for rings R, S the corresponding module categories R-Mod and S-Mod are equivalent (as abelian categories) if there exists a progenerator P for R-Mod such that $S \cong \mathrm{End}_R(P)^{op}$. To compare module categories which are somehow similar without necessarily being Morita equivalent the following definition turns out to be useful.

Definition 4.1. *Let A be a finite dimensional algebra. A finitely generated A-module T is called a* tilting module *if it satisfies the following conditions:*

(i) *T has projective dimension 0 or 1 (i.e., there exists a short exact sequence $0 \to P_1 \to P_0 \to T \to 0$ with P_0, P_1 projective).*

(ii) $\mathrm{Ext}^1_A(T,T) = 0$, *i.e., T does not have selfextensions.*

(iii) *There exists an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_1, T_2 being direct summands of finite direct sums of copies of T.*

About 1980, S. Brenner and M. Butler proved an important result which compares the module categories of A and $B := \mathrm{End}_A(T)^{op}$ where T is a tilting module for A. The module categories are not equivalent in general but there exist certain functors mapping large subcategories equivalently onto each other. For a proof we refer to [B-B]; see also [H-R].

Theorem 4.2 (Brenner-Butler)**.** *Let A be a finite dimensional algebra, T a tilting module, and set $B := \mathrm{End}_A(T)^{op}$. Then the functors $\mathrm{Hom}_A(T,-)$ and $T \otimes_B -$ provide mutually inverse equivalences between the subcategory*

$$\mathcal{T}({}_AT) := \{ \ {}_AM \in A\text{-mod} \mid \mathrm{Ext}^1_A(T,M) = 0\}$$

of A-mod *and the subcategory*

$$\mathcal{Y}(T_B) := \{ \ {}_BN \in B\text{-mod} \mid \mathrm{Tor}^B_1(N,T) = 0\}$$

of B-mod. *Moreover, the functors* $\mathrm{Ext}^1_A(T,-)$ *and* $\mathrm{Tor}^B_1(-,T)$ *provide mutually inverse equivalences between the subcategories*

$$\mathcal{F}({}_AT) := \{\ {}_AM \in A\text{-mod} \mid \mathrm{Hom}_A(T,M) = 0\}$$

and

$$\mathcal{X}(T_B) := \{\ {}_BN \in B\text{-mod} \mid T \otimes_B N = 0\}.$$

Remark 4.3. (1) If T is a tilting module for A then the algebra $B = \mathrm{End}_A(T)^{op}$ is said to be *tilted from* A.

(2) Morita equivalence is a special case of tilting. In fact, any progenerator P for A-Mod is a tilting module. In this case, $\mathcal{T} = A$-mod and $\mathcal{Y} = B$-mod (and $\mathcal{F} = \mathcal{X} = 0$).

(3) For selfinjective algebras (e.g. group algebras, blocks) also the converse holds, i.e., any tilting module is a progenerator. In fact, for selfinjective algebras projective modules are injective which implies that any finite projective resolution splits. Therefore, each non-projective module for a selfinjective algebra has projective dimension ∞. Hence, for selfinjective algebras tilting is the same as Morita equivalence.

Digression: Quivers with relations. A *quiver* is a finite directed graph (possibly with loops and multiple arrows).

Let Q be a quiver and k an algebraically closed field. The *path algebra* kQ has as k-basis all paths in Q, including for each vertex x a trivial path e_x of length 0. Multiplication is induced by concatenation of paths if possible, and zero otherwise.

Example: The path algebra kQ of the quiver

$$\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet}$$

has dimension 6; a basis is given by $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$.

A *system of linear relations on* Q is a twosided ideal I of kQ contained in the ideal generated by all paths of length at least 2. The pair (Q,I) is called a *quiver with relations*. The *path algebra of* (Q,I) is the factor algebra kQ/I.

For the above quiver Q, let $I = \langle \alpha\beta \rangle$ (the ideal generated by $\alpha\beta$). Then (Q,I) is a quiver with relations, and kQ/I is an algebra of dimension 5.

The importance of the concept of quivers with relations becomes apparent from the next result of P. Gabriel [Ga1] published in 1980. A proof can also be found in the textbook [Be2].

Theorem 4.4 (Gabriel)**.** *Let* k *be an algebraically closed field and let* A *be a finite dimensional* k*-algebra. Then there exists a quiver with relations* (Q,I) *such that* A *is Morita equivalent to* kQ/I*.*

The identity element of a path algebra $A = kQ$ is given by the sum of all trivial paths, the trivial paths e_x being primitive idempotents. The projective indecomposable A-modules are then of the form Ae_x with k-basis given by all paths ending in the vertex

x. The corresponding simple A-modules are one-dimensional (spanned by the coset of e_x modulo the radical of Ae_x).

Example: Let $A = kQ$ be the above path algebra of the quiver

$$\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet}$$

Then A has three simple modules L_1, L_2, L_3. For the corresponding projective indecomposable A-modules we have $P_1 = Ae_1 = \langle e_1 \rangle = L_1$, $P_2 = Ae_2 = \langle e_2, \alpha \rangle$ and $P_3 = Ae_3 = \langle e_3, \beta, \alpha\beta \rangle$. In particular, they are uniserial, and their radical series is described by $\mathrm{rad}(P_2) \cong P_1 = L_1$, $\mathrm{rad}(P_3) \cong P_2$.

