CONNECTIONS, CURVATURE, AND COHOMOLOGY

Volume III

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Connections, Curvature, and Cohomology

Volume III

Cohomology of Principal Bundles and Homogeneous Spaces

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VOLUME III

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Respectfully dedicated to the memory of

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Preface

This monograph developed out of the *Abendseminar* of 1958–1959 at the University of Zürich. It was originally a joint enterprise of the first author and H. H. Keller, who planned a brief treatise on connections in smooth fibre bundles. Then, in 1960, the first author took a position in the United States, and geographic considerations forced the cancellation of this arrangement.

The collaboration between the first and third authors began with the former's move to Toronto in 1962; they were joined by the second author in 1965. During this time the purpose and scope of the book grew to its present form: a three-volume study, ab initio, of the de Rham cohomology of smooth bundles. In particular, the material in volume I has been used at the University of Toronto as the syllabus for an introductory graduate course on differentiable manifolds.

During the long history of this book we have had numerous valuable suggestions from many mathematicians. We are especially grateful to the faculty and graduate students of the institutions below.

The proof of Theorem VII in sec. 2.17 is due to J. C. Moore (unpublished), and we thank him for showing it to us. We also thank A. Borel for sending us his unpublished example of a homogeneous space not satisfying the Cartan condition, which we have used in sec. 11.15. The late G. S. Rinehart read an early version of the manuscript and made many valuable suggestions. A. E. Fekete, who prepared the subject index, has our special gratitude.

We are indebted to the institutions whose facilities were used by one or more of us during the writing. These include the Departments of Mathematics of Cornell University, Flinders University, the University of Fribourg, and the University of Toronto, as well as the Institut für theoretische Kernphysik and the Hoffmannhaus, both at Bonn, and the Forschungsinstitut für Mathematik der Eidgenössischen Technischen Hochschule, Zürich.

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Finally, we would like to thank the production and editorial staff at Academic Press for their unfailing helpfulness and cooperation. Their universal patience, while we rewrote the manuscript (ad infinitum), oscillated amongst titles, and ruined production schedules, was in large measure responsible for the completion of this work.

Werner Greub
Stephen Halperin
Ray Vanstone

Toronto, Canada

Introduction

The purpose of this monograph is to develop the theory of de Rham cohomology for manifolds and fibre bundles. The present (and final) volume is an exposition of the work of H. Cartan, C. Chevalley, J.-L. Koszul, and A. Weil, which provides an effective means of calculating the de Rham cohomology of principal bundles and of homogeneous spaces.

In fact, let (P, π, B, G) be a smooth principal bundle with G a compact connected Lie group, and let E denote the Lie algebra of G. Then $H(G)$ is an exterior algebra over the subspace P_G of primitive elements. Let $\vee E^*$ and $(\vee E^*)_I$, respectively, denote the symmetric algebra over E^* and the subalgebra of elements invariant under the adjoint representation.

Now suppose given the following data:

(i) A linear map $\tau: P_G \to (\vee E^*)_I$.

(ii) The graded differential algebra $(A(B), \delta)$ of differential forms on B.

(iii) The Chern–Weil homomorphism $h: (\vee E^*)_I \to H(B)$.

Choose a linear map $\gamma: (\vee E^*)_I \to A(B)$ such that $\gamma(\Phi)$ is a closed form representing the class $h(\Phi)$.

Then a differential algebra $(A(B) \otimes \wedge P_G, \nabla_B)$ is given by

$$\nabla_B(\Psi \otimes x_1 \wedge \cdots \wedge x_p) = \delta_B \Psi \otimes x_1 \wedge \cdots \wedge x_p + (-1)^{\deg \Psi} \sum_{j=1}^{p} (-1)^{j-1} \Psi \wedge \gamma(\tau x_j) \otimes x_1 \wedge \cdots \hat{x}_j \cdots \wedge x_p .$$

A fundamental theorem of C. Chevalley states that for suitable maps τ, called transgressions, there is a homomorphism of graded differential algebras $(A(B) \otimes \wedge P_G, \nabla_B) \to (A(P), \delta)$, which induces an isomorphism of cohomology.

Next let K be a closed connected subgroup of G with Lie algebra F. An analogous theorem identifies the cohomology of G/K with the cohomology of the graded differential algebra $((\vee F^*)_I \otimes \wedge P_G, \nabla)$, where ∇ is given by

$$\nabla(\Psi \otimes x_1 \wedge \cdots \wedge x_p) = -\sum_{i=1}^{p} (-1)^{i-1} \Psi \vee j^\vee(\tau x_i) \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_p .$$

(Here $j^\vee: (\vee E^*)_I \to (\vee F^*)_I$ denotes the homomorphism induced by the inclusion map $j: F \to E$.

These two theorems generalize to a single theorem on the cohomology of a fibre bundle whose fibre is a homogeneous space.

The first part of this volume is chiefly devoted to Koszul complexes. (These are graded differential algebras of the form $(A(B) \otimes \wedge P_G, \nabla_B)$ described above.)

The second part is a careful and complete exposition of the purely algebraic theory (due to H. Cartan) of the operation of a Lie algebra in a graded differential algebra. In particular, the theorems described above are special cases of results about operations. These results reduce problems on operations to problems on Koszul complexes, to which the machinery of Part 1 can then be applied.

The applications to manifolds (including the theorems above and a number of concrete examples) are given separately in the last articles of Chapters V to IX, all of Chapter XI, and the last article of Chapter XII. If these articles (which depend heavily on volumes I and II) are omitted, the rest of this volume is an entirely self-contained unit, accessible to any reader familiar with linear and multilinear algebra. Indeed, Part 2 begins with an account of Lie algebras (Chapter IV) which contains all the necessary definitions and results (but omits some proofs).

Moreover, aside from a single quotation in Chapter VI of a theorem proved early in Chapter II, the results of Chapters II and III are not used until Chapter IX. Thus the interdependence of chapters is given by

$$\begin{tikzcd}
 & \mathrm{II} \arrow[r] & \mathrm{III} \arrow[rrd] & & & & & \\
\mathrm{I} \arrow[ru] \arrow[r] & \mathrm{VI} \arrow[r] & \mathrm{VII} \arrow[r] & \mathrm{VIII} \arrow[r] & \mathrm{IX} \arrow[r] & \mathrm{X} \arrow[r] & \mathrm{XI} \arrow[r] & \mathrm{XII} \\
\mathrm{IV} \arrow[r] & \mathrm{V} \arrow[u] & & & & & &
\end{tikzcd}$$

A more detailed description of the contents appears below. Unlike volumes I and II, this volume contains no problems.

Much of the material in this volume first appeared in the articles by Cartan, Koszul, and Leray in the proceedings of the Colloque de Topologie (espaces fibrés) held at Brussels in 1950 (cf. [53], [168], and [187]). The first account with complete proofs is [9]. In the notes at the back we shall give more historical details and acknowledge the discoverers of the main theorems. We apologize for errors and omissions.

Part 1

In this part all vector spaces and algebras are defined over a commutative field Γ.

Chapter I. Spectral Sequences. This chapter, which has been included only for the sake of completeness, is a self-contained description of the spectral sequence of a filtered differential space. The reader already familiar with this material may omit the whole chapter and simply refer back to it as necessary. Almost all the theorems and proofs in the chapter apply verbatim to filtered modules over a commutative ring.

Chapter II. Koszul Complexes of P-Spaces and P-Algebras. Let $P = \Sigma_k P^k$ be a finite-dimensional positively graded vector space with $P^k = 0$ for even k. Let $\boldsymbol{P}$ denote the evenly graded space given by $\boldsymbol{P}^k = P^{k-1}$. A P-algebra $(S;\sigma)$ is a positively graded associative algebra S together with a linear map $\sigma : P \to S$, homogeneous of degree 1, such that $\sigma(x)$ is in the centre of S, $x \in P$.

The Koszul complex of $(S;\sigma)$ is the graded differential algebra $(S \otimes \wedge P, \nabla_\sigma)$ given by

$$\nabla_\sigma(z \otimes x_1 \wedge \cdots \wedge x_p)$$
$$= (-1)^{\deg z} \sum_{i=1}^{p} (-1)^{i-1} z \cdot \sigma(x_i) \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_p .$$

The gradation $S \otimes \wedge P = \Sigma_k S \otimes \wedge^k P$ induces a gradation, $H(S \otimes \wedge P) = \Sigma_k H_k(S \otimes \wedge P)$ in cohomology. These and other basic facts are established in article 1.

The isomorphism theorems in article 2 show, for example, that $H_+(S \otimes \wedge P) = 0$ if and only if $S \cong H_0(S \otimes \wedge P) \otimes \vee \boldsymbol{P}$. Suppose now that S is connected (i.e., $S^0 = \Gamma$). In this case (article 4) the projection $S \to \Gamma$ induces a homomorphism $H(S \otimes \wedge P) \to \wedge P$ whose image is the exterior algebra over a graded subspace $\hat{P} \subset P$. Moreover, if $\tilde{P}$ is a complementary subspace, then $H(S \otimes \wedge P) \cong H(S \otimes \wedge \tilde{P}) \otimes \wedge \hat{P}$.

Finally, article 5 deals with symmetric P-algebras $(\vee Q;\sigma)$. The main theorem asserts that if $H(\vee Q \otimes \wedge P)$ has finite dimension, then $\dim P = \dim \hat{P} + \dim Q + k$, where k is the greatest integer such that $H_k(\vee Q \otimes \wedge \tilde{P}) \neq 0$. This is applied to determine the complete structure of $H(\vee Q \otimes \wedge P)$ when $\dim P = \dim \hat{P} + \dim Q$. The results of article 3 are used to obtain the Poincaré polynomial for $H(\vee Q \otimes \wedge P)$ in this case.

Chapter III. Koszul Complexes of P-Differential Algebras. A P-differential algebra (or (P, δ)-algebra) is a positively graded associative alternating differential algebra (B, δ_B) together with a linear map $\tau : P \to B$, homogeneous of degree 1, and satisfying $\delta \circ \tau = 0$. The Koszul complex of $(B, \delta_B; \tau)$ is the graded differential algebra $(B \otimes \wedge P, \nabla_B)$, where $\nabla_B = \delta_B \otimes \iota + \nabla_\tau$.

In articles 3 and 4 the isomorphism and structure theorems for P-algebras are generalized to (P, δ)-algebras. Theorem VII in article 4

gives six conditions, each equivalent to the surjectivity of the homomorphism $H(B \otimes \wedge P) \to \wedge P$ induced by the natural projection.

The tensor difference $(B \otimes S,\ \delta_{B \otimes S},\ \tau \ominus \sigma)$ of two (P, δ)-algebras $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ is defined by $\delta_{B\otimes S} = \delta_B \otimes \iota - \omega_B \otimes \delta_S$ and $(\tau \ominus \sigma)(x) = \tau x \otimes 1 - 1 \otimes \sigma x$. ($\omega_B$ is the degree involution of B.) Its basic properties are established in article 2, including the fact that if $S = \vee P$, then $H(B \otimes \vee P \otimes \wedge P) \cong H(B)$.

In article 5 the commutativity of the cohomology diagram,

$$\begin{tikzcd}
\vee P \arrow[r] \arrow[d] & H(B) \arrow[d] \arrow[dr] & \\
H(S) \arrow[r] \arrow[dr] & H(B \otimes S \otimes \wedge P) \arrow[r] \arrow[d] & H(B) \otimes \wedge P) \arrow[d] \\
& H(S \otimes \wedge P) \arrow[r] & \wedge P,
\end{tikzcd}$$

is established. It is also shown that the map $H(B \otimes S \otimes \wedge P) \to H(S \otimes \wedge P)$ is surjective if and only if the graded spaces $H(B \otimes S \otimes \wedge P)$ and $H(B) \otimes H(S \otimes \wedge P)$ are isomorphic. Article 6 gives necessary and sufficient conditions for this to be an algebra isomorphism if S is a symmetric algebra.

Part 2

In this part all vector spaces and algebras are defined over a commutative field Γ of characteristic zero.

Chapter IV. Lie Algebras and Differential Spaces. Article 1 starts with the definition of Lie algebras and their representations and then quotes without proof the basic theorems about reductive Lie algebras and semisimple representations. Some material on Cartan subalgebras and root space decompositions is also included.

Article 2 deals with representations of Lie algebras in differential spaces. The results of this article are fundamental for the following chapters, and complete proofs are given. One such theorem asserts that $H((X \otimes Y)_{\theta=0}) = H(X_{\theta=0}) \otimes H(Y_{\theta=0})$ under suitable hypotheses. (If a Lie algebra is represented in a space X, then $X_{\theta=0}$ denotes the subspace of invariant vectors.)

Chapter V. Cohomology of Lie Algebras and Lie Groups. Let E be a finite dimensional Lie algebra. The adjoint representation of E determines representations of E in $\wedge E$ and $\wedge E^*$. Moreover, an anti-

derivation of square zero δ_E in $\wedge E^*$ is determined by $\langle\delta_E x^*, x\wedge y\rangle = -[x, y]$. Its negative dual ∂_E is a differential operator in $\wedge E$, homogeneous of degree -1. The algebra $H^*(E) = H(\wedge E^*, \delta_E)$ and the space $H_*(E) = H(\wedge E, \partial_E)$ are called, respectively, the cohomology and homology of E.

If E is reductive, then $(\wedge E^*)_{\theta=0} \cong H^*(E)$ and $(\wedge E)_{\theta=0} \cong H_*(E)$ (article 3). Moreover, in this case there are canonical dual subspaces $P_E \subset (\wedge E^*)_{\theta=0}$ and $P_*(E) \subset (\wedge E)_{\theta=0}$ (the primitive subspaces), and the inclusions extend to scalar product preserving isomorphisms of graded algebras

$$\wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0} \qquad \text{and} \qquad \wedge P_*(E) \xrightarrow{\cong} (\wedge E)_{\theta=0}$$

(articles 4, 5, and 6).

Recall from volume II (article 4, Chapter IV) that the cohomology of a compact connected Lie group is isomorphic with the cohomology of its Lie algebra. In article 8 this isomorphism is used to translate the results of the chapter into theorems on the cohomology of Lie groups.

Chapter VI. The Weil Algebra. Let E be a finite dimensional Lie algebra. Let E^* denote the graded space with E^* as underlying vector space and deg $x^* = 2$, $x^* \in E^*$. In article 1 an antiderivation δ_W of square zero is introduced in the algebra $W(E) = \vee E^* \otimes \wedge E^*$; the resulting graded differential algebra is called the Weil algebra of E. The adjoint representation of E induces a representation of E in $(W(E), \delta_W)$. The main result of article 1 states that $H^+(W(E)) = 0$ and $H^+(W(E)_{\theta=0}) = 0$.

Using this result we construct in article 2 a canonical linear map $\varrho_E : (\vee^+ E^*)_{\theta=0} \to (\wedge^+ E^*)_{\theta=0}$, homogeneous of degree -1 (the Cartan map). Theorem II in article 4 asserts that if E is reductive, then

$$\operatorname{Im} \varrho_E = P_E \qquad \text{and} \qquad \ker \varrho_E = (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0}.$$

A transgression in $W(E)_{\theta=0}$ is a linear map $\tau : P_E \to (\vee^+ E^*)_{\theta=0}$ homogeneous of degree 1 such that $\varrho_E \circ \tau = \iota$. Theorem I of article 4 says that a transgression extends to an algebra isomorphism $\vee P_E \xrightarrow{\cong} (\vee E^*)_{\theta=0}$. In particular, $(\vee E^*)_{\theta=0}$ is the symmetric algebra over an evenly graded space whose dimension is equal to that of P_E.

In article 6 these results are applied to determine the algebras $(\wedge E^*)_{\theta=0}$ and $(\vee E^*)_{\theta=0}$, where E is a classical Lie algebra. Finally, in article 7 we determine $H(G)$ for the classical compact Lie groups. Explicit invariant multilinear functions in E and closed differential forms on G are constructed and shown to yield bases of these spaces.

Chapter VII. Operation of a Lie Algebra in a Graded Differential Algebra. An operation of a Lie algebra E in a graded differential

algebra (R, δ) consists of a representation θ of E in (R, δ) together with antiderivations $i(x)$, $x \in E$, in R of degree -1 and such that for $x, y \in E$

$$i(x)^2 = 0, \qquad \theta(x) \circ i(y) - i(y) \circ \theta(x) = i([x, y])$$

and

$$i(x)\,\delta + \delta i(x) = \theta(x).$$

The horizontal and basic subalgebras of R are defined, respectively, by $R_{i=0} = \bigcap_{x \in E} \ker i(x)$ and $R_{i=0,\,\theta=0} = (R_{i=0})_{\theta=0}$. $R_{i=0,\,\theta=0}$ is stable under δ.

An action of a Lie group G on a manifold M determines an operation of its Lie algebra on the algebra of differential forms $(A(M), \delta)$ via the Lie derivatives and the substitution operators for the fundamental vector fields (article 6). The Weil algebra of E is a second example of an operation.

In articles 3 and 4 the fibre projection $\varrho_R : H(R_{\theta=0}) \to (\wedge E^*)_{\theta=0}$ is defined when E is reductive and $H(R_{\theta=0})$ is connected. It reduces to the obvious homomorphism $H(M) \to H(G)$ in the example above if G is compact and M and G are connected. It is shown that $\operatorname{Im} \varrho_R = \wedge \hat{P}$ and $H(R_{\theta=0}) \cong A \otimes \wedge \hat{P}$, where $\hat{P}$ is a subspace of P_E and A is a subalgebra of $H(R_{\theta=0})$.

Chapter VIII. Algebraic Connections and Principal Bundles. An algebraic connection for an operation of E in a graded differential algebra (R, δ) is an E-linear map $\chi : E^* \to R^1$ such that $i(x)\chi(x^*) = \langle x^*, x \rangle$. In article 2 it is shown that an algebraic connection determines an isomorphism $R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$.

The curvature of an algebraic connection is the linear map $\chi : E^* \to R^2_{i=0}$ determined by $\delta(1 \otimes x^*) = 1 \otimes \delta_E x^* + \chi x^* \otimes 1$. It extends to a homomorphism $\chi_\vee : \vee E^* \to R_{i=0}$, inducing the Weil homomorphism $\chi^\# : (\vee E^*)_{\theta=0} \to H(R_{i=0,\,\theta=0})$. Theorem V, article 4, shows that the Weil homomorphism is independent of the algebraic connection.

In article 5 we consider the example $R = A(P)$, where (P, π, B, G) is a principal bundle. Then $A(B) \cong R_{i=0,\,\theta=0}$. There is a one-to-one correspondence between algebraic connections and principal connections. Moreover, the corresponding curvatures determine each other, and the Weil homomorphism corresponds to the Chern–Weil homomorphism of the principal bundle as defined in volume II.

Chapter IX. Cohomology of Operations and Principal Bundles. Suppose an operation of a reductive Lie algebra E in a graded differential algebra (R, δ) admits an algebraic connection χ. Let $\tau : P_E \to (\vee E^*)_{\theta=0}$ be a transgression (cf. Chapter VI) and set $\tau_R = \chi_\vee \circ \tau : P_E \to R_{i=0,\,\theta=0}$. Then $(R_{i=0,\,\theta=0}, \delta; \tau_R)$ is a (P_E, δ)-algebra (cf. Chapter III). A funda-

mental theorem of Chevalley (article 2) gives a homomorphism from the corresponding Koszul complex to $(A(P), \delta)$, which induces an isomorphism of cohomology.

This isomorphism is then used to apply the theorems of Chapter III to operations (articles 3 and 4) and to the cohomology of principal bundles (article 6). As examples, the cohomology algebras of the tangent frame bundles of CP^n and $CP^n \times CP^m$ are determined.

Chapter X. Subalgebras. A Lie algebra pair (E, F) is a Lie algebra E together with a subalgebra F. The inclusion map is denoted by $j : F \to E$. The subalgebra F operates in the graded differential algebra $(\wedge E^*, \delta_E)$. This chapter deals with cohomology of the basic subalgebra; this is denoted by $H(E/F)$.

Let $k^* : H(E/F) \to H(E)$ be the homomorphism induced by the inclusion map. In article 1 it is shown that if E is reductive, then Im k^* is the exterior algebra over a subspace $\hat{P} \subset P_E$, called the Samelson space for the pair (E, F).

Suppose (E, F) is a reductive pair; i.e., E is reductive and F acts semisimply in E. Let $\tau : P_E \to (\vee E^*)_{\theta=0}$ be a transgression, and set $\sigma = j^\vee \circ \tau$. Then $((\vee F^*)_{\theta=0} ; \sigma)$ is a symmetric P_E-algebra. According to a fundamental theorem of Cartan (article 2), there is a homomorphism from the Koszul complex $((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma)$ to $((\wedge E^*)_{i_F=0,\ \theta_F=0}, \delta_E)$ inducing an isomorphism in cohomology.

In article 3 this result is applied to translate the structure theorems of Chapter II into theorems on $H(E/F)$. Article 4 shows that for a reductive pair $\dim P_E \geq \dim P_F + \dim \hat{P}$. If equality holds, (E, F) is called a Cartan pair. For such pairs $H(E/F) \cong A \otimes \wedge \hat{P}$, where $A = (\vee F^*)_{\theta=0}/I$ and I is the ideal generated by $j^\vee(\vee^+ E^*)_{\theta=0}$. Moreover, the Poincaré polynomial of $H(E/F)$ is given explicitly in this case.

A subalgebra $F \subset E$ is called noncohomologous to zero in E if $j^* : H^*(E) \to H^*(F)$ is surjective. These subalgebras are discussed in article 5. Article 6 deals with equal rank pairs ($\dim P_E = \dim P_F$). In particular it is shown that if F is a Cartan subalgebra of E, then (E, F) is an equal rank pair.

In article 7 the results are applied to symmetric pairs, and article 8 gives a relative version of the Poincaré duality theorem.

Chapter XI. Homogeneous Spaces. Suppose $K \subset G$ are compact connected Lie groups with Lie algebras $F \subset E$. Then there is an isomorphism $H(E/F) \cong H(G/K)$ which permits us to translate the theorems of Chapter X into theorems on $H(G/K)$ (articles 1 and 2). In particular the Cartan condition is shown to depend only on the topology of G/K.

Article 3 is devoted to Leray's theorem, which asserts that if T is a maximal torus in G with Lie algebra H, then $j^\vee$ is an isomorphism from

$(\vee E^*)_{\theta=0}$ onto the subalgebra of $\vee H^*$ of elements invariant under the action of the Weyl group.

Finally, in article 4 the cohomology of the standard homogeneous spaces is determined, while article 5 contains examples of non-Cartan pairs.

Chapter XII. Operation of a Lie Algebra Pair. An operation of E in (R, δ) restricts to an operation of any subalgebra F. We shall say that this is an operation of the pair (E, F) if the inclusion map $R_{\theta_E=0} \to R_{\theta_F=0}$ induces an isomorphism in cohomology.

Suppose that the hypotheses of the fundamental theorems of Chapters IX and X are satisfied and let $(R_{i_E=0,\,\theta_E=0} \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla)$ be the Koszul complex of the tensor difference of $(R_{i_E=0,\,\theta_E=0}, \delta; \tau_R)$ and $((\vee F^*)_{\theta=0}; \sigma)$ (cf. Chapter III). The main theorem of this chapter (article 2) asserts that

$$H(R_{i_F=0,\,\theta_F=0}) \cong H(R_{i_E=0,\,\theta_E=0} \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E).$$

In article 4 the theorems of Chapter III on tensor differences are applied to the operation of a pair, and in article 5 these results are used to determine the cohomology of fibre bundles whose fibre is a homogeneous space.

In particular, let (P, π, B, G) be a principal bundle, where G is semisimple, compact, and connected, and let K be a torus in G. Consider the associated bundle $(P/K, \varrho, B, G/K)$. It is shown that if the graded algebras $H(P/K)$ and $H(B) \otimes H(G/K)$ are isomorphic, then all the characteristic classes of the principal bundle are zero.

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Volume I: **De Rham Cohomology of Manifolds and Vector Bundles**

Volume II: **Lie Groups, Principal Bundles, and Characteristic Classes**

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Chapter 0

Algebraic Preliminaries

0.0. Notation. Throughout this book ι_X denotes the identity map of a set X. When it is clear which set we mean, we write simply ι. The empty set is denoted by $\emptyset$. The symbol $x_0 \cdots \hat{x}_i \cdots x_p$ means x_i is to be deleted.

The symbols N, Z, Q, R, and C denote, respectively, the natural numbers, the integers, the rationals, the real and the complex numbers.

Throughout the book Γ will denote a commutative field, and all vector spaces and algebras are defined over Γ unless we explicitly state otherwise. Moreover, from Chapter IV on it is assumed that Γ has characteristic zero.

The group of permutations on n letters is denoted by S^n; if $\sigma \in S^n$, then $\varepsilon_\sigma = 1$ (-1) if σ is even (odd).

Finally, the proofs of the assertions made in this chapter will be found in [4] and [5].

0.1. Linear algebra. We shall assume the fundamentals of linear and multilinear algebra. A pair of vector spaces X^*, X is called *dual* with respect to a bilinear function

$$\langle\ ,\ \rangle : X^* \times X \to \Gamma$$

if

$$\langle x^*, X\rangle = 0 \qquad \text{and} \qquad \langle X^*, x\rangle = 0$$

imply, respectively, that $x^* = 0$ and $x = 0$. $\langle\ ,\ \rangle$ is called the *scalar product.*

If X is finite dimensional, then a pair of *dual bases* for X^* and X is a basis e^{*i} for X^* and a basis e_i for X such that $\langle e^{*i}, e_j\rangle = \delta^i_j$.

If $Y \subset X$ is a subspace, then the *orthogonal complement* $Y^\perp \subset X^*$ is defined by

$$Y^\perp = \{x^* \in X^* \mid \langle x^*, Y\rangle = 0\}.$$

The direct sum of vector spaces X^p is denoted by

$$\sum_p X^p \qquad \text{or} \qquad \bigoplus_p X^p.$$

The space of linear maps from X to Y (X and Y arbitrary vector spaces) is denoted by $L(X; Y)$, and we often write $L(X; X) = L_X$. If $\varphi \in L_X$, then its determinant and trace are written $\det \varphi$ and $\operatorname{tr} \varphi$.

Let X^*, X and Y^*, Y be pairs of dual vector spaces. Then linear transformations $\varphi \in L(X; Y)$ and $\varphi^* \in L(Y^*; X^*)$ are called *dual* if

$$\langle \varphi^* y^*, x \rangle = \langle y^*, \varphi x \rangle, \qquad y^* \in Y^*, \quad x \in X.$$

An *inner product space* $(X, \langle\, , \rangle)$ is a finite-dimensional vector space X together with a symmetric scalar product $\langle\, , \rangle$ between X and itself. $\langle\, , \rangle$ is called the *inner product*. If $(X, \langle\, , \rangle)$ is an inner product space, then the dual φ^* of a linear transformation φ of X is again an element of L_X; φ is called *skew symmetric* if $\varphi^* = -\varphi$.

A *symplectic space* $(X, \langle\, , \rangle)$ is a finite-dimensional vector space X together with a skew symmetric scalar product $\langle\, , \rangle$ between X and itself. $\langle\, , \rangle$ is called the *symplectic metric*. If $\varphi \in L_X$ is skew with respect to $\langle\, , \rangle$ (i.e., if $\varphi = -\varphi^*$), then φ is called *skew symplectic*.

The tensor product of vector spaces X and Y is denoted by $X \otimes Y$. If X^*, X and Y^*, Y are pairs of finite-dimensional dual vector spaces, then $X^* \otimes Y^*$ and $X \otimes Y$ are dual with respect to the scalar product $\langle\, , \rangle$ defined by

$$\langle x^* \otimes y^*, x \otimes y \rangle = \langle x^*, x \rangle \langle y^*, y \rangle,$$
$$x^* \in X^*, \quad y^* \in Y^*, \quad x \in X, \quad y \in Y.$$

It is called the *tensor product* of the original two scalar products.

If X and Y are vector spaces and $y \in Y$, then $X \otimes y$ denotes the subspace of $X \otimes Y$ consisting of vectors of the form $x \otimes y$, $x \in X$.

If X has finite dimension, a canonical isomorphism $\alpha: X^* \otimes Y \xrightarrow{\cong} L(X; Y)$ is defined by

$$\alpha(x^* \otimes y)(x) = \langle x^*, x \rangle y, \qquad x^* \in X^*, \quad x \in X, \quad y \in Y.$$

If $Y = X$ and e^{*i}, e_i is any pair of dual bases for X^*, X, then

$$\alpha\Big(\sum_i e^{*i} \otimes e_i\Big) = \iota_X.$$

0.2. Gradations. A *graded space* is a vector space X together with a direct decomposition $X = \sum_{p \in \mathbb{Z}} X^p$. The subspace X^p is called the *space of homogeneous vectors of degree p*. X is called *positively graded* if $X^p = 0$ for $p < 0$. X is called *evenly graded* if $X^p = 0$, p odd, and *oddly graded* if $X^p = 0$, p even. A subspace $Y \subset X$ is called a *graded subspace* if

$$Y = \sum_p Y \cap X^p.$$

The *degree involution* ω_X of a graded space X is defined by

$$\omega_X(x) = (-1)^p x, \qquad x \in X^p.$$

A graded space X is said to have *finite type* if each X^p has finite dimension. In this case the formal series

$$f_X = \sum_p (\dim X^p) t^p$$

is called the *Poincaré series* for X. If Y is a second graded space of finite type, we write $f_Y \leq f_X$ if $\dim Y^p \leq \dim X^p$, all p. If $\dim X$ is finite, then f_X is a polynomial, called the *Poincaré polynomial.* In this case the alternating sum

$$\chi_X = \sum_p (-1)^p \dim X^p = f_X(-1)$$

is called the *Euler–Poincaré characteristic of X.*

Assume that $(X^p)^*$ and X^p are dual spaces ($p \in \mathbb{Z}$). Then the scalar products extend to the scalar product between the graded spaces $X^* = \sum_p (X^p)^*$ and $X = \sum_p X^p$, defined by

$$\langle (X^p)^*, X^q \rangle = 0, \qquad p \neq q.$$

X^* and X are called *dual graded spaces.*

A *bigraded vector space* $X = \sum_{p,q \in \mathbb{Z}} X^{p,q}$ is defined analogously. If $X = \sum X^{p,q}$ is a bigraded space, then the gradation $X = \sum_r X^{(r)}$ given by

$$X^{(r)} = \sum_{p+q=r} X^{p,q}$$

is called the *induced total gradation.*

A linear map $\varphi: X \to Y$ between graded spaces is called *homogeneous of degree r* if it restricts to linear maps from X^p to Y^{p+r} ($p \in \mathbb{Z}$). Homogeneous maps of bidegree (r, s) between bigraded spaces are defined

analogously, and linear maps homogeneous of degree zero (respectively, homogeneous of bidegree zero) are called *homomorphisms of graded spaces* (respectively, *homomorphisms of bigraded spaces*).

Let $X = \sum_p X^p$ be a graded space. Then we write

$$X^+ = \sum_{p>0} X^p.$$

If $\varphi: X \to Y$ is a homomorphism of graded spaces, its restriction to X^+ is denoted by

$$\varphi^+: X^+ \to Y^+.$$

0.3. Algebras. An *algebra* A over Γ is a vector space, together with a bilinear map $A \times A \to A$ (called the *product*). A *system of generators* for A is a subset $S \subset A$ such that every element of A is a linear combination of products of elements of S.

If X and Y are subsets of A, then $X \cdot Y$ denotes the subspace spanned by the products xy ($x \in X, y \in Y$). An *ideal* I in A is a subspace such that

$$I \cdot A \subset I \quad \text{and} \quad A \cdot I \subset I.$$

The ideal $I \cdot I$ is denoted by I^2.

A *homomorphism of algebras* $\varphi: A \to B$ is a product preserving linear map, while a *derivation* θ in A is a linear transformation of A satisfying

$$\theta(xy) = \theta(x)y + x\theta(y).$$

Given a homomorphism $\varphi: A \to B$, a *φ-derivation* is a linear map $\theta_1: A \to B$ such that

$$\theta_1(xy) = \theta_1(x)\varphi(y) + \varphi(x)\theta_1(y).$$

Homomorphisms, derivations, and φ-derivations are completely determined by their restrictions to any set of generators.

In this book we shall consider associative algebras with identity, and Lie algebras (cf. sec. 4.1). In the first case the identity is written 1 and Γ is identified with the subspace $\Gamma \cdot 1$ (so that scalar multiplication coincides with multiplication in A). Homomorphisms between two associative algebras with identity are always assumed to preserve the identity.

A *graded algebra* A is a graded vector space $A = \sum_p A^p$, together with a product, such that $A^p \cdot A^q \subset A^{p+q}$. A *homomorphism* $\varphi: A \to B$ of

graded algebras is an algebra homomorphism, homogeneous of degree zero.

A graded algebra A with identity element $1 \in A^0$ is called *connected* if

$$A^0 = \Gamma \quad \text{and} \quad A^p = 0, \quad p < 0.$$

An *antiderivation* in a graded algebra A is a linear map $\theta: A \to A$ such that

$$\theta(xy) = \theta(x)y + (-1)^p x\theta(y), \qquad x \in A^p, \quad y \in A.$$

If $\varphi: A \to B$ is an algebra homomorphism, then a *φ-antiderivation* is a linear map $\theta_1: A \to B$ such that

$$\theta_1(xy) = \theta_1(x)\varphi(y) + (-1)^p \varphi(x)\theta_1(y), \qquad x \in A^p, \quad y \in A.$$

An associative graded algebra A is called *anticommutative* if

$$xy = (-1)^{pq} yx, \qquad x \in A^p, \quad y \in A^q.$$

If in addition $x^2 = 0$ for $x \in A^p$, p odd, then A is called *alternating*. (These notions coincide if Γ has characteristic not equal to two.)

If A and B are graded algebras, their *anticommutative* (or *skew*) *tensor product* is the graded algebra, $A \otimes B$, defined by

$$(A \otimes B)^r = \sum_{p+q=r} A^p \otimes B^q,$$

and

$$(x \otimes y)(x_1 \otimes y_1) = (-1)^{qp_1} xx_1 \otimes yy_1,$$
$$x \in A, \quad x_1 \in A^{p_1}, \quad y \in B^q, \quad y_1 \in B.$$

If A and B are anticommutative (alternating), then so is $A \otimes B$.

In this book the tensor product of graded algebras will always mean the anticommutative tensor product, unless explicitly stated otherwise. There is, however, a second possible multiplication in $A \otimes B$; it is called the *canonical tensor product* and is defined by

$$(x \otimes y)(x_1 \otimes y_1) = xx_1 \otimes yy_1, \qquad x, x_1 \in A, \quad y, y_1 \in B.$$

Assume A, B, and C are graded algebras, and that C is anticommutative. Let $\varphi_A: A \to C$ and $\varphi_B: B \to C$ be homomorphisms of graded

algebras. Then a homomorphism

$$\varphi: A \otimes B \to C$$

of graded algebras is defined by $\varphi(a \otimes b) = \varphi_A(a) \cdot \varphi_B(b)$.

0.4. Exterior algebra. The exterior algebra over a vector space X is denoted by $\wedge X$ and the multiplication is denoted by $\wedge$. By assigning $\wedge^p X$ the degree p, we make $\wedge X$ into a graded alternating algebra. If $e_1, \ldots, e_n$ is a basis for X, we write

$$\wedge X = \wedge(e_1, \ldots, e_n).$$

Let A be any associative algebra, and assume that $\varphi: X \to A$ is a linear map such that $(\varphi x)^2 = 0$, $x \in X$. Then φ extends to a unique algebra homomorphism

$$\varphi_\wedge: \wedge X \to A.$$

We sometimes denote $\varphi_\wedge$ by $\wedge\varphi$. (If A is graded and alternating, and $\varphi(X) \subset A^1$, then $(\varphi x)^2 = 0$, $x \in X$ and $\varphi_\wedge$ is a homomorphism of graded algebras.)

A linear map $\psi: X \to \wedge^p X (p \text{ odd})$ extends uniquely to a derivation in $\wedge X$. A linear map $\psi: X \to \wedge^p X$ (p even) extends uniquely to an antiderivation in $\wedge X$.

Now let X^*, X be a pair of dual finite-dimensional vector spaces. Then $\wedge^p X^*$ and $\wedge^p X$ are dual with respect to the scalar product given by

$$\langle x_1^* \wedge \cdots \wedge x_p^*, x_1 \wedge \cdots \wedge x_p \rangle = \det(\langle x_i^*, x_j \rangle), \qquad x_i^* \in X^*, \quad x_j \in X.$$

Thus $\wedge X^*$ and $\wedge X$ are dual graded spaces. Moreover, we identify $\wedge^p X^*$ with the space of p-linear skew symmetric functions in X by writing

$$\Phi(x_1, \ldots, x_p) = \langle \Phi, x_1 \wedge \cdots \wedge x_p \rangle, \qquad \Phi \in \wedge^p X^*, \quad x_i \in X.$$

Suppose Y^*, Y is a second pair of dual finite-dimensional spaces, and let $\varphi \in L(X; Y)$ and $\varphi^* \in L(Y^*; X^*)$ be dual maps. Then the homomorphisms

$$\varphi_\wedge: \wedge X \to \wedge Y \qquad \text{and} \qquad (\varphi^*)_\wedge: \wedge X^* \leftarrow \wedge Y^*$$

are dual. We will denote $(\varphi^*)_\wedge$ by $\varphi^\wedge$.

If $x \in X$, then $i(x)$ denotes the unique antiderivation in $\wedge X^*$ extending the linear map $X^* \to \Gamma$ given by $x^* \mapsto \langle x^*, x\rangle$. It is called the *substitution operator* and is homogeneous of degree -1. The substitution operator is dual to the *multiplication operator* $\mu(x)$ in $\wedge X$ defined by

$$\mu(x)b = x \wedge b, \qquad b \in \wedge X.$$

More generally, if $a \in \wedge X$, then $\mu(a)$ is the multiplication operator given by $\mu(a)b = a \wedge b$. The dual operator is denoted by $i(a)$. Clearly,

$$i(x_1 \wedge \cdots \wedge x_p) = i(x_p) \circ \cdots \circ i(x_1), \qquad x_i \in X.$$

The following result is proved in [5; Prop. II, p. 138].

Proposition I: Let $A \subset \wedge X^*$ be a subalgebra, stable under the operators $i(x)$, $x \in X$. Then

$$A = \wedge(X^* \cap A).$$

Next, suppose $X = Y \oplus Z$. Then an isomorphism

$$\wedge Y \otimes \wedge Z \xrightarrow{\cong} \wedge X$$

of graded algebras is defined by $a \otimes b \mapsto a \wedge b$. If Y^*, Y and Z^*, Z are pairs of dual finite-dimensional spaces, then the isomorphisms

$$\wedge Y^* \otimes \wedge Z^* \xrightarrow{\cong} \wedge X^* \qquad \text{and} \qquad \wedge Y \otimes \wedge Z \xrightarrow{\cong} \wedge X$$

satisfy

$$\langle \Phi \otimes \Psi, a \otimes b\rangle = \langle \Phi, a\rangle\langle \Psi, b\rangle = \langle \Phi \wedge \Psi, a \wedge b\rangle,$$
$$\Phi \in \wedge Y^*, \quad \Psi \in \wedge Z^*, \quad a \in \wedge Y, \quad b \in \wedge Z.$$

Finally, let $X = \sum_{p=1}^{r} X^p$ be an oddly positively graded space. Then

$$\wedge X = \wedge X^1 \otimes \cdots \otimes \wedge X^r.$$

Give $\wedge X$ the gradation defined by

$$(\wedge X)^p = \sum_{p_1+3p_3+\cdots+rp_r=p} (\wedge^{p_1} X^1) \otimes \cdots \otimes (\wedge^{p_r} X^r).$$

It is called the *induced gradation*, and makes $\wedge X$ into an alternating graded algebra.

If $\wedge X = \wedge(e_1, \ldots, e_r)$ and degree $e_i = g_i$, then, clearly $\wedge X = \wedge(e_1) \otimes \cdots \otimes \wedge(e_r)$ and so the Poincaré polynomial for $\wedge X$ is given by

$$f_{\wedge X} = \prod_{i=1}^{r} (1 + t^{g_i}).$$

0.5. Symmetric algebra. The symmetric algebra over a vector space X is denoted by $\vee X$ and the multiplication is denoted by $\vee$. If A is an associative algebra, and $\varphi: X \to A$ is a linear map, such that $\varphi x \cdot \varphi y = \varphi y \cdot \varphi x$, $x, y \in X$, then φ extends to a unique algebra homomorphism

$$\varphi_\vee: \vee X \to A.$$

A linear map $\psi: X \to \vee X$ extends uniquely to a derivation in $\vee X$.

Suppose $X = Y \oplus Z$. Then an isomorphism of vector spaces

$$\vee Y \otimes \vee Z \xrightarrow{\cong} \vee X$$

is defined by

$$a \otimes b \mapsto a \vee b.$$

More generally, let $X = \sum_{p=2}^{r} X^p$ be an *evenly* positively graded space. Set

$$(\vee X)^p = \sum_{2p_2 + \cdots + rp_r = p} (\vee^{p_2} X^2) \otimes \cdots \otimes (\vee^{p_r} X^r);$$

then $\vee X$ becomes a graded anticommutative algebra with respect to this *induced gradation*. (Note that $(\vee X)^p = 0$ if p is odd!) If X is the direct sum of graded subspaces Y and Z, then the isomorphism above is an isomorphism of graded algebras.

Suppose that A is an evenly graded commutative algebra, and that $Q \subset A$ is a graded subspace. Then the inclusion extends to a homomorphism of graded algebras

$$\vee Q \to A.$$

If this homomorphism is an isomorphism, we will write $\vee Q = A$. In particular, if $a_1, \ldots, a_r$ is a homogeneous basis of Q, we write

$$A = \vee(a_1, \ldots, a_r).$$

Since then $A = \vee(a_1) \otimes \cdots \otimes \vee(a_r)$, the Poincaré series for A is given by

$$f_A = \prod_{i=1}^{r} (1 - t^{g_i})^{-1}, \qquad \deg a_i = g_i.$$

Now suppose X^*, X is a pair of dual finite-dimensional vector spaces, and *assume that Γ has characteristic zero.* Then a scalar product between $\vee^p X^*$ and $\vee^p X$ is defined by

$$\langle x_1^* \vee \cdots \vee x_p^*, x_1 \vee \cdots \vee x_p\rangle = \operatorname{perm}(\langle x_i^*, x_j\rangle)$$
$$= \sum_{\sigma \in S^p} \langle x_1^*, x_{\sigma(1)}\rangle \cdot \cdots \cdot \langle x_p^*, x_{\sigma(p)}\rangle.$$

In particular we identify $\vee^p X^*$ with the space of symmetric p-linear functions in X by writing

$$\Psi(x_1, \ldots, x_p) = \langle \Psi, x_1 \vee \cdots \vee x_p\rangle, \qquad \Psi \in \vee^p X^*, \qquad x_i \in X.$$

If Y^*, Y is a second dual pair and $\varphi\colon X \to Y$, $\varphi^*\colon X^* \leftarrow Y^*$ are dual linear maps, then the homomorphisms

$$\varphi_\vee\colon \vee X \to \vee Y \qquad \text{and} \qquad (\varphi^*)_\vee\colon \vee X^* \leftarrow \vee Y^*$$

are dual as well. We write $(\varphi^*)_\vee = \varphi^\vee$.

The *substitution operator* $i_S(x)$ determined by $x \in X$ is the unique derivation in $\vee X^*$ satisfying

$$i_S(x)x^* = \langle x^*, x\rangle, \qquad x^* \in X^*.$$

Its dual is multiplication by x in $\vee X$ and is denoted by $\mu_S(x)$.

Finally, assume $X^* = Y^* \oplus Z^*$ and $X = Y \oplus Z$. Then

$$\langle \Phi \vee \Psi, a \vee b\rangle = \langle \Phi, a\rangle\langle \Psi, b\rangle = \langle \Phi \otimes \Psi, a \otimes b\rangle,$$
$$\Phi \in \vee Y^*, \quad \Psi \in \vee Z^*, \quad a \in \vee Y, \quad b \in \vee Z.$$

0.6. Poincaré duality algebras. A *Poincaré duality algebra* is a finite-dimensional positively graded associative algebra $A = \sum_{p=0}^n A^p$ subject to the following conditions:

(1) $\dim A^n = 1$.

(2) Let e^* be a basis vector of $(A^n)^*$. Then the bilinear functions

$$\langle\ ,\ \rangle\colon A^p \times A^{n-p} \to \Gamma$$

given by

$$\langle a, b\rangle = \langle e^*, ab\rangle, \qquad a \in A^p, \quad b \in A^{n-p},$$

are nondegenerate.

If A is a Poincaré duality algebra, then isomorphisms $D: A^p \xrightarrow{\cong} (A^{n-p})^*$ are given by

$$\langle Da, b\rangle = \langle e^*, ab\rangle.$$

D is called the associated *Poincaré isomorphism.*

Note that a Poincaré duality algebra A satisfies

$$\dim A^p = \dim A^{n-p}, \qquad p = 0, \ldots, n.$$

Examples: 1. *Exterior algebra*: Let E^*, E be dual n-dimensional vector spaces. Then $\wedge E^*$ is a Poincaré duality algebra, with Poincaré isomorphism $D: \wedge E^* \xrightarrow{\cong} \wedge E$ given by

$$D\Phi = i(\Phi)e,$$

where e is a basis vector of $\wedge^n E$.

2. *Tensor products*: Let A and B be finite-dimensional graded algebras. Then $A \otimes B$ is a Poincaré duality algebra if and only if both A and B are.

In fact, write $A = \sum_0^n A^p$ and $B = \sum_0^m B^q$, where $A^n \neq 0$ and $B^m \neq 0$. Then $A \otimes B = \sum_0^{n+m} (A \otimes B)^r$, and

$$\dim(A \otimes B)^{n+m} = \dim A^n \cdot \dim B^m.$$

Thus $\dim(A \otimes B)^{n+m} = 1$ if and only if $\dim A^n = 1 = \dim B^m$.

Now assume this condition holds. If $e^* \in (A^n)^*$ and $f^* \in (B^m)^*$ are basis vectors, then $e^* \otimes f^*$ is a basis vector for $[(A \otimes B)^{n+m}]^*$. It follows that the bilinear function in $A \otimes B$ is the tensor product of the corresponding bilinear functions for A and B (up to sign). In particular, it is nondegenerate if and only if both of them are nondegenerate.

0.7. Differential spaces. A *differential space* (X, δ) is a vector space X together with a linear transformation δ of X such that $\delta^2 = 0$; δ is called the *differential operator*. We write

$$\ker \delta = Z(X), \qquad \operatorname{Im} \delta = B(X), \qquad Z(X)/B(X) = H(X, \delta)$$
$$\text{(or simply } H(X))$$

and call these spaces, respectively, the *cocycle*, *coboundary*, and *cohomology spaces* of X.

A *homomorphism* $\varphi: (X, \delta_X) \to (Y, \delta_Y)$ *of differential spaces* is a linear map $\varphi: X \to Y$ such that $\varphi\delta_X = \delta_Y\varphi$. It restricts to maps between the cocycle and coboundary spaces, and so induces a map, written

$$\varphi^{\#}: H(X) \to H(Y),$$

between the cohomology spaces.

Assume (X^*, δ) and (X, ∂) are finite-dimensional differential spaces such that X^*, X is a pair of dual spaces, and $\delta = \pm\partial^*$. Then the scalar product induces a scalar product between $H(X^*)$ and $H(X)$.

A *graded differential space* (X, δ) is a differential space together with a gradation in X, such that δ is homogeneous of some degree. In this case $Z(X)$ and $B(X)$ are graded subspaces

$$Z(X) = \sum_p Z^p(X) \qquad \text{and} \qquad B(X) = \sum_p B^p(X).$$

The space $H(X)$ is then graded; we write

$$H(X) = \sum_p H^p(X); \qquad H^p(X) = Z^p(X)/B^p(X).$$

Suppose (X, δ_X) and (Y, δ_Y) are graded differential spaces. A *homomorphism of graded differential spaces* is a homomorphism of graded spaces $\varphi: X \to Y$ such that $\varphi\delta_X = \delta_Y\varphi$.

Assume (X, δ) is a finite-dimensional graded differential space. Then the *Euler–Poincaré formula* asserts that $\chi_X = \chi_{H(X)}$ (cf. sec. 0.2); i.e.

$$\sum_p (-1)^p \dim X^p = \sum_p (-1)^p \dim H^p(X).$$

Let

$$0 \longrightarrow (W, \delta_W) \xrightarrow{\varphi} (X, \delta_X) \xrightarrow{\psi} (Y, \delta_Y) \longrightarrow 0$$

be an exact sequence of differential spaces. Then a linear map

$$\partial: H(Y) \to H(W)$$

is defined as follows: Let $\alpha \in H(Y)$ and choose $x \in X$ so that ψx represents α. Then there is a unique cocycle $w \in W$ such that $\varphi w = \delta_X x$. The class $\beta \in H(W)$ represented by w is independent of the choice of x, and ∂ is given by $\partial\alpha = \beta$. ∂ is called the *connecting homomorphism.*

Elementary algebra shows that the triangle

$$\begin{tikzcd} H(W) \arrow[rr, "\varphi^*"] & & H(X) \arrow[dl, "\psi^*"] \\ & H(Y) \arrow[ul, "\partial"] & \end{tikzcd}$$

is exact. If the differential spaces are graded differential spaces, and φ and ψ are homogeneous of degrees k and l, then ∂ is homogeneous and

$$\deg \partial = \deg \delta_X - k - l.$$

In this case we obtain a long exact sequence ($m = \deg \partial$)

$$\longrightarrow H^p(Y) \xrightarrow{\partial} H^{p+m}(W) \xrightarrow{\varphi^*} H^{p+m+k}(X) \xrightarrow{\psi^*} H^{p+m+k+l}(Y) \longrightarrow .$$

Let (X, δ_X) and (Y, δ_Y) be graded differential spaces where δ_X and δ_Y are homogeneous of the same odd degree k. Then their *tensor product* is the graded differential space $(X \otimes Y, \delta_{X\otimes Y})$ given by

$$\delta_{X\otimes Y}(x \otimes y) = \delta_X x \otimes y + (-1)^p x \otimes \delta_Y y, \qquad x \in X^p, \quad y \in Y.$$

We will often write

$$(X \otimes Y, \delta_{X\otimes Y}) = (X, \delta_X) \otimes (Y, \delta_Y).$$

Consider the inclusion

$$Z(X) \otimes Z(Y) \to Z(X \otimes Y).$$

It induces the (algebraic) *Künneth isomorphism*

$$H(X) \otimes H(Y) \xrightarrow{\cong} H(X \otimes Y).$$

0.8. Differential algebras. A *graded differential algebra* (R, δ_R) is a positively graded associative algebra R with identity, together with an antiderivation δ_R, homogeneous of degree 1 and satisfying $\delta_R^2 = 0$. If (R, δ_R) is a graded differential algebra, then $Z(R)$ is a graded subalgebra and $B(R)$ is a graded ideal in $Z(R)$. Thus $H(R)$ becomes a graded algebra.

A ***homomorphism*** $\varphi\colon (R, \delta_R) \to (S, \delta_S)$ of ***graded differential algebras*** is a homomorphism of graded algebras which satisfies $\varphi\delta_R = \delta_S\varphi$. The induced map $\varphi^\#$ is then a homomorphism of graded algebras.

A positively graded differential algebra (R, δ_R) is called c-*connected* if (cf. sec. 0.3) the algebra $H(R)$ is connected.

Finally, assume (R, δ_R) and (S, δ_S) are graded differential algebras. Then so is $(R, \delta_R) \otimes (S, \delta_S)$. Moreover, inclusions

$$j_R\colon (R, \delta_R) \to (R, \delta_R) \otimes (S, \delta_S) \quad \text{and} \quad j_S\colon (S, \delta_S) \to (R, \delta_R) \otimes (S, \delta_S)$$

are given by $j_R(x) = x \otimes 1$ and $j_S(y) = 1 \otimes y$. In this case the Künneth isomorphism is the isomorphism of graded algebras given by

$$\alpha \otimes \beta \mapsto j_R^\#(\alpha) \cdot j_S^\#(\beta), \qquad \alpha \in H(R), \quad \beta \in H(S).$$

Remark: In the literature it is usual to regard cohomology as a contravariant functor and homology as a covariant functor; thus "the cohomology of a topological space is the homology of its cochain complex." Whatever the aesthetic advantages of this convention, it would, in this book, lead to a great deal of artificial lowering and raising of indices.

For example (cf. Chapter V), for a Lie algebra E we would have

$$H^p(E) = H_p(\wedge E^*, \delta_E),$$

while for a manifold M we would have

$$H^p(M) = H_p(A(M), \delta_M),$$

and for a smooth map $\varphi\colon M \to N$ inducing $\varphi^*\colon A(M) \leftarrow A(N)$ we would have

$$(\varphi^*)_\# = \varphi^\#\colon H(M) \leftarrow H(N).$$

For this reason we have arbitrarily declared all differential spaces (with a single exception in sec. 5.5) to have cocycles, coboundaries, and cohomology, and used the notation $\varphi^\#\colon H^p(R) \to H^p(S)$ for the map induced by a map $\varphi\colon R \to S$.

0.9. *n*-regularity. A homomorphism $\varphi\colon X \to Y$ of positively graded spaces is called ***n-regular*** if the restrictions $\varphi^p\colon X^p \to Y^p$ satisfy: φ^p is an isomorphism, $0 \le p \le n$, and φ^{n+1} is injective.

Proposition II: Let $\varphi: (X, \delta_X) \to (Y, \delta_Y)$ be an n-regular homomorphism of positively graded differential spaces. Assume δ_X and δ_Y are homogeneous of degree 1. Then $\varphi^\#$ is n-regular.

Proof: It follows at once from the hypotheses that

$$\varphi: Z^p(X) \xrightarrow{\cong} Z^p(Y), \quad 0 \le p \le n$$

and

$$\varphi: B^p(X) \xrightarrow{\cong} B^p(Y), \quad 0 \le p \le n+1.$$

Since $\varphi: Z^{n+1}(X) \to Z^{n+1}(Y)$ is injective, the proposition follows.

Q.E.D.

0.10. c-equivalent differential algebras. Let (R, δ_R) and (S, δ_S) be graded alternating differential algebras. Then a *cohomological relation* (c-relation) from (R, δ_R) to (S, δ_S) is a homomorphism

$$\varphi: (R, \delta_R) \to (S, \delta_S)$$

of graded differential algebras, such that $\varphi^\#$ is an isomorphism.

If such a c-relation exists, we say that (R, δ_R) is c-*related* to (S, δ_S), and write

$$(R, \delta_R) \xrightarrow[c]{} (S, \delta_S).$$

Note that this relation is not an equivalence. However, it *generates* an equivalence relation in the following way:

Definition: Two alternating graded differential algebras (R, δ_R) and (S, δ_S) are called *cohomologically equivalent* (c-*equivalent*) if there is a sequence $(X_i, \delta_i), i = 1, \ldots, n$ of alternating graded differential algebras, satisfying the following properties:

(1) $(X_1, \delta_1) = (R, \delta_R)$ and $(X_n, \delta_n) = (S, \delta_S)$.

(2) Either $(X_i, \delta_i) \xrightarrow[c]{} (X_{i+1}, \delta_{i+1})$ or $(X_{i+1}, \delta_{i+1}) \xrightarrow[c]{} (X_i, \delta_i)$ $(i = 1, \ldots, n)$.

In this case we write

$$(R, \delta_R) \underset{c}{\sim} (S, \delta_S).$$

A specific choice of the (X_i, δ_i), together with a specific choice of c-relations φ_i between them will be called a c-*equivalence* between (R, δ_R) and (S, δ_S). The isomorphisms $\varphi_i^{\#}$ determine an isomorphism

$$H(R) \xrightarrow{\cong} H(S),$$

which will be called the *isomorphism induced by the given* c-*equivalence.*

An alternating graded differential algebra (R, δ_R) is called *split* if there exists a homomorphism of graded algebras $\varphi\colon H(R) \to Z(R)$ which splits the exact sequence

$$0 \to B(R) \to Z(R) \to H(R) \to 0.$$

Thus, if (R, δ_R) is split, then $(H(R), 0) \xrightarrow[\mathrm{c}]{} (R, \delta_R)$.

More generally, (R, δ_R) will be called *cohomologically split* (c-split), if

$$(R, \delta_R) \underset{\mathrm{c}}{\sim} (H(R), 0).$$

In this case the c-equivalence can always be chosen so that the induced isomorphism of $H(R)$ is the identity. Such a c-equivalence will be called a c-*splitting.*

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PART 1

In this part Γ denotes a commutative field of arbitrary characteristic. All vector spaces and algebras are defined over Γ.

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Chapter I

Spectral Sequences

Most of the results (and proofs) in this chapter continue to hold if Γ, instead of being a field, is allowed to be an arbitrary commutative ring with identity.

§1. Filtrations

1.1. Filtered spaces. A *decreasing filtration* of a vector space M is a family of subspaces $F^p(M)$, indexed by the integers $p \in \mathbb{Z}$, and satisfying the conditions

$$F^p(M) \supset F^{p+1}(M) \qquad \text{and} \qquad M = \bigcup_p F^p(M).$$

(In this book we consider only decreasing filtrations.) The definition of $F^p(M)$ is extended to $p = \pm\infty$ by writing

$$F^{-\infty}(M) = M \qquad \text{and} \qquad F^{\infty}(M) = 0.$$

With a filtered vector space M is associated its *associated graded space* $A_M = \bigoplus_p A_M^p$, defined by

$$A_M^p = F^p(M)/F^{p+1}(M), \qquad -\infty < p < \infty.$$

The canonical projection $F^p(M) \to A_M^p$ is denoted by ϱ_M^p or simply by ϱ^p.

Suppose M and $\tilde{M}$ are filtered vector spaces. A linear map $\varphi: M \to \tilde{M}$ is called *filtration preserving* (or a *homomorphism of filtered spaces*) if it restricts to linear maps $\varphi: F^p(M) \to F^p(\tilde{M})$. In this case φ induces a

unique linear map $\varphi_A\colon A_M \to A_{\tilde{M}}$ of graded spaces such that the diagrams

$$\begin{tikzcd} F^p(M) \arrow[r, "\varphi"] \arrow[d, "\varrho^p"'] & F^p(\tilde{M}) \arrow[d, "\tilde{\varrho}^p"] \\ A_M^p \arrow[r, "\varphi_A"'] & A_{\tilde{M}}^p, \end{tikzcd} \qquad -\infty < p < \infty,$$

commute.

If $M = \sum_p M^p$ is a *graded* vector space, then by setting $F^p(M) = \sum_{\mu \geq p} M^\mu$ we obtain a filtration of M. This filtration is said to be *induced by the gradation.* In this case ϱ^p restricts to an isomorphism $M^p \xrightarrow{\cong} A_M^p$, and these isomorphisms define an isomorphism $M \xrightarrow{\cong} A_M$ of graded spaces.

1.2. Filtered differential spaces. A *filtered differential space* is a differential space (M, δ), together with a filtration $\{F^p(M)\}_{p \in Z}$ of M such that the subspaces $F^p(M)$ are stable under δ. If (M, δ) is a filtered differential space, then a filtration of $H(M)$ is defined by

$$F^p(H(M)) = \pi(F^p(M) \cap Z(M)),$$

where $\pi\colon Z(M) \to H(M)$ denotes the canonical projection. Thus we can form the associated graded space $A_{H(M)}$.

Consider the associated graded space A_M. Since the $F^p(M)$ are stable under δ, an operator δ_A in A_M is induced by δ. Clearly (A_M, δ_A) is a differential space, and δ_A is homogeneous of degree zero.

1.3. The differential spaces (E_i, d_i). Let (M, δ) be a filtered differential space. Define subspaces $Z_i^p \subset M$ by

$$Z_i^p = F^p(M) \cap \delta^{-1}(F^{p+i}(M)), \qquad -\infty < p < \infty, \quad 0 \leq i < \infty,$$

and

$$Z_\infty^p = F^p(M) \cap Z(M).$$

It follows from the definition that

$$Z_i^{p+1} \subset Z_{i+1}^p \subset Z_i^p.$$

Next, define subspaces D_i^p by

$$D_i^p = F^p(M) \cap \delta(F^{p-i}(M)), \qquad -\infty < p < \infty, \quad 0 \leq i < \infty,$$

and

$$D_\infty^p = F^p(M) \cap \delta(M).$$

It follows that $D_i^p \subset D_{i+1}^p \subset D_i^{p-1}$. Since $D_\infty^p \subset Z_\infty^p$, we have the inclusion relations

$$D_0^p \subset D_1^p \subset \cdots \subset D_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_1^p \subset Z_0^p, \qquad -\infty < p < \infty.$$

Now form the factor spaces

$$E_i^p = Z_i^p/(Z_{i-1}^{p+1} + D_{i-1}^p), \qquad 1 \leq i \leq \infty,$$

and extend the definition to $i = 0$ by setting

$$E_0^p = Z_0^p/(F^{p+1}(M) + \delta(F^{p+1}(M))) = A_M^p.$$

The canonical projections of Z_i^p onto E_i^p will be denoted by $\eta_i^p: Z_i^p \to E_i^p$.

It is easy to verify that

$$\delta: Z_i^p \to Z_i^{p+i} \qquad \text{and} \qquad \delta: \ker \eta_i^p \to \ker \eta_i^{p+i}, \qquad 0 \leq i < \infty.$$

Hence, if i is finite, a linear map $d_i^p: E_i^p \to E_i^{p+i}$ is defined by the commutative diagram

$$\begin{tikzcd} Z_i^p \arrow[r, "\delta"] \arrow[d, "\eta_i^p"'] & Z_i^{p+i} \arrow[d, "\eta_i^{p+i}"] \\ E_i^p \arrow[r, "d_i^p"'] & E_i^{p+i}, \end{tikzcd} \qquad -\infty < p < \infty.$$

It is clear that $d_i^{p+i} \circ d_i^p = 0$.

Now consider the direct sums $E_i = \bigoplus_p E_i^p$. If $0 \leq i < \infty$, the operators d_i^p define a differential operator d_i, homogeneous of degree i in the vector space E_i. Hence we can form the vector spaces

$$H(E_i, d_i) = \ker d_i/\operatorname{Im} d_i.$$

Since each vector space E_i is graded by the subspaces E_i^p, an induced

gradation in $H(E_i, d_i)$ is given by

$$H^p(E_i, d_i) = \ker d_i^p/\operatorname{Im} d_i^{p-i}.$$

The projections $\ker d_i^p \to H^p(E_i, d_i)$ will be denoted by π_i^p.

1.4. The spaces E_0 and E_∞. Proposition I: (1) The graded differential space (E_0, d_0) coincides with (A_M, δ_A).

(2) The graded space E_∞ is isomorphic to the associated graded space of $H(M)$ with respect to the filtration induced by the filtration of M.

Proof: (1) follows immediately from the definition. To prove (2) observe that

$$\ker \eta_\infty^p = Z_\infty^{p+1} + D_\infty^p.$$

On the other hand (cf. sec. 1.2), the filtration of $H(M)$ is given by

$$F^p(H(M)) = \pi(Z_\infty^p).$$

Since (clearly) $\pi(D_\infty^p) = 0$, we have

$$F^{p+1}(H(M)) = \pi(Z_\infty^{p+1} + D_\infty^p).$$

Hence a surjective linear map $\sigma_\infty^p : E_\infty^p \to A_{H(M)}^p$ is defined by the commutative diagram

$$\begin{tikzcd} Z_\infty^p \arrow[r, "\pi"] \arrow[d, "\eta_\infty^p"] & F^p(H(M)) \arrow[d, "\varrho_{H(M)}^p"] \\ E_\infty^p \arrow[r, "\sigma_\infty^p"] & A_{H(M)}^p. \end{tikzcd}$$

But, evidently

$$\begin{aligned} \ker \sigma_\infty^p &= \eta_\infty^p(\pi^{-1}(F^{p+1}(H(M))) \cap Z_\infty^p) \\ &= \eta_\infty^p((Z_\infty^{p+1} + \delta(M)) \cap Z_\infty^p) \\ &= \eta_\infty^p(Z_\infty^{p+1} + D_\infty^p) = 0. \end{aligned}$$

It follows that σ_∞^p is an isomorphism.

Q.E.D.

1.5. Homomorphisms. Let $\varphi: (M, \delta) \to (\tilde{M}, \tilde{\delta})$ be a homomorphism of filtered differential spaces. Then the subspaces Z_i^p, D_i^p are mapped into the subspaces $\tilde{Z}_i^p$, $\tilde{D}_i^p$. Hence linear maps $\varphi_i^p: E_i^p \to \tilde{E}_i^p$ are given by the commutative diagrams

$$\begin{tikzcd} Z_i^p \arrow[r, "\varphi"] \arrow[d, "\eta_i^p"] & \tilde{Z}_i^p \arrow[d, "\eta_i^p"] \\ E_i^p \arrow[r, "\varphi_i^p"] & \tilde{E}_i^p, \end{tikzcd} \qquad 0 \le i \le \infty.$$

The φ_i^p define linear maps $\varphi_i: E_i \to \tilde{E}_i$. A simple computation shows that $\varphi_i d_i = \tilde{d}_i \varphi_i$; i.e., φ_i is a homomorphism of graded differential spaces. Thus φ_i induces a linear map

$$\varphi_i^{\#}: H(E_i, d_i) \to H(\tilde{E}_i, \tilde{d}_i) \qquad (0 \le i < \infty)$$

homogeneous of degree zero.

Proposition II: If $\varphi: M \to \tilde{M}$ is a homomorphism of filtered differential spaces, then the linear map $\varphi^{\#}: H(M) \to H(\tilde{M})$ preserves the filtration (the filtration being defined in sec. 1.2).

Moreover, if $(\varphi^{\#})_A: A_{H(M)} \to A_{H(\tilde{M})}$ is the induced homomorphism of the associated graded spaces, then the diagram

$$\begin{tikzcd} A_{H(M)} \arrow[r, "(\varphi^{\#})_A"] \arrow[d, "\sigma_\infty"', "\cong"] & A_{H(\tilde{M})} \arrow[d, "\cong"', "\tilde{\sigma}_\infty"] \\ E_\infty \arrow[r, "\varphi_\infty"'] & \tilde{E}_\infty \end{tikzcd}$$

commutes.

Proof: Since

$$\varphi^{\#}(F^p(H(M))) = \varphi^{\#}(\pi(Z_\infty^p)) = \tilde{\pi}(\varphi(Z_\infty^p)) \subset \tilde{\pi}(\tilde{Z}_\infty^p) = F^p(H(\tilde{M})),$$

$\varphi^{\#}$ is filtration preserving.

Now consider the diagram

$$\begin{tikzcd}
Z^p_\infty \arrow[r, "\pi"] \arrow[drr, "\eta^p_\infty"] \arrow[dd, "\varphi"'] & F^p(H(M)) \arrow[rr, "\varrho^p_{H(M)}"] & & A^p_{H(M)} \arrow[dd, "(\varphi^\#)_A"] \\
& & E^p_\infty \arrow[ur, "\sigma^p_\infty", "\cong"'] \arrow[dd, "\varphi^p_\infty"] & \\
\tilde{Z}^p_\infty \arrow[r, "\tilde{\pi}"] \arrow[drr, "\tilde{\eta}^p_\infty"'] & F^p(H(\tilde{M})) \arrow[rr, "\varrho^p_{H(\tilde{M})}"] & & A^p_{H(\tilde{M})} \\
& & \tilde{E}^p_\infty \arrow[ur, "\tilde{\sigma}^p_\infty"'] &
\end{tikzcd}$$

It follows from the definition of $\varphi^\#$ and $(\varphi^\#)_A$ that the back face commutes. The definition of σ^p_∞ (sec. 1.4) implies that the top and bottom faces commute. The definition of φ^p_∞ (above) shows that the left-hand face commutes. Since $\varrho^p_{H(M)} \circ \pi$ is surjective so is η^p_∞, and the right-hand face must commute.

Q.E.D.

§2. Spectral sequences

1.6. The spectral sequence of a filtered differential space. Definition: *A spectral sequence* is a sequence (E_i, d_i, σ_i), $m \leq i < \infty$, where (E_i, d_i) is a differential space, and σ_i is an isomorphism of E_{i+1} onto $H(E_i, d_i)$. We often omit the σ_i from the notation, and refer simply to the spectral sequence (E_i, d_i).

A homomorphism of spectral sequences

$$\alpha: (E_i, d_i, \sigma_i) \to (\tilde{E}_i, \tilde{d}_i, \tilde{\sigma}_i)$$

is a system of homomorphisms of differential spaces $\alpha_i: (E_i, d_i) \to (\tilde{E}_i, \tilde{d}_i)$, such that the isomorphisms σ_i and $\tilde{\sigma}_i$ identify α_{i+1} with $\alpha_i^{\#}$:

$$\alpha_i^{\#}\sigma_i = \tilde{\sigma}_i\alpha_{i+1}.$$

A spectral sequence (E_i, d_i, σ_i) is said to *collapse at the* kth *term* if $d_i = 0$, $i \geq k$. In this case $H(E_i, d_i) = E_i$, $i \geq k$, and so σ_i is an isomorphism from E_{i+1} to E_i.

Let (M, δ) be a filtered differential space, and consider the sequence $(E_i, d_i)_{i \geq 0}$ of differential spaces constructed in sec. 1.3. We shall now construct isomorphisms

$$\sigma_i: E_{i+1} \xrightarrow{\cong} H(E_i, d_i), \qquad i \geq 0,$$

of graded spaces. The resulting spectral sequence (E_i, d_i, σ_i) will be called the *spectral sequence of the filtered differential space* (M, δ).

The σ_i are defined as follows: Consider the projection $\eta_i^p: Z_i^p \to E_i^p$ (cf. sec. 1.3). In Lemma I below we show that

$$\eta_i^p(Z_{i+1}^p) = \ker d_i^p.$$

Thus composing η_i^p (restricted to Z_{i+1}^p) with the projection $\pi_i^p: \ker d_i^p \to H^p(E_i, d_i)$ yields a surjective linear map

$$\gamma_i^p: Z_{i+1}^p \to H^p(E_i, d_i).$$

Evidently, $\ker \gamma_i^p = Z_{i+1}^p \cap ((\eta_i^p)^{-1}(\operatorname{Im} d_i^{p-i}))$. In Lemma II, below, we show that this space coincides with $Z_i^{p+1} + D_i^p$. It follows that γ_i^p induces an isomorphism

$$\sigma_i^p\colon E_{i+1}^p \xrightarrow{\cong} H^p(E_i, d_i).$$

The σ_i^p define the desired isomorphism σ_i.

Remark: In view of Proposition I, (1), sec. 1.4, σ_0 is an isomorphism:

$$\sigma_0\colon E_1 \xrightarrow{\cong} H(A_M, \delta_A).$$

Lemma I: $\eta_i^p(Z_{i+1}^p) = \ker d_i^p$.

Proof: We show first that

$$(\eta_i^p)^{-1}(\ker d_i^p) = Z_{i+1}^p + Z_{i-1}^{p+1}. \tag{1.1}$$

Fix $z \in Z_i^p$. Since $d_i^p \eta_i^p = \eta_i^{p+i}\delta$, it follows that $d_i^p \eta_i^p z = 0$ if and only if

$$\delta z \in F^{p+i+1}(M) + D_{i-1}^{p+i}.$$

But this occurs if and only if $z \in Z_{i+1}^p + Z_{i-1}^{p+1}$, as follows at once from the definitions. Thus (1.1) is proved.

Now apply η_i^p to (1.1) to obtain

$$\ker d_i^p = \eta_i^p(Z_{i+1}^p) + \eta_i^p(Z_{i-1}^{p+1}) = \eta_i^p(Z_{i+1}^p).$$

Q.E.D.

Lemma II: $Z_i^{p+1} + D_i^p = Z_{i+1}^p \cap [(\eta_i^p)^{-1}(\operatorname{Im} d_i^{p-i})]$.

Proof: It follows at once from the definitions that

$$\operatorname{Im} d_i^{p-i} = \eta_i^p \delta(Z_i^{p-i}) = \eta_i^p(D_i^p).$$

Hence

$$\begin{aligned}(\eta_i^p)^{-1}(\operatorname{Im} d_i^{p-i}) &= D_i^p + \ker \eta_i^p \\ &= D_i^p + D_{i-1}^p + Z_{i-1}^{p+1} \\ &= D_i^p + Z_{i-1}^{p+1}. \end{aligned} \tag{1.2}$$

On the other hand, an easy calculation yields

$$Z_{i-1}^{p+1} \cap Z_{i+1}^{p} = Z_i^{p+1} \qquad \text{and} \qquad D_i^p \subset Z_{i+1}^p .$$

Thus intersecting (1.2) with Z_{i+1}^p yields the lemma.

Q.E.D.

Proposition III: Let $\varphi\colon (M, \delta) \to (\tilde{M}, \tilde{\delta})$ be a homomorphism of filtered differential spaces. Then the maps

$$\varphi_i\colon (E_i, d_i) \to (\tilde{E}_i, \tilde{d}_i)$$

of sec. 1.5 form a homomorphism of the corresponding spectral sequences.

Proof: It has to be shown that $\tilde{\sigma}_i \varphi_{i+1} = \varphi_i^{\#} \sigma_i$, $i \geq 0$. Consider the diagram

$$\begin{tikzcd}
Z_{i+1}^p \arrow[rrr, "\eta_i^p"] \arrow[dr, "\varphi"] \arrow[ddd, "\eta_{i+1}^p"'] & & & \ker d_i^p \arrow[dl, "\varphi_i"'] \arrow[ddd, "\pi_i^p"] \\
& \tilde{Z}_{i+1}^p \arrow[r, "\tilde{\eta}_i^p"] \arrow[d, "\tilde{\eta}_{i+1}^p"'] & \ker \tilde{d}_i^p \arrow[d, "\tilde{\pi}_i^p"] & \\
& \tilde{E}_{i+1}^p \arrow[r, "\cong", "\tilde{\sigma}_i"'] & H^p(\tilde{E}_i) & \\
E_{i+1}^p \arrow[ur, "\varphi_{i+1}"] \arrow[rrr, "\cong", "\sigma_i"'] & & & H^p(E_i). \arrow[ul, "\varphi_i^{\#}"']
\end{tikzcd}$$

The outside and inside squares commute, as follows from the definition of σ_i and $\tilde{\sigma}_i$. The left, right, and upper faces commute, as follows from the definitions of φ_{i+1}, $\varphi_i^{\#}$, and φ_i. Since η_{i+1}^p is surjective the lower face must commute.

Q.E.D.

1.7. Filtrations induced by a gradation. Let $M = \sum_{p \in \mathbb{Z}} M^p$ be a differential space, and consider the induced filtration

$$F^p(M) = \sum_{\mu \geq p} M^{\mu}.$$

Assume that for some $k \geq 0$,

$$\delta\colon M^p \to F^{p+k}(M), \qquad p \in \mathbb{Z}.$$

Then δ can be written uniquely as $\delta = D + \hat{\delta}$, where;

$$D(M^p) \subset M^{p+k} \qquad \text{and} \qquad \hat{\delta}(M^p) \subset F^{p+k+1}(M), \qquad p \in \mathbb{Z}. \tag{1.3}$$

It is easy to see that $D^2 = 0$. As an immediate consequence of formula (1.3), we have the relations

$$Z_i^p = F^p(M), \qquad i \leq k,$$

and

$$D_i^p = \delta(F^{p-i}(M)) \subset F^{p-i+k}(M) \subset F^{p+1}(M), \qquad i \leq k-1.$$

This shows that $\ker \eta_i^p = F^{p+1}(M)$, $i \leq k$, whence

$$Z_i^p = M^p \oplus \ker \eta_i^p, \qquad i \leq k.$$

It follows that for $i \leq k$ the inclusion map $j\colon M^p \to F^p(M)$ can be composed with η_i^p to yield an isomorphism $\xi_i^p\colon M^p \xrightarrow{\cong} E_i^p$. The isomorphisms ξ_i^p define isomorphisms of graded spaces

$$\xi_i\colon M \xrightarrow{\cong} E_i, \qquad 0 \leq i \leq k.$$

It is immediate from the definitions that $d_i = 0$, $0 \leq i < k$. Thus for $0 \leq i < k$ we can regard ξ_i as an isomorphism

$$\xi_i\colon (M, 0) \xrightarrow{\cong} (E_i, d_i), \tag{1.4}$$

of graded differential spaces. Next we show that $\xi_k D = d_k \xi_k$, so that

$$\xi_k\colon (M, D) \xrightarrow{\cong} (E_k, d_k) \tag{1.5}$$

is an isomorphism of graded differential spaces.

In fact, fix $z \in M^p$. Then formula (1.3) yields

$$\eta_k^{p+k}\delta z = \eta_k^{p+k} D z = \xi_k D z.$$

It follows that

$$d_k \xi_k z = d_k \eta_k^p z = \eta_k^{p+k} \delta z = \xi_k D z,$$

and so (1.5) is proved.

Formula (1.5) yields an isomorphism $\xi_k^\# : H(M, D) \xrightarrow{\cong} H(E_k, d_k)$. Combining $\xi_k^\#$ with σ_k^{-1}, we obtain an isomorphism

$$\alpha = \sigma_k^{-1} \circ \xi_k^\# : H(M, D) \xrightarrow{\cong} E_{k+1}. \tag{1.6}$$

As a straightforward consequence of Lemmas I and II (sec. 1.6) or by direct computation we now obtain the formulae

$$Z_{k+1}^p = Z^p(M, D) \oplus F^{p+1}(M)$$

and (1.7)

$$Z_k^{p+1} + D_k^p = D(M^{p-k}) \oplus F^{p+1}(M).$$

Next, assume that $\tilde{M} = \sum_{p \in \mathbb{Z}} \tilde{M}^p$ is a second differential space and let it have the induced filtration. Moreover, assume that $\tilde{\delta}(\tilde{M}^p) \subset F^{p+k}(\tilde{M})$, $p \in \mathbb{Z}$, and let $\tilde{D}: \tilde{M}^p \to \tilde{M}^{p+k}$ be defined as above. Assume that $\varphi: M \to \tilde{M}$ is a homomorphism of graded spaces which satisfies $\varphi \delta = \tilde{\delta} \varphi$. Then

$$\varphi D = \tilde{D} \varphi,$$

and so φ determines a linear map

$$\varphi_D^\# : H(M, D) \to H(\tilde{M}, \tilde{D}).$$

On the other hand, φ preserves the filtrations and so it induces a homomorphism of spectral sequences

$$\varphi_i : (E_i, d_i) \to (\tilde{E}_i, \tilde{d}_i), \qquad i \geq 0.$$

It follows from the definitions that

$$\varphi_i \circ \xi_i = \tilde{\xi}_i \circ \varphi, \qquad 0 \leq i \leq k,$$

where $\tilde{\xi}_i : \tilde{M} \xrightarrow{\cong} \tilde{E}_i$ is the induced isomorphism.

Passing to cohomology and using Proposition III we obtain the relations

$$\varphi_k^\# \circ \xi_k^\# = \tilde{\xi}_k^\# \circ \varphi_D^\# \qquad \text{and} \qquad \varphi_k^\# \circ \sigma_k = \tilde{\sigma}_k \circ \varphi_{k+1}.$$

Thus the diagram

$$
\begin{tikzcd}
H(M, D) \arrow[r, "\alpha", "\cong"'] \arrow[d, "\varphi_D^*"'] & E_{k+1} \arrow[d, "\varphi_{k+1}"] \\
H(\tilde{M}, \tilde{D}) \arrow[r, "\cong", "\alpha"'] & \tilde{E}_{k+1}
\end{tikzcd}
\tag{1.8}
$$

commutes.

1.8. The homogeneous case. Suppose now that (M, δ) is as above but that δ is homogeneous of degree k. Then $D = \delta$. It follows that the spectral sequence collapses at the $(k + 1)$th term.

In fact, the homogeneity of δ implies that $Z_i^p = Z_\infty^p + F^{p+i-k}$, $i \geq k + 1$, and hence for $z \in Z_i^p$ $(i \geq k + 1)$,

$$d_i \eta_i^p z = \eta_i^{p+i} \delta z = 0.$$

Since η_i^p is surjective, this implies that $d_i = 0$, $i \geq k + 1$.

Since $D = \delta$ it follows from formula (1.5), sec. 1.7, that there are natural isomorphisms

$$H(M, \delta) \cong H(E_k, d_k) \cong E_{k+1} \cong E_{k+2} \cong \cdots;$$

i.e.,

$$E_i \cong H(M, \delta), \qquad k < i < \infty.$$

It is simple to show that this relation holds for $i = \infty$ as well.

§3. Graded filtered differential spaces

1.9. Graded filtered spaces. Let $M = \sum_{r \geq 0} M^r$ be a graded space which is filtered by subspaces $F^p(M)$, $p \in \mathbb{Z}$. The filtration of M induces the filtration of the subspaces M^r given by

$$F^p(M^r) = F^p(M) \cap M^r.$$

We will call the filtration of M *compatible with the gradation* if the $F^p(M)$ are *graded* subspaces; i.e., if

$$F^p(M) = \sum_{r \geq 0} F^p(M^r). \tag{1.9}$$

It is standard practice to call such spaces graded filtered spaces. However, in this book a *graded filtered space* will mean a graded space $M = \sum_{r \geq 0} M^r$ filtered by subspaces $F^p(M)$ such that (1.9) holds, *and in addition*

$$F^p(M^r) = 0 \qquad \text{if} \quad p > r. \tag{1.10}$$

Let M be a graded filtered space, and (cf. sec. 1.1) consider the associated graded space A_M. Since $F^p(M)$ and $F^{p+1}(M)$ are graded subspaces of M, a gradation is naturally induced in each A_M^p:

$$A_M^p = \sum_r \varrho^p(F^p(M^r)) = \sum_r F^p(M^r)/F^{p+1}(M^r).$$

We write

$$A^{p,r-p} = \varrho^p(F^p(M^r)).$$

Then

$$A_M^p = \sum_q A_M^{p,q} \qquad \text{and} \qquad A_M = \sum_{p,q} A_M^{p,q},$$

and so A_M is a bigraded space. If $x \in A_M^{p,q}$, we say it has *base* or *filtration degree* p, *fibre degree* q and *total degree* $p + q$. Observe that condition (1.10) implies that $A_M^{p,q} = 0$ if $q < 0$, and that $A_M^{p,0} = F^p(M^p)$.

The total gradation of A_M is given by

$$A_M = \sum_r A_M^{(r)}, \qquad A_M^{(r)} = \sum_{p+q=r} A_M^{p,q}.$$

Note that $A_M^{(r)}$ is the associated graded space of the filtered space M^r.

Suppose $\tilde{M}$ is a second graded filtered space and let $\varphi: M \to \tilde{M}$ be a linear map, homogeneous of degree zero, which preserves the filtration. Then φ is called a *homomorphism of graded filtered spaces*. Every such homomorphism induces a linear map $\varphi_A: A_M \to A_{\tilde{M}}$ (cf. sec. 1.1) which preserves the bigradation.

1.10. Graded filtered differential spaces. Let (M, δ) be a graded differential space and assume that δ is homogeneous of degree 1. Assume further that a filtration is given in M which makes M into a graded filtered space and into a filtered differential space. Then M is called a *graded filtered differential space*. Clearly the induced filtration and gradation on $H(M)$ make $H(M)$ into a graded filtered space.

Now consider the subspaces Z_i^p and D_i^p defined in sec. 1.3. It follows from the homogeneity of δ and formula (1.9) that Z_i^p and D_i^p are graded subspaces of M:

$$Z_i^p = \sum_{r\geq p} Z_i^p \cap M^r, \qquad D_i^p = \sum_{r\geq p} D_i^p \cap M^r.$$

(Note that we have used condition (1.10).) Set

$$Z_i^{p,q} = Z_i^p \cap M^{p+q} \qquad \text{and} \qquad D_i^{p,q} = D_i^p \cap M^{p+q}.$$

Then the relations above become

$$Z_i^p = \sum_{q\geq 0} Z_i^{p,q}, \qquad D_i^p = \sum_{q\geq 0} D_i^{p,q}$$

From sec. 1.3 and the fact the δ is homogeneous of degree 1 we obtain the relations

$$Z_i^{p,q} = F^p(M^{p+q}) \cap \delta^{-1}(F^{p+i}(M^{p+q+1}))$$

and (1.11)

$$D_i^{p,q} = \delta(Z_i^{p-i,q+i-1}), \qquad 0 \leq i < \infty.$$

The relation

$$(Z_{i-1}^{p+1} + D_{i-1}^p) \cap Z_i^{p,q} = Z_{i-1}^{p+1,q-1} + D_{i-1}^{p,q}$$

follows at once from the definitions. It shows that the gradations of Z_i^p and D_i^p induce the gradation in E_i^p given by

$$E_i^p = \sum_{q \geq 0} E_i^{p,q},$$

where

$$E_i^{p,q} = Z_i^{p,q}/(Z_{i-1}^{p+1,q-1} + D_{i-1}^{p,q}).$$

Thus E_i becomes a bigraded space

$$E_i = \sum_{p,q} E_i^{p,q}.$$

As above, p, q, and $p + q$ are respectively called *base degree*, *fibre degree*, and *total degree*. The total gradation of E_i induced by the above bigradation is given by

$$E_i = \sum_r E_i^{(r)}, \qquad E_i^{(r)} = \sum_{p+q=r} E_i^{p,q}.$$

The restriction of η_i^p to $Z_i^{p,q}$ will be denoted by $\eta_i^{p,q}$. Thus

$$\eta_i^{p,q}: Z_i^{p,q} \to E_i^{p,q}$$

is a surjective map, and

$$\ker \eta_i^{p,q} = Z_i^{p+1,q-1} + D_i^{p,q}.$$

Now consider the operators $d_i^p: E_i^p \to E_i^{p+i}$. It is immediate from the definitions that d_i^p maps $E_i^{p,q}$ into $E_i^{p+i,q+1-i}$; i.e., d_i is homogeneous of bidegree $(i, 1 - i)$ and total degree 1. Thus d_i restricts to operators

$$d_i^{p,q}: E_i^{p,q} \to E_i^{p+i,q+1-i}.$$

In particular,

$$d_0^{p,q}: E_0^{p,q} \to E_0^{p,q+1}, \qquad d_1^{p,q}: E_1^{p,q} \to E_1^{p+1,q}$$

and

$$d_2^{p,q}: E_2^{p,q} \to E_2^{p+2,q-1}.$$

The bigradation of E_i determines a bigradation of $H(E_i, d_i)$, written

$$H(E_i, d_i) = \sum_{p,q} H^{p,q}(E_i, d_i).$$

The corresponding total gradation, $H(E_i, d_i) = \sum_r H^{(r)}(E_i, d_i)$, is induced by the total gradation of E_i:

$$H^{(r)}(E_i, d_i) = \sum_{p+q=r} H^{p,q}(E_i, d_i)$$

Finally consider the isomorphisms

$$\sigma_i^p: E_{i+1}^p \xrightarrow{\cong} H^p(E_i, d_i),$$

defined in sec. 1.6. Evidently σ_i^p maps $E_{i+1}^{p,q}$ into $H^{p,q}(E_i, d_i)$. Hence it restricts to isomorphisms

$$\sigma_i^{p,q}: E_{i+1}^{p,q} \xrightarrow{\cong} H^{p,q}(E_i, d_i).$$

In the same way it follows that the isomorphism $\sigma_\infty^p: E_\infty^p \xrightarrow{\cong} A_{H(M)}^p$ (cf. sec. 1.4) is homogeneous of bidegree zero, and hence restricts to isomorphisms

$$\sigma_\infty^{p,q}: E_\infty^{p,q} \xrightarrow{\cong} A_{H(M)}^{p,q}.$$

We close this section with a condition that forces the collapse of a spectral sequence.

