Algebraic Structures Underlying Quantum Information Protocols

Yong Zhang[1]
Institute of Theoretical Physics, Chinese Academy of Sciences
P. O. Box 2735, Beijing 100080, P. R. China

Abstract

In this paper, we describe the teleportation from the viewpoints of the braid group and Temperley–Lieb algebra. We propose the virtual braid teleportation which exploits the teleportation swapping and identifies unitary braid representations with universal quantum gates, and suggest the braid teleportation which is explained in terms of the crossed measurement and the state model of knot theory. In view of the diagrammatic representation for the Temperley–Lieb algebra, we devise diagrammatic rules for an algebraic expression and apply them to various topics around the teleportation: transfer operator and acausality problem; teleportation and measurement; all tight teleportation and dense coding schemes; the Temperley–Lieb algebra and maximally entangled states; entanglement swapping; teleportation and topological quantum computing; teleportation and the Brauer algebra; multipartite entanglements. All examples clearly suggest the Temperley–Lieb algebra under local unitary transformations to be the algebraic structure underlying the teleportation. We show the teleportation configuration to be a fundamental element in the diagrammatic representation for the Brauer algebra and propose a new equivalent approach to the teleportation in terms of the swap gate and Bell measurement. To explain our diagrammatic rules to be a natural diagrammatic language for the teleportation, we compare it with the other two known approaches to the quantum information flow: the teleportation topology and strongly compact closed category, and make essential differences among them as clear as possible.

Key Words: Teleportation, Braid group, Temperley–Lieb algebra

PACS numbers: 03.65.Ud, 02.10.Kn, 03.67.Lx

[1]yong@itp.ac.cn

1 Introduction

Quantum entanglements [1] play the key roles in quantum information [2] and are widely exploited in quantum algorithms [3, 4], quantum cryptography [5, 6] and quantum teleportation [7]. On the other hand, topological entanglements [8] denote topological configurations like links or knots which are closures of braids. There are natural similarities between quantum entanglements and topological entanglements. As a unitary braid has the power of detecting knots or links, it is able to transform a separate quantum state into an entangled one. Hence a nontrivial unitary braid representation can be identified with a universal quantum gate [9]. Recently, a series of papers have been published on the application of knot theory to quantum information, see [10, 11, 12] for universal quantum gates and unitary solutions of the Yang–Baxter equation [13, 14]; see [15, 16, 17] for quantum topology and quantum computation; see [18, 19] for quantum entanglements and topological entanglements.

Especially, Kauffman's work on the teleportation topology [10, 20] motivates our tour of revisiting in a diagrammatic approach all tight teleportation and dense coding schemes in Werner's paper [21]. Under the project of setting up a bridge between knot theory and quantum information, the joint paper with Kauffman and Werner [22] makes a survey of diagrammatic tensor calculus and matrix representations and explores topological and algebraic structures underlying multipartite entanglements and then applies them to the realignment criteria of multipartite entanglements, while the present one focuses on the problem of how to study the teleportation from the viewpoints of the braid group and Temperley–Lieb algebra [23].

The teleportation is an amazing procedure of sending a message from Charlie to Bob under the help of Alice. Alice shares a maximally entangled state with Bob but performs the entangling measurement on the composite system between Charlie and her and then informs results of her measurement to Bob who will know what Charlie wants to pass onto him according to the protocol between Alice and him. The transformation matrix between the Bell states and product basis is found out to satisfy the braid relation and so stimulates us to study the teleportation at the level of the braid group. The teleportation equation in terms of braid at least has two types of interpretations, and we propose both the virtual braid teleportation and the braid teleportation.

The virtual braid group [24] is generated by virtual crossings and braids. A virtual crossing acts like permutation P in the teleportation swapping $(P \otimes Id)(Id \otimes P)$ or $(Id \otimes P)(P \otimes Id)$, Id for identity, which performs the teleportation in terms of the swap gate P. The virtual mixed relation for defining the virtual braid group is a kind of formulation of the teleportation equation. Similar to the braid representation of the Temperley–Lieb algebra [8], the virtual braid representation can be derived in terms of the Brauer algebra [25] or the virtual Temperley–Lieb algebra [22]. It is remarkable that the Brauer algebra has the teleportation configuration in the diagrammatic representation for its axioms.

We formulate the teleportation equation in terms of the Bell matrix and assign the interpretation of the braid teleportation to it. We explain the braid teleportation in the sense of the crossed measurement [26, 27, 28] and explore the configuration for the braid teleportation in the state model [8] which is devised for the braid representation of the Temperley–Lieb algebra. We find that the braid teleportation consists of teleportation and non-teleportation terms and realize that the teleportation term is determined by the Temperley–Lieb algebra.

The maximally entangled state is found to be a projector and form the representation of the Temperley–Lieb algebra. In view of the diagrammatic representation for the Temperley–Lieb algebra, we are inspired to deal with various topics on the teleportation in a diagrammatic approach. We devise our diagrammatic rules to describe a regular algebraic expression of quantum mechanics in terms of a diagram. A maximally entangled vector is denoted by the configuration of a cup (cap) and a local unitary transformation is denoted by a solid point or a small circle on a branch of the cup (cap). A projector, trace and transfer operator are described by specified combination of a cup with a cap. An operator, Dirac bra and Dirac ket are denoted by a middle solid point, a top solid point and a bottom solid point, respectively. Note that our diagrammatic rules are invented on the purpose of describing the teleportation instead of dealing with all subjects of quantum mechanics.

Besides its standard description [7], the teleportation has been viewed in terms of the transfer operator [29] and quantum measurements [27]. We revisit them in our diagrammatic approach. The transfer operator has a configuration involving a top cap and a bottom cup in which the teleportation appears to be a kind of the flow of quantum information and the so called acausality problem arises that the state preparation seems to happen after the measurement. As the state preparation is also a result of the measurement, the full teleportation process has a description in the language of quantum measurement. Its diagrammatic representation exploits a top cup with a bottom cap to denote a projector representing measurement and unifies the teleportation schemes of discrete and continuous variables into a picture.

In the tight teleportation schemes [21], the Hilbert spaces have the same dimension and the teleportation is exactly encoded in an algebraic equation. The diagrammatic description for all tight teleportation schemes naturally contains diagrams for the transfer operator and quantum measurements and sheds a light on our observation of the Temperley–Lieb algebra underlying the teleportation. The teleportation configuration is an element of the diagrammatic representation for the Temperley–Lieb algebra. With cups and caps, the axioms of the Temperley–Lieb algebra are easily verified in a diagrammatic way. We call the Temperley–Lieb category for the Temperley–Lieb algebra under local unitary transformations, which includes the configurations consisting of cups, caps and solid points or small circles.

The teleportation has been exploited in topics such as the entanglement swapping [30] and quantum computing [31]. The entanglement swapping is a way of yielding an entangled state between two systems which never met before and have no physical interactions. Its diagrammatic description, an element of the Temperley–Lieb category, leads to an algebraic equation for the entanglement swapping and has a tight picture similar to that for the tight teleportation schemes. The teleportation is an economic way of performing quantum computation. We show the configurations in the Temperley–Lieb category for the unitary braid gate, swap gate and CNOT gate and argue the quantum simulation of topological quantum computing in terms of devices for the teleportation. Additionally, we discuss how to represent multipartite entangled states in our diagrammatic approach, and state that the teleportation configuration is a fundamental ingredient in the diagrammatic description for the Brauer algebra [25] and is performed in terms of the swap gate and Bell measurement.

Before our diagrammatic rules are invented for the teleportation, there are two well known approaches to the flow of quantum information: Kauffman's teleportation topology [10, 20] and the categorical theory mainly considered by Coecke [32]. They are compared with our diagrammatical approach to stress conceptual differences in both physics and mathematics among them and make it clear that our proposal catches essential points of the teleportation and is a natural diagrammatic language for it. The teleportation topology describes the teleportation also in terms of cups and caps but emphasizes the topological condition which is not satisfied in our case. We denote the quantum information flow by the transfer operator instead of identity in the teleportation topology. The category theory describes the quantum information flow by compositions of maps and identifies it with the strongly compact closed category. We regard the quantum information flow as a part of the entire teleportation process and denote a projector in terms of a top cup with a bottom cap instead of a single cup or cap chosen by the category theory. We propose the Temperley–Lieb algebra or Brauer algebra instead of the category theory for the mathematical description of the teleportation.

The plans of the paper are organized as follows. The section 2 introduces the standard version of the teleportation, rewrites the teleportation equation in terms of the braid representation, and proposes both the virtual braid teleportation and the braid teleportation. The section 3 presents our rules for drawing diagrams and applies them to various topics around the teleportation: the transfer operator and acausality problem; teleportation and quantum measurement; all tight teleportation and dense coding schemes; the Temperley–Lieb algebra and maximally entangled states; entanglement swapping; teleportation and topological quantum computing; teleportation and the Brauer algebra; multipartite entanglements. The section 4 sketches the teleportation topology and categorical theory for the flow of quantum information in order to compare them with our methods. The section 5 is on concluding remarks.

2 Teleportation and braid group

We revisit the standard description of the teleportation [7] and rewrite the teleportation equation in terms of the Bell matrix which is proved to form the braid representation. We propose the teleportation swapping and explain the teleportation in the context of the virtual braid group. We further suggest the braid teleportation and present the interpretations in terms of both the crossed measurement [27] and state model for knot theory [8].

2.1 Notations for the teleportation equation

We fix notations for the Pauli matrices, qubits and the Bell states. The Pauli matrices σ_1, σ_2 and σ_3 have the conventional forms,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

and the quantum state $|0\rangle$ and $|1\rangle$ are identified with the matrices,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

so that we have the following useful formulas,

$$\sigma_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}, \; -i\sigma_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}, \; \sigma_3 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}. \tag{3}$$

The four product basis of two-fold tensor products are chosen to be

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

which fixes the rules for calculating the tensor product of matrices, i.e., embedding the right matrix into the left one. With the product basis $|ij\rangle$, $i, j = 0, 1$, the four mutually orthogonal Bell states have the forms,

$$\begin{aligned} |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad & |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad & |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned} \tag{5}$$

which derive the product basis $|ij\rangle$ in terms of the Bell states,

$$\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle), \qquad & |01\rangle &= \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle), \\ |10\rangle &= \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle), \qquad & |11\rangle &= \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle). \end{aligned} \tag{6}$$

The Bell states are transformed to each other under local unitary transformations,

$$\begin{aligned} |\phi^-\rangle &= (1\!\!1_2 \otimes \sigma_3)|\phi^+\rangle = (\sigma_3 \otimes 1\!\!1_2)|\phi^+\rangle, \\ |\psi^+\rangle &= (1\!\!1_2 \otimes \sigma_1)|\phi^+\rangle = (\sigma_1 \otimes 1\!\!1_2)|\phi^+\rangle, \\ |\psi^-\rangle &= (1\!\!1_2 \otimes -i\sigma_2)|\phi^+\rangle = (i\sigma_2 \otimes 1\!\!1_2)|\phi^+\rangle, \end{aligned} \tag{7}$$

where $1\!\!1_2$ denotes 2×2 unit matrix and so $1\!\!1_d$ for $d \times d$ unit matrix.

The teleportation is an application of both quantum entanglement and quantum measurement to the process of transporting a unknown quantum state $|\psi\rangle_C = (a|0\rangle + b|1\rangle)_C$ from Charlie to Bob without classical communication between them. Let Alice and Bob share the Bell state $|\phi^+\rangle_{AB}$, a maximally entangled state which works as Alice is far away from Bob. Do the following calculation, also see [29],

$$\begin{aligned} & |\psi\rangle_C|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle)_C(|00\rangle + |11\rangle)_{AB} \\ = \;& \frac{1}{2}a(|\phi^+\rangle + |\phi^-\rangle)_{CA}|0\rangle_B + \frac{1}{2}a(|\psi^+\rangle + |\psi^-\rangle)_{CA}|1\rangle_B \\ & +\frac{1}{2}b(|\psi^+\rangle - |\psi^-\rangle)_{CA}|0\rangle_B + \frac{1}{2}b(|\phi^+\rangle - |\phi^-\rangle)_{CA}|1\rangle_B \\ = \;& \frac{1}{2}(|\phi^+\rangle_{CA}|\psi\rangle_B + |\phi^-\rangle_{CA}\sigma_3|\psi\rangle_B + |\psi^+\rangle_{CA}\sigma_1|\psi\rangle_B + |\psi^-\rangle_{CA}(-i\sigma_2)|\psi\rangle_B) \end{aligned} \tag{8}$$

which is called the teleportation equation in our paper and tells how to teleport the qubit $|\psi\rangle_C$ from Charlie to Bob with the help of Alice. Alice performs the Bell measurement in the composite system of Charlie and her and has four kinds of outcomes. When Alice detects the Bell state $|\phi^+\rangle_{CA}$ and informs Bob about it through a classical channel, Bob will know that he has the quantum state $|\psi\rangle_B$ which Charlie wants to send to him. Similarly, when Alice gets the Bell states $|\phi^-\rangle_{CA}$ or $|\psi^+\rangle_{CA}$ or $|\psi^-\rangle_{CA}$, Bob will apply the local unitary transformations σ_3 or σ_1 or $i\sigma_2$ on the quantum state that he has in order to obtain the quantum state $|\psi\rangle_B$. Note that the quantum state $|\psi\rangle_C$ Charlie has is destroyed via Alice's Bell measurement.

We have to admit the teleportation to be a sort of magic and remain mystery until now for who are used to classical physical laws and classical communication: Bob gets a perfect copy of the quantum state that Charlie has without classical communication or interaction between them. Only fundamental laws of quantum mechanics and a little linear algebra are involved in the standard description of the teleportation. However, we believe that there exist beautiful mathematical structures underlying the teleportation. In the paper, we will make the fact clear that the teleportation configuration has been already in the diagrammatic descriptions for the braid group, Temperley–Lieb algebra and Brauer algebra and they were found in the mathematical community far before the proposal of the teleportation. Hence our main statement of this paper is addressed in the way: as we understand the teleportation in the framework of the Temperley–Lieb algebra, it will be natural and clear as it is.

2.2 Teleportation equation in terms of Bell matrix

Let introduce the so called Bell matrix [10, 11, 33], denoted by $B = (B_{ij,lm})$, $i,j,l,m = 0,1$. The B matrix and its inverse B^{-1} or transpose B^T are given by

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad B^{-1} = B^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

It has the exponential formalism given by

$$B = e^{i\frac{\pi}{4}(\sigma_1 \otimes \sigma_2)} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}(\sigma_1 \otimes \sigma_2) \tag{10}$$

with the following interesting properties:

$$B^2 = i\sigma_1 \otimes \sigma_2, \qquad B^4 = -1\!\!1_4, \qquad B^8 = 1\!\!1_4, \qquad B = \frac{1}{\sqrt{2}}(1\!\!1_4 + B^2). \tag{11}$$

The vector $\vec{v}$ includes four product basis $|ij\rangle$. Its transpose has the form,

$$\vec{v}^T = (|00\rangle, |01\rangle, |10\rangle, |11\rangle), \qquad v_{ij} = |ij\rangle, \; i,j = 0,1. \tag{12}$$

In terms of the B matrix and product basis v_{ij}, there are two ways of generating the Bell states, the first type (I) given by

$$\begin{aligned} (I): \qquad & |\phi^+\rangle = B|11\rangle, \qquad |\phi^-\rangle = B|00\rangle, \\ & |\psi^+\rangle = B|01\rangle, \qquad |\psi^-\rangle = -B|10\rangle, \end{aligned} \tag{13}$$

and the second type (II) given by

$$\begin{aligned} (II): \qquad & |\phi^+\rangle = B^T|00\rangle, \qquad |\phi^-\rangle = -B^T|11\rangle, \\ & |\psi^+\rangle = B^T|10\rangle, \qquad |\psi^-\rangle = B^T|01\rangle, \end{aligned} \tag{14}$$

where the Bell operator acts on the basis $|ij\rangle$ in the way,

$$B|ij\rangle = \sum_{k,l=0}^{1} |kl\rangle B_{kl,ij} = \sum_{k,l=0}^{1} |kl\rangle B^T_{ij,kl}. \tag{15}$$

We work on the first type of transformation law (I) between the Bell states and product basis, and rewrite the teleportation equation (8) into a new formalism,

$$\begin{aligned} & (1\!\!1_2 \otimes B)(|\psi\rangle \otimes |11\rangle)_{CAB} \\ & = \frac{1}{2}(B \otimes 1\!\!1_2)(|00\rangle \otimes \sigma_3|\psi\rangle + |01\rangle \otimes \sigma_1|\psi\rangle + |10\rangle \otimes i\sigma_2|\psi\rangle + |11\rangle \otimes |\psi\rangle)_{CAB}, \\ & \equiv (B \otimes 1\!\!1_2)(\vec{v}^T \otimes \frac{1}{2}\vec{\sigma}_{11}|\psi\rangle)_{CAB} \end{aligned} \tag{16}$$

in which the vector $\vec{\sigma}_{11}$ is a convenient notation, its transpose given by

$$\vec{\sigma}_{11}^T = (\sigma_3, \sigma_1, i\sigma_2, 1\!\!1_2), \tag{17}$$

and the calculation of $\vec{v}^T \otimes \vec{\sigma}_{11}$ follows the rule,

$$\vec{v}^T \otimes \vec{\sigma}_{11}|\psi\rangle = |00\rangle \otimes \sigma_3|\psi\rangle + |01\rangle \otimes \sigma_1|\psi\rangle + |10\rangle \otimes i\sigma_2|\psi\rangle + |11\rangle \otimes |\psi\rangle. \tag{18}$$

The remaining three teleportation equations are derived in the same way due to local unitary transformations among the Bell states (7). For example,

$$\begin{aligned} & |\psi\rangle_C|\phi^-\rangle_{AB} = (1\!\!1_2 \otimes B)(|\psi\rangle \otimes |00\rangle)_{CAB} \\ = \; & |\psi\rangle_C \otimes (1\!\!1_2 \otimes \sigma_3)|\phi^+\rangle_{AB} = (1\!\!1_2 \otimes 1\!\!1_2 \otimes \sigma_3)|\psi\rangle_C|\phi^+\rangle_{AB} \\ = \; & (B \otimes 1\!\!1_2)(\vec{v}^T \otimes \frac{1}{2}\sigma_3\vec{\sigma}_{11}|\psi\rangle)_{CAB} \end{aligned} \tag{19}$$

where the local unitary transformation $1\!\!1_2 \otimes 1\!\!1_2 \otimes \sigma_3$ commutes with $B \otimes 1\!\!1_2$. Similarly, we have the other two teleportation equations,

$$\begin{aligned} (1\!\!1_2 \otimes B)(|\psi\rangle \otimes |01\rangle)_{CAB} &= (B \otimes 1\!\!1_2)(\vec{v}^T \otimes \frac{1}{2}\sigma_1\vec{\sigma}_{11}|\psi\rangle)_{CAB}, \\ (1\!\!1_2 \otimes B)(|\psi\rangle \otimes -|10\rangle)_{CAB} &= (B \otimes 1\!\!1_2)(\vec{v}^T \otimes -\frac{1}{2}i\sigma_2\vec{\sigma}_{11}|\psi\rangle)_{CAB}, \end{aligned} \tag{20}$$

and then collect four of them into an equation,

$$\begin{aligned} (1\!\!1_2 \otimes B)(|\psi\rangle \otimes \vec{v}^T)_{CAB} &= (B \otimes 1\!\!1_2)(\vec{v}^T \otimes \frac{1}{2}\vec{\Sigma}|\psi\rangle)_{CAB}, \\ (\vec{v}^T \otimes \frac{1}{2}\vec{\Sigma}|\psi\rangle)_{CAB} &= (B^{-1} \otimes 1\!\!1_2)(1\!\!1_2 \otimes B)(|\psi\rangle \otimes \vec{v}^T)_{CAB}, \end{aligned} \tag{21}$$

in terms of the new matrix $\vec{\Sigma}$, a convenient notation given by

$$\vec{\Sigma} = (\sigma_3, \sigma_2, i\sigma_1, 1\!\!1_2)\vec{\sigma}_{11} \tag{22}$$

where $\vec{v}^T \otimes \vec{\Sigma}$ is defined in the way

$$\vec{v}^T \otimes \vec{\Sigma}|\psi\rangle = (\vec{v}^T \otimes \sigma_3\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes \sigma_2\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes i\sigma_1\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes \vec{\sigma}_{11}|\psi\rangle), \tag{23}$$

and $|\psi\rangle \otimes \vec{v}^T$ has the form

$$|\psi\rangle \otimes \vec{v}^T = (|\psi\rangle \otimes |00\rangle, |\psi\rangle \otimes |01\rangle, |\psi\rangle \otimes |10\rangle, |\psi\rangle \otimes |11\rangle). \tag{24}$$

In the following, the Bell matrix B is verified to form the braid representation and the virtual braid representation. It motivates us to study the problem how to observe the teleportation from the viewpoint of braid.