Now consider the A-module $T := P_2 \oplus P_3 \oplus L_2 \in A\text{-mod}$. Then T is a tilting module for A. In fact, $0 \to P_1 \to P_2 \to L_2 \to 0$ is a projective resolution of L_2, i.e., T has projective dimension 1. Moreover, since Ext^1-groups vanish for projective modules in the first argument, and since the kernel P_1 in the resolution of L_2 is projective we have $\mathrm{Ext}^1_A(T,T) = \mathrm{Ext}^1_A(L_2, T) = \underline{\mathrm{Hom}}_A(P_1, T) = 0$. Finally, we have an exact sequence

$$0 \to A \to P_2 \oplus P_2 \oplus P_3 \to L_2 \to 0$$

where the last two terms are direct summands of direct sums of copies of T. The algebra $B := \mathrm{End}_A(T)^{op}$ tilted from A can be determined by a close look on the homomorphisms between the direct summands of T. Each of the direct summands has endomorphism ring k; the only non-zero homomorphism spaces between different direct summands are $\mathrm{Hom}_A(P_2, P_3)$ and $\mathrm{Hom}_A(P_2, L_2)$, each of which is also one-dimensional. Thus, the algebra $B := \mathrm{End}_A(T)^{op}$ is isomorphic to the path algebra of the quiver

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

Exercise: Let $A = kQ$ for the quiver $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$ as above. Show that the module $T' := P_1 \oplus P_3 \oplus L_3$ is a tilting module for A and determine the tilted algebra $\mathrm{End}_A(T')^{op}$ as the path algebra of a quiver with relations.

Definition 4.5. *Let A, B be rings. Then A and B are called derived equivalent if there exists an equivalence of triangulated categories between the bounded derived categories $\mathbf{D}^b(A\text{-mod}) \xrightarrow{\simeq} \mathbf{D}^b(B\text{-mod})$.*

In particular, Morita equivalent rings are derived equivalent, but the converse does not hold, as will become clear soon. The link between tilting theory and derived categories is provided by the following result of D. Happel [Ha1], published in 1987.

Theorem 4.6 (Happel)**.** *Let k be an algebraically closed field and A a finite dimensional k-algebra. Let T be a tilting module for A and set $B = \mathrm{End}_A(T)^{op}$. Then there exists an equivalence of triangulated categories between the bounded derived categories $\mathbf{D}^b(A\text{-mod}) \xrightarrow{\simeq} \mathbf{D}^b(B\text{-mod})$.*

The next definition is fundamental.

Definition 4.7. *Let R be a ring. A* tilting complex T *over R is a bounded complex of finitely generated projective R-modules with the following properties:*

(i) $\mathrm{Hom}_{\mathbf{D}^b(R)}(T, T[i]) = 0$ *for* $i \neq 0$ *(where* $[.]$ *is the shift).*
(ii) add*(T), the full subcategory of* $\mathbf{K}^b(P_R)$ *consisting of direct summands of direct sums of copies of T, generates* $\mathbf{K}^b(P_R)$ *as a triangulated category.*

Where do tilting complexes occurΓ Recall from Section 3 that in the derived category $\mathbf{D}(A\text{-mod})$ we can identify each module (viewed as a complex concentrated in degree zero) with any of its projective (or injective) resolutions.

Example 4.8. As above, let A be the path algebra of the quiver $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet}$. Take the tilting module $T = P_2 \oplus P_3 \oplus L_2$. In $\mathbf{D}^b(A\text{-mod})$ we may replace the module T by its projective resolution

$$\bar{T}: \; 0 \to P_1 \xrightarrow{(0,0,\alpha)} P_2 \oplus P_3 \oplus P_2 \to 0$$

(if a map is denoted by a path in the quiver then this always means right multiplication by this element). We claim that $\bar{T}$ is a tilting complex over A. We first show that $\mathrm{Hom}_{\mathbf{K}^b}(\bar{T}, \bar{T}[i]) = 0$ for $i \neq 0$ (then this also holds in the localized category $\mathbf{D}^b(A\text{-mod})$). The assertion is clear for $|i| \geq 2$ since $\bar{T}$ is concentrated in two degrees. For $i = 1$ a basis of the non-zero morphisms in the category of complexes is given by the maps $(\alpha, 0, 0)$, $(0, \alpha\beta, 0)$ and $(0, 0, \alpha)$, which are easily checked to be homotopic to zero. For $i = -1$ we already have $\mathrm{Hom}_{\mathbf{C}^b}(\bar{T}, \bar{T}[i]) = 0$ since there are no non-zero homomorphisms from P_2 or P_3 to P_1.

Secondly, we have to check that add$(\bar{T})$ generates $\mathbf{K}^b(P_A)$ as a triangulated category. Recall that $P_i[n]$ denotes the stalk complex with P_i concentrated in degree n. By definition, $P_2[0]$ and $P_3[0]$ are in add$(\bar{T})$ and therefore also $P_2[n]$ and $P_3[n]$ for all n since add$(\bar{T})$ is triangulated. We shall show that $P_1[n]$ also is in add$(\bar{T})$. In fact, there exists a homomorphism of complexes f from $P_2[0]$ to the complex $T_3: \; 0 \to P_1 \xrightarrow{\alpha} P_2 \to 0$ (the projective resolution of L_2) given by id_{P_2} in degree zero. Its mapping cone is $M(f): \; 0 \to P_2 \oplus P_1 \xrightarrow{(id,\alpha)} P_2 \to 0$. Now one can check that $M(f)$ is isomorphic in $\mathbf{K}^b(A\text{-mod})$ to $P_1[1]$ (in fact, define maps $\phi := (0, \mathrm{id}) : M(f) \to P_1[1]$ and $\psi := \begin{pmatrix} -\alpha \\ \mathrm{id} \end{pmatrix} : P_1[1] \to M(f)$ and observe that $\phi\psi = \mathrm{id}$ and $\psi\phi \sim \mathrm{id}$). Hence, there is a triangle in $\mathbf{K}^b(A\text{-mod})$ of the form

$$\underbrace{P_2[0]}_{\in add(\bar{T})} \xrightarrow{f} \underbrace{T_3}_{\in add(\bar{T})} \longrightarrow P_1[1] \longrightarrow \underbrace{P_2[1]}_{\in add(\bar{T})} .$$

It follows that $P_1[1]$ and thus also $P_1[n]$ is in add$(\bar{T})$ for all n. But having all stalk complexes of the projective indecomposable modules in add$(\bar{T})$ implies that add$(\bar{T})$ generates $\mathbf{K}^b(A\text{-mod})$ as a triangulated category (just adapt the above argument for the isomorphism between $M(f)$ and $P_1[1]$ to see that one then can actually construct all bounded complexes of projective modules).

