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Higher Structures in Topological Quantum Field Theory

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Abstract

This thesis is comprised of 7 papers where we investigate aspects of higher structures in the study of Topological Quantum Field Theory (TQFT) in the functorial formalism. In particular, we study boundary conditions and defects in extended TQFTs, and the constraints they induce on the algebraic structures involved. We use techniques from homotopy theory and ∞-categories to understand fully extended TQFTs in the presence of boundary conditions, and their relation to anomalous field theories. We also develeop a systematic study of homotopy actions of topological groups in the framework of bicategories, which we apply to the classification of 2d framed and oriented TQFTs, and which can be regarded as an algebraic counterpart of techniques arising in homotopy theory.

Contents

"Beware of the man who works hard to learn something, learns it, and finds himself no wiser than before. He is full of murderous resentment of people who are ignorant without having come by their ignorance the hard way."

K. Vonnegut, *Cat's Cradle*

CHAPTER 1

Preface

1. Introduction

1.1. Topological Quantum Field Theory. Topological Field Theory (TFT) has been a very active area of interaction between Mathematics and Physics for the past 30 years. The study of Topological Field Theories has led to new insight in the mathematical foundations of Quantum Field Theory. At the same time, ideas originated from TFT have provided deep insight into cobordism categories and the structure of manifolds, emphasizing the relevance and importance of higher mathematical structures.

In the seminal paper [**70**], Witten showed how techniques from field theory could be used to produce manifold invariants. More precisely, he showed that the partition function for the $SU(2)$-Chern-Simons action computed as a *path integral* over the space of connections of a principal $SU(2)$-bundle over a 3-dimensional framed manifold M produces a diffeomorphism invariant. Moreover, he argued that by computing the expectation value for an observable given by considering the trace of the holonomy[1] of a connection along a link L in M one obtains the Jones' polynomial associated to L, evaluated at a root of unity.

The formal properties of the path integral were used by Atiyah [**1**] to give an axiomatic framework for Topological Quantum Field Theories (TQFT), following the work of Segal [**65**] on Conformal Field Theory. Atiyah's definition is best encoded by using the language of categories: namely, a n-dimensional TQFT Z is a symmetric monoidal functor from the category of oriented cobordisms $\mathrm{Cob}^{\mathrm{or}}(n)$ to the category of vector spaces Vect_k, where k is a field, usually $\mathbb{C}$. Briefly, $\mathrm{Cob}^{\mathrm{or}}(n)$ has closed oriented $(n-1)$-manifolds as objects, and n-dimensional cobordisms (modulo diffeomorphisms) as morphisms. Then Z assigns a vector space $Z(\Sigma)$ to a $(n-1)$-dimensional manifold Σ, and a linear map from $Z(\Sigma)$ to $Z(\Sigma')$ to any cobordism between Σ and Σ'. Crucially, the axioms force the vector space $Z(\Sigma)$ to be finite dimensional for any Σ. Since any n-dimensional closed manifold M can be regarded as a cobordism from the empty set $\emptyset$ to itself, as a consequence of the monoidality of Z one obtains an element $Z(M) \in k$, which is regarded as the *partition function* of the TQFT evaluated at M, and it is a diffeomorphism invariant of M.

The functorial approach to Topological Quantum Field Theory is powerful enough to allow for a complete classification in specific cases. For instance, for $n = 1$ a

[1]The holonomy is considered in the fundamental representation.

TQFT is simply given by a *finite dimensional* vector space, while for $n = 2$ it is completely determined by a commutative Frobenius algebra. Both these results rely on the fact that 1 and 2-dimensional compact manifolds with boundaries are somehow easy to describe in terms of surgery, or "cut & paste", which allows for a complete presention of $\mathrm{Cob}^{\mathrm{or}}(n), n \leq 2$, in terms of generators and relations. In higher dimensions, though, the situation becomes dramatically more complicated: indeed, at the moment of writing there exists no classification result for n-dimensional TQFTs for $n > 3$. We stress at this point that the functorial approach to TQFT is genuinely *nonperturbative*, in the sense that it does not rely on any quantization framework, in particular on any perturbative approach to Quantum Field Theory.

In [**23**], Freed proposed that an n-dimensional Topological Quantum Field Theory arising from local classical lagrangians via quantization should assign data also to manifolds of higher codimension. Indeed, locality allows to decompose a n-dimensional manifold M into higher codimensional (and hopefully simpler) pieces, and to compute the manifold invariant via the algebraic data assigned to these pieces. In particular, in [**23**] the case of the 3-dimensional Topological Quantum Field Theory given by Dijkgraaf-Witten theory [**18**] was investigated in detail, and it was shown that it assigns a category to the oriented circle S^1, in such a way that the space of states and partition functions are recovered consistently. Similarly, the 3d TQFT constructed by Reshetikhin and Turaev in [**60**] from a modular tensor category $\mathcal{C}$, and by Turaev and Viro [**68**] via a state sum construction, can both be recast in such a framework.

To axiomatically formulate the notion of Topological Quantum Field Theories which attach data to manifolds of codimension higher than 1, and which are called *extended TQFTs*, the use of *higher category theory* is required. In [**3**], Baez and Dolan argued that n-dimensional Topological Quantum Field Theories which are *fully extended*, i.e. which attach data to higher codimensional manifolds up to the point, are described as representation of a framed cobordism n-category, and, more importantly, were conjectured to be *completely* classifiable: this conjecture is known as *the Cobordism hypothesis*. More precisely, the Baez-Dolan conjecture states that the framed extended cobordism n-category Cob_n^{fr} is the free weak n-category generated by a single *fully dualizable* object. Such a conjecture has eluded a proof for many years, one of the reasons being the lack of a robust definition of a weak n-category for $n > 3$, and a systematic way to perform algebra in such a setting. For 2-dimensional oriented TQFTs[2], a proof of the Cobordism Hypothesis was obtained in [**62**], where it was shown that such TQFTs correspond to separable symmetric Frobenius algebras. The techniques used in [**62**] rely heavily on Morse and Cerf theory for two-dimensional manifolds with boundaries and corners, hence they are quite difficult to directly extend to higher dimensions.

By not only extending down, but also *up*, in [**54**] Lurie was able to give an extensive account of a proof of the Cobordism Hypothesis formulated in the language of (∞, n)-categories, a generalization of the notion of category. Though apparently more

[2]The *coefficent* target is given by the Morita bicategory of associative unital algebras

complicated, the passage to ∞-categories allows for a proof of the Baez-Dolan conjecture via an induction procedure. More precisely, Lurie introduces an (∞, n)-category of framed bordisms Bord_n^{fr}, whose definition is sketched[3] in [**54**], and shows that functors (in a suitable sense) between Bord_n^{fr} and any symmetric monoidal (∞, n)-category $\mathcal{C}$ correspond to fully dualizable objects in $\mathcal{C}$. He also argues that TQFTs which are fully extended and defined on manifolds with additional tangential structures, i.e. orientation, spin, etc., can be recovered by considering homotopy fixed points of the relevant group action on a topological space associated to the subcategory of fully dualizable objects $\mathcal{C}^{fd}$.

Topological Quantum Field Theories arising from classical lagrangians often admit extended observables. For instance, Chern-Simons theory on a manifold M is in general considered in the presence of a *Wilson loop* L, namely a 1-dimensional submanifold $L \subset M$ decorated by a character ρ of the structure group G. The expectation value of the observable (L, ρ) could be equivalently regarded as the partition function of a TQFT defined on a manifold with a *defect*. To make this precise, for each n one can construct a cobordism category[4] $\mathrm{Cob}_D^{def}(n)$ where objects and morphisms are stratified manifold of dimension $n-1$ and n, respectively, which are decorated by suitable algebraic data D. A TQFT with defects is then a symmetric monoidal functor from $\mathrm{Cob}_D^{def}(n)$ to Vect_k. Such structures have been extensively investigated [**16, 48, 50, 56**]: in particular, in [**44**] Kapustin proposed that the algebraic data needed to define a TQFT with defects can be organised into higher categories, with the categorical layers roughly corresponding to the codimension of the defect. A particular type of defect is provided by a *boundary condition*[5]. It allows the partition function of a TQFT to be well defined on a manifold with *constrained* boundary, which is defined as a (part of the) boundary where gluing is not allowed. In [**56**], 2d oriented TQFTs with boundary conditions, also known as *open/closed theories*, are investigated, and a classification result is provided. In particular, they show that in this case boundary conditions form a (finite) Calabi-Yau category.

To better understand the rôle of the material contained in the present thesis, we find useful to illustrate some aspects concerning the state of the art in the higher categorical approach to Topological Quantum Field Theory:

- Though the formulation of the Cobordism Hypothesis via (∞, n)-categories in [**54**] has been really fruitful in providing classification results, often concrete cases are presented in the language of low dimensional higher category and are intrinsically algebraic, which makes them more amenable to computations.
- Topological defects and boundary conditions have recently become of interest and appeared in the framework of extended TQFTs, though not many concrete examples are provided. In particular, a description of the extended 3d bicategory of cobordisms with singularities is not yet available.

[3]A rigorous construction of Bord_n^{fr} is presented in [**10**].

[4]Notice that the cobordism category depends on the decoration data.

[5]We are only treating *topological* boundary conditions and defects.

- The classification of fully extended TQFTs defined on manifolds with additional tangential structures require a notion of homotopy action of topological groups on higher categories, which is still not formalised in general.

The papers presented in this thesis give substantial contributions in particular to the topics mentioned above.

1.2. Short summaries of the papers. Here we give a brief summary for each paper we have included. We will give more details in Section 4.

- In [**31**] we study topological boundary conditions and surface defects for extended 3d TQFTs of Reshethikin-Turaev type based on a modular tensor category $\mathcal{C}$. We argue that there are obstructions for the existence of admissible boundary conditions and defects which are detected by the Witt group of fusion categories. Moreover, we realize boundary conditions and defects in terms of module categories over fusion categories, and apply this description to the case of Abelian Chern-Simons theory, recovering known results in the Physics literature.
- In [**32**] we study boundary conditions and surface defects for extended Dijkgraaf-Witten theory from a geometric point of view. More precisely, we consider the linearisation of categories of relative bundles on manifolds with boundaries, and show that the category of generalized Wilson lines agrees with the one which can be obtained via the formalism developed in [**31**].
- In [**29**] we study symmetries of extended 3d TQFTs in terms of invertible surface defects. In particular, we identify symmetries as elements of Brauer-Picard groups. We apply this analysis to the case of extended Abelian Dijkgraaf-Witten theory with a nontrivial cocyle, and give a complete classification of its symmetries in terms of gauge theoretic quantities.
- In [**21**] we introduce the concept of *TQFT of moduli level m*, and give a formulation of anomalous extended TQFTs in these terms. In particular, we show that a fully extended anomalous TQFT produces a boundary condition for the anomaly field theory. To this aim, we develop a theory of n-characters for ∞-groups and their homotopy fixed points.
- In [**20**] we study central extensions of the geometric realisation of the automorphism stack of a manifold (or stack) equipped with a tangential structure, as appeared in [**54**]. We show then how to obtain from this general framework the $\mathbb{Z}$-extensions of surfaces introduced by Segal in [**65**], which provide the required anomaly cancellation for Chern-Simons theory.
- In [**38**] we study actions of topological groups on bicategories, and give a precise definition of homotopy fixed points of such an action. We then show that the homotopy fixed points of the trivial action of $SO(2)$ on the bicategory of algebras, bimodules and intertwiners are in bijection with Frobenius algebras. We use this result to then show that fully extended oriented 2d TQFTs taking value in 2-Vector spaces are classified by finitely semisimple Calabi-Yau categories.

- In [**39**] we show that the Serre automorphism of a symmetric monoidal bicategory $\mathcal{C}$ introduced in [**54**] has a geometric origin, as expected. More precisely, it arises from the $SO(2)$-action obtained by "rotating the framing" in the framed cobordism bicategory. We study the homotopy fixed points of this action, and discuss its relation to 2d oriented fully extend TQFTs and to invertible field theories.

The results above group around the following topics:

(i) Boundary conditions and defects in extended TQFTs [**21**, **29**, **31**, **32**].
(ii) Fully extended TQFTs and homotopy theory [**20**, **21**].
(iii) Group actions on higher categories [**21**, **38**, **39**].

1.3. Outlook. The results of the papers presented in this thesis, in particular [**21**,**29**,**31**,**32**], constitute the beginning of a systematic study of extended TQFTs with boundary conditions and stratifications, and their symmetries. The observations contained in these works will hopefully lead to a full construction of extended cobordism categories with stratifications. The results on the study of symmetries in [**29**,**31**] can be applied to orbifolded theories. This allows to understand the topological conditions one must impose on boundaries and defects to be able to "gauge" global symmetries of extended TQFTs.
The work contained in [**21**] shows a concrete consequence of the Cobordism Hypothesis with singularities, which we hope will be of relevance in clarifying nonperturbative aspects of Chern-Simons theory as an anomalous theory.
The results in [**38**, **39**] provide a bridge between homotopy theory and algebra. We show that notions like homotopy coherent actions of topological groups and homotopy fixed points provide interesting algebraic results. This convinces us more and more that the interaction between homotopy theory and algebra, unified in the higher categorical framework, is a very fruitful one, and worth pursuing.

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2. Papers included

Here we list the abstracts of the papers which comprise the thesis.

(1) [**31**] *Bicategories for Boundary Conditions and for Surface Defects in 3-d TFT*, with J. Fuchs and C. Schweigert.

We analyze topological boundary conditions and topological surface defects in 3-dimensional topological field theories of Reshetikhin-Turaev type based on arbitrary modular tensor categories. Boundary conditions are described by central functors that lift to trivializations in the Witt group of modular tensor categories. The bicategory of boundary conditions can be described through the bicategory of module categories over any such trivialization. A similar description is obtained for topological surface defects. Using string diagrams for bicategories we also establish a precise relation between special symmetric Frobenius algebras and Wilson lines involving special defects. We compare our results with previous work of Kapustin-Saulina and of Kitaev-Kong on boundary conditions and surface defects in abelian Chern-Simons theories and in Turaev-Viro type TFTs, respectively.

(2) [**32**] *A Geometric Approach to Boundaries and Surface Defects in Dijkgraaf-Witten Theories*, with J. Fuchs and C. Schweigert.
Dijkgraaf-Witten theories are extended three-dimensional topological field theories of Turaev-Viro type. They can be constructed geometrically from categories of bundles via linearization. Boundaries and surface defects or interfaces in quantum field theories are of interest in various applications and provide structural insight. We perform a geometric study of boundary conditions and surface defects in Dijkgraaf-Witten theories. A crucial tool is the linearization of categories of relative bundles. We present the categories of generalized Wilson lines produced by such a linearization procedure. We establish that they agree with the Wilson line categories that are predicted by the general formalism for boundary conditions and surface defects in three-dimensional topological field theories that has been developed in Fuchs et al. (Commun Math Phys 321:543575, 2013)

(3) [**29**] *On the Brauer Groups of Symmetries of Abelian Dijkgraaf-Witten theories.* with J. Fuchs, J. Priel and C. Schweigert.
Symmetries of three-dimensional topological field theories are naturally defined in terms of invertible topological surface defects. Symmetry groups are thus Brauer-Picard groups. We present a gauge theoretic realization of all symmetries of abelian Dijkgraaf-Witten theories. The symmetry group for a Dijkgraaf-Witten theory with gauge group a finite abelian group A, and with vanishing 3-cocycle, is generated by group automorphisms of A, by automorphisms of the trivial Chern-Simons 2-gerbe on the stack of A-bundles, and by partial e-m dualities. We show that transmission functors naturally extracted from extended topological field theories with surface defects give a physical realization of the bijection between invertible bimodule categories of a fusion category and braided auto-equivalences of its Drinfeld center. The latter provides the labels for bulk Wilson lines; it follows that a symmetry is completely characterized by its action on bulk Wilson lines.

(4) [**21**] *Boundary Conditions for Topological Quantum Field Theories, Anomalies and Projective Modular Functors*, with D. Fiorenza.
We study boundary conditions for extended Topological Quantum Field Theories (TQFTs) and their relation to topological anomalies. We introduce the notion of TQFTs with moduli level m, and describe extended anomalous theories as natural transformations of invertible field theories of this type. We show how in such a framework anomalous theories give rise naturally to homotopy fixed points for n-characters on ∞-groups. By using dimensional reduction on manifolds with boundaries, we show how boundary conditions for $n+1$-dimensional TQFTs produce n-dimensional anomalous field theories. Finally, we analyse the case of fully extended TQFTs, and show that any fully extended anomalous theory produces a suitable boundary condition for the anomaly field theory.

(5) [**20**] *Central Extensions of Mapping Class Groups from Characteristic Classes*, with D. Fiorenza and U. Schreiber.
We characterize, for every higher smooth stack equipped with "tangential structure", the induced higher group extension of the geometric realization of its higher automorphism stack. We show that when restricted to smooth manifolds equipped with higher degree topological structures, this produces higher extensions of homotopy types of diffeomorphism groups. Passing to the groups of connected components, we obtain abelian extensions of mapping class groups and we derive sufficient conditions for these being central. We show as a special case that this provides an elegant re-construction of Segal's approach to $\mathbb{Z}$-extensions of mapping class groups of surfaces that provides the anomaly cancellation of the modular functor in Chern-Simons theory. Our construction generalizes Segal's approach to higher central extensions of mapping class groups of higher dimensional manifolds with higher tangential structures, expected to provide the analogous anomaly cancellation for higher dimensional TQFTs.

(6) [**38**] *Frobenius Algebras and Homotopy Fixed Points of Group Actions on Bicategories*, with J. Hesse and C. Schweigert.
We explicitly show that symmetric Frobenius structures on a finite dimensional, semi-simple algebra stand in bijection to homotopy fixed points of the trivial $SO(2)$-action on the bicategory of finite dimensional, semi-simple algebras, bimodules and intertwiners. The results are motivated by the 2-dimensional Cobordism Hypothesis for oriented manifolds, and can hence be interpreted in the realm of Topological Quantum Field Theory.

(7) [**39**] *The Serre Automorphism via Homotopy Actions and the Cobordism Hypothesis for Oriented Manifolds*, with J. Hesse.
We explicitely construct an $SO(2)$-action on a skeletal version of the 2-dimensional framed bordism bicategory. By the 2-dimensional Cobordism Hypothesis for framed manifolds, we obtain an $SO(2)$-action on the core of

fully-dualizable objects of the target bicategory. This action is shown to coincide with the one given by the Serre automorphism. We give an explicit description of the bicategory of homotopy fixed points of this action, and discuss its relation to the classification of oriented 2-d Topological Quantum Field Theories.

3. Background

3.1. Tensor categories and their module categories. Many aspects of Topological Quantum Field Theory can be described in a compact and elegant way by using the language of higher algebra. In the following we will introduce some basic notions which are used throughout the works presented in the thesis. We will mainly follow [**19**] in the exposition.

3.1.1. *Monoidal categories and their functors.* A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ consisting of:

- A category $\mathcal{C}$,
- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product*,
- A natural isomorphism $a : (-\otimes-)\otimes- \xrightarrow{\simeq} -\otimes(-\otimes-)$, called the *associator*,
- An object $\mathbf{1}$ of $\mathcal{C}$, called the *unit object*,
- An isomorphism $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\simeq} \mathbf{1}$,

subject to the following axioms:

a) *Pentagon axiom*: The following diagram

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y}\otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

is commutative for all X, Y, W, Z in $\mathcal{C}$;

b) *Unit axiom*: The functors

$$\begin{array}{rcl} X & \to & X \otimes \mathbf{1} \\ X & \to & \mathbf{1} \otimes X \end{array}$$

are autoequivalences of $\mathcal{C}$.

Remark 3.1. Some authors refer to the pair $(\mathbf{1}, \iota)$ as the unit object of $\mathcal{C}$.

In general, we will refer to $\mathcal{C}$ instead of the quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ in any case where the rest of the structure is clear. Moreover, a monoidal category $\mathcal{C}$ is called *strict* if the associator a and the morphism ι are identity isomorphisms.

Example 3.2. The category Sets of sets is a monoidal category, where the tensor product is given by the cartesian product, the unit object is a one element set, and the associator a and the morphism ι are the obvious one.

Example 3.3. Let k be a field. The category Vect_k of k-vector spaces is a monoidal category, where the tensor product is given by $\otimes_k$, the unit object is given by the vector space k, and the associator a is the usual one.

Example 3.4. Let A be an associative algebra over a field k. The category A-bimod of bimodules over A is a monoidal category, where the tensor product is given by $\otimes_A$, the unit object is given by the algebra A itself, and the associator a is the usual one.

Example 3.5. Let $n > 0$, and consider $\mathrm{Cob}^{\mathrm{or}}(n)$, the category of n-dimensional oriented[6] cobordisms. Namely, the objects of $\mathrm{Cob}^{\mathrm{or}}(n)$ are closed $n-1$-dimensional smooth oriented manifolds, and morphisms are oriented n-dimensional cobordisms[7] up to diffeomorphisms fixing the boundaries. Then $\mathrm{Cob}^{\mathrm{or}}(n)$ is a monoidal category, where the tensor product is given by the disjoint union $\coprod$ of manifolds, the unit object is given by the empty set $\emptyset$ regarded as a $n-1$-dimensional manifold, and where the associator a and the morphism ι are the obvious one.

Example 3.6. Let G be a group, and regard it as a discrete category[8] $\underline{G}$. Then $\underline{G}$ is a monoidal category, where the tensor product is given by group multiplication, the unit object is the identity $e \in G$, and the associator a and the morphism ι are the identity morphisms. Moreover, $\underline{G}$ is a strict monoidal category.

Example 3.7. Let $\mathcal{C}$ be a category, and let $\mathrm{End}(\mathcal{C})$ denote the category of endofunctors of $\mathcal{C}$. We can regard $\mathrm{End}(\mathcal{C})$ as a monoidal category where the tensor product is given by composition of functors, the unit object is given by the identity functor, and the associator a and the morphism ι are natural equivalences. Moreover, $\mathrm{End}(\mathcal{C})$ is a strict category.

Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ and $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{a}, \tilde{\mathbf{1}}, \tilde{\iota})$ be monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ is a pair (F, J), where $F : \mathcal{C} \to \tilde{\mathcal{C}}$ is a functor, and

$$J : \tilde{\otimes} \circ (F \times F) \xrightarrow{\simeq} F \circ \otimes$$

is a natural isomorphism, i.e. a collection of isomorphism

$$J_{X,Y} : F(X) \tilde{\otimes} F(Y) \simeq F(X \otimes Y)$$

[6]Other type of *tangential structures*, i.e. reductions of the structure group of the tangent bundle, are usually considered. See section 3.2.1 for more details.

[7]We consider the cobordisms equipped with collars to be able to compose morphisms by gluing the cobordisms along their common boundary.

[8]A *discrete category* is a category with no morphisms apart from identities.

which are natural in X and Y, such that $F(\mathbf{1})$ is isomorphic to $\tilde{\mathbf{1}}$, and such that the following diagram

$$\begin{tikzcd}
(F(X)\tilde{\otimes}F(Y))\tilde{\otimes}F(Z) \arrow[r, "\tilde{a}_{X,Y,Z}"] \arrow[d, "J_{X,Y}\tilde{\otimes}\mathrm{id}_Z"] & F(X)\tilde{\otimes}(F(Y)\tilde{\otimes}F(Z)) \arrow[d, "\mathrm{id}_{F(X)}\tilde{\otimes}J_{Y,Z}"] \\
F(X\otimes Y)\tilde{\otimes}F(Z) \arrow[d, "J_{X\otimes Y,Z}"] & F(X)\tilde{\otimes}F(Y\otimes Z) \arrow[d, "J_{X,Y\otimes Z}"] \\
F((X\otimes Y)\otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X\otimes(Y\otimes Z))
\end{tikzcd}$$

commutes for all X, Y, Z in $\mathcal{C}$. The above diagrams can be regarded as a compatibility condition between the associators a and $\tilde{a}$, and the natural isomorphism J. A *monoidal equivalence* is a monoidal functor which is also an equivalence of categories.

Remark 3.8. Notice that a monoidal functor is not just a functor between monoidal categories, but it requires an additional piece of structure, namely the natural isomorphism J, which must satisfy some conditions[9]. For a given functor between monoidal categories, there can be more natural isomorphisms J making it into a monoidal functor, or none at all.

Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ and $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{a}, \tilde{\mathbf{1}}, \tilde{\iota})$ be monoidal categories, and (F, J) and (G, L) be monoidal functors between $\mathcal{C}$ and $\tilde{\mathcal{C}}$. A *natural transformation* η between (F, J) and (G, L) is a natural transformation $\eta : F \to G$ such that $\eta_{\mathbf{1}}$ is an isomorphism, and such the following diagram

$$\begin{tikzcd}
F(X)\tilde{\otimes}F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\eta_x\tilde{\otimes}\eta_y"] & F(X\otimes Y) \arrow[d, "\eta_{x\otimes y}"] \\
G(X)\tilde{\otimes}G(Y) \arrow[r, "L_{X,Y}"] & G(X\otimes Y)
\end{tikzcd}$$

commutes for all X, Y in $\mathcal{C}$.
An important result in the theory of monoidal categories is the following result, known as “Mac Lane’s strictness theorem”.

Theorem 3.9. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

The theorem above guarantees that in working with monoidal categories we do not have to “worry too much” about the associator and the other structural morphisms.

Remark 3.10. It is important to stress that Mac Lane’s strictness theorem only guarantees that that a monoidal category is monoidally *equivalent* to a strict one, but not in general *isomorphic*. In other words, we might have to add other objects to the category to achieve strictness.

[9] In other words, monoidality of a functor is a *structure*, and not a *property*.

3.1.2. *Braided categories.* Let $\mathcal{C}$ be a monoidal category. A *braiding* on $\mathcal{C}$ is a natural isomorphism $c : (-\otimes-) \xrightarrow{\simeq} (-\otimes-)\circ\tau$, where τ is the canonical endofunctor of $\mathcal{C}\times\mathcal{C}$ which exchanges the factors, such that the following diagrams

$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y\otimes Z)\otimes X \arrow[dr, "a_{Y,Z,X}"] & \\
(X\otimes Y)\otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c_{X,Y}\otimes\mathrm{id}_Z"'] & & & Y\otimes(Z\otimes X) \\
 & (Y\otimes X)\otimes Z \arrow[r, "a_{Y,X,Z}"] & Y\otimes(X\otimes Z) \arrow[ur, "\mathrm{id}_Y\otimes c_{X,Z}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (X\otimes Y)\otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z\otimes(X\otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"] & \\
X\otimes(Y\otimes Z) \arrow[ur, "a^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X\otimes c_{Y,Z}"'] & & & (Z\otimes X)\otimes Y \\
 & X\otimes(Z\otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X\otimes Z)\otimes Y \arrow[ur, "c_{X,Z}\otimes\mathrm{id}_Y"'] &
\end{tikzcd}$$

commute for all X, Y, Z in $\mathcal{C}$.
A *braided* monoidal category is a pair consisting of a monoidal category and a braiding.

REMARK 3.11. A given monoidal category may admit more braidings.

A braided monoidal category is called *symmetric* if

$$c_{Y,X}\circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$$

commutes for all X, Y in $\mathcal{C}$.

EXAMPLE 3.12. The category Sets,Vect_k and $\mathrm{Cob}^{\mathrm{or}}(n)$ are all example of braided categories with the braiding given by the transposition of factors. Moreover, they are all symmetric.

Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be braided monoidal categories with braidings c and $\tilde{c}$, respectively. A monoidal functor (F, J) between $\mathcal{C}$ and $\tilde{\mathcal{C}}$ is called *braided* if the following diagram

$$\begin{tikzcd}
F(X)\tilde{\otimes}F(Y) \arrow[r, "\tilde{c}_{X,Y}"] \arrow[d, "J_{X,Y}"] & F(Y)\tilde{\otimes}F(X) \arrow[d, "J_{Y,X}"] \\
F(X\otimes Y) \arrow[r, "F(c_{X,Y})"] & F(Y\otimes X)
\end{tikzcd}$$

for all X, Y in $\mathcal{C}$.

REMARK 3.13. For a monoidal functor, to be braided is a property, and not a structure.

A braided monoidal functor between symmetric monoidal category is called a *symmetric monoidal functor*.

Example 3.14. Let $\mathcal{C}$ be an arbitrary category, and let $\mathcal{D}$ be a monoidal category. Then the category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between $\mathcal{C}$ and $\mathcal{D}$ is a monoidal category with product $\star$ defined objectwise, unit object given by the constant functor to $\mathbf{1}_{\mathcal{D}}$, and associator obtained from that of $\mathcal{D}$.

Example 3.15. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and consider the category of monoidal functors $\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ between $\mathcal{C}$ and $\mathcal{D}$. In general, the objectwise product does *not* equip $\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ with a monoidal structure. Neverthless, in the case in which both $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal, we have that the objectwise product induces a structure of braided monoidal category on $\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ which is moreover symmetric.

3.1.3. *The Drinfeld center.* To any monoidal category $\mathcal{C}$, one can associate a braided monoidal category $\mathcal{Z}(\mathcal{C})$, called *the Drinfeld center* of $\mathcal{C}$.

Definition 3.16. Given a monoidal category $\mathcal{C}$, the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is the category where

- objects are pairs (X, Φ), where $X \in \mathcal{C}$, and Φ is a natural isomorphism
$$\Phi : X \otimes (-) \xrightarrow{\simeq} (-) \otimes X$$
such that for all $Y \in \mathcal{C}$ we have
$$\Phi_{Y \otimes Z} = (\mathrm{id} \otimes \Phi_Z) \circ (\Phi_Y \otimes \mathrm{id})$$
- a morphism from (X, Φ) to (Y, Ψ) is a morphism $f : X \to Y$ such that
$$(\mathrm{id} \otimes f) \circ \Phi_Z = \Psi_Z \circ (f \otimes \mathrm{id})$$
for any $Z \in \mathcal{C}$.

The category $\mathcal{Z}(\mathcal{C})$ comes naturally equipped with a tensor product, defined on objects as
$$(X, \Phi) \otimes (Y, \Psi) := (X \otimes Y, (\Phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi))$$
More interestingly, the monoidal category $\mathcal{Z}(\mathcal{C})$ comes equipped also with a braiding, which is given by
$$c_{(X,\Phi),(Y,\Psi)} := \Phi_Y$$
for any (X, Φ) and (Y, Ψ) in $\mathcal{Z}(\mathcal{C})$.

Remark 3.17. The Drinfeld center of a monoidal category is the natural categorification of the notion of the center of a monoid: we consider objects in $\mathcal{C}$ which commute with all the other objects. Notice though that in general $\mathcal{Z}(\mathcal{C})$ is *not* a subcategory of $\mathcal{C}$, since commutativity is promoted from a property to a structure.

3.1.4. *Duality in monoidal categories.* Let $\mathcal{C}$ be a monoidal category. An object A^* in $\mathcal{C}$ is said to be a *left dual* for an object A if there exist morphisms

$$\mathrm{ev}_A : A^* \otimes A \to \mathbf{1}$$
$$\mathrm{coev}_A : \mathbf{1} \to A \otimes A^*$$

called the *evaluation* and *coevaluation*, respectively, and such that the following compositions

$$A \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_A} (A \otimes A^*) \otimes A \xrightarrow{a_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{\mathrm{id}_A \otimes \mathrm{ev}_A} A$$
$$A^* \xrightarrow{\mathrm{id}_{A^*} \otimes \mathrm{coev}_A} A^* \otimes (A \otimes A^*) \xrightarrow{a^{-1}_{A^*,A,A^*}} (A^* \otimes A) \otimes A^* \xrightarrow{\mathrm{ev}_A \otimes \mathrm{id}_{A^*}} A^*$$

are the identity morphisms. The evaluation and coevaluation morphisms should be considered part of the data of a left dual A^*: in other words, the left dual for an object A is a triple $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$.
Similarly, one can define the notion of a *right dual* object ${}^*A$. In particular, left and right dualities are compatible, in the sense that

$${}^*(A^*) \simeq A \simeq ({}^*A)^*$$

Moreover, in any monoidal category the unit object $\mathbf{1}$ is the left and right dual of itself.

Remark 3.18. Let $\mathcal{C}$ be a category, and consider the monoidal category $\mathrm{End}(\mathcal{C})$. A left (respectively, right) dual for a functor $F \in \mathrm{End}(\mathcal{C})$ is the same thing as a left (respectively, left) adjoint to F.

The following result guarantees that, if it exists, we can talk of “the” left (respectively, right) dual of an object

Theorem 3.19. *If $A \in \mathcal{C}$ has a left (respectively, right) dual object, then it is unique up to a unique isomorphism.*

Remark 3.20. The existence of a left (or right) dual for an object imposes strong restriction. For instance, the only objects in Vect_k admitting a (left or right) dual are the *finite* dimensional k-vector spaces.

Let $A, B \in \mathcal{C}$ be objects adimitting left duals A^*, B^*, and let $f : A \to B$ be a morphism. One can then define the left dual f^* of f as the composition

$$B^* \xrightarrow{\mathrm{id}_{B^*} \otimes \mathrm{coev}_A} B^* \otimes (A \otimes A^*) \simeq (B^* \otimes A) \otimes A^* \xrightarrow{(\mathrm{id}_{B^*} \otimes f) \otimes \mathrm{id}_{A^*}} (B^* \otimes B) \otimes A^* \xrightarrow{\mathrm{ev}_B \otimes \mathrm{id}^*_A} A^*$$

Similarly, we can define the right dual of a morphism.
An object $A \in \mathcal{C}$ is said to be *rigid* if it admits both a left and a right dual. A rigid monoidal category $\mathcal{C}$ is a monoidal category where any object is rigid.

Given a monoidal functor (F, J) between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, for any object A with left dual A^* we have that $F(A^*)$ is left dual to $F(A)$ in a canonical way. The following result is relevant

Lemma 3.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, with $\mathcal{C}$ being moreover rigid. Then $\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})$ is a groupoid.*

Finally, if $\mathcal{C}$ is a rigid braided monoidal category, for any $A \in \mathcal{C}$ we have ${}^*A \simeq A^*$, hence we can refer to A^* as the dual object of A.

3.1.5. *Modular tensor categories.* Modular tensor categories can be regarded as linear braided categories which are *nondegenerate*. They have a prominent rôle, since they appear in the construction of 3d TQFTs and knots invariants, and they encode information regarding Rational Conformal Field theories. We will briefly introduce the definition of a modular tensor category, and provide some examples.
Let k be an algebraically closed field. A monoidal category $\mathcal{C}$ which is a locally finite[10] k-linear abelian and rigid category is called a *tensor category* (over k) if the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms, and if $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq k$. A tensor category which is also finitely semisimple is called a *fusion category*.

EXAMPLE 3.22. Let G be a finite group. Then the category $\mathrm{Rep}(G)$ of finite dimensional G-representations is a tensor category. It is also a fusion category if the characteristic of k does not divide $|G|$.

Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* is a monoidal natural equivalence

$$a : \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} (-)^{**}$$

i.e., a family of natural isomorphisms $a_X : X \xrightarrow{\simeq} X^{**}$ for each X in $\mathcal{C}$ such that $a_{X\otimes Y} = a_X \otimes a_Y$ for X, Y in $\mathcal{C}$.
Given a pivotal structure a, we can define the notion of the *dimension* $\dim_a(X)$ of an object X with respect to a as the composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{a_X \otimes \mathrm{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{ev_{X^*}} \mathbf{1}$$

If $\dim_a(X) = \dim_a(X^*)$ for all X in $\mathcal{C}$, then a is called a *spherical structure*, and $\mathcal{C}$ is called a *spherical category.* Notice that the composition above allows to define a *left quantum trace* for any morphism $g : X \to X^{**}$, which we denote with $\mathrm{Tr}^L(g)$. In a spherical category[11] $\mathcal{C}$ we can define a *trace* for any morphism $f : X \to X$ as

$$\mathrm{Tr}(f) := \mathrm{Tr}^L(a_X \circ f)$$

If $\mathcal{C}$ is a spherical fusion category, then $\mathrm{Tr}(f) \in k$ for any morphism $f : X \to X$.
Let $\mathcal{C}$ be a spherical fusion category over an algebraically closed field of characteristic 0, equipped with a braiding c, and let $\{X_i\}_{i\in I}$ a choice of representatives of simple objects. Then to $\mathcal{C}$ we can associate a matrix S, called the *S-matrix* defined as

$$S_{ij} := \mathrm{Tr}(c_{X_j,X_i} \circ c_{X_i,X_j})$$

A braided spherical fusion category $\mathcal{C}$ is called a *modular tensor category* if its S-matrix is invertible.

[10]A k-linear abelian category is called *locally finite* if every Hom space is finite dimensional, and any object X has finite length, i.e. there exists a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that X_i/X_{i-1} is a simple object for all i.

[11]A spherical fusion category guarantees that the left and right traces coincide for any endomorphism.

REMARK 3.23. A different choice of representatives of simple objects produces similar S-matrices, hence the property of being a modular tensor category is well defined.

EXAMPLE 3.24. The category Vect_k is a modular tensor category. In general a spherical category equipped with a symmetric braiding is *never* a modular tensor category, unless it has a single equivalence class of simple objects.

3.1.6. *Invertibility in monoidal categories.* Let $\mathcal{C}$ be a monoidal category. A rigid object $A \in \mathcal{C}$ is said to be *invertible* if ev_A and coev_A are invertible morphisms. For an invertible object A we have that the following properties hold

i) ${}^*A \simeq A^*$ and A^* is invertible.
ii) For B an invertible object, $A \otimes B$ is invertible.
iii) Given a monoidal functor $F : \mathcal{C} \to \mathcal{D}$, $F(A)$ is an invertible object in $\mathcal{D}$.

EXAMPLE 3.25. The invertible objects in $\mathcal{C} = \mathrm{Vect}_k$ are exactly the 1-dimensional k-vector spaces.

EXAMPLE 3.26. Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories. An invertible object in $\mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})$ is the same thing as a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(A)$ and $F(f)$ are invertible in $\mathcal{D}$, for all objects A and all morphisms f in $\mathcal{C}$.

The *Picard groupoid* $\mathrm{Pic}(\mathcal{C})$ of a symmetric monoidal category is the maximal subgroupoid of $\mathcal{C}$ having for objects the invertibles in $\mathcal{C}$. It follows from the properties above that $\mathrm{Pic}(\mathcal{C})$ inherits the structure of a symmetric monoidal category, and it is an example of an abelian 2-group[12]. It is not difficult to show that the set of equivalence classes $\pi_0(\mathrm{Pic}(\mathcal{C}))$ has the structure of an abelian group.

REMARK 3.27. Let R be a commutative ring, and let $\mathcal{C} = R\text{-Mod}$, the category of R-modules equipped with the tensor product provided by $\otimes_R$. In this case $\pi_0(\mathrm{Pic}(\mathcal{C}))$ is known as the *Brauer group* $Br(R)$ of R, and it has an important role in Representation Theory and Algebraic Geometry.

3.1.7. *Module categories.* Let $\mathcal{C}$ be a monoidal category. A *left module category* over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an *action* of $\mathcal{C}$, namely a functor $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a natural isomorphism

$$m_{X,Y,M} : (X \otimes Y) \odot M \xrightarrow{\simeq} X \odot (Y \odot M) \quad X, Y \in \mathcal{C}, M \in \mathcal{M}$$

called the *associativity constraints*, such that the functor $M \to \mathbf{1}_{\mathcal{C}} \odot \mathcal{M}$ is an auto-equivalence of $\mathcal{M}$, and the following pentagon diagram is commutative

[12]A *2-group* is a rigid monoidal category where all objects and all morphisms are invertible.

$$\begin{tikzcd}[column sep=tiny]
 & ((X \otimes Y) \otimes Z) \odot M \arrow[dl, "a_{X,Y,Z} \otimes \mathrm{id}_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \odot M \arrow[d, "m_{X, Y \otimes Z, M}"'] & & (X \otimes Y) \odot (Z \odot M) \arrow[d, "m_{X,Y,Z \odot M}"] \\
X \odot ((Y \otimes Z)) \odot M) \arrow[rr, "\mathrm{id}_X \otimes m_{Y,Z,M}"] & & X \odot (Y \odot (Z \odot M))
\end{tikzcd}$$

for any X, Y, Z in $\mathcal{C}$ and any M in $\mathcal{M}$.
Similarly, one can define the notion of a *right module category* over $\mathcal{C}$. By module category we will mean a left module category.
A different way to define a module category is the following. Let $\mathcal{M}$ be a category. Recall that the category $\mathrm{End}(\mathcal{M})$ of endofunctors of $\mathcal{M}$ is canonically a monoidal category. A structure on $\mathcal{M}$ of module category over $\mathcal{C}$ is then given by a monoidal functor

$$F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$$

One has the following

PROPOSITION 3.28 ([**19**]). *There is a bijective correspondence between structures of $\mathcal{C}$-module category on a category $\mathcal{M}$ and monoidal functors $\mathcal{C} \to \mathrm{End}(\mathcal{M})$.*

REMARK 3.29. Proposition 3.28 is a categorification of the result that a module over a monoid is the same thing as a representation.

We can similarly define a $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$ as a category which is a left $\mathcal{C}$-module and a right $\mathcal{D}$-module, such that the structures are compatible. We omit the compatibility diagrams, since they follow naturally.

EXAMPLE 3.30. Let $\mathcal{C}$ be a monoidal category. Then $\mathcal{M} = \mathcal{C}$ has a canonical structure of left module category over $\mathcal{C}$ given by tensor product.

EXAMPLE 3.31. Let G be a group, and $H \subset G$ a subgroup. Since the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is a tensor functor, the category $\mathrm{Rep}(H)$ is canonically a left module category over $\mathrm{Rep}(G)$.

Let $\mathcal{M}$ and $\mathcal{N}$ module categories over $\mathcal{C}$ with associativity constraints m and n, respectively[13]. A $\mathcal{C}$-*module functor* F from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F : \mathcal{M} \to \mathcal{N}$ together with a natural isomorphism

$$s_{X,M} : F(X \odot M) \xrightarrow{\simeq} X \odot F(M), \quad X \in \mathcal{C}, M \in \mathcal{M}$$

[13]We will use the symbol $\odot$ for both the actions on $\mathcal{M}$ and $\mathcal{N}$.

such that the following diagrams

$$\begin{tikzcd}
& F((X \otimes Y) \odot M) \arrow[dl, "F(m_{X,Y,M})"'] \arrow[dr, "F(s_{X\otimes Y,M})"] & \\
F(X \odot (Y \odot M)) \arrow[d, "s_{X,Y\odot M}"'] & & (X \otimes Y) \odot F(M) \arrow[d, "n_{X,Y\odot F(M)}"] \\
X \odot F(Y \odot M) \arrow[rr, "\mathrm{id}_X \otimes s_{Y,M}"] & & X \odot (Y \odot F(M))
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(\mathbf{1}_{\mathcal{C}} \odot M) \arrow[rr, "s_{\mathbf{1},M}"] \arrow[dr, "F(l_M)"'] & & \mathbf{1}_{\mathcal{C}} \odot F(M) \arrow[dl, "l_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

commute for X, Y in $\mathcal{C}$ and M $\mathcal{M}$, and where $l_M : \mathbf{1}_{\mathcal{C}} \odot M \simeq M$ is the unit isomorphism.

Let (F, s) and (G, t) be $\mathcal{C}$-module functors between $\mathcal{M}$ and $\mathcal{N}$. A *module natural transformation* between (F, s) and (G, t) is a natural transformation $\eta : F \to G$ such that the following diagram

$$\begin{tikzcd}
F(X \odot M) \arrow[r, "s_{X,M}"] \arrow[d, "\eta_{X\odot M}"'] & X \odot F(M) \arrow[d, "\mathrm{id}_X \otimes \eta_M"] \\
G(X \odot M) \arrow[r, "t_{X,M}"] & X \odot G(M)
\end{tikzcd}$$

commutes for X in $\mathcal{C}$ and M in $\mathcal{M}$.
Given $\mathcal{M}$ and $\mathcal{N}$ two $\mathcal{C}$-module categories, we can then consider the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ of $\mathcal{C}$-module functors between $\mathcal{M}$ and $\mathcal{N}$.

3.2. Functorial Topological Quantum Field Theory. The axiomatic approach of Atiyah [**1**] can be compactly described as follows. Let $\mathcal{C}$ be a symmetric monoidal category. An n-dimensional oriented Topological Quantum Field Theory (TQFT) with values in $\mathcal{C}$ is a symmetric monoidal functor

$$Z : \mathrm{Cob}^{\mathrm{or}}(n) \to \mathcal{C}$$

In the case where $\mathcal{C} = \mathrm{Vect}_k$, the functor above corresponds to the following data

- A finite dimensional vector space V_Σ assigned to any oriented closed $(n-1)$-dimensional manifold Σ.
- A linear map $\varphi_M : V_\Sigma \to V_{\Sigma'}$ for any diffeomorphism class of oriented cobordism $M : \Sigma \to \Sigma'$, such that for $M : \Sigma \to \Sigma'$ and $N : \Sigma' \to \Sigma''$ we have that

$$\varphi_N \circ \varphi_M = \varphi_{M \cup_{\Sigma'} N}$$

- An isomorphism $V_{\Sigma \sqcup \Sigma'} \simeq V_\Sigma \otimes V_{\Sigma'}$ compatible with transposition of factors for each pair of closed $(n-1)$-manifolds.
- An isomorphism $V_\emptyset \simeq k$.

Since any closed oriented n-dimensional manifold M can be regarded as a cobordism from the empty manifold to itself, from the data above we have a linear map

$$\varphi_M : k \to k$$

which amounts to an element of k, and it is called the *partition function* of the TQFT Z evaluated on M.

REMARK 3.32. The fact that the value $Z(\Sigma)$ is a *finite dimensional* vector space is a consequence of the fact that any object in $\mathrm{Cob}^{\mathrm{or}}(n)$ has a dual, and the fact that Z is a symmetric monoidal functor.

Since TQFTs are symmetric monoidal functors between symmetric monoidal categories, they naturally form a category, namely we can define

$$\mathrm{TQFT}^{\mathrm{or}}{}_{\mathcal{C}}(n) := \mathrm{Fun}_\otimes(\mathrm{Cob}^{\mathrm{or}}(n), \mathcal{C})$$

Since any object of $\mathrm{Cob}^{\mathrm{or}}(n)$ admits a dual, by Lemma 3.21 we have that $\mathrm{TQFT}^{\mathrm{or}}{}_{\mathcal{C}}(n)$ is a groupoid, hence it can be considered as the *space*[14] of oriented Topological Quantum Field Theories in dimension n with values in $\mathcal{C}$.
The functorial approach to Topological Quantum Field Theory is powerful enough to allow for a complete classification in low dimensions. For instance, in the case $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$, a 1-dimensional oriented TQFT is completely specified by a finite dimensional vector space, which is assigned to the oriented point. In dimension 2, a result by Dijkgraaf shows that any oriented TQFT is determined by a commutative *Frobenius algebra*, namely an algebra equipped with a non-degenerate trace. The classification of oriented 2d TQFTs relies heavily on Morse theory and the classification of compact surfaces. This should suggest that the classification of TQFTs in dimension higher than 2 becomes quickly out of reach. For a recent classification of 3d oriented TQFTs, we refer the reader to [**41**].

3.2.1. *Tangential structures.* Let $BO(n)$ be a classifying space for $O(n)$-bundles. An *n-dimensional tangential structure* is a fibration $\pi(n) : \mathcal{X}(n) \to BO(n)$. Morever, denote by $S(n) \to \mathcal{X}(n)$ the pullback of the universal vector bundle $EO(n) \to BO(n)$ along $\pi(n)$.
Let M be an m-dimensional manifold, with $m \leq n$. A $\mathcal{X}(n)$-structure on M is a (continuous) map $f : M \to \mathcal{X}(n)$ together with an isomorphism

$$\theta : \underline{\mathbb{R}^{n-m}} \oplus TM \xrightarrow{\simeq} f^* S(n)$$

where $\underline{\mathbb{R}^{n-m}}$ denotes the trivial real bundle over M of rank $n-m$.

[14] By space we mean a *homotopy type*.

EXAMPLE 3.33. Consider $\mathcal{X}(n) = BSO(n)$, with $\pi(n) : BSO(n) \to BO(n)$ induced by the inclusion $SO(n) \hookrightarrow O(n)$. Then a $\mathcal{X}(n)$-structure on a m-manifold M is the same thing as an orientation of $\underline{\mathbb{R}^{n-m}} \oplus TM$, which is in turn equivalent to an orientation of M.

EXAMPLE 3.34. Consider $\mathcal{X}(n) = EO(n)$, with $\pi(n) : EO(n) \to BO(n)$ being the universal principal $O(n)$-bundle. Then a $\mathcal{X}(n)$-structure on a m-manifold M is the same thing as an n-framing of M.

REMARK 3.35. Not all tangential structures arise from the reduction of the structure group of the stabilized tangent bundle. Let Γ be a finite group, and consider $\mathcal{X}(n) = BO(n) \times B\Gamma$, with $\pi(n) : \mathcal{X}(n) \to BO(n)$ given by projection on the first factor. Then a $\mathcal{X}(n)$-structure on a m-manifold M is the datum of a principal Γ-bundle $P \to M$.

Given a $\mathcal{X}(n)$-structure on a $(n-1)$-dimensional manifold M, we can define the *opposite* $\mathcal{X}(n)$-structure on M by considering the isomorphism

$$\theta^* : \underline{\mathbb{R}} \oplus TM \xrightarrow{-1 \oplus \mathrm{id}_{TM}} \underline{\mathbb{R}} \oplus TM \xrightarrow{\theta} f^*S(n)$$

A $\mathcal{X}(n)$-structure on a m-dimensional manifold M induces a $\mathcal{X}(n)$-structure on ∂M by restriction. Hence, we can define a cobordism between $(n-1)$-dimensional closed manifolds with $\mathcal{X}(n)$-structure (Σ, f, θ) and (Σ', g, ψ) as an n-dimensional manifold with $\mathcal{X}(n)$-structure (M, F, Θ) together with an equivalence

$$(\partial M, F|_{\partial M}, \Theta|_{\partial M}) \simeq (\Sigma, f, \theta) \amalg (\Sigma', g, \psi^*)$$

of manifolds with $\mathcal{X}(n)$-structures.
Denote with $\mathrm{Cob}^{\mathcal{X}(n)}(n)$ the category of n-dimensional cobordisms with $\mathcal{X}(n)$-structure, with symmetric monoidal structure given by disjoint union.
A TQFT with $\mathcal{X}(n)$-structure taking values in $\mathcal{C}$ is a symmetric monoidal functor

$$Z : \mathrm{Cob}^{\mathcal{X}(n)}(n) \to \mathcal{C}$$

REMARK 3.36. Any object in $\mathrm{Cob}^{\mathcal{X}(n)}(n)$ admits a dual. Namely, the dual of (Σ, f, θ) is provided by (Σ, f, θ^*), with evaluation and coevaluation morphisms given by the incoming and outcoming "bent cylinders", respectively.

3.2.2. *Dijkgraaf-Witten Theories.* For any finite group G and any $n > 0$, one can produce a n-dimensional TQFT valued in Vect_k, with k a field of characteristic not dividing the order of G. The case for $n = 3$ was first studied by Dijkgraaf and Witten in [**18**], hence the name.
Let Σ be a closed $(n-1)$-manifold, and denote with $\mathrm{Bun}^G(\Sigma)$ the groupoid of G-bundles over Σ. We have then the following assignment

$$\Sigma \to V_\Sigma := k[\pi_0(\mathrm{Bun}^G(\Sigma))]$$

Notice that since G is finite and Σ is compact, the vector space V_Σ is finite dimensional. Moreover, we have a canonical isomorphism

$$V_{\Sigma \amalg \Sigma'} \simeq V_\Sigma \otimes V_{\Sigma'}$$

for any Σ, Σ'.
Consider a n-dimensional cobordism M between Σ_0 and Σ_1, with $\iota_i : \Sigma_i \hookrightarrow M, i = 0, 1$ denoting the inclusion maps. We can then consider the follow diagram of groupoids

$$\begin{tikzcd} & \mathrm{Bun}^G(M) \arrow[dl, "\iota_0^*"'] \arrow[dr, "\iota_1^*"] & \\ \mathrm{Bun}^G(\Sigma_0) & & \mathrm{Bun}^G(\Sigma_1) \end{tikzcd}$$

where the functors ι_i are given by restriction of G-bundles. To such a diagram we can assign a linear map from V_{Σ_0} to V_{Σ_1} defined as

$$\begin{aligned} \varphi_M \quad &: V_{\Sigma_0} \to V_{\Sigma_1} \\ &[y] \to \textstyle\sum_{[x]:[x]=[\iota_0^* y]} \dfrac{\iota_1^*[x]}{\#\mathrm{Aut}(x)} \end{aligned}$$

where $y \in \mathrm{Bun}^G(\Sigma_0), x \in \mathrm{Bun}^G(M)$ and $\mathrm{Aut}(x)$ denotes the automorphism group of the bundle x in $\mathrm{Bun}^G(M)$. By using the fact that G-bundles are "local", more precisely that

$$\mathrm{Bun}^G(M \cup_\Sigma N) \simeq \mathrm{Bun}^G(M) \times_{\mathrm{Bun}^G(\Sigma)} \mathrm{Bun}^G(N)$$

we have that

$$\varphi_{M \circ N} := \varphi_N \circ \varphi_M$$

for cobordisms $M : \Sigma_0 \to \Sigma_1$ and $N : \Sigma_1 \to \Sigma_2$. Notice that the construction is compatible with disjoint union of maniolds.
The above assignments provide then a symmetric monoidal functor

$$Z^G : \mathrm{Cob}(n) \to \mathrm{Vect}_k$$

which we refer to as *Dijkgraaf-Witten theory* in dimension n, and it represents one of the best understood examples of Topological Quantum Field Theories in the functorial formalism.
For a closed n-dimensional manifold M, it follows that the partition function of Z^G evaluated at M is given by

$$Z^G(M) = \sum_{[x] \in \mathrm{Bun}^G(M)} \frac{1}{\#\mathrm{Aut}(x)}$$

The expression above can be interpreted as the *Feynman integral* of the constant function 1 over the space of fields given by $\mathrm{Bun}^G(M)$, equipped with its gauge invariant measure.

REMARK 3.37. It is straightforward to establish that for $n > 1$, we have $Z^G(S^n) = 1/\#G$.

REMARK 3.38. In [**18**], an oriented 3d TQFT is constructed from a finite group G and a group 3-cocycle $\alpha \in Z^3(G, \mathbb{C}^*)$. For $\alpha = 0$ it coincides with Z^G once the orientation is forgotten.

REMARK 3.39. As emphasised in [**27**], Dijkgraaf-Witten theory is actually the result of a two-step procedure. Indeed, we first consider the classical theory given by a functor Bun^G from the relevant cobordism category to the category Fam_1 of "spans of groupoids". We then apply a suitable "quantization functor" $\mathrm{Sum}_1 : \mathrm{Fam}_1 \to \mathrm{Vect}_k$ to obtain a TQFT.

3.2.3. *Remarks on Chern-Simons theory.* A motivating example for the axiomatic approach to Topological Quantum Field Theory is provided by Chern-Simons theory, studied by Witten in [**71**].
Let G be a connected, simply connected[15] Lie group, let $\mathfrak{g}$ be its Lie algebra, and let tr be an invariant "trace" on $\mathfrak{g}$. For any closed 3-manifold M we have the *classical Chern-Simons* functional

$$\begin{aligned} S_X : \quad & \mathcal{A}(M) \to \mathbb{R} \\ & A \to \frac{1}{8\pi^2} \int \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \end{aligned}$$

where $\mathcal{A}(M) \simeq \Omega^1(M;\mathfrak{g})$ denotes the space of connections on the *trivial* G-principal bundle over M.
Witten's idea then was to "integrate over" the connection variable, in order to produce a quantity which only depends on the topology[16] of M. More precisely, he suggested to consider a path-integral quantization, namely to consider the quantity

$$Z_k(M) \text{``=''} \int_{\mathcal{A}(M)/\sim} e^{ikS_M(A)} \mathcal{D}A$$

where k is an integer, called the *level*[17], and $\mathcal{D}A$ is supposed to be a measure on the space of connections on M modulo gauge transformations. Such a measure is in general not constructed, hence the expression above remains not defined. There are several approaches to deal with the quantity $Z_k(M)$: the one related to functorial TQFTs as we have presented them in this thesis amounts to avoid constructing directly the measure $\mathcal{D}A$, but first to rather axiomatise the formal properties it has, and then to look for examples of such axioms. For instance, given a compact 3-manifold M with boundary ∂M we should be able to compute the following quantity

$$Z_k(M)(\alpha) \text{``=''} \int_{\mathcal{A}^\alpha(M)/\sim} e^{ikS_M(A)} \mathcal{D}A$$

where $\mathcal{A}^\alpha(M)$ is the subspace of connections whose restriction on ∂M is gauge equivalent to α. We can then regard $Z_k(M)$ as a function on $\mathcal{A}(\partial M)$, hence as an element of a vector space. This property is indeed one of the axioms in Atiyah's formulation of TQFTs. We can heuristically deduce the other axioms from a similar line of reasoning. Notice that though the invariant $Z_k(M)$ suffers from the fact that the measure $\mathcal{D}(A)$ remains undefined, the vector space associated to a closed 2-manifold Σ can be rigorously defined as the geometric quantization of the moduli space of G-local

[15] Lie groups which are not simply connected, e.g. $U(1)$, require a different approach.

[16] More precisely, there is a dependence on the smooth structure of M and its orientation.

[17] The level corresponds to the class of $\mathrm{tr}(F_A \wedge F_A)$ in $H^4(BG;\mathbb{Z}) \simeq \mathbb{Z}$, where F_A denotes the curvature of the connection A.

systems (i.e. flat G-bundles) $\mathrm{Loc}_G(\Sigma)$ on Σ, as in [**2**, **40**].
There are various proposal for a description of Chern-Simons as a TQFT which involve quantum groups, as in the work of Reshetikhin and Turaev [**60**]. That this construction does indeed reproduce the results from geometric quantization is discussed[18] in [**36**].
Another recent approach [**14**] is to investigate Chern-Simons theory using techniques from perturbative Quantum Field Theory, and to show that it fits the framework of functorial TQFTs.

3.2.4. *Representations of mapping class groups.* Given a n-dimensional oriented TQFT taking values in Vect_k, we obtain a system of representations of the mapping class group of manifolds which are "related", in a precise sense. Namely, consider a symmetric monoidal functor $Z : \mathrm{Cob}^{or}(n) \to \mathrm{Vect}_k$. Given an orientation preserving diffeomorphism $f : M \to M$, we can construct its *mapping cylinder*

$$M_f := ([0,1] \times M) \amalg M/\sim$$

where we identify $(1, x)$ and $f(x)$. The mapping cylinder can be equipped with a smooth structure, making it into a cobordism between M and itself. Morever, for a given diffeomorphism f, M_f satisfies the following properties

i) If f is isotopic to a diffeomorphism $g : M \to M$, then M_f is diffeomorphic to M_g with a diffeomorphism which preserves the boundary.
ii) M_f is an invertible morphism in $\mathrm{Cob}^{or}(n)$.
iii) For $f : M \to M$ and $g : M \to M$, we have that $M_{g\circ f}$ is diffeomorphic to $M_g \cup_M M_f$.

Recall that the mapping class group $MCG(M)$ of a manifold M is defined as

$$MCG(M) := \mathrm{Diff}^{or}(M)/\mathrm{Diff}^{or}_0(M)$$

i.e., it the group of orientation preserving diffeomorphisms of M modulo those which are isotopic to the identity. The properties of the mapping cylinder guarantee that for any manifold M we have a group homomorphism

$$\begin{aligned} MCG(M) &\to \mathrm{Aut}(Z(M)) \\ [f] &\to Z(M_f) \end{aligned}$$

i.e. a representation of $MCG(M)$ on the vector space $Z(M)$.
Such representations are "intertwined", in the sense that any diffeomorphism $f : M \to N$ induces an intertwiner of the respective represention.
The case for $n = 2$ is particulartly interesting because it concerns representations of the mapping class groups of surfaces, which in turn are related to flat bundles on the moduli space of Riemann surfaces.
An interesting question is to ask how much information such a system of representations, also called *a modular functor*[19],contains, namely if it is always possible to

[18]The following discussion on Mathoverflow is worth a reading https://mathoverflow.net/questions/86792/why-hasnt-anyone-proved-that-the-two-standard-approaches-to-quantizing-chern-si .

[19]The are several definitions of an (extended) modular functor which are more or less known to be *not* equivalent. See [**35**] for more details.

construct a TQFT Z which induces such a family of representations. For $n = 2$, we refer to [**4**] for a discussion of such topics.

3.3. Elements of Higher Category Theory. Higher categories are a generalization of the notion of a category, where we allow objects, morphisms, morphisms between morphisms, and so on. One way to picture[20] this is to consider that we have a hierarchy of information

$$\bullet_x$$

$$\bullet_x \xrightarrow{f} \bullet_y$$

$$\begin{tikzcd} \bullet_x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet_y \end{tikzcd}$$

$$\vdots$$

which must be provided with laws concerning how we compose arrows, identities, etc. One of the basic principles of higher category theory is the following: in an n-category, any equation between objects, 1-morphisms, 2-morphisms, etc., up to $(n-1)$-morphisms should always be considered to hold only up to invertible 1-morphisms, 2-morphisms, etc., while for n-morphisms equations must hold on the nose. Here invertibility of morphisms must be considered in the higher sense, namely it must be considered as a set of equations holding only up to higher morphisms.
In the following we will give a very brief introduction to higher category, referring the reader to the literature for details.

3.3.1. *Strict n-categories.* Strict higher categories can be defined in a recursive way. More precisely, we declare a strict 0-category to be a set, a strict 1-category to be an ordinary category[21], and for $n \geq 2$ a *strict n-category* to be a category enriched in Cat^{n-1}, the category of strict $(n-1)$-categories.
For $n = 2$, the above definition amounts to the following data

- A collection of objects $A, B, \ldots$.
- For each pair A, B, a category $\mathrm{Hom}(A, B)$
- For each triple A, B, C a functor $\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, called *horizontal composition*
- For each A, an object $\mathbf{1}_A \in \mathrm{Hom}(A, A)$, called the *identity 1-morphism*.

The data above must satisfy the following conditions

[20] Technically speaking, this uses *globes* as higher morphisms. Different shapes can be chosen.
[21] We consider only small categories to avoid set foundational issues.

- Horizontal composition is *strictly* associative, i.e. for any quadruple A, B, C, D of object the following diagram of functors

$$\begin{tikzcd}
\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C)\times\mathrm{Hom}(C,D) \arrow[r, "\mathrm{id}\times\circ"] \arrow[d, "\circ\times\mathrm{id}"'] & \mathrm{Hom}(A,B)\times\mathrm{Hom}(B,D) \arrow[d, "\circ"] \\
\mathrm{Hom}(A,C)\times\mathrm{Hom}(C,D) \arrow[r, "\circ"'] & \mathrm{Hom}(A,D)
\end{tikzcd}$$

  commutes strictly.
- For each A, the object $\mathbf{1}_A$ is *strictly* the identity for the horizontal composition, namely for each A, B the following diagrams of functors

$$\begin{tikzcd}
\mathrm{Hom}(A,B)\times\star \arrow[d, "\mathrm{id}\times\mathbf{1}_B"] \arrow[dr, "\simeq"] & \\
\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,B) \arrow[r, "\circ"] & \mathrm{Hom}(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\star\times\mathrm{Hom}(A,B) \arrow[d, "\mathbf{1}_A\times\mathrm{id}"] \arrow[dr, "\simeq"] & \\
\mathrm{Hom}(A,A)\times\mathrm{Hom}(A,B) \arrow[r, "\circ"] & \mathrm{Hom}(A,B)
\end{tikzcd}$$

  commute strictly, where $\star$ denotes the category with a single object and a single morphism.

For each pair of objects A, B of a 2-category, the objects and morphisms of $\mathrm{Hom}(A, B)$ are called 1-morphisms and 2-morphisms, respectively.
Given $\mathfrak{A}$ and $\mathfrak{B}$ n-categories, a *strict n-functor* $F : \mathfrak{A} \to \mathfrak{B}$ is simply a functor of Cat^{n-1}-enriched categories.

EXAMPLE 3.40. The 2-category Cat^2 has small categories for objects, and for any two such categories $\mathcal{A}$ and $\mathcal{B}$, we define

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) := \mathrm{Fun}(\mathcal{A}, \mathcal{B})$$

, where $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ denotes the category of functors from $\mathcal{A}$ to $\mathcal{B}$, and their natural transformations.
Horizontal composition is given by composition of functors, and the object $\mathbf{1}_{\mathcal{A}}$ is given by the identity functor $\mathrm{id}_{\mathcal{A}}$.

EXAMPLE 3.41. Let $\mathcal{C}$ be a strict monoidal category. We can consider the 2-category $\mathfrak{C}$ given by a single object $\bullet$ and

$$\mathrm{Hom}(\bullet, \bullet) := \mathcal{C}$$

with composition given by the tensor product of $\mathcal{C}$, and $\mathbf{1}_\bullet := \mathbf{1}_{\mathcal{C}}$. A strict 2-functor between $\mathfrak{C}$ and $\mathfrak{C}$ is precisely a strict monoidal endofunctor of $\mathcal{C}$.

REMARK 3.42. An ordinary category $\mathcal{C}$ can be regarded as a 2-category where the objects are the same objects of $\mathcal{C}$, and for any A, B we define

$$\mathrm{Hom}(A, B) := \underline{\mathrm{Hom}_{\mathcal{C}}}(A, B)$$

$\underline{\mathrm{Hom}_{\mathcal{C}}}(A, B)$ denotes the discrete category associated to the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and where the horizontal composition and identities are those provided by $\mathcal{C}$.

3.3.2. *Weak n-categories.* Differently from their strict versions, *weak* higher categories are more difficult to describe. The basic idea of a weak n-category revolves around weakening the various structural relations which hold for their strict counterparts. For instance, we could ask that the associativity of horizontal composition of 1-morphisms only holds up to invertible 2-morphisms, namely that for every composable triple of 1-morphisms f, g, h we provide an *associator*

$$\alpha_{f,g,h} : (fg)h \xrightarrow{\simeq} f(gh)$$

Similarly, we can provide invertible 2-morphisms witnessing the left and right identity constraints. In turn, the associator and the identity constraints must satisfy their own compatibility equations, called "coherence laws", which can be witnessed by invertible 3-morphisms, and so on. For $n > 2$, the task of determining the coherence laws which are needed becomes quickly daunting. For this reason, different notions of weak n-categories are available, including tricategories, ω-categories, simplicial ω-categories, etc. Some of this definitions are not "algebraic", in the sense that to a pair of morphisms f, g they might assign a *space* of compositions, rather than a single composition. We refer the reader to [**51**] for a tour of the zoo of definitions of n-categories.
For $n = 2$, weak 2-categories are also called *bicategories*. In this case, one explicitely spells out the coherence laws that must be satisfied by the associator, the identity constraints, and the compatiblity of vertical and horizontal composition of 2-morphisms: they amount to the "pentagon axiom" and to the "triangle identity".

EXAMPLE 3.43. Let $\mathcal{C}$ be a weak monoidal category. We can then form a bicategory $\mathfrak{C}$ with a single object $\bullet$, and

$$\mathrm{Hom}(\bullet, \bullet) := \mathcal{C}$$

where the composition of 1-morphisms and the identity are given by the tensor product $\otimes$ and $\mathbf{1}_{\mathcal{C}}$ in $\mathcal{C}$, and the associator and left and right identity constraints are given by those of $\mathcal{C}$.

EXAMPLE 3.44. Let k be a field. We can consider then the bicategory Alg_2, where the objects are associative k-algebras, and for any A, B

$$\mathrm{Hom}(A, B) :=_A \mathrm{Mod}_B$$

where ${}_A\mathrm{Mod}_B$ denotes the category of A-B-bimodules and intertwiners, and the composition of 1-morphisms given by tensor products of bimodules. This bicategory is not strict, since associativity of tensor products of bimodules only holds up to an invertible 2-morphism.

A crucial aspect of weak (resp. strict) n-category theory is that the collection of n-categories form themselves a weak (resp. strict) $(n+1)$-category. Hence, this means that to be able to talk about equivalences of n-categories we should know about $(n+1)$-category theory. For instance, the collection of all bicategories form a weak 3-category, called BiCat: natural transformations between 2-functors admit themselves morphisms, called *modifications*. It can be shown that BiCat is not equivalent to a

strict 3-category[22].
A result by Mac Lane ensures that any weak 2-category is equivalent to a strict 2-category. Neverthless, the equivalence itself will be in general *not* strict, which has the consequence that even after a strictification procedure, we have to deal with the weak 3-category BiCat.
The work presented in the present thesis does not address n-categories for $n > 3$ and $n \neq \infty$, so we avoid choosing and presenting in details a definition of weak n-categories.

3.3.3. *Interlude: 2-Vector spaces.* An important example of higher category is that of *2-Vector spaces*, which are a categorification of the notion of k-modules, where k is a field. There are several candidates in the literature which could be considered as an adequate generalization of vector spaces, in the sense that they are symmetric monoidal bicategories[23] $\mathcal{C}$ with the property that $\mathrm{Hom}_{\mathcal{C}}(1,1) \simeq \mathrm{Vect}_k$. We will present some of them for illustrative purpouses, referring to Appendix A in [**6**] for a thorough discussion.
In a certain sense, one of the most general notion of 2-Vector space is provided by Cauchy complete categories [**67**]. Recall that a k-linear category is a category such that the Hom sets are provided with the structure of a k-module, in such a way that the composition of morphisms is linear[24].
A *Cauchy complete category over k* is a k-linear category which admits finite direct sums and where any idempotent splits. We can then consider the bicategory $2\mathrm{Vect}_k$ given by Cauchy complete categories over k, linear functors and natural transformations. Notice that $2\mathrm{Vect}_k$ is actually a 2-category, since the composition of functors is strictly associative.
A different flavour is provided by LinCat_k, the 2-category of finite abelian linear categories, right-exact linear functors, and natural transformations. A *finite* abelian linear category $\mathcal{C}$ is an abelian linear category where all the Hom k-modules are finite dimensional, every object has a finite length, $\mathcal{C}$ has enough projectives, and finitely many isomorphism classes of simple objects.
Finally, we can consider the 2-category KV_k of *Kapranov-Voevodsky vector spaces* [**43**], which is the 2-category of abelian finitely semisimple linear categories over k, right-exact functors, and natural transformations. Alternatively, KV_k is equivalent to the bicategory where the objects are elements of $\mathbb{N}$, the 1-morphisms are $n \times m$ matrices of finite-dimensional k-vector spaces, and the 2-morphisms are matrices of linear maps.

Remark 3.45. We can consider also the bicategory Alg_2 introduced in Example 3.44 as a notion of 2-Vector space. Indeed, $\mathrm{Hom}_{\mathrm{Alg}_2}(k, k)$ is the category of k-bimodules, hence of k-vector spaces.

[22] It is though equivalent to a *Gray category*, which is a certain type of semi-strict 3-category.
[23] See comment in Section 3.3.6.
[24] Notice that we do *not* require the k-modules to be finite dimensional.

3.3.4. *∞-categories.* Informally speaking, an ∞-category[25] is an n-category with $n = \infty$. After Section 3.3.2, it would seem that such a generalization is bound to be completely untractable. On the contrary, ∞-categories are usually easier to manage, since many techniques from topology, specifically homotopy theory, can be used, thus avoiding the whole task of dealing with coherence laws.
One of the reasons to develop a theory of ∞-categories is a proposal by Grothendieck in [**33**], where he suggested that homotopy theory should be a branch of higher category theory. Namely, let X be a topological space. We can think of assigning to X the following higher category $\Pi_{\leq\infty}(X)$, called the *fundamental ∞-groupoid*, which is informally defined as a higher category where

- Objects are points of X.
- A 1-morphism $f : x \to y$ is a (continuous) path from x to y.
- A 2-morphism between f and g is a homotopy h between the corresponding paths.
- A 3-morphism between h_1 and h_2 is a homotopy between homotopies.

  $\vdots$

where composition is given by composition of oriented paths and homotopies. From the wish list above, one of the desired properties of $\Pi_{\leq\infty}(X)$ is a consequence of the fact that paths and homotopies can be traversed in the opposite direction. Any (higher) morphism in $\Pi_{\leq\infty}(X)$ should then be invertible: we call such a higher category a *∞-groupoid*, since it generalizes the properties of an ordinary groupoid. According to [**33**], the whole homotopy type of X should be encoded in $\Pi_{\leq\infty}(X)$. Conversely, any ∞-groupoid should arise in this way, namely it should be "equivalent" to the fundamental ∞-groupoid of some topological space. These requirements are known as the *homotopy hypothesis*, and form a general principle in higher category theory: namely, any reasonable definition of ∞-category should satisfy the homotopy hypothesis. More precisely, the homotopy hypothesis states an equivalence of homotopy theories:

$$|-| : n\text{-groupoids} \Longleftrightarrow \left\{\begin{matrix}\text{Spaces with}\\ \pi_k = 0 \text{ for}\\ k > n\end{matrix}\right\} : \Pi_{\leq n}$$

where $|-|$ and $\Pi_{\leq n}$ are generalizations of the functors given by geometric realisation and the fundamental groupoid, respectively. Recall that a topological space X with $\pi_k(X) = 0$ for $k > n$ is called a homotopy *n-type*. When n goes to ∞, the homotopy hypothesis requires then that the homotopy type of a topological space is completely encoded in its fundamental ∞-groupoid.
An ∞-groupoid can be equivalently defined as an $(\infty, 0)$-category, i.e. a higher category with morphisms up to ∞, such that any morphism is invertible in the higher sense. More generally, one can define an (∞, n)-category as a higher category with morphisms up to ∞, but such that all j-morphisms are invertible for $j > n$.

[25]It has become a custom to denote *quasi-categories* with the term ∞-categories, in particular after [**53**]. We will not follow this convention in the present thesis.

Just for illustrative purpouses, in the following subsection we briefly illustrate a definition of (∞, n)-categories introduced by Rezk [**61**], and popularised by Lurie [**54**]. Other approaches to ∞-categories use simplicial categories, Segal categories, quasi-categories, operads. We refer the reader to [**7, 9**] for a survey of other definitions of ∞-categories, and how they compare.

3.3.5. *Complete n-fold Segal spaces.* A *0-fold Segal space* is defined to be just a topological space. This is consistent with the requirements coming from the homotopy hypothesis, namely that 0-fold Segal spaces should model $(\infty, 0)$-categories, i.e ∞-groupoids.
A *(1-fold) Segal space* is a simplicial space $X = X_\bullet$ such that for any $n, m \geq 0$ the map

$$X_{m+n} \to X_m \times^h_{X_0} X_n$$

is a weak homotopy equivalence. Notice that $\times^h$ denotes the homotopy pullback in topological spaces.

REMARK 3.46. Let $\mathcal{C}$ be a topological category, i.e. a category enriched in topological spaces. Then $X^{\mathcal{C}} := N(C)$ is a (1-fold) Segal space.

We can interpret a Segal space X as an $(\infty, 1)$-category as follows: the points of X_0 can be regarded as objects, while the topological space X_1 can be regarded as the $(\infty, 0)$-category of morphisms, i.e. the objects are 1-morphisms, the paths are 2-morphisms, and so on. More generally, the space X_n can be regarded as the $(\infty, 0)$-category of n-tuples of composable arrows.
To any Segal space X we can associate an ordinary category $\mathrm{H}^{(1)}(X)$, called the *homotopy category* of X, in the following way: the objects of $\mathrm{H}^{(1)}(X)$ are the points of X_0, and for any $x, y \in X_0$ we have

$$\mathrm{Hom}_{\mathrm{H}^{(1)}(X)}(x, y) := \pi_0 \left(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\} \right)$$

For any $x, y \in X_0$ we have the following canonical map

$$\{x\} \times_{X_0} X_1 \times_{X_0} \{y\} \to \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\} \to \pi_0 \left(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\} \right) = \mathrm{Hom}_{\mathrm{H}^{(1)}(X)}(x, y)$$

A point of $\{x\} \times_{X_0} X_1 \times_{X_0} \{y\}$, which we can regard as a morphism f from x to y, is called *invertible* if its image through the map above is an invertible morphism in $\mathrm{H}^{(1)}(X)$.
Denote by X_1^{inv} the subset of X_1 consisting of invertible elements. Notice that the degeneracy map $X_0 \to X_1$ factors through X_1^{inv}.
A Segal space is said to be *complete* if the map $X_0 \to X_1^{inv}$ is a weak equivalence.
An $(\infty, 1)$-category is a complete Segal space.

An n-fold Segal space is an n-fold simplicial space $X = X_{\bullet,\dots,\bullet}$ such that

i) For every $1 \leq i \leq n$, and every $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geq 0$ the simplicial space

$$X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

is a Segal space.

ii) For every $1 \leq i \leq n$ and every $k_1, \ldots, k_{i-1} \geq 0$

$$X_{k_1,\ldots,k_{i-1},0,\bullet,\ldots,\bullet}$$

is essentially constant[26].

A n-fold Segal space X is complete if for every $1 \leq i \leq n$ and every $k_1, \ldots, k_{i-1} \geq 0$ we have that the simplicial space

$$X_{k_1,\ldots,k_{i-1},\bullet,0,\ldots,0}$$

is a complete Segal space.

In the case $n = 2$, we can show how a complete 2-fold Segal space can be interpreted essentially as a bicategory. Namely, we can first "fatten" a globular diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

to a diagram of the type

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The vertices are objects, i.e. elements of $X_{0,0}$, the horizontal arrows are elements of $X_{1,0}$, and the vertical arrows are elements of $X_{0,1}$, which are essentially identities because of condition ii) in the definition of n-fold Segal spaces. Finally, the arrow filling the face can be thought of as a 2-morphism, i.e. an element in $X_{1,1}$.

Example 3.47. An important example of $(\infty, 1)$-category is provided by chain complexes of k-modules $\mathrm{Ch}(k)$. Informally speaking, the objects of $\mathrm{Ch}(k)$ are chain complexes of k-modules, the 1-morphisms are maps of chain complexes, the 2-morphisms are chain homotopies between chain maps, and so on.

Remark 3.48. In the framework of complete n-fold Segal spaces, the fundamental ∞-groupoid $\Pi_{\leq\infty}(X)$ can be *defined* to be the 0-fold Segal space given by the topological space X itself.

3.3.6. *Comments on algebraic structures in higher categories.* The notions of tensor categories and their modules discussed in 3.1 generalize to n-categories and to (∞, n)-categories. In the former case, for $n < 3$ one can give explicit definitions of a monoidal structure, etc. by directly mimicing the diagrammatics for ordinary categories; see [**62**] for details. For the case of (∞, n)-categories a different approach is required, depending on the model used; we refer to [**55**] for details. In the rest of the present work, we will follow the principle that all the notions in categorical algebra can be appropriately lifted to the context of (weak) n-categories and (∞, n)-categories

[26]An m-fold simplicial space X is essentially constant if there is a weak homotopy equivalence $Y \to X$, with Y constant.

as long as we are careful in aknownledging the additional data that equations and commutative diagrams needs to hold.

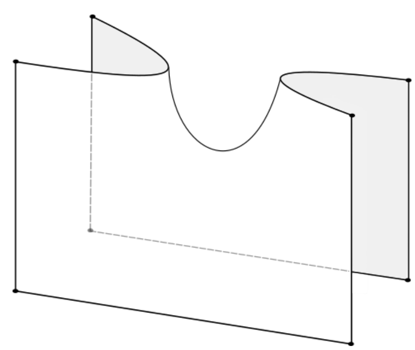
3.4. Extended TQFTs. The definition of a (functorial) d-dimensional Topological Quantum Field Theory presented in Section 3.2 can be generalized, or *extended*, in two possible directions, in order to attach data to manifolds of dimension $\leq d$-2 or/and to the moduli space of diffeomorphisms of d-dimensional manifolds. Proposals for such generalizations have originally appeared in Lawrence [**49**], Freed [**22**,**23**] and Baez and Dolan [**3**].

3.4.1. *Extending down and up.* We can first consider an n-category of oriented cobordism $\mathrm{Cob}_n^{or}(d)$ which can be informally described as the higher category where

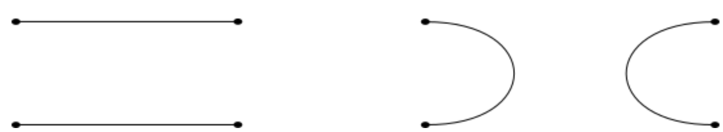
- the objects are closed oriented $(d-n)$-dimensional manifolds;
- the 1-morphisms are $(d-n+1)$-dimensional oriented cobordisms between closed manifolds;
- the 2-morphisms are $(d-n+2)$-dimensional manifolds with corners which are oriented cobordisms between cobordisms between closed manifolds;

  $\vdots$
- the n-morphisms are d-dimensional manifolds with corners which are oriented cobordisms between cobordisms ... between cobordisms between closed manifolds up to diffeomorphisms fixing the orientation and the boundary data,

and where composition of morphisms is given by gluing cobordisms along the common boundaries.
To show how higher morphisms in $\mathrm{Cob}_n^{or}(d)$ look like, we can consider the case $d = 2, n = 2$. The following 2-dimensional manifold[27] with corners



can be regarded as a 2-morphism between the following 1-dimensional manifolds with border



[27]We have omitted to depict orientations for simplicity.

which in turn are 1-morphisms between the 0-dimensional manifold

and itself.

Let $\mathcal{C}$ be a symmetric monoidal n-category. An n-extended TQFT in dimension d with values in $\mathcal{C}$ is a symmetric monoidal functor

$$Z : \mathrm{Cob}_n^{or}(d) \to \mathcal{C}$$

In the following, we will denote $\mathrm{Cob}_n^{or}(n)$ as Cob_n^{or}. In this case, a n-dimensional TQFT with values in $\mathcal{C}$ is called *fully extended.*

REMARK 3.49. As for the ordinary cobordism category, one can consider extended cobordism categories $\mathrm{Cob}_n^{\mathcal{X}(d)}(d)$ with tangential structures.

One of the features of extended n-dimensional TQFTs is that the partition function of the theory evaluated on a n-dimensional closed manifold M can be computed by decomposing M in pieces whose boundaries have strata which are submanifolds of M of codimension higher than 1, and hopefully easier to manage. This can be regarded as a form of *locality*, and it is the main concept behind the Corbordism Hypothesis (see Sect. 3.4.2).

EXAMPLE 3.50. An example of extended 3d TQFT is provided by Dijkgraaf-Witten theory, as shown in [**23**], and more recently in [**57**]. Namely, given a finite group G, it can be regarded as an extended TQFT valued in $2\mathrm{Vect}_k$ which assigns to the circle $\mathrm{Rep}(D(k[G]))$, the category of representations of the Drinfel'd double of $k[G]$, with k a field of characteristic 0.

REMARK 3.51. Examples of extended 3-dimensional TQFTs include those producing invariants of the Reshethikhin-Turaev type [**60**, **68**] obtained from a modular tensor category.

As it should be clear from the example above, tracking all the data in $\mathrm{Cob}_n^{or}(d)$ for higher n and d is a complicated combinatorial problem, which faces the same general issues as weak higher categories discussed in 3.3.2. Indeed, a direct construction is only available for $n = 2$, producing a symmetric monoidal bicategory $\mathrm{Cob}_2^{or}(d)$; we refer to [**62**] for details on the construction.

As mentioned above, an n-extended TQFT carries information about manifolds of higher codimension, so one could hope that a complete classification is within reach. In some cases, this is indeed possible. In [**62**], a complete classification[28] of oriented 2-dimensional fully extended theories with values in Alg_2 is given in terms of symmetric separable Frobenius algebras (not necessarily commutative). In [**5**, **6**] it has

[28] This is an instance of the Cobordism Hypothesis.

been shown that extended oriented 3-dimensional TQFTs are classified by modular tensor objects in symmetric monoidal bicategories.

In [**54**], Lurie introduced a different way of extending the category $\mathrm{Cob}^{or}(n)$, given as follows. We can consider the $(\infty,1)$-category $\mathrm{Cob}^{or,\infty}(n)$ defined informally as follows

- The objects are oriented closed (n-1)-dimensional manifolds.
- The 1-morphisms are n-dimensional oriented cobordisms.
- The 2-morphisms are diffemorphisms of oriented cobordisms preserving the boundary data.
- The 3-morphisms are isotopies of diffeomorphisms.

  $\vdots$

The effect of using $\mathrm{Cob}^{or,\infty}(n)$ instead of $\mathrm{Cob}^{or}(n)$ is visible when the target category is a genuine ∞-category, e.g. $\mathrm{Ch}(k)$. This is for instance the case of Topological Conformal Field Theories, as in [**15**].
We can combine the two extensions, down and up, in a single (∞,n)-category, which we will denote with Bord_n^{or}. The (∞,n)-category Bord_n^{or} (together with some variations) was introduced by Lurie in [**54**], where a sketch of its construction using (n-fold) complete Segal spaces as a model for (∞,n)-categories was given. In [**10**] a fully fledged construction of Bord_n^{or}, together with a symmetric monoidal structure, was given.

3.4.2. *The Cobordism Hypothesis.* First formulated by Baez and Dolan in [**3**], the Cobordism Hypothesis is a statement concerning a presentation of Cob_n^{fr}, the n-category of fully extended framed cobordisms. They conjectured that Cob_n^{fr} is equivalent to the *free stable weak n-category with duals on a single object.* The key element behind the Cobordism Hypothesis is the notion of *adjointable morphism* in a higher category, which we illustrate in the special case of a bicategory.
Let $\mathcal{C}$ be a bicategory. A 1-morphism $f : x \to y$ is *left adjoint* to a 1-morphism $g : y \to x$ if there exist 2-morphisms $e : f \circ g \to \mathrm{id}_y$ and $c : \mathrm{id}_x \to g \circ f$ such that the compositions

$$f = f \circ \mathrm{id}_x \xrightarrow{\mathrm{id}\times c} f \circ g \circ f \xrightarrow{e\times\mathrm{id}} \mathrm{id}_y \circ f = f$$

and

$$g = \mathrm{id}_y \circ g \xrightarrow{c\times\mathrm{id}} g \circ f \circ g \xrightarrow{\mathrm{id}\times e} g \circ \mathrm{id}_x = g$$

are identities. In this case we will say that g is a *right adjoint* to f, and we will refer to e and c as the evaluation and coevaluation of the adjunction, respectively.

Example 3.52. Let $\mathcal{C}$ be the bicategory with a single object, and with Vect_k as the monoidal category of morphisms. Then a 1-morphism which is adjointable corresponds to a finite dimensional vector space. The definition of an adjointable morphism is then intimately related to that of a dualizable object in a monoidal category.

Example 3.53. Let F be a 1-morphism in Cat^2, the 2-category of small categories. Then F is a left adjoint as 1-morphism in Cat^2 exactly when it admits a right adjoint functor.

The definition above can be generalized to a n-category, in which case we will require that the compositions above are identities only up to equivalence. This can be made precise by considering the bicategory $\mathrm{H}^{(k)}(\mathcal{C})$ associated to k-morphisms in an n-category $\mathcal{C}$ where objects are k-morphisms, 1-morphisms are $k+1$-morphisms, and 2-morphisms are equivalence classes of $k+2$-morphisms. We will then say that a k-morphism in $\mathcal{C}$ is left adjointable if it admits a left adjoint in $\mathrm{H}^{(k-1)}(\mathcal{C})$.
We will say that a symmetric monoidal n-category $\mathcal{C}$ is k-fully dualizable for $1 \leq k \leq n-1$ if every object of $\mathcal{C}$ admits a dual object, and if any j-morphism with $j \leq k$ admits a left and right adjoint. For $k = n-1$, we will say that $\mathcal{C}$ is *fully dualizable*. Any symmetric monoidal n-category $\mathcal{C}$ admits a maximal fully dualizable subcategory, which we denote by $\mathcal{C}^{fd}$. The objects of $\mathcal{C}^{fd}$ are called *fully dualizable objects of* $\mathcal{C}$.

EXAMPLE 3.54. When $\mathcal{C} = \mathrm{Vect}_k$ the category of fully dualizable objects is given by the full subcategory of *finite dimensional vector spaces.*

EXAMPLE 3.55. For $\mathcal{C} = \mathrm{Alg}_2$, the bicategory of fully dualizable objects is given by separable *finite* dimensional algebras, finite dimensional bimodules, and intertwiners.

Finally, denote with $\mathcal{K}(\mathcal{C})$ the maximal ∞-subgroupoid of $\mathcal{C}$, which we can obtain by discarding all non invertible morphisms.
Since symmetric monoidal funtors preserve the property of being fully dualizable, the Cobordism Hypothesis can be formulated as following:

(Cobordism Hypothesis) *There is an equivalence of weak n-categories*

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}_n^{fr}, \mathcal{C}) \simeq \mathcal{K}(\mathcal{C}^{fd})$$

for every symmetric monoidal n-category $\mathcal{C}$, induced by the evaluation on the n-framed point.

In other words, the Cobordism Hypotheses provides a complete classification for fully extended framed n-dimensional TQFTs in terms of fully dualizable objects of $\mathcal{C}$. Moreover, it states that they form an n-groupoid, hence a topological space which is a homotopy n-type.
The Cobordism Hypothesis in the n-categorical form stated above has eluded a proof for $n > 3$, mainly due to the complex diagrammatic behind the definition of a weak n-category mentioned in 3.3.2.
A recent leap forward in the proof of the Cobordism Hypothesis is due to Lurie, who reformulated it in term of (∞, n)-categories, and sketched a proof in [**54**]. Namely, we have

(Cobordism Hypothesis: ∞-version) *There is an equivalence of ∞-groupoids*

$$\mathrm{Fun}_{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \simeq \mathcal{K}(\mathcal{C}^{fd})$$

for every symmetric monoidal (∞, n)-category $\mathcal{C}$, induced by the evaluation on the n-framed point.

The reformulation in terms of ∞-categories is crucial for the proof of the Cobordism Hypothesis: the proof involves indeed techniques regarding diffeomorphism groups of manifolds, and uses an induction argument which is not possible in the n-categorical setting.

In [**54**], the case of cobordisms with G-structures is discussed as well. First, it is argued that the topological group $O(n)$ acts homotopically on Bord_n^{fr}: namely, for $k \leq n$, a k-morphism in Bord_n^{fr} is a k-manifold W equipped with an n-framing, i.e. an isomorphism $\alpha : \underline{\mathbb{R}}^{n-k} \oplus TW \simeq \underline{\mathbb{R}}^n$. For any element $g \in O(n)$ we can obtain a new framing by composing α with the isomorphism $\underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$ induced by g, which is referred to as a "framing rotation". This action finally induces an $O(n)$ action on $\mathrm{Fun}_{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C})$, and by the Cobordism Hypothesis on $\mathcal{K}(\mathcal{C}^{fd})$. In the following, $\big(\mathcal{K}(\mathcal{C}^{fd})\big)^{hG}$ denotes the ∞-groupoids of homotopy fixed points of a G-action (see Section 3.7 for more details).
Consider now a group G, and a homorphism $\rho : G \to O(n)$. As discussed in 3.2.1, this gives rise to a tangential structure with $\mathcal{X}(n) = BG$. We denote with Bord_n^G the (∞, n)-category of cobordisms equipped with the tangential structure $\mathcal{X}(n)$.

(Cobordism Hypothesis: G-structure version) *There is an equivalence of ∞-groupoids*

$$\mathrm{Fun}_{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \simeq \Big(\mathcal{K}(\mathcal{C}^{fd})\Big)^{hG}$$

for every symmetric monoidal (∞, n)-category $\mathcal{C}$, induced by the evaluation on the n-framed point.

REMARK 3.56. The homotopy action of $O(n)$ on Bord_n^{fr} remains rather mysterious, in the sense that an explicit description is not available in the literature. This is due to the fact that the theory of group actions on higher categories is still not well developed. The works [**17**,**38**,**39**,**63**] make progress in understanding the $O(n)$-action and the Cobordism Hypothesis in low dimensions.

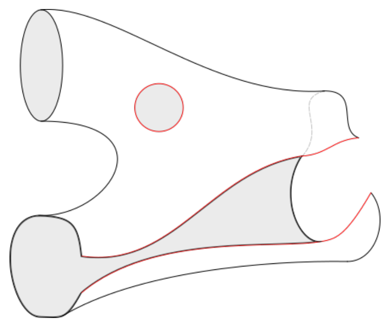
3.5. Boundary conditions and defects. An additional generalization of the notion of TQFT involves considering manifolds with decorated boundaries and decorated stratifications, which are called *boundary conditions* and *defects*. The former case is known in the literature as an *open/closed TQFT*. In the following sections, we discuss some basic aspects of such generalizations, referring the reader to the vast literature on the topic, as for instance [**12**, **31**, **46**, **48**, **54**, **56**].

3.5.1. *Boundary conditions.* The category of n-cobordisms $\mathrm{Cob}^{or}(n)$ has closed $(n-1)$-manifolds as its object. We can consider a "bigger" cobordism category by considering also $(n-1)$-manifolds with boundaries as *objects*. To this aim, we need an appropriate notion of a n-dimensional cobordism between oriented $(n-1)$-manifolds with possibly non-empty boundary. See [**34**] for additional details.
Let Σ_1 and Σ_2 be oriented $(n-1)$-dimensional manifolds. A cobordism with *constrained boundary* between Σ_1 and Σ_2 is a n-dimensional oriented manifold M such that

i) $\partial M = (\partial_{in} M \amalg \partial_{out} M) \cup \partial_c M$
ii) We have $\partial_{in} M \simeq \Sigma_1$ and $\partial_{out} M \simeq \overline{\Sigma_2}$

iii) $(\partial_{in}M \amalg \partial_{out}M) \cap \partial_c M = \partial(\partial_c W) = \partial(\partial_{in}M \amalg \partial_{out}M)$

The first condition tells that the boundary of M might have a component which is not incoming or outcoming, and which we term "constrained", while the third condition implies that the boundary of the constrained part must lie in the incoming or outcoming part of the cobordism. Notice that in general M is a manifold with corners. Morever, we can decorate the connected components of $\partial\Sigma_1$,$\partial\Sigma_2$ and $\partial_c M$ with a set of labels, and ask that such choices are made in the obvious compatible way. We avoid making such a choice notationally explicit, implying that such a choice must be made throughout.
An example of a 2-dimensional cobordism M with constrained boundary is the following



where $\Sigma_1 = S^1 \amalg I$ and $\Sigma_2 = I$, and where the red part is $\partial_c M$. Notice that also the intervals come equipped with a red decoration at their endpoints. Other examples of 2-dimensional cobordisms with constrained boundaries are the following

We can now consider the category $\mathrm{Cob}^{or,\partial}(n)$ of n-dimensional cobordisms with constrained boundaries, where objects are compact $(n-1)$-dimensional manifolds and morphisms are diffeomorphisms classes of n-dimensional cobordisms with constrained boundaries[29]. Composition is given by gluing cobordism along the part of the boundary which is not constrained. It is a symmetric monoidal category under disjoint union of manifolds.
A n-dimensional TQFT with boundary conditions Z with values in a symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal functor

$$Z : \mathrm{Cob}^{or,\partial}(n) \to \mathcal{C}$$

[29]As mentioned above, we assume a choice of a set of labels or *colors* L.

