ELEMENTS OF QUANTUM FINANCE MODELS

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Abstract. Both academic research and practical application of mathematical finance have been extremely fruitful since the seminal work of Black-Scholes-Merton in the early 1970s. In this framework, the prices of financial assets are modeled as stochastic processes in probability spaces inside which the machinery of stochastic calculus is a powerful tool. The fundamental asset pricing theorem states that the absence of arbitrage opportunities in a market is equivalent to the existence of a probability measure, equivalent to the objective probability, under which the discounted prices of the assets become local martingales. This linkage between finance on the one hand and probability theory on the other is the key to the success of mathematical finance. In this note, we show that it is possible to extend a classical probability model to a quantum probability model. Classical stochastic calculus is replaced by its quantum counterpart on a Boson Fock space. In particular, we show that the fundamental asset pricing theorem remains valid in this non-commutative setting.

By its very nature of coping with non-commutative random variables however, many essential elements of classic mathematical finance but in quantum form have not been recovered in this note. A couple that have been established are not in their full generality. For instance, existence of non-trivial self-financing portfolios and preserving of self-financing property under change of numéraire can only be reclaimed subject to technical conditions. This writing puts together a few helpful results for the purpose of future studies of this potentially new research subject.

1. Introduction

Using a stochastic process as a model to describe the price evolution of financial assets goes back as early as 1900. In his thesis [9], Bachelier assumed that the share price movements of an asset followed a Wiener process. By employing the equilibrium arguments, he was able to obtain the prices of Barrier options. In their seminal work, Black and Scholes [10] and Merton [27], assuming Geometric Brownian Motion for the stock prices, applying so-called no-arbitrage argument, established the famous Black-Scholes-Merton formula for option pricing. This no-arbitrage argument has become the central part of mathematical finance. In a financial market, an arbitrage is a risk-less way of making profit. To insist that there is no such arbitrage opportunity in a liquid financial market is a basic theoretical assumption. It is remarkable that this simple and intuitive principle ensures the existence of an equivalent martingale probability measure $\mathbb{Q}$ in a filtrated real world probability space $\left(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathcal{F}, \mathbb{P}\right)$ inside which asset prices live as stochastic processes. This martingale measure together with the stochastic

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calculus allow one to calculate option prices as already being seen in the Black-Scholes-Merton theory. The connection, termed fundamental asset pricing theorem, has become the foundation of the entire theory of mathematical finance. The pioneering work and sub-sequential developments can be found in [16, 17, 20, 21, 24, 39]. A summary of the main results can be found in [32]. The essential mathematical ingredients used in proving the theorem include stochastic integration and Hahn-Banach theorems.

A financial market is undoubtedly stochastic in nature. Some may be content with the probabilistic formulation of it. Others may argue that the system is far more complicated and exhibits quantum phenomenons. Whether markets are quantum is debatable and we do not try to join such debate here nor do we suggest to tackle quantum like issues which may exist in market places. Instead, we have a very simple motivation. It is nevertheless interesting to know whether one can formulate mathematical finance in terms of quantum probability hoping that the existence of such a theory might enrich our understanding of financial markets and possibly provide us with additional tools to manage them.

This amounts to employing a filtrated quantum probability space $\left(\mathcal{A},(\mathcal{A}_t)_{t\geq 0},\rho\right)$ where $\mathcal{A}$ is a von Neumann algebra with filtrated subalgebras $(\mathcal{A}_t)_{t\geq 0}$ and where ρ is a normal state to replace the filtrated abelian von Neumann algebras $\left(L^\infty(\Omega,\mathcal{F}_t,\mathbb{P})\right)_{t\geq 0}$. Furthermore, in order to have a satisfactory theory in continuous time, a quantum stochastic calculus to contain at least quantum stochastic integration, and better to also include elements of quantum stochastic differential equation and a quantum Ito formula, should be available at our disposal. On top of that, a natural specification of a filtration which represents the information structure in a market place should be possible. Fortunately, these mathematical tools do exist. The purpose of this note is to introduce a non-commutative mathematical finance model based on the quantum stochastic calculus developed initially by Hudson and Parthasarathy [22, 26, 28] and later extended by many others (see [3, 4, 5, 7, 8, 25] and the references therein). In this setting, asset prices are modeled by self-adjoint operator processes acting on a infinite dimensional complex Hilbert space Φ called Boson Fock space. They are semimartingales of bounded operator processes in $\left(\mathcal{A},(\mathcal{A}_t)_{t\geq 0},\rho\right)$ with $\mathcal{A}$ contained in the algebra of all bounded operators on Φ, $(\mathcal{A}_t)_{t\geq 0}$ is a von Neumann algebra filtration compatible with the natural filtration induced by the continuous tensor structure of the Fock space Φ and ρ is a faithful normal state in the predual $\mathcal{A}_*$ of the von Neumann algebra $\mathcal{A}$. As a restriction, in this note, asset prices are bounded. This assumption is acceptable in practice. While the quantum finance model in this note is based on Boson or symmetric Fock space, similar models, but built on the free Fock space, can be found in [35].

For possible future studies on its applications, we simply remark that the Fock space quantum stochastic calculus has found various applications which may potentially lend us methods and techniques in calculations (for example [6] [11]). It is also worthy pointing out that quantum theory has found its applications in recent years. For example, quantum information theory, quantum game theory and quantum computer advance have been mentioned in various places. Hence, it might be helpful and even necessary to develop a quantum finance theory.

Quantum mathematical finance models were considered in the context of quantum probability space $\left(\mathcal{A},(\mathcal{A}_t)_{t\geq 0},\tau\right)$ where $\mathcal{A}=M_n(\mathbb{C})$ is the $n\times n$ matrices [12] or where $\mathcal{A}$ is a finite von Neumann algebra with a tracial state τ [13]. In these interesting works, quantum stochastic integration in discrete time became finite sums of operators. Fundamental asset pricing theorems in such settings were established. A quantum binomial model was discussed in [14]. The model we develop in this note is a continuation along this direction and can be viewed as another attempt to provide a quantum framework for interpreting financial markets.

The main result of this note is Theorem 2.16 of fundamental asset pricing theorem. Other basic elements consist of existence of non-trivial self-financing portfolios (Theorem 2.9) and preserving of self-financing property under change of numéraire (Theorem 2.7). While this note has demonstrated that a quantum mathematical finance model can be formulated and the important fundamental asset pricing theorem remains true, it only translates a small part of the classical mathematical finance and has only touched the surface of this potentially new research area. Furthermore, the note, as it stands now, describes one of the many possibilities of extending the commutative mathematical finance model to a non-commutative one. It is our hope that this note will generate interests for further research.

The current paper is an improvement of part of the draft [33]. To increase accessibility, especially for financial industry professionals like the author, we separate [33] into two papers. The first one is [34] which summarizes the quantum stochastic calculus tools that are needed for setting up a model. This paper is the second one which focus on financial modeling.

Effort has been made to provide accurate references. However, in order to better understand the history of the developments of quantum stochastic calculus, operator algebra theory and mathematical finance, we encourage the readers to consult the references in this note and further references they point to. Our quoting of references may not follow the historic origin but rather to point to review papers and books. The author apologizes for any confusion this may create.

2. A Quantum Finance Model

This section exemplifies a mathematical quantum finance model in continuous time. The development of the model follows the footsteps of its classic counterpart. Among numerous references about classic mathematical finance in the literature [23] is the one this note turns to most often for consultation. Quantum stochastic calculus tools one shall depend upon are made available in [34]. The main content of [34] includes Riemann sum attainable triplet which allows one to construct bounded quantum integrals of semi-martingales, with an additional converging property. Quantum regular semimartingales were classified by Parthasarathy and Sinha [28] and by Attal [4]. Quantum stochastic integrals, possibly unbounded, was initially developed by Hudson and Parthasarathy [22, 26, 28] and later extended by many others (see [3, 4, 5, 7, 8, 25] and the references therein). Another useful result in [34] is a quantum fundamental asset pricing theorem in pure von Neumann algebra language which finds direct application here.

2.1. **Riemann sum attainable triplet.** We start off by briefly recalling notations and results from [34] and the references therein, paving the way for setting up the model. Let $\Phi = \Gamma_s(\mathcal{H}) \triangleq \bigoplus_{n=0}^{\infty} \mathcal{H}^{\odot n}$ be the Boson Fock space with $\mathcal{H} = L^2(\mathbb{R}^+, dt)$. There is a continuous tensor decomposition $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ and a filtration $\mathscr{B}\left(\Phi_{t]}\right) \otimes I_{\Phi_{[t}}$ of $\mathscr{B}(\Phi)$, the bounded linear operators on Φ. A quantum market, confined to a finite time horizon $\mathsf{t} < \infty$, is described by the filtrated von Neumann algebra $\mathcal{A} = \overline{\bigcup_{t\in[0,\mathsf{t}]} \mathcal{A}_t}^w$, with $\mathcal{A}_t = \mathscr{B}\left(\Phi_{t]}\right) \otimes I_{\Phi_{[t}}$, $s \le t \in [0,\mathsf{t}]$. A process $(Y(t))_{t\ge 0}$ is adapted, with respect to $(\mathcal{A}_t)_{t\ge0}$, or simply adapted, if $Y(t) \in \mathcal{A}_t$ for $t \in [0,\mathsf{t}]$.

A quantum stochastic integral of densely defined operators on Φ is well-understood (see [22, 3, 28] and the references therein). Let $\{A^\epsilon\}_{\epsilon\in\Sigma}, \Sigma = \{+,-,\circ,\emptyset\}$ be the creation, annihilation, conservation and time processes. Integrals of the form, borrowing the notations from [2, 5],

$$X(t) = \lambda I_\Phi + \sum_{\epsilon\in\Sigma} \int_0^t C_s^\epsilon(X)\, dA_s^\epsilon$$

for suitable adapted processes $\{C_\cdot^\epsilon\}_{\epsilon\in\Sigma}$ and $\lambda \in \mathbb{C}$, the complex numbers, can be defined. Among them, the set of regular semimartingales of bounded integral processes, denoted by $\boldsymbol{\mathcal{S}}$ and characterized in [29, 3], is particularly important to us. In addition, we pay special attention to integrals of the form $\int_0^t L(s)\, dX(s)\, M(s) \in \boldsymbol{\mathcal{S}}$ which can be written as the limits of Riemann sums, under the σ-weakly operator topology, for $X \in \boldsymbol{\mathcal{S}}$ and for pairs of adapted processes (L, M). Such triplets (L, X, M) are labeled Riemann sum attainable triplet (also defined below) in [34]. It is helpful to write

$$Y(t) \equiv \sum_{\epsilon\in\Sigma} \int_0^t C_s^\epsilon(Y)\, dA_s^\epsilon = \sum_{\epsilon\in\Sigma} \int_0^t L(s)\, C_s^\epsilon(X)\, M(s)\, dA_s^\epsilon.$$

A division D of $[0,t] \subseteq [0,\mathsf{t}]$ is represented by $0 = t_0 < \cdots < t_N = t$. Tagged systems $P = (D,\xi), \xi = (\xi_1, \cdots, \xi_N) \in [0,t]^N$ are called left partitions of $[0,t]$ if $\xi_k = t_{k-1}, 1 \le k \le N$. The set of all left partitions of $[0,t]$ is denoted by $\mathcal{P}[0,t]$. The norm of a partition P is $\|P\| = \underset{1\le k\le N}{max}(t_k - t_{k-1})$. A sequence $\{P_n\}_{n=1}^\infty \subset \mathcal{P}[0,t]$ is said converging if $\lim_{n\to\infty} \|P_n\| = 0$.

An adapted step process L, associated with $P \in \mathcal{P}[0,\mathsf{t}]$, has the form

$$L(t) = \sum_{k=1}^N \left[L_{k-1}\chi_{(t_{k-1},t_k)}(t) + \chi_{\{t_{k-1}\}}(t)\, L^{(k-1)}\right], L_{k-1}, L^{(k-1)} \in \mathcal{A}_{t_{k-1}}, t \in [0,\mathsf{t}).$$

The value of $L(\mathsf{t})$ is defined by the left limit. This is a process with both right and left limits in operator norm topology. We shall call a process with such property $norm-regulated$. Similarly, we can define $strong-regulated$ and $weak-regulated$ processes. If $L^{(k-1)} = L_{k-1}$, then L is càdlàg. We can also speak of $norm-$*càdlàg*, $strong-$*càdlàg* and $weak-$*càdlàg*. Corresponding to the notion of predictable process we consider those of left continuous with right limits:

$$L\left(t\right)=\sum_{k=1}^{N}L_{k-1}\chi_{(t_{k-1},t_{k}]}\left(t\right),L_{k-1}\in\mathcal{A}_{t_{k-1}},t\in\left[0,\mathsf{t}\right].$$

For convenience, we introduce the parallel notions of $norm-predictable$, $strong-predictable$ and $weak-predictable$.

For $X=\left(X_{t}\right)_{t\in[0,\mathsf{t}]}\in\boldsymbol{\mathcal{S}}$ and $t\in[t_{n-1},t_{n}),n\leq N$ one can define a finite sum process as

$$\begin{aligned}S\left(t,L,dX,M,P\right)&\triangleq\sum_{k=1}^{n-1}L_{k-1}\left[X_{t_{k}}-X_{t_{k-1}}\right]M_{k-1}+L_{n-1}\left[X_{t}-X_{t_{n-1}}\right]M_{n-1}\\&\equiv\sum_{k=1}^{n-1}L\left(t_{k-1}^{+}\right)\left[X_{t_{k}}-X_{t_{k-1}}\right]M\left(t_{k-1}^{+}\right)+L\left(t_{k-1}^{+}\right)\left[X_{t}-X_{t_{n-1}}\right]M\left(t_{k-1}^{+}\right)\end{aligned}$$

where M is another adapted step process with the representation

$$M\left(t\right)=\sum_{k=1}^{N}M_{k-1}\chi_{(t_{k-1},t_{k}]}\left(t\right),M_{k-1}\in\mathcal{A}_{t_{k-1}},t\in\left[0,\mathsf{t}\right].$$

It is more intuitive to write the sum as $\int_{0}^{t}L_{s}dM_{s}M_{s}$. Notice that the values of L and M on $\{t_{k-1}\}_{k=1}^{N}$ are irrelevant.