The importance of tilting complexes became clear from the work of J. Rickard around 1990, in which he obtained necessary and sufficient conditions for when the derived module categories of two rings are equivalent.

Theorem 4.9 (Rickard, [Ri4]). *For two rings R, S the following conditions are equivalent:*

(a) $\mathbf{K}^-(R\text{-Proj})$ *and* $\mathbf{K}^-(S\text{-Proj})$ *are equivalent as triangulated categories.*
(b) $\mathbf{D}^b(R\text{-Mod})$ *and* $\mathbf{D}^b(S\text{-Mod})$ *are equivalent as triangulated categories.*
(c) $\mathbf{K}^b(R\text{-Proj})$ *and* $\mathbf{K}^b(S\text{-Proj})$ *are equivalent as triangulated categories.*
(b) $\mathbf{K}^b(P_R)$ *and* $\mathbf{K}^b(P_S)$ *are equivalent as triangulated categories.*
(e) *There exists a tilting complex T for R such that* $\mathrm{End}_{\mathbf{D}^b(R)}(T)^{op} \cong S$.

If R, S are finite dimensional algebras over a field then moreover, (a)-(e) are equivalent to

(f) $\mathbf{D}^b(R\text{-mod})$ *and* $\mathbf{D}^b(S\text{-mod})$ *are equivalent as triangulated categories.*

Remark 4.10. Note the similarity of condition (e) with the classical Morita theorem. In Morita theory one can explicitly describe the functors providing the equivalence between the module categories. This could also be achieved by Rickard in the context of equivalences of derived module categories. Roughly speaking, algebras A, B are derived equivalent if and only if there exist a certain bounded complex X of A-B-bimodules such that the equivalence between the bounded derived categories is given by the left derived functor of $X \otimes_B -$. Such a complex of bimodules is then called a *twosided tilting complex.* But even to state the main result precisely would require a lot of additional work, e.g. a discussion of derived functors, which is beyond the scope of this elementary introduction. The interested reader is referred to the original article [Ri2].

Example 4.11 (A_5 vs. A_4 revisited)**.** In Example 3.9 we observed a close connection between the ordinary characters in the principal 2-blocks of the alternating groups A_5 and A_4 which could not be explained by a Morita equivalence. But as we shall show now *the block algebras of the principal 2-blocks of A_5 and A_4 are derived equivalent.* It is again convenient to work with quivers and relations. As above let k be an algebraically closed field of characteristic 2. By Gabriel's theorem the block algebra kA_4 and the principal 2-block B of kA_5 can be described by quivers with relations, up to Morita equivalence. These quivers with relations can be deduced from the structure of the radical layers of the projective indecomposable modules given in Example 3.9. Namely, the group algebra kA_4 is isomorphic to the algebra kQ/I where Q is the quiver

$$\begin{tikzcd}
0 \arrow[rr, shift left, "a"] \arrow[dr, shift left, "f"] & & 1 \arrow[ll, shift left, "b"] \arrow[dl, shift left, "c"] \\
& 2 \arrow[ul, shift left, "e"] \arrow[ur, shift left, "d"] &
\end{tikzcd}$$

and I is the ideal generated by $ac,\ ce,\ ea,\ bf,\ fd,\ db,\ ab - fe,\ cd - ba,\ ef - dc$.

The principal 2-block algebra B of kA_5 is Morita equivalent to the algebra $\bar{B}$ given by the quiver (where the vertex 0 corresponds to the trivial module k)

$$\begin{tikzcd}
1 \arrow[r, shift left, "a"] & 0 \arrow[l, shift left, "b"] \arrow[r, shift left, "c"] & 2 \arrow[l, shift left, "d"]
\end{tikzcd}$$

and the ideal generated by $ab,\ dc,\ bacd - cdba$.

We define a complex of projective $\bar{B}$-modules as follows: Denote by P_i the projective indecomposable $\bar{B}$-module corresponding to the vertex i. Set $T_0:\ 0 \to P_1 \oplus P_2 \xrightarrow{(a,d)} P_0$ (where P_0 is in degree 0), $T_1:\ 0 \to P_1 \to 0$ and $T_2:\ 0 \to P_2 \to 0$ (concentrated in degree -1). Then it can be shown (a good exercise for the more experienced reader; a detailed proof can be found in [Li1]) that

- $T := T_0 \oplus T_1 \oplus T_2$ is a tilting complex for $\bar{B}$.
- $\mathrm{End}_{\mathbf{D}^b}(T)^{op} \cong kA_4$.

By Rickard's theorem it follows that the block algebras of the principal 2-blocks of A_5 and A_4 are derived equivalent. This was first observed by Rickard in [Ri3]; more generally he showed that the principal 2-blocks are derived equivalent as $\mathcal{O}$-algebras where $\mathcal{O}$ is a complete discrete valuation ring having k as residue field. (It is a general fact that any derived equivalence between blocks over a complete discrete valuation ring $\mathcal{O}$ reduces to a derived equivalence over the residue field $k = \mathcal{O}/J(\mathcal{O})$ [Ri2], 2.2.) Such a derived equivalence in characteristic zero then 'explains' the existence of a perfect isometry as observed in Example 3.9 (and the foregoing remarks).

5. Brauer tree algebras

An important class of algebras occurring in the representation theory of finite groups are Brauer tree algebras, to be discussed in this section. A very good reference for this subject is J. Alperin's book [Al1]. Throughout this section let k be an algebraically closed field.

A finite connected quiver Q is called a *Brauer quiver* if

- Q is the union of the cycles contained in it.
- Every vertex belongs to exactly two cycles.
- Any two cycles meet in at most one vertex.
- To each cycle in Q we assign a *multiplicity* $m \in \mathbb{N}$ such that at most one cycle (the exceptional cycle) has a multiplicity greater than one.

From the definition it is clear that the edges of a Brauer quiver can be divided into two families, denoted by α and β, such that edges of different adjacent cycles are not in the same family.