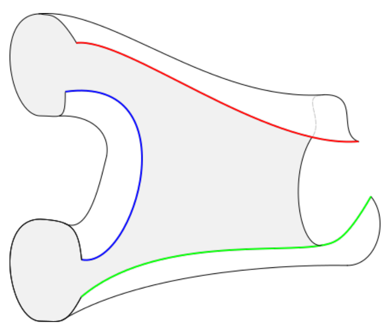
Since any ordinary cobordism between closed manifolds can be trivially regarded as a cobordism with constrained boundaries, we have a non-full embedding $\mathrm{Cob}^{or}(n) \hookrightarrow \mathrm{Cob}^{or,\partial}(n)$. Given an ordinary n-dimensional TQFT $\tilde{Z}$ with values in $\mathcal{C}$, we can then ask when is it possible to *extend* it to a n-dimensional TQFT with boundaries, namely when we have a commutative diagram of monoidal functors

$$\begin{tikzcd} \mathrm{Cob}^{or}(n) \arrow[r, hook] \arrow[dr, "Z"'] & \mathrm{Cob}^{or,\partial}(n) \arrow[d, "\tilde{Z}"] \\ & \mathcal{C} \end{tikzcd}$$

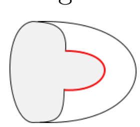
This is in general a hard problem to address: the extensions might be many, or none at all, in which case we will say that an obstruction is present.
In the case $n = 2$, TQFTs valued in $\mathrm{Vect}_{\mathbb{C}}$ with boundary conditions have been at great extent investigated in [**48**,**56**]. In particular, in [**56**] it is argued that the set of of labels or colors L is equipped with a Calabi-Yau category structure. Namely, assume that L is a countable set, whose elements we denote with $a, b, \ldots$, and assume we are given a 2-dimensional TQFTs Z with boundary conditions labelled by L valued in $\mathrm{Vect}_{\mathbb{C}}$. For any a, b in L we can then consider the vector space

$$\mathrm{Hom}_L(a, b) := \mathcal{O}_{ab}$$

where $\mathcal{O}_{ab}$ is the vector space that Z assigns to the interval $I = [0, 1]$, where 0 and 1 are decorated by a and b, respectively. The following cobordism



provides then a linear map $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$, where we have indicated the label a, b and c with the color green, blue, and red, respectively. The gluing properties of cobordism with constraints guarantee that we can interpret this linear map as the composition law for the category having for objects the elements of L, and Hom spaces defined above. Moreover, for each a, the following cobordism



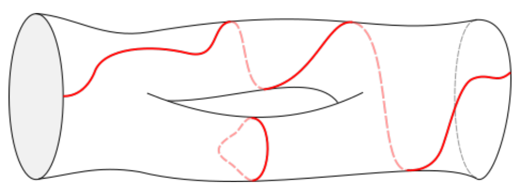
provides a linear map $\mathcal{O}_{aa} \to \mathbb{C}$ which is then shown to be nondegenerate, and which induces a perfect pairing $\mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathbb{C}$.
One of the achievements of [**56**] is also an identification of the category associated to

L with a category of representation theoretic flavour. Namely, it is shown that the category of boundary conditions for a 2-dimensional TQFT Z which assigns to the circle a *semisimple* Frobenius algebra A is equivalent to the category of right modules over a non-necessarily commutative semisimple Frobenius algebra B equipped with an algebra isomorphism $\mathcal{Z}(B) \simeq A$, where $\mathcal{Z}$ denotes the center.

Due to the complexity of n-manifolds with $n > 2$, structural results as in [**48**, **56**] are hard to achieve. Recently, in [**44**] Kapustin argued using a physics heuristic that to a n-dimensional TQFTs with boundary conditions one should be able to assign a $(n-1)$-category of boundary conditions. The work in [**31**, **32**, **46**] give major contributions supporting this proposal in the case of 3-dimensional extended TQFTs.

Remark 3.57. We can consider cobordisms with constrained boundaries equipped with tangential structures which are more general than orientation. Of course, such tangential structures have to satisfy compatibility conditions on the corners of the cobordism at hand.

3.5.2. *Defects.* Defects, in particular topological ones, have been intensively investigated as a way to get hindsight in the structure of Quantum Field Theories. Equivalently, topological defects arise when we consider TQFTs on *stratified manifolds.* Particularly interesting are defects of codimension 1, which provide interfaces to connect different TQFTs defined on a single manifold. An example of a codimension 1 defect for a 2d TQFT is the following



Formally, we consider a category of *decorated* stratified cobordisms, $\mathrm{Cob}_D^{or,def}(n)$, where D denotes the decoration data. A n-dimensional TQFT with defects taking value in a symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal functor

$$Z : \mathrm{Cob}_D^{or,def}(n) \to \mathcal{C}$$

In the following, we illustrate briefly the definition of the *undecorated* stratified cobordism category, referring to [**12**, **13**] for the decorated case.
Recall that a closed stratified n-manifold Σ is a manifold equipped with a finite ascending filtration $\{\mathcal{F}_n\}$ such that

i) $\Sigma_j := \mathcal{F}_j/\mathcal{F}_{j-1}$ is a j-manifold, possibly empty, for all $1 \leq j \leq m$. The connected components of Σ_j, denoted by Σ_j^α, are called the *j-strata*, and are equipped with an orientation;
ii) For all $\Sigma_i^\alpha, \Sigma_j^\beta$ such that $\Sigma_i^\alpha \cap \overline{\Sigma}_j^\beta \neq \emptyset$, we have that $\Sigma_i^\alpha \subset \overline{\Sigma}_j^\beta$;
iii) The total number of strata is finite.

We have also the notion of a morphism between stratified n-manifolds. Namely, it is a smooth map $f : \Sigma \to \Sigma'$ of manifolds such that $f(\Sigma_j) \subset \Sigma'_j$, and such that it preserves the orientation of each strata.
Similarly, we can define stratified manifolds with boundary and their morphisms; see [**12**,**13**]. An example of a stratified manifold with boundary is given by the figure above: it has a single 2-stratum, two 1-strata, and no 0-straum.
We can then construct a symmetric monoidal category $\mathrm{Cob}^{or,def}(n)$ in analogy with the ordinary category of n-dimensional cobordisms, and define a *stratified* TQFT as a functor from $\mathrm{Cob}^{or,def}(n)$ to a symmetric monoidal category $\mathcal{C}$.
When the defects, i.e. the strata, are decorated, it is expected that any stratified TQFT equips the decoration data with the structure of a higher category. Namely, to each n-dimensional stratified TQFT one should assign a weak n-category where objects are (the decoration data for) codimension 0 defects, 1-morphisms are codimension 1 defects, and so on. For $n \leq 3$ such constructions have been discussed in the literature, as in [**11**,**12**,**16**] and references therein. In Section 3.8 we will briefly discuss the relevance of defects in applications to Physics.

EXAMPLE 3.58. Given a modular[30] tensor category $\mathcal{C}$, the TQFT constructed by Reshethikin-Turaev in [**60**,**68**] can be regarded as an oriented 3d TQFT with defects of codimension 2. In this description, the invariant of a knot L in a 3-manifold M is the partition function of the TQFT evaluated on M equipped with the stratification arising from the embedding. Notice that the knot is decorated by an object of $\mathcal{C}$.

3.5.3. *Interlude: the Cobordism Hypothesis with singularities.* In the previous sections we have discussed two possible extensions of the notion of a TQFT involving higher category theory. On one side, we can consider extended cobordism categories, while on the other side we can consider manifolds with additional structures, like decorated boundaries and stratifications. It is then natural to consider the combination of these two extensions, namely to consider extended cobordism categories of manifolds with stratifications. At the time of writing the present work, there is no explicit construction of such cobordism categories, even if their relevance is widely aknowledged. Neverthless, in [**54**] Lurie proposes the notion of a fully extended cobordism category of *manifolds with singularities.* We will not attemp to replicate his definition here, since it is quite involved, but observe that in [**54**] it is claimed that it subsumes the notion of boundary conditions and manifolds with defects[31]. Moreover, a Cobordism Hypothesis *with singularities* is stated: in the case of boundary conditions, Lurie argues that any symmetric monoidal ∞-functor

$$Z : \mathrm{Bord}_n^{fr,\partial} \to \mathcal{C}$$

is given by the choice of an object A in $\mathcal{C}^{fd}$ *and* the choice of a morphism $\mathbf{1} \to A$ in $\mathcal{C}^{fd}$.

[30] For simplicity, we will assume that $\mathcal{C}$ has no central charge. Otherwise, one has to consider manifolds equipped with a p_1-structure.

[31] See Section 4.3, Example 4.3.22, 4.3.23 in [**54**].

The Cobordism Hypothesis with singularities has been used prevalently in [**21**] to obtain results concerning TQFTs with anomalies. For an application to Representation Theory, see [**8**]

3.6. Invertible TQFTs. A particular class of field theories which has enjoyed recent attention is that of *invertible TQFTs*. Invertible TQFTs have been used to give a geometric description of anomalies [**21**, **24**], and they have been proposed as a tool to describe aspects of Condensed Matter Physics [**25**, **26**] (see also Section 3.8). Let $\mathcal{C}$ be a symmetric monoidal category, and let $\mathcal{X}(n)$ be a n-dimensional tangential structure[32]. A TQFT $Z : \mathrm{Cob}^{\mathcal{X}(n)}(n) \to \mathcal{C}$ is said to be *invertible* if it factors through the underlying Picard groupoid of $\mathcal{C}$, namely it fits in a commutative diagram

$$\begin{tikzcd} \mathrm{Cob}^{\mathcal{X}(n)}(n) \arrow[r, "Z"] \arrow[dr, dashed] & \mathcal{C} \\ & \mathrm{Pic}(\mathcal{C}) \arrow[u] \end{tikzcd}$$

of symmetric monoidal functors.
An invertible TQFT assigns invertible data at each level, i.e. to objects and morphisms.
Functors to groupoids always factor through the *groupoid completion*, which we now illustrate. Given a category $\mathcal{C}$, its groupoid completion is a pair $(|\mathcal{C}|, i)$, where $|\mathcal{C}|$ is a groupoid, and where $i : \mathcal{C} \to |\mathcal{C}|$ is a functor satisfing the following universal property: for any groupoid $\mathcal{G}$ and any functor $f : \mathcal{C} \to \mathcal{G}$ there exists a unique functor $\tilde{f} : |\mathcal{C}| \to \mathcal{G}$ that makes the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[dr, "f"'] & {|\mathcal{C}|} \arrow[d, dashed, "\tilde{f}"] \\ & \mathcal{G} \end{tikzcd}$$

commutes.
One can show that for any category $\mathcal{C}$ its groupoid completion does indeed exists: intuitively, one can construct it by "adding" an inverse to each morphism. Moreover, one can show that the groupoid completion of a symmetric monoidal category is again symmetric monoidal.
For any invertible n-dimensional TQFT with tangential structure $\mathcal{X}(n)$ we have then a commutative diagram

$$\begin{tikzcd} \mathrm{Cob}^{\mathcal{X}(n)}(n) \arrow[r, "Z"] \arrow[d] & \mathcal{C} \\ {|\mathrm{Cob}^{\mathcal{X}(n)}(n)|} \arrow[r, "\tilde{Z}"] & \mathrm{Pic}(\mathcal{C}) \arrow[u] \end{tikzcd}$$

[32] Recall that a tangential structure is actually a fibration $\mathcal{X}(n) \to BO(n)$.

Since $\mathrm{Cob}^{\mathcal{X}(n)}(n)$ has duals for each object, the groupoid completion $|\mathrm{Cob}^{\mathcal{X}(n)}(n)|$ is itself a Picard groupoid[33]. The re-writing above is very powerful: indeed, Picard groupoids can be regarded as *spectra*, and any symmetric monoidal functor between Picard groupoids can be regarded as a map of spectra. This allows to lift the powerful techniques of stable homotopy theory to the realm of Topological Quantum Field Theories. We will not dwelve into such a connection to stable homotopy theory in the present thesis: we will content ourselves with providing some suggestive comments in Section 3.8.

EXAMPLE 3.59. For $n = 2$ and any $\lambda \in \mathbb{C}^*$, we can consider the following 2-dimensional invertible oriented TQFT with values in $\mathrm{Vect}_{\mathbb{C}}$. To each closed 1-dimensional manifold S it associates the vector space $\mathbb{C}$, and to any cobordism Σ it associates the linear map given by multiplying with $\lambda^{\chi(\Sigma)-\#}$, where $\chi(\Sigma)$ denotes the Euler number of Σ once all the boundary components have been "filled", and $\#$ denotes the total number of boundary components. The local properties of the Euler number of surfaces guarantee the compatibility under gluing of cobordisms.

The notion of an m-extended invertible TQFT is obtained by substituting the category $\mathrm{Cob}^{\mathcal{X}(n)}(n)$ with its m-extended counterpart $\mathrm{Cob}_m^{\mathcal{X}(n)}(n)$. For extended TQFTs, invertibility imposes strong conditions on the TQFT. Indeed, in [**64**] it is shown that an extended n-dimensional TQFT taking value in a $(\infty, 2)$-category $\mathcal{C}$ is invertible as soon as the value assigned to the $(n-1)$-dimensional torus is invertible in $\mathcal{C}$.

3.7. Group actions on categories. In Section 3.4.2 we have argued that the Cobordism Hypothesis for TQFTs with G-structures requires studying the homotopy fixed points of a G-action on a topological space, namely the core of the ∞-groupoid of fully dualizable objects $\mathcal{K}(\mathcal{C}^{fd})$. Let G be a topological group. Recall that the space of homotopy fixed points of a G-action on a topological space X is given by

$$X^{hG} := \mathrm{Maps}_G(EG, X)$$

where EG is a[34] contractible G-space.

The definition of homotopy fixed points described above requires that we first obtain a topological space X from $\mathcal{K}(\mathcal{C}^{fd})$ together with a G-action, and then compute the space of homotopy fixed points X^{hG}. We obtain back an ∞-groupoid by considering $\Pi_{\leq\infty}(X^{hG})$. We can ask then if there is a way to describe homotopy fixed points of a group action *directly* in a categorical formalism. This is discussed in [**39**] for bicategories, together with a definition of homotopy actions of topological groups. In the following, we discuss topological group actions on categories.

Let G be a group. A compact way to describe the action of G on a set X is to consider a functor

$$\rho^X : \mathrm{B}G \to \mathrm{Set}$$

where $\mathrm{B}G$ is the category with a single object $\star$, and $\rho^X(\star) = X$. Unravelling the definition, we see that ρ^X provides for each g in G an automorphism $\rho^X(g)$ of X,

[33]A dualizable object in a symmetric monoidal groupoid is invertible.

[34]Notice that X^{hG} is defined only up to homotopy.

such that $\rho^X(gh) = \rho^X(g)\rho^X(h)$.
The definition above is powerful enough to be easily generalized. Indeed, we can consider other categories, like Top, for actions on topological spaces.
Given a G-action ρ^X, we can consider the limit[35] $\lim\ \rho^X$ of the diagram provided by ρ^X itself. It is not difficult then to show that

$$X^G \simeq \lim\ \rho^X$$

where X^G denotes the ordinary fixed point subset of X.
The definitions above can be naturally generalized to higher categories. Indeed, we can regard the category $\mathrm{B}G$ as a 2-category with only identities 2-morphisms. An action of a group G on a category $\mathcal{C}$ is then a 2-functor

$$\rho^{\mathcal{C}} : \mathrm{B}G \to \mathrm{Cat}$$

such that $\rho^{\mathcal{C}}(\star) = \mathcal{C}$. Similar we can consider an arbitrary bicategory as a target for our 2-functor.
Notice that the above definition generalizes two aspect of group actions. On one side, we are not asking for the 2-functor to be strictly associative: namely, the action property holds only *coherently*, in the sense that for any $g, h \in G$, $\rho^{\mathcal{C}}(gh)$ will only be isomorphic to $\rho^{\mathcal{C}}(g)\rho^{\mathcal{C}}(h)$, and the isomorphisms satisfy coherence equations. On the other hand, for any $g \in G$, we ask that $\rho^{\mathcal{C}}(g)$ is only an equivalence of categories, and not an isomorphism. In a precise sense, this generalize the notion of a *homotopy coherent action*: namely, if we consider the 2-category Top^h given by topological spaces, continuous maps, and equivalence classes of homotopies, then any 2-functor $\rho^X : \mathrm{B}G \to \mathrm{Top}^h$ induces a homotopy action[36] on X. The converse, though, does not hold in general.
We have still room for an additional layer of generalization: we can take into account the topology of G. Namely, we can define a *topological* G-action on a category as a 2-functor

$$\rho^{\mathcal{C}} : \mathrm{B}\Pi_{\leq 1}(G) \to \mathrm{Cat}$$

such that $\rho^{\mathcal{C}}(\star) = \mathcal{C}$. Recall that $\Pi_{\leq 1}(G)$ denotes the fundamental groupoid of G: since G is a group, $\Pi_{\leq 1}(G)$ is a strict monoidal category. Notice that only the topological information up to the fundamental group of G is involved in the definition.
Whenever the group involved has a topology, we will consider topological actions.
It is natural at this point to associate to an action $\rho^{\mathcal{C}}$ the category

$$\lim\ \rho^{\mathcal{C}}$$

obtained as a *2-limit*, i.e. a (weak) limit in Cat. We will denote such a category with $\mathcal{C}^{hG}$, which is referred to as *the category of homotopy fixed points* of $\rho^{\mathcal{C}}$. For a finite group G, one can prove that the category $\rho^{\mathcal{C}}$ is equivalent to the *G-equivariantization of* $\mathcal{C}$ discussed in [**19**]; see [**37**] for details.

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[35]Recall that Set possesses all limits and colimits.

[36]A *homotopy action* of a group G on a topological space X is a homomorphism $G \to \mathrm{Aut}_{\mathrm{H(Top)}}(X)$, where H(Top) denotes the homotopy category of topological spaces.

REMARK 3.60. Let X be a topological space which is a 1-type, regarded as an object in Top^h. Then given a homotopy coherent G-action ρ^X on X the topological space corresponding to $\lim \rho^X$ is homotopy equivalent to X^{hG}. This is in general not the case if X is not a 1-type.

3.7.1. *To ∞ and beyond.* In the previous Section we have considered topological actions on categories. This is actually an example of topological actions on objects in ∞-categories. Namely, given a topological group G and an $(\infty, 1)$-category $\mathcal{C}$, a topological homotopy action of G on an object X in $\mathcal{C}$ can be defined as an ∞-functor

$$\rho^X : \mathrm{B}\Pi_{\leq\infty}(G) \to \mathcal{C}$$

such that $\rho^X(\star) = X$.
By the homotopy hypothesis, we should be able to encapsulate *all* the homotopy properties of G. This is indeed the case: namely, if we regard Top as an $(\infty, 1)$-category, then we have that $\lim \rho^X$ is indeed (homotopy) equivalent to X^{hG}.

REMARK 3.61. Topological actions on objects of $(\infty, 1)$-categories as defined above should induce by *truncation* the constructions mentioned in the previous Section. Neverthless, the full details of how this works are not available in the literature at the time of writing, since there is a lack of an *equivariant* homotopy hypothesis for n-types.

3.8. Applications to Condensed Matter Theory. In this Section, we will very briefly illustrate how some of the ideas surrounding TQFTs and homotopy theory have found applications to aspects of Condensed Matter theory, more specifically to the description of topological phases of matter. We refer the reader to [**69**] for an introduction to such topics, which we treat here in a very cavalier way.
One source for the appearence of TQFTs and the related categorical machinery is lattice gauge theory. A *quantum lattice gauge theory* on a n-manifold M consists roughly of a triangulation where the $(n-1)$-cells are decorated with some labels, e.g. spin-configurations. A given decoration can be considered as a *field state*. The *state space* $\mathcal{H}$ is then the vector space of complex valued functions over the field states (equipped with some measure). A *quantum Hamiltonian* is a linear operator $H : \mathcal{H} \to \mathcal{H}$. Under the assumption that H is bounded below, we can consider the *space of ground states* as the eigenspace associated to the lowest eigenvalue of H. An example of lattice gauge theory in 2d is given by the *toric code*, whose Hamiltonian is given by

$$H = \sum_{f \in \text{faces}} H_f + \sum_{v \in \text{vertices}} H_v$$

where

$$H_f = \frac{1}{2}(1 - \prod_{e \in \partial f} \sigma_e^x) \qquad H_v = \frac{1}{2}(1 - \prod_{e : v \in \partial e} \sigma_e^z)$$

where σ_e^x and σ_e^z are Pauli matrices.
It may happen that the space of ground states does not depend on the lattice structure, in such a way that it corresponds to the vector space a TQFT Z^H would assign to M. The TQFT Z^H is then referred to as the *low-energy effective field theory* of

the system, which is said to be a *topological state of matter*. This is indeed the case of the toric code, for instance. The Levin-Wen model [**52**] provides another important example. Given a n-dimensional TQFT, we can then ask if it is the low energy effective field theory for some lattice system in $(n-1)$ dimensions: this is often a very complex question.

Another important appearance of TQFTs is in describing aspects of the *quantum Hall liquid*, which is a phase of matter obtained by subjecting a two-dimensional system of electrons to low temperatures and to a strong magnetic field. In particular, it is believed that the *fractional* Quantum Hall liquid is described at low energy by the Witten-Chern-Simons TQFT. Moreover, physicists have argued that the excitation modes in the fractional Quantum Hall state give rise to emergent quasi-particles called *anyons*, whose dynamics turns out to be described by a modular tensor category (see [**58**] for a review).

Recently, techniques from stable homotopy theory have been used in computing invariants of *topological insulators*. In [**25**], Freed formulated a *short-range entanglement hypothesis*: namely, he proposed that the low-energy/long-range effective field theory describing gapped[37] systems with short-range entanglement is a fully extended TQFT which is moreover invertible[38]. As we have argued in Section 3.6, invertible TQFTs can be regarded as maps of spectra. In [**26**], the authors apply this ansatz to study, among other things, phases of topological insulators: by an educated guess on the spectrum describing the Picard ∞-groupoid of the coefficient category, they are able to recover, for instance, the Kane-Mele invariant [**42**]. This has spawned a renewed interested in the application of Topology, and in particular homotopy theory to Physics.

Finally, we want to mention the relevance of boundary conditions and defects in the study of topological phases. In [**47**], the authors study a class of two-dimensional lattice models which describe phases of matter with boundaries which are gapped, together with the bulk. They also discuss *domain walls*, which are "lines" in the lattice where the system can undergo a transition of phase. In the low energy limit, these can be interpreted as a stratification, and the system is then described by a stratified TQFT, as discussed in Section 3.5.

4. Main results

4.1. Paper "Bicategories for Boundary Conditions and for Surface Defects in 3d-TFT" [31], with J. Fuchs and C. Schweigert. Given a three-dimensional manifold with boundaries M, we consider in the bulk the Reshetikhin-Turaev (RT) TQFT associated to a modular tensor category $\mathcal{C}$. We then investigate which mathematical structure encodes possible topological boundary conditions, and which obstructions are present to extend the 3d TQFT from the bulk to the boundary. Similarly, we consider a three-dimensional manifold M equipped with a surface

[37] A quantum system is called *gapped* if its Hamiltonian is bounded below, and the lowest eigenvalue is an isolated point in the spectrum

[38] There are additional assumptions, like *unitarity*, which we prefer to ignore in the present discussion.

defect S separating two regions of M where two RT-type TQFTs are defined, whose modular tensor categories are $\mathcal{C}_1$ and $\mathcal{C}_2$. We then ask what possible topological defect conditions are allowed. Following a physical heuristic, we show that the existence of a topological boundary condition is described by a central functor which lifts the trivialisation of $\mathcal{C}$ in the Witt group of fusion categories. This is equivalent to providing a fusion category $\mathcal{W}$ together with a braided equivalence $\mathcal{C} \simeq \mathcal{Z}(\mathcal{W})$, where $\mathcal{Z}(W)$ denotes the Drinfeld center. The bicategory of boundary conditions can then be described as the bicategory of module categories over $\mathcal{W}$. Similary, a topological surface defect is described by a fusion category $\mathcal{A}$ together with a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{rev} \simeq \mathcal{Z}(\mathcal{A})$. We compare our results with those obtained in [**46**] and [**47**]. More precisely, we show that boundary conditions for abelian toroidal Chern-Simons theory are in bijective correspondence with Lagrangian algebras in the modular tensor category associated to a distinguished quadratic group (Λ, q) which encodes the data needed to define the classical theory. This result is due to the fact that equivalence classes of indecomposable $\mathcal{W}$-modules categories correspond to Lagrangian algebras in $\mathcal{Z}(\mathcal{W})$.
Given a surface defect S between a TQFT described by a modular tensor category $\mathcal{C}$ and itself, we assign to any Wilson line separting S and the transparent defect $T_{\mathcal{C}}$ a (Morita class of) special symmetric Frobenius algebra in $\mathcal{C}$, recovering the construction in [**45**]. This can be regarded as a construction of the gluing conditions for rational CFTs discussed in [**30**].

4.2. Paper "A Geometric Approach to Boundaries and Surface Defects in Dijkgraaf-Witten theories" [32], with J. Fuchs and C. Schweigert. We consider extended 3d Dijkgraaf-Witten(DW) theory on manifolds with decorated boundaries and surface defects. Given a manifold M with such a decorated structure, and given a finite group G, we consider the groupoid of relative (G, H)-bundles over M, with $H \subset G$ a subgroup. We then apply the quantization bifunctor described in [**57**] to obtain a category of generalized Wilson lines. In particular, we explicitely compute the category that corresponds to the interval I, where each of the end points is decorated with a choice of a subgroup H of G, and a 2-group cocyle θ . We show that this category is indeed equivalent to the category expected from the general machinery developed in [**31**], namely a category of functors between module categories over the category of ω-twisted G-graded vector spaces Vect_G^ω. This constitute a concrete and highly non-trivial test for the results in [**31**], and a first step in the complete construction of extended 3d DW-TQFT as a stratified Topological Quantum Field Theory.

4.3. Paper "On the Brauer Groups of Symmetries of Abelian Dijkgraaf-Witten theories" [29], with J. Fuchs, J. Priel and C. Schweigert. We consider extended Dijkgraaf-Witten (DW) theory in three-dimensions based on an abelian group A, and investigate the notion of gauge-theoretic and algebraic symmetries. Namely, algebraic symmetries correspond to the automorphisms of the modular tensor category that DW-TQFT Z^A associates to S^1, while gauge-theoretic symmetries correspond to the automorphisms of the stack of A-bundles equipped with a trivialised 2-gerbe, and to genuine quantum symmetries. The category $\mathcal{C} := Z^A(S^1)$ is equivalent to $D(A)$-mod, the category of modules over the Drinfeld double $D(A)$ of A. Since A is abelian, we have that the equivalence classes of simple objects of $\mathcal{C}$ are in correspondence with elements of $A \oplus A^*$. The group of algebraic symmetries of the theory is then given by $\mathrm{EqBr}(\mathcal{C})$, the group of braided autoequivalences of $\mathcal{C}$. We introduce the notion of a *transmission functor*, namely the functor a stratified 3d TQFT assigns to a cilinder with a circle embedded "in the middle", and decorated with an invertible defect condition. A transmission functor provides a genuine TQFT realisation of a symmetry. By using fusion of surface defects, we obtain a group of equivalence classes of such transmission functors, and we show that such a group is isomorphic to $\mathrm{EqBr}(\mathcal{C})$. Moreover, once we realize $\mathrm{EqBr}(\mathcal{C})$ as $O_q(A \oplus A^*)$, the group of automorphisms of $A \oplus A^*$ preserving the quadratic form q associated to $\mathcal{C}$, we give a presentation of $\mathrm{EqBr}(\mathcal{C})$ in terms of generators. We find that any element in $O_q(A \oplus A^*)$ can be obtained as the composition of three types of symmetries: *universal kinematic symmetries*, corresponding to the automorphisms of the stack of A-bundles Bun^A; *universal dynamic symmetries*, corresponding to automorphisms of the trivialised 2-gerbe on Bun^A; *electric-magnetic dualities*, which are given by symmetries exchanging A and A^*, and which can be interpreted as the groups of electric and magnetic charges.

4.4. Paper "Boundary Conditions for Topological Quantum Field Theories, Anomalies and Projective Modular Functors" [21], with D. Fiorenza. We consider extended TQFTs in the framework of Lurie, and study boundary conditions and their relation to invertible anomalies. We give a definition of *anomalous* TQFT and *anomaly field theory*, which have appeared in a different guise in the literature [**28**, **66**]. We define TQFTs *with moduli level* m as a symmetric monoidal ∞-functors from $\mathrm{Bord}_n^{\mathcal{X}(n)}$ to $(n+m)$-Vect, and we consider in detail the case $m = 1$, which describe anomalous theory. We define the anomaly field theory for a fully extended n-dimensional TQFT as a functor $W : \mathrm{Bord}_n^{\mathcal{X}(n)} \to \mathrm{Pic}((n+1)\text{-Vect})$, and show that an anomalous TQFT is given by a (lax) section of the anomaly field theory. We introduce the notion of n-character for ∞-groups, and show that any (invertible) anomaly field theory gives rise to a 2-character. We show that the homotopy fixed points of such a 2-character correspond to anomalous field theories. We then consider $(n+1)$-extended TQFTs with boundary, and by using dimensional reduction, we show how boundary conditions give rise to n-dimensional anomalous theories. By using the Cobordism Hypothesis with singularities, we argue that the converse is true in the fully extended case, namely that any n-dimensional fully extended anomalous theory can be obtained via dimensional reduction by a boundary condition for the anomaly field theory. This suggestively points towards a description of 3d Chern-Simons theory in its Reshetikhin-Turaev form as a boundary field theory for 4d Crane-Yetter theory, which is expected to hold. We end the paper with a conjecture on this topic.

4.5. Paper "Central Extensions of Mapping Class Groups from Characteristic Classes" [20], with D. Fiorenza and U. Schreiber. We consider

(X, ξ)-framed manifolds as defined by Lurie in [**54**], which are a generalizations of tangential structures, and which we refer to as ρ-structures. We consider smooth ∞-groupoids, which can be regarded as sheaves of Kan complexes over the site of cartesian spaces. This allows to describe in a very natural way a ρ-structure on an n-dimensional manifold M as a morphism in the slice topos $\mathbb{H}^\infty/\mathcal{B}GL(N)$, where $\mathbb{H}^\infty$ denotes an enriched version of the ∞-topos of smooth groupoids, and $\mathcal{B}GL(N)$ denotes the *smooth* stack of $GL(n)$-bundles. We show that any characteristic class gives rise to a ρ-structure by considering its smooth homotopy fiber construction. We study the automorphism ∞-group of a given ρ-structure on M, from which we can extract the ∞-group of diffeomorphisms $\mathrm{Diff}^\rho(M)$ of M preserving the ρ-structure, which is *not* a subgroup of the diffeomorphisms of M, but rather a pullback construction. We then prove a theorem classifying the possible extensions of $\mathrm{Diff}^\rho(M)$ along morphisms of ρ-structures. We show how such extensions give rise extensions of the mapping classing group of ρ-preserving diffeomorphisms, and that in some cases these extensions are short exact sequences. We then study the case of orientation, spin, and in particular p_1-structures, where we show that one obtains a $\mathbb{Z}$-central extention of the oriented mapping class group of a 2-manifold M, which was proposed by Segal in [**65**] as one of the main ingredients in the construction of a projective modular functor.

4.6. Paper "Frobenius Algebras and Homotopy Fixed Points of Group Actions on Bicategories" [38], with J. Hesse and C. Schweigert. We consider the action of a topological group G on a bicategory $\mathcal{C}$ by regarding G as a tricategory via its fundumental 2-groupoid. This allows to incorporate a notion of continuity of the action, and it is a novelty in our definition of group action. On the other hand, the fact that the target is a bicategory means that we deal with a homotopy coherent group action. We construct the bicategory of homotopy fixed points $\mathcal{C}^G$ as a limit of the action, which comes naturally equipped with a forgetting bifunctor $\mathcal{C}^G \to \mathcal{C}$. We also give a direct explicit construction of $\mathcal{C}^G$. Motivated by the Cobordism Hypothesis in 2 dimensions, we specialise to the case $G = SO(2)$. As suggested by Lurie in [**54**], for any symmetric monoidal bicategory $\mathcal{C}$ there is an $SO(2)$-action on $\mathcal{K}(\mathcal{C}^{fd})$, the core of fully dualizable objects of $\mathcal{C}$, which should be generated by the Serre automorphism. An interesting case is provided by bicategories where the Serre automorphism is trivialisable, given rise to trivialisable $SO(2)$-action: this is the case for Alg_2^{fd}, the fully dualizable objects in the bicategory of algebras, bimodules, and intertwiners, as shown in [**17**]. We then study trivial $SO(2)$-actions on arbitrary bicategories, and show that the bicategory of homotopy fixed point is far from trivial. When we specialize these results to the case $\mathcal{C} = \mathrm{Alg}_2$, we obtain that $\mathcal{C}^{SO(2)}$ is the bicategory of semi-simple Frobenius algebras Frob, recovering thus the results in [**17, 27**]. We also consider the case of the trivial $SO(2)$-action on $\mathcal{C} = \mathrm{2Vect}$, more precisely KV-vector spaces: we show in this case that $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of finitely-semisimple Calabi-Yau categories. By using that the representation bifunctor $\mathrm{Rep} : \mathrm{Alg}_2 \to \mathrm{2Vect}$ is $SO(2)$-equivariant, we then obtain that Calabi-Yau categories classify fully extended oriented 2d TQFTs taking value in 2Vect, which had previoulsy enjoyed the status of a "folk theorem".

4.7. Paper "The Serre Automorphism via Homotopy Actions and the Cobordism Hypothesis for Oriented Manifolds" [39], with J. Hesse. We extend the techniques of [**38**] to give a geometric description of the Serre automorphism on an arbitrary symmetric monoidal bicategory, which was predicted by Lurie in [**54**]. We consider a generators and relations presentation of the fully extended bicategory of two-dimensional framed cobordisms $\mathbb{F}_{cfd}$ developed in [**59**]. We construct an $SO(2)$-action using such a presentation, and show that the induced action on a symmetric monoidal bicategory $\mathcal{C}$ via the Cobordism Hypothesis coincides with the action generated by the Serre automorphism. We then explicitely construct the bicategory of homotopy $SO(2)$-fixed points for an *arbitrary* $SO(2)$-action on a symmetric monoidal bicategory $\mathcal{C}$. We show that evaluation on the point induces an equivalence of bicategories between $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)}$ and $\mathcal{K}(\mathcal{C}^{fd})^{SO(2)}$, which can be regarded as a proof of the Cobordism Hypothesis for 2d oriented TQFTs taking value in $\mathcal{C}$. We then consider the case of 2d invertible field theories. We show that the Serre automorphism is *monoidal*, and that the $SO(2)$-action it generates on $\mathcal{K}(\mathcal{C}^{fd})$ induces an action on the Picard 2-groupoid $\mathrm{Pic}(\mathcal{C})$. We give sufficient conditions on $\mathcal{C}$ for the $SO(2)$-action on $\mathrm{Pic}(\mathcal{C})$ to trivialize, and argue that in this case any framed 2d fully extended TQFT can be lifted to an oriented one. These conditions are satisfied for Alg_2 and 2Vect.

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CHAPTER 2

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Bicategories for Boundary Conditions and for Surface Defects in 3-d TFT

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Abstract: We analyze topological boundary conditions and topological surface defects in three-dimensional topological field theories of Reshetikhin-Turaev type based on arbitrary modular tensor categories. Boundary conditions are described by central functors that lift to trivializations in the Witt group of modular tensor categories. The bicategory of boundary conditions can be described through the bicategory of module categories over any such trivialization. A similar description is obtained for topological surface defects. Using string diagrams for bicategories we also establish a precise relation between special symmetric Frobenius algebras and Wilson lines involving special defects. We compare our results with previous work of Kapustin-Saulina and of Kitaev-Kong on boundary conditions and surface defects in abelian Chern-Simons theories and in Turaev-Viro type TFTs, respectively.

1. Introduction

An insight gained in recent years in the study of quantum field theories is that interesting effects are captured when allowing for codimension-one defects, i.e. interfaces between regions on which two different theories are living. Depending on the application, it is sensible to impose specific kinds of conditions on such interfaces; for instance, in integrable field theories, integrable defects, as considered e.g. in [DMS, BCZ], are naturally of interest. In two-dimensional rational conformal field theories, the study of totally transmissive defect lines (see e.g. [Wa,PZ,BDDO,FFRS2]) has produced structural information about non-chiral symmetries and Kramers-Wannier-type dualities. It has also become apparent that boundaries and defects are close relatives.

In this paper we concentrate on topological quantum field theories in three dimensions (TFTs), specifically on theories that include (compact) Chern-Simons theories. While for the latter subclass a Lagrangian formulation is available, in the general case considered here we work within the combinatorial approach of Reshetikhin and Turaev

[RT] type, which associates a TFT to any semisimple modular tensor category. This includes TFTs of Turaev-Viro type as well.

In the three-dimensional situation the simplest codimension-one structures are surfaces that constitute either a defect surface or a two-dimensional boundary. It is worth stressing that here the term boundary refers, as in [KS1], to a brim at which 'the three-dimensional world ends'. Such brim boundaries must in particular not be confused with the cut-and-paste boundaries that commonly occur (see e.g. [Tu]) in these theories. Boundaries of the latter kind arise when a three-manifold is cut into more elementary three-manifolds with boundaries; accordingly, their function is to account for locality and to allow for sewing, or cut-and-paste, procedures. Both classes of boundaries are geometric boundaries of three-manifolds, but cut-and-paste boundaries come with additional local (chiral) degrees of freedom and can support vector spaces of conformal blocks. In contrast, brim boundaries need not involve any of those structures. Note that the distinction between two different kinds of boundaries is not specific to three dimensions. In two dimensions, in the discussion of so-called open-closed theories, such a distinction is standard; see e.g. [Mo, Sect. 3], where intervals corresponding to in- and out-going open strings are distinguished from "free boundaries" corresponding to the ends of an open string "moving along a D-brane."

Among the Reshetikhin-Turaev type theories there are in particular TFTs constructed from lattices, which have e.g. been prominent (see, for instance, [FCGK] for a detailed discussion) in the discussion of universality classes of quantum Hall systems. Thus in this particular case, our results may have applications to topological interfaces with gapped excitations between two quantum Hall fluids. Our discussion applies, however, to arbitrary semisimple modular tensor categories and does not rely on any specific aspects of lattice models.

There is no guarantee that for a given quantum field theory a consistent defect or boundary condition exists at all. In particular there can be theories that make perfect sense in the bulk, but cannot be consistently extended to the boundary. On the other hand, if consistent codimension-one defects, or boundary conditions, do exist, they will typically not be unique. It is then natural to study interfaces between such lower-dimensional regions as well, i.e. interfaces of codimension two. In our case of three-dimensional topological field theories, these are generalized Wilson lines. (In other words, the brim boundaries we consider can contain such Wilson lines. In contrast, this is not possible for cut-and-paste boundaries. On the other hand, bulk Wilson lines can end on either kind of boundary – in the case of cut-and-paste boundaries, they end on marked points.) Again, such generalized Wilson lines need not exist, but again, if they do exist, then they need not be unique, so that the game can be repeated one step further.

Hereby we arrive at a four-layered structure: At the top level, we associate a topological field theory of Reshetikhin-Turaev type to each three-dimensional part of a stratified three-dimensional manifold. For two-dimensional parts we deal with physical boundaries or with two-dimensional defects, for which we must choose a boundary condition, respectively, in the same spirit, an additional datum that describes the type, or 'color' of the defect. Such a datum has been called a *surface operator* in [KS1]; we prefer the term *surface defect* instead. The third layer of structure consists of one-dimensional structures labeled by generalized Wilson lines that separate boundaries or surface defects. And finally, generalized Wilson lines can fuse and split at point-like defects, which may be interpreted as local field insertions and constitute the fourth layer of structure.

The basic questions we are addressing in this paper can thus be posed as follows:

(1) *Given a three-dimensional region with non-empty boundary for which the TFT of Reshetikhin-Turaev type in the interior is labeled by a modular tensor category $\mathcal{C}$, what are the data describing the types of* topological boundary conditions *on the boundary*?
(2) *Given two three-dimensional regions separated by a two-dimensional interface, for which the TFTs of Reshetikhin-Turaev type in the two regions are labeled by modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, what are the data describing the types of* topological surface defects *on the interface*?

The key in our analysis of these issues is the following process: a Wilson line in the three-dimensional bulk can be moved "adiabatically" into the boundary or into a defect surface. This has already been studied in [KS1, Sect. 5.2], and a similar process in two dimensions has been considered in [DKR, Sect. 4.1]. A careful analysis of this process allows us to give a complete answer to both questions, including in particular a criterion for the existence of non-trivial solutions. The analysis yields in particular a model-independent generalization of results that have been obtained in [KS1] for abelian Chern-Simons theories using a Lagrangian description.

Our considerations involve mathematical ingredients that, to the best of our knowledge, have not been applied to Reshetikhin-Turaev type TFTs before. Many of them come from higher category theory, like aspects of fusion categories [ENO1,ENOM] and of braided fusion categories [DrGNO], and specifically the notions [DMNO] of central functors and of the Witt group of non-degenerate fusion categories. This group naturally generalizes the classical Witt group of lattices; it has been originally devised as a tool in the classification of modular tensor categories. Since some familiarity with such concepts is required for appreciating our analysis, we collect the pertinent mathematical background in Sect. 2.

Our results can be summarized as follows.

(1a) For a boundary adjacent to a three-dimensional region that is labeled by a modular tensor category $\mathcal{C}$, and thus with bulk Wilson lines given by $\mathcal{C}$ as well, the central information about a topological boundary condition a is contained in the process of moving Wilson lines to the boundary. It is mathematically described by a central functor $F_{\rightarrow a} : \mathcal{C} \rightarrow \mathcal{W}_a$, with $\mathcal{W}_a$ the fusion category of Wilson lines in the boundary with boundary condition a.
(1b) A careful distinction between the three-dimensional physics in the bulk and the two-dimensional physics in the boundary allows one to argue that the functor $F_{\rightarrow a}$ lifts to a (braided) equivalence $\widetilde{F}_{\rightarrow a} : \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$ between the category $\mathcal{C}$ of bulk Wilson lines and the Drinfeld center of the category $\mathcal{W}_a$ of boundary Wilson lines.
(1c) This equivalence implies that a topological boundary condition exists for a TFT labeled by the modular tensor category $\mathcal{C}$ if and only if the class of $\mathcal{C}$ in the Witt group of modular tensor categories is trivial. Put differently, topological boundary conditions exist if and only if the modular tensor category $\mathcal{C}$ is the Drinfeld center of a fusion category.
(1d) For fixed $\mathcal{C}$, the three-layered structure carried by the boundary conditions and their higher-codimension substructures is a bicategory. It naturally encodes e.g. the fusion of (generalized) Wilson lines.
This bicategory can be constructed from any single boundary condition described by a central functor $F_{\rightarrow a} : \mathcal{C} \rightarrow \mathcal{W}_a$ as the bicategory of module categories over the fusion category $\mathcal{W}_a$. The 1-morphisms of this bicategory – i.e. module functors – describe the possible Wilson lines (one-dimensional defects) on the boundary,

including their fusion. The 2-morphisms describe the possible junctions of Wilson lines.

(2) A similar analysis can be performed for surface defects separating TFTs that are labeled by modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$. There are now two different processes of moving bulk Wilson lines from either $\mathcal{C}_1$ or $\mathcal{C}_2$ into the defect surface with a fusion category $\mathcal{W}_d$ of defect Wilson lines. They yield two central functors, which can be combined into a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d)$. Here $\mathcal{C}_2^{\mathrm{rev}}$ is the modular category with reversed braiding as compared to $\mathcal{C}_2$, and $\boxtimes$ is the Deligne tensor product. Again this equivalence fully captures a surface defect. Thus topological surface defects exist if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are in the same Witt class. Again, once one defect is described by an equivalence $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d)$ of braided fusion categories, the bicategory of all topological surface defects separating $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by the bicategory of $\mathcal{W}_d$-modules.

The description of boundary conditions and surface defects in terms of module categories that is achieved in this paper allows for a rigorous treatment of related issues. For instance, we can show that all module functors appearing in our theory admit ambidextrous adjunctions, which brings the technology of string diagrams for bicategories to our disposal. This way we can e.g. provide mathematical foundations for the constructions in [KS2]; in particular we prove:

(3) To every (special) topological surface defect S separating a TFT labeled by the modular tensor category $\mathcal{C}$ from itself, string diagrams provide, for any Wilson line separating S and the transparent surface defect, an explicit construction of a special symmetric Frobenius algebra in $\mathcal{C}$. Different Wilson lines give Morita equivalent algebras; we realize the Morita context explicitly in terms of string diagrams.

Before proceeding to the main body of the text, a few further remarks seem to be in order:

- A TFT of Reshetikhin-Turaev type based on a Drinfeld center of a fusion category $\mathcal{A}$ is, by the results of [BK,TV], equivalent to a TFT of Turaev-Viro type based on $\mathcal{A}$. Topological boundary conditions for TFTs of Reshetikhin-Turaev type thus only exist if the TFT admits a Turaev-Viro type description.
- Not surprisingly, the description of boundary conditions and defects in three-dimensional theories is one step higher in the categorical ladder than for two-dimensional theories, e.g. two-dimensional CFTs, for which boundary conditions and defect lines form categories of modules and of bimodules, respectively.
  In fact one expects a relation of boundary conditions for the TFT based on the modular tensor category $\mathcal{C}$ and $\mathcal{C}$-module categories. And indeed, as we will explain in Sects. 3 and 4, respectively, the existence of a consistent fusion of bulk and boundary Wilson lines requires such a relation. However, *not* every $\mathcal{C}$-module category describes a topological boundary condition. Rather, the structure we present involves more stringent requirements that are fulfilled only by a subclass of $\mathcal{C}$-module categories. Analogous comments apply to topological surface defects.
- We describe surface defects and boundary conditions as specific objects of a bicategory, not just as isomorphism classes thereof. This opens up the perspective to obtain a vast extension of the entire Reshetikhin-Turaev construction to manifolds with substructures of arbitrary codimension. Here we will not delve into this issue further, but just mention that a first inspection indeed indicates that one can associate the appropriate vector spaces of conformal blocks to cut-and-paste boundaries

of such extended manifolds. Any such construction should respect the known relations between topological field theories of Turaev-Viro and of Reshetikhin-Turaev type and therefore be compatible with the kind of construction that is sketched in [KK].

- We obtain our results separately for boundary conditions and for surface defects. A comparison of the results shows that the two situations are related by a 'folding' procedure. We thus find a three-dimensional realization of the 'folding trick', which in two-dimensional conformal field theory is often invoked as a heuristic tool.
- We finally comment on surface defects separating $\mathcal{C}$ from itself. For the Deligne product $\mathcal{C}\boxtimes\mathcal{C}^{\rm rev}$ of any modular tensor category $\mathcal{C}$ there exists canonically a braided equivalence to the center of a fusion category, namely to the center of $\mathcal{C}$ itself, $\mathcal{C}\boxtimes\mathcal{C}^{\rm rev}\simeq\mathcal{Z}(\mathcal{C})$. Thus there exist topological surface defects separating the TFT labeled by $\mathcal{C}$ from itself. Among them there is in particular the *transparent*, or invisible, surface defect whose presence is equivalent to having no interface at all. It corresponds to $\mathcal{C}$ seen as a module category over itself. The generalized Wilson lines on the transparent surface defect are just the ordinary Wilson lines.

The rest of this paper is organized as follows. We start by providing some mathematical background information in Sect. 2; the reader already familiar with the relevant aspects of monoidal categories can safely skip this part. Afterwards we present details of our proposal for boundary conditions (Sect. 3) and surface defects (Sect. 4). In Sect. 5 we then use the relation between module categories and Lagrangian algebras to show that, in the specific case of abelian Chern-Simons theories, our analysis gives the same results as the Lagrangian analysis of [KS1]. We conclude in Sect. 6 with a model-independent study that extends the results of [KS2] about the relation between Frobenius algebras in a modular tensor category $\mathcal{C}$ and generalized Wilson lines separating the transparent surface defect for $\mathcal{C}$ from an arbitrary surface defect.

2. Mathematical Preliminaries

We start by summarizing some pertinent mathematical background. By $(\mathcal{C},\otimes_{\mathcal{C}},\mathbf{1},a_{\mathcal{C}},l_{\mathcal{C}},r_{\mathcal{C}})$ we denote a monoidal category with tensor product $\otimes_{\mathcal{C}}$, tensor unit $\mathbf{1}$, associativity constraint $a_{\mathcal{C}}$, and left and right unit constraints $l_{\mathcal{C}}, r_{\mathcal{C}}$ that obey the pentagon and triangle constraints. In our discussion we will, however, usually suppress the associativity and unit constraints altogether, as is justified by the coherence theorem. We work over a fixed ground field $\Bbbk$ that is algebraically closed and has characteristic zero. For definiteness we take $\Bbbk$ to be the field $\mathbb{C}$ of complex numbers, which is the case relevant for typical applications. All categories are required to be $\Bbbk$-linear and abelian.

As we are interested in generalizations of the Reshetikhin-Turaev construction, all categories will be finitely semisimple, i.e. all objects are projective, the number of isomorphism classes of simple objects is finite, and the tensor unit is simple. If such a category is also rigid monoidal and has finite-dimensional morphism spaces, it is called a *fusion category*. With some further structure, such categories encode Moore-Seiberg data of chiral conformal field theories. (Examples can be constructed from even lattices, see Sect. 5.) We are particularly interested in *braided* categories, i.e. monoidal categories $\mathcal{C}$ endowed with a natural isomorphism from $\mathcal{C}$ to $\mathcal{C}$ with the opposite tensor product (i.e. with a commutativity constraint) satisfying the hexagon axioms.

Objects U, V of a braided fusion category are said to *centralize* each other iff the monodromy $c_{U,V} \circ c_{V,U}$ is the identity morphism. For $\mathcal{D}$ a fusion subcategory of a braided fusion category $\mathcal{C}$, the *centralizer* $\mathcal{D}'$ of $\mathcal{D}$ is the full subcategory of objects of $\mathcal{C}$ that centralize every object of $\mathcal{D}$. A braided fusion category is called *non-degenerate* iff $\mathcal{C}' \simeq \mathcal{V}ect_{\Bbbk}$ [DMNO, Def. 2.1]; a braided fusion category is called *premodular* iff it is equipped with a twist (or, equivalently, with a spherical structure). A premodular category is *modular*, i.e. its braiding is maximally non-symmetric, iff it is non-degenerate [DrGNO, Prop. 3.7].

2.1. Module categories. Categorification of the standard notion of module over a ring yields the notion of a module category over a monoidal category. Similarly, the notion of a bimodule category is the categorification of the notion of a bimodule.

Definition 2.1. **(i)** *A (left)* **module category** *over a monoidal category* $(\mathcal{A}, \otimes_{\mathcal{A}}, \mathbf{1}, a_{\mathcal{A}}, l_{\mathcal{A}}, r_{\mathcal{A}})$ *or, in short, an* $\boldsymbol{\mathcal{A}}$**-module**, *is a quadruple* $(\mathcal{M}, \otimes, a, l)$, *where* $\mathcal{M}$ *is a* $\Bbbk$*-linear abelian category and* $\otimes\colon \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ *is an exact bifunctor, while* $a = (a_{U,V,M})_{U,V\in\mathcal{A},M\in\mathcal{M}}$ *and* $l = (l_M)_{M\in\mathcal{M}}$ *are natural families of isomorphisms* $a_{U,V,M}\colon (U \otimes_{\mathcal{A}} V) \otimes M \to U \otimes (V \otimes M)$ *and* $l_M\colon \mathbf{1} \otimes M \to M$ *that satisfy pentagon and triangle axioms analogous to those valid for a monoidal category.*[1]

(ii) *In the same spirit, for* $(\mathcal{A}_1, \otimes_{\mathcal{A}_1}, \mathbf{1}_1, a_{\mathcal{A}_1}, l_{\mathcal{A}_1}, r_{\mathcal{A}_1})$ *and* $(\mathcal{A}_2, \otimes_{\mathcal{A}_2}, \mathbf{1}_2, a_{\mathcal{A}_2}, l_{\mathcal{A}_2}, r_{\mathcal{A}_2})$ *monoidal categories, a* $\boldsymbol{\mathcal{A}_1}$-$\boldsymbol{\mathcal{A}_2}$**-bimodule category**, *or* $\boldsymbol{\mathcal{A}_1}$-$\boldsymbol{\mathcal{A}_2}$**-bimodule**, *is a tuple* $(\mathcal{X}, \otimes_1, a_1, l_1, \otimes_2, a_2, r_2, b)$, *where* $\mathcal{X}$ *is a* $\Bbbk$*-linear abelian category,*

$$\otimes_1\colon \quad \mathcal{A}_1 \times \mathcal{X} \to \mathcal{X} \qquad \textit{and} \qquad \otimes_2\colon \quad \mathcal{X} \times \mathcal{A}_2 \to \mathcal{X} \tag{2.1}$$

are bifunctors, while $a_1 = (a_{1;U,V,X})_{U,V\in\mathcal{A}_1,X\in\mathcal{X}}$, $l_1 = (l_{1;X})_{X\in\mathcal{X}}$ *and* $a_2 = (a_{2;X,U,V})_{U,V\in\mathcal{A}_2,X\in\mathcal{X}}$, $l_2 = (l_{2;X})_{X\in\mathcal{X}}$ *as well as* $(b_{U;X;V})_{U\in\mathcal{A}_1,V\in\mathcal{A}_2 X\in\mathcal{X}}$ *are natural families of isomorphisms* $a_{1;U,V,X}\colon (U \otimes_{\mathcal{A}_1} V) \otimes_1 X \to U \otimes_1 (V \otimes_1 X)$, $l_{1;X}\colon \mathbf{1}_1 \otimes_1 X \to X$, $a_{2;X,U,V}\colon X \otimes_2 (U \otimes_{\mathcal{A}_2} V) \to (X \otimes_2 U) \otimes_2 V)$, $l_{2;X}\colon X \otimes_2 \mathbf{1}_2 \to X$ *and* $b_{U;X;V}\colon (U \otimes_1 X) \otimes_2 V \to U \otimes_1 (X \otimes_2 V)$ *that satisfy pentagon and triangle axioms similar to those valid for a monoidal category.*[2]

Remark 2.2. *(i)* Very much like a ring is a left module over itself, any monoidal category $\mathcal{A}$ is naturally a module category over itself; we denote this 'regular' $\mathcal{A}$-module by $\mathcal{A}_{\mathcal{A}}$. Also, via $F \boxtimes A := F(A)$ every category $\mathcal{A}$ is a module over the monoidal category $\mathcal{E}nd(\mathcal{A})$ of endofunctors of $\mathcal{A}$.

(ii) Module categories over $\mathcal{A}$ can be described in terms of algebras in $\mathcal{A}$, i.e. objects A of $\mathcal{A}$ together with a multiplication morphism $m\colon A \otimes A \to A$ and a unit morphism $\eta\colon \mathbf{1} \to A$ that obey associativity and unit axioms. As usual one introduces a category mod-A of right A-modules in $\mathcal{A}$. One easily verifies that the functor $(U, M) \mapsto U \otimes M$ endows the category mod-A with the structure of a module category over $\mathcal{A}$ [Os1, Sect. 3.1]. Conversely, given a module category, algebras can be constructed in terms of internal Homs.

(iii) Algebras that are not isomorphic can yield equivalent module categories. In fact, there is a Morita theory generalizing the classical Morita theory of algebras over commutative rings.

(iv) An $\mathcal{A}$-module $\mathcal{M}$ is the same as a monoidal functor from $\mathcal{A}$ to the monoidal category $\mathcal{E}nd(\mathcal{M})$ of endofunctors of $\mathcal{M}$ [Os1, Prop. 2.2].

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1 For a complete statement of the axioms see e.g. [Os1, Sect. 2.3].

2 A complete statement of the axioms can e.g. be found in [Gr, Def. 2.10 & Prop. 2.12].

(v) We recall that for our purposes we assume all categories to be abelian categories enriched over the category of finite-dimensional complex vector spaces and to be finitely semisimple.

Along with module categories there come corresponding notions of functors and natural transformations.

Definition 2.3. **(i)** *A (strong)* **module functor** *between two $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{M}'$ is an additive functor $F\colon\ \mathcal{M}\to\mathcal{M}'$ together with a natural family $b=(b_{U,M})_{U\in\mathcal{A},M\in\mathcal{M}}$ of isomorphisms $b_{U,M}\colon\ F(U\otimes M)\to U\otimes F(M)$ that satisfy pentagon and triangle axioms analogous to those valid for a monoidal functor.*

(ii) *A* **natural transformation** *between two module functors is a natural transformation of $\Bbbk$-linear additive functors compatible with the module structure.*

(iii) *The corresponding notions for bimodule categories are defined analogously.*

There is also an obvious operation of *direct sum* of $\mathcal{A}$-modules: $\mathcal{M}\oplus\mathcal{M}'$ is the Cartesian product of the categories $\mathcal{M}$ and $\mathcal{M}'$ with coordinate-wise additive and module structure. An *indecomposable* $\mathcal{A}$-module is one that is not equivalent (as $\mathcal{A}$-modules, i.e. via a module functor) to a direct sum of two nontrivial $\mathcal{A}$-modules. Any $\mathcal{A}$-module can be written as a direct sum of indecomposable ones, uniquely up to equivalence.

2.2. Bicategories and Deligne products. Given a monoidal category $\mathcal{A}$, the collection of all $\mathcal{A}$-modules has a three-layered structure, consisting of $\mathcal{A}$-modules, module functors, and module natural transformations. This structure cannot be described any longer in terms of a category; we rather need the notion of a *bicategory*, which is pervasive in this paper. A bicategory has three layers of structure: objects, 1-morphisms and 2-morphisms. The composition of 1-morphisms is not necessarily strictly associative, but only up to 2-isomorphisms; if it *is* strictly associative, one calls the bicategory *strict* (or a 2-category). For 2-morphisms there are two different concatenations, referred to as vertical and horizontal compositions. For details about bicategories see e.g. [Ben].

A standard example for a strict bicategory is the one for which objects are small categories, 1-morphisms are functors and 2-morphisms are natural transformations. An example for a non-strict bicategory is the one whose objects are associative algebras, 1-morphisms are bimodules and 2-morphisms are bimodule maps. Here we are interested, for a given monoidal category $\mathcal{A}$, in its bicatgeory $\mathcal{A}$-$\mathcal{M}od$ of modules, having $\mathcal{A}$-modules as objects, module functors as 1-morphisms and natural transformations between module functors as 2-morphisms. Similarly, for any pair $(\mathcal{A}_1, \mathcal{A}_2)$ of monoidal categories there is the bicatgeory $\mathcal{A}_1$-$\mathcal{A}_2$-$\mathcal{B}imod$.

The universal property of the tensor product of vector spaces allows one to describe bilinear maps in terms of linear maps out of the tensor product. Similarly, the *Deligne tensor product* $\mathcal{C}_1\boxtimes\mathcal{C}_2$ [De, Sect. 5] of abelian categories provides a bijection between bifunctors $F\colon \mathcal{C}_1\times\mathcal{C}_2\to\mathcal{D}$ and functors $\hat F\colon \mathcal{C}_1\boxtimes\mathcal{C}_2\to\mathcal{D}$. If $\mathcal{C}_1=A_1$-mod is the category of (left, say) modules over a finite-dimensional $\Bbbk$-algbera A_1 and $\mathcal{C}_2=A_2$-mod, then $\mathcal{C}_1\boxtimes\mathcal{C}_2$ is equivalent to the category of modules over the $\Bbbk$-algebra $A_1\otimes_\Bbbk A_2$ [De, Prop. 5.5], and if $\mathcal{C}_1$ and $\mathcal{C}_2$ are semisimple with simple objects given by S_i and T_j, respectively, then $\mathcal{C}_1\boxtimes\mathcal{C}_2$ is semisimple as well, with simple objects given by $S_i\boxtimes T_j$.

A significant feature of bimodules over a ring is that they admit a tensor product. The Deligne product can be used in a similar way. Given, say, rings R_1, R_2 and R_3, the tensor product provides us with functors

$$\otimes_{R_2}:\quad R_1\text{-}R_2\text{-bimod}\times R_2\text{-}R_3\text{-bimod}\to R_1\text{-}R_3\text{-bimod} \tag{2.2}$$

describing ‘mixed’ tensor products. The Deligne tensor product categorifies this feature as well and provides bifunctors between bimodule categories. For details we refer to [EGNO, Sect. 1.46]. For a *commutative* ring R, the tensor product of two R-modules is again an R-module. Braided tensor categories are categorifications of commutative rings. Indeed, if $\mathcal{C}$ is a *braided* abelian monoidal category, then the Deligne tensor product endows the bicategory $\mathcal{C}$-$\mathcal{M}od$ with a monoidal structure.

Next we notice that for any $\Bbbk$-algebra A the space $\mathrm{End}_A(A_A)$ of module endomorphisms of A as a module over itself is isomorphic to $\mathrm{Hom}_{\Bbbk}(\Bbbk, A)$ and thus to A. This suggests to study the properties of the category $\mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$ of module endofunctors of $\mathcal{A}$ as a module category over itself. Since endofunctors can be composed, $\mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$ is a monoidal category. Moreover, we have the following categorified version of the classical isomorphism $\mathrm{End}_A(A_A) \cong A$ of algebras:

Proposition 2.4. *Let $\mathcal{A}$ be a $\Bbbk$-linear monoidal category. For any object $U \in \mathcal{A}$ denote by $F_U : \mathcal{A}_{\mathcal{A}} \to \mathcal{A}_{\mathcal{A}}$ the module endofunctor that acts on objects by tensoring with U from the left, $F_U(V) := U \otimes V$. Then the functor*

$$\begin{aligned} F_{\mathcal{A}}: \quad \mathcal{A} &\longrightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{A}_{\mathcal{A}}) \\ U &\longmapsto F_U \end{aligned} \tag{2.3}$$

is an equivalence of monoidal categories.

Proof. We first show that the functor

$$\begin{aligned} G_{\mathcal{A}}: \quad \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}}) &\longrightarrow \mathcal{A} \\ F &\longmapsto F(\mathbf{1}) \end{aligned} \tag{2.4}$$

is an essential inverse of $F_{\mathcal{A}}$. Indeed we have the chain of equalities $G_{\mathcal{A}} \circ F_{\mathcal{A}}(U) = G_{\mathcal{A}}(F_U) = F_U(\mathbf{1}) = U \otimes \mathbf{1} = U$, so that $G_{\mathcal{A}} \circ F_{\mathcal{A}} = Id_{\mathcal{A}}$. Conversely, for any $\varphi \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$ the functor $F_{\mathcal{A}} \circ G_{\mathcal{A}}(\varphi) \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$ acts on $U \in \mathcal{A}_{\mathcal{A}}$ as $(F_{\mathcal{A}} \circ G_{\mathcal{A}}(\varphi))(U) = F_{G_{\mathcal{A}}(\varphi)}(U) = \varphi(\mathbf{1}) \otimes U$. The unit constraint of the module functor φ then provides a natural isomorphism to the identity functor.

It remains to obtain tensoriality constraints for the functor $F_{\mathcal{A}}$. The equalities

$$\begin{aligned} & F_{\mathcal{A}}(U \otimes V)(W) = (U \otimes V) \otimes W \\ & \xmapsto{\alpha} U \otimes (V \otimes W) = F_{\mathcal{A}}(U)(F_{\mathcal{A}}(V)W) = (F_{\mathcal{A}}(U) \circ F_{\mathcal{A}}(V))(W) \end{aligned} \tag{2.5}$$

show that these are afforded by the associativity constraint $a_{\mathcal{A}}$ of $\mathcal{A}$. □

2.3. Drinfeld center and enveloping category. For algebras over fields, a very useful invariant of the Morita class of an algebra is the center. In our situation, i.e. for algebras in a monoidal category $\mathcal{A}$, a similar invariant is at hand which still is an algebra, albeit in a category different from $\mathcal{A}$, namely in the *Drinfeld center* $\mathcal{Z}(\mathcal{A})$. We recall the definition of the Drinfeld center: for $\mathcal{A}$ a monoidal category, the objects of the category $\mathcal{Z}(\mathcal{A})$ are pairs (U, e_U), where $U \in \mathcal{C}$ and e_U is a ‘half-braiding’, i.e. a functorial isomorphism $e_U : U \otimes - \xrightarrow{\simeq} - \otimes U$ satisfying appropriate axioms, see e.g. [Ka, Ch. XIII.4]. $\mathcal{Z}(\mathcal{A})$ has a natural structure of a braided monoidal category. The forgetful functor

$$\begin{aligned} \varphi_{\mathcal{A}}: \quad \mathcal{Z}(\mathcal{A}) &\to \mathcal{A} \\ (U, e_U) &\mapsto U \end{aligned} \tag{2.6}$$

is a tensor functor.

The *reverse* category of a braided monoidal category $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{rev}}$, is the same category with opposite braiding; if $\mathcal{C}$ is even a ribbon category, as in all our applications, we also endow it with the opposite twist. The Deligne product

$$\mathcal{C}^{\mathrm{e}} := \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \tag{2.7}$$

is a categorified version of the enveloping algebra $A^{\mathrm{e}} = A \otimes_{\Bbbk} A^{\mathrm{op}}$ of an associative algebra. Accordingly we call $\mathcal{C}^{\mathrm{e}}$ the *enveloping category* of $\mathcal{C}$. And in the same way as the category of A-bimodules can be described, as an abelian category, in terms of A^{e}-modules, the bicategory $\mathcal{C}_1$-$\mathcal{C}_2$-$\mathcal{B}imod$ is equivalent to the bicategory $(\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}})$-$\mathcal{M}od$.

Suppose now that the monoidal category $\mathcal{C}$ is already braided itself, with braiding c. Then the braiding provides a functor, actually a braided tensor functor, from $\mathcal{C}$ into its center $\mathcal{Z}(\mathcal{C})$ by $U \mapsto (U, c_{U,-})$. We also have a braided tensor functor $\mathcal{C}^{\mathrm{rev}} \to \mathcal{Z}(\mathcal{C})$, which is obtained by the opposite braiding: $U \mapsto (U, c_{-,U}^{-1})$. Using the universal property of the Deligne tensor product we combine the two functors into a tensor functor

$$\begin{aligned} G_{\mathcal{C}}: \quad \mathcal{C}^{\mathrm{e}} &\longrightarrow \mathcal{Z}(\mathcal{C}) \\ U \boxtimes V &\longmapsto (U \otimes V, e_{U \otimes V}), \end{aligned} \tag{2.8}$$

where

$$e_{U \otimes V}(W): \quad U \otimes V \otimes W \xrightarrow{id_U \otimes c_{W,V}^{-1}} U \otimes W \otimes V \xrightarrow{c_{U,W} \otimes id_V} W \otimes U \otimes V. \tag{2.9}$$

The functor $G_{\mathcal{C}}$ has a natural structure of a braided tensor functor. A braided monoidal category is called *factorizable* iff $G_{\mathcal{C}}$ is an equivalence of braided monoidal categories. Representation categories of finite-dimensional factorizable Hopf algebras in the sense of [Dr] are factorizable.

It is natural to ask under what condition the functor $G_{\mathcal{C}}$ is a braided equivalence. This is answered by the

Lemma 2.5 [Mü2,ENO1]. *For $\mathcal{C}$ a semisimple ribbon category, the functor $G_{\mathcal{C}}$ (2.8) is an equivalence between the center $\mathcal{Z}(\mathcal{C})$ and the enveloping category $\mathcal{C}^{e}$ if and only if $\mathcal{C}$ is a modular tensor category.*

Thus in particular in the context of the Reshetikhin-Turaev construction, which takes as an input a *modular* tensor category, the center and enveloping category of $\mathcal{C}$ are equivalent as braided categories, including their spherical structure.

For a braided category $\mathcal{C}$ the obvious functor $\mathcal{C}^{\mathrm{e}} \to \mathcal{C}$ factors through the center of $\mathcal{C}$: composing the functor $G_{\mathcal{C}}$ (2.8) with the forgetful functor, we obtain

$$\mathcal{C}^{\mathrm{e}} \to \mathcal{Z}(\mathcal{C}) \to \mathcal{C}. \tag{2.10}$$

Hereby $\mathcal{C}$ becomes a $\mathcal{C}^{\mathrm{e}}$-module, and any $\mathcal{C}$-module is turned into a $\mathcal{C}$-bimodule.

The following assertion shows again that it is appropriate to regard the Drinfeld center as a categorification of the center of an algebra:

Proposition 2.6 [ENO2, Thm. 3.1, Mü1, Rem. 3.18]. *Let $\mathcal{A}$ and $\mathcal{B}$ be fusion categories. Their centers $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are braided equivalent iff their bicategories $\mathcal{A}$-$\mathcal{M}od$ and $\mathcal{B}$-$\mathcal{M}od$ of module categories are equivalent.*

There is a close relation between module categories and the Drinfeld center [ENOM, Sect. 5.1]. For any indecomposable $\mathcal{A}$-module $\mathcal{M}$ over a fusion category $\mathcal{A}$, the category $\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})$ of $\mathcal{A}$-module endofunctors of $\mathcal{M}$ is a fusion category, and $\mathcal{M}$ can be regarded as a right $\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})$-module, and thus as an $\mathcal{A}\boxtimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})^{\mathrm{rev}}$-module. The $\mathcal{A}\boxtimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})^{\mathrm{rev}}$-module endofunctors of this module category can be identified [DMNO, Sect. 2.6] with the functors of tensoring with an object of the Drinfeld center $\mathcal{Z}(\mathcal{A})$ from the left, or, alternatively, with the functors of tensoring with an object of $\mathcal{Z}(\mathcal{E}nd_{\mathcal{A}}(\mathcal{M}))$ from the right. Comparing the two descriptions of these functors gives the following result:

Proposition 2.7 [Sc]. *For any module $\mathcal{M}$ over a fusion category $\mathcal{A}$ there is a canonical equivalence*

$$\mathcal{Z}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})) \tag{2.11}$$

of braided categories.

2.4. Central functors. In this brief subsection, we recall a notion that will enter crucially into our analysis of boundary conditions and defect surfaces.

Definition 2.8 [Bez, Sect. 2.1]. *A structure of a* **central functor** *on a monoidal functor $F: \mathcal{C} \to \mathcal{A}$ from a braided monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{A}$ is a natural family of isomorphisms*

$$\sigma_{U,V}: \quad F(U) \otimes V \xrightarrow{\cong} V \otimes F(U) \tag{2.12}$$

for U in $\mathcal{C}$ and V in $\mathcal{A}$, satisfying the following compatibility conditions:

(i) *For $X, X' \in \mathcal{C}$ the isomorphism $\sigma_{X,F(X')}$ coincides with the composition*

$$F(X) \otimes F(X') \cong F(X \otimes X') \cong F(X' \otimes X) \cong F(X') \otimes F(X) , \tag{2.13}$$

where the first and the third isomorphisms are the tensoriality constraints of F, while the middle isomorphism comes from the braiding on $\mathcal{C}$.

(ii) *For $Y_1, Y_2 \in \mathcal{A}$ and $X \in \mathcal{C}$ the composition*

$$F(X) \otimes Y_1 \otimes Y_2 \xrightarrow{\sigma_{X,Y_1} \otimes Y_2} Y_1 \otimes F(X) \otimes Y_2 \xrightarrow{Y_1 \otimes \sigma_{X,Y_2}} Y_1 \otimes Y_2 \otimes F(X) \tag{2.14}$$

coincides with the isomorphism $\sigma_{X,Y_1 \otimes Y_2}$.

(iii) *For $Y \in \mathcal{A}$ and $X_1, X_2 \in \mathcal{C}$ the composition*

$$\begin{aligned} F(X_1 \otimes X_2) \otimes Y \;\cong\; F(X_1) \otimes F(X_2) \otimes Y &\xrightarrow{F(X_1) \otimes \sigma_{X_2,Y}} F(X_1) \otimes Y \otimes F(X_2) \\ &\xrightarrow{\sigma_{X_1,Y} \otimes F(X_2)} Y \otimes F(X_1) \otimes F(X_2) \cong Y \otimes F(X_1 \otimes X_2) \end{aligned} \tag{2.15}$$

coincides with $\sigma_{X_1 \otimes X_2, Y}$.

The following result relates central functors into $\mathcal{A}$ to the Drinfeld center $\mathcal{Z}(\mathcal{A})$:

Lemma 2.9 [DMNO, Def. 2.4]. *A structure of central functor on $F:\mathcal{C}\to\mathcal{A}$ is equivalent to a lift of F to a braided tensor functor $\widetilde{F}:\mathcal{C}\to\mathcal{Z}(\mathcal{A})$, i.e. the composition $\varphi_{\mathcal{A}}\circ\widetilde{F}$ with the forgetful functor (2.6) equals F,*

$$\begin{tikzcd} & \mathcal{Z}(\mathcal{A}) \arrow[d, "\varphi_{\mathcal{A}}"] \\ \mathcal{C} \arrow[ur, dashed, "\widetilde{F}"] \arrow[r, "F"'] & \mathcal{A} \end{tikzcd} \tag{2.16}$$

2.5. Lagrangian algebras. In general, for an algebra A in a fusion category $\mathcal{A}$ there is no notion of a center, at least not as an object of $\mathcal{A}$. This is simply because $\mathcal{A}$ is not required to be braided, so that there is no natural concept of commuting factors in a tensor product. As it turns out, the Drinfeld center $\mathcal{Z}(\mathcal{A})$, which *is* braided, is the right recipient for a notion of a center. Keeping in mind that, in classical algebra, Morita equivalent algebras have isomorphic centers, a center should better be associated to a module category over $\mathcal{A}$ rather than to an algebra in $\mathcal{A}$.

Definition 2.10 [DMNO, Defs. 3.1 & 4.6].

- **(i)** *An algebra A in a monoidal category is called* **separable** *iff the multiplication morphism splits as a morphism of A-bimodules.*
- **(ii)** *An algebra in a monoidal category that is also a coalgebra is called* **special** *iff it is separable, with the right-inverse of the product given by a multiple of the coproduct, and the composition $\varepsilon\circ\eta$ of the counit and unit is non-zero.*
- **(iii)** *An* **étale** *algebra in a braided $\mathbb{k}$-linear monoidal category $\mathcal{C}$ is a separable commutative algebra in $\mathcal{C}$.*
- **(iv)** *An étale algebra $A\in\mathcal{C}$ is said to be* **connected** *(or* **haploid***) iff* $\dim_{\mathbb{k}}\mathrm{Hom}(\mathbf{1},A)=1$.
- **(v)** *A* **Lagrangian algebra** *in a non-degenerate braided fusion category $\mathcal{C}$ is a connected étale algebra L in $\mathcal{C}$ for which the category $\mathcal{C}_L^0$ of local L-modules in $\mathcal{C}$ is equivalent to $\mathcal{V}ect_{\mathbb{k}}$ as an abelian category.*

Remark 2.11. (i) A local (or dyslectic) module (M,ρ) over a commutative algebra A is an A-module for which the representation morphism ρ satisfies $\rho\circ c_{A,M}\circ c_{M,A} = \rho$ [Pa, KO, FFRS1]. The full subcategory of dyslectic modules is a braided monoidal category.

(ii) The defining property $\mathcal{C}_L^0\simeq\mathcal{V}ect_{\mathbb{k}}$ of a Lagrangian algebra is equivalent to the equality $(\mathrm{FPdim}(L))^2=\mathrm{FPdim}(\mathcal{C})$ of Perron-Frobenius dimensions [DMNO, Cor. 3.32].

Proposition 2.12 [DNO, Cor. 3.8]. *For $\mathcal{C}$ a non-degenerate braided fusion category, equivalence classes of indecomposable $\mathcal{C}$-modules are in bijection with isomorphism classes of triples (A_1,A_2,Ψ) with A_1 and A_2 connected étale algebras in $\mathcal{C}$ and $\Psi:\mathcal{C}_{A_1}^0\xrightarrow{\simeq}\mathcal{C}_{A_2}^{0\,rev}$ a braided equivalence between the category of local A_1-modules and the reverse of the category of local A_2-modules.*

Remark 2.13. Étale algebras can be obtained from central functors and, conversely, central functors from induction along étale functors:

(i) Given a central functor $F\colon \mathcal{C} \to \mathcal{A}$ from a braided fusion category $\mathcal{C}$ to a fusion category $\mathcal{A}$, denote by R_F its right adjoint functor. The object $\mathrm{R}_F(\mathbf{1}_{\mathcal{A}})$ then has a canonical structure of connected étale algebra in $\mathcal{C}$ [DMNO, Lem. 3.5].

(ii) For $\mathcal{C}$ a braided fusion category and A a connected étale algebra in $\mathcal{C}$, the induction functor $\mathrm{Ind}_A\colon \mathcal{C} \to \mathcal{C}_A$ that acts as $U \mapsto U \otimes A$ admits a natural structure of a central functor [DMNO, Sect. 3.4].

(iii) If in addition $\mathcal{C}$ is non-degenerate and A is Lagrangian, then the lift $\widetilde{\mathrm{Ind}_A}\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$ of the induction functor is a braided tensor equivalence [DMNO, Cor. 4.1(i)].

We are now in a position to relate indecomposable module categories over a fusion category $\mathcal{A}$ and Lagrangian algebras in its center $\mathcal{Z}(\mathcal{A})$. Denote by

$$F: \quad \mathcal{Z}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})) \xrightarrow{\varphi} \mathcal{E}nd_{\mathcal{A}}(\mathcal{M}) \tag{2.17}$$

the composition of the equivalence (2.11) with the forgetful functor. This is, trivially, a central functor, and the image $A_{\mathcal{M}}$ of the tensor unit of the monoidal category $\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})$ under the functor R_F right adjoint to F is an étale algebra and, as it turns out, even a Lagrangian algebra.

The following proposition shows that these Lagrangian algebras can be seen as invariants of indecomposable tensor categories.

Proposition 2.14 [DMNO, Prop. 4.8]. *For any fusion category $\mathcal{A}$ there is a bijection between the sets of isomorphism classes of Lagrangian algebras in $\mathcal{Z}(\mathcal{A})$ and equivalence classes of indecomposable $\mathcal{A}$-modules.*

The proof of this statement is based on Proposition 2.6.

2.6. The Witt group. One step in the long-standing problem of classifying rational conformal field theories is the classification of modular tensor categories. Recently, the following algebraic structure was established in the wider context of non-degenerate braided fusion categories (i.e. without assuming a spherical structure): The quotient of the monoid (with respect to the Deligne product) of non-degenerate braided fusion categories by its submonoid of Drinfeld centers forms a group that contains as a subgroup the group $\mathfrak{W}_{\mathrm{pt}}$ of the classes of non-degenerate pointed braided fusion categories [DMNO, Sect. 5.3]. The latter coincides with the classical Witt group [Wi] of metric groups, i.e. of finite abelian groups equipped with a non-degenerate quadratic form. This motivates the

Definition 2.15 [DMNO, Defs. 5.1 & 5.5].

(i) *Two non-degenerate braided fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are called* **Witt equivalent** *iff there exists a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$ with suitable fusion categories $\mathcal{A}_1$ and $\mathcal{A}_2$.*

(ii) *The* **Witt group** $\mathfrak{W}$ *is the group of Witt equivalence classes of non-degenerate braided fusion categories.*

It is not hard to see [DMNO] that Witt equivalence is indeed an equivalence relation, and that $\mathfrak{W}$ is indeed an abelian group, with multiplication induced by the Deligne product. The neutral element of $\mathfrak{W}$ is the class of all Drinfeld centers, and the inverse of the class of $\mathcal{C}$ is the class of its reverse category $\mathcal{C}^{\mathrm{rev}}$.

As we will see below, in our considerations the Witt group $\mathfrak{W}$ will play an important role. But we will also be interested in the categories themselves rather than in their classes in $\mathfrak{W}$. Moreover, in our context, the categories whose Witt classes are relevant are even modular. Accordingly we set:

Definition 2.16. **(i)** *A modular tensor category $\mathcal{C}$ is called* **Witt-trivial** *iff its class in the Witt group $\mathfrak{W}$ is the neutral element of $\mathfrak{W}$.*

(ii) *A* **Witt-trivialization** *of a modular tensor category $\mathcal{C}$ consists of a fusion category $\mathcal{A}$ and an equivalence*

$$\alpha : \quad \mathcal{C} \to \mathcal{Z}(\mathcal{A}) \tag{2.18}$$

as ribbon categories.

3. Bicategories for Boundary Conditions

We are now ready to formulate our proposal for topological boundary conditions for Reshetikhin-Turaev type topological field theories. Since a topological field theory of Turaev-Viro type based on a fusion category $\mathcal{A}$ has a natural description as a TFT of Reshetikhin-Turaev type based on the Drinfeld center $\mathcal{Z}(\mathcal{A})$, our results cover TFTs of Turaev-Viro type as well.

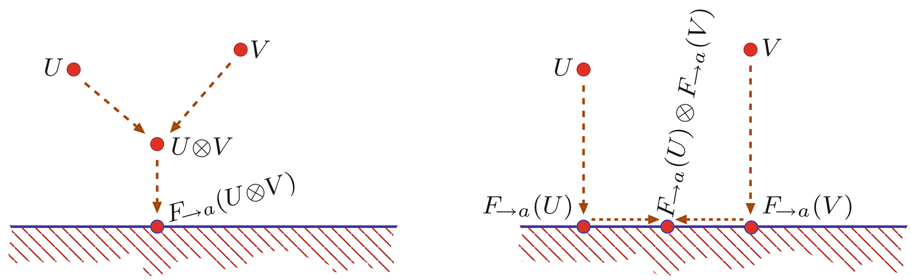
Recall from the Introduction that the boundary conditions we are going to discuss refer to boundaries at which the three-dimensional world ends, rather than cut-and-paste boundaries. As we are working in the Reshetikhin-Turaev framework, in which the categories labeling three-dimensional regions are modular categories and thus in particular finitely semisimple, we only allow for boundary conditions that correspond to finitely semisimple categories as well (though not modular and not even braided, in general, as in two dimensions there is no room for a braiding).

We seize from [KS1, Sect. 5.2] the idea to analyze what happens when Wilson lines in the bulk approach the boundary. We assume that for a given TFT in the bulk there exists a topological boundary condition a at the end of the three-dimensional world. The two-dimensional boundary can contain Wilson lines. These Wilson lines can carry insertions, and for this reason they are labeled by the objects of a category $\mathcal{W}_a$. Boundary Wilson lines can be fused, and accordingly $\mathcal{W}_a$ has the structure of a monoidal category, and moreover, owing to the fact that the Wilson lines are topological, this comes with dualities. On the other hand, this category is not braided, in general, since there does not exist a natural way to 'switch' two boundary Wilson lines without leaving the boundary which is two-dimensional.

However, there are Wilson lines in the nearby bulk as well; they are labeled by some modular tensor category $\mathcal{C}$ (the same that labels the bulk region adjacent to the boundary). The category of bulk Wilson lines is in particular braided, since Wilson lines can be switched in the three-dimensional region. Now part of what is to be meant by a boundary condition is to be able to tell what happens when the boundary is approached from the bulk. Thus we postulate that for a consistent boundary condition there should exist a process of adiabatically moving Wilson lines in the bulk to the boundary, whereby they turn into boundary Wilson lines. Put differently, we postulate that there is a functor

$$F_{\to a} : \quad \mathcal{C} \to \mathcal{W}_a . \tag{3.1}$$

Furthermore, the following two processes should yield equivalent results: On the one hand, first fusing two bulk Wilson lines in the bulk and then bringing the so obtained single bulk Wilson line to the boundary; and on the other hand, first moving the two bulk Wilson lines separately to the boundary and then fusing them as boundary Wilson lines inside the boundary. Schematically, showing a two-dimensional section perpendicular to the boundary, the situation looks as follows:



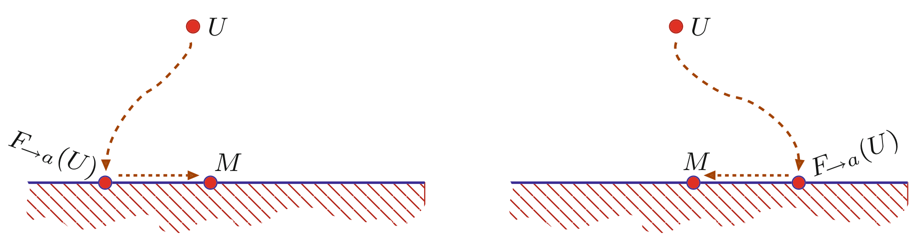
(3.2)

Put differently, the functor (3.1) obeys

$$F_{\to a}(U \otimes V) \cong F_{\to a}(U) \otimes F_{\to a}(V) , \tag{3.3}$$

with coherent isomorphisms, i.e. the functor $F_{\to a}$ has the structure of a tensor functor. From multiple fusion, one concludes the existence of associativity constraints. Moreover, we should get the same result when homotopies are applied to Wilson lines in the boundary as when they are applied in the bulk. Put differently, the functor $F_{\to a}$ should respect dualities.

The next consideration shows that $F_{\mathrm{bulk}\to a}$ has even more structure. Consider again the situation that a bulk Wilson line $U \in \mathcal{C}$ is moved to the boundary, resulting in a boundary Wilson line $F_{\to a}(U) \in \mathcal{W}_a$. Assume in addition that nearby on the boundary there is already another parallel boundary Wilson line $M \in \mathcal{W}_a$. Since the process of moving U to the boundary is supposed to be adiabatic, we should get isomorphic results when we either move U to the *left* of M and then fuse $F_{\to a}(U)$ with M, or else move U to the *right* of M and then fuse $F_{\to a}(U)$ with M, as indicated in the following picture:



(3.4)

Put differently, we expect a natural isomorphism

$$F_{\to a}(U) \otimes M \xrightarrow{\cong} M \otimes F_{\to a}(U). \tag{3.5}$$

The following argument shows that these isomorphisms endow the functor $F_{\to a}$ with the structure of a central functor in the sense of Definition 2.8. Property (i) of a central functor is the statement that for a boundary Wilson line that has been obtained by the adiabatic process, the interchange with another such Wilson line comes from the braiding of bulk Wilson lines. Property (ii) of a central functor, which may be called a boundary Yang-Baxter property, is a consequence of the homotopy equivalence of two different processes in the bulk: either moving the Wilson line in a single step past two boundary Wilson lines, or else doing it in two separate steps. Property (iii) is seen similarly, this time with two bulk Wilson lines involved.

According to Lemma 2.9 such a structure is, in turn, equivalent to a lift of $F_{\mathrm{bulk}\to a}$ to a braided functor

$$\widetilde{F}_{\to a}: \quad \mathcal{C} \to \mathcal{Z}(\mathcal{W}_a) \tag{3.6}$$

from $\mathcal{C}$ to the Drinfeld center of the fusion category $\mathcal{W}_a$.

The two-dimensional physics of the boundary surface does not provide any natural reason for such a half-braiding rule to exist. The only possible natural origin of such a rule is thus that it is related to the half-braiding in the three-dimensional bulk, via the processes encoded in the functor $F_{\to a}$. Accordingly there should not exist any systematic rule of moving a boundary Wilson line M to the other side of a neighbouring boundary Wilson line, except through the fact that M secretly is a bulk Wilson line that has been brought to the boundary (so that the rule comes from the process of first bringing it again into the bulk, moving it around there, and then moving it back to the boundary).

A boundary Wilson line is labeled by an object of $\mathcal{W}_a$; a systematic rule of moving a boundary Wilson line M to the other side of a neighbouring boundary Wilson line constitutes a half-braiding $c_{M,-}$ on $\mathcal{W}_a$ for the object M. The pair $(M, c_{M,-})$ is thus just an object in the Drinfeld center $\mathcal{Z}(\mathcal{W}_a)$. Put differently, the functor (3.6) is essentially surjective.

Similarly, no information about the bulk should be lost when a bulk Wilson line is brought to the boundary, provided one remembers the way the Wilson line can wander within the bulk to the other side of any other boundary Wilson line. This principle applies likewise to insertions on the Wilson lines. In the bulk, such insertions are morphisms in $\mathcal{C}$; for boundary Wilson lines, we can only allow morphisms that are compatible with the rule to switch the boundary Wilson line with any other boundary Wilson line. In other words, we only allow those morphisms of $\mathcal{W}_a$ that are compatible with the half-braiding, i.e. we only consider morphisms in $\mathcal{Z}(\mathcal{W}_a)$. Put differently, the functor $\widetilde{F}_{\to a}$ (3.6) is fully faithful, and thus, being also essentially surjective, it is a braided *equivalence*:

$$\widetilde{F}_{\to a}: \quad \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a). \tag{3.7}$$

From this equivalence, we conclude that boundary conditions of the type we consider can only exist for a bulk theory that is Witt-trivial in the sense of Definition 2.16. Even more, the boundary data are given by a Witt-trivialization.

Once one has understood one boundary condition in a physical theory, frequently the way is open to understand other boundary conditions as well. Thus let us assume that there exists another boundary condition besides a. At this point we do *not*, however, assume that this boundary condition b comes with a central functor $F_{\to b}$ as well, but rather perform an analysis purely within the boundary. Consider a generalized Wilson line that separates the boundary condition a on the left from b on the right. Such Wilson lines can carry local field insertions as well; hence we describe them in terms of a category $\mathcal{W}_{a,b}$. We can fuse such a Wilson line with a Wilson line from $\mathcal{W}_a$ to the left of it. This gives again a Wilson line separating the boundary condition a from b and thus an object in $\mathcal{W}_{a,b}$. We thus get on the category $\mathcal{W}_{a,b}$ the structure of a module category over $\mathcal{W}_a$.

By a similar argument, the category $\mathcal{W}_b$ of boundary Wilson lines separating the boundary condition b from itself has to act on the Wilson lines in $\mathcal{W}_{a,b}$ from the right. Put differently, $\mathcal{W}_{a,b}$ is a right $\mathcal{W}_b$-module. On the other hand, $\mathcal{W}_{a,b}$ is already naturally a right module category over the category $\mathcal{W}^*_{a,b} = \mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})$ of module endofunctors. We now invoke a principle of naturality and require that this category describes the tensor category of generalized boundary Wilson lines for the boundary condition b, i.e. that $\mathcal{W}_b \simeq \mathcal{W}_{a,b}^*$.

The latter postulate can only make sense if the fusion category $\mathcal{W}_{a,b}^*$ comes with a Witt-trivialization of the bulk category $\mathcal{C}$ as well, i.e. if we have a canonical equivalence

$$\mathcal{C} \simeq \mathcal{Z}(\mathcal{W}_{a,b}^*) \tag{3.8}$$

of braided categories. According to Proposition 2.7 this is indeed the case. This can be seen as a justification of our naturality principle by which we identified Wilson lines with module functors.

To obtain another check of our proposal, we next consider a trivalent vertex in a boundary, with one incoming bulk Wilson line labeled by $U \in \mathcal{C}$, one incoming boundary Wilson line labeled by $W_1 \in \mathcal{W}_a$ and one outgoing boundary Wilson line $W_2 \in \mathcal{W}_a$. According to our general picture the three-valent vertex should be labeled by an element of a vector space obtainable as a morphism space. We can realize this vector space in terms of morphisms in the category $\mathcal{W}_a$, provided that there is a mixed tensor product

$$\mathcal{C} \times \mathcal{W}_a \longrightarrow \mathcal{W}_a, \tag{3.9}$$

and take the trivalent vertex to be labeled by an element of $\mathrm{Hom}_{\mathcal{W}_a}(U \otimes W_1, W_2)$. Put differently, we need the structure of a $\mathcal{C}$-module on $\mathcal{W}_a$. To determine what module category is relevant, we invoke topological invariance of the bulk Wilson line so as to have it running parallel with the boundary before it enters the vertex. We then apply the adiabatic process described by the functor $F_{\to a}$ to the piece parallel to the surface, thereby turning the bulk Wilson line with label $U \in \mathcal{C}$ into a boundary Wilson line with label $F_{\to a}(U)$. This way we reduce the problem of a trivalent vertex involving a bulk Wilson line to the one of a trivalent vertex involving only boundary Wilson lines. The relevant vector space is thus $\mathrm{Hom}_{\mathcal{W}_a}(F_{\to a}(U) \otimes W_1, W_2)$. Put differently, we use the $\mathcal{C}$-module structure on $\mathcal{W}_a$ that is induced by pullback of the regular module category along the monoidal functor $F_{\to a} : \mathcal{C} \to \mathcal{W}_a$ or, what is the same, along the monoidal functor

$$\mathcal{C} \xrightarrow{\widetilde{F}_{\to a}} \mathcal{Z}(\mathcal{W}_a) \xrightarrow{\varphi_{\mathcal{W}_a}} \mathcal{W}_a \tag{3.10}$$

from $\mathcal{C}$ to $\mathcal{W}_a$.

It is important to note that this way one does *not* obtain *all* $\mathcal{C}$-modules; thus our results lead to a selection principle that singles out an interesting subclass of $\mathcal{C}$-modules. This can be seen already in simple examples, e.g. when $\mathcal{W}_a$ is the category of finite-dimensional representations of a finite group G, so that $\mathcal{Z}(\mathcal{A})$ is the category of finite-dimensional representations of the double $D(G)$ [Os2, Thm. 3.1]. For instance, for $G = \mathbb{Z}_2$, there are two indecomposable $\mathcal{A}$-modules (called 'rough' and 'smooth' in [KK]), but six indecomposable $\mathcal{Z}(\mathcal{A})$-modules.

What we have managed so far is to use one given topological boundary condition to obtain also other topological boundary conditions. This raises the question of whether we can obtain *all* topological boundary conditions this way. Suppose we are given two different boundary conditions and thus two braided equivalences

$$\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A}_1) \quad \text{and} \quad \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A}_2). \tag{3.11}$$

Then we have a braided equivalence $\mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{Z}(\mathcal{A}_2)$. By the result of [ENO2] on 2-Morita theory that we recalled in Proposition 2.6, this implies that the bicategories of

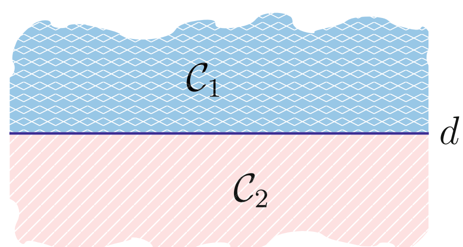
$\mathcal{A}_1$-modules and of $\mathcal{A}_2$-modules are equivalent bicategories. We thus conclude that we can indeed access every boundary condition from any other boundary condition.

We summarize our proposal: Topological boundary conditions for a topological field theory of Reshetikhin-Turaev type, based on a modular tensor category $\mathcal{C}$, are described by Witt-trivializations of $\mathcal{C}$, i.e. by braided equivalences $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A})$. Given any such trivialization, the bicategory of topological boundary conditions can be identified with the bicategory of $\mathcal{A}$-modules.

One should also appreciate that if a TFT of Turaev-Viro type based on the fusion category $\mathcal{A}$ is described as a Reshetikhin-Turaev theory based on the modular tensor category $\mathcal{Z}(\mathcal{A})$, then it comes with a trivialization and the category of topological boundary conditions is naturally identified with the bicategory of $\mathcal{A}$-modules. In the special case of TFTs of Turaev-Viro type, our results thus reproduce results of [KK] about boundary conditions in such TFTs.