Following [34] let $\left(\mathcal{B}\right)G\left(\left[0,\mathsf{t}\right],\mathscr{B}\left(\Phi\right)\right)$ stand for the set of $strong-regulared$ functions. For any strong regulated adapted process L and a left partition P, one can associate it with an adapted $norm-predictable$ step process

$$L^{P}\left(t\right)=\sum_{k=1}^{N}L\left(t_{k-1}^{+}\right)\chi_{(t_{k-1},t_{k}]}\left(t\right),t\in\left[0,\mathsf{t}\right].$$

If M is another such adapted process, the following integral is well-defined: $\int_{0}^{t}L_{s}^{P}dX_{s}M_{s}^{P}=S\left(t,L^{P},dX,M^{P},P\right)$. Following [34] we have the definition.

Definition 2.1. Let $X\in\boldsymbol{\mathcal{S}}$ and let L and M be two strong regulated adapted processes. The triplet (L,X,M) is weakly Riemann sum attainable if $S\left(t,L^{P},dX,M^{P},P\right)$ converges weakly:

$$\lim_{\|P\in\mathcal{P}[0,t]\|\to0}\langle u,S\left(t,L^{P},dX,M^{P},P\right)v\rangle\text{ exists }\forall u,v\in\Phi,\forall t\in\left[0,\mathsf{t}\right].$$

From now on, we shall require that $L,L^{*},M,M^{*}\in\left(\mathcal{B}\right)G\left(\left[0,\mathsf{t}\right],\mathscr{B}\left(\Phi\right)\right)$ and furthermore they are strong left continuous. By Theorem 3.3 of [34], $S\left(t,L^{P_{n}},dX,M^{P_{n}},P_{n}\right)$ converges to an operator on a dense subset of Φ for any left converging partition sequence $\{P_{n}\}_{n=1}^{\infty}$.

Corollary 2.2. *If $S\left(t,L,dX,M,P\right)$ converges weakly, then it converges to a bounded operator $\int_{0}^{t}L\left(s\right)dX\left(s\right)M\left(s\right)$. Furthermore, the convergence is $\sigma-weakly$ and the norms*

$$\left\{\|S\left(t,L^{P},dX,M^{P},P\right)\|\right\}_{P\in\mathcal{P}_{l}[0,t]}$$

is uniformly bounded for $\forall t \in [0, \mathsf{t}]$.

Proof. First, $Y(t) \equiv \int_0^t L(s)\, dX(s)\, M(s)$ is a densely defined operator. Since $Y_n(t) \equiv S\left(t, L^{P_n}, dX, M^{P_n}, P_n\right)$ converges weakly for any left converging partition sequence $\{P_n\}_{n=1}^{\infty}$, the sequence $\{\|Y_n(t)\, u\|\}_{n=1}^{\infty}$ is bounded for any $u \in \Phi$. This is a consequence of the Uniformly Boundedness Principle [31]. By Theorem 3.3 of [34], $Y_n(t) \to Y(t)$ $\sigma - \textit{weakly}$. Following the proof of Corollary 3.11 of [34] we arrive at the desired conclusion. □

Let us now turn our attention to $X \in \boldsymbol{\mathcal{S}}$. First, we notice that $X(t)\, u$ is continuous for any $u \in \Phi$. Following Lemma 14 of [5] this is true for u in a dense subset. The general case follows from the local boundedness of $\|X(t)\|$. Due to the necessity of discounting in finance, one often needs to consider $U^{-\frac{1}{2}} X U^{-\frac{1}{2}}$ for $0 < U \in \boldsymbol{\mathcal{S}}$. We remark that $U > 0$ implies that

$$0 < \alpha < U < \beta, \alpha, \beta \in (0, \infty).$$

By the Example 2.6 of [34], $U^{-\frac{1}{2}} \in \boldsymbol{\mathcal{S}}$ and $U^{-\frac{1}{2}} X U^{-\frac{1}{2}} \in \boldsymbol{\mathcal{S}}$.

Next, we list some conditions for (L, X, M) to become a weakly Riemann sum attainable triplet. The first is to have point-wise adapted bounded variation (Definition 3.10 [34]) on $[0, t], t \in [0, \mathsf{t}]$

$$\sup_{P \in \mathcal{P}_l[0,t]} \|S\left(t, L^P, dX, M^P, P\right)\| < \infty.$$

Corollary 3.11 of [34] assures that (L, X, M) is weakly Riemann sum attainable. In order for $\left(A, U^{-\frac{1}{2}} X U^{-\frac{1}{2}}, B\right)$ to become a weakly Riemann sum attainable triplet we need a stronger condition on X and U being of generalized Lipschitz type (Definition 3.13 [34]), it follows from Lemma 3.14 and Lemma 3.15 [34] that $U^{-\frac{1}{2}} X U^{-\frac{1}{2}}$ is also of generalized Lipschitz type.

Alternately, for any $X, U \in \boldsymbol{\mathcal{S}}$, if L and M are taken from the class of adapted processes with bounded variations (Section 3.4 of Kurzweil-Stiejes integral [34]), then the corresponding Kurzweil-Stiejes integrals coincide with the quantum stochastic ones

$$\int_0^t L(s)\, dX(s)\, M(s) = \int_0^t Ld[X]\, M;$$

$$\int_0^t L(s)\, d\left(U^{-\frac{1}{2}}(s)\, X(s)\, U^{-\frac{1}{2}}(s)\right) M(s) = \int_0^t Ld\left[U^{-\frac{1}{2}} X U^{-\frac{1}{2}}\right] M.$$

In this note, we shall therefore assume that (L, X, M) and $\left(L, U^{-\frac{1}{2}} X U^{-\frac{1}{2}}, M\right)$ are weakly Riemann sum attainable whenever we use the relevant integrals with the understanding that the processes in question are connected with suitable conditions.

2.2. **Quantum portfolio.** A financial instrument in a market $\mathcal{E}$ is often called "asset". Normally, there are two classes of assets. One consists of basic securities which are liquid financial products that are always available for trading. Their market prices are determined by supply and demand and not by models. The price evolution, however, are random. The other class of assets consists of derivative securities, also called contingent claims, for which the key issues are pricing and risk managing. The assets in this class are structured to yield specific cash-flows to match special needs of agents, with underlyings coming from the basic securities.

The values of financial quantities in $\mathcal{E}$ are self-adjoint operator processes adapted to $(\mathcal{A}_t)_{t\in[0,\mathsf{t}]}$. There are finitely many, say $d \in \mathbb{N}$, basic assets whose prices are $\mathbb{S} = \{S_i\}_{i=1}^d \subset \boldsymbol{\mathcal{S}}$. Often, they are further assumed positive. We shall from now on to write $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathbb{S})$ to specify both the filtrated von Neumann algebra $\mathcal{A} = (\mathcal{A}_t)_{t\in[0,\mathsf{t}]}$ and the set of basic assets $\mathbb{S}$, or to be precise, the price processes of them.

For any $S \in \mathbb{S}$ and two adapted processes $L = (L(t))_{t\in[0,\mathsf{t}]}$ and $M = (M(t))_{t\in[0,\mathsf{t}]}$ one may think of

$$[L, S, M]_t \equiv L(t)\, S(t)\, M(t) + M^*(t)\, S(t)\, L^*(t)\,, t \in [0, \mathsf{t}]$$

as the value of (L, M) shares of S at time t. The pair (L, M) is called a position of S. If (L, S, M) is weakly Riemann sum attainable, we denote

$$\int_0^t [L, dS, M]_s \equiv \int_0^t [L(s)\, dS(s)\, M(s) + M^*(s)\, dS(s)\, L^*(s)]\,, t \in [0, \mathsf{t}]\,.$$

Intuitively, it is the value accumulation according to the position (L, M) purely coming from the variation of S. Already at the stage of forming portfolios that non-commutativity of operators brings up a difficulty in combining positions. Suppose that (A, B) is another position of S due to, say, a new trading. One can not add (L, M) and (A, B) the usual way. Instead, a union of the two is the correct answer. Hence, we define a Riemann sum attainable trading strategy of $S_i \in \mathbb{S}$ as a finite collection of positions $\{(L_{ij}, M_{ij})\}_{j=1}^J$ such that each (L_{ij}, S_i, M_{ij}) is weakly Riemann sum attainable. Here J is a strategy specific number in $\mathbb{N}$.

Let $(\mathbb{L}, \mathbb{M})$ be a set of trading strategies, or a trading strategy for short, where $\mathbb{L} = \{L_{ij}\}_{i=1,j=1}^{d,J}$ and $\mathbb{M} = \{M_{ij}\}_{i=1,j=1}^{d,J}$. One can form the wealth and gain portfolios, respectively,

$$V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = \sum_{i=1}^d \sum_{j=1}^J [L_{ij}, S_i, M_{ij}]_t\,;$$

$$V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = \lambda(\mathbb{L}, \mathbb{S}, \mathbb{M}) + \sum_{i=1}^d \sum_{j=1}^J \int_0^t [L_{ij}, dS_i, M_{ij}]_s\,.$$

Here $\lambda(\mathbb{L}, \mathbb{S}, \mathbb{M})$ is a constant of initial investment. Clearly, if there is any hope that the values of these two portfolios are the same, one should have

$$\lambda(\mathbb{L}, \mathbb{S}, \mathbb{M}) = V_w(0, \mathbb{L}, \mathbb{S}, \mathbb{M})\,.$$

We shall denote the integral part of $V_g(t, \mathbb{L}, \mathbb{M})$ as $V_{cg}(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ and call it the capital gain.

Definition 2.3. The strategy $(\mathbb{L}, \mathbb{M})$ is self-financing if $V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ for $\forall t \in [0, \mathsf{t}]$. In this case, we shall call $(\mathbb{L}, \mathbb{M})$ a self-financing strategy and $(\mathbb{L}, \mathbb{S}, \mathbb{M})$ a self-financing Riemann sum attainable triplet.

The financial meaning of a self-financing is clear. While V_w indicates the actual wealth, mathematically V_g plays an important role in various calculations. Trivially, the following is a self-financing equation

$$\sum_{i=1}^{d} a_i S_i(t) = \sum_{i=1}^{d} a_i S_i(0) + \sum_{i=1}^{d} \int_0^t a_i dS_i(s), \{a_i\}_{i=1}^{d} \subset \mathbb{R}^d.$$

Later, we shall show that there exist non-trivial self-financing portfolios.

For convenience sake, we shall use $(\mathbb{L}, \mathbb{M}) = \lambda \in \mathbb{R}$ to indicate that all processes are λI_Φ. For any two adapted processes S and T, the notation $S\mathbb{L}T$ is also self-explanatory

$$S\mathbb{L}T = \{SL_{ij}T\}_{i=1,j=1}^{d,J}.$$

2.3. **Num*é*raire deflated portfolios.** The values of financial assets are often quoted in terms of the values of some other special assets acting as units. Money market account, bond and exchange rate are three simple examples of unit. Taking a money market account as a unit, the relative value is nothing but discounted value in the conventional sense. In fact, we shall allow any positive regular semi-martingale to be a unit, under a more familiar name.

Definition 2.4. An adapted non-constant process $U \in \boldsymbol{\mathcal{S}}$ is called a num*é*raire if $U(t)$ is positive for $t \in [0, \mathsf{t}]$.

Using a num*é*raire U one can specify a set of num*é*raire deflated processes $\mathbb{S}^U = \left\{ S_i^U = U^{-\frac{1}{2}} S_i U^{-\frac{1}{2}} \right\}_{i=1}^{d}$. In general, for any process $(Y(t))_{t\in[0,\mathsf{t}]}$ the notation

$$\left(Y^U(t)\right)_{t\in[0,\mathsf{t}]} = \left(U^{-\frac{1}{2}}(t)\, Y(t)\, U^{-\frac{1}{2}}(t)\right)_{t\in[0,\mathsf{t}]}$$

will be used. If one thinks of $Y(t)$ as a $time-t$ price or forward price, then, by abusing of terminology, $Y^U(t)$ may be viewed as the forward price discounted by $U(t)$.

As a set of deflated basic assets $\mathbb{S}^U$ can also be used to form wealth and gain portfolios. For example, the followings are self-financing portfolios with respect to $\mathbb{S}^U$.

$$\sum_{i=1}^{d} a_i S_i^U(t) = \sum_{i=1}^{d} a_i S_i^U(0) + \sum_{i=1}^{d} \int_0^t a_i dS_i^U(s), \{a_i\}_{i=1}^{d} \subset \mathbb{R}^d.$$

More generally, for a Riemann sum attainable triplet $(\mathbb{L}, \mathbb{S}, \mathbb{M})$ the num*é*raire U deflated wealth portfolio $V_w^U(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ of $V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ can be written as

$$V_w^U(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = V_w\left(t, U^{-\frac{1}{2}} \mathbb{L} U^{\frac{1}{2}}, \mathbb{S}^U, U^{\frac{1}{2}} \mathbb{M} U^{-\frac{1}{2}}\right).$$

The corresponding gain portfolio of the strategy $(\mathbb{L}_U, \mathbb{M}_{U^{-1}}) = \left(U^{-\frac{1}{2}} \mathbb{L} U^{\frac{1}{2}}, U^{\frac{1}{2}} \mathbb{M} U^{-\frac{1}{2}}\right)$ against $\mathbb{S}^U$, can be defined as

$$V_g\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) = \lambda\left(\mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) + \sum_{i=1}^{d} \sum_{j=1}^{J} \int_0^t \left[U^{-\frac{1}{2}} L_{ij} U^{\frac{1}{2}}, dS_i^U, U^{\frac{1}{2}} M_{ij} U^{-\frac{1}{2}}\right]_s.$$

Note that one can also independently define two portfolios $V_w\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$ and $V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$ without referencing $(\mathbb{L},\mathbb{M})$. But implicitly, one has $(\mathbb{L},\mathbb{M})=(\mathbb{A}_{U^{-1}},\mathbb{B}_U)$, provided that all involved integrals exist.

Lemma 2.5. *Let $U\in\mathbb{S}$ be a numéraire, then for a given gain portfolio*

$$V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)=\lambda\left(\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)+\sum_{i=1}^{d}\sum_{j=1}^{J}\int_0^t\left[A_{ij},dS_i^U,B_{ij}\right]_s$$

there exists a self-financing portfolio whose gain portfolio is $V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$.