Proposition IV: Let (M, δ) be a graded filtered differential space with spectral sequence (E_i, d_i). Assume that, for some m, E_m is evenly graded with respect to the total gradation:

$$E_m^{(r)} = 0, \qquad r \text{ odd}.$$

Then the spectral sequence collapses at the mth term.

Proof: Since E_m is evenly graded, and d_m is homogeneous of total degree 1, it follows that $d_m = 0$. Hence $E_m = H(E_m, d_m)$.

Since the isomorphism $\sigma_m: E_{m+1} \xrightarrow{\cong} H(E_m, d_m)$ is homogeneous of degree zero, it follows that E_{m+1} is evenly graded. Now an induction argument shows that $d_i = 0$ for $i \geq m$.

Q.E.D.

1.11. Bigraded differential spaces. Let $M = \sum_{p,q} M^{p,q}$ be a bigraded vector space such that $M^{p,q} = 0$ unless $p + q \geq 0$ and $q \geq 0$,

and consider the induced total gradation

$$M = \sum_{r \geq 0} M^{(r)}, \qquad M^{(r)} = \sum_{p+q=r} M^{p,q}.$$

Then the subspaces

$$F^p(M) = \sum_{\substack{\mu \geq p \\ q \geq 0}} M^{\mu,q}$$

make M into a graded filtered space.

Now assume that δ is a differential operator in M homogeneous of degree 1 with respect to the total degree, and such that for some fixed $k \geq 0$,

$$\delta: M^{p,q} \to F^{p+k}(M), \qquad \text{all} \quad p, q.$$

Let D be the differential operator defined as in sec. 1.7. Then D is homogeneous of bidegree $(k, 1 - k)$. It follows from the results of sec. 1.7 that

$$E_i^{p,q} \cong M^{p,q}, \qquad i \leq k$$

and

$$E_{k+1}^{p,q} \cong H^{p,q}(M, D).$$

1.12. Convergent spectral sequences. Let (M, δ) be a graded filtered space, with spectral sequence (E_i, d_i). Since $F^p(M^r) = 0$, $p > r$, formula (1.11), sec. 1.10, implies that

$$Z_i^{p,q} = Z_\infty^{p,q}, \qquad i \geq q + 2$$

and

$$Z_{i-1}^{p+1,q-1} + D_{i-1}^{p,q} \subset Z_\infty^{p+1,q-1} + D_\infty^{p,q}, \qquad i \geq q + 2.$$

Thus the first equality induces a surjective linear map

$$\gamma_i^{p,q}: E_i^{p,q} \to E_\infty^{p,q}, \qquad i \geq q + 2.$$

Definition: The spectral sequence (E_i, d_i) is said to *converge* if for each (p, q) there is some $i \geq q + 2$ such that $\gamma_i^{p,q}$ is an isomorphism.

Remarks: 1. The linear map $\gamma_i^{p,q}$ is an isomorphism if and only if

$$Z_\infty^{p+1,q-1} + D_{i-1}^{p,q} = Z_\infty^{p+1,q-1} + D_\infty^{p,q}.$$

In this case

$$E_i^{p,q} = E_{i+1}^{p,q} = \cdots = E_\infty^{p,q} \cong A_{H(M)}^{p,q},$$

and $\gamma_i^{p,q}$ is the identity map.

2. Let (E_i, d_i) be a convergent spectral sequence for a graded filtered differential space (M, δ) which collapses at the mth term. Then

$$E_m = E_\infty \cong A_{H(M)}.$$

Proposition V: Let (M, δ) be a graded filtered space. Assume that for each r there is a finite integer $k(r)$ (possibly negative) such that

$$F^p(M^r) = M^r, \qquad p \leq k(r).$$

Then there is a finite integer $i(r)$ with the following property: $\gamma_i^{p,r-p}$ is defined, and an isomorphism, for all p and all $i \geq i(r)$. In particular

$$E_i^{(r)} = E_\infty^{(r)} \cong A_{H(M)}^{(r)}, \qquad i \geq i(r),$$

and the spectral sequence is convergent.

Proof: It follows from our hypothesis that

$$D_i^{p,r-p} = D_\infty^{p,r-p}, \qquad i \geq p - k(r-1).$$

Hence $\gamma_i^{p,r-p}$ is defined, and an isomorphism, whenever both $i \geq r - p + 2$ and $i \geq p - k(r-1)$. In particular if we set $i(r) = \max(r - k(r) + 2, r - k(r-1))$, then $\gamma_i^{p,r-p}$ is an isomorphism for $i \geq i(r)$ and $k(r) \leq p \leq r$.

On the other hand, our hypothesis shows that

$$E_i^{p,r-p} = 0 = E_\infty^{p,r-p}, \qquad 0 \leq i < \infty,$$

whenever $p < k(r)$ or $p > r$. In particular, $\gamma_i^{p,r-p}$ is always an isomorphism for $p < k(r)$ or $p > r$.

Q.E.D.

1.13. The base space. Given a graded filtered differential space (M, δ) consider the graded subspace $B = \sum_p B^p$ defined by

$$B^p = Z_1^{p,0}, \qquad p \in Z.$$

Thus B^p consists of the elements of $F^p(M^p)$ which are mapped into $F^{p+1}(M^{p+1})$ under δ. B is called the *base* of (M, δ). Since (by definition) $\delta(B^p) \subset F^{p+1}(M^{p+1})$, it follows that the base is stable under δ. Hence it is a graded differential space. A homomorphism $\varphi: M \to \tilde{M}$ of graded filtered differential spaces restricts to a linear map between the respective base spaces.

Now let B have the filtration induced by the gradation; i.e., set $\hat{F}^p(B) = \sum_{\mu \geq p} B^\mu$. Since δ is homogeneous of degree 1 we may apply the results of sec. 1.8 to obtain the spectral sequence for this filtration; it is given by

$$(\hat{E}_i, \hat{d}_i) \cong \begin{cases} (B, 0), & i = 0, \\ (B, \delta), & i = 1, \\ (H(B), 0), & i \geq 2. \end{cases}$$

Moreover, in view of sec. 1.11, the bigradation of $\hat{E}_i$ is given by

$$\hat{E}_i^{p,0} \cong B^p, \qquad \hat{E}_i^{p,q} = 0, \qquad q > 0 \quad (i = 0, 1),$$

and

$$\hat{E}_i^{p,0} \cong H^p(B), \qquad \hat{E}_i^{p,q} = 0, \qquad q > 0 \quad (i \geq 2).$$

Next, consider the inclusion $e: B \to M$. It induces a linear map $e^*: H(B) \to H(M)$. Moreover, e is filtration preserving, and so it determines a homomorphism

$$e_i: (\hat{E}_i, \hat{d}_i) \to (E_i, d_i)$$

of spectral sequences.

Proposition VI: The maps

$$e_1^{p,0}: \hat{E}_1^{p,0} \to E_1^{p,0} \qquad \text{and} \qquad e_2^{p,0}: \hat{E}_2^{p,0} \to E_2^{p,0}$$

are isomorphisms. In particular,

$$E_1^{p,0} \cong B^p \qquad \text{and} \qquad E_2^{p,0} \cong H^p(B).$$

Proof: We show first that each $e_1^{p,0}$ is an isomorphism. In fact, by definition $Z_1^{p,0} = B^p$, while

$$Z_0^{p+1,-1} = 0 = D_0^{p,0}$$

(cf. formula (1.11), sec. 1.10). It follows that $\eta_1^{p,0}$ is an isomorphism from B^p onto $E_1^{p,0}$. Now the commutative diagram

$$\begin{tikzcd} B^p \arrow[r, "\iota"] \arrow[d, "\hat{\eta}_1^{p,0}"', "\cong"] & B^p \arrow[d, "\cong"', "\eta_1^{p,0}"] \\ \hat{E}_1^{p,0} \arrow[r, "e_1^{p,0}"'] & E_1^{p,0} \end{tikzcd}$$

implies that $e_1^{p,0}$ is an isomorphism.

Next note that the differential operator d_1 is homogeneous of bidegree $(1, 0)$. Thus, for fixed q, the direct sums $\sum_p E_1^{p,0}$ are stable under d_1, and so

$$H(E_1, d_1) \cong \sum_q H\left(\sum_p E_1^{p,q}, d_1\right). \tag{1.12}$$

Since $e_1 \colon \hat{E}_1 \xrightarrow{\cong} \sum_p E_1^{p,0}$, it follows that

$$e_1^\# \colon H(\hat{E}_1, \hat{d}_1) \xrightarrow{\cong} H\left(\sum_p E_1^{p,0}, d_1\right).$$

In view of formula (1.12), and Proposition III, sec. 1.6, this implies that

$$e_2^{p,0} \colon \hat{E}_2^{p,0} \xrightarrow{\cong} E_2^{p,0}.$$

Q.E.D.

Corollary: The maps

$$e_k^{p,0} \colon H^p(B) \to E_k^{p,0}$$

are surjective for $k \geq 2$.

1.14. Homomorphisms of graded filtered differential spaces. Assume that $\varphi \colon M \to \tilde{M}$ is a homomorphism of graded filtered differential spaces. Then the induced maps $\varphi_i \colon E_i \to \tilde{E}_i$ (cf. sec. 1.5) preserve the bigradation, and hence restrict to linear maps

$$\varphi_i^{p,q} \colon E_i^{p,q} \to \tilde{E}_i^{p,q} \qquad \text{and} \qquad \varphi_i^{(r)} \colon E_i^{(r)} \to \tilde{E}_i^{(r)}.$$

Since the isomorphisms $\sigma_i \colon E_{i+1} \xrightarrow{\cong} H(E_i, d_i)$ and $\sigma_\infty \colon E_\infty \xrightarrow{\cong} A_{H(M)}$

also preserve the bigradation, we have the commutative diagrams

$$\begin{tikzcd} E_{i+1}^{p,q} \arrow[r, "\cong"] \arrow[d, "\varphi_{i+1}^{p,q}"'] & H^{p,q}(E_i) \arrow[d, "(\varphi_i^\#)^{p,q}"] \\ \tilde{E}_{i+1}^{p,q} \arrow[r, "\cong"'] & H^{p,q}(\tilde{E}_i) \end{tikzcd} \tag{1.13}$$

and

$$\begin{tikzcd} E_\infty^{p,q} \arrow[r, "\cong"] \arrow[d, "\varphi_\infty^{p,q}"'] & A_{H(M)}^{p,q} \arrow[d, "\varphi_A^{p,q}"] \\ \tilde{E}_\infty^{p,q} \arrow[r, "\cong"'] & A_{H(\tilde{M})}^{p,q}. \end{tikzcd} \tag{1.14}$$

Recall from sec. 0.9 that a homomorphism $\varphi: \sum_{p\geq 0} A^p \rightarrow \sum_{p\geq 0} B^p$ of graded spaces is called *n-regular*, if $\varphi^p: A^p \rightarrow B^p$ is an isomorphism for $p \leq n$ and injective for $p = n + 1$.

Theorem I (Comparison theorem): Suppose $\varphi: M \rightarrow \tilde{M}$ is a homomorphism of graded filtered differential spaces whose spectral sequences are convergent. Assume that for some i the induced homomorphism $\varphi_i: E_i \rightarrow \tilde{E}_i$ is n-regular (with respect to the total gradation). Then

$$\varphi_j: E_j \rightarrow \tilde{E}_j, \qquad j \geq i,$$

and

$$\varphi^\#: H(M) \rightarrow H(\tilde{M})$$

are n-regular.

Proof: Since φ_i is n-regular, so is $\varphi_i^\#$ (cf. Proposition II, sec. 0.9). But the isomorphisms σ_{i+1} and $\tilde{\sigma}_{i+1}$ identify φ_{i+1} with $\varphi_i^\#$ (cf. Proposition III, sec. 1.6) and so φ_{i+1} is n-regular. It follows by induction that φ_{i+j} is n-regular for all $j \geq 0$. But for any p, q and sufficiently large k we have by hypothesis

$$E_k^{p,q} = E_\infty^{p,q}, \qquad \tilde{E}_k^{p,q} = \tilde{E}_\infty^{p,q}, \qquad \text{and} \qquad \varphi_k^{p,q} = \varphi_\infty^{p,q}.$$

Hence φ_∞ is n-regular. Since σ_∞ identifies φ_∞ with $(\varphi^\#)_A$, this implies

that $(\varphi^{\#})_A$ is n-regular. Now the theorem follows from Proposition VII, below.

Q.E.D.

Corollary: If $\varphi_i\colon E_i \to \tilde{E}_i$ is an isomorphism for a certain i, then

$$\varphi_j\colon\ E_j \to \tilde{E}_j, \qquad j \geq i,$$

and

$$\varphi^{\#}\colon H(M) \to H(\tilde{M}).$$

are isomorphisms.

Proposition VII: Let $\varphi\colon M \to \tilde{M}$ be a homomorphism of graded filtered spaces and consider the induced linear maps

$$\varphi_A^{(r)}\colon A_M^{(r)} \to A_{\tilde{M}}^{(r)}.$$

Suppose $\varphi_A^{(r)}$ is injective (respectively, surjective). Then $\varphi^r\colon M^r \to \tilde{M}^r$ is injective (respectively, surjective). In particular, if $\varphi_A^{(r)}$ is an isomorphism, so is φ^r.

Proof: (1) Assume that $\varphi_A^{(r)}$ is injective, and fix $x \in \ker\varphi^r$. Then, since $M = \bigcup_p F^p(M)$, it follows that $x \in F^p(M^r)$ for *some* p. But if $x \in F^p(M^r)$, then

$$0 = \varrho^p\varphi x = \varphi_A^{(r)}\varrho^p x.$$

Since $\varphi_A^{(r)}$ is injective, this implies that $\varrho^p x = 0$; i.e.,

$$x \in F^{p+1}(M^r).$$

Now any easy induction argument yields

$$x \in \bigcap_p F^p(M^r).$$

Hence $x = 0$ and φ^r is injective.

(2) Assume that $\varphi_A^{(r)}$ is surjective, and fix $x \in \tilde{M}^r$. We show by induction that $x \in F^p(\tilde{M}^r) + \varphi(M^r)$ for all p. As above, we have $x \in F^s(\tilde{M}^r)$ for some s; then $x \in F^s(\tilde{M}^r) + \varphi(M^r)$. Now assume

$$x \in F^p(\tilde{M}^r) + \varphi(M^r).$$

This relation yields (for some $y \in M^r$)

$$x - \varphi y \in F^p(\tilde{M}^r).$$

Since $\varphi_A^{(r)}$ is surjective, it follows that for some $z \in F^p(M^r)$,

$$\tilde{\varrho}^p(x - \varphi y) = \varphi_A \varrho^p(z) = \tilde{\varrho}^p \varphi z.$$

Hence $x - \varphi y - \varphi z \in \ker \tilde{\varrho}^p$; i.e.

$$x - \varphi y - \varphi z \in F^{p+1}(\tilde{M}^r).$$

It follows that $x \in F^{p+1}(\tilde{M}^r) + \varphi(M^r)$.

Proceeding in this way yields

$$x \in F^p(\tilde{M}^r) + \varphi(M^r), \qquad \text{all } p.$$

In particular, since $F^{r+1}(\tilde{M}^r) = 0$, $x \in \varphi(M^r)$. Thus φ^r is surjective.

Q.E.D.

Corollary I: Suppose $\varphi: M \to \tilde{M}$ is a homomorphism of graded filtered differential spaces, and assume that

$$\varphi_\infty^{(r)}: E_\infty^{(r)} \to \tilde{E}_\infty^{(r)}$$

is injective (surjective) for some r. Then the mapping

$$(\varphi^\#)^r: H^r(M) \to H^r(\tilde{M}),$$

is injective (surjective). In particular, if φ_∞ is injective (surjective), then $\varphi^\#$ is injective (surjective).

Proof: It follows from the diagram (1.14) that $\varphi_A^{(r)}$ is injective (surjective) if $\varphi_\infty^{(r)}$ is. Now the corollary follows from Proposition VII, applied to the graded filtered spaces $H(M)$ and $H(\tilde{M})$.

Q.E.D.

Corollary II: Let $\varphi: M \to \tilde{M}$ be a homomorphism of graded filtered differential spaces whose spectral sequences are convergent and collapse at the kth term. Then $(\varphi^\#)^r: H^r(M) \to H^r(\tilde{M})$ is injective (surjective) whenever $\varphi_k^{(r)}: E_k^{(r)} \to \tilde{E}_k^{(r)}$ is injective (surjective).

Corollary III: Let M be a graded filtered differential space with base B. Assume that the spectral sequence of M collapses at the second term and is convergent. Then

$$e^{\#}: H(B) \to H(M)$$

is injective.

Proof: Since $e_2: \hat{E}_2 \to E_2$ is injective (cf. Proposition VI, sec. 1.13) this is an immediate consequence of Corollary II.

Q.E.D.

1.15. Poincaré series. In this section M will denote a graded filtered differential space with a convergent spectral sequence.

Proposition VIII: Let (E_i, d_i) be the spectral sequence for M and assume that for some i, the spaces $E_i^{(r)}$ all have finite dimension. Then the spaces $E_j^{(r)}$ $(i \leq j \leq \infty)$ and $H^r(M)$ all have finite dimension. Moreover

$$f_i \geq f_{i+1} \geq \cdots \geq f_\infty = f_{H(M)}, \tag{1.15}$$

where f_j denotes the Poincaré series of E_j with respect to the total gradation. Finally, the relation

$$f_j = f_{H(M)}$$

holds if and only if the spectral sequence collapses at the jth term.

Proof: The isomorphism σ_j restricts to an isomorphism $E_{j+1}^{(r)} \xrightarrow{\cong} H^{(r)}(E_j)$. Hence, if $E_i^{(r)}$ has finite dimension, so has $E_{i+1}^{(r)}$ and by induction we obtain this for every $j \geq i$ (j finite). Since the spectral sequence is convergent, it follows that $E_\infty^{(r)}$ has finite dimension. The isomorphism $E_\infty^{(r)} \cong A_{H(M)}^{(r)}$ (cf. sec. 1.4) shows that $A_{H(M)}^{(r)}$ has finite dimension. Finally, it is a straightforward exercise in linear algebra (via Proposition VII, sec. 1.14) to construct a linear isomorphism

$$A_{H(M)} \cong H(M), \tag{1.16}$$

homogeneous of degree zero. Hence $H^r(M)$ has finite dimension. This proves the first part of the proposition.

For the proof of the second part we observe that, since $E_{j+1}^{(r)} \cong H^{(r)}(E_j)$,

$$\dim E_j^{(r)} \geq \dim E_{j+1}^{(r)}, \qquad i \leq j < \infty,$$

while, because the spectral sequence converges, for sufficiently large j,

$$\dim E_j^{(r)} = \dim E_\infty^{(r)}$$

It follows that $f_i \geq \cdots \geq f_\infty$.

On the other hand, the existence of the isomorphism (1.16) and the isomorphisms $E_\infty^{(r)} \cong A_{H(M)}^{(r)}$ show that

$$f_{H(M)} = f_{A_{H(M)}} = f_\infty .$$

It remains to prove the last statement. If the sequence collapses at the jth term, it follows from Remark 2, sec. 1.12, that

$$E_j \cong E_\infty \cong A_{H(M)},$$

whence $f_j = f_\infty = f_{H(M)}$. Conversely, assume $f_j = f_{H(M)}$. Then formula (1.15) yields

$$f_j = f_{j+1} = \cdots = f_{H(M)}.$$

In particular,

$$\dim H^{(r)}(E_{j+k}) = \dim E_{j+k+1}^{(r)} = \dim E_{j+k}^{(r)}, \qquad r \geq 0, \quad k \geq 0.$$

This implies that $d_{j+k} = 0$ $(k \geq 0)$, and so the spectral sequence collapses at the jth term.

Q.E.D.

Corollary: Suppose $\dim E_i < \infty$ for some i. Then $\dim H(M) < \infty$, and the spectral sequence collapses at the kth term if and only if $\dim E_k = \dim H(M)$.

1.16. Euler characteristic. Proposition IX: Assume that M is as in sec. 1.15. Suppose that $\dim E_i < \infty$ for some i. Then the Euler–Poincaré characteristics χ_{E_j} satisfy

$$\chi_{E_i} = \chi_{E_{i+1}} = \cdots = \chi_{E_\infty} = \chi_{H(M)}.$$

Proof: Since the isomorphism $E_{i+1} \cong H(E_i)$ preserves the total gradation, it follows from the Euler–Poincaré formula (cf. sec. 0.7) that $\chi_{E_{i+1}} = \chi_{H(E_i)} = \chi_{E_i}$.

Since $\dim E_i < \infty$, $E_i^{(r)} = 0$ for all but finitely many r. Thus because the spectral sequence converges, it follows that for some fixed finite n,

$$E_n^{(r)} = E_\infty^{(r)}, \qquad \text{all } r.$$

Hence $f_n = f_\infty = f_{H(M)}$, and so

$$\chi_{E_n} = \chi_{E_\infty} = \chi_{H(M)}.$$

Q.E.D.

§4. Graded filtered differential algebras

1.17. Filtered algebras. Let R be an algebra (over Γ). A *filtration of the algebra* R is a filtration of the vector space R by subspaces $F^p(R)$ of R which satisfy

$$F^p(R) \cdot F^q(R) \subset F^{p+q}(R). \tag{1.17}$$

Given a filtration of R, consider the associated graded space

$$A_R = \sum_p A_R^p.$$

Relation (1.17) allows us to define a multiplication in A_R by setting

$$\varrho^p x \cdot \varrho^q y = \varrho^{p+q}(x \cdot y), \qquad x \in F^p(R), \quad y \in F^q(R).$$

This multiplication makes A_R into a graded algebra, called the *associated graded algebra* of the filtered algebra R.

Remark: If $F^0(R) = R$ then $F^p(R)$ is an ideal, and so multiplication in $F^p(R)$ induces a product in A_R^p. But this product is trivial for $p \geq 1$. In fact, if $x \in F^p(R)$ and $y \in F^p(R)$, then (if $p \geq 1$) $xy \in F^{2p}(R) \subset F^{p+1}(R)$, whence $\varrho^p(xy) = 0$.

On the other hand, A_R^0 equipped with this multiplication is a subalgebra of the associated graded algebra.

1.18. Graded filtered differential algebras. Let (R, δ) be a graded differential algebra. Suppose R is filtered by subspaces $F^p(R)$ so that the filtration makes R into a filtered algebra (cf. sec. 1.17) and also into a graded filtered differential space (cf. sec. 1.9 and sec. 1.10). Then (R, δ) is called a *graded filtered differential algebra.*

Assume that (R, δ) is a graded filtered differential algebra. Then we have the relations

$$Z_i^p \cdot Z_i^s \subset Z_i^{p+s} \tag{1.18}$$

and

$$Z_i^p \cdot D_{i-1}^s \subset Z_{i-1}^{p+s+1} + D_{i-1}^{p+s}, \qquad 0 \leq i \leq \infty. \tag{1.19}$$

In fact, if $u \in Z_i^{p,q}$ and $v \in Z_i^{s,t}$, then

$$u \cdot v \in F^p(R) \cdot F^s(R) \subset F^{p+s}(R).$$

Moreover,

$$\delta(u \cdot v) = \delta u \cdot v + (-1)^{p+q} u \cdot \delta v \in F^{p+i}(R) \cdot F^s(R) + F^p(R) \cdot F^{s+i}(R) \subset F^{p+s+i}(R),$$

whence (1.18). Formula (1.18) implies in particular that for $p \geq 0$

$$Z_i^p \cdot Z_i^p \subset Z_i^p$$

and so Z_i^p is an algebra if $p \geq 0$.

To prove (1.19) observe first that this relation is trivial for $i = \infty$. Hence we may assume that $i < \infty$. Let $u \in Z_i^{p,q}$ and $v \in Z_{i-1}^{s-i+1,t}$ be any elements. Write

$$u \cdot \delta v = -(-1)^{p+q} \delta u \cdot v + (-1)^{p+q} \delta(u \cdot v).$$

It is easy to see that

$$\delta u \cdot v \in Z_{i-1}^{p+s+1} \qquad \text{and} \qquad \delta(u \cdot v) \in D_{i-1}^{p+s}.$$

Formula (1.19) follows.

In view of relations (1.18) and (1.19) a multiplication is defined in E_i, by

$$(\eta_i^p u) \cdot (\eta_i^q v) = \eta_i^{p+q}(u \cdot v), \qquad u \in Z_i^p, \qquad v \in Z_i^q.$$

In this way the space E_i becomes a bigraded algebra. The operator d_i is an antiderivation with respect to the total gradation of E_i. Thus (E_i, d_i) is a graded differential algebra.

Next observe that the isomorphisms

$$\sigma_i \colon E_{i+1} \xrightarrow{\cong} H(E_i, d_i) \qquad \text{and} \qquad \sigma_\infty \colon E_\infty \xrightarrow{\cong} A_{H(M)}$$

are algebra isomorphisms, as follows directly from the definitions. Moreover, the algebra structures of E_0 and A_R coincide (recall from sec. 1.4 that E_0 and A_R are equal as spaces).

Note also that the basic subspace B is a subalgebra of R, as follows from formula (1.18).

Finally, let R and $\tilde{R}$ be graded filtered differential algebras. A homomorphism of graded differential algebras $\varphi \colon (R, \delta) \to (\tilde{R}, \tilde{\delta})$ which

preserves the filtrations will be called a *homomorphism of graded filtered differential algebras*. The induced mappings

$$\varphi_i\colon (E_i, d_i) \to (\tilde{E}_i, \tilde{d}_i)$$

are homomorphism of graded differential algebras.

§5. Differential couples

1.19. Let $M = \sum_{p \in Z} M^p$ be a graded space and let

$$F^p(M) = \sum_{\mu \geq p} M^\mu$$

be the corresponding filtration of M. Let δ_1, δ_2 be differential operators in M such that for some $k \geq 0$

$$\delta_1: M^p \to M^{p+k} \qquad \text{and} \qquad \delta_2: M^p \to F^{p+k+1}(M). \tag{1.20}$$

Assume further that

$$\delta_1\delta_2 + \delta_2\delta_1 = 0. \tag{1.21}$$

Then (M, δ_1, δ_2) will be called a *differential couple of degree k*. It follows from (1.21) that

$$\delta = \delta_1 + \delta_2$$

is again a differential operator. δ is called the *total differential operator* of the couple (M, δ_1, δ_2).

Relations (1.20) imply that (M, δ) is a filtered differential space satisfying the conditions of sec. 1.7. Moreover, the corresponding homogeneous differential operator D is precisely δ_1.

Next consider the differential space (M, δ_2). It is also a filtered differential space and satisfies the conditions of sec. 1.7. Hence δ_2 determines a differential operator D_2 in M, homogeneous of degree $k + 1$, and such that

$$\delta_2 - D_2: M^p \to F^{p+k+2}(M).$$

It follows from (1.21) and an argument on degrees that

$$\delta_1 D_2 + D_2\delta_1 = 0.$$

Hence D_2 induces a differential operator $D_2^\#$ in $H(M, \delta_1)$, homogeneous of degree $k + 1$.

Theorem II: Let (M, δ_1, δ_2) be a differential couple of degree k. Then the first terms of the spectral sequence for the filtered differential space (M, δ) are given by:

(1) $(E_i, d_i) \cong (M, 0), \qquad 0 \le i < k;$

(2) $(E_k, d_k) \cong (M, \delta_1);$

(3) $(E_{k+1}, d_{k+1}) \cong (H(M, \delta_1), D_2^\#);$

(4) $E_{k+2} \cong H(H(M, \delta_1), D_2^\#).$

Remark: The actual isomorphisms are important, and appear explicitly in the proof.

Proof: The isomorphisms (1) and (2) are constructed in sec. 1.7 (formulae (1.4) and (1.5)).

Moreover, in sec. 1.7 (formula (1.6)) we constructed an isomorphism $\alpha: H(M, \delta_1) \xrightarrow{\cong} E_{k+1}$. It is immediate from the definition of α that the diagram

$$\begin{tikzcd} Z^p(M, \delta_1) \arrow[r, "\text{inclusion}"] \arrow[d, "\pi_1"'] & Z^p_{k+1} \arrow[d, "\eta^p_{k+1}"] \\ H^p(M, \delta_1) \arrow[r, "\cong", "\alpha"'] & E^p_{k+1} \end{tikzcd}$$

commutes. To establish (3) we show that

$$\alpha D_2^\# = d_{k+1}\alpha. \tag{1.22}$$

Fix $z \in Z^p(M, \delta_1)$. Then

$$\alpha D_2^\#(\pi_1 z) = \eta_{k+1}^{p+k+1} D_2 z$$

and

$$d_{k+1}^p \alpha(\pi_1 z) = \eta_{k+1}^{p+k+1}\delta z = \eta_{k+1}^{p+k+1}\delta_2 z.$$

Subtract the second relation from the first to obtain

$$(\alpha D_2^\# - d_{k+1}\alpha)(\pi_1 z) = \eta_{k+1}^{p+k+1}(D_2 z - \delta_2 z).$$

But clearly

$$(D_2 - \delta_2)(z) \in F^{p+k+2}(M) \cap Z(M, \delta_1) \subset \ker \eta_{k+1}^{p+k+1}$$

and so (1.22) follows.

It follows from (1.22) that α induces an isomorphism

$$\alpha^{\#}: H(H(M, \delta_1), D_2^{\#}) \xrightarrow{\cong} H(E_{k+1}, d_{k+1}).$$

Composing $\alpha^{\#}$ with the isomorphism σ_{k+1}^{-1} we obtain the isomorphism (4).

Q.E.D.

A differential couple (M, δ_1, δ_2) of degree k is called *homogeneous* if δ_2 is homogeneous of degree $k+1$,

$$\delta_2: M^p \to M^{p+k+1}.$$

If (M, δ_1, δ_2) is a homogeneous differential couple, it follows that $D_2 = \delta_2$. Hence Theorem II reads

$$(E_{k+1}, d_{k+1}) \cong (H(M, \delta_1), \delta_2^{\#})$$

and

$$E_{k+2} \cong H(H(M, \delta_1), \delta_2^{\#}).$$

1.20. Homomorphisms. Let (M, δ_1, δ_2) and $(\tilde{M}, \tilde{\delta}_1, \tilde{\delta}_2)$ be differential couples of degree k. Then a linear map $\varphi: M \to \tilde{M}$, homogeneous of degree zero, is called a *homomorphism of differential couples* if

$$\varphi\delta_1 = \tilde{\delta}_1\varphi \qquad \text{and} \qquad \varphi\delta_2 = \tilde{\delta}_2\varphi.$$

In particular, setting $\delta = \delta_1 + \delta_2$ and $\tilde{\delta} = \tilde{\delta}_1 + \tilde{\delta}_2$, we see that

$$\varphi\delta = \tilde{\delta}\varphi.$$

Moreover, $\varphi D_2 = \tilde{D}_2\varphi$.

Let $\varphi_{(1)}^{\#}: H(M, \delta_1) \to H(\tilde{M}, \tilde{\delta}_1)$ be the induced linear map. Then since $\varphi D_2 = \tilde{D}_2\varphi$,

$$\varphi_{(1)}^{\#} D_2^{\#} = \tilde{D}_2^{\#}\varphi_{(1)}^{\#}.$$

Thus $\varphi_{(1)}^{\#}$ is a homomorphism of differential spaces. Let

$$(\varphi_{(1)}^{\#})^{\#}: H(H(M, \delta_1), D_2^{\#}) \to H(H(\tilde{M}, \tilde{\delta}_1), \tilde{D}_2^{\#})$$

be the induced map.

Proposition X: Let φ be as above. Then the diagrams

$$\begin{tikzcd} H(M, \delta_1) \arrow[r, "\varphi_{(1)}^{\#}"] \arrow[d, "\alpha"', "\cong"] & H(\tilde{M}, \tilde{\delta}_1) \arrow[d, "\tilde{\alpha}", "\cong"'] \\ E_{k+1} \arrow[r, "\varphi_{k+1}"'] & \tilde{E}_{k+1} \end{tikzcd} \tag{1.23}$$

and

$$\begin{tikzcd} H(H(M, \delta_1), D_2^{\#}) \arrow[r, "(\varphi_{(1)}^{\#})^{\#}"] \arrow[d, "\cong"'] & H(H(\tilde{M}, \tilde{\delta}_1), \tilde{D}_2^{\#}) \arrow[d, "\cong"] \\ E_{k+2} \arrow[r, "\varphi_{k+2}"'] & \tilde{E}_{k+2} \end{tikzcd} \tag{1.24}$$

commute, the isomorphisms being defined in Theorem II.

Proof: Diagram (1.23) follows directly from diagram (1.8) in sec. 1.7. Since α, $\varphi_{(1)}^{\#}$, $\tilde{\alpha}$, and φ_{k+1} commute with the appropriate differential operators, we have

$$\varphi_{k+1}^{\#}\alpha^{\#} = \tilde{\alpha}^{\#}(\varphi_{(1)}^{\#})^{\#}.$$

This, together with (1.23) and the relation $\varphi_{k+1}^{\#}\sigma_{k+1} = \tilde{\sigma}_{k+1}\varphi_{k+2}$ (cf. sec. 1.6), yields (1.24).

Q.E.D.

1.21. Graded differential couples. Let (M, δ_1, δ_2) be a differential couple and assume that the spaces M^p are graded, $M^p = \sum_q M^{p,q}$. Suppose that $M^{p,q} = 0$ unless $p + q \geq 0$ and $q \geq 0$. Then M is a bigraded space $M = \sum_{p,q} M^{p,q}$. As usual we set

$$M^{(r)} = \sum_{p+q=r} M^{p,q},$$

and observe that $M = \sum_{r \geq 0} M^{(r)}$.

The couple (M, δ_1, δ_2) is called a *graded differential couple*, if δ_1 and δ_2 are homogeneous of degree 1 with respect to the total gradation of M. In this case (M, δ) $(\delta = \delta_1 + \delta_2)$ becomes a graded filtered differential space (cf. sec. 1.10).

Remark: If δ_1 and δ_2 are homogeneous of bidegrees (0, 1) and (1, 0) respectively, (M, δ) is a *double complex* in the notation of [2; p. 60].

The operators δ_1 and D_2 of a graded differential couple are bi-homogeneous. Hence a bigradation is induced in the spaces $H(M, \delta_1)$ and $H(H(M, \delta_1), D_2^\#)$. The isomorphisms of Theorem II, sec. 1.19, are homogeneous of bidegree zero:

$$E_i^{p,q} \cong M^{p,q}, \qquad i \leq k,$$

$$E_{k+1}^{p,q} \cong H^{p,q}(M, \delta_1),$$

and

$$E_{k+2}^{p,q} \cong H^{p,q}(H(M, \delta_1), D_2^\#).$$

A *homomorphism of graded differential couples* is a homomorphism of differential couples, homogeneous of bidegree zero.

Proposition XI: Let $\varphi\colon (M, \delta_1, \delta_2) \to (\tilde{M}, \tilde{\delta}_1, \tilde{\delta}_2)$ be a homomorphism of graded differential couples of degree k. Suppose the spectral sequences are convergent. Assume that the homomorphism

$$\varphi_{(1)}^\#\colon H(M, \delta_1) \to H(\tilde{M}, \tilde{\delta}_1)$$

is n-regular. Then so is the homomorphism $\varphi^\#\colon H(M) \to H(\tilde{M})$.

Proof: It follows from the hypothesis and diagram (1.23) in Proposition X that the homomorphism $\varphi_{k+1}\colon E_{k+1} \to \tilde{E}_{k+1}$ is n-regular. Now Theorem I of sec. 1.14 implies that $\varphi^\#$ is n-regular.

Q.E.D.

Corollary: If $\varphi_{(1)}^\#$ is an isomorphism, then so is $\varphi^\#$.

Chapter II

Koszul Complexes of *P*-Spaces and *P*-Algebras

In this chapter $P = \sum_k P^k$ denotes a finite-dimensional positively graded vector space which satisfies

$$P^k = 0 \qquad \text{if} \quad k \text{ is even.}$$

$\mathbb{P}$ denotes the evenly graded vector space defined by $\mathbb{P}^k = P^{k-1}$. Note that P and $\mathbb{P}$ are equal as vector spaces.

The gradations of P and $\mathbb{P}$ determine gradations

$$\wedge P = \sum_j (\wedge P)^j \qquad \text{and} \qquad \vee \mathbb{P} = \sum_j (\vee \mathbb{P})^j$$

in the algebras $\wedge P$ and $\vee \mathbb{P}$; these are defined by the relations

$$x_1 \wedge \cdots \wedge x_p \in (\wedge P)^{i_1 + \cdots + i_p} \qquad \text{if} \quad x_\nu \in P^{i_\nu}$$

and

$$x_1 \vee \cdots \vee x_q \in (\vee \mathbb{P})^{i_1 + \cdots + i_q} \qquad \text{if} \quad x_\nu \in \mathbb{P}^{i_\nu}.$$

§1. *P*-spaces and *P*-algebras

2.1. *P*-spaces. A *P-space* is a positively graded vector space

$$S = \sum_{k \geq 0} S^k$$

together with a bilinear map $S \times P \to S$ (written $(z, x) \mapsto z \circ x$) which satisfies the conditions

$$(z \circ x) \circ y = (z \circ y) \circ x, \qquad z \in S, \quad x, y \in P \tag{2.1}$$

and

$$z \circ x \in S^{p+q+1}, \qquad z \in S^p, \quad x \in P^q. \tag{2.2}$$

A *P-linear map* (or a *homomorphism of P-spaces*) between P-spaces S and T is a linear map $\varphi\colon S \to T$ such that

$$\varphi(z \circ x) = (\varphi z) \circ x, \qquad z \in S, \quad x \in P.$$

If φ is homogeneous of degree zero, it is called a *homomorphism of graded P-spaces.*

An *exact sequence of P-spaces* is an exact sequence

$$\cdots \to S_1 \to S_2 \to S_3 \to \cdots$$

in which S_1, S_2, and S_3 are P-spaces, and the arrows are P-linear maps.

A *P-subspace* of a P-space S is a graded subspace $S_1 \subset S$ such that $z \circ x \in S_1$ whenever $z \in S_1$ and $x \in P$. A P-subspace S_1 is itself a P-space, and the inclusion $S_1 \to S$ is a P-linear map. In particular the subspace of S spanned by the vectors of the form $z \circ x$ ($z \in S$, $x \in P$) is a P-subspace; it is denoted by $S \circ P$.

If S_1 is any P-subspace of S, then the quotient space S/S_1 admits a unique P-space structure for which the projection $\pi\colon S \to S/S_1$ is a P-linear map. The P-space S/S_1 is called the *quotient* or *factor space of S* with respect to S_1. If $\varphi\colon S \to T$ is a homomorphism of graded P-spaces, then $\ker \varphi$ and $\operatorname{Im} \varphi$ are respectively P-subspaces of S and T. Moreover, φ induces an isomorphism

$$S/\ker \varphi \xrightarrow{\cong} \operatorname{Im} \varphi$$

of graded P-spaces.

Given a P-space S, a graded $\vee P$-module structure is defined in S by

$$z \cdot (x_1 \vee \cdots \vee x_p) = z \circ x_1 \circ \cdots \circ x_p, \qquad z \in S, \quad x_\nu \in P,$$

(cf. formulae (2.1) and (2.2)). This establishes a 1-1 correspondence between P-spaces and graded $\vee P$-modules. Evidently, P-subspaces, P-factor spaces, and P-linear maps correspond respectively to submodules, factor modules, and module homomorphisms.

2.2. Koszul complexes. With each P-space S is associated the following differential space: In the tensor product $S \otimes \wedge P$ define a linear operator ∇_S by setting

$$\nabla_S(z \otimes 1) = 0, \qquad z \in S,$$

and

$$\nabla_S(z \otimes x_0 \wedge \cdots \wedge x_p) = \sum_{i=0}^{p} (-1)^{i-q} z \circ x_i \otimes x_0 \wedge \cdots \hat{x}_i \cdots \wedge x_p, \qquad z \in S^q, \quad x_i \in P.$$

The relation $z \circ x_i \circ x_j = z \circ x_j \circ x_i$ (cf. sec. 2.1) implies that $\nabla_S^2 = 0$. Thus $(S \otimes \wedge P, \nabla_S)$ is a differential space; it is called the *Koszul complex associated with the P-space S*. The corresponding cohomology space $H(S \otimes \wedge P, \nabla_S)$ is called *the cohomology space associated with the P-space S*.

The gradations of S and P induce a gradation in $S \otimes \wedge P$. It is written

$$S \otimes \wedge P = \sum_r (S \otimes \wedge P)^r,$$

and is uniquely determined by the following condition: If $z \in S^q$ and $x_i \in P^{p_i}$, then

$$z \otimes x_1 \wedge \cdots \wedge x_m \in (S \otimes \wedge P)^{q+p_1+\cdots+p_m}.$$

It follows from formula (2.2) that ∇_S is homogeneous of degree 1.

On the other hand, a second gradation is defined in $S \otimes \wedge P$ by

$$S \otimes \wedge P = \sum_k (S \otimes \wedge P)_k, \qquad \text{where} \quad (S \otimes \wedge P)_k = S \otimes \wedge^k P.$$

To distinguish it from the first gradation (called, simply, *the gradation*) we call it the *lower gradation*. Evidently ∇_S is homogeneous of degree -1 with respect to the lower gradation.

The two gradations of $S \otimes \wedge P$ define the bigradation given by

$$S \otimes \wedge P = \sum_{k,r} (S \otimes \wedge P)_k^r, \qquad \text{where} \quad (S \otimes \wedge P)_k^r = (S \otimes \wedge^k P)^r.$$

The elements of $(S \otimes \wedge^k P)^r$ are called *homogeneous of degree r, lower degree k*, and *bidegree (r, k)*.

Since ∇_S is homogeneous of bidegree $(1, -1)$, a bigradation is induced in $H(S \otimes \wedge P)$; it is denoted by

$$H(S \otimes \wedge P) = \sum_{k,r} H_k^r(S \otimes \wedge P).$$

Elements of $H_k^r(S \otimes \wedge P)$ are called *homogeneous of degree r, lower degree k*, and *bidegree (r, k)*. We shall write

$$H^r(S \otimes \wedge P) = \sum_k H_k^r(S \otimes \wedge P) \quad \text{and} \quad H_k(S \otimes \wedge P) = \sum_r H_k^r(S \otimes \wedge P).$$

Next consider the inclusion map

$$l_S\colon S \to S \otimes \wedge P$$

given by $l_S(z) = z \otimes 1$. We may regard l_S as an isomorphism

$$S \xrightarrow{\cong} Z_0(S \otimes \wedge P)$$

$(Z(S \otimes \wedge P) = \ker \nabla_S)$. It restricts to an isomorphism

$$S \circ P \xrightarrow{\cong} B_0(S \otimes \wedge P)$$

$(B(S \otimes \wedge P) = \operatorname{Im} \nabla_S)$. Thus l_S induces a commutative diagram

$$\begin{tikzcd} S \arrow[d] \arrow[dr, "l_S^\#"] & \\ S/(S \circ P) \arrow[r, "\bar{l}_S", "\cong"'] & H_0(S \otimes \wedge P) \end{tikzcd}$$

in which $l_S^\#$ is surjective and $\bar{l}_S$ is an isomorphism (both homogeneous of degree zero). We may write simply l, $l^\#$, and $\bar{l}$ for l_S, $l_S^\#$, and $\bar{l}_S$.

Finally, recall from sec. 0.4 that each $x^* \in P^*$ determines a unique antiderivation $i(x^*)$ (substitution operator) in $\wedge P$, homogeneous of degree -1, such that

$$i(x^*)(x) = \langle x^*, x \rangle, \qquad x^* \in P^*, \quad x \in P.$$

We will also denote by $i(x^*)$ the operator in $S \otimes \wedge P$ given by

$$i(x^*)(z \otimes \Phi) = (-1)^p z \otimes i(x^*)\Phi, \qquad z \in S^p, \quad \Phi \in \wedge P.$$

A simple calculation shows that

$$\nabla_S(z \otimes \Phi) = (-1)^p \sum_\nu z \circ e_\nu \otimes i(e^{*\nu})\Phi, \qquad z \in S^p, \Phi \in \wedge P,$$

where e_ν, $e^{*\nu}$ is any pair of dual bases for P and P^*. This relation implies that

$$i(x^*)\nabla_S + \nabla_S i(x^*) = 0, \qquad x^* \in P^*. \tag{2.3}$$

In particular, $i(x^*)$ induces a linear operator $i(x^*)^\#$ in $H(S \otimes \wedge P)$.

Examples. 1. *Direct sums*: Given P-spaces S and T make $S \oplus T$ into a P-space by setting

$$(z, w) \circ x = (z \circ x, w \circ x), \qquad z \in S, \quad w \in T, \quad x \in P.$$

Then

$$(S \oplus T) \otimes \wedge P = (S \otimes \wedge P) \oplus (T \otimes \wedge P), \qquad \nabla_{S\oplus T} = \nabla_S \oplus \nabla_T,$$

and $l_{S\oplus T} = l_S \oplus l_T$. In particular,

$$H((S \oplus T) \otimes \wedge P) = H(S \otimes \wedge P) \oplus H(T \otimes \wedge P).$$

2. *Tensor products*: Let S be a P-space and let F be a graded vector space. Make the graded space $F \otimes S$ into a P-space by setting

$$(a \otimes z) \circ x = a \otimes (z \circ x), \qquad a \in F, \quad z \in S, \quad x \in P.$$

Then the corresponding Koszul complex is $(F \otimes S \otimes \wedge P, \omega_F \otimes \nabla_S)$, where ω_F is the degree involution in F.

It follows that $H(F \otimes S \otimes \wedge P) = F \otimes H(S \otimes \wedge P)$. Moreover,

$$l_{F\otimes S} = \iota_F \otimes l_S \qquad \text{and} \qquad l^{\#}_{F\otimes S} = \iota_F \otimes l^{\#}_S.$$

2.3. Homomorphisms. Let $\varphi: S \to T$ be a homomorphism of P-spaces. Then $\varphi \otimes \iota: S \otimes \wedge P \to T \otimes \wedge P$ is a homomorphism of differential spaces, homogeneous of lower degree zero. Thus it induces a linear map, homogeneous of lower degree zero,

$$(\varphi \otimes \iota)^{\#}: H(S \otimes \wedge P) \to H(T \otimes \wedge P).$$

We denote the restrictions of $(\varphi \otimes \iota)^{\#}$ by

$$(\varphi \otimes \iota)^{\#}_p: H_p(S \otimes \wedge P) \to H_p(T \otimes \wedge P).$$

The P-linear map φ determines commutative diagrams

$$\begin{tikzcd} S \arrow[r, "l^{\#}_S"] \arrow[d, "\varphi"'] & H(S \otimes \wedge P) \arrow[d, "(\varphi\otimes\iota)^{\#}"] \\ T \arrow[r, "l^{\#}_T"'] & H(T \otimes \wedge P) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} S/(S \circ P) \arrow[r, "\bar{l}_S", "\cong"'] \arrow[d, "\bar{\varphi}"'] & H_0(S \otimes \wedge P) \arrow[d, "(\varphi\otimes\iota)^{\#}_0"] \\ T/(T \circ P) \arrow[r, "\cong", "\bar{l}_T"'] & H_0(T \otimes \wedge P). \end{tikzcd}$$

Moreover, if φ is homogeneous of degree zero, then

$$i(x^*) \circ (\varphi \otimes \iota) = (\varphi \otimes \iota) \circ i(x^*)$$

and (2.4)

$$i(x^*)^\# \circ (\varphi \otimes \iota)^\# = (\varphi \otimes \iota)^\# \circ i(x^*)^\#, \qquad x^* \in P^*.$$

If φ is homogeneous of degree k, then so are $\varphi \otimes \iota$ and $(\varphi \otimes \iota)^\#$.

If $\psi: T \to W$ is a second homomorphism of P-spaces, then $\psi \circ \varphi$ is P-linear, and

$$(\psi \circ \varphi \otimes \iota)^\# = (\psi \otimes \iota)^\# \circ (\varphi \otimes \iota)^\#.$$

The identity map of S induces the identity in $S \otimes \wedge P$ and $H(S \otimes \wedge P)$.

Next, consider a short exact sequence

$$0 \longrightarrow S \xrightarrow{\varphi} T \xrightarrow{\psi} W \longrightarrow 0$$

of P-spaces. The induced sequence of differential spaces is again short exact, and so it determines an exact triangle of cohomology spaces. Since the differential operators have lower degree -1, while $\varphi \otimes \iota$ and $\psi \otimes \iota$ have lower degree zero, this triangle yields the long exact sequence

$$\begin{array}{r}
\longrightarrow H_k(S \otimes \wedge P) \xrightarrow{(\varphi\otimes\iota)^\#} H_k(T \otimes \wedge P) \xrightarrow{(\psi\otimes\iota)^\#} H_k(W \otimes \wedge P) \\
\downarrow \qquad\qquad \\
H_{k-1}(S \otimes \wedge P) \\
\downarrow {\scriptstyle (\varphi\otimes\iota)^\#} \qquad \\
H_{k-1}(T \otimes \wedge P) \longrightarrow
\end{array}$$

Now suppose that φ and ψ are homogeneous of degree zero. Then the connecting homomorphism is homogeneous of degree 1, and we have the system of long exact sequences

$$\begin{array}{r}
\longrightarrow H^{r+k}_{-k}(S \otimes \wedge P) \longrightarrow H^{r+k}_{-k}(T \otimes \wedge P) \longrightarrow H^{r+k}_{-k}(W \otimes \wedge P) \\
\downarrow \qquad\qquad \\
H^{r+k+1}_{-k-1}(S \otimes \wedge P) \longrightarrow \cdots, \\
r = 0, 1, 2, \ldots .
\end{array}$$

2.4. *P*-algebras. A *P-algebra* is a pair $(S; \sigma)$, where:

(1) S is a positively graded associative algebra with identity, and

(2) $\sigma: P \to S$ is a linear map, homogeneous of degree 1, which satisfies

$$\sigma(x) \cdot z = z \cdot \sigma(x), \qquad x \in P, \quad z \in S. \tag{2.5}$$

σ is called the ***structure map*** of the P-algebra $(S; \sigma)$.

A *homomorphism of P-algebras* $\varphi: (S; \sigma) \to (T; \tau)$ is an algebra homomorphism $\varphi: S \to T$ which satisfies $\varphi \circ \sigma = \tau$ and $\varphi(1) = 1$. If φ is homogeneous of degree zero, it is called a *homomorphism of graded P-algebras*.

With each P-algebra $(S; \sigma)$ is associated the P-space structure of S given by

$$z \circ x = z \cdot \sigma(x), \qquad z \in S, \quad x \in P.$$

(Observe that a homomorphism of P-algebras is simply a P-linear algebra homomorphism.)

On the other hand, σ extends to a homomorphism

$$\sigma_\vee: \vee P \to S$$

of graded algebras (cf. formula (2.5)). This formula also shows that the image of $\sigma_\vee$ is in the centre of S. Thus $\sigma_\vee$ makes S into a graded $\vee P$-algebra. In this way P-algebras are put in 1-1 correspondence with graded $\vee P$-algebras.

Finally, observe that each $\vee P$-algebra is, in particular, a $\vee P$-module. Evidently,

$$z \cdot \sigma_\vee(\Psi) = z \circ \Psi, \qquad z \in S, \quad \Psi \in \vee P,$$

and so the diagram

$$\begin{array}{ccc} P\text{-algebras} & \longrightarrow & P\text{-spaces} \\ \updownarrow & & \updownarrow \\ \text{graded } \vee P\text{-algebras} & \longrightarrow & \text{graded } \vee P\text{-modules} \end{array}$$

commutes.

2.5. The Koszul complex of a P-algebra. To the Koszul complex of a P-algebra $(S; \sigma)$ we assign a multiplication in the following way:

$$(z \otimes \Phi) \cdot (w \otimes \Psi) = (-1)^{pq} z \cdot w \otimes \Phi \wedge \Psi,$$
$$z \in S, \quad w \in S^q, \quad \Phi \in \wedge^p P, \quad \Psi \in \wedge P,$$

(skew tensor product of algebras, cf. sec. 0.3). This makes $S \otimes \wedge P$ into a bigraded associative algebra with identity. If S is anticommutative, then so is $S \otimes \wedge P$.

The Koszul complex for a P-algebra $(S; \sigma)$ will be denoted by $(S \otimes \wedge P, \nabla_\sigma)$ (i.e., we use ∇_σ rather than ∇_S). A straightforward computation (using formula (2.5), sec. 2.4, and the fact that $P^k = 0$ for even k) shows that ∇_σ is an antiderivation with respect to the gradation of $S \otimes \wedge P$. Thus $(S \otimes \wedge P, \nabla_\sigma)$ is a graded differential algebra, and so $H(S \otimes \wedge P)$ becomes a bigraded algebra. In particular,

$$H(S \otimes \wedge P) = H_0(S \otimes \wedge P) \oplus H_+(S \otimes \wedge P)$$

decomposes $H(S \otimes \wedge P)$ as a direct sum of a graded subalgebra and a graded ideal.

Finally observe that the maps l_S, $l_S^\#$, and $\bar{l}_S$ (cf. sec. 2.2) are all homomorphisms of graded algebras. Moreover, the operators, $i(x^*)$ ($x^* \in P^*$), are antiderivations in $S \otimes \wedge P$ (with respect to the gradation). Hence the operators $i(x^*)^\#$ are antiderivations in $H(S \otimes \wedge P)$.

2.6. The algebra $\vee P$ The identity map $P \to P$ makes $\vee P$ into a P-algebra; the corresponding P-space structure is given by

$$z \circ x = z \vee x, \qquad z \in \vee P, \quad x \in P.$$

We shall show that

$$H(\vee P \otimes \wedge P) = H_0^0(\vee P \otimes \wedge P) = \Gamma. \tag{2.6}$$

In fact, recall that the differential operator ∇ $(= \nabla_{\vee P})$ is an antiderivation in the graded algebra $\vee P \otimes \wedge P$. Define a second antiderivation k in this algebra by setting

$$k(1 \otimes \Phi) = 0$$

and

$$k(y_1 \vee \cdots \vee y_q \otimes \Phi) = \sum_{i=1}^{q} y_1 \vee \cdots \hat{y}_i \cdots \vee y_q \otimes y_i \wedge \Phi, \qquad y_i \in P, \quad \Phi \in \wedge P.$$

Set $\Delta = \nabla k + k\nabla$. Then Δ is a derivation which reduces to the identity map in $(P \otimes 1) \oplus (1 \otimes P)$. It follows that Δ reduces to $(p + q)\iota$ in $\vee^p P \otimes \wedge^q P$. This implies formula (2.6) if Γ has characteristic zero.

If Γ has characteristic different from zero, we obtain the result as follows: Fix a homogeneous basis $x_1, \ldots, x_r$ of P (and hence of P) and let P_i and P_i denote the 1-dimensional subspaces of P and P spanned by x_i. Then the Künneth formula yields

$$H(\vee P \otimes \wedge P) = H(\vee P_1 \otimes \wedge P_1) \otimes \cdots \otimes H(\vee P_r \otimes \wedge P_r),$$

where the differential operator ∇_i in $\vee P_i \otimes \wedge P_i$ is given by

$$\nabla_i(z \otimes x_i + w \otimes 1) = z \vee x_i \otimes 1, \qquad z, w \in \vee P_i.$$

A trivial verification shows that

$$H(\vee P_i \otimes \wedge P_i) = H_0^0(\vee P_i \otimes \wedge P_i) = \Gamma,$$

and so the same formula must hold for $H(\vee P \otimes \wedge P)$.

2.7.* Interpretation of $H(S \otimes \wedge P)$ as Tor. Make Γ into a P-space by setting

$$\lambda \circ x = 0, \qquad \lambda \in \Gamma, \quad x \in P.$$

Let S be any P-space. Then (cf. [2; p. 106] for the definition of Tor)

$$H(S \otimes \wedge P) = \mathrm{Tor}^{\vee P}(S, \Gamma).$$

To see this, consider the Koszul complex $(\vee P \otimes \wedge P, \nabla)$ of sec. 2.6, and let $\varepsilon: \vee P \to \Gamma$ denote the canonical projection with kernel $\vee^+ P$. Then the sequence

$$\longrightarrow \vee P \otimes \wedge^k P \xrightarrow{\nabla} \vee P \otimes \wedge^{k-1} P \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \vee P \otimes 1 \xrightarrow{\varepsilon} \Gamma \longrightarrow 0$$

is exact, as was shown in sec. 2.6. Evidently, $\vee P \otimes \wedge^k P$ is a free $\vee P$-module, and the maps ∇ and ε are linear over $\vee P$. Thus this sequence is a free resolution of the $\vee P$-module Γ. Hence, by definition,

$$\mathrm{Tor}^{\vee P}(S, \Gamma) = H(S \otimes_{\vee P} (\vee P \otimes \wedge P), \iota \otimes \nabla).$$

But clearly,

$$S \otimes_{\vee P} (\vee P \otimes \wedge P) = S \otimes \wedge P \qquad \text{and} \qquad \iota \otimes \nabla = \nabla_S.$$

It follows that $\mathrm{Tor}^{\vee P}(S, \Gamma) = H(S \otimes \wedge P)$.

§2. Isomorphism theorems

2.8. The first isomorphism theorem. In this section we establish

Theorem I: Let $\varphi: S \to T$ be a homomorphism of graded P-spaces. Then the following conditions are equivalent:

(1) φ is an isomorphism.
(2) $(\varphi \otimes \iota)^{\#}$ is an isomorphism.
(3) $(\varphi \otimes \iota)_0^{\#}$ is an isomorphism and $(\varphi \otimes \iota)_1^{\#}$ is surjective.

The proof is preceded by some preliminary results.

Lemma I: Let F be a graded subspace of the vector space S such that $S = F + S \circ P$. Then

$$S = F \circ \vee P.$$

($F \circ \vee P$ denotes the $\vee P$-module generated by F.)

Proof: A simple induction from the hypotheses yields the relations

$$S = \sum_{i=0}^{k} F \circ \vee^i P + S \circ \vee^{k+1} P, \qquad k = 0, 1, 2, \dots .$$

Since $S^j = 0$ and $P^j = 0$ for $j \leq 0$, it follows that

$$(S \circ \vee^{k+1} P)^k = 0.$$

Hence

$$S^k = \sum_{i=0}^{k} (F \circ \vee^i P)^k, \qquad k = 0, 1, \dots .$$

Q.E.D.

Lemma II: Let S be a P-space such that $H_0(S \otimes \wedge P) = 0$. Then $S = 0$.

Proof: Choose a graded subspace F of S so that $S = F \oplus (S \circ P)$. Then (cf. sec. 2.2) $F \cong H_0(S \otimes \wedge P)$, and so $F = 0$. Hence Lemma I yields

$$S = F \circ \vee P = 0.$$

Q.E.D.

Proposition I: Let $\varphi: S \to T$ be a homomorphism of graded P-spaces. Then φ is surjective if and only if $(\varphi \otimes \iota)_0^\#$ is surjective.

Proof: According to the commutative diagram in sec. 2.3, $(\varphi \otimes \iota)_0^\#$ is surjective if and only if the induced map $\bar{\varphi}: S/S \circ P \to T/T \circ P$ is surjective.

If φ is surjective, then, clearly so is $\bar{\varphi}$. Conversely, assume that $\bar{\varphi}$ is surjective. Then

$$T = \varphi(S) + T \circ P.$$

Applying Lemma I (with $F = \varphi(S)$) we find

$$T = \varphi(S) \circ \vee P = \varphi(S \circ \vee P) = \varphi(S),$$

and so φ is surjective.

Q.E.D.

Proof of Theorem I: Clearly, $(1) \Rightarrow (2) \Rightarrow (3)$. Now assume that (3) holds. Then Proposition I implies that we have a short exact sequence

$$0 \longrightarrow K \xrightarrow{i} S \xrightarrow{\varphi} T \longrightarrow 0$$

of graded P-spaces, where $K = \ker \varphi$.

This yields the long exact sequence

$$\begin{array}{l} \longrightarrow H_1(S \otimes \wedge P) \xrightarrow{(\varphi\otimes\iota)_1^\#} H_1(T \otimes \wedge P) \\ \qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \partial \\ \qquad\qquad\qquad\qquad\qquad H_0(K \otimes \wedge P) \xrightarrow{(i\otimes\iota)_0^\#} H_0(S \otimes \wedge P) \xrightarrow{(\varphi\otimes\iota)_0^\#} H_0(T \otimes \wedge P) \end{array}$$

(cf. sec. 2.3). Since $(\varphi \otimes \iota)_1^\#$ is surjective, it follows that $\partial = 0$. Since $(\varphi \otimes \iota)_0^\#$ is injective, $(i \otimes \iota)_0^\# = 0$. These relations imply that $H_0(K \otimes \wedge P) = 0$. Now Lemma II shows that $K = 0$ and so φ is injective; i.e., $(3) \Rightarrow (1)$.

Q.E.D.

2.9. The second isomorphism theorem. Let S be a P-space. Recall that the map $l^\#: S \to H_0(S \otimes \wedge P)$ is surjective (cf. sec. 2.2) and choose a linear injection

$$\gamma: H_0(S \otimes \wedge P) \to S,$$

homogeneous of degree zero, so that $l^\# \circ \gamma = \iota$. Then

$$S = \operatorname{Im} \gamma \oplus S \circ P.$$

Now make $H_0(S \otimes \wedge P) \otimes \vee P$ into a P-space by setting

$$(\alpha \otimes \Phi) \circ x = \alpha \otimes \Phi \vee x, \qquad \alpha \in H_0(S \otimes \wedge P), \quad \Phi \in \vee P, \quad x \in P,$$

(cf. Example 2, sec. 2.2, and sec. 2.6). Then a homomorphism

$$g\colon H_0(S \otimes \wedge P) \otimes \vee P \to S$$

of graded P-spaces is defined by

$$g(\alpha \otimes \Phi) = \gamma(\alpha) \circ \Phi.$$

Theorem II: Let S, γ, and g be as above. Then the following conditions are equivalent:

(1) g is an isomorphism.

(2) $(g \otimes \iota)^\#\colon H(H_0(S \otimes \wedge P) \otimes \vee P \otimes \wedge P) \to H(S \otimes \wedge P)$ is an isomorphism.

(3) $l^\#\colon S \to H(S \otimes \wedge P)$ is surjective.

(4) $H_+(S \otimes \wedge P) = 0$.

(5) $H_1(S \otimes \wedge P) = 0$.

Proof: Theorem I, sec. 2.8, shows that (1) $\Leftrightarrow$ (2). Since $\operatorname{Im} l^\# = H_0(S \otimes \wedge P)$, it follows that (3) $\Leftrightarrow$ (4). Next, assume (2) holds. Using Example 2, sec. 2.2, and sec. 2.6, observe that

$$H_+(H_0(S \otimes \wedge P) \otimes \vee P \otimes \wedge P) = H_0(S \otimes \wedge P) \otimes H_+(\vee P \otimes \wedge P) = 0.$$

Hence (2) $\Rightarrow$ (4). Clearly, (4) $\Rightarrow$ (5).

It remains to be shown that (5) $\Rightarrow$ (2). Assume that (5) holds. Then (obviously) $(g \otimes \iota)_1^\#$ is surjective. Moreover, the diagram

$$\begin{tikzcd}[column sep=tiny] & H_0(S \otimes \wedge P) \arrow[dl, "\cong"'] \arrow[dr, "\cong", "\bar\gamma"'] & \\ (H_0(S \otimes \wedge P) \otimes \vee P)/(H_0(S \otimes \wedge P) \otimes \vee^+ P) \arrow[rr, "\bar g"'] & & S/(S \circ P) \end{tikzcd}$$

commutes; hence, $\bar g$ is an isomorphism. Thus $(g \otimes \iota)_0^\#$ is an isomorphism (cf. the commutative diagram of sec. 2.3).

Since $(g \otimes \iota)_1^\#$ is surjective and $(g \otimes \iota)_0^\#$ is an isomorphism, Theorem I, sec. 2.8, shows that $(g \otimes \iota)^\#$ is an isomorphism. Thus (5) $\Rightarrow$ (2).

Q.E.D.

Theorem III: Let S, γ, and g be as in Theorem II, and assume that S is evenly graded. Then conditions (1)–(5) in Theorem II are equivalent to

(6) $H(S \otimes \wedge P)$ is evenly graded.

Proof: We show that (3) $\Rightarrow$ (6) $\Rightarrow$ (5). Suppose (3) holds. Then, since S is evenly graded and $l^\#$ is surjective, $H(S \otimes \wedge P)$ is evenly graded. Thus (3) $\Rightarrow$ (6).

Now assume that (6) holds. Then, in particular $H_1^r(S \otimes \wedge P) = 0$, r odd. On the other hand, since $P^k = 0$ for even k, while $S^k = 0$ for odd k, it follows that

$$(S \otimes P)^r = 0, \qquad r \text{ even}.$$

This shows that $H_1^r(S \otimes \wedge P) = 0$, r even. Thus $H_1(S \otimes \wedge P) = 0$; i.e., (6) $\Rightarrow$ (5).

Q.E.D.

2.10. *P*-algebras. Let $(S; \sigma)$ be a P-algebra, and let

$$\gamma \colon H_0(S \otimes \wedge P) \to S$$

be a linear map, homogeneous of degree zero such that

$$l^\# \circ \gamma = \iota \qquad \text{and} \qquad \gamma(1) = 1,$$

(cf. sec. 2.9). Let

$$g \colon H_0(S \otimes \wedge P) \otimes \vee P \to S$$

be the corresponding P-linear map, and observe that the diagram

$$\begin{tikzcd} \vee P \arrow[r, "\eta"] \arrow[dr, "\sigma_\vee"'] & H_0(S \otimes \wedge P) \otimes \vee P \arrow[d, "g"] \\ & S \end{tikzcd} \tag{2.7}$$

commutes, where

$$\eta(\Psi) = 1 \otimes \Psi, \qquad \Psi \in \vee P.$$

Proposition II: Let $(S; \sigma)$ be a connected P-algebra. Then

(1) $\sigma_\vee$ is surjective if and only if $(l^\#)^+ = 0$; i.e., if and only if $H_0(S \otimes \wedge P) = H_0^0(S \otimes \wedge P) = \Gamma$.

(2) If $H_1(S \otimes \wedge P) = 0$, then $\sigma_\vee$ is injective.

(3) $\sigma_\vee$ is an isomorphism if and only if $H(S \otimes \wedge P) = \Gamma$.

Proof: (1) If $\sigma_\vee$ is surjective, then

$$S^+ = \sigma_\vee(\vee^+ P) = S \circ P$$

and so $H_0^+(S \otimes \wedge P) = 0$. Conversely, if $H_0^+(S \otimes \wedge P) = 0$, then $S = \Gamma \oplus S \circ P$. Hence, by Lemma I, sec. 2.8,

$$S = 1 \circ \vee P = \operatorname{Im} \sigma_\vee .$$

(2) Assume that $H_1(S \otimes \wedge P) = 0$. Then, according to Theorem II, sec. 2.9, g is an isomorphism. Now commutative diagram (2.7) implies that $\sigma_\vee$ is injective.

(3) This follows at once from (1) and (2) and sec. 2.6.

Q.E.D.

§3. The Poincaré–Koszul series

2.11. Definition: Let $E = \sum_{p,q} E_q^p$ be a bigraded vector space such that

$$\dim \sum_{p+q=r} E_q^p < \infty, \qquad \text{all } r.$$

A simple gradation of E is defined by $E = \sum_p E^p$, where

$$E^p = \sum_q E_q^p.$$

The *Poincaré–Koszul series of* E is the formal series

$$U_E = \sum_{r=0}^{\infty} c_r t^r,$$

where

$$c_r = \sum_{p+q=r} (-1)^q \dim E_q^p.$$

If E has finite dimension, then U_E is a polynomial. In particular, in this case

$$U_E(-1) = \sum_p (-1)^p \dim E^p;$$

i.e., $U_E(-1)$ is the Euler-Poincaré characteristic χ_E of the graded space E.

Example: Suppose that $E = \sum_p E^p$ is a graded vector space of finite type. Define a bigradation in E by setting

$$E_0^p = E^p \qquad \text{and} \qquad E_q^p = 0, \qquad q > 0.$$

Then

$$U_E = \sum_p \dim E^p t^p = f_E,$$

where f_E denotes the Poincaré series for E.

The following lemma is easy to check:

Lemma III: Let $E = \sum_{p,q} E_q^p$ and $F = \sum_{p,q} F_q^p$ be bigraded vector spaces satisfying the conditions above. Then

$$U_{E\oplus F} = U_E + U_F \qquad \text{and} \qquad U_{E\otimes F} = U_E \cdot U_F$$

(Set $(E \otimes F)_s^r = \sum E_q^p \otimes F_n^m$ where the sum is over those p, q, m, and n such that $p + m = r$, and $q + n = s$.)

2.12. The Poincaré–Koszul series of a P-space. Let S be a P-space of finite type. Then $S \otimes \wedge P$ and $H(S \otimes \wedge P)$ are bigraded spaces satisfying the condition of sec. 2.11. Thus we can form the corresponding Poincaré–Koszul series.

Proposition III: The Poincaré–Koszul series of $S \otimes \wedge P$ and $H(S \otimes \wedge P)$ satisfy

$$U_{H(S\otimes\wedge P)} = U_{S\otimes\wedge P} = f_S \cdot U_{\wedge P} = f_S \cdot (f_{\vee P})^{-1},$$

where $\wedge P$ is given the bigradation

$$(\wedge P)_j^i = (\wedge^j P)^i$$

and $f_{\vee P}$ and f_S denote the Poincaré series of $\vee P$ and S.

Proof: Let N^k be the subspace of $S \otimes \wedge P$ given by

$$N^k = \sum_{i+j=k} (S \otimes \wedge P)_i^j.$$

Define a gradation in N^k by

$$N^k = \sum_i N_i^k, \qquad N_i^k = (S \otimes \wedge P)_i^{k-i}.$$

Then N^k is stable under ∇_S, and ∇_S maps N_i^k into N_{i-1}^k.

Applying the Euler–Poincaré formula to the differential space (N^k, ∇_S) we obtain

$$\sum_i (-1)^i \dim H_i(N^k) = \sum_i (-1)^i \dim N_i^k.$$

But $H_i(N^k) = H_i^{k-i}(S \otimes \wedge P)$, and so

$$\sum_i (-1)^i \dim H_i^{k-i}(S \otimes \wedge P) = \sum_i (-1)^i \dim(S \otimes \wedge P)_i^{k-i}$$

This shows that $U_{H(S\otimes\wedge P)} = U_{S\otimes\wedge P}$.

On the other hand, Lemma III, sec. 2.11, and the example of sec. 2.11 yield

$$U_{S\otimes\wedge P} = U_S \cdot U_{\wedge P} = f_S \cdot U_{\wedge P}.$$

Finally, recall from sec. 2.6 that $H(\vee P \otimes \wedge P) = \Gamma$. It follows that

$$1 = U_{H(\vee P \otimes \wedge P)} = f_{\vee P} \cdot U_{\wedge P}.$$

Hence $U_{\wedge P} = (f_{\vee P})^{-1}$, and so

$$U_{H(S \otimes \wedge P)} = U_{S \otimes \wedge P} = f_S \cdot U_{\wedge P} = f_S \cdot (f_{\vee P})^{-1}.$$

Q.E.D.

Corollary I: Suppose that the Poincaré polynomial of P is given by

$$f_P = t^{g_1} + \cdots + t^{g_r}.$$

Then

$$U_{H(S \otimes \wedge P)} = f_S \cdot \prod_{i=1}^{r} (1 - t^{g_i+1}).$$

Corollary II: Assume that $H_+(S \otimes \wedge P) = 0$. Then

$$f_{H(S \otimes \wedge P)} = U_{H(S \otimes \wedge P)},$$

and so

$$f_{H(S \otimes \wedge P)} = f_S \cdot (f_{\vee P})^{-1} = f_S \cdot \prod_{i=1}^{r} (1 - t^{g_i+1}).$$

Corollary III: Let $0 \to S \to T \to W \to 0$ be a short exact sequence of graded P-spaces. Then

$$U_{H(T \otimes \wedge P)} = U_{H(S \otimes \wedge P)} + U_{H(W \otimes \wedge P)}.$$

Corollary IV: The series $U_{H(S \otimes \wedge P)}$ is independent of the P-structure of S.

Finally, suppose $H(S \otimes \wedge P)$ has finite dimension. Then its Euler–Poincaré characteristic is given by

$$\chi_{H(S \otimes \wedge P)} = U_{H(S \otimes \wedge P)}(-1) \tag{2.8}$$

(cf. sec. 2.11). If, in addition, S is evenly graded, then Proposition III shows that

$$U_{H(S \otimes \wedge P)}(t) = U_{H(S \otimes \wedge P)}(-t).$$

Thus, in this case

$$\chi_{H(S \otimes \wedge P)} = U_{H(S \otimes \wedge P)}(1). \tag{2.9}$$

§4. Structure theorems

In this article $(S;\sigma)$ and $(T;\tau)$ denote connected P-algebras. We identify S^0 and T^0 with Γ via the isomorphisms $\lambda \mapsto \lambda \cdot 1$, $\lambda \in \Gamma$. Recall that the Koszul complexes are written $(S \otimes \wedge P, \nabla_\sigma)$ and $(T \otimes \wedge P, \nabla_\tau)$.

2.13. The Samelson projection. Define a linear map

$$\varrho_S \colon S \otimes \wedge P \to \wedge P$$

by setting

$$\varrho_S(1 \otimes \Psi + z \otimes \Phi) = \Psi, \qquad \Phi, \Psi \in \wedge P, \quad z \in S^+.$$

Then, since $S \circ P \subset S^+$, we have $\varrho_S \circ \nabla_S = 0$. Hence ϱ_S induces a homomorphism

$$\varrho_S^\# : H(S \otimes \wedge P) \to \wedge P$$

of graded algebras.

Definition: The homomorphism $\varrho_S^\#$ is called the *Samelson projection for* $(S;\sigma)$, and the graded space

$$\hat{P}_S = P \cap \operatorname{Im} \varrho_S^\#$$

is called the *Samelson subspace of P.*

If $\varphi\colon S \to T$ is a homomorphism of graded P-algebras, then $\varrho_T \circ (\varphi \otimes \iota) = \varrho_S$; thus the diagram

$$\begin{tikzcd}
H(S \otimes \wedge P) \arrow[d, "(\varphi\otimes\iota)^\#"'] \arrow[dr, "\varrho_S^\#"] & \\
H(T \otimes \wedge P) \arrow[r, "\varrho_T^\#"'] & \wedge P
\end{tikzcd}$$

commutes. In particular the Samelson spaces satisfy $\hat{P}_S \subset \hat{P}_T$.

Theorem IV: Let $(S; \sigma)$ be a connected P-algebra. Then

$$\operatorname{Im} \varrho_S^\# = \wedge \hat{P}_S.$$

Proof: Evidently $i(x^*) \circ \varrho_S = \varrho_S \circ i(x^*)$, $x^* \in P^*$, whence

$$i(x^*) \circ \varrho_S^\# = \varrho_S^\# \circ i(x^*)^\#, \qquad x^* \in P^*.$$

This relation shows that the subalgebra $\operatorname{Im} \varrho_S^\#$ is stable under the operators $i(x^*)$. Now Proposition I, sec. 0.4, implies that $\operatorname{Im} \varrho_S^\# = \wedge \hat{P}_S$.

Q.E.D.

Definition: A *Samelson complement* for $(S; \sigma)$ is a graded subspace $\tilde{P}_S$ of P such that $P = \tilde{P}_S \oplus \hat{P}_S$.

Proposition IV: Let $(S; \sigma)$ be a connected P-algebra. Then an element x of P is in $\hat{P}_S$ if and only if

$$\sigma(x) \in S^+ \cdot \sigma(P).$$

Moreover, if $\tilde{P}_S$ is a Samelson complement, then

$$\sigma(\hat{P}_S) \subset S^+ \cdot \sigma(\tilde{P}_S).$$

Proof: (1) Fix $x \in P$. It is immediate from the definitions that

$$\hat{P}_S = \varrho_S(\ker \nabla_\sigma \cap (S \otimes P)).$$

Therefore $x \in \hat{P}_S$ if and only if there are elements $x_i \in P$ and $z_i \in S^+$ such that

$$\nabla_\sigma\Big(1 \otimes x + \sum_i z_i \otimes x_i\Big) = 0;$$

i.e., $x \in \hat{P}_S$ if and only if $\sigma(x)$ has the form

$$\sigma(x) = \sum_i z_i \cdot \sigma(x_i).$$

But this condition is equivalent to $\sigma(x) \in S^+ \cdot \sigma(P)$.

(2) Observe via (1) that

$$\sigma(\hat{P}_S^k) \subset S^+ \cdot \sum_{<k} \sigma(P^j) \subset S^+ \cdot \sigma(\tilde{P}_S) + S^+ \cdot \sum_{j<k} \sigma(\hat{P}_S^j).$$

Now induction on k yields the relation

$$\sigma(\hat{P}_S) \subset S^+ \cdot \sigma(\hat{P}_S).$$

Q.E.D.

2.14. The cohomology sequence. The sequence

$$\vee P \xrightarrow{\sigma_\vee} S \xrightarrow{l_S^\#} H(S \otimes \wedge P) \xrightarrow{\varrho_S^\#} \wedge P$$

is called the *cohomology sequence of the P-algebra* $(S; \sigma)$.

A homomorphism $\varphi: S \to T$ of graded P-algebras induces the commutative diagram

$$\begin{tikzcd}
 & S \arrow[r, "l_S^\#"] \arrow[d, "\varphi"] & H(S \otimes \wedge P) \arrow[d, "(\varphi\otimes\iota)^\#"] \arrow[dr, "\varrho_S^\#"] & \\
\vee P \arrow[ur, "\sigma_\vee"] \arrow[dr, "\tau_\vee"'] & & & \wedge P \\
 & T \arrow[r, "l_T^\#"'] & H(T \otimes \wedge P) \arrow[ur, "\varrho_T^\#"'] &
\end{tikzcd}$$

of cohomology sequences.

On the other hand, let P_1 be a second finite-dimensional positively graded vector space such that $P_1^k = 0$ for even k. Assume that $\alpha: P_1 \to P$ is a linear map, homogeneous of degree zero. Then, setting $\sigma_1 = \sigma \circ \alpha$, we obtain a P_1-algebra $(S; \sigma_1)$. Evidently

$$(\iota \otimes \alpha_\wedge): S \otimes \wedge P_1 \to S \otimes \wedge P$$

is a homomorphism of graded differential algebras, preserving the bigradation. Thus we obtain a commutative diagram of cohomology sequences

$$\begin{tikzcd}
\vee P_1 \arrow[r, "(\sigma_1)_\vee"] \arrow[d, "\alpha_\vee"] & S \arrow[r] \arrow[d, "\iota"] & H(S \otimes \wedge P_1) \arrow[r] \arrow[d, "(\iota\otimes\alpha_\wedge)^\#"] & \wedge P_1 \arrow[d, "\alpha_\wedge"] \\
\vee P \arrow[r, "\sigma_\vee"'] & S \arrow[r] & H(S \otimes \wedge P) \arrow[r] & \wedge P.
\end{tikzcd}$$

Note as well that

$$i(x^*)^\# \circ (\iota \otimes \alpha_\wedge)^\# = (\iota \otimes \alpha_\wedge)^\# \circ i(\alpha^*(x^*))^\#, \qquad x^* \in P^*.$$

Finally we have

Proposition V: In the cohomology sequence for $(S; \sigma)$:

(1) $l_S^\# \circ \sigma^+ = 0$, and $\ker l_S^\#$ coincides with the ideal in S generated by $\operatorname{Im} \sigma_\vee^+$. In particular, $(l_S^\#)^+ = 0$ holds if and only if $\sigma_\vee$ is surjective.

(2) $\varrho_S^\# \circ (l_S^\#)^+ = 0$, and so $\ker \varrho_S^\#$ contains the ideal in $H(S \otimes \wedge P)$ generated by $\operatorname{Im}(l_S^\#)^+$.

Proof: Cf. sec. 2.2, and Proposition II, sec. 2.10. Q.E.D.

2.15. The reduction theorem. Let $\hat{P}$ and $\tilde{P}$ denote the Samelson subspace and a Samelson complement for the P-algebra $(S; \sigma)$. Then multiplication defines an isomorphism

$$g: \wedge\tilde{P} \otimes \wedge\hat{P} \xrightarrow{\cong} \wedge P$$

of graded algebras.

Next, let $(S; \tilde{\sigma})$ be the $\tilde{P}$-algebra obtained by restricting σ to $\tilde{P}$, and denote its Koszul complex by $(S \otimes \wedge\tilde{P}, \nabla_{\tilde{\sigma}})$. Then $\nabla_{\tilde{\sigma}}$ is the restriction of ∇_σ to $S \otimes \wedge\tilde{P}$ (cf. sec. 2.14). Moreover,

$$(S \otimes \wedge\tilde{P} \otimes \wedge\hat{P}, \nabla_{\tilde{\sigma}} \otimes \iota)$$

is a graded differential algebra.

Theorem V (reduction theorem): Suppose that $(S; \sigma)$ is an alternating connected P-algebra with Samelson space $\hat{P}$, and let $\tilde{P}$ be a Samelson complement. Then there is an isomorphism

$$f: (S \otimes \wedge\tilde{P} \otimes \wedge\hat{P}, \nabla_{\tilde{\sigma}} \otimes \iota) \xrightarrow{\cong} (S \otimes \wedge P, \nabla_\sigma)$$

of graded differential algebras, such that the diagram

$$\begin{tikzcd}
& S \otimes \wedge\tilde{P} \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\
S \otimes \wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "f", "\cong"'] \arrow[d, "\tilde{\varrho}_S \otimes \iota"'] & & S \otimes \wedge P \arrow[d, "\varrho_S"] \\
\wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "\cong", "g"'] & & \wedge P
\end{tikzcd}$$

commutes. (λ_1 and λ_2 are the obvious inclusion maps.)

Proof: (1) *Construction of f:* Choose a linear map

$$\beta\colon \hat{P} \to \ker \nabla_\sigma \cap S \otimes P,$$

homogeneous of degree zero, such that $\varrho_S \circ \beta = \iota$. Since S is alternating, so is $S \otimes \wedge P$. Thus, since $\hat{P}^k = 0$ for even k, we have

$$\beta(x)^2 = 0, \qquad x \in \hat{P}.$$

Hence β extends to a homomorphism

$$\beta_\wedge\colon \wedge\hat{P} \to \ker \nabla_\sigma.$$

Now define f by setting

$$f(z \otimes \Phi \otimes \Psi) = (z \otimes \Phi) \cdot \beta_\wedge(\Psi), \qquad z \in S, \quad \Phi \in \wedge\hat{P}, \quad \Psi \in \wedge\hat{P}.$$

(2) *f is a homomorphism of graded differential algebras:* Clearly f is a homomorphism of graded algebras. Moreover, using the relations

$$\nabla_\sigma \circ \beta_\wedge = 0 \qquad \text{and} \qquad \nabla_\sigma|_{S\otimes\wedge\hat{P}} = \nabla_{\hat{\sigma}}$$

we find that for $z \in S$, $\Phi \in \wedge\hat{P}$, and $\Psi \in \wedge\hat{P}$,

$$\begin{aligned}(\nabla_\sigma \circ f)(z \otimes \Phi \otimes \Psi) &= \nabla_\sigma((z \otimes \Phi) \cdot \beta_\wedge(\Psi))\\ &= \nabla_{\hat{\sigma}}(z \otimes \Phi) \cdot \beta_\wedge(\Psi)\\ &= (f \circ (\nabla_{\hat{\sigma}} \otimes \iota))(z \otimes \Phi \otimes \Psi).\end{aligned}$$

Thus $\nabla_\sigma \circ f = f \circ (\nabla_{\hat{\sigma}} \otimes \iota)$.

(3) *f is an isomorphism:* Since $\varrho_S \circ \beta = \iota$, it follows that for $x \in \hat{P}$

$$\beta(x) = 1 \otimes x + w, \qquad w \in S^+ \otimes P.$$

This implies that

$$f - \iota \otimes g\colon S^k \otimes \wedge\hat{P} \otimes \wedge\hat{P} \to \sum_{j>k} S^j \otimes \wedge P. \tag{2.10}$$

Now set

$$I^k = \sum_{j\geq k} S^j.$$

Then the algebras $S \otimes \wedge\hat{P} \otimes \wedge\hat{P}$ and $S \otimes \wedge P$ are filtered respectively by the ideals $I^k \otimes \wedge\hat{P} \otimes \wedge\hat{P}$ and $I^k \otimes \wedge P$. Moreover, $\iota \otimes g$ and $\iota \otimes g^{-1}$ are filtration preserving isomorphisms; hence $\iota \otimes g$ induces

an isomorphism $A_{\iota\otimes g}$ between the associated graded algebras (cf. sec. 1.1). Since $\iota \otimes g$ is filtration preserving, formula (2.10) shows that so is f and that

$$A_f = A_{\iota\otimes g}.$$

Thus A_f is an isomorphism; hence so is f (cf. Proposition VII, sec. 1.14).

(4) *The diagram commutes*: It is trivial that the upper triangle commutes. Moreover, formula (2.10) yields

$$\varrho_S \circ f = \varrho_S \circ (\iota \otimes g) = g \circ (\tilde{\varrho}_S \otimes \iota).$$

Q.E.D.

Corollary I: f induces an isomorphism of graded algebras

$$f^\#: H(S \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \xrightarrow{\cong} H(S \otimes \wedge P)$$

for which the diagram

$$\begin{tikzcd}
 & H(S \otimes \wedge\tilde{P}) \arrow[dl, "\lambda_1^\#"'] \arrow[dr, "\lambda_2^\#"] & \\
H(S \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \arrow[rr, "f^\#", "\cong"'] \arrow[d, "\tilde{\varrho}_S^\# \otimes \iota"'] & & H(S \otimes \wedge P) \arrow[d, "\varrho_S^\#"] \\
\wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "\cong", "g"'] & & \wedge P
\end{tikzcd}$$

commutes.