2.3 Bell matrix and braid representation

The braid representation b-matrix is a $d \times d$ matrix acting on $V \otimes V$ where V is an d-dimensional vector space. The symbol b_i denotes the braid b acting on the tensor product $V_i \otimes V_{i+1}$. The classical braid group B_n is generated by the braids $b_1, b_2, \cdots, b_{n-1}$ satisfying the braid relation,

$$\begin{aligned} b_i b_j &= b_j b_i, \qquad j \neq i \pm 1, \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \qquad i = 1, \cdots, n-2. \end{aligned} \tag{25}$$

The virtual braid group VB_n is an extension of the classical braid group B_n by the symmetric group S_n [24]. It has both the braids b_i and virtual crossings v_i which are defined by the virtual crossing relation,

$$\begin{aligned} v_i^2 &= 1\!\!1, \qquad v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}, \qquad i = 1, \cdots, n-2, \\ v_i v_j &= v_j v_i, \qquad j \neq i \pm 1, \end{aligned} \tag{26}$$

a presentation of the symmetric group S_n, and the virtual mixed relation:

$$\begin{aligned} b_i v_j &= v_j b_i, \qquad j \neq i \pm 1, \\ b_{i+1} v_i v_{i+1} &= v_i v_{i+1} b_i, \qquad i = 1, \cdots, n-2. \end{aligned} \tag{27}$$

Let verify the Bell matrix B satisfying the braid relation. On the right handside of (25), we do the algebraic calculation:

$$\begin{aligned} & (1\!\!1_2 \otimes B)(B \otimes 1\!\!1_2)(1\!\!1_2 \otimes B) \\ &= \frac{1}{2\sqrt{2}}(1\!\!1_8 + 1\!\!1_2 \otimes i\sigma_1 \otimes \sigma_2)(1\!\!1_8 + i\sigma_1 \otimes \sigma_2 \otimes 1\!\!1_2)(1\!\!1_8 + 1\!\!1_2 \otimes i\sigma_1 \otimes \sigma_2) \\ &= \frac{i}{\sqrt{2}}(1\!\!1_2 \otimes \sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \sigma_2 \otimes 1\!\!1_2) = \frac{1}{\sqrt{2}}(1\!\!1_2 \otimes B^2 + B^2 \otimes 1\!\!1_2) \end{aligned} \tag{28}$$

and on its left handside we have the same result, i.e.,

$$(B \otimes 1\!\!1_2)(1\!\!1_2 \otimes B)(B \otimes 1\!\!1_2) = \frac{1}{\sqrt{2}}(1\!\!1_2 \otimes B^2 + B^2 \otimes 1\!\!1_2). \tag{29}$$

Now we prove the Bell matrix B to satisfy the virtual mixed relation (27) as the permutation matrix P is chosen to be a virtual crossing,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P|ij\rangle = |ji\rangle, \qquad i, j = 0, 1. \tag{30}$$

On the left handside of (27), we have

$$(1\!\!1_2 \otimes B)(P \otimes 1\!\!1_2)(1\!\!1_2 \otimes P)(|i\rangle \otimes |j\rangle \otimes |k\rangle) = (1\!\!1_2 \otimes B)(|k\rangle \otimes |ij\rangle) \tag{31}$$

and on its right handside, we do the following calculation,

$$
\begin{aligned}
&(P\otimes 1\!\!1_2)(1\!\!1_2\otimes P)(B\otimes 1\!\!1_2)(|i\rangle\otimes|j\rangle\otimes|k\rangle)\\
&=\sum_{k,l=0}^{1} B_{i'j',ij}(P\otimes 1\!\!1_2)(1\!\!1_2\otimes P)(|i'j'\rangle\otimes|k\rangle)\\
&=\sum_{k,l=0}^{1} B_{i'j',ij}(|k\rangle\otimes|i'j'\rangle)=(1\!\!1_2\otimes B)(|k\rangle\otimes|ij\rangle). \qquad (32)
\end{aligned}
$$

Around the fact that the Bell matrix forms the braid representation, recently, there are interesting research progress: the Bell matrix recognized as a universal quantum gate [10, 33]; Yang–Baxterization of the Bell matrix [11, 12]; joint paper with Jing and Ge on new types of quantum algebras via the RTT relation [34]; the Markov trace [10] and related extraspecial 2-groups [35]. In the following, we will propose two kinds of braid interpretations for the teleportation in view of the braid formulation (21) of the teleportation equation.

2.4 Teleportation swapping and virtual braid group

We describe the teleportation in the framework of the virtual braid group. The braid relation (25) builds a connection between topological entanglements and quantum entanglements, while the virtual mixed relation (27) is a sort of formulation of the teleportation equation (8).

A nontrivial unitary braid detecting knots or links is identified with a universal quantum gate transforming a separate state into an entangled one, see [10, 11, 12]. It acts as a device yielding an entangled source. On the other hand, a virtual crossings v_i is responsible for the teleportation based on our observation called the teleportation swapping. There are two natural teleportation swapping operators $(P\otimes Id)(Id\otimes P)$ and $(Id\otimes P)(P\otimes Id)$, satisfying the teleportation swapping equations,

$$
\begin{aligned}
|k\rangle\otimes|ij\rangle &= (P\otimes Id)(Id\otimes P)(|ij\rangle\otimes|k\rangle),\\
|ij\rangle\otimes|k\rangle &= (Id\otimes P)(P\otimes Id)(|k\rangle\otimes|ij\rangle), \qquad (33)
\end{aligned}
$$

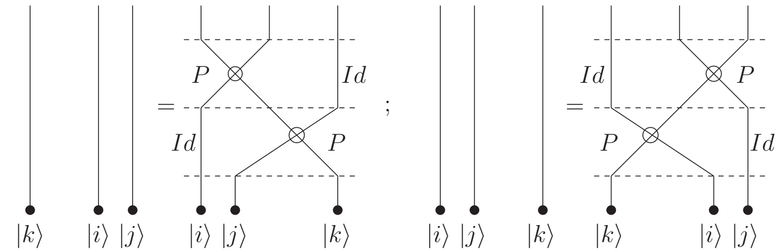
which are shown up in Figure 1, the permutation P represented by a virtual crossing with a small circle at the crossing point.

In the following, we set up the connection between the virtual mixed relation (27) and the teleportation equation (8). In terms of the Bell matrix and teleportation swapping, the left handside of the teleportation equation (8) has a form,

$$
\begin{aligned}
&|\psi\rangle_C\otimes|\phi^+\rangle_{AB}=(1\!\!1_2\otimes B)(|\psi\rangle_C\otimes|11\rangle_{AB})\\
&=(1\!\!1_2\otimes B)(P\otimes 1\!\!1_2)(1\!\!1_2\otimes P)(|11\rangle_{CA}\otimes|\psi\rangle_B), \qquad (34)
\end{aligned}
$$

while its right handside leads to the formalism,

$$
\frac{1}{2}(|\phi^-\rangle_{CA}\sigma_3|\psi\rangle_B+|\psi^-\rangle_{CA}(-i\sigma_2)|\psi\rangle_B+|\psi^+\rangle_{CA}\sigma_1|\psi\rangle_B+|\phi^+\rangle_{CA}|\psi\rangle_B)
$$



$|k\rangle \otimes |ij\rangle = (P \otimes Id)(Id \otimes P)(|ij\rangle \otimes |k\rangle)$ $\qquad$ $|ij\rangle \otimes |k\rangle = (Id \otimes P)(P \otimes Id)(|k\rangle \otimes |ij\rangle)$

Figure 1: Teleportation swapping with permutation P as virtual crossing.

$$
\begin{aligned}
&= \frac{1}{2}(\mathbb{1}_2 \otimes \sigma_3 \otimes \sigma_3 + \mathbb{1}_2 \otimes i\sigma_2 \otimes i\sigma_2 + \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_1 + \mathbb{1}_8)(B \otimes \mathbb{1}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B) \\
&= (\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)(B \otimes \mathbb{1}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B) \qquad (35)
\end{aligned}
$$

where the permutation matrix P is given by

$$P = \frac{1}{2}(\mathbb{1}_4 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3), \qquad (36)$$

and the local unitary transformations (7) among the Bell states take the forms,

$$
\begin{aligned}
|\phi^-\rangle &= B|00\rangle = (\mathbb{1}_2 \otimes \sigma_3)B|11\rangle = (\sigma_3 \otimes \mathbb{1}_2)B|11\rangle, \\
|\psi^+\rangle &= B|01\rangle = (\mathbb{1}_2 \otimes \sigma_1)B|11\rangle = (\sigma_1 \otimes \mathbb{1}_2)B|11\rangle, \\
|\psi^-\rangle &= -B|10\rangle = (\mathbb{1}_2 \otimes -i\sigma_2)B|11\rangle = (i\sigma_2 \otimes \mathbb{1}_2)B|11\rangle. \qquad (37)
\end{aligned}
$$

Hence the teleportation equation (8) is recognized to be a kind of the teleportation swapping,

$$
\begin{aligned}
|\psi\rangle_C \otimes |\phi^+\rangle_{AB} &= (\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)(|\phi^+\rangle_{CA} \otimes |\psi\rangle_B) \\
&= (P \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P)(|\phi^+\rangle_{CA} \otimes |\psi\rangle_B), \qquad (38)
\end{aligned}
$$

and a reformulation of the virtual mixed relation (27),

$$
\begin{aligned}
&(\mathbb{1}_2 \otimes B)(P \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P)(|11\rangle_{CA} \otimes |\psi\rangle_B) \\
&= (\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)(B \otimes \mathbb{1}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B) \\
&= (P \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P)(B \otimes \mathbb{1}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B). \qquad (39)
\end{aligned}
$$

The remaining three teleportation equations are derived in the way by applying the local unitary transformations to the teleportation equation (38). For example, the teleportation equation for the Bell state $|\phi^-\rangle_{AB}$ is obtained to be

$$
\begin{aligned}
&|\psi\rangle_C \otimes |\phi^-\rangle_{AB} = (\mathbb{1}_2 \otimes \sigma_3 \otimes \mathbb{1}_2)(|\psi\rangle_C \otimes |\phi^+\rangle_{AB}) \\
&= (\mathbb{1}_2 \otimes \sigma_3 \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P - \mathbb{1} \otimes \sigma_2 \otimes \sigma_2)(\mathbb{1}_2 \otimes \sigma_3 \otimes \mathbb{1}_2)(|\phi^-\rangle_{CA} \otimes |\psi\rangle_B) \\
&= (\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_1)(|\phi^-\rangle_{CA} \otimes |\psi\rangle_B). \qquad (40)
\end{aligned}
$$

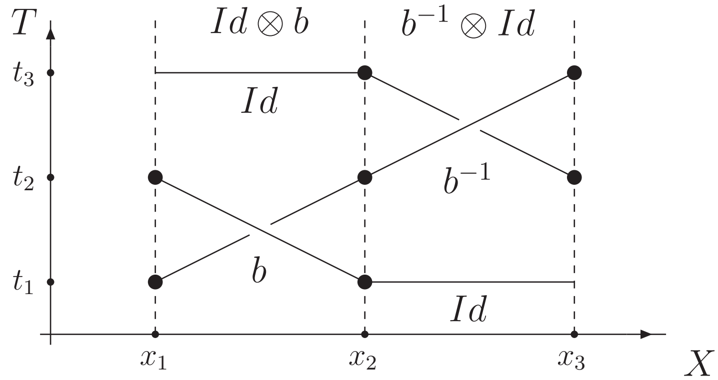


Figure 2: Braid teleportation and crossed measurement.

Similarly, the teleportation equations for the Bell states $|\psi^{\pm}\rangle_{AB}$ have the similar forms as the teleportation swapping,

$$|\psi\rangle_C \otimes |\psi^+\rangle_{AB} = (1\!\!1_2 \otimes P - 1\!\!1_2 \otimes \sigma_3 \otimes \sigma_3)(|\psi^+\rangle_{CA} \otimes |\psi\rangle_B),$$
$$|\psi\rangle_C \otimes |\psi^-\rangle_{AB} = (1\!\!1_2 \otimes P - 1\!\!1_8)(|\psi^-\rangle_{CA} \otimes |\psi\rangle_B), \tag{41}$$

in which the following local unitary transformations of $(1\!\!1_2 \otimes P - 1\!\!1_2 \otimes \sigma_2 \otimes \sigma_2)$ have been exploited,

$$(1\!\!1_2 \otimes \sigma_1 \otimes 1\!\!1_2)(1\!\!1_2 \otimes P - 1\!\!1 \otimes \sigma_2 \otimes \sigma_2)(1\!\!1_2 \otimes \sigma_1 \otimes 1\!\!1_2) = 1\!\!1_2 \otimes P - 1\!\!1_2 \otimes \sigma_3 \otimes \sigma_3,$$
$$(1\!\!1_2 \otimes i\sigma_2 \otimes 1\!\!1_2)(1\!\!1_2 \otimes P - 1\!\!1 \otimes \sigma_2 \otimes \sigma_2)(1\!\!1_2 \otimes i\sigma_2 \otimes 1\!\!1_2) = 1\!\!1_2 \otimes P - 1\!\!1_8. \tag{42}$$

As a remark, we regard the virtual braid group to be a natural language describing the teleportation. Note that the virtual braid teleportation is essentially different from the braid teleportation in the following subsection. A braid is a device for entangling separate states in the virtual braid teleportation, while a braid plays the role of teleportation in the braid teleportation.

2.5 Braid teleportation, crossed measurement and state model

As the concept of braid is a kind of generalization of permutation, the braid teleportation is an extension of the teleportation swapping. In view of the formalism (21) of the teleportation equation, the braid configuration $(b^{-1} \otimes Id)(Id \otimes b)$ has an ability of performing the teleportation of a quantum state from Charlie to Bob. There are two kinds of interpretations for it. The one is Vaidman's crossed measurement [26] and the other is Kauffman's state model [8] for knot theory.

In the crossed measurement, a braid or crossing acts as a device of measurement which is non-local in both space and time. In Figure 2, the two lines of the crossing b

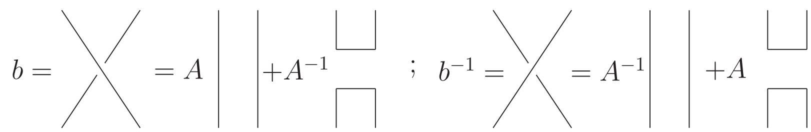


Figure 3: Braid b and its inverse b^{-1} in the state model.

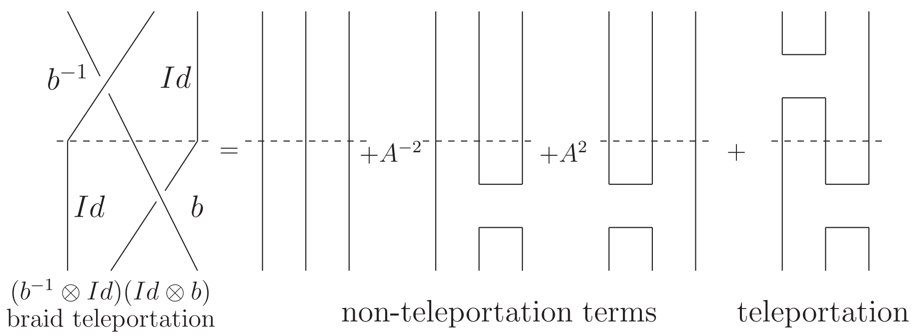


Figure 4: Braid teleportation and state model.

represent two observable operations: the first relating the measurement at the space-time point (x_1, t_1) to that at the other point (x_2, t_2) and the second one relating the measurement at (x_1, t_2) to that at (x_2, t_1). The crossed measurement $(Id \otimes b)$ plays the role of sending the qubit from Charlie to Alice with a possible local unitary transformation. Similarly, the crossed measurement $(b^{-1} \otimes Id)$ transfers the qubit from Alice to Bob with a possible local unitary transformation. This kind of braid teleportation is charming because it is expected to be related to the braid statistics of anyons or topological quantum computing [36]. We will make a further study on the connection between the braid teleportation and crossed measurement.