4. Bicategories for Surface Defects

Next we study what kind of mathematical objects describe topological surface defects, i.e. the topological surface operators considered for abelian Chern-Simons theories in [KS1] or the domain walls in [BSW,KK]. We consider a surface defect d separating two modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and follow the same line of arguments as for boundary conditions in Sect. 3. The situation to be studied is displayed schematically in the following picture, which shows a two-dimensional section perpendicular to the defect surface:



(4.1)

Again we start with a semisimple fusion category $\mathcal{W}_d$ of Wilson lines that are contained in the defect surface. We refer to such Wilson lines also as *defect Wilson lines*. In complete analogy with the case of boundary conditions, we postulate that there are adiabatic processes of moving Wilson lines from the bulk on either side of the defect surface into the defect surface, whereby they yield defect Wilson lines. By the same arguments as for boundaries this leads to a central functor

$$F_{\to d}: \quad \mathcal{C}_1 \to \mathcal{W}_d \tag{4.2}$$

and, accounting for relative orientations, to another central functor

$$F_{d\leftarrow}: \quad \mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d\,. \tag{4.3}$$

According to Lemma 2.9 we thus have two braided functors

$$\widetilde{F}_{\to d}: \quad \mathcal{C}_1 \longrightarrow \mathcal{Z}(\mathcal{W}_d) \qquad \text{and} \qquad \widetilde{F}_{d\leftarrow}: \quad \mathcal{C}_2^{\mathrm{rev}} \longrightarrow \mathcal{Z}(\mathcal{W}_d) \tag{4.4}$$

as in (3.7). Since $\mathcal{Z}(\mathcal{W}_d)$ is braided, the images of these two functors commute. Thus, with the help of the Deligne tensor product, we combine $\widetilde{F}_{\to d}$ and $\widetilde{F}_{d\leftarrow}$ into a single functor

$$\widetilde{F}_{\to d\leftarrow}: \quad \mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \longrightarrow \mathcal{Z}(\mathcal{W}_d). \tag{4.5}$$

We again invoke a principle of naturality to assert that the combined functor $\widetilde{F}_{\to d\leftarrow}$ is an equivalence of braided categories.

Suppose now that we have a defect Wilson line $W \in \mathcal{W}_d$ together with a rule for exchanging W with any other defect Wilson line $W' \in \mathcal{W}_d$. The two-dimensional physics of the defect surface does not provide any natural reason why such a half-braiding rule should exist. The only possible natural origin of such a rule is that it is related to the half-braiding in the three-dimensional parts, using the processes encoded in the two functors $F_{\to d}$ and $F_{d\leftarrow}$. This amounts to the assumption that the defect Wilson line W can be written as a direct sum of fusion products of the form $W_1 \otimes W_2$, where W_1 is a defect Wilson line that has been obtained by the adiabatic process from $\mathcal{C}_1$, i.e. $W_1 = F_{\to d}(L_1)$ for some $L_1 \in \mathcal{C}_1$, and similarly $W_2 = F_{d\leftarrow}(L_2)$ with $L_2 \in \mathcal{C}_2$. This shows essential surjectivity of $\widetilde{F}_{\to d\leftarrow}$; an argument about point-like insertions on Wilson lines that is completely analogous to one used for boundary conditions shows that $\widetilde{F}_{\to d\leftarrow}$ is fully faithful.

We thus arrive at an equivalence

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d) \tag{4.6}$$

of braided categories that, together with the fusion category $\mathcal{W}_d$, is part of the data specifying a surface defect. We immediately conclude that a topological surface defect joining regions labeled by the modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ can only exist if $\mathcal{C}_1$ and $\mathcal{C}_2$ are in the same Witt class. The existence of such an obstruction should not come as a surprise. Similar effects are, for instance, known from two dimensions: conformal line defects (and, a fortiori, topological defects) can only exist if the two conformal field theories joined by the defect have the same Virasoro central charge. In the situation at hand the Witt group – a concept that has been introduced for independent reasons, namely to structure the space of modular tensor categories – turns out to be the right recipient for the obstruction.

Further, as in the case of boundary conditions, we conclude that other possible labels of surface defects separating $\mathcal{C}_1$ and $\mathcal{C}_2$ are described by module categories over the fusion category $\mathcal{W}_d$. This also gives the right bicategorical structure to this collection of surface defects: the one of $\mathcal{W}_d$-modules. By the same type of argument based on Proposition 2.7 as in the case of boundaries, it follows that the other categories of defect Wilson lines come with Witt-trivializations of $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$ as well, and that the bicategorical structure does not depend on the choice of d.

Again we can consider two special cases to compare our results with existing literature. The first – abelian Chern-Simon theories – will be relegated to Sect. 5. The second is that the TFT on either side of the defect surface admits a description of Turaev-Viro type, i.e. that both modular tensor categories are Drinfeld centers of fusion categories, $\mathcal{C}_1 \simeq \mathcal{Z}(\mathcal{A}_1)$ and $\mathcal{C}_2 \simeq \mathcal{Z}(\mathcal{A}_2)$. Using the identifications

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}_1) \boxtimes \mathcal{Z}(\mathcal{A}_2)^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2)\,, \tag{4.7}$$

where we identify left and right half-braidings, shows that in this case the bicategory of $\mathcal{C}_1$-$\mathcal{C}_2$ surface defects can be identified withthe bicategory of $\mathcal{A}_1$-$\mathcal{A}_2$-bimodules.

Thus in the special case of TFTs of Turaev-Viro type our results reproduce those of [KK].

Let us explore some consequences of our results. First we consider the special case that the surface defect separates two regions with the same TFT, i.e. that $\mathcal{C}_1 = \mathcal{C}_2 =: \mathcal{C}$. By the characterization of modular tensor categories given in Definition 2.5, there is then a distinguished Witt trivialization,

$$\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}) , \tag{4.8}$$

which is obtained by using the braiding of the categories $\mathcal{C}$ and of $\mathcal{C}^{\text{rev}}$, respectively, to embed them into $\mathcal{Z}(\mathcal{C})$. This specific surface defect can be interpreted as a *transparent defect*, very much in the way as a Wilson line labeled by the tensor unit can be seen as a transparent Wilson line (and is, for this reason, usually invisible in a graphical calculus), and accordingly we denote it by the symbol $T_\mathcal{C}$. Indeed, the defect Wilson lines for this specific defect are labeled by the objects of $\mathcal{C}$. The central functor

$$F_{\rightarrow T_\mathcal{C}} : \quad \mathcal{C} \rightarrow \mathcal{C} \tag{4.9}$$

describing a specific adiabatic process is, as a functor, just the identity. Its structure of a central functor is then just given by the braiding of $\mathcal{C}$. In physical terms this means that in the adiabatic process labels do not change and the braiding is preserved. Similar statements apply to the functor

$$F_{T_\mathcal{C} \leftarrow} : \quad \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C} , \tag{4.10}$$

where the structure of a central functor is now given by the opposite braiding. Thus defect Wilson lines separating the surface defect $T_\mathcal{C}$ from itself are naturally identified, including the braiding, with ordinary Wilson lines in $\mathcal{C}$. Phrased the other way round: Wilson lines in the three-dimensional chunk labeled by $\mathcal{C}$ can be thought of as being secretly Wilson lines inside a defect surface, namely one labeled by the transparent defect $T_\mathcal{C}$.

We next discuss implications to surface defects separating $\mathcal{C}$ from itself of the result of [DNO], reported in Proposition 2.12, that indecomposable $\mathcal{C}$-modules are in bijection to pairs (A_1, A_2) of étale algebras in $\mathcal{C}$ together with a braided equivalence between their full subcategories of local (or dyslectic) modules in $\mathcal{C}$. This has the following physical interpretation: A generic Wilson line in $\mathcal{C}$ cannot pass through a given surface defect. If, however, a whole package of $\mathcal{C}$-Wilson lines condenses so as to form a local A_1-module, the resulting Wilson line can pass through the surface defect and reappear on the other side as a condensed package of $\mathcal{C}$-Wilson lines that forms a local A_2-module.

This is exactly the type of structure needed in the application of surface operators in the TFT construction of the correlators of rational conformal field theories [FRS], following the suggestions of [KS2]. The process then puts the fact [MS, Sect. 4] that the general structure of the bulk partition function is “automorphism on top of extension” in the appropriate and complete setting. This picture can be easily extended to heterotic theories, for which left- and right-moving degrees are in different module tensor categories $\mathcal{C}_\text{l}$ and $\mathcal{C}_\text{r}$. In particular, the obstruction to the existence of a heterotic TFT construction based on a pair $(\mathcal{C}_\text{l}, \mathcal{C}_\text{r})$ of modular tensor categories is again captured by the Witt group: $\mathcal{C}_\text{l}$ and $\mathcal{C}_\text{r}$ must lie in the same Witt class.

The transmission of (bunches of) Wilson lines should be seen as a three-dimensional analogue of the following process in two dimensions: A topological defect line can wrap around bulk insertions in one full conformal field theory to produce a bulk insertion in another theory. This effect associates a map on bulk fields to any topological defect line. This map has, in turn, been instrumental in obtaining classification results for defects [FGRS] and in understanding their target space formulation [FSW]. We expect that the transmission of Wilson lines can be used to a similar effect in the situation at hand. That the transmission data describe, by Proposition 2.14, the isomorphism class of a module category, is an encouragement for attempting similar classifications as in the two-dimensional case. A first example of a classification will be presented in Sect. 5.

Returning to the case of general pairs $(\mathcal{C}_1, \mathcal{C}_2)$ of modular tensor categories, the forgetful functor $\varphi_{\mathcal{W}_d}$ from the Drinfeld center $\mathcal{Z}(\mathcal{W}_d)$ to the fusion category $\mathcal{W}_d$ provides us with a tensor functor

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d) \xrightarrow{\varphi_{\mathcal{W}_d}} \mathcal{W}_d. \tag{4.11}$$

Via pullback along this functor, the category $\mathcal{W}_d$ of defect Wilson lines comes with a natural structure of a $\mathcal{C}_1$-$\mathcal{C}_2$-bimodule category. This bimodule structure arises naturally when one considers three-valent vertices in the defect surface with two defect Wilson lines and one bulk Wilson line involved. This structure should also enter in the description of fusion of topological surface defects. We leave a detailed discussion of fusion to future work and only remark that the transparent defect $T_{\mathcal{C}}$ must act as the identity under fusion.

We conclude with a word of warning: While the structure of a $\mathcal{C}_1$-$\mathcal{C}_2$-bimodule on the category $\mathcal{W}_d$ of defect Wilson lines can be expected to have a bearing on fusion, the bicategory $\mathcal{C}_1$-$\mathcal{C}_2$-$\mathcal{B}imod$ of $\mathcal{C}_1$-$\mathcal{C}_2$-bimodules can*not* provide the proper mathematical model for the bicategory of surface defects. For instance, taking $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$, the natural candidate for the transparent defect is $\mathcal{C}$ as a bimodule over itself. Using $\mathcal{C}_1$-$\mathcal{C}_2$-$\mathcal{B}imod$ as a model for the surface defects, the Wilson lines separating this transparent defect from itself would correspond to bimodule endofunctors of $\mathcal{C}$, and the category of these endofunctors is equivalent to $\mathcal{Z}(\mathcal{C})$ and thus, $\mathcal{C}$ being modular, to the enveloping category $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$. We would then *not* recover ordinary Wilson lines as defect Wilson lines in the transparent defect. As we will see in the next section, taking $\mathcal{C}_1$-$\mathcal{C}_2$-$\mathcal{B}imod$ as the model for the bicategory of surface defects would also contradict results of [KS1] for abelian Chern-Simons theories, and of [KK] on surface defects in TFTs admitting a description of Turaev-Viro type.

5. Lagrangian Algebras and Abelian Chern-Simons Theories

In this section we describe consequences of our proposal in the special case of abelian Chern-Simons theories and compare our findings with the results of [KS1] for this subclass of TFTs. As a new ingredient our discussion involves Lagrangian algebras in the modular tensor category $\mathcal{C}$ that labels the TFT. Recall from Proposition 2.14 that Lagrangian algebras in the Drinfeld center of a fusion category $\mathcal{A}$ are complete invariants of equivalence classes of indecomposable $\mathcal{A}$-module categories.

If we are just interested in equivalence classes of indecomposable boundary conditions, Lagrangian algebras can be used as follows. The presence of a topological boundary condition for a modular tensor category $\mathcal{C}$ requires the existence of a Witt-trivialization $\mathcal{C} \simeq \mathcal{Z}(\mathcal{A})$ with $\mathcal{A}$ a fusion category providing a reference boundary condition. Indecomposable or elementary boundary conditions are then in bijection with indecomposable $\mathcal{A}$-modules. The latter, in turn, are in bijection with Lagrangian algebras in the Drinfeld center $\mathcal{Z}(\mathcal{A})$, which is just $\mathcal{C}$. We can thus classify elementary topological boundary conditions by classifying Lagrangian algebras in $\mathcal{C}$. This is of considerable practical interest, since it acquits us of the task to find an explicit Witt-trivialization. However, for many explicit constructions it will be important to have the full bicategorical structure at our disposal, and this requires an explicit Witt-trivialization.

The situation for topological surface defects separating modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is analogous: the classification of equivalence classes amounts to classifying Lagrangian algebras in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$. Again, this avoids finding a Witt-trivialization, but does not give direct access to the full bicategorical structure.

To make contact to the situation studied in [KS1] we first recall some basic facts about abelian Chern-Simons theories and their relation to finite groups with quadratic forms. Let Λ be a free abelian group of rank n and $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the corresponding real vector space. Denote by $\mathbb{T}_\Lambda$ the torus V/Λ. The classical abelian Chern-Simons theory with structure group $\mathbb{T}_\Lambda$ is completely determined by the choice of a symmetric bilinear form K on V whose restriction to the additive subgroup Λ is integer-valued and even. We call the pair (Λ, K) an even *lattice* of rank n.

Definition 5.1. **(i)** *A* **bicharacter***, with values in $\mathbb{C}^\times$, on a finite abelian group D is a bimultiplicative map $\beta\colon D \times D \to \mathbb{C}^\times$.*
A **symmetric** *bicharacter, or symmetric bilinear form, on D is a bicharacter β satisfying $\beta(x, y) = \beta(y, x)$ for all $x, y \in D$.*
(ii) *A* **quadratic form** *on a finite abelian group D is a function $q\colon D \to \mathbb{C}^\times$ such that $q(x) = q(x^{-1})$ and such that $\beta(x, y) := q(x \cdot y)/q(x)\, q(y)$ is a symmetric bilinear form. A* **quadratic group** *is a finite abelian group endowed with a quadratic form.*

To the lattice (Λ, K) we associate a finite group with a quadratic form in the following way. Set $\Lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$, and denote by $\mathrm{Im}\, K$ the image of Λ in Λ^* under the canonical map $K\colon \Lambda \to \Lambda^*$. The finite abelian group $D := \Lambda^*/\mathrm{Im}\, K$ is called the *discriminant group* of the lattice (Λ, K). Since the symmetric bilinear form K is integer and even, the group D comes equipped with a quadratic form $q\colon D \to \mathbb{C}^\times$, with $q(\mu) = \exp(2\pi\mathrm{i}\, K(\mu, \mu))$.

Different lattices may give rise to discriminant groups that are isomorphic as quadratic groups. As argued in [BM], many properties of quantum Chern-Simons theory are encoded in the pair (D, q). The pair (D, q) determines, in turn, an equivalence class of braided monoidal categories, which we denote by $\mathcal{C}(D, q)$. For completeness we briefly give some details on how this category is constructed (for more details see [JS, KS1]). First, recall that for any abelian group A there is a bijection

$$H^3_{\mathrm{ab}}(A; \mathbb{C}^\times) \xrightarrow{\simeq} \mathrm{Quad}(A)\,, \tag{5.1}$$

between the group $\mathrm{Quad}(A)$ of quadratic forms on A taking values in $\mathbb{C}^\times$ and the third abelian cohomology group $H^3_{\mathrm{ab}}(A; \mathbb{C}^\times)$ of A [Ma, Thm. 3]. A representative for a class in $H^3_{\mathrm{ab}}(A; \mathbb{C}^\times)$ is given by a pair (ψ, Ω) consisting of a 3-cocycle ψ in the ordinary group cohomology of A and a 2-cochain Ω, satisfying some compatibility conditions (which imply the validity of the hexagon axioms for the braiding (5.2) below). Given a pair (ψ, Ω) representing an abelian 3-cocycle, we obtain a quadratic form q on A by setting $q(a) := \Omega(a, a)$ for $a \in A$. This realizes one direction of the isomorphism (5.1).

On the other hand, given a quadratic form q on A, we obtain a pre-image (ψ, Ω) only upon additional choices; one possible choice is an ordered set of generators of the abelian group A. We will ignore this subtlety in the following and omit the label (ψ, Ω) from the notation.

Consider now a quadratic group (D, q) and choose an abelian 3-cocycle (ψ, Ω) representing the quadratic form q in $H^3_{\mathrm{ab}}(D; \mathbb{C}^\times)$. As an abelian category, $\mathcal{C}(D, q)$ is the category of finite-dimensional complex D-graded vector spaces and graded linear maps. The simple objects of this category are complex lines $\mathbb{C}_x$ labeled by group elements $x \in D$. In particular we have $\mathrm{Hom}(\mathbb{C}_x, \mathbb{C}_x) \cong \mathbb{C}$. We equip the category $\mathcal{C}(D, q)$ with the tensor product of complex vector spaces, but with associator given by the 3-cocyle ψ. The 2-cochain Ω induces a braiding c on this monoidal category; the braiding acts on simple objects as

$$\begin{array}{rrcl} c_{xy}: & \mathbb{C}_x \otimes \mathbb{C}_y & \xrightarrow{\simeq} & \mathbb{C}_y \otimes \mathbb{C}_x \\ & v \otimes w & \longmapsto & \Omega(x, y)\, w \otimes v. \end{array} \tag{5.2}$$

The braided pointed fusion category thus obtained depends, up to equivalence of braided monoidal categories, only on the class $[(\psi, \Omega)]$ in abelian cohomology [JS].

Taking the reverse category amounts to replacing the quadratic form q by the quadratic form q^{-1} which takes inverse values, i.e. $(\mathcal{C}(D, q)^{\mathrm{rev}} \cong \mathcal{C}(D, q^{-1})$, while the Deligne product amounts to taking the direct sums of the groups and of the quadratic forms. In other words, one has

Lemma 5.2. *Let (D_1, q_1) and (D_2, q_2) be finite groups with quadratic forms. Then*

$$\mathcal{C}(D_1, q_1) \boxtimes \mathcal{C}(D_2, q_2)^{\mathrm{rev}} \simeq \mathcal{C}(D_1 \oplus D_2, q_1 \oplus q_2^{-1}) \tag{5.3}$$

as braided monoidal categories.

A quadratic form (D, q) is said to be *non-degenerate* iff the associated symmetric bilinear form is non-degenerate in the sense that the associated group homomorphism $D \to \mathrm{Hom}(D, \mathbb{C}^\times)$ is an isomorphism. A basic fact about categories of the type $\mathcal{C}(D, q)$ is

Lemma 5.3 [DMNO, Sect. 5.3.]. *The braided monoidal category $\mathcal{C}(D, q)$ is modular iff the quadratic form q is non-degenerate.*

In the present context the role of the modular tensor category $\mathcal{C}(D, q)$ is as the category of (bulk) Wilson lines in the Chern-Simons theory corresponding to a lattice with discriminant group (D, q). We now make our proposal for boundary conditions and surface defects explicit in this case. To this end we need an explicit description of Lagrangian algebras.

Definition 5.4 [ENOM, Sect. 2.4].

- **(i)** *A* **metric group** *is a quadratic group $(D; q)$ for which the quadratic form q is non-degenerate.*
- **(ii)** *For U a subgroup of a quadratic group (D, q) with symmetric bilinear form $\beta\colon D \times D \to \mathbb{C}^\times$, the* **orthogonal complement** *$U^\perp$ of U is the set of all $d \in D$ such that $\beta(d, u) = 1$ for all $u \in U$.*
  If q is non-degenerate, $U^\perp$ is isomorphic to D/U, so that $|D| = |U| \cdot |U^\perp|$.

(iii) *Let (D,q) be a metric group. A subgroup U of D is said to be* **isotropic** *iff $q(u)=1$ for all $u\in U$.*
(iv) *For any isotropic subgroup U of a metric group (D,q) there exists an injection $U \hookrightarrow (D/U)^*$, so that $|U|^2 \le |D|$. An isotropic subgroup L of D is called* **Lagrangian** *iff $|L|^2 = |D|$.*

The concept of a Lagrangian subgroup is linked to Lagrangian algebras by the following assertion, which is a corollary of the results in [DrGNO, Sect. 2.8].

Theorem 5.5. *Let D be a finite abelian group with a nondegenerate quadratic form q. There is a bijection between Lagrangian subgroups of D and Lagrangian algebras in $\mathcal{C}(D,q)$.*

We thus arrive at the following two statements:

(1) Elementary topological boundary conditions for the abelian Chern-Simons theory based on the modular tensor category $\mathcal{C}(D;q)$ are in bijection with Lagrangian algebras in $\mathcal{C}(D;q)$ and thus with Lagrangian subgroups of the metric group (D,q).
(2) Elementary topological surface defects separating the abelian Chern-Simons theories based on the modular tensor categories $\mathcal{C}(D_1;q_1)$ and $\mathcal{C}(D_2;q_2)$ are in bijection with Lagrangian subgroups of the metric group $(D_1 \oplus D_2, q_1 \oplus q_2^{-1})$.

The first of these results was established in [KS1] by an explicit analysis using Lagrangian field theory. The second result was then deduced from the first by arguments based on the folding trick.

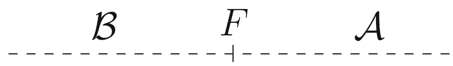
As a particular case, consider the transparent surface defect $T_{\mathcal{C}}$ separating $\mathcal{C}(D,q)$ from itself. It corresponds to the canonical trivialization $\mathcal{C}(D,q) \boxtimes \mathcal{C}(D,q)^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C}(D,q))$. The Lagrangian algebra corresponding to $T_{\mathcal{C}}$ is given by the Cardy algebra $\bigoplus_{X\in\mathrm{Irr}(\mathcal{C})} X \boxtimes X^\vee$ and corresponds to the diagonal subgroup in $D \oplus D$, in accordance with the results in [KS1, Sect. 3.3].

6. Relation with Special Symmetric Frobenius Algebras

In this section we explain how in our framework special symmetric Frobenius algebras can be obtained from certain surface defects S separating a modular tensor category $\mathcal{C}$ from itself and a Wilson line separating S from the transparent defect $T_{\mathcal{C}}$. Our results provide a rigorous mathematical foundation for the ideas of [KS2]. A central tool in our study is string diagrams.

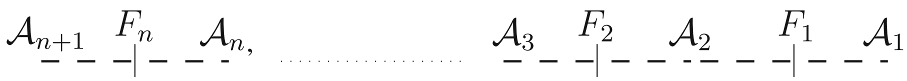
6.1. String diagrams. A *string diagram* is a planar diagram describing morphisms in a bicategory. Such diagrams are Poincaré dual to another type of diagram frequently used for bicategories, in which 2-morphisms are attached to 2-dimensional parts of the diagram. String diagrams are particularly convenient for encoding properties of adjointness and biadjointness in a graphical calculus. String diagrams apply in particular to the bicategory of small categories in which 1-morphisms are functors and 2-morphisms are natural transformations, an example the reader might wish to keep in mind. For more details see e.g. [La, Kh].

We fix a bicategory; in a first step, we only consider objects and 1-morphisms. They can be visualized in one-dimensional diagrams, with one-dimensional segments describing objects and zero-dimensional parts indicating 1-morphisms. In our convention, such diagrams are drawn horizontally and are to be read from right to left. Thus for $\mathcal{A}$ and $\mathcal{B}$ objects of the bicategory and a 1-morphism $F\colon \mathcal{A} \to \mathcal{B}$, we draw the diagram



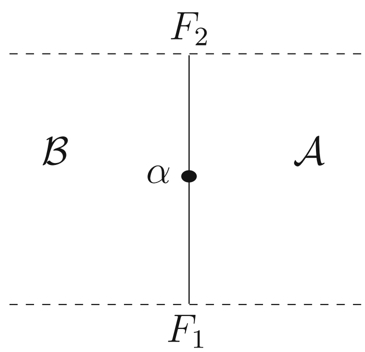
(6.1)

The composition $F_n \cdots F_1 \equiv F_n \circ \cdots \circ F_1\colon \mathcal{A}_1 \to \mathcal{A}_n$ of 1-morphisms $F_i\colon \mathcal{A}_i \to \mathcal{A}_{i+1}$ is represented by horizontal concatenation



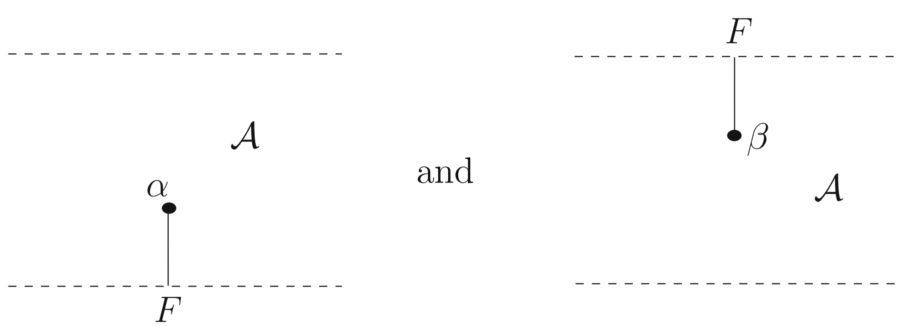
(6.2)

To accommodate also 2-morphisms a second dimension is needed. Objects are now represented by two-dimensional regions and 1-morphisms by one-dimensional vertical segments, while zero-dimensional parts indicate 2-morphisms. In our convention, the vertical direction is to be read from bottom to top. Thus a 2-morphism $\alpha\colon F_1 \Rightarrow F_2$ between 1-morphisms F_1, F_2 from objects $\mathcal{A}$ to $\mathcal{B}$ is depicted by the diagram



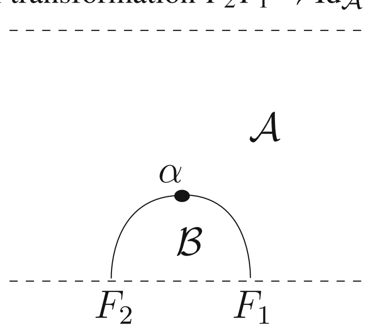
(6.3)

For the moment, we require that the strands always go from bottom to top and do not allow ‘U-turns’. For the identity 2-morphism $\alpha = id_F$ we omit the blob in the diagram. For the identity 1-morphism $Id_{\mathcal{A}}$ we omit any label except for the one referring to the object $\mathcal{A}$. With these conventions, 2-morphisms $\alpha\colon F \Rightarrow Id_{\mathcal{A}}$ and $\beta\colon Id_{\mathcal{A}} \Rightarrow F$ with F an endo-1-morphism of the object $\mathcal{A}$ are drawn as



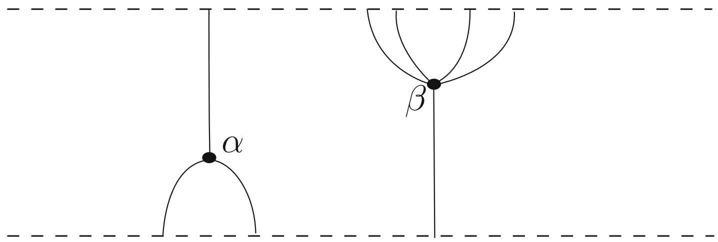
(6.4)

respectively, while a natural transformation $F_2 F_1 \Rightarrow Id_{\mathcal{A}}$ is represented by



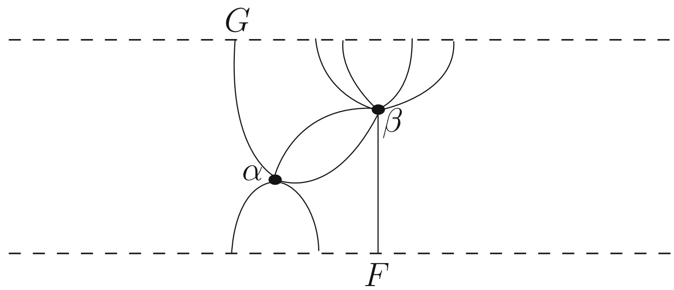
(6.5)

The 2-morphisms of a bicategory can be composed horizontally and vertically. Horizontal composition is depicted as juxtaposition, as in

$\alpha \otimes \beta \;=$ 

(6.6)

Vertical composition is represented as vertical concatenation of diagrams; thus e.g.

$(id_G \otimes \beta) \circ (\alpha \otimes id_F) \;=$ 

(6.7)

In the bicategory of small categories, we have the notion of an adjoint functor. This notion can be generalized to any 1-morphism in a bicategory. Given two 1-morphisms $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, G is said to be *right adjoint* to F, and F *left adjoint* to G, iff there exist 2-morphisms

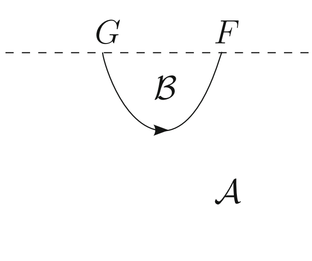
$$\eta : \quad Id_{\mathcal{A}} \Rightarrow GF \qquad \text{and} \qquad \varepsilon : \quad FG \Rightarrow \mathrm{Id}_{\mathcal{B}} \tag{6.8}$$

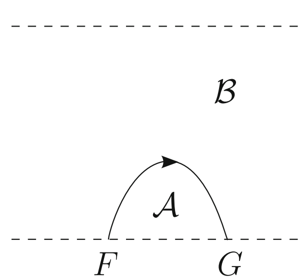
satisfying

$$(id_F \otimes \eta) \circ (\varepsilon \otimes id_F) = id_F \qquad \text{and} \qquad (\eta \otimes id_G) \circ (id_G \otimes \varepsilon) = id_G . \tag{6.9}$$

The 2-morphisms η and ε, if they exist, are not unique. For any number $\lambda \in \mathbb{C}^\times$ we can replace η by $\lambda\,\eta$ and ε by $\lambda^{-1}\varepsilon$ to get another pair of morphisms. For each such pair, η is called a *unit* and ε a *counit* of the *adjoint pair* (F, G).

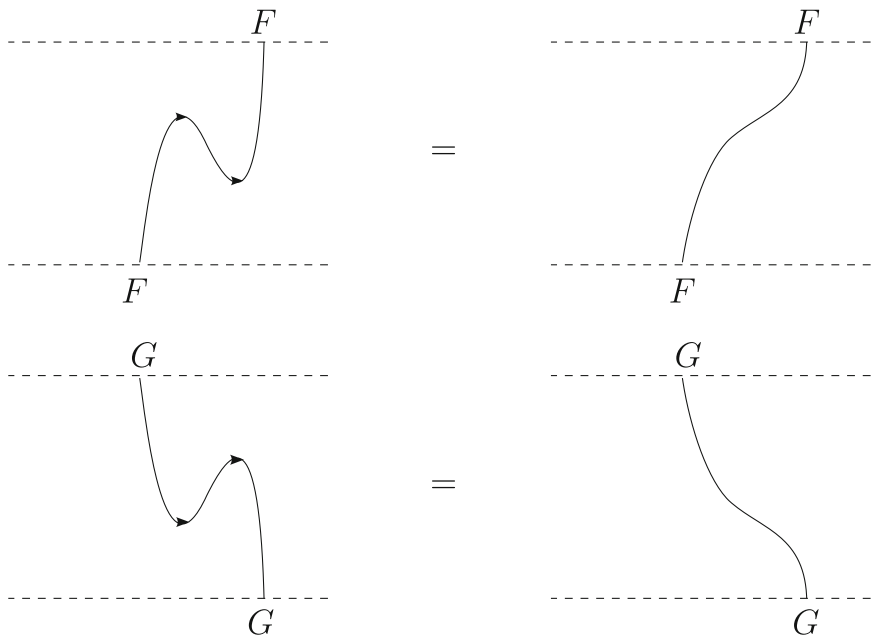
In the diagrammatic description, special notation is introduced for the unit and counit of an adjoint pair of functors: we depict them as

$\eta \;=$ 

and $\varepsilon \;=$ 

(6.10)

The equalities (6.9) amount to the identifications



(6.11)

of diagrams.

In general, the existence of a left adjoint functor does not imply the existence of a right adjoint functor. Even if both adjoints exist, they need not coincide. The same statements hold for left and right adjoints of a 1-morphism in an arbitrary bicategory. It therefore makes sense to give the

Definition 6.1. *A 1-morphism F in a bicategory is called* **biadjoint** *to a 1-morphism G iff it is both a left and a right adjoint of G. Since then G is both left and right adjoint to F as well, such a pair (F, G) of 1-morphisms is called a* **biadjoint pair**. *The adjunction (F, G) is then called* **ambidextrous**.

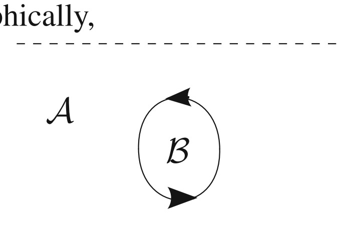
For a biadjoint pair (F, G), we thus have, apart from the 2-morphisms η and ε introduced in formula (6.9), additional 2-morphisms

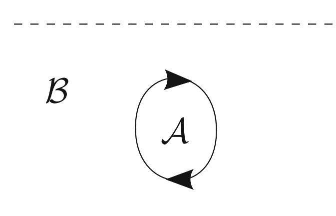
$$\tilde{\eta}: \quad Id_{\mathcal{B}} \Rightarrow FG \qquad \text{and} \qquad \tilde{\varepsilon}: \quad GF \Rightarrow \mathrm{Id}_{\mathcal{A}} \tag{6.12}$$

satisfying zigzag identities analogous to the identities (6.11) for η and ε.

Once we restrict ourselves to string diagrams in which all lines are labeled by 1-morphisms admitting an ambidextruous adjoint, and having fixed adjunction 2-morphisms, we can allow for lines with U-turns in string diagrams with the appropriate one of the four adjunction 2-morphisms at the cups and caps, because the relations we have just presented allow us to consistently apply isotopies to all lines. We thus obtain complete isotopy invariance.

For a biadjoint pair (F, G) one can in particular form the composition $\tilde{\varepsilon} \circ \eta$, which is an endomorphism of the identity functor $Id_{\mathcal{A}}$, as well as $\varepsilon \circ \tilde{\eta}$ which is an endomorphism of $Id_{\mathcal{B}}$. Graphically,

$\tilde{\varepsilon} \circ \eta \ =$ 

$\varepsilon \circ \tilde{\eta} \ =$ 

(6.13)

It should be appreciated that if the adjunction 2-morphisms are rescaled, the endomorphisms of $Id_{\mathcal{A}}$ and $Id_{\mathcal{B}}$ which appear here get rescaled by reciprocal factors.

6.2. Frobenius algebras from string diagrams. So far our discussion concerned general bicategories. We now turn to the bicategory of surface operators separating modular tensor categories. As was argued in [KS2], from a surface defect S separating a modular tensor category $\mathcal{C}$ from itself, we expect to be able to construct a symmetric special Frobenius algebra in $\mathcal{C}$ for each Wilson line separating S and the transparent defect $T_{\mathcal{C}}$. Different Wilson lines should yield Morita equivalent Frobenius algebras. We now give a proof of this fact that is based on our description of surface defects.

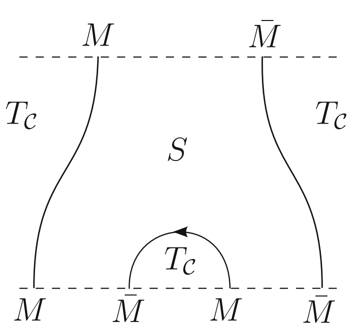
We thus consider a $\mathcal{C}$-module S. Recall that a Wilson line $M \in \mathcal{H}om(S, T_{\mathcal{C}})$ is described by a $\mathcal{C}$-module functor $M\colon\ S \to \mathcal{C}_{\mathcal{C}}$.

Lemma 6.2. *Let $\mathcal{C}$ be a modular tensor category, S an object in $\mathcal{C}$-$\mathcal{M}od$ and $M \in \mathcal{H}om_{\mathcal{C}}(S, T_{\mathcal{C}})$. Then the functor M has a biadjoint as a module functor.*

Proof. M is an additive functor between semisimple $\mathbb{C}$-linear categories. Now as an abelian category, a finitely semisimple $\mathbb{C}$-linear category is equivalent to $(\mathcal{V}ect_{\mathbb{C}})^{\boxtimes n}$ with $n = |\mathrm{Irr}(\mathcal{C})|$ the (finite) number of isomorphism classes of simple objects. Moreover, any additive endofunctor F of $\mathcal{V}ect_{\mathbb{C}}$ is given by tensoring with the vector space $V = F(\mathbb{C})$, and it is ambidextrous, the adjoint being given by tensoring with the dual vector space V^*. It follows that the functor M is equivalent to a functor $\tilde{M}\colon\ (\mathcal{V}ect_{\mathbb{C}})^{\boxtimes n} \to (\mathcal{V}ect_{\mathbb{C}})^{\boxtimes m}$ for some integers n and m and is completely specified by an $n \times m$-matrix of $\mathbb{C}$-vector spaces. Further, both the left and the right adjoint functor to $\tilde{M}$ are then given by the 'adjoint' matrix, and hence $\tilde{M}$ is ambidextrous. As a consequence, M is ambidextrous as a functor. Using arguments from [ENO1], the bi-adjoint of M has two structures of a module functor, from being a left adjoint and right adjoint, respectively. These two structures coincide. □

Since all adjunctions involved are ambidextruous, we can from now on freely use isotopies in the manipulations of string diagrams. We next consider the following construction for any module functor $M \in \mathcal{H}om_{\mathcal{C}}(S, T_{\mathcal{C}})$. Denote by $\bar{M}$ the module functor biadjoint to M and set $A^M := M \circ \bar{M}$. Then $A^M \in \mathcal{H}om(T_{\mathcal{C}}, T_{\mathcal{C}})$, which by Proposition 2.4 is equivalent to $\mathcal{C}$ as a monoidal category. We proceed to equip the object $A^M \in \mathcal{C}$ with the structure of a Frobenius algebra in $\mathcal{C}$. For the product, we introduce the morphism $m_{A^M}\colon\ A^M \otimes A^M \to A^M$ as

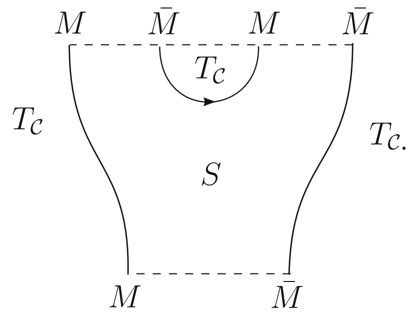
m_{A^M} :=



(6.14)

in terms of string diagrams. Similarly, we introduce the coproduct $\Delta_{A^M} :\ A^M \to A^M \otimes A^M$ as

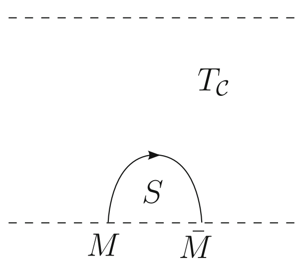
$$\Delta_{A^M} \;:=$$



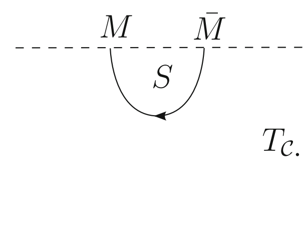
(6.15)

The morphisms for counit ε_{A^M} and unit η_{A^M} are given by

$$\varepsilon_{A^M} \;:=$$



$$\eta_{A^M} \;:=$$

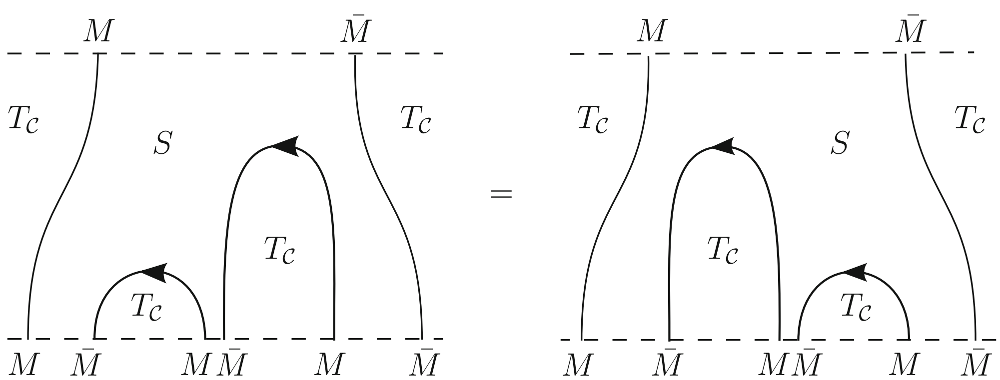


(6.16)

It should be appreciated that rescaling the adjunction morphisms rescales product and coproduct, and unit and counit, by inverse factors.

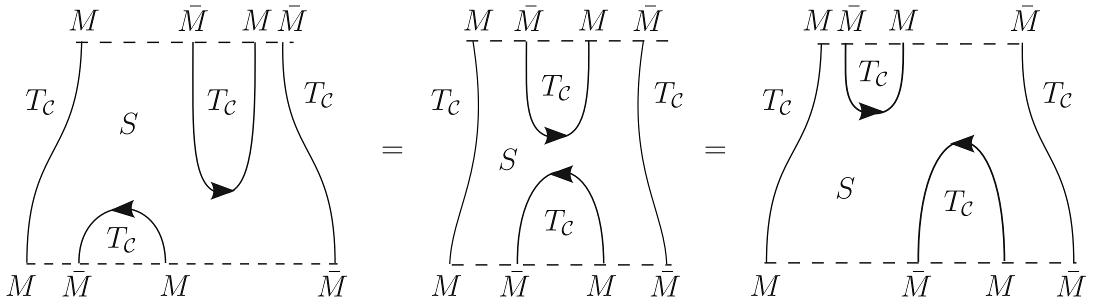
Proposition 6.3. *Let S be an object in $\mathcal{C}$-$\mathcal{M}od$ corresponding to an exact module category. Then for any $M \in \mathcal{H}om(S, \mathcal{C}_{\mathcal{C}})$ the morphisms m_{A^M}, η_{A^M}, Δ_{A^M} and ε_{A^M} just introduced endow the object A^M with the structure of a symmetric Frobenius algebra in $\mathcal{C}$.*

Proof. The equality



(6.17)

which follows from the properties of string diagrams, shows that the product is associative. Coassociativity of Δ_{A^M} is seen in an analogous manner. The equalities



(6.18)

prove the Frobenius property.

Finally, $\mathcal{C}$ is rigid, and left and right duals coincide. It is not difficult to see that by construction the algebra A^M is equal to its dual. The compositions

$$\varepsilon_{A^M} \circ m_{A^M} : \quad A^M \otimes A^M \to \mathbf{1} \qquad \text{and} \qquad \Delta_{A^M} \circ \eta_{A^M} : \quad \mathbf{1} \to A^M \otimes A^M \tag{6.19}$$

give the duality morphisms. □

There exists a particularly interesting subclass of surface defects for which the Frobenius algebra A^M obtained from any Wilson line has additional properties. We need first the

Definition 6.4. *A surface defect S in $\mathcal{C}$-$\mathcal{M}od$ is said to be* **special** *iff*

$$\text{2-Hom}(Id_S, Id_S) \simeq \mathbb{C}. \tag{6.20}$$

The transparent Wilson line inside a special surface defect S can only have multiples of the identity as insertions. Put differently, there are no non-trivial local excitations on a surface defect of type S other than those related to Wilson lines and their junctions.

As an application of this definition, we consider the following situation in a special surface defect: there is a hole punched out, i.e. the surface contains a disk labeled by the transparent defect $T_{\mathcal{C}}$; the label for the boundary of the disk is a Wilson line M. Since there are no local excitations, we can replace the punched-out hole by the surface defect S, provided that we multiply every expression obtained with this replacement by a scalar factor depending on the Wilson line M. For the moment we cannot yet tell whether this scalar factor is non-zero.

Wilson lines separating special defects from the transparent defect should yield Frobenius algebras with a particular property. Recall from Sect. 2 that a *special algebra A* in a monoidal category $\mathcal{C}$ is an object which is both an algebra and a coalgebra and satisfies

$$\varepsilon \circ \eta = \beta_1 \, id_{\mathbf{1}} \qquad \text{and} \qquad m \circ \Delta = \beta_A \, id_A \tag{6.21}$$

with non-zero complex numbers β_1 and β_A.

Proposition 6.5. *Let S be a special surface defect described by a semisimple module category S over a modular tensor category $\mathcal{C}$. Then for any Wilson line $M \in \mathcal{H}om(S, \mathcal{C}_{\mathcal{C}})$ the corresponding symmetric Frobenius algebra A^M in $\mathcal{C}$ is a special algebra.*

Proof. For the algebra A^M, the composition $m \circ \Delta$ of product and coproduct is described by a string diagram with a hole in the surface defect S whose boundary is labeled by M. Since S is special, there are no local excitations and thus the diagram can be replaced, up to a scalar factor β_{A^M}, by a diagram without hole. On the other hand, since the tensor unit of the modular tensor category $\mathcal{C}$ is simple, the composition $\varepsilon \circ \eta$ of counit and unit of A^M is a multiple β_1 of the identity morphism $id_{\mathbf{1}}$.

Both β_1 and β_{A^M} depend on the choices of adjunction and coadjunction 2-morphisms for $\bar{M}$. However, computing the quantum dimension of A^M using the duality morphisms (6.19), we obtain

$$\dim(A^M) = \beta_1\, \beta_{A^M}\,, \tag{6.22}$$

so that the product of the two scalars is independent of the choices of adjunction 2-morphisms. Since the object A^M has been constructed as a composition of a non-vanishing module functor and its adjoint, A^M is not the zero object. As the only object of a modular tensor category having vanishing quantum dimension is the zero object, we conclude that both scalars β_1 and β_{A^M} are non-zero. Hence the algebra A^M is special. □

We next investigate how the Frobenius algebras A^M for a fixed surface defect S depends on the choice of Wilson line M.

Proposition 6.6. *Let S be a surface defect in $\mathcal{C}$-$\mathcal{M}od$ and let $M, M' \in \mathcal{H}om_{\mathcal{C}}(S, \mathcal{C}_{\mathcal{C}})$ be Wilson lines separating S from the transparent defect $T_{\mathcal{C}}$. Then the symmetric Frobenius algebras A^M and $A^{M'}$ are Morita equivalent.*

Proof. We explicitly construct a Morita context. Consider the objects

$$B := M \circ \bar{M}' \qquad \text{and} \qquad \tilde{B} := M' \circ \bar{M} \tag{6.23}$$

in $\mathcal{E}nd_{\mathcal{C}}(\mathcal{C}_{\mathcal{C}}) \simeq \mathcal{C}$. The counit of the adjunction for M provides a morphism

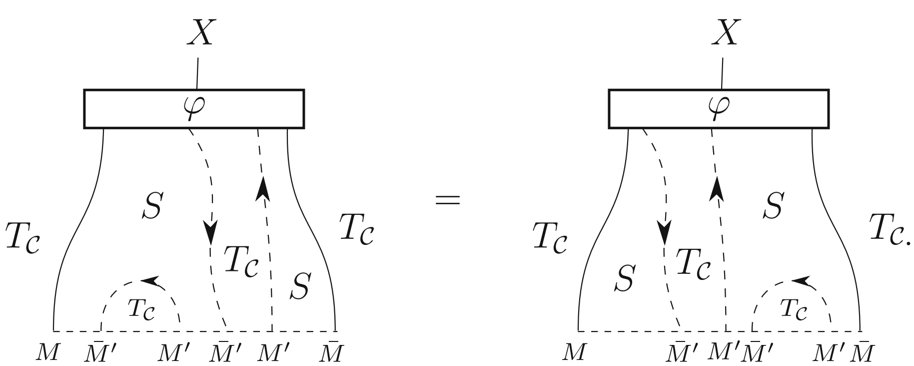
$$M \circ \bar{M} \circ M \circ \bar{M}' \to M \circ \bar{M}'\,, \qquad \text{i.e.} \quad A^M \otimes B \to B. \tag{6.24}$$

With the help of the isotopy invariance of string diagrams, one quickly checks that this morphism obeys the axiom for a left action of A^M on B. This type of argument can be repeated to show that B has a natural structure of an A^M-$A^{M'}$-bimodule, and that $\tilde{B}$ has the structure of an $A^{M'}$-A^M-bimodule.

We next must procure an isomorphism $B \otimes_{A^{M'}} \tilde{B} \to A^M$ of bimodules. This is achieved by showing that the morphism

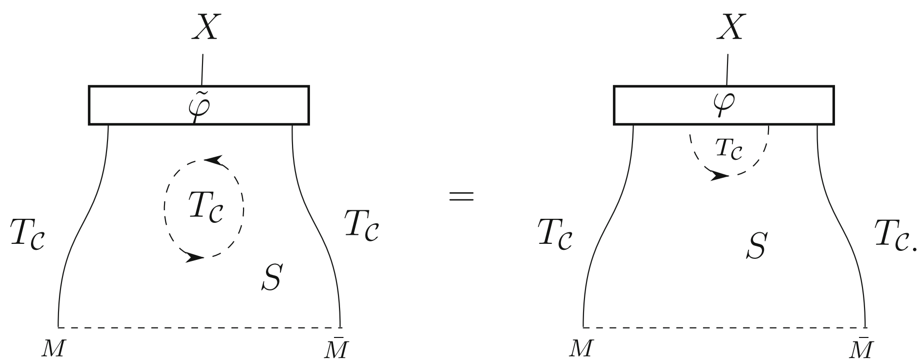
$$M \circ \bar{M}' \circ M' \circ \bar{M} \to M \circ \bar{M}\,, \qquad \text{i.e.} \quad B \otimes \tilde{B} \to A^M \tag{6.25}$$

that is provided by the counit of the adjunction (which is obviously a morphism of bimodules) has the universal property of a cokernel. To this end we select any morphism $\varphi\colon B \otimes A^{M'} \otimes \tilde{B} \to X$, with X any object of $\mathcal{C}$, such that



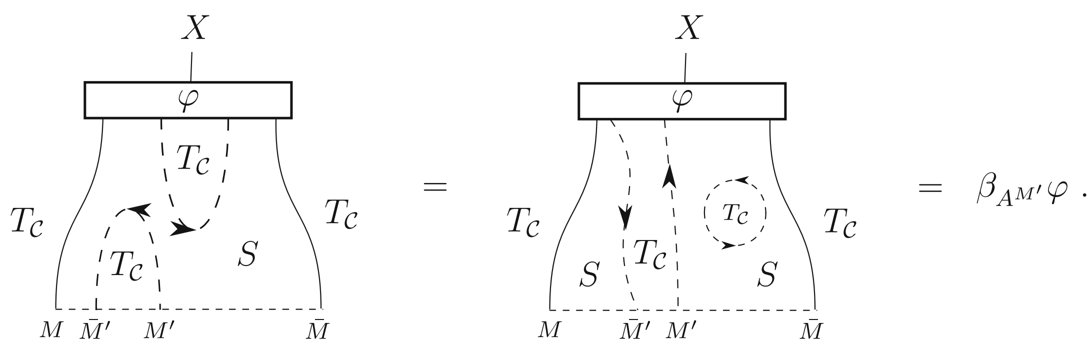
(6.26)

We are looking for a morphism $\tilde{\varphi}\colon B \to X$ such that $\tilde{\varphi} \circ (id_M \otimes \varepsilon_{M'} \otimes id_{\bar{M}}) = \varphi$. Composing this equality with the morphism $id_M \otimes \eta_{M'} \otimes id_{\bar{M}}$ yields



(6.27)

The left hand side of this equality equals $\beta_{A^{M'}}\,\tilde{\varphi}$. This shows that the morphism $\tilde{\varphi}$ is uniquely determined. To establish that A^M is indeed a cokernel, we have to show that the morphism $\beta_{A^{M'}}^{-1}\,\varphi \circ [\, id_M \otimes (\eta_{M'} \circ \varepsilon_{M'}) \otimes id_{\bar{M}}\,]$, which is the composition of $\tilde{\varphi}$ with the cokernel morphism, equals φ. This is established by



(6.28)

Here in the first equality a right action of $A^{M'}$ composed with φ is replaced by a left action, as in (6.26). The second equality uses the fact that the defect S is special, so as to remove the bubble at the expense of a factor of $\beta_{A^{M'}}$. A similar argument shows that $\tilde{B} \otimes_{A^M} B \cong A^{M'}$. This completes the proof. □

We can summarize the findings of this section in the

Theorem 6.7. *Consider the three-dimensional topological field theory corresponding to a modular tensor category. To any surface defect separating the TFT from itself there is associated a Morita equivalence class of special symmetric Frobenius algebras.*

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A Geometric Approach to Boundaries and Surface Defects in Dijkgraaf–Witten Theories

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Abstract: Dijkgraaf–Witten theories are extended three-dimensional topological field theories of Turaev–Viro type. They can be constructed geometrically from categories of bundles via linearization. Boundaries and surface defects or interfaces in quantum field theories are of interest in various applications and provide structural insight. We perform a geometric study of boundary conditions and surface defects in Dijkgraaf–Witten theories. A crucial tool is the linearization of categories of relative bundles. We present the categories of generalized Wilson lines produced by such a linearization procedure. We establish that they agree with the Wilson line categories that are predicted by the general formalism for boundary conditions and surface defects in three-dimensional topological field theories that has been developed in Fuchs et al. (Commun Math Phys 321:543–575, 2013)

1. Introduction

For more than two decades, Dijkgraaf–Witten theories have provided a laboratory for new ideas in mathematical physics. They form a particularly tractable subclass of three-dimensional topological field theories. Since they have a Lagrangian description in which path integrals reduce to counting measures, they also serve as toy models for more complicated classes of topological field theories like Chern–Simons theories.

The defining data of a Dijkgraaf–Witten theory are a finite group G and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. Given these data, a clear geometric construction [Fr, Mor] describes the theory in terms of a linearization of categories of spans of G-bundles. In the present paper we extend this approach by a geometric study of Dijkgraaf–Witten theories on manifolds with boundaries and defects. More specifically, we consider the class of boundary conditions and defects for three-dimensional topological field theories that was investigated in [FSV]. Besides providing new structural insight, such boundary conditions and surface defects are relevant to various applications, ranging from a geometric visualization of the TFT approach to RCFT correlators to universality classes of gapped boundaries and defects in condensed matter systems that are of interest in many areas.

A crucial input in our construction are the concepts of relative manifolds and relative bundles. Via the linearization of relative bundles we obtain categories of generalized Wilson lines for Dijkgraaf–Witten theories with boundaries and defects. Our results perfectly match the general analysis of [FSV], combined with Ostrik's explicit description [Os2] of module categories over the categories of G-graded vector spaces.

The rest of this paper is organized as follows. In Sect. 2 we collect pertinent background information. We start in Sect. 2.1 with a summary of the geometric construction of Dijkgraaf–Witten theories, with emphasis on the implementation of locality, which naturally leads to the use of bicategories. We then present some facts about relative bundles (Sect. 2.3), about groupoid cohomology (Sect. 2.4), and about module categories over the monoidal category G-vect$^{\omega}$ of G-graded vector spaces with associativity constraint twisted by the cocycle ω (Sect. 2.5).

Section 3 contains our results for categories of generalized Wilson lines in Dijkgraaf–Witten theories with defects and boundaries. These categories are associated to one-dimensional manifolds with additional data. In the present paper, we restrict our attention to one-dimensional manifolds, leaving the case of two-dimensional manifolds with boundaries and of three-dimensional manifolds with corners to future work. (The results for two- and three-dimensional manifolds will allow us to make statements about generalized partition functions.) In Sect. 3.1 we discuss the relevant concepts of decorated one-dimensional manifolds and of categories of generalized bundles and use them to obtain the groupoids for the geometric situations of our interest. Afterwards we introduce in Sect. 3.2 the additional data from groupoid cohomology that are needed for the linearization process. From the perspective of Lagrangian field theory, these data are a topological bulk Lagrangian and compatible boundary terms; accordingly we refer to them as Lagrangian data. In Sect. 3.3 we explain how to get 2-cocycles for the groupoids obtained in Sect. 3.1 from Lagrangian data assigned to intervals and circles.

Invoking fusion of defects, all one-dimensional manifolds arising from boundaries and defects can be reduced to two building blocks: the interval without interior marked points, and the circle with a single marked point. The linearization of the groupoids for these two basic situations is described in detail in Sect. 3.5 and 3.7, respectively. A convenient tool in these calculations is a graphical calculus for groupoid cocycles which is inspired by Willerton's work [Wi]. It is introduced in Sect. 3.4. Another input is a concrete description of the transparent surface defect; this is obtained in Sect. 3.6, based crucially on the invariance of the graphical calculus under Pachner moves.

In the considerations in Sects. 3.5 and 3.7 we concentrate on the situation that the relevant group homomorphisms are subgroup embeddings; these lead to indecomposable module categories over G-vect$^{\omega}$. Without this restriction, one obtains decomposable module categories; this is discussed in the Appendix.

2. Background Material

In this section we summarize some background material on the geometric construction of Dijkgraaf–Witten theories and on boundaries and surface defects in three-dimensional topological field theories, and on some aspects of relative bundles.

We fix the following conventions. By vect$_{\Bbbk}$ we denote the category of finite-dimensional vector spaces over a field $\Bbbk$; In the present paper we only consider the case of complex vector spaces, $\Bbbk = \mathbb{C}$. A group is assumed to be finite. Manifolds, including manifolds with boundaries and manifolds with corners, are smooth.

For a finite group G and a smooth manifold X of any dimension, we denote by $\mathrm{Bun}_G(X)$ the category of smooth G-principal bundles, which has maps covering the identity as morphisms. We adopt the convention that the G-action on the fiber of a principal G-bundle is a right action. In particular, a G-bundle over a point is just a right G-torsor. Morphisms of the category $\mathrm{Bun}_G(X)$ are morphisms of G-bundles covering the identity. They are all invertible, i.e. $\mathrm{Bun}_G(X)$ is a groupoid. Diffeomorphisms $f\colon X \to Y$ relate the groupoids by pullback functors, $f^*\colon \mathrm{Bun}_G(Y) \to \mathrm{Bun}_G(X)$. We note that with respect to e.g. surjective submersions, Bun_G becomes a stack on the category of smooth manifolds; we will not use the language of stacks in this paper, though.

2.1. The geometric construction of Dijkgraaf–Witten theories. A classic definition by Atiyah characterizes d-dimensional topological field theories as symmetric monoidal functors from a geometric category, the symmetric monoidal category $\mathrm{cobord}_{d,d-1}$ of d-dimensional cobordisms, to some linear category, e.g. to the symmetric monoidal category $\mathrm{vect}_{\mathbb{C}}$. A classic result states that for $d = 2$ the functor given by

$$\mathrm{tft} \longmapsto \mathrm{tft}(S^1) \tag{2.1}$$

is an equivalence between the category of topological field theories and the category of complex commutative Frobenius algebras.

Dijkgraaf–Witten theories are three-dimensional topological field theories. The Dijkgraaf–Witten theory

$$\mathrm{tft}_G : \quad \mathrm{cobord}_{3,2} \to \mathrm{vect}_{\mathbb{C}} \tag{2.2}$$

based on a finite group G can be characterized as follows. The functor tft_G associates to a closed oriented surface Σ the vector space $\mathrm{tft}_G(\Sigma)$ freely generated by the set of isomorphism classes of principal G-bundles on Σ. To a cobordism

$$\begin{tikzcd} & M & \\ \Sigma \arrow[ur] & & \Sigma' \arrow[ul] \end{tikzcd} \tag{2.3}$$

it associates a linear map $\mathrm{tft}_G(\Sigma) \to \mathrm{tft}_G(\Sigma')$ whose matrix element for principal G-bundles P on Σ and P' on Σ' is the number $|\mathrm{Bun}_G(M, P, P')|$. Here $\mathrm{Bun}_G(M, P, P')$ is the groupoid of G-bundles on M that restrict to a given G-bundle P on Σ and to P' on Σ', and for any groupoid Γ we denote by $|\Gamma|$ the groupoid cardinality, which is the rational number

$$|\Gamma| := \sum_{\gamma \in \pi_0(\Gamma)} \frac{1}{|\mathrm{Aut}(\gamma)|} \tag{2.4}$$

obtained by summing over the set $\pi_0(\Gamma)$ of isomorphism classes of objects of Γ.

The introduction of $d-1$-dimensional manifolds can be seen as a first step towards implementing locality in topological field theories: These submanifolds can be used to cut the d-dimensional manifold into smaller and simpler pieces, which are manifolds with boundary. The boundaries of cobordisms are thus to be thought of as "cut-and-paste boundaries". They must not be mixed up with physical boundaries to be discussed in Sect. 2.2.

Our analysis uses a framework which goes one step further in the implementation of locality and naturally leads to the use of bicategories. We need the following concepts:

Definition 2.1. *(i) The bicategory* 2-vect$_{\mathbb{C}}$ *of complex 2-vector spaces is the bicategory of $\mathbb{C}$-linear finitely semisimple abelian categories. The Deligne product of $\mathbb{C}$-linear categories endows this bicategory with the structure of a symmetric monoidal bicategory.*

(ii) The symmetric monoidal category $\mathrm{cobord}_{3,2,1}$ *has as objects compact oriented smooth one-dimensional manifolds. 1-morphisms are two-dimensional manifolds with boundary; 2-morphisms are three-manifolds with corners, up to diffeomorphisms preserving the orientation and the boundary. (For brevity we suppress collars in our discussion.)*

(iii) An extended three-dimensional topological field theory is a symmetric monoidal functor

$$\mathrm{tft}: \quad \mathrm{cobord}_{3,2,1} \rightarrow 2\text{-}\mathrm{vect}_{\mathbb{C}}. \tag{2.5}$$

We note that, as a consequence of the axioms,

$$\mathrm{tft}(S \sqcup S') \cong \mathrm{tft}(S) \boxtimes \mathrm{tft}(S') \tag{2.6}$$

for any pair (S, S') of one-dimensional manifolds, and $\mathrm{tft}(\emptyset) = \mathrm{vect}_{\mathbb{C}}$, where $\emptyset$ is considered as a one-dimensional manifold and monoidal unit of $\mathrm{cobord}_{3,2,1}$.

The Dijkgraaf–Witten theory based on a finite group G is in fact an extended topological field theory [Fr, Mor]. It assigns to a one-dimensional manifold S the category

$$\mathrm{tft}_G(S) := [\mathrm{Bun}_G(S), \mathrm{vect}_{\mathbb{C}}]. \tag{2.7}$$

Here by $[\,\mathcal{C}_1, \mathcal{C}_2\,]$ we denote the category of functors between two (essentially small) categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

This formula already gives a hint on the general construction of the theory: In a first step, one uses the functor Bun_G that associates to a smooth manifold the groupoid of G-bundles to construct a bifunctor

$$\mathrm{cobord}_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}_G}} \mathrm{SpanGrp} \tag{2.8}$$

to a bicategory of spans of groupoids. In a second step one linearizes by taking functor categories with values in $\mathrm{vect}_{\mathbb{C}}$,

$$\mathrm{tft}_G: \quad \mathrm{cobord}_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}_G}} \mathrm{SpanGrp} \xrightarrow{[-,\mathrm{vect}_{\mathbb{C}}]} 2\text{-}\mathrm{vect}_{\mathbb{C}}. \tag{2.9}$$

The non-extended topological field theory can be obtained from this extended topological field theory by restricting to the endomorphism categories of the monoidal units of $\mathrm{cobord}_{3,2,1}$ and 2-vect$_{\mathbb{C}}$, since $\mathrm{End}_{\mathrm{cobord}_{3,2,1}}(\emptyset) \cong \mathrm{cobord}_{3,2}$ and $\mathrm{End}_{2\text{-}\mathrm{vect}_{\mathbb{C}}}(\mathrm{vect}_{\mathbb{C}}) \cong \mathrm{vect}_{\mathbb{C}}$.

The fact that tft_G involves pure counting measures amounts to considering vanishing Lagrangians. Dijkgraaf and Witten [DW] introduced the following generalization, in which the linearization is only projective. Select a cocycle ω representing a class $[\omega] \in H^3(G, \mathbb{C}^\times)$ in group cohomology. One may think about this class as a 2-gerbe [Wi] on the classifying space BG of G-bundles, which we represent by the action groupoid $*//G$ of G acting on a single object $*$. A G-bundle on a 3-manifold M corresponds to a map into this classifying space. Pulling back the 2-gerbe along this map to M we get a 2-gerbe on M, which for dimensional reasons is trivial. It therefore gives rise to a 3-manifold holonomy, which should be seen as the value of a topological Lagrangian. For this reason, we refer to the cocycle ω (and later on to similar quantities) as a Lagrangian datum.

The second step of the construction of Dijkgraaf–Witten models consists of a linearization of the groupoids obtained in the first step. In general, such a linearization is only projective. The relevant 2-cocycle on the groupoids must be derived from the Lagrangian data. In the case at hand, the 3-cocycle ω can be transgressed [Wi] to a cocycle $\tau(\omega)$ representing a class in $H^2(G/\!/G, \mathbb{C}^\times)$, the groupoid cohomology for the action groupoid $G/\!/_{\mathrm{ad}}G$ with G acting on itself by the adjoint action.

Direct calculation now yields [Mor] $\mathrm{tft}_G(S^1) = \mathcal{D}^\omega(G)$-mod, i.e. the category associated to the circle is the modular tensor category of modules over the twisted Drinfeld double [DPR] of the category of G-graded vector spaces – or, equivalently, of complex representations of the finite group G. This category is the category of bulk Wilson lines. The goal of the present paper is to generalize this construction to more general cobordism categories and to consistently obtain categories of generalized Wilson lines: both bulk and boundary Wilson lines. Our construction requires the use of more general categories of bundles on smooth manifolds.

2.2. Boundaries and defects in three-dimensional TFT. The structure of boundary conditions in two-dimensional topological field theories is well understood [LaP,MoS] in the framework of open/closed topological field theories. In this setting one considers a larger cobordism category $\mathrm{cobord}_{2,1}^{\mathrm{op/cl}}$. Its objects are one-dimensional smooth manifolds with boundary, with a suitable boundary condition fixed for each connected component of the (physical) boundary. Morphisms are now cobordisms with boundary, with each boundary component partitioned into segments each of which is either a physical boundary or a cut-and-paste boundary. An open/closed topological field theory is then a symmetric monoidal functor $\mathrm{cobord}_{2,1}^{\mathrm{op/cl}} \to \mathrm{vect}_{\mathbb{C}}$. It turns out that a boundary condition a gives rise to a (not necessarily commutative) Frobenius algebra W_a whose center is the commutative Frobenius algebra $\mathrm{tft}(S^1)$. Explicitly, a boundary condition is thus a pair consisting of a Frobenius algebra W_a and an isomorphism

$$\mathrm{tft}(S^1) \xrightarrow{\cong} Z(W_a) \tag{2.10}$$

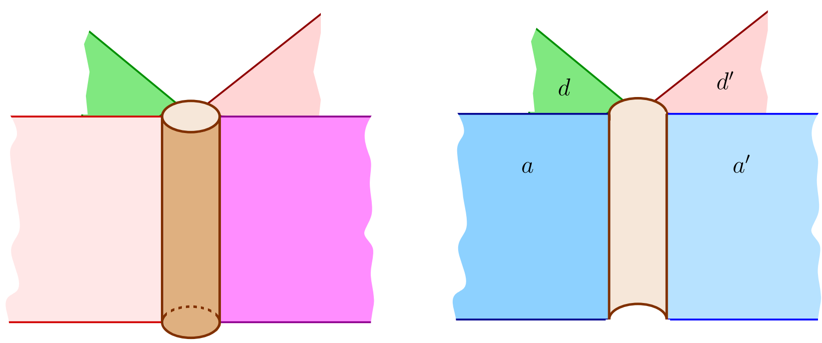
of commutative associative algebras. Once such a Frobenius algebra W_a has been determined, the category of boundary conditions can be described as the category W_a-mod.

We pause for two comments. First, we allow for point-like insertions on boundaries that separate possibly different boundary conditions. As a consequence, boundary conditions form a category rather than a set: The space $\mathrm{Hom}_{W_a\text{-mod}}(M_c, M_d)$ of morphisms between two boundary conditions $M_c, M_d \in W_a$-mod is the vector space of labels for insertions that separate the boundary condition M_c from the boundary condition M_d. Second, distinguishing one boundary condition in the discussion could be avoided, but at the price of using a higher-categorical language: the one of module categories over $\mathrm{vect}_{\mathbb{C}}$. For the three-dimensional topological field theories we are interested in, a Morita invariant treatment would amount to working with three-categories; we prefer an approach that avoids this. For a more detailed analysis of two-dimensional open/closed topological field theories we refer to the literature, in particular to [LaP].

Once one allows for manifolds with boundary, codimension-one defects that partition a manifold into cells supporting possibly different topological field theories are a natural extension of the picture described above. For two-dimensional theories such defects provide a lot of additional insight, in particular about symmetries and dualities [FFRS].

In three-dimensional topological field theories, boundary conditions and defects have been studied only recently. In this case, codimension-one defects are surface defects. Boundaries and surface defects in three-dimensional topological field theories of Reshetikhin–Turaev type appear in a geometric interpretation [KaS] of the TFT approach [SFR] to RCFT correlators and as models for universality classes of gapped boundaries and gapped interfaces for topological phases (see e.g. [KK, WW, Le, BJQ, Ka]), which arise for instance in the study of 2+1-dimensional electron fluids, including certain fractional quantum Hall states.

A model-independent study of boundary conditions and surface defects in such theories [FSV] yields the following results, which can be regarded as a categorified version of the results in two dimensions described above. To any boundary condition a there is associated a fusion category $\mathcal{W}_a$. It describes boundary Wilson lines, i.e. Wilson lines that are confined to the boundary with boundary condition a. Let us recall that, depending on the chosen formalism, Wilson lines are embedded ribbons or tubes with a marked line at the boundary of the tube. In a similar spirit, boundary Wilson lines should be described by half-tubes extending into the three-dimensional bulk, as illustrated by the following picture:



(2.11)

Here the figure on the right shows a boundary Wilson line in the form of a half-tube separating two (possibly different) boundary conditions a and a' and at which two surface defects d and d' end, while the left figure shows a bulk Wilson line in the form of a tube at which four surface defects end.

Since boundary Wilson lines are objects in a two-dimensional theory, the category $\mathcal{W}_a$ is not braided. A boundary condition can now be defined as a pair consisting of a fusion category $\mathcal{W}_a$ and a braided equivalence

$$\mathcal{C} = \mathrm{tft}(S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a), \tag{2.12}$$

where $\mathcal{Z}$ denotes the Drinfeld center of the fusion category $\mathcal{W}_a$, which is a braided monoidal category. We refer to an equivalence of the type (2.12) as a *Witt trivialization* of $\mathcal{C}$. One should note that not any braided category is equivalent to a Drinfeld center. In general three-dimensional topological field theories this is a source of obstructions. But in the case of Dijkgraaf–Witten theories the relevant modular tensor category $\mathcal{C}$ indeed is a Drinfeld double, namely the Drinfeld double of the fusion category G-vect$^{\omega}$ of G-graded vector spaces with associator twisted by ω (see Sect. 2.5)

$$\mathcal{C} = \mathcal{Z}(G\text{-vect}^{\omega}). \tag{2.13}$$

As a consequence, in the case of our interest the existence of boundary conditions is not obstructed.

The collection of all boundary conditions now has the structure of a bicategory: the bicategory of all module categories over the fusion category $\mathcal{W}_a$. (Module categories over a fusion category are a categorification of the notion of a module over a ring; we refer to [Os1] for details.) The category of boundary Wilson lines separating two boundary conditions c and d that are given by two $\mathcal{W}_a$-module categories $\mathcal{M}_c$ and $\mathcal{M}_d$, respectively, is the abelian $\mathbb{C}$-linear category

$$\mathrm{Fun}_{\mathcal{W}_a\text{-mod}}(\mathcal{M}_c, \mathcal{M}_d) \tag{2.14}$$

of $\mathcal{W}_a$-module functors.

A similar analysis can be carried out for surface defects that separate two topological field theories of Reshetikhin–Turaev type, which are labeled by modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$. The category of Wilson lines in a surface defect of type d is now a fusion category $\mathcal{W}_d$ together with a braided equivalence

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d). \tag{2.15}$$

Since the modular categories relevant for Dijkgraaf–Witten theories are already Drinfeld centers themselves, the existence of surface defects between any two Dijkgraaf–Witten theories is not obstructed. The category of Wilson lines separating surface defects that are given by two $\mathcal{W}_d$-module categories $\mathcal{M}_c$ and $\mathcal{M}_d$, respectively, is the abelian $\mathbb{C}$-linear category

$$\mathrm{Fun}_{\mathcal{W}_a\text{-mod}}(\mathcal{M}_c, \mathcal{M}_d) \tag{2.16}$$

of $\mathcal{W}_d$-module functors.

In the special case of defects separating a modular tensor category $\mathcal{C}$ from itself, we can work with the canonical Witt trivialization

$$\mathrm{can}: \quad \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}). \tag{2.17}$$

This functor maps the object $U \boxtimes V \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ to the object $U \otimes V \in \mathcal{C}$ endowed with a half braiding $e_{U\otimes V}$ given by [ENO, Eq. (4.2)]

$$e_{U\otimes V}(X): \quad U \otimes V \otimes X \xrightarrow{c^{-1}} U \otimes X \otimes V \xrightarrow{c} X \otimes U \otimes V. \tag{2.18}$$

With respect to the canonical Witt trivialization (2.17), we describe a defect separating $\mathcal{C}$ from itself by a $\mathcal{C}$-module category. Now $\mathcal{C}$ has a natural structure of module category over itself. This specific $\mathcal{C}$-module category describes a particularly important surface defect, the *transparent* (or *invisible*) surface defect. In fact, one expects a notion of a fusion product of defects, so that the bicategory of surface defects is even a monoidal bicategory. The transparent defect is then the tensor unit of the monoidal bicategory of defects. (At one step lower in the categorical ladder, the tensor unit of the monoidal category of endofunctors of any given defect category describes a Wilson line that is invisible inside the surface. The category of endofunctors of $\mathcal{C}$ describes Wilson lines inside the transparent defect; these are ordinary bulk Wilson lines. In particular, the tensor unit of this monoidal category is the invisible bulk Wilson line.)

Our goal in this paper is to achieve a concrete geometric, Lagrangian construction of some of the categories describing Wilson lines in the presence of boundaries and surface defects in Dijkgraaf–Witten theories in the spirit of [Fr, Mor]. To this end, we need the appropriate geometric objects that form categories whose linearizations enter in the topological field theory.

2.3. Relative bundles. In this section we review the notion of a relative bundle. We restrict our attention to finite groups, which is sufficient for our construction.

Definition 2.2. *Let G and H be finite groups, $\iota\colon H \to G$ a morphism of finite groups, and X a smooth manifold, Then the functor*

$$\mathrm{Ind}_\iota : \quad \mathrm{Bun}_H(X) \to \mathrm{Bun}_G(X) \tag{2.19}$$

is the one that acts on objects as $P_H \mapsto P_H \times_H G$.

Remark 2.3. (i) If the group homomorphism ι injective, then the functor Ind_ι is injective on morphisms.

Indeed, suppose $f_1, f_2\colon P_H \to P'_H$ are two different morphisms of H-bundles on X. Then there exist points $x \in X$ and p in the fiber of P_H over x such that $f_1(p) \neq f_2(p)$. Since both $f_1(p)$ and $f_2(p)$ are in the fiber of P'_H over x, we have a unique $h \in H \setminus \{e\}$ such that $f_1(p) = f_2(p).h$. Suppose that after induction $[f_1(p), g] = [f_2(p), g]$ for some $g \in G$. Then

$$[f_1(p), g] = [f_2(p), g] = [f_1(p).h, g] = [f_1(p), \iota(h) \cdot g]. \tag{2.20}$$

Equality of the left and right hand sides implies $\iota(h) \cdot g = g$, i.e. $\iota(h) = e$. If ι is injective, this is impossible for $h \neq e$.

(ii) Induction commutes with pullback: if $f\colon X_1 \to X_2$ is a morphism of smooth manifolds and if $P_H^{(2)}$ is a H-bundle on X_2, then

$$\mathrm{Ind}_\iota\, f^* P_H^{(2)} = f^* \,\mathrm{Ind}_\iota P_H^{(2)}. \tag{2.21}$$

More abstractly, for any finite group G we have the stack $\mathrm{Bun}_G(-)$ of G-bundles on the category of smooth manifolds with topology given by surjective submersions. Induction is also compatible with descent. Thus Ind_ι gives a morphism $\mathrm{Ind}_\iota\colon \mathrm{Bun}_H \to \mathrm{Bun}_G$ of stacks.

A crucial ingredient for our construction is the notion of relative smooth manifolds and relative bundles. This is as follows, see e.g. [St].

Definition 2.4. *(i) A* relative *(smooth)* manifold *$Y \xrightarrow{j} X$ consists of a pair Y, X of smooth manifolds and a morphism $j\colon Y \to X$ of smooth manifolds.*

A morphism $(Y_1 \xrightarrow{j_1} X_1) \longrightarrow (Y_2 \xrightarrow{j_2} X_2)$ of relative smooth manifolds is a commuting diagram

$$\begin{tikzcd} Y_1 \arrow[r, "f_Y"] \arrow[d, "j_1"'] & Y_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f_X"'] & X_2 \end{tikzcd} \tag{2.22}$$

in the category of smooth manifolds.

(ii) Let $\iota\colon H \to G$ be a homomorphism of finite groups. A relative (G, H)-bundle *on the relative manifold $Y \xrightarrow{j} X$ is a triple consisting of a G-bundle P_G on X, an H-bundle P_H on Y, and an isomorphism*

$$\alpha : \quad \mathrm{Ind}_\iota(P_H) \xrightarrow{\simeq} j^*(P_G) \tag{2.23}$$

of G-bundles on Y.

(iii) A morphism $(P_G, P_H, \alpha) \to (P'_G, P'_H, \alpha')$ *of relative* (G, H)*-bundles on a relative smooth manifold* $Y \xrightarrow{j} X$ *consists of a morphism*

$$\varphi_G : \quad P_G \to P'_G \tag{2.24}$$

of G*-bundles on* X *and of a morphism*

$$\varphi_H : \quad P_H \to P'_H \tag{2.25}$$

of H*-bundles on* Y *such that the diagram*

$$\begin{tikzcd} \mathrm{Ind}_\iota(P_H) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Ind}_\iota \varphi_H"'] & j^* P_G \arrow[d, "j^*\varphi_G"] \\ \mathrm{Ind}_\iota(P'_H) \arrow[r, "\alpha'"] & j^* P'_G \end{tikzcd} \tag{2.26}$$

of morphisms of G*-bundles on* Y *commutes.*
The category of relative (G, H)*-bundles on* (X, Y) *is denoted by* $\mathrm{Bun}_{(G,H)}(Y \to X)$.

Remark 2.5. (i) The category $\mathrm{Bun}_{(G,H)}(Y \to X)$ depends the group homomorphism $\iota \colon H \to G$. The notation $\mathrm{Bun}_{(G,H)}(Y \to X)$ suppresses this dependence and is thus slightly inappropriate.

(ii) The category $\mathrm{Bun}_{(G,H)}(Y \to X)$ inherits from the category of principal bundles the property of being a groupoid: all morphisms of relative bundles are invertible.

(iii) For the special case that $j = id_X$ is the identity on $X = Y$, we obtain the notion of a reduction of a G-bundle to an H-bundle along the group homomorphism ι.

(iv) As an object, a relative bundle is thus a G-bundle P_G on X together with a reduction of its pullback $j^* P_G$ to an H-bundle along the group homomorphism ι. One should note, however, that the *morphisms* in $\mathrm{Bun}_{G,H}(X, Y)$ are *not* simply morphisms of reductions, which would only involve a morphism of G-bundles on the manifold Y. Rather, also a G-morphism on the manifold X is required. (Later on, Y will typically be a submanifold of X; hence we require a morphism on a larger manifold in that case.) In gauge theory terminology, the morphisms are thus gauge transformations on Y and on X, respectively.

(v) If the group homomorphism ι is injective, then by Remark 2.3(i) the morphism φ_H of H-bundles is determined uniquely by φ_G, provided it exists. It is thus not an extra datum. The morphisms of relative (G, H)-bundles are in this situation morphisms of G-bundles that are compatible with the reductions.

(vi) Fix a homomorphism $\iota \colon H \to G$ of finite groups and consider a relative bundle (P^2_G, P^2_H, α^2) on the relative manifold $Y_2 \xrightarrow{j_2} X_2$. We define a pullback of relative bundles along the morphism

$$\begin{tikzcd} Y_1 \arrow[r, "f_Y"] \arrow[d, "j_1"'] & Y_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f_X"'] & X_2 \end{tikzcd} \tag{2.27}$$

of relative manifolds. Since induction and pullback commute by Remark 2.3(ii), we have a canonical isomorphism

$$\mathrm{Ind}_\iota(f_Y^* P_H^2) \cong f_Y^* \,\mathrm{Ind}_\iota P_H^2 \tag{2.28}$$

of bundles. Noting that $f_X \circ j_1 = j_2 \circ f_Y$, we also have another isomorphism

$$j_1^* f_X^* P_G^2 \cong f_Y^* j_2^* P_G^2 \tag{2.29}$$

of G-bundles, and thus an isomorphism

$$f_Y^*(\alpha): \quad \mathrm{Ind}_\iota(f_Y^* P_H^2) \to f_Y^* \mathrm{Ind}_\iota P_H^2 \to f_Y^* f_2^* P_G^2 \to j_1^* f_X^* P_G^2 \tag{2.30}$$

of G-bundles on Y_1. Hence $(f_X^* P_G^2, f_Y^* P_H^2, f_Y^*(\alpha))$ is a relative (G, H)-bundle on (X_1, Y_1).

We have thus a bifunctor $\mathrm{Bun}_{\iota\colon H\to G}$ from the category opposite to the category of relative manifolds to the bicategory of groupoids, i.e. a prestack $\mathrm{Bun}_{(G,H)}$ on the category of relative manifolds.

It should be appreciated that we do not require the group homomorphism $\iota\colon H \to G$ to be injective. For later use, we will consider two examples.

Example 2.6. Consider the case that $X = Y$ is a point. Bundles are then torsors $\underline{H}$ and $\underline{G}$, respectively, which are unique up to isomorphism. The additional datum characterizing a relative bundle is then an isomorphism

$$\alpha: \quad \underline{H} \times_H G \xrightarrow{\cong} \underline{G} \tag{2.31}$$

of torsors. If we fix base points $*_H \in \underline{H}$ and $*_G \in \underline{G}$, then α is determined by the group element $\gamma_\alpha \in G$ such that $\alpha([*_H, e]) = *_G .\gamma_\alpha$.

Morphisms $(\underline{G}, \underline{H}, \alpha) \to (\underline{G}', \underline{H}', \alpha')$ are pairs of morphisms $\varphi_H\colon \underline{H} \to \underline{H}'$ and $\varphi_G\colon \underline{G} \to \underline{G}'$ of torsors. Using the base points $*_H$ and $*'_H$ of $\underline{H}$ and $\underline{H}'$, respectively, and similarly base points of the G-torsors, morphisms are described by group elements $g \in G$ and $h \in H$ such that

$$\varphi_H(*_H) = *'_H . h \qquad \text{and} \qquad \varphi_G(*_G) = *'_G . g. \tag{2.32}$$

The commuting diagram (2.26) requires that

$$\varphi_G(\alpha[*_H, e]) = \varphi_G(*_G.\gamma_\alpha) = *'_G . (g\gamma_\alpha) \tag{2.33}$$

equals

$$\alpha'(\mathrm{Ind}_\iota \varphi_H([*_H, e]) = \alpha'([*'_H h, e]) = \alpha'([*'_H, \iota(h)]) = *'_G . (\gamma_{\alpha'} \iota(h)). \tag{2.34}$$

We thus find the condition

$$g\,\gamma_\alpha = \gamma_{\alpha'}\,\iota(h) \tag{2.35}$$

on the pair (g, h) of group elements. As expected, for ι injective, this determines h in terms of g. Moreover, given any two relative bundles, we can always find group elements g and h such that this relation holds. So there is a single isomorphism class of objects. In particular, we can restrict our attention to just one H-torsor $\underline{H}$ and one G-torsor $\underline{G}$. Then we get a category with objects labeled by $\gamma_\alpha \in G$ and morphisms being pairs (g, h) such that $g\gamma_\alpha = \gamma_{\alpha'}\iota(h)$, or put differently, the action groupoid

$$G \,\backslash\!\!\backslash\, G /\!\!/_{\iota^-} H. \tag{2.36}$$

Here the notation is as follows. We deal with left actions for both G and H. The left action of the group G is simply left multiplication, while the left action of H is right multiplication after applying the group homomorphism ι and taking the inverse, i.e. $(g, h).\gamma = g \cdot \gamma \cdot \iota(h)^{-1}$.

Example 2.7. Take for X a closed interval and for Y the subset consisting of its two end points, which we label by $1, 2$. Since the interval is contractible and G is finite, the category of G-bundles on X is canonically equivalent to the category of G-torsors. Similarly we have H_1- and H_2-torsors, one over each end point. We fix one such torsor for each end point and for the interval itself from now on. We also fix base points $*_{H_1}$, $*_{H_2}$ and $*_G$ for these torsors. Objects in the category are then pairs $(\gamma_{\alpha,1}, \gamma_{\alpha,2}) \in G \times G$ which describe the morphisms of torsors as

$$\alpha_1([*_{H_1}, e]) = *_G.\gamma_{\alpha,1} \qquad \text{and} \qquad \alpha_2([*_{H_2}, e]) = *_G.\gamma_{\alpha,2}. \tag{2.37}$$

The morphisms are described by triples $(h_1, h_2, g) \in H_1 \times H_2 \times G$ satisfying

$$\varphi_{H_1}(*_{H_1}) = *_{H_1}.\, h_1, \quad \varphi_{H_2}(*_{H_2}) = *_{H_2}.\, h_2 \quad \text{and} \quad \varphi_G(*_G) = *_G.\, g. \tag{2.38}$$

Based on the commuting diagram (2.26), we check when a triple (h_1, h_2, g) gives a morphism $(\gamma_{\alpha,1}, \gamma_{\alpha,2}) \to (\gamma'_{\alpha,1}, \gamma'_{\alpha,2})$. As before we compute

$$\varphi_G(\alpha_i[*_{H_i}, e]) = \varphi_G(*_G\gamma_{\alpha,i}) = *_G.\,(g\gamma_{\alpha,i}) \tag{2.39}$$

and

$$\alpha'_i(\mathrm{Ind}_\iota\varphi_H([*_{H_i}, e]) = \alpha'_i([*_{H_i}h_i, e]) = \alpha'([*_{H_i}, \iota(h_i)]) = *_G.\,(\gamma'_{\alpha,i}.\iota(h_i)). \tag{2.40}$$

We thus arrive at the equalities

$$g\,\gamma_{\alpha,i} = \gamma'_{\alpha,i}\,\iota(h_i) \tag{2.41}$$

for $i = 1, 2$. Hence the action groupoid is

$$G \,\backslash\!\backslash\, G \times G /\!/_{\iota_1^- \times \iota_2^-} H_1 \times H_2, \tag{2.42}$$

where the G-action is the diagonal one.

2.4. Groupoid cohomology and gerbes on groupoids. The definition of a Dijkgraaf–Witten theory on a three-manifold requires, as an additional datum besides a finite group G, the choice of a 3-cocycle $\omega \in Z(G, \mathbb{C}^\times)$. This cocycle enters in the linearization. We now describe how this 3-cocycle can be seen geometrically as a 2-gerbe on the groupoid $*/\!/G$.

We first give a brief outline of groupoid cohomology. Given a finite groupoid $\Gamma = (\Gamma_0, \Gamma_1)$, consider its nerve, which is a simplicial set

$$\left(\cdots \overset{\partial_0}{\underset{\partial_3}{\Rrightarrow}} \Gamma_2 \overset{\partial_0}{\underset{\partial_2}{\Rrightarrow}} \Gamma_1 \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \Gamma_0 \right) =: \Gamma_\bullet, \tag{2.43}$$

where for $i \geq 1$, Γ_i consists of i-tuples of composable morphisms of Γ. Applying the functor $\mathrm{Map}(-, \mathbb{C}^\times)$ and taking alternating combinations of the face maps yields a complex

$$\mathrm{Map}(\Gamma_0, \mathbb{C}^\times) \to \mathrm{Map}(\Gamma_1, \mathbb{C}^\times) \to \mathrm{Map}(\Gamma_2, \mathbb{C}^\times) \to \mathrm{Map}(\Gamma_3, \mathbb{C}^\times) \to \cdots \tag{2.44}$$

of groups. A group G gives rise to the groupoid $*/\!/G$ with a single object. In this case the complex (2.44) reduces to the standard bar complex.

It is useful to think about cochains in this complex in a geometric way.

Definition 2.8. *An n-*gerbe *on the groupoid Γ is an $(n{+}1)$-cocycle*

$$\omega \in Z^{n+1}(\Gamma, \mathbb{C}^\times). \tag{2.45}$$

Using standard facts about complexes in small abelian categories one deduces that n-gerbes on a groupoid Γ form an $n{+}1$-category:

- A (-1)-gerbe is an object in degree 0, i.e. an element of the set of objects of Γ.
- A 0-gerbe consists of a 1-cocycle $\omega \in Z^1(\Gamma)$. The morphism sets are

  $$\mathrm{Hom}(\omega, \omega') = \{\eta \in \Gamma_0 \mid \mathrm{d}\eta = \omega' - \omega\}. \tag{2.46}$$

  We thus get a category of 0-gerbes, which we also call line bundles on Γ. Its isomorphism classes are classified by the cohomology group $H^1(\Gamma, \mathbb{C}^\times)$.
- 1-gerbes form a bicategory. Its objects are 2-cocycles, and the set of 1-morphisms between two 2-cocycles ω and ω' is $\{\eta \in \Gamma_1 \mid \mathrm{d}\eta = \omega' - \omega\}$. Given two 1-morphisms $\eta, \eta' \colon \omega \to \omega'$, a 2-morphism $\Phi \colon \eta \Rightarrow \eta'$ is an element $\Phi \in \Gamma_0$ satisfying $\mathrm{d}\Phi = \eta' - \eta$. The isomorphism classes of this bicategory of gerbes are classified by the cohomology group $H^2(\Gamma, \mathbb{C}^\times)$.

For Dijkgraaf–Witten theories based on a finite group G, 2-gerbes on the groupoid $*/\!/G$ are relevant. As we already have pointed out, they should be thought of as a finite version of a Chern–Simons 2-gerbe.

*2.5. Module categories over the fusion category G-*vect$^\omega$. We next discuss category-theoretic and algebraic realizations of group 3-cocycles. A closed 3-cocycle ω on a finite group G allows one to endow the abelian category G-vect of G-graded vector spaces with a non-trivial associativity constraint, defined on simple objects by

$$\begin{aligned} a_{V_{g_1},V_{g_2},V_{g_3}} :\quad & \big(V_{g_1} \otimes V_{g_2}\big) \otimes V_{g_3} \to V_{g_1} \otimes \big(V_{g_2} \otimes V_{g_3}\big) \\ & \qquad v_1 \otimes v_2 \otimes v_3 \mapsto \omega(g_1, g_2, g_3)\, v_1 \otimes v_2 \otimes v_3 . \end{aligned} \tag{2.47}$$

This yields a fusion category, which is denoted by G-vect$^\omega$ (the pentagon axiom is fulfilled because ω is closed). Cohomologous 3-cocycles give rise to monoidally equivalent fusion categories.

The modular tensor category relevant for the Dijkgraaf–Witten theory based on (G, ω) is the Drinfeld center $\mathcal{Z}(G\text{-vect}^\omega)$. (This has been discussed in [DPR]; a helpful more recent exposition is given in [Wi].) It is thus a topological field theory of Reshetikhin–Turaev type. This allows us to compare our geometric results with those obtained in the model independent approach to defects and boundary conditions in [FSV].

The indecomposable module categories over the monoidal category G-vect$^\omega$ have been classified [Os2, Example 2.1]: Consider a subgroup $H \le G$ and a 2-cochain θ on H such that $\mathrm{d}\theta = \omega|_H$. Note that this requires the restriction of ω to the subgroup H to be exact and thus imposes in general restrictions on the subgroup. Rephrased in the language of Sect. 2.4, θ is a 1-morphism from the trivial 2-gerbe on $*/\!/H$ to the pullback 2-gerbe $\iota^*\omega$.

The twisted group algebra $A_{H,\theta} := \mathbb{C}_\theta[H]$ is then a (haploid special symmetric) Frobenius algebra in G-vect$^\omega$. For any 1-cochain χ on H the algebras $A_{H,\theta}$ and $A_{H,\theta+d\chi}$ are isomorphic. Thus, given a subgroup H the isomorphism classes of algebras form a torsor over $H^2(H, \mathbb{C}^\times)$. Indecomposable module categories over G-vect$^\omega$ are given by Morita classes of twisted group algebras. They are thus in bijection with equivalence classes of pairs (H, θ); we denote them by $\mathcal{M}_{H,\theta}$.

Actually, any pair consisting of a group homomorphism $\iota\colon H \to G$ and a 2-cochain θ on H such that $\iota^*\omega = \mathrm{d}\theta$ defines a module category, albeit not an indecomposable one unless ι is injective. For the case that both ω and θ vanish, this is discussed in the Appendix.

3. Categories of Generalized Wilson Lines in Dijkgraaf–Witten Theories

We are now ready to discuss Dijkgraaf–Witten theories with boundaries and defects. Our ultimate goal is to consider such a theory as a 1–2–3-extended topological field theory. Concretely this means:

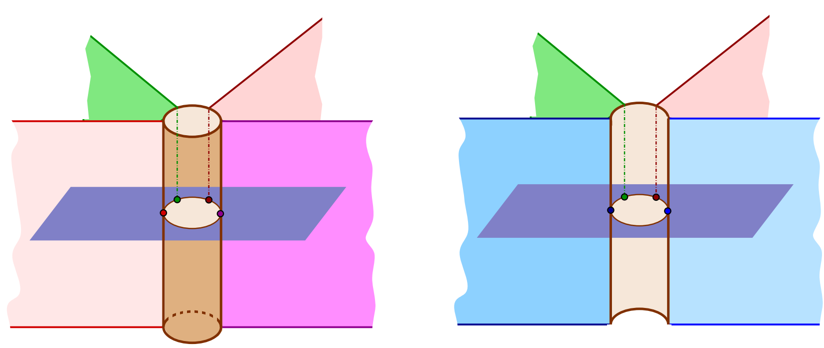
- To a decorated smooth oriented one-dimensional manifold, we have to assign a finitely semisimple $\mathbb{C}$-linear category.
  This category will have the interpretation of a category of (generalized) Wilson lines. The one-dimensional manifold is allowed to have boundaries, corresponding to physical boundaries of the three-dimensional theories, and to have marked points, corresponding to surface defects.
- To a decorated smooth oriented two-dimensional manifold we have to assign a $\mathbb{C}$-linear functor. A two-dimensional manifold can have physical boundaries and lines corresponding to surface defects. Moreover, it can have cut-and-paste boundaries which are one-dimensional manifolds of the type described in the first item. These cut-and-paste boundaries determine the categories which are the source and target for the functor associated to the two-manifold.
- To a decorated three-manifold with corners, we have to associate a natural transformation.

3.1. Decorated one-manifolds and categories of generalized bundles. In the present paper we concentrate on examples and restrict our attention to one-dimensional manifolds. We should also keep in mind that cut-and-paste boundaries have been introduced to implement locality. Accordingly we impose the condition that a cutting is transversal to any additional decoration data such as surface defects or generalized Wilson lines.

This leaves us with two types of connected one-manifolds only:

- An interval which is partitioned by finitely many distinct points in its interior.
- A circle that is partitioned by finitely many distinct points.