Corollary II: The diagram

$$\begin{tikzcd}
 & H(S \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \arrow[dd, "f^\#", "\cong"'] \arrow[dr] & \\
S \arrow[ur, "\tilde{l}_S^\#"] \arrow[dr, "l_S^\#"'] & & \wedge\hat{P} \\
 & H(S \otimes \wedge P) \arrow[ur, "\varrho_S^\#"'] &
\end{tikzcd}$$

commutes. In particular, $(\tilde{\varrho}_S^\#)^+ = 0$.

Proof: The Samelson theorem (sec. 2.13) shows that $\operatorname{Im} \varrho_S^\# = \wedge \hat{P}$. It follows that (cf. Corollary I)

$$(\tilde{\varrho}_S^\#)^+ = 0.$$

Now Corollary II follows from Corollary I.

Q.E.D.

Corollary III: $f^\#$ restricts to an isomorphism

$$f^\#\colon \operatorname{Im} \tilde{\iota}_S^\# \xrightarrow{\cong} \operatorname{Im} \iota_S^\# .$$

Corollary IV: $f^\#$ restricts to an isomorphism

$$f^\#\colon H^+(S \otimes \wedge \hat{P}) \otimes \wedge \hat{P} \xrightarrow{\cong} \ker \varrho_S^\# .$$

2.16. Simplification theorem. The theorem of this section is, in some sense, a generalization of Theorem V. Let $(S; \sigma)$ be an alternating graded connected P-algebra. Assume $\alpha\colon P \to S^+ \cdot \sigma(P)$ is a linear map, homogeneous of degree 1, and define a second P-algebra $(S; \tau)$ by setting $\tau = \sigma + \alpha$.

Theorem VI (simplification theorem): With the hypotheses and notation above, there is an isomorphism

$$f\colon (S \otimes \wedge P, \nabla_\tau) \xrightarrow{\cong} (S \otimes \wedge P, \nabla_\sigma)$$

of bigraded differential algebras, such that the diagram

$$\begin{tikzcd}
& S \otimes \wedge P \arrow[dd, "f", "\cong"'] \arrow[dr, "\varrho_S"] & \\
S \arrow[ur, "\iota_S"] \arrow[dr, "\iota_S"'] & & \wedge P \\
& S \otimes \wedge P \arrow[ur, "\varrho_S"'] &
\end{tikzcd}$$

commutes.

Proof: Choose a linear map $\gamma\colon P \to S^+ \otimes P$, homogeneous of degree zero, and such that $\nabla_\sigma \circ \gamma = \alpha$. Define $\beta\colon P \to S \otimes P$ by

$$\beta(x) = 1 \otimes x + \gamma(x), \quad x \in P,$$

and set

$$f(z \otimes \Phi) = (z \otimes 1) \cdot \beta_\wedge(\Phi), \qquad z \in S, \quad \Phi \in \wedge P.$$

Then it follows, exactly as in the proof of the reduction theorem, that f is an isomorphism making the diagram above commute. Moreover

$$\begin{aligned}(f \circ \nabla_\tau)(1 \otimes x) = \tau(x) \otimes 1 &= \sigma(x) \otimes 1 + \nabla_\sigma(\gamma(x)) \\ &= (\nabla_\sigma \circ f)(1 \otimes x), \qquad x \in P\end{aligned}$$

and

$$(f \circ \nabla_\tau)(z \otimes 1) = 0 = (\nabla_\sigma \circ f)(z \otimes 1), \qquad z \in S.$$

Since $f \circ \nabla_\tau$ and $\nabla_\sigma \circ f$ are f-antiderivations, this implies that $f \circ \nabla_\tau = \nabla_\sigma \circ f$.

Q.E.D.

Corollary I: There is an isomorphism

$$g\colon (S \otimes \wedge P, \nabla_\sigma) \xrightarrow{\cong} (S \otimes \wedge P, \nabla_\tau)$$

which satisfies $g \circ l_S = l_S$ and $\varrho_S \circ g = \varrho_S$.

Corollary II: f induces an isomorphism

$$f^\#\colon H(S \otimes \wedge P, \nabla_\tau) \xrightarrow{\cong} H(S \otimes \wedge P, \nabla_\sigma)$$

of graded algebras, which makes the diagram

$$\begin{tikzcd}
& H(S \otimes \wedge P, \nabla_\tau) \arrow[dd, "f^\#", "\cong"'] \arrow[dr, "\varrho_S^\#"] & \\
S \arrow[ur, "l_S^\#"] \arrow[dr, "l_S^\#"'] & & \wedge P \\
& H(S \otimes \wedge P, \nabla_\sigma) \arrow[ur, "\varrho_S^\#"'] &
\end{tikzcd}$$

commute. (Note that we have used $l_S^\#$ (and $\varrho_S^\#$) to denote two different homomorphisms!)

§5. Symmetric P-algebras

2.17. The main theorem. Let Q be an evenly graded finite-dimensional vector space with $Q^k = 0$ for $k \leq 0$. Let $\vee Q$ have the induced gradation; i.e.,

$$\deg(y_1 \vee \cdots \vee y_q) = \deg y_1 + \cdots + \deg y_q.$$

Then if $\sigma: P \to \vee Q$ is a linear map homogeneous of degree 1, the pair $(\vee Q; \sigma)$ will be called a *symmetric P-algebra.*

Given a symmetric P-algebra $(\vee Q; \sigma)$ with Samelson space $\hat{P}$, define an integer $k(\vee Q; \sigma)$ by

$$k(\vee Q; \sigma) = \dim P - \dim \hat{P} - \dim Q.$$

A main purpose of this article is to establish

Theorem VII: Let $(\vee Q; \sigma)$ be a symmetric P-algebra such that $H(\vee Q \otimes \wedge P)$ has finite dimension. Then $k(\vee Q; \sigma) \geq 0$; i.e.,

$$\dim P \geq \dim \hat{P} + \dim Q.$$

Moreover, if $\tilde{P}$ is a Samelson complement, then

$$H_j(\vee Q \otimes \wedge \tilde{P}) \neq 0, \qquad j = k(\vee Q; \sigma),$$

while

$$H_j(\vee Q \otimes \wedge \tilde{P}) = 0, \qquad j > k(\vee Q; \sigma).$$

In particular, the following conditions are equivalent:

(1) $\dim P = \dim \hat{P} + \dim Q$.

(2) $H_+(\vee Q \otimes \wedge \tilde{P}) = 0$.

The first statement is established in Proposition VI, below. The rest of the theorem is proved in Proposition VII, sec. 2.18.

Proposition VI: If $\dim H(\vee Q \otimes \wedge P) < \infty$, then $k(\vee Q; \sigma) \geq 0$.

Proof: Let $\tilde{P}$ be a Samelson complement. According to Corollary I of the reduction theorem (sec. 2.15),

$$H(\vee Q \otimes \wedge P) \cong H(\vee Q \otimes \wedge \tilde{P}) \otimes \wedge \hat{P}.$$

Thus $H(\vee Q \otimes \wedge \tilde{P})$ has finite dimension; in particular, its Poincaré–Koszul series $U_{H(\vee Q \otimes \wedge \tilde{P})}$ is a polynomial.

We show that $k(\vee Q; \sigma)$ is the multiplicity of 1 as a root of the polynomial $U_{H(\vee Q \otimes \wedge \tilde{P})}$. Write the Poincaré polynomials of $\tilde{P}$ and Q in the form

$$f_{\tilde{P}} = \sum_{i=1}^{\tilde{n}} t^{g_i} \qquad (g_i \text{ odd}), \qquad \tilde{n} = \dim \tilde{P},$$

and

$$f_Q = \sum_{j=1}^{m} t^{l_j} \qquad (l_j \text{ even}), \qquad m = \dim Q.$$

Then

$$f_{\vee \tilde{P}} = \prod_{i=1}^{\tilde{n}} (1 - t^{g_i+1})^{-1} \qquad \text{and} \qquad f_{\vee Q} = \prod_{j=1}^{m} (1 - t^{l_j})^{-1}.$$

Thus Corollary I to Proposition III, sec. 2.12, yields

$$\prod_{j=1}^{m} (1 - t^{l_j}) \cdot U_{H(\vee Q \otimes \wedge \tilde{P})} = \prod_{i=1}^{\tilde{n}} (1 - t^{g_i+1}).$$

Now let r ($r \geq 0$) be the multiplicity of 1 as a root of $U_{H(\vee Q \otimes \wedge \tilde{P})}$. Then the equation above implies that $m + r = \tilde{n}$, whence

$$k(\vee Q; \sigma) = \tilde{n} - m = r.$$

Q.E.D.

Corollary: The relation

$$\dim P = \dim \hat{P} + \dim Q$$

holds if and only if $H(\vee Q \otimes \wedge \tilde{P})$ has nonzero Euler–Poincaré characteristic.

Proof: Since $\vee Q$ is evenly graded,

$$\chi_{H(\vee Q \otimes \wedge \tilde{P})} = U_{H(\vee Q \otimes \wedge \tilde{P})}(1)$$

(cf. sec. 2.12). Now apply the proposition.

Q.E.D.

2.18. Proposition VII: With the hypotheses of Theorem VII,

$$H_j(\vee Q \otimes \wedge \tilde{P}) \neq 0, \quad j = k(\vee Q; \sigma)$$

and

$$H_j(\vee Q \otimes \wedge \tilde{P}) = 0, \quad j > k(\vee Q; \sigma).$$

Proof: Since $\dim \tilde{P} = \dim P - \dim \hat{P}$, and since the Samelson space for the $\tilde{P}$-algebra $(\vee Q; \tilde{\sigma})$ is zero (cf. Corollary II to Theorem V, sec. 2.15), it is clearly sufficient to consider the case $\hat{P} = 0$, $\tilde{P} = P$. We do so from now on.

Define a graded space T by setting $T^p = Q^{p+1}$ ($p = 1, 3, 5, \ldots$). Let $\dim T = \dim Q = m$ and $\dim P = n$; then $k(\vee Q; \sigma) = n - m$. Let k denote the greatest integer such that

$$H_k(\vee Q \otimes \wedge P) \neq 0.$$

We must show that $k = n - m$.

Consider the algebra $R = \vee Q \otimes \wedge P \otimes \wedge T$ (skew tensor product). Define a bigradation in R by setting

$$R = \sum_{p,q} R^{p,q}, \qquad R^{p,q} = \vee Q \otimes \wedge^{n-p} P \otimes \wedge^{m-q} T;$$

denote the corresponding total gradation by

$$R = \sum_r R^{(r)}, \qquad R^{(r)} = \sum_{p+q=r} R^{p,q}.$$

Set $R^{p,*} = \sum_q R^{p,q}$ and $R^{*,q} = \sum_p R^{p,q}$.

Let $(\vee Q \otimes \wedge P; \tau)$ be the T-algebra defined by

$$\tau(y) = y \otimes 1, \qquad y \in T,$$

and denote the corresponding differential operator in $\vee Q \otimes \wedge P \otimes \wedge T$ by ∇_τ. It is homogeneous of bidegree $(0, 1)$. On the other hand, if ∇_σ denotes the differential operator in $\vee Q \otimes \wedge P$, then $\nabla_\sigma \otimes \iota$ is homogeneous of bidegree $(1, 0)$.

Set

$$D = \nabla_\sigma \otimes \iota + \nabla_\tau.$$

Then $D^2 = 0$, and so (R, D) is a graded differential space (with respect to the total gradation defined above). The proposition is now an immediate consequence of the following two lemmas.

Lemma IV: $H^{(r)}(R, D) = 0,\ 0 \le r < m$, and $H^{(m)}(R, D) \neq 0$.

Proof: Define a linear map $\alpha\colon P \to \vee Q \otimes 1 \otimes T$ such that $\nabla_\tau \circ \alpha = \sigma$. Then the linear map $\beta\colon P \to R$ given by

$$\beta(x) = 1 \otimes x \otimes 1 - \alpha(x), \qquad x \in P,$$

extends to an algebra homomorphism $\beta_\wedge\colon \wedge P \to R$.

Now consider the algebra homomorphism $\varphi\colon R \to R$ given by

$$\varphi(z \otimes \Phi \otimes \Psi) = (z \otimes 1 \otimes 1) \cdot \beta_\wedge(\Phi) \cdot 1 \otimes 1 \otimes \Psi.$$

It is easy to verify that φ is an isomorphism of graded algebras (R has the gradation defined just above).

A straightforward computation shows that $D\varphi - \varphi\nabla_\tau$ is zero in $\vee Q \otimes 1 \otimes 1$, $1 \otimes P \otimes 1$, and $1 \otimes 1 \otimes T$. Since these spaces generate the algebra R and since $D\varphi - \varphi\nabla_\tau$ is a φ-antiderivation, it follows that

$$D\varphi = \varphi\nabla_\tau.$$

This implies that φ induces an isomorphism,

$$\varphi^\#\colon H(\vee Q \otimes \wedge P \otimes \wedge T, \nabla_\tau) \xrightarrow{\cong} H(R, D).$$

Since φ is homogeneous of degree zero, and since $H(\vee Q \otimes \wedge T) = \Gamma$ (cf. sec. 2.6) we obtain isomorphisms

$$\wedge^{n-j} P \xrightarrow{\cong} H^{(m+j)}(R, D).$$

The lemma follows.

Q.E.D.

Lemma V: Let k be the greatest integer such that $H_k(\vee Q \otimes \wedge P) \neq 0$. Then

$$H^{(r)}(R, D) = 0, \quad r < n - k, \qquad \text{and} \qquad H^{(n-k)}(R, D) \neq 0.$$

Proof: Filter R by the subspaces

$$F^q(R) = \sum_{j \ge q} R^{*,j} = \vee Q \otimes \wedge P \otimes \sum_{i=0}^{m-q} \wedge^i T.$$

Then the E_2-term of the corresponding spectral sequence is given by

$$E_2 \cong H(H(\vee Q \otimes \wedge P) \otimes \wedge T, \nabla_\tau^\#) \tag{2.11}$$

(cf. Theorem II, sec. 1.19, and sec. 1.21). Here $\nabla_\tau^\#$ is defined by

$$\nabla_\tau^\#(\alpha \otimes 1) = 0$$

and

$$\begin{aligned}\nabla_\tau^\#(\alpha \otimes y_1 \wedge \cdots \wedge y_p) \\ = (-1)^q \sum_{i=1}^{p} (-1)^{i-1} \alpha \cdot l_{\vee Q}^\#(y_i) \otimes y_1 \wedge \cdots \hat{y}_i \cdots \wedge y_p, \\ \alpha \in H^q(\vee Q \otimes \wedge P), \quad y_i \in T.\end{aligned}$$

In particular, the spaces $H_p(\vee Q \otimes \wedge P) \otimes \wedge T$ are stable under $\nabla_\tau^\#$. This implies that

$$\begin{aligned}H(H(\vee Q \otimes \wedge P) \otimes \wedge T, \nabla_\tau^\#) &= \sum_p H(H_p(\vee Q \otimes \wedge P) \otimes \wedge T) \\ &= \sum_{p,q} H_q(H_p(\vee Q \otimes \wedge P) \otimes \wedge T).\end{aligned}$$

Composing this with (2.11) we obtain

$$E_2^{q,p} = H_{m-q}(H_{n-p}(\vee Q \otimes \wedge P) \otimes \wedge T). \tag{2.12}$$

Next we establish the relations

$$E_i^{q,p} = 0, \qquad p < n - k, \quad i \geq 2, \tag{2.13}$$

$$E_2^{0,n-k} \neq 0, \tag{2.14}$$

and

$$E_i^{(n-k)} \neq 0, \qquad i \geq 2. \tag{2.15}$$

Formula (2.13) (for $i = 2$) is an immediate consequence of the definition of k, and the formula for $i > 2$ follows at once from the definition of the $E_i^{q,p}$.

To prove (2.14) choose a non-zero element $a \in \wedge^m T$ and a non-zero element $\alpha \in H_k(\vee Q \otimes \wedge P)$ with maximum degree (with respect to the gradation defined in sec. 2.2). Then $\alpha \cdot l_{\vee Q}^\#(y) = 0$, $y \in T$, and so

$$\nabla_\tau^\#(\alpha \otimes a) = 0.$$

Thus $\alpha \otimes a$ is a nonzero cocycle in $H_k(\vee Q \otimes \wedge P) \otimes \wedge^m T$.

Since T has dimension m,

$$\operatorname{Im} \nabla_\tau^\# \subset \sum_{j=0}^{m-1} H(\vee Q \otimes \wedge P) \otimes \wedge^j T;$$

hence $\alpha \otimes a$ represents a nonzero cohomology class. This implies that

$$H_m(H_k(\vee Q \otimes \wedge P) \otimes \wedge T) \neq 0.$$

Combining this relation with formula (2.12) we obtain formula (2.14).
To prove (2.15) observe that, in view of (2.13), $E_i^{q,p} = 0$ if $p < n - k$ and $i \geq 2$. Since

$$d_i \colon E_i^{q,p} \to E_i^{q+i,p-i+1},$$

a straightforward induction, starting with (2.14), establishes (2.15).
In particular, relations (2.13) and (2.15) imply that

$$H^{(r)}(R, D) = 0, \quad r < n - k, \qquad \text{and} \qquad H^{(n-k)}(R, D) \cong E_\infty^{(n-k)} \neq 0.$$

Q.E.D.

Remark: With the proof of Lemma V, the proofs of Proposition VII and Theorem VII are completed.

2.19. The decomposition theorem. In Theorem VIII below we give a number of conditions on a symmetric P-algebra $(\vee Q; \sigma)$ which are equivalent to conditions (1) and (2) of Theorem VII. Part of Theorem VIII remains true for any P-algebra; however, that part is true in even greater generality (P-differential algebras) and will be established in Chapter III (cf. sec. 3.16).

Theorem VIII: Let $(\vee Q; \sigma)$ be a symmetric P-algebra with Samelson space $\hat{P}$ and a Samelson complement $\tilde{P}$. Assume that $H(\vee Q \otimes \wedge P)$ has finite dimension. Then the following conditions are equivalent:

(1) $\dim P = \dim \hat{P} + \dim Q$.
(2) The kernel of $\varrho_{\vee Q}^\#$ coincides with the ideal generated by $\operatorname{Im}(l_{\vee Q}^\#)^+$
(3) The map $\tilde{l}_{\vee Q}^\# \colon \vee Q \to H(\vee Q \otimes \wedge \tilde{P})$ is surjective.
(4) There is an isomorphism

$$[\vee Q / \vee Q \circ P] \otimes \wedge \hat{P} \xrightarrow{\cong} H(\vee Q \otimes \wedge P),$$

of graded algebras which makes the diagram

$$\begin{tikzcd}
& {[\vee Q/\vee Q \circ P] \otimes \wedge \hat{P}} \arrow[dd, "\cong"] \arrow[dr] & \\
\vee Q \arrow[ur] \arrow[dr, "\iota^*_{\vee Q}"'] & & \wedge \hat{P} \\
& H(\vee Q \otimes \wedge P) \arrow[ur, "\varrho^*_{\vee Q}"'] &
\end{tikzcd}$$

commute.

(5) There is an isomorphism

$$[\vee Q/\vee Q \circ P] \otimes \vee \hat{P} \xrightarrow{\cong} \vee Q$$

of graded *$\tilde{P}$-spaces* which makes the diagram

$$\begin{tikzcd}
& {[\vee Q/\vee Q \circ P] \otimes \vee \tilde{P}} \arrow[dd, "\cong"] \arrow[dr] & \\
\vee \tilde{P} \arrow[ur] \arrow[dr, "\tilde{\sigma}_\vee"'] & & \vee Q/\vee Q \circ P \\
& \vee Q \arrow[ur, "\iota_{\vee Q}"'] &
\end{tikzcd}$$

commute.

(6) $H_1(\vee Q \otimes \wedge \tilde{P}) = 0$.

(7) $H_+(\vee Q \otimes \wedge \tilde{P}) = 0$.

(8) $H(\vee Q \otimes \wedge \tilde{P})$ is evenly graded.

(9) $H(\vee Q \otimes \wedge \tilde{P})$ has nonzero Euler–Poincaré characteristic.

(10) The map $\tilde{\sigma}_\vee : \vee \tilde{P} \to \vee Q$ is injective.

Proof: We show that

$$\begin{array}{cccc}
(1) \Leftrightarrow (7), & (2) \Leftrightarrow (3), & (3) \Leftrightarrow (4), & (5) \Leftrightarrow (7), \\
(6) \Leftrightarrow (7) \Leftrightarrow (3), & (6) \Leftrightarrow (8), & (7) \Leftrightarrow (9), & (7) \Leftrightarrow (10).
\end{array}$$

(1) $\Leftrightarrow$ (7): This is Theorem VII, sec. 2.17.

(2) $\Leftrightarrow$ (3): In view of the Corollaries III and IV to the reduction theorem (sec. 2.15), condition (2) is equivalent to the following condition:

The ideal in $H^+(\vee Q \otimes \wedge \hat{P})$ generated by $\operatorname{Im} \tilde{l}^\#$ is all of $H^+(\vee Q \otimes \wedge \hat{P})$.

But (by the same argument as the one used in Lemma I, sec. 2.8), this happens if and only if

$$\operatorname{Im} \tilde{l}^\#_{\vee Q} = H^+(\vee Q \otimes \wedge \hat{P}).$$

(3) $\Leftrightarrow$ (4): This is an immediate consequence of the reduction theorem and the fact that $\vee Q \circ P = \vee Q \circ \hat{P}$ (cf. Proposition IV, sec. 2.13).

(5) $\Leftrightarrow$ (7): In view of the relation $\vee Q \circ P = \vee Q \circ \hat{P}$, this follows from Theorem II, sec. 2.9.

(6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (3): This corresponds to the equivalent conditions (5), (4), and (3) in Theorem II, sec. 2.9.

(6) $\Leftrightarrow$ (8): This follows from Theorem III, sec. 2.9.

(7) $\Leftrightarrow$ (9): In view of the corollary to Proposition VI, sec. 2.17, (9) is equivalent to the condition $\dim P = \dim \hat{P} + \dim Q$. By Theorem VII, sec. 2.17, this is equivalent to (7).

(7) $\Leftrightarrow$ (10): In view of the results above, (7) $\Rightarrow$ (5) $\Rightarrow$ (10). Conversely, if (10) holds, Lemma VI, below, implies that $\dim \hat{P} \leq \dim Q$. On the other hand, Theorem VII, sec. 2.17, shows that $\dim \hat{P} \geq \dim Q$. Thus $\dim \hat{P} = \dim Q$; hence, again by Theorem VII, $H_+(\vee Q \otimes \wedge \hat{P}) = 0$; i.e., (10) $\Rightarrow$ (7).

Q.E.D.

Corollary: Assume $(\vee Q; \sigma)$ is a symmetric P-algebra such that $H(\vee Q \otimes \wedge P)$ has finite dimension. Then the following conditions are equivalent:

(1) $\dim P = \dim Q$.

(2) $l^\#_{\vee Q}$ is surjective.

(3) $H(\vee Q \otimes \wedge P) \cong \vee Q / \vee Q \circ P$.

(4) There is an isomorphism

$$g: H_0(\vee Q \otimes \wedge P) \otimes \vee P \xrightarrow{\cong} \vee Q$$

of graded P-spaces such that $g(1) = 1$.

(5) $H_+(\vee Q \otimes \wedge P) = 0$.

(6) $H(\vee Q \otimes \wedge P)$ is evenly graded.

(7) The Euler–Poincaré characteristic of $H(\vee Q \otimes \wedge P)$ is nonzero.

(8) $\sigma_\vee$ is injective.

Lemma VI: Let Q_1 and Q_2 be strictly positively graded, finite-dimensional vector spaces. Give $\vee Q_1$ and $\vee Q_2$ the induced gradations and assume that $\varphi\colon \vee Q_1 \to \vee Q_2$ is a linear injection, homogeneous of degree zero. Then

$$\dim Q_1 \le \dim Q_2.$$

Proof: Let

$$f_1 = \sum_{i=1}^{r} t^{k_i} \qquad \text{and} \qquad f_2 = \sum_{j=1}^{s} t^{l_j}$$

be the Poincaré polynomials of Q_1 and Q_2. Then the Poincaré series of $\vee Q_1$ and $\vee Q_2$ are given by

$$f_{\vee Q_1} = \prod_{i=1}^{r} (1 - t^{k_i})^{-1} \qquad \text{and} \qquad f_{\vee Q_2} = \prod_{j=1}^{s} (1 - t^{l_j})^{-1}.$$

These series are absolutely convergent for $0 \le t < 1$.

Moreover, by hypothesis,

$$\dim(\vee Q_1)^p \le \dim(\vee Q_2)^p, \qquad p = 0, 1, \dots .$$

Hence, for $0 \le t < 1$,

$$f_{\vee Q_1}(t) = \sum_{p=0}^{\infty} \dim(\vee Q_1)^p t^p \le \sum_{p=0}^{\infty} \dim(\vee Q_2)^p t^p = f_{\vee Q_2}(t).$$

Thus $f_{\vee Q_1}(t) \le f_{\vee Q_2}(t)$, $0 \le t < 1$.

On the other hand,

$$f_{\vee Q_2}(t)/f_{\vee Q_1}(t) = \prod_{i=1}^{r} (1 - t^{k_i}) \prod_{j=1}^{s} (1 - t^{l_j})^{-1} = (1 - t)^{r-s} g(t),$$

where g is a continuous function in the interval $0 \le t < \infty$. Thus $(1 - t)^{r-s} g(t) \ge 1$ if $0 \le t < 1$. Since g is continuous at 1, it follows that $r \le s$; i.e. $\dim Q_1 \le \dim Q_2$.

Q.E.D.

2.20. Poincaré series. Let $(\vee Q; \sigma)$ by a symmetric P-algebra such that $H(\vee Q \otimes \wedge P)$ has finite dimension and $\dim P = \dim \hat{P} + \dim Q$. Let $\tilde{P}$ be a Samelson complement.

In view of Theorem VIII, sec. 2.19, and Corollary III to the reduction theorem, sec. 2.15, we have isomorphisms

$$H(\vee Q \otimes \wedge \tilde{P}) = H_0(\vee Q \otimes \wedge \tilde{P}) \cong \operatorname{Im} l_{\vee Q}^{\#}$$

and

$$H(\vee Q \otimes \wedge P) \cong \operatorname{Im} l_{\vee Q}^{\#} \otimes \wedge \hat{P}.$$

Thus Corollary II to Proposition III, sec. 2.12, gives

$$f_{\operatorname{Im} l_{\vee Q}^{\#}} = f_{\vee Q} \cdot f_{\vee \hat{P}}^{-1},$$

whence

$$f_{H(\vee Q \otimes \wedge P)} = f_{\operatorname{Im} l_{\vee Q}^{\#}} f_{\wedge \hat{P}} = f_{\vee Q} \cdot f_{\vee \hat{P}}^{-1} \cdot f_{\wedge \hat{P}}.$$

Since $f_{\vee P} = f_{\vee \hat{P}} f_{\vee \tilde{P}}$, these equations yield

$$f_{\operatorname{Im} l_{\vee Q}^{\#}} = f_{\vee Q} \cdot f_{\vee \hat{P}} \cdot f_{\vee P}^{-1} \tag{2.16}$$

and

$$f_{H(\vee Q \otimes \wedge P)} = \frac{f_{\vee Q} f_{\vee \hat{P}} f_{\wedge \hat{P}}}{f_{\vee P}}. \tag{2.17}$$

In particular write the Poincaré polynomials of P, $\hat{P}$, and Q in the form

$$f_P = \sum_{i=1}^{r} t^{g_i}, \qquad f_{\hat{P}} = \sum_{i=s+1}^{r} t^{g_i}, \qquad \text{and} \qquad f_Q = \sum_{i=1}^{s} t^{k_i},$$

(g_i odd, k_i even, $r = \dim P$, $s = \dim Q$, $r - s = \dim \hat{P}$). Then these equations read

$$f_{\operatorname{Im} l_{\vee Q}^{\#}} = \frac{\prod_{i=1}^{s} (1 - t^{g_i+1})}{\prod_{i=1}^{s} (1 - t^{k_i})} \tag{2.18}$$

and

$$f_{H(\vee Q \otimes \wedge P)} = \frac{\prod_{i=1}^{s} (1 - t^{g_i+1}) \prod_{i=s+1}^{r} (1 + t^{g_i})}{\prod_{i=1}^{s} (1 - t^{k_i})}. \tag{2.19}$$

Now consider the Euler–Poincaré characteristics of $\operatorname{Im} l_{\vee Q}^{\#}$ and $H(\vee Q \otimes \wedge P)$. Since $\operatorname{Im} l_{\vee Q}^{\#}$ is evenly graded, we obtain from formula (2.18)

$$\chi_{\operatorname{Im} l_{\vee Q}^{\#}} = \dim \operatorname{Im} l_{\vee Q}^{\#} = \frac{\prod_{i=1}^{s} (g_i + 1)}{\prod_{i=1}^{s} k_i}. \tag{2.20a}$$

Moreover, if $\hat{P} \neq 0$, then $\chi_{H(\vee Q \otimes \wedge P)} = 0$; while if $\hat{P} = 0$, then

$$H(\vee Q \otimes \wedge P) = \operatorname{Im} l_{\vee Q}^{\#}$$

and

$$\chi_{H(\vee Q \otimes \wedge P)} = \dim H(\vee Q \otimes \wedge P) = \frac{\prod_{i=1}^{r} (g_i + 1)}{\prod_{i=1}^{r} k_i}. \tag{2.20b}$$

2.21. A third structure theorem. Theorem IX: Let $(\vee Q; \sigma)$ be a symmetric P-algebra with Samelson space $\hat{P}$ and a Samelson complement $\tilde{P}$. Then the following conditions are equivalent:

(1) $(l_{\vee Q}^{\#})^{+} = 0$.

(2) $\sigma_{\vee}: \vee P \to \vee Q$ is surjective.

(3) $\tilde{\sigma}_{\vee}: \vee \tilde{P} \to \vee Q$ is an isomorphism.

(4) $\varrho_{\vee Q}^{\#}: H(\vee Q \otimes \wedge P) \to \wedge \hat{P}$ is an isomorphism.

(5) The algebra $H(\vee Q \otimes \wedge P)$ is generated by 1 together with elements of odd degree.

If these conditions hold, then

$$\dim P = \dim \hat{P} + \dim Q.$$

Proof: According to Proposition V, (1), sec. 2.14, (1) $\Leftrightarrow$ (2). Now we show that

$$(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1).$$

In fact, suppose (1) holds. Then Corollary III of the reduction theorem (sec. 2.15) shows that $(\tilde{l}_{\vee Q}^{\#})^{+} = 0$. Moreover, the Samelson space for $(\vee Q; \tilde{\sigma})$ is zero, as follows from Corollary I of the reduction theorem. Hence we can apply Lemma VII below (with $\tilde{P}$ replacing P) to show that $\tilde{\sigma}_{\vee}$ is an isomorphism. Thus (1) $\Rightarrow$ (3).

Suppose (3) holds. Then by sec. 2.6, $H(\vee Q \otimes \wedge \tilde{P}) = \Gamma$. Now Corollary I to the reduction theorem implies that

$$f^{\#}: \wedge \hat{P} \xrightarrow{\cong} H(\vee Q \otimes \wedge P).$$

Moreover, it is an easy consequence of the commutative diagram in that corollary that $\varrho_{\vee Q}^{\#} = (f^{\#})^{-1}$. Thus (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (5) is obvious. Suppose that (5) holds. Since $\vee Q$ is evenly graded, so is $H_0(\vee Q \otimes \wedge P)$; thus the elements of odd degree are contained in the ideal $H_{+}(\vee Q \otimes \wedge P)$. It follows that $H_{+}(\vee Q \otimes \wedge P)$, together with 1, generates $H(\vee Q \otimes \wedge P)$. Hence $H_0^{+}(\vee Q \otimes \wedge P) = 0$, and so $(l_{\vee Q}^{\#})^{+} = 0$. This shows that (5) $\Rightarrow$ (1).

Finally, if these conditions hold, then (3) implies that $\dim \tilde{P} = \dim Q$, whence

$$\dim P = \dim \hat{P} + \dim \tilde{P} = \dim \hat{P} + \dim Q.$$

Q.E.D.

Lemma VII: Let $(\vee Q; \sigma)$ be a symmetric P-algebra such that

$$(l^{\#}_{\vee Q})^{+} = 0 \qquad \text{and} \qquad \hat{P} = 0.$$

Then $\sigma_{\vee}: \vee P \to \vee Q$ is an isomorphism.

Proof: By Proposition V, (1), sec. 2.14, the condition $(l^{\#}_{\vee Q})^{+} = 0$ implies that $\sigma_{\vee}$ is surjective. Hence there is a linear injection $\varphi: \vee Q \to \vee P$, homogeneous of degree zero, such that $\sigma_{\vee} \circ \varphi = \iota$. Now Lemma VI, sec. 2.19, yields

$$\dim Q \leq \dim P.$$

On the other hand, let $\pi: \vee^{+}Q \to Q$ be the projection with kernel $(\vee^{+}Q) \cdot (\vee^{+}Q)$. We show that $\pi \circ \sigma: P \to Q$ is injective. In fact, if $\pi(\sigma(x)) = 0$ for some $x \in P$, then

$$\sigma(x) \in (\vee^{+}Q) \cdot (\vee^{+}Q) = \sigma_{\vee}(\vee^{+}P) \cdot \sigma_{\vee}(\vee^{+}P).$$

Now Proposition IV, sec. 2.13, shows that $x \in \hat{P}$. Hence, $x = 0$ and so $\pi \circ \sigma$ is injective.

Since $\dim Q \leq \dim P$, it follows that $\pi \circ \sigma$: $P \to Q$ is an isomorphism of *graded* vector spaces. Hence $\vee P$ and $\vee Q$ have the same Poincaré series. Thus $\sigma_{\vee}$ restricts to linear surjections

$$\sigma_{\vee}: (\vee P)^{k} \to (\vee Q)^{k}, \qquad k = 0, 1, \ldots$$

between spaces of the same dimension. Hence $\sigma_{\vee}$ is an isomorphism.

Q.E.D.

§6. Essential P-algebras

2.22. Essential P-algebras. Let $(\vee Q; \sigma)$ be a symmetric P-algebra. Consider the ideal $\vee^+Q \cdot \vee^+Q$ in $\vee Q$. Define a graded subspace P_1 of P by

$$P_1 = \sigma^{-1}(\vee^+Q \cdot \vee^+Q).$$

It is called the *essential subspace* for the P-algebra $(\vee Q; \sigma)$.

Note that the Samelson space $\hat{P}$ of the P-algebra $(\vee Q; \sigma)$ is contained in the essential subspace P_1: $\hat{P} \subset P_1$. In fact, if $x \in \hat{P}$, then, by Proposition IV, sec. 2.13,

$$\sigma(x) \in \vee^+Q \cdot \sigma(P).$$

In particular $\sigma(x) \in \vee^+Q \cdot \vee^+Q$, and so $x \in P_1$.

A symmetric P-algebra is called *essential* if $P_1 = P$; i.e., if

$$\sigma(P) \subset \vee^+Q \cdot \vee^+Q.$$

2.23. The associated essential P_1-algebra. Given a symmetric P-algebra $(\vee Q; \sigma)$ with essential subspace P_1, we shall construct an essential P_1-algebra $(\vee Q_1; \sigma_1)$ (with Q_1 a graded subspace of Q) such that

$$H(\vee Q \otimes \wedge P) \cong H(\vee Q_1 \otimes \wedge P_1).$$

Choose a graded subspace $P_2 \subset P$ so that

$$P = P_1 \oplus P_2.$$

Let $\pi: \vee^+Q \to Q$ denote the projection with kernel $\vee^+Q \cdot \vee^+Q$. Then the map $\pi \circ \sigma: P \to Q$ restricts to a linear isomorphism

$$\pi \circ \sigma: P_2 \xrightarrow{\cong} \operatorname{Im}(\pi \circ \sigma).$$

Now choose a graded subspace $Q_1 \subset Q$ so that

$$Q = Q_1 \oplus \operatorname{Im}(\pi \circ \sigma).$$

Then a homomorphism of graded algebras

$$\eta: \vee P_2 \otimes \vee Q_1 \to \vee Q$$

is given by

$$\eta(\Psi \otimes \Phi) = \sigma_\vee(\Psi) \vee \Phi, \qquad \Psi \in \vee P_2, \quad \Phi \in \vee Q_1.$$

Lemma VIII: η is an isomorphism of graded algebras.

Proof: Define a linear map $\psi: P_2 \oplus Q_1 \to \vee Q$ by

$$\psi(x, y) = \sigma(x) + y, \qquad x \in P_2, \quad y \in Q_1.$$

Then $\eta = \psi_\vee$ (here $\vee(P_2 \oplus Q_1)$ is identified with $\vee P_2 \otimes \vee Q_1$).

Now filter the algebras $\vee(P_2 \oplus Q_1)$ and $\vee Q$ by the ideals

$$\sum_{j \geq q} \vee^j(P_2 \oplus Q_1) \qquad \text{and} \qquad \sum_{j \geq q} \vee^j Q.$$

Since η is a homomorphism, it is filtration preserving. Denote the induced homomorphism of associated graded algebras by A_η.

Next, observe that the canonical linear isomorphism

$$Q \xrightarrow{\cong} \vee^+ Q / \vee^+ Q \cdot \vee^+ Q$$

extends to an algebra isomorphism $\vee Q \xrightarrow{\cong} A_{\vee Q}$ ($A_{\vee Q}$ the associated graded algebra; cf. sec. 1.17). Similarly we obtain an algebra isomorphism

$$\vee(P_2 \oplus Q_1) \xrightarrow{\cong} A_{\vee(P_2 \oplus Q_1)}.$$

A simple computation shows that the diagram

$$\begin{tikzcd} \vee(P_2 \oplus Q_1) \arrow[r, "\cong"] \arrow[d, "(\pi \circ \psi)_\vee"'] & A_{\vee(P_2 \oplus Q_1)} \arrow[d, "A_\eta"] \\ \vee Q \arrow[r, "\cong"] & A_{\vee Q} \end{tikzcd}$$

commutes.

Since the map $\pi \circ \psi: P_2 \oplus Q_1 \to Q$ is given by

$$(\pi \circ \psi)(x, y) = (\pi \circ \sigma)(x) + y, \qquad x \in P_2, \quad y \in Q_1,$$

it is a linear isomorphism. Hence $(\pi \circ \psi)_\vee$ is an isomorphism and so the

commutative diagram implies that A_η is an isomorphism. It follows now from Proposition VII, sec. 1.14, that η is an isomorphism.

Q.E.D.

The lemma shows that $\vee Q$ admits the direct decomposition

$$\vee Q = \vee Q_1 \oplus \vee Q \cdot \sigma(P_2).$$

This decomposition determines a projection

$$\gamma: \vee Q \to \vee Q_1$$

with kernel $\vee Q \cdot \sigma(P_2)$; γ is a homomorphism of graded algebras.

Next define a linear map $\sigma_1: P_1 \to \vee Q$ by $\sigma_1 = \gamma \circ \sigma$. Then the P_1-algebra $(\vee Q_1; \sigma_1)$ is essential. In fact, since

$$\sigma(P_1) \subset \vee^+ Q \cdot \vee^+ Q \qquad \text{and} \qquad \gamma(\vee^+ Q) \subset \vee^+ Q_1,$$

it follows that

$$\sigma_1(P_1) \subset \gamma(\vee^+ Q \cdot \vee^+ Q) \subset \gamma(\vee^+ Q) \cdot \gamma(\vee^+ Q) \subset \vee^+ Q_1 \cdot \vee^+ Q_1.$$

The P_1-algebra $(\vee Q_1; \sigma_1)$ is called the *associated essential P_1-algebra for* $(\vee Q; \sigma)$.

Note that the associated essential P_1-algebra depends only on the choice of the graded subspaces P_2 of P and Q_1 of Q. In particular, if $(\vee P; \sigma)$ is itself essential, then $P_1 = P$, $Q_1 = Q$, and $\sigma_1 = \sigma$. Thus $(\vee Q_1; \sigma_1) = (\vee Q; \sigma)$ in this case.

To construct an isomorphism between the cohomology algebras of $(\vee Q; \sigma)$ and $(\vee Q_1; \sigma_1)$, let $\beta: P \to P_1$ denote the projection induced by the direct decomposition $P = P_1 \oplus P_2$. Extend β to a homomorphism $\beta_\wedge: \wedge P \to \wedge P_1$. Then

$$\gamma \otimes \beta_\wedge: (\vee Q \otimes \wedge P, \nabla_\sigma) \to (\vee Q_1 \otimes \wedge P_1, \nabla_{\sigma_1})$$

is a homomorphism of graded differential algebras and the diagram

$$\begin{tikzcd}
\vee P \arrow[r, "\sigma_\vee"] \arrow[d, "\beta_\vee"'] & \vee Q \arrow[r, "\iota_{\vee Q}"] \arrow[d, "\gamma"] & \vee Q \otimes \wedge P \arrow[r, "\varrho_{\vee Q}"] \arrow[d, "\gamma\otimes\beta_\wedge"] & \wedge P \arrow[d, "\beta_\wedge"] \\
\vee P_1 \arrow[r, "(\sigma_1)_\vee"'] & \vee Q_1 \arrow[r, "\iota_{\vee Q_1}"'] & \vee Q_1 \otimes \wedge P_1 \arrow[r, "\varrho_{\vee Q_1}"'] & \wedge P_1
\end{tikzcd} \tag{2.21}$$

commutes.

Theorem X: (1) $\gamma \otimes \beta_1$ induces an isomorphism of graded algebras

$$(\gamma \otimes \beta_\wedge)^\#: H(\vee Q \otimes \wedge P) \xrightarrow{\cong} H(\vee Q_1 \otimes \wedge P_1).$$

(2) The diagram

$$\begin{tikzcd}
\vee P \arrow[r, "\sigma_\vee"] \arrow[d, "\beta_\vee"'] & \vee Q \arrow[r, "i^\#_{\vee Q}"] \arrow[d, "\gamma"] & H(\vee Q \otimes \wedge P) \arrow[r, "\varrho^\#_{\vee Q}"] \arrow[d, "\cong"', "(\gamma\otimes\beta_\wedge)^\#"] & \wedge P \arrow[d, "\beta_\wedge"] \\
\vee P_1 \arrow[r, "(\sigma_1)_\vee"'] & \vee Q_1 \arrow[r, "i^\#_{\vee Q_1}"'] & H(\vee Q_1 \otimes \wedge P_1) \arrow[r, "\varrho^\#_{\vee Q_1}"'] & \wedge P_1
\end{tikzcd}$$

commutes.

(3) The Samelson spaces of the P-algebras $(\vee Q; \sigma)$ and $(\vee Q_1; \sigma_1)$ coincide.

Proof: (1) Identify $\vee Q$ with $\vee P_2 \otimes \vee Q_1$ via η, and write

$$\vee Q \otimes \wedge P = \vee P_2 \otimes \vee Q_1 \otimes \wedge P_1 \otimes \wedge P_2.$$

Then the subspaces

$$F^{-p} = \sum_{\mu \le p} \vee P_2 \otimes \vee Q_1 \otimes \wedge^\mu P_1 \otimes \wedge P_2, \qquad p = 0, \ldots, \dim P_1$$

define a filtration of $\vee Q \otimes \wedge P$. The E_1-term of the corresponding spectral sequence is given by

$$E_1 \cong \vee Q_1 \otimes \wedge P_1 \otimes H(\vee P_2 \otimes \wedge P_2)$$

(cf. Theorem II, sec. 1.19). Since $H(\vee P_2 \otimes \wedge P_2) = \Gamma$ (cf. sec. 2.6), it follows that

$$E_1 \cong \vee Q_1 \otimes \wedge P_1.$$

On the other hand, filter $\vee Q_1 \otimes \wedge P_1$ by the subspaces

$$\hat{F}^{-p} = \sum_{\mu \le p} \vee Q_1 \otimes \wedge^\mu P_1.$$

Then $\hat{E}_1 = \vee Q_1 \otimes \wedge P_1$. Moreover, the map $\gamma \otimes \beta_\wedge$ is filtration preserving and the induced homomorphism of E_1-terms is simply the identity map of $\vee Q_1 \otimes \wedge P_1$. Thus, by the comparison theorem (sec. 1.14), $(\gamma \otimes \beta_\wedge)^\#$ is an isomorphism.

(2) This follows immediately from the commutative diagram (2.21).

(3) In view of (2), $\beta_\wedge$ restricts to a surjective linear map between the Samelson spaces $\hat{P}$ and $\hat{P}_1$. But since $\hat{P} \subset P_1$ (cf. sec. 2.22) β is the identity map in $\hat{P}$. This shows that $\hat{P} = \hat{P}_1$.

Q.E.D.

Chapter III

Koszul Complexes of *P*-Differential Algebras

In this chapter $P = \sum_k P^k$ denotes a finite-dimensional positively graded vector space satisfying $P^k = 0$ for even k. $\mathbb{P}$ is the evenly graded space given by $\mathbb{P}^k = P^{k-1}$, and $\wedge P$ and $\vee \mathbb{P}$ are the graded algebras described at the start of Chapter II.

§1. *P*-differential algebras

3.1. Definition: A *P-differential algebra* ((P, δ)-algebra) is a triple $(B, \delta_B; \tau)$ where

(1) (B, δ_B) is an associative, alternating, positively graded, differential algebra with unit element 1.

(2) $\tau: P \to \ker \delta_B$ is a linear map, homogeneous of degree 1.

The differential algebra (B, δ_B) is called the *base* of the (P, δ)-algebra $(B, \delta_B; \tau)$.

Note that if $(B, \delta_B; \tau)$ is a (P, δ)-algebra, then $(B; \tau)$ is a P-algebra.

A *homomorphism of* (P, δ)*-algebras*

$$\varphi: (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$$

is a homomorphism $\varphi: (B, \delta_B) \to (\tilde{B}, \delta_{\tilde{B}})$ of graded differential algebras that satisfies $\varphi(1) = (1)$, and $\varphi \circ \tau = \tilde{\tau}$.

A *semimorphism of* (P, δ)*-algebras* is a homomorphism of graded differential spaces $\psi: (B, \delta_B) \to (\tilde{B}, \delta_{\tilde{B}})$ that satisfies $\psi(1) = 1$, and

$$\psi(b \cdot \tau(x)) = \psi(b) \cdot \tilde{\tau}(x), \qquad b \in B, \quad x \in P.$$

Thus every homomorphism is a semimorphism.

3.2. The Koszul complex. With each (P, δ)-algebra $(B, \delta_B; \tau)$ is associated a graded differential algebra $(B \otimes \wedge P, \nabla_B)$ as follows: $B \otimes \wedge P$ denotes the skew tensor product of the graded algebras B and $\wedge P$, with the gradation given by

$$(B \otimes \wedge P)^r = \sum_{p+q=r} B^p \otimes (\wedge P)^q.$$

Define operators δ_B and ∇_τ in $B \otimes \wedge P$ by $\delta_B = \delta_B \otimes \iota$, and

$$\nabla_\tau(b \otimes 1) = 0,$$

$$\nabla_\tau(b \otimes x_1 \wedge \cdots \wedge x_q) = (-1)^p \sum_{i=1}^{q} (-1)^{i-1} b \cdot \tau(x_i) \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_q, \qquad b \in B^p, \quad x_i \in P.$$

Then $(B \otimes \wedge P, \nabla_\tau)$ is the Koszul complex of the underlying P-algebra $(B; \tau)$; in particular, ∇_τ is an antiderivation of square zero, homogeneous of degree 1. Moreover, since $\delta_B \circ \tau = 0$, it follows that

$$\nabla_\tau \circ \delta_B + \delta_B \circ \nabla_\tau = 0.$$

Now set

$$\nabla_B = \delta_B + \nabla_\tau$$

Then ∇_B is also an antiderivation homogeneous of degree 1 which satisfies $\nabla_B^2 = 0$. Thus $(B \otimes \wedge P, \nabla_B)$ is a graded differential algebra. It is called the *Koszul complex of the* (P, δ)*-algebra* $(B, \delta_B; \tau)$, and ∇_B is called the *differential operator in* $B \otimes \wedge P$. The graded algebra $H(B \otimes \wedge P, \nabla_B)$ is called the *cohomology algebra of the* (P, δ)-algebra $(B, \delta_B; \tau)$. We sometimes denote it simply by $H(B \otimes \wedge P)$.

Next observe that the inclusion map $b \mapsto b \otimes 1$ $(b \in B)$ defines a homomorphism

$$l_B\colon (B, \delta_B) \to (B \otimes \wedge P, \nabla_B),$$

of graded differential algebras; l_B is called the *base inclusion.* It induces a homomorphism

$$l_B^\#\colon H(B) \to H(B \otimes \wedge P)$$

of graded algebras.

Remark: Clearly $l_B^\#$ restricts to an isomorphism $H^0(B) \xrightarrow{\cong} H^0(B \otimes \wedge P)$. Thus $H(B)$ is connected if and only if $H(B \otimes \wedge P)$ is connected. In this case the (P, δ)-algebra $(B, \delta_B; \tau)$ is called c-*connected.*

On the other hand, recall from sec. 2.2 that each element $x^* \in P^*$ determines the linear operator $i(x^*)$ in $B \otimes \wedge P$ given by

$$i(x^*)(b \otimes \Phi) = (-1)^p b \otimes i(x^*)\Phi, \qquad b \in B^p, \quad \Phi \in \wedge P.$$

Evidently $i(x^*)$ is an antiderivation and satisfies

$$i(x^*) \circ \delta_B + \delta_B \circ i(x^*) = 0 \qquad \text{and} \qquad i(x^*)\nabla_\tau + \nabla_\tau i(x^*) = 0$$

(cf. sec. 2.2). These relations imply that

$$i(x^*)\nabla_B + \nabla_B i(x^*) = 0.$$

Hence $i(x^*)$ induces an antiderivation $i(x^*)^\#$ in $H(B \otimes \wedge P)$.

Thus we obtain operators $i(a)$ in $B \otimes \wedge P$ and $i(a)^\#$ in $H(B \otimes \wedge P)$ $(a \in \wedge P^*)$ by setting

$$i(x_1^* \wedge \cdots \wedge x_p^*) = i(x_p^*) \circ \cdots \circ i(x_1^*), \qquad x_i^* \in P^*.$$

Furthermore, since B is alternating and $\tau(P) \subset \sum_{p \text{ even}} B^p$, we have the relations

$$\tau(x) \cdot \tau(y) = \tau(y) \cdot \tau(x), \qquad x, y \in P.$$

Thus τ extends to a homomorphism

$$\tau_\vee : \vee P \to B$$

of graded algebras. Since $\delta_B \circ \tau = 0$ it follows that $\delta_B \circ \tau_\vee = 0$. Hence, composing $\tau_\vee$ with the projection $\ker \delta_B \to H(B)$ we obtain a homomorphism

$$\tau_\vee^\# : \vee P \to H(B)$$

of graded algebras.

Now suppose $\varphi : (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ is a semimorphism. Then the map $\varphi \otimes \iota : B \otimes \wedge P \to \tilde{B} \otimes \wedge P$ commutes with the differential operators ∇_B and $\nabla_{\tilde{B}}$, and so it induces a linear map

$$(\varphi \otimes \iota)^\# : H(B \otimes \wedge P) \to H(\tilde{B} \otimes \wedge P),$$

homogeneous of degree zero. Moreover, the commutative diagram

$$
\begin{tikzcd}
 & B \arrow[r, "l_B"] \arrow[dd, "\varphi"] & B \otimes \wedge P \arrow[dd, "\varphi\otimes\iota"] \\
\vee P \arrow[ur, "\tau_\vee"] \arrow[dr, "\tilde{\tau}_\vee"'] & & \\
 & \tilde{B} \arrow[r, "l_{\tilde{B}}"'] & B \otimes \wedge P
\end{tikzcd}
$$

induces the commutative diagram

$$
\begin{tikzcd}
 & H(B) \arrow[r, "l_B^{\#}"] \arrow[dd, "\varphi^{\#}"] & H(B \otimes \wedge P) \arrow[dd, "(\varphi\otimes\iota)^{\#}"] \\
\vee P \arrow[ur, "\tau_\vee^{\#}"] \arrow[dr, "\tilde{\tau}_\vee^{\#}"'] & & \\
 & H(\tilde{B}) \arrow[r, "l_{\tilde{B}}^{\#}"'] & H(\tilde{B} \otimes \wedge P)
\end{tikzcd}
$$

in cohomology.

The relations $i(x^*) \circ (\varphi \otimes \iota) = (\varphi \otimes \iota) \circ i(x^*)$, $x^* \in P^*$, imply that

$$i(x^*)^{\#} \circ (\varphi \otimes \iota)^{\#} = (\varphi \otimes \iota)^{\#} \circ i(x^*)^{\#}.$$

If φ is a homomorphism, then so are $\varphi \otimes \iota$ and $(\varphi \otimes \iota)^{\#}$.

Finally observe that if $\delta_B = 0$ (so that $(B, \delta_B; \tau)$ is a P-algebra), then the definitions and notation above reduce to those in Chapter II.

3.3. The associated *P*-algebra. Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra. Since $\delta_B \circ \tau = 0$, τ induces a linear map $\tau^{\#}: P \to H(B)$ which makes $H(B)$ into an alternating P-algebra. It is called the *associated P-algebra.* Its Koszul complex is $(H(B) \otimes \wedge P, \nabla_{\tau^{\#}})$, and $\nabla_{\tau^{\#}}$ is the operator induced in $H(B) \otimes \wedge P$ by ∇_τ; i.e., $\nabla_{\tau^{\#}} = (\nabla_\tau)^{\#}$.

If $\varphi: (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ is a semimorphism of (P, δ)-algebras, then $\varphi^{\#}$ is a homomorphism of graded P-spaces. Hence $\varphi^{\#} \otimes \iota$ commutes with the differential operators $\nabla_{\tau^{\#}}$ and $\nabla_{\tilde{\tau}^{\#}}$. It induces a linear map

$$(\varphi^{\#} \otimes \iota)^{\#}: H(H(B) \otimes \wedge P, \nabla_{\tau^{\#}}) \to H(H(\tilde{B}) \otimes \wedge P, \nabla_{\tilde{\tau}^{\#}}).$$

If φ is a homomorphism of (P, δ)-algebras, then $\varphi^{\#}$ is a homomorphism of P-algebras, $\varphi^{\#} \otimes \iota$ is a homomorphism of graded differential algebras, and $(\varphi^{\#} \otimes \iota)^{\#}$ is a homomorphism of graded algebras.

3.4. The spectral sequence. Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra. Filter $B \otimes \wedge P$ by the ideals

$$F^p(B \otimes \wedge P) = \sum_{\mu \geq p} B^\mu \otimes \wedge P. \tag{3.1}$$

Then $(B \otimes \wedge P, \nabla_B)$ becomes a graded filtered differential algebra (cf. sec. 1.18). This filtration leads to a convergent spectral sequence of graded differential algebras; it is called the *spectral sequence of the* (P, δ)-*algebra* $(B, \delta_B; \tau)$.

Observe that the filtration (3.1) arises out of the bigradation

$$B \otimes \wedge P = \sum_{p,q} B^p \otimes (\wedge P)^q.$$

With respect to this bigradation $(B \otimes \wedge P, \delta_B, \nabla_\tau)$ is a graded differential couple of degree 1 (not in general homogeneous) (cf. sec. 1.21). Thus the spectral sequence of $(B, \delta_B; \tau)$ is the spectral sequence of this couple; in particular,

$$E_0 = E_1 \cong B \otimes \wedge P \qquad \text{and} \qquad E_2 \cong H(B) \otimes \wedge P,$$

as follows from Theorem II, sec. 1.19.

The basic subalgebra (cf. sec. 1.13) for this filtration is simply $B \otimes 1$. It follows that l_B induces surjective maps

$$H^p(B) \to E_i^{p,0}, \qquad i \geq 2,$$

homogeneous of degree zero (cf. Proposition VI, sec. 1.13).

Moreover, the operators $i(x^*)$ are filtration preserving, and hence induce antiderivations in each E_i.

Finally, let $\varphi\colon B \to \tilde{B}$ be a semimorphism of (P, δ)-algebras. Then $(\varphi \otimes \iota)\colon B \otimes \wedge P \to \tilde{B} \otimes \wedge P$ is filtration preserving, and so it induces a homomorphism of spectral sequences. The induced homomorphisms of the E_0, E_1, and E_2 terms correspond under the identifications above to $\varphi \otimes \iota$, $\varphi \otimes \iota$, and $\varphi^{\#} \otimes \iota$ respectively.

3.5. The lower spectral sequence. Consider the bigradation of $B \otimes \wedge P$ given by

$$(B \otimes \wedge P)^{(-k,l)} = (B \otimes \wedge^k P)^{-k+l} \tag{3.2}$$

It gives rise to the filtration of $B \otimes \wedge P$ by the subspaces

$$L^{-k}(B \otimes \wedge P) = \sum_{i=0}^{k} B \otimes \wedge^i P, \qquad k = 0, \ldots, n \ (n = \dim P).$$

Then $L^{-k} \cdot L^{-l} \subset L^{-(k+l)}$, and

$$\cdots \supset L^{-k-1} \supset L^{-k} \supset L^{-k+1} \supset \cdots .$$

In this way $(B \otimes \wedge P, \nabla_B)$ becomes a graded filtered differential algebra. The corresponding spectral sequence is denoted by $(E_r^{(k,l)}, d_r)$ and satisfies

$$E_r^{(k,l)} = 0 \qquad (\text{any } r \geq 0)$$

unless $-n \leq k \leq 0$ ($n = \dim P$) and $k + l \geq 0$. Hence it is convergent (cf. Proposition V, sec. 1.12). It is called the *lower spectral sequence of the* (P, δ)*-algebra* $(B, \delta_B; \tau)$.

Remarks: 1. If one used the convention that $X^{-p} = X_p$ to raise and lower the indices of graded spaces, then the bigradation above is determined by the bigradation

$$(B \otimes \wedge P)_q^p = (B \otimes \wedge^q P)^p$$

used for P-spaces in Chapter II (cf. sec. 2.2). This explains the terminology "lower spectral sequence."

2. Note that with the notation of this section, the Poincaré–Koszul series for $B \otimes \wedge P$ is given by

$$U_{B \otimes \wedge P} = \sum_{k,l} (-1)^k \dim(B \otimes \wedge P)^{(k,l)} t^l$$

whenever the right-hand side is defined.

With respect to the bigradation (3.2) the operators δ_B and ∇_τ are homogeneous of bidegrees (0, 1) and (1,0). Thus the lower spectral sequence is the spectral sequence of the graded homogeneous differential couple (or double complex) $(B \otimes \wedge P, \delta_B, \nabla_\tau)$.

In view of Theorem II, sec. 1.19, the first terms of the lower spectral sequence are given by

$$\begin{aligned} (E_0, d_0) &\cong (B \otimes \wedge P, \delta_B) \\ (E_1, d_1) &\cong (H(B) \otimes \wedge P, \nabla_{\tau *}) \end{aligned} \tag{3.3}$$

and

$$E_2 \cong H(H(B) \otimes \wedge P, \nabla_{\tau *}).$$

In particular, the E_1-term is exactly the Koszul complex of the associated P-algebra $(H(B); \tau^\#)$. Moreover, the isomorphisms above restrict to isomorphisms

$$E_1^{(k,l)} \cong (H(B) \otimes \wedge^{-k} P)^{k+l} \quad \text{and} \quad E_2^{(k,l)} \cong H_{-k}^{k+l}(H(B) \otimes \wedge P, \nabla_{\tau^*}). \tag{3.4}$$

Finally, let $\varphi: B \to \tilde{B}$ be a semimorphism of (P, δ)-algebras. Then $\varphi \otimes \iota$ preserves filtrations and so it induces a homomorphism of lower spectral sequences. The induced homomorphisms of the E_0, E_1, and E_2 terms correspond under the isomorphisms (3.3) to $\varphi \otimes \iota$, $\varphi^\# \otimes \iota$, and $(\varphi^\# \otimes \iota)^\#$, respectively.

3.6. Example: Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra where P is a 1-dimensional space homogeneous of degree g. Let x^* be a basis vector of P^*. Then

$$0 \longrightarrow B \xrightarrow{l_B} B \otimes \wedge P \xrightarrow{i(x^*)} B \longrightarrow 0$$

is an exact sequence of differential spaces (up to sign). Hence we have the exact triangle

$$\begin{tikzcd} H(B) \arrow[rr, "l_B^\#"] & & H(B \otimes \wedge P) \arrow[dl, "i(x^*)^\#"] \\ & H(B) \arrow[ul, "\partial"] & \end{tikzcd}$$

called the *Gysin triangle*. The corresponding long exact sequence

$$\xrightarrow{\partial} H^i(B) \xrightarrow{l_B^\#} H^i(B \otimes \wedge P) \xrightarrow{i(x^*)^\#} H^{i-g}(B) \xrightarrow{\partial} H^{i+1}(B) \xrightarrow{l_B^\#}$$

is called the *Gysin sequence*.

To compute ∂, let $x \in P$ satisfy $\langle x^*, x\rangle = 1$. Then, if $\alpha \in H^p(B)$ is represented by a cocycle b, $\partial\alpha$ is represented by the cocycle

$$\nabla_B((-1)^p b \otimes x) = b \cdot \tau(x).$$

Hence,

$$\partial\alpha = \alpha \cdot \tau^\#(x), \qquad \alpha \in H(B). \tag{3.5}$$

The Gysin sequence yields the important short exact sequence

$$0 \longrightarrow \operatorname{coker} \partial \xrightarrow{l_B^\#} H(B \otimes \wedge P) \xrightarrow{i(x^*)^\#} \ker \partial \longrightarrow 0. \tag{3.6}$$

Now consider the associated P-algebra and its Koszul complex $(H(B) \otimes \wedge P, \nabla_{\tau*})$. Then

$$H(H(B) \otimes \wedge P, \nabla_{\tau*}) = H_0(H(B) \otimes \wedge P) \oplus H_1(H(B) \otimes \wedge P).$$

Evidently,

$$H_0(H(B) \otimes \wedge P) = H(B)/(H(B) \circ P) = \text{coker } \partial,$$

and

$$H_1(H(B) \otimes \wedge P) = (H(B) \otimes \wedge^1 P) \cap \ker \nabla_{\tau*} = (\ker \partial) \otimes x.$$

These equations show that there is a linear isomorphism

$$H(H(B) \otimes \wedge P, \nabla_{\tau*}) \cong H(B \otimes \wedge P, \nabla_B)$$

of graded vector spaces; however, in general this isomorphism *cannot* be chosen to preserve products.

§2. Tensor difference

3.7. Definition. Let $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ be (P, δ)-algebras, and consider the skew tensor product of the graded algebras B and S. Set

$$\delta_{B\otimes S} = \delta_B \otimes \iota - \omega_B \otimes \delta_S,$$

where ω_B denotes the degree involution of B. Then $(B \otimes S, \delta_{B\otimes S})$ is a graded, alternating, differential algebra. The multiplication in $B \otimes S$ induces an isomorphism

$$H(B \otimes S, \delta_{B\otimes S}) \cong H(B) \otimes H(S).$$

Now define a linear map

$$\tau \ominus \sigma : P \to B \otimes S$$

by

$$(\tau \ominus \sigma)(x) = \tau(x) \otimes 1 - 1 \otimes \sigma(x).$$

Then $(B \otimes S, \delta_{B\otimes S}; \tau \ominus \sigma)$ becomes a P-differential algebra. It is called the *tensor difference* of $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$.

The differential operator of the Koszul complex $(B \otimes S \otimes \wedge P, \nabla_{B\otimes S})$ is the sum of four anticommuting antiderivations:

$$\nabla_{B\otimes S} = (\delta_B + \nabla_\tau) - (\delta_S + \nabla_\sigma).$$

Here δ_B and δ_S are the obvious extensions to $B \otimes S \otimes \wedge P$, while ∇_τ and ∇_σ denote the operators corresponding to the P-algebras $(B \otimes S; \tau)$ and $(B \otimes S; \sigma)$.

In particular, ∇_τ and ∇_σ restrict to operators in $B \otimes \wedge P$ and $S \otimes \wedge P$, respectively. The Koszul complexes for $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ are given by

$$(B \otimes \wedge P, \delta_B + \nabla_\tau) \qquad \text{and} \qquad (S \otimes \wedge P, \delta_S + \nabla_\sigma).$$

Next, suppose that $\varphi: B \to \tilde{B}$ and $\psi: S \to \tilde{S}$ are semimorphisms (respectively, homomorphisms) of (P, δ)-algebras. Then

$$\varphi \otimes \psi : B \otimes S \to \tilde{B} \otimes \tilde{S}$$

is a semimorphism (respectively, homomorphism) between the tensor differences. Thus it induces a linear map (respectively, a homomorphism)

$$(\varphi \otimes \psi \otimes \iota)^{\#}: H(B \otimes S \otimes \wedge P) \to H(\tilde{B} \otimes \tilde{S} \otimes \wedge P).$$

Example: If $\delta_S = 0$ (i.e., if $(S, \delta_S; \sigma)$ is simply an alternating P-algebra), then the tensor difference of $(B, \delta_B; \tau)$ and $(S; \sigma)$ is simply $(B \otimes S, \delta_B; \tau \ominus \sigma)$. In this case $\nabla_{B\otimes S}$ is given by

$$\nabla_{B\otimes S} = \delta_B + \nabla_\tau - \nabla_\sigma.$$

(Note that we consistently write $\delta_B = \delta_B \otimes \iota = \delta_B \otimes \iota \otimes \iota$.)

3.8. Tensor difference with $\vee P$. Recall the P-algebra $(\vee P; \sigma)$ given by $\sigma(x) = x$, $x \in P$ (cf. sec. 2.6). Suppose $(B, \delta_B; \tau)$ is any (P, δ)-algebra and form the tensor difference $(B \otimes \vee P, \delta_B; \tau \ominus \sigma)$. Then a homomorphism

$$m_B: (B, \delta_B) \to (B \otimes \vee P \otimes \wedge P, \nabla_{B\otimes\vee P})$$

of graded differential algebras is given by $m_B(b) = b \otimes 1 \otimes 1$.

Proposition I: The homomorphism m_B induces an isomorphism

$$m_B^{\#}: H(B) \xrightarrow{\cong} H(B \otimes \vee P \otimes \wedge P),$$

of graded algebras.

Proof: A linear map $\alpha: P \to B \otimes \vee P$ homogeneous of degree zero, is given by

$$\alpha(x) = \tau(x) \otimes 1 + 1 \otimes x, \qquad x \in P.$$

Since $B \otimes \vee P$ is alternating and $\vee P$ is evenly graded, we have $\alpha(x)\alpha(y) = \alpha(y)\alpha(x)$, $x, y, \in P$. Thus α extends to a homomorphism

$$\alpha_\vee: \vee P \to B \otimes \vee P.$$

Now define a homomorphism of graded algebras $\psi: B \otimes \vee P \to B \otimes \vee P$ by setting

$$\psi(b \otimes \Psi) = (b \otimes 1) \cdot \alpha_\vee(\Psi), \qquad \Psi \in \vee P, \quad b \in B.$$

Filter $B \otimes \vee P$ by the subspaces $\sum_{\mu \geq p} B^\mu \otimes \vee P$. Then ψ is filtration

preserving and induces the identity in the associated graded algebra. Hence, in view of Proposition VII, sec. 1.14, ψ is an isomorphism.

Next observe that the relation $\delta_B \circ \tau = 0$ implies that

$$\psi \circ \delta_B = \delta_B \circ \psi.$$

Moreover,

$$\begin{aligned}(\psi \circ (\tau \ominus \sigma))(x) &= \psi(\tau(x) \otimes 1 - 1 \otimes x)\\ &= -1 \otimes x = -\sigma(x), \qquad x \in P.\end{aligned}$$

Thus

$$\psi\colon (B \otimes \vee P, \delta_B; \tau \ominus \sigma) \xrightarrow{\cong} (B \otimes \vee P, \delta_B; -\sigma)$$

is an isomorphism of (P, δ)-algebras. Hence we have an induced isomorphism

$$(\psi \otimes \iota)^\#\colon H(B \otimes \vee P \otimes \wedge P, \delta_B + \nabla_\tau - \nabla_\sigma) \xrightarrow{\cong} H(B \otimes \vee P \otimes \wedge P, \delta_B - \nabla_\sigma).$$

Finally, since $\psi \circ m_B = m_B$, we have the commutative diagram

$$\begin{tikzcd} & H(B \otimes \vee P \otimes \wedge P, \delta_B + \nabla_\tau - \nabla_\sigma) \arrow[dd, "(\psi\otimes\iota)^\#", "\cong"'] \\ H(B) \arrow[ur, "m_B^\#"] \arrow[dr, "m_B^\#"'] & \\ & H(B \otimes \vee P \otimes \wedge P, \delta_B - \nabla_\sigma). \end{tikzcd}$$

But $H(\vee P \otimes \wedge P, \nabla_\sigma) = \Gamma$ (cf. sec. 2.6), and so the Künneth theorem yields

$$H(B \otimes \vee P \otimes \wedge P, \delta_B - \nabla_\sigma) = H(B) \otimes H(\vee P \otimes \wedge P, \nabla_\sigma) = H(B).$$

This shows that the lower $m_B^\#$ in the diagram above is an isomorphism. Hence so is the upper $m_B^\#$. Q.E.D.

Now we shall construct an isomorphism inverse to $m_B^\#$. Define a homomorphism of graded algebras $\varphi\colon B \otimes \vee P \otimes \wedge P \to B$ by

$$\varphi(b \otimes \Psi \otimes 1) = b \cdot \tau_\vee(\Psi)$$

and

$$\varphi(b \otimes \Psi \otimes \Phi) = 0, \qquad b \in B, \quad \Psi \in \vee P, \quad \Phi \in \wedge^+ P.$$

Proposition II: φ is a homomorphism of graded differential algebras: $\varphi \circ (\delta_B + \nabla_\tau - \nabla_\sigma) = \delta_B \circ \varphi$, and satisfies $\varphi \circ m_B = \iota$. In particular,

$$\varphi^{\#} = (m_B^{\#})^{-1}.$$

Proof: Since $\delta_B \circ \tau_\vee = 0$ it follows that

$$\varphi \circ \delta_B = \delta_B \circ \varphi.$$

Moreover,

$$\varphi \circ (\nabla_\tau - \nabla_\sigma)(b \otimes \Psi \otimes 1) = 0, \qquad b \in B, \quad \Psi \in \vee P,$$

and

$$\begin{aligned}\varphi \circ (\nabla_\tau - \nabla_\sigma)(1 \otimes 1 \otimes x) &= \varphi(\tau(x) \otimes 1 \otimes 1 - 1 \otimes x \otimes 1)\\ &= 0, \qquad x \in P.\end{aligned}$$

Since $\varphi \circ (\nabla_\tau - \nabla_\sigma)$ is a φ-antiderivation and since $B \otimes \vee P \otimes \wedge P$ is generated by the elements of the form $b \otimes \Psi \otimes 1$ and $1 \otimes 1 \otimes x$, it follows that

$$\varphi \circ (\nabla_\tau - \nabla_\sigma) = 0.$$

Hence,

$$\varphi \circ (\delta_B + \nabla_\tau - \nabla_\sigma) = \varphi \circ \delta_B = \delta_B \circ \varphi.$$

Finally, it is obvious that $\varphi \circ m_B = \iota$. Thus $\varphi^{\#} \circ m_B^{\#} = \iota$. By Proposition I, $m_B^{\#}$ is an isomorphism. Thus the above relation shows that $\varphi^{\#}$ is the inverse isomorphism.

Q.E.D.

3.9. Spectral sequences of a tensor difference. Consider the tensor difference of two (P, δ)-algebras $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$. Bigrade the Koszul complex by setting

$$(B \otimes S \otimes \wedge P)^{p,q} = B^p \otimes (S \otimes \wedge P)^q.$$

The corresponding filtration of the algebra $B \otimes S \otimes \wedge P$ is given by the ideals

$$F^p(B \otimes S \otimes \wedge P) = \sum_{\mu \geq p} B^\mu \otimes (S \otimes \wedge P)$$

and leads to a convergent spectral sequence of graded differential algebras. It is called the *B-sequence* of the tensor difference and does *not* coincide with the spectral sequence of the (P, δ)-algebra $(B \otimes S, \delta_{B \otimes S}; \tau \ominus \sigma)$.

The B-sequence is the spectral sequence of the graded differential couple $(B \otimes S \otimes \wedge P, -(\delta_S + \nabla_\sigma), \ \delta_B + \nabla_\tau)$. Since

$$\delta_B: B^p \otimes S \otimes \wedge P \to B^{p+1} \otimes S \otimes \wedge P$$

and

$$\nabla_\tau: B^p \otimes S \otimes \wedge P \to \sum_{\mu \geq 2} B^{p+\mu} \otimes S \otimes \wedge P,$$

the first terms of this sequence are given by

$$E_0^{p,q} \cong B^p \otimes (S \otimes \wedge P)^q$$
$$E_1^{p,q} \cong B^p \otimes H^q(S \otimes \wedge P, \nabla_S) \tag{3.7}$$

and

$$E_2^{p,q} \cong H^p(B) \otimes H^q(S \otimes \wedge P)$$

(cf. sec. 1.19 and sec. 1.21). All the above isomorphisms are algebra isomorphisms.

Now let $\varphi: B \to \tilde{B}$ and $\psi: S \to \tilde{S}$ be semimorphisms of (P, δ)-algebras. Then the map $\varphi \otimes \psi \otimes \iota$ preserves the filtrations. The induced homomorphisms of spectral sequences correspond under (3.7) to the linear maps

$$\varphi \otimes \psi \otimes \iota, \qquad \varphi \otimes (\psi \otimes \iota)^\#, \qquad \text{and} \qquad \varphi^\# \otimes (\psi \otimes \iota)^\#,$$

respectively.

Next, define a bigradation in $B \otimes S \otimes \wedge P$ by setting

$$(B \otimes S \otimes \wedge P)^{(k,l)} = \sum_{\mu+\nu=l} B^\mu \otimes S^k \otimes (\wedge P)^\nu.$$

The induced filtration is given by the ideals

$$\hat{F}^p(B \otimes S \otimes \wedge P) = B \otimes \sum_{\mu \geq p} S^\mu \otimes \wedge P.$$

The corresponding spectral sequence is called the *S-sequence* of the tensor difference.

A canonical isomorphism $B \otimes S \otimes \wedge P \xrightarrow{\cong} S \otimes B \otimes \wedge P$ is given by $b \otimes s \otimes \Phi \mapsto (-1)^{pq} s \otimes b \otimes \Phi$ ($b \in B^p$, $s \in S^q$, $\Phi \in \wedge P$). It induces isomorphisms of bigraded algebras

$$E_0^{(k,l)} \cong S^k \otimes (B \otimes \wedge P)^l$$
$$E_1^{(k,l)} \cong S^k \otimes H^l(B \otimes \wedge P, \nabla_B) \tag{3.8}$$

and

$$E_2^{(k,l)} \cong H^k(S) \otimes H^l(B \otimes \wedge P)$$

for the first terms of the S-sequence. Under these isomorphisms the homomorphisms of spectral sequences induced by semimorphisms $\varphi: B \to \tilde{B}$ and $\psi: S \to \tilde{S}$ correspond to the linear maps

$$\psi \otimes (\varphi \otimes \iota), \qquad \psi \otimes (\varphi \otimes \iota)^{\#}, \qquad \text{and} \qquad \psi^{\#} \otimes (\varphi \otimes \iota)^{\#},$$

respectively.

§3. Isomorphism theorems

The purpose of this article is to generalize the theorems of article 2, Chapter II, to (P, δ)-algebras.

3.10. *n*-regularity. Recall that a linear map $\varphi: E \to F$ between graded vector spaces is called *n-regular* if $\varphi^p: E^p \to F^p$ is an isomorphism for $p \leq n$ and injective for $p = n + 1$.

Theorem I: Let $\varphi: (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ be a semimorphism. Then the following conditions are equivalent:

(1) $\varphi^\#: H(B) \to H(\tilde{B})$ is n-regular.

(2) $(\varphi \otimes \iota)^\#: H(B \otimes \wedge P, \nabla_B) \to H(\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$ is n-regular.

(3) For all (P, δ)-algebras $(S, \delta_S; \sigma)$ the linear maps

$$(\varphi \otimes \iota \otimes \iota)^\#: H(B \otimes S \otimes \wedge P, \nabla_{B \otimes S}) \to H(\tilde{B} \otimes S \otimes \wedge P, \nabla_{\tilde{B} \otimes S})$$

are n-regular (cohomology of tensor differences).

Proof: $(1) \Rightarrow (2)$: Apply the comparison theorem of sec. 1.14 to the spectral sequence of the (P, δ)-algebras (cf. sec. 3.4).

$(2) \Rightarrow (3)$: Apply the comparison theorem to the S-sequence of the tensor differences (cf. sec. 3.9).

$(3) \Rightarrow (1)$: Consider the commutative diagram

$$\begin{tikzcd} H(B \otimes \vee P \otimes \wedge P) \arrow[r, "(\varphi\otimes\iota\otimes\iota)^\#"] & H(\tilde{B} \otimes \vee P \otimes \wedge P) \\ H(B) \arrow[u, "m_B^\#"', "\cong"] \arrow[r, "\varphi^\#"'] & H(\tilde{B}) \arrow[u, "\cong"', "m_{\tilde{B}}^\#"] \end{tikzcd}$$

(cf. Proposition I, sec. 3.8). If $(\varphi \otimes \iota \otimes \iota)^\#$ is n-regular, the diagram shows that $\varphi^\#$ is also n-regular.

Q.E.D.

Corollary: The following conditions on the semimorphism φ are equivalent:

(1) $\varphi^{\#}: H(B) \to H(\tilde{B})$ is an isomorphism.

(2) $(\varphi \otimes \iota)^{\#}: H(B \otimes \wedge P, \nabla_B) \to H(\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$ is an isomorphism.

(3) $(\varphi^{\#} \otimes \iota)^{\#}: H(H(B) \otimes \wedge P, \nabla_{\tau^*}) \to H(H(\tilde{B}) \otimes \wedge P, \nabla_{\tilde{\tau}^*})$ is an isomorphism.

(4) $(\varphi^{\#} \otimes \iota)_0^{\#}$ is an isomorphism and $(\varphi^{\#} \otimes \iota)_1^{\#}$ is injective.

Proof: Theorem I above implies that (1) $\Leftrightarrow$ (2). Theorem I, sec. 2.8, applied to the P-linear map $\varphi^{\#}$ shows that (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Q.E.D.

Remark: The corollary generalizes Theorem I, sec. 2.8.

3.11. A second isomorphism theorem. In this section we generalize Theorem II, sec. 2.9. Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra. Recall from sec. 3.2 that the inclusion $l_B: B \to B \otimes \wedge P$ induces a homomorphism $l_B^{\#}: H(B) \to H(B \otimes \wedge P)$. Form the (P, δ)-algebra $(\operatorname{Im} l_B^{\#} \otimes \vee P, 0; \sigma)$, where

$$\sigma(x) = 1 \otimes x, \qquad x \in P.$$

Its Koszul complex is simply $(\operatorname{Im} l_B^{\#} \otimes \vee P \otimes \wedge P, \nabla_\sigma)$.

Next choose a linear map, homogeneous of degree zero,

$$\gamma: \operatorname{Im} l_B^{\#} \to \ker \delta_B,$$

which satisfies

$$l_B^{\#} \circ \pi \circ \gamma = \iota \qquad \text{and} \qquad \gamma(1) = 1.$$

(Here $\pi: \ker \delta_B \to H(B)$ denotes the projection.)

Then define a map

$$g: \operatorname{Im} l_B^{\#} \otimes \vee P \to B$$

by

$$g(\alpha \otimes \Phi) = \gamma(\alpha) \cdot \tau_\vee(\Phi), \qquad \alpha \in \operatorname{Im} l_B^{\#}, \quad \Phi \in \vee P.$$

Evidently g is a semimorphism of (P, δ)-algebras; in particular it satisfies $\delta_B \circ g = 0$.

Theorem II: Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra and let γ, g be as above. Then the following conditions are equivalent:

(1) $g^{\#}: \operatorname{Im} l_B^{\#} \otimes \vee P \to H(B)$ is an isomorphism.

(2) $(g \otimes \iota)^{\#}: H(\operatorname{Im} l_B^{\#} \otimes \vee P \otimes \wedge P) \to H(B \otimes \wedge P)$ is an isomorphism.

(3) $l_B^{\#}: H(B) \to H(B \otimes \wedge P)$ is surjective.

(4) $H_+(H(B) \otimes \wedge P, \nabla_{\tau *}) = 0$.

(5) $H_1(H(B) \otimes \wedge P, \nabla_{\tau *}) = 0$.

Proof: We show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (4) $\Leftrightarrow$ (5), (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (3).

(1) $\Leftrightarrow$ (2): Apply the corollary to Theorem I, sec. 3.10.

(2) $\Leftrightarrow$ (3): It follows from the definitions of g and γ that the diagram

$$\begin{tikzcd} & H(\operatorname{Im} l_B^{\#} \otimes \vee P \otimes \wedge P) \arrow[dd, "(g\otimes\iota)^{\#}"] \\ \operatorname{Im} l_B^{\#} \arrow[ur, "i^{\#}"] \arrow[dr, "j"'] & \\ & H(B \otimes \wedge P) \end{tikzcd}$$

commutes, where i and j are the obvious inclusion maps. Moreover, it follows from the example in sec. 2.2 and from sec. 2.6 that $i^{\#}$ is an isomorphism. Thus $(g \otimes \iota)^{\#}$ is an isomorphism if and only if j is; i.e., if and only if j is surjective. But this is equivalent to $l_B^{\#}$ being surjective.

(4) $\Leftrightarrow$ (5): This is proved in Theorem II, sec. 2.9.

(1) $\Rightarrow$ (4): This follows from Theorem II, sec. 2.9 ((1) $\Rightarrow$ (4)).

(4) $\Rightarrow$ (3): Recall from sec. 2.2 the linear map

$$l_{H(B)}^{\#}: H(B) \to H(H(B) \otimes \wedge P, \nabla_{\tau *}),$$

and denote its image by F. If (4) holds, Theorem II, sec. 2.9, yields a P-linear isomorphism

$$\varphi: F \otimes \vee P \xrightarrow{\cong} H(B)$$

which satisfies $\varphi(1) = 1$.

Next, choose a linear map $\eta: F \to \ker \delta_B$, homogeneous of degree zero, such that $\eta(1) = 1$, and

$$\eta^{\#}(\alpha) = \varphi(\alpha \otimes 1), \qquad \alpha \in F.$$

Then a semimorphism

$$\psi\colon (F \otimes \vee P, 0; \sigma) \to (B, \delta_B; \tau)$$

is defined by

$$\psi(\alpha \otimes \Psi) = \eta(\alpha) \cdot \tau_\vee(\Psi), \qquad \alpha \in F, \quad \Psi \in \vee P.$$

The induced map $\psi^\#\colon F \otimes \vee P \to H(B)$ is given by

$$\psi^\#(\alpha \otimes \Psi) = \eta^\#(\alpha) \cdot \tau_\vee^\#(\Psi) = \varphi(\alpha \otimes \Psi), \qquad \alpha \in F, \quad \Psi \in \vee P;$$

i.e., $\psi^\# = \varphi$. In particular, $\psi^\#$ is an isomorphism.

Hence, by the corollary of Theorem I, sec. 3.10, $(\psi \otimes \iota)^\#$ is an isomorphism. Since the diagram

$$\begin{tikzcd} H(F \otimes \vee P \otimes \wedge P) \arrow[r, "(\psi\otimes\iota)^\#", "\cong"'] & H(B \otimes \wedge P) \\ F \arrow[u, "\cong"] \arrow[r, "\eta^\#"'] & H(B) \arrow[u, "l_B^\#"'] \end{tikzcd}$$

commutes, it follows that $l_B^\#$ is surjective.

Q.E.D.

3.12.* In this section we generalize Theorem III, sec. 2.9, to

Theorem III: Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra. Then the following conditions are equivalent:

(1) $H(B)$ is evenly graded, and the conditions of Theorem II, sec. 3.11 hold.

(2) $H(B \otimes \wedge P)$ is evenly graded.

Proof: (1) ⇒ (2): If (1) holds, then $l_B^\#$ is surjective (cf. Theorem II, (3), sec. 3.11). Since by hypothesis $H(B)$ is evenly graded, so is $H(B \otimes \wedge P)$.

(2) ⇒ (1): Recall from Proposition I, sec. 3.8, that

$$H(B \otimes \vee P \otimes \wedge P) \cong H(B).$$

On the other hand, the E_1-term of the $\vee P$-spectral sequence for the

tensor difference $(B \otimes \vee P, \delta_B; \tau \ominus \sigma)$ is given by

$$E_1 \cong H(B \otimes \wedge P) \otimes \vee P$$

(cf. sec. 3.9). Thus E_1 is evenly graded. It follows that $E_1 = E_\infty$ (cf. Proposition IV, sec. 1.10 and Proposition V, sec. 1.12) and so we have isomorphisms of graded vector spaces

$$H(B) \cong H(B \otimes \vee P \otimes \wedge P) \cong E_\infty \cong E_1.$$

In particular, $H(B)$ is evenly graded.

It remains to show that $l_B^\#: H(B) \to H(B \otimes \wedge P)$ is surjective. Assume first that $\dim P = 1$. Then we have the exact sequence

$$0 \longrightarrow \operatorname{coker} \partial \xrightarrow{l_B^\#} H(B \otimes \wedge P) \xrightarrow{i(x^*)^\#} \ker \partial \longrightarrow 0$$

of sec. 3.6.

All three spaces in this sequence are evenly graded, but $i(x^*)^\#$ is homogeneous of odd degree. It follows that $i(x^*)^\# = 0$ and so, by exactness, $l_B^\#$ is surjective.

In the general case we argue by induction on $\dim P$. Write $P = P_1 \oplus P_2$ where P_1 and P_2 are graded subspaces. Then τ restricts to a linear map $\tau_1: P_1 \to B$ and thus yields a (P_1, δ)-algebra $(B, \delta_B; \tau_1)$ whose Koszul complex will be written $(B \otimes \wedge P_1, \nabla_1)$.

Next define a map $\tau_2: P_2 \to B \otimes \wedge P_1$ by setting

$$\tau_2(x) = \tau(x) \otimes 1, \qquad x \in P_2.$$

This yields a (P_2, δ)-algebra $(B \otimes \wedge P_1, \nabla_1; \tau_2)$. Its Koszul complex is given by

$$(B \otimes \wedge P_1 \otimes \wedge P_2, \nabla_2) = (B \otimes \wedge P, \nabla_B).$$

Finally, since $H((B \otimes \wedge P_1) \otimes \wedge P_2, \nabla_2) = H(B \otimes \wedge P, \nabla_B)$, we have that $H((B \otimes \wedge P_1) \otimes \wedge P_2, \nabla_2)$ is evenly graded. It follows, as above, that $H(B \otimes \wedge P_1, \nabla_1)$ is evenly graded. Thus, by induction, the maps

$$H(B) \to H(B \otimes \wedge P_1) \quad \text{and} \quad H(B \otimes \wedge P_1) \to H(B \otimes \wedge P_1 \otimes \wedge P_2)$$

are surjective. Hence so is their composite, $l_B^\#$. This closes the induction.