However, the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ may not have a fixed configuration like $(P \otimes Id)(Id \otimes P)$ from the point of braid. For example, we rewrite the Bell teleportation $(B^{-1} \otimes 1\!\!1_2)(1\!\!1_2 \otimes B)$ into the other formalism,

$$
\begin{aligned}
&(B^{-1} \otimes 1\!\!1_2)(1\!\!1_2 \otimes B) = \frac{1}{2}((1\!\!1_4 - B^2) \otimes 1\!\!1_2)(1\!\!1_2 \otimes (1\!\!1_4 + B^2)) \\
= \; & \frac{1}{2}(1\!\!1_2 \otimes (1\!\!1_4 + B^2))((1\!\!1_4 - B^2) \otimes 1\!\!1_2) + (1\!\!1_2 \otimes B^2)(B^2 \otimes 1\!\!1_2) \\
= \; & (1\!\!1_2 \otimes B)(B^{-1} \otimes 1\!\!1_2) + (1\!\!1_2 \otimes B^2)(B^2 \otimes 1\!\!1_2) \qquad (43)
\end{aligned}
$$

where we make use of the equation,

$$
(1\!\!1_2 \otimes B^2)(B^2 \otimes 1\!\!1_2) = -(B^2 \otimes 1\!\!1_2)(1\!\!1_2 \otimes B). \qquad (44)
$$

The braid configuration $(1\!\!1_2 \otimes B)(B^{-1} \otimes 1\!\!1_2)$ is different from $(B^{-1} \otimes 1\!\!1_2)(1\!\!1_2 \otimes B)$ and B^2 is not a braid representation. Hence the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ has to be understood at a more fundamental level. We turn to the state model for knot theory [8] and explore the structure of the configuration for the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$.

The state model is an approach to the disentanglement of a knot or link, see Figure 3. A braid b is denoted by a under crossing and its inverse b^{-1} is denoted by an over crossing. Each crossing is identified with a linear combination of two types of configurations. The first configuration is given by two straight lines representing identity. The second is a top cup together with a bottom cap representing a projector. The coefficients A, A^{-1} are specified by a given state model [8]. We rewrite the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ in terms of the state model, see Figure 4. The braid teleportation has four diagrammatic terms. The parts above dashed lines are contributed from $(b^{-1} \otimes Id)$ and the parts under dashed lines are from $(Id \otimes b)$. Obviously, the first three are irrelevant with the teleportation and the fourth one is responsible for the teleportation.

Therefore, the braid group is not a most suitable mathematical structure what we are looking for to describe the teleportation. We will submit our further research work on the braid teleportation elsewhere. In the following, we will focus on the topic how the Temperley–Lieb algebra under local unitary transformations describes various topics around the teleportation in a simple and beautiful way. As a matter of fact, the state model for knot theory was proposed in view of the braid representation of the Temperley–Lieb algebra [8]. The teleportation term in Figure 4 is completely determined by the Temperley–Lieb algebra, as will become transparent in the next section.

3 Teleportation and Temperley–Lieb algebra

In this section, we set up the rules how to draw a diagram for an algebraic expression and apply them to various topics around the teleportation. Firstly, we revisit three types of descriptions for the teleportation: the transfer operator, quantum measurement and tight teleportation scheme, in our diagrammatic approach. Secondly, we propose the Temperley–Lieb algebra or Brauer algebra under local unitary transformations to underlie the teleportaiton. Thirdly, we study the Temperley–Lieb configurations for the entanglement swapping, quantum computing and multipartite entanglements.

3.1 Notations for maximally entangled states

As maximally entangled states play the key roles in the teleportation, how to make diagrammatic representations for them is the bone of our rules. Before devising the rules, we fix our notations for maximally entangled states and sketch their relevant algebraic properties.

The vectors $|e_i\rangle$, $i = 0, 1, \cdots d-1$ form a set of complete and orthogonal basis for a d-dimension Hilbert space $\mathcal{H}$ and the covectors $\langle e_i|$, are chosen for its dual Hilbert space $\mathcal{H}^*$,

$$\sum_{i=0}^{d-1} |e_i\rangle\langle e_i| = 1\!\!1_d, \qquad \langle e_j|e_i\rangle = \delta_{ij}, \qquad i, j = 0, 1, \cdots d-1, \tag{45}$$

where δ_{ij} is the Kronecker symbol. The maximally entangled state $|\Omega\rangle$, a quantum state in the two-fold tensor product $\mathcal{H}\otimes\mathcal{H}$ of the Hilbert space $\mathcal{H}$, and its dual state $\langle\Omega|$ are given by

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i \otimes e_i\rangle, \qquad \langle\Omega| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle e_i \otimes e_i|, \tag{46}$$

The action of a bounded linear operator M in the Hilbert space $\mathcal{H}$ on $|\Omega\rangle$ satisfies

$$\begin{aligned} &|\psi\rangle \equiv (M \otimes 1\!\!1_d)|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} M|e_i\rangle \otimes |e_i\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} |e_j\rangle M_{ji} \otimes |e_i\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} |e_j\rangle \otimes |e_i\rangle M^T_{ij} \\ &= (1\!\!1_d \otimes M^T)|\Omega\rangle, \qquad M_{ij} = \langle e_i|M|e_j\rangle, \qquad M^T_{ij} = M_{ji}, \end{aligned} \tag{47}$$

and so it is permitted to move the local action of the operator M from the Hilbert space to the other Hilbert space as it acts on $|\Omega\rangle$.

The trace of two operators $M^\dagger$ and M' is represented by an inner product between $|\psi\rangle$ and $|\psi'\rangle$,

$$\begin{aligned} \langle\psi|\psi'\rangle &\equiv \langle\Omega|(M^\dagger \otimes 1\!\!1_d)(M' \otimes 1\!\!1_d)|\Omega\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \langle e_i|M^\dagger M'|e_j\rangle\langle e_i|e_j\rangle, \\ &= \frac{1}{d} tr(M^\dagger M'), \qquad |\psi'\rangle = (M' \otimes 1\!\!1_d)|\Omega\rangle, \end{aligned} \tag{48}$$

while an inner product with the action of the operator $N_1 \otimes N_2$ is also a trace,

$$\langle\psi|N_1 \otimes N_2|\psi'\rangle = \frac{1}{d} tr(M^\dagger N_1 M' N_2^T). \tag{49}$$

The transfer operator T_{BC} sending the quantum state from Charlie to Bob,

$$T_{BC}|\psi\rangle_C \equiv T_{BC} \sum_{k=0}^{d-1} a_k|e_k\rangle_C = \sum_{k=0}^{d-1} a_k|e_k\rangle_B = |\psi\rangle_B, \tag{50}$$

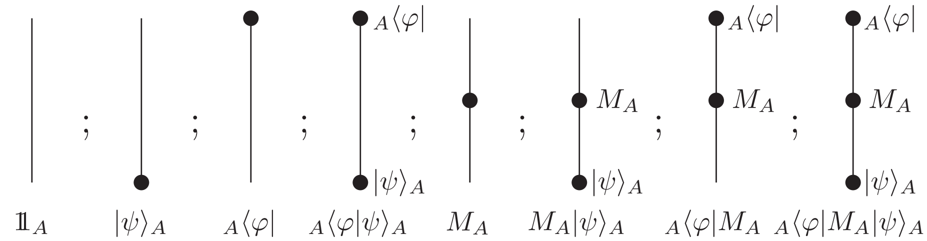


Figure 5: Straight lines without or with points.

is recognized to be an inner product between $_{CA}\langle\Omega|$ and $|\Omega\rangle_{AB}$,

$$\begin{aligned} {}_{CA}\langle\Omega|\Omega\rangle_{AB} &= \frac{1}{d}\sum_{i,j=0}^{d-1}({}_C\langle e_i| \otimes {}_A\langle e_i|)(|e_j\rangle_A \otimes |e_j\rangle_B) \\ &= \frac{1}{d}T_{BC} \equiv \frac{1}{d}\sum_{i=0}^{d-1}|e_i\rangle_B\ {}_C\langle e_i|. \end{aligned} \tag{51}$$

As the above, the maximally entangled state $|\Omega\rangle$ has beautiful properties to be realized at the level of diagrams. Diagrams catch essential points from the global view so that they are able to express complicated algebraic objects in a clear way. The diagrams from our rules uncover the Temperley–Lieb algebra under local unitary transformations behind the teleportation and remove mysteries around the teleportation from both physical and mathematical points.

3.2 Rules and examples for drawing diagrams

Now we present our rules to assign a definite diagram for a given algebraic expression. A diagram consists of straight lines, oblique lines, points, small circles, caps and cups. Every diagrammatic element is mapped to an algebraic term. The rules are divided into three parts: the first for our convention; the second for straight lines and oblique lines; the third for cups and caps.

Rule 1. Read an algebraic expression such as an inner product from the left to the right and draw a diagram from the top to the bottom.

Rule 2. A straight line of type A denotes identity for the system A, which is a linear combination of projectors, i.e., $\mathbb{1}_A = \sum_{i=0}^{d-1}|e_i\rangle_A\ {}_A\langle e_i|$. Straight lines of type A with top or bottom boundary solid points describe the vector $|\psi\rangle_A$, covector $_A\langle\varphi|$, and the inner product $_A\langle\varphi|\psi\rangle_A$ for the system A, respectively, see Figure 5. Straight lines of type A with a middle solid point and top or bottom boundary solid points describe the operator M_A, the covector $_A\langle e_i|M_A$, the vector $M_A|\psi\rangle_A$ and the inner product $_A\langle\varphi|M_A|\psi\rangle_A$, respectively, see Figure 5. An oblique line connecting the

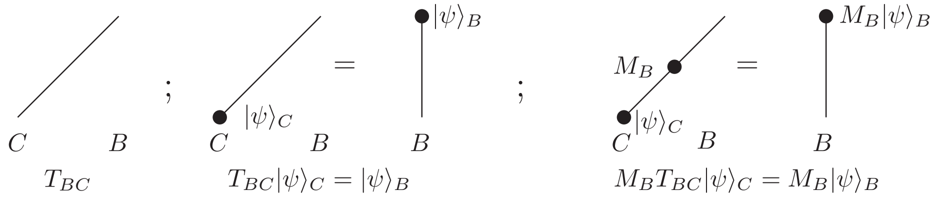


Figure 6: Oblique line for transfer operator.

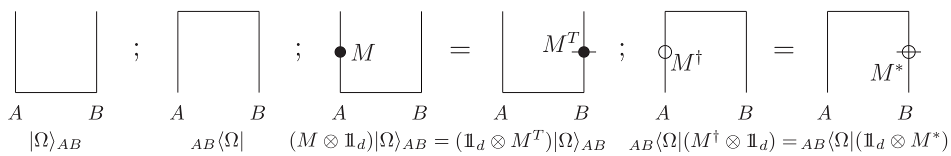


Figure 7: Cups and caps without or with points.

system C to the system B describes the transfer operator T_{BC} and its solid points have the same interpretation as those on a straight line of type A, see Figure 6.

Rule 3. A cup denotes the maximally entangled state $|\Omega\rangle$ and a cap does for its dual $\langle\Omega|$. A cup with a middle solid point at its one branch describes the local action of the operator M on $|\Omega\rangle$ and this solid point can flow to its other branch and becomes a solid point with a cross line representing M^T since we have $(M \otimes \mathbb{1}_d)|\Omega\rangle = (\mathbb{1}_d \otimes M^T)|\Omega\rangle$, see Figure 7. The same things happen for a cap except that a solid point is replaced by a small circle to distinguish the operator M and its transposed and complex version $M^\dagger = (M^T)^*$, see Figure 7. A cup and a cap have at least three types of combinations for three different cases, see Figure 8. As a cup is at the top and a cap is at the bottom for the same composite system, the diagram is assigned to the projector $|\Omega\rangle\langle\Omega|$. As a cap is at the top and a cup is at the bottom for the same composite system, the diagram describes the inner product $\langle\Omega|\Omega\rangle = 1$ by a closed circle. Additionally, as a cup has the local action of the operator M and a cap has the local action of the operator $N^\dagger$, the resulted circle with a solid point for M and a small circle for $N^\dagger$ represents the trace $\frac{1}{d}tr(MN^\dagger)$. As conventions, we describe a trace of operators by a closed circle with solid points or small circles and assign each cap or cup a normalization factor $\frac{1}{\sqrt{d}}$ and a circle a normalization factor d according to the trace of $\mathbb{1}_d$. As a cup is at the bottom for the composite system $\mathcal{H}_C \otimes \mathcal{H}_A$ and a cap is at the top for the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$, the diagram is an oblique line representing the transfer operator T_{BC} from Charlie to Bob, see Figure 8.

In the first example, we make a diagrammatic representation for the teleporta-

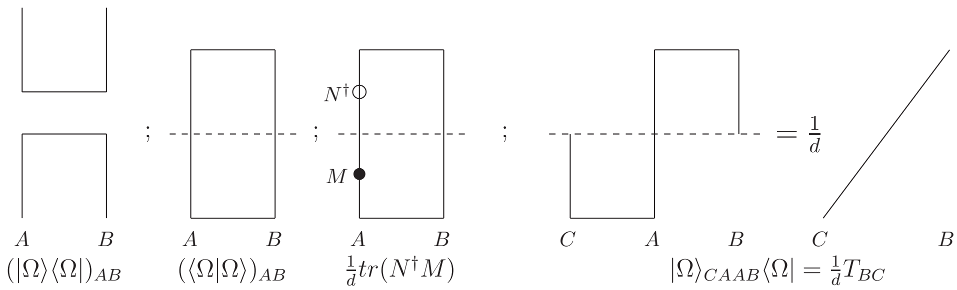


Figure 8: Three kinds of combinations of a cup and a cap.

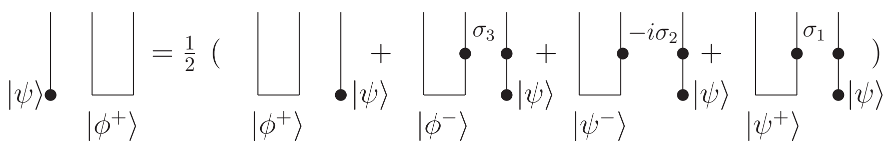


Figure 9: A diagrammatic representation of the teleportation equation (8).

tion equation (8) in Figure 9. The cup denotes the Bell state $|\phi^+\rangle$ and the cups with middle solid points σ_3, $-i\sigma_2$, σ_1 denote the Bell states $|\phi^-\rangle$, $|\psi^-\rangle$ and $|\psi^+\rangle$, respectively. The straight line with a bottom boundary solid point denotes the transported unknown state $|\psi\rangle$ and its middle solid points represent the local unitary transformation on $|\psi\rangle$.

In the second example, we draw a diagram for the inner product between $|\phi\rangle = \sum_{i=0}^{d-1} \phi_i |e_i\rangle$ and $|\psi\rangle = \sum_{j=0}^{d-1} \psi_j |e_j\rangle$. It consists of three terms: the covector $\langle\phi|\otimes\langle\Omega|$, the local operator $1\!\!1_d \otimes M \otimes 1\!\!1_d$ and the vector $|\Omega\rangle \otimes |\psi\rangle$. It is calculated in the following algebraic way,

$$
\begin{aligned}
&\langle \phi \otimes \Omega | (1\!\!1_d \otimes M \otimes 1\!\!1_d) | \Omega \otimes \psi \rangle \\
&= \frac{1}{d} \sum_{i,j=0}^{d-1} \langle \phi \otimes e_i \otimes e_i | (1\!\!1_d \otimes M \otimes 1\!\!1_d) | e_j \otimes e_j \otimes \psi \rangle \\
&= \frac{1}{d} \sum_{i,j=0}^{d-1} \langle \phi | e_j \rangle \langle e_i | M | e_j \rangle \langle e_i | \psi \rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \phi_j^* M_{ij} \psi_i \\
&= \frac{1}{d} \sum_{i,j=0}^{d-1} \phi_j^* M_{ji}^T \psi_i = \frac{1}{d} \langle \phi | M^T | \psi \rangle. \qquad (52)
\end{aligned}
$$

It is also derived in our diagrammatic approach, see Figure 10. The vector $|\psi\rangle$ is represented by a straight line with a bottom boundary solid point and the local

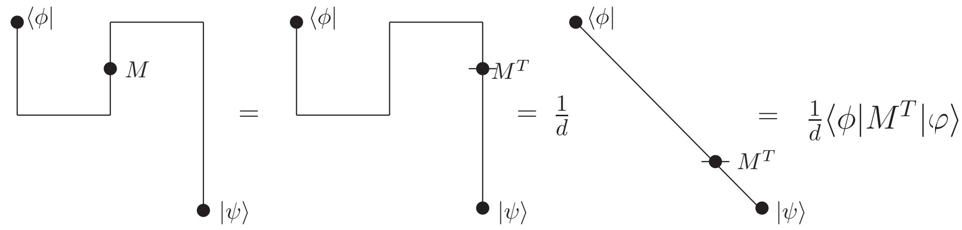


Figure 10: Inner product in terms of a cap and a cup.

operator $1\!\!1_d \otimes M \otimes 1\!\!1_d$ is denoted by a solid point on the cup $|\Omega\rangle$. Move the local operator M from its position to the other branch of the cap, change it to the local operator M^T and then allow the bottom cup and the top cap to vanish into an oblique line denoting the transfer operator with a normalization factor $\frac{1}{d}$.

In the third example, we make the diagrammatic representation for the partial trace of a composite system. The partial trace denotes the summation over a subsystem of a composite system,

$$tr_A(|e_i^C \otimes e_j^A\rangle\langle e_l^A \otimes e_m^B|) = |e_i^C\rangle\langle e_m^B|\delta_{jl}, \qquad tr_A(|e_j^A\rangle\langle e_l^A|) = \delta_{jl}, \tag{53}$$

and so the trace is a sort of partial trace, i.e., the summation over the entire composite system,

$$tr_{CA}(|e_i^C \otimes e_j^A\rangle\langle e_l^C \otimes e_m^A|) = \delta_{il}\delta_{jm}, \qquad i, j, l, m = 0, 1, \cdots d-1. \tag{54}$$

The first diagrammatic term in Figure 11 describes the one type of partial trace by a straight line for identity,

$$tr_A(|\Omega\rangle_{CA\,CA}\langle\Omega|) = \frac{1}{d}\sum_{i,j=0}^{d-1} tr_A(|e_i^C \otimes e_i^A\rangle\langle e_j^C \otimes e_j^A|) = \frac{1}{d}(1\!\!1_d)_C, \tag{55}$$

and the other two terms in Figure 11 represent the other type of partial trace by an oblique line for the transfer operator. The second term in Figure 11 denotes the transfer operator T_{CB} given by

$$tr_A(|\Omega\rangle_{CA\,AB}\langle\Omega|) = \frac{1}{d}\sum_{i,j=0}^{d-1} tr_A(|e_i^C \otimes e_i^A\rangle\langle e_j^A \otimes e_j^B|) = \frac{1}{d}T_{CB} \tag{56}$$

and the third one leads to the transfer operator T_{CB} given by

$$tr_A({}_{CA}\langle\Omega|\Omega\rangle_{AB}) = \frac{1}{d}T_{BC}. \tag{57}$$

In the fourth example, we identify two types of diagrams with the diagrammatic representation for the trace of operator products. In the first case, a top cup with a

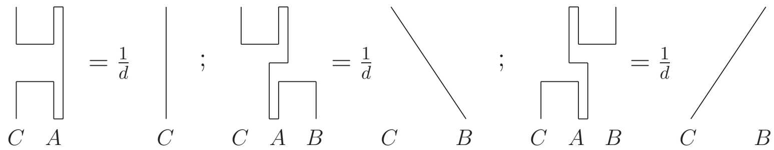


Figure 11: Three kinds of partial traces between a cup and a cap.