Examples: In the figure below the β-arrows are indicated by dotted lines.

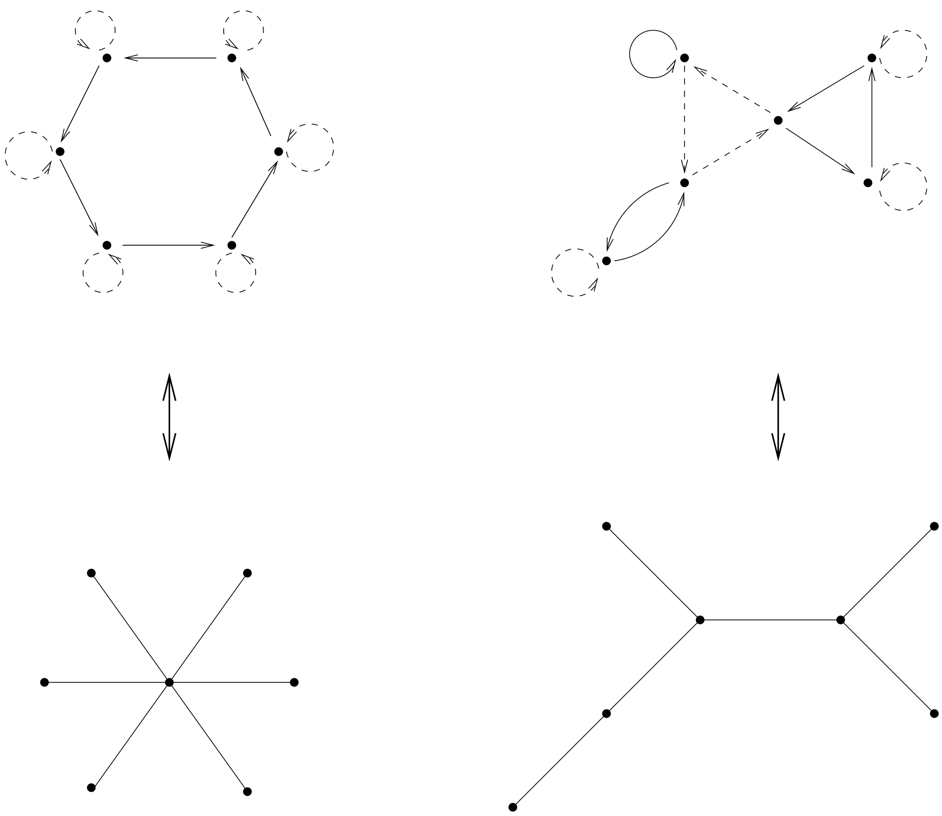


Figure 1

There is an equivalent combinatorial description. To any Brauer quiver Q one can associate a tree T by the following procedure (see Figure 1). Any cycle in Q gives a vertex in T; two vertices in T are joined by an edge if their corresponding cycles in Q have a common vertex. The vertex in T corresponding to the exceptional cycle in Q is the *exceptional vertex*. The tree T has the following properties

(i) T is a finite connected undirected tree.
(ii) Each vertex has a multiplicity $m \in \mathbb{N}$ such that at most one vertex has a multiplicity greater than one (induced from the multiplicities in Q).
(iii) For each vertex in T there is a cyclic ordering of the edges adjacent to it (induced from the orientation of Q).

Any tree with the properties (i)-(iii) is called a *Brauer tree*. A Brauer tree all of whose edges are adjacent to a single vertex is called a *star* (see Figure 1).

Conversely, we leave it to the reader to figure out how to assign in a similar way to any Brauer tree a Brauer quiver. Hence, there is a one-one correspondence between Brauer quivers and Brauer trees.

Definition 5.1. *Let Q be a Brauer quiver. Denote by C the exceptional cycle, and by m its multiplicity.*
(a) The basic Brauer tree algebra corresponding to *Q is defined as kQ/I where the ideal I is generated by the relations*

- $\alpha\beta = 0 = \beta\alpha$
- $\alpha^{x_\alpha} = \beta^{x_\beta}$ *for each vertex $x \notin C$ (where x_α resp. x_β is the length of the α resp. β cycle containing x)*
- $(\alpha^{x_\alpha})^m = \beta^{x_\beta}$ *(resp. $\alpha^{x_\alpha} = (\beta^{x_\beta})^m$) for $x \in C$ and C an α-cycle (resp. β-cycle).*

(b) A k-algebra is called a Brauer tree algebra *if it is Morita equivalent to a basic Brauer tree algebra.*

Brauer tree algebras were first discovered in group representation theory. Let G be a finite group, p a prime dividing the order of G, and k an (algebraically closed) field of characteristic $p > 0$. Assume the group algebra kG has a block B whose defect group D is cyclic of order p^d. Denote by e the number of simple modules in B.

Theorem 5.2 (Dade [Da1]). *B is a Brauer tree algebra with multiplicity $\frac{p^d-1}{e}$.*

For the rest of this section we will discuss the following result of Rickard on derived equivalences for Brauer tree algebras.

Theorem 5.3 (Rickard [Ri1]). *Up to derived equivalence, a Brauer tree algebra is determined by the number of edges of the Brauer tree and the multiplicity of the exceptional vertex.*

The idea of the proof is to show that given an arbitrary Brauer tree, the corresponding algebra is derived equivalent to the Brauer tree algebra for the star with the same number of edges and exceptional vertex in the center with same multiplicity.

Proof: Let B be a Brauer tree algebra, where we can assume that B is basic. For any vertex z in Q denote by $P(z)$ the corresponding projective indecomposable module $P(z) = Be_z$. There is a unique shortest path in Q from some vertex x (depending on z) of the exceptional cycle to z, say

$$x \xrightarrow{\alpha} \dots \xrightarrow{\alpha} z_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} z_2 \to \dots \to z_t \xrightarrow{\alpha/\beta} \dots \xrightarrow{\alpha/\beta} z$$

(where α/β means α or β depending on whether the path ends in an α- or a β-cycle). Define a complex $T(z) \in \mathbf{K}^b(P_B)$ as

$$T(z):\ 0 \to P(x) \xrightarrow{\alpha\dots\alpha} P(z_1) \xrightarrow{\beta\dots\beta} P(z_2) \longrightarrow \dots \longrightarrow P(z_t) \xrightarrow{\alpha/\beta\dots\alpha/\beta} P(z) \to 0$$

where $P(x)$ is in degree zero.