For the situations shown in (2.11) above, the cutting leading to such one-manifolds is indicated in the following picture:



(3.1)

Every subinterval of such a one-manifold is decorated by a Dijkgraaf–Witten theory. The decoration datum for each subinterval is thus a finite group G together with a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. The locality of the geometric construction of Dijkgraaf–Witten theories [Fr, Mor] then suggests that G-bundles on these intervals should appear in our construction.

However, we also must assign data to the end points of a subinterval. Recall from Sect. 2.1 that the general construction of Dijkgraaf–Witten theories consists of two steps: first finding an appropriate stack of bundles, leading to spans of groupoids, which then have to be linearized with the help of Lagrangian data. In the situation at hand, the relevant categories are variants of relative bundles which have been introduced in Sect. 2.3. In the case of an interval without marked points in the interior, the morphism defining the relative manifold is the embedding of the end points.

One might thus pick a group homomorphism $\iota\colon\ H \to G$ and assign H-bundles to the two end points. This is, however, not the most general situation one can consider—for complying with locality we must allow for the possibility to assign different local conditions to the two end points of the interval. Thus we select possibly different groups H_i, $i = 1, 2$, and group homomorphism $\iota_i\colon\ H_i \to G$ separately for each end point p_1, p_2 and consider the following category: an object consists of a G-bundle P_G over the interval, an H_1-bundle P_{H_1} over p_1, a morphism $\mathrm{Ind}_{\iota_1} P_{H_1} \to (P_G)|_{p_1}$ of G-bundles on p_1, an H_2-bundle P_{H_2} over p_2, and a morphism $\mathrm{Ind}_{\iota_2} P_{H_2} \to (P_G)|_{p_2}$ of G-bundles on p_2.

This leads to the following assignment of kinematical data. At the level of groups, we associate to an end point of an interval that is labeled by a group G a group homomorphism $\iota\colon\ H \to G$, with H some finite group. This prescription still needs to be complemented by group cohomological Lagrangian data; these will be introduced in Sect. 3.2.

Example 2.7 allows us to determine directly a finite action groupoid that is relevant for an interval without any marked interior points, labeled by a group G, and with end points labeled by groups H_1, H_2 and group homomorphisms $\iota_1\colon\ H_1 \to G$ and $\iota_2\colon\ H_2 \to G$ respectively: it is given by

$$G \,\backslash\!\!\backslash\, G \times G /\!\!/_{\iota_1^- \times \iota_2^-} H_1 \times H_2. \tag{3.2}$$

Here G acts from the left as the diagonal subgroup, while H_1 is mapped via ι_1 to the first copy of G and acts by right multiplication after taking the inverse; the action of H_2 is analogous, the only difference being that it is mapped by ι_2 into the second copy of G.

Let us describe the structure of this groupoid: its set of objects is given by a Cartesian product of groups, one factor for each pair consisting of a marked point and a neighbouring interval. The group is determined by the interval, since it comes from the morphism of bundles in the corresponding relative bundle. The morphisms in the groupoid are gauge transformations: the G-action describes gauge transformations of the G-bundle on the interval and acts by multiplication from the left. The H_i-actions are by multiplication from the right after having taken the inverse; their origin are H_i-gauge transformations of the H_i-bundles on the respective marked point.

This picture generalizes to marked points in the interior, either of an interval or of a circle. To any such point two intervals are adjacent, which are labeled by gauge groups G_l and G_r, respectively. To describe the resulting relative manifold, consider as an example the closed interval $[0, 1]$ with a marked interior point $p_1 := \frac{1}{2}$. Take for X the disjoint union $X := [0, \frac{1}{2}] \sqcup [\frac{1}{2}, 1]$. One should appreciate that in X the point p_1 is "doubled". By locality, the category of bundles is now defined with separate data for each of the marked points $p_0 = 0$, $p_1 = \frac{1}{2}$ and $p_2 = 1$. For p_0 and p_2 we select again group homomorphisms $\iota_0\colon\ H_0 \to G_\mathrm{l}$ and $\iota_2\colon\ H_2 \to G_\mathrm{r}$. At p_1 we take as a datum a finite group H_1 and a group homomorphism $\iota\colon H_1 \to G_\mathrm{l} \times G_\mathrm{r}$ or, equivalently, a pair of group homomorphisms $\iota_\mathrm{l}\colon H_1 \to G_\mathrm{l}$ and $\iota_\mathrm{r}\colon H_2 \to G_\mathrm{r}$.

We consider thus for a given one-manifold S the following geometric category: an object is the assignment of a G-bundle to each subinterval labeled by a finite group G and of H-bundles to marked points in the interior or end points. The final datum are compatible morphisms from induced bundles to restrictions of bundles at all marked points. We denote this geometric category by $\mathrm{Bun}(S)$.

Definition 3.1. *(i) A one-dimensional* pre-DW manifold *is a smooth one-dimensional manifold S, possibly with boundary, together with the following data:*

- *A finite set P_S of points of S, containing all boundary points of S.*
  We refer to the elements of P_S as marked *points, and to a connected component of $S \setminus P_S$ as a* subinterval *of S. We choose an orientation for each subinterval.*
- *To each subinterval of S we associate a finite group.*
- *To a marked point $p \in P_S$ that is a boundary point and is thus adjacent to a single subinterval I with associated group G, we select a finite group H and a group homomorphism $\iota\colon H \to G$.*
  To a marked point $p \in P_S$ that is not a boundary point of S and is thus adjacent to two subintervals I_1 and I_2, labeled by finite groups G_1 and G_2, respectively, we select a finite group H and a pair of group homomorphisms $\iota_i\colon H \to G_i$.

(ii) To a one-dimensional pre-DW manifold S, we associate the category $\mathrm{Bun}(S)$ *of bundles described above. This is an essentially finite groupoid.*

(iii) Each subinterval of a one-dimensional pre-DW manifold S is endowed with an orientation. Thereby any marked point $p \in P_S$ is either a start point or an end point for any interval I adjacent to p. In the first case, we set $\epsilon(p, I) := +1$, in the latter $\epsilon(p, I) := -1$.

To make contact with the results in [FSV] which use the theory of module categories, we need to find finite groupoids that are equivalent to groupoids $\mathrm{Bun}(S)$ of relative bundles of pre-DW manifolds. This is the goal of the remaining part of this subsection.

As a first example, consider a circle with one marked point, which corresponds to a surface defect. If we associate to the interval the group G, then we have to associate to the defect a group homomorphism $\iota\colon H \to G \times G$, and the resulting action groupoid is

$$G \,\backslash\!\!\backslash\, G \times G /\!\!/_{\iota^-} H. \tag{3.3}$$

Of particular interest is the case that the group homomorphism ι is the embedding homomorphism of the diagonal subgroup $G \leq G \times G$. We denote by $G/\!\!/_{\mathrm{ad}} G$ the action groupoid for the left adjoint action of G on itself. The functor

$$F: \quad G \,\backslash\!\!\backslash\, G \times G /\!\!/ G \;\to\; G/\!\!/_{\mathrm{ad}} G \tag{3.4}$$

that acts on objects as $F(\gamma_1, \gamma_2) = \gamma_1 \gamma_2^{-1}$ and on morphisms as

$$F\left((\gamma_1, \gamma_2) \xrightarrow{(h_1, h_2)} (h_1 \gamma_1 h_2^{-1}, h_1 \gamma_2 h_2^{-1}) \right) = \left(\gamma_1 \gamma_2^{-1} \xrightarrow{h_1} h_1 \gamma_1 \gamma_2^{-1} h_1^{-1} \right) \tag{3.5}$$

is an equivalence of categories. We will see that the linearization of the adjoint action groupoid together with the relevant cocycle (see formula (3.46)) produces the appropriate category associated to the circle without marked points, i.e. the category of ordinary bulk Wilson lines.

As a more involved example, let us discuss a circle with two marked points. We describe the circle as $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and take the marked points to be $\pm\mathrm{i} \in S^1$. For the two intervals that consist of points with positive and negative real parts, respectively, we choose groups $G_>$ and $G_<$, respectively. At the points $\pm\mathrm{i}$, we choose group homomorphisms

$$\iota_+ : \quad H_+ \to G_> \times G_< \qquad \text{and} \qquad \iota_- : \quad H_- \to G_< \times G_> \,. \tag{3.6}$$

The relevant action groupoid is then

$$G_> \times G_< \,\backslash\!\backslash\, G_> \times G_< \times G_< \times G_> /\!/_{\iota_+^- \times \iota_-^-} H_+ \times H_-, \tag{3.7}$$

where the action of $G_>$ and $G_<$ is again diagonal and the left action of $H_\pm$ is again by right multiplication preceded by applying the relevant group homomorphism and taking inverses. This description generalizes in an obvious manner to circles with an arbitrary finite number of marked points. The generalization to intervals with an arbitrary finite number of marked points is easy as well. We have thus succeeded in describing for a specific type of one-dimensional pre-DW manifold the category $\mathrm{Bun}(S)$ by a finite action groupoid.

We discuss again a specific case: suppose that $G_> = G_< =: G$ and that $H_+ \cong G \xrightarrow{d} G \times G$ is the diagonal subgroup, while $\iota_- = \iota\colon\, H \to G \times G$ is an arbitrary group homomorphism. Then the relevant action groupoid is

$$G \times G \,\backslash\!\backslash\, G \times G \times G \times G /\!/_{d^- \times \iota^-} G \times H \tag{3.8}$$

with the first copy of G in the gauge group $G \times G$ acting on the first and forth copies of G in $G \times G \times G \times G$ by left multiplication and the second copy of G acting on the second and third copies. The left action of G on the right is as a subgroup of the first and second copy of G. The action groupoid (3.8) is equivalent to the action groupoid

$$G \,\backslash\!\backslash\, G \times G /\!/_{\iota^-} H \tag{3.9}$$

via the functor F that acts on objects as

$$F(\gamma_1, \gamma_2, \gamma_3, \gamma_4) := (\gamma_1 \gamma_2^{-1} \gamma_3, \gamma_4) \tag{3.10}$$

and maps the morphism

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \xrightarrow{(g_1, g_2, g, h)} (g_1 \gamma_1 g^{-1}, g_2 \gamma_2 g^{-1}, g_2 \gamma_3 h^{-1}, g_1 \gamma_4 h^{-1}) \tag{3.11}$$

in the groupoid (3.8) to the morphism

$$(\gamma_1 \gamma_2^{-1} \gamma_3, \gamma_4) \xrightarrow{(g_1, h)} (g_1 \gamma_1 \gamma_2^{-1} \gamma_3 h^{-1}, g_1 \gamma_4 h^{-1}) \tag{3.12}$$

in (3.9). It is straightforward to check that this functor is surjective and a bijection on morphism spaces and is thus an equivalence of groupoids.

3.2. Lagrangian data and linearization of groupoids. We now proceed to the linearization process. This requires additional data which come from the cohomology of the groupoids that have to be linearized. These data have the physical interpretation of (topological) Lagrangians and appropriate boundary terms.

We introduce such additional data as follows. To an end point of an interval that is adjacent to a subinterval labeled by a finite group G and 3-cocycle ω we associate a group homomorphism $\iota\colon H \to G$ and a 2-cochain $\theta \in C^2(H, \mathbb{C}^\times)$ such that $\mathrm{d}\theta = \iota^*\omega$. It is appropriate to think about θ as a morphism $\mathrm{triv} \to \iota^*\omega$ of 2-gerbes on the groupoid $*//H$. The situation can be regarded as a higher categorical analogue of the role played by gerbe modules in the description of boundary conditions in two-dimensional theories with non-trivial Wess-Zumino terms (see e.g. [FNSW, Sect. 6] for an exposition using gerbes and gerbe modules). In the two-dimensional situation, one has a gerbe module on a submanifold $\iota\colon \Sigma \to M$, which amounts to a 1-morphism $I_\omega \to \iota^*\mathcal{G}$ of gerbes on Σ from a trivial gerbe I_ω to the restriction of the gerbe $\mathcal{G}$ on M. In the present situation we have a module of a 2-gerbe; technical simplifications come from the fact that the groups we deal with are finite and that thus any infinitesimal data related to connections are trivial.

In the case of two intervals adjacent to one another, labeled by (G_1, ω_1) and (G_2, ω_2), respectively, we choose a group homomorphism $\iota = (\iota_1, \iota_2)\colon H \to G_1 \times G_2$ and a 2-cochain θ on H such that $d\theta = (\iota_2^*\omega_2) \cdot (\iota_1^*\omega_1)^{-1}$. Again the situation has an analogue in two dimensions: defects in backgrounds with non-trivial Wess-Zumino term are described by gerbe bimodules and bibranes, see [FSW] and [FNSW, Sect. 7] for a review.

We summarize these prescriptions in the following

Definition 3.2. *A one-dimensional* DW manifold *is a one-dimensional pre-DW manifold S together with the following choice of Lagrangian data:*

- *To each subinterval of S with finite group G, we associate a closed 3-cochain on G.*
- *To a marked boundary point $p \in P_S \cap \partial S$ adjacent to a subinterval with group G and 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ and labeled with a group homomorphism $\iota\colon H \to G$, we assign a 2-cochain $\theta \in C^2(H, \mathbb{C}^\times)$ such that*

$$\mathrm{d}\theta = \iota^*\omega^{\epsilon(p,I)}, \tag{3.13}$$

  with $\epsilon(p, I)$ as defined in Definition 3.1(iii).
- *To a marked interior point $p \in P_S \backslash \partial S$ adjacent to subintervals I_1 and I_2 with group homomorphisms $\iota_i\colon H \to G_i$ we assign a cochain $\theta \in C^2(H, \mathbb{C}^\times)$ such that*

$$\mathrm{d}\theta = \iota_1^*\omega_1^{\epsilon(p,I_1)} \cdot \iota_2^*\omega_2^{\epsilon(p,I_2)}. \tag{3.14}$$

We now use the data of a DW manifold to define twisted linearizations of the groupoids that we constructed in the previous subsection. Let us describe the general idea of a twisted linearization of a finite groupoid $H \backslash\backslash G$ given by a left action of a group H on a set G. The ordinary linearization is the functor category $[H \backslash\backslash G, \mathrm{vect}_{\mathbb{C}}]$. An object of this category is given by

- A finite-dimensional vector space V_γ for each element $\gamma \in G$.
- For each $\gamma \in G$ and $h \in H$ a linear map $\rho_h\colon V_\gamma \to V_{h.\gamma}$ such that the diagram

$$\begin{tikzcd} & V_{h_2.\gamma} \arrow[dr, "\rho_{h_1}"] & \\ V_\gamma \arrow[ur, "\rho_{h_2}"] \arrow[rr, "\rho_{h_1 h_2}"'] & & V_{h_1 h_2.\gamma} \end{tikzcd} \tag{3.15}$$

  commutes for all $\gamma \in G$ and $h_1, h_2 \in H$.

Morphisms in the functor category are natural transformations; explicitly, they are G-homogeneous maps commuting with the H-action.

The additional input datum for a *twisted* linearization is a 2-cocycle τ on the groupoid $H \backslash\backslash G$. This gives rise to the following twisted version of the functor category $[H \backslash\backslash G, \mathrm{vect}_{\mathbb{C}}]$ (see also [Mor, Sect. 5.4]):

Definition 3.3. *The* τ-twisted linearization *of the groupoid* $H \backslash\backslash G$*, denoted by* $[H \backslash\backslash G, \mathrm{vect}_{\mathbb{C}}]^{\tau}$*, is the following category. An object of* $[H \backslash\backslash G, \mathrm{vect}_{\mathbb{C}}]^{\tau}$ *consists of*

- *A finite-dimensional vector space* V_γ *for each* $\gamma \in G$.
- *For each* $h \in H$ *a linear map* $\rho_h \colon V_\gamma \to V_{h.\gamma}$ *such that the composition law of the* H*-action is realized projectively, i.e. up to the scalar factor* $\tau(h_1, h_2; \gamma) \in \mathbb{C}^\times$*. Diagrammatically,*

$$\begin{tikzcd} & V_{h_2.\gamma} \arrow[dr, "\rho_{h_1}"] \arrow[d, phantom, "\Downarrow \tau(h_1,h_2;\gamma)"] & \\ V_\gamma \arrow[ur, "\rho_{h_2}"] \arrow[rr, "\rho_{h_1h_2}"'] & & V_{h_1h_2.\gamma} \end{tikzcd} \tag{3.16}$$

As a formula,

$$\rho_{h_1h_2} = \tau(h_1, h_2; \gamma)\, \rho_{h_1}\, \rho_{h_2}. \tag{3.17}$$

Morphisms of $[H \backslash\backslash G, \mathrm{vect}_{\mathbb{C}}]^{\tau}$ *are* G*-homogeneous maps commuting with the* H*-action.*

3.3. 2-cocycles from Lagrangian data. Our next task is thus to use the Lagrangian data that are part of the data of a one-dimensional DW-manifold. We have assigned them in Definition 3.2 to intervals and circles with marked points to produce 2-cocycles for the groupoids discussed in Sect. 3.1. For brevity we consider in this subsection Lagrangian data for boundaries only; the discussion for surface defects is similar.

Any homomorphism $\iota\colon H \to G$ of finite groups provides a morphism $\iota\colon BH \to BG$ of the corresponding classifying spaces. Assume now that we are given a 3-cocycle $\omega \in Z^3(BG, \mathbb{C}^\times)$ and a 2-cochain $\theta \in C^2(BH, \mathbb{C}^\times)$ such that

$$i^*\omega = \mathrm{d}\theta. \tag{3.18}$$

We recall that a G-bundle on a manifold M can be described by a map from M to the classifying space BG. Morphisms of bundles can be described by homotopies between such maps. Thus for Σ an oriented one-dimensional manifold with boundary, a relative bundle on the relative manifold $(\Sigma, \partial\Sigma)$ leads to the following data (up to homotopy):

- A map $f \in \mathrm{Map}(\Sigma, BG)$ describing a G-bundle on Σ.
- A map $g \in \mathrm{Map}(\partial\Sigma, BH)$ describing an H-bundle on $\partial\Sigma$.
- A homotopy describing the morphism of bundles, i.e. a map $h \in \mathrm{Map}([0, 1], \mathrm{Map}(\partial\Sigma, BG))$, with $[0, 1]$ the standard interval.

We will later need the subset $X_\circ$ consisting of such triples (f, g, h) subject to the condition that h is a homotopy relating the maps $f|_{\partial\Sigma}$ and $\iota \circ g$ from $\partial\Sigma$ to BG,

$$X_\circ := \{\, (f, g, h) \mid f\big|_{\partial\Sigma} \simeq^h i \circ g \,\}. \tag{3.19}$$

Each point of $X_\circ$ describes a relative bundle, i.e. an object of $\mathrm{Bun}_{(G,H)}(\partial\Sigma \to \Sigma)$. Isomorphism classes of relative bundles are in bijection with the set $\pi_0(X_\circ)$ of connected components of $X_\circ$.

From the cohomological data ω and θ we now build a 2-cocycle in $Z^2(X_\circ, \mathbb{C}^\times)$. To this end we use the evaluation map

$$\mathrm{ev}: \quad \Sigma \times \mathrm{Map}(\Sigma, BG) \to BG \tag{3.20}$$

to define a cochain $\tau_\Sigma(\omega) \in C^2(\mathrm{Map}(\Sigma, BG), \mathbb{C}^\times)$ by

$$\tau_\Sigma(\omega) := \int_\Sigma \mathrm{ev}^*\omega, \tag{3.21}$$

where $\int_\Sigma$ denotes the pushforward along the fibration $p_2\colon \Sigma \times \mathrm{Map}(\Sigma, BG) \to \mathrm{Map}(\Sigma, BG)$. As Σ can have a non-empty boundary, there is, in general, no reason that the cochain $\tau_\Sigma(\omega)$ should be closed.

By the same procedure we obtain a 2-cochain $\tau_{\partial\Sigma}(\theta) \in C^2(\mathrm{Map}(\partial\Sigma, BH)), \mathbb{C}^\times)$, as well as a 2-cochain $\tau_{[0,1]}(\tau_{\partial\Sigma}(\omega)) \in C^2(\mathrm{Map}([0,1], \mathrm{Map}(\partial\Sigma, BG)), \mathbb{C}^\times)$. We then consider the product space

$$X := \mathrm{Map}(\Sigma, BG) \times \mathrm{Map}(\partial\Sigma, BH) \times \mathrm{Map}([0,1], \mathrm{Map}(\partial\Sigma, BG)). \tag{3.22}$$

The pullbacks along the canonical projections p_i to the three factors of (3.22) supply us with a 2-cochain on X:

$$\varphi := p_1^*\tau_\Sigma(\omega) - p_2^*\tau_{\partial\Sigma}(\theta) - p_3^*\tau_{[0,1]}(\tau_{\partial\Sigma}(\omega)). \tag{3.23}$$

The space $X_\circ$ introduced in (3.19) to describe relative bundles is by definition a subspace of X (3.22). The central insight is now that the 2-cochain that is obtained by restricting φ to the subspace $X_\circ$ of X is closed,

$$\mathrm{d}\varphi|_{X_\circ} = 0. \tag{3.24}$$

In other words, we have obtained a 2-cocycle $\varphi|_{X_\circ} \in Z^2(X_\circ, \mathbb{C}^\times)$ on the space $X_\circ$ describing relative bundles.

To see that (3.24) holds, we work for the moment with differential forms and consider an arbitrary manifold U. Consider $\alpha \in \Omega^3_{\mathrm{cl}}(\Sigma \times U, \mathbb{R})$ and $\beta \in \Omega^2(\partial\Sigma \times U), \mathbb{R})$ obeying $\alpha|_{\partial\Sigma\times U} = \mathrm{d}\beta$. Taking into account that Σ has a boundary, we have

$$\mathrm{d}(\int_\Sigma \alpha) = \int_\Sigma \mathrm{d}\alpha + \int_{\partial\Sigma} \alpha|_{\partial\Sigma\times U} = \int_\Sigma \mathrm{d}\alpha + \int_{\partial\Sigma} \mathrm{d}\beta = \int_{\partial\Sigma} \mathrm{d}\beta. \tag{3.25}$$

This means that the form

$$\phi := \int_\Sigma \alpha - \int_{\partial\Sigma} \beta \ \in \Omega^2(U, \mathbb{R}) \tag{3.26}$$

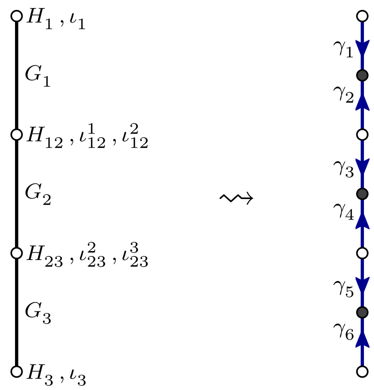
is closed, $\mathrm{d}\phi = 0$. The same argument applies to elements in $Z^3(\Sigma \times U, \mathbb{C}^\times)$ where slant products are used as the analogue of integration along the fiber.

The argument can now be applied to the situation of our interest: The role of $\int_\Sigma \alpha$ is then played by $p_1^*\tau_\Sigma(\omega)|_{X_\circ}$ and the role of $\int_{\partial\Sigma} \beta$ by $(p_2^*\tau_{\partial\Sigma}(\varphi) + p_3^*\tau_{[0,1]}(\tau_{\partial\Sigma}(\omega)))|_{X_\circ}$. Their difference is precisely the combination φ introduced in (3.23). From the relation $\alpha|_{\partial\Sigma\times U} = \mathrm{d}\beta$ we thus obtain the desired equality (3.24).

3.4. Graphical calculus for groupoid cocycles. Generalizing the approach of [Wi], we can achieve a more combinatorial description of the 2-cocycles on the groupoids derived in Sect. 3.1. We formulate it with the help of an algorithm which is based on three-dimensional diagrams and their decomposition into simplices. The diagrams are obtained from a graphical representation of the groupoids involved.

We start with a one-dimensional diagram, drawn vertically, which represents a one-dimensional pre-DW manifold to which we wish to associate a category by linearization. These manifolds are circles or intervals with finitely many marked points, including boundary points in the case of intervals. Each subinterval is marked by a finite group G_i and a 3-cocycle $\omega_i \in Z^3(G_i, \mathbb{C}^\times)$. For each marked point we have a group H_j and group homomorphisms to the groups associated with the adjacent intervals. The data characterizing an object in the associated groupoid described in Sect. 3.1 are then elements in the groups G_i associated to the subinterval, one for each point adjacent to the subinterval.

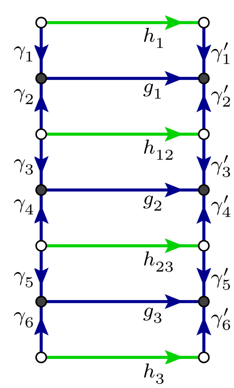
Our convention is now to draw an empty circle for a marked point and to replace the original subintervals by filled circles. Between these circles we draw edges which are labeled by elements of the groups G_i that are part of the data describing a relative bundle. An example is depicted in the following picture:



(3.27)

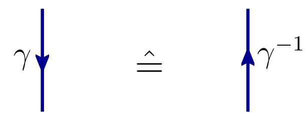
The figure on the left hand side of (3.27) shows the pre-DW-manifold S which is an interval with two interior marked points, together with the relevant groups and group homomorphisms. The labels in the figure on the right hand side are group elements $\gamma_1, \gamma_2 \in G_1$, $\gamma_3, \gamma_4 \in G_2$ and $\gamma_5, \gamma_6 \in G_3$i that specify an object in $\mathrm{Bun}(S)$.

A morphism in the groupoid consists of elements of the groups H_j and G_i describing gauge transformations of the involved bundles. We represent such morphisms by two-dimensional diagrams with oriented edges as follows:



(3.28)

Here horizontal edges connecting empty circles are labeled by elements of the groups H_j, while horizontal edges connecting filled circles are labeled by elements of the groups G_i. For each square in the diagram there is a consistency condition relating the labels of its edges. To formulate this condition, we adopt the convention that orientation reversal amounts to inversion of the group element that labels the edge:



(3.29)

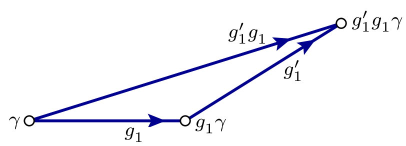
With this convention the product of all group elements (possibly after applying an appropriate group homomorphism $H_j \to G_i$) along a closed curve equals the neutral element; we refer to this relation as the *holonomy condition*. For instance, the holonomy condition for the top square in (3.28) is the equality

$$\gamma_1' \cdot \iota_1(h_1) = g_1 \cdot \gamma_1 \tag{3.30}$$

in G_1. This determines the element γ_1' of G_1, or alternatively γ_1 or g_1, as a function of the three other group elements. Also, in case the homomorphism ι_1 is injective it alternatively fixes $h_1 \in H_1$ in terms of the three other elements.,

We wish to obtain a 2-cocycle on the groupoid we have just described. For a general groupoid $\Gamma = (\Gamma_0, \Gamma_1)$ with sets Γ_0 of objects and Γ_1 of morphisms we define the 2-cocycle by its values $\tau(g_1, g_1'; \gamma)$ for an object $\gamma \in \Gamma_0$ and two compatible morphisms $g_1, g_1' \in \Gamma_1$. We depict these values graphically as triangles,

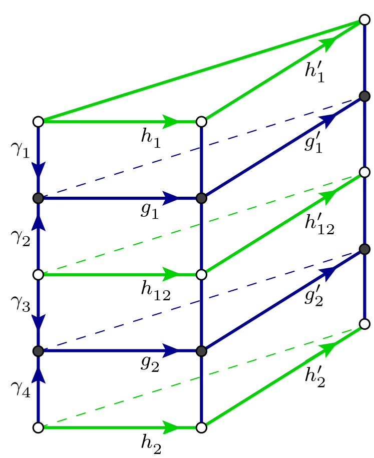
$\tau(g_1, g_1'; \gamma) \;=$



(3.31)

(Again the holonomy condition is in effect: we have $(g_1' g_1)^{-1} g_1' g_1 = e$.)

Now in the situation of our interest, in which we represent objects and morphisms of the groupoid by one-dimensional and two-dimensional graphical elements, respectively, we obtain a graphical representation of the 2-cocycle by a piecewise-linear three-manifold. In the case of an interval considered in (3.28)—but now, for simplicity, with only a single interior marked point—this three-manifold looks as follows:

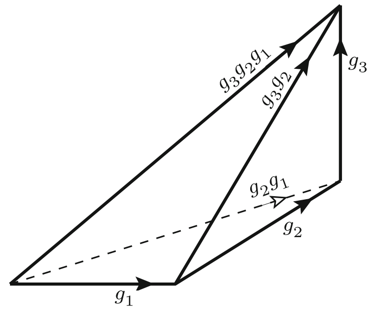


(3.32)

Here the labeling of all lines for which the labels are not indicated explicitly is fixed as a function of the displayed labels by the holonomy condition.

Following the strategy in [Wi], our goal is now to cut the so obtained three-manifolds into standard pieces to which we can naturally assign values in $\mathbb{C}^\times$. The value of the groupoid 2-cocycle is then given by the product of the numbers associated with the various standard pieces into which the three-manifold is decomposed. In our situation, in which also physical boundaries and surface defects are present, there are *two* types of standard pieces:

- First, a 3-simplex whose edges are all labeled by elements $g_1, g_2, g_3, \ldots$ of a group G with 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$, subject to the holonomy condition. To such a 3-simplex

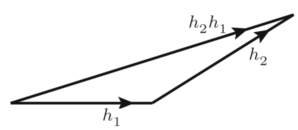


(3.33)

we associate the number

$$\tilde{\omega}(g_1, g_2, g_3) := \omega(g_1^{-1}, g_2^{-1}, g_3^{-1}) \in \mathbb{C}^\times. \tag{3.34}$$

- Second, a horizontal triangle whose edges are correspondingly labeled by elements of a group H with 2-cochain θ. To such a triangle



(3.35)

we associate the number

$$\tilde{\theta}(h_1, h_2) := [\,\theta(h_1^{-1}, h_2^{-1})\,]^{-1} \in \mathbb{C}^\times. \tag{3.36}$$

We require that any horizontal triangle having only empty circles as vertices that is contained in a three-dimensional diagram of our interest must be taken as a face of the simplicial decomposition. The symmetric groups S_4 and S_3 which consist of permutations of the vertices in (3.33) and (3.35), respectively, are realized on $\tilde{\omega}$ and $\tilde{\theta}$ by a sign that depends on the relative orientations of the two bases involved, i.e. we have equalities such as

$$\tilde{\omega}(g_1, g_2, g_3) = \tilde{\omega}(g_1^{-1}g_2^{-1}g_3^{-1}, g_1, g_2)^{-1} = \tilde{\omega}(g_3^{-1}, g_2^{-1}, g_1^{-1}) \tag{3.37}$$

and

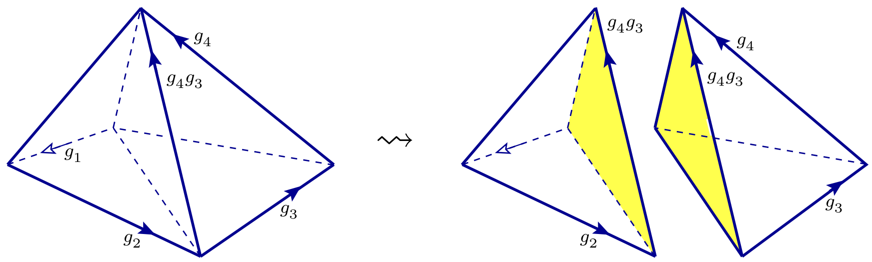
$$\tilde{\theta}(h_1, h_2) = \tilde{\theta}(h_1^{-1}h_2^{-1}, h_1) = \tilde{\theta}(h_2^{-1}, h_1^{-1}) \tag{3.38}$$

etc. We require that $\tilde{\omega}$ and $\tilde{\theta}$ are normalized, i.e.

$$\tilde{\omega}(e, g, g') = 1 \qquad \text{and} \qquad \tilde{\theta}(e, h) = 1. \tag{3.39}$$

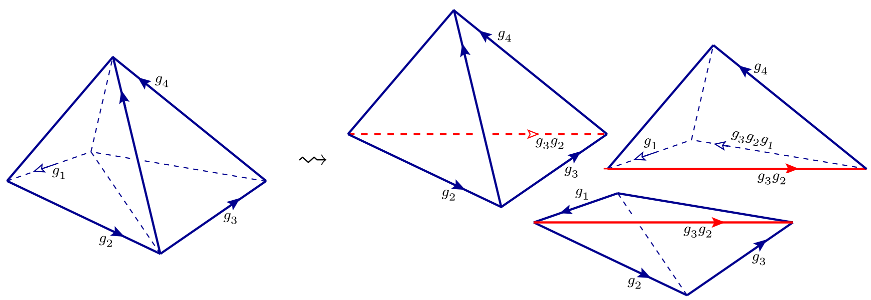
We will freely use the identities (3.37)–(3.39) below.

A simplicial decomposition obtained this way is not unique. We therefore must still verify that the value of the 2-cocycle on the groupoid that is obtained by our prescription is well-defined. When no boundaries or defects (and thus no triangular standard pieces) are involved, there are two situations to be dealt with: First, a gone with 5 vertices, 8 edges, 4 triangles and 1 quadrangle. This gone can be decomposed into tetrahedra in two different ways; the first is a decomposition



(3.40)

into two tetrahedra that share a face (shaded in the picture). The other is a decomposition is into three tetrahedra according to



(3.41)

i.e. the three tetrahedra share an edge (the one labeled by g_3g_2) which intersects transversally the shaded face in (3.40) and pairwise share one of three faces which have the shared edge as a boundary segment.

The two decompositions are related by a 3-2 Pachner move. As is well known, invariance under this move is guaranteed by the closedness of ω. Indeed we have

Lemma 3.4. *The groupoid cocycles obtained from the two decompositions* (3.40) *and* (3.41) *coincide.*

Proof. The decomposition (3.40) gives the number

$$\tau_1 := \tilde{\omega}(g_1, g_2, g_4g_3) \cdot \tilde{\omega}(g_2g_1, g_3, g_4), \tag{3.42}$$

while the decomposition (3.41) yields

$$\tau_2 := \tilde{\omega}(g_1, g_2, g_3) \cdot \tilde{\omega}(g_2, g_3, g_4) \cdot \tilde{\omega}(g_1, g_3g_2, g_4), \tag{3.43}$$

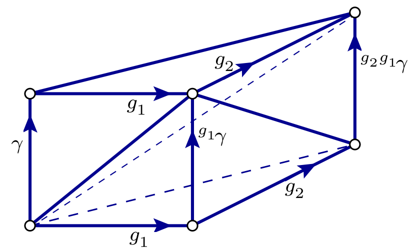
with the three factors being the contributions from the lower, the front, and the back tetrahedron, respectively. Equality of τ_1 and τ_2 amounts to

$$\begin{aligned} &\omega(g_1^{-1}, g_2^{-1}, g_3^{-1}g_4^{-1}) \cdot \omega(g_1^{-1}g_2^{-1}, g_3^{-1}, g_4^{-1}) \\ &\quad = \omega(g_1^{-1}, g_2^{-1}, g_3^{-1}) \cdot \omega(g_2^{-1}, g_3^{-1}, g_4^{-1}) \cdot \omega(g_1^{-1}, g_2^{-1}g_3^{-1}, g_4^{-1}). \end{aligned} \tag{3.44}$$

This is nothing but the statement that ω is closed, and is thus indeed satisfied. □

The second situation to be analyzed corresponds to a 4-1 Pachner move. It can be treated in an analogous manner as the 3-2 move; we leave the details to the reader.

Let us briefly comment on the particular case of the circle without insertions. According to Sect. 3.1, in this case the action groupoid is $G/\!\!/_{\mathrm{ad}}G$ with the adjoint action. This situation is described by the simplex



(3.45)

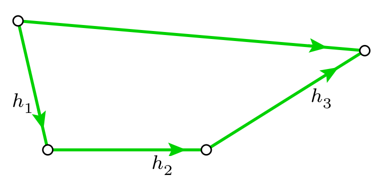
where we indicate the adjoint left action by a superscript, ${}^{g}\gamma = g\gamma g^{-1}$. This yields the cocycle

$$\tau(g_1,g_2;\gamma) = \tilde\omega(g_1,g_2,{}^{g_2g_1}\gamma)\,\tilde\omega(g_1,{}^{g_2}\gamma,g_2)^{-1}\,\tilde\omega(\gamma,g_1,g_2). \tag{3.46}$$

This way we precisely recover the argument given in [Wi] that leads to the 2-cochain found in [DPR, (3.2.5)]. Our formalism thus produces the correct category of bulk Wilson lines.

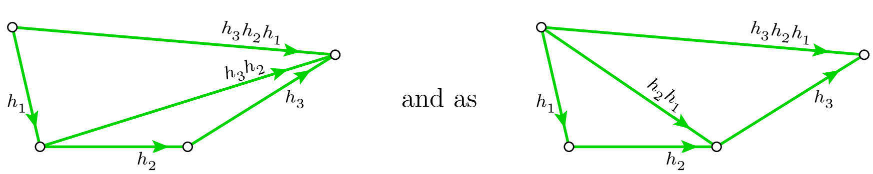
We next consider the case of an interval with no marked interior points. The interior is labeled by a finite group G and $\omega \in Z^3(G;\mathbb{C}^\times)$, while the end points are labeled by group homomorphisms $\iota\colon H_i \to G$ and by 2-cochains $\theta_i \in C^2(H_i,\mathbb{C}^\times)$ such that $\mathrm{d}\theta_i = \iota_i^*\omega$.

Again there is the issue of non-uniqueness of simplicial decomposition, with the new aspect that the boundary of the interval leads to the presence of triangles of the form (3.35) in the decompositions. Thus we must consider tetragons



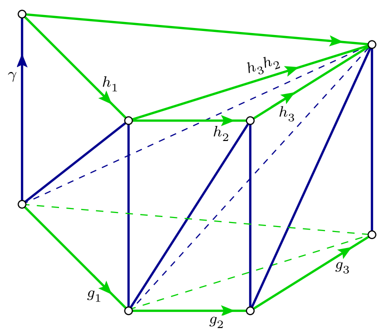
(3.47)

Such a boundary tetragon can be decomposed into triangles in two different ways: as

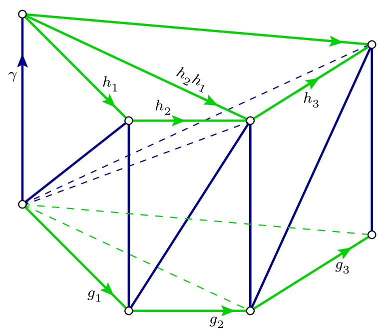


(3.48)

We compare these two decompositions by continuing the situation to the interior of the interval. This leads to the two simplicial decompositions



and



(3.49)

respectively, each consisting of six tetrahedra and of two triangles at the top.

Proposition 3.5. *The complex numbers obtained from the two decompositions in* (3.49) *coincide.*

Proof. Of the six tetrahedra appearing in the two simplices (3.49), only two are different: the ones attached to the top. The simplex on the right hand side of (3.49) gives factors $\tilde\theta(h_1, h_2)$ and $\tilde\theta(h_2h_1, h_3)$ from the triangles at the top and

$$\tilde\omega(\gamma, \iota(h_1), \iota(h_2)) \cdot \tilde\omega(\gamma, \iota(h_2h_1), \iota(h_3)). \tag{3.50}$$

from the two tetrahedra attached to the top triangle, while for the simplex on the left hand side we get $\tilde\theta(h_2, h_3)\,\tilde\theta(h_1, h_3h_2)$ from the top triangles and

$$\tilde\omega(\gamma, \iota(h_1), \iota(h_3h_2)) \cdot \tilde\omega(\iota(h_1)\gamma, \iota(h_2), \iota(h_3)) \tag{3.51}$$

from the attached tetrahedra. Equality of the two expressions yields, after implementing the closedness (3.44) of ω,

$$\tilde\theta(h_1, h_2)\,\tilde\theta(h_2h_1, h_3) = \tilde\theta(h_2, h_3)\,\tilde\theta(h_1, h_3h_2)\,\tilde\omega(\iota(h_1), \iota(h_2), \iota(h_3)), \tag{3.52}$$

or, what is the same

$$\mathrm{d}\theta(h_1^{-1}, h_2^{-1}, h_3^{-1}) = \omega(\iota(h_1^{-1}), \iota(h_2^{-1}), \iota(h_3^{-1})). \tag{3.53}$$

This indeed holds true, owing to $\mathrm{d}\theta = \iota^*\omega$. □

3.5. Wilson line categories for the interval. As already pointed out, by invoking fusion of defects (and of defects to boundaries), among the one-dimensional manifolds there are two fundamental building blocks, the interval without interior marked points and the circle with a single marked point. We now turn to the computation of the categories for these building blocks and then compare them to the model-independent results of [FSV]. In the present subsection we consider an interval without interior marked points. The interior is labeled by (G, ω) with G a finite group and ω a 3-cocycle. For the two boundary points we have group homomorphisms $\iota_i : H_i \to G$ and 2-cochains θ_i on H_i such that $\iota_i^*\omega = \mathrm{d}\theta_i$ for $i = 1, 2$.

Before computing the linearization, we outline what the general formalism of [FSV] predicts for the situation at hand: The data associated to a boundary leads to module categories $\mathcal{M}_i$ over the fusion category G-vect$^\omega$. Such a module category can be decomposed into indecomposable module categories. As described in Sect. 2.5, an indecomposable module category over G-vect$^\omega$ can, in turn, be concretely described [Os1] as the category of modules over an algebra in G-vect$^\omega$. Thus for the description of $\mathcal{M}_i$

it suffices to know such an algebra $A_{H,\theta}$ for any subgroup $H \leq G$ and 2-cochain θ on H satisfying $\mathrm{d}\theta = \omega|_H$. As seen in Sect. 2.5. such algebras can be described as follows. Isomorphism classes of simple objects in G-vect$^\omega$ are in bijection with elements $g \in G$; we fix a set of representatives $(U_g)_{g\in G}$. Then $A_{H,\theta}$ is the object $\bigoplus_{h\in H} U_h$ endowed with the multiplication morphism that is furnished by the cochain θ. This multiplication is associative, due to the relation $\mathrm{d}\theta = \iota^*\omega$. Then the category $\mathcal{M}_{H,\theta} := A_{H,\theta}$-mod is a right module category over G-vect$^\omega$.

By the results of [FSV], such a module category corresponds to an indecomposable boundary condition of the Dijkgraaf–Witten theory based on (G, ω). Given two such boundary conditions, consider the abelian category

$$\mathcal{F} := \mathrm{Fun}_{G\text{-vect}^\omega}(A_{H_2,\theta_2}\text{-mod}, A_{H_1,\theta_1}\text{-mod}) \tag{3.54}$$

of module functors. It has the following physical interpretation: Objects of $\mathcal{F}$ label boundary Wilson lines separating the boundary condition $\mathcal{M}_{H_1,\theta_1}$ from $\mathcal{M}_{H_2,\theta_2}$. Morphisms of $\mathcal{F}$ label point-like insertions on such Wilson lines. $\mathcal{F}$ can be described as the category of A_{H_1,θ_1}-A_{H_2,θ_2}-bimodules in G-vect$^\omega$.

The objects $M = \bigoplus_{g\in G} M_g$ of the category of A_{H_1,θ_1}-A_{H_2,θ_2}-bimodules have been described explicitly in [Os2, Prop. 3.2]: Taking into account that the tensor product on G-vect$^\omega$ realizes the group law strictly, i.e. $U_h \otimes U_g = U_{hg}$, the restriction of the left action of A_{H_1,θ_1} on M to $U_{h_1} \otimes U_g$ leads to an endomorphism of $U_{h_1 g}$ which is a multiple $\rho(h_1, g) \in \mathbb{C}$ of the identity. Analogously the right action of A_{H_2,θ_2} gives us scalars $\reflectbox{$\rho$}(g, h_2) \in \mathbb{C}$. These scalars obey the following conditions.

- That we have a left A_{H_1,θ_1}-action amounts to the relation

$$\rho(h_1' h_1, g) = \theta_1(h_1', h_1)^{-1}\, \omega(h_1', h_1, g)\, \rho(h_1, g)\, \rho(h_1', h_1 g) \tag{3.55}$$

  for all $g \in G$ and all $h_1, h_1' \in H_1$.
- Similarly the right A_{H_2,θ_2}-action gives

$$\reflectbox{ρ}(g, h_2 h_2') = \theta_2(h_2, h_2')^{-1}\, \omega(g, h_2, h_2')^{-1}\, \reflectbox{ρ}(g, h_2)\, \reflectbox{ρ}(g h_2, h_2') \tag{3.56}$$

  for all $g \in G$ and all $h_2, h_2' \in H_2$.
- The condition that left and right actions commute amounts to

$$\rho(h_1, g)\, \reflectbox{ρ}(h_1 g, h_2) = \omega(h_1, g, h_2)\, \reflectbox{ρ}(g, h_2)\, \rho(h_1, g h_2) \tag{3.57}$$

  for all $g \in G$, $h_1 \in H_1$ and $h_2 \in H_2$.
- Finally the unitality of the actions implies the two constraints

$$\rho(e, g) = 1 = \reflectbox{ρ}(g, e) \tag{3.58}$$

  for all $g \in G$.

(Note that θ_1 and θ_2 are normalized because the algebras are strictly unital; (3.58) corresponds to ω being normalized as well.) The objects in the category $\mathcal{F}$ of A_{H_1,θ_1}-A_{H_2,θ_2}-bimodules are thus G-graded vector spaces together with two functions ρ and $\reflectbox{$\rho$}$ that obey the constraints (3.55)–(3.58). Morphisms of $\mathcal{F}$ are G-homogeneous maps, commuting with the actions.

We may also consider, for given $\gamma \in G$, the group

$$H_\gamma := \{(h_1, h_2) \in H_1 \times H_2 \mid h_1\gamma = \gamma h_2\}. \tag{3.59}$$

We can identify H_γ with a subgroup of H_1, which in turn is a subgroup of G. Then $h \in H_\gamma$ acts on the homogeneous component M_γ of M as a scalar multiple

$$\varrho_\gamma(h) := \rho(h, \gamma)\, \reflectbox{ρ}(\gamma, \gamma^{-1}h\gamma)^{-1} \tag{3.60}$$

of the identity. In view of (3.55)–(3.57) this gives rise to a 2-cocycle ϑ_γ on H_γ, given by

$$\begin{aligned}
\vartheta_\gamma(h, h') &:= \varrho_\gamma(hh')^{-1}\, \varrho_\gamma(h)\, \varrho_\gamma(h') \\
&= \rho(hh', \gamma)^{-1} \rho(h, \gamma) \rho(h', \gamma) \reflectbox{ρ}(\gamma, \gamma^{-1}hh'\gamma) \reflectbox{ρ}(\gamma, \gamma^{-1}h\gamma)^{-1} \reflectbox{ρ}(\gamma, \gamma^{-1}h'\gamma)^{-1} \\
&= \theta_1(h, h')\theta_2(\gamma^{-1}h\gamma, \gamma^{-1}h'\gamma)^{-1}\, \omega(h, h', \gamma)^{-1}\, \omega^{-1}(\gamma, \gamma^{-1}h\gamma, \gamma^{-1}h'\gamma) \\
&\qquad\qquad \rho(h, \gamma)\, \rho(h, h'\gamma)^{-1}\, \reflectbox{ρ}(\gamma, \gamma^{-1}h'\gamma)^{-1}\, \reflectbox{ρ}(h\gamma, \gamma^{-1}h'\gamma) \\
&= \theta_1(h, h')\, \theta_2(\gamma^{-1}h'^{-1}\gamma, \gamma^{-1}h^{-1}\gamma) \\
&\qquad\qquad \omega(h, h', \gamma)^{-1}\, \omega(\gamma, \gamma^{-1}h\gamma, \gamma^{-1}h'\gamma)^{-1}\, \omega(h, \gamma, \gamma^{-1}h'\gamma)
\end{aligned} \tag{3.61}$$

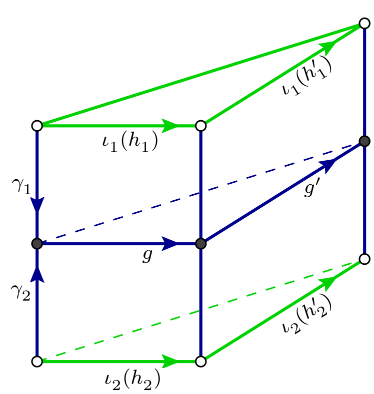
(compare formula (3.1) of [Os2]). Here in the third equality we invoke (3.55) and (3.56), while the last equality uses (3.57).

We now show that the prescription (3.2) indeed produces the expected result:

Proposition 3.6. *Consider the groupoid $\Gamma = G \,\backslash\!\backslash\, G \times G /\!/_{\iota_1^- \times \iota_2^-} H_1 \times H_2$ that according to formula (3.2) is assigned to the interval without interior marked points. If the group homomorphisms $\iota_i\colon\, H_i \to G$ are subgroup embeddings, then the category that is obtained by the projective linearization of Γ for the Lagrangian data θ_1, θ_2 and ω is equivalent, as a $\mathbb{C}$-linear abelian category, to the category of A_{H_1,θ_1}-A_{H_2,θ_2}-bimodules,*

$$[\, G \,\backslash\!\backslash\, G \times G /\!/_{\iota_1^- \times \iota_2^-} H_1 \times H_2,\, \mathrm{vect}_{\mathbb{C}}\,]^{\theta_1,\theta_2,\omega} \;\simeq\; A_{H_1,\theta_1}\text{-}A_{H_2,\theta_2}\text{-Bimod}_{G\text{-vect}^\omega}. \tag{3.62}$$

Proof. The objects of the groupoid in question are pairs (γ_1, γ_2) of elements of G; they label the vertical edges in the following figure:



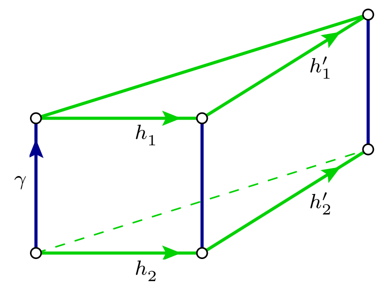
(3.63)

Morphisms are gauge transformations in H_1, H_2 and in G – labeling horizontal edges that connect empty circles and filled circles in (3.63), respectively. Again we consider a pair of compatible morphisms leading to horizontal edges forming the shape of a triangle to get the relevant 2-cocycle on the groupoid Γ. In the sequel we suppress the embedding homomorphisms ι_1 and ι_2.

Observe that the functor

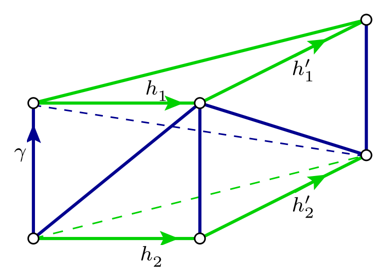
$$G\,\backslash\!\!\backslash\, G \times G/\!\!/_{\iota_1^- \times \iota_2^-} H_1 \times H_2 \;\longrightarrow\; H_1\,{}_{\iota_1}\!\backslash\!\!\backslash\, G/\!\!/_{\iota_2^-} H_2 \tag{3.64}$$

that is defined on objects by $(\gamma_1, \gamma_2) \mapsto \gamma_1^{-1}\gamma_2$ is actually an equivalence of groupoids. Accordingly we set $\gamma := \gamma_1^{-1}\gamma_2$ and obtain from (3.63) a number $\tau(\gamma; h_1, h_1'; h_2, h_2')$ that can be read off from the following slice of pie:



(3.65)

where $\gamma \in G$, $h_1, h_1 \in H_1$ and $h_2, h_2' \in H_2$. There are many equivalent ways to express the so defined numbers in terms of the 2-cocycles θ_i and the 3-cocycle ω; they are related by the various properties of θ_i and ω. Let us choose one such expression that corresponds to the decomposition



(3.66)

of the slice (3.65) into three tetrahedra. This yields

$$\begin{aligned}\tau(\gamma; h_1, h_1'; h_2, h_2') = \tilde\theta_1(h_1, h_1')\, \tilde\theta_2(h_2, h_2')\, \tilde\omega(h_1, h_1', h_2' h_2 \gamma^{-1} h_1^{-1} h_1'^{\,-1})\\ \tilde\omega(h_2, h_2', h_1 \gamma h_2^{-1} h_2'^{\,-1})\, \tilde\omega(\gamma, h_1, h_2' h_2 \gamma^{-1} h_1^{-1}).\end{aligned} \tag{3.67}$$

To make contact to the relations (3.55)–(3.57) for the category of A_{H_1,θ_1}-A_{H_2,θ_2}-bimodules, we consider three special cases of $\tau(\gamma; h_1, h_1'; h_2, h_2')$.

- First we set $h_2 = e = h_2'$; then (3.67) reduces to

$$\begin{aligned}\tau(\gamma; h_1, h_1'; e, e) &= \tilde\theta_1(h_1'^{\,-1}, h_1^{-1})\, \tilde\omega(h_1, h_1', \gamma^{-1} h_1^{-1} h_1'^{\,-1})\, \tilde\omega(\gamma, h_1, \gamma^{-1} h_1^{-1})\\ &= \tilde\theta_1(h_1'^{\,-1}, h_1^{-1})\, \tilde\omega(h_1'^{\,-1}, h_1^{-1}, \gamma^{-1})\\ &= \theta_1(h_1', h_1)^{-1}\, \omega(h_1', h_1, \gamma).\end{aligned} \tag{3.68}$$

This reproduces the factor in the relation (3.55) for the left action of H_1, with $g = \gamma$.

- Next consider the case $h_1 = e = h_1'$; then we get

$$\begin{aligned}\tau(\gamma; e, e; h_2, h_2') &= \tilde{\theta}_2(h_2, h_2')\, \tilde{\omega}(h_2, h_2', \gamma h_2^{-1} h_2'^{-1}) \\ &= \tilde{\theta}_2(h_2, h_2')\tilde{\omega}(\gamma^{-1}, h_2, h_2')^{-1} \\ &= \theta_2(h_2^{-1}, h_2'^{-1})^{-1}\omega(\gamma, h_2^{-1}, h_2'^{-1})^{-1}. \qquad (3.69)\end{aligned}$$

  This is the factor in (3.56), provided we replace the group elements h_2 and h_2' in (3.56) by their inverses, which is precisely what is needed to turn the right action of H_2 in (3.56) to the left action considered here.
- Finally take $h_1' = e = h_2'$. This results in

$$\tau(\gamma; h_1, e, h_2, e) = \tilde{\omega}(\gamma, h_1, h_2\gamma^{-1}h_1^{-1}) = \tilde{\omega}(h_1^{-1}, \gamma^{-1}, h_2) = \omega(h_1, \gamma, h_2^{-1}), \qquad (3.70)$$

  thus reproducing the factor appearing in the bimodule relation (3.57) (again upon putting $g = \gamma$ and inverting h_2). □

Notice that the number $\tilde{\omega}(h_1'^{-1}, h_1^{-1}, \gamma^{-1})$ appearing in the expression (3.68) corresponds to a tetrahedron that can be viewed as the degeneration of the slice (3.65) that results from the degeneration of its bottom triangle to a single point. Similarly, $\tilde{\omega}(\gamma^{-1}, h_2, h_2')^{-1}$ in (3.69) corresponds to the degeneration of the top triangle of (3.65) to a point. And the tetrahedron corresponding to $\tilde{\omega}(h_1^{-1}, \gamma^{-1}, h_2)$ in (3.70) can be obtained by gluing together two quadrangles along their edges which are obtained from the slice (3.65) by degenerating both the top and the bottom triangle to a single edge.

3.6. The transparent defect. We now address aspects of categories associated to DW manifolds with the topology of a circle. Recall that one expects that surface defects can be fused and should thus form a monoidal bicategory. We refer to the monoidal unit of this monoidal bicategory as the *transparent*, or *invisible* surface defect. We have already mentioned in Sect. 2.2 that in the framework of [FSV] the transparent surface defect should correspond to the canonical Witt trivialization (2.17). In the present subsection we are interested in the Lagrangian realization of this distinguished surface defect.

To understand what group homomorphism and 2-cocycle furnish the transparent defect, we consider a circle with any number n of surface defects, one of which is transparent. By fusing all other surface defects to a single one, we can reduce the situation to the case $n = 2$. This situation has already been studied in Sect. 3.1; it leads to the groupoid (3.7). To realize the transparent defect for one of the two marked points we must moreover set $G_> = G_< =: G$ and take the same 3-cocycle ω on either side. Now we claim that the group homomorphism for the transparent defect is the diagonal subgroup embedding, i.e. we have to set $H_+ = G$ with $\iota_+ = d\colon G \to G \times G$ the diagonal embedding. This way we arrive at the action groupoid

$$\Gamma_1 := G \times G \,\backslash\!\backslash\, G \times G \times G \times G /\!/_{d^- \times \iota^-} G \times H \qquad (3.71)$$

which we already considered in (3.8). We further claim that the relevant 2-cochain on $H = G$ is the constant 2-cochain $\theta_d \equiv 1$. Note that this is a valid cochain, as it satisfies $\mathrm{d}\theta_d = 1 = \omega \cdot \omega^{-1}$.

To see that the defect defined by $\iota = d$ and $\theta = 1$ indeed has the relevant properties of the transparent defect, recall first that in (3.10) we have obtained an equivalence $F\colon \Gamma_1 \xrightarrow{\simeq} \Gamma_2$ between Γ_1 and the action groupoid

$$\Gamma_2 := G \,\backslash\!\backslash\, G \times G /\!/_{\iota^-} H \qquad (3.72)$$

introduced in (3.9), and that the latter groupoid is precisely the one relevant for the circle with a single surface defect of arbitrary type. Our prescription also yields 2-cocycles τ_1 on Γ_1 and τ_2 on Γ_2. We need to show that we still get an equivalence after linearization with respect to Lagrangian data. To this end, describe the second defect by (H, θ) with group homomorphisms $\iota_i : H \to G$ and a 2-cochain θ on H satisfying $\mathrm{d}\theta = (\iota_1^* \omega)\, (\iota_2^* \omega)^{-1}$. We then have

Proposition 3.7. *The pullback along the functor* $F : \Gamma_1 \to \Gamma_2$ *described in* (3.10) *yields an equivalence*

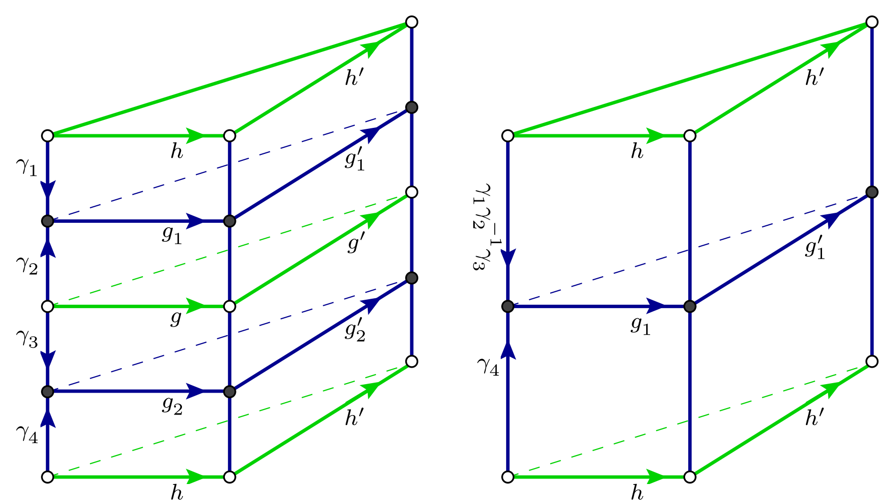
$$\begin{aligned} F^* : \quad [\Gamma_2, \mathrm{vect}]^{\tau_2} &\xrightarrow{\ \simeq\ } [\Gamma_1, \mathrm{vect}]^{\tau_1} \\ \varphi &\longmapsto \varphi \circ F \end{aligned} \tag{3.73}$$

of $\mathbb{C}$*-linear abelian categories.*

Proof. Morphisms in the groupoid Γ_1 have the form (3.11). Pick two such morphisms (g_1, g_2, g, h) and (g_1', g_2', g', h'). Their images under F are morphisms (g_1, h) and (g_1', h') in Γ_2, of the form (3.12). We must show that

$$\tau_1(\gamma_1, \gamma_2, \gamma_3, \gamma_4; g_1, g_2, g, h; g_1', g_2', g', h') = \tau_2(\gamma_1 \gamma_2^{-1} \gamma_3, \gamma_4; g_1, h; g_1', h') \tag{3.74}$$

for all quadruples $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of elements of G. Both sides of (3.74) are obtained by evaluating appropriate diagrams of the form of slices of pie with top and bottom faces identified. The diagram relevant to Γ_1 is similar to the one of figure (3.32), but now with identified top and bottom, so that $h_1 = h_2 =: h$ and $h_1' = h_2' =: h'$, as well as with $h_{12} = g$ and $h_{12}' = g'$ being now elements of G; this diagram is shown on the left hand side of the picture (3.75) below. In the case of Γ_2 there is, besides the identified top and bottom faces, only one horizontal face, with edges labeled by elements g_1 and g_1' of G; this diagram is shown on the right hand side of the picture:



(3.75)

It should be appreciated that the two diagrams only differ in a part that is of the same topology and only involves edges labeled by G. It is easily seen that there is a sequence of Pachner moves relating the decompositions of the two diagrams in (3.75). And as discussed in Sect. 3.4, invariance under Pachner moves holds (as a direct consequence of the axioms of group cohomology) for the decomposition of simplices into tetrahedra. Together it follows that indeed the 2-cocycles on the left and right hand sides of (3.74) have the same value. □

To summarize our findings: The surface defect labeled by $\iota = d$ and $\theta = 1$ can be omitted without changing the category that our linearization procedure associates to the circle. In other words, it has the characteristic property of the monoidal unit for the fusion of surface defects, and thus of the transparent defect.

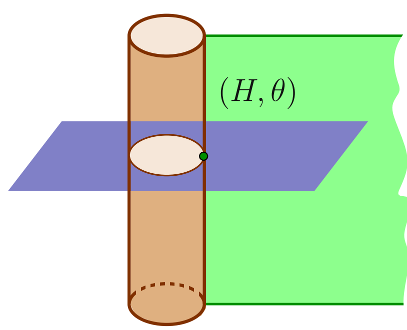
3.7. Wilson line categories for the circle. A one-dimensional DW manifold with the topology of a circle can contain finitely many marked points, corresponding to surface defects. Invoking fusion of defects, the situation with any number of marked points can be reduced to the one with a single marked point, which thereby constitutes one of the two fundamental building blocks. In this subsection we finally compute the category of generalized Wilson lines corresponding to this building block and compare it with the results of [FSV] for defects in topological field theories of Reshetikhin–Turaev type.

Let, as before, the subinterval be labeled by (G, ω) and the defect by a group homomorphism $\iota\colon\, H \to G \times G$ and a 2-cochain θ on H satisfying $\mathrm{d}\theta = (\iota_1^*\omega)\,(\iota_2^*\omega)^{-1}$. We can restrict our attention to indecomposable defects and therefore assume that ι is injective. For this situation our formalism yields in a straightforward manner the groupoid

$$G \,\backslash\!\!\backslash\, G \times G /\!\!/_{\iota^-} H \tag{3.76}$$

that we already encountered in (3.9). Its (projective) linearization, which we denote by $\mathcal{W}_{H,\theta}$, is the abelian category of $G \times G$-graded vector spaces with two commuting left actions (which are, in general, projective): a left action of G such that $g.\, V_{\gamma_1\gamma_2} \subseteq V_{g\gamma_1, g\gamma_2}$ and a left H-action such that $h.\, V_{\gamma_1\gamma_2} \subseteq V_{\gamma_1\iota_1(h)^{-1}, g\gamma_2\iota_2(h)^{-1}}$.

The category $\mathcal{W}_{H,\theta}$ has the interpretation of the category of generalized Wilson lines separating the defect labeled by ι and θ from the transparent defect that we studied in the previous subsection. Pictorially, fusion of surface defects replaces the configuration depicted on the right hand side of figure (3.1), in which four surface defects meet in a generalized Wilson line, by the configuration shown in the following picture, in which the single non-trivial defect is on the right and the transparent defect on the left:



(3.77)

We claim that the category produced by our geometric prescription is the same as the Wilson line category that is obtained in the formalism of [FSV]. Let us thus compute the latter. According to formula (2.12), in the framework of [FSV] a surface defect is described by a Witt trivialization. Now in the case of Dijkgraaf–Witten theories already the modular tensor category $\mathcal{C}$ of bulk Wilson lines is, by definition, Witt trivial. Indeed, $\mathcal{C} = \mathcal{Z}(\mathcal{A})$, where for the theory based on (G, ω), $\mathcal{A}$ is the fusion category of finite-dimensional G-graded vector spaces with associativity constraint given by ω as in (2.47). It is not difficult to verify that the Witt trivialization of $\mathcal{C}$ implies the Witt trivialization

$$\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\;\simeq\;} \mathcal{Z}(\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}), \tag{3.78}$$

where $\mathcal{A}^{\mathrm{op}}$ is the fusion category $\mathcal{A}$ with opposite tensor product.

Indecomposable surface defects separating the modular tensor category $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ from itself correspond [FSV] to indecomposable module categories over $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}$ which is, as an abelian category, the category of $G \times G$-graded vector spaces. According to the results reported in Sect. 2.5, such a module category is described by a subgroup $H \leq G \times G$ and a 2-cochain θ on H. This category can be realized as $\mathcal{M}_{H,\theta} = A_{H,\theta}$-mod, with the algebra $A_{H,\theta}$ as introduced in Sect. 2.5. The category $\mathcal{M}_{H,\theta}$ of $A_{H,\theta}$-modules, seen as a module category over $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}$, describes the non-transparent surface defect in the situation we are considering.

The analogous algebra in $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}$ that is relevant for the transparent defect can be deduced from the discussion in Sect. 3.6: it is the algebra A_d for the diagonal subgroup $G \leq G \times G$ with trivial 2-cocycle $\theta = 1$. The category of Wilson lines described by the linearization of the groupoid (3.76) should therefore be matched to the category

$$\mathrm{Hom}_{\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{M}_{H,\theta}) \tag{3.79}$$

of module functors or, equivalently, to the category of A_d-$A_{H,\theta}$-bimodules in $\mathcal{A} \boxtimes \mathcal{A}^{\mathrm{op}}$. But the latter is nothing else than the category of $G \times G$-graded vector spaces together with projective actions of H and G.

This concludes the match of the categories that are obtained, for the case of the circle, in the present geometric approach and in [FSV].

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A. Module Categories for Non-injective Group Homomorphisms

As described in Sect. 2.5, indecomposable module categories over the fusion category G-vect are given by subgroups $H \leq G$ and group cochains. On the other hand, in the definition of relative bundles a group homomorphism $\iota\colon H \to G$ enters. In the geometric context, it is not natural, and for many purposes, e.g. for the discussion of fusion of surface defects, not appropriate, to require ι to be injective. This raises the question how corresponding module categories decompose into indecomposable ones if the group homomorphism ι is not injective. We discuss this issue in the simplest setting, in particular dropping Lagrangian data.

We consider a morphism $\iota\colon H \to G$ of finite groups and the action groupoid $G /\!/_{\iota^-} H$ with left action $h.\gamma = \gamma\, \iota(h)^{-1}$. The functor category $\mathcal{M} := [G /\!/_{\iota^-} H, \text{vect}]$ is a module category over the monoidal category G-vect as follows. Objects in $\mathcal{M}$ are G-graded vector spaces $V = \bigoplus_{g \in G} V_g$ endowed with a left action of H such that

$$h.V_g \subset V_{g \cdot \iota(h)^{-1}}. \tag{A.1}$$

The simple object W_γ of G-vect acts on such an object of $\mathcal{M}$ by shifting the degrees of the homogeneous components by left multiplication by γ and keeping the action of H:

$$(W_\gamma \otimes V)_g = V_{\gamma \cdot g}. \tag{A.2}$$

Any module category over G-vect can be decomposed into indecomposable module categories. Let us see how this works for the module categories arising in the way considered here. To this end we consider the normal subgroup $K := \ker \iota \leq H$ and the exact sequence

$$1 \to K \to H \xrightarrow{\pi} J \to 1 \tag{A.3}$$

of groups. This sequence is, in general, not split, and H is thus not a semidirect product. Still, we can choose a set-theoretic section $s\colon J \to H$ of π, which for convenience we require to respect neutral elements, $s(e_J) = e_H$. We keep the section s fixed from now on. For each $j \in J$ consider the group automorphism

$$\alpha_j := \mathrm{ad}_{s(j)}|_K \ \in \mathrm{Aut}(K). \tag{A.4}$$

The automorphism α_j is not necessarily inner; its class $[\overline{\alpha}_j] \in \mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K)$ does not depend on the choice of s. Moreover, introduce group elements

$$c_{i,j} := s(i)\, s(j)\, s(ij)^{-1} \ \in K \tag{A.5}$$

for each pair $i, j \in J$. Then one has the relation

$$\alpha_j \circ \alpha_{j'} = \mathrm{ad}_{c_{j,j'}} \circ \alpha_{jj'} \tag{A.6}$$

and obvious coherence conditions on the elements $c_{ij} \in K$; thus $(\alpha_j, c_{i,j})$ defines a *weak action* of the group J on the group K.

We use this observation to rewrite the group H in a convenient way. The map

$$\begin{aligned} \psi: \quad J \times K &\to H \\ (j,k) &\mapsto k \cdot s(j) \end{aligned} \tag{A.7}$$

has the inverse

$$\begin{aligned} \psi^{-1}: \quad H &\to J \times K \\ h &\mapsto (\pi(h), h \cdot (s\pi(h))^{-1}). \end{aligned} \tag{A.8}$$

Define on the set $J \times K$ a composition map

$$(i,k) \cdot (j,k') := (ij, k\alpha_i(k')c_{ij}). \tag{A.9}$$

A direct calculation shows that the map ψ is compatible with the product (A.9) and with the product on H. Thus (A.9) endows the set $J \times K$ with the structure of a finite group isomorphic to H. We denote this group structure by $J \times_\alpha K$, suppressing the group elements c in the notation. We will identify $J \cong G/K$ with a subgroup of G in the sequel.

Thus we now replace H by the isomorphic group $J \times_\alpha K$. Then the left $J \times_\alpha K$-action on $V = \bigoplus_{g \in G} V_g$ satisfies

$$(j,k)(V_g) \subset V_{g.j^{-1}}. \tag{A.10}$$

Moreover, each homogeneous component V_g has a natural structure of a K-module from the action of elements of the form $(e_J, k) \in J \times_\alpha K$.

It is crucial to note that the so obtained K-module structures on different homogeneous components V_g are in general not isomorphic. They are related by the action of elements of the form (j,k) that are *twisted* intertwiners rather than morphisms of K-modules. Comparing the group elements $(e,k) \cdot (j,k') = (j, kk')$ and (j,k') $(e,k'') = (j, k'\alpha_j(k''))$ we deduce that

$$(e,k) \cdot (j,k') = (j,k')(e,k'') \quad \text{with} \quad k'' = \alpha_j^{-1}((k')^{-1}kk'). \tag{A.11}$$

Thus the action by (j, k') is a twisted intertwiner relating a K-module in the isomorphism class $[V_g]$ to a K-module in the class $[V_{g.j}] = \overline{\alpha}_j^{-1}[V_g]$. These two isomorphism classes are different if α_j is outer.

To find the simple objects of the category $[G/\!\!/_{\iota}\!-H, \text{vect}]$, fix representatives $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ for the orbits of the right action of J on G. Then the isomorphism classes of simple objects are in bijection with pairs (γ_i, χ) with $\chi \in \widehat{K}$ a simple character of K.

The action of G-vect on the set of isomorphism classes of simple objects (γ_i, χ) of the category $[G/\!\!/_{\iota}\!-H, \text{vect}]$ and thus its decomposition as a module category over G-vect can now be computed explicitly.

An instructive example is the group homomorphism $\iota\colon H = S_3 \to \mathbb{Z}_2 = G$, with S_3 the symmetric group on three letters, that is given by the sign function. The exact sequence (A.3) of groups is then

$$1 \longrightarrow A_3 \cong \mathbb{Z}_3 \longrightarrow S_3 \xrightarrow{\text{sign}} \mathbb{Z}_2 \longrightarrow 1. \tag{A.12}$$

The simple objects of the resulting linearization $[\mathbb{Z}_2/\!\!/S_3, \text{vect}]$ are labeled by the single orbit of the right action of $\mathbb{Z}_2$ on itself and by one of the three irreducible characters $\{1, \zeta, \zeta^\vee\}$ of $\mathbb{Z}_3$. Since S_3 is a semidirect product, any section $s\colon \mathbb{Z}_2 \to S_3$, e.g. the one mapping the generator of $\mathbb{Z}_2$ to the permutation $\tau_{12} \in S_3$, gives a genuine action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$, rather than only a weak action. Here the generator of $\mathbb{Z}_2$ acts as the outer automorphism of $\mathbb{Z}_3$ which exchanges the non-trivial irreducible characters ζ and $\zeta^\vee$.

This fixes the $\mathbb{Z}_3$-representation on the homogeneous component V_1 in terms of the $\mathbb{Z}_2$-representation on V_0 as shown in the following table:

rep. on V_0	rep. on V_1
1	1
ζ	$\zeta^\vee$
$\zeta^\vee$	ζ

(A.13)

We conclude that the abelian category $[\mathbb{Z}_2/\!\!/S_3, \text{vect}]$ has three isomorphism classes of simple objects, corresponding to the three lines of the table.

To determine the structure of $[\mathbb{Z}_2/\!\!/S_3, \text{vect}]$ as a module category over $\mathbb{Z}_2$-vect we note that the action of the simple object X_g in a non-trivial homogeneous component exchanges the two homogeneous components V_0 and V_1. It therefore preserves the isomorphism class of simple $[\mathbb{Z}_2/\!\!/S_3, \text{vect}]$-objects in the first line of (A.13) and exchanges the two classes in the other two lines. Thus the first line of (A.13) gives us one indecomposable module category over $\mathbb{Z}_2$-vect with a single simple object, which corresponds to $\mathbb{Z}_2$ seen as a subgroup of itself. From the second and third lines of (A.13) we get another indecomposable module category having two simple objects, corresponding to the trivial subgroup $\{e\}$ of $\mathbb{Z}_2$.

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On the Brauer Groups of Symmetries of Abelian Dijkgraaf–Witten Theories

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Abstract: Symmetries of three-dimensional topological field theories are naturally defined in terms of invertible topological surface defects. Symmetry groups are thus Brauer–Picard groups. We present a gauge theoretic realization of all symmetries of abelian Dijkgraaf–Witten theories. The symmetry group for a Dijkgraaf–Witten theory with gauge group a finite abelian group A, and with vanishing 3-cocycle, is generated by group automorphisms of A, by automorphisms of the trivial Chern–Simons 2-gerbe on the stack of A-bundles, and by partial e-m dualities. We show that transmission functors naturally extracted from extended topological field theories with surface defects give a physical realization of the bijection between invertible bimodule categories of a fusion category $\mathcal{A}$ and braided auto-equivalences of its Drinfeld center $\mathcal{Z}(\mathcal{A})$. The latter provides the labels for bulk Wilson lines; it follows that a symmetry is completely characterized by its action on bulk Wilson lines.

1. Symmetries of Abelian Dijkgraaf–Witten Theories

Dijkgraaf–Witten theories are extended topological field theories that have a mathematically precise gauge theoretic formulation with finite gauge group. In that setting, the fields of the Dijkgraaf–Witten theory with gauge group G are obtained by first considering G-bundles, to which, in a second step, a linearization procedure is applied (see [Mo] for a recent description). In the present note we investigate the notion of symmetries of three-dimensional Dijkgraaf–Witten theories, regarded as extended 1-2-3-dimensional topological field theories. To keep the presentation simple, we restrict ourselves to the case in which the gauge group is an abelian group, which we denote by A.

Braided auto-equivalences of bulk Wilson lines. The task of understanding symmetries in Dijkgraaf–Witten theories can be approached from two different angles, either algebraically or gauge theoretically. From a purely algebraic point of view, one would consider the modular category of bulk Wilson lines, which is the representation category

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$\mathcal{D}(A)$-mod of the Drinfeld double of A. Symmetries should then in particular induce braided auto-equivalences of $\mathcal{D}(A)$-mod.

The group of braided auto-equivalences (up to monoidal natural equivalence) can be described as follows. Denote by A^* the group of complex characters of A. The group $A \oplus A^*$ comes with a natural quadratic form $q: A \oplus A^* \to \mathbb{C}^\times$, given by $q(g+\chi) = \chi(g)$ for $g + \chi \in A \oplus A^*$. The automorphism group of $A \oplus A^*$ then has a subgroup, denoted by $\mathrm{O}_q(A\oplus A^*)$, consisting of those group automorphisms φ that preserve this form, i.e., satisfy $q(\varphi(z)) = q(z)$ for all $z \in A \oplus A^*$. Now the group of braided auto-equivalences is isomorphic to this group $\mathrm{O}_q(A\oplus A^*)$ [ENOM]. Simple objects of $\mathcal{D}(A)$-mod, and thus simple labels for bulk Wilson lines of the Dijkgraaf–Witten theory with gauge group A, are in bijection with elements of $A \oplus A^*$; a braided auto-equivalence induces the natural action of the corresponding element of $\mathrm{O}_q(A\oplus A^*)$ on the group $A \oplus A^*$.

In this approach the auto-equivalences of $\mathcal{D}(A)$-mod are not intrinsically realized in the Dijkgraaf–Witten theory as a gauge theory. It is therefore not clear whether every braided auto-equivalence of the category of bulk Wilson lines preserves all aspects of the three-dimensional topological field theory so that it can indeed be regarded as a full-fledged symmetry of the theory. It is not clear either whether a braided auto-equivalence would then describe a symmetry uniquely. There might be several different realizations, or also none at all, of the auto-equivalences on other field theoretic quantities, such as boundary conditions.

Universal kinematical symmetries. It is thus important to find a field theoretic realization of the auto-equivalences, relating to the fact that Dijkgraaf–Witten theories can be formulated as gauge theories. At the same time we then get additional insight into the structure of the group $\mathrm{O}_q(A\oplus A^*)$. From a gauge theoretic point of view it is natural to expect that the symmetries of the stack $\mathrm{Bun}(G)$ of G-bundles are symmetries of both classical and quantum Dijkgraaf–Witten theories.[1] One might call these symmetries universal kinematical symmetries—kinematical, because they are symmetries of the kinematical setting, i.e., G-bundles; and universal, because the manifold on which the G-bundles are defined does not enter.

The symmetries of $\mathrm{Bun}(G)$ form the 2-group $\mathrm{AUT}(G)$, i.e. the category whose objects are group automorphisms $\varphi\colon G \to G$ and whose morphisms $\varphi_1 \to \varphi_2$ are given by group elements $h \in G$ that satisfy $\varphi_2(g) = h\,\varphi_1(g)\,h^{-1}$ for all $g \in G$. Since in the case of our interest the group A is abelian, we can safely ignore the morphisms in the category $\mathrm{AUT}(A)$ and work with the ordinary automorphism group $\mathrm{Aut}(A)$ of the group A.

The group $\mathrm{Aut}(A)$ of symmetries of $\mathrm{Bun}(A)$ can be identified in a natural way with a subgroup of the group $\mathrm{O}_q(A\oplus A^*)$ of braided auto-equivalences. Indeed, for any $\alpha \in \mathrm{Aut}(A)$, the automorphism $\alpha \oplus (\alpha^{-1})^*$ of $A \oplus A^*$ belongs to $\mathrm{O}_q(A\oplus A^*)$, where $(\alpha)^* \in \mathrm{Aut}(A^*)$ is defined by $[\alpha^*\chi](a) := \chi(\alpha(a))$ for all $\chi \in A^*$ and all $a \in A$. But this argument is purely group theoretical, and it is not clear at this point whether the embedding has any physical relevance and relates symmetries of bundles to braided auto-equivalences of bulk Wilson lines.

Universal dynamical symmetries. The realization of Dijkgraaf–Witten theories as gauge theories leads to even more symmetries. Apart from a finite group G, a three-cocycle $\omega \in Z^3(G, U(1))$ is another ingredient of a Dijkgraaf–Witten theory. Geometrically this

[1] Actually, a general Dijkgraaf–Witten theory involves a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. Here we only consider the case of trivial ω, and hence do not expect any compatibility relations between the automorphism and ω.

cocycle is interpreted [Wi] as a (Chern-Simons) 2-gerbe on the stack $\mathrm{Bun}(G)$ of G-bundles, and we may think of ω heuristically as a topological Lagrangian. In the present note we restrict ourselves to the case of vanishing cocycle ω, corresponding to a trivial 2-gerbe. Still, the automorphism group of the trivial 2-gerbe is a non-trivial 3-group: it is the 3-group of 1-gerbes on G. It is thus again natural to expect that this 3-group provides us with symmetries of the Dijkgraaf–Witten theory with gauge group G. We call these symmetries dynamical universal symmetries, as they involve symmetries of the topological Lagrangian.

By the results of [Wi], the objects of the 3-group of 1-gerbes on G are 2-cocycles on G; isomorphism classes are described by elements of the group cohomology $H^2(G, \mathbb{C}^\times)$. The group generated by classical kinematical and dynamical symmetries has the structure of a semi-direct product, $H^2(G, \mathbb{C}^\times) \rtimes \mathrm{Aut}(G)$. By [NR, Prop. 4.1] this group is isomorphic to the automorphism group of the fusion category G-vect of G-graded vector spaces. Indeed, this fusion category enters in the construction of Dijkgraaf–Witten theories as topological field theories of Turaev–Viro type.

If $G = A$ is abelian, cohomology classes in $H^2(A, \mathbb{C}^\times)$ are in bijection with alternating bicharacters. (An alternating bicharacter is a map $\beta\colon A \times A \to \mathbb{C}^\times$ that is a group homomorphism in each argument and satisfies $\beta(a, a) = 1$ for all $a \in A$, and thus $\beta(a_1, a_2) = \beta(a_2, a_1)^{-1}$ for all $a_1, a_2 \in A$.) Again, there is a natural embedding $H^2(A, \mathbb{C}^\times) \hookrightarrow \mathrm{O}_q(A \oplus A^*)$ of finite groups: to a class in $H^2(A, \mathbb{C}^\times)$ described by an alternating bicharacter β, we associate a map $\phi_\beta\colon A \oplus A^* \to A \oplus A^*$ defined by $\phi_\beta(a + \chi) := (a + \beta(a, -) + \chi(-))$. One immediately verifies that ϕ_β is an element of $\mathrm{O}_q(A \oplus A^*)$. Again it remains to be shown, though, that this embedding is of physical relevance in the sense that it relates symmetries of the topological Lagrangian to braided auto-equivalences.

Electric-magnetic dualities. The universal kinematical and dynamical symmetries cannot, however, exhaust the symmetries of Dijkgraaf–Witten models—the subgroup of $\mathrm{O}_q(A \oplus A^*)$ generated by them is a proper subgroup. As an illustration, consider the case that A is the cyclic group $\mathbb{Z}_2$. This group does not admit any non-trivial automorphisms, i.e. $\mathrm{Aut}(A) = 1$. It does not admit any non-trivial alternating bicharacter either, and hence the group generated by the universal dynamical and kinematical symmetries is trivial. On the other hand, the group $\mathrm{Aut}(A \oplus A^*)$ is the symmetric group S_3 that permutes the three order-two elements of $A \oplus A^*$. Its subgroup $\mathrm{O}_q(A \oplus A^*)$ is the subgroup $S_2 \cong \mathbb{Z}_2$ of S_3 whose non-trivial element exchanges the generator of A with the one of A^*; in physics terminology, a transformation of this type is called an electric-magnetic duality, or e-m duality. The presence of such electric-magnetic dualities is a central feature of gauge theories in various dimensions (see e.g. [KaW, KaBSS] for a general discussion). Electric-magnetic dualities have a particularly explicit description in theories that can be realized as lattice models, compare [DW, Ki, BuCKA] and references therein.

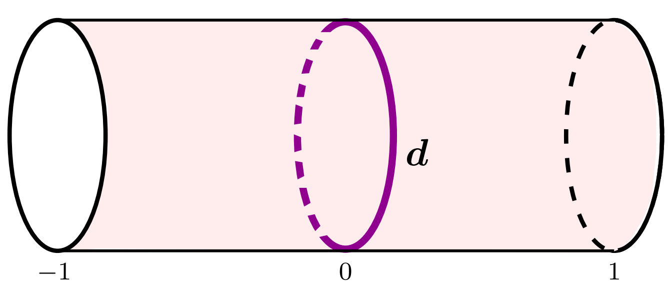
Topological surface defects and bimodule categories. A proper understanding of the situation, including a physical realization of the subgroups described above, calls for a unified field theoretic perspective. In this note we explain that in the present situation, for which no rigorous definition of symmetry for an extended topological field theory has been fully tested out so far, topological surface defects provide such a perspective. In fact, the relation between symmetries and classes of invertible topological codimension-one defects has been established long ago [FFRS1, FFRS2] for the case of *two*-dimensional field theories. But the mechanism that implements symmetries via topological defects is not restricted to the two-dimensional case. One of the virtues of realizing symmetries in terms of topological defects of codimension one is that this realization immediately determines how the symmetries act on all kinds of aspects of the field theory, including in particular labels of boundaries and defects.

Topological surface defects in 3d TFTs have recently attracted increasing interest, see [Bo, KK, KaS, BaMS, EKRS, KhTH, GGP, FSV1] for a selection of recent contributions. The case of three-dimensional topological field theories of Turaev–Viro type is particularly well understood. In particular, it is by now well-established that topological surface defects in Dijkgraaf–Witten theories with gauge group A correspond to bimodule categories over the fusion category $\mathcal{A} = A$-vect of finite-dimensional A-graded vector spaces. Those defects that describe symmetries correspond to invertible bimodule categories; accordingly, we call them invertible defects. Their fusion product with the opposite defect is the monoidal unit for fusion, which is also called the invisible or transparent defect. Invertible defects can alternatively be characterized by the fact that the only bulk Wilson lines that 'condense' on them are the invisible bulk Wilson lines.

The group of (equivalence classes of) invertible bimodule categories, the so-called Brauer–Picard group of $\mathcal{A}$, has been described in [NN, ENOM, DaN]. In particular, a bijection has been established [ENOM, Thm 1.1] between invertible bimodule categories of a fusion category—in our case A-vect—and braided auto-equivalences of its center—in our case the category $\mathcal{D}(A)$-mod of bulk Wilson lines. As a consequence, also (equivalence classes of) invertible bimodule categories are described by the group $\mathrm{O}_q(A \oplus A^*)$.

The transmission functor. The results of [ENOM] are of purely representation theoretic nature. The purpose of the present note is to investigate their consequences and counterparts in Dijkgraaf–Witten theories as gauge theories. The bijection between equivalence classes of invertible bimodule categories and braided auto-equivalences in [ENOM] leads us to consider braided auto-equivalences F_d of $\mathcal{D}(A)$-mod labeled by invertible bimodule categories d over $\mathcal{D}(A)$-mod. Thus to any invertible topological surface defect d we have to associate such a braided auto-equivalence.

Now in an extended three-dimensional topological field theory, functors are obtained from surfaces with boundaries, and there is indeed a natural candidate for the relevant two-dimensional cobordism with defect. Namely, to yield an endofunctor of the category of bulk Wilson lines, the cobordism should have one ingoing and one outgoing boundary; and it should not induce any additional topological information; hence we have to consider a cylinder. The cylinder can be thought of as coming from a cut-and-paste boundary in a three-dimensional topological field theory. Such boundaries have to intersect surface defects transversally. Hence a surface defect results in a line embedded in the cobordism. We are thus lead to consider a cylinder $Z = S^1 \times [-1, 1]$ with a defect line along the circle $D = S^1 \times \{0\} \subset Z$, as shown in the following picture:



(1.1)

In the sequel we regard the circle $S^1 \times \{-1\} \subset Z$ as incoming and the circle $S^1 \times \{1\} \subset Z$ as outgoing. We denote the functor described by the cobordism (1.1) by F_d and call it the *transmission functor* for the defect d. We will show in Sect. 2.3 that for an invertible defect in a general three-dimensional extended topological field theory, the transmission functor F_d is a braided auto-equivalence of the category of bulk Wilson lines. The transmission functor describes what happens to the type of a bulk Wilson line when it passes through the surface defect d.

We note that in some physical applications Wilson lines can be interpreted as world lines of quasi-particles, with the type of the quasi-particle specified by the type of Wilson line. When such a quasi-particle crosses an invertible topological surface defect of type d, the type of quasi-particle is changed according to the transmission functor F_d. In field-theoretic terms, this change is brought about by a so-called Alice string [Sc, ABCMW, DB]. Let us illustrate this interpretation with the situation that the surface defect is a half-plane $\mathbb{R}_{x\geq 0} \times \mathbb{R}$ in three-dimensional space $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$. The boundary of the half-plane consists of a Wilson line that separates the surface defect d from the transparent defect (such Wilson lines always exist). The intersection of the defect d with a plane $\mathbb{R}^2 \times \{t_0\}$ of fixed time is a half-line labeled by d; this half-line constitutes the Alice string. Since the surface defect is topological, the precise position of the half-line does not matter. Whenever a quasi-particle crosses the world surface swept out by the Alice string, i.e., crosses the topological surface defect, it changes its type according to the transmission functor.

In the case of Dijkgraaf–Witten theories, transmission functors are explicitly accessible: there is a gauge theoretic realization of topological surface defects in Dijkgraaf–Witten theories based on relative bundles [FSV2]. As a consequence, topological defects are classified by a subgroup $H \leq A \oplus A$, together with a cohomology class in $H^2(H, \mathbb{C}^\times)$. The formalism developed in [Mo] then allows one to compute the transmission functor.

In this note we provide a set of generators for the group $\mathrm{O}_q(A\oplus A^*)$ of braided auto-equivalences, which implies that universal kinematical and dynamical symmetries together with electric-magnetic dualities generate all symmetries. We then give, for each of these generators of $\mathrm{O}_q(A\oplus A^*)$, a topological defect, compute the resulting transmission functor and show that it acts on simple labels for bulk Wilson lines by the natural action of $\mathrm{O}_q(A\oplus A^*)$ on $A \oplus A^*$. This provides a field theoretic realization in terms of topological surface defects for all braided auto-equivalences. At the same time it establishes that the embeddings of the subgroups of dynamical and kinematical universal symmetries described above are indeed physical. When combined with the results of [ENOM], it also follows that the braided equivalences of bulk Wilson lines are in bijection with field-theoretic symmetries.[2] Hereby we realize all elements of the Brauer–Picard group as gauge-theoretic dualities.

Plan of the paper. The rest of this note is organized as follows. Section 2 collects some background about topological surface defects in Dijkgraaf–Witten theories and provides information about transmission functors arising from invertible defects. In Sect. 3 we construct these defects explicitly for various classes of generators and compute their transmission functors. Finally we show in Sect. 4 that the group of invertible defects is generated by kinematical and dynamical symmetries together with e-m dualities.

[2] This is reminiscent of the situation in two-dimensional rational conformal field theories, where the action of topological line defects on bulk fields characterizes isomorphism classes of defects, so that the action of topological line defects on bulk fields has been used in the classification of defects.

Technically, this is proven as the group-theoretical statement that a certain set of elements of the group $\mathrm{O}_q(A{\oplus}A^*)$ generates this group.

2. Surface Defects in DW Theories and the Transmission Functor

2.1. Surface defects in Dijkgraaf–Witten theories. A model independent analysis of topological surface defects between topological field theories of Reshetikin–Turaev type has been presented in [FSV1]. We summarize the pertinent aspects of that analysis: For $\mathcal{C}$ and $\mathcal{C}'$ modular tensor categories, a topological surface defect separating the Reshetikhin–Turaev theories with bulk Wilson lines labeled by $\mathcal{C}$ and by $\mathcal{C}'$, respectively, exists if and only if the modular category $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}}$ is braided equivalent to the Drinfeld center of some fusion category $\mathcal{A}$;[3] here $(\mathcal{C}')^{\mathrm{rev}}$ is the same monoidal category as $\mathcal{C}'$, but with opposite braiding. We call the corresponding braided equivalence functor $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A})$ a trivialization of $\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}}$. If such a trivialization exists, then the bicategory of defects is equivalent to the bicategory of module categories over the fusion category $\mathcal{A}$.

In the present paper we are interested in the case of defects that separate a Dijkgraaf–Witten theory based on the abelian group A from itself. Thus the category of bulk Wilson lines is already a Drinfeld center, $\mathcal{C}=\mathcal{C}'=\mathcal{Z}(A\text{-vect})$, and accordingly there is a distinguished trivialization

$$\mathcal{C}\boxtimes(\mathcal{C}')^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(A{\oplus}A\text{-vect}). \tag{2.1}$$

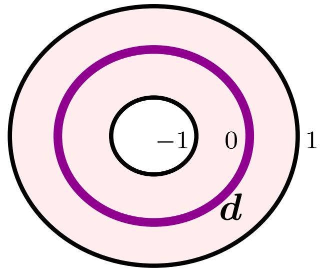
The defects of our interest are thus classified by module categories over the category of $A\oplus A$-graded vector spaces.

Indecomposable $A{\oplus}A$-vect-module categories have been classified in [Os]: they correspond to subgroups $H\le A\oplus A$, together with a two-cocycle on H. That they describe surface defects of Dijkgraaf–Witten theories has been explicitly demonstrated in [FSV2].

2.2. The transmission functor. We want to determine the transmission functor $F_d:\mathcal{C}\to\mathcal{C}$ for an invertible topological surface defect d described by an indecomposable module category over $\mathcal{C}$. The physical interpretation of the transmission functor F_d for an invertible defect is as follows. When a bulk Wilson line labeled by an object $U\in\mathcal{C}$ passes through the surface defect d, its label changes to $F_d(U)\in\mathcal{C}$. (Recall that no bulk Wilson lines condense on the defect.)

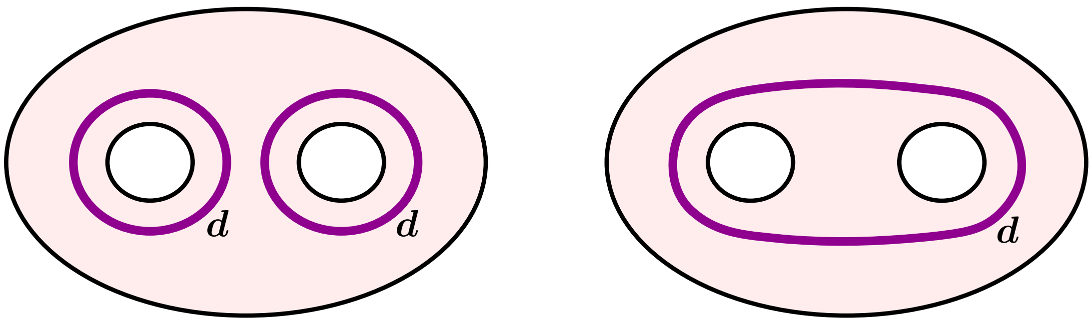
We now explain why the transmission functor for an *invertible* surface defect in an extended three-dimensional topological field theory has a natural structure of a braided auto-equivalence. First of all, by composing the transmission functor F_d for a surface defect d with the transmission functor $F_{\overline{d}}$ for the opposite defect $\overline{d}$ and invoking fusion of defects, we conclude that $F_d\circ F_{\overline{d}}=\mathrm{Id}_{\mathcal{C}}=F_{\overline{d}}\circ F_d$, so that F_d is indeed an auto-equivalence. To proceed, it will be convenient to draw the cylinder (1.1) as an annulus with an embedded defect line, according to

[3] Then the classes of $\mathcal{C}$ and $\mathcal{C}'$ in the Witt group [DaMNO] of modular tensor categories coincide.