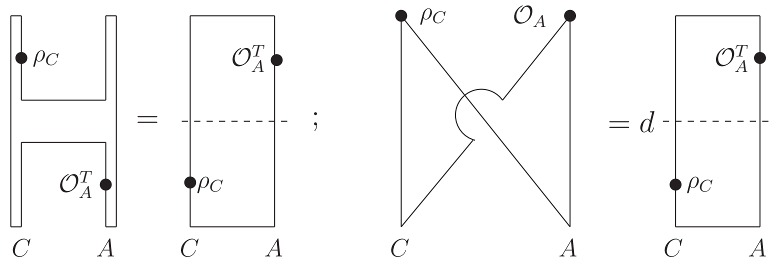


Figure 12: Closed circle via cup and cap, and oblique lines

bottom cap forms the same closed circle as a top cap and bottom cup, see the first term of Figure 12 which represents the algebraic equation given by,

$$tr_{CA}((\rho_C \otimes \mathbb{1}_d|\Omega\rangle_{CA})({}_{CA}\langle\Omega|\mathbb{1}_d \otimes \mathcal{O}_A^T)) = {}_{CA}\langle\Omega|(\mathbb{1}_d \otimes \mathcal{O}_A^T)(\rho_C \otimes \mathbb{1}_d)|\Omega\rangle_{CA}, \quad (58)$$

where ρ_C and $\mathcal{O}_A$ are bounded linear operators in d-dimensional Hilbert space. In the second case, a closed circle formed by two oblique lines for transfer operators denotes the same trace as a top cap with a bottom cup, which becomes clear via the following calculation,

$$\begin{aligned} tr_{CA}((\rho_C T_{CA})(\mathcal{O}_A T_{AC})) &= \sum_{i,j=0}^{d-1} tr_{CA}((\rho_C|e_i\rangle_{CA}\langle e_i|)(\mathcal{O}_A|e_j\rangle_{AC}\langle e_j|)) \\ &= \sum_{i,j=0}^{d-1} tr_{CA}((\rho_C|e_i\rangle_C \otimes \mathcal{O}_A|e_j\rangle_A)({}_C\langle e_j| \otimes {}_A\langle e_i|)) \\ &= \sum_{i,j=0}^{d-1} (\rho_C)_{ji}(\mathcal{O}_A)_{ij} = tr(\rho_C\mathcal{O}_A) \\ &= d \cdot {}_{CA}\langle\Omega|(\rho_C \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathcal{O}_A^T)|\Omega\rangle_{CA} \qquad (59) \end{aligned}$$

which has the diagrammatic representation as the second term of Figure 12.

In the fifth example, we represent the configuration of cup (cap) by compositions of cups and caps. In the first term of Figure 13, the cup $|\Omega\rangle_{AD}$ is the result of

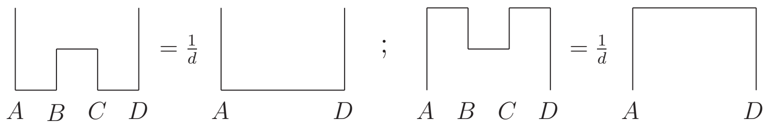


Figure 13: Cup and cap via compositions of cups and caps.

connecting the cap ${}_{BC}\langle\Omega|$ with the cups $|\Omega\rangle_{AB}$ and $|\Omega\rangle_{CD}$, as is verified by

$$
\begin{aligned}
& ({}_{BC}\langle\Omega|)(|\Omega\rangle_{AB})(|\Omega\rangle_{CD}) \\
\equiv\ & (\mathbb{1}_d \otimes {}_{BC}\langle\Omega| \otimes \mathbb{1}_d)(|\Omega\rangle_{AB} \otimes \mathbb{1}_d \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathbb{1}_d \otimes |\Omega\rangle_{CD}) \\
=\ & \frac{1}{d\sqrt{d}} \sum_{i,j,k=0}^{d-1} (\langle e_i^B \otimes e_i^C|)(|e_j^A \otimes e_j^B\rangle)(|e_k^C \otimes e_k^D\rangle) \\
=\ & \frac{1}{d\sqrt{d}} \sum_{i=0}^{d-1} |e_i^A \otimes e_i^D\rangle = \frac{1}{d}|\Omega\rangle_{AD}. \qquad (60)
\end{aligned}
$$

The second term of Figure 12 shows the cap ${}_{AD}\langle\Omega|$ to be the composition of the caps ${}_{AB}\langle\Omega|$, ${}_{CD}\langle\Omega|$ and the cup $|\Omega\rangle_{BC}$,

$$
\begin{aligned}
& ({}_{AB}\langle\Omega|)({}_{CD}\langle\Omega|)(|\Omega\rangle_{BC}) \\
\equiv\ & ({}_{AB}\langle\Omega| \otimes \mathbb{1}_d \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathbb{1}_d \otimes {}_{CD}\langle\Omega|)(\mathbb{1}_d \otimes |\Omega\rangle_{BC} \otimes \mathbb{1}_d) = \frac{1}{d}\ {}_{AD}\langle\Omega|. \quad (61)
\end{aligned}
$$

Note that the above examples are devised to explain how to apply our diagrammatic rules. On the other hand, they will be exploited in the following subsections as known diagrammatic tricks.

3.3 Transfer operator and acausality problem

Besides its standard description [7] for the teleportation equation (8), the teleportation can be formulated via the transfer operator T_{BC} (50) which sends the quantum state from Charlie to Bob in the way: $T_{BC}|\psi\rangle_C = |\psi\rangle_B$. The transfer operator T_{BC} has the form in terms of the maximally entangled state, $\frac{1}{d}T_{BC} = {}_{CA}\langle\Omega|\Omega\rangle_{AB}$, and so has a diagrammatic realization according to our rules, see Figure 8 in which the teleportation is regarded as the flow of quantum information.

The entire teleportation process involves local unitary transformations which are not shown up in the formalism of the transfer operator T_{BC} (51). To be general, therefore, we recall the calculation in [29] to represent the transfer operator in terms of the maximally entangled states $|\Phi(U)\rangle_{CA}$ and $|\Phi(V^T)\rangle_{AB}$ defined by local unitary actions of U and V^T on $|\Omega\rangle$, i.e.,

$$
{}_{CA}\langle\Phi(U)|\Phi(V^T)\rangle_{AB} \equiv {}_{CA}\langle\Omega|(U^\dagger \otimes \mathbb{1}_d)(V^T \otimes \mathbb{1}_d)|\Omega\rangle_{AB}
$$

$$
\begin{aligned}
&\equiv {}_{CAB}\langle \Omega \otimes \mathbb{1}_d | (U^\dagger \otimes \mathbb{1}_d \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes V^T \otimes \mathbb{1}_d) | \mathbb{1}_d \otimes \Omega \rangle_{CAB} \\
&= {}_{CAB}\langle \Omega \otimes \mathbb{1}_d | (\mathbb{1}_d \otimes \mathbb{1}_d \otimes VU^\dagger) | \mathbb{1}_d \otimes \Omega \rangle_{CAB} \\
&= \frac{1}{d} \sum_{i=0}^{d-1} {}_{CB}\langle e_i \otimes \mathbb{1}_d | (\mathbb{1}_d \otimes VU^\dagger) | \mathbb{1}_d \otimes e_i \rangle_{CB} \\
&\equiv \frac{1}{d} \sum_{i=0}^{d-1} {}_{C}\langle e_i | (VU^\dagger)_B | e_i \rangle_B = \frac{1}{d} \sum_{i=0}^{d-1} (VU^\dagger)_B | e_i \rangle_B \; {}_{C}\langle e_i | \\
&= \frac{1}{d} (VU^\dagger)_B \; T_{BC}
\end{aligned} \tag{62}
$$

which has a special case of $U = V$ given by

$$
\frac{1}{d} T_{BC} = {}_{CA}\langle \Phi(U) | \Phi(U^T) \rangle_{AB}. \tag{63}
$$

In the following, we repeat the above algebraic calculation for the transfer operator T_{BC} at the diagrammatic level and then discuss the so called acausality problem which becomes explicit when the teleportation is recognized as the flow of quantum information.

From the left to the right, the inner product ${}_{CA}\langle \Phi(U) | \Phi(V^T) \rangle_{AB}$ has the cap $\langle \Omega |$, identity $\mathbb{1}_d$, local unitary operators U and V^T, identity $\mathbb{1}_d$ and cup $|\Omega\rangle$ which are drawn from the top to the bottom, see Figure 14. Move the local operators $U^\dagger$ and V^T along the lines from their positions to the top boundary point of the system B and obtain the local product $(VU^\dagger)_B$ of unitary operators acting on the quantum state that Bob has. The transfer operator T_{BC} has the normalization factor $\frac{1}{d}$ from the vanishing of a cup and a cap. Hence in our diagrammatic approach it is clear that the teleportation can be viewed as a kind of quantum information flow from Charlie to Bob.

But the result $\frac{1}{d}(VU^\dagger)_B T_{BC}$ seems to argue that the measurement with the unitary operator $U^\dagger$ plays the role before the preparation with the unitary operator V^T. It is not true. Let read Figure 14 in the way where the T-axis denotes the time arrow and the X-axis denotes the space distance. The quantum information flow starts from the preparation of state, goes to the measurement and come backs to the preparation again and finally goes to the measurement. As a result, it flows from the preparation to the measurement without violating the causality principle.

Note that we have to add a rule on how to move operators in our diagrammatic approach: It is forbidden for an operator to cross over another operator. For example, we have the operator product $\frac{1}{d}(VU^\dagger)_B$ instead of $\frac{1}{d}(U^\dagger V)_B$ in Figure 14. Obviously, this rule is related to our analysis on the acausality problem. The violation of the rule leads to the violation of the causality principle. In addition, there are other known approaches to the quantum information flow: the teleportation topology [10, 20] and categorical approach [32]. We will compare them with our diagrammatic approach in the next section.

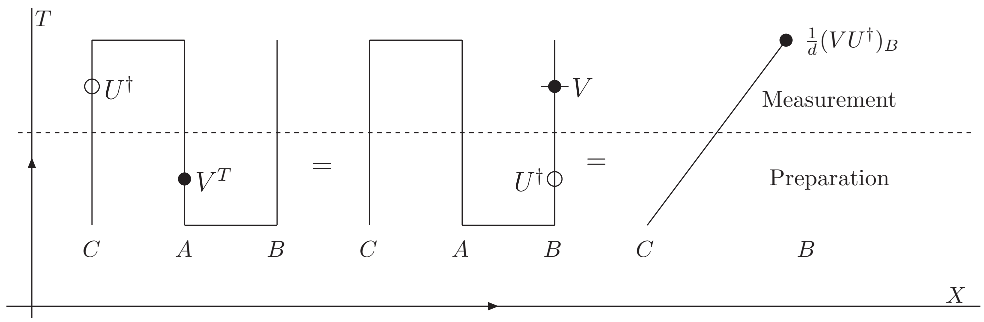


Figure 14: Transfer operator and acausality problem.

3.4 Entanglement via measurement (I): teleportation

Teleportation can be observed from the point of quantum measurement [26, 27, 28, 37]. The only difference from the standard description [7] of the teleportation is that the maximally entangled state $|\Omega\rangle_{AB}$ between Alice and Bob is created in the non-local measurement [26]. Here it is not necessary to go through details in the measurement but simply represent it in terms of the projector $(|\Omega\rangle\langle\Omega|)_{AB}$.

Therefore, the teleportation is determined by two quantum measurements: the one denoted by $(|\Omega\rangle\langle\Omega|)_{AB}$ and the other denoted by $(|\Omega_n\rangle\langle\Omega_n|)_{CA}$, as leads to a new formulation of the teleportation equation,

$$(|\Omega_n\rangle\langle\Omega_n| \otimes 1\!\!1_d)(|\psi\rangle \otimes |\Omega\rangle\langle\Omega|) = \frac{1}{d}(|\Omega_n\rangle \otimes 1\!\!1_d)(1\!\!1_d \otimes (1\!\!1_d \otimes U_n^\dagger|\psi\rangle)\langle\Omega|), \qquad (64)$$

where the maximally entangled state $|\Omega_n\rangle$ is the local unitary transformation of $|\Omega\rangle$, i.e., $|\Omega_n\rangle = (U_n \otimes 1\!\!1_d)|\Omega\rangle$, the set of local unitary operators U_n satisfies the orthogonal relation $tr(U_n^\dagger U_m) = d\,\delta_{nm}$, $n, m = 1, \cdots d^2$, and the lower indices A, B, C are omitted for convenience.

Read the teleportation equation (64) from the left to the right and draw the diagram from the top to the bottom in view of our rules, i.e., Figure 15. There is a natural connection between two formalisms (8) and (64) of the teleportation equation. Choose all unitary matrices U_n in the way so that they satisfy

$$1\!\!1_d = \sum_{i,j=0}^{d-1} |e_i \otimes e_j\rangle\langle e_i \otimes e_j| = \sum_{n=1}^{d^2} |\Omega_n\rangle\langle\Omega_n|, \qquad U_1 = 1\!\!1_d \qquad (65)$$

and then make a summation of all possible teleportation equations like (64) to derive the version (8) of the teleportation equation in the d-dimension Hilbert space,

$$|\psi\rangle \otimes |\Omega\rangle = \frac{1}{d}\sum_{n=1}^{d^2} |\Omega_n\rangle \otimes U_n^\dagger|\psi\rangle. \qquad (66)$$

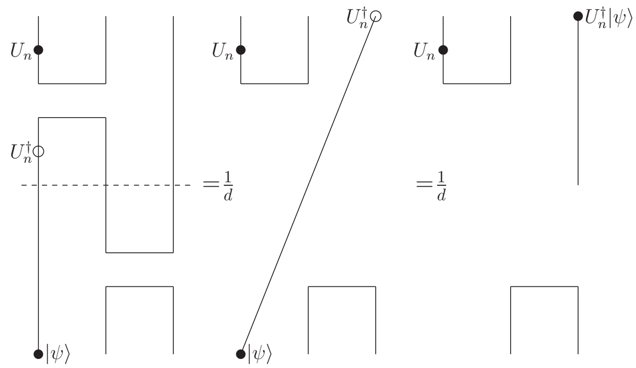


Figure 15: Teleportation based on quantum measurement.

In the case of $d = 2$, the collections of the unitary operators consist of the unit matrix $1\!\!1_2$ and Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. The Bell measurements are denoted by the projectors in terms of the Bell states $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$. They lead to the same kind of the teleportation equations as (64),

$$
\begin{aligned}
(|\phi^-\rangle\langle\phi^-| \otimes 1\!\!1_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\phi^-\rangle \otimes \sigma_3|\psi\rangle), \\
(|\psi^+\rangle\langle\psi^+| \otimes 1\!\!1_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\psi^+\rangle \otimes \sigma_1|\psi\rangle), \\
(|\psi^-\rangle\langle\psi^-| \otimes 1\!\!1_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\psi^-\rangle \otimes -i\sigma_2|\psi\rangle), \\
(|\phi^+\rangle\langle\phi^+| \otimes 1\!\!1_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\phi^+\rangle \otimes |\psi\rangle), \qquad (67)
\end{aligned}
$$

which has the summation to be the teleportation equation (8) since the Bell states $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are verified to satisfy

$$1\!\!1_2 = |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|. \qquad (68)$$

The second example is on the continuous teleportation [27]. The maximally entangled state $|\Omega'\rangle$ and teleportated state $|\Psi\rangle$ in the continuous case have the forms,

$$|\Omega'\rangle = \int dx\ |x, x\rangle, \qquad |\Psi\rangle = \int dx\ \psi(x)\ |x\rangle, \qquad (69)$$

and the other maximally entangled state $|\Omega'_{\alpha\beta}\rangle$ is formulated by the combined action of the $U(1)$ rotation with the translation T on $|\Omega'\rangle$, i.e.,

$$|\Omega'_{\alpha\beta}\rangle = (U_\beta \otimes T_\alpha)|\Omega'\rangle \equiv \int dx \exp(i\beta x)|x, x+\alpha\rangle, \qquad \alpha, \beta \in \mathbb{R} \qquad (70)$$

where $U_\beta|x\rangle = e^{i\beta x}|x\rangle$, $T_\alpha|x\rangle = |x+\alpha\rangle$ and which is a common eigenvector of the location operator $\mathbf{X}\otimes 1\!\!1 - 1\!\!1\otimes\mathbf{X}$ and conjugate momentum operator $\mathbf{P}\otimes 1\!\!1 + 1\!\!1\otimes\mathbf{P}$,

$$(\mathbf{X}\otimes 1\!\!1 - 1\!\!1\otimes\mathbf{X})|\Omega'_{\alpha\beta}\rangle = -\alpha|\Omega'_{\alpha\beta}\rangle, \qquad (\mathbf{P}\otimes 1\!\!1 + 1\!\!1\otimes\mathbf{P})|\Omega'_{\alpha\beta}\rangle = 2\beta|\Omega'_{\alpha\beta}\rangle. \quad (71)$$

The teleportation equation of the type (64) is obtained to be

$$(|\Omega'_{\alpha\beta}\rangle\langle\Omega'_{\alpha\beta}|\otimes 1\!\!1)(|\Psi\rangle\otimes|\Omega'\rangle) = (|\Omega'_{\alpha\beta}\rangle\otimes 1\!\!1)(1\!\!1\otimes 1\!\!1\otimes U_{-\beta}T_\alpha|\Psi\rangle) \qquad (72)$$

which has a similar diagrammatic representation as Figure 15. The translation operator T_α is its own adjoint operator, and is permitted to move along the cup to the top boundary point (see, Figure 15), although it does not behave like the matrix operator M (47).

Note that the continuous teleportation is a simple generalization of the discrete teleportation without essential conceptual changes, as is explicit in our diagrammatic approach. Importantly, the diagrammatic representation for the teleportation based on quantum measurement, Figure 15 is a key clue to propose the Temperley–Lieb algebra to be underlying the teleportation because it is a standard configuration in the diagrammatic representation for the Temperley–Lieb algebra.