Claim: The direct sum of complexes $T = \oplus_{z\in Q} T(z)$ is a tilting complex for B.

1) We first have to show that $\mathrm{Hom}(T, T[i]) = 0$ for all $i \neq 0$. This is clear for $|i| \geq 2$ since $\mathrm{Hom}(P(i), P(j)) = 0$ if i, j do not belong to a common cycle (use the relations $\alpha\beta = 0 = \beta\alpha$).

Consider the case $i = -1$ and let $0 \neq f = (f_i) : T(y) \to T(z)[-1]$ be a homomorphism of complexes, and choose s minimal with the property $f_{s+1} \neq 0$, i.e., we have a diagram of the form

$$\begin{array}{ccccccccccccc} 0 & \to & P(x') & \to & P(y_1) & \to & \dots & \to & P(y_s) & \to & P(y_{s+1}) & \to & \dots \\ & & & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_{s+1}} & & \\ & & 0 & \to & P(x) & \to & \dots & \to & P(z_{s-1}) & \to & P(z_s) & \to & \dots \end{array}$$

From this we can deduce that $y_s = z_s$. In fact, since $f_{s+1} \neq 0$ the vertex z_s lies on a cycle together with y_s and y_{s+1} (use the relations $\alpha\beta = 0 = \beta\alpha$); but then the paths from the exceptional cycle to y_s and z_s have to coincide, i.e. $y_s = z_s$. It follows that the image of the composition $P(y_s) \to P(y_{s+1}) \xrightarrow{f_{s+1}} P(z_s)$ is equal to the socle of $P(z_s)$, i.e., it is non-zero, a contradiction to f being a homomorphism of complexes.

A similar argument works for the case $i = 1$; we leave the details to the reader.

2) Secondly, we have to show that $\mathrm{add}(T)$ generates $\mathbf{K}^b(P_B)$ as a triangulated category. The main (but straightforward) step is left as an exercise.

Exercise: For a vertex z in Q let $P(z)[0]$ be the stalk complex $\dots 0 \to P(z) \to 0 \dots \in \mathbf{K}^b(P_B)$ (concentrated in degree zero). Then $P(z)[0]$ is isomorphic in $\mathbf{K}^b(P_B)$ to the mapping cone of the homomorphism of complexes

$$\begin{array}{cccccccccccccc} T(z)[t]: & 0 & \to & P(x) & \to & P(z_1) & \to & \dots & \to & P(z_t) & \to & P(z) & \to & 0 \\ & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} & & \\ T(z_t)[t]: & 0 & \to & P(x) & \to & P(z_1) & \to & \dots & \to & P(z_t) & \to & 0 & \to & 0 \end{array}$$

Using the result of the exercise we obtain a triangle

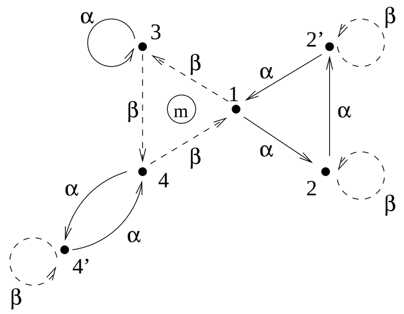
$$\underbrace{T(z)}_{\in add(T)} \longrightarrow \underbrace{T(z_t)}_{\in add(T)} \longrightarrow P(z)[-t] \to \underbrace{T(z)[1]}_{\in add(T)}.$$

By definition $\mathrm{add}(T)$ is a triangulated category, so it follows that also $P(z) \in \mathrm{add}(T)$.

As we now have shown that all stalk complexes of all projective indecomposable modules are in $\mathrm{add}(T)$, the homotopy category is generated by $\mathrm{add}(T)$ as a triangulated category, i.e., T is indeed a tilting complex for B.

Now, Rickard's main theorem states that the Brauer tree algebra B is derived equivalent to the endomorphism ring $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$. We shall show that the latter is isomorphic to the Brauer tree algebra for the star, as desired. In these notes we do not intend to include a complete proof. This could be found in [Ri1]; but note that this reference may be a bit unsatisfactory for the novice as the argument there uses at a crucial point some deep results of Gabriel and Riedtmann (on algebras stably equivalent to Brauer tree algebras). Instead of that we shall discuss a particular example in a more elementary way, in order to get across the flavour of the proof.

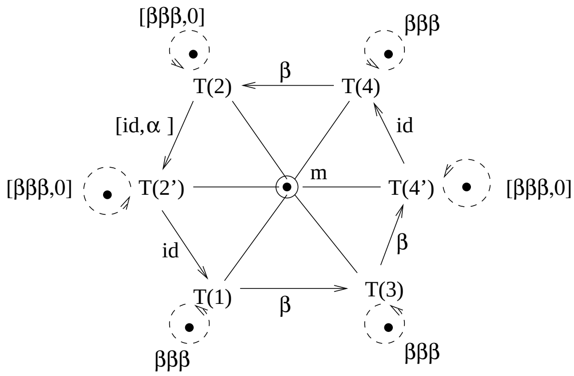
Consider the (basic) Brauer tree algebra for the Brauer quiver