Proof. Let us assume that $U=S_1$. Then $U^U\equiv 1$ and $dS_1^U\equiv 0$. Hence, the given gain portfolio will not change if we assume that all (A_{1j},B_{1j}) are zero. Denote the self-adjoint element

$$Y(t)=V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)-V_w\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right).$$

It has a Jordan decomposition (cf. [36] p20) $Y(t)=Y^+(t)-Y^-(t)$ with $Y^+(t)\,Y^-(t)=0$. Define

$$A=\frac{\left(\sqrt{Y^+}+\sqrt{Y^-}\right)}{\sqrt{2}},B=\frac{\left(\sqrt{Y^+}-\sqrt{Y^-}\right)}{\sqrt{2}}.$$

Then

$$AB=\frac{Y}{2};\quad B^*A^*=\frac{Y}{2}.$$

Let us define a new strategy $\left(\overline{\mathbb{A}},\overline{\mathbb{B}}\right)$ as

$$\overline{A}_{ij}=\begin{cases}A & i=j=1\\ A_{ij} & \text{otherwise}\end{cases};\overline{B}_{ij}=\begin{cases}B & i=j=1\\ B_{ij} & \text{otherwise}\end{cases}.$$

Clearly, $V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)=V_g\left(t,\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)$. As desired,

$$V_w\left(t,\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)=V_w\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)+Y(t)=V_g\left(t,\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right).$$

The time zero constant $\lambda\left(\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)$ can be calculated as

$$\lambda\left(\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)=V_w\left(0,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)+Y(0)=\lambda\left(\mathbb{A},\mathbb{S}^U,\mathbb{B}\right).$$

□

Here one takes advantage of $U^U\equiv 1$ which at the same time makes the result less useful. Nevertheless, it turns out that this simple result helps to construct non-trivial self-financing portfolios with respect to $\mathbb{S}$.

Definition 2.6. Let $(\mathbb{L},\mathbb{M})$ be a self-financing strategy with respect to $\mathbb{S}$. A numéraire U is said to preserve the self-financing property of $(\mathbb{L},\mathbb{M})$ if

$$V_w\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right).$$

The definition is not empty. For example, $(\mathbb{L},\mathbb{M})=\lambda\in\mathbb{R}$ is one. We now manufacture a class of self-financing portfolios out of an existing one. Let $(\mathbb{L},\mathbb{M})$ be a trading strategy such that $(\mathbb{L}_U,\mathbb{M}_{U^{-1}})$ is self-financing

$$V_w\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)$$

for some numéraire U . In this particular case, we allow $U\equiv 1$. To search $Z\in\boldsymbol{\mathcal{S}}$ as a numéraire such that

$$V_w\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right)=V_g\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right),$$

let $Y(t)=Z^{-\frac{1}{2}}(t)\,U^{\frac{1}{2}}(t)\in\boldsymbol{\mathcal{S}}$. One checks that

$$\begin{aligned}
&Y\mathbb{L}_U=\mathbb{L}_UY, Y^*\mathbb{M}_{U^{-1}}=\mathbb{M}_{U^{-1}}Y^*\\
&\Leftrightarrow Y^*\mathbb{L}=\mathbb{L}Y^*, Y\mathbb{M}=\mathbb{M}Y\\
&\Leftrightarrow \mathbb{L}_U=\mathbb{L}_Z, \mathbb{M}_{U^{-1}}=\mathbb{M}_{Z^{-1}}.
\end{aligned}$$

In fact, we need one more commuting condition. Since $Y\in\boldsymbol{\mathcal{S}}$, it can be written as

$$dY=\sum_{\epsilon\in\Sigma}C_s^{\epsilon}(Y)\,dA_s^{\epsilon}.$$

The condition we shall use is the following

$$\mathbb{L}_UdY=dY\mathbb{L}_U, \mathbb{M}_{U^{-1}}dY^*=dY^*\mathbb{M}_{U^{-1}}.$$

This can be achieved by demanding $\mathbb{L}_U$ evaluated at s to commute with $Y(t)$ and $\mathbb{M}_{U^{-1}}$ evaluated at s to commutes with $Y^*(t)$ for $s\leq t$. For future use, we say that an adapted process $S=(S(t))_{t\in[0,\mathrm{t}]}$ commutes with the history of another adapted process $T=(T(t))_{t\in[0,\mathrm{t}]}$ if $S(t)\,T(s)=T(s)\,S(s)$ for all $s\leq t$. Using this terminology, Y commutes with the histories of $\mathbb{L}_U$ and Y^* commutes with the histories of $\mathbb{M}_{U^{-1}}$.

Theorem 2.7. *Let $(\mathbb{L},\mathbb{S},\mathbb{M})$ be a Riemann sum attainable triplet and let U be a numéraire such that $\left(\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)$ is a self-financing Riemann sum attainable triplet*

$$V_w\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right).$$

If Z is another numéraire such that $\left(\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right)$ is Riemann sum attainable and if $Y=Z^{-\frac{1}{2}}U^{\frac{1}{2}}$ commutes with the histories of $\mathbb{L}_U$ and Y^* commutes with the histories of $\mathbb{M}_{U^{-1}}$, then

$$V_w\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right)=V_g\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right).$$

Proof. To simplify the notation, denote $\mathbb{A}=\mathbb{L}_U$, $\mathbb{B}=\mathbb{M}_{U^{-1}}$ and $\mathbb{X}=\mathbb{S}^U$. In addition, we shall also utilize the symbols

$$\begin{aligned}
&V_w(t,\Omega)=\sum_{i,j=1}^{d,J}\left[A_{ij}(t)\,d\Omega_i(t)\,B_{ij}(t)+B_{ij}^*(t)\,d\Omega_i(t)\,A_{ij}^*(t)\right], \Omega=\{\Omega_i\}_{i=1}^d,\\
&\Omega_i=X_i, X_iY^*, dX_i, d(X_iY^*), d(YX_iY^*), d[X_i,Y^*], d[Y,X_iY^*]
\end{aligned}$$

for the fixed coefficients $\mathbb{A}$ and $\mathbb{B}$. Taking the differential form [5] of the wealth portfolio we obtain

$$\begin{aligned}&dV_w\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right)=dV_w\left(t,\mathbb{L}_U,\mathbb{S}^Z,\mathbb{M}_{U^{-1}}\right)=d\left[Y(t)\,V_w(t,\mathbb{X})\,Y^*(t)\right]\\&=dY(t)\left(V_w(t,\mathbb{X})\,Y^*(t)\right)+Y(t)\,d\left(V_w(t,\mathbb{X})\,Y^*(t)\right)+d\left[Y,V_w(t,\mathbb{X})\,Y^*\right]_t.\end{aligned}$$

The second term can be further decomposed as

$$Y(t)\,V_w(t,d\mathbb{X})\,Y^*(t)+Y(t)\,V_w(t,\mathbb{X})\,dY^*(t)+Y(t)\,d\left[V_w(t,\mathbb{X}),Y^*\right]_t.$$

For the gain portfolio, we have

$$\begin{aligned}&dV_{cg}\left(t,\mathbb{L}_Z,\mathbb{S}^Z,\mathbb{M}_{Z^{-1}}\right)=dV_{cg}\left(t,\mathbb{L}_U,(Y\mathbb{X}Y^*),\mathbb{M}_{U^{-1}}\right)\\&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}A_{ij}\left\{dY\left(X_iY^*\right)+Y\,d\left(X_iY^*\right)+d\left[Y,X_iY^*\right]\right\}B_{ij}\\+B_{ij}^*\left\{dY\left(X_iY^*\right)+Y\,d\left(X_iY^*\right)+d\left[Y,X_iY^*\right]\right\}A_{ij}^*\end{array}\right](t)\\&=dY(t)\left(V_w(t,\mathbb{X})\,Y^*(t)\right)+Y(t)\,V_w\left(t,d\left(\mathbb{X}Y^*\right)\right)+V_w\left(t,d\left[Y,\mathbb{X}Y^*\right]\right).\end{aligned}$$

Here the relations

$$\begin{aligned}(dY)\,A_{ij}=A_{ij}dY;\,(dY)\,B_{ij}^*=B_{ij}^*dY;\\YA_{ij}=A_{ij}Y;\,YB_{ij}^*=B_{ij}^*Y\end{aligned}$$

are used. The middle term can be further decomposed as

$$Y(t)\,V_w\left(t,(d\mathbb{X})\right)Y^*(t)+Y(t)\,V_w(t,\mathbb{X})\,dY^*(t)+Y(t)\,V_w\left(t,d\left[\mathbb{X},Y^*\right]\right).$$

We are left to show that

$$\begin{aligned}&Y(t)\,d\left[V_w(t,\mathbb{X}),Y^*\right]_t+d\left[Y,V_w(t,\mathbb{X})\,Y^*\right]_t\\&=Y(t)\,V_w\left(t,d\left[\mathbb{X},Y^*\right]\right)+V_w\left(t,d\left[Y,\mathbb{X}Y^*\right]\right).\end{aligned}$$

The bracket $[\bullet,\bullet]$ defined in [3] is linear on each of its two arguments. Hence, the above equation becomes

$$\begin{aligned}&Y(t)\,d\left[V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}),Y^*\right]_t+d\left[Y,V_w(t,\mathbb{X})\,Y^*\right]_t\\&=Y(t)\,V_w\left(t,d\left[\mathbb{X},Y^*\right]\right)+V_w\left(t,d\left[Y,\mathbb{X}Y^*\right]\right).\end{aligned}$$

Let us examine the first term of each side

$$\begin{aligned}&Y(t)\,V_w\left(t,d\left[\mathbb{X},Y^*\right]\right)\\&=Y(t)\sum_{i,j=1}^{d,J}\left\{A_{ij}(t)\,d\left[X_i,Y^*\right]_tB_{ij}(t)+B_{ij}^*(t)\,d\left[X_i,Y^*\right]_tA_{ij}^*(t)\right\}.\end{aligned}$$

For the one on the other side

$$\begin{aligned}&Y(t)\,d\left[V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}),Y^*\right]_t\\&=Y(t)\sum_{i,j=1}^{d,J}d\left[\int\left(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\right),Y^*\right]_t.\end{aligned}$$

Note that for $\epsilon,\eta\in\Sigma$,

$$\begin{aligned}&C_t^\epsilon\left[\int\left(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\right)\right]C_t^\eta\left(Y^*\right)\\&=A_{ij}(t)\left[C_t^\epsilon\left(X_i\right)C_t^\eta\left(Y^*\right)\right]B_{ij}(t)+B_{ij}^*(t)\left[C_t^\epsilon\left(X_i\right)C_t^\eta\left(Y^*\right)\right]A_{ij}^*(t).\end{aligned}$$

Summing such terms over chosen ϵ and η according to the definition of $[\bullet,\bullet]$ we obtain

$$\begin{aligned}&d\left[\int\big(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\big),Y^*\right]_t\\&=A_{ij}(t)\,d\left[X_i,Y^*\right]_t B_{ij}(t)+B_{ij}^*(t)\,d\left[X_i,Y^*\right]_t A_{ij}^*(t).\end{aligned}$$

Finally, we are left to compare $d\left[Y,V_w(t,\mathbb{X})\,Y^*\right]_t$ with $V_w(t,d[Y,\mathbb{X}Y^*])$. Denote $\lambda=\lambda(\mathbb{A},\mathbb{X},\mathbb{B})$. Note that

$$\begin{aligned}d\left[Y,V_w(t,\mathbb{X})\,Y^*\right]_t&=\lambda d\left[Y,Y^*\right]_t+d\left[Y,V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*\right]_t;\\V_w(t,d[Y,\mathbb{X}Y^*])&=\sum_{i,j=1}^{d,J}\left\{\begin{array}{c}A_{ij}(t)\,d\left[Y,X_iY^*\right]_t B_{ij}(t)\\+B_{ij}^*(t)\,d\left[Y,X_iY^*\right]_t A_{ij}^*(t)\end{array}\right\}.\end{aligned}$$

By the definition of [3]

$$\begin{aligned}&d\left[Y,V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*\right]_t\\&=\left[\begin{array}{c}C_t^\circ(Y)\,C_t^\circ(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}))\,dA_t^\circ+C_t^-(Y)\,C_t^\circ(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}))\,dA_t^-\\+C_t^\circ(Y)\,C_t^+(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}))\,dA_t^++C_t^-(Y)\,C_t^+(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B}))\,dA_t^\emptyset\end{array}\right].\end{aligned}$$

We need $C_t^\circ(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*)$ and $C_t^+(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*)$. To expand the term

$$C_t^\epsilon(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*)=\sum_{i,j=1}^{d,J}C_t^\epsilon\left(\left\{\int\left[A_{ij},dX_i,B_{ij}\right]_s\right\}Y^*\right)$$

let us compute

$$\begin{aligned}&d\left[\left\{\int_0^t\big(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\big)\right\}Y^*(t)\right]\\&=\big(A_{ij}(t)\,dX_i(t)\,B_{ij}(t)+B_{ij}^*(t)\,dX_i(t)\,A_{ij}^*(t)\big)\,Y^*(t)\\&+\left\{\int_0^t\big(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\big)\right\}dY^*(t)\\&+d\left[\int\big(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\big),Y^*\right]_t\\&=\sum_{\epsilon'\in\Sigma}\left\{A_{ij}(t)\,C_t^{\epsilon'}(X_i)\,B_{ij}(t)+B_{ij}^*(t)\,C_t^{\epsilon'}(X_i)\,A_{ij}^*(t)\right\}Y^*(t)\,dA_t^{\epsilon'}\\&+\sum_{\epsilon'\in\Sigma}\left\{\int_0^t\big(A_{ij}(s)\,dX_i(s)\,B_{ij}(s)+B_{ij}^*(s)\,dX_i(s)\,A_{ij}^*(s)\big)\right\}C_t^{\epsilon'}(Y^*)\,dA_t^{\epsilon'}\\&+A_{ij}(t)\left[\begin{array}{c}C_t^\circ(X_i)\,C_t^\circ(Y^*)\,B_{ij}(t)\,dA_t^\circ+C_t^-(X_i)\,C_t^\circ(Y^*)\,B_{ij}(t)\,dA_t^-\\+C_t^\circ(X_i)\,C_t^+(Y^*)\,B_{ij}(t)\,dA_t^++C_t^-(X_i)\,C_t^+(Y^*)\,B_{ij}(t)\,dA_t^\emptyset\end{array}\right]\\&+B_{ij}^*(t)\left[\begin{array}{c}C_t^\circ(X_i)\,C_t^\circ(Y^*)\,A_{ij}^*(t)\,dA_t^\circ+C_t^-(X_i)\,C_t^\circ(Y^*)\,A_{ij}^*(t)\,dA_t^-\\+C_t^\circ(X_i)\,C_t^+(Y^*)\,A_{ij}^*(t)\,dA_t^++C_t^-(X_i)\,C_t^+(Y^*)\,A_{ij}^*(t)\,dA_t^\emptyset\end{array}\right].\end{aligned}$$