3.5 Tight teleportation and dense coding schemes

In the tight teleportation and dense coding schemes [21], all involved finite Hilbert spaces are d dimensional and the classical channel distinguishes d^2 signals. All examples we treated in the above belong to the tight class. We exploit the same notations as in [21]. The density operator ρ is a positive operator with a normalized trace. Charlie has his density operator $\rho_C = (|\phi_1\rangle\langle\phi_2|)_C$ which denotes the quantum state to be sent to Bob. Alice and Bob share the maximally entangled state $\omega_{AB} = (|\Omega\rangle\langle\Omega|)_{AB}$. The set of observables ω_n, $n = 1, 2, \cdots d^2$ over an output parameter space is a collection of bounded linear operators in the Hilbert space $\mathcal{H}$. Alice makes the Bell measurement in the composite system between Charlie and her, and she chooses her observables $(\omega_n)_{CA}$ to be local unitary transformations on the maximally entangled state $|\Omega\rangle\langle\Omega|$,

$$(\omega_n)_{CA} = (|\Omega_n\rangle\langle\Omega_n|)_{CA}, \qquad |\Omega_n\rangle = (U_n\otimes 1\!\!1_d)|\Omega\rangle, \qquad n = 1, 2, \cdots d^2 \qquad (73)$$

where the set of unitary operators U_n is chosen in the way so that $|\Omega_n\rangle$ and ω_n have the properties

$$\langle\Omega_n|\Omega_m\rangle = \delta_{nm}, \qquad \sum_{n=1}^{d^2}\omega_n = 1\!\!1_d, \qquad n, m = 1, \cdots d^2. \qquad (74)$$

As Bob gets the message denoted by n from Alice and then applies the local unitary transformation T_n on his observable $\mathcal{O}_B$, which are given by

$$T_n(\mathcal{O}_B) = U_n^\dagger\mathcal{O}_B U_n, \qquad \mathcal{O}_B = (|\psi_1\rangle\langle\psi_2|)_B, \qquad n = 1, 2, \cdots d^2. \qquad (75)$$

The operator T_n defined this way is called a channel, a complete positive linear operator and normalized as $T_n(1\!\!1_d) = 1\!\!1_d$.

In terms of ρ_C, ω_{AB}, $(\omega_n)_{CA}$ and $T_n(\mathcal{O}_B)$, the tight teleportation scheme is summarized in the equation

$$\sum_{n=1}^{d^2} tr((\rho \otimes \omega)(\omega_n \otimes T_n(\mathcal{O}))) = tr(\rho\mathcal{O}), \tag{76}$$

where lower indices A, B, C are neglected for convenience. It catches the aim of a successful teleportation, i.e., Charlie makes the measurement in his system as he does in Bob's system although they are far away from each other. It can be proved after some algebraic calculation. The term containing the message n has the form denoted by $term_n$ in the following,

$$\begin{aligned} & term_n = tr((|\phi_1\rangle\langle\phi_2| \otimes |\Omega\rangle\langle\Omega|)(|\Omega_n\rangle\langle\Omega_n| \otimes U_n^\dagger|\psi_1\rangle\langle\psi_2|U_n)) \\ & = \langle\Omega_n \otimes \psi_2 U_n|\phi_1 \otimes \Omega\rangle\langle\phi_2 \otimes \Omega|\Omega_n \otimes U_n^\dagger\psi_1\rangle \\ & = (\frac{1}{d}\langle\psi_2|\phi_1\rangle)(\frac{1}{d}\langle\phi_2|\psi_1\rangle) = \frac{1}{d^2} tr(\rho\mathcal{O}), \end{aligned} \tag{77}$$

where the inner product (52) has been applied twice. There are d^2 distinguished messages labeled by n, so we prove the tight teleportation equation (76).

At the diagrammatic level, we have two ways of deriving the equation (76), see the left diagrammatic term of Figure 16. The first way moves the operators $U_n^\dagger$, $\mathcal{O}$ and U_n from the branch of the cap to the other branch and then applies the known diagrammatic tricks suggested by the first term of Figure 11 and the first term of Figure 12. The second one makes use of the tricks given by the second and third terms of Figure 11 and the second term of Figure 12, as has been ensured by our calculation (59) for the closed circle formed by two oblique lines. In addition, the number of classical channel, n^2 is the number of all possible teleportation diagrams like the left one in Figure 16. Note that it is this type of diagrams that shed us the insight of proposing the Temperley–Lieb algebra behind the teleportation.

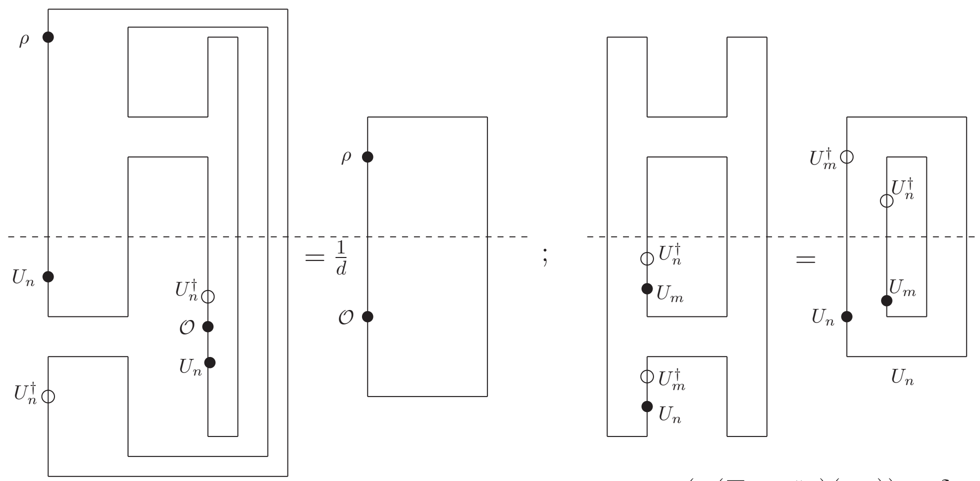
Similarly, all the tight dense coding schemes are concluded in the equation,

$$tr(\omega(T_n \otimes 1\!\!1_d)(\omega_m)) = \delta_{nm} \tag{78}$$

which is explained as follows. When Alice and Bob share the maximally entangled state $|\Omega\rangle_{AB}$, Alice transforms her state by the channel T_n to encode the message n and then Bob makes the measurement on observables ω_m of his system. At $n = m$, Bob gets the message. The entire process of dense coding is performed in the way,

$$\begin{aligned} & tr(|\Omega\rangle\langle\Omega|(U_n^\dagger \otimes 1\!\!1_d)|\Omega_m\rangle\langle\Omega_m|(U_n \otimes 1\!\!1_d)) \\ & = \langle\Omega|U_n^\dagger \otimes 1\!\!1_d|\Omega_m\rangle\langle\Omega_m|U_n \otimes 1\!\!1_d|\Omega\rangle = \frac{1}{d^2}(tr(U_n^\dagger U_m))^2 = \delta_{nm} \end{aligned} \tag{79}$$

which is the tight dense coding equation (78) and also proved in our diagrammatic approach, see the right term of Figure 16.



$tr((\rho \otimes \omega)(\omega_n \otimes T_n(\mathcal{O}))) = \frac{1}{d^2} tr(\rho \mathcal{O})$

$tr(\omega(T_n \otimes 1\!\!1_d)(\omega_m)) = \delta_{nm}$

Figure 16: All tight teleportation and dense coding schemes.

Note that the tight teleportation scheme includes all elements of the teleportation and unifies them from the point of the global view into an algebraic or diagrammatic equation. It differs from the other approaches such as the standard description [7], the transfer operator [29] and quantum measurement [27], which all observe the teleportation from the local point of view.

3.6 Temperley–Lieb algebra and maximally entangled states

We propose the Temperley–Lieb algebra under local unitary transformations to underlie the teleportation and make it clear in the following. The Temperley–Lieb algebra TL_n is generated by identity Id and $n-1$ hermitian projectors e_i satisfying

$$\begin{aligned} e_i^2 &= e_i, \qquad (e_i)^\dagger = e_i, \;\; i = 1, \ldots, n-1, \\ e_i e_{i\pm 1} e_i &= \lambda^{-2} e_i, \qquad e_i e_j = e_j e_i, \;\; |i-j| > 1, \end{aligned} \tag{80}$$

in which λ is called loop parameter [2]. Let set up the representation of the $TL_n(\lambda)$ algebra in terms of the maximally entangled state ω, a projector,

$$\omega = |\Omega\rangle\langle\Omega| = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj|, \qquad \omega^2 = \omega \tag{81}$$

by defining the idempotents e_i in the way

$$e_i = (Id)^{\otimes(i-1)} \otimes \omega \otimes (Id)^{\otimes(n-i-1)}, \qquad i = 1, \cdots n-1. \tag{82}$$

[2] The notation e_i for the idempotent of the Temperley–Lieb algebra is easily confused with that in the basis vector $|e_i\rangle$. When they appear at the same time, we denote $|e_i\rangle$ by $|i\rangle$.

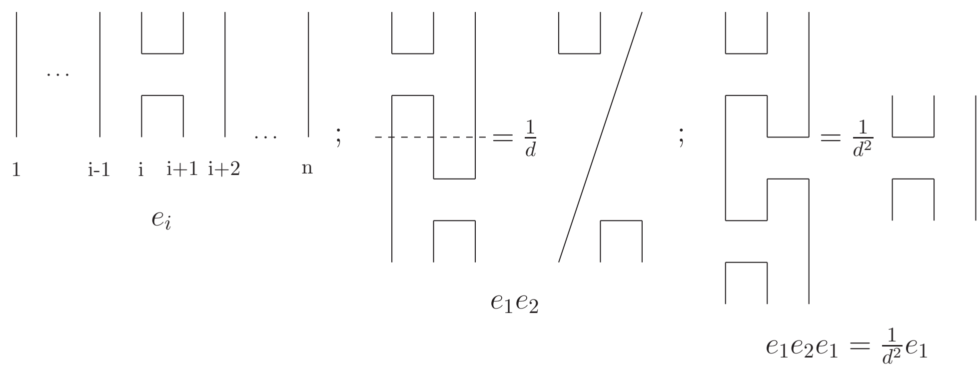


Figure 17: Generator e_i, multiplication e_1e_2 and axiom for the $TL_3(d)$ algebra.

For example, the $TL_3(d)$ algebra is generated by two idempotents e_1 and e_2,

$$e_1 = \omega \otimes Id, \qquad e_2 = Id \otimes \omega, \tag{83}$$

which are proved to satisfy the axiom $e_1e_2e_1 = \frac{1}{d^2}e_1$ in the way

$$e_1e_2e_1|\alpha\beta\gamma\rangle = \frac{1}{d}\sum_{l=0}^{d-1} e_1e_2|ll\gamma\rangle\delta_{\alpha\beta} = \frac{1}{d^3}\sum_{n=0}^{d-1}|nn\gamma\rangle\delta_{\alpha\beta} = \frac{1}{d^2}e_1|\alpha\beta\gamma\rangle \tag{84}$$

and satisfy the axiom $e_2e_1e_2 = \frac{1}{d^2}e_2$ via similar calculation.

To build an apparent connection between the Temperley–Lieb algebra and teleportation, let make rules for the diagrammatic representation of the Temperley–Lieb algebra which is also called the Brauer diagram [25] or Kauffman diagram [38] in literature. It is a planar (n, n) diagram including a hidden rectangle in the plane with hidden $2n$ distinct points: n on its top edge and n on its bottom edge which are connected by disjoint strings drawn within in the rectangle. The identity is the diagram with all strings vertical, while e_i has its ith and $i+1$th top (and bottom) boundary points connected and all other strings vertical. The multiplication e_ie_j identifies bottom points of e_i with corresponding top points of e_j, removes the common boundary and replaces each obtained loop with a factor λ. The adjoint of e_i is an image under mirror reflection of e_i on a horizontal line.

In Figure 17, there are diagrammatic representations for e_i, e_1e_2 and $e_1e_2e_1 = \frac{1}{d^2}e_1$ with loop parameter d. Therefore, the cup (cap) introduced in our diagrammatic approach is a connected line between top (and bottom) boundary points. Each cup (cap) with a normalization factor $d^{-\frac{1}{2}}$ leads to an additional normalization factor $d^{-\frac{1}{2}N}$ as the number of vanishing cups and caps is N, while a closed circle yields a normalization factor $d = tr(\mathbb{1}_d)$. For examples, in Figure 17, e_1e_2 has a normalization factor $\frac{1}{d}$ from a vanishing cup and a vanishing cap, and $e_1e_2e_1$ has a factor $\frac{1}{d^2}$ from four vanishing cups and caps.

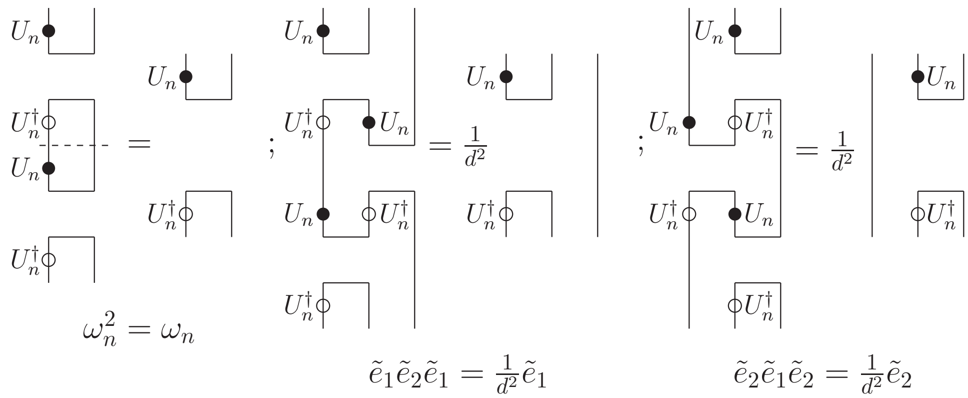


Figure 18: The $TL_3(d)$ algebra under local unitary transformations.

In terms of the density matrix ω_n (73) which is the local unitary transformation U_n of the maximally entangled state ω, we can set up the representation of the Temperley–Lieb algebra, too. For example, the $TL_3(d)$ algebra is generated by $\tilde{e}_1$ and $\tilde{e}_2$,

$$\tilde{e}_1 = \omega_n \otimes Id, \qquad \tilde{e}_2 = Id \otimes \omega_n, \tag{85}$$

which are proved to satisfy the axioms of the Temperley–Lieb algebra in a diagrammatic approach, see Figure 18. Therefore the axioms of the Temperley–Lieb algebra are invariant under local unitary transformations as the idempotents e_i are generated by the maximally entangled state ω.

Furthermore, observed from the tight teleportation scheme, the $TL_3(d)$ algebra is also built in the way,

$$e_1' = \rho \otimes \omega, \qquad e_2' = \omega \otimes \rho, \qquad \rho = |\phi\rangle\langle\psi|, \ tr(\rho) = 1. \tag{86}$$

because ρ and ω and the tensor products between them are all projectors. The axioms of the Temperley–Lieb algebra can be checked in both algebraic and diagrammatic approaches. Do calculation,

$$\begin{aligned}(\omega \otimes \rho)(\rho \otimes \omega) &= \frac{1}{d^2}\sum_{i,j=0}^{d-1}(|ii\rangle\langle jj| \otimes |\phi\rangle\langle\psi|)\sum_{l,m=0}^{d-1}(|\phi\rangle\langle\psi| \otimes |ll\rangle\langle mm|) \\ &= \frac{1}{d^2}\sum_{i,j=0}^{d-1}\sum_{l,m=0}^{d-1}(|ii\phi\rangle\langle jj\psi|)(|\phi ll\rangle\langle\psi mm|) = \frac{1}{d^2}\sum_{i,j=0}^{d-1}|ii\phi\rangle\langle\psi jj|\end{aligned} \tag{87}$$

which leads to the proof for the axiom $e_2'e_1'e_2' = \frac{1}{d^2}e_2'$,

$$(\omega \otimes \rho)(\rho \otimes \omega)(\omega \otimes \rho) = \frac{1}{d^3}\sum_{i,j=0}^{d-1}\sum_{l,m=0}^{d-1}|ii\phi\rangle\langle\psi jj|ll\phi\rangle\langle mm\psi|$$

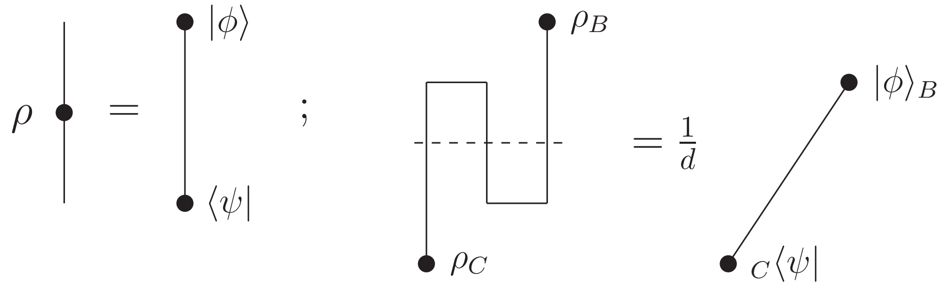


Figure 19: The density state and the transfer operator.

$$= \frac{1}{d^2} \sum_{i,j=0}^{d-1} |ii\phi\rangle\langle jj\psi| = \frac{1}{d^2}(\omega \otimes \rho). \tag{88}$$

Similarly to prove $e'_1 e'_2 e'_1 = \frac{1}{d^2} e'_1$. The one thing has to be noted, $\rho = |\phi\rangle\langle\psi|$ so that the transfer operator T_{BC} sends a half of ρ_C, i.e., $|\phi\rangle_C$ from Charlie to Bob to form a unit inner product with ${}_B\langle\psi|$, a half of ρ_B, see Figure 19.

Moreover, in terms of ρ and ω_n, the $TL_3(d)$ algebra is set up in the way,

$$\tilde{e}'_1 = \rho \otimes \omega_n, \qquad \tilde{e}'_2 = \omega_n \otimes \rho, \qquad n = 1, \cdots d^2, \tag{89}$$

if and only if the local unitary transformation U_n is a symmetric matrix, as is clear from the diagrammatic point of view, see Figure 18,

$$U_n^T = U_n, \qquad U_n^T U_n^\dagger = U_n^* U_n = 1\!\!1_d. \tag{90}$$

Note that the Temperley–Lieb diagrams with solid points or small circles in our diagrammatic approach are found to present simple and clear pictures for various topics around the teleportation. For simplicity, we name the Temperley–Lieb category for the Temperley–Lieb algebra under local unitary transformations.