Q.E.D.

§4. Structure theorems

In this article all (P, δ)-algebras will be assumed to be c-connected. The purpose of this article is to generalize the theorems of article 4, Chapter II, to (P, δ)-algebras.

3.13. The Samelson theorem. Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra and consider the projection $B \to B^0$ with kernel B^+. It gives rise to a projection

$$\varrho_B: B \otimes \wedge P \to B^0 \otimes \wedge P.$$

Lemma I: ϱ_B satisfies the conditions

$$\varrho_B \circ \nabla_B = 0 \qquad \text{and} \qquad \varrho_B(\ker \nabla_B) \subset 1 \otimes \wedge P.$$

Proof: The first relation is obvious. To prove the second, fix $z \in \ker \nabla_B$ and write

$$z = \varrho_B z + z_1, \qquad z_1 \in B^+ \otimes \wedge P.$$

Then

$$\nabla_B z_1 + \nabla_\tau \varrho_B z \in \sum_{j \geq 2} B^j \otimes \wedge P,$$

while $\delta_B \varrho_B z \in B^1 \otimes \wedge P$.

Since $\delta_B \varrho_B z + \nabla_\tau \varrho_B z + \nabla_B z_1 = \nabla_B z = 0$, these relations imply that $\delta_B \varrho_B z = 0$. Thus

$$\varrho_B z \in (\ker \delta_B)^0 \otimes \wedge P = H^0(B) \otimes \wedge P = 1 \otimes \wedge P.$$

Q.E.D.

In view of Lemma I there is a unique homomorphism

$$\varrho_B^\#: H(B \otimes \wedge P) \to \wedge P,$$

which makes the diagram

$$\begin{tikzcd} \ker \nabla_B \arrow[d] \arrow[dr, "\varrho_B"] & \\ H(B \otimes \wedge P) \arrow[r, "\varrho_B^\#"'] & \wedge P \end{tikzcd}$$

commute. $\varrho_B^\#$ is called the *Samelson projection for* $(B, \delta_B; \tau)$ and the space $\hat{P}_B = P \cap (\operatorname{Im} \varrho_B^\#)$ is called the *Samelson subspace.* A graded complement of $\hat{P}_B$ in P will be called a *Samelson complement for* $(B, \delta_B; \tau)$.

Note that these definitions reduce to the definitions of sec. 2.13 if $\delta_B = 0$. Exactly the same argument as given in Theorem IV, sec. 2.13, establishes

Theorem IV: Let $(B, \delta_B; \tau)$ be a (P, δ)-algebra. Then

$$\operatorname{Im} \varrho_B^\# = \wedge \hat{P}_B.$$

Finally, observe that if $\varphi: (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ is a semimorphism, then

$$\varrho_{\tilde{B}}^\# \circ (\varphi \otimes \iota)^\# = \varrho_B^\#$$

In particular, $\hat{P}_B \subset \hat{P}_{\tilde{B}}$.

3.14. The cohomology sequence. The *cohomology sequence of a (c-connected) (P, δ)-algebra* $(B, \delta_B; \tau)$ is the sequence

$$\vee P \xrightarrow{\tau_\vee^\#} H(B) \xrightarrow{\iota_B^\#} H(B \otimes \wedge P) \xrightarrow{\varrho_B^\#} \wedge P.$$

A semimorphism $\varphi: (B, \delta_B; \tau_B) \to (\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ determines the commutative diagram

$$\begin{tikzcd}
 & H(B) \arrow[r, "\iota_B^\#"] \arrow[d, "\varphi^\#"] & H(B \otimes \wedge P) \arrow[d, "(\varphi\otimes\iota)^\#"] \arrow[dr, "\varrho_B^\#"] & \\
\vee P \arrow[ur, "\tau_\vee^\#"] \arrow[dr, "\tilde{\tau}_\vee^\#"'] & & & \wedge P. \\
 & H(\tilde{B}) \arrow[r, "\iota_{\tilde{B}}^\#"'] & H(\tilde{B} \otimes \wedge P) \arrow[ur, "\varrho_{\tilde{B}}^\#"'] &
\end{tikzcd}$$

Next, suppose P_1 is a second graded space satisfying the same conditions as P (cf. the beginning of this chapter) and let $\alpha: P_1 \to P$ be a linear map homogeneous of degree zero. Define a map $\tau_1: P_1 \to B$ by

$$\tau_1 = \tau \circ \alpha.$$

Then $(B, \delta_B; \tau_1)$ is a (P_1, δ)-algebra, and we have the commutative diagram

$$\begin{tikzcd} \vee P_1 \arrow[r, "(\tau_1)^\#_\vee"] \arrow[d, "\alpha_\vee"'] & H(B) \arrow[r] \arrow[d, "\iota"] & H(B \otimes \wedge P_1) \arrow[r] \arrow[d, "(\iota\otimes\alpha_\wedge)^\#"] & \wedge P_1 \arrow[d, "\alpha_1"] \\ \vee P \arrow[r, "\tau^\#_\vee"'] & H(B) \arrow[r] & H(B \otimes \wedge P) \arrow[r] & \wedge P. \end{tikzcd} \tag{3.9}$$

Note as well that for $x^* \in P^*$,

$$i(x^*) \circ (\iota \otimes \alpha_\wedge)^\# = (\iota \otimes \alpha_\wedge)^\# \circ i(\alpha^*(x^*)).$$

Proposition III: Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Then (1) $l_B^\# \circ (\tau_\vee^\#)^+ = 0$ and so $\ker l_B^\#$ contains the ideal generated by $\mathrm{Im}(\tau_\vee^\#)^+$.

(2) $\varrho_B^\# \circ (l_B^\#)^+ = 0$ and so $\ker \varrho_B^\#$ contains the ideal generated by $\mathrm{Im}(l_B^\#)^+$.

Proof: (1) It is sufficient to show that $l_B^\# \circ \tau^\# = 0$. But

$$l_B \tau(x) = \tau(x) \otimes 1 = \nabla_B(1 \otimes x), \qquad x \in P.$$

(2) This is obvious.

Q.E.D.

3.15. The reduction theorem. In this section we generalize the results of sec. 2.15.

Let $\hat{P}_B$ be the Samelson subspace of a c-connected (P, δ)-algebra $(B, \delta_B; \tau)$. Choose a Samelson complement $\tilde{P}$. Then multiplication defines an isomorphism

$$g: \wedge\hat{P} \otimes \wedge\tilde{P} \xrightarrow{\cong} \wedge P$$

of graded algebras.

Next, let $(B, \delta_B; \tilde{\tau})$ be the $(\tilde{P}, \delta)$-algebra determined by restricting τ to $\tilde{P}$, and denote its Koszul complex by $(B \otimes \wedge\tilde{P}, \tilde{\nabla}_B)$. (Then $\tilde{\nabla}_B$ is the restriction of ∇_B to $B \otimes \wedge\tilde{P}$.) In particular, $(B \otimes \wedge\tilde{P} \otimes \wedge\hat{P}, \tilde{\nabla}_B \otimes \iota)$ is a graded differential algebra.

Theorem V (reduction theorem): Suppose that $(B, \delta_B; \tau)$ is a c-connected (P, δ)-algebra. Let $\hat{P}$ be the Samelson subspace and let $\tilde{P}$ be

a Samelson complement. Then there is an isomorphism

$$f\colon (B \otimes \wedge\tilde{P} \otimes \wedge\hat{P},\ \tilde{\nabla}_B \otimes \iota) \xrightarrow{\cong} (B \otimes \wedge P, \nabla_B),$$

of graded differential algebras, such that the diagram

$$\begin{tikzcd}
 & B \otimes \wedge\tilde{P} \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\
B \otimes \wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "f", "\cong"'] \arrow[d, "\tilde{\varrho}_B \otimes \iota"'] & & B \otimes \wedge P \arrow[d, "\varrho_B"] \\
B^0 \otimes \wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "\iota \otimes g", "\cong"'] & & B^0 \otimes \wedge P
\end{tikzcd}$$

commutes (λ_1 and λ_2 are the obvious inclusions).

Proof: Choose a linear map

$$\beta\colon \hat{P} \to \ker \nabla_B,$$

homogeneous of degree zero, and such that

$$(\varrho_B \circ \beta)(x) = 1 \otimes x, \qquad x \in \hat{P}.$$

Then the proof of Theorem V, sec. 2.15, with trivial modifications (using *this* map, β) establishes this theorem as well. Q.E.D.

Corollary I: f induces an isomorphism of graded algebras

$$f^{\#}\colon H(B \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \xrightarrow{\cong} H(B \otimes \wedge P)$$

which makes the diagram

$$\begin{tikzcd}
 & H(B \otimes \wedge\tilde{P}) \arrow[dl] \arrow[dr] & \\
H(B \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \arrow[rr, "f^{\#}", "\cong"'] \arrow[d, "\tilde{\varrho}_B^{\#} \otimes \iota"'] & & H(B \otimes \wedge P) \arrow[d, "\varrho_B^{\#}"] \\
\wedge\tilde{P} \otimes \wedge\hat{P} \arrow[rr, "\cong", "g"'] & & \wedge P
\end{tikzcd}$$

commute.

Corollary II: The diagram

$$\begin{tikzcd}
 & H(B \otimes \wedge \tilde{P}) \otimes \wedge \hat{P} \arrow[dd, "f^\#", "\cong"'] \arrow[dr] & \\
H(B) \arrow[ur, "\tilde{l}_B^\#"] \arrow[dr, "l_B^\#"'] & & \wedge \hat{P} \\
 & H(B \otimes \wedge P) \arrow[ur, "\varrho_B^\#"'] &
\end{tikzcd}$$

commutes. In particular, $(\tilde{\varrho}_B^\#)^+ = 0$.

Proof: Apply the Samelson theorem (sec. 3.13) and Corollary I.
Q.E.D.

Corollary III: $f^\#$ restricts to an isomorphism

$$f^\#: \operatorname{Im} \tilde{l}_B^\# \xrightarrow{\cong} \operatorname{Im} l_B^\# .$$

Corollary IV: $f^\#$ restricts to an isomorphism

$$f^\#: H^+(B \otimes \wedge \tilde{P}) \otimes \wedge \hat{P} \xrightarrow{\cong} \ker \varrho_B^\# .$$

3.16. The decomposition theorem. The results of this section generalize much of Theorem VIII, sec. 2.19.

Theorem VI: Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Then the kernel of $\varrho_B^\#$ contains the ideal generated by $\operatorname{Im}(l_B^\#)^+$. Moreover, if $\hat{P}$ and $\tilde{P}$ denote respectively the Samelson subspace and a Samelson complement, then the following conditions are equivalent:

(1) The kernel of $\varrho_B^\#$ coincides with the ideal generated by $\operatorname{Im}(l_B^\#)^+$.

(2) The map $\tilde{l}_B^\#: H(B) \to H(B \otimes \wedge \tilde{P})$ is surjective.

(3) There is an isomorphism of graded algebras

$$\operatorname{Im} l_B^\# \otimes \wedge \hat{P} \xrightarrow{\cong} H(B \otimes \wedge P)$$

making the diagram

$$\begin{tikzcd}
& \operatorname{Im} l_B^\# \otimes \wedge\hat{P} \arrow[dd, "\cong"] \arrow[dr] & \\
H(B) \arrow[ur] \arrow[dr, "l_B^\#"'] & & \wedge\hat{P} \\
& H(B \otimes \wedge P) \arrow[ur, "\varrho_B^\#"'] &
\end{tikzcd}$$

commute.

(4) There is an isomorphism $\operatorname{Im} l_B^\# \otimes \vee\hat{P} \xrightarrow{\cong} H(B)$ of graded $\hat{P}$-spaces, which makes the diagram

$$\begin{tikzcd}
& \operatorname{Im} l_B^\# \otimes \vee\hat{P} \arrow[dd, "\cong"] \arrow[dr] & \\
\vee\hat{P} \arrow[ur] \arrow[dr, "\tau_\vee^\#"'] & & \operatorname{Im} l_B^\# \\
& H(B) \arrow[ur, "l_B^\#"'] &
\end{tikzcd}$$

commute.

Remark: For further equivalent conditions see Theorem II, sec. 3.11, and Theorem III, sec. 3.12.

Proof: (1) $\Leftrightarrow$ (2): In view of Corollaries II and III, sec. 3.15, we may at once reduce to the case $\hat{P} = 0$. In this case we have to prove that $l_B^\#$ is surjective if and only if

$$\operatorname{Im}(l_B^\#)^+ \cdot H(B \otimes \wedge P) = H^+(B \otimes \wedge P).$$

This follows by an elementary degree argument (in the same way as Lemma I, sec. 2.8).

(2) $\Rightarrow$ (3): This follows from Corollary II, sec. 3.15.

(3) $\Rightarrow$ (1): This is obvious.

(2) $\Leftrightarrow$ (4): This is proved in Theorem II, sec. 3.11.

Q.E.D.

Corollary I: If the conditions of the theorem hold, then $\ker l_B^\#$ coincides with the ideal generated by $\operatorname{Im}(\tau_\vee^\#)^+$.

Proof: In view of condition (2) of the theorem, we may apply Theorem II, sec. 3.11, to the $(\tilde{P}, \delta)$-algebra $(B, \delta_B; \tilde{\tau})$ where $\tilde{\tau}$ denotes the restriction of τ to $\tilde{P}$. This yields a semimorphism

$$g\colon \operatorname{Im} \bar{l}_B^\# \otimes \vee \tilde{P} \to B$$

of $(\tilde{P}, \delta)$-algebras which induces a commutative diagram

$$\begin{tikzcd}
\operatorname{Im} \bar{l}_B^\# \otimes \vee \tilde{P} \arrow[r, "g^\#", "\cong"'] \arrow[d, "l^\#"'] & H(B) \arrow[d, "\bar{l}_B^\#"] \\
H(\operatorname{Im} \bar{l}_B^\# \otimes \vee \tilde{P} \otimes \wedge \tilde{P}) \arrow[r, "\cong", "(g\otimes\iota)^\#"'] & H(B \otimes \wedge \tilde{P}).
\end{tikzcd}$$

Now recall that if S is a $\tilde{P}$-space, then the kernel of $l_S^\#$ is the space $S \circ P$ (cf. sec. 2.2). Thus in the diagram above

$$\ker l^\# = (\operatorname{Im} \bar{l}_B^\# \otimes \vee \tilde{P}) \circ \tilde{P}.$$

It follows that

$$\ker \bar{l}_B^\# = g^\#(\ker l^\#) = H(B) \circ \tilde{P} = H(B) \cdot \tau^\#(\tilde{P}).$$

Finally, Corollary II of the reduction theorem, sec. 3.15, shows that $\ker l_B^\# = \ker \bar{l}_B^\#$. Thus we may combine the relation above with Proposition III, sec. 3.14, to obtain

$$\ker l_B^\# = H(B) \cdot \tau^\#(\tilde{P}) \subset H(B) \cdot \tau_\vee^\#(\vee^+ P) \subset \ker l_B^\#.$$

Q.E.D.

Corollary II: If the conditions of the theorem hold, then there is a commutative diagram

$$\begin{tikzcd}
 & (H(B)/H(B)\cdot \operatorname{Im} \tau^\#) \otimes \wedge \hat{P} \arrow[dd, "\cong"'] \arrow[dr] & \\
H(B) \arrow[ur] \arrow[dr, "l_B^\#"'] & & \wedge \hat{P} \\
 & H(B \otimes \wedge P) & \arrow[ul, "\varrho_B^\#"']
\end{tikzcd}$$

in which the vertical arrow is an isomorphism of graded algebras.

Proof: Note that by Corollary I

$$\operatorname{Im} l_B^\# \cong H(B)/\ker l_B^\# = H(B)/H(B)\cdot \operatorname{Im} \tau^\#.$$

Now apply part (3) of the theorem.

Q.E.D.

3.17. (P, δ)-algebras with $\wedge P$ noncohomologous to zero. Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Then $\wedge P$ is said to be *noncohomologous to zero in* $B \otimes \wedge P$ (n.c.z.) if the homomorphism

$$\varrho_B^\#\colon H(B \otimes \wedge P) \to \wedge P$$

is surjective.

Theorem VII: Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Then the following conditions are equivalent:

(1) $\varrho_B^\#$ is surjective.

(2) There is a linear isomorphism of graded vector spaces

$$H(B) \otimes \wedge P \xrightarrow{\cong} H(B \otimes \wedge P, \nabla_B)$$

which makes the diagram

$$\begin{tikzcd} & H(B) \otimes \wedge P \arrow[dr] \arrow[dd, "\cong"] & \\ H(B) \arrow[ur] \arrow[dr, "l_B^\#"'] & & \wedge P \\ & H(B \otimes \wedge P) \arrow[ur, "\varrho_B^\#"'] & \end{tikzcd} \tag{3.10}$$

commute.

(3) $l_B^\#$ is injective.

(4) $\tau^\# = 0$.

(5) The spectral sequence for $(B, \delta_B; \tau)$ (cf. sec. 3.4) collapses at the E_2-term.

(6) There is an isomorphism of graded differential algebras

$$f\colon (B \otimes \wedge P, \delta_B) \xrightarrow{\cong} (B \otimes \wedge P, \nabla_B)$$

such that

$$f(b \otimes 1) = b \otimes 1, \quad f(1 \otimes x) - 1 \otimes x \in B^+ \otimes 1, \quad b \in B, \quad x \in P,$$

and

$$i(x^*) \circ f = f \circ i(x^*), \qquad x^* \in P^*.$$

(7) There is an isomorphism of graded algebras

$$g: H(B) \otimes \wedge P \xrightarrow{\cong} H(B \otimes \wedge P, \nabla_B)$$

which makes the diagram (3.10) commute, and satisfies

$$i(x^*)^\# \circ g = g \circ i(x^*)^\#, \qquad x^* \in P^*.$$

Proof: We show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$$

and

$$(6) \Rightarrow (5) \Rightarrow (3).$$

$(1) \Rightarrow (2)$: If (1) holds, then the Samelson subspace is all of P. In this case (2) follows from Corollary II of the reduction theorem (cf. sec. 3.15).

$(2) \Rightarrow (3)$: This is obvious.

$(3) \Rightarrow (4)$: This follows from the relation $l_B^\# \circ (\tau_\vee^\#)^+ = 0$ (cf. Proposition III, sec. 3.14).

$(4) \Rightarrow (6)$: If $\tau^\# = 0$, then there is a linear map $\alpha: P \to B^+$ homogeneous of degree zero, and such that

$$\tau = -\delta_B \circ \alpha.$$

Define $\beta: P \to B \otimes \wedge P$ by

$$\beta(x) = \alpha(x) \otimes 1 + 1 \otimes x, \qquad x \in P.$$

Then, clearly, $\nabla_B \circ \beta = 0$.

Extend β to a homomorphism $\beta_\wedge: \wedge P \to B \otimes \wedge P$ and define a homomorphism of graded algebras

$$f: B \otimes \wedge P \to B \otimes \wedge P$$

by

$$f(b \otimes \Phi) = (b \otimes 1) \cdot (\beta_\wedge \Phi), \qquad b \in B, \quad \Phi \in \wedge P.$$

Exactly as in the proof of Theorem V, sec. 2.15, it follows that f is an isomorphism.

Finally, since $\nabla_B \circ \beta = 0$, we have

$$\nabla_B \circ f = f \circ \delta_B .$$

The relations of (6) are trivial consequences of the definition of f.

(6) $\Rightarrow$ (7) $\Rightarrow$ (1): This is obvious.

(6) $\Rightarrow$ (5): The ideals $F^p(B \otimes \wedge P)$ which determine the spectral sequence are given by

$$F^p(B \otimes \wedge P) = \sum_{\mu \geq p} B^\mu \otimes \wedge P = \left(\sum_{\mu \geq p} B^\mu\right) \cdot (B \otimes \wedge P).$$

Thus if (6) holds, then f is an isomorphism of filtered graded differential algebras; i.e., f and f^{-1} preserve the filtration. Hence f induces an isomorphism of spectral sequences. But the spectral sequence for $(B \otimes \wedge P, \delta_B)$ collapses at the second term (cf. sec. 1.8).

(5) $\Rightarrow$ (3): Recall from sec. 3.4 that $B \otimes 1$ is the basic subalgebra of $B \otimes \wedge P$ with respect to the given filtration. Thus, if the spectral sequence collapses at the E_2-term, $l_B^\#$ is injective (cf. Corollary III to Proposition VII, sec. 1.14).

Q.E.D.

Corollary: If $\wedge P$ is n.c.z. in $B \otimes \wedge P$, then $H(B \otimes \wedge P)$ is generated by the classes which can be represented by cocycles of the form

$$z \otimes 1 \qquad \text{or} \qquad w \otimes 1 + 1 \otimes x, \qquad z, w \in B, \quad x \in P.$$

Finally, assume $\wedge P$ is n.c.z. in $B \otimes \wedge P$, and let $\eta: \wedge P \to H(B \otimes \wedge P)$ be a linear map, homogeneous of degree zero, and such that

$$\varrho_B^\# \circ \eta = \iota.$$

Define

$$h: H(B) \otimes \wedge P \to H(B \otimes \wedge P)$$

by setting

$$h(\alpha \otimes \Phi) = l_B^\#(\alpha) \cdot \eta(\Phi), \qquad \alpha \in H(B), \quad \Phi \in \wedge P.$$

Proposition IV: The map h is an isomorphism of graded spaces. Moreover, if η is a homomorphism, then h is an isomorphism of graded algebras.

Proof: Let $g\colon H(B) \otimes \wedge P \xrightarrow{\cong} H(B \otimes \wedge P)$ be the isomorphism of part 7 of Theorem VII. Then $g^{-1} \circ h$ is an endomorphism of $H(B) \otimes \wedge P$ satisfying

$$(g^{-1} \circ h - \iota)\colon H^p(B) \otimes \wedge P \to \sum_{j>p} H^j(B) \otimes \wedge P.$$

Filter $H(B) \otimes \wedge P$ by the subspaces $\sum_{j \geq p} H^j(B) \otimes \wedge P$; observe that $g^{-1} \circ h$ induces the identity in the associated graded space; conclude via Proposition VII, sec. 1.14, that $g^{-1} \circ h$ is an isomorphism. Since g is an isomorphism, so is h.

Q.E.D.

3.18. Poincaré series. In this section the Poincaré series of a graded vector space X will be written f_X. The relation $f_X \leq f_Y$ will mean that

$$\dim X^p \leq \dim Y^p, \qquad \text{each } p.$$

Proposition V: Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Then $H(B)$ is of finite type if and only if $H(B \otimes \wedge P)$ is of finite type. In this case the Poincaré series satisfy the relations

$$f_{H(B \otimes \wedge P)} \leq f_{H(B)} f_{\wedge P} \tag{3.11}$$

and

$$f_{H(B)} \leq f_{\vee P} f_{H(B \otimes \wedge P)}. \tag{3.12}$$

Equality holds in (3.11) if and only if $\wedge P$ is n.c.z. in $B \otimes \wedge P$.

Proof: Suppose $H(B)$ is of finite type. The E_2-term of the spectral sequence for $B \otimes \wedge P$ is given by

$$E_2 \cong H(B) \otimes \wedge P$$

(cf. sec. 3.4). Hence E_2 is of finite type and

$$f_{E_2} = f_{H(B)} f_{\wedge P}.$$

Now Proposition VIII, sec. 1.15, implies that: (1) $H(B \otimes \wedge P)$ has finite type, (2) relation (3.11) holds, and (3) equality holds in (3.11) if and only if the spectral sequence collapses at the E_2-term. In view of Theorem VII, sec. 3.17, this last is equivalent to $\wedge P$ being n.c.z. in $B \otimes \wedge P$.

Conversely, assume that $H(B \otimes \wedge P)$ is of finite type. Recall from sec. 3.9 that the E_2-term of the $\vee P$-spectral sequence of $B \otimes \vee P \otimes \wedge P$ is given by

$$E_2 \cong \vee P \otimes H(B \otimes \wedge P).$$

It follows that E_2 has finite type and that

$$f_{E_2} = f_{\vee P} f_{H(B \otimes \wedge P)}.$$

But according to Proposition I, sec. 3.8, $H(B) \cong H(B \otimes \vee P \otimes \wedge P)$. Hence, $H(B)$ has finite type and (3.12) holds.

Q.E.D.

Corollary: If $H(B)$ has finite dimension, then so does $H(B \otimes \wedge P)$. In this case the Euler–Poincaré characteristic of $H(B \otimes \wedge P)$ is zero. Moreover,

$$\dim H(B \otimes \wedge P) \leq \dim H(B) \dim \wedge P$$

and equality holds if and only if $\wedge P$ is n.c.z. in $B \otimes \wedge P$.

Proof: Apply Proposition IX, sec. 1.16, to the spectral sequence of sec. 3.4 to obtain $\chi_{H(B \otimes \wedge P)} = 0$. The rest follows at once from the proposition.

Q.E.D.

§5. Cohomology diagram of a tensor difference

In this article $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ denote c-connected (P, δ)-algebras. $(B \otimes S, \delta_{B\otimes S}; \tau \ominus \sigma)$ denotes their tensor difference (cf. sec. 3.7) and $(B \otimes S \otimes \wedge P, \nabla_{B\otimes S})$ denotes the corresponding Koszul complex.

3.19. The homomorphisms $p_B^\#$ and $p_S^\#$. Recall from sec. 3.14 the cohomology sequences

$$\vee P \xrightarrow{\tau_\vee^\#} H(B) \xrightarrow{l_B^\#} H(B \otimes \wedge P) \xrightarrow{\varrho_B^\#} \wedge P$$

and

$$\vee P \xrightarrow{\sigma_\vee^\#} H(S) \xrightarrow{l_S^\#} H(S \otimes \wedge P) \xrightarrow{\varrho_S^\#} \wedge P.$$

In this section we construct homomorphisms

$$p_B^\#: H(B \otimes S \otimes \wedge P) \to H(S \otimes \wedge P)$$

and

$$p_S^\#: H(B \otimes S \otimes \wedge P) \to H(B \otimes \wedge P).$$

Extend the projection $B \to B^0$ to the projection

$$p_B: B \otimes S \otimes \wedge P \to B^0 \otimes S \otimes \wedge P.$$

Then, clearly, $p_B \circ \nabla_{B\otimes S} = -(\iota \otimes \nabla_S) \circ p_B$, and so we have an induced homomorphism

$$\eta_B: H(B \otimes S \otimes \wedge P) \to B^0 \otimes H(S \otimes \wedge P).$$

The image of this homomorphism is contained in $1 \otimes H(S \otimes \wedge P)$. In fact, let $\alpha \in H(B \otimes S \otimes \wedge P)$. Lemma II, below, shows that α can be represented by a cocycle Φ such that

$$\Phi \in (1 \otimes S \otimes \wedge P) \oplus (B^+ \otimes S \otimes \wedge P).$$

Thus $p_B(\Phi) \in 1 \otimes S \otimes \wedge P$, and represents $\eta_B(\alpha)$. It follows that $\eta_B(\alpha) \in 1 \otimes H(S \otimes \wedge P)$.

Since $\operatorname{Im} \eta_B \subset 1 \otimes H(S \otimes \wedge P)$, there is a unique homomorphism $p_B^\#: H(B \otimes S \otimes \wedge P) \to H(S \otimes \wedge P)$ such that

$$\eta_B(\alpha) = 1 \otimes p_B^\#(\alpha), \qquad \alpha \in H(B \otimes S \otimes \wedge P).$$

Similarly, extend the projection $S \to S^0$ to a homomorphism

$$p_S: B \otimes S \otimes \wedge P \to B \otimes S^0 \otimes \wedge P.$$

Exactly as above we obtain a homomorphism

$$p_S^\#: H(B \otimes S \otimes \wedge P) \to H(B \otimes \wedge P).$$

Thus each $\alpha \in H(B \otimes S \otimes \wedge P)$ can be represented by a cocycle Φ in $(B \otimes 1 \otimes \wedge P) \oplus (B \otimes S^+ \otimes \wedge P)$, and $p_S(\Phi)$ represents $p_S^\#(\alpha)$.

Lemma II: Let $\Omega \in B \otimes S \otimes \wedge P$ be any cocycle. Then there is an element $\Psi \in B \otimes S \otimes \wedge P$ such that

$$\Omega - \nabla_{B\otimes S}\Psi \in (1 \otimes S \otimes \wedge P) \oplus (B^+ \otimes S \otimes \wedge P).$$

Proof: Write $\Omega = \Omega_0 + \Omega_1 + \Omega_2$, where

$$\Omega_0 \in B^0 \otimes S \otimes \wedge P, \;\; \Omega_1 \in B^1 \otimes S \otimes \wedge P, \;\; \text{and} \;\; \Omega_2 \in \sum_{j\geq 2} B^j \otimes S \otimes \wedge P.$$

Denote $\omega_B \otimes \nabla_S$ by ∇_S. Then an argument on degrees shows that

$$\nabla_S\Omega_0 = 0 \qquad \text{and} \qquad \delta_B\Omega_0 - \nabla_S\Omega_1 = 0. \tag{3.13}$$

Now choose a subspace $C \subset B^0$ so that $B^0 = \Gamma \cdot 1 \oplus C$, and let $\pi: B^0 \to \Gamma$ be the corresponding projection. Since $H(B)$ is connected, there is a linear map $h: B^1 \to B^0$ such that

$$\pi - \iota = h \circ \delta_B. \tag{3.14}$$

Since $\pi \otimes \iota \otimes \iota$ and $h \otimes \iota \otimes \iota$ commute up to sign with ∇_S in $B \otimes S \otimes \wedge P$, relations (3.13) and (3.14) yield

$$\Omega_0 - (\nabla_S \circ (h \otimes \iota \otimes \iota))\Omega_1 \in 1 \otimes S \otimes \wedge P.$$

Hence

$$\Omega - (\nabla_{B\otimes S} \circ (h \otimes \iota \otimes \iota))\Omega_1 \in (1 \otimes S \otimes \wedge P) \oplus (B^+ \otimes S \otimes \wedge P).$$

Q.E.D.

3.20. The cohomology diagram. Consider the inclusion maps

$$m_B: B \to B \otimes S \otimes \wedge P \qquad \text{and} \qquad m_S: S \to B \otimes S \otimes \wedge P$$

given by $m_B(b) = b \otimes 1 \otimes 1$ and $m_S(z) = 1 \otimes z \otimes 1$. They are homomorphisms of graded differential algebras and hence induce homomorphisms

$$m_B^\sharp: H(B) \to H(B \otimes S \otimes \wedge P) \qquad \text{and} \qquad m_S^\sharp: H(S) \to H(B \otimes S \otimes \wedge P)$$

of graded algebras. Obviously, $p_B^\sharp \circ (m_S^\sharp)^+ = 0$ and $p_S^\sharp \circ (m_B^\sharp)^+ = 0$.

Combining these homomorphisms with the cohomology sequences for $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ we obtain the diagram

$$\begin{tikzcd}
\vee P \arrow[r, "\tau_\vee^\#"] \arrow[d, "\sigma_\vee^\#"'] & H(B) \arrow[d, "m_B^\#"] \arrow[dr, "l_B^\#"] & \\
H(S) \arrow[r, "m_S^\#"] \arrow[dr, "l_S^\#"'] & H(B \otimes S \otimes \wedge P) \arrow[r, "p_S^\#"] \arrow[d, "p_B^\#"] & H(B \otimes \wedge P) \arrow[d, "\varrho_B^\#"] \\
& H(S \otimes \wedge P) \arrow[r, "\varrho_S^\#"'] & \wedge P.
\end{tikzcd}$$

It is called the *cohomology diagram for the tensor difference.*

Proposition VI: The cohomology diagram commutes.

Proof: It is immediate from the definitions that

$$p_S \circ m_B = l_B, \qquad p_B \circ m_S = l_S, \qquad \text{and} \qquad \varrho_B \circ p_S = \varrho_S \circ p_B.$$

In view of these relations the two triangles and the lower square of the diagram commute.

It remains to show that $m_B^\sharp \circ \tau_\vee^\sharp = m_S^\sharp \circ \sigma_\vee^\sharp$. Since these maps are homomorphisms it is sufficient to verify that they agree in P. But for $x \in P$ we have

$$\begin{aligned}(m_B\tau_\vee - m_S\sigma_\vee)(x) &= \tau(x) \otimes 1 \otimes 1 - 1 \otimes \sigma(x) \otimes 1 \\ &= \nabla_{B \otimes S}(1 \otimes 1 \otimes x).\end{aligned}$$

Q.E.D.

Example: Suppose that $(S, \delta_S; \sigma)$ is given by $S = \vee P$, $\delta_S = 0$, and $\sigma(x) = x$, $x \in P$, (cf. sec. 3.8). Then the cohomology diagram yields the commutative diagram

$$\begin{tikzcd}
 & H(B) \arrow[dr, "l_B^\#"] \arrow[dd, shift right, "m_B^\#"'] & \\
\vee P \arrow[ur, "\tau_\vee^\#"] \arrow[dr, "m_{\vee P}^\#"'] & & H(B \otimes \wedge P) \\
 & H(B \otimes \vee P \otimes \wedge P) \arrow[uu, shift right, "\varphi^\#"'] \arrow[ur, "p_{\vee P}^\#"'] &
\end{tikzcd}$$

where $m_B^\#$ and $\varphi^\#$ are the inverse isomorphisms of Propositions I and II, sec. 3.8.

3.21. Tensor difference with $S \otimes \wedge P$ noncohomologous to zero. Suppose $(B, \delta_B; \tau)$ and $(S, \delta_S: \sigma)$ are c-connected (P, δ)-algebras. Then $S \otimes \wedge P$ is called *noncohomologous to zero in* $B \otimes S \otimes \wedge P$ if the projection

$$p_B^\#: H(B \otimes S \otimes \wedge P) \to H(S \otimes \wedge P)$$

(cf. sec. 3.19) is surjective.

Remark: This is *not* the same as saying that $\wedge P$ is n.c.z. in $B \otimes S \otimes \wedge P$ (cf. sec. 3.17).

Example: If the map $l_S^\#: H(S) \to H(S \otimes \wedge P)$ is surjective, then $S \otimes \wedge P$ is n.c.z. in $B \otimes S \otimes \wedge P$.

In fact, since $l_S^\# = p_B^\# \circ m_S^\#$, $p_B^\#$ is surjective.

Theorem VIII: Suppose $(B, \delta_B; \tau)$ and $(S, \delta_S; \sigma)$ are c-connected (P, δ)-algebras. Then the following conditions are equivalent:

(1) $p_B^\#$ is surjective.

(2) There is a linear isomorphism of graded vector spaces

$$f: H(B) \otimes H(S \otimes \wedge P) \xrightarrow{\cong} H(B \otimes S \otimes \wedge P)$$

which satisfies $f(\alpha \otimes \beta) = m_B^\#(\alpha) \cdot f(1 \otimes \beta)$ and makes the diagram

$$\begin{tikzcd}
& H(B) \otimes H(S \otimes \wedge P) \arrow[dd, "\cong"', "f"] \arrow[dr] & \\
H(B) \arrow[ur] \arrow[dr, "m_B^\#"'] & & H(S \otimes \wedge P) \\
& H(B \otimes S \otimes \wedge P) \arrow[ur, "p_B^\#"'] &
\end{tikzcd}$$

commute.

(3) The B-spectral sequence for the tensor difference collapses at the E_2-term.

Proof: We show that

$$(1) \Rightarrow (2), \qquad (2) \Rightarrow (1), \qquad (1) \Rightarrow (3), \qquad (3) \Rightarrow (1).$$

$(1) \Rightarrow (2)$: Let $\pi\colon Z(S \otimes \wedge P, \nabla_S) \to H(S \otimes \wedge P)$ be the projection. Since $p_B^\#$ is surjective, there is a linear map, homogeneous of degree zero,

$$\theta\colon H(S \otimes \wedge P) \to \ker \nabla_{B \otimes S},$$

such that

(i) $p_B \circ \theta\colon H(S \otimes \wedge P) \to 1 \otimes Z(S \otimes \wedge P)$.
(ii) $\pi \circ p_B \circ \theta = \iota$.
(iii) $\theta(1) = 1$.

Define a linear map

$$\varphi\colon B \otimes H(S \otimes \wedge P) \to B \otimes S \otimes \wedge P$$

by setting $\varphi(b \otimes \beta) = m_B(b) \cdot \theta(\beta)$, $b \in B$, $\beta \in H(S \otimes \wedge P)$. Since $\nabla_{B \otimes S} \circ \theta = 0$, we have

$$\varphi \circ (\delta_B \otimes \iota) = \nabla_{B \otimes S} \circ \varphi.$$

Thus φ induces a linear map

$$\varphi^\#\colon H(B) \otimes H(S \otimes \wedge P) \to H(B \otimes S \otimes \wedge P).$$

It follows from (ii), (iii), and the definition that

$$\varphi^\#(\alpha \otimes \beta) = m_B^\#(\alpha) \cdot \varphi^\#(1 \otimes \beta),$$

and that the diagram

$$\begin{tikzcd}[column sep=small]
 & H(B) \otimes H(S \otimes \wedge P) \arrow[dd, "\varphi^{\#}"] \arrow[dr] & \\
H(B) \arrow[ur] \arrow[dr, "m_B^{\#}"'] & & H(S \otimes \wedge P) \\
 & H(B \otimes S \otimes \wedge P) \arrow[ur, "p_B^{\#}"'] &
\end{tikzcd}$$

commutes.

To show that $\varphi^{\#}$ is an isomorphism, filter $B \otimes H(S \otimes \wedge P)$ by the subspaces

$$\hat{F}^p(B \otimes H(S \otimes \wedge P)) = \sum_{\mu \geq p} B^\mu \otimes H(S \otimes \wedge P).$$

Then φ is filtration preserving with respect to this filtration and the filtration of $B \otimes S \otimes \wedge P$ giving rise to the B-spectral sequence (cf. sec. 3.9).

Thus φ induces homomorphisms φ_i: $(\hat{E}_i, \hat{d}_i) \to (E_i, d_i)$ of the spectral sequences. In particular, φ_0 is the homomorphism

$$\varphi_0: B \otimes H(S \otimes \wedge P) \to B \otimes S \otimes \wedge P$$

given by

$$\varphi_0(b \otimes \beta) = b \otimes p_B\theta(\beta), \qquad b \in B, \quad \beta \in H(S \otimes \wedge P).$$

(To see this observe that $p_B \circ \theta$ is a linear map from $H(S \otimes \wedge P)$ to $Z(S \otimes \wedge P)$ and that

$$\theta(\beta) - 1 \otimes p_B\theta(\beta) \in B^+ \otimes S \otimes \wedge P.)$$

Since $\varphi_1 = \varphi_0^{\#}$, it follows from the relation above that $\varphi_1 = \iota$. Hence, by the comparison theorem (cf. sec. 1.14), $\varphi^{\#}$ is an isomorphism.

(2) $\Rightarrow$ (1): This is obvious.

(1) $\Rightarrow$ (3): Let φ be the linear map constructed above and consider the induced maps φ_i: $(\hat{E}_i, \hat{d}_i) \to (E_i, d_i)$. Since φ_1 is an isomorphism so is each φ_i, $i \geq 1$. Now the spectral sequence $(\hat{E}_i, \hat{d}_i)$ for $B \otimes H(S \otimes \wedge P)$ collapses at the E_2-term. Hence so does the B-sequence (E_i, d_i).

(3) $\Rightarrow$ (1): Recall from sec. 3.9 that the E_2-term of the B-sequence is given by

$$E_2^{p,q} \cong H^p(B) \otimes H^q(S \otimes \wedge P).$$

Let $\beta \in H^q(S \otimes \wedge P)$ be arbitrary, and choose a representing cocycle $z \in S \otimes \wedge P$. Then, in the notation of sec. 1.10,

$$1 \otimes z \in Z_2^{0,q}.$$

Since by hypothesis $E_2 \cong E_\infty$, it follows that

$$Z_2^{0,q} = Z_\infty^{0,q} + Z_1^{1,q-1} + D_\infty^{0,q} = Z_\infty^{0,q} + Z_1^{1,q-1}.$$

Thus we can write $1 \otimes z = z_1 + z_2$, where

$$z_1 \in Z_\infty^{0,q} \quad (\subset Z^q(B \otimes S \otimes \wedge P))$$

and

$$z_2 \in Z_1^{1,q-1} \; (\subset B^+ \otimes S \otimes \wedge P).$$

In particular, $p_B(z_1) = z$.

Thus if $\alpha \in H(B \otimes S \otimes \wedge P)$ is the element represented by z_1, then $p_B^\#(\alpha) = \beta$. This shows that $p_B^\#$ is surjective.

Q.E.D.

Corollary I: Assume that $S \otimes \wedge P$ is n.c.z. in $B \otimes S \otimes \wedge P$ and let $\eta: H(S \otimes \wedge P) \to H(B \otimes S \otimes \wedge P)$ be a linear map, homogeneous of degree zero, and such that $p_B^\# \circ \eta = \iota$. Then an isomorphism of graded spaces

$$h: H(B) \otimes H(S \otimes \wedge P) \xrightarrow{\cong} H(B \otimes S \otimes \wedge P)$$

is defined by $h(\alpha \otimes \beta) = m_B^\#(\alpha) \cdot \eta(\beta)$.

Proof: This follows in exactly the same way as Proposition IV, sec. 3.17.

Q.E.D.

Corollary II: Assume that $l_S^\#: H(S) \to H(S \otimes \wedge P)$ is surjective and let $\gamma: H(S \otimes \wedge P) \to H(S)$ be a linear map, homogeneous of degree zero, such that $l_S^\# \circ \gamma = \iota$. Then an isomorphism of graded spaces

$$h: H(B) \otimes H(S \otimes \wedge P) \xrightarrow{\cong} H(B \otimes S \otimes \wedge P)$$

is given by

$$h(\alpha \otimes \beta) = m_B^\#(\alpha) \cdot m_S^\#(\gamma(\beta)), \qquad \alpha \in H(B), \quad \beta \in H(S \otimes \wedge P).$$

Proof: Observe that

$$(p_B^\# \circ m_S^\# \circ \gamma)(\beta) = (l_S^\# \circ \gamma)(\beta) = \beta, \qquad \beta \in H(S \otimes \wedge P),$$

and apply Corollary I.

Q.E.D.

Finally, consider the homomorphism

$$\psi: H(B) \otimes H(S) \to H(B \otimes S \otimes \wedge P)$$

given by $\psi(\alpha \otimes \beta) = m_B^\#(\alpha) \cdot m_S^\#(\beta)$. It follows from the commutativity of the cohomology diagram that

$$\psi(\tau^\#(x) \otimes 1 - 1 \otimes \sigma^\#(x)) = 0, \qquad x \in P.$$

Let I denote the ideal in $H(B) \otimes H(S)$ generated by elements of the form $\tau^\#(x) \otimes 1 - 1 \otimes \sigma^\#(x)$, $x \in P$. Then ψ factors to yield an algebra homomorphism

$$\bar{\psi}: \frac{H(B) \otimes H(S)}{I} \to H(B \otimes S \otimes \wedge P).$$

Corollary III: Assume that $l_S^\#: H(S) \to H(S \otimes \wedge P)$ is surjective. Then $\bar{\psi}$ is an isomorphism of graded algebras.

Proof: Consider the (P, δ)-algebra $(B \otimes S, \delta_{B\otimes S}; \tau \ominus \sigma)$. Evidently,

$$\psi = l_{B\otimes S}^\#.$$

By Corollary II, $l_{B\otimes S}^\#$ is surjective. Now Corollary I to Theorem VI, sec. 3.16, applies, and shows that

$$\ker l_{B\otimes S}^\# = [H(B) \otimes H(S)] \cdot (\tau \ominus \sigma)^\#(P) = I.$$

Thus $\bar{\psi}$ is injective.

Q.E.D.

Corollary IV: Assume that $H(B)$ and $H(S \otimes \wedge P)$ are of finite type. Then so is $H(B \otimes S \otimes \wedge P)$, and the Poincaré series satisfy

$$f_{H(B\otimes S\otimes\wedge P)} \leq f_{H(B)} \cdot f_{H(S\otimes\wedge P)}.$$

Equality holds if and only if $S \otimes \wedge P$ is n.c.z. in $B \otimes S \otimes \wedge P$.

Corollary V: If $H(B)$ and $H(S \otimes \wedge P)$ are finite dimensional, then so is $H(B \otimes S \otimes \wedge P)$, and

$$\dim H(B \otimes S \otimes \wedge P) \leq \dim H(B) \cdot \dim H(S \otimes \wedge P).$$

Equality holds if and only if $S \otimes \wedge P$ is n.c.z. in $B \otimes S \otimes \wedge P$.

Corollary VI: If $H(B)$ and $H(S \otimes \wedge P)$ are finite dimensional, then the Euler–Poincaré characteristic of $H(B \otimes S \otimes \wedge P)$ is given by

$$\chi_{H(B \otimes S \otimes \wedge P)} = \chi_{H(B)} \chi_{H(S \otimes \wedge P)}.$$

§6. Tensor difference with a symmetric P-algebra

In this article $(B, \delta_B; \tau)$ denotes a c-connected (P, δ)-algebra. $(\vee Q; \sigma)$ is a symmetric P-algebra and $(B \otimes \vee Q \otimes \wedge P, \nabla_{B\otimes\vee Q})$ denotes the Koszul complex of their tensor difference.

We shall carry over all the notation of article 6, Chapter II, unchanged. In particular, $P_1 \subset P$ denotes the essential subspace for $(\vee Q; \sigma)$ and we write (cf. Lemma VIII, sec. 2.23)

$$P = P_1 \oplus P_2 \qquad \text{and} \qquad \vee Q = \vee P_2 \otimes \vee Q_1 .$$

$\hat{P}$ will denote the Samelson space for $(\vee Q; \sigma)$.

Recall that Theorem VIII, sec. 3.21, gives a necessary and sufficient condition for a linear isomorphism

$$H(B) \otimes H(\vee Q \otimes \wedge P) \cong H(B \otimes \vee Q \otimes \wedge P)$$

which makes the appropriate diagram commute. A main result of this article (Theorem IX) gives necessary and sufficient conditions for this to be an *algebra isomorphism*.

Note that this contrasts with the situation for (P, δ)-algebras where the existence of a linear isomorphism $H(B) \otimes \wedge P \cong H(B \otimes \wedge P)$ implies the existence of an algebra isomorphism (cf. sec. 3.17).

3.22. Cohomology of the tensor difference. Recall from Theorem X, sec. 2.23, that $H(\vee Q \otimes \wedge P) \cong H(\vee Q_1 \otimes \wedge P_1)$. In this section that theorem will be generalized to yield an isomorphism

$$H(B \otimes \vee Q \otimes \wedge P) \cong H(B \otimes \vee Q_1 \otimes \wedge P_1, \nabla_1).$$

Unfortunately, ∇_1 is more complicated than the differential operator corresponding to the tensor difference of $(B, \delta_B; \tau|_{P_1})$ and $(\vee Q_1; \sigma_1)$.

First define a homomorphism

$$\varphi: B \otimes \vee Q \to B \otimes \vee Q_1$$

by setting

$$\varphi(b \otimes \Psi \otimes \Phi) = b \cdot \tau_\vee \Psi \otimes \Phi, \qquad b \in B, \quad \Psi \in \vee P_2, \quad \Phi \in \vee Q_1.$$

It satisfies $\varphi \circ \delta_B = \delta_B \circ \varphi$.

Next, define a linear map

$$\tau_1 : P_1 \to B \otimes \vee Q_1$$

by $\tau_1(x) = \tau(x) \otimes 1 - \varphi(1 \otimes \sigma(x))$, $x \in P_1$. Then $(B \otimes \vee Q_1, \delta_B; \tau_1)$ is a (P_1, δ)-algebra. Denote its Koszul complex by $(B \otimes \vee Q_1 \otimes \wedge P_1, \nabla_1)$.

The inclusion map $m_1 : B \to B \otimes \vee Q_1 \otimes \wedge P_1$ is a homomorphism of graded differential algebras. Moreover, since $H(B)$ is connected, the projection $p_1 : B \otimes \vee Q_1 \otimes \wedge P_1 \to B^0 \otimes \vee Q_1 \otimes \wedge P_1$ induces a homomorphism

$$p_1^\# : H(B \otimes \vee Q_1 \otimes \wedge P_1) \to H(\vee Q_1 \otimes \wedge P_1, \nabla_{\sigma_1})$$

in exactly the way described in sec. 3.19.

Finally, let $\beta : P \to P_1$ be the projection with kernel P_2, and extend it to homomorphisms

$$\beta_\wedge : \wedge P \to \wedge P_1 \qquad \text{and} \qquad \beta_\vee : \vee P \to \vee P_1.$$

Proposition VII: With the hypotheses and notation above:

(1) $\varphi \otimes \beta_\wedge : (B \otimes \vee Q \otimes \wedge P, \nabla_{B \otimes \vee Q}) \to (B \otimes \vee Q_1 \otimes \wedge P_1, \nabla_1)$ is a homomorphism of graded differential algebras.

(2) $(\varphi \otimes \beta_\wedge)^\#$ is an isomorphism.

(3) The diagram

$$\begin{tikzcd}
 & H(B \otimes \vee Q \otimes \wedge P) \arrow[r, "p_B^\#"] \arrow[d, "(\varphi\otimes\beta_\wedge)^\#", "\cong"'] & H(\vee Q \otimes \wedge P) \arrow[d, "(\gamma\otimes\beta_\wedge)^\#", "\cong"'] \\
H(B) \arrow[ur, "m_B^\#"] \arrow[dr, "m_1^\#"'] & & \\
 & H(B \otimes \vee Q_1 \otimes \wedge P_1) \arrow[r, "p_1^\#"'] & H(\vee Q_1 \otimes \wedge P_1)
\end{tikzcd}$$

commutes, where $\gamma : \vee Q \to \vee Q_1$ denotes the projection defined in sec. 2.23.

Proof: (1) and (3) are straightforward consequences of the definitions. The proof of (2) is essentially the same as the proof in Theorem X, sec. 2.23, that $(\gamma \otimes \beta_\wedge)^\#$ is an isomorphism. The necessary modifications are left to the reader.

Q.E.D.

3.23. The algebra isomorphism theorem. Theorem IX: Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra, and let $(\vee Q; \sigma)$ be a symmetric P-algebra. Then the following conditions are equivalent:

(1) There is a homomorphism $\psi: \vee Q \to H(B) \otimes \vee Q/\vee Q \circ P$ of graded algebras, which makes the diagram

$$\begin{tikzcd}
\vee P \arrow[r, "\tau_\vee^\#"] \arrow[d, "\sigma_\vee"'] & H(B) \arrow[d] \\
\vee Q \arrow[r, "\psi"] \arrow[dr, "\bar{l}"'] & H(B) \otimes \vee Q/\vee Q \circ P \arrow[d] \\
 & \vee Q/\vee Q \circ P
\end{tikzcd}$$

commute ($\bar{l}$ is the projection).

(2) There is an isomorphism

$$f: (B \otimes \vee Q \otimes \wedge P, \nabla_{B\otimes \vee Q}) \xrightarrow{\cong} (B \otimes \vee Q \otimes \wedge P, \delta_B - \nabla_\sigma)$$

of graded differential algebras, which makes the diagram

$$\begin{tikzcd}
 & B \otimes \vee Q \otimes \wedge P \arrow[dd, "\cong"', "f"] \arrow[dr, "p_B"] & \\
B \arrow[ur, "m_B"] \arrow[dr, "m_B"'] & & B^0 \otimes \vee Q \otimes \wedge P \\
 & B \otimes \vee Q \otimes \wedge P \arrow[ur, "p_B"'] &
\end{tikzcd}$$

commute.

(3) There is an isomorphism

$$g: H(B \otimes \vee Q \otimes \wedge P, \nabla_{B\otimes \vee Q}) \xrightarrow{\cong} H(B) \otimes H(\vee Q \otimes \wedge P)$$

of graded algebras, which makes the diagram

$$\begin{tikzcd}
 & H(B) \otimes H(\vee Q \otimes \wedge P) \arrow[dr] & \\
H(B) \arrow[ur] \arrow[dr, "m_B^\#"'] & & H(\vee Q \otimes \wedge P) \\
 & H(B \otimes \vee Q \otimes \wedge P) \arrow[uu, "g", "\cong"'] \arrow[ur, "p_B^\#"'] &
\end{tikzcd}$$

commute.

Remark: Consider the tensor difference as a generalization of (P, δ)-algebras with $\vee Q \otimes \wedge P$ replacing $\wedge P$. Then Theorem VIII, sec. 3.21, and Theorem IX together generalize Theorem VII, sec. 3.17. In fact, if one sets $Q = S = 0$ in Theorems VIII and IX, one finds that the conditions Theorem VIII: (1), (2), (3) and Theorem IX: (1), (2), (3) reduce respectively to the conditions Theorem VII: (1), (2), (5), (4), (6), (7).

However, in general the conditions in Theorem VIII are not equivalent to those in Theorem IX (cf. the corollary to Proposition VIII, sec. 3.25, and sec. 12.31).

3.24. Proof of Theorem IX: $(1) \Rightarrow (2)$: In view of the commutative diagram of (1), we may write

$$\psi(\Phi) = \tilde{\psi}(\Phi) + 1 \otimes \bar{l}(\Phi), \qquad \Phi \in Q,$$

where $\tilde{\psi}\colon Q \to H^+(B) \otimes \vee Q/\vee Q \circ P$. Choose a linear map

$$\tilde{\eta}\colon Q \to Z^+(B) \otimes \vee Q,$$

homogeneous of degree zero, so that $\pi \circ \tilde{\eta} = \tilde{\psi}$. (Here $\pi\colon Z(B) \otimes \vee Q \to H(B) \otimes \vee Q/\vee Q \circ P$ is the projection.)

Define $\eta\colon Q \to Z(B) \otimes \vee Q$ by

$$\eta(\Phi) = \tilde{\eta}(\Phi) + 1 \otimes \Phi, \qquad \Phi \in Q,$$

and extend η to a homomorphism $\eta_\vee\colon \vee Q \to Z(B) \otimes \vee Q$. Then

$$\pi \circ \eta_\vee = \psi \quad \text{and} \quad \eta_\vee(\Phi) - 1 \otimes \Phi \in Z^+(B) \otimes \vee Q, \quad \Phi \in \vee Q. \qquad (3.15)$$

On the other hand, the commutative diagram of (1) shows that

$$\tau^{\#}(x) \otimes 1 = \psi\sigma(x), \qquad x \in P.$$

Since $\pi \circ \eta_\vee = \psi$, it follows that

$$\tau(x) \otimes 1 - \eta_\vee\sigma(x) \in \ker \pi;$$

i.e.,

$$\tau(x) \otimes 1 - \eta_\vee\sigma(x) \in \delta_B(B) \otimes \vee Q + Z(B) \otimes (\vee Q \circ P).$$

Next observe that since $\tau(x)$ and $\eta_\vee\sigma(x)$ have even degree we can write

$$\begin{aligned} \tau(x) \otimes 1 - \eta_\vee\sigma(x) &\in \delta_B\Big(\sum_{p \text{ odd}} B^p\Big) \otimes \vee Q + Z(B) \otimes (\vee Q \circ P) \\ &\subset \delta_B(B^+) \otimes \vee Q + Z(B) \otimes (\vee Q \circ P), \qquad x \in P. \end{aligned}$$

Thus we obtain from formula (3.15) that

$$\tau(x) \otimes 1 - (\eta_\vee\sigma(x) - 1 \otimes \sigma(x)) \in \delta_B(B^+) \otimes \vee Q + Z^+(B) \otimes (\vee Q \circ P), \qquad x \in P.$$

It follows that there are linear maps

$$\theta_1 \colon P \to B^+ \otimes \vee Q \qquad \text{and} \qquad \theta_2 \colon P \to Z^+(B) \otimes \vee Q \otimes P,$$

homogeneous of degree zero, such that

$$\tau(x) \otimes 1 - \eta_\vee\sigma(x) + 1 \otimes \sigma(x) = \delta_B\theta_1(x) - \nabla_\sigma\theta_2(x), \qquad x \in P. \tag{3.16}$$

Define a linear map $\theta \colon P \to B \otimes \vee Q \otimes \wedge P$ by setting

$$\theta(x) = 1 \otimes 1 \otimes x + \theta_1(x) \otimes 1 + \theta_2(x), \qquad x \in P.$$

Extend θ to a homomorphism $\theta_\wedge \colon \wedge P \to B \otimes \vee Q \otimes \wedge P$. Finally, define a homomorphism

$$f \colon B \otimes \vee Q \otimes \wedge P \to B \otimes \vee Q \otimes \wedge P$$

by

$$f(b \otimes \Psi \otimes \Phi) = (b \otimes 1 \otimes 1) \cdot (\eta_\vee\Psi \otimes 1) \cdot (\theta_\wedge\Phi), \qquad b \in B, \quad \Psi \in \vee Q, \quad \Phi \in \wedge P.$$

Since (cf. formula (3.15))

$$\eta_\vee(\Psi) \otimes 1 - 1 \otimes \Psi \otimes 1 \in B^+ \otimes \vee Q \otimes \wedge P, \qquad \Psi \in \vee Q,$$

and (by definition)

$$\theta_\wedge(\Phi) - 1 \otimes 1 \otimes \Phi \in B^+ \otimes \vee Q \otimes \wedge P, \qquad \Phi \in \wedge P,$$

it follows that

$$(f - \iota)\colon \sum_{\mu \geq p} B^\mu \otimes \vee Q \otimes \wedge P \to \sum_{\mu \geq p+1} B^\mu \otimes \vee Q \otimes \wedge P, \quad p = 0, 1, \ldots . \tag{3.17}$$

This implies (as in the proof of Theorem V, sec. 2.15) that f is an isomorphism.

The commutativity of the diagram of (2) is an immediate consequence of formula (3.17) and the definition of f. Finally, a simple computation, using formula (3.16) and the relation $\delta_B \circ \eta_\vee = 0$ shows that

$$f \circ \nabla_{B \otimes \vee Q} = (\delta_B - \nabla_\sigma) \circ f.$$

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Recall from sec. 2.2 that $H_0(\vee Q \otimes \wedge P) = \vee Q / \vee Q \circ P$. Thus we have a commutative diagram of algebra homomorphisms

$$\begin{tikzcd}
H(B) \otimes H(\vee Q \otimes \wedge P) \arrow[r, "\iota \otimes \varrho"] \arrow[d] & H(B) \otimes \vee Q / \vee Q \circ P \arrow[d] \\
H(\vee Q \otimes \wedge P) \arrow[r, "\varrho"] & \vee Q / \vee Q \circ P \\
& \vee Q \arrow[ul, "l_{\vee Q}^{\#}"] \arrow[u, "l"']
\end{tikzcd}$$

where ϱ denotes the projection with kernel $H_+(\vee Q \otimes \wedge P)$.

Now set

$$\psi = (\iota \otimes \varrho) \circ g \circ m_{\vee Q}^{\#}.$$

Then, combining the cohomology diagram for the tensor difference (sec. 3.20) with the commutative diagram of (3) and the commutative diagram above, we obtain the commutative diagram of (1).

Q.E.D.

Corollary: If $\tau^{\#} = 0$, then the conditions of Theorem IX hold.

Proof: Set $\psi(\Psi) = 1 \otimes \bar{l}(\Psi)$, $\Psi \in \vee Q$, and observe that the diagram of (1) commutes.

Q.E.D.

3.25. The homomorphism ψ. In this section we study the commutative diagram in Theorem IX, (1).

Proposition VIII: Let $(\vee Q; \sigma)$ be an essential symmetric P-algebra such that $\vee Q/\vee Q \circ P$ has finite dimension. Let A be a connected graded anticommutative algebra. Suppose

$$\begin{tikzcd}
\vee P \arrow[r, "\varphi"] \arrow[d, "\sigma_\vee"'] & A \arrow[d, "i"] \\
\vee Q \arrow[r, "\psi"] \arrow[dr, "\bar{l}"'] & A \otimes \vee Q/\vee Q \circ P \arrow[d, "\varrho"] \\
& \vee Q/\vee Q \circ P
\end{tikzcd}$$

is a commutative diagram of homomorphisms of graded algebras, where i and ϱ are the obvious inclusion and projection.

Then, if Γ has characteristic zero, (1) $\operatorname{Im} \psi^+ \subset A \otimes (\vee Q/\vee Q \circ P)^+$ and (2) $\varphi^+ = 0$.

Proof: In view of the commutative diagram, (2) is a direct consequence of (1). To establish (1), let $\bar{\varphi}: \vee Q \to A$ be the (unique) homomorphism satisfying

$$\psi(\Psi) - \bar{\varphi}(\Psi) \otimes 1 \in A \otimes (\vee Q/\vee Q \circ P)^+, \qquad \Psi \in \vee Q.$$

Then (1) is equivalent to the relation

$$\bar{\varphi}(\vee^+ Q) = 0. \tag{3.18}$$

To prove formula (3.18) we show by induction on k that

$$\bar{\varphi}(Q^p) = 0, \qquad 1 \le p \le k. \tag*{$(3.18)_k$}$$

Formula $(3.18)_1$ is trivially correct. Now assume that formula $(3.18)_{k-1}$ holds.

Let I be the ideal in $\vee Q$ generated by $\sigma(P) + \sum_{j<k} Q^j$, and let

$$\pi\colon \vee Q/\vee Q \circ P \to \vee Q/I$$

be the corresponding projection. Further, set $R = A/\sum_{j>k} A^j$, and let $\pi_R\colon A \to R$ be the projection. Set

$$\psi_R = (\pi_R \otimes \pi) \circ \psi, \qquad \varphi_R = \pi_R \circ \varphi,$$
$$\bar{\psi}_R = \pi_R \circ \bar{\psi}, \qquad l = \pi \circ \bar{l}.$$

Then the diagram (of algebra homomorphisms)

$$\begin{tikzcd} \vee P \arrow[r, "\varphi_R"] \arrow[d, "\sigma_\vee"'] & R \arrow[d] \\ \vee Q \arrow[r, "\psi_R"] \arrow[dr, "l"'] & R \otimes \vee Q/I \arrow[d] \\ & \vee Q/I \end{tikzcd}$$

commutes.

Since $(\vee Q/I)^p = 0$, $1 \le p < k$, it follows that

$$\psi_R(\Psi) = \bar{\psi}_R(\Psi) \otimes 1 + 1 \otimes l(\Psi), \qquad \Psi \in Q^k.$$

Fix $\Psi \in Q^k$ and let n ($n \ge 1$) denote the least integer such that $[l(\Psi)]^n = 0$. (Recall that $\vee Q/\vee Q \circ P$ has finite dimension.) Since $R^p = 0$, $p > k$, $[\bar{\psi}_R(\Psi)]^2 = 0$. Thus we obtain

$$\begin{aligned} \psi_R(\Psi^n) &= n\bar{\psi}_R(\Psi) \otimes [l(\Psi)]^{n-1} + 1 \otimes [l(\Psi)]^n \\ &= n\bar{\psi}_R(\Psi) \otimes [l(\Psi)]^{n-1}. \end{aligned} \tag{3.19}$$

On the other hand, since

$$l(\Psi^n) = [l(\Psi)]^n = 0,$$

it follows that $\Psi^n \in I$. Hence, Lemma III, below, implies that

$$\psi_R(\Psi^n) = 0. \tag{3.20}$$

Combining formulae (3.19) and (3.20) yields $\bar{\psi}_R(\Psi) = 0$ (since Γ has characteristic zero).

Finally, observe that $\pi_R\colon A^k \xrightarrow{\cong} R^k$. Since $\pi_R\bar{\psi}(\Psi) = 0$, it follows that $\bar{\psi}(\Psi) = 0$. Thus formula $(3.18)_k$ is established, and the induction is closed.

Q.E.D.

Lemma III: Assume formula $(3.18)_{k-1}$ holds. Then (with the notation established in the proof of Proposition VIII)

$$I \subset \ker \psi_R.$$

Proof: Since $R^p = 0$, $p > k$, we have

$$\bar{\psi}_R(Q^k) \cdot \bar{\psi}_R(Q^k) = 0 \qquad \text{and} \qquad \bar{\psi}_R(Q^p) = 0, \qquad p > k.$$

Moreover, formula $(3.18)_{k-1}$ implies that $\bar{\psi}_R(Q^p) = 0$, $p < k$. These relations show that

$$\bar{\psi}_R(\vee^+ Q) \cdot \bar{\psi}_R(\vee^+ Q) = 0.$$

Now the commutative diagram of the proposition implies that

$$\psi_R(\sigma(P)) \subset R \otimes 1$$

and so $\psi_R(\sigma(P)) = \bar{\psi}_R(\sigma(P)) \otimes 1$. Since $(\vee Q; \sigma)$ is essential, we have $\sigma(P) \subset \vee^+ Q \cdot \vee^+ Q$; hence

$$\psi_R(\sigma(P)) \subset \bar{\psi}_R(\vee^+ Q) \cdot \bar{\psi}_R(\vee^+ Q) \otimes 1 = 0. \tag{3.21}$$

On the other hand, since $I \supset Q^p$, $p < k$, we have $(\vee Q/I)^p = 0$, $0 < p < k$. It follows that

$$\psi_R(Q^p) \subset R^p \otimes 1, \qquad p < k.$$

This, together with formula $(3.18)_{k-1}$ yields

$$\psi_R(\Psi) = \bar{\psi}_R(\Psi) \otimes 1 = 0, \qquad \Psi \in \sum_{p<k} Q^p. \tag{3.22}$$

Relations (3.21) and (3.22) show that

$$\ker \psi_R \supset \sum_{p<k} Q^p + \sigma(P).$$

But the space on the right generates the ideal I; thus since $\ker \psi_R$ is an ideal we have $\ker \psi_R \supset I$.

Q.E.D.

In view of Theorem IX, sec. 3.23, and its corollary, we obtain the following

Corollary: Let $(\vee Q; \sigma)$ be as in the proposition, and let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra. Assume that Γ has characteristic zero. Then there is an algebra isomorphism

$$g: H(B \otimes \vee Q \otimes \wedge P) \xrightarrow{\cong} H(B) \otimes H(\vee Q \otimes \wedge P),$$

making the diagram of Theorem IX,(3) commute, if and only if $\tau^{\#} = 0$.

3.26. Theorem X: Let $(\vee Q; \sigma)$ be a symmetric P-algebra with essential subspace P_1 such that $H(\vee Q \otimes \wedge P)$ has finite dimension. Let $(B, \delta_B; \tau)$ be a c-connected (P, δ)-algebra, and assume that the tensor difference satisfies the conditions of Theorem IX, sec. 3.23.

Then P_1 is contained in the Samelson space for $(B, \delta_B; \tau)$:

$$P_1 \subset \operatorname{Im} \varrho_B^{\#}.$$

Lemma IV: With the hypotheses of Theorem X

$$\tau^{\#}(P_1) \subset H^{+}(B) \cdot \tau^{\#}(P_2)$$

(where $P = P_1 \oplus P_2$).

Proof: In view of Theorem IX, (1) we have the commutative diagram

$$\begin{tikzcd} \vee P \arrow[r, "\tau_{\vee}^{\#}"] \arrow[d, "\sigma_{\vee}"'] & H(B) \arrow[d] \\ \vee Q \arrow[r, "\psi"] \arrow[dr, "l_{\vee Q}"'] & H(B) \otimes \vee Q / \vee Q \circ P \arrow[d] \\ & \vee Q / \vee Q \circ P. \end{tikzcd} \tag{3.23}$$

Moreover, the projection $\vee Q \to \vee Q_1$ induced by the decomposition $\vee Q = \vee P_2 \otimes \vee Q_1$ induces an isomorphism

$$\vee Q / \vee Q \circ P \xrightarrow{\cong} \vee Q_1 / \vee Q_1 \cdot \sigma_1(P_1).$$

We identify these algebras under this isomorphism. Observe that $\sigma_1 \colon P_1 \to \vee Q_1$ is the linear map of sec. 2.23.

Denote by A the factor algebra

$$A = H(B)/H(B) \cdot \tau^{\#}(P_2)$$

and let $\pi_A \colon H(B) \to A$ be the projection. Define

$$\psi_1 \colon \vee Q_1 \to A \otimes \frac{\vee Q_1}{\vee Q_1 \cdot \sigma_1(P_1)}$$

by $\psi_1(\Psi) = (\pi_A \otimes \iota)(\psi(\Psi))$, $\Psi \in \vee Q_1$.

We show that the diagram

$$\begin{tikzcd}
\vee P_1 \arrow[r, "\pi_A \circ \tau_\vee^{\#}"] \arrow[d, "(\sigma_1)_\vee"'] & A \arrow[d] \\
\vee Q_1 \arrow[r, "\psi_1"] \arrow[dr] & A \otimes \vee Q_1/\vee Q_1 \cdot \sigma_1(P_1) \arrow[d] \\
& \vee Q_1/\vee Q_1 \cdot \sigma_1(P_1)
\end{tikzcd}$$

commutes.

The commutativity of the triangle is obvious. To prove that the square commutes, recall that $\vee Q = \vee P_2 \otimes \vee Q_1$ and that the restriction of $\sigma_\vee$ to $\vee P_2$ is given by

$$\sigma_\vee(\Phi) = \Phi \otimes 1, \qquad \Phi \in \vee P_2.$$

Moreover, $\sigma(x) \in \vee^+ Q \cdot \vee^+ Q$, $x \in P_1$.

Thus we can write

$$\sigma(x) = \bar{\sigma}(x) \otimes 1 + \sum_i \Phi_i \otimes \Psi_i + 1 \otimes \sigma_1(x), \qquad x \in P_1,$$

where

$$\bar{\sigma}(x) \in (\vee^+ P_2) \cdot (\vee^+ P_2), \qquad \Phi_i \in \vee^+ P_2, \qquad \text{and} \qquad \Psi_i \in \vee^+ Q_1.$$

It follows that (cf. diagram (3.23))

$$\begin{aligned}
\tau^{\#}(x) \otimes 1 &= \psi\sigma(x) \\
&= \tau_\vee^{\#}\bar{\sigma}(x) \otimes 1 + \sum_i (\tau_\vee^{\#}(\Phi_i) \otimes 1) \cdot \psi(\Psi_i) + \psi(\sigma_1(x)), \quad x \in P_1.
\end{aligned}$$

Hence

$$\pi_A \tau^{\#}(x) \otimes 1 = (\pi_A \otimes \iota) \circ \psi(\sigma_1(x)) = \psi_1(\sigma_1(x)), \qquad x \in P_1.$$

This shows that the square commutes.

According to sec. 2.23, $(\vee Q_1, \sigma_1)$ is an essential P-algebra. Thus Proposition VIII, sec. 3.25, implies that

$$\psi_1(\vee^+ Q_1) \subset A \otimes (\vee Q_1 / \vee Q_1 \cdot \sigma_1(P_1))^+.$$

It follows that

$$\psi(Q_1) \subset H(B) \otimes (\vee Q/\vee Q \circ P)^+ + (H(B) \cdot \tau^{\#}(P_2)) \otimes \vee Q/\vee Q \circ P.$$

Since $\psi(P_2) \subset \tau^{\#}(P_2) \otimes 1$, this implies that

$$\psi(\vee^+ Q) \subset H(B) \otimes (\vee Q/\vee Q \circ P)^+ + (H(B) \cdot \tau^{\#}(P_2)) \otimes \vee Q/\vee Q \circ P. \tag{3.24}$$

Now let $\bar{\psi}: \vee Q \to H(B)$ be the homomorphism obtained by composing ψ with the projection $H(B) \otimes \vee Q/\vee Q \circ P \to H(B)$. Then formula (3.24) yields

$$\bar{\psi}(\vee^+ Q) \subset H(B) \cdot \tau^{\#}(P_2),$$

whence

$$\bar{\psi}(\vee^+ Q \cdot \vee^+ Q) \subset H^+(B) \cdot \tau^{\#}(P_2).$$

But for $x \in P_1$, $\sigma(x) \in \vee^+ Q \cdot \vee^+ Q$, and so

$$\tau^{\#}(x) = \bar{\psi}(\sigma(x)) \in H^+(B) \cdot \tau^{\#}(P_2).$$

Q.E.D.

Proof of Theorem X: In view of Lemma IV, there is a linear map

$$\theta_1: P_1 \to Z^+(B) \otimes P_2 + B^+ \otimes 1,$$

such that $\nabla_B \theta_1(x) = -\tau(x) \otimes 1$, $x \in P_1$. Hence

$$\nabla_B(1 \otimes x + \theta_1(x)) = 0 \qquad \text{and} \qquad \varrho_B(1 \otimes x + \theta_1(x)) = x, \qquad x \in P_1.$$

This shows that $P_1 \subset \operatorname{Im} \varrho_B^{\#}$.

Q.E.D.

§7. Equivalent and c-equivalent (P, δ)-algebras

3.27. Equivalent (P, δ)-algebras. Two (P, δ)-algebras $(B, \delta_B; \tau)$ and $(\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ are called *equivalent* if $\tilde{B} = B$, $\delta_{\tilde{B}} = \delta_B$, and $\tilde{\tau}^{\#} = \tau^{\#}$.

Proposition IX: Let $(B, \delta_B; \tau)$ and $(B, \delta_B; \tilde{\tau})$ be equivalent (P, δ)-algebras. Denote their Koszul complexes by $(B \otimes \wedge P, \nabla_B)$ and $(B \otimes \wedge P, \tilde{\nabla}_B)$. Then there is an isomorphism of graded differential algebras

$$f\colon (B \otimes \wedge P, \nabla_B) \xrightarrow{\cong} (B \otimes \wedge P, \tilde{\nabla}_B)$$

such that $f \circ i(x^*) = i(x^*) \circ f$, $x^* \in P^*$, and the diagram

$$\begin{tikzcd}
& B \otimes \wedge P \arrow[dd, "f", "\cong"'] \arrow[dr, "\varrho_B"] & \\
B \arrow[ur, "\iota_B"] \arrow[dr, "\iota_B"'] & & B^0 \otimes \wedge P \\
& B \otimes \wedge P \arrow[ur, "\varrho_B"'] &
\end{tikzcd}$$

commutes.

In particular, $f^{\#}$ is an isomorphism.

Proof: Since $\tau^{\#} = \tilde{\tau}^{\#}$, there is a linear map $\alpha\colon P \to B^+$, homogeneous of degree zero, such that

$$\tau - \tilde{\tau} = \delta_B \circ \alpha.$$

Define $\beta\colon P \to B \otimes \wedge P$ by

$$\beta(x) = 1 \otimes x + \alpha(x) \otimes 1, \qquad x \in P,$$

and define $f\colon B \otimes \wedge P \to B \otimes \wedge P$ by

$$f(b \otimes \Phi) = (b \otimes 1) \cdot \beta_\wedge(\Phi), \qquad b \in B, \quad \Phi \in \wedge P.$$

It follows exactly as in the proof of Theorem V, sec. 2.15, that f is an isomorphism. The remaining properties are straightforward consequences of the definition of f.

Q.E.D.

Remark: Theorem VII, sec. 3.17 ((4) $\Rightarrow$ (6)) is a special case of the proposition above.

Proposition X: Let $(B, \delta_B; \tau)$ and $(\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$ be (P, δ)-algebras. Assume that $\varphi: B \to \tilde{B}$ is a homomorphism of graded differential algebras such that $\varphi^{\#} \circ \tau^{\#} = \tilde{\tau}^{\#}$.

Then there is a homomorphism of graded differential algebras

$$\psi: (B \otimes \wedge P, \nabla_B) \to (\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$$

such that $\psi \circ i(x^*) = i(x^*) \circ \psi$, $x^* \in P^*$, and the diagram

$$\begin{tikzcd} B \arrow[r, "\iota_B"] \arrow[d, "\varphi"'] & B \otimes \wedge P \arrow[r, "\varrho_B"] \arrow[d, "\psi"] & B^0 \otimes \wedge P \arrow[d, "\varphi \otimes \iota"] \\ \tilde{B} \arrow[r, "\iota_{\tilde{B}}"'] & \tilde{B} \otimes \wedge P \arrow[r, "\varrho_{\tilde{B}}"'] & \tilde{B}^0 \otimes \wedge P \end{tikzcd}$$

commutes.

Proof: Consider the map $\hat{\tau}: P \to \tilde{B}$ given by $\hat{\tau} = \varphi \circ \tau$. Then $\varphi: (B, \delta_B; \tau) \to (\tilde{B}, \delta_{\tilde{B}}; \hat{\tau})$ is a homomorphism of (P, δ)-algebras; hence

$$\varphi \otimes \iota: (B \otimes \wedge P, \nabla_B) \to (\tilde{B} \otimes \wedge P, \delta_{\tilde{B}} + \nabla_{\hat{\tau}})$$

is a homomorphism of the Koszul complexes.

On the other hand $(\tilde{B}, \delta_{\tilde{B}}; \hat{\tau})$ is equivalent to $(\tilde{B}, \delta_{\tilde{B}}; \tilde{\tau})$; thus Proposition IX yields an isomorphism

$$f: (\tilde{B} \otimes \wedge P, \delta_{\tilde{B}} + \nabla_{\hat{\tau}}) \xrightarrow{\cong} (\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$$

of Koszul complexes. Now set $\psi = f \circ (\varphi \otimes \iota)$.

Q.E.D.

3.28. c-equivalent (P, δ)-algebras. A (P, δ)-algebra $(B, \delta; \tau)$ will be called *cohomologically related* (c-*related*) to a (P, δ)-algebra $(\tilde{B}, \tilde{\delta}; \tilde{\tau})$ if there is a homomorphism of (P, δ)-algebras $\varphi: (B, \delta; \tau) \to (\tilde{B}, \tilde{\delta}; \tilde{\tau})$

such that $\varphi^{\#}: H(B) \to H(\tilde{B})$ is an isomorphism. In this case we write

$$(B, \delta; \tau) \xrightarrow[c]{} (\tilde{B}, \tilde{\delta}; \tilde{\tau})$$

and call φ a c-*relation.* (Note that then $(B, \delta) \xrightarrow[c]{} (\tilde{B}, \tilde{\delta})$; cf. sec. 0.10).

Two (P, δ)-algebras $(B, \delta; \tau)$ and $(\tilde{B}, \tilde{\delta}; \tilde{\tau})$ will be called *cohomologically equivalent* (c-*equivalent*) if there is a sequence of (P, δ)-algebras, $(B_i, \delta_i; \tau_i)$ $(i = 1, \ldots, n)$ such that

(1) $(B_1, \delta_1; \tau_1) = (B, \delta; \tau)$ and $(B_n, \delta_n; \tau_n) = (\tilde{B}, \tilde{\delta}, \tilde{\tau})$.

(2) For each i $(1 \le i \le n-1)$, either $(B_i, \delta_i; \tau_i) \xrightarrow[c]{} (B_{i+1}, \delta_{i+1}; \tau_{i+1})$, or $(B_{i+1}, \delta_{i+1}; \tau_{i+1}) \xrightarrow[c]{} (B_i, \delta_i; \tau_i)$. This is an equivalence relation; it is denoted by

$$(B, \delta; \tau) \underset{c}{\sim} (\tilde{B}, \tilde{\delta}; \tilde{\tau}).$$

A specific choice of the $(B_i, \delta_i; \tau_i)$, together with a specific choice of the c-relations between them, is called a c-*equivalence* between $(B, \delta; \tau)$ and $(\tilde{B}, \tilde{\delta}; \tilde{\tau})$.

Let $\{(B_i, \delta_i; \tau_i), \varphi_i\}$ be a fixed c-equivalence. Then $\{(B_i, \delta_i), \varphi_i\}$ is a c-equivalence between the graded differential algebras (B, δ) and $(\tilde{B}, \tilde{\delta})$ (cf. sec. 0.10). On the other hand, Theorem I, sec. 3.10, implies that $\{(B_i \otimes \wedge P, \nabla_{B_i}), \varphi_i \otimes \iota\}$ is a c-equivalence between the Koszul complexes $(B \otimes \wedge P, \nabla_B)$ and $(\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$.

The resulting isomorphisms

$$H(B) \cong H(\tilde{B}) \tag{3.25}$$

and

$$H(B \otimes \wedge P) \cong H(\tilde{B} \otimes \wedge P), \tag{3.26}$$

will be called the *isomorphisms induced by the* c-*equivalence* $\{(B_i, \delta_i; \tau_i), \varphi_i\}$.

It is immediate from the definitions that the isomorphisms (3.25) and (3.26) make the diagram

$$\begin{tikzcd}
 & H(B) \arrow[r, "l_B^{\#}"] \arrow[d, "\cong"] & H(B \otimes \wedge P) \arrow[d, "\cong"] \arrow[dr, "\varrho_B^{\#}"] & \\
\vee P \arrow[ur, "\tau_\vee^{\#}"] \arrow[dr, "\tilde{\tau}_\vee^{\#}"'] & & & \wedge P \\
 & H(\tilde{B}) \arrow[r, "l_{\tilde{B}}^{\#}"'] & H(\tilde{B} \otimes \wedge P) \arrow[ur, "\varrho_{\tilde{B}}^{\#}"'] &
\end{tikzcd}$$

commute. (The right-hand triangle is to be omitted unless $H(B)$ and $H(\tilde{B})$ are connected.)

Moreover, the c-equivalence between $(B \otimes \wedge P, \nabla_B)$ and $(\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$ determines an isomorphism of their spectral sequences, and of their lower spectral sequences (cf. sec. 3.4 and sec. 3.5).

Finally, let $(S, \delta_S; \sigma)$ be any (P, δ)-algebra. Then

$$\{(B_i \otimes S \otimes \wedge P, \nabla_{B_i \otimes S}), \varphi_i \otimes \iota \otimes \iota\}$$

is a c-equivalence between the Koszul complexes $(B \otimes S \otimes \wedge P, \nabla_{B \otimes S})$ and $(\tilde{B} \otimes S \otimes \wedge P, \nabla_{\tilde{B} \otimes S})$ for the tensor differences. The induced isomorphism

$$H(B \otimes S \otimes \wedge P) \cong H(\tilde{B} \otimes S \otimes \wedge P), \tag{3.27}$$

together with the isomorphisms (3.25) and (3.26), defines an isomorphism of cohomology diagrams. Moreover, there is an induced isomorphism of B- and S-spectral sequences (cf. sec. 3.9).

3.29. Let $(B, \delta; \tau)$ and $(\tilde{B}, \tilde{\delta}; \tilde{\tau})$ be (P, δ)-algebras. Suppose there is a c-equivalence between the differential algebras (B, δ) and $(\tilde{B}, \tilde{\delta})$ with induced isomorphism

$$\gamma\colon H(B) \xrightarrow{\cong} H(\tilde{B}).$$

Proposition XI: Assume $\gamma \circ \tau^{\#} = \tilde{\tau}^{\#}$. Then there is a c-equivalence of (P, δ)-algebras

$$(B, \delta; \tau) \underset{c}{\sim} (\tilde{B}, \tilde{\delta}; \tilde{\tau}),$$

such that the induced isomorphism $H(B) \xrightarrow{\cong} H(\tilde{B})$ coincides with γ.

Proof: It is easy to reduce to the case that there is a homomorphism of graded differential algebras

$$\varphi\colon (B, \delta) \to (\tilde{B}, \tilde{\delta})$$

such that $\varphi^{\#} = \gamma$.

Let $\psi\colon (B \otimes \wedge P, \nabla_B) \to (\tilde{B} \otimes \wedge P, \nabla_{\tilde{B}})$ be the homomorphism constructed from φ in Proposition X, sec. 3.27. Extend ψ in the obvious way to a homomorphism

$$\hat{\psi}\colon B \otimes \vee P \otimes \wedge P \to \tilde{B} \otimes \vee P \otimes \wedge P,$$

such that $\hat{\psi}(1 \otimes \Psi \otimes 1) = 1 \otimes \Psi \otimes 1$. Then the relations $\psi \circ i(x^*) = i(x^*) \circ \psi$, $x^* \in P^*$, (cf. Proposition X) imply that

$$\hat{\psi} \circ \nabla_{B \otimes \vee P} = \nabla_{\tilde{B} \otimes \vee P} \circ \hat{\psi}.$$

(Here $\nabla_{B \otimes \vee P}$ and $\nabla_{\tilde{B} \otimes \vee P}$ denote the differential operators in the tensor differences—cf. sec. 3.8.)

Now consider the (P, δ)-algebras, $(B \otimes \vee P \otimes \wedge P, \nabla_{B \otimes \vee P}; \sigma)$ and $(\tilde{B} \otimes \vee P \otimes \wedge P, \nabla_{\tilde{B} \otimes \vee P}; \tilde{\sigma})$, where

$$\sigma(x) = 1 \otimes x \otimes 1 \qquad \text{and} \qquad \tilde{\sigma}(x) = 1 \otimes x \otimes 1, \qquad x \in P.$$

Then $\hat{\psi}$ is a homomorphism of (P, δ)-algebras. Moreover

$$\hat{\psi} \circ m_B = m_{\tilde{B}} \circ \varphi,$$

as follows from the construction of ψ.

Since (cf. Proposition I, sec. 3.8) $m_B^\#$ and $\tilde{m}_{\tilde{B}}^\#$ are isomorphisms, and since $\varphi^\#$ is an isomorphism by hypothesis, it follows that $\hat{\psi}^\#$ is an isomorphism. Thus

$$(B \otimes \vee P \otimes \wedge P, \nabla_{B \otimes \vee P}; \sigma) \xrightarrow[c]{} (\tilde{B} \otimes \vee P \otimes \wedge P, \nabla_{\tilde{B} \otimes \vee P}; \tilde{\sigma}).$$

Finally, observe that Proposition II, sec. 3.8, implies that

$$(B \otimes \vee P \otimes \wedge P, \nabla_{B \otimes \vee P}; \sigma) \xrightarrow[c]{} (B, \delta; \tau)$$

and

$$(\tilde{B} \otimes \vee P \otimes \wedge P, \nabla_{\tilde{B} \otimes \vee P}; \tilde{\sigma}) \xrightarrow[c]{} (\tilde{B}, \tilde{\delta}; \tilde{\tau}).$$

These three c-relations define a c-equivalence between $(B, \delta; \tau)$ and $(\tilde{B}, \tilde{\delta}; \tilde{\tau})$.

Moreover (cf. Proposition II, sec. 3.8), the induced isomorphism β between $H(B)$ and $H(\tilde{B})$ is given by

$$\beta = (m_{\tilde{B}}^\#)^{-1} \circ \hat{\psi}^\# \circ m_B^\# = \varphi^\# = \gamma.$$

Q.E.D.

Corollary: Equivalent (P, δ)-algebras are c-equivalent.

Example: Suppose $(B, \delta; \tau)$ is a c-connected (P, δ)-algebra such that the differential algebra (B, δ) is c-split. Then we can apply Proposition XI to a c-splitting

$$(B, \delta) \underset{c}{\sim} (H(B), 0),$$

(cf. sec. 0.10).