Combining them together we have

$$\begin{aligned}
&C_t^\circ\left(V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right)\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^\circ\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^\circ\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+\left\{\int_0^t\left(A_{ij}\left(s\right)dX_i\left(s\right)B_{ij}\left(s\right)+B_{ij}^*\left(s\right)dX_i\left(s\right)A_{ij}^*\left(s\right)\right)\right\}C_t^\circ\left(Y^*\right)\\
+A_{ij}\left(t\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right];\\
&C_t^+\left(V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right)\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^+\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^+\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+\left\{\int_0^t\left(A_{ij}\left(s\right)dX_i\left(s\right)B_{ij}\left(s\right)+B_{ij}^*\left(s\right)dX_i\left(s\right)A_{ij}^*\left(s\right)\right)\right\}C_t^+\left(Y^*\right)\\
+A_{ij}\left(t\right)C_t^\circ\left(X_i\right)C_t^+\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^\circ\left(X_i\right)C_t^+\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right].
\end{aligned}$$

As a result,

$$d\left[Y,V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right]\equiv\sum_{\epsilon\in\Sigma}H_t^\epsilon dA_t^\epsilon,$$

where

$$\begin{aligned}
&H_t^\circ=C_t^\circ\left(Y\right)C_t^\circ\left(V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right)\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+C_t^\circ\left(Y\right)\left\{\int_0^t\left(A_{ij}\left(s\right)dX_i\left(s\right)B_{ij}\left(s\right)+B_{ij}^*\left(s\right)dX_i\left(s\right)A_{ij}^*\left(s\right)\right)\right\}C_t^\circ\left(Y^*\right)\\
+A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right]\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right]\\
&+C_t^\circ\left(Y\right)V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)C_t^\circ\left(Y^*\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&H_t^-=C_t^-\left(Y\right)C_t^\circ\left(V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right)\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^-\left(Y\right)C_t^\circ\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^-\left(Y\right)C_t^\circ\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+A_{ij}\left(t\right)C_t^-\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^-\left(Y\right)C_t^\circ\left(X_i\right)C_t^\circ\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right]\\
&+C_t^-\left(Y\right)V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)C_t^\circ\left(Y^*\right).
\end{aligned}$$

$$\begin{aligned}
&H_t^+=C_t^\circ\left(Y\right)C_t^+\left(V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)Y^*\right)\\
&=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}
\left(A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^+\left(X_i\right)B_{ij}\left(t\right)+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^+\left(X_i\right)A_{ij}^*\left(t\right)\right)Y^*\left(t\right)\\
+A_{ij}\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^+\left(Y^*\right)B_{ij}\left(t\right)\\
+B_{ij}^*\left(t\right)C_t^\circ\left(Y\right)C_t^\circ\left(X_i\right)C_t^+\left(Y^*\right)A_{ij}^*\left(t\right)
\end{array}\right]\\
&+C_t^\circ\left(Y\right)V_{cg}\left(t,\mathbb{A},\mathbb{X},\mathbb{B}\right)C_t^+\left(Y^*\right).
\end{aligned}$$

$$\begin{aligned}&H_t^{\emptyset} = C_t^{-}(Y)\,C_t^{+}(V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,Y^*)\\ &= \sum_{i,j=1}^{d,J}\left[\begin{array}{c}\left(A_{ij}(t)\,C_t^{-}(Y)\,C_t^{+}(X_i)\,B_{ij}(t)+B_{ij}^*(t)\,C_t^{-}(Y)\,C_t^{+}(X_i)\,A_{ij}^*(t)\right)Y^*(t)\\ +A_{ij}(t)\,C_t^{-}(Y)\,C_t^{\circ}(X_i)\,C_t^{+}(Y^*)\,B_{ij}(t)\\ +B_{ij}^*(t)\,C_t^{-}(Y)\,C_t^{\circ}(X_i)\,C_t^{+}(Y^*)\,A_{ij}^*(t)\end{array}\right]\\ &+C_t^{-}(Y)\,V_{cg}(t,\mathbb{A},\mathbb{X},\mathbb{B})\,C_t^{+}(Y^*)\,.\end{aligned}$$

On the other hand,

$$d\,[Y,\mathbb{X}Y^*]_t = \left[\begin{array}{c} C_t^{\circ}(Y)\,C_t^{\circ}(\mathbb{X}Y^*)\,dA_t^{\circ}+C_t^{-}(Y)\,C_t^{\circ}(\mathbb{X}Y^*)\,dA_t^{-}\\ +C_t^{\circ}(Y)\,C_t^{+}(\mathbb{X}Y^*)\,dA_t^{+}+C_t^{-}(Y)\,C_t^{+}(\mathbb{X}Y^*)\,dA_t^{\emptyset}\end{array}\right]\quad .$$

Again, we need $C^{\circ}(\mathbb{X}Y^*)$ and $C^{+}(\mathbb{X}Y^*)$. Let us calculate the term $C^{\epsilon}(\mathbb{X}Y^*)$. Write

$$\begin{aligned}&d\,(X_i(t)\,Y^*(t)) = d\,(X_i(t))\,Y^*(t)+X_i(t)\,dY^*(t)+d\,[X_i,Y^*]_t\\ &=\left(\sum_{\epsilon'\in\Sigma}C_t^{\epsilon'}(X_i)\,Y^*(t)\,dA_t^{\epsilon'}\right)+\left(\sum_{\epsilon'\in\Sigma}X_i(t)\,C_t^{\epsilon'}(Y^*)\,dA_t^{\epsilon'}\right)\\ &+\left[\begin{array}{c}C_t^{\circ}(X_i)\,C_t^{\circ}(Y^*)\,dA_t^{\circ}+C_t^{-}(X_i)\,C_t^{\circ}(Y^*)\,dA_t^{-}\\ +C_t^{\circ}(X_i)\,C_t^{+}(Y^*)\,dA_t^{+}+C_t^{-}(X_i)\,C_t^{+}(Y^*)\,dA_t^{\emptyset}\end{array}\right].\end{aligned}$$

It follows that

$$\begin{aligned}&C_t^{\circ}(X_iY^*) = C_t^{\circ}(X_i)\,Y^*(t)+X_i(t)\,C_t^{\circ}(Y^*)+C_t^{\circ}(X_i)\,C_t^{\circ}(Y^*)\,;\\ &C_t^{+}(X_iY^*) = C_t^{+}(X_i)\,Y^*(t)+X_i(t)\,C_t^{+}(Y^*)+C_t^{\circ}(X_i)\,C_t^{+}(Y^*)\,.\end{aligned}$$

Let us compute

$$\begin{aligned}&V_w(t,d\,[Y,\mathbb{X}Y^*]) \equiv \sum_{\epsilon\in\Sigma}G_t^{\epsilon}dA_t^{\epsilon}\\ &=\sum_{i,j=1}^{d,J}\left\{\begin{array}{c}\left[\begin{array}{c}A_{ij}(t)\,C_t^{\circ}(Y)\,C_t^{\circ}(X_iY^*)\,B_{ij}(t)\,dA_t^{\circ}\\ +B_{ij}^*(t)\,C_t^{\circ}(Y)\,C_t^{\circ}(X_iY^*)\,A_{ij}^*(t)\,dA_t^{\circ}\end{array}\right]\\ \left[\begin{array}{c}+A_{ij}(t)\,C_t^{-}(Y)\,C_t^{\circ}(X_iY^*)\,B_{ij}(t)\,dA_t^{-}\\ +B_{ij}^*(t)\,C_t^{-}(Y)\,C_t^{\circ}(X_iY^*)\,A_{ij}^*(t)\,dA_t^{-}\end{array}\right]\\ \left[\begin{array}{c}+A_{ij}(t)\,C_t^{\circ}(Y)\,C_t^{+}(X_iY^*)\,B_{ij}(t)\,dA_t^{+}\\ +B_{ij}^*(t)\,C_t^{\circ}(Y)\,C_t^{+}(X_iY^*)\,A_{ij}^*(t)\,dA_t^{+}\end{array}\right]\\ \left[\begin{array}{c}A_{ij}(t)\,C_t^{-}(Y)\,C_t^{+}(X_iY^*)\,B_{ij}(t)\,dA_t^{\emptyset}\\ +B_{ij}^*(t)\,C_t^{-}(Y)\,C_t^{+}(X_iY^*)\,A_{ij}^*(t)\,dA_t^{\emptyset}\end{array}\right]\end{array}\right\},\end{aligned}$$

where

$$G_t^\circ = \sum_{i,j=1}^{d,J} \left[\begin{array}{c} A_{ij}(t)\, C_t^\circ(Y) \{C_t^\circ(X_i)\, Y^*(t) + X_i(t)\, C_t^\circ(Y^*) + C_t^\circ(X_i)\, C_t^\circ(Y^*)\}\, B_{ij}(t) \\ B_{ij}^*(t)\, C_t^\circ(Y) \{C_t^\circ(X_i)\, Y^*(t) + X_i(t)\, C_t^\circ(Y^*) + C_t^\circ(X_i)\, C_t^\circ(Y^*)\}\, A_{ij}^*(t) \end{array} \right]$$

$$= \sum_{i,j=1}^{d,J} \left[\begin{array}{c} C_t^\circ(Y) \left\{ A_{ij}(t)\, C_t^\circ(X_i)\, B_{ij}(t) + B_{ij}^*(t)\, C_t^\circ(X_i)\, A_{ij}^*(t) \right\} Y^*(t) \\ +C_t^\circ(Y) \left\{ A_{ij}(t)\, X_i(t)\, B_{ij}(t) + B_{ij}^*(t)\, X_i(t)\, A_{ij}^*(t) \right\} C_t^\circ(Y^*) \\ +C_t^\circ(Y)\, A_{ij}(t)\, C_t^\circ(X_i)\, B_{ij}(t)\, C_t^\circ(Y^*) \\ +C_t^\circ(Y)\, B_{ij}^*(t)\, C_t^\circ(X_i)\, A_{ij}^*(t)\, C_t^\circ(Y^*) \end{array} \right]$$

$$= \sum_{i,j=1}^{d,J} \left[\begin{array}{c} C_t^\circ(Y) \left\{ A_{ij}(t)\, C_t^\circ(X_i)\, B_{ij}(t) + B_{ij}^*(t)\, C_t^\circ(X_i)\, A_{ij}^*(t) \right\} Y^*(t) \\ +C_t^\circ(Y)\, A_{ij}(t)\, C_t^\circ(X_i)\, B_{ij}(t)\, C_t^\circ(Y^*) \\ +C_t^\circ(Y)\, B_{ij}^*(t)\, C_t^\circ(X_i)\, A_{ij}^*(t)\, C_t^\circ(Y^*) \end{array} \right]$$

$$+ C_t^\circ(Y)\, V_w(t, \mathbb{A}, \mathbb{X}, \mathbb{B})\, C_t^\circ(Y^*);$$

Note that

$$C_t^\circ(Y)\, V_w(t, \mathbb{X})\, C_t^\circ(Y^*) = C_t^\circ(Y)\, V_{cg}(t, \mathbb{A}, \mathbb{X}, \mathbb{B})\, C_t^\circ(Y^*) + \lambda C_t^\circ(Y)\, C_t^\circ(Y^*).$$

So

$$G_t^\circ = H_t^\circ + \lambda C_t^\circ(Y)\, C_t^\circ(Y^*).$$

Similarly

$$G_t^- = H_t^- + \lambda C_t^-(Y)\, C_t^\circ(Y^*);$$
$$G_t^+ = H_t^+ + \lambda C_t^\circ(Y)\, C_t^+(Y^*);$$
$$G_t^\emptyset = H_t^\emptyset + \lambda C_t^-(Y)\, C_t^+(Y^*).$$

Putting them together one has,

$$dV_w(t, d[Y, \mathbb{X}Y^*]) = d[Y, V_{cg}(t, \mathbb{A}, \mathbb{X}, \mathbb{B})\, Y^*]_t + d[Y, Y^*]_t.$$

This completes the proof of $dV_w\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_Z\right) = dV_g\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_Z\right)$. Finally,

$$V_w\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_Z\right) = V_g\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_Z\right) + Y(0)\, V_w\left(0, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right).$$

We remark that some of the terms appear in the calculations may only be densely defined. However, this does not affect our result. □

Corollary 2.8. *If a numéraire U commutes with the histories of $\mathbb{L}$ and $\mathbb{M}$ then*

$$V_w\left(t, \mathbb{L}, \mathbb{S}^U, \mathbb{M}\right) = V_g\left(t, \mathbb{L}, \mathbb{S}^U, \mathbb{M}\right)$$

if and only if

$$V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}).$$

As a consequence, if one of the above holds

$$V_w\left(t, \mathbb{L}, \mathbb{S}^V, \mathbb{M}\right) = V_g\left(t, \mathbb{L}, \mathbb{S}^V, \mathbb{M}\right)$$

for any numéraire V which commutes with the histories of $\mathbb{L}$ and $\mathbb{M}$.

Proof. Suppose that $V_w\left(t, \mathbb{L}, \mathbb{S}^U, \mathbb{M}\right) = V_g\left(t, \mathbb{L}, \mathbb{S}^U, \mathbb{M}\right)$. Taking $Z = 1$ one has $Y = Z^{-\frac{1}{2}} U^{\frac{1}{2}} = U^{\frac{1}{2}}$. Conversely, suppose that $V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$. Taking $Z = U$ one also obtains desired result. □

Using the theorem we just obtained, one can construct non-trivial self-financing portfolios in terms of $\mathbb{S}$.