(where the cycle indicated by the encircled multiplicity m is the exceptional cycle). We follow the above construction for the complexes $T(z)$ (the direct summands of the tilting complex T) and we consider the following homomorphisms between them

$$\begin{array}{cccccccc}
T(2): & 0 & \rightarrow & P(1) & \stackrel{\alpha}{\longrightarrow} & P(2) & \rightarrow & 0 \\
 & & & \downarrow \mathrm{id} & & \downarrow \alpha & & \\
T(2'): & 0 & \rightarrow & P(1) & \stackrel{\alpha\alpha}{\longrightarrow} & P(2') & \rightarrow & 0 \\
 & & & \downarrow \mathrm{id} & & & & \\
T(1): & 0 & \rightarrow & P(1) & \rightarrow & 0 & & \\
 & & & \downarrow \beta & & & & \\
T(3): & 0 & \rightarrow & P(3) & \rightarrow & 0 & & \\
 & & & \downarrow \beta & & & & \\
T(4'): & 0 & \rightarrow & P(4) & \stackrel{\alpha}{\longrightarrow} & P(4') & \rightarrow & 0 \\
 & & & \downarrow \mathrm{id} & & & & \\
T(4): & 0 & \rightarrow & P(4) & \rightarrow & 0 & & \\
 & & & \downarrow \beta & & & & \\
T(2): & 0 & \rightarrow & P(1) & \stackrel{\alpha}{\longrightarrow} & P(2) & \rightarrow & 0
\end{array}$$

The homomorphisms given in this diagram will be denoted by $\tilde{\alpha}$. Moreover, on each $T(z)$ there exists an endomorphism $\tilde{\beta}$ as indicated in the following figure.



This means that we have found arrows in the quiver of $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ giving a circular Brauer quiver (with exceptional cycle in the middle). What can be said about the relationsΓ

Exercise: Check that the following relations hold up to homotopy: $\tilde{\alpha}^{6m+1} = 0$, $\tilde{\alpha}\tilde{\beta} = 0 = \tilde{\beta}\tilde{\alpha}$.

In order to actually pin down the structure of $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ as a Brauer tree algebra we need to know that there are no further defining relations. For this the following result is often useful which gives a general method for computing the Cartan invariants of endomorphism rings of tilting complexes from the Cartan invariants of the algebra B. A proof (which is not too difficult but would lead us astray from our main topic) can be found in [Ha2], III.1.3, III.1.4.

Proposition 5.4. *For a k-algebra B let $Q = (Q^r)_{r\in\mathbb{Z}}$ and $R = (R^s)_{s\in\mathbb{Z}}$ be bounded complexes of projective B-modules. Then*

$$\sum_i (-1)^i \dim_k \mathrm{Hom}_{\mathbf{D}^b(B)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim_k \mathrm{Hom}_B(Q^r, R^s).$$

In particular, if Q and R are direct summands of tilting complexes then

$$\dim_k \mathrm{Hom}_{\mathbf{D}^b(B)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim_k \mathrm{Hom}_B(Q^r, R^s).$$

With this proposition at hand we can continue with our example. The algebra B has the following Cartan matrix (the rows and columns indexed by the simple modules 1,2,2',3,4,4'):

$$\begin{pmatrix} m+1 & 1 & 1 & m & m & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ m & 0 & 0 & m+1 & m & 0 \\ m & 0 & 0 & m & m+1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Using the preceding proposition we can compute the Cartan matrix of the endomorphism ring of T to be

$$\begin{pmatrix} m+1 & m & m & m & m & m \\ m & m+1 & m & m & m & m \\ m & m & m+1 & m & m & m \\ m & m & m & m+1 & m & m \\ m & m & m & m & m+1 & m \\ m & m & m & m & m & m+1 \end{pmatrix}$$

This is exactly the Cartan matrix of the Brauer tree algebra $B(6,m)$ with Brauer tree a star with 6 edges and with exceptional vertex in the center with multiplicity m.

Looking back, we have defined homomorphisms between summands of the tilting complex T giving a part of the Brauer quiver of $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ being the same as the Brauer quiver for $B(6,m)$. Moreover, we have seen in the Exercise that the defining relations of $B(6,m)$ hold in $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ and that the Cartan matrices of $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ and $B(6,m)$ coincide. Hence, there can not be any further defining relations in $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ (each one would decrease the dimension), i.e., we obtain that $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op} \cong B(6,m)$, as desired.

This example already indicates the general behaviour. Starting with an arbitrary Brauer tree algebra B, define the tilting complex $T = \oplus_{z\in Q} T(z)$. Then there is always a circular ordering of the direct summands of T, and homomorphisms $\tilde{\alpha}$, $\tilde{\beta}$ between them, proving $\mathrm{End}_{\mathbf{D}^b(B)}(T)^{op}$ to be a Brauer tree algebra of the form $B(e,m)$. □

If B is a block of a modular group algebra kG with cyclic defect group D, then B is a Brauer tree algebra. Let b be the Brauer correspondent of B in the group algebra of the normalizer $N_G(D)$, i.e., b is also a block with defect group D (which is now normal in $N_G(D)$), hence a Brauer tree algebra. But a cyclic block whose defect group is normal has Brauer tree a star with exceptional vertex in the center ([Al1], Theorem 19.1). So, passing from an arbitrary cyclic block B to its Brauer correspondent means on the level of Brauer trees to pass from an arbitrary Brauer tree to a star with exceptional vertex in the center where the number of edges and the multiplicity remain unchanged. Hence, Rickard's result on derived equivalence for Brauer tree algebras has the following consequence for blocks with cyclic defect group.

Corollary 5.5. *Let B be a block of a modular group algebra kG with cyclic defect group D and let b be the Brauer correspondent of B in $N_G(D)$. Then B and b are derived equivalent.*

This is a special case of a very important conjecture of M. Broué stating that derived equivalences should occur for a much larger class of blocks (and thus explaining a lot of similarities between blocks which are not necessarily Morita equivalent).

Conjecture 5.6 (Broué). *Assume char $k = p > 0$, and let B be a block of kG with* abelian *defect group D and b its Brauer correspondent in $kN_G(D)$. Then B and b are derived equivalent.*

Broué's conjecture is very hard to check, even on examples. The following list contains those cases (to the best of the author's knowledge) where Broué's conjecture has been verified up to now.