3.7 Entanglement via measurement (II): entanglement swapping

Entanglement swapping [30] is a sort of experimental technique realizing the entanglement between two independent systems to be a consequence of quantum measurement instead of physical interaction. Let make an example for its theoretical interpretation in terms of projector representing measurement. Alice has a bipartite entangled state $|\Omega_l\rangle_{ab}^A$ for particles a, b and Bob has $|\Omega_m\rangle_{cd}^B$ for particles c, d. They are independently created and do not share common history. Alice applies the measurement denoted by $Id \otimes |\Omega_n\rangle\langle\Omega_n| \otimes Id$ to the product state of $|\Omega_l\rangle_{ab}^A$ and $|\Omega_m\rangle_{cd}^B$ so that the output called the entanglement swapped state $|\Omega_{lnm}\rangle_{ad}^{AB}$ is a bipartite

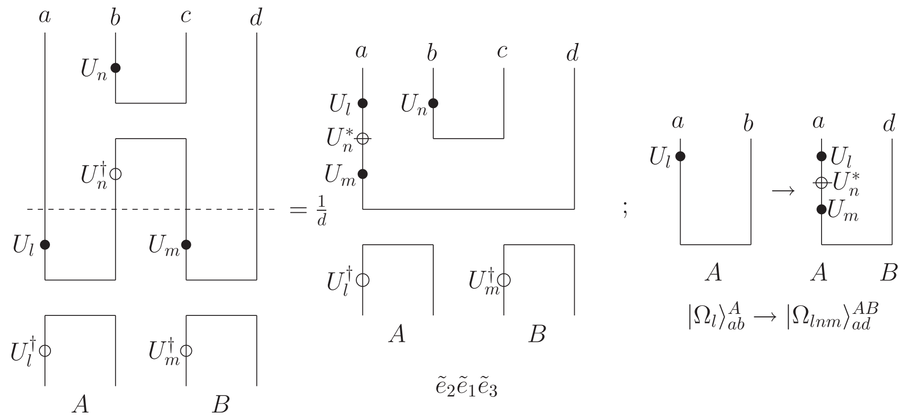


Figure 20: Entanglement swapping and teleportation of entanglement.

entangled state shared by Alice and Bob for particles a, d, i.e.,

$$\begin{aligned}
&(Id \otimes |\Omega_n\rangle\langle\Omega_n| \otimes Id)(|\Omega_l\rangle_{ab}^A \otimes |\Omega_m\rangle_{cd}^B) \\
&= \frac{1}{d}(Id \otimes |\Omega_n\rangle \otimes Id)\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}(U_l U_n^* U_m |e_i\rangle_a^A \otimes Id \otimes Id \otimes |e_i\rangle_d^B) \\
&\equiv \frac{1}{d}(Id \otimes |\Omega_n\rangle \otimes Id)\ |\Omega_{lnm}\rangle_{ad}^{AB}.
\end{aligned} \tag{91}$$

In other words, the entanglement swapping reduces a four-particle state $|\Omega_l\rangle_{ab}^A \otimes |\Omega_m\rangle_{cd}^B$ to a bipartite entangled state $|\Omega_{lnm}\rangle_{ad}^{AB}$ via the entangling measurement.

Read the entanglement swapping equation (91) from the left to the right and draw a diagram from the top to the bottom according to our diagrammatic rules. It represents a diagrammatic description for an element $\tilde{e}_2\tilde{e}_1\tilde{e}_3$ of the Temeperly–Lieb category, i.e.,

$$\tilde{e}_2\tilde{e}_1\tilde{e}_3 = (Id \otimes \omega_n \otimes Id)(\omega_l \otimes Id \otimes Id)(Id \otimes Id \otimes \omega_m), \tag{92}$$

which is shown up in the left term of Figure 20. Also, we view the entanglement swapping from the point of the teleportation and hence it changes the entangled state $|\Omega_l\rangle_{ab}^A$ shared by particles a, b in Alice's system to the entangled state $|\Omega_{lnm}\rangle_{ad}^{AB}$ shared by particles a, d in Alice and Bob's composite system, see the right term of Figure 20.

Furthermore, as the teleportation is denoted by the transfer operator T_{BC}, the entanglement swapping can be called the teleportation via the cup state. Alice measures the Bell state $|\Omega\rangle_{AB}$ shared by Bob and her with the projector $|\psi\rangle_A\ {}_A\langle\psi|$ so that she transfers her quantum state to Bob in the way,

$$|\psi\rangle_A\ {}_A\langle\psi|\Omega\rangle_{AB} = \frac{1}{\sqrt{d}}|\psi\rangle_A \sum_{i,j=0}^{d-1} {}_{AB}\langle e_j \otimes Id|\psi_j^*|e_i \otimes e_i\rangle_{AB}$$

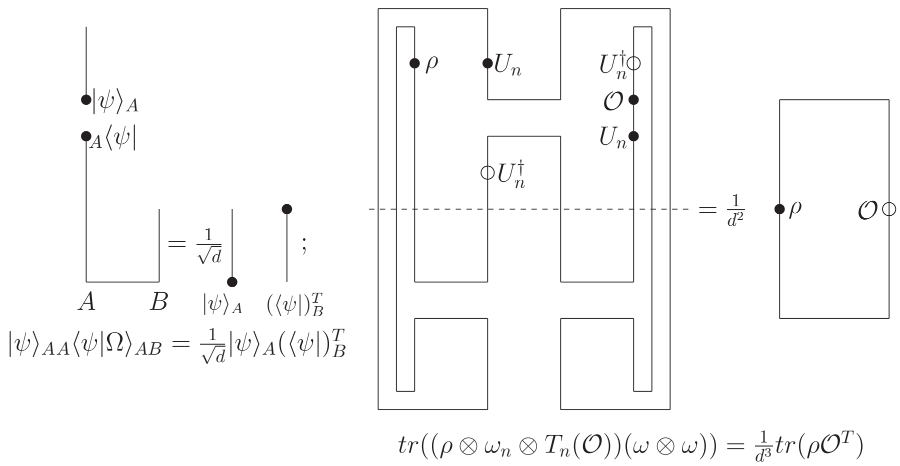


Figure 21: Teleportation via cup and tight entanglement swapping.

$$= \frac{1}{\sqrt{d}}|\psi\rangle_A \sum_{i=0}^{d-1} \psi_i^* |e_i\rangle_B = \frac{1}{\sqrt{d}}|\psi\rangle_A(\langle\psi|)^T_B, \tag{93}$$

which has a diagrammatic representation, see the left term of Figure 21 with $(\langle e_i|)^T = |e_i\rangle$, similar to the two-way teleportation performed by the crossed measurement [27, 37].

Moreover, the tight entanglement swapping equation is derived by following the procedure of setting up all tight teleportation and dense coding schemes [21], i.e.,

$$\sum_{n=1}^{d^2} tr((\rho\otimes\omega_n\otimes T_n(\mathcal{O}))(\omega\otimes\omega)) = \frac{1}{d}tr(\rho\mathcal{O}^T) \tag{94}$$

where the density operator ρ for the particle a, observable $\mathcal{O}$ for the particle d and quantum channel $T_n(\mathcal{O})$ for the particle d are respectively given by

$$\rho = |\phi_1\rangle\langle\phi_2|, \;\; \mathcal{O} = |\psi_1\rangle\langle\psi_2|, \;\; T_n(\mathcal{O}) = U_n^\dagger \mathcal{O} U_n \tag{95}$$

and the transpose $\mathcal{O}^T$ of the density operator $\mathcal{O}$ is defined in the way

$$\mathcal{O}^T = \sum_{i,j=0}^{d-1} \psi_{1i}\psi_{2j}^*(|e_i\rangle\langle e_j|)^T = \sum_{i,j=0}^{d-1} \psi_{1i}\psi_{2j}^*|e_j\rangle\langle e_i|, \;\; (\langle e_i|)^T = |e_i\rangle. \tag{96}$$

The tight entanglement swapping equation (94) is easily proved in our diagrammatic approach, see the right term of Figure 21. We exploit the diagrammatic trick suggested by Figure 13 to derive the same configuration as the first term of Figure

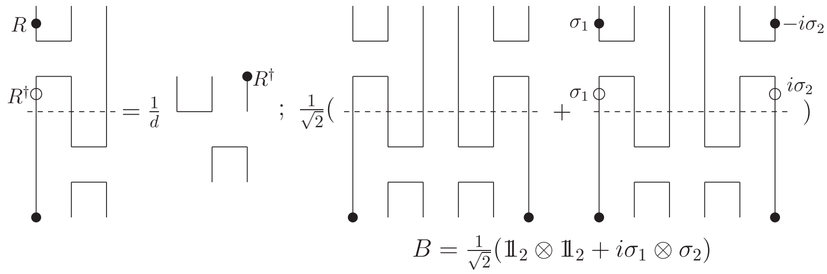


Figure 22: Fault-tolerant gate and braid gate via teleportation.

12 which denotes the trace of operator products. We also verify it in the following algebraic way, the $term_n$ given by

$$
\begin{aligned}
& term_n \equiv tr((|\phi_1\rangle\langle\phi_2| \otimes |\Omega_n\rangle\langle\Omega_n| \otimes U_n^\dagger|\psi_1\rangle\langle\psi_2|U_n)(|\Omega\rangle\langle\Omega| \otimes |\Omega\rangle\langle\Omega|)) \\
& = \langle\phi_2 \otimes \Omega_n \otimes \psi_2 U_n|\Omega \otimes \Omega\rangle\langle\Omega \otimes \Omega|\phi_1 \otimes \Omega_n \otimes U_n^\dagger\psi_1\rangle \\
& = \frac{1}{d^3}(\phi_2^* \cdot \psi_2^*)(\phi_1 \cdot \psi_1)
\end{aligned} \tag{97}
$$

which is found to be

$$
\frac{1}{d^3} tr(\rho\mathcal{O}^T) = \frac{1}{d^3}\sum_{i,j=0}^{d-1} \psi_{1i}\psi_{2j}^* \langle e_i|\phi_1\rangle\langle\phi_2|e_j\rangle = term_n, \tag{98}
$$

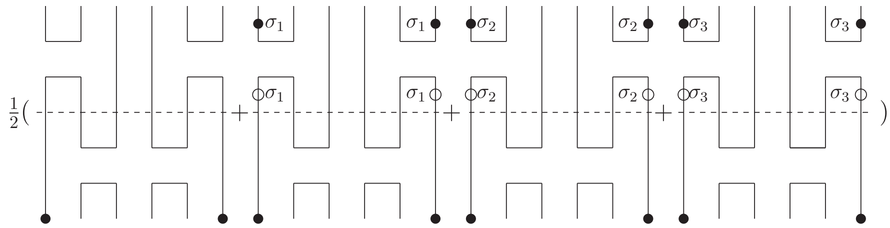
as proves (94) since there are d^2 $term_n$ and each $term_n$ is independent of n.

Note that the entanglement swapped state $|\Omega_{lnm}\rangle_{ad}^{AB}$ plays a special role in the comparison of quantum mechanics with classical physics since it is a quantum state in the quantum world but produced without any direct classical physical interactions.

3.8 Teleportation and topological quantum computing

Teleportation has been considered as a universal quantum computational primitive [31]. Under such a proposal, there are both theoretical observations and experimental motivations. The teleported state permits the action of local unitary transformations and so the teleportation realizes single qubit gates as local unitary transformations and two-qubit gates as linear combinations of products of single qubit gates. In the experimental sense, single qubit transformations and Bell measurements, basic elements of the teleportation, have been worked out in labs. Additionally, as single qubit transformations are performed fault-tolerantly, we will have the fault-tolerant quantum computation [39, 40].

A fault-tolerant gate R, an element of the Clifford group [39, 40], enters the teleportation via the entangling measurement and then is transported in the form of



$$P = \frac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)$$

Figure 23: Swap gate in terms of teleportation.

its conjugation $R^\dagger$, see the left term of Figure 22. The braid gate (9) has the form given by

$$B = \frac{1}{\sqrt{2}}(\mathbb{1}_2 \otimes \mathbb{1}_2 + i\sigma_1 \otimes \sigma_2) \tag{99}$$

which is performed in the way shown up in the right term of Figure 22. It has two diagrammatic terms and each one consists of two teleportation processes for sending two qubits respectively. The swap gate P, denoted by (36), is an element of the Temperley–Lieb category, see Figure 23, which has four diagrammatic terms and each one represents an algebraic term in (36). The CNOT gate, a linear combination of products of Pauli matrices,

$$\begin{aligned} C &= (|0\rangle\langle 0| \otimes \mathbb{1}_2 + |1\rangle\langle 1| \otimes \sigma_1) = \frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes \mathbb{1}_2 + \frac{1}{2}(\mathbb{1}_2 - \sigma_3) \otimes \sigma_1, \\ &= \frac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_1 + \sigma_3 \otimes \mathbb{1}_2 - \sigma_3 \otimes \sigma_1) \end{aligned} \tag{100}$$

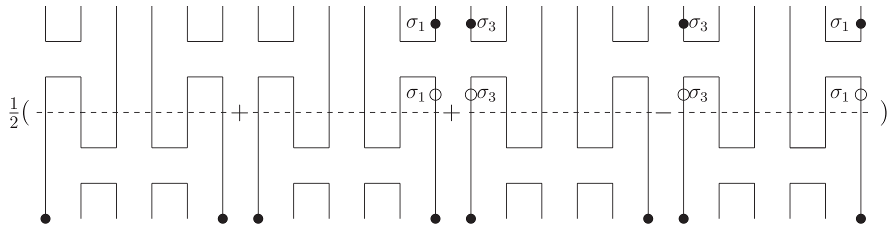
which satisfies the basic properties of the CNOT gate,

$$C|00\rangle = |00\rangle,\ C|01\rangle = |01\rangle,\ C|10\rangle = |11\rangle,\ C|11\rangle = |10\rangle, \tag{101}$$

has a diagrammatic representation, Figure 24.

In terms of a unitary braid gate [10, 11, 12], knot polynomial can be calculated through quantum simulation of knot on quantum computer, as is different from the way proposed in [41] for computing the Jones polynomial by a devised approximate quantum algorithm. With unitary braid gates and swap gate, virtual knots can be simulated via a quantum program. Also, unitary solutions of the Yang–Baxter equation with spectral parameters [11, 12] can be realized via the Temperley–Lieb category, as leads to the possibility of realiztion of exactly solvable two dimensional quantum field theories or statistical models [13, 14] on quantum computer.

Topological quantum computing have two known types of realizations. The one operates with anyons [42], the other is settled in the Chern-Simons theory [43] and both involve the unitary braid representation as quantum gates acting on particles like anyons. Our realization of a unitary braid via the teleportation provides



$$C = \frac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_1 + \sigma_3 \otimes \mathbb{1}_2 - \sigma_3 \otimes \sigma_1)$$

Figure 24: CNOT gate in terms of teleportation.

a theoretical or conceptual lab for them. We do not presume any specific models such as the Chern-Simons theory or anyons theory since our construction of a unitary braid gate is only based on its linear combination of products of local unitary transformations.

Note that the Temperley–Lieb category, a name for the Temperley–Lieb algebra under local unitary transformations, is an interesting mathematical subject since it contains abundant objects such as braids, permutation, and so on, and provides a kind of realization of theoretical quantum computer.

3.9 Teleportation and Brauer algebra

The teleportation has the same diagrammatic representation as the element e_1e_2 (e_2e_1) of the Temperley–Lieb algebra, as has been emphasized and repeated in the above subsections. But there exists a natural question which has to be answered. The teleportation plays the fundamental roles in quantum information, but the product e_1e_2 (e_2e_1) seems to be only an element of the Temperley–Lieb algebra. Therefore, it is natural to find out in which case the configuration of e_1e_2 (e_2e_1) is crucial in the mathematical sense. Let present the axioms of the Brauer algebra [25] and explain that the teleportation configuration is a bone of this algebra.

The Brauer algebra $D_n(\lambda)$ is an extension of the Temperley–Lieb algebra with virtual crossings, λ called loop parameter. It has two types of generators: idempotents e_i of the Temperley–Lieb algebra $TL_n(\lambda)$ satisfying (80) and virtual crossings v_i satisfying (26), $i = 1, \cdots n-1$. Both generators have to satisfy the mixed relations to determine the Brauer algebra,

$$\begin{aligned} (ev/ve): \;\; & e_i v_i = v_i e_i = e_i, \qquad e_i v_j = v_j e_i, \qquad j \neq i \pm 1, \\ (vve): \;\; & v_{i\pm1} v_i e_{i\pm1} = \lambda e_i e_{i\pm1}, \qquad (evv): \;\; e_i v_{i\pm1} v_i = \lambda e_i e_{i\pm1}. \end{aligned} \tag{102}$$

For example, the permutation P and maximally entangled state ω form the Brauer algebra $D_2(d)$ with the loop parameter d as P is the virtual crossing and ω

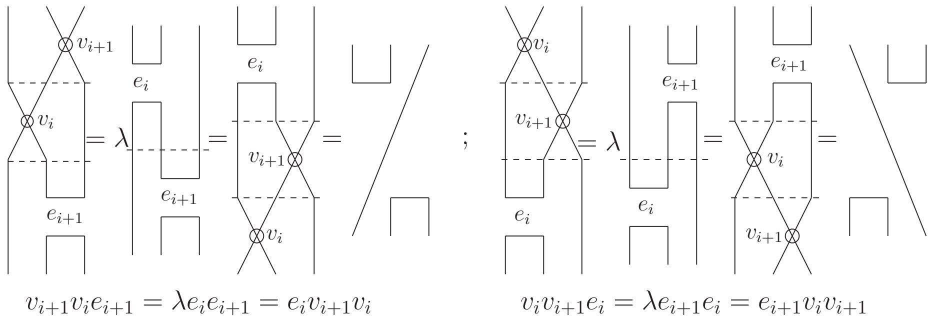


Figure 25: Teleportation, teleportation swapping and Brauer algebra.

is the idempotent. The mixed relations (102) are obviously verified by substituting

$$P = \sum_{i,j=0}^{d-1} |i \otimes j\rangle\langle j \otimes i|, \qquad \omega = \frac{1}{d}\sum_{i=0}^{d-1} |i \otimes i\rangle\langle j \otimes j| \tag{103}$$

into the axiom (ev/ve),

$$P\omega = \frac{1}{d}\sum_{i,j=0}^{d-1}\sum_{i',j'=0}^{d-1} |i \otimes j\rangle\langle j \otimes i|i' \otimes i'\rangle\langle j' \otimes j'| = \omega = \omega P, \tag{104}$$

and into the axioms (vve) and (eev),

$$\begin{aligned}(1\!\!1_d \otimes P)(P \otimes 1\!\!1_d)(1\!\!1_d \otimes \omega) = d(\omega \otimes 1\!\!1_d)(1\!\!1_d \otimes \omega) = (\omega \otimes 1\!\!1_d)(1\!\!1_d \otimes P)(P \otimes 1\!\!1_d),\\ (P \otimes 1\!\!1_d)(1\!\!1_d \otimes P)(\omega \otimes 1\!\!1_d) = d(1\!\!1_d \otimes \omega)(\omega \otimes 1\!\!1_d) = (1\!\!1_d \otimes \omega)(P \otimes 1\!\!1_d)(1\!\!1_d \otimes P),\end{aligned} \tag{105}$$

which are proved in a diagrammatic approach, see Figure 25 where $\lambda = d$.