Theorem 2.9. *Let* $(\mathbb{L}, \mathbb{S}, \mathbb{M})$ *be a Riemann sum attainable triplet where*

$$(\mathbb{L}, \mathbb{M}) = (L_{ij}, M_{ij})_{i,j=1}^{d,J}.$$

If the numéraire $U = S_1 \in \mathbb{S}$ *commutes with the histories of* $\mathbb{L}$, $\mathbb{M}$ *and* $\mathbb{S}$*, then there exists a self-financing Riemann sum attainable triplet* $\left(\overline{\mathbb{L}}, \mathbb{S}, \overline{\mathbb{M}}\right)$ *such that* S_1 *commutes with the histories of* $\left(\overline{\mathbb{L}}, \overline{\mathbb{M}}\right) = \left(\overline{L}_1, \overline{M}_1\right) \cup (\mathbb{L}, \mathbb{M})$ *and*

$$\begin{aligned}
&V_w\left(t, \overline{\mathbb{L}}, \mathbb{S}, \overline{\mathbb{M}}\right) = V_g\left(t, \overline{\mathbb{L}}, \mathbb{S}, \overline{\mathbb{M}}\right);\\
&T(t) = V_g\left(t, \mathbb{L}_U, \mathbb{X}^U, \mathbb{M}_{U^{-1}}\right) - V_w\left(t, \mathbb{L}_U, \mathbb{X}^U, \mathbb{M}_{U^{-1}}\right);\\
&V_w\left(t, \overline{\mathbb{L}}, \mathbb{S}, \overline{\mathbb{M}}\right) = V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) + T(t) S_1(t);\\
&V_g\left(t, \overline{\mathbb{L}}, \mathbb{S}, \overline{\mathbb{M}}\right) = V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) + \int_0^t T(s)\, dS_1(s).
\end{aligned}$$

Proof. Let us introduce a set of new assets $X_i = S_1 S_i, 1 \le i \le d$. Taking $U = X_1 = S_1^2$ and define a gain portfolio with respect to the strategy $(\mathbb{L}_U, \mathbb{M}_{U^{-1}})$

$$V_g\left(t, \mathbb{L}_U, \mathbb{X}^U, \mathbb{M}_{U^{-1}}\right) = \lambda + V_{cg}\left(t, \mathbb{L}_U, \mathbb{X}^U, \mathbb{M}_{U^{-1}}\right).$$

Note that $\mathbb{L}_U = \mathbb{L}$ and $\mathbb{M}_{U^{-1}} = \mathbb{M}$. The difference T defined above has a Jordan decomposition (cf. [31] p20) $T(t) = T^+(t) - T^-(t)$. Define

$$\overline{L}_1 = \frac{\left(\sqrt{T^+} + \sqrt{T^-}\right)}{\sqrt{2}}, \overline{M}_1 = \frac{\left(\sqrt{T^+} - \sqrt{T^-}\right)}{\sqrt{2}}.$$

Based on the new strategy $\left(\overline{\mathbb{L}}, \overline{\mathbb{M}}\right)$ one can define $\left(\overline{\mathbb{L}}_U, \overline{\mathbb{M}}_{U^{-1}}\right)$,

$$V_g\left(t, \mathbb{L}_U, \mathbb{X}^U, \mathbb{M}_{U^{-1}}\right) = V_g\left(t, \overline{\mathbb{L}}_U, \mathbb{X}^U, \overline{\mathbb{M}}_{U^{-1}}\right)$$

since $dX_1 = 0$. Also,

$$V_w\left(t, \overline{\mathbb{L}}_U, \mathbb{X}^U, \overline{\mathbb{M}}_{U^{-1}}\right) = V_g\left(t, \overline{\mathbb{L}}_U, \mathbb{X}^U, \overline{\mathbb{M}}_{U^{-1}}\right).$$

It is important to point out that $\left(\overline{L}_1, \overline{M}_1\right)$ may not be zero.

We now set a numéraire Z in such a way that $X_i^Z = S_i, 1 \le i \le d$. Clearly, $Z = S_1$. Denote by $Y = Z^{-\frac{1}{2}} U^{\frac{1}{2}} = \sqrt{S_1}$. Since Y commutes with the histories of $\left(\overline{\mathbb{L}}_U, \overline{\mathbb{M}}_{U^{-1}}\right)$,

$$V_w\left(t, \overline{\mathbb{L}}_Z, \mathbb{X}^Z, \overline{\mathbb{M}}_{Z^{-1}}\right) = V_g\left(t, \overline{\mathbb{L}}_Z, \mathbb{X}^Z, \overline{\mathbb{M}}_{Z^{-1}}\right).$$

□

The condition placed on S_1 is unquestionably restrictive. Yet it is not impossible. Recall that in the Black-Scholes-Merton model, the interest rates are assumed deterministic. Thus, the so induced money market account as a numéraire is the discount extracted from the time zero discount curve of the valuation day. A deterministic variable is surely in the center of $\mathcal{A}$. Another drawback of this result is that both $V_w(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ and $V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$ have been altered unless $(\mathbb{L}, \mathbb{M})$ is already self-financing.

2.4. **Portfolio numéraire.** Given the importance of self-financing portfolio in this quantum finance model, we construct another class of numéraire deflated self-financing portfolios, possibly larger than the class formed in Lemma 2.5.

Definition 2.10. A numéraire U is a $\mathbb{S}$ portfolio numéraire if there exists a strategy $(\mathbb{L},\mathbb{M})$ such that $U(t)=V_w(t,\mathbb{L},\mathbb{S},\mathbb{M})$. U is said to have trivial self-deflated gain portfolio if the portfolio $V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)$ is a constant in $\mathbb{R}$ independent of t.

Before proceeding, let us examine the definition. Clearly, $U^U\equiv 1$. The gain portfolio

$$V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=\lambda\left(\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)+V_{cg}\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)$$

is a constant is equivalent to the self-financing condition

$$V_w\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)$$

which says that

$$\sum_{i,j=1}^{d,J}\left[U^{-\frac{1}{2}}L_{ij}U^{\frac{1}{2}},dS_i,U^{\frac{1}{2}}M_{ij}U^{-\frac{1}{2}}\right]_t=0$$

but $(\mathbb{L},\mathbb{M})\neq 0$. There are examples of portfolio numéraires whose self-deflated gain portfolios are trivial. In fact, they are also self-financing numéraires.

Example 2.11. Each $\alpha_i S_i\in\mathbb{S},\alpha_i\in(0,\infty)$ is a $\mathbb{S}$ portfolio self-financing numéraire. The non-negative linear combination

$$U(t)=\sum_{i=1}^{d}\alpha_i S_i(t)\,,\alpha_i\in[0,\infty),\sum_{i=1}^{d}\alpha_i>0$$

is a $\mathbb{S}$ portfolio self-financing numéraire. They all have trivial self-deflated gain portfolios.

Proof. Let $J=1$ and $L_{i1}=\frac{\alpha_i}{2},M_{i1}=1$. □

For those familiar with mathematical finance, the annuity numéraire used in swaption pricing is of the above form. That being said, one observes that this example is just a positive linear combination of $\{S_i\}_{i=1}^d$ which is covered in Lemma 2.5. It would seem though that the equation

$$\sum_{i,j=1}^{d,J}\left[U^{-\frac{1}{2}}L_{ij}U^{\frac{1}{2}},dS_i,U^{\frac{1}{2}}M_{ij}U^{-\frac{1}{2}}\right]_t=0$$

for nonzero $(\mathbb{L},\mathbb{M})$ may lead to other examples.

Theorem 2.12. *Let U be a portfolio numéraire with trivial self-deflated gain portfolio*

$$\begin{aligned}&U(t)=V_w(t,\mathbb{L},\mathbb{S},\mathbb{M})\,;\\&1=U^U(t)=V_w\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right)=V_g\left(t,\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\right).\end{aligned}$$

Then for any Riemann sum attainable triplet $\left(\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$ and its corresponding gain portfolio $V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$, there exists a new Riemann sum attainable triplet $\left(\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)$ such that

$$V_w\left(t,\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)=V_g\left(t,\overline{\mathbb{A}},\mathbb{S}^U,\overline{\mathbb{B}}\right)=V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$$

with $\lambda\left(\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)=V_w\left(0,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$.

Proof. Let

$$Y(t)=V_g\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)-V_w\left(t,\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$$

with its Jordon decomposition $Y=Y^+-Y^-$. Note that $Y(0)=0$ determines $\lambda\left(\mathbb{A},\mathbb{S}^U,\mathbb{B}\right)$. Let us introduce two sets of strategies

$$\left(\sqrt{Y^+}\mathbb{L}_U,\mathbb{M}_{U^{-1}}\sqrt{Y^+}\right)\cup\left(-\sqrt{Y^-}\mathbb{L}_U,\mathbb{M}_{U^{-1}}\sqrt{Y^-}\right).$$

The wealth portfolio obtained from these strategies is Y. It remains to show that these strategies make no contribution to the gain portfolio. First, one has

$$0=dU^U(s)=\sum_{i,j=1}^{d,J}\left[\begin{array}{c}(L_U)_{ij}(s)\left[dS_i^U(s)\right](M_{U^{-1}})_{ij}(s)\\+(M_{U^{-1}})_{ij}^*(s)\left[dS_i^U(s)\right](L_U)_{ij}^*(s)\end{array}\right].$$

This equation leads to

$$\begin{aligned}0&=\sum_{i,j=1}^{d,J}\sqrt{Y^+(s)}\left[\begin{array}{c}(L_U)_{ij}(s)\left[dS_i^U(s)\right](M_{U^{-1}})_{ij}(s)\\+(M_{U^{-1}})_{ij}^*(s)\left[dS_i^U(s)\right](L_U)_{ij}^*(s)\end{array}\right]\sqrt{Y^+(s)}\\&\Rightarrow\lambda=V_{cg}\left(t,\sqrt{Y^+}\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\sqrt{Y^+}\right),\lambda\in\mathbb{R}.\end{aligned}$$

Evaluating at $t=0$ leads to $\lambda=0$. Similarly,

$$V_{cg}\left(t,\sqrt{Y^-}\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\sqrt{Y^-}\right)=0.$$

Since $Y(0)=0$, we have

$$V_g\left(\sqrt{Y^+}\mathbb{L}_U\cup\sqrt{Y^-}\mathbb{L}_U,\mathbb{S}^U,\mathbb{M}_{U^{-1}}\sqrt{Y^+}\cup\mathbb{M}_{U^{-1}}\sqrt{Y^-}\right)=0.$$

The conclusion follows if we set $\left(\overline{\mathbb{A}},\overline{\mathbb{B}}\right)$ to

$$\left(\sqrt{Y^+}\mathbb{L}_U\cup\left(-\sqrt{Y^-}\mathbb{L}_U\right)\cup\mathbb{A},\mathbb{S}^U,\mathbb{M}_{U^{-1}}\sqrt{Y^+}\cup\left(-\mathbb{M}_{U^{-1}}\sqrt{Y^-}\right)\cup\mathbb{B}\right).$$

$\square$

It is desirable that the above theorem is valid for $U\equiv 1$. However, the method used in the proof can not be applied. If U^{-1} is also of the same kind portfolio numéraire with respect to $\mathbb{S}^U$, namely there exists a strategy $(\mathbb{G},\mathbb{H})$ such that

$$\begin{aligned}&U^{-1}(t)=V_w\left(t,\mathbb{G},\mathbb{S}^U,\mathbb{H}\right);\\&1=\left(U^{-1}\right)^{U^{-1}}(t)=V_w\left(t,\mathbb{G}_{U^{-1}},\mathbb{S}^{U^{-1}},\mathbb{M}_U\right)=V_g\left(t,\mathbb{G}_{U^{-1}},\mathbb{S}^{U^{-1}},\mathbb{M}_U\right),\end{aligned}$$

then we can find a strategy $\left(\hat{\mathbb{A}},\hat{\mathbb{B}}\right)$ such that

$$V_w\left(t,\hat{\mathbb{A}},\mathbb{S},\hat{\mathbb{B}}\right)=V_g\left(t,\hat{\mathbb{A}},\mathbb{S},\hat{\mathbb{B}}\right)=V_g\left(t,\mathbb{A}_{U^{-1}},\mathbb{S},\mathbb{B}_U\right).$$

2.5. **Martingale state.** In this section we turn our attention to quantum probability measures. The following notion is taken from [12].

Definition 2.13. Let $(T_t)_{t\in[0,\mathsf{t}]}$ be an operator process adapted to a filtrated von Neumann algebra $\mathcal{A} = (\mathcal{A}_t)_{t\in[0,\mathsf{t}]}$ and let $\rho \in \mathcal{A}_*$, the predual of $\mathcal{A}$, be a faithful normal state. The process $(T_t)_{t\in[0,\mathsf{t}]}$ is a ρ-martingale with respect to $(\mathcal{A}_t)_{t\in[0,\mathsf{t}]}$, or simply a ρ-martingale, if for $\forall s \leq t \in [0,\mathsf{t}]$ and $\forall A \in \mathcal{A}_s$,

$$\rho\left(AT_tA^*\right) = \rho\left(AT_sA^*\right).$$

Or equivalently,

$$\rho\left(AT_tB\right) = \rho\left(AT_sB\right), A, B \in \mathcal{A}_s.$$

It turns out that ρ-martingale has a special unique property.

Proposition 2.14. *Let $(V_t)_{t\in[0,\mathsf{t}]}$ and $(W_t)_{t\in[0,\mathsf{t}]}$ be two ρ-martingales. If $V_{t_0} = W_{t_0}$ for some $t_0 \in [0,\mathsf{t}]$, then $V_t = W_t$ for all $t \leq t_0$.*

Proof. First, let us assume that both processes are self-adjoint. For any $A \in \mathcal{A}_s$, $s \leq t_0$, one has

$$\rho\left(AV_sA^*\right) = \rho\left(AV_{t_0}A^*\right) = \rho\left(AW_{t_0}A^*\right) = \rho\left(AW_sA^*\right).$$

Hence $\rho\left(A\left(V_s - W_s\right)A^*\right) = 0$ for $\forall A \in \mathcal{A}_s$. The self-adjoint element $V_s - W_s$ has a Jordon decomposition $V_s - W_s = Y^+ - Y^-$ with $Y^+Y^- = 0$. Note that $Y^+, Y^- \in \mathcal{A}_s$. Taking $A = Y^+$ and $A = Y^-$, respectively, one arrives at $\rho\left(\left(Y^+\right)^3\right) = 0$ and $\rho\left(\left(Y^-\right)^3\right) = 0$. Since ρ is faithful, one obtains $Y^+ = Y^- = 0$.

Since V is a ρ-martingale, V^* is also a ρ-martingale. As a consequence the real part $V^{(R)} = \frac{V+V^*}{2}$ and the imaginary part $V^{(I)} = \frac{V-V^*}{2i}$ are all ρ-martingales. Furthermore, $V^{(R)}_{t_0} = W^{(R)}_{t_0}$ and $V^{(I)}_{t_0} = W^{(I)}_{t_0}$. The conclusion now follows. ☐

Definition 2.15. Let $\rho \in \mathcal{A}_*$ be a faithful state and let U be a numéraire . The pair (U, ρ) is called a numéraire pair for the economy $\mathcal{E} = (\mathcal{A}, \mathbb{S})$ if every element in $\mathbb{S}^U$ is a ρ-martingale. By abusing of terminology we shall say that $\mathbb{S}^U$ is a ρ-martingale.

It turns out that for any numéraire U there exists a faithful state $\rho \in \mathcal{A}_*$ such that $\mathbb{S}^U$ is a ρ-martingale, provided that a geometrical condition holds. In the classic theory, the mentioned condition is interpreted as “no free lunch with vanishing risk”.