This yields a c-equivalence

$$(B, \delta; \tau) \underset{c}{\sim} (H(B), 0; \tau^{\#})$$

inducing the identity in $H(B)$. Thus $(B, \delta; \tau)$ is c-equivalent to its associated P-algebra $(H(B); \tau^{\#})$ (cf. sec. 3.3).

The commutative diagram of sec. 3.28 reads

$$\begin{tikzcd}
 & H(H(B) \otimes \wedge P, \nabla_{\tau^{\#}}) \arrow[dd, "\cong"] \arrow[dr, "\varrho^{\#}_{H(B)}"] & \\
H(B) \arrow[ur, "l^{\#}_{H(B)}"] \arrow[dr, "l^{\#}_B"'] & & \wedge P \\
 & H(B \otimes \wedge P, \nabla_B) \arrow[ur, "\varrho^{\#}_B"'] &
\end{tikzcd}$$

(The vertical arrow is the isomorphism induced by the c-equivalence.) In particular, it follows that

$$\ker l^{\#}_{H(B)} = \ker l^{\#}_B.$$

3.30. Symmetric P-algebras. Theorem XI: Let $(\vee Q; \sigma)$ be a symmetric P-algebra with Samelson space $\hat{P}$. Then the graded differential algebra $(\vee Q \otimes \wedge P, \nabla_\sigma)$ is c-split if and only if

$$\dim P = \dim Q + \dim \hat{P}. \tag{3.28}$$

Proof: First assume that (3.28) holds. Let $\tilde{P}$ be a Samelson complement. The reduction theorem (sec. 2.15) shows that

$$(\vee Q \otimes \wedge P, \nabla_\sigma) \underset{c}{\sim} (\vee Q \otimes \wedge \tilde{P}, \nabla_\sigma) \otimes (\wedge \hat{P}, 0).$$

Define a homomorphism of graded differential algebras

$$\psi\colon (\vee Q \otimes \wedge \tilde{P}, \nabla_\sigma) \to (\vee Q/\vee Q \circ \tilde{P}, 0)$$

by setting

$$\psi(\Psi \otimes 1) = \bar{l}(\Psi) \qquad \text{and} \qquad \psi(\Psi \otimes \Phi) = 0, \qquad \Psi \in \vee Q, \quad \Phi \in \wedge^+ P.$$

Since $\dim \tilde{P} = \dim Q$, Theorem VIII, sec. 2.19, implies that $H_+(\vee Q \otimes \wedge \tilde{P}) = 0$, and it follows easily that $\psi^{\#}$ is an isomorphism.

Thus

$$(\vee Q \otimes \wedge \tilde{P}, \nabla_\sigma) \underset{c}{\sim} (\vee Q/\vee Q \circ \tilde{P}, 0),$$

and so $(\vee Q \otimes \wedge \tilde{P}, \nabla_\sigma)$ is c-split. Hence so is $(\vee Q \otimes \wedge P, \nabla_\sigma)$.

Conversely, assume that $(\vee Q \otimes \wedge P, \nabla_\sigma)$ is c-split. Define an oddly graded space $T = \sum_k T^k$ by

$$T^k = Q^{k+1}, \qquad k = 1, 3, \dots,$$

and consider the (T, δ)-algebra $(\vee Q \otimes \wedge P, \nabla_\sigma; \tau)$, where

$$\tau(x) = x \otimes 1, \qquad x \in T.$$

Since the base of this (T, δ)-algebra is c-split, the example of sec. 3.29 (with $B = \vee Q \otimes \wedge P$, $\delta = \nabla_\sigma$) yields the commutative diagram

$$\begin{tikzcd} & H(H(\vee Q \otimes \wedge P) \otimes \wedge T, \nabla_{\tau*}) \arrow[d, "\cong"] \\ H(\vee Q \otimes \wedge P) \arrow[ur, "l^{\#}_{H(\vee Q \otimes \wedge P)}"] \arrow[r, "l^{\#}_{\vee Q \otimes \wedge P}"'] & H(\vee Q \otimes \wedge P \otimes \wedge T, \nabla_B). \end{tikzcd}$$

Next, let $\varphi: \vee Q \otimes \wedge P \otimes \wedge T \to \wedge P$ be the obvious projection. It satisfies $\varphi \circ \nabla_B = 0$ and an easy spectral sequence argument shows that it induces an isomorphism

$$\varphi^{\#}: H(\vee Q \otimes \wedge P \otimes \wedge T, \nabla_B) \xrightarrow{\cong} \wedge P.$$

Moreover, if $\varrho_{\vee Q}: \vee Q \otimes \wedge P \to \wedge P$ denotes the projection, we have

$$\varphi \circ l_{\vee Q \otimes \wedge P} = \varrho_{\vee Q},$$

whence $\varphi^{\#} \circ l^{\#}_{\vee Q \otimes \wedge P} = \varrho^{\#}_{\vee Q}$.

Combining this with the commutative diagram above yields the commutative diagram

$$\begin{tikzcd} & H(H(\vee Q \otimes \wedge P) \otimes \wedge T) \arrow[d, "\cong"] \\ H(\vee Q \otimes \wedge P) \arrow[ur, "l^{\#}_{H(\vee Q \otimes \wedge P)}"] \arrow[r, "\varrho^{\#}_{\vee Q}"'] & \wedge P. \end{tikzcd}$$

It follows that

$$\ker l^{\#}_{H(\vee Q\otimes\wedge P)} = \ker \varrho^{\#}_{\vee Q}. \tag{3.29}$$

Finally, observe that $(H(\vee Q \otimes \wedge P) \otimes \wedge T, \nabla_{\tau^{\#}})$ is the Koszul complex of the T-algebra $(H(\vee Q \otimes \wedge P), \tau^{\#})$. Thus, in view of Proposition V, (1), sec. 2.14, the kernel of $l^{\#}_{H(\vee Q\otimes\wedge P)}$ coincides with the ideal generated by $\mathrm{Im}(\tau^{\#}_{\vee})^{+}$. Since

$$\tau_{\vee} = l_{\vee Q}\colon \vee Q \to \vee Q \otimes \wedge P$$

(as follows from the definition of τ), we have $\tau^{\#}_{\vee} = l^{\#}_{\vee Q}$. Thus $\ker l^{\#}_{H(\vee Q\otimes\wedge P)}$ is the ideal generated by $\mathrm{Im}(l^{\#}_{\vee Q})^{+}$.

Now relation (3.29) shows that $\ker \varrho^{\#}_{\vee Q}$ coincides with the ideal generated by $\mathrm{Im}(l^{\#}_{\vee Q})^{+}$. Thus, applying Theorem VIII, sec. 2.19, ((2) $\Rightarrow$ (1)), we obtain (3.28).

Q.E.D.

PART 2

In this part Γ denotes a commutative field of characteristic zero, and all vector spaces and algebras are defined over Γ.

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Chapter IV

Lie Algebras and Differential Spaces

§1. Lie algebras

4.1. Basic concepts. A *Lie algebra* is a vector space E together with a bilinear map

$$[\ ,\ ]: E \times E \to E$$

which satisfies the conditions

$$[x, x] = 0, \qquad x \in E,$$

and

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \qquad x, y, z \in E$$
(Jacobi identity).

A *homomorphism* $\varphi: E \to F$ *of Lie algebras* is a linear map φ such that

$$\varphi[x, y] = [\varphi(x), \varphi(y)], \qquad x, y \in E.$$

A *subalgebra* of E is a subspace $E_1 \subset E$, which satisfies $[x, y] \in E_1$, $x, y \in E_1$. A subspace I of a Lie algebra E is called an *ideal* if $[x, y] \in I$, $x \in E, y \in I$.

If I is an ideal in E, there is a unique multiplication in E/I making E/I into a Lie algebra, such that $\pi: E \to E/I$ is a homomorphism of Lie algebras. The Lie algebra E/I is called the *factor algebra* of E with respect to the ideal I. The *direct sum* of two Lie algebras E and F is the vector space $E \oplus F$ with Lie product given by $[x_1 \oplus y_1, x_2 \oplus y_2] = [x_1, x_2] \oplus [y_1, y_2]$, $x_i \in E$, $y_i \in F$. E and F are ideals in $E \oplus F$.

If $\varphi: E \to F$ is a homomorphism of Lie algebras, then the kernel of φ is an ideal in E.

The elements $z \in E$ which satisfy

$$[x, z] = 0, \qquad x \in E,$$

form an ideal Z_E in E, as follows from the Jacobi identity. Z_E is called the *centre* of E. A Lie algebra E is called *abelian*, if it coincides with its centre; i.e. if $[x, y] = 0$, $x, y \in E$.

The *derived algebra* E' of a Lie algebra is the subspace of E spanned by the products $[x, y]$, $x, y \in E$. E' is an ideal in E.

If $\{I_\alpha\}$ is a family of ideals in E, then the spaces $\bigcap_\alpha I_\alpha$ and $\sum_\alpha I_\alpha$ are again ideals. If E_1 and E_2 are subspaces, we denote by $[E_1, E_2]$ the subspace of E spanned by the products $[x, y]$, $x \in E_1$, $y \in E_2$. If I_1 and I_2 are ideals, then $[I_1, I_2]$ is an ideal and $[I_1, I_2] \subset I_1 \cap I_2$.

4.2. Representations. Let V be a vector space and consider the space L_V of linear transformations $\varphi: V \to V$. Then the bilinear map $L_V \times L_V \to L_V$ given by $(\varphi, \psi) \mapsto \varphi \circ \psi - \psi \circ \varphi$ makes L_V into a Lie algebra.

A *representation* of a Lie algebra E in a vector space V is a homomorphism of Lie algebras

$$\theta: E \to L_V.$$

Given two representations $\theta_V: E \to L_V$ and $\theta_W: E \to L_W$ of a Lie algebra E, a linear map $\varphi: V \to W$ is called *E-linear* if

$$\varphi \circ \theta_V(x) = \theta_W(x) \circ \varphi, \qquad x \in E.$$

Let $\theta: E \to L_V$ be a representation of E in V. Then a subspace $W \subset V$ is called *stable under* θ or simply *E-stable* if

$$\theta(x)w \in W, \qquad x \in E, \quad w \in W.$$

In this case θ induces a representation of E in W. It is called the *restriction of* θ *to* W. If $W \subset V$ is a stable subspace, there is a unique representation of E in V/W such that the projection $V \to V/W$ is E-linear.

On the other hand, if F is a subalgebra of E, then θ restricts to a homomorphism $\theta_F: F \to L_V$. This representation is called the *restriction of* θ *to* F.

A vector $v \in V$ is called *invariant under* θ if

$$\theta(x)v = 0, \qquad x \in E.$$

The invariant vectors form a stable subspace of V, denoted by $V_{\theta=0}$, and called the *invariant subspace*.

A second stable subspace of V is the vector space generated by the vectors of the form $\theta(x)v$, $x \in E$, $v \in V$. It is denoted by $\theta(V)$.

A representation θ of E in V is called *faithful* if the homomorphism $\theta: E \to L_V$ is injective.

If θ, θ_V, and θ_W are representations of E in U, V, and W, we obtain representations of E in $V \oplus W$, $V \otimes W$, U^*, $\wedge U$, $\vee U$ given respectively by

$$\begin{aligned}
\theta_{V\oplus W}(x) &= \theta_V(x) \oplus \theta_W(x) \\
\theta_{V\otimes W}(x) &= \theta_V(x) \otimes \iota + \iota \otimes \theta_W(x) \\
\theta^\natural(x) \quad &= -\theta(x)^* \\
\theta_\wedge(x)(u_1 \wedge \cdots \wedge u_p) &= \sum_{i=1}^{p} u_1 \wedge \cdots \theta(x)u_i \cdots \wedge u_p \\
\theta_\vee(x)(u_1 \vee \cdots \vee u_p) &= \sum_{i=1}^{p} u_i \vee \cdots \theta(x)u_i \cdots \vee u_p .
\end{aligned}$$

The representations θ and $\theta^\natural$ are called *contagredient*. Evidently,

$$\theta^\natural_{V\oplus W}(x) = \theta^\natural_V(x) \oplus \theta^\natural_W(x), \qquad \theta^\natural_{V\otimes W}(x) = \theta^\natural_V(x) \otimes \iota + \iota \otimes \theta^\natural_W(x),$$
$$(\theta_\wedge)^\natural = (\theta^\natural)_\wedge \qquad \text{and} \qquad (\theta_\vee)^\natural = (\theta^\natural)_\vee .$$

We shall write

$$\theta^\natural_\wedge = \theta^\wedge \qquad \text{and} \qquad \theta^\natural_\vee = \theta^\vee.$$

A *representation θ of E in a graded space* is a representation such that the operators $\theta(x)$ are homogeneous of degree zero. A *representation in an algebra* is a representation such that each $\theta(x)$ is a derivation. A *representation in a graded algebra* is a representation such that each $\theta(x)$ is a derivation, homogeneous of degree zero.

Let θ be a representation of E in a finite-dimensional vector space V. Then the *trace form of θ* is the bilinear function $T_\theta: E \times E \to \Gamma$ given by

$$T_\theta(x, y) = \operatorname{tr} \theta(x) \circ \theta(y).$$

It satisfies

$$T_\theta([x, y], z) + T_\theta(y, [x, z]) = 0, \qquad x, y, z \in E.$$

The *adjoint representation* of a Lie algebra E is the representation $\operatorname{ad}: E \to L_E$, given by

$$(\operatorname{ad} x)y = [x, y], \qquad x, y \in E.$$

The Jacobi identity implies that this is indeed a representation of E. Observe that each map ad x is a derivation in E; i.e.,

$$\operatorname{ad} x([y, z]) = [\operatorname{ad} x(y), z] + [y, \operatorname{ad} x(z)], \qquad x, y, z \in E.$$

The trace form of the adjoint representation of a finite dimensional Lie algebra E is called the *Killing form of* E, and is denoted by $(x, y) \mapsto K(x, y)$.

4.3. Semisimple representations. A linear transformation φ of a vector space V is called *semisimple* if whenever W is a subspace stable under φ, then there is a second φ-stable subspace W_1 such that $V = W \oplus W_1$.

A representation θ of E in V is called *semisimple* if, for every E-stable subspace $W \subset V$, there is a second E-stable subspace $W_1 \subset V$ such that $V = W \oplus W_1$. As an immediate consequence of the definitions we obtain

Lemma I: Let θ be a semisimple representation of E in V and assume that W is a stable subspace. Then

(1) $V = V_{\theta=0} \oplus \theta(V)$.

(2) If dim V is finite, then $\theta^\natural$ is a semisimple representation in V^*.

(3) The restriction of θ to W and the induced representation in V/W are semisimple.

(4) $W_{\theta=0} = W \cap V_{\theta=0}$ and $\theta(W) = W \cap \theta(V)$.

(5) $(V/W)_{\theta=0} = V_{\theta=0}/W_{\theta=0}$ and $\theta(V/W) = \theta(V)/\theta(W)$.

(6) If Z is another vector space, then the representation $x \mapsto \theta(x) \otimes \iota$ in $V \otimes Z$ is semisimple.

A representation θ of E in V is called *quasi-semisimple* if the restriction of θ to every finite-dimensional stable subspace is semisimple.

Proposition I: Let θ_V and θ_W be representations of a Lie algebra E in vector spaces V and W, and assume that θ_W is quasi-semisimple. Let θ denote the representation in $V \otimes W$ induced by θ_V and θ_W, and let $\Psi \in (V \otimes W)_{\theta=0}$.

Then there are finite-dimensional subspaces $Y \subset V$, $Z \subset W$ with the following properties:

(1) Y (respectively, Z) is stable under the operators $\theta_V(x)$ (respectively, $\theta_W(x)$), $x \in E$.

(2) The induced representations θ_Y and θ_Z of E in Y and Z are semisimple.

(3) $\Psi \in (Y \otimes Z)_{\theta=0}$.

Proof: Write

$$\Psi = \sum_{i=1}^{m} v_i \otimes w_i, \qquad v_i \in V, \quad w_i \in W,$$

where the v_i and the w_i are linearly independent. Let Y denote the space spanned by the vectors v_i and let Z denote the space spanned by the vectors w_i. Then (3) is obvious.

We show first that the spaces Y and Z are E-stable. In fact, since $\theta(x)\Psi = 0$, $x \in E$, we have

$$\sum_i \theta_V(x)v_i \otimes w_i = -\sum_i v_i \otimes \theta_W(x)w_i.$$

Thus

$$\sum_i \theta_V(x)v_i \otimes w_i \in (V \otimes Z) \cap (Y \otimes W) = Y \otimes Z.$$

Since the w_i form a basis for Z, it follows that

$$\theta_V(x)v_i \in Y, \qquad x \in E, \quad i = 1, \dots, m.$$

Thus Y is E-stable. Similarly, Z is E-stable. Let θ_Y and θ_Z denote the restrictions of θ_V and θ_W to Y and Z.

Now the canonical isomorphism α: $Y \otimes Z \xrightarrow{\cong} L(Y^*; Z)$ satisfies

$$\alpha(\theta(x)\Phi) = \theta_Z(x) \circ \alpha(\Phi) - \alpha(\Phi) \circ \theta_Y^\natural(x), \qquad \Phi \in Y \otimes Z, \quad x \in E.$$

It follows that $\alpha(\Psi)$ is an E-linear map from Y^* to Z. Elementary linear algebra shows that $\alpha(\Psi)$ is an isomorphism. But since θ_W is quasi-semisimple, and Z is finite dimensional, θ_Z is semisimple. Since $\alpha(\Psi)$ is an E-linear isomorphism, it follows that $\theta_Y^\natural$ is semisimple. Hence so is θ_Y.

Q.E.D.

4.4. Semisimple Lie algebras. A Lie algebra is called *simple* if it is nonabelian and contains no proper nontrivial ideals.

Theorem I: Let E be a finite-dimensional Lie algebra. Then the following conditions are equivalent:

(1) The Killing form of E is nondegenerate.

(2) E is the direct sum of simple ideals.

(3) Every representation of E in a finite-dimensional vector space is semisimple.

Proof: Cf. [1; Theorem I, p. 71, Prop. 2, p. 74, Theorem 2, p. 74].

A Lie algebra that satisfies the (equivalent) conditions above is called *semisimple.* Theorem I, (1) shows that if E is a semisimple Lie algebra (over Γ) and Ω is an extension field of Γ, then $E \otimes_\Gamma \Omega$ is a semisimple Lie algebra (over Ω).

If E is semisimple, then $Z_E = 0$ and $E' = E$.

A finite-dimensional Lie algebra E is called *reductive* if

$$E = Z_E \oplus E'$$

and E' is semisimple. It follows from Theorem I that E is reductive if and only if the adjoint representation of E is semisimple.

A finite-dimensional Lie algebra E over $\mathbb{R}$ is called *compact* if it admits a negative definite inner product $\langle\ ,\ \rangle$ which satisfies

$$\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0, \qquad x, y, z \in E.$$

It follows from Proposition XVI, sec. 1.17, volume II, that the Lie algebra of a compact Lie group is compact. Evidently every compact Lie algebra is reductive.

Theorem II: Let E be a finite-dimensional Lie algebra. The following conditions are equivalent:

(1) E is reductive.

(2) E admits a faithful, finite-dimensional representation with nondegenerate trace form.

(3) E admits a faithful, finite-dimensional, semisimple representation.

Proof: Cf. [1; Proposition 5, p. 78].

Theorem III: A representation θ of a reductive Lie algebra E in a finite-dimensional vector space V is semisimple if and only if each transformation $\theta(x)$, $x \in Z_E$, is semisimple.

Proof: Cf. [1; Theorem 4, p. 81].

In the rest of this article all Lie algebras are assumed to be finite dimensional.

Let F be a subalgebra of a Lie algebra E. Restricting the adjoint representation of E to F yields a representation of F in E; it is called the *adjoint representation of F in E* and denoted by $\mathrm{ad}_{E,F}$. If this representation is semisimple, then F is called *reductive in E*.

If F is reductive in E, then clearly F is reductive. Moreover, every transformation $\mathrm{ad}\, y\colon E \to E$ $(y \in Z_F)$ is semisimple. Conversely, if F is a reductive subalgebra of a Lie algebra E and each $\mathrm{ad}\, y (y \in Z_F)$ is semisimple, then Theorem III shows that F is reductive in E.

Example: Let F be reductive in E and assume that F is abelian and that Γ is algebraically closed. Fix $\alpha \in F^*$ and set

$$E_\alpha = \{x \in E \mid (\mathrm{ad}\, h)x = \alpha(h)x,\ h \in F\}.$$

In particular,

$$E_0 = \{x \in E \mid [h, x] = 0,\ h \in F\},$$

and so $F \subset E_0$. Moreover,

$$E = E_0 \oplus \sum_\alpha E_\alpha,$$

where the sum is extended over all nonzero α. Finally, observe that

$$[E_\alpha, E_\beta] \subset E_{\alpha+\beta}, \qquad \alpha, \beta \in F^*.$$

4.5. Cartan subalgebras. Given a Lie algebra E, define ideals $E^{(k)} \subset E$ inductively by

$$E^{(0)} = E \qquad \text{and} \qquad E^{(k)} = [E, E^{(k-1)}].$$

E is called *nilpotent*, if $E^{(k)} = 0$ for some k.

A *Cartan subalgebra* of a Lie algebra E is a nilpotent subalgebra H such that

$$H = \{y \in E \mid [y, H] \subset H\}.$$

It follows from the definition that every Cartan subalgebra of E contains the centre Z_E. According to [6; Theorem I, p. 59], every Lie algebra contains a Cartan subalgebra.

Now assume that E is reductive and let H be a Cartan subalgebra. Then H has the following properties (cf. [6; §1, §2, Chap. IV]):

(1) H is abelian.
(2) H is reductive in E.
(3) $E = H \oplus [H, E]$.

Moreover, if the coefficient field Γ is algebraically closed, then we have the direct decomposition

$$E = H \oplus \sum_{\alpha \neq 0} E_\alpha \tag{4.1}$$

and every nonzero subspace E_α has dimension 1. If $E_\alpha \neq 0$, α is called a *root*, and E_α is called the corresponding *root space*. (4.1) is called the *root space decomposition* of the reductive Lie algebra E with respect to the Cartan subalgebra H.

Lemma II: Let E be a Lie algebra and let F be an abelian subalgebra which is reductive in E. Let E_0 be the subalgebra of E given by

$$E_0 = \{y \in E \mid [y, F] = 0\}.$$

Then F is contained in every Cartan subalgebra of E_0, and every Cartan subalgebra of E_0 is a Cartan subalgebra of E.

Proof: Evidently $F \subset Z_{E_0}$ and so F is contained in every Cartan subalgebra of E_0.

Now let H be a Cartan subalgebra of E_0. We must show that H is a Cartan subalgebra of E. Clearly H is nilpotent and so it is sufficient to prove that if $x \in E$ satisfies $[x, H] \subset H$, then $x \in H$.

First observe that (since F is reductive in E)

$$E = E_0 \oplus [F, E].$$

Now assume that $[x, H] \subset H$. Then

$$[x, F] \subset [x, H] \subset H \subset E_0.$$

On the other hand,

$$[x, F] \subset [F, E].$$

It follows that $[x, F] = 0$ and so $x \in E_0$.

But H is a Cartan subalgebra of E_0 and so the relations

$$[x, H] \subset H \qquad \text{and} \qquad x \in E_0$$

imply that $x \in H$.

Q.E.D.

Proposition II: Let E be a reductive Lie algebra and let F be an abelian subalgebra which is reductive in E. Let E_0 be the subalgebra of E given by

$$E_0 = \{x \in E \mid [x, F] = 0\}.$$

Then E_0 is reductive in E.

Proof: First, set $E_1 = [F, E]$; then $E = E_0 \oplus E_1$. Now we show that E_0 and E_1 are orthogonal with respect to the Killing form K of E. For this we may assume that Γ is algebraically closed, and write (cf. the example of sec. 4.4)

$$E_1 = \sum_{\substack{\alpha \in F^* \\ \alpha \neq 0}} E_\alpha .$$

The relations $[E_\alpha, E_\beta] \subset E_{\alpha+\beta}$ imply that

$$\operatorname{ad} x \circ \operatorname{ad} y \colon E_\beta \to E_{\alpha+\beta}, \qquad x \in E_\alpha, \quad y \in E_0.$$

Hence $K(x, y) = 0$.

Now observe that $E' = (E_0 \cap E') \oplus E_1$. Since E' is semisimple, K restricts to a nondegenerate bilinear form in E', and so, by what we have just proved, the restriction of K to $E_0 \cap E'$ is again nondegenerate. But this is the trace form of the faithful representation of $E_0 \cap E'$ in E'; thus Theorem II, sec. 4.4, implies that $E_0 \cap E'$ is reductive. Now the decomposition $E_0 = Z_E \oplus (E_0 \cap E')$ shows that E_0 is reductive.

Finally, let H_0 be a Cartan subalgebra of E_0. According to Lemma II, H_0 is then a Cartan subalgebra of E; in particular, H_0 acts semisimply in E. But since $Z_{E_0} \subset H_0$, Z_{E_0} acts semisimply in E. Now Theorem III, sec. 4.4, implies that E_0 is reductive in E.

Q.E.D.

4.6. Examples. 1. Consider the Lie algebra L_V, where V is a finite-dimensional vector space. Then $Z_{L_V} = \Gamma \cdot \iota$ and $(L_V)'$ is the Lie algebra $L_0(V)$ of transformations with trace zero. It follows that $L_V = Z_{L_V} \oplus L_V'$.

Moreover, the Killing form of $L_0(V)$ is given by

$$K(\alpha, \beta) = \operatorname{tr} \alpha \circ \beta, \qquad \alpha, \beta \in L_0(V),$$

and so $L_0(V)$ is semisimple. It follows that L_V is reductive.

2. Let V be a finite-dimensional Euclidean space. Then the linear transformations σ which satisfy $\tilde{\sigma} = -\sigma$ ($\tilde{\sigma}$ denotes the adjoint trans-

formation with respect to the inner product) form a Lie algebra, which will be denoted by Sk_V. The Killing form of Sk_V is negative definite, and so Sk_V is semisimple.

4.7. Semisimple representations. Proposition III. Let θ be a semisimple representation of a Lie algebra E in a finite-dimensional vector space V. Let F be a subalgebra which is reductive in E. Then the restriction θ_F of θ to F is semisimple.

Proof: Cf. [1; Corollary 1 to Proposition 7, p. 84].

Corollary: Assume that H is reductive in F and that F is reductive in E. Then H is reductive in E.

Proposition IV: Let θ be a faithful representation of E in a finite-dimensional vector space V. Let $F \subset E$ be a subalgebra, and assume that the restriction θ_F of θ to F is semisimple. Then F is reductive in E.

Proof: Without loss of generality we may consider F and E as subalgebras of L_V. To show that F is reductive in E, it is clearly sufficient to show that F is reductive in L_V.

Since F admits a faithful semisimple representation, it is reductive (cf. Theorem II, sec. 4.4). Thus we need only show that each transformation $\operatorname{ad} y\colon L_V \to L_V$ $(y \in Z_F)$ is semisimple. But the canonical isomorphism $V^* \otimes V \xrightarrow{\cong} L_V$ identifies the transformation $-\theta_F^*(y) \otimes \iota + \iota \otimes \theta_F(y)$ with $\operatorname{ad} y$. Since θ_F is semisimple and $y \in Z_F$, the transformation $\theta_F(y)$ is semisimple (cf. Theorem III, sec. 4.4). Hence so is $\operatorname{ad} y$.

Q.E.D.

4.8. Involutions. Proposition V: Let ω be an involutive isomorphism of a reductive Lie algebra E. Then the subalgebra F given by $F = \{x \in E \mid \omega(x) = x\}$ is reductive in E.

Proof: Since ω preserves Z_E and E', we have the direct decomposition

$$F = F \cap Z_E \oplus F \cap E'.$$

Thus it is sufficient to consider the case that E is semisimple.

Write $E_- = \{x \in E \mid \omega(x) = -x\}$. Then the relations

$$E = E_- \oplus F, \qquad [E_-, F] \subset E_-, \qquad [E_-, E_-] \subset F$$

imply that F and E_- are orthogonal with respect to the Killing form of E. Hence the restriction of this Killing form to F is nondegenerate, and so F is reductive (cf. Theorem II). Thus we have only to show that each transformation $\operatorname{ad} y: E \to E$ $(y \in Z_F)$ is semisimple (cf. Theorem III, sec. 4.4). Without loss of generality, we may assume that Γ is algebraically closed.

Case I: F is nonabelian. Then $F' \neq 0$. Since F is reductive, F' is semisimple. Hence, by Theorem I, F' is reductive in E. Now let H be a Cartan subalgebra of F'. Then H is reductive in F'. Hence, by the corollary to Proposition III, sec. 4.7, H is reductive in E.

Now let $E_0 \subset E$ be the subalgebra given by

$$E_0 = \{x \in E \mid (\operatorname{ad} x)H = 0\}.$$

Then Proposition II, sec. 4.5, shows that E_0 is reductive in E. Moreover, $E_0 \neq E$. In fact, since E is semisimple, $Z_E = 0$, while $Z_{E_0} \supset H \neq 0$.

Next observe that since $H \subset F$, ω restricts to an involution of E_0 with fixed point subalgebra $F_0 = F \cap E_0$.

Since E_0 is a proper reductive subalgebra of E, we may assume by induction on $\dim E$ that F_0 is reductive in E_0 and hence reductive in E (cf. Proposition III, sec. 4.7). But evidently $Z_F \subset Z_{F_0}$ and so it follows that F is reductive in E.

Case II: F is abelian. Define subspaces $E_\lambda \subset E$ $(\lambda \in F^*)$ by $x \in E_\lambda$ if and only if

$$(\operatorname{ad} y - \lambda(y)\iota)^n(x) = 0, \qquad y \in F, \quad n = \dim E.$$

Then

$$E = E_0 \oplus \sum_{\lambda \neq 0} E_\lambda \qquad \text{and} \qquad [E_\lambda, E_\mu] \subset E_{\lambda+\mu}.$$

Hence the restriction of the Killing form of E to E_0 is nondegenerate, and so E_0 is reductive.

Now we show that

$$F \subset Z_{E_0}. \tag{4.2}$$

Since $F \subset E_0$ and E_0 is stable under ω, we have

$$E_0 = F \oplus (E_0 \cap E_-).$$

Now assume that (4.2) fails. Then there is a least integer p $(p \geq 2)$ such that

$$(\operatorname{ad} y)^p(x) = 0, \qquad y \in F, \quad x \in E_0.$$

Let $Z \subset E_0$ be the subspace given by

$$Z = \{x \in E_0 \mid (\operatorname{ad} y)^{p-1}x = 0, y \in F\}.$$

Then clearly

$$[F, E_0 \cap E_-] \subset Z \qquad \text{and} \qquad [E_0 \cap E_-, E_0 \cap E_-] \subset F \subset Z.$$

Hence $(E_0)' \subset Z$, and so Z is an ideal in E_0. Since E_0 is reductive, we obtain the direct decomposition

$$E_0 = Z \oplus Z_1, \qquad [Z, Z_1] = 0.$$

Since $F \subset Z$, it follows that $(\operatorname{ad} y)^{p-1}(x) = 0$, $y \in F, x \in E_0$. This contradiction proves (4.2).

Finally, choose a Cartan subalgebra H of E_0. Then (4.2) implies that $F \subset H$. Thus, if $z \in E$ satisfies $[z, H] \subset H$, then certainly $[z, F] \subset E_0$. This implies that $z \in E_0$, and so, since H is a Cartan subalgebra of E_0, $z \in H$. It follows that H is a Cartan subalgebra of E. In particular, $F \subset H$ is reductive in E.

Q.E.D.

Example: Let $\langle\, , \rangle$ be a nondegenerate bilinear form in V which is either symmetric or skew symmetric. Then an involution $\varphi \mapsto -\varphi^*$ in the Lie algebra L_V is defined by

$$\langle -\varphi^* x, y\rangle = -\langle x, \varphi y\rangle, \qquad x, y \in V.$$

The fixed point subalgebra consists of the transformations which are skew with respect to $\langle\, , \rangle$ and, by the proposition, this subalgebra is reductive in L_V.

§2. Representation of a Lie algebra in a differential space

4.9. Let E be a Lie algebra and let (M, δ_M) be a differential space. A *representation of E in the differential space* (M, δ_M) is a representation θ_M of E in M such that

$$\theta_M(x)\delta_M = \delta_M\theta_M(x), \qquad x \in E.$$

In particular the cocycle space $Z(M)$ and the coboundary space $B(M)$ are stable under such a representation. Hence each map $\theta_M(x)$ induces a linear map

$$\theta_M(x)^\#: H(M) \to H(M).$$

Observe that the correspondence $x \mapsto \theta_M(x)^\#$ defines a representation $\theta_M^\#$ of E in $H(M)$. It is called the *induced representation in the cohomology space*. The corresponding invariant subspace $(H(M))_{\theta^\#=0}$ will be simply denoted by $H(M)_{\theta^\#=0}$.

A *representation of E in a graded differential space* (M, δ_M) is a representation θ_M in the differential space (M, δ_M) such that each map $\theta_M(x)$ is homogeneous of degree zero. In this case $\theta_M^\#(x)$ is also homogeneous of degree zero.

A *representation of E in a graded differential algebra* (R, δ_R) is a representation of E in the graded differential space (R, δ_R) such that each $\theta_R(x)$ is a derivation in R. In this case each $\theta_R^\#(x)$ is a derivation in $H(R)$ and so $\theta_R^\#$ is a representation in the graded algebra $H(R)$.

Let θ_M be a representation of E in a differential space (M, δ_M). Then, evidently, the invariant subspace $M_{\theta=0}$ and the subspace $\theta(M)$ are stable under δ. Hence we can form the cohomology spaces $H(M_{\theta=0})$ and $H(\theta(M))$. The inclusion map $M_{\theta=0} \to M$ induces a linear map $H(M_{\theta=0}) \to H(M)$. Clearly we may regard this as a map into $H(M)_{\theta^\#=0}$.

If θ_M is a representation of E in a graded differential space (respectively, in a graded differential algebra), then these maps are homomorphisms of graded vector spaces (respectively, of graded algebras).

4.10. Semisimple representations. Let θ_M be a semisimple representation of a Lie algebra E in a differential space (M, δ_M). Since $Z(M)$ and $B(M)$ are E-stable subspaces and θ_M is semisimple, we can

find E-stable subspaces

$$C_M \subset M \qquad \text{and} \qquad A_M \subset Z(M)$$

such that

$$M = Z_M \oplus C_M \qquad \text{and} \qquad Z(M) = A_M \oplus B(M).$$

In particular, the projection $Z(M) \to H(M)$ restricts to an E-linear isomorphism $A_M \xrightarrow{\cong} H(M)$. It follows that $\theta_M^\#$ is a semisimple representation.

Theorem IV: Let θ_M be a semisimple representation of a Lie algebra E in a differential space (M, δ_M). Then the inclusion $M_{\theta=0} \to M$ induces an isomorphism,

$$H(M_{\theta=0}) \xrightarrow{\cong} H(M)_{\theta^\#=0}.$$

Proof: Since θ_M is semisimple, we have

$$Z(\theta(M)) = Z(M) \cap \theta(M) = \theta(Z(M)).$$

It follows that the inclusion $\theta(M) \to M$ induces a map $H(\theta(M)) \to \theta(H(M))$. Moreover, because θ_M is semisimple, $M = M_{\theta=0} \oplus \theta(M)$. Since $\theta_M^\#$ must also be semisimple, it follows that

$$H(M_{\theta=0}) \oplus H(\theta(M)) = H(M) = H(M)_{\theta^\#=0} \oplus \theta(H(M)).$$

But $H(M_{\theta=0}) \subset H(M)_{\theta^\#=0}$ and $H(\theta(M)) \subset \theta(H(M))$. The theorem follows.

Q.E.D.

Corollary I: If $\theta_M^\# = 0$, then $H(M_{\theta=0}) \xrightarrow{\cong} H(M)$.

Corollary II: If $\theta_M^\# = 0$, then $H(\theta(M)) = 0$.

Proposition VI: Let θ_M be a semisimple representation of a Lie algebra E in a graded differential space (M, δ_M). Then there are E-linear maps

$$\lambda_M\colon H(M) \to M, \qquad \pi_M\colon M \to H(M), \qquad h_M\colon M \to M,$$

respectively homogeneous of degrees 0, 0, and -1 and having the following properties:

(i) $\pi_M \delta_M = 0$, $\delta_M \lambda_M = 0$, $\pi_M \lambda_M = \iota$.

(ii) $\iota - \lambda_M \pi_M = h_M \delta_M + \delta_M h_M$.

Proof: Choose E-stable subspaces $C_M \subset M$ and $A_M \subset Z(M)$ such that

$$M = Z(M) \oplus C_M \qquad \text{and} \qquad Z(M) = A_M \oplus B(M).$$

Then the projection $Z(M) \to H(M)$ restricts to an E-linear isomorphism $\alpha\colon A_M \xrightarrow{\cong} H(M)$, while δ_M restricts to an E-linear isomorphism $\bar{\delta}\colon C_M \xrightarrow{\cong} B(M)$.

Now set

$$\pi_M(a \oplus b \oplus c) = \alpha(a), \qquad \lambda_M(\gamma) = \alpha^{-1}(\gamma), \qquad h_M(a \oplus b \oplus c) = \bar{\delta}^{-1}(b),$$
$$a \in A_M, \quad b \in B(M), \quad c \in C_M, \quad \gamma \in H(M).$$

Q.E.D.

Now suppose that θ_N is a semisimple representation of E in a second graded differential space N. Let

$$\varphi\colon M \to N \qquad \text{and} \qquad \psi\colon M \to N$$

be homomorphisms of graded differential spaces such that $\varphi - \psi$ is E-linear.

Proposition VII: Assume that $\varphi^{\#} = \psi^{\#}$. Then there is an E-linear map $k\colon M \to N$, homogeneous of degree -1, such that

$$\varphi - \psi = k\delta_M + \delta_N k.$$

Proof: Write $\varphi - \psi = \chi$. Using the notation of Proposition VI, define a linear map

$$\chi_A\colon A_M \to A_N$$

by $\chi_A(a) = \lambda_N \pi_N \chi(a)$, $a \in A_M$. Then the diagram

$$\begin{tikzcd} A_M \arrow[r, "\chi_A"] \arrow[d, "\cong"'] & A_N \arrow[d, "\cong"] \\ H(M) \arrow[r, "\chi^{\#}"'] & H(N) \end{tikzcd}$$

commutes. Since $\chi^{\#} = 0$, it follows that $\chi_A = 0$.

On the other hand, the linear map $\lambda_M \circ \pi_M : M \to M$, has image A_M, so that

$$(\lambda_N \pi_N) \circ \chi \circ (\lambda_M \pi_M) = \chi_A \circ (\lambda_M \pi_M) = 0.$$

It follows that

$$\begin{aligned}
\chi &= \chi - \lambda_N \pi_N \chi \lambda_M \pi_M = \lambda_N \pi_N \chi(\iota - \lambda_M \pi_M) + (\iota - \lambda_N \pi_N)\chi \\
&= \lambda_N \pi_N \chi(h_M \delta_M + \delta_M h_M) + (h_N \delta_N + \delta_N h_N)\chi \\
&= \delta_N k + k \delta_M,
\end{aligned}$$

where $k = \lambda_N \pi_N \chi h_M + h_N \chi$.

Q.E.D.

Corollary I: If $H(M) = 0$, then there exists an E-linear map $k: M \to M$, homogeneous of degree -1, and satisfying

$$\iota = \delta_M k + k \delta_M.$$

Corollary II: If $\theta_M^\# = 0$, then there is an E-linear map $k: \theta(M) \to \theta(M)$ homogeneous of degree -1 such that

$$\iota = \delta_M k + k \delta_M.$$

Proof: By Corollary II to Theorem I, we have $H(\theta(M)) = 0$. Now apply Corollary I above to the differential space $(\theta(M), \delta_M)$.

Q.E.D.

4.11. Representations in a tensor product. Let E be a Lie algebra, (X, δ_X) a graded differential space with δ_X homogeneous of degree 1, and let θ_X be a semisimple representation of E in (X, δ_X) such that $\theta_X^\# = 0$. Let θ_Y be a representation of E in a second graded differential space (Y, δ_Y) and consider the differential space $(X \otimes Y, \delta)$, where

$$\delta(u \otimes v) = \delta_X u \otimes v + (-1)^p u \otimes \delta_Y v, \qquad u \in X^p, \quad v \in Y.$$

Finally, let θ denote the induced representation in $X \otimes Y$.

Theorem V: With the notation and hypotheses above, the inclusion map

$$X_{\theta=0} \otimes Y_{\theta=0} \to (X \otimes Y)_{\theta=0},$$

induces an isomorphism

$$H(X_{\theta=0}) \otimes H(Y_{\theta=0}) \xrightarrow{\cong} H((X \otimes Y)_{\theta=0}).$$

Proof: Since θ_X is semisimple, we have the direct decomposition $X = X_{\theta=0} \oplus \theta(X)$, whence

$$(X \otimes Y)_{\theta=0} = (X_{\theta=0} \otimes Y_{\theta=0}) \oplus (\theta(X) \otimes Y)_{\theta=0}.$$

Since δ commutes with θ_X and θ_Y (and hence with θ) all terms in this relation are stable under δ. This implies (via the Künneth formula) that

$$H((X \otimes Y)_{\theta=0}) = H(X_{\theta=0}) \otimes H(Y_{\theta=0}) \oplus H((\theta(X) \otimes Y)_{\theta=0}).$$

Hence it has to be shown that

$$H((\theta(X) \otimes Y)_{\theta=0}) = 0. \tag{4.3}$$

Corollary II to Theorem IV, sec. 4.10, implies that $H(\theta(X)) = 0$. Hence, by Corollary II to Proposition VII, sec. 4.10, there is an E-linear operator $k\colon \theta(X) \to \theta(X)$, homogeneous of degree -1, such that

$$\delta_X k + k\delta_X = \iota.$$

Now set $\tilde{k} = k \otimes \iota$. Then $\tilde{k}$ is E-linear, and satisfies

$$\delta\tilde{k} + \tilde{k}\delta = \iota. \tag{4.4}$$

Hence $\tilde{k}$ restricts to an operator $\tilde{k}_{\theta=0}$ in $(\theta(X) \otimes Y)_{\theta=0}$. Relation (4.4) implies that

$$\delta\tilde{k}_{\theta=0} + \tilde{k}_{\theta=0}\delta = \iota,$$

and so (4.3) follows.

Q.E.D.

Chapter V

Cohomology of Lie Algebras and Lie Groups

In this chapter E denotes a finite-dimensional Lie algebra. The multiplication operators determined by $a \in \wedge E$ and $\Phi \in \wedge E^*$ will be denoted by $\mu(a)$ and $\mu(\Phi)$:

$$\mu(a)(b) = a \wedge b \qquad \text{and} \qquad \mu(\Phi)(\Psi) = \Phi \wedge \Psi.$$

The (dual) substitution operators are denoted either by $i_E(a)\colon \wedge E^* \to \wedge E^*$ and $i_E(\Phi)\colon \wedge E \to \wedge E$, or simply by $i(a)$ and $i(\Phi)$.

§1. Exterior algebra over a Lie algebra

5.1. The operators $\theta_E(x)$ and $\theta^E(x)$. Given a Lie algebra E consider the dual space E^*. The linear maps $\operatorname{ad} x$ $(x \in E)$ extend to unique derivations $\theta^E(x)$ in the algebra $\wedge E$. Similary, $-(\operatorname{ad} x)^*$ extends to a derivation $\theta_E(x)$ in $\wedge E^*$.

θ^E and θ_E are contragredient representations of E in the graded algebras $\wedge E$ and $\wedge E^*$, extending the contragredient representations ad and $\operatorname{ad}^\natural$. The derivation property of these operators is expressed by the formulae

$$\theta^E(x)\mu(a) - \mu(a)\theta^E(x) = \mu(\theta^E(x)a), \qquad a \in \wedge E, \quad x \in E,$$

and

$$\theta_E(x)\mu(\Phi) - \mu(\Phi)\theta_E(x) = \mu(\theta_E(x)\Phi), \qquad \Phi \in \wedge E^*,\ x \in E.$$

Dualizing we obtain

$$\theta_E(x)i(a) - i(a)\theta_E(x) = i(\theta^E(x)a)$$

and

$$\theta^E(x)i(\Phi) - i(\Phi)\theta^E(x) = i(\theta_E(x)\Phi).$$

In particular,

$$\theta_E(x)i_E(y) - i_E(y)\theta_E(x) = i_E([x, y]), \qquad x, y \in E. \tag{5.1}$$

The representations θ^E and θ_E determine the invariant subalgebras $(\wedge E)_{\theta=0}$ and $(\wedge E^*)_{\theta=0}$, as well as the stable subspaces $\theta(\wedge E)$ and $\theta(\wedge E^*)$ (cf. sec. 4.2). If a and Φ are invariant, the relations above reduce to simple commutation formulae. In particular, the invariant subalgebras and the subspaces $\theta(\wedge E)$ and $\theta(\wedge E^*)$ are stable under multiplication and substitution by invariant elements.

Since θ^E and θ_E are contragredient, we have the relations

$$(\wedge E)^{\perp}_{\theta=0} = \theta(\wedge E^*) \qquad \text{and} \qquad (\wedge E^*)^{\perp}_{\theta=0} = \theta(\wedge E).$$

Moreover, since the restriction of θ^E to E is the adjoint representation, it follows that

$$(\wedge^1 E)_{\theta=0} = Z_E \qquad \text{and} \qquad \theta(\wedge^1 E) = E'.$$

Finally, let e_ν, $e^{*\nu}$ $(\nu = 1, \dots, n)$ be a pair of dual bases for E and E^*. Then

$$\theta_E(x) = -\sum_\nu \mu(e^{*\nu})i([x, e_\nu]) \qquad \text{and} \qquad \theta^E(x) = \sum_\nu \mu([x, e_\nu])i(e^{*\nu}). \tag{5.2}$$

(Since both sides are derivations, these formulae have only to be verified in E and E^*, where they are immediate consequences of the definitions.)

5.2. The operators $\boldsymbol{\delta_E}$ and $\boldsymbol{\partial_E}$. Consider the linear map $\nu\colon \wedge^2 E \to E$ given by

$$\nu(x \wedge y) = [x, y], \qquad x, y \in E.$$

Extend the negative dual,

$$-\nu^*\colon \wedge^2 E^* \leftarrow E^*,$$

to an antiderivation

$$\delta_E\colon \wedge E^* \leftarrow \wedge E^*,$$

homogeneous of degree 1 (cf. [5; p. 111]).

We shall establish the fundamental relations

$$\begin{aligned} i_E(x)\delta_E + \delta_E i_E(x) &= \theta_E(x) \\ \delta_E^2 &= 0 \end{aligned} \tag{5.3}$$

and

$$\theta_E(x)\delta_E = \delta_E\theta_E(x), \qquad x \in E.$$

Since

$$\begin{aligned}\langle (i_E(x)\delta_E + \delta_E i_E(x))x^*, y\rangle &= \langle \delta_E x^*, x \wedge y\rangle \\ &= \langle x^*, -[x, y]\rangle \\ &= \langle \theta_E(x)x^*, y\rangle, \qquad x, y \in E, \quad x^* \in E^*,\end{aligned}$$

the first relation holds when applied to elements in E^*. Since both sides are derivations in $\wedge E^*$, it must be true in general.

To prove the second relation observe that the first relation yields

$$\begin{aligned}&\langle \delta_E(x^* \wedge y^*), x \wedge y \wedge z\rangle \\ &\quad = \langle i_E(x)\delta_E(x^* \wedge y^*), y \wedge z\rangle \\ &\quad = -\langle x^* \wedge y^*, \theta^E(x)(y \wedge z)\rangle - \langle \delta_E i_E(x)(x^* \wedge y^*), y \wedge z\rangle \\ &\quad = -\langle x^* \wedge y^*, [x, y] \wedge z + y \wedge [x, z] - x \wedge [y, z]\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad x, y, z \in E, \quad x^* \in E^*.\end{aligned}$$

Thus

$$\langle \delta_E \Phi, x \wedge y \wedge z\rangle = -\langle \Phi, [x, y] \wedge z + y \wedge [x, z] - x \wedge [y, z]\rangle, \qquad \Phi \in \wedge^2 E^*, \quad x, y, z \in E.$$

In particular,

$$\begin{aligned}\langle \delta_E^2 x^*, x \wedge y \wedge z\rangle &= \langle x^*, [[x, y], z] + [[z, x], y] + [[y, z], x]\rangle \\ &= 0, \qquad x^* \in E^*, \quad x, y, z \in E.\end{aligned}$$

Since δ_E^2 is a derivation, it follows that $\delta_E^2 = 0$.

Finally, the last relation is established by applying δ_E to both sides of the first, and using the fact that $\delta_E^2 = 0$.

It follows from the relations (5.3) that $(\wedge E^*, \delta_E)$ is a graded differential algebra, that θ_E represents E in $(\wedge E^*, \delta_E)$, and that $\theta_E^\# = 0$.

Next, let $\partial_E: \wedge E \to \wedge E$ be the linear map, homogeneous of degree -1, given by $\partial_E = -\delta_E^*$. Then

$$\partial_E(x \wedge y) = [x, y], \qquad \partial_E x = 0, \qquad \text{and} \qquad \partial_E(\lambda) = 0, \qquad x, y \in E, \ \lambda \in \Gamma.$$

Dualizing formulae (5.3) yields the formulae

$$\mu(x)\partial_E + \partial_E \mu(x) = \theta^E(x), \qquad \partial_E^2 = 0$$

and

$$\theta^E(x)\partial_E = \partial_E \theta^E(x), \qquad x \in E.$$

In particular, θ^E represents E in the graded differential space $(\wedge E, \partial_E)$. Note that ∂_E is *not* in general an antiderivation in the algebra $\wedge E$.

5.3. The Koszul formula. In this section we shall establish the *Koszul formula*

$$\delta_E = \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})\theta_E(e_\nu), \tag{5.4}$$

where e_ν, $e^{*\nu}$ ($\nu = 1, \ldots, n$) is a pair of dual bases for E and E^*. Since δ_E and the operator on the right-hand side are both antiderivations, it is sufficient to verify this formula for elements in E^*.

Let $x^* \in E^*$ and $x \in E$ be arbitrary. Then, using relations (5.3) and (5.1), we find that

$$i_E(x)\delta_E x^* = \theta_E(x)x^*,$$

while

$$\begin{aligned} i_E(x)\, \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})\theta_E(e_\nu)x^* &= \tfrac{1}{2}\theta_E(x)x^* - \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})i_E(x)\theta_E(e_\nu)x^* \\ &= \tfrac{1}{2}\theta_E(x)x^* - \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})i_E([x, e_\nu])x^*. \end{aligned}$$

Now formula (5.2) yields

$$i_E(x)\Big(\tfrac{1}{2} \sum_\nu \mu(e^{*\nu})\theta_E(e_\nu)x^*\Big) = \theta_E(x)x^* = i_E(x)\delta_E x^*,$$

which completes the proof.

Dualizing the Koszul formula we obtain the equation

$$\partial_E = \tfrac{1}{2} \sum_\nu \theta^E(e_\nu)i_E(e^{*\nu}) \tag{5.5}$$

(*contravariant Koszul formula*). As an immediate consequence we obtain

$$\partial_E(x_0 \wedge \cdots \wedge x_p) = \sum_{\nu<\mu} (-1)^{\nu+\mu+1}[x_\nu, x_\mu] \wedge x_0 \wedge \cdots \hat{x}_\nu \cdots \hat{x}_\mu \cdots \wedge x_p, \qquad x_i \in E. \tag{5.6}$$

Thus, if $\wedge^p E^*$ is interpreted as the space of p-linear skew functions in E, then δ_E is given by

$$\begin{aligned} &(\delta_E \Phi)(x_0, \ldots, x_p) \\ &\quad = \sum_{\nu<\mu} (-1)^{\nu+\mu}\Phi([x_\nu, x_\mu], x_0, \ldots, \hat{x}_\nu, \ldots, \hat{x}_\mu, \ldots, x_p). \end{aligned} \tag{5.7}$$

5.4. The product formula for ∂_E. In this section we derive the relations

$$\begin{aligned}\partial_E(a \wedge b) &= \partial_E a \wedge b + (-1)^p a \wedge \partial_E b + \sum_\nu i_E(e^{*\nu})a \wedge \theta^E(e_\nu)b \\ \partial_E(a \wedge b) &= \partial_E a \wedge b + (-1)^p a \wedge \partial_E b + (-1)^p \sum_\nu \theta^E(e_\nu)a \wedge i_E(e^{*\nu})b\end{aligned} \tag{5.8}$$

and

$$\partial_E(a \wedge b) = -\partial_E a \wedge b + (-1)^p a \wedge \partial_E b + \sum_\nu \theta^E(e_\nu)(i_E(e^{*\nu})a \wedge b), \qquad a \in \wedge^p E, \quad b \in \wedge E, \tag{5.9}$$

where $e_\nu, e^{*\nu}$ $(\nu = 1, \ldots, n)$ are dual bases for E and E^*.

To prove these, use the contravariant Koszul formula to obtain

$$\begin{aligned}\partial_E(a \wedge b) &= \tfrac{1}{2}\sum_\nu \theta^E(e_\nu)i_E(e^{*\nu})(a \wedge b) \\ &= \sum_\nu \tfrac{1}{2}\{\theta^E(e_\nu)i_E(e^{*\nu})a \wedge b + i_E(e^{*\nu})a \wedge \theta^E(e_\nu)b \\ &\quad +(-1)^p\theta^E(e_\nu)a \wedge i_E(e^{*\nu})b + (-1)^p a \wedge \theta^E(e_\nu)i_E(e^{*\nu})b\} \\ &= \partial_E a \wedge b + (-1)^p a \wedge \partial_E b + \tfrac{1}{2}\sum_\nu i_E(e^{*\nu})a \wedge \theta^E(e_\nu)b \\ &\quad +\tfrac{1}{2}\sum_\nu (-1)^p\theta^E(e_\nu)a \wedge i_E(e^{*\nu})b.\end{aligned}$$

The two last terms in this relation are equal. In fact for $x_i, y_j \in E$, we have

$$\begin{aligned}&\sum_\nu [i_E(e^{*\nu})(x_1 \wedge \cdots \wedge x_p)] \wedge [\theta^E(e_\nu)(y_1 \wedge \cdots \wedge y_q)] \\ &\quad= \sum_{i,j} (-1)^{i-1}x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_p \wedge y_1 \cdots \wedge [x_i, y_j] \wedge \cdots \wedge y_q \\ &\quad= \sum_\nu (-1)^p[\theta^E(e_\nu)(x_1 \wedge \cdots \wedge x_p)] \wedge [i_E(e^{*\nu})(y_1 \wedge \cdots \wedge y_q)].\end{aligned}$$

Thus the relations (5.8) follow.

To verify (5.9) observe that, in view of the derivation property of $\theta^E(e_\nu)$,

$$\sum_\nu \theta^E(e_\nu)(i_E(e^{*\nu})a \wedge b) = 2\partial_E a \wedge b + \sum_\nu i_E(e^{*\nu})a \wedge \theta^E(e_\nu)b.$$

Subtracting this from (5.8) yields (5.9).

5.5. Cohomology and homology of a Lie algebra. Let E be a Lie algebra of dimension n, and recall the graded differential algebra $(\wedge E^*, \delta_E)$ introduced in sec. 5.2. The cocycle (respectively, coboundary) algebras of this differential algebra will be denoted by

$$Z^*(E) = \sum_p Z^p(E) \qquad \text{and} \qquad B^*(E) = \sum_p B^p(E).$$

The corresponding cohomology algebra

$$H^*(E) = \sum_p H^p(E)$$

is called the *cohomology algebra of the Lie algebra E*. Notice that $H^0(E) = \Gamma$.

Next consider the graded differential space $(\wedge E, \partial_E)$. The cycle (respectively boundary) subspaces will be denoted by

$$Z_*(E) = \sum_p Z_p(E) \qquad \text{and} \qquad B_*(E) = \sum_p B_p(E).$$

The corresponding homology space

$$H_*(E) = \sum_p H_p(E)$$

is called the *homology space of the Lie algebra E*.

For example, $Z_1(E) = E$, $B_1(E) = \partial_E(\wedge^2 E) = E'$, and so

$$H_1(E) = E/E'.$$

Since $\partial_E = -\delta_E^*$, it follows that

$$B_*(E) = Z^*(E)^\perp \qquad \text{and} \qquad Z_*(E) = B^*(E)^\perp,$$

and so a natural duality is determined between the spaces $H^*(E)$ and $H_*(E)$. This duality restricts to a duality between each pair $H^p(E)$, $H_p(E)$ $(p = 0, \ldots, n)$. In particular, $\dim H^p(E) = \dim H_p(E)$, $p = 0, \ldots, n$.

The integers

$$b_p = \dim H^p(E), \qquad p = 0, \ldots, n,$$

are called the pth *Betti numbers of E* and the polynomial

$$f_{H(E)} = \sum_{p=0}^n b_p t^p$$

is called the *Poincaré polynomial of* $H^*(E)$. Since $Z^p(E) \subset \wedge^p E^*$, it follows that

$$b_p \leq \binom{n}{p} \qquad (n = \dim E).$$

5.6. Homomorphisms. Let $\varphi: F \to E$ be a homomorphism of Lie algebras, and consider the dual map $\varphi^*: F^* \leftarrow E^*$. Extend φ and φ^* to (dual) homomorphisms

$$\varphi_\wedge: \wedge F \to \wedge E \qquad \text{and} \qquad \varphi^\wedge: \wedge F^* \leftarrow \wedge E^*.$$

The relations

$$\varphi \circ \operatorname{ad} y = \operatorname{ad} \varphi y \circ \varphi, \qquad y \in F,$$

imply that

$$\varphi_\wedge \circ \theta^F(y) = \theta^E(\varphi y) \circ \varphi_\wedge, \qquad y \in F.$$

(since both sides are $\varphi_\wedge$-derivations). Dualizing we obtain

$$\theta_F(y) \circ \varphi^\wedge = \varphi^\wedge \circ \theta_E(\varphi y).$$

In particular, $\varphi^\wedge$ restricts to a homomorphism

$$\varphi^\wedge_{\theta=0}: (\wedge F^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0}.$$

Moreover, if φ is surjective, then $\varphi_\wedge$ maps invariant elements into invariant elements.

Finally, use the expressions (5.6) and (5.7) for ∂_E and δ_E (at the end of sec. 5.3) to obtain

$$\varphi^\wedge \circ \delta_E = \delta_F \circ \varphi^\wedge \qquad \text{and} \qquad \varphi_\wedge \circ \partial_F = \partial_E \circ \varphi_\wedge.$$

Thus $\varphi^\wedge$ (respectively, $\varphi_\wedge$) is a homomorphism of graded differential algebras (respectively, graded differential spaces).

The induced maps in cohomology and homology are denoted by

$$\varphi^\#: H^*(F) \leftarrow H^*(E) \qquad \text{and} \qquad \varphi_\#: H_*(F) \to H_*(E).$$

They are dual, and homogeneous of degree zero. Moreover, $\varphi^\#$ is an algebra homomorphism.

5.7. The space $(\wedge^3 E^*)_{\theta=0}$. Let E be a Lie algebra. Extend the representation of E in E^* to the representation θ_S of E in $\vee^2 E^*$ given by

$$\theta_S(x)(x^* \vee y^*) = -(\operatorname{ad} x)^* x^* \vee y^* + x^* \vee (-\operatorname{ad} x)^* y^*, \qquad x \in E, \quad x^*, y^* \in E^*.$$

We shall construct a canonical linear map

$$\varrho: (\vee^2 E^*)_{\theta=0} \to (\wedge^3 E^*)_{\theta=0}.$$

In fact, let $\Psi \in (\vee^2 E^*)_{\theta=0}$. Then a 3-linear function Φ is defined in E by

$$\Phi(x, y, z) = \Psi([x, y], z), \qquad x, y, z \in E.$$

Since Ψ is invariant, it follows that

$$\Psi([x, y], z) = \Psi([z, x], y) = \Psi([y, z], x).$$

This relation shows that Φ is skew symmetric.

Moreover, the Jacobi identity yields

$$\begin{aligned}(\theta_E(w)\Phi)(x, y, z) &= (\theta_S(w)\Psi)([x, y], z)\\ &= 0, \qquad x, y, z, w \in E.\end{aligned}$$

Thus Φ is invariant.

The correspondence $\Psi \mapsto \Phi$ defines a linear map

$$\varrho: (\vee^2 E^*)_{\theta=0} \to (\wedge^3 E^*)_{\theta=0}.$$

Proposition I: Let E be a Lie algebra such that $H_1(E) = 0$ and $H_2(E) = 0$. Then ϱ is a linear isomorphism.

Proof: To show that ϱ is injective assume that $\Psi \in \ker \varrho$. Then

$$\Psi([x, y], z) = 0, \qquad x, y, z \in E.$$

This implies that $\Psi(u, z) = 0$, $u \in E'$, $z \in E$. But since $H_1(E) = 0$, $E = E'$ (cf. sec. 5.5). Thus $\Psi = 0$.

Now we show that ϱ is surjective. Let $\Phi \in (\wedge^3 E^*)_{\theta=0}$. Then $\langle \Phi, a \rangle = 0$, $a \in \theta(\wedge^3 E)$.

The Koszul formula (5.5), sec. 5.3, shows that $\operatorname{Im} \partial_E \subset \theta(\wedge E)$. On the other hand, the relations at the end of sec. 5.2 yield

$$y \wedge \partial_E w = \theta^E(y) w - \partial_E(y \wedge w), \qquad y \in E, \quad w \in \wedge^3 E.$$

It follows that $\partial_E(y \wedge w) \in \theta(\wedge E)$ and $y \wedge \partial_E w \in \theta(\wedge E)$, whence

$$\langle \Phi, \partial_E(y \wedge w)\rangle = 0 \qquad \text{and} \qquad \langle \Phi, y \wedge \partial_E w\rangle = 0.$$

Now since $H_1(E) = 0$, ∂_E maps $\wedge^2 E$ onto E. Set

$$\Psi(x, y) = \langle \Phi, u \wedge y\rangle, \qquad x, y \in E,$$

where u is an element such that $\partial_E u = x$. To show that Ψ is well-defined, suppose $\partial_E u = \partial_E v$. Then $\partial_E(u - v) = 0$. Since $H_2(E) = 0$, there is an element $w \in \wedge^3 E$ such that $\partial_E w = u - v$. Thus

$$\langle \Phi, u \wedge y\rangle - \langle \Phi, v \wedge y\rangle = \langle \Phi, \partial_E w \wedge y\rangle = 0.$$

It follows that Ψ is a well-defined bilinear function in E.

Next, use formula (5.9), sec. 5.4, to obtain the relation

$$\begin{aligned}\Psi(\partial_E u, \partial_E v) &= \langle \Phi, u \wedge \partial_E v\rangle = \langle \Phi, \partial_E u \wedge v\rangle \\ &= \Psi(\partial_E v, \partial_E u), \qquad u, v \in \wedge^2 E.\end{aligned}$$

Since $\partial_E \colon \wedge^2 E \to E$ is surjective, it follows that Ψ is symmetric. The invariance of Ψ is immediate from the definition.

Finally, it follows directly from the definitions that $\varrho\Psi = \Phi$. Hence ϱ is surjective.

Q.E.D.

5.8. Abelian Lie algebras. Let E be an abelian Lie algebra of dimension n. Then the operators $\theta_E(x)$, $\theta^E(x)$, δ_E, and ∂_E are all zero. Thus we have

$$(\wedge E^*)_{\theta=0} = H^*(E) = \wedge E^* \qquad \text{and} \qquad (\wedge E)_{\theta=0} = H_*(E) = \wedge E.$$

In particular $b_p(E) = \binom{n}{p}$, $0 \le p \le n$, and so the Poincaré polynomial of E is given by $f_E = (1 + t)^n$.

5.9. Direct sums. Assume E and F are Lie algebras and consider the direct sum $E \oplus F$. Consider the isomorphisms (of graded algebras)

$$\wedge E \otimes \wedge F \xrightarrow{\cong} \wedge(E \oplus F) \qquad \text{and} \qquad \wedge E^* \otimes \wedge F^* \xrightarrow{\cong} \wedge(E \oplus F)^*,$$

given by $a \otimes b \mapsto a \wedge b$ and $\Phi \otimes \Psi \mapsto \Phi \wedge \Psi$. These algebras will be

identified via these isomorphisms. The scalar products are given by

$$\langle \Phi \otimes \Psi, a \otimes b \rangle = \langle \Phi, a \rangle \langle \Psi, b \rangle,$$
$$\Phi \in \wedge E^*, \quad \Psi \in \wedge F^*, \quad a \in \wedge E, \quad b \in \wedge F.$$

With these identifications we have

$$\theta_{E\oplus F}(x \oplus y) = \theta_E(x) \otimes \iota + \iota \otimes \theta_F(y), \qquad x \in E, \quad y \in F.$$

It follows that

$$(\wedge(E \oplus F)^*)_{\theta=0} = (\wedge E^*)_{\theta=0} \otimes (\wedge F^*)_{\theta=0}$$

and

$$\theta(\wedge(E \otimes F))^* = \theta(\wedge E^*) \otimes \wedge F^* + \wedge E^* \otimes \theta(\wedge F^*).$$

The analogous formulae hold in $\wedge(E \oplus F)$.

Next we establish the relations

$$\delta_{E\oplus F} = \delta_E \otimes \iota + \omega_E \otimes \delta_F \qquad \text{and} \qquad \partial_{E\oplus F} = \partial_E \otimes \iota + \omega_E \otimes \partial_F.$$

(ω_E denotes the degree involution in $\wedge E^*$ and in $\wedge E$.)

In fact, it follows at once from the definition of the Lie product in $E \oplus F$ that $\partial_{E\oplus F}$ agrees with $\partial_E \otimes \iota + \omega_E \otimes \partial_F$ in $\wedge^2(E \oplus F)$. Dualize and conclude that $\delta_{E\oplus F}$ and $\delta_E \otimes \iota + \omega_E \otimes \delta_F$ agree in $(E \oplus F)^*$; since both operators are antiderivations, they coincide. Dualize again to obtain that $\partial_{E\oplus F} = \partial_E \otimes \iota + \omega_E \otimes \partial_F$.

In view of these relations and the Künneth formula it follows that $H^*(E \oplus F) \cong H^*(E) \otimes H^*(F)$ and $H_*(E \oplus F) \cong H_*(E) \otimes H_*(F)$. Moreover, the first isomorphism is the algebra isomorphism given by

$$\alpha \otimes \beta \mapsto (\pi_1^\# \alpha) \cdot (\pi_2^\# \beta), \qquad \alpha \in H^*(E), \quad \beta \in H^*(F),$$

where $\pi_1: E \oplus F \to E$ and $\pi_2: E \oplus F \to F$ are the projections.

Now consider the case $F = E$. Then the *diagonal map* $\Delta: E \to E \oplus E$ given by

$$\Delta(x) = x \oplus x, \qquad x \in E,$$

is a Lie algebra homomorphism. The induced homomorphism of graded differential algebras $\Delta^\wedge: \wedge E^* \leftarrow \wedge E^* \otimes \wedge E^*$ is given by

$$\Delta^\wedge(\Phi \otimes \Psi) = \Phi \wedge \Psi, \qquad \Phi, \Psi \in \wedge E^*.$$

It follows that the induced homomorphisms

$$\Delta^{\wedge}_{\theta=0}\colon (\wedge E^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$$

and

$$\Delta^{\#}\colon H^*(E) \leftarrow H^*(E) \otimes H^*(E)$$

are given, respectively, by

$$\Delta^{\wedge}_{\theta=0}(\Phi \otimes \Psi) = \Phi \wedge \Psi \qquad \text{and} \qquad \Delta^{\#}(\alpha \otimes \beta) = \alpha \cdot \beta,$$
$$\Phi, \Psi \in (\wedge E^*)_{\theta=0}, \quad \alpha, \beta \in H^*(E).$$

§2. Unimodular Lie algebras

5.10. Unimodular Lie algebras. Consider the vector $t^* \in E^*$ defined by

$$\langle t^*, x\rangle = \operatorname{tr} \operatorname{ad} x, \qquad x \in E.$$

The vector t^* is invariant, as follows from the relation

$$\langle \theta_E(x)t^*, y\rangle = -\operatorname{tr} \operatorname{ad}[x, y] = -\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x) = 0,$$
$$x, y \in E.$$

In terms of dual bases we have

$$t^* = \sum_\nu \theta_E(e_\nu)e^{*\nu}. \tag{5.10}$$

In fact, for $x \in E$,

$$\sum_\nu \langle \theta_E(e_\nu)e^{*\nu}, x\rangle = -\sum_\nu \langle e^{*\nu}, [e_\nu, x]\rangle = \sum_\nu \langle e^{*\nu}, \operatorname{ad} x(e_\nu)\rangle$$
$$= \operatorname{tr} \operatorname{ad} x.$$

Relation (5.10) implies, in view of the formulae in sec. 5.1, that

$$i(t^*) = \sum_\nu \theta^E(e_\nu)i(e^{*\nu}) - \sum_\nu i(e^{*\nu})\theta^E(e_\nu). \tag{5.11}$$

A Lie algebra is called *unimodular* if

$$\operatorname{tr} \operatorname{ad} x = 0, \qquad x \in E;$$

this is equivalent to $t^* = 0$. If E is unimodular, formula (5.11) reduces to

$$\sum_\nu \theta^E(e_\nu)i(e^{*\nu}) = \sum_\nu i(e^{*\nu})\theta^E(e_\nu).$$

Hence the contravariant Koszul formula (cf. formula (5.5), sec. 5.3) can be written in the form

$$\partial_E = \tfrac{1}{2} \sum_\nu i(e^{*\nu})\theta^E(e_\nu). \tag{5.12a}$$

Dualizing we obtain

$$\delta_E = \tfrac{1}{2} \sum_\nu \theta_E(e_\nu)\mu(e^{*\nu}). \tag{5.12b}$$

Proposition II: Let E be a Lie algebra. Then

$$(\wedge E^*)_{\theta=0} \subset Z^*(E) \qquad \text{and} \qquad B_*(E) \subset \theta(\wedge E).$$

If E is unimodular, then also

$$(\wedge E)_{\theta=0} \subset Z_*(E) \qquad \text{and} \qquad B^*(E) \subset \theta(\wedge E^*).$$

Proof: The first statement follows from the Koszul formulae of sec. 5.3. The second follows from formulae (5.12a) and (5.12b) above.
Q.E.D.

Every reductive Lie algebra is unimodular. In fact, if E is reductive, we have the direct decomposition $E = Z_E \oplus E'$. Since

$$\operatorname{tr} \operatorname{ad}[x, y] = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x) = 0, \qquad x, y \in E,$$

this decomposition implies that E is unimodular.

5.11. Poincaré duality. Let E be a unimodular Lie algebra of dimension n, and let $e \in \wedge^n E$. Then

$$\theta^E(x)e = \operatorname{tr}(\operatorname{ad} x)e = 0, \qquad x \in E,$$

and so e is an invariant element.

Now choose a fixed nonzero vector $e \in \wedge^n E$. Then the *Poincaré inner product* in $\wedge E^*$ determined by e is the nondegenerate bilinear function $(\,,\,)$ defined by

$$(\Phi, \Psi) = \langle \Phi \wedge \Psi, e \rangle, \qquad \Phi, \Psi \in \wedge E^*.$$

The corresponding Poincaré isomorphism $D\colon \wedge E^* \xrightarrow{\cong} \wedge E$ is given by $D(\Phi) = i(\Phi)e$ (cf. sec. 0.6).

Since e is invariant, the formulae of sec. 5.1 imply that

$$D \circ \theta_E(x) = \theta^E(x) \circ D, \qquad x \in E.$$

Hence D restricts to an isomorphism

$$D_{\theta=0}\colon (\wedge E^*)_{\theta=0} \xrightarrow{\cong} (\wedge E)_{\theta=0}.$$

On the other hand, since $\partial_E(e) = 0$ (cf. Proposition II, sec. 5.10), the antiderivation property of δ_E yields

$$\partial_E \circ D = D \circ \delta_E \circ \omega_E.$$

(ω_E denotes the degree involution in $\wedge E^*$.)

Hence, D induces an isomorphism

$$D^\#: H^*(E) \xrightarrow{\cong} H_*(E).$$

Now let $\varepsilon \in H_n(E)$ be the class represented by e. Then, in view of the definition,

$$\langle \alpha \cdot \beta, \varepsilon \rangle = \langle \beta, D^\# \alpha \rangle, \qquad \alpha, \beta \in H^*(E).$$

Hence a nondegenerate bilinear function, denoted by $\mathscr{P}_E$, is determined in $H^*(E)$ by

$$\mathscr{P}_E(\alpha, \beta) = \langle \alpha \cdot \beta, \varepsilon \rangle, \qquad \alpha, \beta \in H^*(E).$$

Thus $H^*(E)$ is a Poincaré algebra (cf. sec. 0.6).

Finally, observe that D restricts to isomorphisms

$$\wedge^p E^* \xrightarrow{\cong} \wedge^{n-p} E, \qquad (\wedge^p E^*)_{\theta=0} \xrightarrow{\cong} (\wedge^{n-p} E)_{\theta=0},$$

while

$$D^\#: H^{n-p}(E) \xrightarrow{\cong} H_p(E).$$

In particular, $b_p(E) = b_{n-p}(E)$ $(0 \leq p \leq n)$.

§3. Reductive Lie algebras

5.12. Let E be a reductive Lie algebra. Since $\operatorname{ad} x = 0$, $x \in Z_E$, it follows that

$$\theta_E(x) = 0 \qquad \text{and} \qquad \theta^E(x) = 0, \qquad x \in Z_E.$$

Hence, by Theorem III, sec. 4.4, the representations θ_E and θ^E are semisimple. In particular, we have the direct decompositions

$$\wedge E^* = (\wedge E^*)_{\theta=0} \oplus \theta(\wedge E^*) \qquad \text{and} \qquad \wedge E = (\wedge E)_{\theta=0} \oplus \theta(\wedge E).$$

On the other hand, the duality of θ_E and θ^E implies that

$$\theta(\wedge E^*)^\perp = (\wedge E)_{\theta=0} \qquad \text{and} \qquad \theta(\wedge E)^\perp = (\wedge E^*)_{\theta=0}. \tag{5.13}$$

Thus, in the decomposition above, the scalar product between $\wedge E^*$ and $\wedge E$ restricts to a scalar product between $(\wedge E^*)_{\theta=0}$ and $(\wedge E)_{\theta=0}$ as well as between $\theta(\wedge E^*)$ and $\theta(\wedge E)$.

Lemma I: Let E be a reductive Lie algebra. Then

(1) $Z^*(E) = (\wedge E^*)_{\theta=0} \oplus B^*(E)$.
(2) $Z_*(E) = (\wedge E)_{\theta=0} \oplus B_*(E)$.
(3) $B^*(E) = \theta(Z^*(E)) = Z^*(E) \cap \theta(\wedge E^*)$.
(4) $B_*(E) = \theta(Z_*(E)) = Z_*(E) \cap \theta(\wedge E)$.

Proof: First we establish (3). In fact, the relations

$$\theta_E(x) = i_E(x)\delta_E + \delta_E i_E(x), \qquad x \in E,$$

of sec. 5.2 imply that

$$\theta(Z^*(E)) \subset B^*(E).$$

On the other hand, in view of Proposition II, sec. 5.10,

$$B^*(E) \subset Z^*(E) \cap \theta(\wedge E^*).$$

Since θ_E is semisimple, $Z^*(E)$ has an E-stable complementary subspace in $\wedge E^*$. It follows that

$$\theta(Z^*(E)) = Z^*(E) \cap \theta(\wedge E^*),$$

and these three relations prove (3).

To establish (1), observe from the Koszul formula (formula (5.4), sec. 5.3) that $(\wedge E^*)_{\theta=0} \subset Z^*(E)$. Thus $(\wedge E^*)_{\theta=0} = (Z^*(E))_{\theta=0}$. But, since θ_E is semisimple, so is its restriction to $Z^*(E)$. Thus

$$Z^*(E) = (Z^*(E))_{\theta=0} \oplus \theta(Z^*(E)) = (\wedge E^*)_{\theta=0} \oplus \theta(Z^*(E)),$$

and so (1) is a consequence of (3). Relations (2) and (4) are proved in the same way.

Q.E.D.

Theorem I: Let E be a reductive Lie algebra. Then the projections

$$Z^*(E) \to H^*(E) \qquad \text{and} \qquad Z_*(E) \to H_*(E)$$

restrict to linear isomorphisms

$$\pi_E\colon (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H^*(E) \qquad \text{and} \qquad \pi_*\colon (\wedge E)_{\theta=0} \longrightarrow H_*(E).$$

Moreover, π_E is an algebra isomorphism.

Proof: Apply Lemma I.

Q.E.D.

Remarks: 1. The projections π_E and π_* satisfy

$$\langle \pi_E \Phi, \pi_* a\rangle = \langle \Phi, a\rangle, \qquad \Phi \in (\wedge E^*)_{\theta=0}, \quad a \in (\wedge E)_{\theta=0}.$$

2. If $\varphi\colon F \to E$ is a homomorphism of reductive Lie algebras, then the diagram

$$\begin{tikzcd}
(\wedge F^*)_{\theta=0} \arrow[d, "\pi_F"', "\cong"] & (\wedge E^*)_{\theta=0} \arrow[l, "\varphi^\wedge_{\theta=0}"'] \arrow[d, "\cong"', "\pi_E"] \\
H^*(F) & H^*(E), \arrow[l, "\varphi^*"]
\end{tikzcd}$$

commutes.

3. The diagram

$$\begin{array}{ccc}
(\wedge E^*)_{\theta=0} & \xrightarrow[\cong]{D_{\theta=0}} & (\wedge E)_{\theta=0} \\
\pi_E \downarrow \cong & & \cong \downarrow \pi_F \\
H^*(E) & \xrightarrow[D^\#]{\cong} & H_*(E)
\end{array}$$

commutes, where $D_{\theta=0}$ and $D^\#$ are the isomorphisms of sec. 5.11.

5.13. The Pontrjagin algebra. Let E be reductive. Then there is a unique multiplication in $H_*(E)$ which makes the linear isomorphism $\pi_*: (\wedge E)_{\theta=0} \xrightarrow{\cong} H_*(E)$ into an isomorphism of graded algebras. With this multiplication $H_*(E)$ becomes a graded anticommutative algebra, called the *Pontrjagin algebra of E.*

Remark: Note that in general $H_*(E)$ does not have a natural algebra structure since ∂_E is not an antiderivation.

Let $\varphi: F \to E$ be a homomorphism between reductive Lie algebras. The map $\varphi_\wedge: \wedge F \to \wedge E$ (in general) does *not* restrict to a map from $(\wedge F)_{\theta=0}$ to $(\wedge E)_{\theta=0}$. In this section we shall, nonetheless, construct a homomorphism

$$\varphi_*: (\wedge F)_{\theta=0} \to (\wedge E)_{\theta=0}.$$

In fact, consider the projections

$$\eta_F: \wedge F \to (\wedge F)_{\theta=0} \qquad \text{and} \qquad \eta_E: \wedge E \to (\wedge E)_{\theta=0},$$

with kernels $\theta(\wedge F)$ and $\theta(\wedge E)$. Set

$$\varphi_* = \eta_E \circ \varphi_\wedge.$$

Proposition III: With the notation and hypotheses above:

(1) φ_* is an algebra homomorphism.

(2) φ_* is dual to the homomorphism,

$$\varphi^\wedge_{\theta=0}: (\wedge F^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0}.$$

(3) The diagram

$$\begin{array}{ccc}(\wedge F)_{\theta=0} & \xrightarrow{\varphi_*} & (\wedge E)_{\theta=0} \\ {\scriptstyle \pi_*}\downarrow\cong & & \cong\downarrow{\scriptstyle \pi_*} \\ H_*(F) & \xrightarrow[\varphi_*]{} & H_*(E)\end{array}$$

commutes.

Lemma II: Let E be reductive. Then

$$\eta_E(z_1 \wedge z_2) = \eta_E(z_1) \wedge \eta_E(z_2), \qquad z_1, z_2 \in Z_*(E).$$

Proof: Lemma I, (4), sec. 5.12, implies that $\eta_E \circ \partial_E = 0$. Thus applying η_E to formula (5.9), sec. 5.4, yields

$$\eta_E(\partial_E a \wedge b) = (-1)^p \eta_E(a \wedge \partial_E b), \qquad a \in \wedge^p E, \quad b \in \wedge E.$$

In particular,

$$\eta_E(u \wedge \partial_E v) = 0, \qquad u \in Z_*(E), \quad v \in \wedge E.$$

On the other hand, in view of Lemma I, (2), sec. 5.12, we can write (for $z_1, z_2 \in Z_*(E)$)

$$z_i = \eta_E z_i + \partial_E a_i, \qquad i = 1, 2.$$

Now the relation above gives

$$\begin{aligned}\eta_E(z_1 \wedge z_2) &= \eta_E(\eta_E z_1 \wedge \eta_E z_2) + \eta_E(z_1 \wedge \partial_E a_2) + \eta_E(\partial_E a_1 \wedge \eta_E z_2) \\ &= \eta_E z_1 \wedge \eta_E z_2.\end{aligned}$$

Q.E.D.

Proof of Proposition III: (1) Let $u, v \in (\wedge F)_{\theta=0}$. Then by Lemma I, sec. 5.12, u and v are ∂_F cycles. Since φ is a homomorphism of Lie algebras, it follows that

$$\partial_E \varphi_\wedge(u) = \varphi_\wedge \partial_F(u) = 0.$$

Similarly, $\partial_E \varphi_\wedge(v) = 0$. Thus, in view of Lemma II,

$$\begin{aligned}\varphi_*(u \wedge v) &= \eta_F(\varphi_\wedge(u) \wedge \varphi_\wedge(v)) = \eta_F \varphi_\wedge(u) \wedge \eta_F \varphi_\wedge(v) \\ &= \varphi_*(u) \wedge \varphi_*(v).\end{aligned}$$

(2) In view of formula (5.13), sec. 5.12, we have

$$\langle \Phi, \varphi_*(a) \rangle = \langle \Phi, \varphi_\wedge(a) \rangle = \langle \varphi^\wedge_{\theta=0}(\Phi), a \rangle,$$
$$\Phi \in (\wedge E^*)_{\theta=0}, \qquad a \in (\wedge F)_{\theta=0}.$$

(3) This follows immediately from the definitions, and Lemma I, (2), sec. 5.12.

Q.E.D.

Corollary: The induced map $\varphi_\#: H_*(F) \to H_*(E)$ is a homomorphism between the Pontrjagin algebras.

§4. The structure theorem for $(\wedge E)_{\theta=0}$

In this article E denotes a reductive Lie algebra.

5.14. The primitive subspace. Consider the diagonal map

$$\Delta: E \to E \oplus E$$

(cf. sec. 5.9). Since

$$(\wedge(E \oplus E))_{\theta=0} = (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0},$$

the induced homomorphism Δ_* (cf. Proposition III, sec. 5.13) is a homomorphism

$$\Delta_*: (\wedge E)_{\theta=0} \to (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0}.$$

Lemma III: Let $a \in (\wedge^+ E)_{\theta=0}$. Then

$$\Delta_*(a) = a \otimes 1 + b + 1 \otimes a,$$

where $b \in (\wedge^+ E)_{\theta=0} \otimes (\wedge^+ E)_{\theta=0}$.

Proof: Write

$$\Delta_*(a) = a_1 \otimes 1 + b + 1 \otimes a_2,\ b \in (\wedge^+ E)_{\theta=0} \otimes (\wedge^+ E)_{\theta=0}.$$

Then by Proposition III, (2), sec. 5.13, and sec. 5.9,

$$\langle \Phi, a_1 \rangle = \langle \Phi \otimes 1, \Delta_*(a) \rangle = \langle \Delta^{\wedge}_{\theta=0}(\Phi \otimes 1), a \rangle = \langle \Phi, a \rangle, \quad \Phi \in (\wedge E^*)_{\theta=0}.$$

Now the duality between $(\wedge E^*)_{\theta=0}$ and $(\wedge E)_{\theta=0}$ (cf. sec. 5.12) implies that $a_1 = a$. Similarly, $a_2 = a$.

Q.E.D.

Definition: An element $a \in (\wedge^+ E)_{\theta=0}$ is called *primitive* if

$$\Delta_*(a) = a \otimes 1 + 1 \otimes a.$$

The primitive elements form a graded subspace

$$P_*(E) = \sum_{j=1}^{n} P_j(E) \qquad (n = \dim E)$$

of $(\wedge^+ E)_{\theta=0}$. It is called the *primitive subspace.* The dimension of $P_*(E)$ is called the *rank of E.*

Lemma IV: (1) Every homogeneous primitive element has odd degree.

(2) If $a_1, \ldots, a_p$ are linearly independent homogeneous primitive elements, then

$$a_1 \wedge \cdots \wedge a_p \neq 0.$$

Proof: (1) Let a be a homogeneous primitive element of even degree. Since Δ_* is a homomorphism (cf. Proposition III, sec. 5.13),

$$\Delta_*(a^k) = (\Delta_*(a))^k = (a \otimes 1 + 1 \otimes a)^k, \qquad k = 1, 2, \ldots .$$

Since a has even degree, the elements $a \otimes 1$ and $1 \otimes a$ commute. Thus the binomial theorem yields

$$\Delta_*(a^k) = \sum_{i=0}^{k} \binom{k}{i} a^i \otimes a^{k-i}.$$

Now choose k to be the least integer such that $a^k = 0$. (Since E has finite dimension, this integer exists.) We show that $k = 1$. In fact, assume that $k > 1$. Then (for degree reasons) the elements $a, \ldots, a^{k-1}$ are linearly independent, which contradicts the formula above. Thus $k = 1$. It follows that $a = 0$ and so (1) is established.

(2) Set $\deg a_i = k_i$ and number the a_i so that $k_1 \leq \cdots \leq k_p$. Since Δ_* is a homomorphism and the a_i are primitive,

$$\Delta_*(a_1 \wedge \cdots \wedge a_p) = (a_1 \otimes 1 + 1 \otimes a_1) \wedge \cdots \wedge (a_p \otimes 1 + 1 \otimes a_p).$$

In particular, the component w of $\Delta_*(a_1 \wedge \cdots \wedge a_p)$ in $(\wedge^{k_1} E)_{\theta=0} \otimes (\wedge E)_{\theta=0}$ is given by

$$w = \sum_{j=1}^{q} (-1)^{j-1} a_j \otimes a_1 \wedge \cdots \hat{a}_j \cdots \wedge a_p,$$

where $a_1, \ldots, a_q$ are the elements of degree k_1.

By induction on p we may assume that

$$a_1 \wedge \cdots \hat{a}_j \cdots \wedge a_p \neq 0, \qquad j = 1, \ldots, q.$$

Since the a_i are linearly independent, it follows that $w \neq 0$. Thus $\Delta_*(a_1 \wedge \cdots \wedge a_p) \neq 0$, and so $a_1 \wedge \cdots \wedge a_p \neq 0$.