Figure 25 clearly shows that the teleportation is a fundamental configuration for defining the Brauer algebra. It presents an equivalent realization of the teleportation via the swap gate P and Bell measurement, as would be amazing in both theoretical and experimental senses. It also suggests that the Brauer algebra underlies our previous proposal of the teleportation swapping $(P \otimes Id)(Id \otimes P)$ and virtual braid teleportation. As the braid representation is set up in terms of the Temeperley–Lieb algebra, the virtual braid representation is built with the so called virtual Temperley–Lieb algebra [22], i.e., the Brauer algebra.

3.10 Comments on multipartite entanglements

This section starts with our diagrammatic rules and then applies them to various kinds of examples in order to propose that the Temperley–Lieb algebra (or the

Brauer algebra) under local unitary transformations underlies the teleportation. In a general sense, Bell measurements and local unitary transformations are crucial points for the application of our diagrammatic rules. Here we study additional examples for comments on how to deal multipartite entangled states like the GHZ state or the state $|\chi\rangle$ with our rules.

The GHZ state $|GHZ\rangle$ is rewritten into the form of local unitary transformations on the Bell state,

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |00\rangle + |1\rangle \otimes |11\rangle) \\ &= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |\phi^+\rangle + \frac{1}{2}(|0\rangle - |1\rangle) \otimes |\phi^-\rangle \\ &= \frac{1}{2}(1\!\!1_8 + \sigma_3 \otimes 1\!\!1_2 \otimes \sigma_3)(|\alpha\rangle \otimes |\phi^+\rangle) \end{aligned} \tag{106}$$

where $|\alpha\rangle = |0\rangle + |1\rangle$ is for conveniences. Similarly, the four-particle state $|\chi\rangle$ [31], in the construction of the CNOT gate via the teleportation, has the form

$$\begin{aligned} |\chi\rangle = \frac{1}{\sqrt{2}}&(|00\rangle + |11\rangle)|00\rangle + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)|11\rangle \\ = \quad & \frac{1}{\sqrt{2}}|\phi^+\rangle(|\phi^+\rangle + |\phi^-\rangle) + \frac{1}{\sqrt{2}}|\psi^+\rangle(|\phi^+\rangle - |\phi^-\rangle) \\ = \quad & \frac{1}{\sqrt{2}}(1\!\!1_{16} + 1\!\!1_8 \otimes \sigma_3 + 1\!\!1_2 \otimes \sigma_1 \otimes 1\!\!1_4 - 1\!\!1_2 \otimes \sigma_1 \otimes 1\!\!1_2 \otimes \sigma_3)|\phi^+\rangle|\phi^+\rangle. \end{aligned} \tag{107}$$

As a remark, therefore, we expect our diagrammatic rules to be applied to topics like Bell inequalities, quantum cryptography and so on in which Bell measurements and local unitary transformations play the fundamental roles.

4 Comparisons with known approaches

We propose the Temperley–Lieb algebra under local unitary transformations to be a suitable mathematical framework for the teleportation. To support it, we collect various topics around the teleportation together, deal them with our diagrammatic rules in a systematic way and find all involved configurations are elements of the Temperley–Lieb category. To make our proposal more reasonable, we compare it with two known approaches to the quantum information flow: the teleportation topology [10, 20] and strongly compact closed category theory [32].

4.1 Teleportation topology

Teleportation topology [10, 20] explains the teleportation as a kind of topological amplitude satisfying the topological condition. There are one to one correspondences between quantum amplitude and topological amplitude. The state preparation (the Dirac ket) describes the creation of two particles from the vacuum and has a diagrammatic representation of the cup state $|Cup\rangle$, while the measurement process

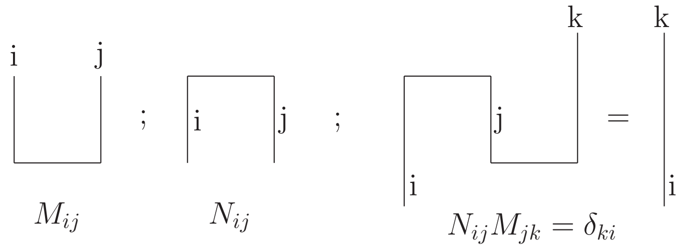


Figure 26: Teleportation topology: cup, cap and topological condition.

(the Dirac bra) denotes the annihilation of two particles and is related to the cap state $\langle Cap|$. The cup state and cap state are associated with the matrices M and N in the way,

$$|Cup\rangle = \sum_{i,j=0}^{d-1} M_{ij}|e_i \otimes e_j\rangle, \qquad \langle Cap| = \sum_{i,j=0}^{d-1} \langle e_i \otimes e_j|N_{ij} \tag{108}$$

which have to satisfy the topological condition, i.e., the concatenation of a cup and a cap is a straight line denoted by the identity matrix $N_{ij}M_{jk} = \delta_{ik}$, see Figure 26.

However, our diagrammatic teleportation does not have the interpretation of the teleportation topology. The concatenation of a cup and a cap is formulated by the transfer operator which is not identity required by the topological condition. Also, our cup and cap states are the normalized maximally entangled states given by

$$|Cup\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i \otimes e_i\rangle, \qquad \langle Cap| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle e_i \otimes e_i| \tag{109}$$

which assign a normalization factor $\frac{1}{d}$ to a straight line from the concatenation of a cup and a cap in the teleportation topology. More essentially, our approach underlies the Temperley–Lieb algebra and involves all kinds of combinations of a cup and a cap. For example, we represent a projector by a cup with a cap which is not considered by the teleportation topology.

Note that the configurations like cups and caps are well known for their applications to the Temperley–Lieb algebra, the braid representation, knot theory and statistics mechanics [8].

4.2 Quantum information flow in terms of maps

The teleportation is a kind of information protocol transporting a unkown quantum state from Charlie to Bob under the help of Alice. To describe it in a unified mathematical formalism, we have to integrate standard quantum mechanics with classical features since the outcomes of measurements are sent to Bob from Alice via classical channels and then Bob carries out a required unitary operation. The one approach has been proposed by Abramsky and Coecke in recent research. It applies the category theory to quantum protocols and describes the quantum information flow by strongly compact closed categories, see [44, 45] for abstract physical traces; see [32, 46] for quantum information flow; see [47, 48] for the categorical description of quantum protocols; see [49] for diagrammatic quantum mechanics.

To sketch the quantum information flow in the form of compositions of a series of maps which are central topics of the category theory, we study an example in details. Set five Hilbert spaces $\mathcal{H}_i$ and its dual $\mathcal{H}_i^*$, $i = 1, \cdots, 5$ and define eight bipartite projectors $P_\alpha = |\Phi_\alpha\rangle\langle\Phi_\alpha|$, $\alpha = 1, \cdots, 8$ in which the bipartite vector $|\Phi_\alpha\rangle$ is an element of $\mathcal{H}_i \otimes \mathcal{H}_{i+1}$, $i = 1, \cdots, 4$. In the left diagram of Figure 27, every box represents a bipartite projector P_α and the vector $|\phi_C\rangle \in \mathcal{H}_1$ that Charlie owns will be transported to Bob who is supposed to obtain the vector $|\phi_B\rangle \in \mathcal{H}_5$ through the quantum information flow. The projectors P_1 and P_2 pick up the incoming vector in $\mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$ and the projectors P_7 and P_8 determine the outgoing vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$. The right diagram in Figure 27 shows the quantum information flow from $|\phi_C\rangle$ to $|\phi_B\rangle$ in a clear way. It is drawn according to the permitted and forbidden rules [46]: the flow is forbidden to go through a box from the one side to the other side, and is forbidden to be reflected at the incoming point, and has to change its direction from an incoming flow to an outgoing flow as it passes through a box. Obviously, if these rules are not imposed there will be many possible pathes from $|\phi_C\rangle$ to $|\phi_B\rangle$.

Let set up the one to one correspondence between a bipartite vector and a map. There are a d_1-dimension Hilbert space $\mathcal{H}_{(1)}$ and a d_2-dimension Hilbert space $\mathcal{H}_{(2)}$. The bipartite vector $|\Phi\rangle$ has the form in terms of the product basis $|e_i^{(1)}\rangle \otimes |e_j^{(2)}\rangle$ of $\mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)}$,

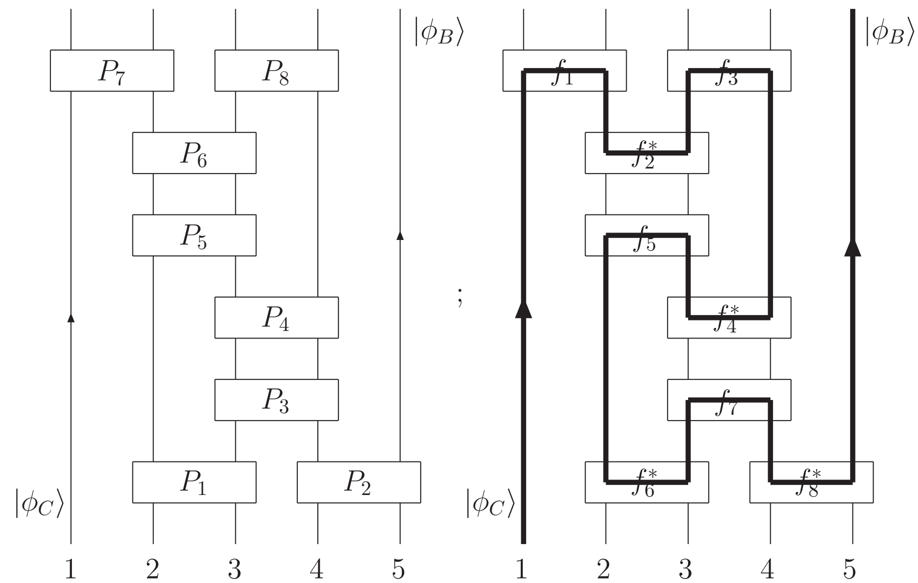
$$|\Phi\rangle = \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} m_{ij}|e_i^{(1)}\rangle \otimes |e_j^{(2)}\rangle, \qquad \langle\Phi| = \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} m_{ij}^*\langle e_i^{(1)}| \otimes \langle e_j^{(2)}| \tag{110}$$

where $\langle\Phi|$ denotes the dual vector of $|\Phi\rangle$ in the dual product space $\mathcal{H}_{(1)}^* \otimes \mathcal{H}_{(2)}^*$ with the basis $\langle e_i^{(1)}| \otimes \langle e_j^{(2)}|$. Also, once the product basis are fixed, the bipartite vectors $|\Phi\rangle$ or $\langle\Phi|$ are determined by a $d_1 \times d_2$ matrix $M_{d_1\times d_2} = (m_{ij})$. Defining two types of maps f and f^* in the way,

$$\begin{aligned} f : \mathcal{H}_1 \to \mathcal{H}_2^*, \qquad & f(\cdot) = \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} m_{ij}\langle e_i^{(1)}|\cdot\rangle\langle e_j^{(2)}|, \\ f^* : \mathcal{H}_1^* \to \mathcal{H}_2, \qquad & f^*(\cdot) = \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} m_{ij}|e_j^{(2)}\rangle\langle\cdot|e_i^{(1)}\rangle, \end{aligned} \tag{111}$$

we have the following bijective correspondences,

$$|\Phi\rangle \approx \langle\Phi| \approx M \approx f \approx f^* \tag{112}$$



$$|\phi_B\rangle = f_8^* \circ f_7 \circ f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1 |\phi_C\rangle$$

Figure 27: Quantum information flow in the categorical approach.

which suggests that we can label the bipartite project box in Figure 27 by the map f or f^* or matrix M.

Let work out the formalism of the quantum information flow in the categorical approach. Consider the projector $P_7 = |\Phi_7\rangle\langle\Phi_7|$ and introduce the map f_1 to represent the action of $\langle\Phi_7|$, the half of P_7,

$$f_1 : \mathcal{H}_1 \to \mathcal{H}_2^*, \qquad f_1(\phi_C) = \langle\Phi_7|\phi_C\rangle. \tag{113}$$

Similarly, the remaining seven boxes are labeled by the maps f_2^*, f_3, f_4^*, f_5, f_6^*, f_7 and f_8^*, respectively defined by

$$\begin{aligned}
&f_2^* : \mathcal{H}_2^* \to \mathcal{H}_3, \qquad f_2^* \circ f_1(\phi_C) = \langle\Phi_7|\phi_C \otimes \Phi_6\rangle, \\
&f_3 : \mathcal{H}_3 \to \mathcal{H}_4^*, \qquad f_3 \circ f_2^* \circ f_1(\phi_C) = \langle\Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6\rangle, \\
&f_4^* : \mathcal{H}_4^* \to \mathcal{H}_3, \qquad f_4^* \circ f_3 \circ f_2^* \circ f_1(\phi_C) = \langle\Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4\rangle, \\
&f_5 : \mathcal{H}_3 \to \mathcal{H}_2^*, \qquad f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1(\phi_C) = \langle\Phi_5 \otimes \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4\rangle, \\
&f_6^* : \mathcal{H}_2^* \to \mathcal{H}_3, \qquad f_7 : \mathcal{H}_3 \to \mathcal{H}_4^*, \qquad f_8^* : \mathcal{H}_4^* \to \mathcal{H}_5, \\
&f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1(\phi_C) = \langle\Phi_5 \otimes \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4 \otimes \Phi_1\rangle, \\
&f_7 \circ f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1(\phi_C) = \langle\Phi_3 \otimes \Phi_5 \otimes \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4 \otimes \Phi_1\rangle,
\end{aligned} \tag{114}$$

and so the quantum information flow is encoded in the the form,

$$
\begin{aligned}
& f_8^* \circ f_7 \circ f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1(\phi_C) \\
& = \langle \Phi_3 \otimes \Phi_5 \otimes \Phi_7 \otimes \Phi_8 | \phi_C \otimes \Phi_6 \otimes \Phi_4 \otimes \Phi_1 \otimes \Phi_2 \rangle,
\end{aligned} \tag{115}
$$

namely, it is given by the composition of a series of maps,

$$
|\phi_B\rangle = f_8^* \circ f_7 \circ f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1 |\phi_C\rangle \tag{116}
$$

where we identify the tensor product $|\Phi\rangle \otimes 1\!\!1_d \otimes \cdots \otimes 1\!\!1_d$ with $|\Phi\rangle$. Additionally, following the rules of the teleportation topology [10, 20] and assigning the matrices M, N to a cup and a cap respectively, we have the quantum information flow in the matrix teleportation,

$$
|\phi_B\rangle = M_8 \cdot N_7 \cdot M_6 \cdot N_5 \cdot M_4 \cdot N_3 \cdot M_2 \cdot N_1 |\phi_C\rangle. \tag{117}
$$

Now it is time to make differences clear in a straight way between our diagrammatic teleportation and quantum information flow in the categorical approach. These essential differences are conceptual: both physical and mathematical. On the mathematical side, we think the braid group and Temperley–Lieb algebra behind the teleportation instead of various maps in the category theory because we are looking for a real and stable bridge between knot theory and quantum information [22]. In our approach, the bipartite projector is regarded as an idempotent of the Temperley–Lieb algebra but in the categorical approach only its half, a bipartite vector, has been seriously exploited. On the physical side, we choose the tight teleportation scheme [21] as our definition for the teleportation, i.e., we stick to concepts like the Hilbert space, state, vector and local unitary transformation which are basic ingredients of standard quantum mechanics described by the von Neumann's axioms. But the categorical approach aims at setting up a high-level approach beyond the von Neummann's axioms to describe quantum information theory in a unified mathematical framework.

To explain these differences in details, we revisit the example in Figure 28 and redraw the diagram according to our diagrammatical rules. In Figure 29, every projector consists of a cup and a cap denoting the maximally entangled states $|\Omega\rangle$ and $\langle\Omega|$. The solid points $1, \cdots 8$ on the left branches of cups denote the local unitary transformations $U_1, \cdots, U_8$ and small circles on the left branches of caps denote their adjoint operators $U_1^\dagger, \cdots, U_8^\dagger$, respectively. Following our rules, the quantum information flow from $|\phi_C\rangle$ to $|\phi_B\rangle$ is determined by the transfer operator,

$$
|\phi_B\rangle = \frac{1}{d^6} tr(U_2^\dagger U_5) tr(U_4^\dagger U_7)(U_8^T U_7^\dagger U_6^T U_5^* U_4 U_3^\dagger U_2^T U_1^\dagger)|\phi_C\rangle \tag{118}
$$

where the normalization factor $\frac{1}{d^6}$ is contributed from six vanishing cups and six vanishing caps and two traces from two closed circles.

There are at least three remarkable things to be mentioned as Figure 28 is compared with Figure 27. They are related to three questions respectively: why do we denote the projector by itself instead of its half? How to deal with the acausality problem? which type of bijective correspondence between a bipartite vector and a map or matrix is chosen? In the categorical approach, only the half of a projector is considered in order to make use of the bijective correspondence between a bipartite vector and a map. But obviously observed from our diagrammatic description in Figure 28, our approach not only derives the quantum information flow from $|\phi_C\rangle$ to $|\phi_B\rangle$ in a clear way but also yields a normalization factor from the closed circles. The normalization factor contributed by vanishing cups and caps is crucial for the quantum formation flow. For examples, setting eight local unitary operators U_i to be identity leads to $|\phi_B\rangle = \frac{1}{d^4}|\phi_C\rangle$ and assuming U_2 and U_5 (or U_4 and U_7) orthogonal to each other causes a zero vector to be sent to Bob, $|\phi_B\rangle = 0$, no flow! Also, we want to explain the teleportation from the point of quantum measurement since a projector represents a process of measurement in standard quantum mechanics.