Let us define a real vector subspace $\mathcal{K} \subset \mathcal{A}_{sa}$ where $\mathcal{A}_{sa}$ is the space of all self-adjoint elements. Denote an one point partition by $P\left(t_0\right) = \{t_0\}, t_0 \in [0,\mathsf{t}]$. A step *norm−predictable* process $\left(L_{P(t_0)}\left(t\right)\right)_{t\in[0,\mathsf{t}]}$ corresponding to this partition has the form

$$L_{P(t_0)}\left(t\right) = \begin{cases} L_{P(t_0)}\left(t\right) & t \in (t_0, \mathsf{t}] \\ 0 & t \in [0, t_0] \end{cases}, L_{P(t_0)} \in \mathcal{A}_{t_0}, t \in [0,\mathsf{t}].$$

For $\left(L_{P(t_0)}(t)\right)_{t\in[0,\mathsf{t}]}$ and $\left(M_{P(t_0)}(t)\right)_{t\in[0,\mathsf{t}]}$, we have the integral

$$\int_0^t \left[L_{P(t_0)}, S_i^U, M_{P(t_0)}\right]_s = \left[\begin{array}{c} S\left(t, T_{P(t_0)}, dS_i^U, M_{P(t_0)}, P(t_0)\right) \\ +S\left(t, M^*_{P(t_0)}, dS_i^U, M^*_{P(t_0)}, P(t_0)\right) \end{array}\right]$$

$$= \begin{cases} \left[\begin{array}{c} T_{P(t_0)}(t_0)\left[S_i^U(t) - S_i^U(t_0)\right] M_{P(t_0)}(t_0) \\ +M^*_{P(t_0)}(t_0)\left[S_i^U(t) - S_i^U(t_0)\right] L^*_{P(t_0)}(t_0) \end{array}\right] & t_0 \le t \le \mathsf{t}; \\ 0 & t < t_0. \end{cases}$$

Define $\mathcal{K}(U)$ as the vector subspace of real linear span of all the one point integrals of the above type with respect to all $S_i^U \in \mathbb{S}^U$. For any $\rho \in \mathcal{A}_*$, S_i^U is a ρ-martingale if and only if ρ vanishes on $\mathcal{K}(U)$. Let $P = \{t_k\}_{k=0}^n$ be a partition of $[0,\mathsf{t}]$, clearly $\mathcal{K}$ contains the following integrals

$$\left[\begin{array}{c} \sum_{k=1}^n L_{k-1}\left[S_i^U(t_k) - S_i^U(t_{k-1})\right] M_{k-1} \\ +\sum_{k=1}^n M^*_{k-1}\left[S_i^U(t_k) - S_i^U(t_{k-1})\right] L^*_{k-1} \end{array}\right], \forall L_{k-1}, M_{k-1} \in \mathcal{A}_{t_{k-1}}.$$

Using $\mathcal{K}(U)$ one can construct a convex cone

$$\mathcal{C}(\mathcal{K}(U)) = \{T \in \mathcal{A}_{sa} : \exists K \in \mathcal{K}(U) \text{ such that } T \le K\} \subset \mathcal{A}_{sa}.$$

The $\sigma(\mathcal{A}, \mathcal{A}_*)$ closure of this set, denoted by $\overline{\mathcal{C}(\mathcal{K}(U))}^{\sigma(\mathcal{A},\mathcal{A}_*)}$ is convex and closed under the $\sigma(\mathcal{A}, \mathcal{A}_*)$ topology. It is a cone since $\mathcal{K}(U)$ is a real subspace. Note that

$$\overline{\mathcal{K}(U)}^{\sigma(\mathcal{A},\mathcal{A}_*)} \subset \overline{\mathcal{C}(\mathcal{K}(U))}^{\sigma(\mathcal{A},\mathcal{A}_*)}.$$

This closed subspace contains integrals of all weakly Riemann sum attainable triplets $\left(L, S_i^U, M\right)$.

The following quantum fundamental asset pricing theorem (Theorem 4.1 [34]) provides a so-called no free lunch condition for $\mathbb{S}^U$ being a ρ-martingale.

Theorem 2.16. *The following two conditions are equivalent*

(i) The condition holds $\overline{\mathcal{C}(\mathcal{K}(U))}^{\sigma(\mathcal{A},\mathcal{A}_*)} \cap \mathcal{A}_+ = \{0\}$.

(ii) There exists a faithful normal state ρ on $\mathcal{A}$ such that $\rho = 0$ on $\mathcal{K}(U)$.

Evidently

$$\rho\left[V_{cg}\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)\right] = 0$$

for any weakly Riemann sum attainable $\left(\mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$.

2.6. **Arbitrage.** It is supposedly agreed that derivatives pricing only makes sense in arbitrage free markets. An arbitrage in an economy is a risk-less strategy to yield profit without injecting and withdrawing capital after the initial investment.

Definition 2.17. The economy $\mathcal{E}(\mathcal{A}, \mathbb{S})$ is arbitrage-free if there exists a numéraire pair (U, ρ_U) for $\mathbb{S}^U$ such that for any strategy $(\mathbb{L}, \mathbb{M})$ none of the following conditions holds: (i) $V_g\left(0, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) < 0$ and $V_g\left(T, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) \ge 0$; (ii) $V_g\left(0, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) \le 0$ and $V_g\left(T, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) > 0$ for some $T > 0$. The case $U \equiv 1$ is permitted.

Theorem 2.18. *The economy $\mathcal{E}$ is arbitrage-free if there exists a numéraire pair (U, ρ_U).*

Proof. Suppose $(\mathbb{L}, \mathbb{M})$ is an arbitrage strategy for the economy. Since $\mathbb{S}^U$ is a set of ρ-martingales, the capital gain$V_{cg}\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right)$, as a limit of Riemann sums, has zero value under ρ_U. Hence, for case (i)

$$0 > V_g\left(0, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right) = \rho_U\left[V_g\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_U\right)\right] \geq 0.$$

This is a contradiction. Similar proof applies to case (ii) with the additional assumption that ρ_U is faithful. □

2.7. **Price process.** In classic mathematical finance, self-financing portfolios with desired properties are used to define the price processes of financial derivatives, thanks to the powerful Radon-Nikodým derivative and semi-martingale representation theorem. However, these crucial tools are not available in quantum probability spaces in general. Therefore, we are not as lucky when coming to assign prices to financial derivatives. Nevertheless, we shall follow the same or similar approach before a new formulation comes into existence. While it is certainly counter-intuitive to have multiple price processes for a single derivatives, we should also recognize that we are in a quantum world. As a matter of fact, no unique price exists in market places. With this observation in mind, we move on to investigate what we can and can not achieve in derivatives pricing.

Let $C(T) \in \mathcal{A}_T$ be the terminal payoff of a financial derivatives or contingent claim for some $T \in (0, \mathsf{t}]$. This is the $time-T$ price of the claim. Its functional form, in terms of the underlyings, is known when the claim is structured. A fundamental problem is to find its $time-t$ prices $C(t)$, for $t \leq T$, if they exist. In particular, the presence of a consistent time zero price would enable one to mark the claim to the market and to estimate its risks.

Due to the lack of a representation theory, we have no choice but to hypothesize a replication of $C(T)$ of some kind. Let us first accept that $\mathbb{S}$ is a τ-martingale and $C(T) = V_g(T, \mathbb{L}, \mathbb{S}, \mathbb{M})$ for some strategy $(\mathbb{L}, \mathbb{M})$. A logical definition of $C(t)$ would be $V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M})$. Such definition is indeed consistent. If $(\mathbb{G}, \mathbb{H})$ is another strategy to reproduce $C(T)$ by the portfolio $V_g(t, \mathbb{G}, \mathbb{S}, \mathbb{H})$ at time T, then $V_g(t, \mathbb{L}, \mathbb{S}, \mathbb{M}) = V_g(t, \mathbb{G}, \mathbb{S}, \mathbb{H})$ for $0 \leq t \leq T$. Here, we cautiously not to insist that $(\mathbb{L}, \mathbb{M})$ is self-financing. This is because that we do not know whether we can do so without altering $C(T)$. While self-financing portfolios about $\mathbb{S}$ are abundant, this fact offers no assistant in this regard.

Another way to define price processes is through discounting. Suppose that (U, ρ_U) is a numéraire pair and

$$U^{-\frac{1}{2}}(T)\, C(T)\, U^{-\frac{1}{2}}(T) = V_g\left(T, \mathbb{A}, \mathbb{S}^U, \mathbb{B}\right)$$

for some strategy $(\mathbb{A}, \mathbb{B})$. It is convenient to write $(\mathbb{A}, \mathbb{B}) = (\mathbb{L}_U, \mathbb{M}_{U^{-1}})$. In another word, $U(T)$ discounted terminal value can be duplicated by a gain portfolio of $\mathbb{S}^U$. The process $C^U(t) = V_g\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$ satisfies the expectation of a discounted price process. The corresponding forward price process will be $V_g^{U^{-1}}\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$. Again, both the discounted and forward price processes are independent of strategies.

Disagreement may arise if there is another numéraire pair (Z, ρ_Z) and

$$Z^{-\frac{1}{2}}(T)\, C(T)\, Z^{-\frac{1}{2}}(T) = V_g\left(T, \mathbb{G}_Z, \mathbb{S}^Z, \mathbb{H}_{Z^{-1}}\right).$$

While one can assign unique discounted and forward price processes, respectively, under Z, there is no assurance that the forward price process $V_g^{U^{-1}}\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$ agrees with $V_g^{Z^{-1}}\left(t, \mathbb{G}_Z, \mathbb{S}^Z, \mathbb{H}_{Z^{-1}}\right)$. For this reason, we may write $C(t, U)$ to announce its dependency on the numéraire U. This remark also applies to the particular case $U \equiv 1$ or $Z \equiv 1$.

If $V_g\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right) = V_w\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$ and $Y = Z^{-\frac{1}{2}} U^{\frac{1}{2}}$ commutes with the histories of $(\mathbb{L}, \mathbb{M})$, then $V_g\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_{Z^{-1}}\right) = V_w\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_{Z^{-1}}\right)$. But

$$\begin{aligned} V_g\left(T, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_{Z^{-1}}\right) &= V_w^Z\left(T, \mathbb{L}, \mathbb{S}, \mathbb{M}\right) = Y(T)\, V_w\left(T, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right) Y^*(T) \\ &= Y(T)\, V_g\left(T, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right) Y^*(T) = C^Z(T, Z) = V_g\left(T, \mathbb{G}_Z, \mathbb{S}^Z, \mathbb{H}_{Z^{-1}}\right). \end{aligned}$$

Thus,

$$V_g\left(t, \mathbb{L}_Z, \mathbb{S}^Z, \mathbb{M}_{Z^{-1}}\right) = V_g\left(t, \mathbb{G}_Z, \mathbb{S}^Z, \mathbb{H}_{Z^{-1}}\right), t \leq T.$$

This equation leads to

$$Z^{-\frac{1}{2}}(t)\, C(t, Z)\, Z^{-\frac{1}{2}}(t) = Z^{-\frac{1}{2}}(t)\, C(t, U)\, Z^{-\frac{1}{2}}(t).$$

2.8. **Semi-commutative model.** In this section, we examine the situation that $\mathbb{S}$ commutes with the histories of all possible strategies. To set up the model, let us denote by

$$\mathbb{S}_t = vN\left\{S_i(x) : t \leq x \leq \mathsf{t}, i = 1, \cdots, d\right\}.$$

This is a net of decreasing von Neumann algebras indexed by $\mathbb{R}^+$. The commutant $\mathcal{S}_t \equiv \mathbb{S}_t' \cap \mathcal{A}_t$ forms a filtration of von Neumann algebras. For a trading strategy $(\mathbb{L}, \mathbb{M})$, we require that L_{ij} and M_{ij} are adapted to $(\mathcal{S}_t)_{t\in[0,\mathsf{t}]}$. We also insist that all numéraires U are adapted to $(\mathbb{S}_t)_{t\in[0,\mathsf{t}]}$. According to this requirement,

$$U = \sum_{i=1}^{d} \alpha_i S_i, \alpha_i \in [0, \mathsf{t}], \sum_{i=1}^{d} \alpha_i > 0$$

is a numéraire. We remark that one could enlarge $\mathbb{S}$ by adding desired numéraires. This construction is to ensure that any S_i or any numéraire commutes with the histories of strategies.

Clearly, we can take self-adjoint $\mathbb{L} = \{L_i\}_{i=1}^d$ and $\mathbb{M} = I_\Phi$. The wealth and gain portfolios, with respect to $\mathbb{S}^U$ for a numéraire U, take the forms, respectively,

$$V_w\left(t, \mathbb{L}, \mathbb{S}^U\right) = \sum_{i=1}^{d} L_i(t)\, S_i^U(t);$$

$$V_g\left(t, \mathbb{L}, \mathbb{S}^U\right) = \lambda\left(\mathbb{L}, \mathbb{S}^U\right) + \sum_{i=1}^{d} \int_0^t L_i(s)\, dS_i^U(s).$$

If there exists a numéraires U, possibly with $U \equiv 1$, such that $V_w\left(t, \mathbb{L}, \mathbb{S}^U\right) = V_g\left(t, \mathbb{L}, \mathbb{S}^U\right)$, then $V_w\left(t, \mathbb{L}, \mathbb{S}^Z\right) = V_g\left(t, \mathbb{L}, \mathbb{S}^Z\right)$ for any numéraire Z.

While trading strategies are more restrictive, we shall continue to use the previous definition of ρ-martingale. Hence, we retain the uniqueness of ρ-martingales. With this consideration, we now once again try to define a price process to a contingent claims.

Proposition 2.19. *Let $C(T)$ be the terminal value of a contingent claim at time $T \in (0,T]$. Suppose that the economy $\mathcal{E}(\mathcal{A},\mathbb{S})$ is arbitrage free. If there exists a numéraires U and a self-financing strategy $\mathbb{L}$ such that*

$$C^U(T) = V_w\left(T,\mathbb{L},\mathbb{S}^U\right) \text{ or } C(T) = V_w(T,\mathbb{L},\mathbb{S}),$$

we define a forward price as

$$C(t,U) = U^{\frac{1}{2}}(t)\,V_w\left(t,\mathbb{L},\mathbb{S}^U\right)U^{\frac{1}{2}}(t) = V_w(t,\mathbb{L},\mathbb{S}), t \le T.$$

Then, so defined price process is unique.