- [Ri1], [Li2] p arbitrary, D a cyclic group
- [Er], [Li1], [Li3] $p = 2$, D a Klein four group.
- [Ro] $G = SL_2(8)$, principal 2-block.
- [Ro] $G = PSL_2(9)$, principal 3-block.
- [Ok] B principal 3-block of one of:
  alternating groups A_6, A_7, A_8,
  symmetric group S_6,
  Mathieu groups M_{11}, M_{21}, M_{22}, M_{23},
  Higman-Sims group HS.
- [Ho3] $G = HS$, B non-principal 3-block of maximal defect.
- [Ch] All blocks of symmetric groups whose defect groups have order p^2.

6. Applications and Outlook

In this section we intend to discuss some of the main applications of derived equivalences. This will not at all be an exhaustive treatment but only an overview without giving proofs but pointing to the relevant literature.

Invariants of the derived category. A derived equivalence, although being a far weaker notion than e.g. Morita equivalence, still preserves some important properties and invariants of rings and algebras. In this section we list some of these invariants of the derived module category.

Assume Λ, Γ are rings for which there exists an equivalence of triangulated categories between their bounded derived module categories $\mathbf{D}^b(\Lambda\text{-Mod}) \xrightarrow{\simeq} \mathbf{D}^b(\Gamma\text{-Mod})$. Then

- (Grothendieck groups, [Ri4]) $K_0(\Lambda) \cong K_0(\Gamma)$
- (centers, [Ri4]) $Z(\Lambda) \cong Z(\Gamma)$ as rings
- (Hochschild cohomology, [Ri2]) $HH^*(\Lambda) \cong HH^*(\Gamma)$ as rings
- (cyclic homology, [Ke3]) $HC_*(\Lambda) \cong HC_*(\Gamma)$.

Let k be a field and A, B derived equivalent k-algebras. Then

- ([Ri2]) A symmetric $\Longrightarrow$ B symmetric
- ([Ha2]) A has finite global dimension $\Longrightarrow$ B has finite global dimension
- (number of simple modules, [Ha2]) $l(A) = l(B)$.

Let $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$ be p-blocks of finite groups, where $(K, \mathcal{O}, k)$ is a (splitting) p-modular system. Assume that A and B are derived equivalent. Then

- KGe and KHf have the same number of (ordinary) irreducible characters.
- kGe and kHf have the same number of simple modules.

Broué's conjecture vs. other conjectures. Modular representation theory is an area in which there are surprisingly many important longstanding open conjectures. (For a very readable survey on various conjectures in modular representation theory see [Ku].) One important feature of Broué's conjecture is that it is related to some of these conjectures.

Let G be a finite group. Moreover, let B a block of kG with defect group D, and b its Brauer correspondent in $kN_G(D)$. For a block B of kG denote by $l(B)$ the number of simple kG-modules lying in B.

The block version of Alperin's conjecture claims that the number of simple modules in B is determined by the number of simple projective modules in various local blocks, i.e., blocks of normalizers of p-subgroups (see [Al2] for a precise statement). In the case of blocks with abelian defect groups this conjecture takes a particularly simple form.

Conjecture 6.1. (Alperin's conjecture for blocks with abelian defect groups)
Let B be a block of kG with abelian defect group D and b its Brauer correspondent in $kN_G(D)$. Then $l(B) = l(b)$.

Another important open conjecture is the Alperin-McKay conjecture. Again, we only state the block version in the case of abelian defect.

Conjecture 6.2. (Alperin–McKay's conjecture for blocks with abelian defect groups)
Let B be a block of kG with abelian defect group D and b the Brauer correspondent in $kN_G(D)$. Then the centers $Z(B)$ and $Z(b)$ have the same dimension over k.

Both conjectures are known to hold for solvable groups G; in fact, in this case the blocks are Morita equivalent by a result of Dade [Da2]. But the conjectures are open in general (even for blocks with abelian defect groups). Comparing them with the statement of Broué's conjecture we see from the results of the previous section that the validity of Broué's conjecture implies Alperin's conjecture and the Alperin-McKay conjecture for blocks with abelian defect groups. In fact, derived equivalent blocks have the same number of simple modules, and their centers have to be isomorphic (even as rings).

Derived equivalence vs. stable equivalence. Let A be a selfinjective algebra. For X, Y in A-Mod denote by $\mathrm{PHom}(X, Y)$ the morphisms which factor through a projective (=injective) module. The *stable module category* A-$\underline{\mathrm{mod}}$ has as objects the same as A-Mod, but the morphisms are equivalence classes $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}(X, Y)/\mathrm{PHom}(X, Y)$. Algebras are called *stably equivalent* if they have equivalent stable module categories. The stable module category of a selfinjective algebra carries the structure of a triangulated category [Ha2].

A full triangulated subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C}$ is called an *épaisse subcategory* if whenever $X \to Y$ is a morphism in $\mathcal{C}$ contained in a distinguished triangle $X \to Y \to Z \to X[-1]$ where Z is in $\mathcal{D}$, and if the map $X \to Y$ factors through an object of $\mathcal{D}$, then X and Y are in $\mathcal{D}$. For the theory of taking quotients of triangulated categories by épaisse subcategories we refer to [Ve].

Consider the natural embedding $\mathbf{K}^b(P_A) \to \mathbf{D}^b(A\text{-mod})$; the essential image of this map, i.e., the full subcategory of $\mathbf{D}^b(A\text{-mod})$ consisting of all objects isomorphic to objects of $\mathbf{K}^b(P_A)$, is an épaisse subcategory of $\mathbf{D}^b(A\text{-mod})$ [Ri1]. Consider the diagram

$$\begin{tikzcd} A\text{-mod} \arrow[r, "I"] \arrow[dr, "G"'] & \mathbf{D}^b(A\text{-mod}) \arrow[r, "L"] & \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(P_A) \\ & A - \underline{\text{mod}} \arrow[ur, "F"'] & \end{tikzcd}$$

where I is the natural embedding, L and G the respective quotient functors. The composition $L \circ I$ sends projective modules to zero, hence it factors over the stable module category, i.e., there exists the functor F as indicated, making the diagram commutative. The crucial result now is that the stable module category can in this context be described as a quotient of the derived module category (thus also giving a new description of the triangulated structure of the stable module category via the triangulated structure of the derived category). For a proof see [Ri1], 2.1.