(2.2)

To discuss monoidality we then have to compare the functors corresponding to the two ‘trinion’ surfaces shown in the following picture:

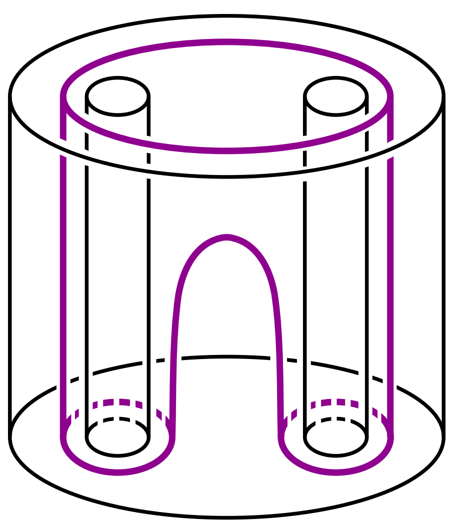


(2.3)

For a general defect, the functors associated to these two trinions are not isomorphic and the transmission functor is not monoidal; one rather obtains monoidal functors between categories of local modules over braided-commutative algebras in $\mathcal{C}$. But if the defect is invertible, then the functors corresponding to the two trinions are isomorphic. In fact, a natural isomorphism

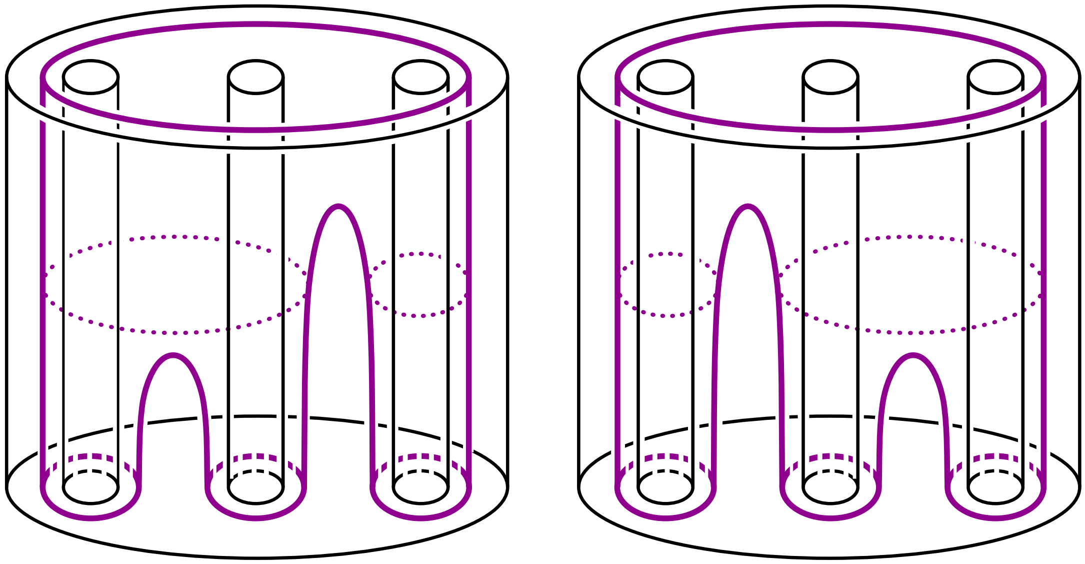
$$\otimes \circ (F_d \times F_d) \Longrightarrow F_d \circ \otimes \tag{2.4}$$

of functors $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is furnished by the following three-manifold with corners and defects:



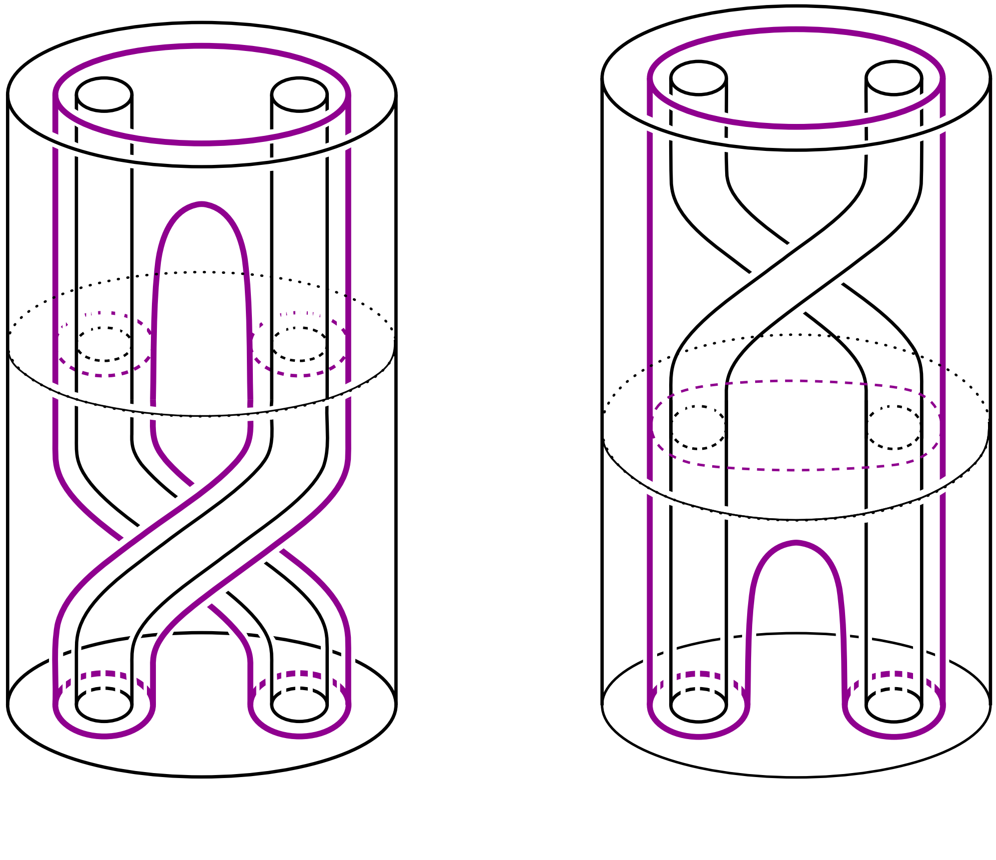
(2.5)

Such a three-manifold with corners is to be read as a span of manifolds from the bottom lid to the top lid. To show that this natural transformation provides a monoidal structure on the functor F_d, one needs to check an identity of natural transformations. This identity follows from the fact that the following two three-manifolds with corners and defects are related by a homotopy relative to the boundary:



(2.6)

(This homotopy, restricted to the surface defect, looks like the homotopy used in two-dimensional topological field theories to show associativity of the algebras assigned to circles, but its role is rather different.) In a similar manner the property that the monoidal structure on F_d is braided can be deduced from the fact that the following two three-manifolds with corners and defects are homotopic as well:



(2.7)

Remark 2.1. In passing, we mention another physical application: According to [KaS], surface defects provide an interpretation of the so-called TFT construction (see [SFR] for a review) of correlators of two-dimensional rational conformal field theories associated with the category $\mathcal{C}$. Thereby a surface defect d in particular determines a modular invariant torus partition function Z^d of the conformal field theory. For an invertible defect d with transmission functor F_d, the resulting torus partition function is of automorphism type; its coefficient matrix reads

$$\mathrm{Z}^d_{ij} = \delta_{[U_i],[F_d(U_j^\vee)]}\,, \tag{2.8}$$

where $\{U_i\}$ is a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$.

2.3. Transmission functors for Dijkgraaf–Witten theories. In the case of our interest the modular tensor category $\mathcal{C}$ is the representation category of the (untwisted) Drinfeld double $\mathcal{D}(A)$ of a finite abelian group A, and a topological surface defect is described by a subgroup H of $A \oplus A$ and a two-cocycle on H. To obtain the relevant groupoids of bundles we follow the prescription of [FSV2] to find the appropriate relative bundles: For a defect associated to the subgroup $H \le A \oplus A$ with two-cocycle $\theta \in Z^2(H, \mathbb{C}^\times)$, the objects of the category of relative bundles consist of an A-bundle $P_A^\pm$ on each of the two cylinders $Z_- : S^1 \times [-1, 0]$ and $Z_+ := S^1 \times [0, 1]$, an H-bundle P_H on D and an isomorphism

$$\alpha: \quad \mathrm{Ind}_H^{A\oplus A} P_H \xrightarrow{\simeq} (P_A^+)\big|_D \times (P_A^-)\big|_D \tag{2.9}$$

of $A\oplus A$-bundles on D. Using that the cylinders $Z_\pm$ are homotopic to the circle D, one can describe all bundles appearing in (2.9) by bundles on a circle. And since α is an isomorphism, one can work with an equivalent groupoid in which only the H-bundles appear as data. As a consequence the category of relative bundles can be replaced by the action groupoid $H/\!/_{\mathrm{ad}} H$ for the adjoint action of H on itself. The objects of this groupoid are group elements $h \in H$, which can be thought of as holonomies of the H-bundle on the defect circle with respect to some fixed base point; the morphisms of the groupoid correspond to gauge transformations.

According to the general picture of Dijkgraaf–Witten theories [Mo], for the cylinder we thus get a span of groupoids. For each boundary circle, we have the category of A-bundles on S^1, which we replace by the equivalent action groupoid $A/\!/_{\mathrm{ad}} A$. The relevant functor is restriction of bundles to the boundary components. To describe it, consider the group homomorphisms obtained from the canonical projections $p_{1,2}$ for $A \oplus A$ to its two summands,

$$\pi_i: \quad H \hookrightarrow A \oplus A \xrightarrow{p_i} A\,. \tag{2.10}$$

These give rise to functors

$$\hat\pi_i: \quad H/\!/_{\mathrm{ad}} H \to A/\!/_{\mathrm{ad}} A \tag{2.11}$$

on action groupoids, acting both on objects and morphisms like π_i. We thus can replace the span of groupoids of categories of bundles and relative bundles by the equivalent span

$$\begin{tikzcd} & H/\!/_{\mathrm{ad}} H \arrow[dl, "\hat\pi_1"'] \arrow[dr, "\hat\pi_2"] & \\ A/\!/_{\mathrm{ad}} A & & A/\!/_{\mathrm{ad}} A \end{tikzcd} \tag{2.12}$$

of finite groupoids. Next we linearize, i.e. for each groupoid consider the category of functors from the groupoid to vect. This gives us two pullbacks

$$\hat{\pi}_i^* : \quad [A /\!/_{\mathrm{ad}} A, \text{vect}] \to [H /\!/_{\mathrm{ad}} H, \text{vect}] , \tag{2.13}$$

as well as pushforwards

$$\hat{\pi}_{i\,*} : \quad [H /\!/_{\mathrm{ad}} H, \text{vect}] \to [A /\!/_{\mathrm{ad}} A, \text{vect}] \tag{2.14}$$

as their two-sided adjoints. Note that the category $[A /\!/_{\mathrm{ad}} A, \text{vect}] \cong \mathcal{D}(A)\text{-mod} \cong \mathcal{C}$ is the category of labels for bulk Wilson lines of the Dijkgraaf–Witten theory.

We finally construct a functor $[H /\!/_{\mathrm{ad}} H, \text{vect}] \to [H /\!/_{\mathrm{ad}} H, \text{vect}]$ from the two-cocycle θ, following [Mo, Sect. 5.4]. To this end we first transgress $\theta \in Z^2(H, \mathbb{C}^\times)$ to $\omega_\theta \in Z^1(H /\!/_{\mathrm{ad}} H, \mathbb{C}^\times)$, a one-cocycle for the loop groupoid $H /\!/_{\mathrm{ad}} H \cong [*/\!/\mathbb{Z}, */\!/G]$. According to [Wi, Thm. 3] this is the commutator

$$\omega_\theta(h_1; h_2) = \frac{\theta(h_1, h_2)}{\theta(h_2, h_1)} , \tag{2.15}$$

which is an alternating bicharacter on the abelian group A. (As is well known, alternating bicharacters for an abelian group A are in bijection with the group cohomology $H^2(A, \mathbb{C}^\times)$.) The groupoid algebra $\mathbb{C}[H /\!/_{\mathrm{ad}} H]$ has as a basis the morphisms $b_{\gamma;h} \colon \gamma \xrightarrow{h} \gamma$ in $H /\!/_{\mathrm{ad}} H$; its product is composition of morphisms, wherever this is defined, and zero else. We can canonically identify

$$\mathbb{C}[H /\!/_{\mathrm{ad}} H]\text{-mod} \simeq [H /\!/_{\mathrm{ad}} H, \text{vect}]. \tag{2.16}$$

The two-cocycle ω_θ gives an algebra automorphism

$$\begin{aligned} \varphi_\theta : \quad \mathbb{C}[H /\!/_{\mathrm{ad}} H] &\to \mathbb{C}[H /\!/_{\mathrm{ad}} H] , \\ b_{\gamma;h} &\mapsto \omega_\theta(\gamma; h)\, b_{\gamma;h} , \end{aligned} \tag{2.17}$$

which in turn provides us with the desired functor

$$\varphi_\theta^* : \quad \mathbb{C}[H /\!/_{\mathrm{ad}} H]\text{-mod} \to \mathbb{C}[H /\!/_{\mathrm{ad}} H]\text{-mod}. \tag{2.18}$$

The transmission functor $F_{H,\theta}$ is now obtained [Mo, Sect. 5.4] by pre- and post-composing this functor with the pullback and pushforward functors obtained above:

$$F_{H,\theta} : \quad [A /\!/_{\mathrm{ad}} A, \text{vect}] \xrightarrow{\hat{\pi}_1^*} [H /\!/_{\mathrm{ad}} H, \text{vect}] \xrightarrow{\varphi_\theta^*} [H /\!/_{\mathrm{ad}} H, \text{vect}] \xrightarrow{(\hat{\pi}_2)_*} [A /\!/_{\mathrm{ad}} A, \text{vect}]. \tag{2.19}$$

In particular the transmission functor is explicitly computable. Thus for any given invertible surface defect $(H \le A \oplus A, \theta)$ of the Dijkgraaf–Witten theory with gauge group A we can find the corresponding braided equivalence $F_{H,\theta}$ explicitly. From these explicit expressions, it is clear that the transmission functor only depends on the cohomology class of θ.

2.4. Action of the transmission functor on simple objects. Let us determine the action of the transmission functor on the isomorphism classes of simple objects. For the double of a general finite group G these classes are in bijection with pairs consisting of a conjugacy class c of G and an irreducible representation χ of the centralizer of a representative of c. If $G = A$ is abelian, this reduces to pairs (a, χ) consisting of a group element $a \in A$ and an irreducible character $\chi \in A^*$; thus the isomorphism classes of simple objects are

$$\pi_0([A/\!/_{\mathrm{ad}} A, \mathrm{vect}]) \cong A \oplus A^*. \tag{2.20}$$

The group structure on $\pi_0([A/\!/_{\mathrm{ad}} A, \mathrm{vect}])$ coming from the monoidal structure on the category $[A/\!/_{\mathrm{ad}} A, \mathrm{vect}]$ coincides with the natural group structure on $A \oplus A^*$.

It is straightforward to determine the action of each of the three functors (2.13), (2.14) and (2.18) on such pairs. First, for the pullback along $\hat{\pi}_1$ maps we find

$$\begin{aligned} [\hat{\pi}_1^*]: \quad \pi_0([A/\!/_{\mathrm{ad}} A, \mathrm{vect}]) &\to \pi_0([H/\!/_{\mathrm{ad}} H, \mathrm{vect}])\,, \\ (a\,, \chi) &\mapsto \bigoplus_{h \in p_1^{-1}(a)} (h\,,\ p_1^* \chi) \end{aligned} \tag{2.21}$$

with p_1^* defined by $[p_1^* \chi](h) := \chi(p_1(h))$. Second, the functor φ_θ^* acts as

$$\begin{aligned} [\varphi_\theta^*]: \quad \pi_0([H/\!/_{\mathrm{ad}} H, \mathrm{vect}]) &\to \pi_0([H/\!/_{\mathrm{ad}} H, \mathrm{vect}])\,, \\ (h\,, \psi) &\mapsto (h\,,\ \psi + \omega_\theta(h; -)) \end{aligned} \tag{2.22}$$

with ω_θ as in (2.15). And third, the pushforward along $\hat{\pi}_2$

$$\begin{aligned} [(\hat{\pi}_2)_*]: \quad \pi_0([H/\!/_{\mathrm{ad}} H, \mathrm{vect}]) &\to \pi_0([A/\!/_{\mathrm{ad}} A, \mathrm{vect}])\,, \\ (h\,, \psi) &\mapsto \bigoplus_{\chi_2 \in A^*} (p_2(h)\,, \chi_2)\, \delta_{p_2^* \chi_2, \psi}\,. \end{aligned} \tag{2.23}$$

3. Realizing the Symmetries

Various aspects of the group $\mathrm{O}_q(A \oplus A^*)$ have been described in [DaN, Sect. 5C]. Here we present a set of generators of this group: As discussed in detail in Sect. 4, the group $\mathrm{O}_q(A \oplus A^*)$ is generated by the following elements:

1. The *kinematical universal symmetries*, which come from automorphisms of the stack of A-bundles. They are given by the subgroup $S_{\mathrm{kin}} := \{\alpha \oplus \alpha^{-1*} \mid \alpha \in \mathrm{Aut}(A)\}$, which is isomorphic to $\mathrm{Aut}(A)$.
2. The *dynamical universal symmetries*, which can be identified with the group of (equivalence classes of) 1-gerbes on the stack of A-bundles. They are given by the group of alternating bicharacters on A. In the terminology of quantum field theory [SW], the connection on a 1-gerbe is called a B-field. Accordingly we refer to the subgroup of alternating bicharacters as B-fields and denote it by S_{B}.
3. *Partial electric-magnetic* (or e-m, for short) *dualities*. Such a symmetry is given by the exchange, in $A \oplus A^*$, of a cyclic summand C of A with its character group C^*. More explicitly, for A written in the form $A = A' \oplus C$ with C a cyclic subgroup, it acts on $A \oplus A^* = A' \oplus C \oplus (A')^* \oplus C^*$ as $id_{A'} \oplus \delta \oplus id_{(A')^*} \oplus \delta^{-1}$, with $\delta: C \to C^*$ any isomorphism from C to C^*.
   If one fixes a decomposition of A into a direct sum of cyclic groups C_i, together with an isomorphism $\delta_i: C_i \to C_i^*$ for each cyclic summand, then the corresponding partial e-m dualities generate a subgroup of $\mathrm{O}_q(A \oplus A^*)$, which we denote by $S_{\text{e-m}}$.

For each type of generator, we will now specify the subgroup H of $A \oplus A$ and cocycle θ that label the corresponding invertible surface defect.

Remark 3.1. In principle, for any element of the group $\mathrm{O}_q(A{\oplus}A^*)$ the subgroup $H \leq A \oplus A$ and the cocycle θ can be computed from the results in [ENOM, Sect. 10.2]. However, Theorem 1.1. of [ENOM] ensures that there is a bijection between equivalence classes of invertible topological surface defects and equivalence classes of braided equivalences. Hence it is sufficient to verify that a given defect described by a pair (H, θ) reproduces the correct braided equivalence.

We will make use of the following fact (see Corollary 3.6.3 of [Da] and Proposition 5.2 of [NR]):

Corollary 3.2. *The A-*vect*-bimodule category associated with the pair (H, θ) is invertible iff*

$$H \cdot (A \oplus \{0\}) = A \oplus A = (A \oplus \{0\}) \cdot H \tag{3.1}$$

and the restriction of the commutator cocycle ω_θ (2.15) to

$$H_\cap := \big(H \cap (A \oplus \{0\})\big) \times \big((A \oplus \{0\}) \cap H\big) \tag{3.2}$$

is non-degenerate.

3.1. Kinematical symmetries: group automorphisms. The automorphisms in S_{kin} are the symmetries of the stack $\mathrm{Bun}(A)$ and are thus symmetries of the classical configurations.

A group automorphism $\alpha\colon A \to A$ induces a group automorphism $\alpha^*\colon A^* \to A^*$ acting on $\chi \in A^*$ as

$$[\alpha^* \chi](a) := \chi(\alpha(a)) \tag{3.3}$$

for all $a \in A$. The combined group automorphism $\widetilde{\alpha} := \alpha \oplus (\alpha^{-1})^*\colon A \oplus A^* \to A \oplus A^*$ satisfies

$$\begin{aligned} q\big(\widetilde{\alpha}(a{+}\chi)\big) &= q\big(\alpha(a) + \alpha^{-1*}(\chi)\big) \\ &= [\alpha^{-1*}(\chi)](\alpha(a)) = \chi(\alpha^{-1}\alpha(a)) = \chi(a) = q(a{+}\chi)\,, \end{aligned} \tag{3.4}$$

i.e. preserves the quadratic form q and is thus an element of $\mathrm{O}_q(A{\oplus}A^*)$.

We claim that the surface defect whose transmission functor corresponds to the automorphism $\widetilde{\alpha}$ is the following: For the subgroup, we take the graph of α, i.e.

$$H_\alpha := \{(a, \alpha(a)) \,|\, a \in A\} \;<\; A \oplus A\,, \tag{3.5}$$

and for two-cocycle on H_α the trivial two-cocycle $\theta_\circ$. (We could actually take any exact two-cocycle; for the transmission functor only the cohomology class matters.) For instance, for $\alpha = id$, H is the diagonal subgroup of $A \oplus A$, which describes the invisible defect, while for the 'charge conjugation' $a \mapsto a^{-1}$ it is the antidiagonal subgroup.

Let us first check that the pair $(H_\alpha, \theta_\circ)$ defines an invertible surface defect. We have

$$H_\alpha \cdot (A \oplus \{0\}) = \{(ab\,, \alpha(a) \,|\, a, b \in A\} = A \oplus A \tag{3.6}$$

and analogously $(A \oplus \{0\}) \cdot H_\alpha = A \oplus A$. Moreover,

$$\big(H_\alpha \cap (A \oplus \{0\})\big) = \{0\} = \big((A \oplus \{0\}) \cap H_\alpha\big)\,, \tag{3.7}$$

so that trivially the restriction of $\omega_{\theta_\circ}$ to $(H_\alpha)_\cap$ is non-degenerate. Thus both conditions in Corollary 3.2 are satisfied, and hence the defect labeled by $(H_\alpha, \theta_\circ)$ is indeed invertible.

Next we compute the action of the transmission functor $F_{H_\alpha,\theta_\circ}$ on isomorphism classes of simple objects. The functor $\varphi^*_{\theta_\circ}$ is the identity, so that the transmission functor is the composition

$$(a;\chi) \xmapsto{[\hat{\pi}_1^*]} (a,\alpha(a);\tilde{\chi}) \xmapsto{[\hat{\pi}_{2*}]} (\alpha(a), \alpha^{-1*}(\chi)) \tag{3.8}$$

where $\tilde{\chi} \in H_\alpha^*$ is defined by $\tilde{\chi}(a,\alpha(a)) = \chi(a)$. Thus indeed $F_{H_\alpha,\theta_\circ}$ acts on isomorphism classes of simple objects by $\widetilde{\alpha} \in \mathrm{O}_q(A{\oplus}A^*)$.

3.2. Dynamical symmetries: B-fields. These symmetries come from automorphisms of the trivial 2-gerbe on $\mathrm{Bun}(A)$. They are thus symmetries of the classical action.

A dynamical symmetry is described by an alternating bicharacter $\beta\colon A \times A \to \mathbb{C}^\times$. To such a bicharacter β we associate the group homomorphism $\xi_\beta\colon A \to A^*$ that acts as $[\xi_\beta(a)](b) = \beta(a,b)$ for $a, b \in A$. The automorphism $\widetilde{\beta}$ for a dynamical symmetry is then given by

$$\widetilde{\beta}(a{+}\chi) = a + \xi_\beta(a) + \chi\,. \tag{3.9}$$

This is an automorphism because ξ_β is a group homomorphism, and it is in $\mathrm{O}_q(A{\oplus}A^*)$ because β is in addition antisymmetric:

$$[\xi_\beta(a)](a) = \beta(a,a) = 1 \tag{3.10}$$

for all $a \in A$, which implies $\beta(a,b) = (\beta(b,a))^{-1}$ for $a, b \in A$.

We claim that the surface defect whose transmission functor reproduces $\widetilde{\beta} \in \mathrm{O}_q(A{\oplus}A^*)$ looks as follows: The relevant subgroup is the diagonal subgroup $A_{\mathrm{diag}} \le A \oplus A$ (independently of the particular choice of β), and the relevant two-cocycle θ_β on $A_{\mathrm{diag}} \cong A$ is characterized by the fact that its commutator cocycle ω_{θ_β} is β. (Recall that for the transmission functor only the cohomology class of the two-cocycle matters; the alternating bicharacters are in bijection to these classes.)

Now notice that we have $A_{\mathrm{diag}} = H_{\alpha=id}$ with H_α as in (3.5), so that as a special case of (3.6) and of (3.7) we see that

$$A_{\mathrm{diag}} \cdot (A \oplus \{0\}) = A \oplus A = (A \oplus \{0\}) \cdot A_{\mathrm{diag}} \tag{3.11}$$

and

$$\big(A_{\mathrm{diag}} \cap (A \oplus \{0\})\big) = \{0\} = \big((A \oplus \{0\}) \cap A_{\mathrm{diag}}\big)\,, \tag{3.12}$$

respectively. Thus precisely as in Sect. 3.1 we can conclude that the surface defect labeled by the pair $(A_{\mathrm{diag}}, \theta_\beta)$ is invertible.

The action of the transmission functor $F_{A_{\mathrm{diag}},\theta_\beta}$ on isomorphism classes of simple objects is obtained as follows:

$$(a;\chi) \xmapsto{[\hat{\pi}_1^*]} (a,a;\tilde{\chi}) \xmapsto{[\varphi^*_{\theta_\delta}]} \big(a,a;\tilde{\chi}{+}\xi_\beta(a)\big) \xmapsto{[\hat{\pi}_{2*}]} \big(a;\chi{+}\xi_\beta(a)\big) = \widetilde{\beta}(a{+}\chi)\,, \tag{3.13}$$

where $\tilde{\chi} \in A_{\mathrm{diag}}$ is defined by $\tilde{\chi}(a,a) = \chi(a)$. Thus $F_{A_{\mathrm{diag}},\theta_\beta}$ acts on isomorphism classes precisely by $\widetilde{\beta} \in \mathrm{O}_q(A{\oplus}A^*)$.

3.3. Partial e-m dualities. The partial e-m dualities appear as symmetries of quantized Dijkgraaf–Witten theories. Every partial e-m duality can be obtained in the following manner. Suppose that A is written as a direct sum $A \cong A' \oplus C$ with C a cyclic subgroup (allowing for the case that A' is the trivial subgroup). This induces a similar decomposition of the character group A^*: denoting by C^* the subgroup of A^* of characters that vanish on A', and by $(A')^*$ the subgroup of characters vanishing on C, we have $A^* \cong (A')^* \oplus C^*$.

As abstract groups, C and C^* are isomorphic. Fix an isomorphism $\delta_C \colon C \to C^*$ and define the automorphism δ of the group $A \oplus A^* \cong A' \oplus C \oplus (A')^* \oplus C^*$ as follows: δ is the identity on the summands A' and $(A')^*$, while on $C \oplus C^*$ it acts as

$$(\, c \,, \psi \,) \longmapsto \big(\, \delta_C^{-1}(\psi) \,, \delta_C(c) \big). \tag{3.14}$$

That δ preserves the quadratic form is seen by calculating

$$\begin{aligned} q\big(\delta(a'{+}c{+}\chi'{+}\psi)\big) &= q\big(a' + \delta_C^{-1}(\psi) + \chi' + \delta_C(c)\big) \\ &= \chi'(a') \cdot [\delta_C(c)](\delta_C^{-1}(\psi)) = \chi'(a') \cdot \psi(c) = q(a'{+}c{+}\chi'{+}\psi). \end{aligned} \tag{3.15}$$

We claim that the surface defect whose transmission functor corresponds to $\delta \in \mathrm{O}_q(A{\oplus}A^*)$ is as follows: The relevant subgroup of $A \oplus A$ is the group $H_\delta := A'_{\mathrm{diag}} \oplus C \oplus C$, where A'_{diag} is embedded diagonally into the summand $A' \oplus A'$ of $A \oplus A$, while the cocycle θ_δ on H_δ is characterized by its commutator cocycle, which is defined to act as

$$\omega_{\theta_\delta}((a', c_1, c_2), (\tilde{a}', \tilde{c}_1, \tilde{c}_2)) := [\delta_C(c_1)](\tilde{c}_2) \cdot ([\delta_C(c_2)](\tilde{c}_1))^{-1} \tag{3.16}$$

(this is obviously an alternating bicharacter on H_δ). For determining the transmission functor, it again suffices to know this bicharacter.

To verify invertibility, note that

$$A'_{\mathrm{diag}} \cdot (A' \oplus \{0\}) = \{(a'b', b') \,|\, a', b' \in A'\} = A' \oplus A' \tag{3.17}$$

and analogously $(A' \oplus \{0\}) \cdot A'_{\mathrm{diag}} = A' \oplus A'$, which implies that $(A \oplus \{0\}) \cdot H_\delta = A \oplus A = H_\delta \cdot (A' \oplus \{0\})$. Moreover, we have

$$A'_{\mathrm{diag}} \cap (A' \oplus \{0\}) = \{0\} = (A' \oplus \{0\}) \cap A'_{\mathrm{diag}} \,, \tag{3.18}$$

which implies that $(H_\delta)_\cap = (C \oplus C) \times (C \oplus C)$. To see that ω_{θ_δ} restricted to $(H_\delta)_\cap$ is non-degenerate, we fix a generator a of C and denote by $\psi \in C^*$ the character with value $\psi(a) = \mathrm{e}^{2\pi \mathrm{i}/N}$, with $N = |C|$. Then $\delta_C(a) = \psi^l$ with l such that $(l, N) = 1$, and we find

$$\omega_{\theta_\delta}(a^{p_1}; a^{q_1}, a^{p_2}, a^{q_2}) = \mathrm{e}^{2\pi \mathrm{i} l (p_1 q_2 - q_1 p_2)/N}. \tag{3.19}$$

Now $\mathrm{e}^{2\pi \mathrm{i} l/N}$ is a primitive N-th root of unity, so that for any pair (p_1, q_1) we can find $(p_2, q_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $p_1 q_2 - q_1 p_2 \neq 0 \bmod N$. Hence ω_{θ_δ} is non-degenerate. We can thus again invoke Corollary 3.2 to conclude that the defect labeled by $(H_\delta, \theta_\delta)$ is invertible.

To compute the action of the transmission functor on simple objects, we note that the problem splits into a part involving only the subgroup A' and another part involving only the cyclic group C. The first problem reduces to the computation of the transmission functor for the defect associated with the identity automorphism, which was treated in Sect. 3.1. Thus we can restrict ourselves to the case that $A = C$ is cyclic.

In this case the action of the pullback functor on the simple object (c, χ) with $b \in C$ and $\chi \in C^*$ reads

$$(c; \chi) \xmapsto{[\hat{\pi}_1^*]} \bigoplus_{\tilde{c} \in C} \big(c, \tilde{c}; \chi^{[1]}\big) \tag{3.20}$$

with the character $\chi^{[1]} \in (C \oplus C)^*$ taking the values $\chi^{[1]}(d, \tilde{d}) = \chi(d)$ for $d, \tilde{d} \in C$. Next we note that the functor $\varphi^*_{\theta_\delta}$ acts on simple objects of $\mathcal{D}(C \oplus C)$-mod as

$$(c, \tilde{c}, \chi^{[1]}) \xmapsto{[\varphi^*_{\theta_\delta}]} (c, \tilde{c}, \chi^{[2]}) \tag{3.21}$$

with the character $\chi^{[2]} \in (C \oplus C)^*$ taking the values $\chi^{[2]}(d, \tilde{d}) = \chi(d)\,[\delta_C(c)](\tilde{d})/[\delta_C(\tilde{c})](d)$ for $d, \tilde{d} \in C$. This is, in turn, mapped by the pushforward functor $[\hat{\pi}_{2*}]$ to those characters $\chi^{[3]} \in C^*$ for which $p_2^* \chi^{[3]} = \chi^{[2]}$. This condition amounts to the identity

$$\chi^{[3]}(\tilde{d}) = \chi^{[2]}(d, \tilde{d}) = \chi(d)\, \frac{[\delta_C(c)](\tilde{d})}{[\delta_C(\tilde{c})](d)} \tag{3.22}$$

for all $d, \tilde{d} \in C$. Considering the dependence of both sides of this equality on $\tilde{d}$ determines $\chi^{[3]} = \delta_C(c)$, while the fact that the dependence on d on the right hand side must be trivial shows that we need $\chi(d) = [\delta_C(\tilde{c})](d)$ for all $d \in C$. This means that in the summation over $\tilde{c}$ in (3.20) only the term with $\delta_C(\tilde{c}) = \chi$ survives the pushforward. We conclude that the composition of the three functors maps the simple object (c, χ) to a single simple object, as befits an equivalence. Concretely,

$$(c, \chi) \longmapsto (\delta_C^{-1}(\chi), \delta_C(c))\,, \tag{3.23}$$

and thus the defect realizes an e-m duality.

4. Generators of $\mathrm{O}_q(A \oplus A^*)$

It remains to be shown that the three types of group elements discussed in the preceding section—corresponding to kinematical and dynamical classical symmetries and to partial e-m dualities—indeed constitute a set of generators for the Brauer–Picard group $\mathrm{O}_q(A \oplus A^*)$. To this end we have to show that an arbitrary element of $\mathrm{O}_q(A \oplus A^*)$ can be expressed as a product of elements in a suitable explicitly specified set of generators. This description turns out to be similar to the description of symplectic or orthogonal groups over the integers (see e.g. [HuR,SW]) and the proof involves a variant of the Euclidean algorithm similar as in the proof of Bruhat decompositions (see e.g. [Re]). Technical complications arise from the need to respect the divisibility properties of the orders of the generators.

We start by introducing pertinent notation. Any finite abelian group A can be presented as $A = \bigoplus_p A^{(p)}$ with the sum being over all primes and $A^{(p)}$ a direct sum of cyclic groups of order a power of p. To analyze the group A we present it in terms of some arbitrary, but fixed, ordered family $(a_i \mid i = 1, 2, \ldots, r)$ of generators such that (writing the group product additively)

$$A^{(p)} = \bigoplus_{i=1}^{r} \langle a_i \mid p^{\ell_i} a_i = 0 \rangle = \bigoplus_{i=1}^{r} \mathbb{Z}_{p^{\ell_i}} = \bigoplus_s (\mathbb{Z}_{p^s})^{\oplus n_s} \tag{4.1}$$

with non-negative integers n_s, $r = \sum_s n_s$ and ℓ_i. It will be convenient to order the generators such that the powers of p appear in ascending order, i.e. $\ell_i \le \ell_j$ for $i < j$. It is easy to see that

$$\mathrm{Aut}(A) = \times_{p \text{ prime}} \mathrm{Aut}(A^{(p)}) \quad \text{as well as} \quad \mathrm{O}_q(A \oplus A^*) = \times_{p \text{ prime}} \mathrm{O}_q(A^{(p)} \oplus A^{(p)*}). \tag{4.2}$$

As a consequence we can, and will, restrict our attention to a single prime p. By a slight abuse of notation, in the sequel we will just write A for $A^{(p)}$.

In terms of the generators, a general group element is a linear combination $\sum_{i=1}^r \bar\gamma_i\, a_i$ with $\bar\gamma_i \in \mathbb{Z} \bmod p^{\ell_i}$. In the sequel we freely replace such classes $\bar\gamma_i$ by representatives $\gamma_i \in \mathbb{Z}$; also, we denote by $\gamma_i^{-1} \in \mathbb{Z}$ a representative of the inverse of γ_i modulo p^{ℓ_i}. For the character group A^* we choose generators x_i in such a way that $x_i(a_i)$ is a primitive p^{ℓ_i}th root of unity while $x_i(a_j) = 1$ for $i \neq j$, so that the quadratic form q is given by

$$q\big(\textstyle\sum_{i=1}^r (\gamma_i\, a_i + \zeta_i\, x_i)\big) = \exp\big(2\pi\mathrm{i} \displaystyle\sum_{i=1}^r p^{-\ell_i}\, \gamma_i\, \zeta_i\big)\,, \tag{4.3}$$

and in particular $\mathrm{ord}(x_i) = \mathrm{ord}(a_i)$. With these conventions, an element g of $\mathrm{O}_q(A \oplus A^*)$ (or, for that matter, of $\mathrm{End}(A \oplus A^*)$) is determined by the expressions

$$g(a_i) = \sum_{j=1}^r \big(\alpha^g_{i,j}\, a_j + \xi^g_{i,j}\, x_j\big) \qquad \text{and} \qquad g(x_i) = \sum_{j=1}^r \big(\beta^g_{i,j}\, a_j + \eta^g_{i,j}\, x_j\big) \tag{4.4}$$

for $i = 1, 2, \dots, r$, with suitable constraints on the coefficients $\alpha^g_{i,j},\ \xi^g_{i,j},\ \beta^g_{i,j},\ \eta^g_{i,j} \in \mathbb{Z}$ which, however, we do not need to spell out.

We introduce three subgroups of $\mathrm{Aut}(A \oplus A^*)$:

$$\begin{aligned} S_{\text{kin}} &:= \big\{\alpha \oplus \alpha^{-1*} \,|\, \alpha \in \mathrm{Aut}(A)\big\} \;\cong\; \mathrm{Aut}(A)\,, \\ S_{\text{B}} &:= \big\langle b_{i,j} \mid 1 \le i < j \le r\big\rangle \qquad \text{and} \qquad S_{\text{e-m}} := \textstyle\bigoplus_{i=1}^r D_i \;\cong\; \mathbb{Z}_2^{\oplus r}. \end{aligned} \tag{4.5}$$

Here $D_i \cong \mathbb{Z}_2$ is generated by the automorphism d_i that exchanges a_i and x_i and leaves all other generators fixed, while $b_{i,j}$ is given by

$$b_{i,j}\;:\quad \begin{cases} a_i \,\mapsto\, a_i + x_j\,, \\ a_j \,\mapsto\, a_j - x_i\,, \\ a_k \,\mapsto\, a_k \quad \text{for } k \notin \{i,j\}\,, \\ x_k \,\mapsto\, x_k. \end{cases} \tag{4.6}$$

It is not hard to check that the groups (4.5) are actually subgroups of $\mathrm{O}_q(A \oplus A^*) < \mathrm{Aut}(A \oplus A^*)$. The groups (4.5) describe kinematical universal symmetries (S_{kin}), dynamical symmetries or B-fields (S_{B}), and partial e-m-dualities associated to the direct sum decomposition (4.1) of A ($S_{\text{e-m}}$), respectively.

We will also be interested in two particular types of elements of S_{kin}: for $i \neq j$ satisfying $\mathrm{ord}(a_i) = \mathrm{ord}(a_j)$ we set

$$t_{i,j}\;:\quad \begin{cases} a_j \,\mapsto\, a_j - a_i\,, \\ a_k \,\mapsto\, a_k \quad \text{for } k \neq j\,, \\ x_i \,\mapsto\, x_i + x_j\,, \\ x_k \,\mapsto\, x_k \quad \text{for } k \neq i\,, \end{cases} \tag{4.7}$$

and for $\gamma \neq 0 \bmod p$ and any j

$$\omega_{j;\gamma}: \quad \begin{cases} a_j \mapsto \gamma^{-1} a_j\,, \\ a_k \mapsto a_k \quad \text{for } k \neq j\,, \\ x_j \mapsto \gamma\, x_j\,, \\ x_k \mapsto x_k \quad \text{for } k \neq j. \end{cases} \tag{4.8}$$

We further introduce a separate notation for those elements of S_{kin} that act as a transposition on pairs of generators of some fixed order and leave all other generators fixed, according to

$$\tau_{i,j}: \quad \begin{cases} a_i \leftrightarrow a_j\,, \\ a_k \mapsto a_k \quad \text{for } k \notin \{i, j\}\,, \\ x_i \leftrightarrow x_j\,, \\ x_k \mapsto x_k \quad \text{for } k \notin \{i, j\} \end{cases} \tag{4.9}$$

with $\text{ord}(a_j) = \text{ord}(a_i)$. These generate a subgroup $\mathfrak{S} = \bigoplus_s \mathfrak{S}_{n_s} \leq S_{\text{kin}}$ consisting of elements that permute pairs (a_i, x_i) of generators of the same order. Below, for convenience we allow for $i = j$ in (4.9), i.e. for any i, $\tau_{i,i}$ is just the unit element of $\text{O}_q(A \oplus A^*)$.

We now establish the following

Fact 4.1. $\text{O}_q(A \oplus A^*)$ *is generated by the subgroups* (4.5).

Proof. Step 1: Given $g \in \text{O}_q(A \oplus A^*)$ we show that multiplying g with suitable elements of the subgroups (4.5) yields a group element that leaves the last generator x_r invariant.

Step 1a: Describe g as in (4.4). We first consider the case that $\text{ord}(\eta^g_{r,i}\, x_i) < \text{ord}(x_r)$ for all i. Then, since g must preserve the order of x_r, there exists at least one value of i such that $\text{ord}(\beta^g_{r,i}\, a_i) = \text{ord}(x_r)$. Take one such value (say, the largest one satisfying the equality) and denote it by $k(r)$ or, for brevity, just by k. Then the group element $g' := d_k \circ g$ acts as

$$g'(x_r) \equiv d_k(g(x_r)) = g(x_r) + (\beta^g_{r,k} - \eta^g_{r,k})\,(x_r - a_r)\,, \tag{4.10}$$

so that in particular $\text{ord}(\eta^{g'}_{r,k}\, x_k) = \text{ord}(\beta^{g'}_{r,k}\, x_k) = \text{ord}(x_r)$.

Step 1b: By step 1a we can assume that g satisfies $\text{ord}(\eta^g_{r,k}\, x_k) = \text{ord}(x_r)$, i.e. $\text{ord}(x_k) = \text{ord}(x_r)$ and $\eta^g_{r,k} \neq 0 \bmod p$. It follows that $\tau_{k,r} \in \mathfrak{S} \leq S_{\text{kin}}$ and that there exists a $\gamma \in \mathbb{Z}$ such that $\gamma\, \eta^g_{r,k} = 1 \bmod p$. Then the group element $g' := \omega_{r;\gamma} \circ \tau_{r,k} \circ g$ acts as in (4.4) with $\eta^{g'}_{r,r} = 1$.

Step 1c: Invoking step 1b we assume from now on that g satisfies $\eta^g_{r,r} = 1$. Further, for $i \neq r$ the element $g' := (b_{i,r})^{\beta^g_{r,i}}$ satisfies $\beta^{g'}_{r,i} = \beta^g_{r,i} - \beta^g_{r,i} = 0$. Hence by composing g successively, for all $i = 1, 2, \ldots, r-1$, with the group element $b_{i,r}$ raised to the power $\beta^g_{r,i}$ one obtains a group element $\tilde{g}$ satisfying

$$\tilde{g}(x_r) = \beta^g_{r,r}\, a_r + x_r + \sum_{i=1}^{r-1} \xi^g_{r,i}\, x_r. \tag{4.11}$$

Now by construction, $\tilde{g} \in \mathrm{O}_q(A \oplus A^*)$, while on the other hand $q(\tilde{g}(x_r)) = \exp(2\pi\mathrm{i}\, p^{-\ell_r} \beta^g_{r,r})$. Thus in fact we must have $\beta^g_{r,r} = 0 \bmod p^{\ell_r}$.

Step 1d: By step 1c we can assume that g satisfies $\beta^g_{r,i} = 0$ for all $i = 1, 2, \dots, r$. Further, for $i \neq r$ the element $g' := (t_{r,i})^{\eta^g_{r,i}}$ satisfies $\eta^{g'}_{r,i} = \eta^g_{r,i} - \eta^g_{r,i} = 0$. Hence by composing g successively, for all $i = 1, 2, \dots, r-1$, with the group element $t_{r,i}$ raised to the power $\eta^g_{r,i}$ one obtains a group element $\tilde{g}$ satisfying $\tilde{g}(x_r) = x_r$.

Step 2: By step 1 we can assume that $g(x_r) = x_r$. Now consider the image $g(x_{r-1})$ of the group element x_{r-1}. We manipulate it in full analogy with what we did with $g(x_r)$ in step 1, just replacing $r \mapsto r-1$ everywhere, but with the following amendment: In case that in the analogue of step 1a the label $k = k(r-1)$ should turn out to take the value r, before proceeding to replacing $g \mapsto d_k \circ g$ we consider instead of g the group element

$$g' := t_{r-1,r} \circ g. \tag{4.12}$$

After this replacement we can assume that $k \leq r-1$. As a consequence, afterwards one never will have to compose with elements from (4.5) of the form $\omega_{r;\gamma}$ or $b_{r,j} \circ d_r$ which would potentially alter the input relation $g(x_r) = x_r$. Thus by further proceeding along the lines of step 1 we end up with a group element $\tilde{g}$ satisfying both $\tilde{g}(x_r) = x_r$ and $\tilde{g}(x_{r-1}) = x_{r-1}$.

Steps $3, 4, \dots, r$: Proceed iteratively for $g(x_{r-j})$ for $j = 2, 3, \dots, r-1$, where in the jth iteration the role of $t_{r-1,r}$ in (4.12) is taken over by $t_{r-j,r-l}$ for suitable $l < j$. The result is a group element $\tilde{g}$ satisfying $\tilde{g}(x_i) = x_i$ for all $i = 1, 2, \dots, r$.

Step r+1: By step r we can assume that $g(x_i) = x_i$ for all $i = 1, 2, \dots, r$. We show that this in fact implies that $g(a_i) = a_i + \sum_{j=1}^r \xi^g_{i,j}\, x_j$ for all $i = 1, 2, \dots, r$. Indeed, from [HiR] we know that in order for g to belong to $\mathrm{Aut}(A \oplus A^*)$, the matrix $M(g) = \begin{pmatrix} \alpha^g & \xi^g \\ \beta^g & \eta^g \end{pmatrix}$ with block matrices $\alpha^g,\ \xi^g,\ \beta^g,\ \eta^g$ consisting of the coefficients in (4.4), must satisfy $\det(M(g) \bmod p) \neq 0$.
Now for g of the form considered here we have $\eta^g = 1\!\!1_{r\times r}$ and $\beta^g = 0$; this implies in particular that $0 \neq \det(M(g) \bmod p) = \det(\alpha^g \bmod p)$, and thus that $\alpha^g \in \mathrm{Aut}(A)$. As a consequence, together with g also the product $g' := g \circ \big((\alpha^g)^{-1} \oplus (\alpha^g)^*\big)$ is an element of $\mathrm{O}_q(A \oplus A^*)$. On the other hand, we have explicitly

$$g'(a_i) = a_i + \sum_j \xi^{g'}_{i,j}\, x_j \qquad \text{and} \qquad g'(x_i) = \sum_j \eta^{g'}_{i,j}\, x_j. \tag{4.13}$$

Hence the fact that g' belongs to $\mathrm{O}_q(A \oplus A^*)$ amounts in particular to the following restrictions, which together are also sufficient:

$$\begin{aligned} q(g'(a_i)) = q(a_i) \ &\Longrightarrow\ \xi^{g'}_{i,i} = 0\,, \\ q(g'(a_i{+}a_j)) = q(a_i{+}a_j) &\Longrightarrow \xi^{g'}_{i,j} + \xi^{g'}_{j,i} = 0 \ \text{ for } \ i \neq j\,, \\ q(g'(a_i{+}x_i)) = q(a_i{+}x_i) \ &\Longrightarrow \eta^{g'}_{i,i} = 1\,, \\ q(g'(a_j{+}x_i)) = q(a_j{+}x_i) &\Longrightarrow \eta^{g'}_{i,j} = 0 \ \text{ for } \ i \neq j. \end{aligned} \tag{4.14}$$

Together, these restrictions just say that $g' \in S_{\mathrm{B}}$.
This concludes the proof. □

In the following example we illustrate how the result follows from an explicit analysis in a particularly simple case.

Example 4.2. $A = \mathbb{Z}_p$.
It is not hard to see that in order for (4.4) to be in $\mathrm{O}_q(A \oplus A^*)$, it is necessary and sufficient that the numbers $\alpha,\ \beta,\ \xi,\ \eta$ satisfy $\alpha\xi = 0 = \beta\eta$ and $\alpha\eta + \beta\xi = 1$ modulo p. These constraints are solved by

$$\xi = 0 = \beta\,, \quad \eta = \alpha^{-1} \qquad \text{and by} \qquad \alpha = 0 = \eta\,, \quad \beta = \xi^{-1}. \tag{4.15}$$

Among the solutions of the second type is in particular the case $\xi = \beta = 1$, which gives the (unique) e-m duality, while all other solutions of this type are obtained from one of the first type by composing with the e-m duality. In short, we have

$$\mathrm{O}_q(\mathbb{Z}_p \oplus \mathbb{Z}_p^*) = S_{\mathrm{kin}} \rtimes S_{\text{e-m}} \tag{4.16}$$

with

$$S_{\mathrm{kin}} = \mathrm{Aut}(\mathbb{Z}_p) = GL_1(\mathbb{F}_p) = \mathbb{F}_p^{\times} \cong \mathbb{Z}_{p-1} \qquad \text{and} \qquad S_{\text{e-m}} = \mathbb{Z}_2. \tag{4.17}$$

In particular, $|\mathrm{O}_q(\mathbb{Z}_p \oplus \mathbb{Z}_p^*)| = 2(p-1)$.

The transmission functor for the automorphism given by $\alpha \in \mathbb{F}_p^{\times}$ is

$$\begin{aligned} \mathbb{Z}_p \oplus \mathbb{Z}_p^* &\to \mathbb{Z}_p \oplus \mathbb{Z}_p^*\,, \\ (a, \chi) &\mapsto (\alpha a, \chi(\alpha^{-1} \cdot)) \end{aligned} \tag{4.18}$$

and the transmission functor corresponding to the single non-trivial e-m duality is given by

$$\begin{aligned} \mathbb{Z}_p \oplus \mathbb{Z}_p^* &\to \mathbb{Z}_p \oplus \mathbb{Z}_p^*\,, \\ (a, \chi) &\mapsto (\delta(\chi)^{-1}, \delta(a)). \end{aligned} \tag{4.19}$$

Example 4.3. $A = \mathbb{Z}_p^2$.
As another example consider the non-cyclic group $A = \mathbb{Z}_p \oplus \mathbb{Z}_p$. In this case the second cohomology group $H^2(A, \mathbb{C}^{\times})$ is non-trivial. A choice (e_1, e_2) of generators of A allows us to describe the subgroups generating the symmetry group $\mathrm{O}_q(\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^{*2})$ explicitly:

$$S_{\mathrm{kin}} = \mathrm{Aut}(\mathbb{Z}_p^2) \cong GL_2(\mathbb{F}_p)\,, \qquad S_{\text{e-m}} \cong \mathbb{Z}_2^2\,, \qquad S_{\mathrm{B}} \cong \mathbb{Z}_p. \tag{4.20}$$

Transmission functors for the kinematical symmetries and e-m dualities are similar to the previous example. An alternating bicharacter β can be described by

$$\beta_\ell(e_i, e_i) = 1 \qquad \text{and} \qquad \beta_\ell(e_1, e_2) = \beta_\ell(e_2, e_1)^{-1} = \mathrm{e}^{2\pi\mathrm{i}\ell/p} \tag{4.21}$$

with $\ell \in \{0, 1, \ldots, p-1\}$. This yields a map $b_\ell : \mathbb{Z}_p \to \mathbb{Z}_p^*$ with $b(a)(b) = \exp(2\pi\mathrm{i}\ell ab/p)$. We then have the following transmission functor:

$$\begin{aligned} \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^{*2} &\to \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^{*2}\,, \\ (a_1, a_2, \chi_1, \chi_2) &\mapsto (a_1, a_2, \chi_1 + b(a_2), \chi_2 - b(a_1)). \end{aligned} \tag{4.22}$$

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Boundary Conditions for Topological Quantum Field Theories, Anomalies and Projective Modular Functors

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Abstract: We study boundary conditions for extended topological quantum field theories (TQFTs) and their relation to topological anomalies. We introduce the notion of TQFTs with moduli level m, and describe extended anomalous theories as natural transformations of invertible field theories of this type. We show how in such a framework anomalous theories give rise naturally to homotopy fixed points for n-characters on ∞-groups. By using dimensional reduction on manifolds with boundaries, we show how boundary conditions for $n+1$-dimensional TQFTs produce n-dimensional anomalous field theories. Finally, we analyse the case of fully extended TQFTs, and show that any fully extended anomalous theory produces a suitable boundary condition for the anomaly field theory.

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1. Introduction

In recent years, the study of boundary conditions for topological quantum field theories (TQFTs) has attracted much interest, both in the physics and mathematics literature; see for instance [10,15,22,23,25–28,39,49], among others. Namely, given an n-dimensional TQFT, from the mathematical point of view it is a sensible question to ask when does such a theory produce genuine numerical invariants of an n-dimensional manifold with boundary, rather than vectors in a state space associated to it. This is possible if we can regard the boundary not as arising from a "cut-and-paste" procedure implementing locality, but rather as a "constrained" part of the manifold. In general, there will be obstructions in extending a TQFT to manifolds with boundaries: the case of Reshetikhin–Turaev and Turaev–Viro TQFTs has been recently investigated in [22]. Both Reshetikhin–Turaev [40] and Turaev–Viro [47] TQFTs are extended topological field theories, namely these theories assign data also to manifolds of codimension 2. In the present work, we focus our attention on TQFTs that are extended down to codimension k, and at the same time, most importantly, extended *up to infinity* to include diffeomorphisms, and their isotopies. This is the framework pioneered in [35], which makes extensive use of the language of ∞-categories, and which we find particularly suitable for our aims. Indeed, by regarding n-categories as ∞-categories, we can introduce the notion of a *TQFT with moduli level* m: these are topological field theories that also detect information about the homotopy type of the diffeomorphisms group of manifolds up to a certain level m.

Our main motivation to introduce and study such field theories is due to the fact that they provide a very natural and elegant description of *anomalous* TQFTs. It is well known, for instance, that the Reshetikhin–Turaev construction produces from a modular tensor category $\mathcal{C}$ a TQFT that is defined on a central extension of the extended 3-dimensional cobordism category [48]: namely, it gives rise only to a *projective* representation of the 2-tier extended cobordism category $\mathrm{Cob}_2(3)$ taking values in 2-Vect, and the anomaly, in this context, is represented via a 2-cocycle on the modular groupoid [2,3,5,46]. In a more modern approach, (topological) anomalies are themselves field theories in higher dimensions, and of a special kind, namely they are *invertible*; anomalous TQFTs are then realised as *truncated morphisms* from the trivial theory $\underline{1}$ to the given anomaly. We refer the reader to very recent works [16,17] detailing this point of view. In the present work, we realise the anomaly theory as an invertible TQFT of moduli level 1 of the *same* dimension as the anomalous TQFT. Namely, taking the higher morphisms into account there is no need for the involved TQFTs to be truncations of TQFTs defined in one dimension higher; rather, truncated TQFTs are a very particular example of moduli level 1 TQFTs. This provides a unified language to describe anomalous theories extended down to codimension k, and their category: given an anomaly theory W, it is the $(\infty, k-1)$-category of natural transformations between the trivial theory and W. Moreover, this description allows for more general anomaly theories, as explained in the text, and it has a strong representation theoretic flavour: anomalous n-dimensional TQFTs extended down to codimension k give rise to homotopy fixed points for $k+1$-characters, a suitable and natural generalisation of group characters to the setting of ∞-groups. In codimension 1, these provide projective representations of the mapping class group of $n-1$-closed manifolds.

Anomalous TQFTs and boundary conditions are expected to intertwine in a subtle relationship. The most striking example is provided by Chern–Simons theory, which should best be regarded as a field theory living on the boundary of a 4-dimensional TQFT [20,48,50]. Similarly, the Reshetikhin–Turaev theory arising from a modular tensor category $\mathcal{C}$ is induced by a 4-dimensional Crane–Yetter theory [11,48]. By basically using a *dimensional reduction* procedure, we show that from a boundary condition of an (invertible) $n+1$-dimensional theory Z one can obtain an anomalous TQFT, where the anomaly is induced by Z itself. One sensible question to ask concerns the converse statement, i.e., the possibility of producing a boundary condition for an $n+1$-dimensional theory from the datum of an anomalous TQFT. In general, we do not expect this to hold: indeed, an anomalous TQFT with anomaly W contains too little information to determine a boundary condition $\tilde{Z}$. Nevertheless, when Z is a fully extended theory the situation is much more amenable to treatment: via the cobordism hypothesis *for manifolds with singularities*, we show that anomalous TQFTs with anomaly given by a fully extended TQFT Z do indeed produce boundary conditions for Z. In other words, in the fully extended situation, "truncated morphisms" of TQFTs are just a shadow of something richer, namely TQFTs with genuine boundary conditions. This is particularly clear thanks to the formalism used to describe anomalies, namely as morphisms of TQFTs of moduli level 1.

The present work is organised as follows.

In Sect. 2 we present a very gentle introduction to the language of ∞-categories, in the amount necessary to allow the reader acquainted with category theory to follow the rest of the paper. We also include some results we were not able to retrieve from the literature.

In Sect. 3 we give some basic notions concerning cobordism categories, with emphasis on properties available once we consider extension "up to infinity".

In Sect. 4 we introduce the notion of an extended TQFT with moduli level m, and provide some examples; we show also how we recover ordinary extended TQFTs. The fully extended case is discussed as well in this section.

In Sect. 5, we introduce anomalies and anomalous TQFTs via the language developed in Sect. 4. For consistency, we also discuss invertible theories, and some properties of the Picard groupoid of n-vector spaces.

In Sect. 6 we take a little detour to introduce n-characters and their homotopy fixed points, which is a subject in its own. We present the basic definitions and results needed to provide a description of anomalous TQFTs as homotopy fixed points, and we show how anomalous n-dimensional TQFTs in codimension 1 give rise to projective representations of the mapping class group of closed $n-1$-dimensional manifolds, hence to projective modular functors.

In Sect. 7 we finally introduce boundary condition for TQFTs, providing examples in the simplest situations, and comparisons with the existing literature when needed.

In Sect. 8 we show how boundary conditions for invertible TQFTs give rise to anomalous theories. Moreover, we show that in the fully extended case also the contrary holds. We conclude with some remarks on recent results on 4-dimensional field theories arising from modular tensor categories.

Not to burden the present work with technicalities of Higher Category theory, we have in several places appealed to intuition, and hence have preferred to give "sketches" of definitions, rather than full blown ones. We do feel the need then to be clearer concerning which aspects of our results should be regarded as rigorously established, and which ones still require a solid foundation, or at least technical details to be filled in. In the following we try to concisely state which tools we require: most of them are contained in [35], which, though lacking some amount of rigor in certain points, has had a wide influence in the study of TQFTs, in particular concerning their classification. See, for instance, [20].

First, for any nonnegative integer n and any group homomorphism $G \to O(n)$ we assume there exists a symmetric monoidal (∞, n)-category $\mathrm{Bord}(n)^G$ of G-framed cobordism. Next, for any nonnegative integer n, we assume there exists a notion of a symmetric monoidal n-category n-Vect of n-vector spaces over a field $\mathbb{K}$, which, for

$n = 1$, reproduces the usual monoidal category of vector spaces over $\mathbb{K}$. Moreover, we require a natural equivalence of symmetric monoidal $(\infty, n-1)$-categories $\Omega(n\text{-Vect}) \cong (n-1)\text{-Vect}$. In the last part of the present work, we assume also the cobordism hypothesis to hold, namely that a symmetric monoidal functor $Z\colon \mathrm{Bord}(n)^G \to n\text{-Vect}$ is completely determined by its value on the G-framed point, and that this value can be any G-invariant fully dualizable object of n-Vect. Finally, we assume a robust notion of lax natural transformations between strong monoidal ∞-functors between symmetric (∞, n)-categories. All the other results in the article are mathematically derived by these assumptions, and so they should be considered as mathematically established as soon as one is confident in assuming that in any rigorous foundation of the theory of symmetric monoidal (∞, n)-categories, all of the above assumptions will have to be true. This is widely expected to be so in the extended TQFTs/Higher Categories communities.

Nevertheless, for $n \leq 2$ all the constructions we present here can be entirely reformulated using the language of ordinary categories, or the well established language of 2-categories and bicategories (see, e.g., [9]). Indeed, the reader who is uncomfortable with the theory of ∞-category *tout court* can safely substitute k and m in the paper with 1, and only have to deal with bicategories for the $(n \leq 2)$-version of the results presented here. In particular, the main results of this article, i.e., the construction of projective representations of the mapping class groups of manifolds from anomalous TQFTs, and that boundary conditions for extended (invertible) TQFTs do produce anomalous topological theories can be both entirely expressed within a bicategorical language. On the other hand, we have preferred to use the language of ∞-category because the naturality of the ideas contained in the present work become visibly clearer. Moreover, it allows us to "see far" in the landscape of topological quantum field theories, and permits indeed interesting speculations, like the conjectural relation between Reshetikhin–Turaev anomalous 3d TQFT, and the 4-category $\mathrm{Braid}^{\otimes}$ we present in the final part of the article. These could certainly be seen as additional motivations to pursue the consolidation of the foundation of ∞-category theory in all its aspects.

2. Preliminary Notions on Higher Category Theory

In this section we will collect relevant results concerning higher category theory, and in particular ∞-categories, which we will use in the paper, mainly following [6,35], to which we direct the reader for details. The experienced reader, instead, can skip this section altogether.

An n-category can be informally thought of as a mathematical structure generalizing the notion of a category: we not only have objects and morphisms, but also morphisms between morphisms, morphisms between morphisms between morphisms, and so on, up to n. In the case $n = 2$, a precise definition can be given (see, e.g., [9,42]), where the crucial difference arises between *strict* and *weak* 2-category. Once we notice that a strict 2-category is equivalent to a category enriched in Cat, we can give a recursive definition for strict n-categories as follows: for $n \geq 2$, a strict n-category is a category enriched in Cat^{n-1}, the category of strict $n-1$-categories. The problem arises when we try to extend the above definition to obtain weak n-categories, i.e. an n-category where associativity for k-morphisms, etc. is only preserved up to $k+1$-morphisms, for $1 \leq k \leq n$, which obey the necessary coherence diagrams. A rigorous definition of weak n-category can nevertheless be given, and there are even different equivalent ways of formalizing this notion. Basic references are [7,8]. It goes without saying that weak n-categories are those of relevance in the mathematical world.

Example 1. An important example of (weak) n-category is that of n-vector spaces over a fixed characteristic 0 base field $\mathbb{K}$. For $n = 0$, the 0-category (i.e., the set) 0-Vect is the field $\mathbb{K}$; for $n = 1$ the 1-category (i.e., the ordinary category) 1-Vect is the category of (finite dimensional) vector spaces over $\mathbb{K}$. For $n = 2$, the 2-category 2-Vect comes in various flavours: by 2-Vect one can mean the 2-category of Kapranov–Voevodsky 2-vector spaces [29], or the 2-category of (finite) $\mathbb{K}$-linear categories with linear functors as morphisms and $\mathbb{K}$-linear natural transformations as 2-morphisms, or the 2-category of (finite dimensional) $\mathbb{K}$-algebras (to be thought as placeholders for their categories of right modules), with (finite dimensional) bimodules as 1-morphisms and morphisms of bimodules as 2-morphisms, as in [43].[1] This latter incarnation of 2-Vect suggests an iterative definition of n-Vect, see [20]. For instance one can define 3-Vect as the 3-category whose objects are tensor categories over $\mathbb{K}$, whose morphisms are bimodule categories, and so on. In any of its incarnations, n-Vect is an example of symmetric monoidal n-category. For instance, for $n = 2$ the symmetric monoidal structure on the 2-category of finite $\mathbb{K}$-linear categories is induced by Deligne's tensor product [12].

When one has k-morphisms for any k up to infinity, one speaks of an ∞-category. Just to settle the notation, we give the following

Definition 1. For $n \geq 0$, a (∞, n)-category is a ∞-category in which every k-morphisms is invertible for $k > n$.

Details, and a rigorous definition of an (∞, n)-category as an n-fold complete Segal space can be found in [7]; see also [8,35,41]. Notice that in the "definition" above, invertibility of k-morphisms must be understood recursively in the higher categorical sense, i.e. up to invertible $k + 1$-morphisms. In particular, any n-category can be extended to an n-*discrete* (∞, n)-category, i.e., an (∞, n)-category in which all k-morphisms for $k > n$ are identities. We will often pass tacitly from n-categories to n-discrete ∞ categories in what follows. Moreover, given an (∞, n)-category and objects $x, y \in \mathcal{C}$, there is a $(\infty, n-1)$-category $\mathrm{Mor}_{\mathcal{C}}(x, y)$ of 1-morphisms.

Example 2. The prototypical example of ∞-category arises from homotopy theory. Indeed, let X be a topological space. Then there is an ∞-category $\pi_{\leq\infty}(X)$, with objects given by the points of X, 1-morphisms given by continuous paths in X, 2-morphisms given by homotopies of paths with fixed end-points, 3-morphisms given by homotopies between homotopies, and so on. Since the composition of paths is only associative up to homotopy, i.e. up to a 2-morphism, $\pi_{\leq\infty}(X)$ is necessarily a weak ∞-category. Nevertheless, the 2-morphism above, which is part of the data, is invertible up to 3-morphisms. Indeed, all k-morphisms in $\pi_{\leq\infty}(X)$ are invertible, hence it is a $(\infty, 0)$-category, which is usually called a ∞-groupoid. The guiding principle behind ∞-categories is that also the converse should be true, i.e. any ∞-groupoid arises as $\pi_{\leq\infty}(X)$ for some topological space, hence the theory of $(\infty, 0)$-categories can be defined via homotopy theory.

Example 3. A genuine example of an (∞, n)-category with $n > 0$ is given by $\mathrm{Bord}(n)$, the ∞-category of cobordisms, which can be informally described as consisting of having points as objects, 1-dimensional bordisms as 1-morphisms, 2-dimensional bordisms between bordisms as 2-morphisms, and so on until we arrive at n-dimensional bordisms as n-morphisms, from where higher morphisms are given by diffeomorphisms

[1] The 2-category of Kapranov–Voevodsky 2-vector spaces can be seen as the full subcategory of the 2-category of $\mathbb{K}$-algebras and bimodules on the $\mathbb{K}$-algebras of the form $\mathbb{K}^{\oplus m}$, for $m \in \mathbb{N}$.

and isotopies: more precisely, the $(n+1)$-morphisms are diffeomorphisms which fix the boundaries, $(n+2)$-morphisms are isotopies of diffeomorphisms, $(n+3)$-morphisms are isotopies of isotopies, and so on. This is an example of a (∞, n)-symmetric monoidal category, see [35]. A rigorous and detailed construction of $\mathrm{Bord}(n)$ as an (∞, n)-symmetric monoidal category can be found in [41].

Remark 1. The (∞, n)-category $\mathrm{Bord}(n)$ comes also in other "flavours", depending on the additional structures we equip the manifolds with: for instance orientation and n-framing give (∞, n)-categories $\mathrm{Bord}(n)^{or}$ and $\mathrm{Bord}(n)^{fr}$, respectively. More precisely, let $G \to GL(n;\mathbb{R})$ be a group homomorphism. For any $k \leq n$, a k-manifold M is naturally equipped with the $GL(n;\mathbb{R})$-bundle $TM \oplus \mathbb{R}^{n-k}$, and a G-framing for M is the datum of a reduction of the structure group of $TM \oplus \mathbb{R}^{n-k}$ from $GL(n;\mathbb{R})$ to G. Just as in the non-framed case, G-framed k-manifolds with $k \leq n$ are the k-morphisms for a symmetric monoidal (∞, n)-category $\mathrm{Bord}(n)^G$, called the (∞, n)-category of G-cobordism. Notice that one can consider an equivalent category of G-cobordisms, where our manifolds are equipped with a $O(n)$-structure on the stable tangent bundle, and its G-reductions. The equivalence comes from the fact that $O(n)$ is a retract of $GL(n;\mathbb{R})$. We will implicitly make this identification later on.

In particular, when G is the trivial group, one writes $\mathrm{Bord}(n)^{fr}$ for $\mathrm{Bord}(n)^{\{e\}}$, and calls it the (∞, n)-category of framed cobordism, while when G is $SO(n)$ one writes $\mathrm{Bord}(n)^{or}$ for $\mathrm{Bord}(n)^{SO(n)}$, and calls it the (∞, n)-category of oriented cobordism. The unoriented case $\mathrm{Bord}(n)$ is obtained when G is $O(n)$. We will use $\mathrm{Bord}(n)$ generically to indicate one of these G-framed versions, unless explicitly specified.

As for any mathematical structure, there is a notion of morphisms between ∞-category, which are given by ∞-functors. Informally speaking, an ∞–functor F between two ∞–categories $\mathcal{C}$ and $\mathcal{D}$ is a rule assigning to each k-morphism in $\mathcal{C}$ a k-morphism in $\mathcal{D}$ in a way respecting sources, targets and (higher) compositions. For instance, if one adopts the simplicial model for $(\infty, 1)$-categories, i.e., if one looks at $(\infty, 1)$-categories as simplicial sets with internal horn-filling conditions (with k-morphisms corresponding to k-simplices), then an ∞-functor between $(\infty, 1)$-categories is precisely a morphism of simplicial sets. See [36, Chapter 1] and [35] for details. In particular, given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we have an ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. It is immediate to see that, if $\mathcal{D}$ is n-discrete, then also $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is n-discrete (or, more precisely, it is equivalent to an n-discrete ∞-category).

Given an (∞, n)-category $\mathcal{C}$ we can obtain an ordinary category $\pi_{\leq 1}\mathcal{C}$, called the *homotopy category* of $\mathcal{C}$, with objects given by the objects of $\mathcal{C}$, and morphisms given by equivalence classes of 1-morphisms up to invertible 2-morphisms in $\mathcal{C}$, where invertibility is understood in the ∞ setting. Similarly, for $k \geq 2$ we can associate to $\mathcal{C}$ a k-category $\pi_{\leq k}\mathcal{C}$, called the *homotopy k-category* of $\mathcal{C}$, with objects and morphisms up to $k-1$-morphisms given by those of $\mathcal{C}$, and k-morphisms given by equivalence classes of k-morphisms up to invertible $k+1$-morphisms. By the usual identification of k-categories with k-discrete ∞-categories, we have then the following

Lemma 1. *The formation of the* homotopy n-category *is the adjoint ∞-functor to the inclusion of n-discrete categories into (∞, n)-categories, i.e., if $\mathcal{C}$ and $\mathcal{D}$ are (∞, n)-categories, with $\mathcal{D}$ discrete, then one has a natural equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}(\pi_{\leq n}\mathcal{C}, \mathcal{D}). \tag{1}$$

In more colloquial terms, this is just the statement that if $\mathcal{D}$ is n-discrete then an ∞-functor $\mathcal{C} \to \mathcal{D}$ naturally factors as $\mathcal{C} \to \pi_{\leq n}\mathcal{C} \to \mathcal{D}$.

For any (∞, n)-category $\mathcal{C}$ and an object $x \in \mathcal{C}$, we have that $\mathrm{End}_{\mathcal{C}}(x) = \mathrm{Hom}_{\mathcal{C}}(x, x)$ is a monoidal $(\infty, n-1)$-category. In particular, to a monoidal (∞, n)-category $\mathcal{C}$ we can canonically assign a monoidal $(\infty, n-1)$-category $\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$, where $1_{\mathcal{C}}$ denotes the monoidal unit of $\mathcal{C}$. We will refer to $\Omega\mathcal{C}$ as the (based) loop space of $\mathcal{C}$. It can be seen as the homotopy pullback

$$\begin{tikzcd} \Omega\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d] \\ \mathbf{1} \arrow[r] & \mathcal{C} \end{tikzcd} \tag{2}$$

where $\mathbf{1}$ is the trivial monoidal category, and $\mathbf{1} \to \mathcal{C}$ is the unique monoidal functor from $\mathbf{1}$ to $\mathcal{C}$. We can reiterate the construction to obtain a monoidal $(\infty, n-k)$-category, which we denote with $\Omega^k\mathcal{C}$. If $\mathcal{C}$ is also symmetric, then $\Omega^k\mathcal{C}$ is symmetric as well. We will denote with $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ the (∞, n)-category of monoidal ∞-functors between $\mathcal{C}$ and $\mathcal{D}$. Any monoidal ∞-functor F from $\mathcal{C}$ to $\mathcal{D}$ induces a monoidal ∞-functor $\Omega^k F$ from $\Omega^k\mathcal{C}$ to $\Omega^k\mathcal{D}$.

Example 4. One has $\Omega(n\text{-Vect}) \simeq (n-1)\text{-Vect}$ for any $n \geq 1$. For instance, the monoidal unit of the category 1-Vect is the field $\mathbb{K}$ seen as a vector space over itself, hence

$$\Omega(1\text{-Vect}) = \mathrm{End}_{1\text{-Vect}}(\mathbb{K}) = \mathbb{K} = 0\text{-Vect}. \tag{3}$$

Similarly, the monoidal unit of the 2-category 2-Vect is the category Vect, while its category of endomorphisms is the category of linear functors from Vect to Vect, which can be canonically identified with Vect itself.

Lemma 2. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category, and let $\mathcal{D}$ be a symmetric monoidal $(\infty, n+1)$-category. Then*

$$\mathrm{End}_{\mathrm{Fun}^{\otimes}(\mathcal{C},\mathcal{D})}(\underline{1}_{\mathcal{D}}) \simeq \mathrm{Fun}^{\otimes}(\mathcal{C}, \Omega\mathcal{D}) \tag{4}$$

where $\underline{1}_{\mathcal{D}} \colon \mathcal{C} \to \mathcal{D}$ denotes the trivial monoidal functor, mapping all objects of $\mathcal{C}$ to the monoidal unit $1_{\mathcal{D}}$ of $\mathcal{D}$, and all morphisms in $\mathcal{C}$ to identities.

Proof. The trivial monoidal functor $\underline{1}_{\mathcal{D}}$ is the composition $\mathcal{C} \to \mathbf{1} \to \mathcal{D}$. It follows from this description that $\mathrm{End}_{\mathrm{Fun}^{\otimes}(\mathcal{C},\mathcal{D})}(\underline{1}_{\mathcal{D}})$ is the ∞-category of homotopy commutative diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & \mathbf{1} \arrow[d] \\ \mathbf{1} \arrow[r] & \mathcal{D} \end{tikzcd} \tag{5}$$

By the universal property of the homotopy pullback, this is equivalent to $\mathrm{Fun}^{\otimes}(\mathcal{C}, \Omega\mathcal{D})$.

On the other hand, given a monoidal (∞, n)-category $\mathcal{C}$ we can obtain an $(\infty, n+1)$-category $B\mathcal{C}$ with a single object, and $\mathcal{C}$ as the ∞-category of morphisms. We will refer to $B\mathcal{C}$ as the classifying space of $\mathcal{C}$. The relationship between B and Ω is given by the following

Lemma 3. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category, and let $\mathcal{D}$ be a symmetric monoidal $(\infty, n+1)$-category. Then*

$$\mathrm{Fun}^{\otimes}(B\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\otimes}(\mathcal{C}, \Omega\mathcal{D}) \tag{6}$$

Proof. Let $F \in \mathrm{Fun}^{\otimes}(B\mathcal{C}, \mathcal{D})$. Since $B\mathcal{C}$ is an ∞-category with a single object $\star$, and F is a monoidal functor, then necessarily $F(\star) = 1_{\mathcal{D}}$. Hence, to any k-morphism in $B\mathcal{C}$, corresponding to a $(k-1)$-morphism in $\mathcal{C}$, is assigned by F a k-morphism from $1_{\mathcal{D}}$ to $1_{\mathcal{D}}$ in $\mathcal{D}$, i.e., a $(k-1)$-morphism in the symmetric monoidal (∞, n)-category $\Omega\mathcal{D} = \mathrm{End}_{\mathcal{D}}(1_{\mathcal{D}})$.

3. Cobordism (∞, k)-Categories

In this section we will recall some basic properties concerning ∞-categories of cobordisms. We will mainly refer to oriented cobordisms, unless otherwise stated. Via the mapping cylinder construction, we obtain a monoidal embedding

$$i : \mathrm{Bord}(n) \hookrightarrow \mathrm{Bord}(n+1) \tag{7}$$

Let us briefly recall how this works. Given a (orientation preserving) diffeomorphism $f : \Sigma_1 \to \Sigma_2$ between closed n-dimensional oriented manifolds, the mapping cylinder of f is the oriented manifold M_f with boundary obtained as

$$M_f := (([0,1] \times \Sigma_1) \amalg \Sigma_2)/\sim \tag{8}$$

where $\sim$ is the equivalence relation generated by $(0, x) \sim f(x), \forall x \in \Sigma_1$. In particular, we have that $\partial M_f = \Sigma_1 \amalg \overline{\Sigma}_2$, where $\overline{\Sigma}_2$ denotes the manifold Σ_2 endowed with the opposite orientation, so that M_f represents a (oriented) cobordism between Σ_1 and Σ_2. This means that $f \mapsto M_f$ maps an $(n+1)$-morphism in $\mathrm{Bord}(n)$ to an $(n+1)$-morphism in $\mathrm{Bord}(n+1)$. Moreover, the mapping cylinder construction is compatible with composition of diffeomorphisms in the following sense: if $g : \Sigma_1 \to \Sigma_2$ and $f : \Sigma_2 \to \Sigma_3$ are diffeomorphisms between closed oriented n-dimensional manifolds, then we have a canonical diffeomorphism

$$M_{fg} \simeq M_f \circ M_g. \tag{9}$$

In other words, $f \mapsto M_f$ behaves functorially with respect to the composition of $(n+1)$-morphism. Moreover, the mapping cylinder is compatible with isotopies of diffeomorphisms. Namely, an isotopy h between orientation preserving diffeomorphisms $f, g \colon \Sigma_1 \to \Sigma_2$ induces a orientation preserving diffeomorphism

$$h_* : M_f \xrightarrow{\simeq} M_g. \tag{10}$$

Hence the mapping cylinder construction maps an $(n+2)$-morphism in $\mathrm{Bord}(n)$ to an $(n+2)$-morphism in $\mathrm{Bord}(n+1)$, and also in this case one can verify the compatibility with composition. Similarly, isotopies between isotopies of diffeomorphisms produce correspondent isotopies of diffeomorphisms of the mapping cylinders. One has natural generalisations to unoriented and to G-framed cobordism, and so on, so that the mapping cylinder construction actually gives an ∞-functor $\mathrm{Bord}(n) \to \mathrm{Bord}(n+1)$, which is immediately seen to be compatible with disjoint unions, i.e., with the monoidal structure of cobordism categories. Details on the properties of the functor i can be found in [35]: interestingly, the proof of the fact that i is actually a (not full) embedding of ∞-categories is at the core of the Cobordism Hypothesis.

Remark 2. One has natural generalisations of (7) to unoriented, and to G-framed cobordisms.

Applying the iterated loop space construction to the symmetric monoidal (∞, n)-category $\mathrm{Bord}(n)$ we obtain the following important

Definition 2. For any $0 \leq k \leq n$, the (∞, k)-symmetric monoidal category $\mathrm{Cob}_k^\infty(n)$ is defined as

$$\mathrm{Cob}_k^\infty(n) := \Omega^{n-k}\mathrm{Bord}(n) \tag{11}$$

It will be called the (∞, k)-category of n-dimensional cobordism extended down to codimension k.

In a similar way, one can define G-framed cobordism categories $\mathrm{Cob}_k^{\infty,G}(n)$.

Note that $\mathrm{Bord}(n) = \mathrm{Cob}_n^\infty(n)$, the (∞, n)-category of n-dimensional cobordism extended down to codimension n. We will refer to $\mathrm{Bord}(n)$ as the *fully extended n-dimensional cobordism category.*

Notice that if $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, then also $\Omega(F) : \Omega\mathcal{C} \to \Omega\mathcal{D}$ is monoidal. This in particular implies that the monoidal embedding $i : \mathrm{Bord}(n) \hookrightarrow \mathrm{Bord}(n+1)$ induces monoidal embeddings

$$\mathrm{Cob}_k^\infty(n) \hookrightarrow \mathrm{Cob}_{k+1}^\infty(n+1) \tag{12}$$

for any $k \geq 0$.

Remark 3. The homotopy category $\pi_{\leq 1}\mathrm{Cob}_1^\infty(n)$ is the usual category of n-dimensional cobordism: it has $(n-1)$-closed manifolds as objects and diffeomorphism classes of n-dimensional cobordisms as morphisms. In the following, we will refer to this category simply as $\mathrm{Cob}(n)$

Remark 4. The $(\infty, 0)$-category $\mathrm{Cob}_0^\infty(n)$ is the ∞-groupoid having closed n-manifolds as objects, diffeomorphisms between them as 1-morphisms, isotopies between diffeomorphisms as 2-morphisms and so on.

Let Σ be a closed n-dimensional manifold. By slight abuse of notation, we will denote by $B\Gamma^\infty(\Sigma)$ the connected component of Σ in $\mathrm{Cob}_0^\infty(n)$. The homotopy category $\pi_{\leq 1}\mathrm{Cob}_0^\infty(n)$ is the groupoid usually denoted Γ_n, see [5], while $\pi_{\leq 1}B\Gamma^\infty(\Sigma)$ is the (one-object groupoid associated with the) mapping class group $\Gamma(\Sigma)$ of Σ. To emphasise the G-framing, we will occasionally write $\Gamma^G(\Sigma)$ for the mapping class group of a G-framed manifold Σ. For instance, if Σ is a closed oriented surface, then $\Gamma^{SO(2)}(\Sigma)$ is the mapping class group of isotopy classes of oriented diffeomorphisms one encounters in Teichmüller theory. If Σ is a closed oriented surface endowed with a spin structure, i.e., with a lift of the structure group $SO(2)$ of the tangent bundle to the double cover $SO(2) \xrightarrow{2:1} SO(2)$, then $\Gamma^{\mathrm{Spin}}(\Sigma)$ is the spin-framed mapping class group of Σ considered in [31].

4. Topological Quantum Field Theories

In this section we introduce the notion of a topological quantum field theory with moduli level m.

4.1. TQFTs with moduli level. Since both $\mathrm{Cob}_k^\infty(n)$ and r-Vect are symmetric monoidal ∞-categories, it is meaningful to consider symmetric monoidal functors between them. This leads us to the main definition in the present work.

Definition 3. An n-dimensional TQFT extended down to codimension k with moduli level m is a symmetric monoidal functor

$$Z\colon \mathrm{Cob}_k^\infty(n) \to (m+k)\text{-Vect}. \tag{13}$$

Remark 5. One main feature of r-Vect, whichever realisation of r-vector spaces one considers, is that $\Omega(r\text{-Vect}) \cong (r-1)\text{-Vect}$. This, together with the equivalence $\Omega\mathrm{Cob}_k^\infty(n) \cong \mathrm{Cob}_{k-1}^\infty(n)$, implies that by looping an n-dimensional TQFT extended down to codimension k we obtain an n-dimensional TQFT extended down to codimension $k-1$ with the same moduli level:

$$\Omega Z\colon \mathrm{Cob}_{k-1}^\infty(n) \to (m+k-1)\text{-Vect}. \tag{14}$$

On the other hand, pulling back along the inclusion $\mathrm{Cob}_{k-1}^\infty(n-1) \hookrightarrow \mathrm{Cob}_k^\infty(n)$ one can restrict an n-dimensional TQFT extended down to codimension k with moduli level m to a $(n-1)$-dimensional TQFT extended down to codimension $k-1$ with moduli level $m+1$,

$$Z\big|_{k-1}\colon \mathrm{Cob}_{k-1}^\infty(n-1) \to (m+k)\text{-Vect}. \tag{15}$$

We will refer to $Z\big|_{k-1}$ as the $(n-1)$-dimensional *truncation* of Z.

The terminology used in Definition 3 is due to the fact that a TQFT of moduli level greater than 0 produces in general more refined manifold invariants than an ordinary TQFT, namely it can detect the moduli space of diffeomorphisms. As we will illustrate in the following examples, from a TQFT of moduli level k we can obtain in specific situations the notion of ordinary and extended TQFTs.

Example 5. An n-dimensional TQFT extended down to codimension 1 with moduli level 0 is a TQFT in the sense of Atiyah and Segal [4,44]. Namely, since 1-Vect is 1-discrete, a symmetric monoidal functor $Z\colon \mathrm{Cob}_1^\infty(n) \to$ 1-Vect factors through the category $\mathrm{Cob}(n)$ of n-dimensional cobordism $\pi_{\leq 1}\mathrm{Cob}_1^\infty(n)$; see Remark 3. It is interesting to notice that, even if one does not a priori imposes any finite dimensionality condition on the objects in 1-Vect, i.e., if one takes 1-Vect to be the category of all vector spaces over some fixed field $\mathbb{K}$, then, as an almost immediate corollary of the definition, the vector space $Z(M)$ that an Atiyah n-dimensional TQFT assigns to a closed $n-1$-manifold M must be finite dimensional, see [5,30].

Example 6. Similarly, an n-dimensional TQFT extended down to codimension 2 with moduli level 0 is equivalently a symmetric monoidal 2-functor

$$Z : \mathrm{Cob}_2(n) \to 2\text{-Vect} \tag{16}$$

where $\mathrm{Cob}_2(n) = \pi_{\leq 2}\mathrm{Cob}_2^\infty(n)$ is the so-called 2-category of extended cobordism. Its objects are $(n-2)$-dimensional closed manifolds, its 1-morphisms are $(n-1)$-dimensional cobordisms, and its 2-morphisms are diffeomorphism classes of n-dimensional cobordisms. Such a monoidal functor is sometimes called a (2-tier) extended n-dimensional TQFT, see [24,38]. Notice that applying the loop construction to an extended TQFT one obtains an n-dimensional TQFT in the sense of Atiyah and Segal.

Remark 6. 2-tier extended TQFTs have been the subject of great investigation, in particular in 3-dimension. Indeed, historically it was 3-dimensional Chern–Simons theory which motivated the notion of an extended field theory. Particularly relevant are the extended 3d TQFTs known as of *Reshetikhin–Turaev* type [40] obtained by the algebraic data encoded in a modular tensor category, and those of *Turaev–Viro* type [47], which are constructed from a spherical fusion category.[2]

Example 7. The *categorified field theories* in [13] are an example of topological quantum field theories extended down to codimension 2 with moduli level 1.

4.2. Fully extended TQFTs. It is easy to see that a 1-dimensional TQFT in the sense of Atiyah and Segal [4,44] is completely determined by the vector space V^+ it assigns to the oriented point pt^+. Moreover, the category of 1-dimensional Atiyah-Segal TQFTs, i.e. the category

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}_1^{\infty}(1), 1\text{-Vect}) \tag{17}$$

turns out to be equivalent to the groupoid obtained from the category of finite dimensional vector spaces by discarding all the noninvertible morphisms. This can be seen as follows. Given a monoidal natural transformation $\varphi\colon Z_1 \to Z_2$ between two 1-dimensional Atiyah-Segal TQFTs, then we have a linear morphism $\varphi(\mathrm{pt}^+) : V_1^+ \to V_2^+$. The compatibility of φ with the evaluation and coevaluation morphisms forces V_1^+ and V_2^+ to have the same dimension, and $\varphi(\mathrm{pt}^+)$ to be a linear isomorphism. By the same argument one can show that n-dimensional Atiyah-Segal TQFTs as well form a groupoid. See [18] for details.

The rigidity of the 1-dimensional example illustrated above comes from the fact that the involved TQFT is a moduli level 0 *fully extended* TQFT. Indeed, these TQFTs encode so much information that they can be completely classified. This is indeed the content of the *cobordism hypothesis*, which can be stated as follows.[3]

Theorem 1 (Lurie–Hopkins). *A moduli level 0 fully extended n-dimensional framed TQFT is completely determined by a fully dualizable n-vector space. More precisely, let $(n\text{-Vect})_{\mathrm{fd}}$ be the full subcategory of n-Vect of* fully dualizable *objects, and let $(n\text{-Vect})_{\mathrm{fd}}^{(\infty,0)}$ be the underlying $(\infty,0)$-groupoid, i.e., the $(\infty,0)$-groupoid obtained from $(n\text{-Vect})_{\mathrm{fd}}$ by discarding all the non-invertible morphisms. Then there is an equivalence of ∞-categories*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{fr}(n), n\text{-Vect}) \simeq (n\text{-Vect})_{\mathrm{fd}}^{(\infty,0)} \tag{18}$$

induced by the evaluation functor $Z \mapsto Z(\mathrm{pt}^+)$. More generally, if $G \to O(n)$ is a reduction of structure group for n-dimensional manifolds, then there is a natural action of G on $(n\text{-Vect})_{\mathrm{fd}}$ and $Z \mapsto Z(\mathrm{pt}^+)$ induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{G}(n), n\text{-Vect}) \simeq (n\text{-Vect})_{\mathrm{fd}}^{G\,(\infty,0)} \tag{19}$$

where $(n\text{-Vect})_{\mathrm{fd}}^{G}$ denotes the full subcategory on the homotopy fixed points for the induced G-action on $(n\text{-Vect})_{\mathrm{fd}}$.

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[2] In general, the Turaev–Viro construction produces oriented theories, while Reshetikhin–Turaev theories require a *framing* to be defined.

[3] Here we are formulating the cobordism hypothesis for TQFTs with target higher vector spaces; one can give a more general formulation with target an arbitrary (∞, n)-symmetric monoidal category, see [35].

Remark 7. The G-action on $(n\text{-Vect})_{\mathrm{fd}}^{(\infty,0)}$ in Theorem 1 is obtained as follows. First, notice that $O(n)$ acts on the n-framings of a k-dimensional manifold M, and hence it gives an action on $\mathrm{Bord}^{fr}(n)$. Consequently, $O(n)$ acts on $\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{fr}(n), n\text{-Vect})$. By the equivalence in Eq. (18), we obtain an induced action of $O(n)$ on $(n\text{-Vect})_{\mathrm{fd}}^{(\infty,0)}$ and so, for any homomorphism $G \to O(n)$, we have a corresponding G-action on $(n\text{-Vect})_{\mathrm{fd}}^{(\infty,0)}$. The equivalence in Eq. (19) is then obtained as a consequence of the equivalence between $\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{fr}(n), n\text{-Vect})^G$ and $\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{G}(n), n\text{-Vect})$.

Example 8. A fully extended 2-dimensional oriented TQFT Z is the datum of a semisimple Frobenius algebra A. To the oriented point pt^+ it is assigned the linear category Mod_A of finite dimensional right A-modules, while the closed oriented 1-manifold S^1 is sent to the center of A, which is a commutative Frobenius algebra. See [42] for details. This is consistent with what one should have expected: the looped TQFT ΩZ is a 2-dimensional Atiyah-Segal TQFT, and these are equivalent to the category of commutative Frobenius algebras; see [30]. Note, however, that not every 2-dimensional Atiyah-Segal TQFT is obtained a the looping of a fully extended 2-dimensional TQFT, as a commutative Frobenius algebra need not to be semisimple.

Example 9. As a particular case of Example 8, one can show that to any finite group G is associated an extended 2-dimensional TQFT Z_G, mapping pt^+ to the category of finite dimensional representations of G, and S^1 to the algebra $\mathbb{K}[G//G]$ of class functions on G. For a review, see [34].

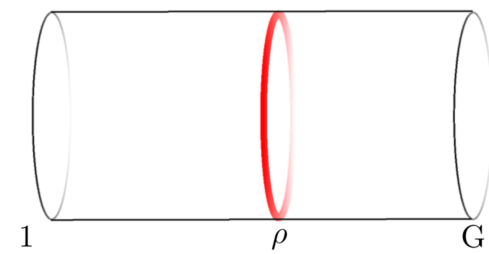
The cobordism hypothesis tells us that the ∞-category of fully extended n-dimensional TQFTs of moduli level 0 constitutes an ∞-groupoid. This is in general no longer true when the moduli level is higher than 0. In particular, this means that if Z_1 and Z_2 are two TQFTs with moduli level greater than 0, it is possible to have nontrivial (i.e., non-invertible) morphisms between Z_1 and Z_2, as in Example 10 below. This possibility will be particularly relevant in the forthcoming sections.

Remark 8. A useful mechanism to produce fully extended n-dimensional TQFTs of moduli level 1 is to start from a fully extended $(n+1)$-dimensional TQFT of moduli level 0 and consider a truncation, as in Remark 5. If Z_1 and Z_2 are moduli level 0 fully extended $(n+1)$-dimensional TQFTs and

$$\eta\colon Z_1\big|_n \to Z_2\big|_n \tag{20}$$

is a morphism between their n-dimensional truncations, then, due to the cobordism hypothesis, η will not in general lift to a morphism between Z_1 and Z_2. At the level of fully extended $(n+1)$-dimensional TQFTs, the morphism η can be considered as a *codimension 1 defect*, also known as a *domain wall*.

Example 10. Let $\underline{1} : \mathrm{Bord}^{or}(2) \to 2\text{-Vect}$ be the trivial extended 2-dimensional oriented TQFT, which assigns to the oriented point the linear category of finite dimensional vector spaces, to S^1 the vector space $\mathbb{K}$, and to closed 2-manifolds the element 1 in $\mathbb{K}$. Let Z_G be the 2-tier extended 2-dimensional oriented TQFT associated with a finite group G, see Example 9. Then, a morphism $\rho : \underline{1}\big|_1 \to Z_G\big|_1$ is the datum of a finite dimensional representation ρ of G, and in the fully extended 2-dimensional TQFT "with defects" lifting it, the representation ρ becomes a domain wall and the cylinder



corresponds to the character of ρ. The cylinder equipped with a circle defect depicted above appears in the literature with the name of *transmission functor*, and plays an important role in the study of symmetries of topological quantum field theories [21].

Since from the literature we are not aware of the any characterization of fully extended TQFTs with moduli level greater than 0, we conclude this section with a conjecture.

Conjecture 1 (Cobordism hypothesis for TQFTs with moduli level m). *For any $m \geq 0$ there is an equivalence of ∞-categories*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}^G(n), (m+n)\text{-Vect}) \simeq ((m+n)\text{-Vect})_{\mathrm{fd}}^{G\,(\infty,m)} \tag{21}$$

induced by the evaluation functor $Z \to Z(\mathrm{pt}^+)$.

In the above conjecture $((m+n)\text{-Vect})_{\mathrm{fd}}^{G\,(\infty,m)}$ denotes the (∞, m) groupoid obtained from $((m+n)\text{-Vect})_{\mathrm{fd}}^{G}$ by discarding all non-invertible k-morphisms with $k > m$.

Example 11. As a supporting evidence for the above conjecture, let us expand Example 10 above. In the same notations as in Example 10, we have seen that any finite dimensional representation ρ of G gives rise to a 1-morphism F_ρ between the moduli level 1 1-dimensional TQFTs $\underline{1}\big|_1$ and $Z_G\big|_1$. From conjecture, we should expect that a morphism of representations $f : \rho_1 \to \rho_2$ induces a 2-morphism $F_{\rho_1} \to F_{\rho_2}$ if and only if f is an isomorphism. This is actually true: looking at the data associated with the 1-dimensional manifold S^1, we see that $F_{\rho_1} \to F_{\rho_2}$ induces a morphism in $\mathbb{K}[G/\!/G]$ between the character of ρ_1 and the character of ρ_2. But since the only morphisms in the vector space $\mathbb{K}[G/\!/G]$ (seen as a 0-category) are identities, this means that the representations ρ_1 and ρ_2 have the same character, and therefore they are isomorphic.