Q.E.D.

5.15. The ideal $D^*(E)$. In this section we shall obtain a different description of the primitive subspace $P_*(E)$. Let $D^*(E) = \sum_j D^j(E)$ denote the graded ideal in $(\wedge E^*)_{\theta=0}$ given by

$$D^*(E) = (\wedge^+ E^*)_{\theta=0} \cdot (\wedge^+ E^*)_{\theta=0}.$$

If F is a second reductive Lie algebra, we have the relation

$$D^*(E \oplus F) = (D^*(E) \otimes 1) \oplus [(\wedge^+ E^*)_{\theta=0} \otimes (\wedge^+ F^*)_{\theta=0}] \oplus (1 \otimes D^*(F)); \qquad (5.14)$$

this follows by squaring the formula

$$(\wedge^+(E \oplus F))^*_{\theta=0} = [(\wedge^+ E^*)_{\theta=0} \otimes (\wedge F^*)_{\theta=0}] + [(\wedge E^*)_{\theta=0} \otimes (\wedge^+ F^*)_{\theta=0}],$$

(cf. sec. 5.9).

Lemma V: The primitive subspace $P_*(E)$ is the orthogonal complement of $D^*(E)$ with respect to the duality between $(\wedge^+ E^*)_{\theta=0}$ and $(\wedge^+ E)_{\theta=0}$,

$$P_*(E) = D^*(E)^\perp.$$

Proof: Let $a \in P_*(E)$. Then the duality between the maps $\Delta^\wedge_{\theta=0}$ and Δ_* (cf. Proposition III, (2), sec. 5.13) shows that for $\Phi, \Psi \in (\wedge^+ E^*)_{\theta=0}$,

$$\begin{aligned}\langle \Phi \wedge \Psi, a\rangle &= \langle \Delta^\wedge_{\theta=0}(\Phi \otimes \Psi), a\rangle = \langle \Phi \otimes \Psi, \Delta_*(a)\rangle \\ &= \langle \Phi \otimes \Psi, a \otimes 1 + 1 \otimes a\rangle = 0.\end{aligned}$$

It follows that $a \in D^*(E)^\perp$.

Conversely, suppose that $a \in D^*(E)^\perp$. Since $\Delta^\wedge_{\theta=0}$ is a homomorphism, it restricts to a homomorphism

$$D^*(E \oplus E) \to D^*(E).$$

Hence the dual map Δ_* restricts to a linear map

$$\Delta_* : D^*(E \oplus E)^\perp \leftarrow D^*(E)^\perp.$$

Thus $\Delta_*(a) \in D^*(E \oplus E)^\perp$.

But in view of formula (5.14) (applied with $F = E$),

$$D^*(E \oplus E)^\perp = D^*(E)^\perp \otimes 1 + 1 \otimes D^*(E)^\perp.$$

This, together with Lemma III, implies that $\Delta_*(a) = a \otimes 1 + 1 \otimes a$, and so a is primitive.

Q.E.D.

Corollary I: If E and F are reductive Lie algebras, then

$$P_*(E \oplus F) = (P_*(E) \otimes 1) \oplus (1 \otimes P_*(F)).$$

Corollary II: If $\varphi: E \to F$ is a homomorphism of reductive Lie algebras, then φ_* restricts to a linear map

$$(\varphi_*)_P: P_*(E) \to P_*(F).$$

5.16. The structure theorem for $(\wedge E)_{\theta=0}$. In view of Lemma IV, (1), sec. 5.14, we have

$$a \wedge a = 0, \qquad a \in P_*(E).$$

Thus the inclusion map $P_*(E) \to (\wedge E)_{\theta=0}$ extends to a homomorphism of algebras

$$\varkappa_* : \wedge(P_*(E)) \to (\wedge E)_{\theta=0}.$$

If $\wedge P_*(E)$ is given the gradation induced by that of $P_*(E)$, then $\varkappa_*$ is homogeneous of degree zero.

Theorem II: Let E be a reductive Lie algebra. Then

$$\varkappa_* : \wedge(P_*(E)) \to (\wedge E)_{\theta=0}$$

is an isomorphism of graded algebras.

Thus the invariant subalgebra of $\wedge E$ (and hence the Pontrjagin algebra of E) are exterior algebras over graded vector spaces with odd gradations.

Proof: To avoid confusion, the product in $\wedge(P_*(E))$ will be denoted by $u \mathbin{\blacktriangle} v$.

(1) $\varkappa_*$ *is injective*: Let $a_1, \ldots, a_r$ be a homogeneous basis of $P_*(E)$. Then, by Lemma IV, (2), sec. 5.14,

$$\varkappa_*(a_1 \mathbin{\blacktriangle} \cdots \mathbin{\blacktriangle} a_r) = a_1 \wedge \cdots \wedge a_r \neq 0.$$

Now let u be a nonzero element in $\wedge(P_*(E))$. Then, for some $v \in \wedge(P_*(E))$,

$$u \mathbin{\blacktriangle} v = a_1 \mathbin{\blacktriangle} \cdots \mathbin{\blacktriangle} a_r$$

(cf. Example 1, sec. 0.6). Hence $\varkappa_*(u) \wedge \varkappa_*(v) \neq 0$ and so $\varkappa_*(u) \neq 0$. This shows that $\varkappa_*$ is injective.

(2) $\varkappa_*$ *is surjective*: Since $\varkappa_*$ is injective it is sufficient to show that

$$\dim \wedge(P_*(E)) \geq \dim(\wedge E)_{\theta=0}.$$

But

$$\dim \wedge(P_*(E)) = 2^r \, (r = \dim P_*(E)), \text{ and } \dim(\wedge E)_{\theta=0} = \dim(\wedge E^*)_{\theta=0}.$$

Thus we have only to show that

$$2^r \geq \dim(\wedge E^*)_{\theta=0}.$$

Choose a graded subspace $U \subset (\wedge^+ E^*)_{\theta=0}$ so that

$$(\wedge^+ E^*)_{\theta=0} = U \oplus D^*(E).$$

Then, by Lemma V, sec. 5.15, U is dual to $P_*(E)$. Thus, all the homogeneous elements of U have odd degree. It follows that

$$\Phi \wedge \Phi = 0, \qquad \Phi \in U.$$

Hence the inclusion map $i\colon U \to (\wedge^+ E^*)_{\theta=0}$, extends to a homomorphism

$$i_\wedge\colon \wedge U \to (\wedge E^*)_{\theta=0}.$$

It satisfies the relation

$$\operatorname{Im}(i_\wedge^+) + (\wedge^+ E^*)_{\theta=0} \cdot (\wedge^+ E^*)_{\theta=0} = (\wedge^+ E^*)_{\theta=0}.$$

Squaring both sides yields

$$\operatorname{Im}(i_\wedge^+) \cdot \operatorname{Im}(i_\wedge^+) + ((\wedge^+ E^*)_{\theta=0})^3 = ((\wedge^+ E^*)_{\theta=0})^2,$$

whence

$$\operatorname{Im}(i_\wedge^+) + ((\wedge^+ E^*)_{\theta=0})^3 = (\wedge^+ E^*)_{\theta=0}.$$

Repeating this argument shows that

$$\operatorname{Im}(i_\wedge^+) + ((\wedge^+ E^*)_{\theta=0})^p = (\wedge^+ E^*)_{\theta=0}, \qquad p = 2, 3, \ldots .$$

Since $((\wedge^+ E^*)_{\theta=0})^{n+1} = 0$ ($n = \dim E$), it follows that $\operatorname{Im}(i_\wedge^+) = (\wedge^+ E^*)_{\theta=0}$; i.e., $i_\wedge$ is surjective.

Finally, since U and $P_*(E)$ are dual,

$$\dim U = \dim P_*(E) = r.$$

Since $i_\wedge$ is surjective, $\dim \wedge U \geq \dim(\wedge E^*)_{\theta=0}$; i.e.

$$2^r \geq \dim(\wedge E^*)_{\theta=0}.$$

Q.E.D.

Corollary: Let $\varphi: F \to E$ be a homomorphism of reductive Lie algebras, and let $(\varphi_*)_P: P_*(F) \to P_*(E)$ be the restriction of φ_* to $P_*(F)$ (cf. Corollary II, to Lemma V, sec. 5.15). Then the diagram

$$\begin{tikzcd}
\wedge(P_*(F)) \arrow[r, "\wedge(\varphi_*)_P"] \arrow[d, "\kappa_*"', "\cong"] & \wedge(P_*(E)) \arrow[d, "\cong"', "\kappa_*"] \\
(\wedge F)_{\theta=0} \arrow[r, "\varphi_*"'] & (\wedge E)_{\theta=0},
\end{tikzcd}$$

commutes.

§5. The structure of $(\wedge E^*)_{\theta=0}$

In this article, E again denotes a reductive Lie algebra.

5.17. The comultiplication in $(\wedge E^*)_{\theta=0}$. Let $\mu: E \oplus E \to E$ be the linear map given by

$$\mu(x, y) = x + y, \qquad x, y \in E.$$

Then the homomorphism

$$\mu_\wedge: \wedge E \otimes \wedge E \to \wedge E$$

is simply multiplication. Hence it restricts to the multiplication map

$$(\mu_\wedge)_{\theta=0}: (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0} \to (\wedge E)_{\theta=0}.$$

The linear map dual to $(\mu_\wedge)_{\theta=0}$ will be written

$$\gamma_E: (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0}.$$

It is called the *comultiplication map for E.*

Let

$$\eta: \wedge(E \oplus E)^* \to (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$$

denote the projection with kernel $\theta(\wedge(E \oplus E)^*)$, and let

$$\mu^\wedge: \wedge(E \oplus E)^* \leftarrow \wedge E^*$$

be the homomorphism extending μ^*. Then

$$\gamma_E(\Phi) = (\eta \circ \mu^\wedge)(\Phi), \qquad \Phi \in (\wedge E^*)_{\theta=0}.$$

Lemma VI: (1) Let $\varphi: F \to E$ be a homomorphism of reductive

Lie algebras. Then the diagram

$$\begin{tikzcd}
(\wedge F^*)_{\theta=0} \arrow[d, "\gamma_F"'] & & (\wedge E^*)_{\theta=0} \arrow[ll, "\varphi^\wedge_{\theta=0}"'] \arrow[d, "\gamma_E"] \\
(\wedge F^*)_{\theta=0} \otimes (\wedge F^*)_{\theta=0} & & (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \arrow[ll, "\varphi^\wedge_{\theta=0} \otimes \varphi^\wedge_{\theta=0}"]
\end{tikzcd}$$

commutes.

(2) γ_E is an algebra homomorphism.

Proof: (1) In view of Proposition III, (1), sec. 5.13,

$$\varphi_*: (\wedge F)_{\theta=0} \to (\wedge E)_{\theta=0}$$

is an algebra homomorphism. Hence, since $(\mu_\wedge)_{\theta=0}$ is multiplication,

$$\varphi_* \circ (\mu_\wedge)_{\theta=0} = (\mu_\wedge)_{\theta=0} \circ (\varphi_* \otimes \varphi_*).$$

Dualizing this relation yields (1), as follows from Proposition III, (2), sec. 5.13.

(2) An automorphism Q of $(\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0}$ is given by

$$Q(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = (-1)^{pq} a_1 \otimes a_3 \otimes a_2 \otimes a_4, \qquad (\deg a_2 = p, \quad \deg a_3 = q).$$

The dual automorphism of $(\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$ is given by

$$Q^*(\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4) = (-1)^{pq} \Phi_1 \otimes \Phi_3 \otimes \Phi_2 \otimes \Phi_4 \qquad (\deg \Phi_2 = p, \quad \deg \Phi_3 = q).$$

Now the multiplication maps for $(\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0}$ and for $(\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$ are given by

$$(\mu_\wedge^{E \oplus E})_{\theta=0} = ((\mu_\wedge)_{\theta=0} \otimes (\mu_\wedge)_{\theta=0}) \circ Q$$

and by

$$(\Delta^\wedge_{E \oplus E})_{\theta=0} = (\Delta^\wedge_{\theta=0} \otimes \Delta^\wedge_{\theta=0}) \circ Q^*.$$

Dualizing the first relation yields

$$\gamma_{E\oplus E} = Q^* \circ (\gamma_E \otimes \gamma_E).$$

Thus, (1), applied with $\varphi = \Delta$, shows that

$$\gamma_E \circ \Delta^\wedge_{\theta=0} = (\Delta^\wedge_{\theta=0} \otimes \Delta^\wedge_{\theta=0}) \circ \gamma_{E\oplus E} = (\Delta^\wedge_{E\oplus E})_{\theta=0} \circ (\gamma_E \otimes \gamma_E).$$

Hence

$$\begin{aligned}\gamma_E(\Phi \wedge \Psi) &= (\gamma_E \circ \Delta^\wedge_{\theta=0})(\Phi \otimes \Psi) = (\Delta^\wedge_{E\oplus E})_{\theta=0}(\gamma_E(\Phi) \otimes \gamma_E(\Psi)) \\ &= \gamma_E(\Phi) \wedge \gamma_E(\Psi), \qquad \Phi, \Psi \in (\wedge E^*)_{\theta=0},\end{aligned}$$

and so γ_E is a homomorphism.

Q.E.D.

5.18. The primitive subspace of $(\wedge E^*)_{\theta=0}$. Exactly as in Lemma III, sec. 5.14, it follows that

$$\gamma_E(\Phi) = \Phi \otimes 1 + \Psi + 1 \otimes \Phi, \qquad \Phi \in (\wedge^+ E^*)_{\theta=0},$$

where $\Psi \in (\wedge^+ E^*)_{\theta=0} \otimes (\wedge^+ E^*)_{\theta=0}$. The invariant elements Φ in $(\wedge^+ E^*)_{\theta=0}$ which satisfy

$$\gamma_E(\Phi) = \Phi \otimes 1 + 1 \otimes \Phi$$

are called *primitive*. They form a graded subspace

$$P_E = \sum_j P_E^j$$

of $(\wedge^+ E^*)_{\theta=0}$, called the *primitive subspace*.

If $\varphi: F \to E$ is a homomorphism of reductive Lie algebras, then by Lemma VI, sec. 5.17, $\varphi^\wedge_{\theta=0}$ restricts to a linear map

$$\varphi_P: P_F \leftarrow P_E.$$

The following lemma is proved in exactly the same way as Lemma IV, sec. 5.14, and Lemma V, sec. 5.15.

Lemma VII: (1) The homogeneous primitive elements of $(\wedge E^*)_{\theta=0}$ have odd degree.

(2) If $\Phi_1, \ldots, \Phi_p$ are linearly independent homogeneous primitive elements, then $\Phi_1 \wedge \cdots \wedge \Phi_p \neq 0$.

(3) If $D_*(E)$ denotes the ideal $((\wedge^+E)_{\theta=0})^2$ in $(\wedge E)_{\theta=0}$, then $P_E = D_*(E)^\perp$ (with respect to the duality between $(\wedge^+E^*)_{\theta=0}$ and $(\wedge^+E)_{\theta=0}$).

Lemma VII implies that

$$\Phi \wedge \Phi = 0, \qquad \Phi \in P_E.$$

Thus the inclusion map $P_E \to (\wedge^+E^*)_{\theta=0}$ extends to a homomorphism

$$\varkappa_E\colon \wedge P_E \to (\wedge E^*)_{\theta=0}.$$

If $\wedge P_E$ is given the gradation induced from that of P_E, then $\varkappa_E$ is homogeneous of degree zero.

Theorem III: Let E be a reductive Lie algebra. Then

$$\varkappa_E\colon \wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0}$$

is an isomorphism of graded algebras.

Thus $(\wedge E^*)_{\theta=0}$ and $H^*(E)$ are exterior algebras over graded subspaces with odd gradation.

Proof: The theorem is established with the aid of Lemma VII in exactly the same way as Theorem II, sec. 5.16, was proved from Lemmas IV and V.

Q.E.D.

5.19. Corollary I: If $\varphi\colon F \to E$ is a homomorphism of reductive Lie algebras, then the diagram

$$\begin{array}{ccccc}
\wedge P_E & \xrightarrow[\cong]{\varkappa_E} & (\wedge E^*)_{\theta=0} & \xrightarrow[\cong]{\pi_E} & H^*(E) \\
\Big\downarrow{\scriptstyle \wedge\varphi_P} & & \Big\downarrow{\scriptstyle \varphi^\wedge_{\theta=0}} & & \Big\downarrow{\scriptstyle \varphi^*} \\
\wedge P_F & \xrightarrow[\varkappa_F]{\cong} & (\wedge F^*)_{\theta=0} & \xrightarrow[\pi_F]{\cong} & H^*(F)
\end{array}$$

commutes.

Proof: Apply Theorem III, and the second remark after Theorem I, sec. 5.12, noting that all maps are homomorphisms.

Q.E.D.

Corollary II: The Poincaré polynomial of $(\wedge E^*)_{\theta=0}$ (and hence the Poincaré polynomial of $H^*(E)$) has the form

$$f = (1 + t^{g_1}) \cdots (1 + t^{g_r}).$$

Here the exponents are odd and satisfy

$$\sum_{i=1}^{r} g_i = n \qquad (n = \dim E).$$

Proof: Let $\sum_{i=1}^{r} t^{g_i}$ be the Poincaré polynomial of P_E. Then the g_i are odd and the product above is the Poincaré polynomial of $\wedge P_E$.

To prove the second statement, observe the elements of top degree in $\wedge P_E$ have degree $g_1 + \cdots + g_r$, while the elements of top degree in $(\wedge E^*)_{\theta=0}$ have degree n.

Q.E.D.

Corollary III: The Betti numbers of a reductive Lie algebra satisfy

$$\sum_{p=0}^{n} (-1)^p b_p = 0 \qquad \text{and} \qquad \sum_{p=0}^{n} b_p = 2^r \qquad (n = \dim E, \quad r = \dim P_E).$$

Moreover, $n \equiv r \pmod 2$.

Proof: Apply Corollary II, noting that $g_i \equiv 1 \pmod 2$.

Q.E.D.

Finally, let $\gamma_P \colon P_E \to P_E \oplus P_E$ denote the diagonal map

$$\gamma_P(\Phi) = \Phi \oplus \Phi, \qquad \Phi \in P_E.$$

Extend it to a homomorphism $\wedge\gamma_P \colon \wedge P_E \to \wedge P_E \otimes \wedge P_E$.

Proposition IV: Suppose E is a reductive Lie algebra. Then the diagram

$$\begin{tikzcd}
\wedge P_E \arrow[r, "\wedge\gamma_P"] \arrow[d, "\kappa_E"', "\cong"] & \wedge P_E \otimes \wedge P_E \arrow[d, "\cong"', "\kappa_E \otimes \kappa_E"] \\
(\wedge E^*)_{\theta=0} \arrow[r, "\gamma_E"'] & (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}
\end{tikzcd}$$

commutes.

Proof: In fact, for $\Phi \in P_E$ we have (by the definition of P_E)

$$(\gamma_E \circ \varkappa_E)(\Phi) = \Phi \otimes 1 + 1 \otimes \Phi = (\varkappa_E \otimes \varkappa_E)(\gamma_P \Phi).$$

Since all the maps are homomorphisms, the proposition follows.

Q.E.D.

5.20. The invariant subspaces of low dimensions. Let E be a semisimple Lie algebra with Poincaré polynomial

$$f_{H(E)} = \prod_{i=1}^{r} (1 + t^{g_i})$$

(cf. Corollary II, sec. 5.19). Since $E' = E$, it follows that $b_1(E) = 0$.

Thus, because the g_i are odd, $g_i \geq 3$ $(i = 1, \dots, r)$. It follows that $b_2(E) = 0$.

These equations in turn imply that

$$(\wedge^3 E^*)_{\theta=0} = P_E^3$$

Moreover, they show that the hypotheses of Proposition I, sec. 5.7, are satisfied. Hence the linear map ϱ defined in that section is an isomorphism

$$\varrho : (\vee^2 E^*)_{\theta=0} \xrightarrow{\cong} P_E^3.$$

In particular, a nonzero primitive element $\Phi \in P_E^3$ is given by $\Phi = \varrho(K)$ (K, the Killing form); i.e.,

$$\Phi(x, y, z) = \operatorname{tr}(\operatorname{ad}[x, y] \circ \operatorname{ad} z), \qquad x, y, z \in E. \tag{5.15}$$

Furthermore it follows that $b_3(E) = \dim(\vee^2 E^*)_{\theta=0}$.

Proposition V: Let $E = E_1 \oplus \cdots \oplus E_m$ be the decomposition of a semisimple Lie algebra E into simple ideals. Then:

(1) $b_1(E) = b_2(E) = 0$.

(2) $b_3(E) \geq m$.

(3) If either Γ is algebraically closed, or $\Gamma = \mathbb{R}$ and the Killing form is negative definite, then

$$b_3(E) = m.$$

Proof: (1) is proved above. In view of (1), the Künneth formula (cf. sec. 5.9) implies that

$$H^3(E) = \sum_{i=1}^{m} H^3(E_i).$$

It is thus sufficient to consider the case that E is simple; i.e., $m = 1$.

To prove (2), observe that formula (5.15) determines a nonzero element in $(\wedge^3 E^*)_{\theta=0}$. Thus $b_3(E) \geq 1$.

It remains to establish (3). Assume that Γ is algebraically closed, and let $\Psi \in (\vee^2 E^*)_{\theta=0}$. Since the Killing form is nondegenerate, Ψ determines a linear transformation $\psi: E \to E$ such that

$$\Psi(x, y) = K(\psi x, y), \qquad x, y \in E.$$

In view of the invariance of Ψ we have

$$\psi \circ (\operatorname{ad} x) = (\operatorname{ad} x) \circ \psi, \qquad x \in E.$$

It follows that if λ is an eigenvalue of ψ, then $\ker(\psi - \lambda\iota)$ is a nonzero ideal in E. Since E is simple, this implies that $\ker(\psi - \lambda\iota) = E$; i.e., $\psi = \lambda\iota$. This shows that $\Psi = \lambda K$, whence $\dim(\vee^2 E^*)_{\theta=0} = 1$.

Finally, suppose $\Gamma = \mathbb{R}$ and K is negative definite. Proceed as above, observing that ψ is self-adjoint with respect to K, and so has real eigenvalues.

Q.E.D.

Corollary I: Let E be a reductive Lie algebra over an algebraically closed field. Let $l = \dim Z_E$ and denote by m the number of simple ideals in E. Then

$$b_1(E) = l,\ b_2(E) = \binom{l}{2},\ b_3(E) = \binom{l}{3} + m, \text{ and } b_4(E) = \binom{l}{4} + ml.$$

Corollary II: Let E be a simple Lie algebra of rank 2 over an algebraically closed field. Then the Poincaré polynomial of $H(E)$ is

$$f = (1 + t^3)(1 + t^{n-3}) \qquad (n = \dim E).$$

§6. Duality theorems

In this article, E denotes a reductive Lie algebra.

5.21. The duality between primitive spaces. Proposition VI: Let E be a reductive Lie algebra. Then the scalar product, $\langle\ ,\ \rangle$, between $(\wedge E^*)_{\theta=0}$ and $(\wedge E)_{\theta=0}$ restricts to a scalar product between the primitive subspaces P_E and $P_*(E)$. Moreover, it satisfies

$$\langle \Phi, a\rangle = i_E(a)\Phi, \qquad a \in P_*(E), \quad \Phi \in P_E. \tag{5.16}$$

Proof: The isomorphism $\varkappa_E$ (cf. sec. 5.18) restricts to an isomorphism

$$\varkappa_E\colon (\wedge^+ P_E)\cdot(\wedge^+ P_E) \xrightarrow{\cong} (\wedge^+ E^*)_{\theta=0}\cdot(\wedge^+ E^*)_{\theta=0}.$$

Thus (cf. sec. 5.15)

$$(\wedge^+ E^*)_{\theta=0} = P_E \oplus D^*(E).$$

On the other hand, by Lemma V, sec. 5.15,

$$P_*(E) = D^*(E)^\perp.$$

This shows that the spaces P_E and $P_*(E)$ are dual with respect to the restriction of $\langle\ ,\ \rangle$.

Next observe that formula (5.16) holds if Φ and a are homogeneous of the same degree. Thus we need only show that

$$i_E(a)\Phi = 0, \qquad a \in P_j(E), \quad \Phi \in P_E^k, \quad k > j.$$

But, by Lemma VII, sec. 5.18, $P_E = D_*(E)^\perp$. This implies that for $b \in (\wedge^{k-j} E)_{\theta=0}$,

$$\langle i_E(a)\Phi, b\rangle = \langle \Phi, a \wedge b\rangle = 0.$$

Hence $i_E(a)\Phi = 0$.

Q.E.D.

5.22. Duality theorems. As we observed in the preceding section, the scalar product between $(\wedge E^*)_{\theta=0}$ and $(\wedge E)_{\theta=0}$ restricts to a scalar product between P_E and $P_*(E)$. This scalar product in turn induces a scalar product between $\wedge P_E$ and $\wedge P_*(E)$.

Theorem IV: Let E be a reductive Lie algebra. Then the isomorphisms

$$\varkappa_E: \wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0} \qquad \text{and} \qquad \varkappa_*: \wedge P_*(E) \xrightarrow{\cong} (\wedge E)_{\theta=0},$$

preserve the scalar product; i.e.,

$$\langle \varkappa_E \Phi, \varkappa_* a \rangle = \langle \Phi, a \rangle, \qquad \Phi \in \wedge P_E, \quad a \in \wedge P_*(E).$$

Proof: We observe first that this relation holds for $\Phi \in P_E$ and $a \in P_*(E)$ by definition. Now let

$$\varkappa_E^*: \wedge P_*(E) \leftarrow (\wedge E)_{\theta=0}$$

be the linear map dual to $\varkappa_E$ with respect to the given scalar products. It has to be shown that $\varkappa_E^* = \varkappa_*^{-1}$.

Observe first that $\langle \varkappa_E \Phi, \varkappa_* a \rangle = \langle \Phi, a \rangle$ if $a \in P_*(E)$. Indeed, if Φ is primitive this is true by definition, while if Φ is a product of primitives both sides are zero (cf. Lemma V, sec. 5.15). It follows that

$$\varkappa_E^*(a) = \varkappa_*^{-1}(a), \qquad a \in P_*(E).$$

Thus to prove $\varkappa_E^* = \varkappa_*^{-1}$ we have only to show that $\varkappa_E^*$ is a homomorphism. For this purpose we establish two lemmas.

Lemma VIII: Let $a \in P_*(E)$. Then the operator

$$i_E(a): (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0}$$

is an antiderivation.

Proof: Since the map $\Delta_*: (\wedge E)_{\theta=0} \to (\wedge E)_{\theta=0} \otimes (\wedge E)_{\theta=0}$ is a homomorphism (cf. Proposition III, (1), sec. 5.13), we have

$$\mu(\Delta_*(a)) \circ \Delta_* = \Delta_* \circ \mu(a).$$

Dualizing this relation we obtain

$$\Delta_{\theta=0}^{\wedge} \circ i_{E\oplus E}(\Delta_*(a)) = i_E(a) \circ \Delta_{\theta=0}^{\wedge}.$$

Thus

$$i_E(a)(\Phi \wedge \Psi) = i_E(a)\Delta^{\wedge}_{\theta=0}(\Phi \otimes \Psi) = \Delta^{\wedge}_{\theta=0} \circ i_{E\oplus E}(\Delta_*(a))(\Phi \otimes \Psi),$$
$$\Phi, \Psi \in (\wedge E^*)_{\theta=0}.$$

But, since a is primitive, it follows that $\Delta_*(a) = a \otimes 1 + 1 \otimes a$. Hence

$$i_{E\oplus E}(\Delta_*(a))(\Phi \otimes \Psi) = i_E(a)\Phi \otimes \Psi + (-1)^p \Phi \otimes i_E(a)\Psi,$$
$$\Phi \in (\wedge^p E^*)_{\theta=0}, \quad \Psi \in (\wedge E^*)_{\theta=0},$$

and so

$$i_E(a)(\Phi \wedge \Psi) = i_E(a)\Phi \wedge \Psi + (-1)^p \Phi \wedge i_E(a)\Psi.$$

Q.E.D.

Lemma IX: Denote by $i_P(a)$ the substitution operator in the algebra $\wedge P_E$ induced by $a \in P_*(E)$. Then

$$\varkappa_E \circ i_P(a) = i_E(a) \circ \varkappa_E.$$

Proof: Since i_P is an antiderivation in $\wedge P_E$, $\varkappa_E \circ i_P(a)$ is a $\varkappa_E$-antiderivation. On the other hand, Lemma VIII shows that $i_E(a)$ is an antiderivation in the algebra $(\wedge E^*)_{\theta=0}$. Thus $i_E(a) \circ \varkappa_E$ is a $\varkappa_E$-antiderivation. Hence it is sufficient to prove that

$$\varkappa_E \circ i_P(a)(\Phi) = i_E(a) \circ \varkappa_E(\Phi), \qquad \Phi \in P_E.$$

But in P_E, $\varkappa_E$ reduces to the identity. Thus in view of the definitions of the scalar products and Proposition VI, sec. 5.21,

$$\varkappa_E \circ i_P(a)(\Phi) = \langle \Phi, a \rangle = i_E(a)\Phi = i_E(a) \circ \varkappa_E(\Phi), \qquad \Phi \in P_E.$$

Q.E.D.

5.23. Proof of Theorem IV: Dualizing the formula in Lemma IX gives

$$\mu_P(a) \circ \varkappa_E^\# = \varkappa_E^\# \circ \mu(a),$$

where $\mu_P(a)$ denotes the multiplication operator (induced by $a \in P_*(E)$) in the algebra $\wedge P_*(E)$. Since $a = \varkappa_E^\# a$, this can be rewritten as

$$\varkappa_E^\#(a) \wedge \varkappa_E^\#(b) = \varkappa_E^\#(a \wedge b), \qquad a \in P_*(E), \quad b \in (\wedge E)_{\theta=0}.$$

Clearly this yields

$$\varkappa_E^{\#}(a_1 \wedge \cdots \wedge a_p) = \varkappa_E^{\#}(a_1) \wedge \cdots \wedge \varkappa_E^{\#}(a_p), \qquad a_i \in P_*(E).$$

Since $P_*(E)$ generates $(\wedge E)_{\theta=0}$, $\varkappa_E^{\#}$ must be a homomorphism. In view of sec. 5.22, Theorem IV is now proved.

Q.E.D.

Corollary: For any $b \in \wedge P_*(E)$,

$$\varkappa_E \circ i_P(b) = i_E(\varkappa_* b) \circ \varkappa_E .$$

§7. Cohomology with coefficients in a graded Lie module

In this article E denotes an arbitrary finite-dimensional Lie algebra, and θ_M denotes a representation of E in a graded vector space $M = \sum_p M^p$.

5.24. The space $M \otimes \wedge E^*$. Consider the graded vector space

$$M \otimes \wedge E^* = \sum_r (M \otimes \wedge E^*)^r, \quad \text{where } (M \otimes \wedge E^*)^r = \sum_{p+q=r} M^p \otimes \wedge^q E^*.$$

Extend the multiplication operators $\mu(\Phi)$ ($\Phi \in \wedge^q E^*$, $q = 0, 1, \ldots$) to operators $\mu(\Phi)$ in $M \otimes \wedge E^*$ by setting

$$\mu(\Phi)(z \otimes \Psi) = (-1)^{pq} z \otimes \Phi \wedge \Psi, \qquad \Phi \in \wedge^q E^*, \quad \Psi \in \wedge E^*, \quad z \in M^p.$$

Extend the substitution operators $i_E(a)$ ($a \in \wedge E$) to the operators $i_E(a)$ in $M \otimes \wedge E^*$ defined by

$$i_E(a)(z \otimes \Psi) = (-1)^{pq} z \otimes i_E(a)\Psi, \qquad a \in \wedge^q E, \quad \Psi \in \wedge E^*, \quad z \in M^p.$$

In particular, $i_E(x)$ ($x \in E$) is homogeneous of degree -1.

Similarly denote the operators $\theta_M(x) \otimes \iota$ and $\iota \otimes \theta_E(x)$ simply by $\theta_M(x)$ and $\theta_E(x)$. Then θ_M and θ_E are representations of E in the graded space $M \otimes \wedge E^*$. They determine the representation θ given by

$$\theta(x) = \theta_M(x) + \theta_E(x), \qquad x \in E.$$

More explicitly,

$$\theta(x)(z \otimes \Psi) = \theta_M(x)z \otimes \Psi + z \otimes \theta_E(x)\Psi, \quad x \in E, \ z \in M, \ \Psi \in \wedge E^*.$$

The symbols $\theta(M \otimes \wedge E^*)$ and $(M \otimes \wedge E^*)_{\theta=0}$ will refer to this representation (cf. sec. 4.2).

In view of sec. 5.1,

$$\theta(x)i_E(a) - i_E(a)\theta(x) = i_E(\theta^E(x)a), \qquad x \in E, \quad a \in \wedge E.$$

In particular, if a is invariant,

$$\theta(x)i_E(a) = i_E(a)\theta(x), \qquad x \in E.$$

Next, extend δ_E to the operator (again denoted by δ_E) in $M \otimes \wedge E^*$ given by

$$\delta_E(z \otimes \Psi) = (-1)^p z \otimes \delta_E \Psi, \qquad z \in M^p, \quad \Psi \in \wedge E^*.$$

Then the relations (5.3) of sec. 5.2 yield the formulae

$$\begin{aligned} i_E(x)\delta_E + \delta_E i_E(x) &= \theta_E(x), \\ \delta_E^2 &= 0, \end{aligned} \tag{5.3}$$

and

$$\delta_E \theta_E(x) = \theta_E(x)\delta_E$$

for the corresponding operators in $M \otimes \wedge E^*$.

Clearly, $\delta_E \theta_M(x) = \theta_M(x)\delta_E$, and so $\delta_E \theta(x) = \theta(x)\delta_E$, $x \in E$.

5.25. The operators δ_θ and δ. Define an operator δ_θ in $M \otimes \wedge E^*$ by

$$\delta_\theta(z \otimes \Psi) = (-1)^p \sum_\nu \theta_M(e_\nu) z \otimes e^{*\nu} \wedge \Psi, \qquad z \in M^p, \quad \Psi \in \wedge E^*,$$

where $e^{*\nu}$, e_ν are dual bases for E^* and E. Evidently, δ_θ is independent of the choice of the dual bases. Moreover,

$$\delta_\theta = \sum_\nu \mu(e^{*\nu})\theta_M(e_\nu) = \sum_\nu \theta_M(e_\nu)\mu(e^{*\nu}).$$

Next, define an operator

$$\delta: M \otimes \wedge E^* \to M \otimes \wedge E^*,$$

by

$$\delta = \delta_E + \delta_\theta.$$

Observe that δ_E, δ_θ, δ are all homogeneous of degree 1. If $M \otimes \wedge^p E^*$ is identified with the space $L(\wedge^p E; M)$ $(0 \le p \le n)$ by the equation

$$\Omega(a) = i_E(a)\Omega, \qquad \Omega \in M \otimes \wedge^p E^*, \quad a \in \wedge^p E,$$

then δ is given by the formula

$$\begin{aligned} (\delta\Omega)(x_0, \dots, x_p) &= \sum_{i=0}^p (-1)^i \theta_M(x_i)(\Omega(x_0, \dots, \hat{x}_i, \dots, x_p)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \Omega([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_p), \\ &\qquad x_i \in E, \quad i = 0, \dots, p, \quad \Omega \in L(\wedge^p E; M), \end{aligned}$$

(cf. formula (5.7), sec. 5.3).

It will now be shown that δ satisfies the relations

$$\begin{aligned} i_E(x)\delta + \delta i_E(x) = \theta(x), \qquad \delta^2 = 0, \\ \delta\theta(x) = \theta(x)\delta, \qquad x \in E. \end{aligned} \tag{5.17}$$

In fact, it follows from the definition that

$$i_E(x)\delta_\theta + \delta_\theta i_E(x) = \theta_M(x).$$

This, together with (5.3), yields the first relation.

To establish the second, let $e^{*\nu}$, e_ν be a pair of dual bases for E^* and E. Observe that for $x, y \in E$,

$$\begin{aligned} \sum_{\nu<\mu} \langle e^{*\nu} \wedge e^{*\mu}, x \wedge y\rangle [e_\nu, e_\mu] &= \sum_{\nu<\mu} \langle e^{*\nu} \wedge e^{*\mu}, x \wedge y\rangle \partial_E(e_\nu \wedge e_\mu) \\ &= \partial_E(x \wedge y), \end{aligned}$$

and

$$\sum_\nu \langle \delta e^{*\nu}, x \wedge y\rangle e_\nu = -\sum_\nu \langle e^{*\nu}, \partial_E(x \wedge y)\rangle e_\nu = -\partial_E(x \wedge y).$$

It follows that, in $E \otimes \wedge^2 E^*$,

$$\sum_{\nu<\mu} [e_\nu, e_\mu] \otimes e^{*\nu} \wedge e^{*\mu} = -\sum_\nu e_\nu \otimes \delta_E e^{*\nu}.$$

Now an easy computation gives

$$\begin{aligned} \delta_\theta^2(z \otimes \Psi) &= \sum_{\nu<\mu} \theta_M([e_\nu, e_\mu])z \otimes e^{*\nu} \wedge e^{*\mu} \wedge \Psi \\ &= -\sum_\nu \theta_M(e_\nu)z \otimes \delta_E e^{*\nu} \wedge \Psi \\ &= -(\delta_E\delta_\theta + \delta_\theta\delta_E)(z \otimes \Psi), \qquad z \in M, \quad \Psi \in \wedge E^*. \end{aligned}$$

Since $\delta_E^2 = 0$, it follows that

$$\delta^2 = \delta_E^2 + \delta_\theta^2 + \delta_E\delta_\theta + \delta_\theta\delta_E = 0.$$

Finally, applying δ on the left and right of the first relation of (5.17) yields the third relation.

As an immediate consequence of (5.3) and (5.17) we have the relation

$$\delta_\theta\theta(x) = \theta(x)\delta_\theta, \qquad x \in E.$$

5.26. The cohomology and the invariant cohomology. It follows from the relations of sec. 5.25 that $(M \otimes \wedge E^*, \delta)$ is a graded differential space. The corresponding cohomology space $H(M \otimes \wedge E^*, \delta)$ is called the *cohomology of* E *with coefficients in* M, and is denoted by $H^*(E; M)$.

If θ_M is the trivial representation, then $\delta_\theta = 0$ and so $\delta = \delta_E$. In this case we have

$$H^*(E; M) = M \otimes H^*(E).$$

On the other hand, relation (5.17) of sec. 5.25 implies that θ represents E in the graded differential space $(M \otimes \wedge E^*, \delta)$. Thus the invariant subspace is stable under δ. The corresponding cohomology is called the *invariant cohomology of* E *with coefficients in* M and is denoted by $H((M \otimes \wedge E^*)_{\theta=0}, \delta)$.

Finally, the Koszul formula of sec. 5.3, together with the definition of δ_θ, gives

$$2\delta_E + \delta_\theta = \sum_\nu \mu(e^{*\nu})\theta(e_\nu). \tag{5.18}$$

It follows that in the invariant subspace $(M \otimes \wedge E^*)_{\theta=0}$, δ_θ reduces to $-2\delta_E$, while δ reduces to $-\delta_E$. Thus

$$H((M \otimes \wedge E^*)_{\theta=0}, \delta) = H((M \otimes \wedge E^*)_{\theta=0}, \delta_E).$$

5.27. Representation in graded algebras. Assume that θ_M is a representation of E in a graded algebra M. Then the operators $\theta_M(x)$ (in M) are derivations, homogeneous of degree zero. Give $M \otimes \wedge E^*$ the algebra structure defined by the anticommutative tensor product. Then a simple verification shows that:

(1) $i_E(x)$ $(x \in E)$ is an antiderivation, homogeneous of degree -1.

(2) $\theta_E(x)$, $\theta_M(x)$, and $\theta(x)$ $(x \in E)$ are derivations, homogeneous of degree zero.

(3) δ_E, δ_θ, and δ are antiderivations, homogeneous of degree 1.

Thus $M \otimes \wedge E^*$ and $(M \otimes \wedge E^*)_{\theta=0}$ become graded differential algebras, and so the corresponding cohomology spaces become graded algebras.

5.28. Reductive Lie algebras. Proposition VII: Let E be a reductive Lie algebra, and let θ_M be a representation of E in a graded vector space M. Then

(1) The inclusion map $M_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (M \otimes \wedge E^*)_{\theta=0}$ induces an isomorphism

$$M_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H((M \otimes \wedge E^*)_{\theta=0}, \delta).$$

(2) If θ_M is a semisimple representation, then the inclusion $(M \otimes \wedge E^*)_{\theta=0} \to M \otimes \wedge E^*$ induces an isomorphism

$$H((M \otimes \wedge E^*)_{\theta=0}, \delta) \xrightarrow{\cong} H^*(E; M).$$

Proof: (1) Recall from sec. 5.26 that in $(M \otimes \wedge E^*)_{\theta=0}$, $\delta = -\delta_E$. Thus apply Theorem V, sec. 4.11 with

$$(X, \delta_X) = (\wedge E^*, \delta_E) \qquad \text{and} \qquad (Y, \delta_Y) = (M, 0).$$

(2) It follows from formula (5.17), sec. 5.25, that the representation $\theta^\#$ of E in $H(M \otimes \wedge E^*)$ is trivial. Thus (2) is a consequence of Theorem IV, sec. 4.10.

Q.E.D.

§8. Applications to Lie groups

Throughout this article, E denotes a Lie algebra of a Lie group G.

5.29. The algebras $H^*(E)$ and $H^*(G)$. Recall from sec. 1.2, volume II, that each vector $h \in E$ determines a unique left invariant vector field X_h on G such that $X_h(e) = h$. Moreover, we have the operators $i(X_h)$, $\theta(X_h)$, and δ (substitution operator, Lie derivative, and exterior derivative) in the algebra $A(G)$ of differential forms on G.

These operators restrict to operators $i_L(h)$, $\theta_L(h)$ and δ_L in the algebra $A_L(G)$ of left invariant forms (cf. sec. 4.5, volume II). Moreover, an isomorphism

$$\tau_L: A_L(G) \xrightarrow{\cong} \wedge E^*$$

is defined by $\tau_L(\Phi) = \Phi(e)$ (cf. Proposition II, sec. 4.5, volume II). Under this isomorphism, the operators $i_L(h)$, $\theta_L(h)$, and δ_L correspond, respectively, to the operators $i_E(h)$, $\theta_E(h)$, and δ_E defined in sec. 5.1 and sec. 5.2. (To see this, apply Proposition III, sec. 4.6, volume II, and formula (5.7), sec. 5.3.)

In particular, τ_L induces an isomorphism

$$\tau_L^\#: H(A_L(G), \delta_L) \xrightarrow{\cong} H^*(E)$$

(which, in volume II, is denoted by $(\tau_L)_\#$).

Composing $\tau_L^{-1}: \wedge E^* \xrightarrow{\cong} A_L(G)$ with the inclusion $A_L(G) \to A(G)$, we obtain a homomorphism

$$\varepsilon_G: (\wedge E^*, \delta_E) \to (A(G), \delta)$$

of graded differential algebras. Let

$$\varepsilon_G^\#: H^*(E) \to H(G)$$

denote the induced homomorphism.

On the other hand, if G is connected, then τ_L restricts to an isomorphism

$$\tau_I: A_I(G) \xrightarrow{\cong} (\wedge E^*)_{\theta=0},$$

where $A_I(G)$ is the algebra of bi-invariant forms (cf. sec. 4.9, volume II). In view of sec. 4.10, volume II, we have the commutative diagram

$$\begin{tikzcd} A_I(G) \arrow[r] \arrow[d, "\tau_I"', "\cong"] & H_L(G) \arrow[r] \arrow[d, "\tau_L^\#"', "\cong"] & H(G) \\ (\wedge E^*)_{\theta=0} \arrow[r, "\pi_E"'] & H^*(E) \arrow[ur, "\varepsilon_G^\#"'] & \end{tikzcd} \tag{5.19}$$

Here π_E denotes the homomorphism induced by the inclusion $(\wedge E^*)_{\theta=0} \to Z^*(E)$; if E is reductive, it coincides with the isomorphism π_E in Theorem I, sec. 5.12.

The composite $\varepsilon_G^\# \circ \pi_E$ will be denoted by α_G,

$$\alpha_G\colon (\wedge E^*)_{\theta=0} \to H(G).$$

Note that, for $\Phi \in (\wedge E^*)_{\theta=0}$, $\alpha_G(\Phi)$ is represented by the (closed) bi-invariant form $\tau_I^{-1}(\Phi)$. It follows from Theorem III, sec. 4.10, volume II, that if G is compact and connected, then all the maps in diagram (5.19) are isomorphisms of graded algebras.

Next, let $\varphi\colon K \to G$ be a homomorphism of connected Lie groups, and let $\varphi'\colon F \to E$ be the induced homomorphism of Lie algebras (cf. sec. 1.3, volume II).

From φ we obtain homomorphisms

$$\varphi^*\colon A(K) \leftarrow A(G), \qquad \varphi_L^*\colon A_L(K) \leftarrow A_L(G), \qquad \varphi_I^*\colon A_I(K) \leftarrow A_I(G),$$

as well as homomorphisms $\varphi^\#\colon H(K) \leftarrow H(G)$ and $\varphi_L^\#\colon H_L(K) \leftarrow H_L(G)$, induced by φ^* and φ_L^*. On the other hand, φ' induces homomorphisms $(\varphi')^\wedge$, $(\varphi')^\wedge_{\theta=0}$, and $(\varphi')^\#$, as described in sec. 5.6. It follows from the results of sec. 4.7, volume II, that the diagram

$$\begin{tikzcd} (\wedge E^*)_{\theta=0} \arrow[r, "\pi_E"] \arrow[d, "(\varphi')^\wedge_{\theta=0}"'] & H^*(E) \arrow[r, "\varepsilon_G^\#"] \arrow[d, "(\varphi')^\#"] & H(G) \arrow[d, "\varphi^\#"] \\ (\wedge F^*)_{\theta=0} \arrow[r, "\pi_F"'] & H^*(F) \arrow[r, "\varepsilon_K^\#"'] & H(K) \end{tikzcd}$$

commutes. This shows that

$$\alpha_K \circ (\varphi')^\wedge_{\theta=0} = \varphi^\# \circ \alpha_G. \tag{5.20}$$

Example: *Products*: Let K and G be connected Lie groups with Lie algebras F and E. Let $\pi_1: K\times G \to K$ and $\pi_2: K\times G \to G$ be the projections. Then π_1' and π_2' are just the projections from $F \oplus E$ to F and E.

Now write $(\wedge(F\oplus E)^*)_{\theta=0} = (\wedge F^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$, as in sec. 5.9. If $\Phi \in (\wedge F^*)_{\theta=0}$, $\Psi \in (\wedge E^*)_{\theta=0}$, then

$$\Phi \otimes \Psi = (\Phi \otimes 1) \wedge (1 \otimes \Psi) = (\pi_1')^\wedge_{\theta=0}(\Phi) \wedge (\pi_2')^\wedge_{\theta=0}(\Psi).$$

On the other hand, recall that the Künneth homomorphism

$$\varkappa_\#: H(K) \otimes H(G) \to H(K\times G)$$

is given by $\varkappa_\#(\alpha \otimes \beta) = \pi_1^\#(\alpha) \cdot \pi_2^\#(\beta)$ (cf. sec. 5.17, volume I).

Proposition VIII: If K and G are connected, then the diagram

$$\begin{tikzcd}
(\wedge F^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \arrow[r, "\cong"] \arrow[d, "\alpha_K\otimes\alpha_G"'] & (\wedge(F\oplus E)^*)_{\theta=0} \arrow[d, "\alpha_{K\times G}"] \\
H(K)\otimes H(G) \arrow[r, "\varkappa_\#"'] & H(K\times G)
\end{tikzcd}$$

commutes.

Proof: Fix $\Phi \in (\wedge F^*)_{\theta=0}$ and $\Psi \in (\wedge E^*)_{\theta=0}$. Since $\alpha_{K\times G}$ is a homomorphism, we have

$$\alpha_{K\times G}(\Phi \otimes \Psi) = \alpha_{K\times G}((\pi_1')^\wedge_{\theta=0}(\Phi)) \cdot \alpha_{K\times G}((\pi_2')^\wedge_{\theta=0}(\Psi)).$$

Now apply formula (5.20), sec. 5.29, to the homomorphisms π_1 and π_2, to obtain

$$\begin{aligned}\alpha_{K\times G}(\Phi\otimes\Psi) &= \pi_1^\#(\alpha_K(\Phi)) \cdot \pi_2^\#(\alpha_G(\Psi)) \\ &= \varkappa_\# \circ (\alpha_K \otimes \alpha_G)(\Phi\otimes\Psi).\end{aligned}$$

Q.E.D.

5.30. The map φ. Let G be a connected Lie group with reductive Lie algebra E. Define a smooth map $\varphi: G\times G \to G$ by

$$\varphi(x,y) = xy^{-1}, \qquad x, y \in G.$$

It induces a homomorphism $\varphi^\#: H(G\times G) \leftarrow H(G)$.

Since $\varphi(e, e) = e$, the derivative $d\varphi$ restricts to a linear map $\varphi' : E \oplus E \to E$. Evidently $\varphi'(h \oplus k) = h - k$, $h, k \in E$. Now extend $(\varphi')^*$ to a homomorphism

$$(\varphi')^\wedge : \wedge E^* \to \wedge(E \oplus E)^*.$$

Let

$$\eta_{E\oplus E} : \wedge(E \oplus E)^* \to (\wedge(E \oplus E)^*)_{\theta=0}$$

be the projection with kernel $\theta_{E\oplus E}(\wedge(E \oplus E)^*)$, and define a linear map,

$$\varphi^\natural : (\wedge E^*)_{\theta=0} \to (\wedge(E \oplus E)^*)_{\theta_{E\oplus E}=0},$$

by setting $\varphi^\natural(\Phi) = \eta_{E\oplus E} \circ (\varphi')^\wedge(\Phi)$, $\Phi \in (\wedge E^*)_{\theta=0}$.

Proposition IX: With the hypotheses above, the diagram

$$\begin{array}{ccc} (\wedge E^*)_{\theta=0} & \xrightarrow{\varphi^\natural} & (\wedge(E \oplus E)^*)_{\theta=0} \\ \downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_{G\times G}} \\ H(G) & \xrightarrow[\varphi^\#]{} & H(G \times G) \end{array}$$

commutes.

Proof: Observe that

$$\varphi(ax, by) = a \cdot \varphi(x, y) \cdot b^{-1}, \qquad a, b, x, y \in G.$$

It follows that the map $\varphi^* : A(G \times G) \leftarrow A(G)$ restricts to a homomorphism

$$\varphi_I^* : A_L(G \times G) \leftarrow A_I(G).$$

Now fix $\Phi \in (\wedge E^*)_{\theta=0}$ and define $\Psi \in A_I(G)$ by $\Psi = \tau_I^{-1}(\Phi)$ (cf. sec. 5.29). Then (again cf. sec. 5.29) Ψ is closed, and represents $\alpha_G(\Phi)$. Thus $\varphi_I^*(\Psi)$ represents $\varphi^\# \alpha_G(\Phi)$.

On the other hand, since $\varphi_I^*(\Psi)$ is left invariant, and

$$(\varphi_I^* \Psi)(e, e) = (\varphi')^\wedge(\Psi(e)) = (\varphi')^\wedge(\Phi),$$

it follows that

$$\tau_L(\varphi_I^* \Psi) = (\varphi')^\wedge \Phi.$$

Thus

$$\delta_{E\oplus E}((\varphi')^\wedge\Phi) = \tau_L\varphi_I^*(\delta\Psi) = 0.$$

Now Theorem I, sec. 5.12, (applied to $E \oplus E$) shows that for some $\Omega \in \wedge(E \oplus E)^*$,

$$(\varphi')^\wedge\Phi = \varphi^\natural\Phi + \delta_{E\oplus E}\Omega.$$

Finally, apply τ_L^{-1} to this equation to obtain

$$\varphi_I^*\Psi = \tau_I^{-1}(\varphi^\natural\Phi) + \delta(\tau_L^{-1}\Omega).$$

Thus, $\tau_I^{-1}(\varphi^\natural\Phi)$ also represents $\varphi^\#\alpha_G(\Phi)$; i.e.,

$$(\alpha_{G\times G} \circ \varphi^\natural)\Phi = (\varphi^\#\alpha_G)\Phi.$$

Q.E.D.

5.31. The comultiplication. Let G be a connected Lie group with reductive Lie algebra E. Then the multiplication map $\mu_G\colon G\times G \to G$ (given by $\mu_G(x, y) = xy$) determines a homomorphism of graded differential algebras $\mu_G^\#\colon H(G\times G) \leftarrow H(G)$.

On the other hand, we have the homomorphism

$$\gamma_E\colon (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0},$$

defined in sec. 5.17. Identify $(\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$ with $(\wedge(E \oplus E)^*)_{\theta=0}$ as in sec. 5.29 and sec. 5.9.

Proposition X: With the hypotheses above, the diagram

$$\begin{array}{ccc} (\wedge(E\oplus E)^*)_{\theta=0} & \xleftarrow{\gamma_E} & (\wedge E^*)_{\theta=0} \\ \downarrow{\scriptstyle \alpha_{G\times G}} & & \downarrow{\scriptstyle \alpha_G} \\ H(G\times G) & \xleftarrow[\mu_G^\#]{} & H(G) \end{array}$$

commutes.

Proof: Define a diffeomorphism $f\colon G\times G \to G\times G$ by

$$f(x, y) = (x, y^{-1}), \qquad x, y \in G.$$

Then

$$f(axb, cyd) = (axb, d^{-1}y^{-1}c^{-1}), \qquad a, b, c, d, x, y \in G.$$

These equations imply that the induced isomorphism f^* restricts to an isomorphism

$$f_I^*: A_I(G\times G) \xleftarrow{\cong} A_I(G\times G).$$

Next observe that $f(e, e) = e$ and that the derivative df restricts to the linear isomorphism f' of $E \oplus E$ given by

$$f'(h, k) = (h, -k).$$

It follows that the automorphism $(f')^\wedge$ of $\wedge(E \oplus E)^*$ commutes with the operators $\theta_{E\oplus E}(h, k)$. Hence it restricts to an automorphism $(f')^\wedge_{\theta=0}$ of the invariant subalgebra.

It is immediate from the definition that the diagram

$$\begin{tikzcd}
A(G\times G) & A_I(G\times G) \arrow[l] \arrow[r, "\tau_I", "\cong"'] & (\wedge(E\oplus E)^*)_{\theta=0} \\
A(G\times G) \arrow[u, leftarrow, "f^*"', "\cong"] & A_I(G\times G) \arrow[l] \arrow[r, "\cong", "\tau_I"'] \arrow[u, leftarrow, "\cong", "f_I^*"'] & (\wedge(E\oplus E)^*)_{\theta=0} \arrow[u, leftarrow, "\cong", "(f')^\wedge_{\theta=0}"']
\end{tikzcd}$$

commutes. It follows that

$$f^\# \circ \alpha_{G\times G} = \alpha_{G\times G} \circ (f')^\wedge_{\theta=0}.$$

Finally, observe that the maps μ_G and φ are connected by

$$\mu_G = \varphi \circ f,$$

where φ is the map defined in sec. 5.30. Thus,

$$\mu_G^\# \circ \alpha_G = f^\# \circ \varphi^\# \circ \alpha_G = \alpha_{G\times G} \circ (f')^\wedge_{\theta=0} \circ \varphi^\natural \tag{5.21}$$

(cf. Proposition IX, sec. 5.30).

On the other hand, since $(f')^\wedge$ commutes with the operators $\theta_{E\oplus E}(h, k)$, it follows that $(f')^\wedge$ commutes with the projection $\eta_{E\oplus E}$ defined in sec. 5.30. This implies that for $\Phi \in (\wedge E^*)_{\theta=0}$,

$$((f')^\wedge_{\theta=0} \circ \varphi^\natural)\Phi = (\eta_{E\oplus E} \circ (\mu'_G)^\wedge)\Phi.$$

Since $\mu'_G(h, k) = h + k$, it follows from sec. 5.17 that

$$\gamma_E = \eta_{E\oplus E} \circ (\mu'_G)^\wedge,$$

whence

$$\gamma_E = (f')^\wedge_{\theta=0} \circ \varphi^\natural. \tag{5.22}$$

The proposition follows from relations (5.21) and (5.22).

Q.E.D.

5.32. The primitive subspace. Now assume that G is compact and connected. Then the Lie algebra E is reductive (cf. sec. 4.4). Use the Künneth isomorphism to identify $H(G\times G)$ with $H(G) \otimes H(G)$ (cf. Theorem VI, sec. 5.20, volume I). Combining the example of sec. 5.29 with Proposition X, sec. 5.31, and observing that α_G is an isomorphism (cf. Theorem III, sec. 4.10, volume II) we obtain the commutative diagram

$$\begin{tikzcd}
(\wedge E^*)_{\theta=0} \arrow[r, "\gamma_E"] \arrow[d, "\alpha_G"', "\cong"] & (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \arrow[d, "\cong"', "\alpha_G\otimes\alpha_G"] \\
H(G) \arrow[r, "\mu_G^\#"'] & H(G) \otimes H(G)
\end{tikzcd}$$

In sec. 4.12, volume II, we defined the primitive subspace $P_G \subset H^+(G)$ to be the subspace of classes $\alpha \in H^+(G)$ satisfying

$$\mu_G^\#(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha.$$

It follows that α_G restricts to an isomorphism

$$\alpha_P\colon P_E \xrightarrow{\cong} P_G.$$

Thus there is a commutative diagram

$$\begin{tikzcd}
\wedge P_E \arrow[r, "\varkappa_E", "\cong"'] \arrow[d, "\wedge\alpha_P"', "\cong"] & (\wedge E^*)_{\theta=0} \arrow[d, "\cong"', "\alpha_G"] \\
\wedge P_G \arrow[r, "\cong", "\lambda_G"'] & H(G),
\end{tikzcd}$$

of algebra isomorphisms, where $\varkappa_E$ is the isomorphism defined in sec.

5.18 while λ_G is defined in sec. 4.12, volume II. This diagram shows that in the compact case Theorem III, sec. 5.18, is equivalent to Theorem IV, sec. 4.12, volume II.

Moreover, Theorem IV, sec. 4.12, volume II, asserts that

$$\dim P_G = \dim T,$$

where T is a maximal torus in G. It follows that the rank of the Lie algebra of a compact Lie group is equal to the dimension of a maximal torus. An algebraic version of this result will be established in Chapter X (sec. 10.23).

Chapter VI

The Weil Algebra

In this chapter E denotes a finite-dimensional Lie algebra.

§1. The Weil algebra

6.1. The algebras $\vee \mathbb{E}^*$ and $\vee \mathbb{E}$. Consider the graded vector space $\mathbb{E}^*$ which is defined as follows: $\mathbb{E}^*$ is equal to E^* as a vector space and the gradation of $\mathbb{E}^*$ is given by

$$\deg x^* = 2, \qquad x^* \in \mathbb{E}^*.$$

The induced gradation of the symmetric algebra $\vee \mathbb{E}^*$ is given by

$$(\vee \mathbb{E}^*)^{2q} = \vee^q E^*, \qquad (\vee \mathbb{E}^*)^{2q+1} = 0, \qquad q \geq 0.$$

Thus $\vee \mathbb{E}^*$ is evenly graded, and so it is a graded anticommutative algebra.

The representation

$$x \mapsto -(\operatorname{ad} x)^*$$

of E in E^* gives rise, evidently, to a representation of E in $\mathbb{E}^*$. Extending the linear transformations $-(\operatorname{ad} x)^*$ to derivations $\theta_S(x)$, we obtain a representation of E in $\vee \mathbb{E}^*$.

In a similar way we form the graded space $\mathbb{E}$ ($\mathbb{E} = E$ as a vector space; $\deg x = 2$, $x \in \mathbb{E}$) and the symmetric algebra $\vee \mathbb{E}$. Then $\vee \mathbb{E}$ is a graded anticommutative algebra and the adjoint representation of E in $\mathbb{E}$ extends uniquely to a representation θ^S of E in the graded algebra $\vee \mathbb{E}$. The representations θ_S and θ^S are contragredient with respect to the induced scalar product between $\vee \mathbb{E}^*$ and $\vee \mathbb{E}$. Thus we have

$$(\vee \mathbb{E}^*)_{\theta=0} = \theta(\vee \mathbb{E})^{\perp} \qquad \text{and} \qquad \theta(\vee \mathbb{E}^*) = (\vee \mathbb{E})^{\perp}_{\theta=0}.$$

Next, let $\varphi: F \to E$ be a homomorphism of Lie algebras. Then φ and its dual φ^* extend to homomorphisms of graded algebras

$$\varphi_\vee: \vee F \to \vee E \qquad \text{and} \qquad \varphi^\vee: \vee F^* \leftarrow \vee E^*.$$

Clearly we have

$$\varphi^\vee \circ \theta_S(\varphi y) = \theta_S(y) \circ \varphi^\vee, \qquad y \in F,$$

and so $\varphi^\vee$ restricts to a homomorphism

$$\varphi^\vee_{\theta=0}: (\vee F^*)_{\theta=0} \leftarrow (\vee E^*)_{\theta=0}.$$

Now suppose E is reductive. Since

$$\theta_S(x) = \theta^S(x) = 0, \qquad x \in Z_E,$$

it follows that the representations θ_S and θ^S are semisimple (cf. Theorem III, sec. 4.4). Hence in particular

$$\vee E^* = (\vee E^*)_{\theta=0} \oplus \theta(\vee E^*) \qquad \text{and} \qquad \vee E = (\vee E)_{\theta=0} \oplus \theta(\vee E).$$

It follows from these decompositions that the scalar product between $\vee E^*$ and $\vee E$ restricts to a scalar product between $(\vee E^*)_{\theta=0}$ and $(\vee E)_{\theta=0}$.

6.2. The algebra $W(E)$. Consider the anticommutative graded algebra $W(E) = \sum_{r \ge 0} W^r(E)$ given by

$$W(E) = \vee E^* \otimes \wedge E^*,$$

$$W^r(E) = \sum_{p+q=r} (\vee E^*)^p \otimes \wedge^q E^* = \sum_{2p+q=r} \vee^p E^* \otimes \wedge^q E^*.$$

Thus $W^0(E) = \Gamma$, $W^1(E) = 1 \otimes E^*$, and $W^2(E) = E^* \otimes 1 \oplus 1 \otimes \wedge^2 E^*$.

We shall now translate some of the results of article 7 of the preceding chapter, with $\vee E^* = M$ and $\theta_S = \theta_M$. In fact, as in sec. 5.24, extend the operators $\mu(\Phi)$ ($\Phi \in \wedge E^*$), $i_E(a)$ ($a \in \wedge E$), $\theta_E(x)$, and $\theta_S(x)$ to $W(E)$ by setting

$$\mu(\Phi) = \iota \otimes \mu(\Phi), \qquad i_E(a) = \iota \otimes i_E(a),$$

$$\theta_S(x) = \theta_S(x) \otimes \iota, \qquad \theta_E(x) = \iota \otimes \theta_E(x).$$

(Note that $\vee E^*$ is evenly graded.)

Then θ_E and θ_S are representations of E in $W(E)$. Next set

$$\theta_W(x) = \theta_E(x) + \theta_S(x), \qquad x \in E.$$

Then θ_W is also a representation of E in the graded algebra $W(E)$ (it corresponds to the representation θ of sec. 5.24). The invariant subalgebra and the "θ"-subspace for this representation will be written $W(E)_{\theta=0}$ and $\theta(W(E))$ respectively.

Assume E is reductive. Since $\theta_W(x) = 0$, $x \in Z_E$, it follows from Theorem III, sec. 4.4, that the representation θ_W is semisimple. In particular,

$$W(E) = W(E)_{\theta=0} \oplus \theta(W(E)).$$

Next, extend δ_E to the operator $\delta_E = \iota \otimes \delta_E$ in $W(E)$, and let δ_θ be the operator determined by the representation θ_S (cf. sec. 5.25). Then

$$\delta_\theta = \sum_\nu \mu(e^{*\nu})\theta_S(e_\nu),$$

where $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E.

The operators δ_E and δ_θ are antiderivations, homogeneous of degree 1. In view of formula (5.17), sec. 5.25, we have the relations

$$\begin{aligned} i_E(x)(\delta_E + \delta_\theta) + (\delta_E + \delta_\theta)i_E(x) &= \theta_W(x), \\ (\delta_E + \delta_\theta)^2 &= 0, \end{aligned} \tag{6.1}$$

and

$$(\delta_E + \delta_\theta)\theta_W(x) = \theta_W(x)(\delta_E + \delta_\theta), \qquad x \in E.$$

On the other hand, formula (5.18), sec. 5.26, yields

$$2\delta_E + \delta_\theta = \sum_\nu \mu(e^{*\nu})\theta_W(e_\nu). \tag{6.2}$$

6.3. The antiderivations h and k. Define operators h and k in $W(E)$ by

$$h(\Psi \otimes x^{*1} \wedge \cdots \wedge x^{*p}) = \sum_{j=1}^{p} (-1)^{j-1} x^{*j} \vee \Psi \otimes x^{*1} \wedge \cdots \widehat{x^{*j}} \cdots \wedge x^{*p}$$

$$h(\Psi \otimes 1) = 0, \qquad \Psi \in \vee E^*, \quad x^{*i} \in E^*,$$

and by

$$k(x^{*1} \vee \cdots \vee x^{*p} \otimes \Phi) = \sum_{j=1}^{p} x^{*1} \vee \cdots \widehat{x^{*j}} \cdots \vee x^{*p} \otimes x^{*j} \wedge \Phi$$

$$k(1 \otimes \Phi) = 0, \qquad \Phi \in \wedge E^*, \quad x^{*i} \in E^*.$$

Then h and k are antiderivations in $W(E)$, homogeneous of degrees 1 and -1 respectively. In terms of dual bases $e^{*\nu}$, e_ν we can write

$$h = \sum_\nu \mu_S(e^{*\nu}) \otimes i_E(e_\nu) \qquad \text{and} \qquad k = \sum_\nu i_S(e_\nu) \otimes \mu(e^{*\nu}),$$

where $\mu_S(x^*)$ and $i_S(x)$ are the multiplication and substitution operators in $\vee E^*$.

The operators h and k satisfy

$$(hk + kh)\Omega = (p + q)\Omega, \qquad \Omega \in \vee^p E^* \otimes \wedge^q E^*. \tag{6.3}$$

In fact, since

$$h(1 \otimes x^*) = x^* \otimes 1, \qquad h(x^* \otimes 1) = 0$$

and

$$k(1 \otimes x^*) = 0, \qquad k(x^* \otimes 1) = 1 \otimes x^*, \qquad x^* \in E^*,$$

it follows that $hk + kh$ reduces to the identity in $E^* \otimes 1$ and in $1 \otimes E^*$. But $hk + kh$ is a derivation and so (6.3) follows.

Similar arguments show that

$$\begin{aligned} h^2 = 0; \qquad k^2 &= 0 \\ h\theta_W(x) &= \theta_W(x)h \\ k\theta_W(x) &= \theta_W(x)k \end{aligned} \tag{6.4}$$

and

$$i_E(x)h + hi_E(x) = 0, \qquad x \in E.$$

6.4. The antiderivation δ_W. Set

$$\delta_W = \delta_E + \delta_\theta + h.$$

Then δ_W is an antiderivation in $W(E)$, homogeneous of degree 1. If E is abelian, then $\delta_E = \delta_\theta = 0$ and so δ_W reduces to h.

We shall now establish the relations

$$\begin{aligned} i_E(x)\delta_W + \delta_W i_E(x) &= \theta_W(x) \\ \delta_W^2 &= 0 \end{aligned} \tag{6.5}$$

and

$$\delta_W\theta_W(x) = \theta_W(x)\delta_W, \qquad x \in E.$$

In fact, the first and third relation follow at once from formula (6.1), sec. 6.2, and formula (6.4), sec. 6.3. Moreover, since $(\delta_E + \delta_\theta)^2 = 0$ and $h^2 = 0$, the second relation is equivalent to

$$(\delta_E + \delta_\theta)h + h(\delta_E + \delta_\theta) = 0.$$

To prove this, we may restrict ourselves to elements of the form $x^* \otimes 1$ and $1 \otimes x^*$, $x^* \in E^*$, because both sides are derivations. Now

$$\begin{aligned}[(\delta_E + \delta_\theta)h + h(\delta_E + \delta_\theta)](x^* \otimes 1) &= h\delta_\theta(x^* \otimes 1)\\ &= \sum_\nu \theta_S(e_\nu)x^* \vee e^{*\nu}.\end{aligned}$$

But

$$\begin{aligned}\left\langle \sum_\nu \theta_S(e_\nu)x^* \vee e^{*\nu}, x \vee y \right\rangle &= \langle \theta_S(x)x^*, y\rangle + \langle \theta_S(y)x^*, x\rangle\\ &= \langle x^*, -[x, y] - [y, x]\rangle = 0, \qquad x, y \in E,\end{aligned}$$

and thus

$$[(\delta_E + \delta_\theta)h + h(\delta_E + \delta_\theta)](x^* \otimes 1) = 0.$$

On the other hand,

$$\begin{aligned}&[(\delta_E + \delta_\theta)h + h(\delta_E + \delta_\theta)](1 \otimes x^*)\\ &\quad = \sum_\nu (\theta_S(e_\nu)x^* \otimes e^{*\nu} + e^{*\nu} \otimes i_E(e_\nu)\delta_E x^*)\\ &\quad = \sum_\nu (\theta_S(e_\nu)x^* \otimes e^{*\nu} + e^{*\nu} \otimes \theta_E(e_\nu)x^*).\end{aligned}$$

This is a vector in $E^* \otimes E^*$. Its scalar product with $x \otimes y$ ($\in E \otimes E$) is given by

$$\langle x^*, -[y, x]\rangle + \langle x^*, -[x, y]\rangle = 0.$$

Hence

$$[(\delta_E + \delta_\theta)h + h(\delta_E + \delta_\theta)](1 \otimes x^*) = 0.$$

This completes the proof of (6.5).

The graded differential algebra $(W(E), \delta_W)$ is called the *Weil algebra of the Lie algebra E.*

Formulae (6.5) show that θ_W is a representation of E in the graded differential algebra $(W(E), \delta_W)$ and that $\theta_W^\# = 0$. In particular, the invariant subalgebra $W(E)_{\theta=0}$ is stable under δ_W. Moreover, the relation (6.2), sec. 6.2, shows that the restriction of δ_W to $W(E)_{\theta=0}$ is given by

$$\delta_W = h - \delta_E.$$

6.5. Homomorphisms. Let $\varphi: F \to E$ be a homomorphism of Lie algebras. Then φ induces a homomorphism

$$\varphi_W = \varphi^\vee \otimes \varphi^\wedge: W(F) \leftarrow W(E).$$

It follows from sec. 5.6 and sec. 6.1 that

$$\varphi_W \theta_W(\varphi y) = \theta_W(y)\varphi_W, \qquad y \in F,$$

and so φ_W restricts to a homomorphism,

$$(\varphi_W)_{\theta=0}: W(F)_{\theta=0} \leftarrow W(E)_{\theta=0}.$$

On the other hand, φ_W is a homomorphism of differential algebras: $\varphi_W \delta_W = \delta_W \varphi_W$. In fact, we know from sec. 5.6 that $\varphi_W \delta_E = \delta_E \varphi_W$. It is immediate from the definitions that $\varphi_W h = h\varphi_W$.

Finally, observe that

$$\varphi_W \delta_\theta(1 \otimes x^*) = 0 = \delta_\theta \varphi_W(1 \otimes x^*)$$

while

$$\begin{aligned} i_F(y)\varphi_W\delta_\theta(x^* \otimes 1) &= \varphi_W i_E(\varphi y)\delta_\theta(x^* \otimes 1) \\ &= \varphi_W(\theta_S(\varphi y)x^* \otimes 1) \\ &= \theta_S(y)\varphi_W(x^* \otimes 1) \\ &= i_F(y)\delta_\theta\varphi_W(x^* \otimes 1), \qquad y \in F, \quad x^* \in E^*. \end{aligned}$$

This implies that

$$\varphi_W \delta_\theta(x^* \otimes 1) = \delta_\theta \varphi_W(x^* \otimes 1), \qquad x^* \in E^*.$$

Since $\varphi_W \delta_\theta$ and $\delta_\theta \varphi_W$ are φ_W-antiderivations, it follows that $\varphi_W \delta_\theta = \delta_\theta \varphi_W$, and so

$$\varphi_W \delta_W = \delta_W \varphi_W.$$

6.6. The cohomology of the Weil algebra. The purpose of this section is to establish the following

Proposition I: Let E be a Lie algebra. Then the cohomology of $W(E)$ and of $W(E)_{\theta=0}$ is trivial,

$$H^+(W(E), \delta_W) = 0, \qquad H^0(W(E), \delta_W) = \Gamma,$$

and

$$H^+(W(E)_{\theta=0}, \delta_W) = 0, \qquad H^0(W(E)_{\theta=0}, \delta_W) = \Gamma.$$

Recall the definition of the antiderivation k in sec. 6.3. Define a derivation Δ in $W(E)$, homogeneous of degree zero, by

$$\Delta = \delta_W k + k\delta_W.$$

Lemma I: Let $\Omega \in W^r(E)$, $r \geq 1$. Then

$$\Omega = \frac{(-1)^{r-1}}{r!} \sum_{\lambda=0}^{r-1} c_\lambda \Delta^{r-\lambda}\Omega$$

where $c_0 = 1$ and

$$c_\lambda = (-1)^\lambda \sum_{1 \leq l_1 < \cdots < l_\lambda \leq r} l_1 \bullet \cdots \bullet l_\lambda.$$

In particular, the restriction of Δ to $W^+(E)$ is an isomorphism.

Proof: Set

$$\Delta_1 = (\delta_E + \delta_\theta)k + k(\delta_E + \delta_\theta) = \Delta - (hk + kh).$$

Then

$$\Delta_1 \colon \vee^q E^* \otimes \wedge E^* \to \vee^{q-1} E^* \otimes \wedge E^*, \qquad q = 0, 1, \ldots .$$

Next, recall from sec. 6.3 that

$$(hk + kh)\Omega = (p + q)\Omega, \qquad \Omega \in \vee^q E^* \otimes \wedge^p E^*.$$

Define operators T_q $(q = 0, 1, \ldots)$ in $W(E)$ by

$$T_q(\Omega) = (r - q)\Omega, \qquad \Omega \in W^r(E).$$

Then

$$T_q(\Omega) = (hk + kh)\Omega, \qquad \Omega \in \vee^q E^* \otimes \wedge E^*,$$

and so it follows that

$$\Delta - T_q \colon \vee^q E^* \otimes \wedge E^* \to \vee^{q-1} E^* \otimes \wedge E^*.$$

Iterating this process we obtain

$$(\Delta - T_0) \circ \cdots \circ (\Delta - T_q)(\Omega) = 0, \qquad \Omega \in \vee^q E^* \otimes \wedge E^*. \tag{6.6}$$

On the other hand, since T_q reduces to scalar multiplication in each $W^r(E)$, it commutes with every operator homogeneous of degree zero. Thus

$$\Delta T_q = T_q \Delta \qquad \text{and} \qquad T_p T_q = T_q T_p.$$

In particular, the relation (6.6) continues to hold for

$$\Omega \in \sum_{j \le q} \vee^j E^* \otimes \wedge E^*.$$

Now let $\Omega \in W^r(E)$, $r \ge 1$. Then $\Omega \in \sum_{j \le r-1} \vee^j E^* \otimes \wedge E^*$, and hence

$$(\Delta - T_0) \circ \cdots \circ (\Delta - T_{r-1})\Omega = 0.$$

Expanding this relation and using the fact that

$$T_{i_1} \cdots T_{i_p}(\Omega) = (r - i_1) \cdots (r - i_p)\Omega,$$

we obtain the formula of the lemma.

This formula shows that Δ restricts to surjective maps

$$W^r(E) \to W^r(E), \qquad r \ge 1.$$

Since $W^r(E)$ has finite dimension, Δ must be an isomorphism in each $W^r(E)$.

Q.E.D.

Proof of the proposition: Since $\delta_W^2 = 0$ and $k^2 = 0$, we have

$$\Delta^p = (\delta_W k)^p + (k\delta_W)^p, \qquad p \ge 1.$$

Thus if $\Omega \in W^r(E)$, $r \ge 1$, and $\delta_W \Omega = 0$, Lemma I yields $\Omega = \delta_W \Omega_1$ with

$$\Omega_1 = \frac{(-1)^{r-1}}{r!} k \sum_{\lambda=0}^{r-1} c_\lambda (\delta_W k)^{r-\lambda-1}\Omega.$$

It follows that $H^r(W(E)) = 0$, $r \ge 1$. On the other hand, we have, trivially, $H^0(W(E)) = \Gamma$.

Finally, suppose $\Omega \in W^r(E)_{\theta=0}$, $r \ge 1$, and $\delta_W \Omega = 0$. Since k and δ_W commute with the operators $\theta_W(x)$, $x \in E$, it follows that Ω_1 is invariant. Hence, $H^+(W(E)_{\theta=0}) = 0$.

Q.E.D.

§2. The canonical map ρ_E

In this article we shall construct a canonical linear map

$$\varrho_E\colon (\vee^+ E^*)_{\theta=0} \to (\wedge^+ E^*)_{\theta=0}$$

for an arbitrary Lie algebra E.

6.7. Definition: Let $\pi_E\colon W(E) \to \wedge E^*$ denote the projection defined by

$$\pi_E(1 \otimes \Phi) = \Phi \quad \text{and} \quad \pi_E(\Psi \otimes \Phi) = 0, \qquad \Phi \in \wedge E^*, \quad \Psi \in \vee^+ E^*.$$

Then π_E is a homomorphism of graded algebras. It satisfies the relations

$$\pi_E \delta_W = \delta_E \pi_E \quad \text{and} \quad \pi_E \theta_W(x) = \theta_E(x) \pi_E, \quad x \in E.$$

In particular, π_E restricts to a homomorphism

$$(\pi_E)_{\theta=0}\colon W(E)_{\theta=0} \to (\wedge E^*)_{\theta=0}.$$

Since (cf. formula (6.2), sec. 6.2) δ_W reduces to $\frac{1}{2}\delta_\theta + h$ in $W(E)_{\theta=0}$, and $\pi_E h = \pi_E \delta_\theta = 0$, it follows that

$$(\pi_E)_{\theta=0} \circ \delta_W = 0. \tag{6.7}$$

Moreover, if $\varphi\colon F \to E$ is a homomorphism of Lie algebras, then

$$\varphi^\wedge \circ \pi_E = \pi_F \circ \varphi_W.$$

Lemma II: Let $\Psi \in (\vee^+ E^*)_{\theta=0}$. Then there is a unique element $\Phi \in (\wedge^+ E^*)_{\theta=0}$ such that for some $\Omega \in W^+(E)_{\theta=0}$,

$$\pi_E \Omega = \Phi \quad \text{and} \quad \delta_W \Omega = \Psi \otimes 1.$$

Proof: Evidently, $\delta_W(\Psi \otimes 1) = 0$. Hence, by Proposition I, sec. 6.6, there exists an element $\Omega \in W^+(E)_{\theta=0}$ such that

$$\delta_W \Omega = \Psi \otimes 1.$$

Set $\Phi = \pi_E(\Omega)$.

If $\Omega_1 \in W^+(E)_{\theta=0}$ is another element such that $\delta_W \Omega_1 = \Psi \otimes 1$, then

$$\delta_W(\Omega - \Omega_1) = 0.$$

Hence, again by Proposition I,

$$\Omega - \Omega_1 = \delta_W \hat{\Omega}, \qquad \text{for some} \quad \hat{\Omega} \in W^+(E)_{\theta=0}.$$

It follows that

$$\pi_E \Omega - \pi_E \Omega_1 = (\pi_E)_{\theta=0} \delta_W(\hat{\Omega}) = 0$$

(cf. formula (6.7) above). Thus Φ is independent of the choice of Ω.
Q.E.D.

The correspondence $\Psi \mapsto \Phi$ defines a linear map

$$\varrho_E \colon (\vee^+ E^*)_{\theta=0} \to (\wedge^+ E^*)_{\theta=0},$$

homogeneous of degree -1. It will be called the *Cartan map for E.* In view of the definition we have

$$\varrho_E \Psi = \pi_E \Omega, \qquad \delta_W \Omega = \Psi \otimes 1.$$

In sec. 6.14 it will be shown that if E is reductive, then

$$\ker \varrho_E = (\vee^+ E^*)^2_{\theta=0} \qquad \text{and} \qquad \operatorname{Im} \varrho_E = P_E.$$

Example: Let E be an abelian Lie algebra. Then the Cartan map is given by

$$\varrho_E \Psi = \Psi, \quad \Psi \in E^*; \qquad \varrho_E \Psi = 0, \quad \Psi \in (\vee^+ E^*) \cdot (\vee^+ E^*).$$

In fact, in this case $\delta_E = \delta_\theta = 0$. Thus, if $\Psi \in \vee^p E^*$, then

$$\Psi \otimes 1 = \frac{1}{p} hk(\Psi \otimes 1) = \delta_W\Big(\frac{1}{p} k(\Psi \otimes 1)\Big).$$

Hence

$$\varrho_E \Psi = \frac{1}{p} \pi_E k(\Psi \otimes 1).$$

If $p \geq 2$, then $k(\Psi \otimes 1) \in \vee^+ E^* \otimes \wedge E^*$ and so $\varrho_E \Psi = 0$. If $p = 1$, then $k(\Psi \otimes 1) = 1 \otimes \Psi$ and so

$$\varrho_E \Psi = \pi_E(1 \otimes \Psi) = \Psi.$$

Proposition II: If $\varphi: F \to E$ is a homomorphism of Lie algebras, then

$$\varphi^{\wedge}_{\theta=0} \circ \varrho_E = \varrho_F \circ \varphi^{\vee}_{\theta=0}.$$

Proof: Let $\Psi \in (\vee^+ E^*)_{\theta=0}$ and let $\Omega \in W^+(E)_{\theta=0}$ satisfy

$$\delta_W \Omega = \Psi \otimes 1.$$

Then $\varphi^{\vee}\Psi \otimes 1 = \varphi_W \delta_W \Omega = \delta_W \varphi_W \Omega$, and hence

$$\varrho_F \varphi^{\vee} \Psi = \pi_F \varphi_W \Omega = \varphi^{\wedge} \pi_E \Omega = \varphi^{\wedge} \varrho_E \Psi.$$

Q.E.D.

6.8. Explicit formula for ρ_E. In this section we give an explicit expression for ϱ_E.

Proposition III: The Cartan map ϱ_E for a Lie algebra E is given by (cf. sec. 6.3 for k)

$$\varrho_E \Psi = \frac{(q-1)!}{(2q-1)!} k(\delta_E k)^{q-1}(\Psi \otimes 1), \qquad \Psi \in (\vee^q E^*)_{\theta=0}.$$

Lemma III: Let $\Psi \in (\vee^q E^*)_{\theta=0}$. Then

$$\Psi \otimes 1 = \delta_W \left\{ \frac{(-1)^{q-1}(q-1)!}{(2q-1)!} \sum_{\lambda=0}^{q-1} a_\lambda k(\delta_W k)^{q-\lambda-1}(\Psi \otimes 1) \right\},$$

where $a_0 = 1$ and

$$a_\lambda = (-1)^\lambda \sum_{1 \le i_1 < \cdots < i_\lambda \le q} (2q - i_1) \cdots (2q - i_\lambda).$$

Proof: We adopt the notation of sec. 6.6, and show first that

$$(\Delta - T_1) \cdots (\Delta - T_q)(\Psi \otimes 1) = 0. \tag{6.8}$$

As in Lemma I, sec. 6.6, observe that

$$(\Delta - T_1) \cdots (\Delta - T_q)(\Psi \otimes 1) = 1 \otimes \Phi,$$

where $\Phi \in (\wedge^{2q} E^*)_{\theta=0}$. Also note that the space $W^{2q}(E)_{\theta=0}$ is stable under Δ and that each T_i restricts to scalar multiplication in $W^{2q}(E)_{\theta=0}$. It follows that there are constants α_ν such that

$$1 \otimes \Phi = \sum_\nu \alpha_\nu \Delta^\nu (\Psi \otimes 1) = \sum_\nu \alpha_\nu (\delta_W k)^\nu (\Psi \otimes 1).$$

Hence

$$1 \otimes \Phi = \alpha_0(\Psi \otimes 1) + \delta_W \Omega, \qquad \text{some} \quad \Omega \in W(E)_{\theta=0}.$$

It follows that (cf. formula (6.7), sec. 6.7)

$$\Phi = \pi_E(1 \otimes \Phi) = (\pi_E)_{\theta=0}\delta_W(\Omega) = 0.$$

This proves formula (6.8).

Finally, obtain the lemma by expanding formula (6.8) in the same way as in the proof of Lemma I, sec. 6.6.

Q.E.D.

Proof of Proposition III: Recall from sec. 6.4 that in $W(E)_{\theta=0}$, $\delta_W = h - \delta_E$. Further note that

$$k\colon \vee^p E^* \otimes \wedge E^* \to \vee^{p-1} E^* \otimes \wedge E^*, \; h\colon \vee^p E^* \otimes \wedge E^* \to \vee^{p+1} E^* \otimes \wedge E^*,$$

and

$$\delta_E\colon \vee^p E^* \otimes \wedge E^* \to \vee^p E^* \otimes \wedge E^*.$$

These formulae imply that for $\Psi \in (\vee^q E^*)_{\theta=0}$,

$$\pi_E k(\delta_W k)^p(\Psi \otimes 1) = 0, \qquad p < q - 1$$

and

$$\begin{aligned}\pi_E k(\delta_W k)^{q-1}(\Psi \otimes 1) &= (-1)^{q-1}\pi_E k(\delta_E k)^{q-1}(\Psi \otimes 1)\\ &= (-1)^{q-1}k(\delta_E k)^{q-1}(\Psi \otimes 1).\end{aligned}$$

Using these relations and Lemma III we obtain the proposition.

Q.E.D.

Next we shall derive a second explicit formula for ϱ_E, considering $\vee E^*$ and $\wedge E^*$ as the spaces of symmetric and skew symmetric multilinear functions in E.