Proof. Let us assume that (U_0,ρ_{U_0}) is a numéraire pair. Let Z be a numéraire and $\mathbb{M}$ be another self-financing strategy such that

$$C^Z(T) = V_w\left(T,\mathbb{M},\mathbb{S}^Z\right) \text{ or } C(T) = V_w(T,\mathbb{M},\mathbb{S}).$$

By definition, we have

$$C(t,Z) = Z^{\frac{1}{2}}(t)\,V_w\left(t,\mathbb{M},\mathbb{S}^Z\right)Z^{\frac{1}{2}}(t) = V_w(t,\mathbb{M},\mathbb{S}), t \le T.$$

Now we have two self-financing equations

$$\begin{aligned} V_w\left(t,\mathbb{L},\mathbb{S}^U\right) &= V_g\left(t,\mathbb{L},\mathbb{S}^U\right), \\ V_w\left(t,\mathbb{M},\mathbb{S}^Z\right) &= V_g\left(t,\mathbb{M},\mathbb{S}^Z\right), \end{aligned}$$

which can be translated into

$$\begin{aligned} V_w\left(t,\mathbb{L},\mathbb{S}^{U_0}\right) &= V_g\left(t,\mathbb{L},\mathbb{S}^{U_0}\right), \\ V_w\left(t,\mathbb{M},\mathbb{S}^{U_0}\right) &= V_g\left(t,\mathbb{M},\mathbb{S}^{U_0}\right). \end{aligned}$$

But

$$V_w\left(T,\mathbb{L},\mathbb{S}^{U_0}\right) = V_w\left(T,\mathbb{M},\mathbb{S}^{U_0}\right) = C^{U_0}(T).$$

Hence

$$\begin{aligned} & V_w\left(t,\mathbb{L},\mathbb{S}^{U_0}\right) = V_w\left(t,\mathbb{M},\mathbb{S}^{U_0}\right) \\ & \Leftrightarrow V_w(t,\mathbb{L},\mathbb{S}) = V_w(t,\mathbb{M},\mathbb{S}), t \le T. \end{aligned}$$

□

We end the note by an easy example to price a forward rate agreement (FRA). Let $[a,b] \subset (0,\infty)$ be an interval and denote the time fraction $\Delta = b - a$. The Libor rate for the period $[a,b]$, ignoring market conventions, can be defined as

$$L(t) = B_b^{-\frac{1}{2}}(t)\left[\frac{B_a(t) - B_b(t)}{\Delta}\right]B_b^{-\frac{1}{2}}(t)$$

where $B_a(t)$ is the *time*$-t$ price of a bond maturing at time a and $B_b(t)$ is the *time*$-t$ price of a bond maturing at time b:

$$B_a(t) = I_\Phi, a \le t \le b; B_b(b) = I_\Phi.$$

A forward rate agreement (FRA) is a contract paying $\Delta\left[L\left(a\right)-K\right]$ amount at time b, where K is a predetermined strike. The $U=B_b^{-1}$ discounted value at time $t\leq a$ is

$$\begin{aligned}C^U\left(t\right)&=B_b^{\frac{1}{2}}\left(t\right)\Delta\left[L\left(t\right)-K\right]B_b^{\frac{1}{2}}\left(t\right)\\&=B_b^{\frac{1}{2}}\left(t\right)B_b^{-\frac{1}{2}}\left(t\right)\left[\frac{B_a\left(t\right)-B_b\left(t\right)}{\Delta}\right]B_b^{-\frac{1}{2}}\left(t\right)B_b^{\frac{1}{2}}\left(t\right)\\&=B_a\left(t\right)-\left(1+K\Delta\right)B_b\left(t\right).\end{aligned}$$

Here we set $\mathbb{S}=\{B_a,B_b\}$ and $\mathbb{L}=(1,-1-K\Delta)$. Time zero discounted price is

$$B_a\left(0\right)-\left(1+K\Delta\right)B_b\left(0\right)=\left[L\left(0\right)-K\right]\Delta B_b\left(0\right).$$

3. Conditional Expectations

For a pair of von Neumann algebras $\mathcal{N}\subset\mathcal{M}$, a conditional expectation $\mathbb{E}_{\mathcal{N}}$ from $\mathcal{M}$ to $\mathcal{N}$ is a completely positive unital linear map satisfying the $\mathcal{N}$ bi-module property;

$$\mathbb{E}_{\mathcal{N}}\left(AXB\right)=A\mathbb{E}_{\mathcal{N}}\left(X\right)B,\forall A,B\in\mathcal{N},\forall X\in\mathcal{M}.$$

It is well-known that such map may not exist. In our setting of $\mathcal{M}=\mathcal{A}_t$ and $\mathcal{N}=\mathcal{A}_s$, however, there always exist conditional expectations $\mathbb{E}_{s,t}$ from $\mathcal{A}_t$ to $\mathcal{A}_s$ for $s\leq t\in[0,\mathsf{t}]$. In this section, we shall characterize them. As usual, we call an adapted process $(X_t)_{t\in[0,\mathsf{t}]}$ $\mathbb{E}$-martingale if $\mathbb{E}_{s,t}\left(X_t\right)=X_s, s\leq t\in[0,\mathsf{t}]$. To fix the notation, let us write

$$\begin{aligned}\Phi_{\mathsf{t}]}&=\Phi_{s]}\otimes\Phi_{[s,t]}\otimes\Phi_{[t,\mathsf{t}]};\\\mathcal{A}&=\mathscr{B}\left(\Phi_{s]}\right)\otimes\mathscr{B}\left(\Phi_{[s,t]}\right)\otimes\mathscr{B}\left(\Phi_{[t,\mathsf{t}]}\right);\\\mathcal{A}_t&=\mathcal{A}_s\otimes\mathscr{B}\left(\Phi_{[s,t]}\right)\otimes I_{[t,\mathsf{t}]};\\\mathcal{A}_{[s,t]}&=I_{s]}\otimes\mathscr{B}\left(\Phi_{[s,t]}\right)\otimes I_{[t,\mathsf{t}]};\\\mathcal{A}_{[t,\mathsf{t}]}&=I_{t]}\otimes\mathscr{B}\left(\Phi_{[s,\mathsf{t}]}\right).\end{aligned}$$

The family of conditional expectations $(\mathbb{E}_t:\mathcal{A}\to\mathcal{A}_t)_{t\in[0,\mathsf{t}]}$, is said consistent if $\mathbb{E}_s\circ\mathbb{E}_t=\mathbb{E}_s, s\leq t\in[0,\mathsf{t}]$. In this case, one can define a conditional expectation $\mathbb{E}_{s,t}$ from $\mathcal{A}_t$ to $\mathcal{A}_s$ such that $\mathbb{E}_{s,t}\circ\mathbb{E}_t=\mathbb{E}_s$. Since $\mathcal{A}_0=\mathbb{C}$, $\mathbb{E}_0$ is a state. Furthermore, it is compatible with $\mathbb{E}_t$ in the sense that $\mathbb{E}_{0,t}\circ\mathbb{E}_t=\mathbb{E}_0$. Clearly, $\mathbb{E}_{0,t}$ is a state defined on $\mathcal{A}_t$ and it is the restriction of $\mathbb{E}_0$ defined on $\mathcal{A}$. Hence, we shall also denote $\mathbb{E}_{0,t}$ by $\mathbb{E}_0$. The state compatible with $(\mathbb{E}_t)_{t\in[0,\mathsf{t}]}$ is unique since $\rho\circ\mathbb{E}_t=\rho, t\in[0,\mathsf{t}]$ implies that $\rho\circ\mathbb{E}_0=\rho$ or $\mathbb{E}_0=\rho$. Clearly, if $\mathbb{E}_0$ is faithful, $\mathbb{E}_t$ must be faithful for all $t\in[0,\mathsf{t}]$. It is easy to see that comptatible implies consistent if $\mathbb{E}_0$ is faithful.

Proposition 3.1. *If $(X_t)_{t\in[0,\mathsf{t}]}$ is an $\mathbb{E}$-martingale for a family of compatible conditional expectations $(\mathbb{E}_t)_{t\in[0,\mathsf{t}]}$, then $(X_t)_{t\in[0,\mathsf{t}]}$ is an $\mathbb{E}_0$-martingale. Conversely, if $(X_t)_{t\in[0,\mathsf{t}]}$ is an $\mathbb{E}_0$-martingale and $\mathbb{E}_0$ is faithful, then $(X_t)_{t\in[0,\mathsf{t}]}$ is an $\mathbb{E}$-martingale.*

Proof. If $(X_t)_{t\in[0,\mathsf{t}]}$ is an $\mathbb{E}$-martingale, then $\forall A_s,B_s\in\mathcal{A}_s$

$$\mathbb{E}_0\left[A_s\left(X_t-X_s\right)B_s\right]=\mathbb{E}_0\left\{\mathbb{E}_s\left[A_s\left(X_t-X_s\right)B_s\right]\right\}=0.$$

Conversely, if $(X_t)_{t\in[0,\mathrm{t}]}$ is an $\mathbb{E}_0$-martingale and $\mathbb{E}_0$ is faithful, then for $\forall A_s, B_s \in \mathcal{A}_s$

$$\mathbb{E}_0\left\{\mathbb{E}_s\left[A_s\left(X_t - X_s\right)B_s\right]\right\} = \mathbb{E}_0\left\{A_s\left[\mathbb{E}_s\left(X_t\right) - X_s\right]B_s\right\} = 0$$

implies that $\mathbb{E}_s\left(X_t\right) = X_s$. □

3.1. **Martingale of operators.** Denote by P_t the projection from $\Phi_{[t}$ to $\mathbb{C}\Omega_{[t}$. Then $I_{t]}\otimes P_t$ is a projection from Φ onto $\Phi_{t]}\otimes\Omega_{[t}$. Here, by abuse of notation we identify, for example, $\Phi_{[t,\mathrm{t}]}$ with $\Phi_{[t}$. Note that $I_{t]}\otimes P_t \notin \mathcal{A}_t$. For convenience, we shall write $I_{t]}\otimes P_t$ as P_t. Recall that an adapted process of bounded operators $(X_t)_{t\in[0,\mathrm{t}]}$ is a martingale of operators if, for $s\le t$ and $u\otimes\Omega_{[s}, v\otimes\Omega_{[s}\in\Phi_{s]}\otimes\Omega_{[s}$,

$$\langle u\otimes\Omega_{[s}, X_t v\otimes\Omega_{[s}\rangle = \langle u\otimes\Omega_{[s}, X_s v\otimes\Omega_{[s}\rangle.$$

Proposition 3.2. *Let $(X_t)_{t\in[0,\mathrm{t}]}$ be an adapted process of bounded operators. Then the followings are equivalent:*

(a) $(X_t)_{t\in[0,\mathrm{t}]}$ *is a martingale of operators;*
(b) $(X_t)_{t\in[0,\mathrm{t}]}$ *is a ρ_Ω-martingale;*
(c) $P_s\left(X_t\right) = P_s\left(X_s\right), s\le t$. *Namely,* $P_sX_tP_s = P_sX_sP_s$.

Proof. (a)⇔(b): Assume first that $(X_t)_{t\in[0,\mathrm{t}]}$ is a martingale of operators. For $s\le t$ and $u\otimes\Omega_{[s}, v\otimes\Omega_{[s}\in\Phi_{s]}\otimes\Omega_{[s}$,

$$\langle u\otimes\Omega_{[s}, X_t v\otimes\Omega_{[s}\rangle = \langle u\otimes\Omega_{[s}, X_s v\otimes\Omega_{[s}\rangle.$$

Replacing $u\otimes\Omega_{[s}$ and $v\otimes\Omega_{[s}$ by $\left(A_{s]}^*\otimes I_{[s}\right)\Omega$ and $\left(B_{s]}\otimes I_{[s}\right)\Omega$, respectively, one has

$$\langle\Omega, A_{s]}\otimes I_{[s}X_tB_{s]}\otimes I_{[s}\Omega\rangle = \langle\Omega, A_{s]}\otimes I_{[s}X_sB_{s]}\otimes I_{[s}\Omega\rangle$$

which is

$$\rho_\Omega\left(A_{s]}\otimes I_{[s}X_tB_{s]}\otimes I_{[s}\right) = \rho_\Omega\left(A_{s]}\otimes I_{[s}X_sB_{s]}\otimes I_{[s}\right).$$

(a)⇔(c): Let $P_s\boldsymbol{\varepsilon}(u) = \alpha\left(\boldsymbol{\varepsilon}(u_{[s})\right)\boldsymbol{\varepsilon}(u_{s]})\otimes\Omega_{[s}$ where $\alpha\left(\boldsymbol{\varepsilon}(u_{[s})\right)\in\mathbb{C}$. Then

$$\begin{aligned}
&\langle\boldsymbol{\varepsilon}(u_{s]})\otimes\Omega_{[s}, X_t\boldsymbol{\varepsilon}(v_{s]})\otimes\Omega_{[s}\rangle = \langle\boldsymbol{\varepsilon}(u_{s]})\otimes\Omega_{[s}, X_s\boldsymbol{\varepsilon}(v_{s]})\otimes\Omega_{[s}\rangle\\
&\Leftrightarrow \alpha\left(\boldsymbol{\varepsilon}(u_{[s})\right)\alpha\left(\boldsymbol{\varepsilon}(v_{[s})\right)\langle\boldsymbol{\varepsilon}(u_{s]})\otimes\Omega_{[s}, X_t\boldsymbol{\varepsilon}(v_{s]})\otimes\Omega_{[s}\rangle\\
&= \alpha\left(\boldsymbol{\varepsilon}(u_{[s})\right)\alpha\left(\boldsymbol{\varepsilon}(v_{[s})\right)\langle\boldsymbol{\varepsilon}(u_{s]})\otimes\Omega_{[s}, X_s\boldsymbol{\varepsilon}(v_{s]})\otimes\Omega_{[s}\rangle\\
&\Leftrightarrow \langle\boldsymbol{\varepsilon}(u), P_sX_tP_s\boldsymbol{\varepsilon}(v)\rangle = \langle\boldsymbol{\varepsilon}(u), P_sX_sP_s\boldsymbol{\varepsilon}(v)\rangle.
\end{aligned}$$

Passing to finite sums and noting that operators involved are bounded we conclude that (a)⇔(c). □

There is a large class of martingale of operators , called regular martingales, characterized in [29]. They are elements of $\boldsymbol{\mathcal{S}}$ which are integrals with respect to $+,-,\circ$,

$$X_t = \lambda + \int_0^t \sum_{\epsilon=+,-,\circ} H_s^\epsilon dA_s^\epsilon, t\in[0,\mathrm{t}].$$

Similarly, for a pure state $\rho_{\boldsymbol{\varepsilon}(u)}$ defined as

$$\rho_{\boldsymbol{\varepsilon}(u)}\left(T\right) = \langle\boldsymbol{\varepsilon}(u), T\boldsymbol{\varepsilon}(u)\rangle, T\in\mathcal{A},$$

$(Y_t)_{t\in[0,\mathsf{t}]}$ is a $\rho_{\boldsymbol{\varepsilon}(u)}$-martingale if and only if $P_{\boldsymbol{\varepsilon}(u_{[s})}Y_tP_{\boldsymbol{\varepsilon}(u_{[s})} = P_{\boldsymbol{\varepsilon}(u_{[s})}Y_sP_{\boldsymbol{\varepsilon}(u_{[s})}$. Here $P_{\boldsymbol{\varepsilon}(u_{[s})}$ is the projection from Φ onto $\Phi_{s]}\otimes\boldsymbol{\varepsilon}(u_{[s})$.