The acausality problem becomes explicit in the known approaches to the quantum information flow. But the quantum information flow is only one part of the entire diagram in our diagrammatical approach. In other words, it is not reasonable to argue such questions on the teleportation without considering the whole process from the global view. Hence we choose to represent the projector as the combination of a cup and a cap instead of a single cup or cap. Furthermore, we apply the bijective correspondence between a local unitary transformation and a bipartite vector, as is different from the choice preferred by the categorical approach. For example, we have

$$|\psi(U)\rangle = (U \otimes 1\!\!1_d)|\Omega\rangle, \qquad |\psi(U)\rangle \approx U \approx |\psi(U)\rangle\langle\psi(U)| \tag{119}$$

and so the Bell states (7) are represented by identity or the Pauli matrices

$$|\phi^+\rangle \approx 1\!\!1_2, \ \ |\phi^-\rangle \approx \sigma_3, \ \ |\psi^+\rangle \approx \sigma_1, \ \ |\psi^-\rangle \approx i\sigma_2. \tag{120}$$

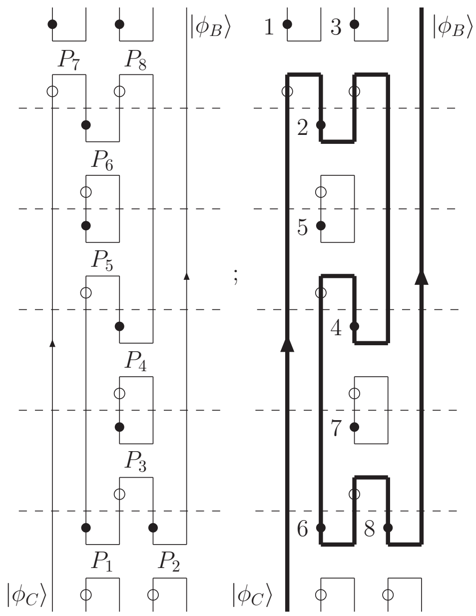
If we label a projector by a local unitary transformation, then we call the equation (118) the quantum information flow in terms of local unitary transformations.

Note that in the categorical approach the quantum information flow is created in view of additional permitted and forbidden rules [46] but in the Temperley–Lieb category it is derived in a natural way without imposed rules, as is clearly observed by comparing Figure 27 with Figure 28.

4.3 Teleportation in the strongly compact closed category

The quantum information flow in terms of compositions of maps naturally leads to its description in the category theory. Here we aim at showing one-to-one correspondences between the quantum information flow and strongly compact category theory in an intuitive way. We will not involve mathematical details about the category theory since they have been treated in various ways in a series of papers by Abramsky, Coecke and Duncan, see [32, 44, 45, 46, 47, 48, 49, 50, 51, 52].

To transport Charlie's unknown quantum state $|\psi\rangle_C$ to Bob, the teleportation has to complete all the operations: preparation of $|\psi\rangle_C$; creation of $|\Omega\rangle_{AB}$ in Alice



$$|\phi_B\rangle = \tfrac{1}{d^6} tr(U_2^\dagger U_5) tr(U_4^\dagger U_7)(U_8^T U_7^\dagger U_6^T U_5^* U_4 U_3^\dagger U_2^T U_1^\dagger)|\phi_C\rangle$$

Figure 28: Quantum information flow in the Temperley–Lieb category.

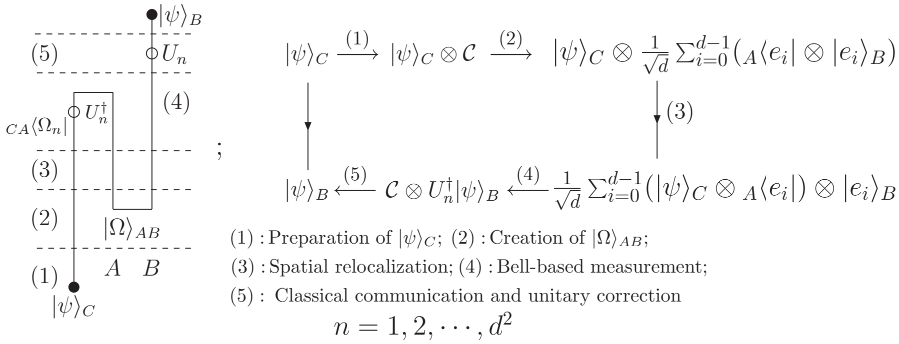


Figure 29: Quantum information flow and strongly compact closed category.

and Bob's systems; Bell-based measurement $_{CA}\langle\Omega_n|$ in Charlie and Alice's systems; classical communication between Alice and Bob; Bob's local unitary correction. These steps divide the quantum information flow into six pieces which are shown up in the left diagrammatic term of Figure 29 where the third piece represents the process bringing Alice and Charlies' particles together for the entangling measurement. In the category theory, every step or piece is denoted by a specific map which satisfies the axioms of the strongly compact closed category theory. The crucial point is to identify the bijective correspondence between the Bell state and a map from the dual Hilbert space $\mathcal{H}^*$ to the Hilbert space $\mathcal{H}$, i.e.,

$$\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i\rangle_A \otimes |e_i\rangle_B \approx \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} {}_A\langle e_i| \otimes |e_i\rangle_B, \qquad \mathcal{H}_A \otimes \mathcal{H}_B \approx \mathcal{H}_A^* \otimes \mathcal{H}_B \tag{121}$$

so that the strongly compact closed category theory has a physical realization in the form of the quantum information flow, see the right diagrammatic term of Figure 29 where the symbol $\mathcal{C}$ denotes the complex field and we admit

$$|\psi\rangle_C \approx |\psi\rangle_C \otimes \mathcal{C}, \qquad |\psi\rangle_B \approx \mathcal{C} \otimes |\psi\rangle_B \tag{122}$$

and create a bipartite state from a complex number $\mathcal{C}$ and also annihilate it into $\mathcal{C}$.

In the paper, we apply the Temperley–Lieb category to various topics around the teleportation and obtain nice pictures and useful formulas. It is obviously different from the strongly compact closed category. The essential conceptual differences have been listed clearly in the above subsection by comparing Figure 27 with Figure 28. Additionally, the appearance of $\mathcal{H}^* \otimes \mathcal{H}$ is not require by the teleportation or even the quantum information flow but is imposed by the axioms of the strongly compact closed category theory, see Figure 29. The quantum information flow (its description of the strongly compact closed category theory) is only a part of the Temperley-Lieb diagrammatic representation for the teleportation, as is observed by reading Figure 27, Figure 28 and Figure 29 together.

As a remark, the category theory [47, 50, 53, 54, 55] is expected to play the roles in quantum information theory and will be involved in our forthcoming research.

5 Concluding remarks

In the paper, we apply the braid group, the Temperley–Lieb algebra and the Brauer algebra to the mathematical description of the teleportation. Besides the proposals for the virtual braid teleportation and braid teleportation, We design diagrammatic rules for describing algebraic expressions in a diagrammatic way in order to show the Temperley–Lieb algebra under local unitary transformations to underly the teleportation. Our diagrammatic approach has been applied to various topics around the teleportation: the transfer operator and acausality problem, teleportation and quantum measurement, all tight teleportation and dense coding schemes, the Temperley–Lieb algebra and maximally entangled states; entanglement swapping; teleportation and topological quantum computing; teleportation and the Brauer algebra; multipartite entanglements. A crucial point is to recognize the teleportation configuration to be a fundamental element of the Brauer algebra and suggest an equivalent realization of the teleportation in terms of the swap gate and Bell measurements. Also, we compare our diagrammatic approach with the teleportation topology and categorical descriptions for the quantum information flow so that those essential differences among them are made clear.

As concluding remarks, in this paper we do not intend to bring new physics or new mathematics to the field of quantum information. We strongly believe in the existence of beautiful mathematical structures behind entanglement and teleportation such as the braid group and Temperley–Lieb algebra which are well known to the community of knot theory for a long time. They not only simplify complicated algebraic calculation in an intuitive diagrammatic way but also oblige us to accept that quantum phenomena like entanglement and teleportation should be no mysterious at all for the mankind living in the classical world once we are equipped with suitable mathematical tools. As Abramsky and Coecke suggested [56], we are also looking for a powerful mathematical framework to describe various topics of quantum information in a natural and unified way which has to be beyond the von Neumman formalism for standard quantum mechanics.

Acknowledgements

I am indebted to X.Y. Li for his constant encouragements and supports, and grateful to L.H. Kauffman and R.F. Werner for their helpful comments on the teleportation topology and tight teleportation schemes. I thank M.L. Ge for helpful discussions on the rules for oblique lines, and thank L. Vaidman for helpful email correspondences on the crossed measurement. I especially thank R.F. Werner for his email correspondence from Dresden on the project of setting up a bridge between knot theory and quantum information. This work is in part supported by NSFC–10447134 and

SRF for ROCS, SEM.

References

[1] R.F. Werner, *Quantum States with Einstein-Podolsky-Rosen Correlations Admitting a Hidden-Variable Model*, Phys. Rev. A **4**0 (1989) 4277.

[2] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 1999).

[3] P.W. Shor, *Algorithms for Quantum Computation: Discrete Logrithms and Factoring*. In S. Goldwasser, editor, *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, pp. 124-134, Los Alamitos, CA, 1994. IEEE Computer Society Press.

[4] L.K. Grover, *Quantum Mechanics Helps in Searching for a Needle in a Haystack*, Phys. Rev. Lett. **78** (1997) 325-328.

[5] C.H. Bennett, G. Brassard, *Quantum Cryptography: Public Key Distribution and Coin Tossing*, Int. Conf. Computers, Systems & Signal Processing, Bangalore, India, December 10-12, 1984, pp. 175-179.

[6] A.K. Ekert, *Quantum Cryptography Based on Bell's Theorem*, Phys. Rev. Lett. **67** (1991) 661-663.

[7] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. **70** (1993) 1895-1899.

[8] L.H. Kauffman, *Knots and Physics* (World Scientific Publishers, 2002).

[9] J.L. Brylinski and R. Brylinski, *Universal quantum gates,* in *Mathematics of Quantum Computation*, Chapman & Hall/CRC Press, Boca Raton, Florida, 2002 (edited by R. Brylinski and G. Chen).

[10] L.H. Kauffman and S.J. Lomonaco Jr., *Braiding Operators are Universal Quantum Gates*, New J. Phys. **6** (2004) 134. Arxiv: quant-ph/0401090.

[11] Y. Zhang, L.H. Kauffman and M.L. Ge, *Universal Quantum Gate, Yang–Baxterization and Hamiltonian*. Int. J. Quant. Inform., Vol. 3, **4** (2005) 669-678. Arxiv: quant-ph/0412095.

[12] Y. Zhang, L.H. Kauffman and M.L. Ge, *Yang–Baxterizations, Universal Quantum Gates and Hamiltonians*. Quant. Inf. Proc. **4** (2005) 159-197. Arxiv: quant-ph/0502015.

[13] C.N. Yang, *Some Exact Results for the Many Body Problems in One Dimension with Repulsive Delta Function Interaction*, Phys. Rev. Lett. **19** (1967) 1312-1314.

[14] R.J. Baxter, *Partition Function of the Eight-Vertex Lattice Model*, Annals Phys. **70** (1972) 193-228.

[15] L.H. Kauffman, *Quantum Computation and the Jones Polynomial*, in *Quantum Computation and Information*, S. Lomonaco, Jr. (ed.), AMS CONM/305, 2002, pp. 101-137. Arxiv: math. QA/0105255.

[16] L.H. Kauffman, *Quantum Topology and Quantum Computing*, in *Quantum Computation*, S. Lomonaco (ed.), AMS PSAPM/58, 2002, pp. 273–303.

[17] L. H. Kauffman and S. J. Lomonaco Jr., *Quantum Knots*, in E. Donkor, A.R. Pirich and H.E. Brandt (eds.), Quantum Information and Computation II, Spie Proceedings, (12 -14 April, Orlando, FL, 2004), Vol. 5436, pp. 268-284. Arxiv: quant-ph/0403228.

[18] L.H. Kauffman and S.J. Lomonaco Jr., *Quantum Entanglement and Topological Entanglement*, New J. Phys. **4** (2002) 73.1–73.18.

[19] L.H. Kauffman and S.J. Lomonaco Jr., *Entanglement Criteria–Quantum and Topological*, in E. Donkor, A.R. Pirich and H.E. Brandt (eds.), Quantum Information and Computation – Spie Proceedings, (21-22 April, Orlando, FL, 2003), Vol. 5105, pp. 51-58. Arxiv: quan-ph/0304091.

[20] L.H. Kauffman, *Teleportation Topology*. Opt. Spectrosc. **9** (2005) 227-232. Arxiv: quan-ph/0407224.

[21] R. F. Werner, *All Teleportation and Dense Coding Schemes*, J. Phys. A **35** (2001) 7081–7094. Arxiv: quant-ph/0003070.

[22] Y. Zhang, L.H. Kauffman and R.F. Werner, *Permutation and its Partial Transpose* (in preparation).

[23] H.N.V. Temperley and E.H. Lieb, *Relations between the 'Percolation' and 'Colouring' Problem and Other Graph-Theoretical Problems Associated with Regular Planar Lattices: Some Exact Results for the 'Percolation' Problem*, Proc. Roy. Soc. A **322** (1971) 251-280.

[24] L.H. Kauffman, *Virtual Knot Theory*, European J. Comb. **20** (1999) 663-690.

[25] R. Brauer, *On Algebras Which are Connected With the Semisimple Continuous Groups*, Ann. of Math. **38** (1937) 857-872.

[26] Y. Aharonov, D.Z. Albert, and L. Vaidman, *Measurement Process in Relativistic Quantum Theory*, Phys. Rev. **D** 34 (1986) 1805-1813.

[27] L. Vaidman, *Teleportation of Quantum States*, Phys. Rev. **A** 49 (1994) 1473-1475.

[28] L. Vaidman, *Instantaneous Measurement of Nonlocal Variables*, Phys. Rev. Lett. 90 (2003) 010402

[29] J. Preskill, *Quantum Information and Computation*, Lecture Notes for Ph219/CS219, Chapter 4, pp. 26-35.

[30] M. Żukowski, A. Zeilinger, M.A. Horne and A.K. Ekert, *'Event-Ready-Detectors' Bell Experiment via Entanglement Swapping*. Phys. Rev. Let. **71** (1993) 4287–4290.

[31] D. Gottesman and I.L. Chuang, *Quantum Teleportation is a Universal Computational Primitive*. Nature **402** (1999) 390–393. Arxiv:quant-ph/9908010.

[32] B. Coecke, *The Logic of Entanglement. An Invitation*. Oxford University Computing Laboratory Research Report nr. PRG-RR-03-12. An 8 page short version is at Arxiv:quant-ph/0402014. The full 160 page version is at `web.comlab.ox.ac.uk/oucl/ publications/tr/rr-03-12.html`.

[33] H.A. Dye, *Unitary Solutions to the Yang–Baxter Equation in Dimension Four*, Quant. Inf. Proc. **2** (2003) 117-150. Arxiv: quant-ph/0211050.

[34] Y. Zhang, N.H. Jing and M.L. Ge, *New Quantum Algebras via RRT Relation on Eight–Vertex Models*, (in preparation).

[35] J. Franko, E.C. Rowell and Z.H. Wang, *Extraspecial 2-Groups and Images of Braid Group Representations*. Arxiv: math.RT/0503435.

[36] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, 1990).

[37] N. Erez, *Teleportation from a Projection Operator Point of View*. Arxiv: quant-ph/0510130.

[38] L.H. Kauffman and S.L. Lin, *Temperley–Lieb Recoupling Theory and Invariants of Three-Manifold*, Ann. of Math. Stud. **114** (Princeton Univ. Press, 1994).

[39] P.W. Shor, *Fault-Tolerant Quantum Computation*. In Proceedings, 35th Annual Symposium on Fundamentals of Computer Science (IEEE Press, Los Alamitos, 1996) 56-65. Arxiv: quant-ph/9605011.

[40] J. Preskill, *Fault-Tolerant Quantum Computation*. Arxiv: quant-ph/9712048.

[41] D. Aharonov, V. Jones and Z. Landau, *A Polynomial Quantum Algorithm for Approximating the Jones Polynomial*. Arxiv: quant-ph/0511096.

[42] A. Yu. Kitaev, *Fault-Tolerant Quantum Computation by Anyons*, Annals Phys. **303** (2003) 2-30. Arxiv: quant-ph/9707021.

[43] M.H. Freedman, M.J. Larsen and Z. Wang, *The Two-Eigenvalue Problem and Density of Jones Representation of Braid Groups*, Comm. Math. Phys. **228** (2002) 177-199.

[44] S. Abramsky, and B. Coecke, *Physical Traces: Quantum vs. Classical Information Processing*. Electronic notes on Theoretical Computer Science **69** (special issue: Proceedings of Category Theory in Computer Science 2002). Arxiv:cs/0207057.

[45] S. Abramsky, and B. Coecke, *Abstract Physical Traces*. Theory and Applications of Categories **14** (2003) 111–124.

[46] B. Coecke, *Quantum Information-Flow, Concretely, and Axiomatically.* In: Proceedings of Quantum Informatics 2004, pp. 15–29, Y. I. Ozhigov, Ed., Proceedings of Spie, Vol. 5833. Arxiv:quant-ph/0506132.

[47] S. Abramsky, and B. Coecke, *A Categorical Semantics of Quantum Protocols*. In: Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LiCS'04), IEEE Computer Science Press. Arxiv:quant-ph/0402130.

[48] B. Coecke, *De-linearizing Linearity: Projective Quantum Axiomatics from Strong Compact Closure*. Electronic Notes in Theoretical Computer Science (special issue: Proceedings of the 3rd International Workshop on Quantum Programming Languages). Arxiv:quant-ph/0506134.

[49] B. Coecke, *Kindergarten Quantum Mechanic–leture notes*. In: *Quantum Theory: Reconstructions of the Foundations III*, pp. 81-98, A. Khrennikov, American Institute of Physics Press. Arxiv: quant-ph/0510032.

[50] S. Abramsky and R. Duncan, *A Categorical Quantum Logic*. Arxiv: quant-ph/0512114.

[51] B. Coecke, *Introducing Categories to the Practicing Physicist* (to appear).

[52] B. Coecke and D. Pavlovic, *Quantum Measurements Without Sums* (to appear).

[53] J. Baez, *Quantum Quandaries: a Category-Theoretic Perspective.* In: S. French et al. (Eds.) *Structural Foundations of Quantum Gravity*, Oxford University Press. Arxiv:quant-ph/0404040.

[54] P. Selinger, *Dagger Compact Closed Categories and Completely Positive Maps.* Electronic Notes in Theoretical Computer Science (special issue: Proceedings of the 3rd International Workshop on Quantum Programming Languages).

[55] L.H. Kauffman and S.J. Lomonaco Jr., *q-Deformed Spin Networks, Knot Polynomials and Anyonic Topological Quantum Computation* (in preparation).

[56] S. Abramsky, *High-Level Methods for Quantum Computation and Quantum Information*. Pleanary talk at 19th Annual IEEE Symposium on Logic in Computer Science (LiCS'04). Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, IEEE Computer Science Press.