Proposition IV: The Cartan map for a Lie algebra E is given by

$$\begin{aligned}&(\varrho_E \Psi)(x_1, \ldots, x_{2q-1})\\ &= \frac{(-1)^{q-1}(q-1)!}{2^{q-1}(2q-1)!} \sum_{\sigma \in S^{2q-1}} \varepsilon_\sigma \Psi(x_{\sigma(1)}, [x_{\sigma(2)}, x_{\sigma(3)}], \ldots, [x_{\sigma(2q-2)}, x_{\sigma(2q-1)}]),\\ &\qquad\qquad \Psi \in (\vee^q E^*)_{\theta=0}, \quad x_i \in E.\end{aligned}$$

Proof: Define a linear map $\varphi: \vee^q E^* \to \wedge^{2q-1} E^*$ by

$$\varphi(\Psi) = \frac{(q-1)!}{(2q-1)!} k(\delta_E k)^{q-1}(\Psi \otimes 1), \qquad \Psi \in \vee^q E^*.$$

Then, in view of Proposition III above, φ restricts to ϱ_E in $(\vee E^*)_{\theta=0}$.

On the other hand, since k is an antiderivation, a simple computation yields

$$k(\delta_E k)^{q-1}(x_1^* \vee \cdots \vee x_q^* \otimes 1) = 1 \otimes \sum_{\tau \in S^q} x_{\tau(1)}^* \wedge \delta_E x_{\tau(2)}^* \wedge \cdots \wedge \delta_E x_{\tau(q)}^*, \qquad x_i^* \in E^*.$$

It follows that for $x_i \in E$,

$$\begin{aligned}
&\langle k(\delta_E k)^{q-1}(x_1^* \vee \cdots \vee x_q^*), x_1 \wedge \cdots \wedge x_{2q-1} \rangle \\
&= \frac{1}{2^{q-1}} \sum_{\substack{\tau \in S^q \\ \sigma \in S^{2q-1}}} \varepsilon_\sigma \langle x_{\tau(1)}^*, x_{\sigma(1)} \rangle \langle \delta_E x_{\tau(2)}^*, x_{\sigma(2)} \wedge x_{\sigma(3)} \rangle \cdots \\
&\qquad \cdots \langle \delta_E x_{\tau(q)}^*, x_{\sigma(2q-2)} \wedge x_{\sigma(2q-1)} \rangle \\
&= \frac{(-1)^{q-1}}{2^{q-1}} \sum_{\substack{\tau \in S^q \\ \sigma \in S^{2q-1}}} \varepsilon_\sigma \langle x_{\tau(1)}^*, x_{\sigma(1)} \rangle \langle x_{\tau(2)}^*, [x_{\sigma(2)}, x_{\sigma(3)}] \rangle \cdots \\
&\qquad \cdots \langle x_{\tau(q)}^*, [x_{\sigma(2q-2)}, x_{\sigma(2q-1)}] \rangle \\
&= \frac{(-1)^{q-1}}{2^{q-1}} \sum_\sigma \varepsilon_\sigma \langle x_1^* \vee \cdots \vee x_q^*, x_{\sigma(1)} \vee [x_{\sigma(2)}, x_{\sigma(3)}] \vee \cdots \\
&\qquad \cdots \vee [x_{\sigma(2q-2)}, x_{\sigma(2q-1)}] \rangle.
\end{aligned}$$

Hence for $\Psi \in \vee^q E^*$,

$$\varphi\Psi(x_1, \ldots, x_{2q-1}) = \frac{(-1)^{q-1}(q-1)!}{2^{q-1}(2q-1)!} \sum_\sigma \varepsilon_\sigma \Psi(x_{\sigma(1)}, [x_{\sigma(2)}, x_{\sigma(3)}], \ldots).$$

Q.E.D.

Corollary: The linear map $\varrho: (\vee^2 E^*)_{\theta=0} \to (\wedge^3 E^*)_{\theta=0}$ of sec. 5.7 satisfies

$$\varrho = -2\varrho_E.$$

§3. The distinguished transgression

In this article E denotes a reductive Lie algebra with primitive space $P_E \subset (\wedge^+ E^*)_{\theta=0}$. We shall construct a linear map

$$\tau_E \colon P_E \to (\vee^+ E^*)_{\theta=0},$$

homogeneous of degree 1, such that

$$\varrho_E \circ \tau_E = \iota.$$

6.9. The space $W(E)_{i_I=0}$. Fix an element $a \in (\wedge^p E)_{\theta=0}$, $p \geq 1$, and consider the operator $i_E(a)$ in $W(E)$ (cf. sec. 6.2). Since a is invariant, the relations of sec. 5.1 yield

$$i_E(a)\theta_W(x) = \theta_W(x) i_E(a), \qquad x \in E.$$

Moreover, dualizing formula (5.8), sec. 5.4, and observing that a is invariant we find that

$$i_E(a)\delta_E = -i(\partial_E a) + (-1)^p \delta_E i_E(a).$$

Since E is reductive and hence unimodular, $\partial_E a = 0$ (sec. 5.10) and so the formula above becomes

$$i_E(a)\delta_E = (-1)^p \delta_E i_E(a).$$

Clearly,

$$i_E(a) \circ h = (\iota \otimes i_E(a)) \circ \sum_\nu \mu_S(e^{*\nu}) i_E(e_\nu) = (-1)^p h \circ i_E(a).$$

Now define a graded subspace $W(E)_{i_I=0}$, of $W(E)$ by

$$W(E)_{i_I=0} = \{\Omega \in W(E) \mid i_E(a)\Omega = 0,\ a \in (\wedge^+ E)_{\theta=0}\}.$$

The relations above imply that θ_W restricts to a representation of E in $W(E)_{i_I=0}$, and that the space $W(E)_{i_I=0}$ is stable under δ_E and h. Hence the invariant subspace

$$W(E)_{i_I=0,\theta=0} = W(E)_{i_I=0} \cap W(E)_{\theta=0}$$

is stable under δ_E and h. Since δ_W coincides with $h - \delta_E$ in $W(E)_{\theta=0}$ (cf. sec. 6.4) it follows that $W(E)_{i_I=0,\theta=0}$ is stable under δ_W.

Proposition V: Let E be a reductive Lie algebra. Then the inclusion map j: $(\vee E^*)_{\theta=0} \to W(E)_{i_I=0,\theta=0}$ induces an isomorphism

$$j^{\#}: (\vee E^*)_{\theta=0} \xrightarrow{\cong} H(W(E)_{i_I=0,\theta=0}, \delta_W).$$

Lemma IV: The inclusion map $(\vee E^*)_{\theta=0} \to W(E)_{i_I=0,\theta=0}$ induces an isomorphism

$$(\vee E^*)_{\theta=0} \xrightarrow{\cong} H(W(E)_{i_I=0,\theta=0}, \delta_E).$$

Proof: Let $(\wedge E^*)_{i_I=0}$ be the subspace of $\wedge E^*$ given by

$$(\wedge E^*)_{i_I=0} = \bigcap_{a \in (\wedge^+ E)_{\theta=0}} \ker i_E(a).$$

Since $\theta_E(x)i_E(a) = i_E(a)\theta_E(x)$, $x \in E$, the representation of E in $\wedge E^*$ restricts to a representation θ in $(\wedge E^*)_{i_I=0}$. Moreover, since the operators $i_E(x)$, $\theta_E(x)$, and δ_E commute (up to sign) with $i_E(a)$ for an invariant element a, the relations $\theta_E(x) = i_E(x)\delta_E + \delta_E i_E(x)$ restrict to $(\wedge E^*)_{i_I=0}$ and imply that $\theta^{\#} = 0$.

Hence, applying Theorem V, sec. 4.11, and observing that the restriction of δ_E to $(\wedge E^*)_{i_I=0,\theta=0}$ is zero, we find that the inclusion map induces an isomorphism

$$(\vee E^*)_{\theta=0} \otimes (\wedge E^*)_{i_I=0,\theta=0} \xrightarrow{\cong} H(W(E)_{i_I=0,\theta=0}, \delta_E).$$

But, in view of Lemma IX, sec. 5.22,

$$(\wedge E^*)_{i_I=0,\theta=0} = \Gamma,$$

and so the lemma follows.

Q.E.D.

Proof of the proposition: Set

$$M^p = [(\vee E^*)^p \otimes \wedge E^*]_{i_I=0,\theta=0},$$

and filter $W(E)_{i_I=0,\theta=0}$ by the subspaces $F^p = \sum_{j \geq p} M^j$. Then

$$\delta_E: M^p \to M^p \qquad \text{and} \qquad h: M^p \to M^{p+1}, \qquad p \geq 0.$$

Hence, in view of formula (1.6), sec. 1.7, the E_1-term of the corresponding spectral sequence is given by

$$E_1 \cong H(W(E)_{i_I=0,\theta=0}, \delta_E).$$

Next we filter $(\vee E^*)_{\theta=0}$ by the subspaces $\hat{F}^p = \sum_{j\geq p}(\vee E^*)^j_{\theta=0}$. Then, giving $(\vee E^*)_{\theta=0}$ the zero differential operator, we obtain a spectral sequence with

$$\hat{E}_1 = (\vee E^*)_{\theta=0}.$$

Finally, observe that the inclusion map $(\vee E^*)_{\theta=0} \to W(E)_{i_I=0,\theta=0}$, is filtration preserving. The induced map $\hat{E}_1 \to E_1$ is precisely the isomorphism of Lemma IV (cf. sec. 1.7). Thus the proposition follows from Theorem I, sec. 1.14.

Q.E.D.

6.10. The map τ_E. Lemma V: Let $\Phi \in P_E$. Then

$$\delta_W(1 \otimes \Phi) \in W(E)_{i_I=0,\theta=0}.$$

Proof: Since $1 \otimes \Phi$ is invariant, we have for $a \in (\wedge^p E)_{\theta=0}$, $p \geq 1$, that

$$i_E(a)\delta_W(1 \otimes \Phi) = i_E(a)(h - \delta_E)(1 \otimes \Phi) = (-1)^p \delta_W(1 \otimes i_E(a)\Phi).$$

On the other hand, since $P_*(E)$ generates $(\wedge E)_{\theta=0}$ (cf. Theorem II, sec. 5.16) Proposition VI, sec. 5.21 implies that $i_E(a)\Phi \in \Gamma$. Thus

$$\delta_W(1 \otimes i_E(a)\Phi) = 0.$$

Q.E.D.

In view of Lemma V, a linear map

$$\beta\colon P_E \to Z(W(E)_{i_I=0,\theta=0}, \delta_W)$$

is given by

$$\beta(\Phi) = \delta_W(1 \otimes \Phi), \qquad \Phi \in P_E.$$

Let $\beta^\#\colon P_E \to H(W(E)_{i_I=0,\theta=0}, \delta_W)$ be the induced map, and define

$$\tau_E\colon P_E \to (\vee E^*)_{\theta=0}$$

by

$$\tau_E = (j^\#)^{-1} \circ \beta^\#,$$

where $j^{\#}: (\vee E^*)_{\theta=0} \xrightarrow{\cong} H(W(E)_{i_I=0,\theta=0}, \delta_W)$ is the isomorphism of Proposition V, sec. 6.9.

Definition: τ_E is called the *distinguished transgression* for the reductive Lie algebra E.

Proposition VI: Let E be a reductive Lie algebra. Then the distinguished transgression has the following properties:

(1) It is homogeneous of degree 1.

(2) For each $\Phi \in P_E$, there exists an $\Omega \in W^+(E)_{i_I=0,\theta=0}$ such that

$$\delta_W(1 \otimes \Phi + \Omega) = \tau_E(\Phi) \otimes 1.$$

(3) $\varrho_E \circ \tau_E = \iota$.

Moreover, τ_E is uniquely determined by these properties.

Proof: (1) and (2) are immediate from the definitions. To prove (3), recall that $W(E)_{i_I=0} = \vee E^* \otimes (\wedge E^*)_{i_I=0}$, and so

$$W^+(E)_{i_I=0,\theta=0} = [\vee^+ E^* \otimes \wedge E^*]_{i_I=0,\theta=0}$$

(because $(\wedge^+ E^*)_{i_I=0,\theta=0} = 0$—cf. the proof of Lemma IV, sec. 6.9).

Now let $\Phi \in P_E$ and write

$$\tau_E \Phi \otimes 1 = \delta_W(1 \otimes \Phi + \Omega), \qquad \Omega \in W^+(E)_{i_I=0,\theta=0}.$$

Our calculation above shows that $\pi_E \Omega = 0$. Thus

$$\varrho_E \tau_E(\Phi) = \pi_E(1 \otimes \Phi) = \Phi.$$

Finally, suppose $\tau: P_E \to (\vee E^*)_{\theta=0}$ is any linear map satisfying properties (1)–(3). Then, for $\Phi \in P_E$,

$$[\tau(\Phi) - \tau_E(\Phi)] \otimes 1 = \delta_W \hat{\Omega}, \qquad \hat{\Omega} \in W^+(E)_{i_I=0,\theta=0}.$$

Now Proposition V, sec. 6.9, implies that $\tau(\Phi) = \tau_E(\Phi)$.

Q.E.D.

6.11. Homomorphisms. Let $\varphi: F \to E$ be a homomorphism of reductive Lie algebras and consider the induced maps

$$\varphi^{\wedge}_{\theta=0}: P_F \leftarrow P_E \qquad \text{and} \qquad \varphi^{\vee}_{\theta=0}: (\vee F^*)_{\theta=0} \leftarrow (\vee E^*)_{\theta=0}.$$

Then, in general, the diagram

$$\begin{tikzcd} P_E \arrow[r, "\tau_E"] \arrow[d, "\varphi^\wedge_{\theta=0}"'] & (\vee E^*)_{\theta=0} \arrow[d, "\varphi^\vee_{\theta=0}"] \\ P_F \arrow[r, "\tau_F"'] & (\vee F^*)_{\theta=0} \end{tikzcd} \tag{6.9}$$

does *not* commute (cf. Example 3, sec. 6.16).

Proposition VII: If $\varphi: F \to E$ is a surjective homomorphism of reductive Lie algebras, then the diagram (6.9) commutes.

Proof: We have the relations

$$\theta_W(y) \circ \varphi_W = \varphi_W \circ \theta_W(\varphi y) \text{ and } i_F(b) \circ \varphi_W = \varphi_W \circ i_E(\varphi_\wedge b) \quad y \in F, b \in \wedge F.$$

Since φ is surjective, $\varphi_\wedge$ maps $(\wedge F)_{\theta=0}$ into $(\wedge E)_{\theta=0}$. It follows that the map φ_W: $W(F) \leftarrow W(E)$ restricts to a homomorphism

$$(\varphi_W)_{i_I=0,\theta=0}: W(F)_{i_I=0,\theta=0} \leftarrow W(E)_{i_I=0,\theta=0}.$$

The diagram

$$\begin{tikzcd} (\vee E^*)_{\theta=0} \arrow[r, "\cong"] \arrow[d, "\varphi^\vee_{\theta=0}"'] & H(W(E)_{i_I=0,\theta=0}) \arrow[d, "(\varphi_W)^\#_{i_I=0,\theta=0}"] & P_E \arrow[l, "\beta_E^\#"'] \arrow[d, "\varphi^\wedge_{\theta=0}"] \\ (\vee F^*)_{\theta=0} \arrow[r, "\cong"'] & H(W(F)_{i_I=0,\theta=0}) & P_F \arrow[l, "\beta_F^\#"] \end{tikzcd}$$

clearly commutes, and the proposition follows.

Q.E.D.

§4. The structure theorem for $(\vee E^*)_{\theta=0}$

In this article E denotes a reductive Lie algebra.

6.12. The filtration of $W(E)_{\theta=0}$. Consider the ideals

$$F^p(W(E)_{\theta=0}) = \sum_{j\geq p} [(\vee E^*)^j \otimes \wedge E^*]_{\theta=0}.$$

They make $W(E)_{\theta=0}$ into a graded filtered differential algebra (cf. sec. 1.18).

Since in $W(E)_{\theta=0}$, $\delta_W = h - \delta_E$, and since

$$\delta_E\colon [(\vee E^*)^p \otimes \wedge E^*]_{\theta=0} \to [(\vee E^*)^p \otimes \wedge E^*]_{\theta=0}$$

and

$$h\colon [(\vee E^*)^p \otimes \wedge E^*]_{\theta=0} \to [(\vee E^*)^{p+2} \otimes \wedge E^*]_{\theta=0},$$

it follows that the spectral sequence associated with the filtration begins with

$$(E_0, d_0) = (W(E)_{\theta=0}, -\delta_E) \tag{6.10}$$

(cf. sec. 1.7). Hence

$$E_1 \cong H(W(E)_{\theta=0}, -\delta_E).$$

As an immediate consequence of Theorem V, sec. 4.11, we have

Lemma VI: The inclusion map $(\vee E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to W(E)_{\theta=0}$ induces an isomorphism

$$(\vee E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H(W(E)_{\theta=0}, -\delta_E).$$

6.13. Transgression. A *transgression* in the filtered differential algebra $W(E)_{\theta=0}$ is a linear map

$$\tau\colon P_E \to (\vee E^*)_{\theta=0}$$

with the following properties:

(1) τ is homogeneous of degree 1.

(2) For every $\Phi \in P_E$, there is an element $\Omega \in W^+(E)_{\theta=0}$ such that

$$\delta_W \Omega = \tau\Phi \otimes 1 \qquad \text{and} \qquad 1 \otimes \Phi - \Omega \in F^1(W(E)_{\theta=0}).$$

Lemma VII: (1) A linear map $\tau: P_E \to (\vee E^*)_{\theta=0}$, homogeneous of degree 1, is a transgression if and only if it satisfies $\varrho_E \circ \tau = \iota$.

(2) The distinguished transgression τ_E defined in sec. 6.10 is a transgression. In particular, a transgression always exists.

Proof: (1) follows from the relation

$$\ker(\pi_E)_{\theta=0} = F^1(W(E)_{\theta=0})$$

(cf. sec. 6.7). (2) is a consequence of Proposition VI, sec. 6.10.

Q.E.D.

Theorem I: Let E be reductive and let $\tau: P_E \to (\vee E^*)_{\theta=0}$ be a transgression. Let $\tilde{P}_E$ be the evenly graded space defined by $\tilde{P}_E^h = P_E^{h-1}$. Then the induced homomorphism

$$\tau_\vee: \vee \tilde{P}_E \to (\vee E^*)_{\theta=0}$$

is an isomorphism of graded algebras.

In particular, $(\vee E^*)_{\theta=0}$ is a symmetric algebra over an evenly graded vector space whose dimension is the rank of E.

Proof: Consider the P_E-algebra $((\vee E^*)_{\theta=0}; \tau)$. In view of Theorem I, sec. 2.8, it is sufficient to show that

$$(\tau_\vee \otimes \iota)^\#: H(\vee \tilde{P}_E \otimes \wedge P_E) \to H((\vee E^*)_{\theta=0} \otimes \wedge P_E, \nabla_\tau)$$

is an isomorphism. But

$$H(\vee \tilde{P}_E \otimes \wedge P_E) = H^0(\vee \tilde{P}_E \otimes \wedge P_E) = \Gamma$$

(cf. sec. 2.6), and so we are reduced to proving that

$$H^+((\vee E^*)_{\theta=0} \otimes \wedge P_E, \nabla_\tau) = 0. \tag{6.11}$$

Since τ is a transgression there is a linear map $\alpha: P_E \to W(E)_{\theta=0}$, homogeneous of degree zero, such that

$$\delta_W \alpha(\Phi) = \tau(\Phi) \otimes 1 \qquad \text{and} \qquad \alpha(\Phi) - 1 \otimes \Phi \in F^1(W(E)_{\theta=0}). \tag{6.12}$$

Extend α to a homomorphism $\alpha_\wedge: \wedge P_E \to W(E)_{\theta=0}$, and define a homomorphism

$$\sigma: (\vee E^*)_{\theta=0} \otimes \wedge P_E \to W(E)_{\theta=0},$$

by setting

$$\sigma(\Psi \otimes \Phi) = (\Psi \otimes 1) \cdot \alpha_\wedge(\Phi).$$

A straightforward computation, using (6.12), shows that

$$\sigma \circ \nabla_\tau = \delta_W \circ \sigma.$$

Thus σ induces a homomorphism

$$\sigma^\#: H((\vee E^*)_{\theta=0} \otimes \wedge P_E, \nabla_\tau) \to H(W(E)_{\theta=0}, \delta_W).$$

Now filter $(\vee E^*)_{\theta=0} \otimes \wedge P_E$ by the ideals

$$\hat{F}^p = \sum_{j \geq p} (\vee E^*)^j_{\theta=0} \otimes \wedge P_E.$$

The corresponding spectral sequence starts off with

$$(\hat{E}_0, \hat{d}_0) = ((\vee E^*)_{\theta=0} \otimes \wedge P_E, 0).$$

Moreover, σ is filtration preserving with respect to the filtration of sec. 6.12 and the filtration above. To compute the map $\sigma_0: \hat{E}_0 \to E_0$, first write it in the form

$$\sigma_0: ((\vee E^*)_{\theta=0} \otimes \wedge P_E, 0) \to (W(E)_{\theta=0}, -\delta_E)$$

(cf. formula (6.10), sec. 6.12). Use the isomorphism $\varkappa_E$ of Theorem III, sec. 5.18, to identify $\wedge P_E$ with $(\wedge E^*)_{\theta=0}$. Then, in view of formula (6.12),

$$\sigma_0(\Psi \otimes \Phi) = \Psi \otimes \Phi, \qquad \Psi \in (\vee E^*)_{\theta=0}, \quad \Phi \in (\wedge E^*)_{\theta=0}.$$

Thus σ_0 is simply the inclusion map

$$(\vee E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to W(E)_{\theta=0},$$

and so, by Lemma VI, sec. 6.12, $\sigma_0^\#$ is an isomorphism. Hence, by the comparison theorem (sec. 1.14), $\sigma^\#$ is an isomorphism.

Finally, observe that, since $\sigma^\#$ is an isomorphism,

$$H^+((\vee E^*)_{\theta=0} \otimes \wedge P_E, \nabla_\tau) = H^+(W(E)_{\theta=0}, \delta_W) = 0$$

(cf. Proposition I, sec. 6.6), and so (6.11) is proved.

Q.E.D.

Corollary: If $f_{H(E)} = \prod_{i=1}^{r} (1 + t^{g_i})$ is the Poincaré polynomial of $(\wedge E^*)_{\theta=0}$, then the Poincaré series of $(\vee E^*)_{\theta=0}$ is given by

$$f_{(\vee E^*)_{\theta=0}} = \prod_{i=1}^{r} (1 - t^{g_i+1})^{-1}.$$

6.14. The image and kernel of ρ_E. Theorem II: Let E be reductive. Then the image and the kernel of the Cartan map are given by

$$\ker \varrho_E = (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0} \qquad \text{and} \qquad \operatorname{Im} \varrho_E = P_E.$$

Proof: We first show that

$$(\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0} \subset \ker \varrho_E. \tag{6.13}$$

Let $\Psi_1, \Psi_2 \in (\vee^+ E^*)_{\theta=0}$. Choose $\Omega_1 \in W^+(E)_{\theta=0}$ so that $\delta_W \Omega_1 = \Psi_1 \otimes 1$. Then since $\delta_W(\Psi_2 \otimes 1) = 0$,

$$\delta_W(\Omega_1 \cdot (\Psi_2 \otimes 1)) = \Psi_1 \vee \Psi_2 \otimes 1,$$

whence

$$\varrho_E(\Psi_1 \vee \Psi_2) = \pi_E \Omega_1 \wedge \pi_E(\Psi_2 \otimes 1) = 0.$$

This proves (6.13).

Next let τ be a transgression in $W(E)_{\theta=0}$. Then Lemma VII, sec. 6.13, and Theorem I, sec. 6.13, yield respectively the relations

$$\varrho_E \circ \tau = \iota \qquad \text{and} \qquad (\vee^+ E^*)_{\theta=0} = \tau(P_E) \oplus (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0}.$$

Theorem II is an immediate consequence of these relations and formula (6.13).

Q.E.D.

Corollary: A linear map τ homogeneous of degree 1 is a transgression if and only if

$$\tau - \tau_E : P_E \to (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0},$$

where τ_E is the distinguished transgression (cf. sec. 6.10).

Proof: Since $\ker \varrho_E = (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0}$, the condition of the corollary is equivalent to

$$\varrho_E \tau = \varrho_E \tau_E = \iota.$$

Now apply Lemma VII, sec. 6.13.

Q.E.D.

6.15. Homomorphisms. Proposition VIII: Let $\varphi: F \to E$ be a homomorphism of reductive Lie algebras. Then the following conditions are equivalent:

(1) $\varphi^{\#}: H^*(F) \leftarrow H^*(E)$ is surjective.

(2) $\varphi^{\wedge}_{\theta=0}: (\wedge F^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0}$ is surjective.

(3) $\varphi_P: P_F \leftarrow P_E$ is surjective.

(4) $\varphi^{\vee}_{\theta=0}: (\vee F^*)_{\theta=0} \leftarrow (\vee E^*)_{\theta=0}$ is surjective.

(5) There are transgressions σ and τ in $W(F)_{\theta=0}$ and $W(E)_{\theta=0}$ such that

$$\varphi^{\vee}_{\theta=0} \circ \tau = \sigma \circ \varphi^{\wedge}_{\theta=0}.$$

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): This follows from Corollary I to Theorem III, sec. 5.19.

(3) $\Rightarrow$ (4): Choose a linear map $\alpha: P_F \to P_E$, homogeneous of degree zero, so that

$$\varphi_P \circ \alpha = \iota.$$

Let τ be any transgression in $W(E)_{\theta=0}$ and define a linear map $\sigma: P_F \to W(F)_{\theta=0}$ by

$$\sigma = \varphi^{\vee}_{\theta=0} \circ \tau \circ \alpha.$$

Then σ is homogeneous of degree 1, and by Proposition II, sec. 6.7,

$$\varrho_F \circ \sigma = \varphi^{\wedge}_{\theta=0} \circ \varrho_E \circ \tau \circ \alpha = \varphi_P \circ \alpha = \iota.$$

Thus, by Lemma VII, sec. 6.13, σ is a transgression in $W(F)_{\theta=0}$.

It follows from Theorem I that $\sigma(P_F)$ generates $(\vee F^*)_{\theta=0}$. Since

$$\sigma(P_F) \subset \operatorname{Im} \varphi^{\vee}_{\theta=0},$$

this shows that $\varphi^{\vee}_{\theta=0}$ is surjective.

(4) $\Rightarrow$ (3): Observe that, by Proposition II, sec. 6.7,

$$\varphi^{\wedge}_{\theta=0} \circ \varrho_E = \varrho_F \circ \varphi^{\vee}_{\theta=0}.$$

Since $\varphi^{\vee}_{\theta=0}$ is surjective and $\operatorname{Im} \varrho_E = P_E$, $\operatorname{Im} \varrho_F = P_F$ (cf. Theorem II, sec. 6.14), it follows that

$$\varphi_P(P_E) = \varphi^{\wedge}_{\theta=0}(P_E) = P_F.$$

Thus φ_P is surjective.

$(4) \Rightarrow (5)$: Let τ_1 be any transgression in $W(E)_{\theta=0}$. Then, for $\Phi \in \ker \varphi_P$,

$$\varrho_F \varphi^{\vee}_{\theta=0} \tau_1(\Phi) = \varphi^{\wedge}_{\theta=0} \varrho_E \tau_1(\Phi) = \varphi^{\wedge}_{\theta=0}(\Phi) = 0.$$

Hence, in view of Theorem II, sec. 6.14,

$$\varphi^{\vee}_{\theta=0} \circ \tau_1 : \ker \varphi_P \to (\vee^+ F^*)_{\theta=0} \cdot (\vee^+ F^*)_{\theta=0}.$$

Since $\varphi^{\vee}_{\theta=0}$ is surjective, it follows that there is a linear map

$$\beta : \ker \varphi_P \to (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0},$$

homogeneous of degree 1, and such that

$$\varphi^{\vee}_{\theta=0}(\tau_1(\Phi) + \beta(\Phi)) = 0, \qquad \Phi \in \ker \varphi_P.$$

Now choose a graded subspace $P \subset P_E$ so that

$$P_E = P \oplus \ker \varphi_P.$$

Define $\tau : P_E \to (\vee^+ E^*)_{\theta=0}$ by

$$\tau(\Phi) = \tau_1(\Phi), \qquad \Phi \in P,$$

and

$$\tau(\Phi) = \tau_1(\Phi) + \beta(\Phi), \qquad \Phi \in \ker \varphi_P.$$

Then, in view of the corollary to Theorem II, sec. 6.14, τ is a transgression in $W(E)_{\theta=0}$. Moreover, by the definition of β,

$$\tau : \ker \varphi_P \to \ker \varphi^{\vee}_{\theta=0}.$$

On the other hand, since $(4) \Rightarrow (3)$, φ_P is surjective. Thus φ_P restricts to an isomorphism $P \xrightarrow{\cong} P_F$. Let $\alpha : P_F \xrightarrow{\cong} P$ denote the inverse isomorphism. Define $\sigma : P_F \to (\vee F^*)_{\theta=0}$ by

$$\sigma = \varphi^{\vee}_{\theta=0} \circ \tau \circ \alpha.$$

Then $\varrho_F \circ \sigma = \iota$ (as in the proof that $(3) \Rightarrow (4)$) and so σ is a transgression in $W(F)_{\theta=0}$.

Finally, since

$$(\varphi^{\vee}_{\theta=0} \circ \tau)\Phi = 0, \qquad (\sigma \circ \varphi^{\wedge}_{\theta=0})\Phi = 0, \qquad \Phi \in \ker \varphi_P$$

and

$$(\varphi^{\vee}_{\theta=0} \circ \tau)\Phi = (\varphi^{\vee}_{\theta=0} \circ \tau) \circ (\alpha \circ \varphi_P)\Phi = (\sigma \circ \varphi^{\wedge}_{\theta=0})\Phi, \qquad \Phi \in P,$$

it follows that $\varphi^{\vee}_{\theta=0} \circ \tau = \sigma \circ \varphi^{\wedge}_{\theta=0}$.

(5) $\Rightarrow$ (4): Suppose τ and σ are transgressions satisfying (5). Then $\operatorname{Im} \sigma \subset \operatorname{Im} \varphi^{\vee}_{\theta=0}$. But by Theorem II, sec. 6.14, $\operatorname{Im} \sigma$ generates $(\vee F^*)_{\theta=0}$. Hence $\varphi^{\vee}_{\theta=0}$ is surjective.

Q.E.D.

6.16. Examples. 1. Let E be the Lie algebra of linear transformations with trace zero in a 2-dimensional vector space. Then, with respect to a suitable basis, h, e, f, of E,

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Since E is semisimple, it follows from Proposition V, sec. 5.20, that P_E is a 1-dimensional subspace of degree 3. Moreover, if K denotes the Killing form of E, then $\varrho_E(K) = -\frac{1}{2}\varrho(K) \neq 0$ (cf. sec. 6.8 and Proposition I, sec. 5.7). Thus, by Theorem I, sec. 6.13, $(\vee E^*)_{\theta=0}$ consists of the polynomials in K; i.e., $1, K, K^2, \dots, K^p, \dots$ is a basis for $(\vee^* E)_{\theta=0}$.

2. Let F be the abelian subalgebra of E (cf. Example 1) spanned by h. Then F is reductive in E. Moreover,

$$(\wedge F^*)_{\theta=0} = \wedge F^* = \wedge(h^*) \qquad \text{and} \qquad (\vee F^*)_{\theta=0} = \vee F^* = \vee(h^*).$$

Thus $(\vee F^*)_{\theta=0}$ consists of the polynomials in h^*.

Now consider the inclusion map $\varphi: F \to E$. Then, since $P_E = P_E^3$ and $P_F = P_F^1$, it follows that the restriction of $\varphi^{\wedge}_{\theta=0}$ to P_E is zero, $\varphi_P = 0$.

On the other hand, let e^*, f^*, h^* be the basis for E^* dual to e, f, h. Then

$$K = 4(h^* \vee h^* + e^* \vee f^*),$$

and hence

$$\varphi^{\vee}_{\theta=0}(K) = 4(h^*)^2.$$

Note that in this case none of the conditions of Proposition VIII, sec. 6.15, can hold.

3. Consider the Lie algebra $L = E \oplus F$, where E and F are the Lie algebras of Examples 1 and 2. Define $\psi: F \to L$ and $\pi: L \to F$ by

$$\psi(y) = \varphi y \oplus y, \quad y \in F \qquad \text{and} \qquad \pi(x \oplus y) = y, \quad x \in E, \quad y \in F.$$

Then $\pi \circ \psi = \iota$ and so $\psi^\vee_{\theta=0} \circ \pi^\vee_{\theta=0} = \iota$.

In particular, $\psi^\vee_{\theta=0}$ is surjective. Hence, Proposition VIII, sec. 6.15, shows that, for appropriate transgressions τ and σ in $W(L)_{\theta=0}$ and $W(F)_{\theta=0}$,

$$\psi^\vee_{\theta=0} \circ \tau = \sigma \circ \psi^\wedge_{\theta=0}.$$

Nonetheless $\psi^\vee_{\theta=0} \circ \tau_L \neq \tau_F \circ \psi^\wedge_{\theta=0}$, where τ_L and τ_F are the distinguished transgressions. Indeed, write

$$P_L = P_E \oplus P_F \qquad \text{and} \qquad (\vee L^*)_{\theta=0} = (\vee E^*)_{\theta=0} \otimes (\vee F^*)_{\theta=0}.$$

Then it follows immediately from the definitions that

$$\tau_L(\Phi_1 \oplus \Phi_2) = \tau_E(\Phi_1) \otimes 1 + 1 \otimes \tau_F(\Phi_2), \qquad \Phi_1 \in P_E, \quad \Phi_2 \in P_F.$$

Hence

$$(\psi^\vee_{\theta=0} \circ \tau_L)(\Phi_1 \oplus \Phi_2) = (\varphi^\vee_{\theta=0} \circ \tau_E)\Phi_1 + \tau_F(\Phi_2).$$

On the other hand,

$$(\tau_F \circ \psi^\wedge_{\theta=0})(\Phi_1 \oplus \Phi_2) = (\tau_F \circ \varphi^\wedge_{\theta=0})\Phi_1 + \tau_F(\Phi_2) = \tau_F(\Phi_2).$$

Thus, if $\Phi_1 \neq 0$, it follows from Example 2 that

$$(\psi^\vee_{\theta=0} \circ \tau_L)(\Phi_1 \oplus \Phi_2) \neq (\tau_F \circ \psi^\wedge_{\theta=0})(\Phi_1 \oplus \Phi_2).$$

§5. The structure theorem for $(\vee E)_{\theta=0}$, and duality

In this article E denotes a reductive Lie algebra.

6.17. The structure of $(\vee E)_{\theta=0}$. Recall from sec. 6.1 the representation θ^S of E in $\vee E$.

Theorem III: Let E be a reductive Lie algebra. Then the graded algebras $(\vee E)_{\theta=0}$ and $(\vee E^*)_{\theta=0}$ are isomorphic. In particular, $(\vee E)_{\theta=0}$ is a symmetric algebra over an evenly graded vector space, whose dimension is the rank of E.

Proof: By Theorem II, sec. 4.4, there is an inner product $(\ ,\ )$ in E such that

$$([x, y], z) + (y, [x, z]) = 0, \qquad x, y, z \in E.$$

Denote the corresponding linear isomorphism by $\alpha: E \xrightarrow{\cong} E^*$,

$$\langle \alpha(x), y \rangle = (x, y), \qquad x, y \in E.$$

Then $\alpha \circ \operatorname{ad} x = -(\operatorname{ad} x)^* \circ \alpha$, whence

$$\alpha_\vee \circ \theta^S(x) = \theta_S(x) \circ \alpha_\vee, \qquad x \in E.$$

Thus $\alpha_\vee$ restricts to an isomorphism $(\vee E)_{\theta=0} \xrightarrow{\cong} (\vee E^*)_{\theta=0}$. Now the theorem follows from Theorem I, sec. 6.13.

Q.E.D.

6.18. Duality. In this section it will be shown that, in contrast with the results on exterior algebra (cf. article 6, Chapter V), it is in general impossible to choose dual generating subspaces $U^* \subset (\vee E^*)_{\theta=0}$ and $U \subset (\vee E)_{\theta=0}$ such that the isomorphisms

$$\vee U^* \xrightarrow{\cong} (\vee E^*)_{\theta=0} \qquad \text{and} \qquad \vee U \xrightarrow{\cong} (\vee E)_{\theta=0}$$

preserve the scalar products.

First recall from sec. 5.9 that the diagonal map $\Delta: E \to E \oplus E$ is a homomorphism of Lie algebras. Hence Δ induces a homomorphism

$$\Delta_{\theta=0}^{\vee}: (\vee E^*)_{\theta=0} \leftarrow (\vee E^*)_{\theta=0} \otimes (\vee E^*)_{\theta=0}.$$

In view of the duality between $(\vee E^*)_{\theta=0}$ and $(\vee E)_{\theta=0}$ there is a dual map

$$(\Delta_{\theta=0}^{\vee})^*: (\vee E)_{\theta=0} \xrightarrow{\cong} (\vee E)_{\theta=0} \otimes (\vee E)_{\theta=0}.$$

Proposition IX: Let E be a reductive Lie algebra. Assume that there are subspaces $U^* \subset (\vee E^*)_{\theta=0}$ and $U \subset (\vee E)_{\theta=0}$ such that the inclusion maps induce isomorphisms

$$\psi: \vee U^* \xrightarrow{\cong} (\vee E^*)_{\theta=0} \qquad \text{and} \qquad \varphi: \vee U \xrightarrow{\cong} (\vee E)_{\theta=0},$$

which preserve the scalar products. Then the map $(\Delta_{\theta=0}^{\vee})^*$ is a homomorphism.

Proof: The diagonal map $D: U \to U \oplus U$ and its dual D^* extend to dual homomorphisms

$$D_{\vee}: \vee U \to \vee U \otimes \vee U \qquad \text{and} \qquad D^{\vee}: \vee U^* \leftarrow \vee U^* \otimes \vee U^*.$$

Moreover, since $D^{\vee}(u^* \otimes v^*) = u^* \vee v^*$, $u^*, v^* \in \vee U^*$, it follows that

$$\psi(u^*) \vee \psi(v^*) = \psi D^{\vee}(u^* \otimes v^*).$$

On the other hand, for $\Phi, \Psi \in (\vee E^*)_{\theta=0}$,

$$\Phi \vee \Psi = \Delta_{\theta=0}^{\vee}(\Phi \otimes \Psi).$$

Hence we have

$$\psi D^{\vee}(u^* \otimes v^*) = \Delta_{\theta=0}^{\vee}(\psi u^* \otimes \psi v^*),$$

and thus $\psi D^{\vee} = \Delta_{\theta=0}^{\vee} \circ (\psi \otimes \psi)$.

Dualizing this formula and observing that $\psi^* = \varphi^{-1}$, we obtain

$$(\Delta_{\theta=0}^{\vee})^* = (\varphi \otimes \varphi) \circ D_{\vee} \circ \varphi^{-1}.$$

This shows that $(\Delta_{\theta=0}^{\vee})^*$ preserves products.

Q.E.D.

Proposition X: If E is semisimple, then the map $(\Delta_{\theta=0}^{\vee})^*$ is not a homomorphism.

Proof: First observe that

$$(\Delta^{\vee}_{\theta=0})^{*} = \pi \circ \Delta_{\vee},$$

where $\pi: \vee E \otimes \vee E \to (\vee E)_{\theta=0} \otimes (\vee E)_{\theta=0}$ denotes the projection with kernel $\theta_{E\oplus E}(\vee E \otimes \vee E)$.

Next, since the Killing form of E is nondegenerate, it induces, as in the previous section, an E-linear isomorphism

$$\alpha_{\vee}: \vee E \xrightarrow{\cong} \vee E^{*}.$$

In particular, $\alpha_{\vee}$ restricts to an isomorphism $(\vee E)_{\theta=0} \xrightarrow{\cong} (\vee E^{*})_{\theta=0}$.

Let $u \in (\vee^2 E)_{\theta=0}$ be the element given by $u = \alpha_{\vee}^{-1}(K)$, K the Killing form. A straightforward computation shows that

$$\Delta_{\vee}(u) = u \otimes 1 + w + 1 \otimes u,$$

where $w \in E \otimes E \subset \vee E \otimes \vee E$. Since E is semisimple, $E = \theta(E)$. Hence

$$E \otimes E = \theta_{E\oplus E}(E \otimes E) \subset \ker \pi,$$

and so $\pi(w) = 0$. It follows that

$$(\Delta^{\vee}_{\theta=0})^{*}(u) = u \otimes 1 + 1 \otimes u.$$

On the other hand, since $\Delta_{\vee}$ is an algebra homomorphism

$$\begin{aligned}\Delta_{\vee}(u \vee u) &= (u \otimes 1 + w + 1 \otimes u)^2 \\ &= (u \otimes 1 + 1 \otimes u)^2 + 2(u \otimes 1 + 1 \otimes u) \vee w + w^2.\end{aligned}$$

Applying π to this equation we obtain

$$(\Delta^{\vee}_{\theta=0})^{*}(u \vee u) = (u \otimes 1 + 1 \otimes u)^2 + \pi(w^2).$$

Hence it remains to be shown that $\pi(w^2) \neq 0$; i.e., that w^2 is not orthogonal to the space $(\vee E)_{\theta=0} \otimes (\vee E)_{\theta=0}$ with respect to the scalar product $(\ ,\ )$ induced by the Killing form. We shall prove that

$$(u \otimes u, w^2) \neq 0.$$

In fact, let e_{λ}, e^{μ} be a pair of dual bases of E (with respect to the Killing form). Then

$$u = \tfrac{1}{2} \sum_{\mu} e_{\mu} \vee e^{\mu},$$

and so

$$w = \tfrac{1}{2} \sum_{\mu} (e_\mu \otimes e^\mu + e^\mu \otimes e_\mu) = \sum_{\mu} e_\mu \otimes e^\mu.$$

It follows that

$$w^2 = \sum_{\mu,\lambda} e_\mu \vee e^\lambda \otimes e^\mu \vee e_\lambda .$$

Finally, it follows from the definitions that $(u, x \vee y) = (x, y)$. Hence

$$(u \otimes u, w^2) = \sum_{\mu,\lambda} (e_\mu , e^\lambda)(e^\mu, e_\lambda) = \dim E \neq 0.$$

Thus $\pi(w^2) \neq 0$, and so $(\varDelta^{\vee}_{\theta=0})^*$ is not a homomorphism.

Q.E.D.

Corollary: If E is a semisimple Lie algebra, it is not possible to choose dual subspaces $U^* \subset (\vee E^*)_{\theta=0}$ and $U \subset (\vee E)_{\theta=0}$ such that the isomorphisms $\vee U^* \xrightarrow{\cong} (\vee E^*)_{\theta=0}$ and $\vee U \xrightarrow{\cong} (\vee E)_{\theta=0}$ preserve the scalar products.

§6. Cohomology of the classical Lie algebras

6.19. The Lie algebra $L(n)$. Let X be an n-dimensional vector space and consider the Lie algebra $L(n)$ of linear transformations of X. According to Example 1, sec. 4.6, $L(n)$ is reductive.

In sec. A.2 and sec. A.3 the invariant, symmetric p-linear functions C_p, $\mathrm{Tr}_p \in (\vee^p L(n)^*)_{\theta=0}$ ($p = 1, \ldots, n$), are introduced. On the other hand, consider the invariant skew symmetric functions

$$\Phi_{2p-1} \in (\wedge^{2p-1} L(n)^*)_{\theta=0} \ (p = 1, \ldots, n)$$

defined by

$$\Phi_{2p-1}(\alpha_1, \ldots, \alpha_{2p-1}) = \sum_{\sigma \in S^{2p-1}} \varepsilon_\sigma \operatorname{tr} \alpha_{\sigma(1)} \circ \cdots \circ \alpha_{\sigma(2p-1)}, \qquad \alpha_i \in L(n).$$

It follows directly from Proposition IV, sec. 6.8, and the definition of Tr_p that

$$\varrho_{L(n)}(\mathrm{Tr}_p) = (-1)^{p-1} \frac{p!(p-1)!}{(2p-1)!} \Phi_{2p-1}, \qquad 1 \leq p \leq n, \quad (6.14)$$

where $\varrho_{L(n)}$ is the Cartan map for $L(n)$. In particular (cf. Theorem II, sec. 6.14) the Φ_{2p-1} are primitive.

Using that same theorem, and Proposition III, sec. A.3, we see that

$$C_p + \frac{(-1)^p}{p} \mathrm{Tr}_p \in (\vee^+ L(n)^*)_{\theta=0} \cdot (\vee^+ L(n)^*)_{\theta=0} = \ker \varrho_{L(n)}.$$

It follows that

$$\varrho_{L(n)}(C_p) = \frac{((p-1)!)^2}{(2p-1)!} \Phi_{2p-1}. \qquad (6.15)$$

We shall show that the Φ_{2p-1} are a basis for the primitive space of $L(n)$ (cf. Theorem IV below). First we need

Lemma VIII: $\Phi_{2p-1} \neq 0$, $1 \leq p \leq n$.

Proof: Let e^{*i}, e_i be dual bases for X^* and X. Define $\alpha, \beta_i, \gamma_i \in L(n)$ by

$$\alpha(x) = \langle e^{*1}, x\rangle e_1, \ \beta_i(x) = \langle e^{*i-1}, x\rangle e_i, \ \gamma_i(x) = \langle e^{*i}, x\rangle e_{i-1}, \ 2 \leq i \leq p.$$

Then the only nonzero products of these transformations are given by

$$\beta_2 \circ \alpha, \quad \alpha \circ \gamma_2, \quad \beta_{i+1} \circ \beta_i, \quad \gamma_i \circ \beta_i, \quad \gamma_i \circ \gamma_{i+1}, \quad \beta_i \circ \gamma_i. \tag{6.16}$$

Now consider a nonzero product of these $2p - 1$ transformations (each occurring once) with α the pth factor. In view of (6.16) it must be

$$\beta_p \circ \beta_{p-1} \circ \cdots \circ \beta_2 \circ \alpha \circ \gamma_2 \circ \cdots \circ \gamma_p.$$

It follows that, in the formula for $\Phi_{2p-1}(\beta_p, \ldots, \alpha, \ldots, \gamma_p)$, the only terms that occur come from cyclic permutations of this order.

Since cyclic permutations of an odd number of elements are even, and since the trace is unaffected by a cyclic permutation, this implies that

$$\Phi_{2p-1}(\beta_p, \ldots, \alpha, \ldots, \gamma_p) = (2p - 1) \operatorname{tr} \beta_p \circ \cdots \alpha \cdots \circ \gamma_p = 2p - 1.$$

Hence $\Phi_{2p-1} \neq 0$.

Q.E.D.

Assign $(\vee L(n)^*)_{\theta=0}$ the even gradation of sec. 6.1.

Theorem IV: (1) The elements Φ_{2p-1} $(1 \leq p \leq n)$ form a basis for the primitive space of $L(n)$. In particular, $L(n)$ has rank n.

(2) $(\vee L(n)^*)_{\theta=0}$ is the symmetric algebra over the graded subspace C spanned by the elements $C_1, \ldots, C_n$. Similarly, $(\vee L(n)^*)_{\theta=0}$ is the symmetric algebra over the subspace T spanned by the elements $\mathrm{Tr}_1, \ldots, \mathrm{Tr}_n$.

(3) The Poincaré polynomial for $H^*(L(n))$ and the Poincaré series for $(\vee L(n)^*)_{\theta=0}$ are given, respectively, by

$$\prod_{p=1}^{n} (1 + t^{2p-1}) \qquad \text{and} \qquad \prod_{p=1}^{n} (1 - t^{2p})^{-1}.$$

Proof: (1) Since the Φ_{2p-1} are nonzero (cf. Lemma VIII) and have different degrees, they are linearly independent. Since they are primitive, and since

$$\sum_{p=1}^{n} \deg \Phi_{2p-1} = \sum_{p=1}^{n} (2p - 1) = n^2 = \dim L(n),$$

they form a basis of the primitive space.

(2) In view of (1), formulae (6.14) and (6.15) show that there are linear isomorphisms, homogeneous of degree 1,

$$\tau_C \colon P_{L(n)} \xrightarrow{\cong} C \qquad \text{and} \qquad \tau_T \colon P_{L(n)} \xrightarrow{\cong} T,$$

such that $\varrho_{L(n)} \circ \tau_C = \iota$ and $\varrho_{L(n)} \circ \tau_T = \iota$. Lemma VII, (1), sec. 6.13, implies that these maps are transgressions. Now apply Theorem I, sec. 6.13.

(3) This follows immediately from (1) and (2).

Q.E.D.

Example: The primitive space of $L(2)$ is spanned by the elements Φ_1 and Φ_3, where

$$\Phi_1(\alpha) = \operatorname{tr} \alpha \qquad \text{and} \qquad \Phi_3(\alpha, \beta, \gamma) = \operatorname{tr}(\alpha \circ [\beta, \gamma]), \qquad \alpha, \beta, \gamma \in L(n).$$

Thus the Poincaré polynomial for $H^*(L(2))$ and the Poincaré series for $(\vee L(2)^*)_{\theta=0}$ are given by

$$(1 + t)(1 + t^3) = 1 + t + t^3 + t^4$$

and

$$(1 - t^2)^{-1}(1 - t^4)^{-1} = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + \cdots .$$

It follows that

$$\dim(\vee^{2p} L(2)^*)_{\theta=0} = p + 1 = \dim(\vee^{2p+1} L(n)^*)_{\theta=0}.$$

6.20. The Lie algebra $L^0(n)$. Let $L^0(n)$ be the Lie algebra of linear transformations of X ($\dim X = n$) with trace zero. Denote the restrictions of Tr_p, C_p, and Φ_{2p-1} to $L^0(n)$ by Tr_p^0, C_p^0, and Φ_{2p-1}^0. Then the direct decomposition $L(n) = (\iota) \oplus L^0(n)$ together with Theorem IV, sec. 6.19, yields (with $(\vee L^0(n)^*)_{\theta=0}$ graded as in sec. 6.1)

Theorem V: (1) The elements Φ_{2p-1}^0 ($2 \le p \le n$) form a basis of the primitive space of $L^0(n)$. In particular, $L^0(n)$ has rank $n - 1$.

(2) $(\vee L^0(n)^*)_{\theta=0}$ is the symmetric algebra over the graded subspace spanned by the elements $C_2^0, \dots, C_n^0$. It is also the symmetric algebra over the graded subspace spanned by $\operatorname{Tr}_2^0, \dots, \operatorname{Tr}_n^0$.

(3) The Poincaré polynomial for $H^*(L^0(n))$ and the Poincaré series for $(\vee L^0(n)^*)_{\theta=0}$, respectively, are given by

$$\prod_{p=2}^{n} (1 + t^{2p-1}) \qquad \text{and} \qquad \prod_{p=2}^{n} (1 - t^{2p})^{-1}.$$

Corollary: The Lie algebra $L^0(n)$ is simple.

Proof: Apply Proposition V, sec. 5.20.

Q.E.D.

6.21. The Lie algebra Sk(n). In this and the next two sections we consider Euclidean spaces. The results obtained, however, (and the proofs) are valid for any inner product space over an arbitrary field of characteristic zero. (Although, in that setting, the Lie algebra depends on more than the dimension of the space!)

Let $(X, \langle\ ,\ \rangle)$ be an n-dimensional Euclidean space, and denote by $\mathrm{Sk}(n)$ the Lie algebra of skew linear transformations of X. According to Example 2, sec. 4.6, $\mathrm{Sk}(n)$ is reductive.

Let $j\colon \mathrm{Sk}(n) \to L(n)$ be the inclusion map, and write (cf. sec. 6.19)

$$C_p^{SO} = j_{\theta=0}^{\vee}(C_p), \quad \mathrm{Tr}_p^{SO} = j_{\theta=0}^{\vee}(\mathrm{Tr}_p), \quad \Phi_{2p-1}^{SO} = j_{\theta=0}^{\wedge}(\Phi_{2p-1}), \quad 1 \le p \le n.$$

(We use the superscript SO because $\mathrm{Sk}(n)$ is the Lie algebra of $SO(n)$ —cf. sec. 6.29.) Since $j_{\theta=0}^{\wedge}$ maps primitive elements to primitive elements (cf. sec. 5.18), the Φ_{2p-1}^{SO} are primitive.

Remark: Let $K \in (\vee^2 \mathrm{Sk}(n)^*)_{\theta=0}$ be the Killing form of $\mathrm{Sk}(n)$. An easy computation yields

$$K = -(n-2)C_2^{SO}.$$

Lemma IX: (1) If p is odd, then $C_p^{SO} = 0$, $\mathrm{Tr}_p^{SO} = 0$, and $\Phi_{2p-1}^{SO} = 0$.
(2) $\Phi_{4p-1}^{SO} \neq 0$, $3 \le 2p + 1 \le n$.

Proof: (1) Since for odd p, $C_p(\varphi) = 0$ and $\mathrm{tr}\,\varphi^p = 0$ ($\varphi \in \mathrm{Sk}(n)$), it follows that $C_p^{SO} = 0$ and $\mathrm{Tr}_p^{SO} = 0$ in this case. Now Proposition II, sec. 6.7, and formula (6.15), sec. 6.19, imply that $\Phi_{2p-1}^{SO} = 0$.

(2) Choose an orthonormal basis $e_0, e_1, \dots, e_{n-1}$ of X. Define skew transformations α_i and β_λ by

$$\alpha_i(x) = \langle x, e_i\rangle e_0 - \langle x, e_0\rangle e_i, \qquad i = 1, \dots, 2p,$$

and

$$\beta_\lambda(x) = \langle x, e_\lambda\rangle e_1 - \langle x, e_1\rangle e_\lambda, \qquad \lambda = 2, \dots, 2p.$$

We show that $\Phi_{4p-1}^{SO}(\alpha_1, \dots, \alpha_{2p}, \beta_2, \dots, \beta_{2p}) \neq 0$.

First observe that for distinct (i, j, k) and distinct (λ, μ, ν),

$$\alpha_i \circ \alpha_j \circ \alpha_k = 0, \qquad \beta_\lambda \circ \beta_\mu \circ \beta_\nu = 0, \qquad \text{and} \qquad \beta_\mu \circ \alpha_i \circ \beta_\nu = 0.$$

Now let φ be a transformation obtained by composing all the α_i and all the β_λ in some order, and assume that $\operatorname{tr}\varphi \neq 0$. Then all the transformations obtained by cyclically permuting the given order, and then composing, have the same trace, and so are nonzero. Hence it follows from the relations above that φ is obtained from a product of the form

$$\psi = \alpha_{i_1} \circ \alpha_{i_2} \circ \beta_{\lambda_1} \circ \beta_{\lambda_2} \circ \alpha_{i_3} \circ \alpha_{i_4} \circ \beta_{\lambda_3} \circ \beta_{\lambda_4} \circ \cdots \circ \alpha_{i_{2p-1}} \circ \alpha_{i_{2p}} \circ \beta_{\lambda_{2p-1}}.$$

by a cyclic permutation. Since $\operatorname{tr}\psi = \operatorname{tr}\varphi \neq 0$, it follows that $\psi \neq 0$.

Next observe that

$$\alpha_i \circ \beta_\lambda = 0 = \beta_\lambda \circ \alpha_i \qquad \text{unless} \qquad i = \lambda \quad \text{or} \quad i = 1,$$

whence $\alpha_i \circ \beta_\lambda \circ \alpha_k = 0$ unless $i = 1$, $k = \lambda$ or $i = \lambda$, $k = 1$. Thus ψ must have one of the following forms: Either

$$\psi = \psi_1 = \alpha_1 \circ \alpha_{i_2} \circ \beta_{i_2} \circ \beta_{i_3} \circ \alpha_{i_3} \circ \alpha_{i_4} \circ \beta_{i_4} \circ \cdots \circ \alpha_{i_{2p-1}} \circ \alpha_{i_{2p}} \circ \beta_{i_{2p}},$$

or

$$\psi = \psi_2 = \alpha_{i_{2p}} \circ \alpha_{i_2} \circ \beta_{i_2} \circ \beta_{i_3} \circ \alpha_{i_3} \circ \cdots \circ \alpha_{i_{2p-1}} \circ \alpha_1 \circ \beta_{i_{2p}}.$$

But $\operatorname{tr}\psi_1 = 1$, and ψ_1 is obtained from $\alpha_1, \ldots, \alpha_{2p}, \beta_2, \ldots, \beta_{2p}$ by an even permutation; while $\operatorname{tr}\psi_2 = -1$, and ψ_2 is obtained from $\alpha_1, \ldots, \alpha_{2p}, \beta_2, \ldots, \beta_{2p}$ by an odd permutation.

Now write $\beta_j = \alpha_{2p+j-1}$ $(2 \leq j \leq 2p)$. Our arguments above show that for every permutation σ,

$$\operatorname{tr}\alpha_{\sigma(1)} \circ \cdots \circ \alpha_{\sigma(4p-1)} = 0 \quad \text{or} \quad \varepsilon_\sigma.$$

Hence, since $\operatorname{tr}\psi_1 \neq 0$,

$$\Phi^{SO}_{4p-1}(\alpha_1, \ldots, \alpha_{4p-1}) = \sum_\sigma \varepsilon_\sigma \operatorname{tr}\alpha_{\sigma(1)} \circ \cdots \circ \alpha_{\sigma(4p-1)} \neq 0.$$

Q.E.D.

6.22. The skew Pfaffian. Let $(X, \langle\,,\,\rangle)$ be a $2m$-dimensional Euclidean space. An isomorphism $\alpha\colon \mathrm{Sk}(2m) \xrightarrow{\cong} \wedge^2 X$ is given by

$$\langle \alpha(\varphi), x \wedge y\rangle = \langle \varphi x, y\rangle, \qquad \varphi \in \mathrm{Sk}(2m), \quad x, y \in X.$$

It is inverse to the isomorphism β defined in sec. A.5.

Now fix an orientation in X and let e be the unique unit vector in $\wedge^{2m} X$ which represents the orientation. Then the *skew Pfaffian* is the invariant skew symmetric $(2m-1)$-linear function $\mathrm{Sf} \in (\wedge^{2m-1}\mathrm{Sk}(2m)^*)_{\theta=0}$,

given by

$$\mathrm{Sf}(\varphi_1, \ldots, \varphi_{2m-1}) = \sum_{\sigma \in S^{2m-1}} \varepsilon_\sigma \langle e, \alpha(\varphi_{\sigma(1)}) \wedge \alpha([\varphi_{\sigma(2)}, \varphi_{\sigma(3)}]) \wedge \cdots \\ \cdots \wedge \alpha([\varphi_{\sigma(2m-2)}, \varphi_{\sigma(2m-1)}])\rangle.$$

On the other hand, consider the Pfaffian $\mathrm{Pf} \in (\vee^m \mathrm{Sk}(2m)^*)_{\theta=0}$, of the oriented Euclidean space X defined in sec. A.7. It follows directly from Proposition IV, sec. 6.8, that

$$\varrho_{\mathrm{Sk}(2m)}(\mathrm{Pf}) = \frac{(-1)^{m-1}(m-1)!}{2^{m-1}(2m-1)!}\,\mathrm{Sf}. \tag{6.17}$$

In particular (cf. Theorem II, sec. 6.14) Sf is primitive.

Lemma X: Sf is nonzero, and linearly independent of Φ^{SO}_{2m-1}.

Proof: Write $X = (x_0) \oplus Y$ where $x_0 \neq 0$ and Y is the orthogonal complement of x_0. Let $x_1, \ldots, x_{2m-1}$ be a basis of Y and define $\varphi_i \in \mathrm{Sk}(2m)$, $i = 1, \ldots, 2m-1$, by

$$\varphi_i = \beta(x_0 \wedge x_i).$$

Then a simple computation shows that

$$\alpha([\varphi_i, \varphi_j]) = \langle x_0, x_0\rangle x_i \wedge x_j.$$

It follows that

$$\mathrm{Sf}(\varphi_1, \ldots, \varphi_{2m-1}) = (2m-1)!\langle x_0, x_0\rangle^{m-1}\langle e, x_0 \wedge x_1 \wedge \cdots \wedge x_{2m-1}\rangle.$$

Hence $\mathrm{Sf} \neq 0$.

To prove the second part of the lemma choose the x_i ($i = 0, \ldots, 2m-1$) mutually orthogonal. Then $\varphi_i(x_j) = 0$ if $i \neq j$ ($j = 1, \ldots, 2m-1$). It follows that for distinct i, j, k, $\varphi_i \circ \varphi_j \circ \varphi_k = 0$. Thus

$$\Phi^{SO}_{2m-1}(\varphi_1, \ldots, \varphi_{2m-1}) = 0$$

and so Sf is linearly independent of Φ^{SO}_{2m-1}.

Q.E.D.

6.23. The cohomology of Sk(n). Assign $(\vee \mathrm{Sk}(n)^*)_{\theta=0}$ the even gradation of sec. 6.1.

Case I: $n = 2m + 1$. Lemma IX, together with a word by word repetition of the proof of Theorem IV, sec. 6.19, yields

Theorem VI: (1) The elements Φ^{SO}_{4p-1} $(1 \leq p \leq m)$ are a basis of the primitive space for $\mathrm{Sk}(2m+1)$. In particular $\mathrm{Sk}(2m+1)$ has rank m.

(2) $(\vee \mathrm{Sk}(2m+1)^*)_{\theta=0}$ is the symmetric algebra over the graded subspace spanned by the elements C^{SO}_{2p} $(1 \leq p \leq m)$. It is also the symmetric algebra over the subspace spanned by the elements Tr^{SO}_{2p} $(1 \leq p \leq m)$.

(3) The Poincaré polynomial for $H^*(\mathrm{Sk}(2m+1))$ and the Poincaré series for $(\vee \mathrm{Sk}(2m+1)^*)_{\theta=0}$, respectively, are given by

$$\prod_{p=1}^{m} (1 + t^{4p-1}) \qquad \text{and} \qquad \prod_{p=1}^{m} (1 - t^{4p})^{-1}.$$

Corollary: $\mathrm{Sk}(2m+1)$ is simple.

Proof: Apply Proposition V, sec. 5.20.

Q.E.D.

Case II: $n = 2m$. In view of Lemmas IX and X, the same argument used to prove Theorem IV, sec. 6.19, establishes

Theorem VII: (1) The elements Φ^{SO}_{4p-1} $(1 \leq p < m)$ and Sf are a basis of the primitive space for $\mathrm{Sk}(2m)$. In particular, $\mathrm{Sk}(2m)$ has rank m.

(2) $(\vee \mathrm{Sk}(2m)^*)_{\theta=0}$ is the symmetric algebra over the graded subspace spanned by the elements C^{SO}_{2p} $(1 \leq p < m)$ and Pf. It is also the symmetric algebra over the subspace spanned by Tr^{SO}_{2p} $(1 \leq p < m)$ and Pf.

(3) The Poincaré polynomial for $H^*(\mathrm{Sk}(2m))$ and the Poincaré series for $(\vee \mathrm{Sk}(2m)^*)_{\theta=0}$ are respectively given by

$$(1 + t^{2m-1}) \prod_{p=1}^{m-1} (1 + t^{4p-1}) \qquad \text{and} \qquad (1 - t^{2m})^{-1} \prod_{p=1}^{m-1} (1 - t^{4p})^{-1}.$$

Corollary: If $m > 2$, then $\mathrm{Sk}(2m)$ is simple.

Proof: Apply Proposition V, sec. 5.20.

Q.E.D.

Examples: 1. *Sk(3)*: The Poincaré polynomial for $H^*(\mathrm{Sk}(3))$ is $1 + t^3$, and $(\vee \mathrm{Sk}(3)^*)_{\theta=0}$ is a polynomial algebra in a single indeterminate of degree four. Thus, in view of sec. 5.20, a basis element of $(\vee \mathrm{Sk}(3)^*)^4_{\theta=0}$ is given by the Killing form:

$$K(x, y) = \operatorname{tr} \operatorname{ad} x \circ \operatorname{ad} y, \qquad x, y \in \mathrm{Sk}(3).$$

A straightforward computation shows that (cf. sec. 6.21)

$$K = -C_2^{SO}.$$

2. *Sk(4)*: The Poincaré polynomial for $H^*(\mathrm{Sk}(4))$ is $1 + 2t^3 + t^6$. Thus (cf. Proposition V, sec. 5.20), Sk(4) is the direct sum of two ideals I_1 and I_2.

Identify $\mathbb{R}^4$ with the space of quaternions (cf. sec. 6.30). Then according to [4; p. 248] every skew transformation, φ determines unique quaternions $p, q \in (e)^\perp$ such that

$$\varphi x = px - xq.$$

The ideals I_1 and I_2 consist respectively of the transformations of the form

$$\varphi x = px \qquad \text{and} \qquad \varphi x = -xq.$$

Now the adjoint representation of I_λ defines an isomorphism of Lie algebras $I_\lambda \cong \mathrm{Sk}(3)$. Moreover, since $\mathrm{Sk}(4) = I_1 \oplus I_2$,

$$(\vee \mathrm{Sk}(4)^*)_{\theta=0} = (\vee I_1^*)_{\theta=0} \otimes (\vee I_2^*)_{\theta=0},$$

whence

$$\begin{aligned}(\vee \mathrm{Sk}(4)^*)^4_{\theta=0} &= \{(\vee I_1^*)^4_{\theta=0} \otimes 1\} \oplus \{1 \otimes (\vee I_2^*)^4_{\theta=0}\} \\ &= (K_1) \oplus (K_2).\end{aligned}$$

(K_λ is the Killing form of I_λ.)

A straightforward computation shows that

$$4\mathrm{Pf} = K_2 - K_1 \qquad \text{and} \qquad 2C_2^{SO} = -(K_1 + K_2).$$

6.24. The Lie algebra Sy(m). Let $(X, \langle\ ,\ \rangle)$ be a symplectic space (cf. sec. 0.1). Then dim X is even, dim $X = 2m$. The Lie algebra Sy(m) of *skew symplectic transformations* is the Lie algebra of those linear transformations of X which are skew with respect to $\langle\ ,\ \rangle$. Note that dim Sy(m) $= m(2m + 1)$. It follows from the example in sec. 4.8 that Sy(m) is reductive.

Let $j\colon \mathrm{Sy}(m) \to L(2m)$ be the inclusion, and set

$$C_p^{\mathrm{Sy}} = j^{\vee}_{\theta=0}(C_p), \qquad \mathrm{Tr}_p^{\mathrm{Sy}} = j^{\vee}_{\theta=0}(\mathrm{Tr}_p),$$

and

$$\Phi_{2p-1}^{\mathrm{Sy}} = j^{\wedge}_{\theta=0}(\Phi_{2p-1}), \qquad 1 \le p \le 2m.$$

As in sec. 6.21, it follows that the $\Phi_{2p-1}^{\mathrm{Sy}}$ are primitive elements.

Lemma XI: (1) C_p^{Sy}, $\mathrm{Tr}_p^{\mathrm{Sy}}$ and $\Phi_{2p-1}^{\mathrm{Sy}}$ are zero if p is odd.
(2) $\Phi_{4p-1}^{\mathrm{Sy}} \neq 0$, $1 \le p \le m$.

Proof: (1) This follows in exactly the same way as Lemma IX, (1), sec. 6.21.

(2) Choose a basis $a_1, \dots, a_m, b_1, \dots, b_m$, of X so that

$$\langle a_i, a_j\rangle = 0, \qquad \langle b_i, b_j\rangle = 0, \qquad \text{and} \qquad \langle a_i, b_j\rangle = \delta_{ij}.$$

Define $\omega, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathrm{Sy}(m)$ by

$$\begin{aligned}
\omega(x) &= \langle x, b_1\rangle a_1 + \langle x, a_1\rangle b_1 \\
\alpha_i(x) &= \langle x, b_i\rangle a_{i-1} + \langle x, a_{i-1}\rangle b_i, && i = 2, \dots, p \\
\beta_i(x) &= -\langle x, a_i\rangle a_i, && i = 1, \dots, p \\
\gamma_i(x) &= \langle x, b_i\rangle b_i, && i = 1, \dots, p
\end{aligned}$$

and

$$\delta_i(x) = \langle x, b_{i-1}\rangle a_i + \langle x, a_i\rangle b_{i-1}, \qquad i = 2, \dots, p, \quad x \in X.$$

We show that

$$\Phi_{4p-1}^{\mathrm{Sy}}(\omega, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \dots, \alpha_p, \beta_p, \gamma_p, \delta_p) \neq 0. \tag{6.18}$$

In fact, denote these transformations (in this order) by $\xi_1, \dots, \xi_{4p-1}$, and, for each permutation $\sigma \in S^{4p-1}$, write

$$\varphi_\sigma = \xi_{\sigma(1)} \circ \cdots \circ \xi_{\sigma(4p-1)}.$$

We show by induction on p that

$$\operatorname{tr} \varphi_\sigma = \varepsilon_\sigma \quad \text{or} \quad 0. \tag{6.19}$$

If $p = 1$, $\xi_1 = \omega$, $\xi_2 = \beta_1$, $\xi_3 = \gamma_1$, and (6.19) is clear. Assume $p \geq 2$ and that (6.19) holds for $p - 1$.

First observe that following a permutation σ by a cyclic permutation changes neither $\operatorname{tr} \varphi_\sigma$ nor ε_σ (since $4p - 1$ is odd). Now choose any σ so that $\operatorname{tr} \varphi_\sigma \neq 0$. In view of this remark we may assume that $\xi_{\sigma(4p-2)} = \gamma_p$. Because $\varphi_\sigma \neq 0$, the relations

$$\xi_i \circ \gamma_p \circ \xi_j = 0, \qquad \begin{array}{lllll} \text{unless} & \xi_i = \beta_p & \text{and} & \xi_j = \delta_p, \\ \text{or} & \xi_i = \delta_p & \text{and} & \xi_j = \beta_p \end{array}$$

imply that

$$\xi_{\sigma(4p-3)} = \beta_p, \quad \xi_{\sigma(4p-1)} = \delta_p \qquad \text{or} \qquad \xi_{\sigma(4p-3)} = \delta_p, \quad \xi_{\sigma(4p-1)} = \beta_p.$$

Consider the first case. (The second case is treated in the same way.) Among the remaining ξ_i the only one which satisfies $\xi_i \beta_p \neq 0$ is α_p. Hence

$$\xi_{\sigma(4p-4)} = \alpha_p,$$

and so φ_σ is of the form

$$\varphi_\sigma = \xi_{\tau(1)} \circ \cdots \circ \xi_{\tau(4p-5)} \circ \alpha_p \circ \beta_p \circ \gamma_p \circ \delta_p, \qquad \text{some} \quad \tau \in S^{4p-5}.$$

Write $\chi = \alpha_p \circ \beta_p \circ \gamma_p \circ \delta_p$. Then

$$\chi(a_{p-1}) = a_{p-1}, \qquad \chi(a_i) = 0, \quad i \neq p - 1 \qquad \text{and} \qquad \chi(b_j) = 0, \quad \text{all } j.$$

On the other hand, write $\psi_\tau = \xi_{\tau(1)} \circ \cdots \circ \xi_{\tau(4p-5)}$. Because $\varphi_\sigma = \psi_\tau \circ \chi$, the relations above for χ show that

$$\psi_\tau(a_{p-1}) = (\operatorname{tr} \varphi_\sigma) a_{p-1}.$$

Finally, observe that $\operatorname{rank} \psi_\tau \leq 1$ (because each β_i has rank 1). Since $\operatorname{tr} \varphi_\sigma \neq 0$, it follows that

$$\psi_\tau(a_i) = 0, \quad i \neq p - 1 \qquad \text{and} \qquad \psi_\tau(b_j) = 0 \quad \text{all } j.$$

This gives

$$\psi_\tau(a_{p-1}) = (\operatorname{tr} \psi_\tau) a_{p-1}.$$

Thus, in view of our induction hypothesis,

$$\operatorname{tr} \varphi_\sigma = \operatorname{tr} \psi_\tau = \varepsilon_\tau = \varepsilon_\sigma,$$

which proves (6.19). Formula (6.18) follows since

$$\operatorname{tr} \eta_p \circ \cdots \circ \eta_2 \circ \beta_1 \circ \gamma_1 \circ \omega \circ \alpha_2 \circ \cdots \circ \alpha_p \neq 0,$$

where $\eta_i = \beta_i \circ \gamma_i \circ \delta_i$.

Q.E.D.

Now assign $(\vee \mathrm{Sy}(m)^*)_{\theta=0}$ the even gradation of sec. 6.1. In view of Lemma XI word by word repetition of the proof of Theorem IV, sec. 6.19, yields

Theorem VIII: (1) The elements $\Phi^{\mathrm{Sy}}_{4p-1}$, $1 \leq p \leq m$, are a basis of the primitive space for $\mathrm{Sy}(m)$. In particular $\mathrm{Sy}(m)$ has rank m.

(2) $(\vee \mathrm{Sy}(m)^*)_{\theta=0}$ is the symmetric algebra over the subspace spanned by $C^{\mathrm{Sy}}_{2p}(m)$, $1 \leq p \leq m$, and over the subspace spanned by $\mathrm{Tr}^{\mathrm{Sy}}_{2p}(m)$, $1 \leq p \leq m$.

(3) The Poincaré polynomial for $H^*(\mathrm{Sy}(m))$ and the Poincaré series for $(\vee \mathrm{Sy}(m)^*)_{\theta=0}$ are given respectively by

$$\prod_{p=1}^{m} (1 + t^{4p-1}) \qquad \text{and} \qquad \prod_{p=1}^{m} (1 - t^{4p})^{-1}.$$

Corollary: The Lie algebra $\mathrm{Sy}(m)$ $(m \geq 1)$ is simple.

Proof: Apply Proposition V, sec. 5.20.

Q.E.D.

§7. The compact classical Lie groups

6.25. Complexification of a real Lie algebra. The *complexification* of a real Lie algebra E is the complex vector space $E^C = C \otimes E$, together with the Lie product

$$[\lambda \otimes x, \mu \otimes y] = \lambda\mu \otimes [x, y], \qquad \lambda, \mu \in C, \quad x, y \in E.$$

Evidently, $\dim_C E^C = \dim E$. The symbols $(E^C)^*$, $\wedge E^C$, $\vee E^C$ will denote the dual space, the exterior algebra, and the symmetric algebra over the complex space E^C.

Regard the elements of $(E^C)^*$ as complex linear functions in E^C; by restricting them to E we obtain a linear isomorphism

$$(E^C)^* \cong L(E; C).$$

On the other hand, there is a canonical isomorphism

$$L(E; C) \cong C \otimes E^*,$$

(cf. sec. 0.1). Combining these yields an isomorphism

$$C \otimes E^* \cong (E^C)^*.$$

It identifies E^* with the set of elements in $(E^C)^*$ taking real values in E.

This isomorphism extends to the isomorphisms of graded algebras

$$\alpha\colon C \otimes \wedge E^* \xrightarrow{\cong} \wedge(E^C)^* \qquad \text{and} \qquad \beta\colon C \otimes \vee E^* \xrightarrow{\cong} \vee(E^C)^*,$$

given by

$$\alpha(\lambda \otimes x_1^* \wedge \cdots \wedge x_p^*) = \lambda x_1^* \wedge \cdots \wedge x_p^*$$

and

$$\beta(\lambda \otimes x_1^* \vee \cdots \vee x_p^*) = \lambda x_1^* \vee \cdots \vee x_p^*, \qquad \lambda \in C, \quad x_i^* \in E^*.$$

Thus $\wedge(E^C)^*$ (respectively, $\vee(E^C)^*$) is identified with the skew symmetric (respectively, symmetric) complex p-linear functions in E.

Since $\theta(1 \otimes x) \circ \alpha = \alpha \circ \theta(x)$ and $\theta(1 \otimes x) \circ \beta = \beta \circ \theta(x)$ $(x \in E)$, α and β restrict to isomorphisms

$$\alpha_{\theta=0}\colon \mathbb{C} \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} (\wedge(E^{\mathbb{C}})^*)_{\theta=0}$$

and

$$\beta_{\theta=0}\colon \mathbb{C} \otimes (\vee E^*)_{\theta=0} \longrightarrow (\vee(E^{\mathbb{C}})^*)_{\theta=0}.$$

Further, we have $\alpha \circ (\iota \otimes \delta_E) = \delta_{E^{\mathbb{C}}} \circ \alpha$, and so α induces an isomorphism

$$\alpha^{\#}\colon \mathbb{C} \otimes H^*(E) \xrightarrow{\cong} H^*(E^{\mathbb{C}}).$$

In particular, $H^*(E)$ and $H^*(E^{\mathbb{C}})$ have the same Poincaré polynomials.

Now assume that E is reductive. Then $\alpha_{\theta=0}$ restricts to an isomorphism of graded vector spaces

$$\alpha_P\colon \mathbb{C} \otimes P_E \xrightarrow{\cong} P_{E^{\mathbb{C}}}$$

as follows immediately from the definitions. Clearly, the diagram

$$\begin{tikzcd}
\mathbb{C} \otimes (\vee^+ E^*)_{\theta=0} \arrow[r, "\beta_{\theta=0}", "\cong"'] \arrow[d, "\iota\otimes\varrho_E"'] & (\vee^+(E^{\mathbb{C}})^*)_{\theta=0} \arrow[d, "\varrho_{E^{\mathbb{C}}}"] \\
\mathbb{C} \otimes P_E \arrow[r, "\cong", "\alpha_P"'] & P_{E^{\mathbb{C}}}
\end{tikzcd}$$

commutes.

Finally, let Q be a graded subspace of $(\vee E^*)_{\theta=0}$ and use β to define an inclusion $\mathbb{C} \otimes Q \to (\vee(E^{\mathbb{C}})^*)_{\theta=0}$. Then the induced homomorphism

$$\vee Q \to (\vee E^*)_{\theta=0}$$

is an isomorphism if and only if the homomorphism

$$\vee(\mathbb{C} \otimes Q) \to (\vee(E^{\mathbb{C}})^*)_{\theta=0}$$

is. (This follows from the fact that $\beta_{\theta=0}$ is an isomorphism.)

6.26. Linear groups. In this section $\Gamma = \mathbb{R}$ or $\mathbb{C}$. Recall from sec. 2.5 and sec. 2.6 (volume II) that $GL(n; \Gamma)$ is the Lie group of linear automorphisms of an n-dimensional vector space Γ^n over Γ.

The Lie algebra of $GL(n;\Gamma)$ is the Lie algebra $L(n;\Gamma)$ of linear transformations of Γ^n. Since (obviously) $GL(n;\Gamma)$ is an open submanifold of $L(n;\Gamma)$, the tangent bundle of $GL(n;\Gamma)$ is given by

$$T_{GL(n;\Gamma)} = GL(n;\Gamma)\times L(n;\Gamma).$$

With this identification, left translation in $T_{GL(n;\Gamma)}$ by an element τ in $GL(n;\Gamma)$ is given by

$$L_\tau(\sigma,\alpha) = (\tau\circ\sigma,\ \tau\circ\alpha), \qquad \sigma\in GL(n;\Gamma),\quad \alpha\in L(n;\Gamma),$$

(cf. Example 2, sec. 1.4, volume II).

A closed subgroup of $GL(n;\Gamma)$ (which, by Theorem I, sec. 2.1, volume II, is a Lie group) is called a *linear group*. Let G be a linear group with Lie algebra $E\subset L(n;\Gamma)$. Then the tangent space of G at τ is given by

$$T_\tau(G) = L_\tau(E) = \tau\circ E.$$

Hence if $\Phi\in\wedge^pE^*$, the corresponding left invariant form $\tau_L^{-1}(\Phi)$ in G (cf. sec. 5.29) is given by

$$(\tau_L^{-1}\Phi)(\tau;\alpha_1,\ldots,\alpha_p) = \Phi(\tau^{-1}\circ\alpha_1,\ldots,\tau^{-1}\circ\alpha_p),$$
$$\tau\in G,\quad \alpha_i\in T_\tau(G). \qquad (6.20)$$

In particular, if $\Phi\in(\wedge^pE^*)_{\theta=0}$, then the corresponding p-form on G is biinvariant and hence closed.

6.27. The group $U(n)$. Consider the compact, connected Lie group $U(n)$ of isometries of an n-dimensional complex vector space C^n with Hermitian inner product (,). Its Lie algebra $\mathrm{Sk}(n;C)$ is the *real* Lie algebra consisting of the complex linear transformations α of C^n which satisfy

$$(\alpha x, y) + (x, \alpha y) = 0, \qquad x, y\in C^n.$$

For simplicity denote $\mathrm{Sk}(n;C)$ by E.

Next recall the elements C_p, Tr_p, Φ_{2p-1} defined in sec. 6.19. Observe that the restrictions of these multilinear functions to E take only real (respectively, only purely imaginary) values if p is even (respectively, odd). Hence elements

$$C_p^U,\ \mathrm{Tr}_p^U\in(\vee^pE^*)_{\theta=0}, \qquad \text{and} \qquad \Phi_{2p-1}^U\in(\wedge^{2p-1}E^*)_{\theta=0}$$

are defined by

$$C_p^U(\alpha_1, \dots, \alpha_p) = \frac{1}{i^p} C_p(\alpha_1, \dots, \alpha_p),$$

$$\mathrm{Tr}_p^U(\alpha_1, \dots, \alpha_p) = \frac{1}{i^p} \mathrm{Tr}_p(\alpha_1, \dots, \alpha_p),$$

and

$$\Phi_{2p-1}^U(\alpha_1, \dots, \alpha_{2p-1}) = \frac{1}{i^p} \Phi_{2p-1}(\alpha_1, \dots, \alpha_{2p-1}),$$
$$\alpha_j \in E, \quad p = 1, \dots, n.$$

The element Φ_{2p-1}^U extends to a unique closed, biinvariant form on $U(n)$. We denote this form by Φ_{2p-1}^U as well; it is given explicitly by

$$\begin{aligned}\Phi_{2p-1}^U&(\tau; \alpha_1, \dots, \alpha_{2p-1}) \\ &= \frac{1}{i^p} \sum_{\sigma \in S^{2p-1}} \varepsilon_\sigma \operatorname{tr}(\tau^{-1} \circ \alpha_{\sigma(1)}) \circ \cdots \circ (\tau^{-1} \circ \alpha_{\sigma(2p-1)}), \\ &\qquad \tau \in U(n), \quad \alpha_j \in T_\tau(U(n)) \qquad (6.21)\end{aligned}$$

(cf. sec. 6.19 and formula (6.20), sec. 6.26). The cohomology class represented by Φ_{2p-1}^U is denoted by $[\Phi_{2p-1}^U]$.

Let $P_{U(n)}$ denote the primitive subspace of $H(U(n))$ (cf. sec. 4.12, volume II, and sec. 5.32). Recall that $E = \mathrm{Sk}(n; \mathbb{C})$.

Theorem IX: (1) The elements Φ_{2p-1}^U $(1 \le p \le n)$ are a basis of P_E, and so $(\wedge E^*)_{\theta=0} = \wedge(\Phi_1^U, \dots, \Phi_{2n-1}^U)$.

(2) $(\vee E^*)_{\theta=0} = \vee(C_1^U, \dots, C_n^U) = \vee(\mathrm{Tr}_1^U, \dots, \mathrm{Tr}_n^U)$.

(3) The cohomology classes $[\Phi_{2p-1}^U]$ $(1 \le p \le n)$ are a basis of $P_{U(n)}$, and so

$$H(U(n)) = \wedge([\Phi_1^U], \dots, [\Phi_{2n-1}^U]).$$

(4) The Poincaré polynomial for $(\wedge E^*)_{\theta=0}$, $H^*(E)$, and $H(U(n))$ is

$$\prod_{p=1}^n (1 + t^{2p-1}).$$

Proof: (1) Observe that an isomorphism of complex Lie algebras

$$\mathbb{C} \otimes E \xrightarrow{\cong} L(n; \mathbb{C})$$

is given by $\lambda \otimes \varphi \mapsto \lambda\varphi$, $\lambda \in \mathbb{C}$, $\varphi \in E$. This isomorphism induces an

isomorphism (as described in sec. 6.25)

$$\alpha_{\theta=0}: C \otimes (\wedge E^*)_{\theta=0} \to (\wedge L(n;\, C)^*)_{\theta=0},$$

which restricts to an isomorphism

$$\alpha_P: C \otimes P_E \xrightarrow{\cong} P_{L(n;C)}.$$

But by definition,

$$\alpha_{\theta=0}(1 \otimes \Phi^U_{2p-1}) = \frac{1}{i^p}\, \Phi_{2p-1}, \qquad 1 \le p \le n.$$

Now it follows from Theorem IV, (1), sec. 6.19, that the Φ^U_{2p-1} are a basis of P_E.

(2) This follows in exactly the same way as (1) from Theorem IV, (2), sec. 6.19 (as described in sec. 6.25).

(3) According to sec. 5.32, the isomorphism

$$\alpha_{U(n)}: (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H(U(n))$$

restricts to an isomorphism from P_E to $P_{U(n)}$. Since $\alpha_{U(n)}(\Phi^U_{2p-1}) = [\Phi^U_{2p-1}]$, $1 \le p \le n$, (3) follows from (1).

(4) This follows at once from (1) and (3) (cf. sec. 5.32).

Q.E.D.

6.28. The Lie group $SU(n)$. Recall from Example 4, sec. 2.6, volume II, that $SU(n)$ is the closed subgroup of $U(n)$ consisting of the isometries with determinant 1. Denote its Lie algebra by E.

Let Φ^{SU}_{2p-1} denote the restriction to E of the element Φ^U_{2p-1} of sec. 6.27, and let Φ^{SU}_{2p-1} also denote the corresponding biinvariant form on $SU(n)$. The latter is the restriction to $SU(n)$ of the biinvariant form Φ^U_{2p-1} on $U(n)$.

Let C^{SU}_p and Tr^{SU}_p denote the restrictions of C^U_p and Tr^U_p to E. Then the argument of the previous section, applied to Theorem V, sec. 6.20, yields

Theorem X: (1) The elements Φ^{SU}_{2p-1} $(2 \le p \le n)$ are a basis of P_E, and so $(\wedge E^*)_{\theta=0} = \wedge(\Phi^{SU}_3, \dots, \Phi^{SU}_{2n-1})$.

(2) $(\vee E^*)_{\theta=0} = \vee(C^{SU}_2, \dots, C^{SU}_n) = \vee(\mathrm{Tr}^{SU}_2, \dots, \mathrm{Tr}^{SU}_n)$.

(3) The cohomology classes $[\Phi_{2p-1}^{SU}]$ $(2 \le p \le n)$ are a basis of $P_{SU(n)}$, and so

$$H(SU(n)) = \wedge([\Phi_3^{SU}], \ldots, [\Phi_{2n-1}^{SU}]).$$

(4) The Poincaré polynomial for $(\wedge E^*)_{\theta=0}$, $H^*(E)$, and $H(SU(n))$ is

$$\prod_{p=2}^{n} (1 + t^{2p-1}).$$

6.29. The Lie group *SO*(*n*). Recall from Example 3, sec. 2.5, volume II, that $SO(n)$ is the compact connected Lie group of proper rotations of Euclidean n-space. Its Lie algebra is the Lie algebra $\mathrm{Sk}(n)$ described in sec. 6.21.