3.2. **Conditional expectations.** We now proceed to show that consistent conditional expectations do exist. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras. For any $T\in\mathcal{M}$, one can define an operator L_T on $\mathcal{M}^*$ such that

$$L_T\left(f\right)\left(X\right) = f\left(TX\right), \forall f\in\mathcal{M}^*, \forall X\in\mathcal{M}.$$

By Theorem 2 [15], $\mathbb{E}$ is a conditional expectation from $\mathcal{M}\otimes\mathcal{N}$ to $\mathcal{M}\otimes I_{\mathcal{N}}$ if and only if there exists a generalized channel $\mathbb{E}_*$ from $\left(\mathcal{M}\otimes I_{\mathcal{N}}\right)_*$ to $\left(\mathcal{M}\otimes\mathcal{N}\right)_*$ and

$$\mathbb{E}_*L_T = L_T\mathbb{E}_*, \forall T\in\mathcal{M}\otimes I_{\mathcal{N}}.$$

Furthermore, $\left(\mathbb{E}_*\right)^* = \mathbb{E}$.

To apply, we set $\mathcal{M} = \mathscr{B}\left(\Phi_{t]}\right)$ and $\mathcal{N} = \mathscr{B}\left(\Phi_{[t,\mathsf{t}]}\right)$ and first assume that $\mathbb{E}_t = \mathbb{E}$ exists. For $\forall A_{t]}\otimes B_{[t,\mathsf{t}]}\in\mathcal{A}$ one has the commuting relation

$$\left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right) = \left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right).$$

Hence, $\mathbb{E}_t\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)$ commutes with all elements in $\mathscr{B}\left(\Phi_{t]}\right)\otimes I_{[t,\mathsf{t}]}$. As a consequence, $\mathbb{E}_t\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\in\mathbb{C}$ and $\mathbb{E}_t$ is a state,denoted by $\varphi_{[t,\mathsf{t}]}$, on $\mathcal{A}_{[t,\mathsf{t}]}$. Similarly, one has two other states $\varphi_{t]}$ and $\varphi_{]s,t[}$ defined on $\mathcal{A}_{t]}$ and $\mathcal{A}_{[s,t]}$, respectively. Clearly

$$\mathbb{E}_0\left(A_{t]}\otimes B_{[t,\mathsf{t}]}\right) = \varphi_{t]}\left[\left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\right]\varphi_{[t,\mathsf{t}]}\left[\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\right].$$

A state $\rho\in\left(\mathcal{M}\otimes\mathcal{N}\right)_*$ with $\rho = \rho_{\mathcal{M}}\otimes\rho_{\mathcal{N}}$ where $\rho_{\mathcal{M}}$ defined on $\mathcal{M}\otimes I_{\mathcal{N}}$ and $\rho_{\mathcal{N}}$ defined on $I_{\mathcal{M}}\otimes\mathcal{N}$ is called a product state. For any $A\otimes B\in\mathcal{M}\otimes\mathcal{N}$, one has

$$\rho\left(A\otimes B\right) = \rho_{\mathcal{M}}\left(A\otimes I_{\mathcal{N}}\right)\rho_{\mathcal{N}}\left(I_{\mathcal{M}}\otimes B\right).$$

Taking $A = I_{\mathcal{M}}$ and $B = I_{\mathcal{N}}$, respectively, one has

$$\rho\left(I_{\mathcal{M}}\otimes B\right) = \rho_{\mathcal{N}}\left(I_{\mathcal{M}}\otimes B\right); \rho\left(A\otimes I_{\mathcal{N}}\right) = \rho_{\mathcal{M}}\left(A\otimes I_{\mathcal{N}}\right).$$

Namely, the two factors can be regarded as the restrictions of ρ: $\rho = \rho|_{\mathcal{M}\otimes I_{\mathcal{N}}}\otimes\rho|_{I_{\mathcal{M}}\otimes\mathcal{N}}$. Now we conclude that $\mathbb{E}_0$ is a product state, for $\forall t\in[0,\mathsf{t}]$, on $\mathcal{A} = \mathcal{A}_t\otimes\mathcal{A}_{[t,\mathsf{t}]}$. Conversely, if $\varphi_{[t,\mathsf{t}]}\in\left(I_{t]}\otimes\mathcal{A}_{[t,\mathsf{t}]}\right)_*$ is a state. Then $\mathbb{E}_t$ defined by

$$\mathbb{E}_t\left(A_{t]}\otimes B_{[t,\mathsf{t}]}\right) = \left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\varphi_{[t,\mathsf{t}]}\left[\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\right]$$

can be extended to a σ-weakly conditional expectation from $\mathcal{A}$ to $I_{t]}\otimes\mathcal{A}_{[t,\mathsf{t}]}$(Theorem 1 [38]).

Proposition 3.3. *A state $\rho\in\mathcal{A}_*$ is a product state for $t\in[0,\mathsf{t}]$ if and only if the family $\left(\mathbb{E}_t\right)_{t\in[0,\mathsf{t}]}$ of conditional expectations defined by*

$$\mathbb{E}_t\left(A_{t]}\otimes B_{[t,\mathsf{t}]}\right) = \left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\rho\left[\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\right]$$

is consistent.

Proof. Note that $\mathbb{E}_0 = \rho$. If $\rho = \varphi_{t]}\otimes\varphi_{[t,\mathsf{t}]}$, then

$$\rho\left[\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\right] = \varphi_{[t,\mathsf{t}]}\left[\left(I_{t]}\otimes B_{[t,\mathsf{t}]}\right)\right];$$
$$\rho\left[\left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\right] = \varphi_{t]}\left[\left(A_{t]}\otimes I_{[t,\mathsf{t}]}\right)\right].$$

This says that

$$\rho\left[\mathbb{E}_t\left(A_{t]}\otimes B_{[t,\mathrm{t}]}\right)\right]=\varphi_{t]}\left[\left(A_{t]}\otimes I_{[t,\mathrm{t}]}\right)\right]\varphi_{t]}\left[\left(I_{t]}\otimes B_{[t,\mathrm{t}]}\right)\right].$$

Hence, $\rho\circ\mathbb{E}_t=\rho$ is compatible. If in addition ρ is assumed to be faithful, $(\mathbb{E}_t)_{t\in[0,\mathrm{t}]}$ is consistent. In general, using the fact that $\rho=\varphi_{s]}\otimes\varphi_{[s,t]}\otimes\varphi_{[t,\mathrm{t}]}$ one shows that $(\mathbb{E}_t)_{t\in[0,\mathrm{t}]}$ is consistent. □

For any $g\in L^2(\mathbb{R}^+)$ such that $\boldsymbol{\varepsilon}(g)$ is unital, one has $\boldsymbol{\varepsilon}(g)=\boldsymbol{\varepsilon}(g_{t]})\otimes\boldsymbol{\varepsilon}(g_{[t})$ which induces a product state. The vacuum state Ω is a spacial case.

3.3. **Product states.** If $\phi\in\mathcal{A}_*$ is a product state for $t\in[0,\mathrm{t}]$, then there exists a family of consistent conditional expectations $(\mathbb{E}_t)_{t\in[0,\mathrm{t}]}$ such that $\phi\circ\mathbb{E}_t=\phi$ and $\phi=\mathbb{E}_0$. So far we only know examples of product pure vector states. These states are not faithful. Now we give a condition to determine whether $\phi\in\mathcal{A}_*$ is a product state.

First, we observe that it is sufficient to show that ϕ is a product state on self-adjoint elements $A\otimes B\in\left(\mathcal{A}_{t]}\right)_{sa}\otimes\left(\mathcal{A}_{[t,\mathrm{t}]}\right)_{sa}$. Let us write

$$A=A_R+iA_I,B=B_R+iB_I$$
$$A\otimes B=A_R\otimes B_R-A_I\otimes B_I+i\left(A_R\otimes B_I+A_I\otimes B_R\right).$$

Assuming ϕ to be product on self-adjoint elements we obtain

$$\begin{aligned}\phi\left(A\otimes B\right)&=\left[\begin{array}{c}\phi\left(A_R\otimes B_R\right)-\phi\left(A_I\otimes B_I\right)\\ i\phi\left(A_R\otimes B_I\right)+i\phi\left(A_I\otimes B_R\right)\end{array}\right]\\ &=\left[\begin{array}{c}\phi\left(A_R\otimes I_{[t,\mathrm{t}]}\right)\phi\left(I_{t]}\otimes B_R\right)-\phi\left(A_I\otimes I_{[t,\mathrm{t}]}\right)\phi\left(I_{t]}\otimes B_I\right)\\ i\phi\left(A_R\otimes I_{[t,\mathrm{t}]}\right)\phi\left(I_{t]}\otimes B_I\right)+i\phi\left(A_I\otimes I_{[t,\mathrm{t}]}\right)\phi\left(I_{t]}\otimes B_R\right)\end{array}\right]\\ &=\left[\phi\left(A_R\otimes I_{[t,\mathrm{t}]}\right)+i\phi\left(A_I\otimes I_{[t,\mathrm{t}]}\right)\right]\left[\phi\left(B_R\otimes I_{[t,\mathrm{t}]}\right)+i\phi\left(B_I\otimes I_{[t,\mathrm{t}]}\right)\right]\\ &=\phi\left(A\otimes I_{[t,\mathrm{t}]}\right)\phi\left(B\otimes I_{[t,\mathrm{t}]}\right).\end{aligned}$$

For fixed ϕ let us define $\mathcal{K}\subset\mathcal{A}_{sa}$ as a real vector subspace, where $\mathcal{A}_{sa}$ is the space of all self-adjoint elements:

$$\mathcal{K}=sp_{\mathbb{R}}\left\{A_{t]}\otimes B_{[t,\mathrm{t}]}-A_{t]}\otimes I_{[t,\mathrm{t}]}\phi\left(I_{t]}\otimes B_{[t,\mathrm{t}]}\right)\in\left(\mathcal{A}_{t]}\right)_{sa}\otimes\left(\mathcal{A}_{[t,\mathrm{t}]}\right)_{sa}\right\}.$$

Using $\mathcal{K}$ one can construct a convex cone

$$\mathcal{C}^{(\mathcal{K})}=\{T\in\mathcal{A}_{sa}:\exists K\in\mathcal{K}\text{ such that }T\leq K\}\subset\mathcal{A}_{sa}.$$

The $\sigma(\mathcal{A},\mathcal{A}_*)$ closure of this set, denoted by $\overline{\mathcal{C}^{(\mathcal{K})}}^{\sigma(\mathcal{A},\mathcal{A}_*)}$, is convex. It is a cone since $\mathcal{K}$ is a real subspace. It follows from Theorem 4.1 [34].

Proposition 3.4. *The following two conditions are equivalent*

(i) *The condition holds* $\overline{\mathcal{C}^{(\mathcal{K})}}^{\sigma(\mathcal{A},\mathcal{A}_*)}\cap\mathcal{A}_+=\{0\}$.

(ii) *There exists a faithful normal state* ρ *on* $\mathcal{A}$ *such that* $\rho=0$ *on* $\mathcal{K}$.

The condition (ii) gives us

$$\rho\left(A_{t]}\otimes B_{[t,\mathrm{t}]}\right)=\rho\left(A_{t]}\otimes I_{[t,\mathrm{t}]}\right)\phi\left(I_{t]}\otimes B_{[t,\mathrm{t}]}\right).$$

Taking $A=I_{t]}$ one obtains

$$\rho\left(I_{t]}\otimes B_{[t,\mathrm{t}]}\right)=\phi\left(I_{t]}\otimes B_{[t,\mathrm{t}]}\right).$$

Finally, we have $\rho = \phi$ and

$$\rho\left(A_{t]} \otimes B_{[t,\mathbf{t}]}\right) = \rho\left(A_{t]} \otimes I_{[t,\mathbf{t}]}\right) \rho\left(I_{t]} \otimes B_{[t,\mathbf{t}]}\right).$$

If there exists a family of faithful conditional expectations, it may be natural to define the price process $C(t) = \mathbb{E}_{t,T}(C(T))$ of a terminal payoff of a contingent claim $C(T) \in \mathcal{A}_T$. In the case that $C(T)$ can be replicated by a gain process $V_g\left(T, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right)$ with respect to a family of $\mathbb{E}$-martingales $\mathbb{S}^U = \left\{S_i^U = U^{-\frac{1}{2}} S_i U^{-\frac{1}{2}}\right\}_{i=1}^{d}$, we have

$$\mathbb{E}_{t,T}(C(T)) = V_g\left(t, \mathbb{L}_U, \mathbb{S}^U, \mathbb{M}_{U^{-1}}\right).$$

3.4. **Generalized conditional expectation.** In general, for any given faithful normal state $\rho \in \mathcal{A}_*$ there is a completely positive unital map $\mathbb{F}_t$ from $\mathcal{A}$ to $\mathcal{A}_t$ such that $\rho \circ \mathbb{F}_t = \rho$. Furthermore, for $s \leq t$, there exists a completely positive map $\mathbb{F}_{s,t}$ from $\mathcal{A}_t$ to $\mathcal{A}_s$ such that $\rho \circ \mathbb{F}_{s,t} = \rho$ and $\mathbb{F}_s = \mathbb{F}_{s,t} \circ \mathbb{F}_t$ [1]. Unfortunately, we do not have the following $\mathcal{A}_s$ bi-module property

$$\mathbb{F}_s(A_s X B_s) = A_s \mathbb{F}_s(X) B_s, A_s, B_s \in \mathcal{A}_s, X \in \mathcal{A}.$$

In another word, $\mathbb{F}_s$ is not a projection [1]. While we still define $C(t) = \mathbb{F}_{t,T}(C(T))$ which is unique, the following may not be zero

$$\mathbb{F}_s\left[\sum_{i=1}^{d}\sum_{j=1}^{J}\int_s^t \left[L_{ij}, dS_i^U, M_{ij}\right]_x\right] \neq 0.$$

Nevertheless,

$$\rho\left(\mathbb{F}_s\left[\sum_{i=1}^{d}\sum_{j=1}^{J}\int_s^t \left[L_{ij}, dS_i^U, M_{ij}\right]_x\right]\right) = 0.$$

This may not be satisfactory, however we do find a way to project $C(T)$ to $\mathcal{A}_t$ for $0 \leq t \leq T$. More investigation is needed for this approach.

Disclaimer: The views expressed here are those of the author and not necessarily those of the CTBC Bank.

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