5. Anomalies in Topological Quantum Field Theories

We consider now a particular type of TQFT, called *invertible*, which will be relevant in the description of anomalies we present later.

5.1. Invertible TQFTs.

To be able to define invertible TQFTs, we first need to introduce the following

Definition 4. The Picard ∞-groupoid $\mathrm{Pic}(n\text{-Vect})$ is defined as the ∞-category with objects given by the invertible objects in n-Vect, and k-morphisms given by the invertible k-morphisms for any k.

Notice that the Picard ∞-groupoid $\mathrm{Pic}(n\text{-Vect})$ is a symmetric monoidal (∞, n)-subcategory of n-Vect. Moreover, Definition 4 can be extended to any symmetric monoidal (∞, n)-category $\mathcal{C}$.

Example 12. The Picard groupoid Pic(0-Vect) is the group $\mathbb{K}^*$ of invertible elements of the field $\mathbb{K}$, and identities as morphisms. The Picard groupoid Pic(1-Vect) is the groupoid with objects given by complex vector spaces of dimension 1, 1-morphisms given by invertible linear maps, and identities for k-morphisms, for $k > 1$. The Picard 2-groupoid Pic(2-Vect) can be realized as the 2-groupoid with objects given by Vect-module categories of rank 1, 1-morphisms given by invertible module functors, 2-morphisms given by invertible module natural transformation, and identities for higher k-morphisms. See [14].

An invertible TQFT is essentially an ∞-functor assigning objects to invertible objects, and morphisms to invertible morphisms. More precisely

Definition 5. An n-dimensional Topological Quantum Field Theory extended to codimension k and with moduli level m

$$Z\colon \mathrm{Cob}_k^\infty(n) \to (m+k)\text{-Vect} \tag{22}$$

is said to be invertible iff it factors as

$$\begin{tikzcd} \mathrm{Cob}_k^\infty(n) \arrow[r, "Z"] \arrow[dr] & (m+k)\text{-Vect} \\ & \mathrm{Pic}((m+k)\text{-Vect}) \arrow[u] \end{tikzcd} \tag{23}$$

From every symmetric monoidal (∞, n)-category $\mathcal{C}$ one obtains a symmetric monoidal $(\infty, n+1)$-category $B\mathcal{C}$ by taking the ∞-category with a single object, and with $\mathcal{C}$ as the ∞-category of morphisms. It is immediate to see that $B\mathrm{Pic}(n\text{-Vect})$ is naturally identified with the full subcategory of $\mathrm{Pic}((n+1)\text{-Vect})$ on the tensor unit of $(n+1)$-Vect. This gives a natural embedding

$$B\mathrm{Pic}(n\text{-Vect}) \hookrightarrow \mathrm{Pic}((n+1)\text{-Vect}). \tag{24}$$

This observation leads us to the following

Definition 6. An invertible TQFT with moduli level m

$$Z\colon \mathrm{Cob}_k^\infty(n) \to \mathrm{Pic}((m+k)\text{-Vect}) \hookrightarrow (m+k)\text{-Vect} \tag{25}$$

is said to be *semitrivialized* if it is given a factorization of Z through $B\mathrm{Pic}((m+k-1)\text{-Vect})$.

Remark 9. For $m+k = 1, 2$ the inclusion $B\mathrm{Pic}((m+k-1)\text{-Vect}) \hookrightarrow \mathrm{Pic}((m+k)\text{-Vect})$ is an equivalence of $(m+k)$-groupoids. Therefore, an invertible TQFT with moduli level m can always be (non canonically) semitrivialized as soon as $m+k \leq 2$. It is presently not clear whether this result holds true for $m+k > 2$.

Remark 10. An important aspect of invertible TQFTs is that they can be described as maps of spectra. Namely, an invertible TQFT factorizes through the "groupoid ∞-completion" $|\mathrm{Cob}_k^\infty(n)|$, which can be proven to be a spectrum in low dimensions. See [16, 17] for details.

We will not push in this direction in the present article.

5.2. Anomalies. Invertible TQFTs of moduli level 1 will be particularly relevant to the present work: they will indeed describe anomalies.

Definition 7. An n-dimensional *anomaly* is an invertible TQFT of moduli level 1

$$W\colon \mathrm{Cob}_k^\infty(n) \to \mathrm{Pic}((k+1)\text{-Vect}) \hookrightarrow (k+1)\text{-Vect}. \tag{26}$$

Remark 11. A natural way of producing an n-dimensional anomaly is by *truncating* a $(n+1)$-dimensional TQFT with moduli level 0, i.e., by considering the composition

$$\mathrm{Cob}_k^\infty(n) \hookrightarrow \mathrm{Cob}_{k+1}^\infty(n+1) \to \mathrm{Pic}((k+1)\text{-Vect}) \hookrightarrow (k+1)\text{-Vect}. \tag{27}$$

Example 13. Let us make explicit the data of a semitirivialized n-dimensional anomaly for $k=1$. By definition, this is a symmetric monoidal functor

$$W\colon \mathrm{Cob}_1^\infty(n) \to B\mathrm{Pic}(1\text{-Vect}) \hookrightarrow \mathrm{Pic}(2\text{-Vect}) \hookrightarrow 2\text{-Vect}. \tag{28}$$

Therefore, to each n-dimensional cobordism M a complex line W_M is assigned, together with an isomorphism $W_{M\circ M'} \simeq W_M \otimes W_{M'}$, whenever $M\circ M'$ exists. This isomorphism, which we denote with $\psi_{MM'}$, is part of the structure of W, and hence has to obey the natural coherence conditions. In particular, to the trivial cobordism $\Sigma\times[0,1]$ is assigned the complex vector space $\mathbb{C}$.

Remark 12. Recall from Remark 4 that $B\Gamma^\infty(\Sigma)$ denotes the ∞-groupoid associated to $\mathrm{Diff}(\Sigma)$,[4] namely $B\Gamma^\infty(\Sigma)$ is the connected component of Σ in $\mathrm{Cob}_0^\infty(n-1)$. Let W be as in Example 13. By the mapping cylinder construction, we have the ∞-functor

$$B\Gamma^\infty(\Sigma) \hookrightarrow \mathrm{Cob}_0^\infty(n-1) \hookrightarrow \mathrm{Cob}_1^\infty(n) \to B\mathrm{Pic}(1\text{-Vect}) \tag{29}$$

where the last arrow is given by the factorisation of W through $B\mathrm{Pic}(1\text{-Vect})$. In the terminology of Sect. 6, W gives rise to a 2-character for $\Gamma^\infty(\Sigma)$.

We can now introduce the definition of anomalous TQFTs with given anomaly W. These are called W-twisted field theories in [45] and relative field theories in [24].

Definition 8. Let $W\colon \mathrm{Cob}_k^\infty(n) \to \mathrm{Pic}((k+1)\text{-Vect}) \hookrightarrow (k+1)\text{-Vect}$ be an n-dimensional anomaly. An anomalous n-dimensional extended TQFT with anomaly W is a morphism of n-dimensional TQFTs with moduli level 1

$$Z_W\colon \underline{1} \to W, \tag{30}$$

where $\underline{1}\colon \mathrm{Cob}_k^\infty(n) \to (k+1)\text{-Vect}$ is the trivial TQFT mapping all objects to the monoidal unit and all morphisms to identities.

Lemma 4. *Let W be the trivial n-dimensional anomaly, i.e., let $W=\underline{1}$. Then an n-dimensional extended anomalous TQFT with anomaly W is equivalent to an ordinary n-dimensional extended TQFT.*

Proof. Immediate from Lemma 2.

[4] Here we are omitting the explicit reference to the framing $G\to O(n)$: the manifold Σ here is (as always in this article) endowed with a G-framing of its stabilised tangent bundle, and $\mathrm{Diff}(\Sigma)$ denotes the group of G-framing preserving diffeomorphisms of Σ.

Remark 13. Strictly speaking, we have defined above a TQFT with *incoming* anomaly, and one could also consider *outgoing* anomalies by taking morphisms $W \to \underline{1}$, see, e.g., [24]. Although this distinction is relevant, e.g., for oriented theories, where one can also have both kinds of anomalies at the same time, we will not elaborate on this here.

To get the flavour of these TQFTs with anomaly, let us spell out the data of an n-dimensional TQFT with semitrivialized anomaly in the $k = 1$ case. As expected, we obtain a structure resembling an n-dimensional TQFT a lá Atiyah-Segal, but with a "twisting" coming from the anomaly W. Namely, if

$$W : \mathrm{Cob}_1^{\infty}(n) \to B\mathrm{Pic}(1\text{-Vect}) \hookrightarrow \mathrm{Pic}(2\text{-Vect}) \hookrightarrow 2\text{-Vect} \tag{31}$$

is a semitrivialized anomaly, then a morphism $Z_W : \underline{1} \to W$ consists of the following collection of data:

(a) To each closed $(n-1)$-dimensional manifold Σ it is assigned a vector space V_Σ, with $V_\emptyset \simeq \mathbb{K}$ and with functorial isomorphisms $V_{\Sigma \sqcup \Sigma'} \simeq V_\Sigma \otimes V_{\Sigma'}$;
(b) To each cobordism M between Σ and Σ' it is assigned a linear map $\varphi_M : W_M \otimes V_\Sigma \to V_{\Sigma'}$; for M the trivial cobordism, the corresponding linear map is the natural isomorphism $\varphi_{\Sigma\times[0,1]} : \mathbb{K} \otimes V_\Sigma \to V_\Sigma$.

Moreover, these data satisfy the following compatibilities:

(i) Let $f_{MM'} : M \to M'$ be a diffeomorphism fixing the boundaries between two cobordisms M and M' between Σ and Σ'. Then the following diagram commutes:

$$\begin{tikzcd} W_M \otimes V_\Sigma \arrow[r, "\varphi_M"] \arrow[d, "f_{MM'*}\otimes \mathrm{id}"'] & V_{\Sigma'} \\ W_{M'} \otimes V_\Sigma \arrow[ur, "\varphi_{M'}"'] & \end{tikzcd} \tag{32}$$

where $f_{MM'*} : W_M \to W_{M'}$ denotes the isomorphism induced by $f_{MM'}$.
(ii) For any cobordism M between Σ and Σ', and M' between Σ' and Σ'', the following diagram commutes

$$\begin{tikzcd} W_{M'} \otimes W_M \otimes V_\Sigma \arrow[r, "\mathrm{id}\otimes\varphi_M"] \arrow[d, "\psi_{M'M}\otimes\mathrm{id}"', "\wr"] & W_{M'} \otimes V_{\Sigma'} \arrow[d, "\varphi_{M'}"] \\ W_{M'\circ M} \otimes V_\Sigma \arrow[r, "\varphi_{M'\circ M}"] & V_{\Sigma''} \end{tikzcd} \tag{33}$$

In general, an anomalous TQFT as defined above will give rise to projective representations of diffeomorphsims of closed manifolds. In order to give a precise statement, in the following section we will take a detour into projective representations of ∞-groups as homotopy fixed points of higher characters.

6. *n*-Characters and Projective Representations

In this section we will introduce the notion of an n-character for an ∞-group (e.g., the Poincaré ∞-groupoid $\pi_{\leq\infty}(G_{\mathrm{top}})$ of a topological group G_{top}), and its homotopy fixed points. This is a natural higher generalisation of the notion of a $\mathbb{C}^*$-group character.

Hence, as a warm up, we will first discuss the case of a discrete group G, and show how this recovers the category of (finite dimensional) projective representations of G. This is well known in geometric representation theory, but since we are not able to point the reader to a specific treatment in the literature, we will provide the necessary amount of detail here.

6.1. Discrete groups. Let G be a (discrete) group, and let BG denote the 1-object groupoid with G as group of morphisms, regarded as an ∞-groupoid with only identity k-morphisms for $k > 1$.

Definition 9. A 2-character for G with values in Vect is a 2-functor

$$\rho : BG \to B\mathrm{Pic}(\mathrm{Vect}) \tag{34}$$

Explicitly, a 2-character ρ consists of a family of complex lines W_g^ρ, one for each $g \in G$, and isomorphisms

$$\psi_{g,h}^\rho : W_g^\rho \otimes W_h^\rho \xrightarrow{\simeq} W_{gh}^\rho \tag{35}$$

satisfying the associativity condition

$$\psi_{gh,j}^\rho \circ (\psi_{g,h}^\rho \otimes \mathrm{id}) = \psi_{g,hj}^\rho \circ (\mathrm{id} \otimes \psi_{h,j}^\rho) \tag{36}$$

for any $g, h, j \in G$. When no confusion is possible we will simply write W_g for W_g^ρ and $\psi_{g,h}$ for $\psi_{g,h}^\rho$.

For a given group G, 2-characters form a category, given by the groupoid $[BG, B\mathrm{Pic}(\mathrm{Vect})]$ of functors between BG and $B\mathrm{Pic}(\mathrm{Vect})$, and their natural transformations. Explicitly, a morphism $\rho \to \tilde{\rho}$ is a collection of isomorphisms of complex lines $\xi_g \colon W_g \xrightarrow{\sim} \tilde{W}_g$ such that

$$\psi_{g,h} \circ (\xi_g \otimes \xi_h) = \xi_{gh} \circ \psi_{g,h},$$

for any $g, h \in G$.

The assignment $W \to W \otimes (-)$ induces an equivalence of groupoids

$$\mathrm{Pic}(\mathrm{Vect}) \simeq \mathrm{Aut}(\mathrm{Vect}), \tag{37}$$

where Aut(Vect) denotes the groupoid of linear auto-equivalences of Vect, i.e. of linear invertible functors from Vect to itself. As a consequence, a 2-character defines an action of G by functors on the linear category Vect. As for any action of a group, we can investigate the structure of its fixed points. Since we are in a categorical setting, though, we can ask that points are fixed at most up to isomorphisms. This motivates the following

Definition 10. Let $\rho = \{W_g; \psi_{g,h}\}$ be a 2-character for a (discrete) group G. A homotopy fixed point for ρ is given by an object $V \in \mathrm{Vect}$ and a family $\{\varphi_g\}_{g\in G}$ of isomorphisms

$$\varphi_g : W_g \otimes V \xrightarrow{\simeq} V \tag{38}$$

satisfying the compatibility condition

$$\varphi_{gh} \circ (\psi_{g,h} \otimes \mathrm{id}) = \varphi_g \circ (\mathrm{id} \otimes \varphi_h) \tag{39}$$

Remark 14. A convenient way to encapsulate the data in Definition 10 is the following. By using the equivalence (37), a 2-character ρ induces a 2-functor $W : BG \to$ 2-Vect, which assigns to the single object in BG the category Vect.[5] If we denote by $\underline{1}$ the trivial 2-functor from BG to 2-Vect, we have then that a homotopy fixed point is equivalently a morphism, i.e. a natural transformation of 2-functors, $\underline{1} \to W$.

Remark 15. Homotopy fixed points for a given 2-character ρ form a category in a natural way, which we denote with Vect^{ρ}. It is immediate to see that, up to equivalence, Vect^{ρ} depends only on the isomorphism class of ρ.

In the following, we will show that 2-characters for a group G are related to group 2-cocycles for G, and that homotopy fixed points are related to projective representations.

Recall that to a group G we can assign its groupoid of group 2-cocycles with values in $\mathbb{K}^*$, which we denote by $\underline{Z}^2_{grp}(G;\mathbb{K}^*)$. This is, essentially by definition, the 2-groupoid $[BG, B^2\mathbb{K}^*]$ of 2-functors from BG to $B^2\mathbb{K}^*$. Since $B^2\mathbb{K}^*$ is the simplicial set with a single 0-simplex, a single 1-simplex, 2-simplices indexed by elements in $\mathbb{K}^*$, and 3-simplices corresponding to those configurations of 2-simplices the indices of whose boundary faces satisfy the 2-cocycle condition, a 2-functor $F\colon BG \to B^2\mathbb{K}^*$ is precisely a group 2-cocycle on G with coefficients in $\mathbb{K}^*$.

The equivalence $B\mathbb{K}^* \xrightarrow{\simeq} \mathrm{Pic}(1\text{-Vect})$ induces an equivalence $B^2\mathbb{K}^* \xrightarrow{\simeq} B\mathrm{Pic}(1\text{-Vect})$, and so an equivalence

$$T : \underline{Z}^2_{grp}(G;\mathbb{K}^*) \xrightarrow{\simeq} [BG, B(\mathrm{Pic}(1\text{-Vect}))] \tag{40}$$

for any finite group G. In particular, every 2-cocycle α naturally induces (and is actually equivalent to) a 2-character $T(\alpha)$. Note that $W_g^{T(\alpha)} = \mathbb{K}$ for any $g \in G$. The morphisms $\psi^{T(\alpha)}_{g,h} : W_g^{T(\alpha)} \otimes W_h^{T(\alpha)} \xrightarrow{\simeq} W_{gh}^{T(\alpha)}$ are given by

$$W_g^{T(\alpha)} \otimes W_h^{T(\alpha)} = \mathbb{K}\otimes\mathbb{K} \cong \mathbb{K} \xrightarrow{\alpha(g,h)} \mathbb{K} = W_{gh}^{T(\alpha)}. \tag{41}$$

Recall that a *projective representation* for a group G with 2-cocycle α is given by a vector space V, and a family of isomorphisms

$$\varphi_g^{\alpha} : V \xrightarrow{\simeq} V, \quad \forall g \in G \tag{42}$$

satisfying the condition

$$\varphi^{\alpha}_{gh} = \alpha(g,h)\,\varphi^{\alpha}_g \circ \varphi^{\alpha}_h, \quad \forall g, h \in G \tag{43}$$

Projective representations for a given 2-cocycle α form naturally a category, which we denote with $\mathrm{Rep}^{\alpha}(G)$.

Given any projective representation (V, φ^{α}) with 2-cocycle α, the vector space V is naturally a homotopy fixed point for $T(\alpha)$: consider the family of isomorphisms

$$\varphi_g^{T(\alpha)}\colon W_g^{T(\alpha)} \otimes V = \mathbb{K}\otimes V \cong V \xrightarrow{\varphi_g^{\alpha}} V, \quad \forall g \in G. \tag{44}$$

Then condition (43) assures that the family of isomorphisms $\{\varphi_g^{T(\alpha)}\}$ realises V as a homotopy fixed point for $T(\alpha)$. It is immediate to check that this construction is functorial and therefore defines a "realisation as homotopy fixed point" functor $H^{\alpha}\colon \mathrm{Rep}^{\alpha}(G) \to \mathrm{Vect}^{T(\alpha)}$, for any 2-cocycle α.

[5] In other words, W is a 2-representation of G of rank 1.

Lemma 5. *The functor* $H^\alpha\colon \mathrm{Rep}^\alpha(G) \to \mathrm{Vect}^{T(\alpha)}$ *is an equivalence of categories.*

Proof. It is immediate to see that H^α is faithful and full. To see that it is essentially surjective, take a homotopy fixed point (V, φ) for $T(\alpha)$, and define φ^α as

$$\varphi^\alpha_g\colon V \cong \mathbb{K} \otimes V = W_g^{T(\alpha)} \otimes V \xrightarrow{\varphi_g} V. \tag{45}$$

Then the compatibility condition (39) ensures then that (V, φ^α) is a projective representation with 2-cocycle α, with $H^\alpha(V, \varphi^\alpha) \simeq (V, \varphi)$.

6.2. 2-characters for ∞-groups. In this subsection we will see how the notion of a 2-character for a finite group immediately generalises to the notion of $(n+1)$-character for an ∞-group (i.e., for a monoidal ∞-groupoid whose objects are invertible for the monoidal structure) G, for any $n \geq 0$.

Since an ∞-group G is in particular a monoidal ∞-category, it has a classifying monoidal ∞-category BG. The fact that G is not just any monoidal ∞-category but an ∞-group can then be expressed by saying that BG is a one-object ∞-groupoid. The ∞-group structure on G induces a (discrete) group structure on the set $\pi_0(G)$ of the isomorphism classes of objects of G, and one has a natural equivalence of groupoids $B\pi_0(G) \cong \pi_{\leq 1}BG$.

Example 14. The basic example of an ∞-group is the fundamental ∞-groupoid of a topological group G_{top}. Namely, since G_{top} is a group, the ∞-groupoid $\pi_{\leq\infty}(G_{\mathrm{top}})$ has a natural monoidal structure for which all the objects are invertible, given by the product in G_{top}. Moreover, one has $\pi_0(\pi_{\leq\infty}(G_{\mathrm{top}})) = \pi_0(G_{\mathrm{top}})$, the (discrete) group of (path-)connected components of the topological group G_{top}.

Example 15. A second fundamental example of an ∞-group is the ∞-group $\Gamma^\infty(\Sigma)$ of diffeomorphisms of a smooth manifold Σ. Here the objects are the diffeomorphisms of Σ, 1-morphisms are isotopies between diffeomorphism, 2-morphism are isotopies between isotopies, and so on. For oriented manifolds one can analogously consider the ∞-group of oriented diffeomorphisms, and more generally for G-framed manifolds one can consider the ∞-group of G-framed diffeomorphisms. The π_0 of the ∞-group $\Gamma^\infty(\Sigma)$ is the mapping class group $\Gamma(\Sigma)$ of the (G-framed) manifold Σ.

Definition 11. Let G be an ∞-group. A $n+1$-character for G is a ∞-functor

$$\rho : BG \to B(\mathrm{Pic}(n\text{-Vect})) \tag{46}$$

The definition given above is very flexible and compact, and can be easily generalised by taking an arbitrary symmetric monoidal (∞, n)-category in place of n-Vect.

Remark 16. A 2-character for an ∞-group contains (in general) more information than a 2-character for a discrete group (which can be seen as a very particular case of an ∞-group). Namely, for G an ∞-group, a 2-character ρ is given by an assignment to each object $g \in G$ of a complex line W_g, of a family $\psi_{g,h}$ of isomorphisms

$$\psi_{g,h} : W_g \otimes W_h \xrightarrow{\sim} W_{gh}, \quad \forall g, h \in G \tag{47}$$

and of isomorphisms

$$\psi_f : W_g \to W_h \tag{48}$$

for any path (i.e., 1-morphism) f connecting g to h. The above isomorphisms must obey coherence conditions which encode the fact that ρ is an ∞-functor. In particular, the isomorphism ψ_f depends only on the isomorphism class of the 1-morphism f. In the particular case of a discrete group, the only paths in G are the identities and one is reduced to Definition 9.

Example 16. Let G_{Lie} be a Lie group, and let L be a multiplicative line bundle over G_{Lie}, equipped with a compatible flat connection ∇. From L one obtains a 2-character ρ for $\pi_{\leq\infty}(G_{\mathrm{Lie}})$ as follows: to each g in G_{Lie}, one assigns the vector space given by the fiber L_g, and for each path γ connecting g and h one takes the isomorphism $\psi_\gamma : L_g \to L_h$ induced by the connection via parallel transport (this depends only the homotopy class of γ, since ∇ is flat). Finally, the fact that L is multiplicative and the compatibility of ∇ with the multiplicative structure imply that this assignment does define a 2-character.

For any n, the $(n+1)$-group $\mathrm{Pic}(n\text{-Vect})$ acts $(n+1)$-linearly on n-Vect. This means that any $(n+1)$-character $\rho\colon BG \to B\mathrm{Pic}(n\text{-Vect})$ can naturally be seen as an ∞-functor $W\colon BG \to (n+1)$-Vect, mapping the unique object of BG to n-Vect. We will denote by $\underline{1}\colon BG \to (n+1)$-Vect the trivial ∞-functor, mapping the unique object of BG to the monoidal unit of $(n+1)$-Vect (i.e., to n-Vect), and all morphisms in BG to identities.

Having introduced this notation, we can give the following definition of homotopy fixed point for an $(n+1)$-character, generalizing the definition of homotopy fixed points for a 2-character of a discrete group seen above.

Definition 12. Let ρ be an $(n+1)$-character for an ∞-group G, and let $W\colon BG \to (n+1)$-Vect be the corresponding ∞-functor. A homotopy fixed point for ρ is a morphism of ∞-functors $\underline{1} \to W$.

Homotopy fixed points for a $(n+1)$-character ρ form naturally an n-category, which we denote by $n\text{-Vect}^\rho$.

Remark 17. Since a 2-character for a ∞-group contains more information than a 2-character for a discrete group (see Remark 16), being a homotopy fixed point is a more restrictive condition (in general) in the ∞-group case. Namely, with respect to the compatibility conditions in Definition 10, one has in addition that the following diagram

$$\begin{tikzcd}
W_g \otimes V \arrow[r, "\varphi_g"] \arrow[dr, "\psi_f \otimes \mathrm{id}"'] & V \\
& W_h \otimes V \arrow[u, "\varphi_h"']
\end{tikzcd} \tag{49}$$

has to commute, for any two objects g and h in G and any 1-morphism $f\colon g \to h$ between them.

Remark 18. Homotopy fixed points for a 2-character for a topological group are a special case of the following construction. Let X be a ∞-groupoid, and let L be a ∞-functor from X to $B(\mathrm{Pic}(1\text{-Vect}))$. A *module* for L is given by an ∞-functor $E : X \to \mathrm{Vect}$, and isomorphisms $L_f \otimes E_x \simeq E_y$ for any 1-morphism $f : x \to y$, where L_f is the complex line assigned to f, and E_x is the vector space assigned to x by E. Higher morphisms must also be taken into account, and together with the above family of isomorphisms they must obey natural coherence conditions. The case of a homotopy fixed point for a 2-character for a topological group G corresponds to $X = BG$. Another geometrically interesting case is when X is the groupoid $Y^{[2]} \rightrightarrows Y$ for a surjective submersion $Y \to M$: in this case an ∞-functor $L\colon X \to B(\mathrm{Pic}(1\text{-Vect}))$ is given by a bundle gerbe with a flat connection over X, while a module E over L is given by a (flat) gerbe module over L.

If G is a (discrete) group and ρ is a 1-character, i.e., a group homomorphism $G \to \mathbb{K}^*$, a homotopy fixed point is then nothing but a fixed point for the natural linear action of G on $\mathbb{K}$ via ρ. Notice how the existence of a nonzero fixed point imposes a very strong constraint on the character ρ in this case: if there exists a nonzero fixed point, then ρ is the trivial character.

An analogous phenomenon happens for $(n+1)$-characters of ∞-groups, for any $n \geq 0$. Here we will investigate in detail the case of 2-characters, due to its relevance to anomalous TQFTs. To do this, it is convenient to introduce the following terminology: we say that a 2-character $\rho\colon BG \to B\mathrm{Pic}(1\text{-Vect})$ has *trivial holonomy* if it factors through the natural projection $BG \to B\pi_0(G)$. The origin of this terminology is clear from Example 16. There, the 2-character ρ factors through $B\pi_{\leq\infty}(G_{\mathrm{Lie}}) \to B\pi_0(G_{\mathrm{Lie}})$ precisely when the connection ∇ has trivial holonomy. We have then the following

Lemma 6. *Let V be a non-zero homotopy fixed point for a 2-character ρ. Then ρ has trivial holonomy.*

Proof. Since V is a homotopy fixed point for ρ, by Remark 17 we have the commutative diagram (49) for any 2-morphism $f\colon g \to h$ in BG (i.e., for any 1-morphism $f\colon g \to h$ in G). Since φ_g and φ_h are isomorphisms, we have

$$\psi_f \otimes \mathrm{id} = \varphi_h^{-1} \circ \varphi_g, \tag{50}$$

and so $\psi_f \otimes \mathrm{id}$ is independent of f. Since V is nonzero, this implies that ψ_f is actually independent of f. This means that all the complex lines W_g with g ranging over a connected component (i.e., an isomorphism class of objects) of G are canonically isomorphic to each other, and so ρ factors through $B\pi_0(G)$.

Summing up the results in this section, we have the following

Proposition 1. *Let ρ be a 2-character on an ∞-group G, and let V be a nontrivial homotopy fixed point for ρ. Then there exist a 2-cocycle α_ρ on $\pi_0(G)$, unique up to equivalence, such that V is isomorphic to (the homotopy fixed point realisation of) a projective representation of $\pi_0(G)$ with 2-cocycle α_ρ.*

Proof. Since ρ has a nontrivial homotopy fixed point, ρ has trivial holonomy by Lemma 6. Therefore, by definition of trivial holonomy, ρ is (equivalent to) a 2-character on the discrete group $\pi_0(G)$. The statement then follows from Equation (40) and Lemma 5.

6.3. Projective representations from TQFTs. We can finally apply the results on $(k+1)$-characters to anomalous TQFTs. Indeed, consider a semitrivialized anomaly $W\colon \mathrm{Cob}_k^\infty(n) \to B\mathrm{Pic}(k\text{-Vect}) \hookrightarrow (k+1)\text{-Vect}$, and let Z_W be an n-dimensional anomalous TQFT extended down to codimension k, with anomaly W. Reasoning as in Remark 12, the anomaly W induces, for any closed (oriented) $(n-k)$-dimensional manifold Σ, a 2-character ρ_Σ for the ∞-group of (oriented) diffeomorphisms $\Gamma^\infty(\Sigma)$, as in the following diagram

$$\begin{tikzcd} B\Gamma^\infty(\Sigma) \arrow[r, hook] \arrow[rrr, bend right, "\rho_\Sigma"'] & \mathrm{Cob}_0^\infty(n-k) \arrow[r, hook] & \mathrm{Cob}_k^\infty(n) \arrow[r] \arrow[rr, bend left, "W"] & B\mathrm{Pic}(k\text{-Vect}) \arrow[r] & (k+1)\text{-Vect} \end{tikzcd} \tag{51}$$

The k-vector space $Z_W(\Sigma)$ associated by the anomalous TQFT Z_W to the (oriented) $(n-k)$-dimensional manifold Σ is, by definition, a homotopy fixed point for ρ_Σ. In particular, for $k = 1$, by Proposition 1, the vector space $Z_W(\Sigma)$ associated to an $(n-1)$-dimensional manifold Σ is a projective representation of the mapping class group $\Gamma(\Sigma)$ as soon as $Z_W(\Sigma)$ is nonzero. In other words, for any $(n-1)$-dimensional manifold Σ we obtain a central extension

$$1 \to \mathbb{K}^* \to \widetilde{\Gamma}(\Sigma) \to \Gamma(\Sigma) \to 1$$

and a linear representation $\widetilde{\Gamma}(\Sigma) \to \mathrm{Aut}(Z_W(\Sigma))$. This can be neatly described by noticing that for $k = 1$ the data for an anomalous TQFT with anomaly W are a homotopy commutative diagram of the form

$$\begin{tikzcd} \mathrm{Cob}_1^\infty(n) \arrow[r] \arrow[d, "W"'] \arrow[dr, phantom, "\swarrow Z_W"] & \mathbf{1} \arrow[d] \\ B\mathrm{Pic}(\mathrm{Vect}) \arrow[r] & 2\text{-Vect} \end{tikzcd} .$$

Such a diagram can be interpreted as the datum of a section Z_W of the 2-line bundle $\mathcal{L}$ over $\mathrm{Cob}_1^\infty(n)$ associated with W. The "graph" of this section is a ∞-category $\widetilde{\mathrm{Cob}_1^\infty}(n)$ over $\mathrm{Cob}_1^\infty(n)$ whose objects are pairs consisting of an $(n-1)$-dimensional manifold Σ together with the choice of an object in the fibre $\mathcal{L}_\Sigma$. The mapping class group for such a pair is the $\mathbb{K}^*$-central extension of $\Gamma(\Sigma)$ described above. Notice the striking similarity with Segal's description of projective modular functors via central extensions of the cobordism category [44], with the remarkable difference that anomalies in the sense of the present article induce $\mathbb{K}^*$-central extensions whereas in Segal's extended cobordism one deals with $\mathbb{Z}$-central extensions.

Remark 19. As we have seen above, having a semitrivialized anomaly W produces projective representations of the mapping class groups of all closed $(n-k)$-dimensional manifolds at once. If one is interested in a single $(n-k)$-dimensional manifold Σ, though, there is no need for a semitrivialization of the anomaly: indeed, one can produce a projective representation of $\Gamma(\Sigma)$ from any anomalous TQFT Z_W, as soon as the invertible $(k+1)$-vector space $W(\Sigma)$ is equivalent to the "trivial" $(k+1)$-vector space k-Vect. As already observed in Remark 9, this is always possible, although non canonically, for any invertible $(k+1)$-vector space, with $k = 0, 1$. Namely, choosing an equivalence between $W(\Sigma)$ and k-Vect amounts to give a homotopy commutative diagram

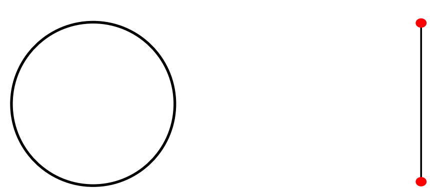
$$\begin{tikzcd} B\mathrm{Aut}(W(\Sigma)) \arrow[r, "W(\Sigma)"] \arrow[d] \arrow[r, phantom, shift right=3ex, "\Psi\,\searrow"] & (k+1)\text{-Vect}\,, \\ B\mathrm{Pic}(k\text{-Vect}) \arrow[ur] & \end{tikzcd}$$

where the top horizontal arrow picks the $(k+1)$-vector space $W(\Sigma)$, while the diagonal arrows is the canonical embedding of $B\mathrm{Pic}(k\text{-Vect})$ into $(k+1)$-Vect, which picks the $(k+1)$-vector space k-Vect. The construction of the projective representation of the mapping class group of Σ follows from the very same arguments as above: indeed, just notice that in diagram (51) it is inessential to have the arrow $\mathrm{Cob}_k^\infty(n) \to B\mathrm{Pic}(k\text{-Vect})$ if we are interested in a single manifold Σ, while at the same time the morphism $B\Gamma^\infty(\Sigma) \to (k+1)$-Vect naturally factors through $B\mathrm{Aut}(W(\Sigma))$. We therefore obtain the following variant of diagram (51), which induces the same considerations as above:

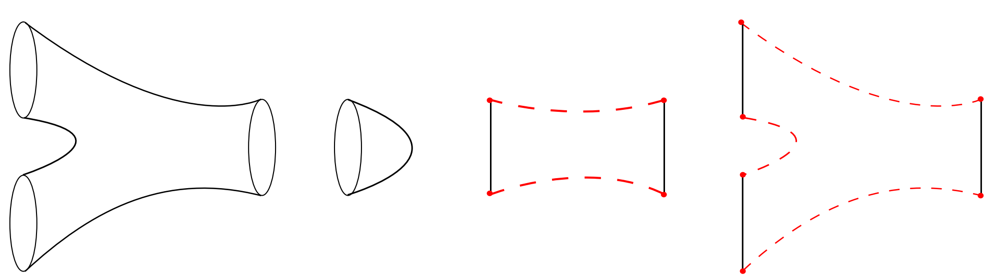
$$
\begin{tikzcd}
B\Gamma^\infty(\Sigma) \arrow[r, hook] \arrow[drr, bend right, "\rho_\Sigma"'] & \mathrm{Cob}_0^\infty(n-k) \arrow[r, hook] \arrow[dr, phantom, "\sim"] & \mathrm{Cob}_k^\infty(n) \arrow[rr, "W"] \arrow[rr, bend left=40, "\underline{1}"] \arrow[rr, phantom, bend left=20, "\Downarrow z_W"] & & (k+1)\text{-Vect} \\
& & B\mathrm{Aut}(W(\Sigma)) \arrow[r] \arrow[urr, "W(\Sigma)"] & B\mathrm{Pic}(k\text{-Vect}) \arrow[ur, bend right] \arrow[ur, phantom, "\Psi"] &
\end{tikzcd}
\tag{52}
$$

7. Boundary Conditions for TQFTs

7.1. Boundary conditions. The n-dimensional TQFTs defined in Sect. 4 assign diffeomorphism invariants to *closed* n-manifolds. Nevertheless, n-manifolds with boundaries have also invariants, usually obtained via relative constructions. One possibility to incorporate invariants of manifolds with boundaries is to enlarge the cobordism category with morphisms represented by manifolds with *constrained* boundaries. The guiding example is given by 2-dimensional open/closed topological field theory [32,33,37], where the authors enlarge the category $\mathrm{Cob}_1(2) = \pi_{\leq 1}\mathrm{Cob}_1^\infty(2)$ of 2-dimensional cobordism by adding to it 1- and 2-dimensional manifolds with part of the boundary declared to be constrained, meaning that it is not possible to glue along. If we denote by $\mathrm{Cob}_1^\partial(2)$ this enlarged category, we will have the following 1-manifolds (and disjoint union of) as objects



and the following 2-manifolds as some of the morphisms



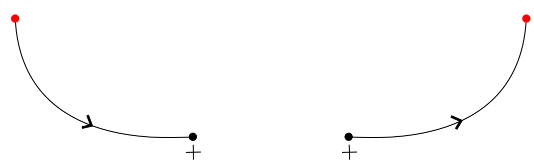
where we denote the constrained boundary with a dashed red line. Notice that, differently from [37], we are here using only *one* type of constrained boundary, which we label/color red. The general case will be discussed in Remark 24 below.

Inspired by the description of $\mathrm{Cob}^{\partial}(2)$ sketched above, let us define iteratively a *constrained bordism* between two constrained d-dimensional manifolds Σ_0 and Σ_1 as a $(d+1)$-dimensional manifold[6] M whose boundary ∂M can be decomposed as $\Sigma_0 \cup \Sigma_1 \cup \partial_{const} M$, where $\partial_{const} M$ is a cobordism from $\partial_{const}\Sigma_0$ to $\partial_{const}\Sigma_1$. Constrained cobordisms come with smooth collars around the part of the boundary which is unconstrained, in order to be able to glue them. With this premise, we can give the following informal definition, a rigorous version of which can be found in [35, Section 4.3].

Definition 13. The symmetric monoidal (∞, n)-category $\mathrm{Bord}^{\partial}(n)$ has points as objects, 1-dimensional constrained bordisms as 1-morphisms, 2-dimensional constrained bordisms between constrained bordisms as 2-morphisms, and so on until we arrive at n-dimensional constrained bordisms as n-morphisms, from where higher morphisms are given by diffeomorphisms fixing the unconstrained boundaries and isotopies between these (and isotopies between isotopies, and so on).

Remark 20. Exactly as $\mathrm{Bord}(n)$, also $\mathrm{Bord}^{\partial}(n)$ comes in different flavours corresponding to the various possible G-framings of the cobordisms. In this section we will be interested in the general features of TQFTs with boundary conditions, and in their relation to anomalous field theories. Hence in what follows, we will always leave the G-marking unspecified, unless stated otherwise.

Example 17. The following 1-dimensional constrained cobordisms are examples of 1-morphisms in $\mathrm{Bord}^{\partial,or}(n)$, for any $n \geq 1$.



The one on the left represents a 1-morphism $\emptyset \to \mathrm{pt}^+$, which cannot be realized in $\mathrm{Bord}(n)$. Similarly, the morphism on the right represents a 1-morphism $\mathrm{pt}^+ \to \emptyset$, which is also not present in $\mathrm{Bord}(n)$.

In analogy with the notation used in the unconstrained case, we will set

$$\mathrm{Cob}_k^{\partial,\infty}(n) = \Omega^{n-k}\mathrm{Bord}^{\partial}(n). \tag{53}$$

With this notation, we have that the category of 2-dimensional constrained cobordism mentioned above is given by $\mathrm{Cob}_1^{\partial}(2) = \pi_{\leq 1}\mathrm{Cob}_1^{\partial,\infty}(2)$. There is a canonical (non full) embedding $\mathrm{Bord}(n) \hookrightarrow \mathrm{Bord}^{\partial}(n)$, hence for any $k \geq 0$ we have a natural (non full) embeddings

$$i : \mathrm{Cob}_k^{\infty}(n) \hookrightarrow \mathrm{Cob}_k^{\partial,\infty}(n). \tag{54}$$

This allows us to give the following

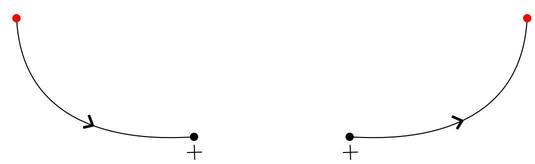
6 Here manifold more precisely means "manifold with corners".

Definition 14. Let $Z\colon \mathrm{Cob}_k^{\infty}(n) \to (k+m)\text{-Vect}$ be an n-dimensional TQFT with moduli level m. A *boundary condition* for Z is a symmetric monoidal extension

$$\begin{tikzcd} \mathrm{Cob}_k^{\partial,\infty}(n) \arrow[r, "\tilde{Z}"] & (k+m)\text{-Vect} \\ \mathrm{Cob}_k^{\infty}(n) \arrow[u, "i"] \arrow[ur, "Z"'] & \end{tikzcd} \tag{55}$$

Remark 21. It is important to notice that boundary conditions for an invertible TQFT are *not* required to be invertible. This is reminiscent of the definition of an anomalous TQFT, where the morphism $\underline{1} \to W$ is not required to be an isomorphism. We will come back to this in Sect. 8.

Example 18. The definition above can be made completely explicit for an Atiyah-Segal 1-dimensional TQFT, i.e., for $Z\colon \mathrm{Cob}_1^{\infty}(1) \to \mathrm{Vect}$. Indeed, in the same way as Z factors through $\mathrm{Cob}_1(1)$, $\tilde{Z}$ will factor through $\mathrm{Cob}_1^{\partial}(1) = \pi_{\leq 1}\mathrm{Cob}_1^{\partial,\infty}(1)$. The objects of $\mathrm{Cob}^{\partial}(1)$ are oriented points, and the morphisms are given by those in $\mathrm{Cob}(1)$, and in addition the following constrained morphisms



and their duals. Therefore, if the 1-dimensional TQFT Z is given by the finite-dimensional vector space V, then a boundary condition $\tilde{Z}$ for Z is the datum of a pair (v, φ), where v is a vector in V and φ is an element in the dual space V^*. We will call these a *left* and a *right* boundary condition, respectively. In the unoriented situation the two morphisms above are identified, and a boundary condition reduces to the datum of the vector v, which also plays the role of a linear functional on V via the symmetric nondegenerate inner product on V.

What makes the description of the boundary conditions so simple in the example above is the fact that we are dealing with a fully extended theory. Indeed, one has the following extension of the cobordism hypothesis to cobordisms with constrained boundaries [35].

Theorem 2 (Lurie–Hopkins). *Let $Z : \mathrm{Bord}^{fr}(n) \to n\text{-Vect}$ be a fully extended TQFT with moduli level 0. Then there is an equivalence*

$$\{\textit{(Left) boundary conditions for } Z\} \cong \mathrm{Hom}_{n\text{-Vect}}((n-1)\text{-Vect}, Z(\mathrm{pt}^+)) \cong Z(\mathrm{pt}^+) \tag{56}$$

induced by the evaluation of $\tilde{Z}$ on the decorated interval on the left in Example 17.

This description of (left) boundary conditions is strongly reminiscent of an anomalous TQFT as in Definition 8. In the following we will see how a TQFT with (left) boundary conditions naturally induces an anomalous TQFT.

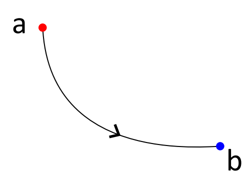
Remark 22. For TQFTs with values in an arbitrary symmetric monoidal (∞, n)-category $\mathcal{C}$, one still has that the $(\infty, n-1)$-category of boundary conditions is equivalent to the hom-space $\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, Z(\mathrm{pt}^+))$, where $1_{\mathcal{C}}$ is the monoidal unit of $\mathcal{C}$. However in general this hom-space is not equivalent to $Z(\mathrm{pt}^+)$.

Remark 23. An analogue statement is likely to hold for cobordisms with a reduction $G \to O(n)$ of the structure group of n-dimensional manifolds, by suitably taking into account the homotopy $O(n)$-action on the homotopy G-fixed point $Z(\mathrm{pt}^+)$. For instance, in the oriented situation one has $O(n)/SO(n) = \mathbb{Z}/2\mathbb{Z}$, and the full boundary conditions data consist of a left boundary condition $(n-1)$-Vect $\to Z(\mathrm{pt}^+)$ and a right boundary condition $Z(\mathrm{pt}^+) \to (n-1)$-Vect. Yet, for $n \geq 2$, every n-vector space V realized as a linear $(n-1)$-category comes naturally equipped with a distinguished inner product given by the Hom bifunctor

$$\mathrm{Hom} : V^{op} \boxtimes V \to (n-1)\text{-Vect} \tag{57}$$

With this choice of inner product, left boundary conditions automatically determine right boundary conditions as in the unoriented case.

Remark 24. One can consider more than a single boundary condition at once, by replacing $\mathrm{Bord}^{\partial}(n)$ by the larger symmetric monoidal (∞, n)-category $\mathrm{Bord}^{\partial_J}(n)$, where constrained boundaries are labelled by indices from a set J of *colours*. An extension $\tilde{Z}$ of a TQFT Z to $\mathrm{Cob}_k^{\partial_J,\infty}(n)$ is then the assignment of a boundary condition to each colour $j \in J$, in such a way that the constraints imposed by requiring $\tilde{Z}$ to be a monoidal symmetric functor are satisfied. One can in particular make the tautological choice $J = \mathrm{objects}(\mathcal{B}_Z)$, where $\mathcal{B}_Z$ denotes the category of boundary conditions for Z. In this way we recover the open/closed field theory framework as in [32,33,37]. Namely, we recall from Example 8 that an extended 2-dimensional *oriented* TQFT Z is the datum of a semisimple Frobenius algebra A, to be seen as a placeholder for its category of finite dimensional right modules. Using the Hom functor as an inner product on ${}_A\mathrm{Mod}$ reduces boundary conditions to left boundary conditions (see Remark 23). Therefore one has constrained boundaries decorated by right A-modules, and the boundary condition $\tilde{Z}$ associates with the oriented segment with constrained boundaries



decorated by the A-modules R_a and R_b the vector space $\mathcal{O}_{ab} = \mathrm{Hom}_A(R_a, R_b)$. See [1] for a treatment of open/closes 2d *nonoriented* TQFTs.

Remark 25. As an intermediate symmetric monoidal (∞, k)-category between $\mathrm{Cob}_k^{\infty}(n)$ and $\mathrm{Cob}_k^{\partial,\infty}(n)$, one can consider the *closed sector* $\mathrm{Cob}_{k,\mathrm{cl}}^{\partial,\infty}(n)$, defined as the *full* (∞, k)-subcategory generated by $\mathrm{Cob}_k^{\infty}(n)$ inside $\mathrm{Cob}_k^{\partial,\infty}(n)$. Namely, objects in $\mathrm{Cob}_{k,\mathrm{cl}}^{\partial,\infty}(n)$ are closed k-manifolds, as in $\mathrm{Cob}_k^{\infty}(n)$. Notice that in $\mathrm{Cob}_k^{\partial,\infty}(n)$ we allow for more objects, since one can consider k-manifolds with completely constrained boundary. For instance, of the two objects in $\mathrm{Cob}_1^{\partial,\infty}(2)$ depicted at the beginning of this section, only S^1 is an object in the closed sector.

One can therefore also consider closed sector boundary conditions, i.e., extensions of a TQFT to the closed sector

$$\begin{tikzcd} \mathrm{Cob}^{\partial,\infty}_{k,\mathrm{cl}}(n) \arrow[r, "\tilde{Z}_{\mathrm{cl}}"] & (k+m)\text{-Vect} \\ \mathrm{Cob}^{\infty}_{k}(n) \arrow[u, "i"] \arrow[ur, "Z"'] & \end{tikzcd} \tag{58}$$

These are expected to be particularly simple in the $k = n - 1$ case. Indeed, since S^1 is the only closed 1-dimensional manifold up to cobordisms, closed sector boundary conditions for a TQFT $Z\colon \mathrm{Cob}^{\infty}_{n-1}(n) \to (n-1)$-Vect should reduce to a $(n-1)$-linear morphism $(n-2)$-Vect $\to Z(S^1)$, i.e., to an object in $Z(S^1)$. This is in agreement with the findings in the literature on extended 3-dimensional TQFTs, where boundary decorations for a 2-dimensional surface Σ with boundary components are objects in the modular tensor category the TQFT associates to S^1 [5].

8. From Boundary Conditions to Anomalous TQFTs

As mentioned in the previous section, there is a close relation between boundary conditions for invertible TQFTs and anomalous TQFTs. In the present section we will exploit this relation in detail.

Let $\tilde{Z}$ be a boundary condition for an $(n+1)$-dimensional invertible TQFT Z extended up to codimension $k+1$ with moduli level 0. In other words, we have the following commutative diagram

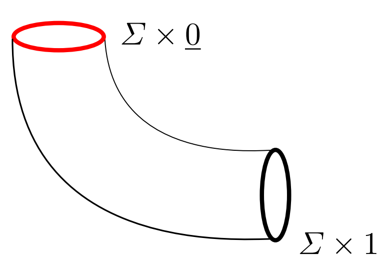
$$\begin{tikzcd} \mathrm{Cob}^{\partial,\infty}_{k+1}(n+1) \arrow[r, "\tilde{Z}"] & (k+1)\text{-Vect} \\ \mathrm{Cob}^{\infty}_{k+1}(n+1) \arrow[u, "i"] \arrow[r, "Z"] & \mathrm{Pic}((k+1)\text{-Vect}) \arrow[u] \end{tikzcd} \tag{59}$$

As mentioned in Remark 11, the restriction of Z to $\mathrm{Cob}_k(n) \hookrightarrow \mathrm{Cob}_{k+1}(n+1)$ is an n-dimensional anomaly, which we will denote W^Z.

Let $[\underline{0}, 1]$ denote the oriented interval $[0, 1]$ with $\{0\}$ being a constrained component of the boundary, as in the figure in Example 17, on the left. Then for any m-morphism Σ in $\mathrm{Cob}_k(n)$, with $k \geq 0$, i.e. for any $(n-k+m)$-dimensional manifold Σ, possibly with boundary, the product manifold $\Sigma \times [\underline{0}, 1]$ can be seen as a $(m+1)$-morphism from $\emptyset$ to Σ in $\mathrm{Cob}^{\partial,\infty}_{k+1}(n+1)$:

$$\emptyset \xrightarrow{\Sigma\times[\underline{0},1]} \Sigma, \tag{60}$$

We can graphically depict the morphism above as follows



Moreover, given an $(n-k+m+1)$-cobordism M between Σ and Σ', we have that the coloured manifold $M \times [\underline{0}, 1]$ induces a cobordism between $\Sigma \times [\underline{0}, 1]$ and $\Sigma' \times [\underline{0}, 1]$.

Evaluating $\tilde{Z}$ on $\Sigma \times [\underline{0}, 1]$ gives us a $(m+1)$-morphism in $(k+1)$-Vect between the unit (in the correct degree) and $\tilde{Z}(\Sigma) = Z(\Sigma) = W^Z(\Sigma)$.

Recall that a $(k+1)$-morphism in $\mathrm{Cob}_k(n)$ is a diffeomorphism $\varphi\colon \Sigma_1 \to \Sigma_2$ of n-dimensional manifolds fixing the boundaries. By combining it with the identity of $[\underline{0}, 1]$, one gets a diffeomorphism of $(n+1)$-dimensional manifolds, which realizes a $(k+2)$-morphism in $\mathrm{Cob}_{k+1}^{\partial,\infty}(n+1)$ between the empty set and the mapping cylinder of φ. Applying $\tilde{Z}$ we get a morphism from the unit to $\tilde{Z}(M_\varphi) = Z(M_\varphi) = W^Z(\varphi)$. This pattern continues with no changes to isotopies between diffeomorphisms, isotopies between isotopies, etc. Hence we have that

$$\tilde{Z}_{W^Z} := \tilde{Z}(- \times [\underline{0}, 1]) \tag{61}$$

defines a morphism $\tilde{Z}_{W^Z} : \underline{1} \to W^Z$, i.e. an anomalous TQFT in the sense of Definition 8.

We can assemble the argument above in the following

Proposition 2. *Let Z be a $(n+1)$-dimensional invertible TQFT extended down to codimension $k+1$ with moduli level 0, and let W^Z denote the $n+1$-dimensional anomaly induced by Z. Then any boundary condition $\tilde{Z}$ for Z induces an n-dimensional anomalous TQFT $\tilde{Z}_{W^Z}$ with anomaly W^Z.*

The above argument shows that we have a "forgetful map"

$$\{\text{boundary conditions on invertible TQFTs}\} \rightsquigarrow \{\text{anomalous TQFTs}\} \tag{62}$$

In general, we do not expect the converse to hold. Namely, an anomalous TQFT with anomaly W contains too little information to determine a boundary condition $\tilde{Z}$. Nevertheless, in the case of fully extended TQFTs the situation is rather different.

Remark 26. The procedure of taking "cartesian products" with the constrained interval can be seen as a form of dimensional reduction for manifolds with boundaries. It is completely analogous to dimensional reduction over S^1, which allows to obtain a $n-1$-dimensional extended TQFT from an n-dimensional one, preserving the tiers of extension.

8.1. Boundary conditions for fully extended TQFTs. For simplicity, in the following we will consider the framed case. Let Z be a $(n+1)$-dimensional fully extended invertible TQFT, namely an ∞-functor $Z : \mathrm{Bord}^{fr}(n+1) \to (n+1)$-Vect which factors through $\mathrm{Pic}((n+1)\text{-Vect})$. As mentioned in Remark 11, from Z we obtain an n-character W^Z. Let Z_{W^Z} be an anomalous TQFT with anomaly W^Z, namely a morphism $\underline{1} \to W^Z$, which contains in particular the datum of a 1-morphism

$$n\text{-Vect} \to W^Z(\mathrm{pt}^+) = Z(\mathrm{pt}^+) \tag{63}$$

By Theorem 2, we have then that Z_W induces a boundary condition $\tilde{Z}$ of Z, and an equivalence

$$Z_{W^Z} \simeq \tilde{Z}_{W^Z} \tag{64}$$

of 1-morphisms $\underline{1} \to W^Z$, where $\tilde{Z}_{W^Z}$ is the anomalous TQFT as from Proposition 2. This argument can be assembled in the following

Theorem 3. *Let Z be a fully extended invertible $(n+1)$-dimensional TQFT. Any n-dimensional anomalous TQFT Z_{W^Z} with respect to W^Z gives rise to a boundary condition $\tilde{Z}$ of Z.*

Hence in the fully extended case, an anomalous TQFT with respect to an anomaly obtained by restriction of a higher dimensional TQFT Z contains enough information to allow Z to be extended on manifolds with boundaries.

We conclude this section with an observation we find intriguing. In [20] a 4-category with duals $\mathrm{Braid}^{\otimes}$ of braided tensor categories has been introduced, as follows:

– objects are given by braided tensor categories $\mathcal{C}$;
– 1-morphisms between $\mathcal{C}$ and $\mathcal{D}$ are pairs $(\mathcal{A}, q)$, with $\mathcal{A}$ a fusion category, and q a braided functor $\mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{D} \to \mathcal{Z}(\mathcal{A})$, where $\mathcal{Z}(\mathcal{A})$ is the Drinfel'd centre of $\mathcal{A}$;
– 2-morphisms are $\mathcal{A}$-$\mathcal{B}$ bimodules M;
– 3-morphisms are bimodule functors;
– 4-morphisms are bimodule natural transformations;

Recently [19], the invertible objects in $\mathrm{Braid}^{\otimes}$ have been investigated: they are exactly the modular tensor categories. They are also fully dualizable. Let then $\mathcal{C}$ be a modular tensor category, and consider the invertible fully extended 4-dimensional TQFT Z induced by $\mathcal{C}$. Also, let $(\mathcal{A}, q)$ be a 1-morphism from Vect (i.e., from the monoidal unit of $\mathrm{Braid}^{\otimes}$) to $\mathcal{C}$, i.e., let q be a braided functor $\mathcal{C} \cong \mathrm{Vect}^{\mathrm{op}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{A})$ for some fusion category $\mathcal{A}$. By the results above[7], to $(\mathcal{A}, q)$ there corresponds a boundary condition $\tilde{Z}$ of Z, and consequently a fully extended 3-dimensional anomalous theory with respect to W^Z with values in $\Omega\mathrm{Braid}^{\otimes}$. We will denote with $Z^{(\mathcal{A},q)}$ this anomalous theory. Notice that if we apply the loop operator to the morphism $Z^{(\mathcal{A},q)}$ we obtain a 3-dimensional anomalous TQFT extended up to codimension 2 with values in $\Omega^2\mathrm{Braid}^{\otimes} \simeq$ 2-Vect.

On the other hand, given a modular tensor category $\mathcal{C}$, the Reshetikhin–Turaev construction also produces an anomalous 3-dimensional TQFT extended up to codimension 2, which we denote by $Z^{\mathrm{RT}}_{\mathcal{C}}$. It is very tempting then to state the following

Conjecture 2. *Let $\mathcal{C}$ be a modular tensor category. Then, any isomorphism $(\mathcal{A}, q)$ between* Vect *and $\mathcal{C}$ in* $\mathrm{Braid}^{\otimes}$*, i.e., any equivalence $q : \mathcal{C} \to \mathcal{Z}(\mathcal{A})$, induces a natural equivalence*

$$Z^{\mathrm{RT}}_{\mathcal{C}} \simeq \Omega(Z^{(\mathcal{A},q)}). \tag{65}$$

The conjecture above is compatible with findings in [22], which studies obstructions to the existence of boundary conditions for Reshetikhin–Turaev TQFTs.

Remark 27. In Conjecture 2, Reshetikhin–Turaev TQFT is regarded as an anomalous theory with respect to the 4-dimensional Crane–Yetter theory, i.e. a natural transformation of (higher) functors, rather than a functor on a central extension of $\mathrm{Cob}_2^{or}(3)$. In other words, we trade the additional structures on 1-, 2-, and 3-manifolds needed to

[7] In the main body of the paper we have been considering only n-Vect as a target for a TQFT. The constructions presented there generalise to an arbitrary symmetric monoidal (∞, n)-category with duals $\mathcal{C}$ as a target, see [35]. More precisely, when $\mathcal{C}$ takes the role of n-Vect, then $\Omega\mathcal{C}$ takes the role of $(n-1)$-Vect, and so on, down to $\Omega^n\mathcal{C}$ taking the role of the base field $\mathbb{K}$. In particular, it is meaningful to have the symmetric monoidal 4-category $\mathrm{Braid}^{\otimes}$ as a target, as we are doing here.

define Reshetikhin–Turaev TQFT as functors, as for instance in [46,48], with looking at them as natural transformations.

8.2. Further applications and outlook. An interesting playground to test and apply the language and results developed in this article is provided by the quantisation of classical Lagrangian field theories, as in [20,38,39]. In this case the TQFT is obtained via a linearisation of the (higher) stack of classical fields over ∞-categories of groupoid correspondences: we expect therefore the anomalous theory to retain some "classical" properties concerning the anomaly. A particularly amenable situation is given by (higher) Dijkgraaf–Witten theories: indeed, in this case we expect to reproduce the results obtained in [23] in 3-dimensions, which would provide a purely quantum field theoretic support to the ansatz therein proposed.

On a closely related topic, we remark that there is a version of the cobordism hypothesis to incorporate *defects* between fully extended TQFTs. Indeed, a boundary condition for Z as presented in this article can be regarded as a defect between the trivial theory and Z. One can then investigate morphisms between two arbitrary n-dimensional TQFTs of moduli level m, with $m > 0$: we expect the structure involved in this case to be richer than the case $m = 0$, where the $(\infty, n-1)$-category of morphisms forms a groupoid.

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CENTRAL EXTENSIONS OF MAPPING CLASS GROUPS FROM CHARACTERISTIC CLASSES

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Abstract. We consider higher extensions of diffeomorphism groups and show how these naturally arise as the group stacks of automorphisms of manifolds that are equipped with higher degree topological structures, such as those appearing in topological field theories. Passing to the groups of connected components, we obtain abelian extensions of mapping class groups and investigate when they are central. As a special case, we obtain in a natural way the $\mathbb{Z}$-central extension needed for the anomaly cancellation of 3d Chern-Simons theory.

"Everything in its right place"
Kid A, Radiohead

Contents

1. Introduction

In higher (stacky) geometry, there is a general and fundamental class of higher (stacky) group extensions:The authors would like to thank Oscar Randal-Williams and Chris Schommer-Pries for useful comments" for $\psi : Y \to B$ any morphism between higher stacks, the automorphism group stack of Y over B extends the automorphisms of Y itself by the loop object of the mapping stack $[Y, B]$ based at ψ. This is not hard to prove [Sc13], but as a general abstract fact it has many non-trivial incarnations. In [FRS13] it is shown how for B a universal moduli stack for ordinary differential cohomology, these extensions generalize the Heisenberg-Kirillov-Kostant-Souriau-extension from prequantum line bundles to higher "prequantum gerbes" which appear in the local (or "extended") geometric quantization of higher dimensional field theories. Here we consider a class of examples at the other extreme: we consider the case in which B is geometrically discrete (i.e., it is a locally constant ∞-stack), and particularly the case that B is the homotopy type of the classifying space of the general linear group. In this special case, due to the fact that geometric realization of smooth ∞-stacks happens to preserve homotopy fibers over geometrically discrete objects [Sc13], the general extension theorem essentially passes along geometric realization. Hence, where the internal extension theorem gives extensions of smooth diffeomorphism groups by higher homotopy types, after geometric realization we obtain higher extensions of the homotopy type of diffeomorphism groups, and in particular of mapping class groups.

A key application where extensions of the mapping class group traditionally play a role is anomaly cancellation in 3-dimensional topological field theories, e.g., in 3d Chern-Simons theory, see, e.g., [Wi89]. The results presented here naturally generalize this to higher extensions relevant for higher dimensional topological quantum field theories (TQFTs). More precisely, by functoriality, a 3d TQFT associates to any connected oriented surface Σ a vector space V_Σ which is a linear representation of the oriented mapping class group $\Gamma^{or}(\Sigma)$ of Σ. However, if the 3d theory has an "anomaly", then the vector space V_Σ fails to be a genuine representation of $\Gamma^{or}(\Sigma)$, and it rather is only a projective representation. One way to think of this phenomenon is to look at anomalous theories as relative theories, that intertwine between the trivial theory and an invertible theory, namely the anomaly. See, e.g. [FT12, FV14]. In particular, for an anomalous TQFT of the type obtained from modular tensor categories with nontrivial central charge [Tu94, BK01], the vector space V_Σ can be naturally realised as a genuine representation of a $\mathbb{Z}$-central extension

$$0 \to \mathbb{Z} \to \widehat{\Gamma(\Sigma)} \to \Gamma(\Sigma) \to 1 \tag{1}$$

of the mapping class group $\Gamma(\Sigma)$. As suggested in Segal's celebrated paper on conformal field theory [Se04], these data admit an interpretation as a genuine functor where one replaces 2-dimensional and 3-dimensional manifolds by suitable "enriched" counterparts, in such a way that the automorphism group of an enriched connected surface is the relevant $\mathbb{Z}$-central extension of the mapping class group of the underlying surface. Moreover, the set of (equivalence classes of) extensions of a 3-manifold with prescribed (connected) boundary behaviour is naturally a $\mathbb{Z}$-torsor. In [Se04] the extension consists in a "rigging" of the 3-manifold, a solution which is not particularly simple, and which is actually quite ad hoc for the 3-dimensional case. Namely, riggings are based on the contractibility of Teichmüller spaces, and depend on the properties of the η-invariant for Riemannian metrics on 3-manifolds with boundary. On the other hand, in [Se04] it is suggested that simpler variants of this construction should exist, the *leitmotiv* being that of associating functorially to any connected surface a space with fundamental group $\mathbb{Z}$. Indeed, there is a well known realization of extended surfaces as surfaces equipped with a choice of a Lagrangian subspace in their first real cohomology group. This is the point of view adopted, e.g., in [BK01]. The main problem with this approach is the question of how to define a corresponding notion for an extended 3-manifold.

In the present work we show how a natural way of defining enrichments of 2-and-3-manifolds, which are topological (or better homotopical) in nature, and in particular do not rely on special features of the dimensions 2 and 3. Moreover, they have the advantage of being immediately adapted to a general TQFT framework. Namely, we consider enriched manifolds as (X, ξ)-framed manifolds in the sense of [Lu09]. In this way, we in particular recover the fact that the simple and natural notion of p_1-structure, i.e. a trivialization of the first Pontryagin class, provides a very simple realization of Segal's prescription by showing how it naturally drops out as a special case of the "higher modularity" encoded in the (∞, n)-category of framed cobordisms.

Finally, if one is interested in higher dimensional Chern-Simons theories, the notable next case being 7-dimensional Chern-Simons theory [FSaS12], then the above discussion gives general means for determining and constructing the relevant higher extensions of diffeomorphism groups of higher dimensional manifolds. More on this is going to be discussed elsewhere.

The present paper is organised as follows. In section 2 we discuss the ambient homotopy theory $\mathbf{H}^\infty$ of smooth higher stacks, and we discuss how smooth manifolds and homotopy actions of ∞-groups can be naturally regarded as objects in its slice ∞-category over the homotopy type $\mathscr{B}GL(n;\mathbb{R})$ of the mapping stack $\mathbf{B}GL(n;\mathbb{R})$ of principal $GL(n;\mathbb{R})$-bundles.

In section 3 we introduce the notion of a ρ-framing (or ρ-structure) over a smooth manifold, and study extensions of their automorphism ∞-group. We postpone the proof of the extension result to the Appendix.

In section 4 we discuss the particular but important case of ρ-structures arising from the homotopy fibers of morphisms of ∞-stacks, which leads to Proposition 4.1, the main result of the present paper. In this section we also consider the case of a manifold with boundaries.

In section 5, we apply the abstract machinery developed in the previous sections to the concrete case of the mapping class group usually encountered in relation to topological quantum field theories.

The Appendix contains a proof of the extension result in section 4.

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2. Framed manifolds

2.1. **From framed cobordism to (X,ξ)-manifolds.** The principal player in Lurie's formalization and proof of the cobordism hypothesis [Lu09] are the (∞,n)-categories of framed cobordisms. These framings come in various flavours, from literal n-framings, i.e., trivialisations of the (stabilized) tangent bundle to more general and exotic framings, which Lurie calls (X,ξ)-structures. Presumably to keep the note at the lowest possible technical level, Lurie avoids to say explicitly that he is working in a slice. However, this is what he is secretely doing, and the slice over $\mathscr{B}GL(n;\mathbb{R})$ is the unifying principle governing all the framings in [Lu09]. Here we make the role played by $\mathscr{B}GL(n;\mathbb{R})$ more explicit. This will allow us not only to see Lurie's framings from a unified perspective, but also to consider apparently more exotic (but actually completely natural) framings given by characteristic classes for the orthogonal group.

2.1.1. *Homotopies, homotopies, homotopies everywhere.* The natural ambient category where all the constructions presented in this note take place is an alternative enrichment $\mathbf{H}^\infty$ of the ∞-topos $\mathbf{H}$ of smooth higher stacks[1]. We will not go into the technicalities of higher toposes or higher smooth stacks in the present work: at any point where one might be unsure on what is precisely going on, mumbling several times the mantra "$\mathbf{B}G$ is a smooth stack" will make everything appear suddenly clear. The reader who is skeptical of the effectiveness of these transcendental methods will find a complete and fully rigorous treatment of the theory of higher smooth stacks in [Sc13]. Also the first sections of [FScS12] can serve as a friendly introduction to the subject. Also, a rigorous construction of $\mathbf{H}^\infty$ is beyond the aims of this note, and will be presented in detail elsewhere: here, we will content ourself with an informal description, which will suffice to motivate and justify the construction.

The reason we need to refine $\mathbf{H}$ is that $\mathbf{H}$ itself is too rigid (or, in other words, the homotopy type of its hom-spaces is too simple) for our aims. For instance, given two smooth manifolds Σ_1 and Σ_2, the ∞-groupoid $\mathbf{H}(\Sigma_1,\Sigma_2)$ is 0-truncated, i.e., it is just a set. Namely, $\mathbf{H}(\Sigma_1,\Sigma_2)$ is the set of smooth maps from Σ_1 and Σ_2 and there are no nontrivial morphisms between smooth maps in $\mathbf{H}(\Sigma_1,\Sigma_2)$. In other words, two smooth maps between Σ_1 and Σ_2 either are equal or they are different: in this hom-space there's no such thing as "a smooth map can be smoothly deformed into another smooth map", which however is a kind of relation that geometry naturally suggests. To take it into account, we make the topology (or, even better, the smooth structure) of Σ_1 and Σ_2 come into play, and we use it to informally define $\mathbf{H}^\infty(\Sigma_1,\Sigma_2)$ as the ∞-groupoid whose objects are smooth maps between Σ_1 and Σ_2, much as for $\mathbf{H}(\Sigma_1,\Sigma_2)$, but whose 1-morphism are the smooth homotopies between smooth maps, and we also have 2-morphisms given by homotopies between homotopies, 3-morphisms given by homotopies between homotopies between homotopies, and so on. A formal definition is

$$\mathbf{H}^\infty(\Sigma_1,\Sigma_2) := \Pi([\Sigma_1,\Sigma_2]) \tag{2}$$

where $[\,,\,]$ denotes the internal-hom in $\mathbf{H}$ and ΠX is the smooth Poincaré ∞-groupoid of X. Similarly we write $\mathbf{Aut}^\infty(\Sigma)$ for the sub-object of invertible objects in $\mathbf{H}^\infty(\Sigma,\Sigma)$.

Here is another example. For G a Lie group, we will write $\mathbf{B}G$ for the smooth stack of principal G-bundles. This means that for Σ a smooth manifold, a morphism $f\colon \Sigma\to\mathbf{B}G$ is precisely a G-principal bundle over Σ. So, in particular, $\mathbf{B}GL(n;\mathbb{R})$ is the smooth stack of principal $GL(n;\mathbb{R})$-bundles. Identifying a principal $GL(n;\mathbb{R})$-bundle with its associated rank n real vector bundle, $\mathbf{B}GL(n;\mathbb{R})$ is equivalently the smooth stack of rank n real vector bundles and their isomorphisms. In particular, a map $\Sigma\to\mathbf{B}GL(n;\mathbb{R})$ is precisely the datum of a rank n vector bundle on the smooth manifold Σ. Again, for a given smooth manifold Σ, the homotopy type of $\mathbf{H}(\Sigma,\mathbf{B}G)$ is too rigid for our aims: the ∞-groupoid $\mathbf{H}(\Sigma,\mathbf{B}G)$ is actually a 1-groupoid. This means that we have objects, which are the principal G-bundles over Σ, and 1-morphism between these objects, which are isomorphisms of principal G-bundles, and then nothing else: we do not have nontrivial morphisms between the morphisms, and there's no such a thing like "a morphism can be smoothly deformed into another morphism", which again is something very natural to consider from a geometric point of view. Making the smooth structure of the group G come into play we get the following description of the ∞-groupoid $\mathbf{H}^\infty(\Sigma,\mathbf{B}G)$: its objects are the principal G-bundles over Σ and its 1-morphism are the

[1]The construction presented here is possible since $\mathbf{H}$ is *cohesive* as an ∞-topos: this guarantees that the ∞-functor Π from $\mathbf{H}$ to ∞-groupoids does indeed exist, and preserves products. Notice that the ordinary enrichment of $\mathbf{H}$ is instead given by $\mathbf{H}(\Sigma_1,\Sigma_2)=\flat([\Sigma_1,\Sigma_2])$, where $\flat$ is the right adjoint to Π. See [Sc13] for details.

isomorphisms of principal G-bundles, much as for $\mathbf{H}(\Sigma, \mathbf{B}G)$, but then we have also 2-morphisms given by isotopies between isomorphisms, 3-morphisms given by isotopies between isotopies, and so on. Notice that we have a canonical ∞-functor[2]

$$\mathbf{H}(\Sigma, \mathbf{B}G) \longrightarrow \mathbf{H}^{\infty}(\Sigma, \mathbf{B}G). \tag{3}$$

This is nothing but saying that for $j \geq 2$, the j-morphisms in $\mathbf{H}(\Sigma, \mathbf{B}G)$ are indeed very special j-morphisms in $\mathbf{H}^{\infty}(\Sigma, \mathbf{B}G)$, namely the identities. Moreover, when G happens to be a discrete group, this embedding is actually an equivalence of ∞-groupoids.

2.2. **Geometrically discrete ∞-stacks and the homotopy type $\mathscr{B}GL(n)$.** The following notion will be of great relevance for the results of this note. We have an inclusion

$$\mathrm{LConst} : \infty\mathrm{Grpd} \to \mathbf{H} \tag{4}$$

given by regarding an ∞-groupoid $\mathcal{G}$ as a constant presheaf over Cartesian spaces. We will say that an object in $\mathbf{H}$ is a *geometrically discrete* ∞-stack if it belongs to the essential image of LConst. An example of a geometrically discrete object in $\mathbf{H}$ is given by the 1-stack $\mathbf{B}G$, with G a discrete group. More generally, for A an abelian discrete group the (higher) stacks $\mathbf{B}^n A$ of principal A-n-bundles are geometrically discrete. The importance of considering geometrically discrete ∞-stacks is that the functor Π introduced before is left adjoint to LConst. In particular we have a canonical counit morphism

$$\mathrm{id}_{\mathbf{H}} \to \mathrm{LConst} \circ \Pi \tag{5}$$

which is the canonical morphism from a smooth stack to its homotopy type (and which corresponds to looking at points of a smooth manifold Σ as constant paths into Σ). In particular, for G a group, we will write $\mathscr{B}G$ for the homotopy type of $\mathbf{B}G$, i.e., we set $\mathscr{B}G := \mathrm{LConst}(\Pi(\mathbf{B}G))$. (Notice that since LConst is a fully faithful inclusion, there is no harm in suppressing it notationally, which we will freely do.) This is equivalently the traditional classifying space for the group G (or rather of its principal bundles). The counit then becomes a canonical morphism

$$\mathbf{B}G \to \mathscr{B}G, \tag{6}$$

which is an equivalence for a discrete group G. This tells us in particular that any object over $\mathbf{B}G$ is naturally also an object over $\mathscr{B}G$. For instance (and this example will be the most relevant for what follows), a choice of a rank n vector bundle over a smooth manifold Σ realises Σ as an object over $\mathscr{B}GL(n;\mathbb{R})$.
Notice how we have a canonical morphism

$$\mathbf{H}^{\infty}(\Sigma, \mathbf{B}G) \longrightarrow \mathbf{H}^{\infty}(\Sigma, \mathscr{B}G) \tag{7}$$

obtained by composing the canonical morphism $\mathbf{H}(\Sigma, \mathbf{B}G) \to \mathbf{H}^{\infty}(\Sigma, \mathbf{B}G)$ mentioned in the previous section with the push forward morphism $\mathbf{H}^{\infty}(\Sigma, \mathbf{B}G) \to \mathbf{H}^{\infty}(\Sigma, \mathscr{B}G)$, The main reason to focus on geometrically discrete stacks is that, though Π preserves finite products, it does *not* in general preserve homotopy pullbacks. Neverthless, Π does indeed preserve homotopy pullbacks of diagrams whose tip is a geometrically discrete object in $\mathbf{H}$ [Sc13].

2.2.1. *Working in the slice.* Let now n be a fixed nonnegative integer and let $0 \leq k \leq n$. Any k-dimensional smooth manifold M_k comes canonically equipped with a rank n real vector bundle given by the stabilized tangent bundle $T^{\mathrm{st}}M_k = TM_k \oplus \underline{\mathbb{R}}^{n-k}_{M_k}$, where $\underline{\mathbb{R}}^{n-k}_{M_k}$ denotes the trivial rank $(n-k)$ real vector bundle over M_k. We can think of the stabilised tangent bundle[3] as a morphism

$$M_k \xrightarrow{T^{\mathrm{st}}} \mathscr{B}GL(n) \tag{8}$$

where $GL(n)$, as in the following, denotes $GL(n;\mathbb{R})$.
Namely, we can regard any smooth manifold of dimension at most n as an object *over* $\mathscr{B}GL(n)$. This suggests that a natural setting to work in is the slice topos $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$, which in the following we will refer

[2] In terms of cohesion this is a component of the canonical points-to-pieces-transform $\flat[\Sigma, \mathbf{B}G] \to [\Sigma, \mathbf{B}G] \to \Pi[\Sigma, \mathbf{B}G]$.
[3] To be precise, T^{st} is the map of stacks induced by the frame bundle of the stabilised tangent bundle to M_k.

to simply as "the slice": in other words, all objects involved will be equipped with morphisms to $\mathscr{B}GL(n)$, and a morphism between $X \xrightarrow{\varphi} \mathscr{B}GL(n)$ and $Y \xrightarrow{\psi} \mathscr{B}GL(n)$ will be a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\swarrow\eta"] & Y \arrow[dl, "\psi"] \\ & \mathscr{B}GL(n) & \end{tikzcd} . \tag{9}$$

More explicitly, if we denote by E_φ and E_ψ the rank n real vector bundles over X and Y corresponding to the morphisms φ and ψ, respectively, then we see that a morphism in the slice between $X \xrightarrow{\varphi} \mathscr{B}GL(n)$ and $Y \xrightarrow{\psi} \mathscr{B}GL(n)$ is precisely the datum of a morphism $f\colon X \to Y$ together with an *isomorphism* of vector bundles over X,

$$\eta : f^*E_\psi \xrightarrow{\simeq} E_\varphi. \tag{10}$$

Notice that these are precisely the same objects and morphisms as if we were working in the slice over $\mathbf{B}GL(n)$ in $\mathbf{H}$. Neverthless, as we will see in the following sections, where the use of $\mathbf{H}^\infty$ makes a difference is precisely in allowing nontrivial higher morphisms. Also, the use of the homotopy type $\mathscr{B}GL(n)$ in place of the smooth stack $\mathbf{B}GL(n)$ will allow us to make all constructions work "up to homotopy", and to identify, for instance, $\mathscr{B}GL(n)$ with $\mathscr{B}O(n)$.

Example 2.1. The inclusion of the trivial group into $GL(n)$ induces a natural morphism $* \to \mathscr{B}GL(n)$, corresponding to the choice of the trivial bundle. If M_k is a k-dimensional manifold, then a morphism

$$\begin{tikzcd} M_k \arrow[rr] \arrow[dr, "T^{\mathrm{st}}"'] & \arrow[d, phantom, "\swarrow\eta"] & * \arrow[dl] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{11}$$

is precisely a trivialisation of the stabilised tangent bundle of M_k, i.e., an n-framing of M.

Example 2.2. Let X be a smooth manifold, and let ζ be a rank n real vector bundle over X, which we can think of as a morphism $\rho_\zeta\colon X \to \mathscr{B}GL(n)$. Then a morphism

$$\begin{tikzcd} M_k \arrow[rr, "f"] \arrow[dr, "T^{\mathrm{st}}"'] & \arrow[d, phantom, "\swarrow\eta"] & X \arrow[dl, "\rho_\zeta"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{12}$$

is precisely the datum of a smooth map $f\colon M_k \to X$ and of an isomorphism $\eta\colon f^*\zeta \to TM \oplus \underline{\mathbb{R}}^{n-k}_{M_k}$. These are the data endowing M_k with a (X,ζ)-structure in the terminology of [Lu09].

The examples above suggest to allow X to be not only a smooth manifold, but a smooth ∞-stack. While choosing such a general target (X,ζ) could at first seem like a major abstraction, this is actually what one commonly encounters in everyday mathematics. For instance a lift through $\mathbf{B}O(n) \to \mathbf{B}GL(n)$ is precisely a (n-stable) Riemannian structure. Generally, for $G \hookrightarrow GL(n)$ any inclusion of Lie groups, or even more generally for $G \to GL(n)$ any morphism of Lie groups, then a lift through $\mathbf{B}G \to \mathbf{B}GL(n)$ is a (n-stable) *G-structure*, e.g., an almost symplectic structure, an almost complex structure, etc. (one may also phrase integrable G-structures in terms of slicing, using more of the axioms of cohesion than we need here). For instance, the inclusion of the connected component of the identity $GL^+(n) \hookrightarrow GL(n)$ corresponds to a morphism of higher stacks $\iota\colon \mathbf{B}GL^+(n) \to \mathbf{B}GL(n)$, and a morphism in the slice from (M_k, T^{st}) to $(\mathbf{B}GL^+(n), \iota)$ is precisely the choice of a (stabilised) orientation on M_k. For G a higher connected cover of $O(n)$ then lifts through $\mathbf{B}G \to \mathbf{B}O(n) \to \mathbf{B}GL(n)$ correspond to spin structures, string structures, etc.
On the other hand, since $\mathscr{B}O(n) \to \mathscr{B}GL(n)$ is an equivalence, a lift through $\mathscr{B}O(n) \to \mathscr{B}GL(n)$ is no additional structure on a smooth manifold M_k, and the stabilized tangent bundle of M_k can be equally seen as a morphism to $\mathscr{B}O(n)$. Similarly, for $G \to GL(n)$ any morphism of Lie groups, lifts of T^{st} through $\mathscr{B}G \to \mathscr{B}GL(n)$ correspond to (n-stable) *topological* G-structures.

2.3. **From homotopy group actions to objects in the slice.** We will mainly be interested in objects of $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$ obtained as a homotopy group action of a smooth (higher) group G on some stack X, when G is equipped with a ∞-group morphism to $GL(n)$. We consider then the following

Definition 2.3. *A homotopy action of a smooth ∞-group G on X is the datum of a smooth ∞-stack $X//_h G$ together with a morphism $\rho\colon X//_h G \to \mathscr{B}G$ satisfying the following homotopy pullback diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X//_h G \arrow[d, "\rho"'] \\ * \arrow[r] & \mathscr{B}G \end{tikzcd} \tag{13}$$

Unwinding the definition, one sees that a homotopy action of G is nothing but an action of the homotopy type of G and that $X//_h G$ is realised as the stack quotient $X//\Pi(G)$. See [NSS12a] for details. Since G is equipped with a smooth group morphism to $GL(n)$, and since this induces a morphism of smooth stacks $\mathscr{B}G \to \mathscr{B}GL(n)$, the stack $X//_h G$ is naturally an object over $\mathscr{B}GL(n)$. In particular, when X is a deloopable object, i.e., when there exists a stack Y such that $\Omega Y \cong X$, then one obtains a homotopy G-action out of any morphism $c\colon \mathscr{B}G \to Y$. Indeed, in this situation one can define $X//_h G \to \mathscr{B}G$ by the homotopy pullback

$$\begin{tikzcd} X//_h G \arrow[r] \arrow[d, "\rho_c"'] & * \arrow[d] \\ \mathscr{B}G \arrow[r, "c"] & Y \end{tikzcd} \tag{14}$$

By using the pasting law for homotopy pullbacks, we can see that X, $X//_h G$, and the morphism ρ_c fit in a homotopy pullback diagram as in (13).

Example 2.4. Let c be a degree $d+1$ characteristic class for the group $SO(n)$. Then c can be seen as the datum of a morphism of stacks $c\colon \mathscr{B}SO(n) \to \mathscr{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d+1}\mathbb{Z}$, where $\mathbf{B}^{d+1}\mathbb{Z}$ is the smooth stack associated by the Dold-Kan correspondence to the chain complex with $\mathbb{Z}$ concentrated in degree $d+1$, i.e., the stack (homotopically) representing degree $d+1$ integral cohomology. Notice how the discreteness of the abelian group $\mathbb{Z}$ came into play to give the equivalence $\mathscr{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d+1}\mathbb{Z}$. Since we have $\Omega\mathbf{B}^{d+1}\mathbb{Z} \cong \mathbf{B}^{d}\mathbb{Z}$, the characteristic class c defines a homotopy action

$$\rho_c\colon \mathbf{B}^d\mathbb{Z}//_h SO(n) \to \mathscr{B}SO(n) \tag{15}$$

and so it realises $\mathbf{B}^d\mathbb{Z}//_h SO(n)$ as an object in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$. For instance, the first Pontryagin class p_1 induces a homotopy action

$$\rho_{p_1}\colon \mathbf{B}^3\mathbb{Z}//_h SO(n) \to \mathscr{B}SO(n). \tag{16}$$

3. ρ-framed manifolds and their automorphisms ∞-group

We can now introduce the main definition in the present work.

Definition 3.1. *Let M be a k-dimensional manifold, and let $\rho\colon X \to \mathscr{B}GL(n)$ be a morphisms of smooth ∞-stacks, with $k \le n$. Then a* ρ-framing *(or* ρ-structure*) on M is a lift of the stabilised tangent bundle seen as a morphism $T^{\mathrm{st}}\colon M \to \mathscr{B}GL(n)$ to a morphism $\sigma\colon M \to X$, namely a homotopy commutative diagram of the form*

$$\begin{tikzcd} M \arrow[rr, "\sigma"] \arrow[dr, "T^{\mathrm{st}}"'] & \arrow[d, phantom, "\Swarrow\eta"] & X \arrow[dl, "\rho"] \\ & \mathscr{B}GL(n) & \end{tikzcd} \tag{17}$$

By abuse of notation, we will often say that the morphism σ is the ρ-framing, omitting the explicit reference to the homotopy η, which is, however, always part of the data of a ρ-framing.

Since the morphism $\rho\colon X \to \mathscr{B}GL(n)$ is an object in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$, we can consider the slice over ρ:

$(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}$. Although this double slice may seem insanely abstract at first, it is something very natural. Its objects are homotopy commutative diagrams, namely 2-simplices

$$
\begin{tikzcd}
Y \arrow[rr, "a"] \arrow[dr, "\tilde{\rho}"'] & \arrow[d, phantom, "\Swarrow\eta"] & X \arrow[dl, "\rho"] \\
& \mathscr{B}GL(n) &
\end{tikzcd}
\tag{18}
$$

while its morphisms are homotopy commutative 3-simplices

$$
\begin{tikzcd}
Y \arrow[rr, "a"] \arrow[dr, "f"] \arrow[ddr, "\tilde{\rho}"'] & & X \arrow[ddl, "\rho"] \\
& Z \arrow[ur, "b"] \arrow[d, "\hat{\rho}"] & \\
& \mathscr{B}GL(n) &
\end{tikzcd}
\tag{19}
$$

where for readability we have omitted the homotopies decorating the faces and the interior of the 3-simplex, and similarly, additional data must be provided for higher morphisms.

In particular we see that a ρ-framing σ on M is naturally an object in the double slice $(\mathbf{H}_{/\mathscr{B}GL(n)})_{/\rho}$. Moreover, the collection of all k-dimensional ρ-framed manifolds has a natural ∞-groupoid structure which is compatible with the forgetting of the framing, and with the fact that any ρ-framed manifold is in particular an object in the double slice $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}$. More precisely, let $\mathscr{M}_k$ denote the ∞-groupoid whose objects are k-dimensional smooth manifolds, whose 1-morphisms are diffeomorphisms of k-dimensional manifolds whose 2-morphisms are isotopies of diffeomorphisms, and so on[4]. There is then an ∞-groupoid $\mathscr{M}_k^\rho$ of ρ-framed k-dimensional manifolds which is a ∞-subcategory of $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}$, and comes equipped with a forgetful ∞-functor

$$\mathscr{M}_k^\rho \to \mathscr{M}_k. \tag{20}$$

Namely, since the differential of a diffeomorphism between k-dimensional manifolds M and N can naturally be seen as an invertible 1-morphism between M and N as objects over $\mathbf{B}GL(n)$, we have a natural (not full) embedding

$$\mathscr{M}_k \hookrightarrow \mathbf{H}^\infty_{/\mathscr{B}GL(n)}. \tag{21}$$

Consider then the forgetful functor

$$(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho} \to \mathbf{H}^\infty_{/\mathscr{B}GL(n)} \tag{22}$$

We have then the following important

Definition 3.2. *Let $\rho\colon X \to \mathscr{B}GL(n)$ be an object in $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$. The ∞-groupoid $\mathscr{M}_k^\rho$ is then defined as the homotopy pullback diagram*

$$
\begin{tikzcd}
\mathscr{M}_k^\rho \arrow[r] \arrow[d] & (\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho} \arrow[d] \\
\mathscr{M}_k \arrow[r] & \mathbf{H}^\infty_{/\mathscr{B}GL(n)}
\end{tikzcd}
\tag{23}
$$

[4]The ∞-groupoid $\mathscr{M}_k$ can be rigorously defined as $\Omega(\mathrm{Cob_t(k)})$, where $\mathrm{Cob_t(k)}$ is the $(\infty,1)$-category defined in [Lu09] in the context of topological field theory.

Given two ρ-framed k-dimensional manifolds (M,σ,η) and (N,τ,ϑ), the ∞-groupoid $\mathscr{M}_k^\rho((M,\sigma,\eta),(N,\tau,\vartheta))$ is the homotopy pullback

(24)
$$\begin{tikzcd}
\mathscr{M}_k^\rho((M,\sigma,\eta),(N,\tau,\vartheta)) \arrow[r] \arrow[d] & (\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}(\sigma,\tau) \arrow[d] \\
\mathscr{M}_k(M,N) \arrow[r] & \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M, T^{\mathrm{st}}_N)
\end{tikzcd}$$

In particular, if we denote with $\mathrm{Diff}(M)$ the ∞-groupoid of diffeomorphisms of M, namely the automorphism ∞-group of M as an object in $\mathscr{M}_k$, and we accordingly write $\mathrm{Diff}^\rho(M,\sigma)$ for the automorphisms ∞-group of (M,σ) as an object in $\mathscr{M}_k^\rho$, then we have a homotopy pullback

(25)
$$\begin{tikzcd}
\mathrm{Diff}^\rho(M,\sigma,\eta) \arrow[r] \arrow[d] & \mathbf{Aut}^\infty_{/\rho}(\sigma) \arrow[d] \\
\mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M)
\end{tikzcd}$$

where $\mathbf{Aut}^\infty_{(-)}(-)$ denotes the homotopy type of the relevant $\mathbf{H}$-internal automorphisms ∞-group. In particular, to abbreviate the notation, we will denote with $\mathbf{Aut}^\infty_\rho(\sigma)$ the automorphism ∞-group of σ in $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}$.
More explicitly, an element in $\mathrm{Diff}^\rho(M,\sigma,\eta)$ is a diffeomorphism $\varphi\colon M\to M$ together with an isomorphism $\alpha\colon \varphi^*\sigma \xrightarrow{\simeq} \sigma$, and a filler β for the 3-simplex

(26)
$$\begin{tikzcd}
M \arrow[rrrr, "\sigma"] \arrow[drr, "\varphi"'] \arrow[ddrr, "T^{st}"'] & & & & X \arrow[ddll, "\rho"] \\
& & M \arrow[urr, "\sigma"'] \arrow[d, "T^{st}"] \arrow[u, phantom, "\alpha\, \Nearrow"] & & \\
& & \mathscr{B}GL(n) \arrow[ul, phantom, "d\varphi\, \Downarrow"] \arrow[ur, phantom, "\eta\, \Downarrow"'] & &
\end{tikzcd}$$

3.1. **Functoriality and homotopy invariance of $\mathscr{M}_k^\rho$.** In this section we will explore some of the properties of $\mathscr{M}_k^\rho$, which will be useful in the following.
It immediately follows from the definition that the forgetful functor $\mathscr{M}_k^\rho \to \mathscr{M}_k$ is a equivalence for $\rho\colon X\to \mathscr{B}GL(n)$ an equivalence in $\mathbf{H}^\infty(X,\mathscr{B}GL(n))$. In particular, if ρ is the identity morphism of $\mathscr{B}GL(n)$ and we write $\mathscr{M}_k^{GL(n)}$ for $\mathscr{M}_k^{\mathrm{id}_{\mathscr{B}GL(n)}}$ then we have $\mathscr{M}_k^{GL(n)} \cong \mathscr{M}_k$. Less trivially, if $X = \mathscr{B}O(n)$, and ρ is the natural morphism

(27)
$$\iota_{O(n)}\colon \mathscr{B}O(n) \to \mathscr{B}GL(n)$$

induced by the inclusion of $O(n)$ in $GL(n)$, then ρ is again an equivalence, and we get $\mathscr{M}_k^{O(n)} \cong \mathscr{M}_k$, where we have denoted $\mathscr{M}_k^{\iota_{O(n)}}$ with $\mathscr{M}_k^{O(n)}$.
More generally, if ρ and $\tilde\rho$ are equivalent objects in the slice $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$, then we have equivalent ∞-groupoids $\mathscr{M}_k^\rho$ and $\mathscr{M}_k^{\tilde\rho}$. For instance, the inclusion of $SO(n)$ into $GL(n)^+$ induces an equivalence between $\mathscr{B}SO(n)$ and $\mathscr{B}GL(n)^+$ over $\mathscr{B}GL(n)$, and so we have a natural equivalence $\mathscr{M}_k^{SO(n)} \cong \mathscr{M}_k^{GL(n)^+}$. Since the objects in the ∞-groupoid $\mathscr{M}_k^{GL(n)^+}$ are k-dimensional manifolds whose stabilised tangent bundle is equipped with a lift to an $SO(n)$-bundle, the objects of $\mathscr{M}_k^{GL(n)^+}$ are oriented k-manifolds. Moreover the pullback defining $\mathscr{M}_k^{GL(n)^+}$ precisely picks up oriented diffeomorphisms, hence the forgetful morphism $\mathscr{M}_k^{GL(n)^+} \to \mathscr{M}_k$ induces an equivalence between $\mathscr{M}_k^{GL(n)^+}$ and the ∞-groupoid $\mathscr{M}_k^{\mathrm{or}}$ of oriented k-dimensional manifolds with orientation preserving diffeomorphisms between them. As a consequence, one has a natural equivalence

(28)
$$\mathscr{M}_k^{SO(n)} \cong \mathscr{M}_k^{\mathrm{or}}$$

Let $\psi\colon \rho \to \tilde{\rho}$ be a morphism in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$ between $\rho\colon X \to \mathscr{B}GL(n)$ and $\tilde{\rho}\colon Y \to \mathscr{B}GL(n)$. Then one has an induced push-forward morphism

$$\psi_*\colon \mathscr{M}_k^{\rho} \to \mathscr{M}_k^{\tilde{\rho}}, \tag{29}$$

which (by (24), and using the pasting law) fits into the homotopy pullback diagram

$$\begin{tikzcd}
\mathscr{M}_k^{\rho} \arrow[r] \arrow[d, "\psi_*"'] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\rho} \arrow[d, "\Psi_*"] \\
\mathscr{M}_k^{\tilde{\rho}} \arrow[r] & (\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}
\end{tikzcd} \tag{30}$$

where Ψ_* denotes the base changing ∞-functor on the slice topos.
The homotopy equivalences illustrated above are particular cases of this functoriality: indeed, when ψ is invertible, then ψ_* is invertible as well (up to coherent homotopies, clearly).
Recall from Example 2.4 that for any characteristic class c of $SO(n)$ we obtain an object ρ_c in the slice $\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$. In this way we obtain natural morphisms $\mathscr{M}_k^{\rho_c} \to \mathscr{M}_k^{SO(n)}$. In particular, by considering the first Pontryagin class $p_1\colon \mathscr{B}SO(n) \to \mathbf{B}^4\mathbb{Z}$, we obtain a canonical morphism

$$\mathscr{M}_k^{\rho_{p_1}} \to \mathscr{M}_k^{\mathrm{or}}. \tag{31}$$

3.2. **Extensions of ρ-diffeomorphism groups.** We are now ready for the extension theorem, which is the main result of this note. Not to break the flow of the exposition, we will postpone the details of the proof to the Appendix.
Let

$$\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, "\rho"'] & \arrow[d, phantom, "\swarrow\Psi"] & Y \arrow[dl, "\tilde{\rho}"] \\
& \mathscr{B}GL(n) &
\end{tikzcd} \tag{32}$$

be a morphism in the slice over $\mathscr{B}GL(n)$, as at the end of the previous section, and let

$$\begin{tikzcd}
M \arrow[rr, "\tau"] \arrow[dr, "T_M^{\mathrm{st}}"'] & \arrow[d, phantom, "\swarrow T"] & Y \arrow[dl, "\tilde{\rho}"] \\
& \mathscr{B}GL(n) &
\end{tikzcd} \tag{33}$$

be a $\tilde{\rho}$-structure on M. Then, arguing as in Section 3, associated to any lift

$$\begin{tikzcd}
M \arrow[rr, "\sigma"] \arrow[dr, "\tau"] \arrow[ddr, "T^{st}"'] & \arrow[d, phantom, "\Leftarrow\alpha"] & X \arrow[dl, "\psi"'] \arrow[ddl, "\rho"] \\
\arrow[r, phantom, "\Downarrow T"] & Y \arrow[d, "\tilde{\rho}"] \arrow[r, phantom, "\Rightarrow\Psi"] & \\
& \mathscr{B}GL(n) &
\end{tikzcd} \tag{34}$$

(where we are not displaying the label Σ on the back face, nor the filler β of the 3-simplex) of T to a ρ-structure Σ on M, we have a homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\Sigma) \arrow[d, "\psi_*"] \\
\mathrm{Diff}^{\tilde{\rho}}(M,T) \arrow[r] & \mathbf{Aut}^{\infty}_{/\tilde{\rho}}(T)
\end{tikzcd} \tag{35}$$

By the pasting law for homotopy pullbacks and from the pasting of homotopy pullback diagrams we have the following homotopy diagram (see Appendix for the proof)

(36)
$$\begin{tikzcd}
\Omega_\beta(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T,\Psi) \arrow[r] \arrow[d] & \Omega_\Sigma\mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M,\rho) \arrow[r] \arrow[d] & \mathbf{Aut}^\infty_{/\rho}(\Sigma) \arrow[d, "\psi_*"] \\
* \arrow[r] & \Omega_T\mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M,\tilde\rho) \arrow[r] \arrow[d] & \mathbf{Aut}^\infty_{/\tilde\rho}(T) \arrow[d] \\
 & * \arrow[r] & \mathbf{Aut}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M)
\end{tikzcd}$$

We therefore obtain the homotopy pullback diagram

(37)
$$\begin{tikzcd}
\Omega_\beta(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T,\Psi) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M,\Sigma) \arrow[d, "\psi_*"] \\
* \arrow[r] & \mathrm{Diff}^{\tilde\rho}(M,T)
\end{tikzcd}$$

presenting $\mathrm{Diff}^\rho(M,\Sigma)$ as an extension of $\mathrm{Diff}^{\tilde\rho}(M,T)$ by the ∞-group $\Omega_\beta(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T,\Psi)$, i.e., by the loop space (at a given lift β) of the space $(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T,\Psi)$ of lifts of the $\tilde\rho$-structure T on M to a ρ-structure Σ. Now notice that, by the Kan condition, we have a natural homotopy equivalence

(38)
$$(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\tilde\rho}(T,\Psi) \cong \mathbf{H}^\infty_{/Y}(\tau,\psi).$$

Namely, since T and Ψ are fixed, the datum of the filler α is homotopically equvalent to the datum of the full 3-simplex, as T,Ψ and α together give the datum of the horn at the vertex Y. As a consequence we see that the space of lifts of the $\tilde\rho$-structure T to a ρ-structure Σ is homotopy equivalent to the space of lifts

(39)
$$\begin{tikzcd}
 & X \arrow[d, "\psi"] \\
M \arrow[ur, "\sigma"] \arrow[r, "\tau"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] & Y
\end{tikzcd}$$

of τ to a morphism $\sigma\colon M \to X$. We refer the the reader to the Appendix for a rigorous proof of equivalence (38).
The arguments above lead directly to

Proposition 3.3. *Let $\rho : X \to \mathscr{B}GL(n)$ and $\tilde\rho : Y \to \mathscr{B}GL(n)$ be morphisms of ∞-stacks, and let $(\psi,\Psi) : \rho \to \tilde\rho$ be a morphism in $\mathbf{H}^\infty_{/\mathscr{B}GL(n)}$. Let (M,T) be a $\tilde\rho$-framed manifold, and let Σ be a ρ-structure on M lifting T through (α,β). We have then the following homotopy pullback*

(40)
$$\begin{tikzcd}
\Omega_\alpha\mathbf{H}^\infty_{/Y}(\tau,\psi) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M,\Sigma) \arrow[d, "\psi_*"] \\
* \arrow[r] & \mathrm{Diff}^{\tilde\rho}(M,T)
\end{tikzcd}$$

Proof. Combine diagram (37) with equivalence (38), which preserves homotopy pullbacks. □

Remark 3.4. Proposition 3.3 gives a presentation of $\mathrm{Diff}^\rho(M,\Sigma)$ as an extension of $\mathrm{Diff}^{\tilde\rho}(M,T)$ by the ∞-group $\Omega_\alpha\mathbf{H}^\infty_{/Y}(\tau,\psi)$. Notice how, for (T,τ) the identity morphism, i.e.