Theorem 6.3 ([Ri1])**.** *Let A be a selfinjective algebra. Then the above functor*

$$F : A\text{-}\underline{\text{mod}} \xrightarrow{\simeq} \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(P_A)$$

is an equivalence of triangulated categories.

For our purposes a consequence of this theorem is of particular interest.

Corollary 6.4. *Let A and B be selfinjective algebras. If A and B are derived equivalent then they are stably equivalent.*

Proof of the Corollary: Rickard's main theorem states that an equivalence between the derived categories $\mathbf{D}^b(A\text{-mod})$ and $\mathbf{D}^b(B\text{-mod})$ implies an equivalence between the triangulated categories $\mathbf{K}^b(P_A)$ and $\mathbf{K}^b(P_B)$. By the preceding theorem and by taking quotients we obtain equivalences

$$A\text{-}\underline{\text{mod}} \xrightarrow{\simeq} \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(P_A) \xrightarrow{\simeq} \mathbf{D}^b(B\text{-mod})/\mathbf{K}^b(P_B) \xleftarrow{\simeq} B\text{-}\underline{\text{mod}}$$

which compose to give the desired equivalence of the stable module categories. $\square$

Representation type. Let k be an algebraically closed field. To any finite dimensional k-algebra A one associates a *representation type*. We do not intend to give the precise definitions, but only the main idea. An algebra A is of finite representation type if there are only finitely many indecomposable A-modules (up to isomorphism); otherwise A is of infinite representation type. The infinite case splits up again. Roughly speaking, the algebra A is of tame representation type if for a fixed dimension the indecomposable A-modules come in finitely many one parameter families with finitely many exceptions. This means that in principle there should be a classification of the indecomposable modules for an algebra of tame representation type (although for a particular example it can be a very hard problem to find such a classification). Otherwise, we say that A is of wild representation type.

In general, determining the representation type of an algebra is a difficult task, and in fact there is a lot of research currently going on which is centered around this question. But for particular classes of finite dimensional algebras there are complete answers. We mention two examples.

For blocks of group algebras the representation type is determined by the defect groups.

Theorem 6.5 (Bondarenko–Drozd [B-D]). *Let B be a p-block of a group algebra kG with defect group D. Then B is of ...*

- *...finite representation type if and only if D is a cyclic group.*
- *...tame representation type if and only if $p = 2$ and D is a dihedral, semidihedral or quaternion group.*
- *...wild representation type otherwise.*

For path algebras of quivers the representation type is determined by the underlying undirected graph.

Theorem 6.6 (Gabriel, [Ga2]). *Let Q be a connected quiver without oriented cycles and $\bar{Q}$ the undirected graph underlying Q. The path algebra kQ is of finite representation type if and only if $\bar{Q}$ is a Dynkin diagram. It is of tame representation type if and only if $\bar{Q}$ is an extended Dynkin diagram.*

We now want to establish a criterion in which derived equivalences can be used to determine the representation type of an algebra. In the preceding section we mentioned Rickard's result saying that if selfinjective algebras are derived equivalent then they are stably equivalent. Recently, H. Krause proved that algebras (not necessarily selfinjective) which are stably equivalent have the same representation type [Kr]. Putting these results together one obtains the following result.

Proposition 6.7. *For selfinjective algebras the representation type is preserved under derived equivalence.*

This new criterion can actually be helpful in determining the representation type for certain classes of algebras, even in cases where the 'classical' methods available so far could not give a complete answer. For example, we considered the algebras of dihedral, semidihedral and quaternion type, as defined by K. Erdmann [Er]. These are classes of algebras containing all blocks with tame representation type. Erdmann was able to classify these algebras up to Morita equivalence, i.e., she gave a list of basic algebras such that any algebra of dihedral, semidihedral or quaternion type is Morita equivalent to an algebra in this list [Er]. But it remained open in Erdmann's work whether all algebras in this list are actually of tame representation type ([Dr], Q4.3). Recently, we obtained a classification of the algebras of dihedral, semidihedral and quaternion type up to derived equivalence [Ho2], [Ho1]. As an application we could then answer the question on the representation type affirmatively, by using the above proposition. It is our hope that such a strategy could also be used for other classes of algebras.

Suggestions for further reading. Almost all topics mentioned in these notes (and much more!) are discussed in detail in the book of S. König and A. Zimmermann (with contributions by B. Keller, M. Linckelmann, J. Rickard and R. Rouquier) [K-Z]. It provides a very readable treatise about derived module categories in the representation theory of groups and finite-dimensional algebras, including a lot of recent work which is available for the first time in bookform. Moreover, the book contains an extensive up-to-date bibliography.

We can also recommend some very well-written survey articles by B. Keller on derived categories (from a more abstract point of view) [Ke1], [Ke2].

A recent survey article by J. Rickard on Broué's conjecture is contained in the Proceedings of ICM 98 [Ri4].

Apart from the original articles already quoted in the course of these notes there are several textbooks dealing with particular topics and which can also be recommended for further study. Classical Morita theory is developed in several books, e.g. [A-F], [C-R] or [Be1]. A very readable introduction into derived categories is given in the first chapter of the book of Kashiwara–Schapira [K-S] (but be aware that it is rather concise so you should be willing to fill in some details). A more detailed treatment of derived categories is presented in Gelfand–Manin's book on Homological Algebra [G-M]. A wealth of material about triangulated categories and about tilting theory can be found in Happel's book [Ha2].

Of course, there are many other interesting sources where one can learn about *Derived Categories, Derived Equivalences and Representation Theory* which we did not mention in this text. So the best recommendation could be: take one of the bibliographies and find your own way through the literature...

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Otto–von–Guericke–Universität Magdeburg, Institut für Algebra und Geometrie, Postfach 4120, D–39016 Magdeburg, Germany

E-mail address: `thorsten.holm@mathematik.uni-magdeburg.de`