(41)
$$\begin{tikzcd}
Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dr, "\tilde\rho"'] & \arrow[d, phantom, "\Swarrow\mathrm{Id}"] & Y \arrow[dl, "\tilde\rho"] \\
 & \mathscr{B}GL(n) &
\end{tikzcd}$$

the space $\mathbf{H}^{\infty}_{/Y}(\tau, \mathrm{id}_Y)$ is contractible since id_Y is the terminal object in the slice $\mathbf{H}^{\infty}_{/Y}$ and so one finds that the extension of $\mathrm{Diff}^{\tilde{\rho}}(M, T)$ is the trivial one in this case, as expected.

4. Lifting ρ-structures along homotopy fibres

In this section we will investigate a particularly simple and interesting case of the lifting procedure of ρ-structures, and of extensions of ρ-diffeomorphisms ∞-groups, namely the case when $\psi\colon X \to Y$ is the homotopy fibre in $\mathbf{H}^{\infty}$ of a morphism $c\colon Y \to Z$ from Y to some pointed stack Z.
In this case, by the universal property of the homotopy pullback, the space $\mathbf{H}^{\infty}_{/Y}(\tau, \psi)$ of lifts of the $\tilde{\rho}$-structure τ to a ρ-structure σ is given by the space of homotopies between the composite morphism $c \circ \tau$ and the trivial morphism $M \to Z$ given by the constant map on the marked point of Z:

(42)
$$\begin{tikzcd} M \arrow[dr, dashed, "\sigma"] \arrow[drr, bend left] \arrow[ddr, bend right, "\tau"'] & & \\ & X \arrow[r] \arrow[d, "\psi"'] & * \arrow[d] \\ & Y \arrow[r, "c"] & Z \end{tikzcd}$$

This fact has two important consequences:

- a lift σ of τ exists if and only if the class of $c \circ \tau$ in $\pi_0 \mathbf{H}^{\infty}(M, Z)$ is the trivial class (the class of the constant map on the marked point z of Z);
- when a lift exists, the space $\mathbf{H}^{\infty}_{/Y}(\tau, \psi)$ is a torsor for the ∞-group of self-homotopies of the constant map $M \to Z$, i.e., for the ∞-group object $\Omega \mathbf{H}^{\infty}(M, Z)$. In particular, as soon as $\mathbf{H}^{\infty}_{/Y}(\tau, \psi)$ is nonempty, any lift σ of τ induces an equivalence of ∞-groupoids $\mathbf{H}^{\infty}_{/Y}(\tau, \psi) \cong \Omega \mathbf{H}^{\infty}(M, Z)$ and so an equivalence

(43)
$$\Omega_{\alpha} \mathbf{H}^{\infty}_{/Y}(\tau, \psi) \cong \Omega^2 \mathbf{H}^{\infty}(M, Z).$$

Moreover, as soon as (Z, z) is a geometrically discrete pointed ∞-stack, we have $\Omega \mathbf{H}^{\infty}(M, Z) \cong \mathbf{H}^{\infty}(M, \Omega Z)$, where ΩZ denotes the loop space of Z in $\mathbf{H}$ at the distinguished point z. In other words, for a geometrically discrete ∞-stack Z, the loop space of Z in $\mathbf{H}$ also provides a loop space object for Z in $\mathbf{H}^{\infty}$. Namely, by definition of $\mathbf{H}^{\infty}$, showing that

(44)
$$\begin{tikzcd} \mathbf{H}^{\infty}(W, \Omega Z) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbf{H}^{\infty}(W, Z) \end{tikzcd}$$

is a homotopy pullback of ∞-groupoids for any ∞-stack W amounts to showing that

(45)
$$\begin{tikzcd} \Pi[W, \Omega Z] \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Pi[W, Z] \end{tikzcd}$$

is a homotopy pullback, and this in turn follows from the fact that $[W, -]$ preserves homotopy pullbacks and geometrical discreteness, and that Π preserves homotopy pullbacks along morphisms of geometrically discrete stacks [Sc13]. If the pointed stack (Z, z) is geometrically discrete, then so is the stack ΩZ (pointed at the constant loop at z), and so

(46)
$$\Omega^2 \mathbf{H}^{\infty}(M, Z) \cong \Omega \mathbf{H}^{\infty}(M, \Omega Z) \cong \mathbf{H}^{\infty}(M, \Omega^2 Z).$$

Therefore, we can assemble the general considerations of the previous section in the following

Proposition 4.1. *Let $\psi\colon X \to Y$ be the homotopy fibre of a morphisms of smooth ∞-stacks $Y \to Z$, where Z is pointed and geometrically discrete. For any $\tilde{\rho}$-structured manifold (M, τ), we have a sequence of natural homotopy pullbacks*

(47)
$$\begin{tikzcd}
\mathbf{H}^\infty(M, \Omega^2 Z) \arrow[r] \arrow[d] & \mathrm{Diff}^\rho(M, \sigma) \arrow[r] \arrow[d, "\psi_*"] & * \arrow[d] \\
* \arrow[r] & \mathrm{Diff}^{\tilde{\rho}}(M, \tau) \arrow[r] & \mathbf{H}^\infty(M, \Omega Z)
\end{tikzcd}$$

whenever a lift to of τ to a ρ-structure σ exists.

4.1. **The case of manifolds with boundary.** With an eye to topological quantum field theories, it is interesting to consider also the case of k-dimensional manifolds with boundary $(M, \partial M)$. Since the boundary ∂M comes with a collar in M, i.e. with a neighbourhood in M diffeomorphic to $\partial M \times [0, 1)$ the restriction of the tangent bundle of M to ∂M splits as $TM|_{\partial M} \cong T\partial M \oplus \underline{\mathbb{R}}_{\partial M}$ and this gives a natural homotopy commutative diagram

(48)
$$\begin{tikzcd}
\partial M \arrow[rr, "\iota"] \arrow[dr, "T^{\mathrm{st}}"'] & & M \arrow[dl, "T^{\mathrm{st}}"] \\
& \mathscr{B}GL(n) &
\end{tikzcd}$$

for any $n \geq k$. In other words, the embedding of the boundary, $\iota\colon \partial M \to M$ is naturally a morphism in the slice over $\mathscr{B}GL(n)$. This means that any $\tilde{\rho}$-framing on M can be pulled back to a $\tilde{\rho}$-framing on ∂M:

(49)
$$\iota^*\colon \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \tilde{\rho}) \to \mathbf{H}^\infty_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}\big|_{\partial M}, \tilde{\rho}).$$

That is, for any $\tilde{\rho}$-framing on M we have a natural homotopy commutative diagram

(50)
$$\begin{tikzcd}
\partial M \arrow[rr, "\iota"] \arrow[dr, "\tau|_{\partial M}"] \arrow[ddr, "T^{\mathrm{st}}|_{\partial M}"'] & & M \arrow[dl, "\tau"'] \arrow[ddl, "T^{\mathrm{st}}"] \\
& Y \arrow[d] & \\
& \mathscr{B}GL(n) &
\end{tikzcd}$$

realizing ι as a morphism in the slice over Y. Therefore we have a further pullback morphism

(51)
$$\iota^*\colon \mathbf{H}^\infty_{/Y}(\tau, \psi) \to \mathbf{H}_{/Y}(\tau|_{\partial M}, \psi)$$

for any morphism $\psi\colon (X, \rho) \to (Y, \tilde{\rho})$ in the slice over $\mathscr{B}GL(n)$. For any fixed ρ-framing ж on ∂M we can then form the space of ρ-framings on the $\tilde{\rho}$-framed manifold M extending ж. This is the homotopy fibre of ι^* at ж:

(52)
$$\begin{tikzcd}
\mathbf{H}^{\infty;\text{ж}}_{/Y}((M, \partial M, \tau), (X, \psi)) \arrow[r] \arrow[d] & * \arrow[d, "\text{ж}"] \\
\mathbf{H}_{/Y}(\tau, \psi) \arrow[r, "\iota^*"] & \mathbf{H}_{/Y}(\tau|_{\partial M}), \psi)
\end{tikzcd}$$

Reasoning as in Section 4, when the morphism $\psi\colon X \to Y$ is the homotopy fibre of a morphism $c\colon Y \to Z$ one sees that, as soon as the ρ-structure ж on ∂M can be extended to a ρ-structure on M, then the space $\mathbf{H}^{\infty;\text{ж}}_{/Y}((M, \partial M, \tau), (X, \psi))$ of such extensions is a torsor for the ∞-group $\mathbf{H}^{\infty,\mathrm{rel}}(M, \partial M; \Omega Z)$ defined by the homotopy pullback

(53)
$$\begin{tikzcd}
\mathbf{H}^{\infty,\mathrm{rel}}(M, \partial M; \Omega Z) \arrow[r] \arrow[d] & * \arrow[d, "\mathbf{0}"] \\
\mathbf{H}^\infty(M, \Omega Z) \arrow[r, "\iota^*"] & \mathbf{H}^\infty(\partial M, \Omega Z)
\end{tikzcd}$$

In particular, for $Z = \mathbf{B}^n A$ for some discrete abelian group A, the space $\mathbf{H}^{\infty,\mathrm{rel}}(M, \partial M; \mathbf{B}^{n-1}A)$ is the space whose set of connected components is the $(n-1)$-th relative cohomology group of $(M, \partial M)$:

$$\pi_0 \mathbf{H}^{\infty,\mathrm{rel}}(M, \partial M; \mathbf{B}^{n-1}A) \cong H^{n-1}(M, \partial M; A). \tag{54}$$

Moreover, since $\mathbf{B}^n A$ is $(n-1)$-connected, we see that any homotopy from $c \circ \tau|_{\partial M} : \partial M \to \mathbf{B}^n A$ to the trivial map can be extended to a homotopy from $c \circ \tau : M \to \mathbf{B}^n A$ to the trivial map, as soon as $\dim M < n$. In other words, for $Z = \mathbf{B}^n A$, if $k < n$ every ρ-structure on ∂M can be extended to a ρ-structure on M.

The space $\mathbf{H}^{\infty,ж}_{/Y}((M, \partial M, \tau), (X, \psi))$ has a natural interpretation in terms of ρ-framed cobordism: it is the space of morphisms from the empty manifold to the ρ-framed manifold $(\partial M, ж)$, whose underlying non-framed cobordism is M. As such, it carries a natural action of the ∞-group of ρ-framings on the cylinder $\partial M \times [0, 1]$ which restrict to the ρ-framing ж both on $\partial M \times \{0\}$ and on $\partial M \times \{1\}$. These are indeed precisely the ρ-framed cobordisms lifting the trivial non-framed cobordism. Geometrically this action is just the glueing of such a ρ-framed cylinder along ∂M, as a collar in M. On the other hand, by the very definition of $\mathbf{H}^\infty$, this ∞-group of ρ-framed cylinders is nothing but the loop space $\Omega_ж(\mathbf{H}^\infty_{/\mathscr{B}GL(n)})_{/\rho}(T^{\mathrm{st}}\big|_{\partial M}, \psi)$, i.e., the loop space at ж of the space of ρ-structures on ∂M lifting the $\tilde{\rho}$-structure $\tau|_{\partial M}$. Comparing this to the diagram (37), we see that the space of ρ-structures on M extending a given ρ-structure on ∂M comes with a natural action of the ∞-group which is the centre of the extension $\mathrm{Diff}^{\tilde{\rho}}(\partial M, ж)$ of $\mathrm{Diff}^{\tilde{\rho}}(M, \tau|_{\partial M})$.[5] In the case $\psi : X \to Y$ is the homotopy fibre of a morphism $c : Y \to \mathbf{B}^n A$, passing to equivalence classes we find the natural action of $H^{n-2}(\partial M, A)$ on the relative cohomology group $H^{n-1}(M, \partial M; A)$ given by the suspension isomorphism $H^{n-2}(\partial M, A) \cong H^{n-1}(\partial M \times [0, 1], \partial M \times \{0, 1\}, A)$ combined with the natural translation action

$$H^{n-1}(M, \partial M; A) \times H^{n-1}(\partial M \times [0, 1], \partial M \times \{0, 1\}, A) \to H^{n-1}(M, \partial M; A). \tag{55}$$

For instance, if M is a connected oriented 3-manifold with connected boundary ∂M and we choose $n = 4$ and $A = \mathbb{Z}$, then we get the translation action of $\mathbb{Z}$ on itself.[6]

5. Mapping class groups of ρ-framed manifolds

In this final section, we consider an application of the general notion of ρ-structure developed in the previous sections to investigate extensions of the mapping class group of smooth manifolds.
Inspired by the classical notion of mapping class group, see for instance [Ha12], we consider the following

Definition 5.1. *Let M be a k-dimensional manifold, and let $\rho : X \to \mathscr{B}GL(n)$ be a morphisms of smooth ∞-stacks, with $k \le n$. The* mapping class group $\Gamma^\rho(M, \sigma)$ *of a ρ-framed manifold (M, σ) is the group of connected components of the ρ-diffeomorphism ∞-group of (M, σ), namely*

$$\Gamma^\rho(M, \sigma) := \pi_0 \mathrm{Diff}^\rho(M, \sigma) \tag{56}$$

In the setting of the Section 4, we consider the case in which the ∞-stack X is the homotopy fiber of a morphism $Y \to Z$, with Z a geometrically discrete ∞-stack. Then, induced by diagram (47), we have the following long exact sequence in homotopy

$$\cdots \to \pi_1 \mathrm{Diff}^\rho(M, \sigma) \to \pi_1 \mathrm{Diff}^{\tilde{\rho}}(M, \tau) \to \pi_2 \mathbf{H}^\infty(M, Z) \to \Gamma^\rho(M, \sigma) \to \Gamma^{\tilde{\rho}}(M, \tau) \to \pi_1 \mathbf{H}^\infty(M, Z). \tag{57}$$

Notice that the morphism

$$\Gamma^{\tilde{\rho}}(M, \tau) \to \pi_1 \mathbf{H}^\infty(M, Z) \tag{58}$$

is a homomorphism at the π_0 level, so it is only a morphism of pointed sets and *not* a morphism of groups. It is the morphism that associates with a ρ-diffeomorphism f the pullback of the lift σ of τ. In other words, it is the morphism of pointed sets from the set of isotopy classes of ρ-diffeomorphisms to the set of equivalence classes of lifts induced by the natural action

$$\Gamma^{\tilde{\rho}}(M, \tau) \times \{\text{(equivalence classes of) lifts of } \tau\} \to \{\text{(equivalence classes of) lifts of } \tau\} \tag{59}$$

[5] This should be compared to Segal's words in [Se04]: "An oriented 3-manifold Y whose boundary ∂Y is rigged has itself a set of riggings which form a principal homogeneous set under the group $\mathbb{Z}$ which is the centre of the central extension of $\mathrm{Diff}(\partial Y)$."

[6] Again, compare to Segal's prescription on the set of riggings on a oriented 3-manifold.

once one picks a distinguished element σ in the set (of equivalence classes of) of lifts and uses it to identify this set with $\pi_0\mathbf{H}^\infty(M,\Omega Z) \cong \pi_1\mathbf{H}^\infty(M,Z)$. A particularly interesting situation is the case when c is a degree d characteristic class for Y, i.e., when $c\colon Y \to \mathbf{B}^d A$ for some discrete abelian group A, and M is a closed manifold. Since $\mathbf{B}^d A$ is a geometrically discrete ∞-stack, we have that $\mathbf{H}^\infty(M,\mathbf{B}^d A)$ is equivalent, as an ∞-groupoid, to $\mathbf{H}(M,\mathbf{B}^d A)$. Consequently, we obtain that $\pi_k\mathbf{H}^\infty(M,\mathbf{B}^d A) = H^{d-k}(M,A)$ for $0 \leq k \leq d$ (and zero otherwise): in particular, the obstruction to lifting a $\tilde{\rho}$-framing τ on M to a ρ-framing σ is given by an element in $H^d(M,A)$. When this obstruction vanishes, hence when a lift σ of τ does exist, the long exact sequence above reads as

$$\cdots \to \pi_1\mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to H^{d-2}(M,A) \to \Gamma^{\rho}(M,\sigma) \to \Gamma^{\tilde{\rho}}(M,\tau) \to H^{d-1}(M,A) \tag{60}$$

for $d \geq 2$, and simply as

$$\cdots \to \pi_1\mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to 1 \to \Gamma^{\rho}(M,\sigma) \to \Gamma^{\tilde{\rho}}(M,\tau) \to H^0(M,A) \tag{61}$$

for $d = 1$.

Remark 5.2. The long exact sequences (60) and (61) are a shadow of Proposition 4.1, which is a more general extension result for the *whole* ∞-group $\mathrm{Diff}^\rho(M,\sigma)$.

The morphism of pointed sets $\Gamma^{\tilde{\rho}}(M,\tau) \to H^{d-1}(M,A)$ is easily described: once a lift σ for τ has been chosen, the space of lifts is identified with $\mathbf{H}^\infty(M,\mathbf{B}^{d-1}A)$ and the natural pullback action of the $\tilde{\rho}$-diffeomorphism group of M on the space of maps from M to $\mathbf{B}^{d-1}A$ induces the morphism

$$\begin{aligned} \mathrm{Diff}^{\tilde{\rho}}(M,\tau) &\to \mathbf{H}^\infty(M,\mathbf{B}^{d-1}A) \\ f &\mapsto f^*\sigma - \sigma \end{aligned} \tag{62}$$

where we have written $f^*\sigma - \sigma$ for the element in $\mathbf{H}^\infty(M,\mathbf{B}^{d-1}A)$ which represents the "difference" between $f^*\sigma$ and σ in the space of lifts of τ seen as a $\mathbf{H}^\infty(M,\mathbf{B}^{d-1}A)$-torsor. The morphism $\Gamma^{\tilde{\rho}}(M,\tau) \to H^{d-1}(M,A)$ is obtained by passing to π_0's and so we see in particular from the long exact sequence (60) that the image of $\Gamma^{\rho}(M,\tau)$ into $\Gamma^{\tilde{\rho}}(M,\tau)$ consist of precisely the isotopy classes of those $\tilde{\rho}$-diffeomorphisms of $(M,\tilde{\rho})$ which fix the ρ-structure σ up to homotopy.
Similarly, for $d \geq 2$, the morphism of groups $\pi_1\mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to H^{d-2}(M,A)$ in sequence (60) can be described explicitly as follows. A closed path γ based at the identity in $\mathrm{Diff}^{\tilde{\rho}}(M,\tau)$ defines then a morphism $\gamma^\#\colon M \times [0,1] \to \mathbf{B}^{d-1}A$, as the composition

$$M \times [0,1] \to M \xrightarrow{\mathbf{0}} \mathbf{B}^{d-1}A, \tag{63}$$

where the first arrow is the homotopy from the identity of M to itself and where $\mathbf{0}\colon M \to \mathbf{B}^{d-1}A$ is the *collapsing* morphism, namely the morphism obtained as the composition $M \to * \to \mathbf{B}^{d-1}A$ (here we are using that $\mathbf{B}^{d-1}A$ comes naturally equipped with a base point). The image of $[\gamma]$ in $H^{d-2}(M,A)$ is then given by the element $[\gamma^\#]$ in the relative cohomology group

$$H^{d-1}(M \times [0,1], M \times \{0,1\}, A) \cong H^{d-1}(\mathbf{\Sigma} M, A) \cong H^{d-2}(M,A)\,. \tag{64}$$

By construction, $[\gamma^\#]$ is the image in $H^{d-1}(M \times [0,1], M \times \{0,1\}, A) \cong H^{d-2}(M,A)$ of the zero class in $H^{d-1}(M,A)$ via the pullback morphism $M \times [0,1] \to M$, so it is the zero class in $H^{d-1}(M \times [0,1], M \times \{0,1\}, A)$. That is, the morphism $\pi_1\mathrm{Diff}^{\tilde{\rho}}(M,\tau) \to H^{d-2}(M,A)$ is the zero morphism, and we obtain the short exact sequence

$$1 \to H^{d-2}(M,A) \to \Gamma^{\rho}(M,\sigma) \to \Gamma^{\tilde{\rho}}(M,\tau) \to H^{d-1}(M,A) \tag{65}$$

showing that $\Gamma^\rho(M,\sigma)$ is a $H^{d-2}(M,A)$-extension of a subgroup of $\Gamma^{\tilde{\rho}}(M,\tau)$: namely, the subgroup is the $\Gamma^{\tilde{\rho}}(M,\tau)$-stabilizer of the element of $H^{d-1}(M,A)$ corresponding to the lift σ of τ. The action of this stabiliser on $H^{d-2}(M,A)$ is the pullback action of $\tilde{\rho}$-diffeomorphisms of M on the $(d-2)$-th cohomology group of M with coefficients in A. Since this action is not necessarily trivial, the $H^{d-2}(M,A)$-extension $\Gamma^\rho(M,\sigma)$ of the stabiliser of σ is not a central extension in general.

5.1. **Oriented and spin manifolds, and r-spin surfaces.** Before discussing p_1-structures and their modular groups, which is the main goal of this note, let us consider two simpler but instructive examples: oriented manifolds and spin curves.

Since the ∞-stack $\mathscr{B}SO(n)$ is the homotopy fibre of the first Stiefel-Whitney class

$$w_1\colon \mathscr{B}O(n) \to \mathbf{B}\mathbb{Z}/2\mathbb{Z} \tag{66}$$

an n-dimensional manifold can be oriented if and only if $[w_1 \circ T_M]$ is the trivial element in $\pi_0\mathbf{H}^\infty(M, \mathbf{B}\mathbb{Z}/2\mathbb{Z}) = H^1(M, \mathbb{Z}/2\mathbb{Z})$. When this happens, the space of possible orientations on M is equivalent to $\mathbf{H}^\infty(M, \mathbb{Z}/2\mathbb{Z})$, so when M is connected it is equivalent to a 2-point set. For a fixed orientation on M, we obtain from (61) with $A = \mathbb{Z}/2\mathbb{Z}$ the exact sequence

$$1 \to \Gamma^{\mathrm{or}}(M) \to \Gamma(M) \to \mathbb{Z}/2\mathbb{Z} \tag{67}$$

where $\Gamma^{\mathrm{or}}(M)$ denotes the mapping class group of oriented diffeomorphisms of M, and where the rightmost morphism is induced by the action of the diffeomorphism group of M on the set of its orientations. The oriented mapping class group of M is therefore seen to be a subgroup of order 2 in $\Gamma(M)$ in case there exists at least an orientation reversing diffeomorphism of M, and to be the whole $\Gamma(M)$ when such a orientation reversing diffeomorphism does not exist (e.g., for $M = \mathbb{P}^{n/2}\mathbb{C}$, for $n \equiv 0 \mod 4$).

Consider now the ∞-stack $\mathscr{B}\mathrm{Spin}(n)$ for $n \geq 3$. It can be realised as the homotopy fibre of the second Stiefel-Whitney class

$$w_2\colon \mathscr{B}SO(n) \to \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}. \tag{68}$$

An oriented n-dimensional manifold M will then admit a spin structure if and only if $[w_2 \circ T_M]$ is the trivial element in $\pi_0\mathbf{H}^\infty(M, \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}) = H^2(M, \mathbb{Z}/2\mathbb{Z})$. When this happens, the space of possible orientations on M is equivalent to $\mathbf{H}^\infty(M, \mathbf{B}\mathbb{Z}/2\mathbb{Z})$, and we obtain, for a given spin structure σ on M lifting the orientation of M, the exact sequence

$$1 \to H^0(M, \mathbb{Z}/2\mathbb{Z}) \to \Gamma^{\mathrm{Spin}}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M, \mathbb{Z}/2\mathbb{Z}). \tag{69}$$

In particular, if M is connected, we get the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma^{\mathrm{Spin}}(M, \sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M, \mathbb{Z}/2\mathbb{Z}). \tag{70}$$

Since, for a connected M, the pullback action of oriented diffeomorphisms on $H^0(M, \mathbb{Z}/2\mathbb{Z})$ is trivial, we see that in this case the group $\Gamma^{\mathrm{Spin}}(M, \sigma)$ is a $\mathbb{Z}/2\mathbb{Z}$-central extension of the subgroup of $\Gamma^{\mathrm{or}}(M)$ consisting of (isotopy classes of) orientation preserving diffeomorphisms of M which fix the spin structure σ (up to homotopy). The group $\Gamma^{\mathrm{Spin}}(M, \sigma)$ and its relevance to Spin TQFTs are discussed in detail in [Ma96].

For $n = 2$, the homotopy fibre of $w_2\colon \mathscr{B}SO(2) \to \mathbf{B}^2\mathbb{Z}/2\mathbb{Z}$ is again $\mathscr{B}SO(2)$ with the morphism $\mathscr{B}SO(2) \to \mathscr{B}SO(2)$ induced by the group homomorphism

$$\begin{aligned} SO(2) &\to SO(2) \\ x &\mapsto x^2 \end{aligned} \tag{71}$$

Since the second Stiefel-Withney class of an oriented surface M is the mod 2 reduction of the first Chern class of the holomorphic tangent bundle of M (for any choice of a complex structure compatible with the orientation), and $\langle c_1(T^{\mathrm{hol}})M|[M]\rangle = 2 - 2g$, where g is the genus of M, one has that $[w_2 \circ T_M]$ is always the zero element in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ for a compact oriented surface, and so the orientation of M can always be lifted to a spin structure. More generally, one can consider the group homomorphism $SO(2) \to SO(2)$ given by $x \mapsto x^r$, with $r \in \mathbb{Z}$. We have then a homotopy fibre sequence

$$\begin{tikzcd} \mathscr{B}SO(2) \arrow[r] \arrow[d, "\rho_{1/r}"'] & * \arrow[d] \\ \mathscr{B}SO(2) \arrow[r, "c_{(x\to x^r)}"] & \mathbf{B}^2\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{72}$$

In this case one sees that an *r-spin structure* on an oriented surface M, i.e. a lift of the orientation of M through $\rho_{1/r}$, exists if and only if $2-2g \equiv 0 \mod r$. When this happens, one obtains the exact sequence

$$1 \to \mathbb{Z}/r\mathbb{Z} \to \Gamma^{1/r}(M,\sigma) \to \Gamma^{\mathrm{or}}(M) \to H^1(M,\mathbb{Z}/r\mathbb{Z}), \tag{73}$$

which exhibits the r-spin mapping class group $\Gamma^{1/r}(M,\sigma)$ as a $\mathbb{Z}/r\mathbb{Z}$-central extension of the subgroup of $\Gamma^{\mathrm{or}}(M)$ consisting of isotopy classes of orientation preserving diffeomorphisms of M fixing the r-spin structure σ (up to homotopy). The group $\Gamma^{1/r}(M,\sigma)$ appears as the fundamental group of the moduli space of r-spin Riemann surfaces, see [R-W12, R-W14].

5.2. **p_1-structures on oriented surfaces.** Let now finally specialise the general construction above to the case of p_1-structures on closed oriented surfaces, to obtain the $\mathbb{Z}$-central extensions considered in [Se04] around page 476. In particular we will see, how p_1-structures provide a simple realisation of Segal's idea of extended surfaces and 3-manifolds (see also [BN09, CHMV95]).[7]To this aim, our stack Y will be the stack $\mathscr{B}SO(n)$ for some $n \geq 3$, the stack Z will be $\mathbf{B}^4\mathbb{Z}$ and the morphism c will be the first Pontryagin class $p_1\colon \mathscr{B}SO(n) \to \mathbf{B}^4\mathbb{Z}$. the stack X will be the homotopy fiber of p_1, and so the morphism ψ will be the morphism

$$\rho_{p_1}\colon \mathbf{B}^3\mathbb{Z}/\!/_h SO(n) \to \mathscr{B}SO(n). \tag{74}$$

of example 2.4. A lift σ of an orientation on a manifold M of dimension at most 3 to a morphism $M \to \mathbf{B}^3\mathbb{Z}/\!/_h SO(n)$ over $\mathscr{B}O(n)$ will be called a p_1-struture on M. That is, a pair (M,σ) is the datum of a smooth oriented manifold M together with a trivialisation of its first Pontryagin class. Note that, since p_1 is a degree four cohomology class, it can always be trivialised on manifolds of dimension at most 3. In particular, when M is a closed connected oriented 3-manifold, we see that the space of lifts of the orientation of M to a p_1 structure, is equivalent to the space $\mathbf{H}(M,\mathbf{B}^3\mathbb{Z})$ and so its set of connected components is

$$\pi_0\mathbf{H}(M,\mathbf{B}^3\mathbb{Z}) = H^3(M,\mathbb{Z}) \cong \mathbb{Z}. \tag{75}$$

In other words, there is a $\mathbb{Z}$-torsor of equivalence classes of p_1-strctures on a connected oriented 3-manifold. Similarly, in the relative case, i.e., when M is a connected oriented 3-manifold with boundary, the set of equivalence classes of p_1-strctures on M extending a given p_1-structure on ∂M is nonempty and is a torsor for the relative cohomology group

$$H^3(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}, \tag{76}$$

in perfect agreement with the prescription in [Se04, page 480].[8]
We can now combine the results of the previous section in the following

Proposition 5.3. *Let M be a connected oriented surface, and let σ be a p_1-structure on M. We have then the following central extension*

$$1 \to \mathbb{Z} \to \Gamma^{p_1}(M,\sigma) \to \Gamma^{\mathrm{or}}(M) \to 1, \tag{77}$$

where Γ^{p_1} as a shorthand notation for $\Gamma^{\rho_{p_1}}$.

Proof. Since M is oriented, we have a canonical isomorphism $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$ induced by Poincaré duality. Moreover, since M is connected, from 65 we obtaine the following short exact sequence

$$1 \to \mathbb{Z} \to \Gamma^{\rho_{p_1}}(M,\sigma) \to \Gamma^{\mathrm{or}}(M) \to 1 \tag{78}$$

Finally, since the oriented diffeomorphisms action on $H^2(M,\mathbb{Z})$ is trivial for a connected oriented surface M, this short exact sequence is a $\mathbb{Z}$-central extension. □

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[7]In [Se04], the extension is defined in terms of "riggings", a somehow ad hoc construction depending on the contractiblity of Teichmüller spaces and on properties of the η-invariant of metrics on 3-manifolds. Segal says: "I've not been able to think of a less sophisticated definition of a rigged surface, although there are many possible variants. The essential idea is to associate *functorially* to a smooth surface a space -such as $\mathcal{P}_X$- which has fundamental group $\mathbb{Z}$."

[8]The naturality of the appearance of this $\mathbb{Z}$-torsor here should be compared to Segal's words in [Se04]: "An oriented 3-manifold Y whose boundary ∂Y is rigged has itself a set of riggings which form a principal homogeneous set under the group $\mathbb{Z}$ which is the centre of the central extension of $\mathrm{Diff}(\partial Y)$. I do not know an altogether straightforward way to define a rigging of a 3-manifold." Rigged 3-manifolds are then introduces by Segal in terms of the space of metrics on the 3-manifold Y and of the η-invariant of these metrics.

Appendix: proof of the extension theorem

Here we provide the details for proof of the existence of the homotopy fibre sequence (36), which is the extension theorem this note revolves around. All the notations in this Appendix are taken from Section 3.2.

Lemma A.1. *We have a homotopy pullback diagram*

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d, "\psi_*"] \\ \mathrm{Diff}^{\tilde\rho}(M,T) \arrow[r] & \mathbf{Aut}^{\infty}_{/\tilde\rho}(\tau) \end{tikzcd} \tag{79}$$

Proof. By definition of (equation (25)), we have homotopy pullback diagrams

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{80}$$

and

$$\begin{tikzcd} \mathrm{Diff}^{\tilde\rho}(M,T) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde\rho}(\tau) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{81}$$

By pasting them together as

$$\begin{tikzcd} \mathrm{Diff}^{\rho}(M,\Sigma) \arrow[r] \arrow[d, "\psi_*"'] & \mathbf{Aut}^{\infty}_{/\rho}(\sigma) \arrow[d, "\psi_*"] \\ \mathrm{Diff}^{\tilde\rho}(M,T) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde\rho}(\tau) \arrow[d] \\ \mathrm{Diff}(M) \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M) \end{tikzcd} \tag{82}$$

and by the 2-out-of-3 law for homotopy pullbacks the claim follows. ☐

We need the following basic fact [Lu06, Lemma 5.5.5.12]:

Lemma A.2. *Let* $\mathbf{C}$ *be an* ∞*-category,* $\mathbf{C}_{/x}$ *its slice over an object* $x \in \mathbf{C}$*, and let* $f\colon a \to x$ *and* $g\colon b \to x$ *be two morphisms into* x*. Then the hom space* $\mathbf{C}_{/x}(f,g)$ *in the slice is expressed in terms of that in* $\mathbf{C}$ *by the fact that there is a homotopy pullback (in* $\infty\mathrm{Grpd}$*) of the form*

$$\begin{tikzcd} \mathbf{C}_{/x}(f,g) \arrow[r] \arrow[d] & \mathbf{C}(a,b) \arrow[d, "g\circ(-)"] \\ * \arrow[r, "{[f]}"] & \mathbf{C}(a,x) \end{tikzcd}$$

where the right morphism is composition with g*, and where the bottom morphism picks* f *regarded as a point in* $\mathbf{C}(a,x)$*.*

Lemma A.3. *We have homotopy pullback diagrams*

(83)
$$\begin{tikzcd}
\Omega_T \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M, \tilde{\rho}) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{/\tilde{\rho}}(T) \arrow[d] \\
{*} \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\Omega_\Sigma \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M, \rho) \arrow[r] \arrow[d] & \mathbf{Aut}^{\infty}_{\rho}(\Sigma) \arrow[d] \\
{*} \arrow[r] & \mathbf{Aut}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M)
\end{tikzcd}$$

Proof. Let $\mathbf{C}$ be an $(\infty, 1)$-category, and let $f\colon x \to y$ be a morphism in $\mathbf{C}$. Then by Lemma A.2 and using 2-out-of-3 for homotopy pullbacks, the forgetful morphism $\mathbf{C}_{/y} \to \mathbf{C}$ from the slice over y to $\mathbf{C}$ induces a morphism of ∞-groups $\mathbf{Aut}_{\mathbf{C}_{/y}}(f) \to \mathbf{Aut}_{\mathbf{C}}(x)$ sitting in a pasting of homotopy pullbacks like this:

(84)
$$\begin{tikzcd}
\Omega_f \mathbf{C}(x, y) \arrow[r] \arrow[d] & \mathbf{Aut}_{\mathbf{C}_{/y}}(f) \arrow[r] \arrow[d] & {*} \arrow[d, "{[f]}"] \\
{*} \arrow[r, "{[\mathrm{id}]}"] \arrow[rr, bend right, "{[f]}"'] & \mathbf{Aut}_{\mathbf{C}}(x) \arrow[r, "f\circ(-)"] & \mathbf{C}(x, y)
\end{tikzcd}$$

By taking here $\mathbf{C} = \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$, $x = T^{\mathrm{st}}_M$, $y = \tilde{\rho}$ (resp., $y = \rho$), and $f = T$ (resp., $f = \Sigma$), the left square yields the first (resp., the second) diagram in the statement of the lemma. $\square$

Lemma A.4. *We have a homotopy pullback diagram*

(85)
$$\begin{tikzcd}
\Omega_\beta(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T, \Psi) \arrow[r] \arrow[d] & \Omega_\Sigma \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M, \rho) \arrow[d] \\
{*} \arrow[r] & \Omega_T \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}_M, \tilde{\rho})
\end{tikzcd}$$

Proof. If we take $\mathbf{C} = \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}$, $g = (\psi, \Psi)$, $a = T^{\mathrm{st}}_M$, $f = T$, $b = \rho$ and $x = \tilde{\rho}$ in Lemma A.2, we find the homotopy fibre sequence

(86)
$$\begin{tikzcd}
(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}}(T, \Psi) \arrow[r] \arrow[d] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \rho) \arrow[d, "\psi_*"] \\
{*} \arrow[r] & \mathbf{H}^{\infty}_{/\mathscr{B}GL(n)}(T^{\mathrm{st}}, \tilde{\rho})
\end{tikzcd}$$

By looping the above diagram, the claim follows. $\square$

Lemma A.5. *We have an equivalence of* $(\infty, 1)$*-categories*

(87)
$$(\mathbf{H}^{\infty}_{/\mathscr{B}GL(n)})_{/\tilde{\rho}} \cong \mathbf{H}^{\infty}_{/Y}.$$

Proof. Let $\mathbf{C}$ be an $(\infty, 1)$-category, and let $f : b \to x$ be a 1-morphism in $\mathbf{C}$. By abuse of notation, we can regard f as a diagram $f : \Delta^1 \to \mathbf{C}$. We have then a morphism

(88)
$$\varphi : (\mathbf{C}_{/x})_{/f} \to \mathbf{C}_{/b}$$

induced by the ∞-functor $\Delta^0 \hookrightarrow \Delta^1$ induced by sending 0 to 1. Since 1 is an initial object in Δ^1, the opposite ∞-functor is a cofinal map. By noticing that $\mathbf{C}^{op}_{x/}$ is canonically equivalent to $\mathbf{C}_{/x}$, then by [Lu06, Proposition 4.1.1.8] we have that φ is an equivalence of ∞-categories. Therefore, if we take $\mathbf{C} = \mathbf{H}^{\infty}$, and $f = \tilde{\rho} : Y \to \mathscr{B}GL(n)$, we have that the claim follows. $\square$

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FROBENIUS ALGEBRAS AND HOMOTOPY FIXED POINTS OF GROUP ACTIONS ON BICATEGORIES

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ABSTRACT. We explicitly show that symmetric Frobenius structures on a finite-dimensional, semi-simple algebra stand in bijection to homotopy fixed points of the trivial $SO(2)$-action on the bicategory of finite-dimensional, semi-simple algebras, bimodules and intertwiners. The results are motivated by the 2-dimensional Cobordism Hypothesis for oriented manifolds, and can hence be interpreted in the realm of Topological Quantum Field Theory.

1. Introduction

While fixed points of a group action on a set form an ordinary subset, homotopy fixed points of a group action on a category as considered in [Kir02, EGNO15] provide additional structure.

In this paper, we take one more step on the categorical ladder by considering a topological group G as a 3-group via its fundamental 2-groupoid. We provide a detailed definition of an action of this 3-group on an arbitrary bicategory $\mathcal{C}$, and construct the bicategory of homotopy fixed points $\mathcal{C}^G$ as a suitable limit of the action. Contrarily from the case of ordinary fixed points of group actions on sets, the bicategory of homotopy fixed points $\mathcal{C}^G$ is strictly "larger" than the bicategory $\mathcal{C}$. Hence, the usual fixed-point condition is promoted from a property to a structure.

Our paper is motivated by the 2-dimensional Cobordism Hypothesis for oriented manifolds: according to [Lur09b], 2-dimensional oriented fully-extended topological quantum field theories are classified by homotopy fixed points of an $SO(2)$-action on the core of fully-dualizable objects of the symmetric monoidal target bicategory. In case the target bicategory of a 2-dimensional oriented topological field theory is given by Alg_2, the bicategory of algebras, bimodules and intertwiners, it is claimed in [FHLT10, Example 2.13] that the additional structure of a homotopy fixed point should be given by the structure of a symmetric Frobenius algebra.

As argued in [Lur09b], the $SO(2)$-action on Alg_2 should come from rotating the 2-framings in the framed cobordism category. By [Dav11, Proposition 3.2.8], the induced action on the core of fully-dualizable objects of Alg_2 is actually trivializable. Hence,

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instead of considering the action coming from the framing, we may equivalently study the *trivial* $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$.

Our main result, namely Theorem 4.1, computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the trivial $SO(2)$-action on an arbitrary bicategory $\mathcal{C}$. It follows then as a corollary that the bicategory $(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}$ consisting of homotopy fixed points of the trivial $SO(2)$-action on the core of fully-dualizable objects of Alg_2 is equivalent to the bicategory Frob of semisimple symmetric Frobenius algebras, compatible Morita contexts, and intertwiners. This bicategory, or rather bigroupoid, classifies 2-dimensional oriented fully-extended topological quantum field theories, as shown in [SP09]. Thus, unlike fixed points of the trivial action on a set, homotopy fixed-points of the trivial $SO(2)$-action on Alg_2 are actually interesting, and come equipped with the additional structure of a symmetric Frobenius algebra.

If Vect_2 is the bicategory of linear abelian categories, linear functors and natural transformations, we show in corollary 4.8 that the bicategory $(\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)}$ given by homotopy fixed points of the trivial $SO(2)$-action on the core of the fully dualizable objects of Vect_2 is equivalent to the bicategory of Calabi-Yau categories, which we introduce in Definition 4.6.

The two results above are actually intimately related to each other via natural considerations from representation theory. Indeed, by assigning to a finite-dimensional, semisimple algebra its category of finitely-generated modules, we obtain a functor $\mathrm{Rep} : \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}) \to \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$. This 2-functor turns out to be $SO(2)$-equivariant, and thus induces a morphism on homotopy fixed point bicategories, which is moreover an equivalence. More precisely, one can show that a symmetric Frobenius algebra is sent by the induced functor to its category of representations equipped with the Calabi-Yau structure given by the composite of the Frobenius form and the Hattori-Stallings trace. These results have appeared in [Hes16].

The present paper is organized as follows: we recall the concept of Morita contexts between symmetric Frobenius algebras in section 2. Although most of the material has already appeared in [SP09], we give full definitions to mainly fix the notation. We give a very explicit description of compatible Morita contexts between finite-dimensional semi-simple Frobenius algebras not present in [SP09], which will be needed to relate the bicategory of symmetric Frobenius algebras and compatible Morita contexts to the bicategory of homotopy fixed points of the trivial $SO(2)$-action. The expert reader might wish to at least take notice of the notion of a compatible Morita context between symmetric Frobenius algebras in definition 2.4 and the resulting bicategory Frob in definition 2.9.

In section 3, we recall the notion of a group action on a category and of its homotopy fixed points, which has been named "equivariantization" in [EGNO15, Chapter 2.7]. By categorifying this notion, we arrive at the definition of a group action on a bicategory and its homotopy fixed points. This definition is formulated in the language of tricategories. Since we prefer to work with bicategories, we explicitly spell out the definition in Remark 3.13.

In section 4, we compute the bicategory of homotopy fixed points of the trivial $SO(2)$-

action on an arbitrary bicategory. Corollaries 4.3 and 4.8 then show equivalences of bicategories

$$\begin{aligned}(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} &\cong \mathrm{Frob}\\ (\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)} &\cong \mathrm{CY}\end{aligned} \tag{1.1}$$

where CY is the bicategory of Calabi-Yau categories. We note that the bicategory Frob has been proven to be equivalent [Dav11, Proposition 3.3.2] to a certain bicategory of 2-functors. We clarify the relationship between this functor bicategory and the bicategory of homotopy fixed points $(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}$ in Remark 4.4.

Throughout the paper, we use the following conventions: all algebras considered will be over an algebraically closed field $\mathbb{K}$. All Frobenius algebras appearing will be symmetric.

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2. Frobenius algebras and Morita contexts

In this section we will recall some basic notions regarding Morita contexts, mostly with the aim of setting up notations. We will mainly follow [SP09], though we point the reader to Remark 2.5 for a slight difference in the statement of the compatibility condition between Morita context and Frobenius forms.

2.1. Definition. *Let A and B be two algebras. A Morita context $\mathcal{M}$ consists of a quadruple $\mathcal{M} := ({}_BM_A, {}_AN_B, \varepsilon, \eta)$, where ${}_BM_A$ is a (B, A)-bimodule, ${}_AN_B$ is an (A, B)-bimodule, and*

$$\begin{aligned}\varepsilon &: {}_AN \otimes_B M_A \to {}_AA_A\\ \eta &: {}_BB_B \to {}_BM \otimes_A N_B\end{aligned} \tag{2.1}$$

are isomorphisms of bimodules, so that the two diagrams

$$\begin{tikzcd} {}_BM \otimes_A N_B \otimes_B M_A \arrow[r, "\mathrm{id}_M \otimes \varepsilon"] & {}_BM \otimes_A A_A \arrow[d] \\ {}_BB \otimes_B M_A \arrow[u, "\eta \otimes \mathrm{id}_M"] \arrow[r] & {}_BM_A \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} {}_AN \otimes_B M \otimes_A N_B \arrow[d, "\varepsilon\otimes\mathrm{id}_N"] & {}_AN \otimes_B B_B \arrow[l, "\mathrm{id}_N\otimes\eta"'] \arrow[d] \\ {}_AA \otimes_A N_B \arrow[r] & {}_AN_B \end{tikzcd} \tag{2.3}$$

commute.

Note that Morita contexts are the adjoint 1-equivalences in the bicategory Alg_2 of algebras, bimodules and intertwiners. These form a category, where the morphisms are given by the following:

2.2. DEFINITION. *Let $\mathcal{M} := ({}_BM_A, {}_AN_B, \varepsilon, \eta)$ and $\mathcal{M}' := ({}_BM'_A, {}_AN'_B, \varepsilon', \eta')$ be two Morita contexts between two algebras A and B. A morphism of Morita contexts consists of a morphism of (B, A)-bimodules $f : M \to M'$ and a morphism of (A, B)-bimodules $g : N \to N'$, so that the two diagrams*

$$\begin{tikzcd} {}_BM \otimes_A N_B \arrow[r, "f\otimes g"] & {}_BM' \otimes_A N'_B \\ B \arrow[u, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd} \qquad \begin{tikzcd} {}_AN \otimes_B M_A \arrow[r, "g\otimes f"] \arrow[d, "\varepsilon"'] & {}_AN' \otimes_B M'_A \arrow[dl, "\varepsilon'"] \\ A & \end{tikzcd} \tag{2.4}$$

commute.

If the algebras in question have the additional structure of a symmetric Frobenius form $\lambda : A \to \mathbb{K}$, we would like to formulate a compatibility condition between the Morita context and the Frobenius forms. We begin with the following two observations: if A is an algebra, the map

$$\begin{aligned} A/[A, A] &\to A \otimes_{A\otimes A^{\mathrm{op}}} A \\ [a] &\mapsto a \otimes 1 \end{aligned} \tag{2.5}$$

is an isomorphism of vector spaces, with inverse given by $a \otimes b \mapsto [ab]$. Furthermore, if B is another algebra, and $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ is a Morita context between A and B, there is a canonical isomorphism of vector spaces

$$\begin{aligned} \tau : (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) &\to (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) \\ n \otimes m \otimes n' \otimes m' &\mapsto m \otimes n' \otimes m' \otimes n. \end{aligned} \tag{2.6}$$

Using the results above, we can formulate a compatibility condition between Morita context and Frobenius forms, as in the following lemma.

2.3. LEMMA. *Let A and B be two algebras, and let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between A and B. Then, there is a canonical isomorphism of vector spaces*

$$\begin{aligned} f : A/[A, A] &\to B/[B, B] \\ [a] &\mapsto \sum_{i,j} \left[\eta^{-1}(m_j.a \otimes n_i)\right] \end{aligned} \tag{2.7}$$

where n_i and m_j are defined by

$$\varepsilon^{-1}(1_A) = \sum_{i,j} n_i \otimes m_j \in N \otimes_B M. \tag{2.8}$$

PROOF. Consider the following chain of isomorphisms:

$$\begin{aligned} f : A/[A,A] &\cong A \otimes_{A \otimes A^{\mathrm{op}}} A && \text{(by equation 2.5)} \\ &\cong (N \otimes_B M) \otimes_{A \otimes A^{\mathrm{op}}} (N \otimes_B M) && \text{(using } \varepsilon \otimes \varepsilon) \\ &\cong (M \otimes_A N) \otimes_{B \otimes B^{\mathrm{op}}} (M \otimes_A N) && \text{(by equation 2.6)} \\ &\cong B \otimes_{B \otimes B^{\mathrm{op}}} B && \text{(using } \eta \otimes \eta) \\ &\cong B/[B,B] && \text{(by equation 2.5)} \end{aligned} \tag{2.9}$$

Chasing through those isomorphisms, we can see that the map f is given by

$$f([a]) = \sum_{i,j} \left[\eta^{-1}(m_j.a \otimes n_i)\right] \tag{2.10}$$

as claimed. ∎

The isomorphism f described in Lemma 2.3 allows to introduce the following relevant definition.

2.4. DEFINITION. *Let (A, λ^A) and (B, λ^B) be two symmetric Frobenius algebras, and let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between A and B. Since the Frobenius algebras are symmetric, the Frobenius forms necessarily factor through $A/[A,A]$ and $B/[B,B]$. We call the Morita context* compatible *with the Frobenius forms, if the diagram*

$$\begin{tikzcd} A/[A,A] \arrow[rr, "f"] \arrow[dr, "\lambda^A"'] & & B/[B,B] \arrow[dl, "\lambda^B"] \\ & \mathbb{K} & \end{tikzcd} \tag{2.11}$$

commutes.

2.5. REMARK. The definition of compatible Morita context of [SP09, Definition 3.72] requires another compatibility condition on the coproduct of the unit of the Frobenius algebras. However, a calculation using proposition 2.8 shows that the condition of [SP09] is already implied by our condition on Frobenius form of definition 2.4; thus the two definitions of compatible Morita context do coincide.

For later use, we give a very explicit way of expressing the compatibility condition between Morita context and Frobenius forms: if (A, λ^A) and (B, λ^B) are two finite-dimensional semi-simple symmetric Frobenius algebras over an algebraically closed field

$\mathbb{K}$, and $({}_BM_A, {}_BN_A, \varepsilon, \eta)$ is a Morita context between them, the algebras A and B are isomorphic to direct sums of matrix algebras by Artin-Wedderburn:

$$A \cong \bigoplus_{i=1}^{r} M_{d_i}(\mathbb{K}), \qquad \text{and} \qquad B \cong \bigoplus_{j=1}^{r} M_{n_j}(\mathbb{K}). \tag{2.12}$$

By Theorem 3.3.1 of [EGH+11], the simple modules $(S_1, \ldots, S_r)$ of A and the simple modules $(T_1, \ldots, T_r)$ of B are given by $S_i := \mathbb{K}^{d_i}$ and $T_i := \mathbb{K}^{n_i}$, and every module is a direct sum of copies of those. Since simple finite-dimensional representations of $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ are given by tensor products of simple representations of A and B^{op} by Theorem 3.10.2 of [EGH+11], the most general form of ${}_BM_A$ and ${}_AN_B$ is given by

$$\begin{aligned} {}_BM_A &:= \bigoplus_{i,j=1}^{r} \alpha_{ij}\, T_i \otimes_{\mathbb{K}} S_j \\ {}_AN_B &:= \bigoplus_{k,l=1}^{r} \beta_{kl}\, S_k \otimes_{\mathbb{K}} T_l \end{aligned} \tag{2.13}$$

where α_{ij} and β_{kl} are multiplicities. First, we show that the multiplicities are trivial:

2.6. Lemma. *In the situation as above, the multiplicities are trivial after a possible re-ordering of the simple modules:* $\alpha_{ij} = \delta_{ij} = \beta_{ij}$ *and the two bimodules M and N are actually given by*

$$\begin{aligned} {}_BM_A &= \bigoplus_{i=1}^{r} T_i \otimes_{\mathbb{K}} S_i \\ {}_AN_B &= \bigoplus_{j=1}^{r} S_j \otimes_{\mathbb{K}} T_j. \end{aligned} \tag{2.14}$$

Proof. Suppose for a contradiction that there is a term of the form $(T_i \oplus T_j) \otimes S_k$ in the direct sum decomposition of M. Let $f : T_i \to T_j$ be a non-trivial linear map, and define $\varphi \in \mathrm{End}_A((T_i \oplus T_j) \otimes S_k)$ by setting $\varphi((t_i + t_j) \otimes s_k) := f(t_i) \otimes s_k$. The A-module map φ induces an A-module endomorphism on all of ${}_AM_B$ by extending φ with zero on the rest of the direct summands. Since $\mathrm{End}_A({}_BM_A) \cong B$ as algebras by Theorem 3.5 of [Bas68], the endomorphism φ must come from left multiplication, which cannot be true for an arbitrary linear map f. This shows that the bimodule M is given as claimed in equation (2.14). The statement for the other bimodule N follows analogously. ∎

Lemma 2.6 shows how the bimodules underlying a Morita context of semi-simple algebras look like. Next, we consider the Frobenius structure.

2.7. Lemma. [Koc03, Lemma 2.2.11] *Let (A, λ) be a symmetric Frobenius algebra. Then, every other symmetric Frobenius form on A is given by multiplying the Frobenius form with a central invertible element of A.*

By Lemma 2.7, we conclude that the Frobenius forms on the two semi-simple algebras A and B are given by

$$\lambda^A = \bigoplus_{i=1}^{r} \lambda_i^A \operatorname{tr}_{M_{d_i}(\mathbb{K})} \qquad \text{and} \qquad \lambda^B = \bigoplus_{i=1}^{r} \lambda_i^B \operatorname{tr}_{M_{n_i}(\mathbb{K})} \tag{2.15}$$

where λ_i^A and λ_i^B are non-zero scalars. We can now state the following proposition, which will be used in the proof of corollary 4.3.

2.8. PROPOSITION. *Let (A, λ^A) and (B, λ^B) be two finite-dimensional, semi-simple symmetric Frobenius algebras and suppose that $\mathcal{M} := (M, N, \varepsilon, \eta)$ is a Morita context between them. Let λ_i^A and λ_j^B be as in equation (2.15), and define two invertible central elements*

$$\begin{aligned} a &:= (\lambda_1^A, \dots, \lambda_r^A) \in \mathbb{K}^r \cong Z(A) \\ b &:= (\lambda_1^B, \dots, \lambda_r^B) \in \mathbb{K}^r \cong Z(B) \end{aligned} \tag{2.16}$$

Then, the following are equivalent:

1. *The Morita context $\mathcal{M}$ is compatible with the Frobenius forms in the sense of definition 2.4.*
2. *We have $m.a = b.m$ for all $m \in {}_BM_A$ and $n.b^{-1} = a^{-1}.n$ for all $n \in {}_AN_B$.*
3. *For every $i = 1, \dots, r$, we have that $\lambda_i^A = \lambda_i^B$.*

PROOF. With the form of M and N determined by equation (2.14), we see that the only isomorphisms of bimodules $\varepsilon : N \otimes_B M \to A$ and $\eta : B \to M \otimes_A N$ must be given by multiples of the identity matrix on each direct summand:

$$\begin{aligned} \varepsilon : N \otimes_A M \cong \bigoplus_{i=1}^{r} M(d_i \times d_i, \mathbb{K}) &\to \bigoplus_{i=1}^{r} M(d_i \times d_i, \mathbb{K}) = A \\ \sum_{i=1}^{r} M_i &\mapsto \sum_{i=1}^{r} \varepsilon_i M_i \end{aligned} \tag{2.17}$$

Similarly, η is given by

$$\begin{aligned} \eta : B = \bigoplus_{i=1}^{r} M(n_i \times n_i, \mathbb{K}) \mapsto M \otimes_A B &\cong \bigoplus_{i=1}^{r} M(n_i \times n_i, \mathbb{K}) \\ \sum_{i=1}^{r} M_i &\mapsto \sum_{i=1}^{r} \eta_i M_i \end{aligned} \tag{2.18}$$

Here, ε_i and η_i are non-zero scalars. The condition that this data should be a Morita context then demands that $\varepsilon_i = \eta_i$, as a short calculation in a basis confirms. By calculating the action of the elements a and b defined above in a basis, we see that conditions (2) and (3) of the above proposition are equivalent.

Next, we show that (1) and (3) are equivalent. In order to see when the Morita context is compatible with the Frobenius forms, we calculate the map $f : A/[A,A] \to B/[B,B]$ from equation (2.11). One way to do this is to notice that $[A,A]$ consists precisely of trace-zero matrices (cf. [AM57]); thus

$$\begin{aligned} A/[A,A] &\to \mathbb{K}^r \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto (\mathrm{tr}(A_1), \cdots, \mathrm{tr}(A_r)) \end{aligned} \tag{2.19}$$

is an isomorphism of vector spaces. Using this identification, we see that the map f is given by

$$\begin{aligned} f : A/[A,A] &\to B/[B,B] \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto \bigoplus_{i=1}^{r} \mathrm{tr}_{M_{d_i}}(A_i) \left[E_{11}^{(n_i \times n_i)}\right] \end{aligned} \tag{2.20}$$

Note that this map is independent of the scalars ε_i and η_i coming from the Morita context. Now, the two Frobenius algebras A and B are Morita equivalent via a compatible Morita context if and only if the diagram in equation (2.11) commutes. This is the case if and only if $\lambda_i^A = \lambda_i^B$ for all i, as a straightforward calculation in a basis shows. ∎

Having established how compatible Morita contexts between semi-simple algebras over an algebraic closed field look like, we arrive at following definition.

2.9. Definition. *Let* $\mathbb{K}$ *be an algebraically closed field. Let* Frob *be the bicategory where*

- *objects are given by finite-dimensional, semisimple, symmetric Frobenius* $\mathbb{K}$*-algebras,*
- *1-morphisms are given by compatible Morita contexts, as in definition 2.4,*
- *2-morphisms are given by isomorphisms of Morita contexts.*

Note that Frob *has got the structure of a symmetric monoidal bigroupoid, where the monoidal product is given by the tensor product over the ground field, which is the monoidal unit.*

The bicategory Frob will be relevant for the remainder of the paper, due to the following theorem.

2.10. Theorem. [Oriented version of the Cobordism Hypothesis, [SP09]] *The weak 2-functor*

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}^{or}, \mathrm{Alg}_2) &\to \mathrm{Frob} \\ Z &\mapsto Z(+) \end{aligned} \tag{2.21}$$

is an equivalence of bicategories.

3. Group actions on bicategories and their homotopy fixed points

For a group G, we denote with BG the category with one object and G as morphisms. Similarly, if $\mathcal{C}$ is a monoidal category, $B\mathcal{C}$ will denote the bicategory with one object and $\mathcal{C}$ as endomorphism category of this object. Furthermore, we denote by $\underline{G}$ the discrete monoidal category associated to G, i.e. the category with the elements of G as objects, only identity morphisms, and monoidal product given by group multiplication.

Recall that an action of a group G on a set X is a group homomorphism $\rho : G \to \mathrm{Aut}(X)$. The set of fixed points X^G is then defined as the set of all elements of X which are invariant under the action. In equivalent, but more categorical terms, a G-action on a set X can be defined to be a functor $\rho : BG \to \mathrm{Set}$ which sends the one object of the category BG to the set X.

If $\Delta : BG \to \mathrm{Set}$ is the constant functor sending the one object of BG to the set with one element, one can check that the set of fixed points X^G stands in bijection to the set of natural transformations from the constant functor Δ to ρ, which is exactly the limit of the functor ρ. Thus, we have bijections of sets

$$X^G \cong \lim_{*//G} \rho \cong \mathrm{Nat}(\Delta, \rho). \tag{3.1}$$

3.1. Remark. A further equivalent way of providing a G-action on a set X is by giving a monoidal functor $\rho : \underline{G} \to \underline{\mathrm{Aut}}(X)$, where we regard both G and $\mathrm{Aut}(X)$ as categories with only identity morphisms. This definition however does not allow us to express the set of homotopy fixed points in a nice categorical way as in equation (3.1), and thus turns out to be less useful for our purposes.

Categorifying the notion of a G-action on a set yields the definition of a discrete group acting on a category:

3.2. Definition. *Let G be a discrete group and let $\mathcal{C}$ be a category. Let $B\underline{G}$ be the 2-category with one object and $\underline{G}$ as the category of endomorphisms of the single object. A G-action on $\mathcal{C}$ is defined to be a weak 2-functor $\rho : B\underline{G} \to \mathrm{Cat}$ with $\rho(*) = \mathcal{C}$.*

Note that just as in remark 3.1, we could have avoided the language of 2-categories and have defined a G-action on a category $\mathcal{C}$ to be a monoidal functor $\rho : \underline{G} \to \mathrm{Aut}(\mathcal{C})$.

Next, we would like to define the homotopy fixed point category of this action to be a suitable limit of the action, just as in equation (3.1). The appropriate notion of a limit of a weak 2-functor with values in a bicategory appears in the literature as a *pseudo-limit* or *indexed limit*, which we will simply denote by lim. We will only consider limits indexed by the constant functor. For background, we refer the reader to [Lac10], [Kel89], [Str80] and [Str87].

We are now in the position to introduce the following definition:

3.3. Definition. *Let G be a discrete group, let $\mathcal{C}$ be a category, and let $\rho : B\underline{G} \to \mathrm{Cat}$ be a G-action on $\mathcal{C}$. Then, the category of homotopy fixed points $\mathcal{C}^G$ is defined to be the pseudo-limit of ρ.*

Just as in the 1-categorical case in equation (3.1), it is shown in [Kel89] that the limit of any weak 2-functor with values in Cat is equivalent to the category of pseudo-natural transformations and modifications $\mathrm{Nat}(\Delta, \rho)$. Hence, we have an equivalence of categories

$$\mathcal{C}^G \cong \lim \rho \cong \mathrm{Nat}(\Delta, \rho). \tag{3.2}$$

Here, $\Delta : B\underline{G} \to \mathrm{Cat}$ is the constant functor sending the one object of $B\underline{G}$ to the terminal category with one object and only the identity morphism. By spelling out definitions, one sees:

3.4. Remark. Let $\rho : B\underline{G} \to \mathrm{Cat}$ be a G-action on a category $\mathcal{C}$, and suppose that $\rho(e) = \mathrm{id}_{\mathcal{C}}$, i.e. the action respects the unit strictly. Then, the homotopy fixed point category $\mathcal{C}^G$ is equivalent to the "equivariantization" introduced in [EGNO15, Definition 2.7.2].

3.5. G-actions on bicategories. Next, we would like to step up the categorical ladder once more, and define an action of a group G on a bicategory. Moreover, we would also like to account for the case where our group is equipped with a topology. This will be done by considering the fundamental 2-groupoid of G, referring the reader to [HKK01] for additional details.

3.6. Definition. *Let G be a topological group. The fundamental 2-groupoid of G is the monoidal bicategory $\Pi_2(G)$ where*

- *objects are given by points of G,*
- *1-morphisms are given by paths between points,*
- *2-morphisms are given by homotopy classes of homotopies between paths, called* 2-tracks.

The monoidal product of $\Pi_2(G)$ is given by the group multiplication on objects, by pointwise multiplication of paths on 1-morphisms, and by pointwise multiplication of 2-tracks on 2-morphisms. Notice that this monoidal product is associative on the nose, and all other monoidal structure like associators and unitors can be chosen to be trivial.

We are now ready to give a definition of a G-action on a bicategory. Although the definition we give uses the language of tricategories as defined in [GPS95] or [Gur07], we provide a bicategorical description in Remark 3.9.

3.7. Definition. *Let G be a topological group, and let $\mathcal{C}$ be a bicategory. A G-action on $\mathcal{C}$ is defined to be a trifunctor*

$$\rho : B\Pi_2(G) \to \mathrm{Bicat} \tag{3.3}$$

with $\rho(*) = \mathcal{C}$. Here, $B\Pi_2(G)$ is the tricategory with one object and with $\Pi_2(G)$ as endomorphism-bicategory, and Bicat *is the tricategory of bicategories.*

3.8. Remark. If $\mathcal{C}$ is a bicategory, let $\mathrm{Aut}(\mathcal{C})$ be the bicategory consisting of auto-equivalences of bicategories of $\mathcal{C}$, pseudo-natural isomorphisms and invertible modifications. Observe that $\mathrm{Aut}(\mathcal{C})$ has the structure of a monoidal bicategory, where the monoidal product is given by composition. Since there are two ways to define the horizontal composition of pseudo-natural transformation, which are *not* equal to each other, there are actually two monoidal structures on $\mathrm{Aut}(\mathcal{C})$. It turns out that these two monoidal structures are equivalent; see [GPS95, Section 5] for a discussion in the language of tricategories.

With either monoidal structure of $\mathrm{Aut}(\mathcal{C})$ chosen, note that as in Remark 3.1 we could equivalently have defined a G-action on a bicategory $\mathcal{C}$ to be a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$.

Since we will only consider trivial actions in this paper, the hasty reader may wish to skip the next remark, in which the definition of a G-action on a bicategory is unpacked. We will, however use the notation introduced here in our explicit description of homotopy fixed points in remark 3.13.

3.9. Remark. [Unpacking Definition 3.7] Unpacking the definition of a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, as for instance in [SP09, Definition 2.5], or equivalently of a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$, as in [GPS95, Definition 3.1], shows that a G-action on a bicategory $\mathcal{C}$ consists of the following data and conditions:

- For each group element $g \in G$, an equivalence of bicategories $F_g := \rho(g) : \mathcal{C} \to \mathcal{C}$,
- For each path $\gamma : g \to h$ between two group elements, the action assigns a pseudo-natural isomorphism $\rho(\gamma) : F_g \to F_h$,
- For each 2-track $m : \gamma \to \gamma'$, the action assigns an invertible modification $\rho(m) : \rho(\gamma) \to \rho(\gamma')$.
- There is additional data making ρ into a weak 2-functor, namely: if $\gamma_1 : g \to h$ and $\gamma_2 : h \to k$ are paths in G, we obtain invertible modifications

$$\phi_{\gamma_2\gamma_1} : \rho(\gamma_2) \circ \rho(\gamma_1) \to \rho(\gamma_2 \circ \gamma_1) \tag{3.4}$$

- Furthermore, for every $g \in G$ there is an invertible modification $\phi_g : \mathrm{id}_{F_g} \to \rho(\mathrm{id}_g)$ between the identity endotransformation on F_g and the value of ρ on the constant path id_g.

  There are three compatibility conditions for this data: one condition making ϕ_{γ_2,γ_1} compatible with the associators of $\Pi_2(G)$ and $\mathrm{Aut}(\mathcal{C})$, and two conditions with respect to the left and right unitors of $\Pi_2(G)$ and $\mathrm{Aut}(\mathcal{C})$.
- Finally, there are data and conditions for ρ to be monoidal. These are:
  - A pseudo-natural isomorphism

$$\chi : \rho(g) \otimes \rho(h) \to \rho(g \otimes h) \tag{3.5}$$

– A pseudo-natural isomorphism

$$\iota : \mathrm{id}_{\mathcal{C}} \to F_e \tag{3.6}$$

– For each triple (g, h, k) of group elements, an invertible modification ω_{ghk} in the diagram

$$\begin{tikzcd} F_g \otimes F_h \otimes F_k \arrow[r, "\chi_{gh}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\chi_{hk}"'] & F_{gh} \otimes F_k \arrow[d, "\chi_{gh,k}"] \arrow[dl, Rightarrow, "\omega_{ghk}"'] \\ F_g \otimes F_{hk} \arrow[r, "\chi_{g,hk}"] & F_{ghk} \end{tikzcd} \tag{3.7}$$

– An invertible modification γ in the triangle below

$$\begin{tikzcd} & F_e \otimes F_g \arrow[dr, "\chi_{e,g}"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ \mathrm{id}_{\mathcal{C}} \otimes F_g \arrow[ur, "\iota\otimes\mathrm{id}"] \arrow[rr, "\mathrm{id}_{F_g}"'] & {} & F_g \end{tikzcd} \tag{3.8}$$

– Another invertible modification δ in the triangle

$$\begin{tikzcd} & F_g \otimes F_e \arrow[dr, "\chi_{g,e}"] \arrow[d, phantom, "\Uparrow\delta"] & \\ F_g \otimes \mathrm{id}_{\mathcal{C}} \arrow[ur, "\mathrm{id}\otimes\iota"] \arrow[rr, "\mathrm{id}_{F_g}"'] & {} & F_g \end{tikzcd} \tag{3.9}$$

The data $(\rho, \chi, \iota, \omega, \gamma, \delta)$ then has to obey equations (HTA1) and (HTA2) in [GPS95, p. 17].

Just as in the case of a group action on a set and a group action on a category, we would like to define the bicategory of homotopy fixed points of a group action on a bicategory as a suitable limit. However, the theory of trilimits is not very well established. Therefore we will take the description of homotopy fixed points as natural transformations as in equation (3.1) as a definition, and define homotopy fixed points of a group action on a bicategory as the bicategory of pseudo-natural transformations between the constant functor and the action.

3.10. Definition. *Let G be a topological group and $\mathcal{C}$ a bicategory. Let*

$$\rho : B\Pi_2(G) \to \mathrm{Bicat} \tag{3.10}$$

be a G-action on $\mathcal{C}$. The bicategory of homotopy fixed points $\mathcal{C}^G$ is defined to be

$$\mathcal{C}^G := \mathrm{Nat}(\Delta, \rho) \tag{3.11}$$

Here, Δ is the constant functor which sends the one object of $B\Pi_2(G)$ to the terminal bicategory with one object, only the identity 1-morphism and only identity 2-morphism. The bicategory $\mathrm{Nat}(\Delta, \rho)$ *then has objects given by tritransformations $\Delta \to \rho$, 1-morphisms are given by modifications, and 2-morphisms are given by perturbations.*

3.11. Remark. The notion of the "equivariantization" of a strict 2-monad on a 2-category has already appeared in [MN14, Section 6.1]. Note that definition 3.10 is more general than the definition of [MN14], in which some modifications have been assumed to be trivial.

3.12. Remark. In principle, even higher-categorical definitions are possible: for instance in [FV15] a homotopy fixed point of a higher character ρ of an ∞-group is defined to be a (lax) morphism of ∞-functors $\Delta \to \rho$.

3.13. Remark. [Unpacking objects of $\mathcal{C}^G$] Since unpacking the definition of homotopy fixed points is not entirely trivial, we spell it out explicitly in the subsequent remarks, following [GPS95, Definition 3.3]. In the language of bicategories, a homotopy fixed point consists of:

- an object c of $\mathcal{C}$,
- a pseudo-natural equivalence

$$\begin{tikzcd} \Pi_2(G) \arrow[r, bend left=50, "\Delta_c"{name=U}] \arrow[r, bend right=50, "\mathrm{ev}_c \circ \rho"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D, "\Theta"] \end{tikzcd} \tag{3.12}$$

  where Δ_c is the constant functor which sends every object to $c \in \mathcal{C}$, and ev_c is the evaluation at the object c. In components, the pseudo-natural transformation Θ consists of the following:

  – for every group element $g \in G$, a 1-equivalence in $\mathcal{C}$

$$\Theta_g : c \to F_g(c) \tag{3.13}$$

  – and for each path $\gamma : g \to h$, an invertible 2-morphism Θ_γ in the diagram

$$\begin{tikzcd} c \arrow[r, "\Theta_g"] \arrow[d, "\mathrm{id}_c"'] & F_g(c) \arrow[d, "\rho(\gamma)_c"] \arrow[dl, Rightarrow, "\Theta_\gamma"] \\ c \arrow[r, "\Theta_h"'] & F_h(c) \end{tikzcd} \tag{3.14}$$

  which is natural with respect to 2-tracks.

- an invertible modification Π in the diagram

$$
\begin{tikzcd}
\Pi_2(G)\times\Pi_2(G) \arrow[rr, bend left, "\Delta_c"] \arrow[dr, "\rho\times\Delta_c"] \arrow[ddr, bend right, "\rho\times\rho"'] \arrow[ddd, bend right=60, "\otimes"'] \arrow[dr, phantom, bend right, "\Swarrow 1\times\Theta"'] & \arrow[d, phantom, "\Downarrow \Theta\times 1"] & \mathcal{C} \\
& \mathrm{Aut}(\mathcal{C})\times\mathcal{C} \arrow[ur, "\mathrm{ev}"'] & \\
& \mathrm{Aut}(\mathcal{C})\times\mathrm{Aut}(\mathcal{C}) \arrow[u, "\mathrm{id}\times\mathrm{ev}_c"] \arrow[dr, "\otimes"] \arrow[dl, Rightarrow, "\chi"] \arrow[ur, phantom, "\cong"] & \\
\Pi_2(G) \arrow[rr, bend right, "\rho"'] & & \mathrm{Aut}(\mathcal{C}) \arrow[uuu, "\mathrm{ev}_c"']
\end{tikzcd}
\tag{3.15}
$$

$$\Downarrow \Pi$$

$$
\begin{tikzcd}
\Pi_2(G)\times\Pi_2(G) \arrow[r, "\Delta_c"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\cong"] & \mathcal{C} \\
\Pi_2(G) \arrow[r, "\rho"'] \arrow[ur, "\Delta_c" description] & \mathrm{Aut}(\mathcal{C}) \arrow[u, "\mathrm{ev}_c"'] \arrow[ul, phantom, "\Searrow \Theta"]
\end{tikzcd}
$$

which in components means that for every tuple of group elements (g,h) we have an invertible 2-morphism Π_{gh} in the diagram

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[rrr, bend right, "\Theta_{gh}"'] & F_g(c) \arrow[r, "F_g(\Theta_h)"] \arrow[d, phantom, "\Downarrow \Pi_{gh}"] & F_g(F_h(c)) \arrow[r, "\chi^c_{gh}"] & F_{gh}(c) \\
& {} & &
\end{tikzcd}
\tag{3.16}
$$

- for the unital structure, another invertible modification M, which only has the component given in the diagram

$$\begin{tikzcd} c \arrow[r, bend left=50, "\Theta_e"{name=U}] \arrow[r, bend right=50, "\iota_c"'{name=D}] & F_e(c) \arrow[Rightarrow, from=U, to=D, "M"] \end{tikzcd} \tag{3.17}$$

with ι as in equation (3.6). The data (c, Θ, Π, M) of a homotopy fixed point then has to obey the following three conditions. Using the equation in [GPS95, p.21-22] we find the condition

$$\begin{tikzcd}
& F_xF_y\,c \arrow[rr, "F_xF_y(\Theta_z)"] \arrow[dr, "\chi^c_{xy}"] \arrow[d, phantom, "\Pi_{xy}\Downarrow"] & & F_xF_yF_z\,c \arrow[dr, "\chi^{F_z(c)}_{xy}"] \arrow[d, phantom, "\cong"] & \\
F_x\,c \arrow[ur, "F_x(\Theta_y)"] & & F_{xy}\,c \arrow[rr, "F_{xy}(\Theta_z)"] \arrow[dr, phantom, "\Pi_{xy,z}\Downarrow"] & & F_{xy}F_z\,c \arrow[dl, "\chi_{xy,z}"] \\
& c \arrow[ul, "\Theta_x"] \arrow[ur, "\Theta_{xy}"'] \arrow[rr, "\Theta_{xyz}"'] & & F_{xyz}\,c &
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
& F_xF_y\,c \arrow[rr, "F_xF_y(\Theta_z)"] \arrow[d, phantom, "\Downarrow F_x(\Pi_{yz})"] & & F_xF_yF_z\,c \arrow[dl, "F_x(\chi^c_{yz})"'] \arrow[dr, "\chi^{F_z(c)}_{xy}"] & \\
F_x\,c \arrow[ur, "F_x(\Theta_y)"] \arrow[rr, "F_x(\Theta_{yz})"] & & F_xF_{yz}\,c \arrow[dr, "\chi^c_{x,yz}"] & & F_{xy}F_z\,c \arrow[ll, phantom, "\Leftarrow\omega_{xyz}"] \arrow[dl, "\chi_{xy,z}"] \\
& c \arrow[ul, "\Theta_x"] \arrow[u, phantom, "\Pi_{x,yz}\Downarrow"] \arrow[rr, "\Theta_{xyz}"'] & & F_{xyz}\,c &
\end{tikzcd} \tag{3.18}$$

whereas the equation on p.23 of [GPS95] demands that we have

$$
\begin{tikzcd}
& F_e\,c \arrow[rr, "F_e(\Theta_x)"] \arrow[rr, phantom, shift right=4ex, "\Downarrow \Pi_{ex}"] & & F_eF_x\,c \arrow[dr, "\chi_{ex}"] & \\
c \arrow[ur, "\Theta_e"] \arrow[rrrr, bend left=40, "\Theta_x"] \arrow[rrrr, phantom, shift left=4ex, "\cong"] \arrow[rr, "\Theta_x"'] & & F_x\,c \arrow[rr, "\mathrm{id}_{F_x(c)}"'] & & F_x\,c
\end{tikzcd}
$$

$$
\| \tag{3.19}
$$

$$
\begin{tikzcd}
& F_e\,c \arrow[rr, "F_e(\Theta_x)"] & & F_eF_x\,c \arrow[dr, "\chi_{ex}"] \arrow[d, phantom, "\Downarrow \gamma"] & \\
c \arrow[ur, bend left=40, "\Theta_e"] \arrow[ur, bend right=20, "\iota_c"'] \arrow[ur, phantom, "\Rightarrow M"] \arrow[rr, "\Theta_x"'] \arrow[urr, phantom, "\cong"] & & F_x\,c \arrow[ur, "\iota_{F_x(c)}"] \arrow[rr, "\mathrm{id}_{F_x(c)}"'] & & F_x\,c
\end{tikzcd}
$$

and finally the equation on p.25 of [GPS95] demands that

$$
\begin{tikzcd}
F_x\,c \arrow[r, "F_x(\Theta_e)"] \arrow[dr, phantom, "\Downarrow \Pi_{xe}"] & F_xF_e\,c \arrow[d, "\chi_{xe}"] \\
c \arrow[u, "\Theta_x"] \arrow[r, "\Theta_x"'] & F_x\,c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_x\,c \arrow[r, "F_x(\iota_c)"] \arrow[r, bend left=50, "F_x(\Theta_e)"] \arrow[r, phantom, shift left=3ex, "\Downarrow F_x(M)"] \arrow[dr, "\mathrm{id}_{F_x(c)}"] & F_xF_e\,c \arrow[d, "\chi_{xe}"] \arrow[dl, phantom, "\Leftarrow \delta^{-1}"] \\
c \arrow[u, "\Theta_x"] \arrow[r, "\Theta_x"'] \arrow[ur, phantom, "\cong"] & F_x\,c
\end{tikzcd}
\tag{3.20}
$$

3.14. Remark. Suppose that (c, Θ, Π, M) and (c', Θ', Π', M') are homotopy fixed points. A 1-morphism between these homotopy fixed points consists of a trimodification. In detail, this means:

- A 1-morphism $f : c \to c'$,
- An invertible modification m in the diagram

$$
\begin{tikzcd}[column sep=large]
\Pi_2(G) \arrow[r, bend left=60, "\Delta_c"{name=U}] \arrow[r, "\mathrm{ev}_c \circ \rho"{name=M, description}] \arrow[r, bend right=60, "\mathrm{ev}_{c'} \circ \rho"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=M, "\Theta"'] \arrow[Rightarrow, from=M, to=D, "\mathrm{ev}_f * \mathrm{id}"'] & \overset{m}{\Rrightarrow} & \Pi_2(G) \arrow[r, bend left=60, "\Delta_c"{name=U2}] \arrow[r, "\Delta_{c'}"{name=M2, description}] \arrow[r, bend right=60, "\mathrm{ev}_{c'} \circ \rho"'{name=D2}] & \mathcal{C} \arrow[Rightarrow, from=U2, to=M2, "\Delta_f"'] \arrow[Rightarrow, from=M2, to=D2, "\Theta'"']
\end{tikzcd}
\tag{3.21}
$$

In components, m_g is given by

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "f"'] & F_g(c) \arrow[d, "F_g(f)"] \arrow[dl, Rightarrow, "m_g"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
\tag{3.22}
$$

The data (f, m) of a 1-morphism of homotopy fixed points has to satisfy the following two equations as on p.25 and p. 26 of [GPS95]:

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[ddd, "f"'] \arrow[rrr, bend right, "\Theta_{gh}" description] & F_g(c) \arrow[r, "F_g(\Theta_h)"] & F_g(F_h(c)) \arrow[r, "\chi^c_{gh}"] \arrow[d, phantom, "\Downarrow \Pi_{gh}"] & F_{gh}(c) \arrow[ddd, "F_{gh}(f)"] \\
 & & {} \arrow[d, phantom, "\Downarrow m_{gh}"] & \\
 & & {} & \\
c' \arrow[rrr, "\Theta'_{gh}"'] & & & F_{gh}(c')
\end{tikzcd}
$$

$$
\begin{gathered}
\Vert \\
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[dd, "f"'] & F_g(c) \arrow[r, "F_g(\Theta_h)"] \arrow[d, "F_g(f)"'] & F_g(F_h(c)) \arrow[r, "\chi^c_{gh}"] \arrow[d, "F_g(F_h(f))"] \arrow[dl, Rightarrow, "F_g(m_h)"] & F_{gh}(c) \arrow[dd, "F_{gh}(f)"] \\
{} \arrow[r, phantom, "\overset{m_g}{\Leftarrow}"] & F_g(c') \arrow[r, "F_g(\Theta'_h)"'] \arrow[d, phantom, "\Downarrow \Pi'_{gh}"] & F_g(F_h(c')) \arrow[dr, "\chi^{c'}_{gh}"'] \arrow[r, phantom, "\cong"] & {} \\
c' \arrow[ur, "\Theta'_g"'] \arrow[rrr, "\Theta'_{gh}"'] & & & F_{gh}(c')
\end{tikzcd}
\end{gathered}
\tag{3.23}
$$

whereas the second equation reads

$$
\begin{tikzcd}
c \arrow[r, "\iota_c" description] \arrow[r, bend left=60, "\Theta_e"] \arrow[r, phantom, bend left=30, "\Downarrow M"] \arrow[d, "f"'] & F_e(c) \arrow[d, "F_e(f)"] \arrow[dl, Rightarrow, "\iota_f"] \\
c' \arrow[r, "\iota_{c'}"'] & F_e(c')
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "\Theta_e"] \arrow[d, "f"'] & F_e(c) \arrow[d, "F_e(f)"] \arrow[dl, Rightarrow, "m_e"] \\
c' \arrow[r, "\Theta'_e"] \arrow[r, bend right=60, "\iota_{c'}"'] \arrow[r, phantom, bend right=30, "\Downarrow M'"] & F_e(c')
\end{tikzcd}
\tag{3.24}
$$

3.15. Remark. The condition saying that m, as introduced in equation (3.21), is a modification will be vital for the proof of Theorem 4.1 and states that for every path $\gamma : g \to h$ in G, we must have the following equality of 2-morphisms in the two diagrams:

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[dr, "f"'] \arrow[ddr, bend right=40, "\mathrm{id}_c"'] & F_g(c) \arrow[r, "F_g(f)"] \arrow[d, phantom, "\Downarrow m_g"] & F_g(c') \arrow[r, "\rho(\gamma)^{c'}"] \arrow[d, phantom, "\Downarrow \Theta'_\gamma"] & F_h(c') \\
& c' \arrow[ur, "\Theta'_g"] \arrow[r, "\mathrm{id}_{c'}"'] \arrow[dr, phantom, "\cong"] & c' \arrow[ur, "\Theta'_h"'] & \\
& c \arrow[r, "f"'] & c' \arrow[uur, bend right=40, "\Theta'_h"'] &
\end{tikzcd}
$$

$$
\| \tag{3.25}
$$

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[ddr, bend right=40, "\mathrm{id}_c"'] & F_g(c) \arrow[r, "F_g(f)"] \arrow[d, "\rho(\gamma)^c"] & F_g(c') \arrow[r, "\rho(\gamma)^{c'}"] \arrow[d, phantom, "\Downarrow \rho(\gamma)_f^{-1}"] & F_h(c') \\
\arrow[r, phantom, "\overset{\Theta_\gamma}{\Leftarrow}"] & F_h(c) \arrow[urr, bend right=20, "F_h(f)"'] \arrow[r, phantom, "\Downarrow m_h"'] & {} & \\
& c \arrow[u, "\Theta_h"] \arrow[r, "f"'] & c' \arrow[uur, bend right=40, "\Theta'_h"'] &
\end{tikzcd}
$$

Next, we come to 2-morphisms of the bicategory $\mathcal{C}^G$ of homotopy fixed points:

3.16. Remark. Let $(f, m), (\xi, n) : (c, \Theta, \Pi, M) \to (c', \Theta', \Pi', M')$ be two 1-morphisms of homotopy fixed points. A 2-morphism of homotopy fixed points consists of a perturbation between those trimodifications. In detail, a 2-morphism of homotopy fixed points consists of a 2-morphism $\alpha : f \to \xi$ in $\mathcal{C}$, so that

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "f"] \arrow[d, bend right=60, "\xi"'] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}", shift right=3] & F_g(c) \arrow[d, "F_g(f)"] \arrow[dl, phantom, "\Swarrow m_g"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "\xi"'] & F_g(c) \arrow[d, "F_g(\xi)"'] \arrow[d, bend left=60, "F_g(f)"] \arrow[d, phantom, "\overset{F_g(\alpha)}{\Leftarrow}", shift left=3] \arrow[dl, phantom, "\Swarrow n_g"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
\tag{3.26}
$$

Let us give an example of a group action on bicategories and its homotopy fixed points:

3.17. EXAMPLE. Let G be a discrete group, and let $\mathcal{C}$ be any bicategory. Suppose $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ is the trivial G-action. Then, by remark 3.13 a homotopy fixed point, i.e. an object of $\mathcal{C}^G$ consists of

- an object c of $\mathcal{C}$,
- a 1-equivalence $\Theta_g : c \to c$ for every $g \in G$,
- a 2-isomorphism $\Pi_{gh} : \Theta_h \circ \Theta_g \to \Theta_{gh}$,
- a 2-isomorphism $M : \Theta_e \to \mathrm{id}_c$.

This is exactly the same data as a functor $B\underline{G} \to \mathcal{C}$, where $B\underline{G}$ is the bicategory with one object, G as morphisms, and only identity 2-morphisms. Extending this analysis to 1- and 2-morphisms of homotopy fixed points shows that we have an equivalence of bicategories

$$\mathcal{C}^G \cong \mathrm{Fun}(B\underline{G}, \mathcal{C}). \tag{3.27}$$

When one specializes to $\mathcal{C} = \mathrm{Vect}_2$, the functor bicategory $\mathrm{Fun}(B\underline{G}, \mathcal{C})$ is also known as $\mathrm{Rep}_2(G)$, the bicategory of 2-representations of G. Thus, we have an equivalence of bicategories $\mathrm{Vect}_2^G \cong \mathrm{Rep}_2(G)$. This result generalizes the 1-categorical statement that the homotopy fixed point 1-category of the trivial G-action on Vect is equivalent to $\mathrm{Rep}(G)$, cf. [EGNO15, Example 4.15.2].

4. Homotopy fixed points of the trivial SO(2)-action

We are now in the position to state and prove the main result of the present paper. Applying the description of homotopy fixed points in Remark 3.13 to the trivial action of the topological group $SO(2)$ on an arbitrary bicategory yields Theorem 4.1. Specifying the bicategory in question to be the core of the fully-dualizable objects of the Morita-bicategory Alg_2 then shows in corollary 4.3 that homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ are given by symmetric, semi-simple Frobenius algebras.

4.1. THEOREM. *Let $\mathcal{C}$ be a bicategory, and let $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathcal{C})$ be the trivial $SO(2)$-action on $\mathcal{C}$. Then, the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory where*

- *objects are given by pairs (c, λ) where c is an object of $\mathcal{C}$, and $\lambda : \mathrm{id}_c \to \mathrm{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \to (c', \lambda')$ are given by 1-morphisms $f : c \to c'$ in $\mathcal{C}$, so that the*

diagram of 2-morphisms

$$\begin{tikzcd} f \arrow[r, "\sim"] \arrow[d, "\wr"'] & f \circ \mathrm{id}_c \arrow[r, "\mathrm{id}_f * \lambda"] & f \circ \mathrm{id}_c \arrow[d, "\wr"] \\ \mathrm{id}_{c'} \circ f \arrow[r, "\lambda' * \mathrm{id}_f"'] & \mathrm{id}_{c'} \circ f \arrow[r, "\sim"'] & f \end{tikzcd} \tag{4.1}$$

commutes, where $*$ denotes horizontal composition of 2-morphisms. The unlabeled arrows are induced by the canonical coherence isomorphisms of $\mathcal{C}$.

- *2-morphisms of $\mathcal{C}^G$ are given by 2-morphisms $\alpha : f \to f'$ in $\mathcal{C}$.*

PROOF. First, notice that we do not require any conditions on the 2-morphisms of $\mathcal{C}^{SO(2)}$. This is due to the fact that the action is trivial, and that $\pi_2(SO(2)) = 0$. Hence, all naturality conditions with respect to 2-morphisms in $\Pi_2(SO(2))$ are automatically fulfilled.

To start, we observe that the fundamental 2-groupoid $\Pi_2(SO(2))$ is equivalent to the bicategory consisting of only one object, $\mathbb{Z}$ worth of morphisms, and only identity 2-morphisms which we denote by $B\underline{\mathbb{Z}}$. Thus, it suffices to consider the homotopy fixed point bicategory of the trivial action $B\underline{\mathbb{Z}} \to \mathrm{Aut}(\mathcal{C})$. In this case, the definition of a homotopy fixed point as in 3.10 reduces to

- An object c of $\mathcal{C}$,
- A 1-equivalence $\Theta := \Theta_* : c \to c$,
- For every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \mathrm{id}_c \circ \Theta \to \Theta \circ \mathrm{id}_c$. Since Θ is a pseudo-natural transformation, it is compatible with respect to composition of 1-morphisms in $B\underline{\mathbb{Z}}$. Therefore, Θ_{n+m} is fully determined by Θ_n and Θ_m, cf. [SP09, Figure A.1] for the relevant commuting diagram. Thus, it suffices to specify Θ_1.

  By using the canonical coherence isomorphisms of $\mathcal{C}$, we see that instead of giving Θ_1, we can equivalently specify an invertible 2-morphism
  $$\tilde{\lambda} : \Theta \to \Theta. \tag{4.2}$$
  which will be used below.
- A 2-isomorphism
  $$\mathrm{id}_c \circ \Theta \circ \Theta \to \Theta \tag{4.3}$$
  which is equivalent to giving a 2-isomorphism
  $$\Pi : \Theta \circ \Theta \to \Theta. \tag{4.4}$$
- A 2-isomorphism
  $$M : \Theta \to \mathrm{id}_c. \tag{4.5}$$

Note that equivalently to the 2-isomorphism $\tilde{\lambda}$, one can specify an invertible 2-isomorphism

$$\lambda : \mathrm{id}_c \to \mathrm{id}_c \tag{4.6}$$

where

$$\lambda := M \circ \tilde{\lambda} \circ M^{-1}. \tag{4.7}$$

with M as in equation (4.5). This data has to satisfy the following three equations: Equation (3.18) says that we must have

$$\Pi \circ (\mathrm{id}_\Theta * \Pi) = \Pi \circ (\Pi * \mathrm{id}_\Theta) \tag{4.8}$$

whereas equation (3.19) demands that Π equals the composition

$$\Theta \circ \Theta \xrightarrow{\mathrm{id}_\Theta * M} \Theta \circ \mathrm{id}_c \cong \Theta \tag{4.9}$$

and finally equation (3.20) tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \mathrm{id}_\Theta} \mathrm{id}_c \circ \Theta \cong \Theta. \tag{4.10}$$

Hence Π is fully specified by M. An explicit calculation using the two equations above then confirms that equation (4.8) is automatically fulfilled. Indeed, by composing with Π^{-1} from the right, it suffices to show that $\mathrm{id}_\Theta * \Pi = \Pi * \mathrm{id}_\Theta$. Suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Then,

$$\begin{aligned} \mathrm{id}_\Theta * \Pi &= \mathrm{id}_\Theta * (M * \mathrm{id}_\Theta) && \text{by equation (4.10)} \\ &= (\mathrm{id}_\Theta * M) * \mathrm{id}_\Theta \\ &= \Pi * \mathrm{id}_\Theta && \text{by equation (4.9)} \end{aligned} \tag{4.11}$$

Adding appropriate associators shows that this is true in a general bicategory.

If $(c, \Theta, \lambda, \Pi, M)$ and $(c', \Theta', \lambda', \Pi', M')$ are two homotopy fixed points, the definition of a 1-morphism of homotopy fixed points reduces to

- A 1-morphism $f : c \to c'$ in $\mathcal{C}$,
- A 2-isomorphism $m : f \circ \Theta \to \Theta' \circ f$ in $\mathcal{C}$

satisfying two equations. The condition due to equation (3.24) demands that the following isomorphism

$$f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \tag{4.12}$$

is equal to the isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \mathrm{id}_f} \mathrm{id}_{c'} \circ f \cong f \tag{4.13}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \cong \mathrm{id}_{c'} \circ f \xrightarrow{M'^{-1} * \mathrm{id}_f} \Theta' \circ f \right). \tag{4.14}$$

Thus, m is fully determined by M and M'. The condition due to equation (3.23) reads

$$m \circ (\mathrm{id}_f * \Pi) = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \tag{4.15}$$

and is automatically satisfied, as an explicit calculation confirms. Indeed, if $\mathcal{C}$ is a strict 2-category we have that

$$\begin{aligned}
&(\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \\
&\quad = (\Pi' * \mathrm{id}_f) \circ \left[\mathrm{id}_{\Theta'} * (M'^{-1} * \mathrm{id}_f \circ \mathrm{id}_f * M)\right] \circ \left[(M'^{-1} * \mathrm{id}_f \circ \mathrm{id}_f * M) * \mathrm{id}_\Theta\right] \\
&\quad = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * M'^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \\
&\qquad\qquad\qquad \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * M * \mathrm{id}_\Theta) \\
&\quad = (\Pi' * \mathrm{id}_f) \circ (\Pi'^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * \Pi) \\
&\quad = (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * \Pi) \\
&\quad = (M^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_f * M) \circ (\mathrm{id}_f * \Pi) \\
&\quad = m \circ (\mathrm{id}_f * \Pi)
\end{aligned}$$

as desired. Here, we have used equation (4.14) in the first and last line, and equations (4.9) and (4.10) in the third line. Adding associators shows this for an arbitrary bicategory.

The condition that m is a modification as spelled out in equation (3.25) demands that

$$(\tilde{\lambda}' * \mathrm{id}_f) \circ m = m \circ (\mathrm{id}_f * \tilde{\lambda}) \tag{4.16}$$

as equality of 2-morphisms between the two 1-morphisms

$$f \circ \Theta \to \Theta' \circ f. \tag{4.17}$$

Using equation (4.14) and replacing $\tilde{\lambda}$ by λ as in equation (4.7), we see that this requirement is equivalent to the commutativity of diagram (4.1).

If (f, m) and (g, n) are two 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-morphisms $\alpha : f \to g$. The condition coming from equation (3.26) then demands that the diagram

$$\begin{tikzcd}
f \circ \Theta \arrow[r, "m"] \arrow[d, "\alpha * \mathrm{id}_\Theta"'] & \Theta' \circ f \arrow[d, "\mathrm{id}_{\Theta'} * \alpha"] \\
g \circ \Theta \arrow[r, "n"'] & \Theta' \circ g
\end{tikzcd} \tag{4.18}$$

commutes. Using the fact that both m and n are uniquely specified by M and M', one quickly confirms that the diagram commutes automatically.

Our analysis shows that the forgetful functor U which forgets the data M, Θ and Π on objects, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. Indeed, let (c, λ) be an object in the strictified homotopy fixed point bicategory. Choose $\Theta := \mathrm{id}_c$, $M := \mathrm{id}_\Theta$ and Π as in equation (4.9). Then, $U(c, \Theta, M, \Pi, \lambda) = (c, \lambda)$. This shows that the forgetful functor is essentially surjective on objects. Since m is fully determined by M and M', it is clear that the forgetful functor is essentially surjective on 1-morphisms. Since (4.18) commutes automatically, the forgetful functor is bijective on 2-morphisms and thus an equivalence of bicategories. ∎

In the following, we specialise Theorem 4.1 to the case of symmetric Frobenius algebras and Calabi-Yau categories.

4.2. Symmetric Frobenius algebras as homotopy fixed points. In order to state the next corollary, recall that the fully-dualizable objects of the Morita bicategory Alg_2 consisting of algebras, bimodules and intertwiners are precisely given by the finite-dimensional, semi-simple algebras [SP09]. Furthermore, recall that the core $\mathscr{K}(\mathcal{C})$ of a bicategory $\mathcal{C}$ consists of all objects of $\mathcal{C}$, the 1-morphisms are given by 1-equivalences of $\mathcal{C}$, and the 2-morphisms are restricted to be isomorphisms.

4.3. Corollary. *Suppose $\mathcal{C} = \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$, and consider the trivial $SO(2)$-action on $\mathcal{C}$. Then $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of finite-dimensional, semi-simple symmetric Frobenius algebras* Frob, *as defined in definition 2.9. This implies a bijection of isomorphism-classes of symmetric, semi-simple Frobenius algebras and homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$.*

Proof. Indeed, by Theorem 4.1, an object of $\mathcal{C}^{SO(2)}$ is given by a finite-dimensional semisimple algebra A, together with an isomorphism of Morita contexts $\mathrm{id}_A \to \mathrm{id}_A$. By definition, a morphism of Morita contexts consists of two intertwiners of (A, A)-bimodules $\lambda_1, \lambda_2 : A \to A$. The diagrams in definition 2.2 then require that $\lambda_1 = \lambda_2^{-1}$. Thus, λ_2 is fully determined by λ_1. Let $\lambda := \lambda_1$. Since λ is an automorphism of (A, A)-bimodules, it is fully determined by $\lambda(1_A) \in Z(A)$. This gives A, by Lemma 2.7, the structure of a symmetric Frobenius algebra.

We analyze the 1-morphisms of $\mathcal{C}^{SO(2)}$ in a similar way: if (A, λ) and (A', λ') are finite-dimensional semi-simple symmetric Frobenius algebras, a 1-morphism in $\mathcal{C}^{SO(2)}$ consists of a Morita context $\mathcal{M} : A \to A'$ so that (4.1) commutes.

Suppose that $\mathcal{M} = ({}_{A'}M_A, {}_AN_{A'}, \varepsilon, \eta)$ is a Morita context, and let $a := \lambda(1_A)$ and $a' := \lambda'(1_{A'})$. Then, the condition that (4.1) commutes demands that

$$\begin{aligned} m.a &= a'.m \\ a^{-1}.n &= n.a'^{-1} \end{aligned} \tag{4.19}$$

for every $m \in M$ and every $n \in N$. By proposition 2.8 this condition is equivalent to the fact that the Morita context is compatible with the Frobenius forms as in definition 2.4.

It follows that the 2-morphisms of $\mathcal{C}^{SO(2)}$ and Frob are equal to each other, proving the result. ■

4.4. Remark. In [Dav11, Proposition 3.3.2], the bigroupoid Frob of corollary 4.3 is shown to be equivalent to the bicategory of 2-functors $\mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))$. Assuming a homotopy hypothesis for bigroupoids, as well as an equivariant homotopy hypothesis in a bicategorical framework, this bicategory of functors should agree with the bicategory of homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ in corollary 4.3. Concretely, one might envision the following strategy for an alternative proof of corollary 4.3, which should roughly go as follows:

1. By [Dav11, Proposition 3.3.2], there is an equivalence of bigroupoids
$$\mathrm{Frob} \cong \mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})).$$
2. Then, use the homotopy hypothesis for bigroupoids. By this, we mean that the fundamental 2-groupoid should induce an equivalence of tricategories
$$\Pi_2 : \mathrm{Top}_{\leq 2} \to \mathrm{BiGrp}\,. \tag{4.20}$$
Here, the right hand-side is the tricategory of bigroupoids, whereas the left hand side is a suitable tricategory of 2-types. Such an equivalence of tricategories induces an equivalence of bicategories
$$\mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})) \cong \Pi_2(\mathrm{Hom}(BSO(2), X)), \tag{4.21}$$
where X is a 2-type representing the bigroupoid $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$.
3. Now, consider the trivial homotopy $SO(2)$-action on the 2-type X. Using the fact that we work with the trivial $SO(2)$-action, we obtain a homotopy equivalence $\mathrm{Hom}(BSO(2), X) \cong X^{hSO(2)}$, cf. [Dav11, Page 50].
4. In order to identify the 2-type $X^{hSO(2)}$ with our definition of homotopy fixed points, we additionally need an equivariant homotopy hypothesis: namely, we need to use that a homotopy action of a topological group G on a 2-type Y is equivalent to a G-action on the bicategory $\Pi_2(Y)$ as in definition 3.7 of the present paper. Furthermore, we also need to assume that the fundamental 2-groupoid is G-equivariant, namely that there is an equivalence of bicategories $\Pi_2(Y^{hG}) \cong \Pi_2(Y)^G$. Using this equivariant homotopy hypothesis for the trivial $SO(2)$-action on the 2-type X then should give an equivalence of bicategories
$$\Pi_2(X^{hSO(2)}) \cong \Pi_2(X)^{SO(2)} \cong (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}. \tag{4.22}$$

Combining all four steps gives an equivalence of bicategories between the bigroupoid of Frobenius algebras and homotopy fixed points:

$$\mathrm{Frob} \underset{(1)}{\cong} \mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})) \underset{(2)}{\cong} \Pi_2(\mathrm{Hom}(BSO(2), X)) \underset{(3)}{\cong} \Pi_2(X^{hSO(2)}) \underset{(4)}{\cong} (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}.$$

In order to turn this argument into a full proof, we would need to provide a proof of the homotopy hypothesis for bigroupoids in equation (4.20), as well as a proof for the equivariant homotopy hypothesis in equation (4.22). While the homotopy hypothesis as formulated in equation (4.20) is widely believed to be true, we are not aware of a proof of this statement in the literature. A step in this direction is [MS93], which proves that the homotopy categories of 2-types and 2-groupoids are equivalent. We however really need the full tricategorical version of this statement as in equation (4.20), since we need to identify the (higher) morphisms in BiGrp with (higher) homotopies. Notice that statements of this type are rather subtle, see [KV91, Sim98].

While certainly interesting and conceptually illuminating, a proof of the equivariant homotopy hypothesis in a bicategorical language in equation (4.22) is beyond the scope of the present paper, which aims to give an *algebraic* description of homotopy fixed points on bicategories. Although an equivariant homotopy hypothesis for ∞-groupoids follows from [Lur09a, Theorem 4.2.4.1], we are not aware of a proof of the bicategorical statement in equation (4.22).

Next, we compute homotopy fixed points of the trivial $SO(2)$-action on $\mathrm{Vect}_2^{\mathrm{fd}}$ and show that they are given by Calabi-Yau categories. This result is new and has not yet appeared in the literature.

4.5. Calabi-Yau categories as homotopy fixed points. We now apply Theorem 4.1 to Calabi-Yau categories, as considered in [MS06]. Let Vect_2 be the bicategory consisting of linear, abelian categories, linear functors, and natural transformations.

Recall that a $\mathbb{K}$-linear, abelian category $\mathcal{C}$ is called finite, if is has finite-dimensional Hom-spaces, every object has got finite length, the category $\mathcal{C}$ has got enough projectives, and there are only finitely many isomorphism classes of simple objects.

The fully-dualizable objects of Vect_2 are then precisely the finite, semi-simple linear categories, cf. [BDSV15, Appendix A]. For convenience, we recall the definition of a finite Calabi-Yau category.

4.6. Definition. *Let $\mathbb{K}$ be an algebraically closed field. A Calabi-Yau category $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ is a $\mathbb{K}$-linear, finite, semi-simple category $\mathcal{C}$, together with a family of $\mathbb{K}$-linear maps*

$$\mathrm{tr}_c^{\mathcal{C}} : \mathrm{End}_{\mathcal{C}}(c) \to \mathbb{K} \tag{4.23}$$

for each object c of $\mathcal{C}$, so that:

1. *for each $f \in \mathrm{Hom}_{\mathcal{C}}(c, d)$ and for each $g \in \mathrm{Hom}_{\mathcal{C}}(d, c)$, we have that*

$$\mathrm{tr}_c^{\mathcal{C}}(g \circ f) = \mathrm{tr}_d^{\mathcal{C}}(f \circ g), \tag{4.24}$$

2. *for each $f \in \mathrm{End}_{\mathcal{C}}(x)$ and each $g \in \mathrm{End}_{\mathcal{C}}(d)$, we have that*

$$\mathrm{tr}_{c \oplus d}^{\mathcal{C}}(f \oplus g) = \mathrm{tr}_c^{\mathcal{C}}(f) + \mathrm{tr}_d^{\mathcal{C}}(g), \tag{4.25}$$

3. for all objects c of $\mathcal{C}$, the induced pairing

$$\begin{aligned}\langle - , - \rangle_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(c,d) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(d,c) &\to \mathbb{K} \\ f \otimes g &\mapsto \mathrm{tr}_c^{\mathcal{C}}(g \circ f)\end{aligned} \tag{4.26}$$

is a non-degenerate pairing of $\mathbb{K}$-vector spaces.

We will call the collection of morphisms $\mathrm{tr}_c^{\mathcal{C}}$ a trace *on $\mathcal{C}$.*

An equivalent way of defining a Calabi-Yau structure on a linear category $\mathcal{C}$ is by specifying a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(c,d) \to \mathrm{Hom}_{\mathcal{C}}(d,c)^*, \tag{4.27}$$

cf. [Sch13, Proposition 4.1].

4.7. DEFINITION. *Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{tr}^{\mathcal{D}})$ be two Calabi-Yau categories. A linear functor $F : \mathcal{C} \to \mathcal{D}$ is called a Calabi-Yau functor, if*

$$\mathrm{tr}_c^{\mathcal{C}}(f) = \mathrm{tr}_{F(c)}^{\mathcal{D}}(F(f)) \tag{4.28}$$

for each $f \in \mathrm{End}_{\mathcal{C}}(c)$ and for each $c \in \mathrm{Ob}(\mathcal{C})$. Equivalently, one may require that

$$\langle Ff, Fg \rangle_{\mathcal{D}} = \langle f, g \rangle_{\mathcal{C}} \tag{4.29}$$

for every pair of morphisms $f : c \to d$ and $g : d \to c$ in $\mathcal{C}$.

If F, $G : \mathcal{C} \to \mathcal{D}$ are two Calabi-Yau functors between Calabi-Yau categories, a Calabi-Yau natural transformation is just an ordinary natural transformation.

This allows us to define the symmetric monoidal bicategory CY *consisting of Calabi-Yau categories, Calabi-Yau functors and natural transformations. The monoidal structure is given by the Deligne tensor product of abelian categories.*

4.8. COROLLARY. *Suppose $\mathcal{C} = \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$, and consider the trivial $SO(2)$-action on $\mathcal{C}$. Then $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of Calabi-Yau categories.*

PROOF. Indeed, by Theorem 4.1 a homotopy fixed point consists of a category $\mathcal{C}$, together with a natural transformation $\lambda : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$. Let $X_1, \ldots, X_n$ be the simple objects of $\mathcal{C}$. Then, the natural transformation $\lambda : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is fully determined by giving an endomorphism $\lambda_X : X \to X$ for every simple object X. Since λ is an invertible natural transformation, the λ_X must be central invertible elements in $\mathrm{End}_{\mathcal{C}}(X)$. Since we work over an algebraically closed field, Schur's Lemma shows that $\mathrm{End}_{\mathcal{C}}(X) \cong \mathbb{K}$ as vector spaces. Hence, the structure of a natural transformation of the identity functor of $\mathcal{C}$ boils down to choosing a non-zero scalar for each simple object of $\mathcal{C}$. This structure is equivalent to giving $\mathcal{C}$ the structure of a Calabi-Yau category.

Now note that by equation (4.1) in Theorem 4.1, 1-morphisms of homotopy fixed points consist of equivalences of categories $F : \mathcal{C} \to \mathcal{C}'$ so that $F(\lambda_X) = \lambda'_{F(X)}$ for every object X of $\mathcal{C}$. This is exactly the condition saying that F must a Calabi-Yau functor.

Finally, one can see that 2-morphisms of homotopy fixed points are given by natural isomorphisms of Calabi-Yau functors. ■

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THE SERRE AUTOMORPHISM VIA HOMOTOPY ACTIONS AND THE COBORDISM HYPOTHESIS FOR ORIENTED MANIFOLDS

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Abstract. We explicitely construct an $SO(2)$-action on a skeletal version of the 2-dimensional framed bordism bicategory. By the 2-dimensional Cobordism Hypothesis for framed manifolds, we obtain an $SO(2)$-action on the core of fully-dualizable objects of the target bicategory. This action is shown to coincide with the one given by the Serre automorphism. We give an explicit description of the bicategory of homotopy fixed points of this action, and discuss its relation to the classification of oriented 2-d topological quantum field theories.

1. Introduction

According to [Lur09], fully-extended 2-dimensional oriented topological quantum field theories are classified by homotopy fixed points of an $SO(2)$-action on the core of fully-dualizable objects of the target space. This action is supposed to be induced by an $SO(2)$-action on the framed bordism bicategory. In this paper, we aim to make these statements precise, along with developing new results concerning actions of groups on bicategories.

We first clarify the situation on the algebraic side by giving a detailed description of the $SO(2)$-action of the Serre automorphism on the core of fully-dualizable objects $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ of an arbitrary symmetric monoidal bicategory $\mathcal{C}$.

We then explicitly construct an $SO(2)$-action on a skeletal version of the framed bordism bicategory, which has been obtained in terms of generators and relations in [Pst14].

By the Cobordism Hypothesis for framed manifolds, which has been proven in the setting of bicategories in [Pst14], there is an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}}). \tag{1.1}$$

This equivalence allows us to transport the $SO(2)$-action on the framed bordism bicategory to the core of fully-dualizable objects of $\mathcal{C}$. We then prove that this induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is given precisely by the Serre automorphism, showing that the Serre automorphism has indeed a geometric origin, as expected from [Lur09].

In the last section we comment on the relation between the oriented bordism bicategory and the bicategory of homotopy co-invariants of the $SO(2)$-action on the framed bordism bicategory. In fact, we argue that exhibiting the oriented bordism bicategory as the colimit of the action is equivalent to proving the Cobordism Hypothesis for oriented manifolds.

The paper is organized as follows.

In Section 2 we recall the notion of a fully-dualizable object in a symmetric monoidal bicategory $\mathcal{C}$. For each such an object X, we define the Serre automorphism as a certain 1-endomorphism of X. We show that the Serre automorphism is a pseudo-natural transformation of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which is moreover monoidal. This suffices to define an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

Section 3 investigates when a group action on a bicategory $\mathcal{C}$ is equivalent to the trivial action. We obtain a general criterion for when such an action is trivializable.

In Section 4, we compute the bicategory of homotopy fixed points of an $SO(2)$-action coming from a pseudo-natural transformation of the identity functor of an arbitrary bicategory $\mathcal{C}$. This generalizes the main result in [HSV16], which computes homotopy fixed points of the trivial $SO(2)$-action on $\mathrm{Alg}^{\mathrm{fd}}_2$. Our more general theorem allows us to give an explicit description of the bicategory of homotopy fixed points of the Serre automorphism.

In Section 5, we introduce a skeletal version of the framed bordism bicategory by generators and relations, and define a non-trivial $SO(2)$-action on this bicategory. By the framed Cobordism Hypothesis, as

in Equation (1.1), we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we prove to be given by the Serre automorphism.
In Section 6 we discuss 2d invertible field theories, providing a general criterion for the trivialization of the $SO(2)$-action in this case.
In Section 7, we give an outlook on *homotopy co-invariants* of the $SO(2)$-action, and argue about their relation to the Cobordism Hypothesis for oriented manifolds.

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2. Fully-dualizable objects and the Serre automorphism

The aim of this section is to introduce the main objects of the present paper. On the algebraic side, these are fully-dualizable objects in a symmetric monoidal bicategory $\mathcal{C}$, and the Serre automorphism. Though some of the following material has already appeared in the literature, we recall the relevant definitions in order to fix notation. For details, we refer the reader to [Pst14].

Definition 2.1. *A dual pair in a symmetric monoidal bicategory $\mathcal{C}$ consists of an object X, an object X^*, two 1-morphisms*

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to 1 \\ \mathrm{coev}_X &: 1 \to X^* \otimes X \end{aligned} \tag{2.1}$$

and two invertible 2-morphisms α and β in the diagrams below.

$$\begin{tikzcd} & X \otimes 1 \arrow[r, "\mathrm{id}_X \otimes \mathrm{coev}_X"] & X \otimes (X^* \otimes X) \arrow[r, "a"] \arrow[d, phantom, "\Downarrow\alpha"] & (X \otimes X^*) \otimes X \arrow[r, "\mathrm{ev}_X \otimes \mathrm{id}_X"] & 1 \otimes X \arrow[dr, "l"] & \\ X \arrow[ur, "r"] \arrow[rrrrr, "\mathrm{id}_X"'] & & {} & & & X \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} & 1 \otimes X^* \arrow[r, "\mathrm{coev}_X \otimes \mathrm{id}_{X^*}"] & (X^* \otimes X) \otimes X^* \arrow[r, "a"] \arrow[d, phantom, "\Downarrow\beta"] & X^* \otimes (X \otimes X^*) \arrow[r, "\mathrm{id}_{X^*} \otimes \mathrm{ev}_X"] & X^* \otimes 1 \arrow[dr, "r"] & \\ X^* \arrow[ur, "l"] \arrow[rrrrr, "\mathrm{id}_{X^*}"'] & & {} & & & X^* \end{tikzcd} \tag{2.3}$$

We call an object X of $\mathcal{C}$ dualizable if it can be completed to a dual pair. A dual pair is said to be coherent if the "swallowtail" equations are satisfied, as in [Pst14, Def. 2.6].

Remark 2.2. Given a dual pair, it is always possible to modify the 2-cell β in such a way that the swallowtail are fulfilled, cf. [Pst14, Theorem 2.7].

Dual pairs can be organized into a bicategory by defining appropriate 1- and 2-morphisms between them. The bicategory of dual pairs turns out to be a 2-groupoid. Moreover, the bicategory of coherent dual pairs is equivalent to the core of dualizable objects in $\mathcal{C}$. In particular, this shows that any two coherent dual pairs over the same dualizable object are equivalent.
We now come to the stronger concept of fully-dualizability.

Definition 2.3. *An object X in a symmetric monoidal bicategory is called fully-dualizable if it can be completed into a dual pair and the evaluation and coevaluation maps admit both left- and right adjoints.*

Note that if left- and right adjoints exists, the adjoint maps will have adjoints themselves, since we work in a bicategorical setting [Pst14]. Thus, Definition 2.3 agrees with the definition of [Lur09] in the special case of bicategories.

2.1. **The Serre automorphism.** Recall that by definition, the evaluation morphism for a fully dualizable object X admits both a right-adjoint ev_X^R and a left adjoint ev_X^L. We use these adjoints to define the Serre-automorphism of X:

Definition 2.4. *Let X be a fully-dualizable object in a symmetric monoidal bicategory. The Serre automorphism of X is the following composition of 1-morphisms:*

$$S_X : X \cong X \otimes 1 \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X^R} X \otimes X \otimes X^* \xrightarrow{\tau_{X,X} \otimes \mathrm{id}_{X^*}} X \otimes X \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes 1 \cong X. \tag{2.4}$$

Notice that the Serre automorphism is actually a 1-equivalence of X, since an inverse is given by the 1-morphism

$$S_X^{-1} = (\mathrm{id}_X \circ \mathrm{ev}_X) \circ (\tau_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes \mathrm{ev}_X^L), \tag{2.5}$$

cf. [Lur09, DSS13].

The next lemma is well-known [Lur09, Pst14], and is straightforward to show graphically.

Lemma 2.5. *Let X be fully-dualizable in $\mathcal{C}$. Then, there are 2-isomorphisms*

$$\begin{aligned} \mathrm{ev}_X^R &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X) \circ \mathrm{coev}_X \\ \mathrm{ev}_X^L &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X^{-1}) \circ \mathrm{coev}_X\,. \end{aligned} \tag{2.6}$$

Next, we show that the Serre automorphism is actually a pseudo-natural transformation of the identity functor on the maximal subgroupoid of $\mathcal{C}$, as suggested in [Sch13]. To the best of our knowledge, a proof of this statement has not appeared in the literature so far, hence we illustrate the details in the following. We begin by showing that the evaluation 1-morphism is "dinatural".

Lemma 2.6. *Let X be dualizable in $\mathcal{C}$. The evaluation 1-morphism ev_X is "dinatural": for every 1-morphism $f : X \to Y$ between dualizable objects, there is a natural 2-isomorphism ev_f in the diagram below.*

$$\begin{tikzcd} X \otimes Y^* \arrow[r, "\mathrm{id}\otimes f^*"] \arrow[d, "f\otimes\mathrm{id}"'] & X \otimes X^* \arrow[d, "\mathrm{ev}_X"] \arrow[dl, Rightarrow, "\mathrm{ev}_f"] \\ Y \otimes Y^* \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd} \tag{2.7}$$

Proof. We explicitly write out the definition of f^* and define ev_f to be the composition of the 2-morphisms in the diagram below.

(2.8)

$$\mathrm{ev}_F := \begin{tikzcd} X\,1\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{coev}_X\,\mathrm{id}"] & X\,X^*\,X\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{id}\,f\,\mathrm{id}"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\cong"] & X\,X^*\,Y\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{id}\,\mathrm{ev}_Y"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\cong"] & X\,X^*\,1 \arrow[r, "\mathrm{id}\,r"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}"] \arrow[dr, phantom, "\Swarrow r_{\mathrm{ev}_X}"] & X\,X^* \arrow[dd, bend left, "\mathrm{ev}_X"] \\ & 1\,X\,Y^* \arrow[r, "\mathrm{id}\,f\,\mathrm{id}"] \arrow[d, "l\,\mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\Swarrow l_f\,\mathrm{id}_{\mathrm{id}}"] & 1\,Y\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{ev}_Y"] \arrow[d, "l\,\mathrm{id}"] \arrow[dr, phantom, "\Swarrow l_{\mathrm{ev}_Y}"] & 1\,1 \arrow[dr, bend left, "r"] \arrow[dr, bend right, "l"'] \arrow[dr, phantom, "\cong"] & \\ X\,Y^* \arrow[uu, bend left, "\mathrm{id}\,l"] \arrow[uu, bend right, "r\,\mathrm{id}"'] \arrow[uu, phantom, "\cong"] \arrow[ur, phantom, "\Swarrow \alpha\,\mathrm{id}_{\mathrm{id}}"] \arrow[r, "\mathrm{id}\,\mathrm{id}"] \arrow[rr, bend right, "f\,\mathrm{id}"'] \arrow[rr, phantom, bend right=20, "\cong"] & X\,Y^* \arrow[r, "f\,\mathrm{id}"] & Y\,Y^* \arrow[rr, "\mathrm{ev}_Y"] & & 1 \end{tikzcd}$$

□

In order to show that the Serre automorphism is pseudo-natural, we also need to show the dinaturality of the right adjoint of the evaluation.

Lemma 2.7. *For a fully-dualizable object X of $\mathcal{C}$, the right adjoint ev^R of the evaluation is "dinatural" with respect to 1-equivalences: for every 1-equivalence $f : X \to Y$ between fully-dualizable objects, there is a natural 2-isomorphism ev_f^R in the diagram below.*

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{ev}_X^R"] \arrow[d, "\mathrm{ev}_Y^R"'] & X \otimes X^* \arrow[d, "f\otimes\mathrm{id}"] \arrow[dl, Rightarrow, "\mathrm{ev}_f^R"] \\ Y \otimes Y^* \arrow[r, "\mathrm{id}\otimes f^*"'] & Y \otimes X^* \end{tikzcd} \tag{2.9}$$

Proof. In a first step, we show that $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is a right-adjoint to $\mathrm{ev}_X \circ (f^{-1} \otimes f^*)$. In formula:

$$(\mathrm{ev}_X \circ f^{-1} \otimes f^*)^R = f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R \,. \tag{2.10}$$

Indeed, let

$$\begin{aligned} \eta_X &: \mathrm{id}_{X \otimes X^*} \to \mathrm{ev}_X^R \circ \mathrm{ev}_X \\ \varepsilon_X &: \mathrm{ev}_X \circ \mathrm{ev}_X^R \to \mathrm{id}_1 \end{aligned} \tag{2.11}$$

be the unit and counit of the right-adjunction of ev_X and its right adjoint ev_X^R. We construct unit and counit for the adjunction in Equation (2.10). Let

$$\begin{aligned} &\tilde{\varepsilon} : \mathrm{ev}_X \circ (f^{-1} \otimes f^*) \circ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \cong \mathrm{ev}_X \circ \mathrm{ev}_X^R \xrightarrow{\varepsilon_X} \mathrm{id}_1 \\ &\tilde{\eta} : \mathrm{id}_{Y \otimes Y^*} \cong (f \otimes (f^*)^{-1}) \circ (f^{-1} \otimes f^*) \xrightarrow{\mathrm{id} * \eta_X * \mathrm{id}} (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \circ \mathrm{ev}_X \circ (f^{-1} \otimes f^*). \end{aligned} \tag{2.12}$$

Now, one checks that the quadruple

$$(\mathrm{ev}_X \circ (f^{-1} \otimes f^*),\, (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R,\, \tilde{\varepsilon}, \tilde{\eta}) \tag{2.13}$$

fulfills indeed the axioms of an adjunction. This follows from the fact that the quadruple $(\mathrm{ev}_X, \mathrm{ev}_X^R, \varepsilon_X, \eta_X)$ is an adjunction. This shows Equation (2.10).
Now, notice that due to the dinaturality of the evaluation in Lemma 2.6, we have a natural 2-isomorphism

$$\mathrm{ev}_Y \cong \mathrm{ev}_X \circ (f^{-1} \otimes f^*). \tag{2.14}$$

Combining this 2-isomorphism with Equation (2.10) shows that the right adjoint of ev_Y is given by $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$. Since all right-adjoints are isomorphic the 1-morphism $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is isomorphic to ev_Y^R, as desired. ☐

We can now prove the following

Proposition 2.8. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Denote by $\mathscr{K}(\mathcal{C})$ the maximal sub-bigroupoid of $\mathcal{C}$. The Serre automorphism S is a pseudo-natural isomorphism of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.*

Proof. Let $f : X \to Y$ be a 1-morphism in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We need to provide a natural 2-isomorphism in the diagram

$$\begin{tikzcd} X \arrow[r, "S_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \arrow[dl, Rightarrow, "S_f"] \\ Y \arrow[r, "S_Y"'] & Y \end{tikzcd} \tag{2.15}$$

By spelling out the definition of the Serre automorphism, we see that this is equivalent to filling the following diagram with natural 2-cells:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X\,1 \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X^R"] \arrow[d, "f\ \mathrm{id}"] & X\,X\,X^* \arrow[r, "\tau_{X,X}\ \mathrm{id}_{X^*}"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,X\,X^* \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,1 \arrow[r] \arrow[d, "f\ \mathrm{id}"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y\,1 \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y^R"'] & Y\,Y\,Y^* \arrow[r, "\tau_{Y,Y}\ \mathrm{id}_{Y^*}"'] & Y\,Y\,Y^* \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y"'] & Y\,1 \arrow[r] & Y \end{tikzcd} \tag{2.16}$$

The first, the last and the middle square can be filled with a natural 2-cell due to the fact that $\mathcal{C}$ is a symmetric monoidal bicategory. The square involving the evaluation commutes up to a 2-cell using the mate of the 2-cell of Lemma 2.6, while the square involving the right adjoint of the evaluation commutes up a 2-cell using the mate of the 2-cell of Lemma 2.7. ☐

2.2. **Monoidality of the Serre automorphism.** In this section we show that the Serre automorphism respects the monoidal structure. We begin with the following two lemma

Lemma 2.9. *Let $\mathcal{C}$ be a monoidal bicategory. Let X and Y be dualizable objects of $\mathcal{C}$. Then, there is a 1-equivalence $\xi : (X \otimes Y)^* \cong Y^* \otimes X^*$.*

Proof. Define two 1-morphisms in $\mathcal{C}$ by setting

$$(\mathrm{id}_{Y^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{ev}_{X\otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_{(X\otimes Y)^*}) : (X\otimes Y)^* \to Y^* \otimes X^* \tag{2.17}$$

$$(\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}^*_Y \otimes \mathrm{id}_{X^*}) : Y^* \otimes X^* \to (X\otimes Y)^*. \tag{2.18}$$

These two 1-morphisms are (up to invertible 2-cells) inverse to each other. This shows the claim. □

Now, we show that the evaluation 1-morphism respects the monoidal structure:

Lemma 2.10. *For a dualizable object X of a symmetric monoidal bicategory $\mathcal{C}$, the evaluation 1-morphism is monoidal. More precisely: the following diagram commutes up to a natural 2-cell.*

$$\begin{tikzcd}
(X\otimes Y)\otimes (X\otimes Y)^* \arrow[rr, "\mathrm{ev}_{X\otimes Y}"] \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes\xi"'] & & 1 \arrow[d] \\
(X\otimes Y)\otimes Y^*\otimes X^* \arrow[r, "\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*}"'] & X\otimes X^*\otimes Y\otimes Y^* \arrow[r, "\mathrm{ev}_X\otimes\mathrm{ev}_Y"'] & 1\otimes 1
\end{tikzcd} \tag{2.19}$$

Here, the 1-equivalence ξ is due to Lemma 2.9.

Proof. Consider the diagram in figure 1 on page 20: here, the composition of the horizontal arrows at the top, together with the two arrows on the vertical right are exactly the 1-morphism in Equation (2.19). The other arrow is given by $\mathrm{ev}_{X\otimes Y}$. We have not written down the tensor product, and left out isomorphisms of the form $1\otimes X \cong X \cong X\otimes 1$ for readability. □

We can now establish the monoidality of the right adjoint of the evaluation via the following

Lemma 2.11. *Let $\mathcal{C}$ a symmetric monoidal bicategory, and let X and Y be fully-dualizable objects. Then, the right adjoint of the evaluation is monoidal. More precisely: if $\xi : (X\otimes Y)^* \to Y^*\otimes X^*$ is the 1-equivalence of Lemma 2.9, the following diagram commutes up to a natural 2-cell.*

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{ev}^R_{X\otimes Y}"] \arrow[d, "\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y"'] & X\otimes Y\otimes (X\otimes Y)^* \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes\xi"] \\
X\otimes X^*\otimes Y\otimes Y^* \arrow[r, "\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*}"'] & X\otimes Y\otimes Y^*\otimes X^*
\end{tikzcd} \tag{2.20}$$

Proof. In a first step, we show that the right adjoint of the 1-morphism

$$(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \tag{2.21}$$

is given by the 1-morphism

$$(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\circ\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y). \tag{2.22}$$

Indeed, if

$$\begin{aligned} \eta_X &: \mathrm{id}_{X\otimes X^*} \to \mathrm{ev}^R_X \circ \mathrm{ev}_X \\ \varepsilon_X &: \mathrm{ev}_X \circ \mathrm{ev}^R_X \to \mathrm{id}_1 \end{aligned} \tag{2.23}$$

are the unit and counit of the right-adjunction of ev_X and its right adjoint ev^R_X, we construct adjunction data for the adjunction in equations (2.21) and (2.22) as follows. Let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be the following 2-morphisms:

$$\begin{aligned} \tilde{\varepsilon} : &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi)\circ(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1}) \\ &\qquad\qquad\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ \cong\ &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ \cong\ &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \xrightarrow{\varepsilon_X\otimes\varepsilon_Y} \mathrm{id}_1 \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} \tilde{\eta} : \mathrm{id}_{X\otimes Y\otimes(X\otimes Y)^*} &\cong (\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \\ &\cong (\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \\ &\xrightarrow{\mathrm{id}\otimes\eta_X\otimes\eta_Y\otimes\mathrm{id}} (\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y) \\ &\qquad\circ(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \end{aligned} \tag{2.25}$$

One now shows that the two 1-morphisms in Equation (2.21) and (2.22), together with the two 2-morphisms $\tilde{\varepsilon}$ and $\tilde{\eta}$ form an adjunction. This gives that the two 1-morphisms in Equations (2.21) and (2.22) are adjoint.

Next, notice that the 1-morphism in Equation (2.21) is isomorphic to the 1-morphism $\mathrm{ev}_{X\otimes Y}$ by Lemma 2.10. Thus, the right adjoint of $\mathrm{ev}_{X\otimes Y}$ is given by the right adjoint of the 1-morphism in Equation (2.21), which is the 1-morphism in Equation (2.22) by the argument above. Since all adjoints are equivalent, this shows the lemma. □

We are now ready to prove that the Serre automorphism is a monoidal pseudo-natural transformation.

Proposition 2.12. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, the Serre automorphism is a* monoidal *pseudo-natural transformation of* $\mathrm{Id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$.

Proof. By definition (cf. [SP09, Definition 2.7]) we have to provide invertible 2-cells

$$\begin{aligned} \Pi &: S_{X\otimes Y} \to S_X \otimes S_Y \\ M &: \mathrm{id}_1 \to S_1. \end{aligned} \tag{2.26}$$

By the definition of the Serre automorphism in Definition 2.4, it suffices to show that the evaluation and its right adjoint are monoidal, since the braiding τ will be monoidal by definition. The monoidality of the evaluation is proven in Lemma 2.10, while the monoidality of its right adjoint follows from Lemma 2.11. These two lemmas thus provide an invertible 2-cell $S_{X\otimes Y} \cong S_X \otimes S_Y$. The second 2-cell $\mathrm{id}_1 \to S_1$ can be constructed in a similar way, by noticing that $1 \cong 1^*$. □

3. Monoidal homotopy actions

In this section, we investigate homotopy actions on symmetric monoidal bicategories. In particular, we are interested in the case when the group action is compatible with the monoidal structure. By a (homotopy) action of a topological group G on a bicategory $\mathcal{C}$, we mean a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, where $\Pi_2(G)$ is the fundamental 2-groupoid of G, and $\mathrm{Aut}(\mathcal{C})$ is the bicategory of auto-equivalences of $\mathcal{C}$. For details on homotopy actions of groups on bicategories, we refer the reader to [HSV16].

In order to simplify the exposition, we introduce the following

Definition 3.1. *Let G be a topological group. We will say that G is 2-truncated if $\pi_2(G,x)$ is trivial for every base point $x \in G$.*

Moreover, we will need also the following

Definition 3.2. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. We will say that $\mathcal{C}$ is 1-connected if it is monoidally equivalent to B^2H, for some abelian group H.*

In the following, we denote with $\mathrm{Aut}^{\otimes}(\mathcal{C})$ the bicategory of auto-equivalences of $\mathcal{C}$ which are compatible with the monoidal structure.

Definition 3.3. *Let $\mathcal{C}$ be a symmetric monoidal category and G be a topological group. A* monoidal *homotopy action of G on $\mathcal{C}$ is a monoidal morphism $\rho : \Pi_2(G) \to \mathrm{Aut}^{\otimes}(\mathcal{C})$.*

We now prove a general criterion for when monoidal homotopy actions are trivializable.

Proposition 3.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let G be a path connected topological group. Assume that G is 2-truncated, and $\mathrm{Aut}^{\otimes}(\mathcal{C})$ is 1-connected, with an abelian group H. If $H^2_{grp}(\pi_1(G,e),H) \simeq 0$, then any monoidal homotopy action of G on $\mathcal{C}$ is pseudo-naturally isomorphic to the trivial action.*

Proof. Let $\rho : \Pi_2(G) \to \mathrm{Aut}^{\otimes}(\mathcal{C})$ be a weak monoidal 2-functor. Since $\mathrm{Aut}^{\otimes}(\mathcal{C})$ was assumed to be monoidally equivalent to B^2H for some abelian group H, the group action ρ is equivalent to a 2-functor $\rho : \Pi_2(G) \to B^2H$. Due to the fact that G is path connected and 2-truncated, we have that $\Pi_2(G) \simeq B\pi_1(G,e)$, where $\pi_1(G,e)$ is regarded as a discrete monoidal category. Thus, the homotopy action ρ is equivalent to a weak monoidal 2-functor $B\pi_1(G,e) \to B^2H$. We claim that such functors are classified by $H^2_{grp}(\pi_1(G,e),H)$ up to pseudo-natural isomorphism. Indeed, going through the definition as in [SP09, Definition 2.5], we find that the only non-trivial datum of a monoidal 2-functor $F : B\pi_1(G,e) \to B^2H$ is given by an invertible endo-modification ω of $\mathrm{id}_F \otimes \mathrm{id}_F$, which is nothing else than a 2-cocycle. One now checks that a monoidal pseudo-natural transformations between two such functors is exactly a 2-coboundary, which shows the claim. Since we assumed that $H^2_{grp}(\pi_1(G,e),H) \simeq 0$, the original action ρ must be trivializable. □

Next, we show that the bicategory $\mathrm{Alg}_2^{\mathrm{fd}}$ of finite-dimensional, semi-simple algebras, bimodules and intertwiners, equipped with the monoidal structure given by the *direct sum* fulfills the conditions of Proposition 3.4.

Lemma 3.5. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ be the bicategory where objects are given by finite-dimensional, semi-simple algebras, equipped with the monoidal structure given by the direct sum. Then, $\mathrm{Aut}^{\otimes}(\mathcal{C})$ is equivalent to $B^2\mathbb{K}^*$.*

Proof. Let $F : \mathrm{Alg}_2^{\mathrm{fd}} \to \mathrm{Alg}_2^{\mathrm{fd}}$ be a weak monoidal 2-equivalence, and let A be a finite-dimensional, semi-simple algebra. Then A is isomorphic to a direct sum of matrix algebras. Calculating up to Morita equivalence and using that F has to preserve the single simple object $\mathbb{K}$ of Alg_2, we have

$$F(A) \cong F\left(\bigoplus_i M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F\left(M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F(\mathbb{K}) \cong \bigoplus_i \mathbb{K} \cong \bigoplus_i M_{n_i}(\mathbb{K}) \cong A. \tag{3.1}$$

Thus, the functor F is pseudo-naturally isomorphic to the identity functor on $\mathrm{Alg}_2^{\mathrm{fd}}$.
Now, let $\eta : F \to G$ be a monoidal pseudo-natural isomorphism between two endofunctors of Alg_2. Since both F and G are pseudo-naturally isomorphic to the identity, we may consider instead a pseudo-natural isomorphism $\eta : \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}} \to \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}$. We claim that up to an invertible modification, the 1-equivalence $\eta_A : A \to A$ must be given by the bimodule ${}_AA_A$, which is the identity 1-morphism on A in Alg_2. Indeed, since η_A is assumed to be linear, it suffices to consider the case of $A = M_n(\mathbb{K})$ and to take direct sums. It is well-known that the only simple modules of A are given by $\mathbb{K}^n$. Thus,

$$\eta_A = (\mathbb{K}^n)^{\alpha} \otimes_{\mathbb{K}} (\mathbb{K}^n)^{\beta}, \tag{3.2}$$

where α and β are multiplicities. Now, [HSV16, Lemma 2.6] ensures that these multiplicities are trivial, and thus we have $\eta_A = {}_AA_A$ up to an invertible intertwiner. This shows that up to invertible modifications, all 1-morphisms in $\mathrm{Aut}^{\otimes}(\mathrm{Alg}_2^{\mathrm{fd}})$ are identities.
Now, let m be an invertible endo-modification of the pseudo-natural transformation $\mathrm{id}_{\mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}}$. Then, the component $m_A : {}_AA_A \to {}_AA_A$ is an element of $\mathrm{End}_{(A,A)}(A) \cong \mathbb{K}$. This show that the 2-morphisms of $\mathrm{Aut}^{\otimes}(\mathrm{Alg}_2^{\mathrm{fd}})$ stand in bijection to $\mathbb{K}^*$. □

Remark 3.6. Notice that the symmetric monoidal structure on $\mathrm{Alg}_2^{\mathrm{fd}}$ considered above is *not* the standard one, which is instead the one induced by the tensor product of algebras, and which is the monoidal structure relevant for the remainder of the paper.

The last lemmas imply the following

Lemma 3.7. *Any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ equipped with the monoidal structure given by the direct sum is trivial.*

Proof. Since $\pi_1(SO(2), e) \simeq \mathbb{Z}$, and $H^2_{grp}(\mathbb{Z}, \mathbb{K}^*) \simeq H^2(S^1, \mathbb{K}^*) \simeq 0$, Proposition 3.4 and Lemma 3.5 ensure that any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ is trivializable. □

Corollary 3.8. *Since $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$ are equivalent as additive categories, any $SO(2)$-action on $\mathrm{Vect}_2^{\mathrm{fd}}$ via linear morphisms is trivializable.*

Remark 3.9. The last two results rely on the fact that $\mathrm{Aut}^{\otimes}(\mathrm{Alg}_2^{\mathrm{fd}})$ and $\mathrm{Aut}^{\otimes}(\mathrm{Vect}_2^{\mathrm{fd}})$ are 1-connected as additive categories. This is due to the fact that fully-dualizable part of either Alg_2 or Vect_2 is semi-simple. An example in which the conditions in Proposition 3.4 do *not* hold is provided by the bicategory of Landau-Ginzburg models.

4. Computing homotopy fixed points

In this Section, we explicitly compute the bicategory of homotopy fixed points of an $SO(2)$-action which is induced by an arbitrary pseudo-natural equivalence of the identity functor of an arbitrary bicategory $\mathcal{C}$. Recall that a G-action on a bicategory $\mathcal{C}$ is a monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, or equivalently a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The bicategory of homotopy fixed points $\mathcal{C}^G$ is then given by the tri-limit of this trifunctor.
In Bicat, the tricategory of bicategory, this trilimit can be computed as follows: if $\Delta : B\Pi_2(G) \to \mathrm{Bicat}$ is the constant functor assigning to the one object $*$ the terminal bicategory with one object, the trilimit of the action functor ρ is given by

$$\mathcal{C}^G := \lim \rho = \mathrm{Nat}(\Delta, \rho), \tag{4.1}$$

the bicategory of tri-transformations between ρ and Δ. This definition is explicitly spelled out in [HSV16, Remark 3.11]. We begin by defining an $SO(2)$-action on an arbitrary symmetric monoidal bicategory, starting from a pseudo-natural transformation of the identity functor on $\mathcal{C}$.

Definition 4.1. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Since $\Pi_2(SO(2))$ is equivalent to the bicategory with one object, $\mathbb{Z}$ worth of morphisms, and only identity 2-morphisms, we may define an $SO(2)$-action $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathcal{C})$ by the following data:*

- *For every group element $g \in SO(2)$, we assign the identity functor of $\mathcal{C}$.*
- *For the generator $1 \in \mathbb{Z}$, we assign the pseudo-natural transformation of the identity functor given by α.*
- *Since there are only identity 2-morphisms in $\mathbb{Z}$, we have to assign these to identity 2-morphisms in $\mathcal{C}$.*
- *For composition of 1-morphisms, we assign the identity modifications $\rho(a + b) := \rho(a) \circ \rho(b)$.*
- *In order to make ρ into a monoidal 2-functor, we have to assign additional data which we can choose to be trivial. In detail, we set $\rho(g \otimes h) := \rho(g) \otimes \rho(h)$, and $\rho(e) := \mathrm{id}_{\mathcal{C}}$. Finally, we choose ω, γ and δ as in* [HSV16, Remark 3.8] *to be identities.*

Our main example is the action of the Serre automorphism on the core of fully-dualizable objects:

Example 4.2. If $\mathcal{C}$ is a symmetric monoidal bicategory, consider $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the core of the fully-dualizable objects of $\mathcal{C}$. By Proposition 2.8, the Serre automorphism defines a pseudo-natural equivalence of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we denote by ρ^S.

The next theorem computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the action in Definition 4.1. This theorem generalizes [HSV16, Theorem 4.1], which only computes the bicategory of homotopy fixed points of the *trivial* $SO(2)$-action.

Theorem 4.3. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Let ρ be the $SO(2)$-action on $\mathcal{C}$ as in Definition 4.1. Then, the bicategory of homotopy fixed points $\mathcal{C}^G$ is equivalent to the bicategory with*

- *objects: (c, λ) where c is an object of $\mathcal{C}$ and $\lambda : \alpha_c \to \mathrm{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \to (c', \lambda')$ in $\mathcal{C}^G$ are given by 1-morphisms $f : c \to c'$ in $\mathcal{C}$, so that the diagram*

$$\begin{tikzcd} \alpha_{c'} \circ f \arrow[d, "\lambda' * \mathrm{id}_f"'] & f \circ \alpha_c \arrow[l, "\alpha_f"'] \arrow[r, "\mathrm{id}_f * \lambda"] & f \circ \mathrm{id}_c \arrow[d] \\ \mathrm{id}_c \circ f \arrow[rr] & & f \end{tikzcd} \tag{4.2}$$

  commutes,
- *2-morphisms of $\mathcal{C}^G$ are given by 2-morphisms in $\mathcal{C}$.*

Proof. In order to prove the theorem, we need to explicitly unpack the definition of the bicategory of homotopy fixed points $\mathcal{C}^G$. This is done in [HSV16, Remark 3.11 - 3.14]. In the following, we will use the notation introduced in [HSV16].

The idea of the proof is to show that the forgetful functor which on objects of $\mathcal{C}^G$ forgets the data Θ, Π and M is an equivalence of bicategories. In order to show this, we need to analyze the bicategory of homotopy fixed points. We start with the objects of $\mathcal{C}^G$.

By definition, a homotopy fixed point of this action consists of

- An object $c \in \mathcal{C}$,
- A 1-equivalence $\Theta : c \to c$,
- For every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \alpha_c^n \circ \Theta \to \Theta \circ \mathrm{id}_c$ so that (Θ, Θ_n) fulfill the axioms of a pseudo-natural transformation,
- A 2-isomorphism $\Pi : \Theta \circ \Theta \to \Theta$ which obeys the modification square,
- Another 2-isomorphism $M : \Theta \to \mathrm{id}_c$

so that the following equations hold: Equation 3.18 of [HSV16] demands that

$$\Pi \circ (\mathrm{id}_\Theta * \Pi) = \Pi \circ (\Pi * \mathrm{id}_\Theta) \tag{4.3}$$

whereas Equation 3.19 of [HSV16] demands that Π equals the composition

$$\Theta \circ \Theta \xrightarrow{\mathrm{id}_\Theta * M} \Theta \circ \mathrm{id}_c \cong \Theta \tag{4.4}$$

and finally Equation 3.20 of [HSV16] tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \mathrm{id}_\Theta} \mathrm{id}_c \circ \Theta \cong \Theta. \tag{4.5}$$

Hence Π is fully specified by M. An explicit calculation using the two equations above then confirms that Equation (4.3) is automatically fulfilled. Indeed, by composing with Π^{-1} from the right, it suffices to show that $\mathrm{id}_\Theta * \Pi = \Pi * \mathrm{id}_\Theta$. Suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Then,

$$\begin{aligned} \mathrm{id}_\Theta * \Pi &= \mathrm{id}_\Theta * (M * \mathrm{id}_\Theta) && \text{by equation (4.5)} \\ &= (\mathrm{id}_\Theta * M) * \mathrm{id}_\Theta \\ &= \Pi * \mathrm{id}_\Theta && \text{by equation (4.4).} \end{aligned} \tag{4.6}$$

Adding appropriate associators shows that this is true in a general bicategory.
Note that by using the modification M, the 2-morphism $\Theta_n : \alpha_n^c \to \Theta \circ \mathrm{id}_c$ is equivalent to a 2-morphism $\lambda_n : \alpha_c \to \mathrm{id}_c$. Here, α_c^n is the n-times composition of 1-morphism α_c. Indeed, define λ_n by setting

$$\lambda_n := \left(\alpha_c \cong \alpha_c \circ \mathrm{id}_c \xrightarrow{\mathrm{id}_{\alpha_c} * M^{-1}} \alpha_c \circ \Theta \xrightarrow{\Theta_n} \Theta \circ \mathrm{id}_c \cong \Theta \xrightarrow{M} \mathrm{id}_c \right). \tag{4.7}$$

In a strict 2-category, the fact that Θ is a pseudo-natural transformation requires that $\lambda_0 = \mathrm{id}_c$ and that $\lambda_n = \lambda_1 * \cdots * \lambda_1$. In a bicategory, similar equations hold by adding coherence morphisms. Thus, λ_n is fully determined by λ_1. In order to simplify notation, we set $\lambda := \lambda_1 : \alpha_c \to \mathrm{id}_c$.
A 1-morphism of homotopy fixed points $(c, \Theta, \Theta_n, \Pi, M) \to (c', \Theta', \Theta'_n, \Pi', M')$ consists of:

- a 1-morphism $f : c \to c'$,
- an invertible 2-morphism $m : f \circ \Theta \to \Theta' \circ f$ which fulfills the modification square. Note that m is equivalent to a 2-isomorphism $m : f \to f'$ which can be seen by using the 2-morphism M.

The condition due to Equation 3.24 of [HSV16] demands that the following 2-isomorphism

$$f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \tag{4.8}$$

is equal to the 2-isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \mathrm{id}_f} \mathrm{id}_{c'} \circ f \cong f \tag{4.9}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \cong \mathrm{id}_{c'} \circ f \xrightarrow{M'^{-1} * \mathrm{id}_f} \Theta' \circ f \right) \tag{4.10}$$

Thus, m is fully determined by M and M'. The condition due to Equation 3.23 of [HSV16] reads

$$m \circ (\mathrm{id}_f * \Pi) = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \tag{4.11}$$

and is automatically satisfied, as an explicit calculation in [HSV16] confirms. Now, it suffices to look at the modification square of m, in Equation 3.25 of [HSV16]. This condition is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \alpha_{c'} \circ f \circ \Theta \arrow[d, "\mathrm{id}_{\alpha_{c'}} * m"'] & f \circ \alpha_c \circ \Theta \arrow[l, "\alpha_f * \mathrm{id}_\Theta"'] \arrow[r, "\mathrm{id}_f * \Theta_1"] & f \circ \Theta \arrow[d, "m"] \\ \alpha_{c'} \circ \Theta' \circ f \arrow[rr, "\Theta'_1 * \mathrm{id}_f"'] & & \Theta' \circ f \end{tikzcd} \tag{4.12}$$

Substituting m as in Equation (4.10) and Θ_1 for $\lambda := \lambda_1$ as defined in Equation (4.7), one confirms that this diagram commutes if and only if the diagram in Equation (4.2) commutes.
If (f, m) and (g, n) are 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-isomorphism $\beta : f \to g$ in $\mathcal{C}$. The condition coming from Equation 3.26 of [HSV16] then demands that the diagram

$$\begin{tikzcd} f \circ \Theta \arrow[r, "m"] \arrow[d, "\beta * \mathrm{id}_\Theta"'] & \Theta' \circ f \arrow[d, "\mathrm{id}_{\Theta'} * \beta"] \\ g \circ \Theta \arrow[r, "n"'] & \Theta' \circ g \end{tikzcd} \tag{4.13}$$

commutes. Using the fact that both m and n are uniquely specified by M and M', one quickly confirms that this diagram commutes automatically.

Our detailed analysis of the bicategory $\mathcal{C}^G$ shows that the forgetful functor which forgets the data Θ, M, and Π on objects and assigns Θ_1 to λ, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. □

Corollary 4.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action of the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ as in Example 4.2. Then, the bicategory of homotopy fixed points $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ is equivalent to a bicategory where*

- *objects are given by pairs (X, λ_X) with X a fully-dualizable object of $\mathcal{C}$ and $\lambda_X : S_X \to \mathrm{id}_X$ is a 2-isomorphism which trivializes the Serre automorphism,*
- *1-morphisms are given by 1-equivalences $f : X \to Y$ in $\mathcal{C}$, so that the diagram*

$$\begin{tikzcd} S_Y \circ f \arrow[d, "\lambda_Y * \mathrm{id}_f"'] & f \circ S_X \arrow[l, "S_f"'] \arrow[r, "\mathrm{id}_f * \lambda_X"] & f \circ \mathrm{id}_X \arrow[d] \\ \mathrm{id}_X \circ f \arrow[rr] & & f \end{tikzcd} \tag{4.14}$$

  commutes, and
- *2-morphisms are given by 2-isomorphisms in $\mathcal{C}$.*

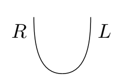
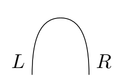
Remark 4.5. Recall that we have defined the bicategory of homotopy fixed points $\mathcal{C}^G$ as the tri-limit of the action considered as a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$. Since we only consider symmetric monoidal bicategories, we actually obtain an action with values in SymMonBicat, the tricategory of symmetric monoidal bicategories. It would be interesting to compute the limit of the action in this tricategory. We expect that this trilimit computed in SymMonBicat is given by $\mathcal{C}^G$ as a bicategory, with the symmetric monoidal structure induced by the symmetric monoidal structure of $\mathcal{C}$.

Remark 4.6. By [Dav11], the action via the Serre automorphism on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ is trivializable. The category of homotopy fixed points $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)}$ is then equivalent to the bigroupoid of symmetric, semi-simple Frobenius algebras.
Similarly, the action of the Serre automorphism on Vect_2 is trivializable. The bicategory of homotopy fixed points of this action is equivalent to the bicategory of finite Calabi-Yau categories, cf. [HSV16].

5. The 2-dimensional framed bordism bicategory

In this Section, we introduce a skeleton of the framed bordism bicategory $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$: this symmetric monoidal bicategory $\mathbb{F}_{cfd}$ is the free bicategory of a coherent fully-dual pair as introduced in [Pst14, Definition 3.13].
Using this presentation, we define a non-trivial $SO(2)$-action on this skeleton. If $\mathcal{C}$ is an arbitrary symmetric monoidal bicategory, the action on $\mathbb{F}_{cfd}$ will induce an action on the functor bicategory $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ of symmetric monoidal functors. Using the Cobordism Hypothesis for framed manifolds, which has been proven in the bicategorical framework in [Pst14], we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We show that this induced action coming from the framed bordism bicategory is exactly the action given by the Serre automorphism.
We begin by recalling the definition of $\mathbb{F}_{cfd}$.

Definition 5.1. *The symmetric monoidal bicategory $\mathbb{F}_{cfd}$ consists of*

- *2 generating objects L and R,*
- *4 generating 1-morphisms, given by*
  - *a 1-morphism* $\mathrm{coev} : 1 \to R \otimes L$, *which we write as*  ,
  - $\mathrm{ev} : L \otimes R \to 1$ *which we write as*  ,
  - *a 1-morphism $q : L \to L$,*
  - *another 1-morphism $q^{-1} : L \to L$,*
- *12 generating 2-cells given by*

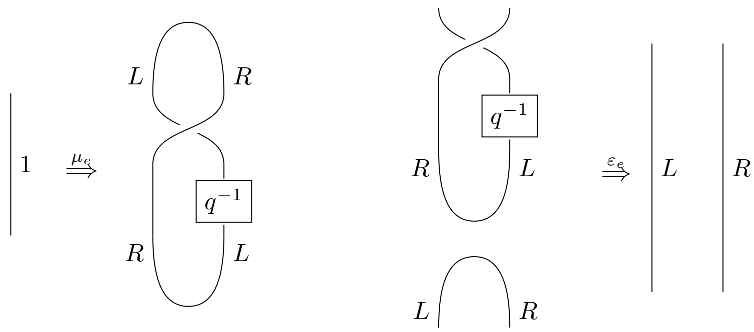
– *isomorphisms* α, β, α^{-1} *and* β^{-1} *as in Definition 2.1, which in pictorial form are given as follows:*

(5.1)

– *isomorphisms* $\psi : qq^{-1} \cong \mathrm{id}_L : \psi^{-1}$ *and* $\phi : q^{-1}q \cong \mathrm{id}_L : \phi^{-1}$

– *2-cells* $\mu_e : \mathrm{id}_1 \to \mathrm{ev} \circ \mathrm{ev}^L$ *and* $\varepsilon_e : \mathrm{ev}^L \circ \mathrm{ev} \to \mathrm{id}_{L\otimes R}$*, where* $\mathrm{ev}^L := \tau \circ (\mathrm{id}_R \otimes q^{-1}) \circ \mathrm{coev}$ *which in pictorial form are given as follows:*

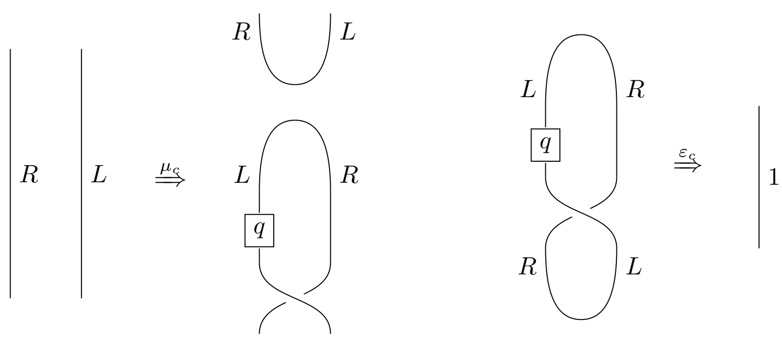
(5.2)



– *2-cells* $\mu_c : \mathrm{id}_{R\otimes L} \to \mathrm{coev} \circ \mathrm{coev}^L$ *and* $\varepsilon_c : \mathrm{coev}^L \circ \mathrm{coev} \to \mathrm{id}_1$*, where* $\mathrm{coev}^L := \mathrm{ev} \circ (q \otimes \mathrm{id}_R) \circ \tau$

which in pictorial form are given as follows:

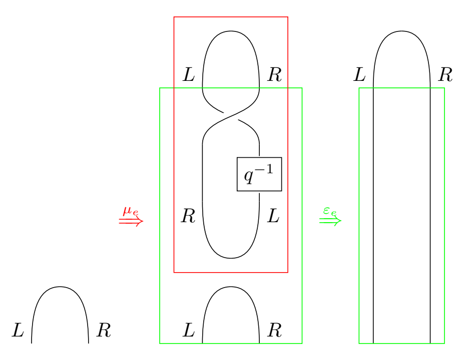
(5.3)



so that the following relations hold:

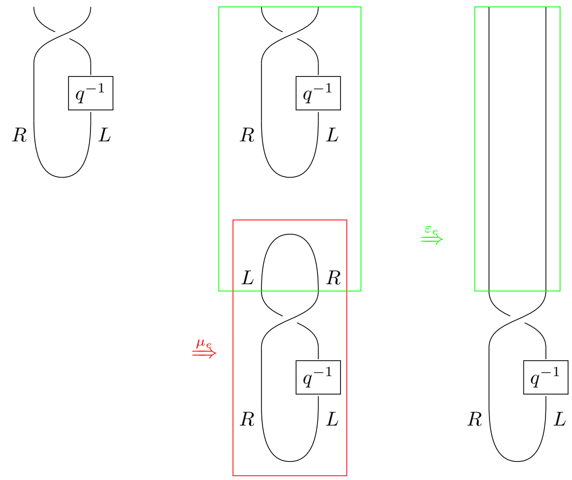
- α *and* α^{-1}, β *and* β^{-1}, ϕ *and* ϕ^{-1}, ψ *and* ψ^{-1} *are inverses to each other,*
- μ_e *and* ε_e *satisfy the two Zorro equations, which in pictorial form demand that the following composition of 2-morphisms*

(5.4)



is equal to $\mathrm{id}_{\mathrm{ev}}$*, and that the following composition of 2-morphisms*

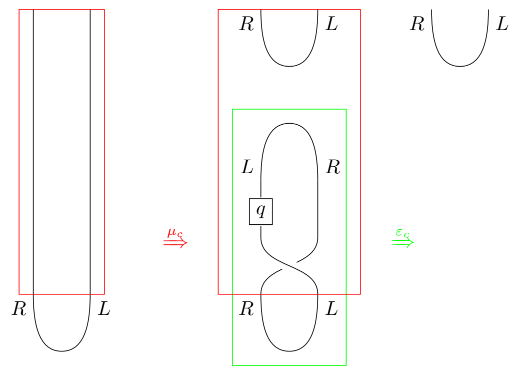
(5.5)



is equal to $\mathrm{id}_{\mathrm{ev}^L}$.

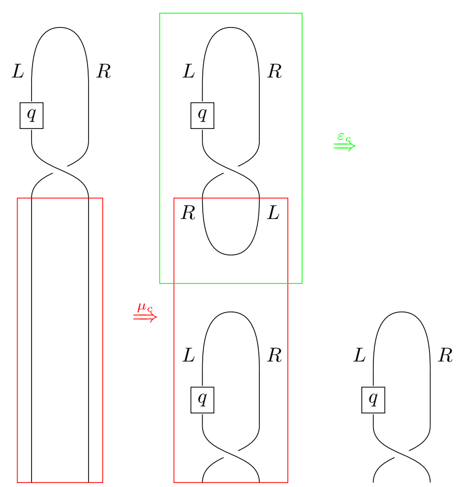
- μ_c *and* ε_c *satisfy the two Zorro equations, which in pictorial form demand that the composition*

(5.6)



is equal to $\mathrm{id}_{\mathrm{coev}}$*, and the composition of the following 2-morphisms*

(5.7)



is equal to $\mathrm{id}_{\mathrm{coev}^L}$.

- ϕ *and* ψ *satisfy triangle identities,*
- *the cusp-counit equations in figure 5 and 6 on p.33 of* [Pst14] *are satisfied,*
- *the swallowtail equations in figure 3 and 4 on p.15 of* [Pst14] *are satisfied.*

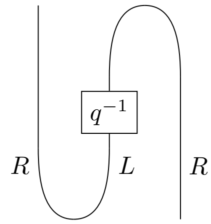
5.1. **Action on the framed bordism bicategory.** We can now proceed to construct an $SO(2)$-action on $\mathbb{F}_{cfd}$. This action will be vital for the remainder of the paper.

By Definition 4.1 it suffices to construct a pseudo-natural equivalence of the identity functor on $\mathbb{F}_{cfd}$ in order to construct an $SO(2)$-action. This pseudo-natural transformation is given as follows:

Definition 5.2. *Let $\mathbb{F}_{cfd}$ be the free symmetric monoidal bicategory on a coherent fully-dual object as in Definition 5.1. We construct a pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ of the identity functor on $\mathbb{F}_{cfd}$ as follows:*

- *For every object c of $\mathbb{F}_{cfd}$, we need to provide a 1-equivalence $\alpha_c : c \to c$.*
  - *For the object L of $\mathbb{F}_{cfd}$, we define $\alpha_L := q : L \to L$,*
  - *for the object R of $\mathbb{F}_{cfd}$, we set $\alpha_R := (q^{-1})^*$, which in pictorial form is given by*

(5.8) $(q^{-1})^* :=$



- *for every 1-morphism $f : c \to d$ in $\mathbb{F}_{cfd}$, we need to provide a 2-isomorphism*

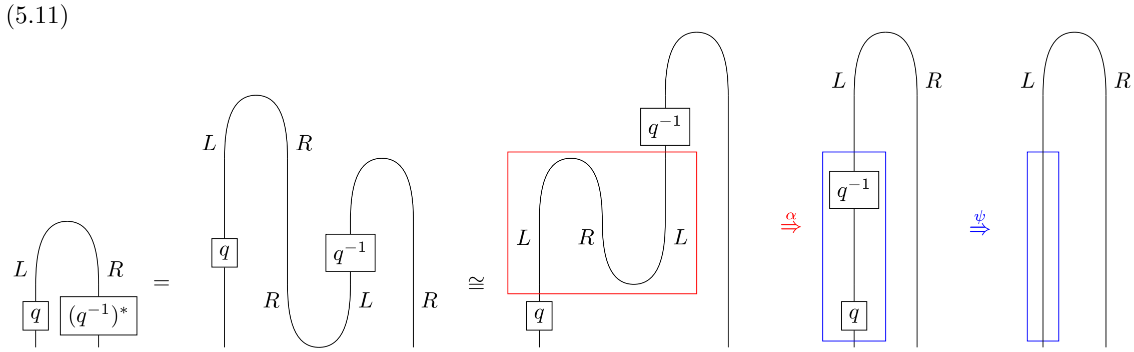
$$\alpha_f : f \circ \alpha_c \to \alpha_d \circ f. \tag{5.9}$$

  - *For the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ we define the 2-isomorphism $\alpha_q := \mathrm{id}_{q \circ q}$.*
  - *For the 1-morphism $q^{-1} : L \to L$ we define the 2-isomorphism*

$$\alpha_{q^{-1}} := \left(q^{-1} \circ q \xrightarrow{\phi} \mathrm{id}_L \xrightarrow{\psi^{-1}} q \circ q^{-1} \right). \tag{5.10}$$

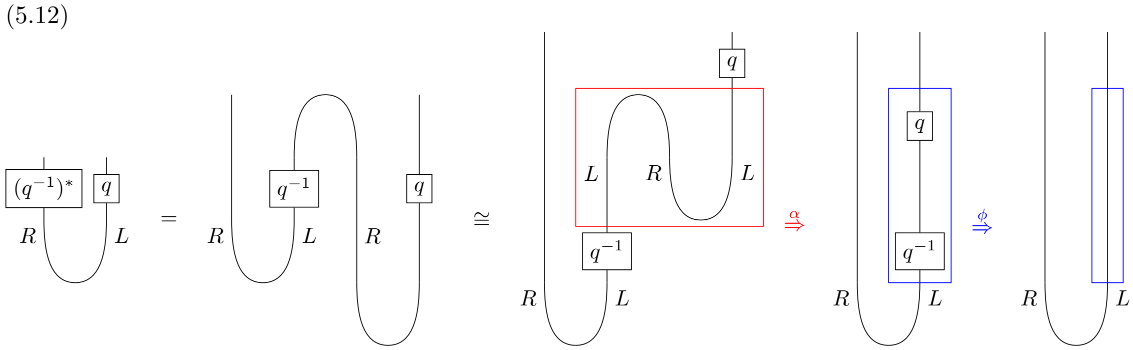
  - *For the evaluation $\mathrm{ev} : L \otimes R \to 1$, we define the 2-isomorphism α_{ev} to be the following composition:*

(5.11)



  - *For the coevaluation $\mathrm{coev} : 1 \to R \otimes L$, we define the 2-isomorphism α_{coev} to be the composition*

(5.12)



One now checks that this defines a pseudo-natural transformation of $\mathrm{id}_{\mathbb{F}_{cfd}}$. Using Definition 4.1 gives us a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$.

Remark 5.3. Note that the $SO(2)$-action on $\mathbb{F}_{cfd}$ does *not* send generators to generators: for instance, the 1-morphism $(q^{-1})^*$ in Equation (5.8) is not part of the generating data of $\mathbb{F}_{cfd}$.

Remark 5.4. Notice that the pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ constructed in Definition 5.2 is a *symmetric monoidal* pseudo-natural transformation. This follows from the fact that $\mathbb{F}_{cfd}$ is the *free* symmetric monoidal bicategory generated by a coherent fully-dual pair. Thus, we obtain an $SO(2)$-action on $\mathbb{F}_{cfd}$ via symmetric monoidal morphisms.

5.2. **Induced action on functor categories.** Starting from the action defined on $\mathbb{F}_{cfd}$, we induce an action on the bicategory of functors $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ for an arbitrary bicategory $\mathcal{C}$. The construction of the induced action on the bicategory of functors is a general construction. We provide details in the following.

Definition 5.5. *Let $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ be a G-action on a bicategory $\mathcal{C}$, and let $\mathcal{D}$ be another bicategory. The G-action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ induced by ρ is defined as follows:*

- *On objects $g \in G$, we define an endofunctor $\tilde{\rho}(g)$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on objects F on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by $\tilde{\rho}(g)(F) := F \circ \rho(g^{-1})$. If $\alpha : F \to G$ is a 1-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we define*

$$\tilde{\rho}(g)(\alpha) := \quad \begin{tikzcd} F\rho(g^{-1})c \arrow[r, "\alpha_{\rho(g^{-1})(c)}"] \arrow[d, "F\rho(g^{-1})(f)"'] & G\rho(g^{-1})c \arrow[d, "G\rho(g^{-1})(f)"] \arrow[dl, Rightarrow, "\alpha_{\rho(g^{-1})(f)}"] \\ F\rho(g^{-1})d \arrow[r, "\alpha_{\rho(g^{-1})(d)}"'] & G\rho(g^{-1})d \end{tikzcd} \tag{5.13}$$

If $m : \alpha \to \beta$ is a 2-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the value of $\tilde{\rho}(\gamma)$ is given by

$$\tilde{\rho}(\gamma)(m)_x := m_{\rho(g^{-1})(x)}. \tag{5.14}$$

- *on 1-morphisms $\gamma : g \to h$ of $\Pi_2(G)$, we define a 1-morphism $\tilde{\rho}(\gamma)$ in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ between the two endofunctors $F \mapsto F \circ \rho(g^{-1})$ and $F \mapsto F \circ \rho(h^{-1})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*
  Explicitly, this means:
  - *For each 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural transformation $\tilde{\rho}(\gamma)_F : F \circ \rho(g^{-1}) \to F \circ \rho(h^{-1})$ which we define via the diagram*

$$\begin{tikzcd} F\rho(g^{-1})x \arrow[r, "F(\rho(\gamma^{-1})_x)"] \arrow[d, "F\rho(g^{-1})(f)"'] & F\rho(h^{-1})x \arrow[d, "F\rho(h^{-1})(f)"] \arrow[dl, Rightarrow, "F(\rho(\gamma^{-1})_f)"] \\ F\rho(g^{-1})y \arrow[r, "F(\rho(\gamma^{-1})_y)"'] & F\rho(h^{-1})y \end{tikzcd} \tag{5.15}$$

  Here, γ^{-1} is the "inverse" path of γ given by $t \mapsto \gamma(t)^{-1}$, and $f : x \to y$ is a 1-morphism in $\mathcal{C}$.
  - *For every pseudo-natural transformation $\alpha : F \to G$, we need to provide a modification $\tilde{\rho}(\gamma)_\alpha$ in the diagram*

$$\begin{tikzcd} \tilde{\rho}(g)(F) \arrow[r, "\tilde{\rho}(\gamma)_F"] \arrow[d, "\tilde{\rho}(g)(\alpha)"'] & \tilde{\rho}(h)(F) \arrow[d, "\tilde{\rho}(h)(\alpha)"] \arrow[dl, Rightarrow, "\tilde{\rho}(\gamma)_\alpha"] \\ \tilde{\rho}(g)(G) \arrow[r, "\tilde{\rho}(\gamma)_G"'] & \tilde{\rho}(h)(G) \end{tikzcd} \tag{5.16}$$

  which we define by

$$\tilde{\rho}(\gamma)_\alpha := \alpha^{-1}_{\rho(\gamma^{-1})_x}. \tag{5.17}$$

- *For the 2-morphisms in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ we proceed in a similar fashion: if $m : \gamma \to \gamma'$ is a 2-track, we have to provide a 2-morphism $\tilde{\rho}(m) : \tilde{\rho}(\gamma) \to \tilde{\rho}(\gamma')$ which can be done by explicitly writing down diagrams as above.*

The rest of the data of a monoidal functor $\tilde{\rho}$ is induced from the data of the monoidal functor ρ.

For $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal bicategories, the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})$ acquires a monoidal structure by "pointwise evaluation" of functors. Such a monoidal structure is also symmetric, see [SP09]. The following result is straightforward.

Lemma 5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal bicategories, and let ρ be a monoidal action of a group G on $\mathcal{C}$. Then ρ induces a monoidal action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}^{\otimes}(\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D}))$.*

Example 5.7. Our main example for induced actions is the $SO(2)$-action on $\mathbb{F}_{cfd}$ as in Definition 5.2. This action only depends on a pseudo-natural equivalence α of the identity functor on $\mathrm{id}_{\mathbb{F}_{cfd}}$. Consequently, the induced action on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ also only depends on a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. Using the definition above, we construct this induced pseudo-natural equivalence $\tilde{\alpha}$ as follows.

- For every 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural equivalence $\tilde{\alpha}_F : F \to F$, which is given by the diagram

$$\tilde{\alpha}_F := \begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "F(f)"'] & Fx \arrow[d, "F(f)"] \arrow[dl, phantom, "\swarrow F(\alpha_f^{-1})"] \\ Fy \arrow[r, "F(\alpha^{-1})_y"'] & Fy \end{tikzcd} \tag{5.18}$$

- for every pseudo-natural transformation $\beta : F \to G$, we need to give a modification $\tilde{\alpha}_\beta$, which we define by the diagram

$$\begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "\beta_x"'] & Fx \arrow[d, "\beta_x"] \arrow[dl, phantom, "\swarrow \beta^{-1}_{(\alpha_x^{-1})}"] \\ Gx \arrow[r, "G(\alpha^{-1})_x"'] & Gx \end{tikzcd} \tag{5.19}$$

This defines a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. By Definition 4.1, we obtain an $SO(2)$-action on $\mathrm{Fun}(\mathbb{F}_{cfd},\mathcal{C})$. Note that $\mathbb{F}_{cfd}$ is even a *symmetric monoidal* bicategory. The $SO(2)$-action on $\mathbb{F}_{cfd}$ of Definition 5.2 is via symmetric monoidal homomorphisms by Remark 5.4. Hence, if $\mathcal{C}$ is also symmetric monoidal, then Lemma 5.6 provides a monoidal action on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$.

5.3. **Induced action on the core of fully-dualizable objects.** In this subsection, we compute the $SO(2)$-action on the core of fully-dualizable objects coming from the $SO(2)$-action on $\mathbb{F}_{cfd}$. Starting from the $SO(2)$-action on $\mathbb{F}_{cfd}$ as by Definition 5.2, we have shown in the previous subsection how to induce an $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ for $\mathcal{C}$ some symmetric monoidal bicategory. By the Cobordism Hypothesis for framed manifolds, we obtain an induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. More precisely, denote by

$$\begin{aligned} \mathrm{ev}_L : \mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C}) &\to \mathscr{K}(\mathcal{C}^{\mathrm{fd}}) \\ Z &\mapsto Z(L) \end{aligned} \tag{5.20}$$

the evaluation map. The Cobordism Hypothesis for framed manifolds in two dimensions [Pst14, Lur09] states that ev_L is an equivalence of symmetric monoidal bicategories. Hence, the composition of the $SO(2)$-action on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ and (the inverse of) ev_L provides an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. The next proposition shows that this action is equivalent to the action ρ^S induced by the Serre automorphism which is illustrated in Example 4.2.

Proposition 5.8. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. By Definition 5.5, we obtain a monoidal $SO(2)$-action on* $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$*. Then, the monoidal $SO(2)$-action induced by the evaluation in Equation* (5.20) *on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is equivalent to ρ^S.*

Proof. Let

$$\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})) \tag{5.21}$$

be the $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ as in Example 5.7. This action only depends on a pseudo-natural transformation α on the identity functor on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$. By [Pst14], the 2-functor in Equation (5.20) which evaluates a framed field theory on the object L is an equivalence of bicategories. Thus, we obtain an $SO(2)$-action ρ' on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. This action is given as follows. By Definition 4.1, we only need to provide a pseudo-natural transformation of the identity functor of $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. In order to write down this pseudo-natural transformation, note that the functor

$$\begin{aligned} \mathrm{Aut}(\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})) &\to \mathrm{Aut}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \\ F &\mapsto \mathrm{ev}_L \circ F \circ \mathrm{ev}_L^{-1} \end{aligned} \tag{5.22}$$

is an equivalence. Hence, the induced pseudo-natural transformation of $\mathrm{id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$ is given as follows:

- For each fully-dualizable object c of $\mathcal{C}$, we assign the 1-morphism $\alpha'_c : c \to c$ defined by

$$\alpha'_c := \mathrm{ev}_L\left(\alpha_{\left(\mathrm{ev}_L^{-1}(c)\right)}\right) \tag{5.23}$$

- for each 1-equivalence $f : c \to d$ between fully-dualizable objects of $\mathcal{C}$, we define a 2-isomorphism $\alpha'_f : f \circ \alpha'_c \to \alpha'_d \circ f$ by the formula

$$\alpha'_f := \mathrm{ev}_L\left(\alpha_{\left(\mathrm{ev}_L^{-1}(f)\right)}\right). \tag{5.24}$$

Here, α is the pseudo-natural transformation as in Example 5.7. In order to see that α'_c is given by the Serre automorphism of the fully-dualizable object c, note that the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ is mapped to the Serre automorphism $S_{Z(L)}$ by the equivalence in Equation (5.20). □

Corollary 5.9. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in Definition 5.2, and let $\mathcal{C}$ be a symmetric monoidal bicategory. Consider the $SO(2)$-action ρ^S on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ induced by the Serre automorphism. Then the evaluation morphism ev_L induces an equivalence of bicategories*

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}. \tag{5.25}$$

Proof. By Proposition 5.8, the equivalence of Equation (5.20) is $SO(2)$-equivariant. Thus, it induces an equivalence on homotopy fixed points, cf. [Hes16, Definition 5.3] for an explicit description. It is also possible to construct this equivalence directly: by theorem 4.3, the bicategory of homotopy fixed points $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)}$ is equivalent to the bicategory where

- objects are given by symmetric monoidal functors $Z : \mathbb{F}_{cfd} \to \mathcal{C}$, together with a modification $\lambda_Z : \tilde{\alpha}_Z \to \mathrm{id}_Z$. Explicitly, this means: if α is the endotransformation of the identity functor of $\mathbb{F}_{cfd}$ as in Definition 5.2, we obtain two 2-isomorphisms in $\mathcal{C}$:

$$\begin{aligned} &\lambda_L : Z(q^{-1}) \to \mathrm{id}_{Z(L)} \\ &\lambda_R : Z\left(\left((q^{-1})^*\right)^{-1}\right) \to \mathrm{id}_{Z(R)} \end{aligned} \tag{5.26}$$

  which are compatible with evaluation and coevaluation,
- 1-morphisms are given by symmetric monoidal pseudo-natural transformations $\mu : Z \to Z'$, so that the analogue of the diagram in Equation (4.2) commutes,
- 2-morphisms are given by symmetric monoidal modifications.

Now notice that $Z(q)$ is precisely the Serre automorphism of the object $Z(L)$. Thus, λ_L provides a trivialization of (the inverse of) the Serre automorphism. Applying theorem 4.3 again to the action of the Serre automorphism on the core of fully-dualizable objects shows that the functor $Z \mapsto (Z(L), \lambda_L)$ is an equivalence of homotopy fixed point bicategories. □

6. Invertible Field Theories

In the section, we consider the case of 2-dimensional oriented *invertible* topological field theories: such theories are in many ways easier to describe than arbitrary TQFTs, and play an important role in condensed matter physics and homotopy theory, as suggested in [Fre14a, Fre14b].

Denote with $\mathrm{Pic}(\mathcal{C})$ the *Picard groupoid* of a symmetric monoidal bicategory $\mathcal{C}$: it is defined as the maximal subgroupoid of $\mathcal{C}$ where the objects are invertible with respect to the monoidal structure of $\mathcal{C}$. Recall that $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ is equipped with a monoidal structure which is defined pointwise.

Definition 6.1. *An invertible framed TQFT with values in $\mathcal{C}$ is an invertible object in $\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})$. The space of invertible framed TQFTs with values in $\mathcal{C}$ is given by $\mathrm{Pic}(\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C}))$.*

Remark 6.2. Equivalently, an invertible TQFT assigns to the point in $\mathrm{Cob}_{2,1,0}$ an invertible object in $\mathcal{C}$, and to any 1- and 2-dimensional manifold it assigns invertible 1- and 2-morphisms.

Since the Cobordism Hypothesis provides a *monoidal* equivalence between $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}, \mathcal{C})$ and $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the space of invertible framed TQFTs is given by $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))$, since taking the Picard groupoid behaves well with respect to monoidal equivalences.

We begin by proving the following:

Lemma 6.3. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, there is an equivalence of symmetric monoidal bicategories*

$$\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}). \tag{6.1}$$

Proof. First note that $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is a monoidal 2-groupoid, so there is an equivalence of bicategories $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}^{\mathrm{fd}})$. Now, it suffices to show that every object X in $\mathrm{Pic}(\mathcal{C})$ is already fully-dualizable. Indeed, denote the tensor-inverse of X by X^{-1}. By definition, we have 1-equivalences $X \otimes X^{-1} \cong 1$ and $1 \cong X^{-1} \otimes X$, which serve as evaluation and coevaluation. These maps may be promoted to adjoint 1-equivalences by [SP09, Proposition A.27]. Thus, the evaluation and coevaluation also admit adjoints, which suffices for fully-dualizability. $\square$

Notice that given a monoidal bicategory $\mathcal{C}$, any monoidal autoequivalence of $\mathcal{C}$ preserves the Picard groupoid of $\mathcal{C}$, since it preserves invertibility of objects and (higher) morphisms. In particular, we have a monoidal 2-functor

$$\mathrm{Aut}^{\otimes}(\mathcal{C}) \to \mathrm{Aut}^{\otimes}(\mathrm{Pic}(\mathcal{C})) \tag{6.2}$$

obtained by restriction. Since the $SO(2)$-action induced by the action on $\mathrm{Cob}_{2,1,0}$ is monoidal, it induces an action on $\mathrm{Pic}(\mathcal{C})$. To proceed, we need the following

Lemma 6.4. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that $\mathrm{Pic}(\mathcal{C})$ is monoidally equivalent to $B^2\mathbb{K}^*$. Then*

$$\mathrm{Aut}^{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*) \tag{6.3}$$

where the category on the right hand side is regarded as a discrete category.

Proof. Since $\mathrm{Pic}(\mathcal{C}) \simeq B^2\mathbb{K}^*$, we have to describe the Picard groupoid of the category of monoidal functors from $B^2\mathbb{K}^*$ to $B^2\mathbb{K}^*$. First, notice that the monoidal bicategory $B^2\mathbb{K}^*$ is computadic in the sense of [SP09]: it admits a presentation with only one object, only the identity 1-morphism, $\mathbb{K}^*$ as the set of 1-morphisms, and no relations between the 2-morphisms. The cofibrancy theorem in [SP09, Theorem 2.78] ensures that every monoidal 2-functor out of a computadic monoidal bicategory is equivalent to a *strict* monoidal functor. It is clear that *strict* monoidal auto-equivalences of $B^2\mathbb{K}$ are classified by $\mathrm{Iso}(\mathbb{K}^*)$ up to natural isomorphism. In order to see that the 1- and 2- morphisms of $\mathrm{Aut}^{\otimes}(B^2\mathbb{K}^*)$ are trivial, we use the cofibrancy theorem again to strictify monoidal pseudo-natural transformations. In detail, if $F, F' : B^2\mathbb{K}^* \to B^2\mathbb{K}^*$ are two weak monoidal 2-functors, and $\eta : F \to F'$ is a monoidal pseudo-natural equivalence, the confibrancy theorem ensures that η is equivalent to a *strict* monoidal pseudo-natural transformation, which means we may choose the data Π and M in [SP09, Figure 2.7] to be identity 2-morphisms. Thus, η is fully determined by a 1-morphism $\eta_* : F(*) \to F(*)$ in $B^2\mathbb{K}^*$ which has to be the identity, and by a 2-morphism η_{id_*} in $B^2\mathbb{K}^*$ filling the naturality square. This 2-morphism however is also fixed to be trivial by the unitality conditions of a monoidal pseudo-natural transformation in [SP09, Axiom MBTA2 and Axiom MBTA3]. We now come to the last layer of information: any monoidal modification between two monoidal pseudo-natural transformations is fixed to be the identity modification by the unitality requirement in [SP09, Axiom BMBM2]. $\square$

Examples of symmetric monoidal bicategories satisfying the assumption of Lemma 6.4 are $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$. In general cases, we have the following

Lemma 6.5. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that $\mathrm{Pic}(\mathcal{C})$ is monoidally equivalent to $B^2\mathbb{K}^*$. Then any monoidal $SO(2)$-action on $\mathrm{Pic}(\mathcal{C})$ is trivializable.*

Proof. Since we have monoidal equivalences $\Pi_2(SO(2)) \simeq B\mathbb{Z}$ and $\mathrm{Aut}^{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*)$, monoidal actions correspond to monoidal 2-functors $B\mathbb{Z} \to \mathrm{Iso}(\mathbb{K}^*)$. Monoidality implies that the single object of $B\mathbb{Z}$ is sent to the identity isomorphism of $\mathbb{K}^*$, which correspond to the identity functor on $\mathrm{Pic}(\mathcal{C})$. This forces the functor to be trivial on objects. It is clear that the action is also trivial on 1- and 2-morphisms. Since there are no nontrivial morphisms in $\mathrm{Iso}(\mathbb{K}^*)$, the monoidal structure on the action ρ must also be trivial. $\square$

Finally, we need the following

Lemma 6.6. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let ρ_S be the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism as in Example 4.2. Since this action is monoidal, it induces an action on $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C})$ by Lemma 6.3. We have then an equivalence of monoidal bicategories*

$$\mathrm{Pic}\left((\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}\right) \cong \mathrm{Pic}(\mathcal{C})^{SO(2)}. \tag{6.4}$$

Proof. Theorem 4.3 allows us to compute the two bicategories of homotopy fixed points explicitly: we see that both bicategories have invertible objects X of $\mathcal{C}$, together with the choice of a trivialization of the Serre automorphism as objects. The 1-morphisms of both bicategories are given by 1-equivalences between invertible objects of $\mathcal{C}$, so that the diagram in equation (4.14) commutes, while 2-morphisms are given by 2-isomorphisms in $\mathcal{C}$. □

The implication of the above lemmas is the following: when $\mathcal{C}$ is a symmetric monoidal bicategory with $\mathrm{Pic}(\mathcal{C}) \cong B^2\mathbb{K}^*$, the action of the Serre-automorphism on framed, invertible field theories with values in $\mathcal{C}$ is trivializable. Thus *all* framed invertible 2d TQFTs wih values in $\mathcal{C}$ can be turned into orientable ones.

7. Comments on Homotopy Orbits

So far, we have constructed an $SO(2)$-action on the bicategory $\mathbb{F}_{cfd}$. We have shown how the action on $\mathbb{F}_{cfd}$ induces an action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$, and that via the (framed) Cobordism Hypothesis the induced action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ for framed manifolds agrees with the action of the Serre automorphism. As a consequence, we are able to provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{7.1}$$

in Corollary 5.9. We could then in principle deduce the Cobordism Hypothesis for oriented manifolds from 7.1, once we provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathrm{Fun}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}). \tag{7.2}$$

The above equivalence can be proven directly by using a presentation of the oriented bordism bicategory via generators and relations, given in [SP09], and the notion of a Calabi-Yau object internal to a bicategory. The details will appear in [HVar].

Here, we want instead to comment on an alternative approach. Namely, in order to provide an equivalence as in (7.2), it suffices to identify the oriented bordism bicategory with the *colimit* of the $SO(2)$-action on $\mathbb{F}_{cfd}$. Indeed, recall that one may define a G-action on a bicategory $\mathcal{C}$ to be a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. The tricategorical colimit of this functor will then be the bicategory of co-invariants or *homotopy orbits* of the G-action, denoted by $\mathcal{C}_G$. By Definition of the tricategorical colimit, and the fact that colimits are sent to limits by the Hom functor, we then obtain an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathcal{C}_G, \mathcal{D}) \cong \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})^G. \tag{7.3}$$

The following conjecture is then natural:

Conjecture 7.1. *The bicategory of co-invariants of the $SO(2)$-action on $\mathbb{F}_{cfd}$ is monoidally equivalent to the oriented bordism bicategory, i.e. we have a monoidal equivalence*

$$(\mathbb{F}_{cfd})_{SO(2)} \cong \mathrm{Cob}^{\mathrm{or}}_{2,1,0}\,. \tag{7.4}$$

Furthermore, the colimit is compatible with the monoidal structure.

Remark 7.2. We believe that this is not an isolated phenomenon, in the sense that any higher bordism category equipped with additional tangential structure should be obtained by taking an appropriate colimit of a G-action on the framed bordism category.

Given Conjecture 7.1 and Equation 7.3, we obtain the following sequence of monoidal equivalences

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) \cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \cong \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}. \tag{7.5}$$

Hence Conjecture 7.1 implies the Cobordism Hypothesis for oriented 2-manifolds. Notice that the chain of equivalences in 7.5 is natural in $\mathcal{C}$.

On the other hand, the Cobordism Hypothesis for oriented manifolds in 2-dimensions implies Conjecture 7.1 . Indeed, by using a tricategorical version of the Yoneda Lemma, as developed for instance in [Buh15], the chain of equivalences

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \end{aligned} \tag{7.6}$$

implies that $\mathrm{Cob}^{\mathrm{or}}_{2,1,0}$ is equivalent to $(\mathbb{F}_{cfd})_{SO(2)}$, due to the uniqueness of representable objects.

We summarise the above arguments in the following

Lemma 7.3. *The Cobordism Hypothesis for oriented 2-dimensional manifolds is equivalent to Conjecture 7.1.*

It would then be of great interest to develop concrete constructions of homotopy co-invariants of actions of groups on bicategories, in the same spirit of [HSV16] and the present work, in order to verify directly the equivalence in Conjecture 7.1, and to extend the above arguments to general tangential G-structures.

$$
\begin{tikzcd}
X\,Y(X\,Y)^* \arrow[r, "\mathrm{id}_{X\,Y}\ \mathrm{coev}_Y\ \mathrm{id}_{(X\,Y)^*}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{\mathrm{id}_X}\,\alpha_Y\,\mathrm{id}_{\mathrm{id}_{(X\,Y)^*}}"] & X\,Y\,Y^*\,Y(X\,Y)^* \arrow[r, "\mathrm{id}_{X\,Y\,Y^*}\ \mathrm{coev}_X\,\mathrm{id}_{Y\,(X\,Y)^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{Y\,(X\,Y)^*}"] \arrow[dr, phantom, "\cong"] & X\,Y\,Y^*\,X^*\,X\,Y\,(X\,Y)^* \arrow[r, "\mathrm{id}_{X\,Y\,Y^*\,X^*}\ \mathrm{ev}_{X\,Y}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{X^*\,X\,Y\,(X\,Y)^*}"] \arrow[dr, phantom, "\cong"] & X\,Y\,Y^*\,X^* \arrow[r, "\mathrm{id}_X\,\tau_{Y\,Y^*,X^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\,\mathrm{id}_{X^*}"] \arrow[dr, phantom, "\cong"] & X\,X^*\,Y\,Y^* \arrow[d, "\mathrm{ev}_X\ \mathrm{ev}_Y"] \\
X\,Y(X\,Y)^* \arrow[r, "\mathrm{id}_{X\,Y\,(X\,Y)^*}"'] \arrow[drr, bend right, equal] \arrow[dr, phantom, "\alpha_X\,\mathrm{id}_{\mathrm{id}_{(X\,Y)^*}}"] & X\,Y\,(X\,Y)^* \arrow[r, "\mathrm{id}_X\ \mathrm{coev}_X\ \mathrm{id}_{Y\,(XY)^*}"'] & X\,X^*\,X\,Y\,(X\,Y)^* \arrow[r, "\mathrm{id}_{X\,X^*}\ \mathrm{ev}_{X\,Y}"'] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}_{X\,Y\,(X\,Y)^*}"] \arrow[drr, phantom, "\cong"] & X\,X^* \arrow[r, "\mathrm{ev}_X"'] & 1 \arrow[d, equal] \\
 & & X\,Y\,(X\,Y)^* \arrow[rr, "\mathrm{ev}_{X\,Y}"] & & 1
\end{tikzcd}
$$

Figure 1. Diagram for the proof of Lemma 2.10

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