Recall the primitive elements $\Phi_{4p-1}^{SO} \in P_{Sk(n)}$ $(3 \le 2p + 1 \le n)$ introduced in sec. 6.21. Their extensions to biinvariant closed forms on $SO(n)$ (also denoted by Φ_{4p-1}^{SO}) are given explicitly by (cf. sec. 6.26)

$$\Phi_{4p-1}^{SO}(\tau; \alpha_1, \ldots, \alpha_{4p-1}) = \sum_{\sigma \in S^{4p-1}} \varepsilon_\sigma \operatorname{tr}(\tau^{-1} \circ \alpha_{\sigma(1)}) \circ \cdots \circ (\tau^{-1} \circ \alpha_{\sigma(4p-1)}),$$
$$\tau \in SO(n), \quad \alpha_j \in T_\tau(SO(n)). \tag{6.22}$$

Moreover, if $n = 2m$, the skew Pfaffian Sf defined in sec. 6.22 extends to a biinvariant form (also written Sf) on $SO(n)$. We may use sec. 5.32 to translate Theorems VI and VII, sec. 6.23. This yields

Theorem XI: The cohomology classes $[\Phi_{4p-1}^{SO}]$ $(1 \le p \le m)$ are a basis of $P_{SO(2m+1)}$. Thus

$$H(SO(2m + 1)) = \wedge([\Phi_3^{SO}], \ldots, [\Phi_{4m-1}^{SO}]),$$

and the Poincaré polynomial of $H(SO(2m + 1))$ is

$$\prod_{p=1}^{m} (1 + t^{4p-1}).$$

Theorem XII: The cohomology classes $[\Phi_{4p-1}^{SO}]$ $(1 \le p < m)$ and [Sf] are a basis of $P_{SO(2m)}$. Thus

$$H(SO(2m)) = \wedge([\Phi_3^{SO}], \ldots, [\Phi_{4m-5}^{SO}], [\mathrm{Sf}]),$$

and the Poincaré polynomial of $H(SO(2m))$ is

$$(1 + t^{2m-1}) \prod_{p=1}^{m-1} (1 + t^{4p-1}).$$

6.30. The Lie group $Q(n)$. Let H denote the algebra of quaternions. Recall from sec. 0.2, volume II, that H is an oriented 4-dimensional Euclidean space with a multiplication defined as follows: Choose a unit vector e and give the Euclidean 3-space $(e)^\perp$ the induced orientation (i.e., a basis e_1, e_2, e_3 of $(e)^\perp$ is positive if and only if the basis e, e_1, e_2, e_3 of H is positive). Now set

$$pe = p = ep, \qquad p \in H$$

and

$$pq = -\langle p, q\rangle e + p \times q, \qquad p, q \in (e)^\perp,$$

where $\times$ denotes the cross product in $(e)^\perp$.

Observe that every quaternion p is uniquely of the form $p = \lambda e + p_1$ ($\lambda \in R$, $p_1 \in (e)^\perp$); the *conjugate* of p is defined by $\bar{p} = \lambda e - p_1$. The identity element of H is e.

Now choose a fixed unit vector $i \in (e)^\perp$. Then $i^2 = -e$ and so the subspace of H spanned by e and i is a subalgebra of H, canonically isomorphic to C. We identify this subalgebra with C and write

$$H = C \oplus C^\perp.$$

Note that $C^\perp$ is stable under left multiplication by elements of C and hence it is a (1-dimensional) complex vector space.

Next, let $(R^n, \langle\,,\,\rangle_{R^n})$ be an n-dimensional Euclidean space and consider the quaternionic vector space $X = H \otimes_R R^n$ with scalar multiplication given by

$$p(q \otimes x) = pq \otimes x, \qquad p, q \in H, \quad x \in R^n.$$

Define a quaternionic inner product in X by setting

$$\langle p \otimes x, q \otimes y\rangle = p\bar{q}\langle x, y\rangle_{R^n}, \qquad p, q \in H, \quad x, y \in R^n.$$

Since $C \subset H$, restriction of scalar multiplication to C yields a $2n$-dimensional complex vector space X_C. Use the decomposition $H = C \oplus C^\perp$ to write

$$\langle u, v\rangle = \langle u, v\rangle_C + \langle u, v\rangle_{C^\perp}, \qquad u, v \in X.$$

Then $\langle\,,\,\rangle_C$ is a Hermitian inner product in X_C.

Finally, fix a unit vector $j \in C^\perp$. Then the complex bilinear function $(\,,\,)$ given by

$$(u, v) = \langle u, jv\rangle_C, \qquad u, v \in X_C,$$

makes X_C into a complex symplectic space (cf. sec. 0.1).

Now recall from Example 4, sec. 2.7, volume II, that the *quaternionic group* $Q(n)$ is the compact connected Lie subgroup of $GL(X_C)$ consisting of those transformations τ which satisfy

$$\tau(pu) = p\tau(u) \qquad \text{and} \qquad \langle \tau u, \tau v\rangle = \langle u, v\rangle, \qquad u, v \in X, \quad p \in H.$$

Its Lie algebra $\mathrm{Sk}(n; H)$ is the *real* subalgebra of L_{X_C} consisting of the transformations α such that

$$\alpha(pu) = p(\alpha u) \qquad \text{and} \qquad \langle \alpha u, v\rangle + \langle u, \alpha v\rangle = 0, \qquad p \in H, \quad u, v \in X.$$

Denote this Lie algebra by E.

Next observe that the inclusions $Q(n) \to GL(X_C)$ and $E \to L_{X_C}$ are in fact inclusions

$$Q(n) \to U(2n) \qquad \text{and} \qquad E \to \mathrm{Sk}(2n; \mathbb{C})$$

(with respect to the Hermitian inner product $\langle\ ,\ \rangle_C$). Thus we may restrict the elements Φ^U_{4p-1} $(1 \le p \le n)$ in $P_{Sk(2n;C)}$ to E; the resulting primitive elements will be written

$$\Phi^Q_{4p-1} \in P_E, \qquad 1 \le p \le n.$$

Moreover, the extension of Φ^Q_{4p-1} to a biinvariant $(4p-1)$-form Φ^Q_{4p-1} on $Q(n)$ coincides with the restriction to $Q(n)$ of the biinvariant form Φ^U_{4p-1} on $U(2n)$.

Finally, let C^Q_{2p} and Tr^Q_{2p} $(1 \le p \le n)$ denote the restrictions of C^U_{2p} and Tr^U_{2p} to E (cf. sec. 6.27).

Theorem XIII: (1) The elements Φ^Q_{4p-1}, $1 \le p \le n$, are a basis of P_E; in particular, $(\wedge E^*)_{\theta=0} = \wedge(\Phi^Q_3, \dots, \Phi^Q_{4n-1})$.

(2) $(\vee E^*)_{\theta=0} = \vee(C^Q_2, \dots, C^Q_{2n}) = \vee(\mathrm{Tr}^Q_2, \dots, \mathrm{Tr}^Q_{2n})$.

(3) The cohomology classes $[\Phi^Q_{4p-1}]$ $(1 \le p \le n)$ are a basis of $P_{Q(n)}$, and so

$$H(Q(n)) = \wedge([\Phi^Q_3], \dots, [\Phi^Q_{4n-1}]).$$

(4) The Poincaré polynomial of $(\wedge E^*)_{\theta=0}$, $H^*(E)$, and $H(Q(n))$ is

$$\prod_{p=1}^{n} (1 + t^{4p-1}).$$

Proof: (1) First observe that if $\varphi \in L_{X_C}$, then φ is in E if and only if φ is skew with respect to both $\langle\ ,\ \rangle_C$ and $(\ ,\ )$; i.e.

$$E = \mathrm{Sk}(2n; \mathbb{C}) \cap \mathrm{Sy}(n; \mathbb{C}).$$

It follows that we have a commutative diagram

$$\begin{tikzcd} C \otimes E \arrow[r, "\cong"] \arrow[d] & \mathrm{Sy}(n;\, C) \arrow[d] \\ C \otimes \mathrm{Sk}(2n;\, C) \arrow[r, "\cong"'] & L(2n;\, C) \end{tikzcd}$$

in which the vertical arrows are the obvious inclusions, while the horizontal arrows are the isomorphisms of complex Lie algebras given by $\lambda \otimes \varphi \mapsto \lambda\varphi$.

This leads (cf. sec. 6.25) to the commutative diagram

$$\begin{tikzcd} C \otimes P_E & P_{\mathrm{Sy}(n;C)} \arrow[l, "\varepsilon_1"', "\cong"] \\ C \otimes P_{\mathrm{Sk}(2n;C)} \arrow[u, "\gamma_1"] & P_{L(2n;C)} \arrow[l, "\cong"', "\varepsilon_2"] \arrow[u, "\gamma_2"'] \end{tikzcd}.$$

Thus (cf. sec. 6.27 and sec. 6.24)

$$\begin{aligned} 1 \otimes \Phi^Q_{4p-1} &= \gamma_1(1 \otimes \Phi^U_{4p-1}) \\ &= \gamma_1\varepsilon_2((-1)^p\Phi_{4p-1}) \\ &= \varepsilon_1((-1)^p\Phi^{\mathrm{Sy}}_{4p-1}). \end{aligned}$$

Now, since ε_1 is an isomorphism, (1) follows from Theorem VIII, (1), sec. 6.24.

(2) This follows from Theorem VIII, (2), via the isomorphism

$$C \otimes (\vee E^*)_{\theta=0} \cong (\vee \mathrm{Sy}(n;\, C)^*)_{\theta=0},$$

(as described in sec. 6.25).

(3) and (4): In view of sec. 5.32, these assertions are immediate consequences of (1) and (2).

Q.E.D.

Example: $G = Q(1)$. In this case E is the Lie algebra of pure quaternions, and a simple calculation shows that the Killing form K satisfies $K = 4C_2^Q$.

Now in Example 2, sec. 6.23, we have $I_\lambda \cong E$, $\lambda = 1, 2$. Let $C_2^{(\lambda)} \in (\vee I_\lambda^*)_{\theta=0}$ correspond to C_2^Q, under these isomorphisms. Then

$$\mathrm{Pf} = C_2^{(2)} - C_2^{(1)} \qquad \text{and} \qquad C_2^{SO} = -2(C_2^{(1)} + C_2^{(2)}).$$

Chapter VII

Operation of a Lie Algebra in a Graded Differential Algebra

§1. Elementary properties of an operation

7.1. Definition. An *operation of a Lie algebra in a graded differential algebra* is a 5-tuple $(E, i, \theta, R, \delta)$ where:

(1) E is a finite dimensional Lie algebra, and (R, δ) is a graded anticommutative differential algebra $R = \sum_{p \geq 0} R^p$.

(2) θ is a representation of E in the graded algebra R; that is, for each $x \in E$, $\theta(x)$ is a derivation in R, homogeneous of degree zero, and

$$\theta([x, y]) = \theta(x) \circ \theta(y) - \theta(y) \circ \theta(x), \qquad x, y \in E.$$

(3) i is a linear map from E to the space of antiderivations of R, such that each $i(x)$ is homogeneous of degree -1.

(4) The following relations hold:

$$i(x)^2 = 0, \tag{7.1}$$

$$i([x, y]) = \theta(x) \circ i(y) - i(y) \circ \theta(x), \tag{7.2}$$

and

$$\theta(x) = i(x) \circ \delta + \delta \circ i(x), \qquad x, y \in E. \tag{7.3}$$

Suppose $(E, i, \theta, R, \delta)$ is an operation of a Lie algebra E in a graded differential algebra (R, δ). Applying δ on the left and on the right hand sides of formula (7.3) we obtain

$$\delta \circ \theta(x) = \theta(x) \circ \delta, \qquad x \in E. \tag{7.4}$$

This relation shows that θ is a representation of E in the graded *differential* algebra (R, δ). Moreover, condition (7.3) implies that θ induces the zero representation in the cohomology algebra $H(R)$.

If $(E, i, \theta, R, \delta)$ is an operation of E, it follows from formula (7.1)

that i extends to a unique linear map

$$i\colon \wedge E \to L(R; R),$$

such that $i(a \wedge b) = i(b) \circ i(a)$, $a, b \in \wedge E$, and $i(1) = \iota$. In particular,

$$i(x_1 \wedge \cdots \wedge x_p) = i(x_p) \circ \cdots \circ i(x_1), \qquad x_i \in E,$$

so that $i(a)$ is homogeneous of degree $-p$ when $a \in \wedge^p E$.

Assume that operations of E in graded differential algebras (R, δ_R) and (S, δ_S) are given. Then a *homomorphism of operations*

$$\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$$

is a homomorphism $\varphi\colon (R, \delta_R) \to (S, \delta_S)$ of graded differential algebras such that

$$\varphi \circ \theta_R(x) = \theta_S(x) \circ \varphi \qquad \text{and} \qquad \varphi \circ i_R(x) = i_S(x) \circ \varphi, \qquad x \in E.$$

Clearly

$$\varphi \circ i_R(a) = i_S(a) \circ \varphi, \qquad a \in \wedge E.$$

Observe that we define only a homomorphism of operations of a fixed Lie algebra.

Clearly the composite of two homomorphisms of operations is again a homomorphism of operations.

7.2. Important formulae. In this section we obtain some relations connecting the operators $i(a)$, $\theta(x)$, and δ in R (for an operation $(E, i, \theta, R, \delta)$). They generalize formulae developed in article 1, Chapter V.

Recall the representation θ^E of E in $\wedge E$ (sec. 5.1) and the differential operator ∂_E (sec. 5.2). The relations to be established are

$$\theta(x) \circ i(a) - i(a) \circ \theta(x) = i(\theta^E(x)a), \qquad x \in E, \quad a \in \wedge E, \tag{7.5}$$

$$\begin{aligned} &i(x_1 \wedge \cdots \wedge x_p) \circ \delta + (-1)^{p-1}\delta \circ i(x_1 \wedge \cdots \wedge x_p) \\ &\quad = \sum_{\nu=1}^{p} (-1)^{\nu-1} i(x_p) \circ \cdots \circ i(x_{\nu+1}) \circ \theta(x_\nu) \circ i(x_{\nu-1}) \circ \cdots \circ i(x_1), \quad x_i \in E, \end{aligned} \tag{7.6}$$

$$i(a)\delta + (-1)^{p-1}\delta i(a) = -i(\partial_E a) + \sum_\varrho \theta(e_\varrho) i(i_E(e^{*\varrho})a), \quad a \in \wedge^p E, \tag{7.7}$$

and

$$i(a)\delta + (-1)^{p-1}\delta i(a) = i(\partial_E a) + \sum_\varrho i(i_E(e^{*\varrho})a)\theta(e_\varrho), \quad a \in \wedge^p E. \tag{7.8}$$

(Here $e^{*\varrho}$, e_ϱ denotes a pair of dual bases for E^* and E.)

Proof of (7.5) and (7.6): Use formula (7.2) to obtain

$$\begin{aligned}
&\theta(x) \circ i(x_1 \wedge \cdots \wedge x_p) - i(x_1 \wedge \cdots \wedge x_p)\theta(x) \\
&\quad = \sum_{\nu=1}^{p} i(x_p) \circ \cdots \circ i(x_{\nu+1}) \circ [\theta(x)i(x_\nu) - i(x_\nu)\theta(x)] \circ i(x_{\nu-1}) \circ \cdots \circ i(x_1) \\
&\quad = i(\theta^E(x)(x_1 \wedge \cdots \wedge x_p)).
\end{aligned}$$

Formula (7.6) is proved in exactly the same way, with the aid of (7.3).

Proof of (7.7): Without loss of generality we may assume that $a = x_1 \wedge \cdots \wedge x_p$, $x_i \in E$. Then, in view of formula (7.6), we have only to prove that

$$\begin{aligned}
&-i(\partial_E a) + \sum_\varrho \theta(e_\varrho) \circ i(i_E(e^{*\varrho})a) \\
&\quad = \sum_{\nu=1}^{p} (-1)^{\nu-1} i(x_p) \circ \cdots \circ \theta(x_\nu) \circ \cdots \circ i(x_1).
\end{aligned}$$

But formula (7.5) yields

$$\begin{aligned}
&\sum_{\nu=1}^{p} (-1)^{\nu-1} i(x_p) \circ \cdots \circ \theta(x_\nu) \circ \cdots \circ i(x_1) \\
&\quad = \sum_{\nu=1}^{p} (-1)^{\nu-1} \theta(x_\nu) \circ i(x_p) \circ \cdots \circ \widehat{i(x_\nu)} \circ \cdots \circ i(x_1) \\
&\qquad - \sum_{\nu=1}^{p} (-1)^{\nu-1} i(\theta^E(x_\nu)(x_{\nu+1} \wedge \cdots \wedge x_p)) \circ i(x_{\nu-1}) \circ \cdots \circ i(x_1) \\
&\quad = \sum_\varrho \theta(e_\varrho) \circ i(i_E(e^{*\varrho})(x_1 \wedge \cdots \wedge x_p)) \\
&\qquad - \sum_{\nu<\mu} (-1)^{\nu-1} i(x_1 \wedge \cdots \hat{x}_\nu \cdots \wedge [x_\nu, x_\mu] \wedge \cdots \wedge x_p).
\end{aligned}$$

It remains to be shown that

$$\partial_E(x_1 \wedge \cdots \wedge x_p) = \sum_{\nu<\mu} (-1)^{\nu-1} x_1 \wedge \cdots \hat{x}_\nu \cdots \wedge [x_\nu, x_\mu] \cdots \wedge x_p.$$

But this follows at once from formula (5.6) sec. 5.3.

Proof of (7.8): Formula (7.5) and formula (5.5) of sec. 5.3 yield

$$\begin{aligned}
\sum_\varrho [\theta(e_\varrho) \circ i(i_E(e^{*\varrho})a) - i(i_E(e^{*\varrho})a)\theta(e_\varrho)] &= \sum_\varrho i(\theta^E(e_\varrho) i_E(e^{*\varrho})a) \\
&= 2i(\partial_E a).
\end{aligned}$$

Now (7.8) follows from (7.7).

7.3. The invariant, horizontal, and basic subalgebras. Let $(E, i, \theta, R, \delta)$ be an operation. Since θ is a representation of E in the graded differential algebra (R, δ), the invariant subspace

$$R_{\theta=0} = \bigcap_{x \in E} \ker \theta(x),$$

is a graded subalgebra, stable under δ. It is called the *invariant subalgebra*.

The inclusion map $R_{\theta=0} \to R$ induces a homomorphism

$$H(R_{\theta=0}) \to H(R).$$

Since θ induces the zero representation in $H(R)$, Theorem IV, sec. 4.10, gives

Proposition I: Let $(E, i, \theta, R, \delta)$ be an operation of E such that the representation θ is semisimple. Then the homomorphism

$$H(R_{\theta=0}) \to H(R)$$

is an isomorphism.

Next consider the subspace

$$R_{i=0} = \bigcap_{x \in E} \ker i(x).$$

Since the operators $i(x)$, $x \in E$, are homogeneous antiderivations, $R_{i=0}$ is a graded subalgebra of R. Moreover, it follows immediately from formula (7.2) that $R_{i=0}$ is stable under the operators $\theta(x)$, $x \in E$. Hence θ restricts to a representation of E in $R_{i=0}$. $R_{i=0}$ is called the *horizontal subalgebra* of R. It is simple to verify the formula

$$i(a)(u \cdot v) = (i(a)u) \cdot v, \qquad a \in \wedge E, \quad u \in R, \quad v \in R_{i=0}. \tag{7.9}$$

Observe that the horizontal subalgebra is, in general, not stable under δ.

Finally consider the subspace $(R_{i=0})_{\theta=0}$ $(= R_{i=0} \cap R_{\theta=0})$. It will be denoted by $R_{i=0,\theta=0}$. Since the operators $\theta(x)$, $x \in E$, are homogeneous derivations in $R_{i=0}$, $R_{i=0,\theta=0}$ is a graded subalgebra of R. It is called the *basic subalgebra.*

The basic subalgebra is stable under δ. In fact, let $z \in R_{i=0,\theta=0}$. Then formulae (7.4) and (7.3) yield

$$\theta(x)\delta z = \delta\theta(x)z = 0 \qquad \text{and} \qquad i(x)\delta z = \theta(x)z - \delta i(x)z = 0,$$

whence $\delta z \in R_{i=0,\theta=0}$.

The inclusion map $e_R\colon R_{i=0,\theta=0} \to R_{\theta=0}$ induces a homomorphism

$$e_R^{\#}\colon H(R_{i=0,\theta=0}) \to H(R_{\theta=0}).$$

Now let $\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations. Then φ restricts to homomorphisms

$$\varphi_{\theta=0}\colon R_{\theta=0} \to S_{\theta=0}, \qquad \varphi_{i=0}\colon R_{i=0} \to S_{i=0},$$

and

$$\varphi_{i=0,\theta=0}\colon R_{i=0,\theta=0} \to S_{i=0,\theta=0},$$

as follows from the definitions. Moreover, $\varphi_{\theta=0}$ and $\varphi_{i=0,\theta=0}$ are homomorphisms of graded differential algebras. Thus we have the commutative diagram

$$\begin{tikzcd}
H(R_{i=0,\theta=0}) \arrow[r, "\varphi^{\#}_{i=0,\theta=0}"] \arrow[d, "e_R^{\#}"'] & H(S_{i=0,\theta=0}) \arrow[d, "e_S^{\#}"] \\
H(R_{\theta=0}) \arrow[r, "\varphi^{\#}_{\theta=0}"'] & H(S_{\theta=0}).
\end{tikzcd}$$

§2. Examples of operations

7.4. Examples. 1. *Actions of Lie groups*: Let $M \times G \to M$ (or $G \times M \to M$) be a smooth action of a Lie group on a manifold M. This action induces an operation of the Lie algebra of G in the algebra of differential forms on M. This particular example is discussed in detail in article 6.

2. *The operation on* $\wedge E^*$: Let E be a Lie algebra and recall the definition of the operators $i_E(x)$, $\theta_E(x)$, and δ_E in $\wedge E^*$ (cf. secs. 5.1, 5.2). Then formulae (5.1) and (5.3) imply that $(E, i_E, \theta_E, \wedge E^*, \delta_E)$ is an operation of E in $\wedge E^*$. In this case we have

$$(\wedge E^*)_{i=0} = (\wedge E^*)_{i=0,\theta=0} = \Gamma.$$

3. *Restriction*: Let $(E, i, \theta, R, \delta)$ be an operation of a Lie algebra E. Let F be a subalgebra of E. Restricting i and θ to F, we obtain an operation $(F, i_F, \theta_F, R, \delta)$ of F in (R, δ), which will be called the *restriction* of the original operation to F. The invariant, horizontal, and basic subalgebras for $(F, i_F, \theta_F, R, \delta)$ will be denoted by $R_{\theta_F=0}$, $R_{i_F=0}$, and $R_{i_F=0,\theta_F=0}$.

A homomorphism of operations $\varphi: (E, i, \theta, R, \delta) \to (E, \tilde{i}, \tilde{\theta}, \tilde{R}, \tilde{\delta})$ may be considered as a homomorphism of operations of F.

4. *Subalgebras*: Let F be a subalgebra of a Lie algebra E. Then the restriction of the operation $(E, i_E, \theta_E, \wedge E^*, \delta_E)$ to F is denoted by $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. This operation will be studied extensively in Chapter X.

5. *The tensor product operation*: Let $(E, i_R, \theta_R, R, \delta_R)$ and $(E, i_S, \theta_S, S, \delta_S)$ be operations of E. Then an operation $(E, i, \theta, R \otimes S, \delta)$ is defined by

$$i(x) = i_R(x) \otimes \iota + \omega_R \otimes i_S(x), \qquad \theta(x) = \theta_R(x) \otimes \iota + \iota \otimes \theta_S(x),$$

and

$$\delta = \delta_R \otimes \iota + \omega_R \otimes \delta_S, \qquad x \in E,$$

(where ω_R denotes the degree involution in R). It is called the *tensor product operation.*

6. *The operation in the Weil algebra*: Let $W(E)$ be the Weil algebra of a Lie algebra E. Then $(E, i, \theta_W, W(E), \delta_W)$ is an operation of E (cf. formula (6.5), sec. 6.4). The horizontal and the basic subalgebras are given by

$$W(E)_{i=0} = \vee E^* \otimes 1 \qquad \text{and} \qquad W(E)_{i=0,\theta=0} = (\vee E^*)_{\theta=0} \otimes 1.$$

7. Let θ_T be a representation of E in a graded anticommutative algebra T and let θ denote the induced representation in $T \otimes \wedge E^*$

$$\theta(x) = \theta_T(x) \otimes \iota + \iota \otimes \theta_E(x), \qquad x \in E.$$

Define a differential operator δ in $T \otimes \wedge E^*$ by $\delta = \delta_E + \delta_\theta$ (cf. sec. 5.25). Then $(E, i_E, \theta, T \otimes \wedge E^*, \delta)$ is an operation, as follows from formula (5.17), sec. 5.25. In this case we have

$$(T \otimes \wedge E^*)_{i=0} = T \otimes 1 \qquad \text{and} \qquad (T \otimes \wedge E^*)_{i=0,\theta=0} = T_{\theta=0} \otimes 1.$$

7.5. The associated semisimple operation. Let $(E, i, \theta, R, \delta)$ be an operation of a reductive Lie algebra. We shall construct a graded subalgebra R_S of R with the following properties:

(i) R_S is stable under the operators $i(x)$, $\theta(x)$ $(x \in E)$, and δ.

(ii) If $\theta_S(x)$ denotes the restriction of $\theta(x)$ to R_S $(x \in E)$, then θ_S is a semisimple representation of E.

A subspace $X \subset R$ will be called *admissible* if

(i) $\dim X < \infty$,

(ii) X is stable under $\theta(x)$, $x \in E$, and

(iii) The restrictions $\theta_X(x)$ of $\theta(x)$ to X $(x \in X)$ define a semisimple representation of E in X.

Given two subspaces $X \subset R$ and $Y \subset R$ we shall denote by $X \cdot Y$ the subspace spanned by the products $u \cdot v$ $(u \in X,\ v \in Y)$.

Lemma I: (1) If X and Y are admissible subspaces of R, then so are $X + Y$ and $X \cdot Y$.

(2) If X is an admissible subspace of R, then so is $\varrho^p(X)$, $p = 0, 1, \ldots$, where $\varrho^p \colon R \to R^p$ is the projection with kernel $\sum_{j \neq p} R^j$.

Proof: (1) Addition and multiplication define surjective linear maps

$$X \oplus Y \to X + Y \qquad \text{and} \qquad X \otimes Y \to X \cdot Y.$$

Moreover, these maps are E-linear with respect to the representations $\theta_{X \oplus Y}$ and $\theta_{X \otimes Y}$ determined by θ_X and θ_Y (cf. sec. 4.2).

But it follows from Theorem III, sec. 4.4, that $\theta_{X \oplus Y}$ and $\theta_{X \otimes Y}$ are semisimple representations, since θ_X and θ_Y are, and E is reductive. Hence $X + Y$ and $X \cdot Y$ are admissible.

(2) Observe that ϱ^p restricts to a surjective E-linear map from X to $\varrho^p(X)$.

Q.E.D.

Lemma II: Let $X \subset R$ be an admissible subspace. Then $X + \delta(X)$ is admissible.

Proof: Apply the argument of Lemma I to the linear map $X \oplus X \to X + \delta(X)$ given by

$$u \oplus v \mapsto u + \delta v, \qquad u, v \in X.$$

Q.E.D.

Lemma III: Let $X \subset R$ be an admissible subspace, and suppose $\beta\colon E \otimes X \to R$ is the linear map given by

$$\beta(x \otimes u) = i(x)u, \qquad x \in E, \quad u \in X.$$

Then $\operatorname{Im} \beta$ is admissible.

Proof: Use formula (7.2), sec. 7.1, to show that

$$\beta \circ (\operatorname{ad} x \otimes \iota + \iota \otimes \theta(x)) = \theta(x) \circ \beta, \qquad x \in E$$

and apply the argument of Lemma I.

Q.E.D.

A vector $z \in R$ is called *admissible* if there exists an admissible subspace $X \subset R$ such that $z \in X$.

Proposition II: The admissible vectors form a graded subalgebra R_S of R. This subalgebra is stable under the operators $i(x)$, $\theta(x)$ $(x \in E)$, and δ.

Proof: It follows immediately from Lemma I that the admissible vectors form a graded subalgebra R_S of R. Clearly, R_S is stable under the operators $\theta(x)$ $(x \in E)$. Lemmas II and III show that R_S is stable under the operators δ and $i(x)$.

Q.E.D.

It follows from Proposition II that the representation θ restricts to a representation θ_S of E in R_S.

Proposition III: The representation θ_S is semisimple.

Proof: Let $X \subset R_S$ be any stable subspace. Choose a subspace $Y \subset R_S$ which is maximal with respect to the properties

$$X \cap Y = 0 \qquad \text{and} \qquad \theta_S(x)(Y) \subset Y, \qquad x \in E.$$

We shall show that $X \oplus Y = R_S$.

Let $z \in R_S$ and choose an admissible subspace, $Z \subset R_S$, such that $z \in Z$. Then $Z \cap (X \oplus Y)$ is an E-stable subspace of R_S. Since θ_Z is semisimple, there is an E-stable subspace U such that

$$Z = Z \cap (X \oplus Y) \oplus U.$$

This implies that $(X \oplus Y) \cap U = 0$, and hence the maximality of Y yields $U = 0$.

It follows that $z \in Z \subset X \oplus Y$, and so $R_S = X \oplus Y$.

Q.E.D.

Definition: Let $i_S(x)$, $\theta_S(x)$ $(x \in E)$ and δ_S denote the restrictions of $i(x)$, $\theta(x)$, and δ to R_S. Then the operation $(E, i_S, \theta_S, R_S, \delta_S)$ is called the *semisimple operation associated with* $(E, i, \theta, R, \delta)$.

Since every element $z \in R_{\theta=0}$ generates a 1-dimensional admissible subspace, we have $R_{\theta=0} \subset R_S$. This relation implies that

$$R_{\theta=0} = (R_S)_{\theta=0} \qquad \text{and} \qquad R_{i=0,\theta=0} = (R_S)_{i=0,\theta=0}.$$

Because θ_S is semisimple, the homomorphism $H(R_{\theta=0}) \to H(R_S)$ induced by inclusion is an isomorphism (cf. Proposition I, sec. 7.3).

7.6. Tensor products. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a Lie algebra. Let θ_M be a representation of E in a graded differential

space (M, δ_M). Define operators $\theta(x)$ $(x \in E)$ and δ in $M \otimes R$ by

$$\theta(x) = \theta_M(x) \otimes \iota + \iota \otimes \theta_R(x) \qquad \text{and} \qquad \delta = \delta_M \otimes \iota + \omega_M \otimes \delta_R,$$

where ω_M denotes the degree involution in M.

Proposition IV: Assume that E is reductive and that at least one of the representations θ_R and θ_M is quasi-semisimple (cf. sec. 4.3). Then the inclusion map

$$g\colon M_{\theta=0} \otimes R_{\theta=0} \to (M \otimes R)_{\theta=0}$$

induces an isomorphism

$$g^{\#}\colon H(M_{\theta=0}) \otimes H(R_{\theta=0}) \xrightarrow{\cong} H((M \otimes R)_{\theta=0}).$$

Proof: We show first that

$$(M \otimes R)_{\theta=0} = (M \otimes R_S)_{\theta=0}, \tag{7.10}$$

where $(E, i_S, \theta_S, R_S, \delta_R)$ denotes the associated semisimple operation (cf. sec. 7.5). In fact, clearly

$$(M \otimes R_S)_{\theta=0} \subset (M \otimes R)_{\theta=0}.$$

On the other hand, let $\Psi \in (M \otimes R)_{\theta=0}$. Then Proposition I, sec. 4.3, shows that there is a finite dimensional subspace $Z \subset R$, stable under the operators $\theta_R(x)$, and such that $\Psi \in (M \otimes Z)_{\theta=0}$, and the induced representation θ_Z of E in Z is semisimple. Thus $\Psi \in (M \otimes R_S)_{\theta=0}$, and so (7.10) is proved.

Since also $(R_S)_{\theta=0} = R_{\theta=0}$, we may replace R by R_S. Thus in view of Proposition III, sec. 7.5, we may assume that the representation θ_R is semisimple.

On the other hand, the equations

$$\theta_R(x) = i_R(x)\delta_R + \delta_R i_R(x), \qquad x \in E,$$

imply that each $\theta_R(x)^{\#} = 0$. Now the proposition follows from Theorem V, sec. 4.11 (with $M = Y$ and $R = X$).

Q.E.D.

Corollary: If E is semisimple, $(E, i_R, \theta_R, R, \delta_R)$ is any operation, and θ_M is a representation of E in a graded differential space (M, δ_M), then the map

$$g^{\#}: H(M_{\theta=0}) \otimes H(R_{\theta=0}) \to H((R \otimes M)_{\theta=0})$$

is an isomorphism.

Proof: Observe (via Theorem I, sec. 4.4) that every representation of a semisimple Lie algebra is quasi-semisimple and apply the proposition.

Q.E.D.

§3. The structure homomorphism

In this article $(E, i_R, \theta_R, R, \delta_R)$ denotes an operation; ω_R denotes the degree involution in R.

7.7. The structure operation. Consider the (skew) tensor product of graded algebras $R \otimes \wedge E^*$, and define operators $i_{R\otimes E}(x)$, $\theta_{R\otimes E}(x)$ $(x \in E)$, and $\delta_{R\otimes E}$ in $R \otimes \wedge E^*$ by

$$i_{R\otimes E}(x) = \omega_R \otimes i_E(x), \qquad \theta_{R\otimes E}(x) = \theta_R(x) \otimes \iota + \iota \otimes \theta_E(x), \qquad x \in E,$$

and

$$\delta_{R\otimes E} = \delta_R \otimes \iota + \delta_\theta + \omega_R \otimes \delta_E.$$

Here δ_θ is the operator defined in sec. 5.25 (with $R = M$); i.e.,

$$\delta_\theta = \sum_\nu \omega_R \theta_R(e_\nu) \otimes \mu(e^{*\nu}).$$

Lemma IV: The 5-tuple $(E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E})$ is an operation.

Proof: Clearly $\theta_{R\otimes E}$ is a representation. Relations (7.1), (7.2), and (7.3) follow from an easy computation (cf. sec. 5.1 and sec. 5.25). Thus we have only to verify that $\delta^2_{R\otimes E} = 0$.

Evidently $\delta_R^2 = 0$. Next, observe that the equations

$$\delta_R \theta_R(x) = \theta_R(x) \delta_R, \qquad x \in E,$$

imply that

$$(\delta_R \otimes \iota) \circ (\delta_\theta + \omega_R \otimes \delta_E) + (\delta_\theta + \omega_R \otimes \delta_E) \circ (\delta_R \otimes \iota) = 0.$$

Moreover, in sec. 5.25 it was shown that

$$(\delta_\theta + \omega_R \otimes \delta_E)^2 = 0.$$

Hence, $\delta^2_{R\otimes E} = 0$.

Q.E.D.

The operation $(E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E})$ is called the *structure operation* for $(E, i_R, \theta_R, R, \delta_R)$.

On the other hand, form the tensor product operation for

$$(E, i_R, \theta_R, R, \delta_R) \qquad \text{and} \qquad (E, i_E, \theta_E, \wedge E^*, \delta_E)$$

(cf. Example 5, sec. 7.4). It is given by $(E, i, \theta, R \otimes \wedge E^*, \delta)$, where

$$i(x) = i_R(x) \otimes \iota + \omega_R \otimes i_E(x), \qquad \theta(x) = \theta_R(x) \otimes \iota + \iota \otimes \theta_E(x),$$

and

$$\delta = \delta_R \otimes \iota + \omega_R \otimes \delta_E.$$

We shall construct an isomorphism of operations

$$(E, i, \theta, R \otimes \wedge E^*, \delta) \xrightarrow{\cong} (E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E}).$$

First, define a derivation α homogeneous of degree zero in $R \otimes \wedge E^*$, by

$$\alpha = \sum_\nu \omega_R i_R(e_\nu) \otimes \mu(e^{*\nu}),$$

where $e^{*\nu}$, e_ν is a pair of dual bases of E^* and E. Since $\alpha^p = 0$ ($p > \dim E$), we can form the map

$$\beta = \sum_{p=0}^{\infty} \frac{1}{p!} \alpha^p.$$

It is an automorphism of the graded algebra $R \otimes \wedge E^*$.

The following explicit form for β will be useful: For $a \in \wedge^p E$, define $j(a)\colon R \to R$ by

$$j(a)(z) = (-1)^{p(q-1)} i_R(a)(z), \qquad z \in R^q.$$

Then an easy computation yields

$$j(a \wedge b) = j(a) \circ j(b). \tag{7.11}$$

Lemma V: (1) Let Φ^λ, a_λ be a pair of dual bases for $\wedge E^*$ and $\wedge E$. Then

$$\beta = \sum_\lambda j(a_\lambda) \otimes \mu(\Phi^\lambda).$$

(2) Let φ be a linear transformation of a finite-dimensional vector space X, and let x_λ, $x^{*\lambda}$ be a pair of dual bases for X and X^*. Then

$$\sum_\lambda \varphi x_\lambda \otimes x^{*\lambda} = \sum_\lambda x_\lambda \otimes \varphi^* x^{*\lambda}.$$

In particular, if $X = \wedge E$, then

$$\sum_\lambda j(\varphi a_\lambda) \otimes \mu(\Phi^\lambda) = \sum_\lambda j(a_\lambda) \otimes \mu(\varphi^*\Phi^\lambda).$$

Proof: (1) First observe that the expression $\sum_\lambda j(a_\lambda) \otimes \mu(\Phi^\lambda)$ is independent of the choice of dual bases. Now fix a basis $e_1, \dots, e_n$ of E and extend it to the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$, $i_1 < \cdots < i_p$ of $\wedge E$. The dual basis is $\{e^{*i_1} \wedge \cdots \wedge e^{*i_p}\}$, $i_1 < \cdots < i_p$, where $\langle e^{i*}, e_j \rangle = \delta^i_j$.

Evidently

$$\alpha = \sum_\nu j(e_\nu) \otimes \mu(e^{*\nu}).$$

It follows (via formula (7.11)) that

$$\alpha^p = \sum j(e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}) \otimes \mu(e^{*\nu_1} \wedge \cdots \wedge e^{*\nu_p}),$$

where the sum is over all p-tuples $(\nu_1, \dots, \nu_p)$ with $1 \le \nu_i \le n$, $i = 1, \dots, p$. Thus

$$\frac{1}{p!}\alpha^p = \sum_{1 \le \nu_1 < \cdots < \nu_p \le n} j(e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}) \otimes \mu(e^{*\nu_1} \wedge \cdots \wedge e^{*\nu_p}),$$

and so (1) follows.

(2) This is obvious.

Q.E.D.

Proposition V: With the notation above β is an isomorphism of operations

$$\beta\colon (E, i, \theta, R \otimes \wedge E^*, \delta) \xrightarrow{\cong} (E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E}).$$

Proof: We must check that β converts the operators $i(x)$, $\theta(x)$, and δ into $i_{R\otimes E}(x)$, $\theta_{R\otimes E}(x)$, and $\delta_{R\otimes E}$, respectively.

(1) *The operator* $i(x)$: Observe that

$$\begin{aligned}(\omega_R \otimes i_E(x)) \circ \alpha - \alpha \circ (\omega_R \otimes i_E(x)) &= \sum_\nu i_R(e_\nu) \otimes [i_E(x)\mu(e^{*\nu}) + \mu(e^{*\nu})i_E(x)] \\ &= i_R(x) \otimes \iota.\end{aligned}$$

It follows by induction on p that

$$(\omega_R \otimes i_E(x)) \circ \alpha^p = \alpha^p \circ (\omega_R \otimes i_E(x)) + p\alpha^{p-1} \circ (i_R(x) \otimes \iota), \qquad p \ge 1.$$

Hence

$$i_{R\otimes E}(x)\circ\beta=(\omega_R\otimes i_E(x))\circ\sum_{p=0}^{\infty}\frac{1}{p!}\alpha^p$$
$$=\left(\sum_{p=0}^{\infty}\frac{1}{p!}\alpha^p\right)\circ(i_R(x)\otimes\iota+\omega_R\otimes i_E(x))$$
$$=\beta\circ i(x).$$

(2) *The operator* $\theta(x)$: Observe that $\theta(x)=\theta_{R\otimes E}(x)=\theta_R(x)\otimes\iota+\iota\otimes\theta_E(x)$. Moreover

$$(\theta_R(x)\otimes\iota+\iota\otimes\theta_E(x))\circ\alpha-\alpha\circ(\theta_R(x)\otimes\iota+\iota\otimes\theta_E(x))$$
$$=\sum_\nu\omega_R i_R([x,e_\nu])\otimes\mu(e^{*\nu})+\sum_\nu\omega_R i_R(e_\nu)\otimes\mu(\theta_E(x)e^{*\nu})$$
$$=0$$

(by Lemma V, (2) with $X=E$ and $\varphi=\operatorname{ad}x$). Hence

$$\theta_{R\otimes E}(x)\circ\beta=\beta\circ\theta(x).$$

(3) *The operator* δ: Recall from sec. 7.2 that

$$i_R(a)\delta_R+(-1)^{p-1}\delta_R i_R(a)=-i_R(\partial_E a)+\sum_\varrho\theta_R(e_\varrho)i_R(i_E(e^{*\varrho})a),\quad a\in\wedge^pE.$$

It follows that

$$j(a)\delta_R-\delta_R j(a)=-\omega_R j(\partial_E a)+\sum_\varrho\omega_R\theta_R(e_\varrho)j(i_E(e^{*\varrho})a),\qquad a\in\wedge^pE.$$

Now apply Lemma V, (1) and (2), to obtain

$$\beta\circ\delta_R-\delta_R\circ\beta$$
$$=-\sum_\lambda\omega_R j(\partial_E a_\lambda)\otimes\mu(\Phi^\lambda)+\sum_{\lambda,\varrho}\omega_R\theta_R(e_\varrho)j(i_E(e^{*\varrho})a_\lambda)\otimes\mu(\Phi^\lambda)$$
$$=\sum_\lambda\omega_R j(a_\lambda)\otimes\mu(\delta_E\Phi^\lambda)+\sum_{\lambda,\varrho}\omega_R\theta_R(e_\varrho)j(a_\lambda)\otimes\mu(e^{*\varrho})\mu(\Phi^\lambda).$$

On the other hand, Lemma V, (1), also yields

$$\beta\circ(\omega_R\otimes\delta_E)-(\omega_R\otimes\delta_E)\circ\beta=-\sum_\lambda\omega_R j(a_\lambda)\otimes\mu(\delta_E\Phi^\lambda)$$

and

$$-\delta_\theta\circ\beta=-\sum_{\lambda,\varrho}\omega_R\theta_R(e_\varrho)j(a_\lambda)\otimes\mu(e^{*\varrho})\mu(\Phi^\lambda).$$

Adding these three relations, we find that

$$\beta \circ (\delta_R \otimes \iota + \omega_R \otimes \delta_E) = (\delta_R \otimes \iota + \delta_\theta + \omega_R \otimes \delta_E) \circ \beta.$$

Q.E.D.

7.8. The structure homomorphism. Define a homomorphism

$$\gamma_R \colon R \to R \otimes \wedge E^*$$

by

$$\gamma_R(z) = \beta(z \otimes 1), \qquad z \in R.$$

Proposition V, sec. 7.7, shows that the map

$$\gamma_R \colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E})$$

is a homomorphism of operations. It is called the *structure homomorphism* for the operation $(E, i_R, \theta_R, R, \delta_R)$.

If $\varphi\colon R \to S$ is a homomorphism of operations, then the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\gamma_R"'] & S \arrow[d, "\gamma_S"] \\ R \otimes \wedge E^* \arrow[r, "\varphi\otimes\iota"'] & S \otimes \wedge E^* \end{tikzcd}$$

commutes, as follows from Lemma V, (1), sec. 7.7.

Examples. 1. *The operation* $(E, i_E, \theta_E, \wedge E^*, \delta_E)$: Let $\mu\colon E \oplus E \to E$ be the addition map

$$\mu(x, y) = x + y, \qquad x, y \in E.$$

Then μ induces a homomorphism $\mu^\wedge\colon \wedge E^* \to \wedge E^* \otimes \wedge E^*$.

It follows from the definitions that

$$\mu^\wedge(x^*) = x^* \otimes 1 + 1 \otimes x^* = \gamma_{\wedge E^*}(x^*), \qquad x^* \in E^*.$$

Since E^* generates $\wedge E^*$, this implies that

$$\mu^\wedge = \gamma_{\wedge E^*}.$$

2. ***The homomorphism*** $\gamma_{R\otimes\wedge E^*}$: Consider the structure homomorphism

$$\gamma_{R\otimes\wedge E^*}: R \otimes \wedge E^* \to R \otimes \wedge E^* \otimes \wedge E^*.$$

Since

$$i_{R\otimes E}(x) = \omega_R \otimes i_E(x), \qquad x \in E,$$

it follows that

$$\gamma_{R\otimes\wedge E^*} = \iota \otimes \gamma_{\wedge E^*} = \iota \otimes \mu^\wedge$$

(cf. Example 1). Thus the commutative diagram above, applied with $\varphi = \gamma_R$ and $S = R \otimes \wedge E^*$, yields the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\gamma_R"] \arrow[d, "\gamma_R"'] & R \otimes \wedge E^* \arrow[d, "\iota\otimes\mu^\wedge"] \\
R \otimes \wedge E^* \arrow[r, "\gamma_R\otimes\iota"'] & R \otimes \wedge E^* \otimes \wedge E^*.
\end{tikzcd}$$

This diagram shows that γ_R makes R into a comodule over the coalgebra $\wedge E^*$.

7.9. The cohomology structure homomorphism. In this section we assume E to be reductive. Since the structure homomorphism is a homomorphism of operations, it induces a homomorphism

$$(\gamma_R)^\#_{\theta=0}: H(R_{\theta=0}) \to H((R \otimes \wedge E^*)_{\theta=0}, \delta_{R\otimes E}).$$

On the other hand, we have the inclusion $g: R_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (R \otimes \wedge E^*)_{\theta=0}$. Since E is reductive, Proposition IV, sec. 7.6 (applied with $R = M$ and $\wedge E^* = R$), implies that g induces an isomorphism

$$g^\#: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H((R \otimes \wedge E^*)_{\theta=0}, \delta_{R\otimes E})$$

of graded algebras. (Note that $\delta_{R\otimes E}$ reduces to $\delta_R \otimes \iota - \omega_R \otimes \delta_E$ in $(R \otimes \wedge E^*)_{\theta=0}$.)

Thus we may compose $(g^\#)^{-1}$ with $(\gamma_R)^\#_{\theta=0}$ to obtain a homomorphism

$$\hat{\gamma}_R: H(R_{\theta=0}) \to H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}.$$

Definition: $\hat{\gamma}_R$ is called the *cohomology structure homomorphism* for the operation $(E, i_R, \theta_R, R, \delta_R)$.

Suppose $\varphi: R \to S$ is a homomorphism of operations of E. Then the diagram

$$\begin{tikzcd} H(R_{\theta=0}) \arrow[r, "\varphi^\#_{\theta=0}"] \arrow[d, "\hat{\gamma}_R"'] & H(S_{\theta=0}) \arrow[d, "\hat{\gamma}_S"] \\ H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \arrow[r, "\varphi^\#_{\theta=0}\otimes\iota"'] & H(S_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \end{tikzcd} \tag{7.12}$$

commutes, as follows from the analogous property for γ_R and γ_S (cf. sec. 7.8).

Proposition VI: Let $q: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \to H(R_{\theta=0})$ be the projection with kernel $H(R_{\theta=0}) \otimes (\wedge^+ E^*)_{\theta=0}$. Then $q \circ \hat{\gamma}_R = \iota$.

In particular, $\hat{\gamma}_R$ is injective.

Proof: Consider the projection

$$p: R \otimes \wedge E^* \to R$$

with kernel $R \otimes \wedge^+ E^*$. Then p induces a homomorphism

$$p^\#_{\theta=0}: H((R \otimes \wedge E^*)_{\theta=0}, \delta_{R\otimes E}) \to H(R_{\theta=0}).$$

Since (clearly) $p \circ \gamma_R = \iota$, it follows that

$$p^\#_{\theta=0} \circ (\gamma_R)^\#_{\theta=0} = \iota.$$

Thus

$$q \circ \hat{\gamma}_R = q \circ (g^\#)^{-1} \circ (\gamma_R)^\#_{\theta=0} = p^\#_{\theta=0} \circ (\gamma_R)^\#_{\theta=0} = \iota,$$

and the proposition is proved.

Q.E.D.

Example: *The operation* $(E, i_E, \theta_E, \wedge E^*, \delta_E)$. The cohomology structure homomorphism for this operation coincides with the comultiplication in $(\wedge E^*)_{\theta=0}$ as defined in sec. 5.17,

$$\hat{\gamma}_{\wedge E^*} = \gamma_E: (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}.$$

To see this, let $\eta: \wedge E^* \otimes \wedge E^* \to (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}$ be the projection with kernel

$$\theta_{E\oplus E}(\wedge E^* \otimes \wedge E^*) = \theta(\wedge E^*) \otimes \wedge E^* + \wedge E^* \otimes \theta(\wedge E^*).$$

Then $\gamma_E = \eta \circ \mu^\wedge_{\theta=0} = \eta \circ (\gamma_{\wedge E^*})_{\theta=0}$ (cf. Example 1, sec. 7.8).

On the other hand, note that

$$\eta \circ (\delta_E \otimes \iota) = 0, \qquad \eta \circ \delta_\theta = 0, \qquad \text{and} \qquad \eta \circ (\omega_E \otimes \delta_E) = 0.$$

Hence η induces a linear map

$$\eta^{\#}: H((\wedge E^* \otimes \wedge E^*)_{\theta_E=0}, \delta_{\wedge E^* \otimes E}) \to (\wedge E^*)_{\theta=0} \otimes (\wedge E^*)_{\theta=0}.$$

Moreover, because $\eta \circ g = \iota$, it follows that $\eta^{\#} = (g^{\#})^{-1}$. Thus

$$\hat{\gamma}_{\wedge E^*} = \eta^{\#} \circ (\gamma_{\wedge E^*})^{\#}_{\theta=0} = [\eta \circ (\gamma_{\wedge E^*})_{\theta=0}]^{\#} = \gamma_E.$$

Remark: This example provides a second proof that the comultiplication γ_E is an algebra homomorphism (cf. sec. 5.17).

§4. Fibre projection

In this article $(E, i_R, \theta_R, R, \delta_R)$ denotes an operation of a reductive Lie algebra. The algebra $H(R_{\theta=0})$ is assumed to be connected.

7.10. Definition. Recall from sec. 7.9 the cohomology structure homomorphism

$$\hat{\gamma}_R: H(R_{\theta=0}) \to H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}.$$

Since $H(R_{\theta=0})$ is connected, we have the canonical projection $H(R_{\theta=0}) \to \Gamma$; it induces a homomorphism

$$\pi_R: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0}.$$

Hence a homomorphism

$$\varrho_R: H(R_{\theta=0}) \to (\wedge E^*)_{\theta=0}$$

is given by $\varrho_R = \pi_R \circ \hat{\gamma}_R$. It is called the *fibre projection for the operation.*

Since

$$\gamma_R \circ e_R(z) = \beta(z \otimes 1) = z \otimes 1 = g(z \otimes 1), \qquad z \in R_{i=0,\theta=0},$$

(cf. Lemma V, (1), sec. 7.7), it follows that $\varrho_R \circ (e_R^{\#})^+ = 0$.

If $\varphi: R \to S$ is a homomorphism of operations (with $H(S_{\theta=0})$ connected), we have

$$\varrho_S \circ \varphi_{\theta=0}^{\#} = \varrho_R$$

(cf. sec. 7.8 and 7.9).

7.11. Projectable operations. The operation of E in R will be called *projectable* if there is a linear map $q: R \to \Gamma$ (not necessarily an algebra homomorphism) which satisfies the conditions

(i) $q(z) = 0, \; z \in R^+$,

(ii) $q(1) = 1$,

and

(iii) $q \circ \theta(x) = 0, \; x \in E$.

Thus the operation is projectable if and only if $1 \notin \theta(R^0)$.

In particular, the operation is projectable if one of the following three conditions holds:

(1) $R^0 = R^0_{\theta=0} \oplus \theta(R^0)$;

(2) the representation of E in R^0 is semisimple;

(3) R is connected; i.e., $R^0 = \Gamma$.

Now assume the operation $(E, i_R, \theta_R, R, \delta_R)$ is projectable, with projection $q: R \to \Gamma$. Consider the linear map

$$q \otimes \iota: R \otimes \wedge E^* \to \wedge E^*.$$

It satisfies

$$\theta_E(x) \circ (q \otimes \iota) = (q \otimes \iota) \circ \theta_{R\otimes E}(x), \qquad x \in E$$

and

$$\delta_E \circ (q \otimes \iota) = (q \otimes \iota) \circ \delta_{R\otimes E}.$$

Hence it induces a linear map

$$(q \otimes \iota)^{\#}_{\theta=0}: H((R \otimes \wedge E^*)_{\theta=0}, \delta_{R\otimes E}) \to (\wedge E^*)_{\theta=0}.$$

Proposition VII: The fibre projection of a projectable operation is given by

$$\varrho_R = (q \otimes \iota)^{\#}_{\theta=0} \circ (\gamma_R)^{\#}_{\theta=0}.$$

Proof: Observe that q restricts to a linear map $q_{\theta=0}: R_{\theta=0} \to \Gamma$, and that

$$\pi_R = q^{\#}_{\theta=0} \otimes \iota.$$

Hence $\varrho_R = (q^{\#}_{\theta=0} \otimes \iota) \circ (g^{\#})^{-1} \circ (\gamma_R)^{\#}_{\theta=0}$.

On the other hand, clearly

$$(q \otimes \iota)_{\theta=0} \circ g = q_{\theta=0} \otimes \iota,$$

whence $(q \otimes \iota)^{\#}_{\theta=0} = (q^{\#}_{\theta=0} \otimes \iota) \circ (g^{\#})^{-1}$. This in turn implies that $\varrho_R = (q \otimes \iota)^{\#}_{\theta=0} \circ (\gamma_R)^{\#}_{\theta=0}$.

Q.E.D.

Example: *The operation* $(E, i_E, \theta_E, \wedge E^*, \delta_E)$. Since $\wedge E^*$ is connected, this operation is projectable. We show that its fibre projection is the identity map

$$\varrho_{\wedge E^*} = \iota: (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0}.$$

In fact, let $q: \wedge E^* \to \Gamma$ be the projection. Since (cf. Example 1, sec. 7.8) $\gamma_{\wedge E^*} = \mu^\wedge$, we have

$$(q \otimes \iota) \circ \gamma_{\wedge E^*} = \iota.$$

Now Proposition VII implies that $\varrho_{\wedge E^*} = \iota$.

7.12. The operators $i_R(a)^\#$. Fix an element $a \in (\wedge^p E)_{\theta=0}$ and consider the operator $i_R(a)$ in R (cf. sec. 7.1). In view of formula (7.5), the invariant subalgebra is stable under this operator. Moreover, formula (7.8) implies that

$$i_R(a)\delta_R(z) + (-1)^{p-1}\delta_R i_R(a)(z) = 0, \qquad z \in R_{\theta=0}.$$

Thus $i_R(a)$ induces an operator

$$i_R(a)^\#: H(R_{\theta=0}) \to H(R_{\theta=0}),$$

homogeneous of degree $-p$.

Proposition VIII: (1) The cohomology structure homomorphism $\hat{\gamma}_R$ satisfies

$$\hat{\gamma}_R \circ i_R(a)^\# = (\omega_R^\# \otimes i_E(a)) \circ \hat{\gamma}_R, \qquad a \in (\wedge E)_{\theta=0}$$

(cf. sec. 7.9).

(2) The fibre projection satisfies

$$\varrho_R \circ i_R(a)^\# = i_E(a) \circ \varrho_R.$$

(3) If a is primitive (i.e., if $a \in P_*(E)$—cf. sec. 5.14), then $i_R(a)^\#$ is an antiderivation.

(4) $i_R(a)^\# \circ e_R^\# = 0$, $a \in (\wedge^+ E)_{\theta=0}$.

Proof: (1) Since $\gamma_R: R \to R \otimes \wedge E^*$ is a homomorphism of operations, we have

$$\gamma_R \circ i_R(a) = i_{R\otimes E}(a) \circ \gamma_R = (\omega_R \otimes i_E(a)) \circ \gamma_R, \qquad a \in (\wedge E)_{\theta=0}.$$

On the other hand, the inclusion $g: R_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (R \otimes \wedge E^*)_{\theta=0}$ commutes with $\omega_R \otimes i_E(a)$, and so (1) follows.

(2) This follows from (1), and the relation $\varrho_R = \pi_R \circ \hat{\gamma}_R$ (cf. sec. 7.10).

(3) This is an immediate consequence of (1), together with the fact

that $i_E(a)$ is an antiderivation (cf. Lemma VIII, sec. 5.22) and the fact that $\hat{\gamma}_R$ is injective (cf. Proposition VI, sec. 7.9).

(4) This follows from the relation $i_R(a) \circ e_R = 0$.

Q.E.D.

7.13. The Samelson subspace. Identify $\wedge P_E$ with $(\wedge E^*)_{\theta=0}$ under the canonical isomorphism $\varkappa_E$ of sec. 5.18. Then the fibre projection for the operation $(E, i_R, \theta_R, R, \delta_R)$ is a homomorphism

$$\varrho_R: H(R_{\theta=0}) \to \wedge P_E.$$

Definition: The graded subspace $\hat{P}_R \subset P_E$ given by

$$\hat{P}_R = P_E \cap \operatorname{Im} \varrho_R$$

is called the *Samelson subspace for the operation* $(E, i_R, \theta_R, R, \delta_R)$.

Theorem I: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra, with $H(R_{\theta=0})$ connected. Then the subalgebra $\operatorname{Im} \varrho_R$ is the exterior algebra over the Samelson subspace

$$\operatorname{Im} \varrho_R = \wedge \hat{P}_R.$$

Proof: Let $i_P(a): \wedge P_E \to \wedge P_E$ $(a \in P_*(E))$ be the substitution operator. In view of Lemma IX, sec. 5.22, and Proposition VIII, sec. 7.12, we have

$$\varrho_R \circ i_R(a)^\# = i_P(a) \circ \varrho_R, \qquad a \in P_*(E).$$

Thus the subalgebra $\operatorname{Im} \varrho_R$ is stable under the operators $i_P(a)$. Now it follows from Proposition I, sec. 0.4, that

$$\operatorname{Im} \varrho_R = \wedge(\operatorname{Im} \varrho_R \cap P_E) = \wedge \hat{P}_R.$$

Q.E.D.

7.14. Basic factors. A graded subalgebra $A \subset H(R_{\theta=0})$ will be called a *basic factor* for the operation $(E, i_R, \theta_R, R, \delta_R)$ if it satisfies the following conditions:

(1) $\operatorname{Im} e_R^\# \subset A$.

(2) There is an isomorphism of graded algebras

$$f: A \otimes \wedge \hat{P}_R \xrightarrow{\cong} H(R_{\theta=0}),$$

where $A \otimes \wedge \hat{P}_R$ denotes the skew symmetric tensor product.

(3) The diagram

$$\begin{tikzcd}
& A \otimes \wedge \hat{P}_R \arrow[r] \arrow[dd, "\cong"', "f"] & \wedge \hat{P}_R \arrow[dd] \\
A \arrow[ur] \arrow[dr] & & \\
& H(R_{\theta=0}) \arrow[r, "\varrho_R"] & \wedge P_E
\end{tikzcd}$$

commutes.

Two basic factors A_1 and A_2 will be called *equivalent* if there is an isomorphism of graded algebras $A_1 \xrightarrow{\cong} A_2$ which restricts to the identity in $\operatorname{Im} e_R^\#$.

It follows immediately from the definition that such an isomorphism extends to an automorphism of $H(R_{\theta=0})$, which makes the diagram

$$\begin{tikzcd}
& H(R_{\theta=0}) \arrow[dr, "\varrho_R"] \arrow[dd, "\cong"'] & \\
H(R_{i=0,\theta=0}) \arrow[ur, "e_R^*"] \arrow[dr, "e_R^*"'] & & (\wedge E^*)_{\theta=0} \\
& H(R_{\theta=0}) \arrow[ur, "\varrho_R"'] &
\end{tikzcd}$$

commute.

Theorem II (reduction theorem): Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra E with $H(R_{\theta=0})$ connected. Then the operation admits a basic factor, and any two basic factors are equivalent.

Proof: *Existence*: Identify $\wedge \hat{P}_R$ with $\operatorname{Im} \varrho_R$ via $\varkappa_E$, so that ϱ_R becomes a surjection onto $\wedge \hat{P}_R$. Let $P \subset P_*(E)$ be a graded subspace dual to $\hat{P}_R$ (cf. sec. 5.21). Then it follows from Proposition VIII, sec. 7.12, that

(1) $\varrho_R\colon H(R_{\theta=0}) \to \wedge \hat{P}_R$ is a surjective algebra homomorphism, and satisfies

$$\varrho_R \circ i_R(a)^\# = i_P(a) \circ \varrho_R, \qquad a \in P.$$

(2) The operators $i_R(a)^\#$ $(a \in P)$ are antiderivations and satisfy $(i_R(a)^\#)^2 = 0$.

(3) $\operatorname{Im} e_R^\# \subset \bigcap_{a\in P} \ker i_R(a)^\#$.

Thus the hypotheses of Proposition X, sec. 7.16, in article 5 below are satisfied. It follows that the subalgebra

$$A = \bigcap_{a\in P} \ker i_R(a)^\#$$

is a basic factor.

Uniqueness: Suppose A_1 and A_2 are two basic factors. Then we have the commutative diagram

$$\begin{tikzcd}
 & & A_1 \otimes \wedge \hat{P}_R \arrow[dd, "\cong"'] \arrow[ddrr] & & \\
 & A_1 \arrow[ur] \arrow[dr] & & & \\
H(R_{i=0,\theta=0}) \arrow[ur] \arrow[dr] & & H(R_{\theta=0}) \arrow[rr] & & \wedge \hat{P}_R \\
 & A_2 \arrow[ur] \arrow[dr] & & & \\
 & & A_2 \otimes \wedge \hat{P}_R \arrow[uu, "\cong"] \arrow[uurr] & &
\end{tikzcd}$$

which yields the commutative diagram

$$\begin{tikzcd}
 & A_1 \otimes \wedge \hat{P}_R \arrow[dd, "\cong"'] \arrow[dr] & \\
H(R_{i=0,\theta=0}) \arrow[ur] \arrow[dr] & & \wedge \hat{P}_R \\
 & A_2 \otimes \wedge \hat{P}_R \arrow[ur] &
\end{tikzcd}$$

Now the equivalence of A_1 and A_2 follows from Proposition XI, sec. 7.18, in article 5, below.

Q.E.D.

7.15. The cohomology structure homomorphism. In this section we express the homomorphism $\hat{\gamma}_R$ (cf. sec. 7.9) in terms of the operators

$i_R(a)^\#$. Recall from sec. 5.22 that the canonical isomorphisms

$$\varkappa_*: \wedge P_*(E) \xrightarrow{\cong} (\wedge E)_{\theta=0} \qquad \text{and} \qquad \varkappa_E: \wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0}$$

preserve scalar products. As in sec. 7.7 define a derivation $\hat{\alpha}$ in the algebra $H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}$ by

$$\hat{\alpha} = \sum_\nu \omega_R^\# i_R(a_\nu)^\# \otimes \mu(a^{*\nu}),$$

where $a^{*\nu}$, a_ν is a pair of dual bases of P_E and $P_*(E)$. Then an automorphism $\hat{\beta}$ of $H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}$ is defined by

$$\hat{\beta} = \sum_{p=0}^{\infty} \frac{1}{p!} \hat{\alpha}^p.$$

Proposition IX: With the notation above, the cohomology structure homomorphism is given by

$$\hat{\gamma}_R(\zeta) = \hat{\beta}(\zeta \otimes 1), \qquad \zeta \in H(R_{\theta=0}).$$

Proof: A simple computation (as in the proof of Proposition V, sec. 7.7) shows that

$$(\omega_R^\# \otimes i_E(a)) \circ \hat{\beta}(\zeta \otimes 1) = \hat{\beta}(i_R(a)^\# \zeta \otimes 1),$$
$$a \in P_*(E), \quad \zeta \in H(R_{\theta=0}).$$

On the other hand, by Proposition VIII, (1),

$$(\omega_R^\# \otimes i_E(a)) \circ \hat{\gamma}_R(\zeta) = \hat{\gamma}_R \circ i_R(a)^\#(\zeta).$$

Now let $\zeta \in H^p(R_{\theta=0})$. We proceed by induction on p. For $p = 0$, the proposition is obvious. Assume now that it holds for $\eta \in H^s(R_{\theta=0})$, $s < p$. Then, for $a \in P_*(E)$,

$$(\omega_R^\# \otimes i_E(a))(\hat{\beta}(\zeta \otimes 1) - \hat{\gamma}_R(\zeta)) = \hat{\beta}(i_R(a)^\# \zeta \otimes 1) - \hat{\gamma}_R(i_R(a)^\# \zeta) = 0,$$

whence

$$\hat{\beta}(\zeta \otimes 1) - \hat{\gamma}_R(\zeta) \in H^p(R_{\theta=0}) \otimes 1.$$

It follows that $\hat{\beta}(\zeta \otimes 1) - \hat{\gamma}_R(\zeta) = q(\hat{\beta}(\zeta \otimes 1) - \hat{\gamma}_R(\zeta))$, where

$$q: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \to H(R_{\theta=0})$$

is the projection with kernel $H(R_{\theta=0}) \otimes (\wedge^+E^*)_{\theta=0}$. Since (clearly) $q(\hat{\beta}(\zeta \otimes 1)) = \zeta$ and (by Proposition VI) $q(\hat{\gamma}_R\zeta) = \zeta$, it follows that

$$\hat{\beta}(\zeta \otimes 1) - \hat{\gamma}_R(\zeta) = \zeta \otimes 1 - \zeta \otimes 1 = 0.$$

This closes the induction and completes the proof.

Q.E.D.

§5. Operation of a graded vector space on a graded algebra

7.16. Operation of a graded space. Let $X = \sum_k X^k$ be a finite-dimensional graded space satisfying $X^k = 0$ for even k, and let M be a graded anticommutative algebra. An *operation of X in M* is a linear map

$$i: X \to L_M$$

such that

(1) $i(x)$ is an antiderivation, homogeneous of degree $-k$ ($x \in X^k$), and

(2) $i(x)^2 = 0$, $x \in X$.

An operation of X in M determines, as in sec. 7.1, the operators $i(a)$ ($a \in \wedge X$) in M given by

$$i(x_1 \wedge \cdots \wedge x_p) = i(x_p) \circ \cdots \circ i(x_1), \qquad x_i \in X.$$

Given an operation of X in M, set

$$M_{i=0} = \bigcap_{x \in M} \ker i(x).$$

Since, for a homogeneous element x, $i(x)$ is a homogeneous antiderivation in M, it follows that $M_{i=0}$ is a graded subalgebra of M.

Next, consider the dual graded space $X^* = \sum_k (X^k)^*$ and let $i_X(x)$ ($x \in X$) denote the substitution operator in $\wedge X^*$.

Proposition X: Suppose X operates in a connected positively graded anticommutative algebra M. Assume that there is a surjective homomorphism of graded algebras $\varrho: M \to \wedge X^*$, satisfying

$$\varrho \circ i(x) = i_X(x) \circ \varrho, \qquad x \in X.$$

Then there is an isomorphism of graded algebras

$$f: M_{i=0} \otimes \wedge X^* \xrightarrow{\cong} M,$$

which makes the diagram

$$\begin{tikzcd}
 & M_{i=0} \otimes \wedge X^* \arrow[dd, "f", "\cong"'] \arrow[dr] & \\
M_{i=0} \arrow[ur] \arrow[dr] & & \wedge X^* \\
 & M \arrow[ur, "\varrho"'] &
\end{tikzcd}$$

commute.

For the proof we first establish

Lemma VI: Assume in addition to the hypotheses of Proposition X that all the vectors of X are homogeneous of some degree k (k odd). Then the conclusion of Proposition X is correct.

Proof: Choose a linear map $\chi\colon X^* \to M^k$ such that $\varrho \circ \chi = \iota$. Since M is anticommutative and k is odd, χ extends to a homomorphism $\chi_\wedge\colon \wedge X^* \to M$. Define

$$f\colon M_{i=0} \otimes \wedge X^* \to M$$

by

$$f(z \otimes \Phi) = z \cdot \chi_\wedge(\Phi).$$

Then the diagram of Proposition X certainly commutes.

It remains to show that f is an isomorphism. First observe that, since $M^0 = \Gamma$ and $X = X^k$, $i(x) \circ \chi$ maps X^* into Γ. Hence

$$i(x)\chi(x^*) = \varrho i(x)\chi(x^*) = i_X(x)(x^*) = \langle x^*, x\rangle, \qquad x^* \in X^*, \quad x \in X. \tag{7.13}$$

Since $i(x)$ is an antiderivation, it follows that

$$f \circ (\omega_M \otimes i_X(x)) = i(x) \circ f, \qquad x \in X,$$

where ω_M denotes the degree involution. This yelds

$$f \circ (\omega_M^p \otimes i_X(a)) = i(a) \circ f, \qquad a \in \wedge^p X. \tag{7.14}$$

(1) f *is injective*: Choose a nonzero element $\Omega \in M_{i=0} \otimes \wedge X^*$.

Then, for some $p \geq 0$ and some $a \in \wedge^p X$,

$$(\omega_M^p \otimes i_X(a))\Omega = z \otimes 1, \qquad z \neq 0.$$

Hence, in view of (7.14),

$$i(a)f(\Omega) = f(z \otimes 1) = z \neq 0.$$

This shows that $f(\Omega) \neq 0$ and so f is injective.

(2) *f is surjective*: Define an operator $Y: M \to M$ by setting

$$Y(z) = \sum_\nu \chi(e^{*\nu}) \cdot i(e_\nu)z, \qquad z \in M,$$

where $e^{*\nu}$, e_ν is a pair of dual bases for X^* and X. Use (7.13) to obtain

$$i(x)Y - Yi(x) = i(x), \qquad x \in X.$$

Conclude that

$$i(a)Y - Yi(a) = p \cdot i(a), \qquad a \in \wedge^p X. \tag{7.15}$$

Next, define subspaces $F^p \subset M$ by

$$F^p = \{z \in M \mid i(a)z = 0, a \in \wedge^p X\}.$$

Then

$$M_{i=0} = F^1 \subset F^2 \subset \cdots \subset F^{n+1} = M \qquad (n = \dim X).$$

Moreover, $i(x): F^{p+1} \to F^p$, $x \in X$. It follows that

$$Y: F^{p+1} \to \operatorname{Im} f \cdot F^p.$$

On the other hand, relation (7.15) implies that

$$\frac{1}{p} Y - \iota: F^{p+1} \to F^p.$$

These equations yield

$$F^{p+1} \subset \operatorname{Im} f \cdot F^p + F^p = \operatorname{Im} f \cdot F^p, \qquad p \geq 1,$$

whence $F^{n+1} \subset \operatorname{Im} f \cdot F^1$. But $F^{n+1} = M$ and $F^1 = M_{i=0} \subset \operatorname{Im} f$. Thus f is surjective.

Q.E.D.

7.17. Proof of Proposition X: We proceed by induction on $\dim X$. Let k be the least integer such that $X^k \neq 0$. Write

$$Y = X^k \qquad \text{and} \qquad Z = \sum_{\mu > k} X^\mu.$$

Then $X = Z \oplus Y$ and $X^* = Z^* \oplus Y^*$, whence

$$\wedge X = \wedge Z \otimes \wedge Y \qquad \text{and} \qquad \wedge X^* = \wedge Z^* \otimes \wedge Y^*.$$

Let $p\colon \wedge X^* \to \wedge Y^*$ be the corresponding projection. Then

$$p \circ i_X(y) = i_Y(y) \circ p, \qquad y \in Y.$$

Finally, define a graded subalgebra $N \subset M$ by

$$N = \{z \in M \mid i(y)z = 0, y \in Y\}.$$

Then, applying Lemma VI, sec. 7.16, to the action of Y on M, we obtain an isomorphism of graded algebras

$$g\colon N \otimes \wedge Y^* \xrightarrow{\cong} M,$$

which makes the diagram

$$\begin{tikzcd}
 & N \otimes \wedge Y^* \arrow[dd, "\cong"', "g"] \arrow[dr] & \\
N \arrow[ur] \arrow[dr] & & \wedge Y^* \\
 & M \arrow[ur, "p \circ \varrho"'] &
\end{tikzcd} \tag{7.16}$$

commute.

Since $\varrho \circ i(x) = i_X(x) \circ \varrho$, ϱ restricts to a homomorphism $N \to (\wedge X^*)_{i_Y = 0}$; i.e.,

$$\varrho\colon N \to \wedge Z^*.$$

Moreover, since $(\wedge X^*)^k = Y^*$,

$$\varrho \circ g(1 \otimes y^*) = (p \circ \varrho) \circ g(1 \otimes y^*) = y^*, \qquad y^* \in Y^*.$$

These equations together with (7.16) imply that the diagram

$$\begin{tikzcd} & N \otimes \wedge Y^* \arrow[r, "\varrho\otimes\iota"] \arrow[d, "g", "\cong"'] & \wedge Z^* \otimes \wedge Y^* \arrow[d, "\cong"] \\ M_{i=0} \arrow[ur] \arrow[dr] & & \\ & M \arrow[r, "\varrho"'] & \wedge X^* \end{tikzcd}$$

commutes.

In particular, the hypotheses of the proposition are satisfied for the operation of Z on N. Thus, by induction hypothesis, there is an isomorphism

$$h\colon M_{i=0} \otimes \wedge Z^* \xrightarrow{\cong} N.$$

(Note that $M_{i=0} = N_{i_Z=0}$.) Now define an isomorphism

$$f\colon M_{i=0} \otimes \wedge X^* \xrightarrow{\cong} M,$$

by $f = g \circ (h \otimes \iota)$. It follows from the construction that f makes the diagram of Proposition X commute.

Q.E.D.

7.18. Proposition XI: Let A_1, A_2, B, S be connected positively graded anticommutative algebras and suppose $i_\lambda\colon B \to A_\lambda$ $(\lambda = 1, 2)$ are homomorphisms homogeneous of degree zero. Assume that

$$\begin{tikzcd} & A_1 \otimes S \arrow[dd, "\varphi", "\cong"'] \arrow[dr, "\varrho_1"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & S \\ & A_2 \otimes S \arrow[ur, "\varrho_2"'] & \end{tikzcd} \tag{7.17}$$

is a commutative diagram of homomorphisms of graded algebras in which ϱ_1, ϱ_2 are the obvious projections, and φ is an isomorphism. Then there is an isomorphism $\psi\colon A_1 \xrightarrow{\cong} A_2$ of graded algebras such that $\psi \circ i_1 = i_2$.

Proof: Define a homomorphism of graded algebras

$$\varphi_1: A_1 \otimes S \to A_2 \otimes S$$

by setting

$$\varphi_1(a \otimes z) = \varphi(a \otimes 1) \cdot (1 \otimes z), \qquad a \in A_1, \quad z \in S.$$

Then the diagram (7.17) still commutes if φ is replaced by φ_1.

Now we show that φ_1 is an isomorphism. In fact, write

$$\varphi_2 = \varphi^{-1} \circ \varphi_1: A_1 \otimes S \to A_1 \otimes S.$$

Then $\varphi_2(a \otimes 1) = a \otimes 1$, $a \in A_1$.

Moreover, $\varrho_1 \circ \varphi_2 = \varrho_1$, and so

$$\varphi_2(1 \otimes z) - 1 \otimes z \in A_1^+ \otimes S, \qquad z \in S.$$

It follows that for $a \in A_1^p$, $z \in S$,

$$\varphi_2(a \otimes z) - a \otimes z = (a \otimes 1) \cdot (\varphi_2(1 \otimes z) - 1 \otimes z) \in \sum_{j \geq p+1} A_1^j \otimes S.$$

Now set $F^p = \sum_{j \geq p} A_1^j \otimes S$. Then the relation above shows that

$$\varphi_2 - \iota: F^p \to F^{p+1}.$$

Thus φ_2 is a filtration preserving map and induces the identity in the associated graded algebras. Hence, Proposition VII, sec. 1.14, implies that φ_2 is an isomorphism. Thus so is $\varphi_1 = \varphi \circ \varphi_2$.

Finally observe that

$$A_\lambda \otimes S^+ = (A_\lambda \otimes S) \cdot (1 \otimes S^+), \qquad \lambda = 1, 2.$$

Since $\varphi_1(1 \otimes z) = 1 \otimes z$, $z \in S$, it follows that φ_1 restricts to an isomorphism

$$\varphi_1: A_1 \otimes S^+ \xrightarrow{\cong} A_2 \otimes S^+.$$

Thus it induces an isomorphism between the factor algebras. Now the

proposition follows from the commutative diagram

$$\begin{tikzcd}
 & A_1 \arrow[r, "j_1", "\cong"'] & (A_1 \otimes S)/(A_1 \otimes S^+) \arrow[dd, "\cong"] \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \\
 & A_2 \arrow[r, "\cong", "j_2"'] & (A_2 \otimes S)/(A_2 \otimes S^+),
\end{tikzcd}$$

where j_λ ($\lambda = 1, 2$) denotes the isomorphism induced by the inclusion map $A_\lambda \to A_\lambda \otimes S$.

Q.E.D.

§6. Transformation groups

7.19. Action of a Lie group. Let G be a Lie group with Lie algebra E. Suppose $M \times G \to M$ is a right action of G on a manifold M. Recall from sec. 3.9, volume II, that this action associates with every vector $h \in E$ the fundamental vector field Z_h on M given by

$$Z_h(f)(z) = \frac{d}{dt} f(z \exp th)_{t=0}, \qquad z \in M, \quad f \in \mathscr{S}(M).$$

Moreover the map $E \to \mathscr{X}(M)$ given by

$$h \mapsto Z_h,$$

is a homomorphism of real Lie algebras (cf. Proposition IV, sec. 3.9, volume II).

Now consider the algebra $A(M)$ of differential forms on M and define operators $i(h)$, $\theta(h)$ in $A(M)$ by

$$i(h) = i(Z_h), \qquad \theta(h) = \theta(Z_h),$$

where $i(Z_h)$ (respectively, $\theta(Z_h)$) is the substitution operator (respectively, Lie derivative) in $A(M)$ associated with the vector field Z_h. In sec. 3.13, volume II, we established the formulae:

$$\begin{aligned} i(h)^2 &= 0 \\ i([h, k]) &= \theta(h)i(k) - i(k)\theta(h) \\ \theta([h, k]) &= \theta(h)\theta(k) - \theta(k)\theta(h) \end{aligned}$$

and

$$\theta(h) = i(h)\delta + \delta i(h), \qquad h, k \in E.$$

Moreover, $i(h)$ (respectively, $\theta(h)$) is an antiderivation of degree -1 (respectively, a derivation of degree zero) in $A(M)$.

It follows that $(E, i, \theta, A(M), \delta)$ is an operation of E in the graded differential algebra $(A(M), \delta)$. It is called the *operation of the Lie algebra associated with the action of G on M.*

Next, let $N \times G \to N$ be an action of G on a second manifold N. Suppose $\varphi: M \to N$ is a smooth equivariant map:

$$\varphi(z \cdot a) = (\varphi z) \cdot a, \qquad z \in M, \quad a \in G.$$

Then the induced homomorphism $\varphi^*: A(M) \leftarrow A(N)$ is a homomorphism of operations (cf. sec. 3.14, volume II).

7.20. The invariant subalgebra. Let $T: M \times G \to M$ be a smooth action of a Lie group G on a manifold M, and let $(E, i_M, \theta_M, A(M), \delta_M)$ be the associated operation of the Lie algebra E of G. Recall from sec. 3.12, volume II, that the invariant subalgebra of $A(M)$ is given by

$$A_I(M) = \{\Phi \in A(M) \mid T_a^*\Phi = \Phi, a \in G\}.$$

Proposition XII: $A_I(M) \subset A(M)_{\theta=0}$. If G is connected, then $A_I(M) = A(M)_{\theta=0}$.

Proof: This is Proposition VI, sec. 3.13, volume II.

Q.E.D.

Proposition XIII: If G is compact, then the inclusion map $A(M)_{\theta=0} \to A(M)$ induces an isomorphism

$$H(A(M)_{\theta=0}) \xrightarrow{\cong} H(M).$$

Proof: Let G^0 be the 1-component of G; then $A(M)_{\theta=0}$ is the algebra of differential forms invariant under G^0. Now apply Theorem I, sec. 4.3, volume II, to the action of G^0 on M.

Q.E.D.

7.21. Example: Consider the multiplication map $\mu_G: G \times G \to G$ of a Lie group as a right action of G on itself. Then the fundamental vector field corresponding to a vector $h \in E$ is the left invariant vector field X_h generated by h. Let $(E, i_G, \theta_G, A(G), \delta_G)$ denote the corresponding operation of E.

Now consider the algebra $A_L(G)$ of left invariant differential forms on G. According to sec. 4.5, volume II, the algebra $A_L(G)$ is stable under the operators $i(X_h)$ and $\theta(X_h)$. Thus restricting these operators to $A_L(G)$, we obtain a second operation of E, called the *left invariant operation*

and denoted by $(E, i_L, \theta_L, A_L(G), \delta_L)$. Clearly the injection

$$l_G \colon A_L(G) \to A(G)$$

is a homomorphism of operations.

On the other hand, as we observed in sec. 5.29, an isomorphism $\tau_L \colon A_L(G) \xrightarrow{\cong} \wedge E^*$ is defined by $\tau_L(\Phi) = \Phi(e)$. Moreover, under this isomorphism $i_L(h)$, $\theta_L(h)$, and δ_L correspond respectively to $i_E(h)$, $\theta_E(h)$, and δ_E; thus

$$\tau_L \colon (E, i_L, \theta_L, A_L(G), \delta_L) \xrightarrow{\cong} (E, i_E, \theta_E, \wedge E^*, \delta_E)$$

is an isomorphism of operations.

In particular, the homomorphism $\varepsilon_G = l_G \circ \tau_L^{-1}$ (defined in sec. 5.29) is a homomorphism

$$\varepsilon_G \colon (E, i_E, \theta_E, \wedge E^*, \delta_E) \to (E, i_G, \theta_G, A(G), \delta_G)$$

of operations. Thus we obtain the commutative diagram

$$\begin{tikzcd}
(\wedge E^*)_{\theta=0} \arrow[r, "(\varepsilon_G)^{\#}_{\theta=0}"] \arrow[d, "\pi_E"'] \arrow[dr, "\alpha_G"] & H(A(G)_{\theta=0}) \arrow[d] \\
H^*(E) \arrow[r, "\varepsilon_G^{\#}"'] & H(G)
\end{tikzcd} \tag{7.18}$$

(cf. sec. 5.29).

Proposition XIV: If G is a connected Lie group with reductive Lie algebra E, then the map

$$(\varepsilon_G)^{\#}_{\theta=0} \colon (\wedge E^*)_{\theta=0} \to H(A(G)_{\theta=0})$$

is an isomorphism.

Proof: Since G is connected, and since $\theta(h) = \theta(X_h)$ where X_h denotes the fundamental field associated with the right action of G on itself, we have

$$A(G)_{\theta=0} = A_R(G)$$

($A_R(G)$ is the algebra of right invariant differential forms on G).

Consider the map $\nu: G \to G$ given by $\nu(x) = x^{-1}$. The isomorphism $\nu^*: A(G) \leftarrow A(G)$ restricts to an isomorphism

$$A_L(G) \xleftarrow{\cong} A_R(G)$$

of graded differential algebras. Clearly the diagram

$$\begin{tikzcd} A_I(G) \arrow[r, "\omega"] \arrow[d, "i"'] & A_I(G) \arrow[d, "j"] \\ A_R(G) \arrow[r, "\nu^*"'] & A_L(G) \end{tikzcd}$$

commutes, where i and j are inclusion maps and ω is the degree involution (cf. Lemma V, sec. 4.9, volume II).

Since $A(G)_{\theta=0} = A_R(G)$, we may pass to cohomology in the commutative diagram above to obtain the commutative diagram

$$\begin{tikzcd} A_I(G) \arrow[r, "\cong"'] \arrow[d, "i^*"'] & A_I(G) \arrow[r, "\tau_I", "\cong"'] \arrow[d, "j^*"] & (\wedge E^*)_{\theta=0} \arrow[d, "\pi_E"] \\ H(A(G)_{\theta=0}) \arrow[r, "\cong", "\nu^*"'] & H(A_L(G)) \arrow[r, "\cong", "\tau_L^\#"'] & H^*(E). \end{tikzcd}$$

Because E is reductive, π_E is an isomorphism (cf. Theorem I, sec. 5.12); it follows that so is i^*. But $(\varepsilon_G)_{\theta=0}^\# = i^\# \circ \tau_I^{-1}$.

Q.E.D.

Remark: If G is compact and connected, then all the maps in diagram (7.18) are isomorphisms (cf. diagram (5.19), sec. 5.29).

7.22. Fibre projection. Let $M \times G \to M$ be a smooth action on a connected manifold M. Let $A_z: G \to M$ be the smooth map given by

$$A_z(a) = z \cdot a, \qquad a \in G,$$

where z is a given point in M. According to sec. 4.2, volume II, the homomorphism

$$A_z^\#: H(G) \leftarrow H(M)$$

is independent of the choice of z.

On the other hand, we have

Lemma VII: The graded algebra $H(A(M)_{\theta=0})$ is connected.

Proof: Observe that

$$H^0(A(M)_{\theta=0}) \subset Z^0(A(M), \delta) = H^0(M).$$

Since M is connected, $H^0(M) = \mathbb{R}$ (cf. p. 177, volume I). It follows that $H^0(A(M)_{\theta=0}) = \mathbb{R}$.

Q.E.D.

Now assume the Lie algebra E of G is reductive. Then, in view of the lemma, the fibre projection

$$\varrho_{A(M)}: H(A(M)_{\theta=0}) \to (\wedge E^*)_{\theta=0},$$

is defined.

Proposition XV: Let $M \times G \to M$ be a smooth right action of a connected Lie group on a connected manifold M. Assume the Lie algebra E of G is reductive. Then the diagram

$$\begin{array}{ccc} H(A(M)_{\theta=0}) & \xrightarrow{\varrho_{A(M)}} & (\wedge E^*)_{\theta=0} \\ \downarrow & & \downarrow{\scriptstyle \alpha_G} \\ H(M) & \xrightarrow[A_z^*]{} & H(G) \end{array}$$

commutes.

Remark: Suppose G is also compact. Then (cf. Theorem I, sec. 4.3, volume II, and sec. 5.29) the vertical arrows of the diagram are isomorphisms.

Proof: Since $A_z(ab) = A_z(a) \cdot b$, $a, b \in G$, it follows that A_z is equivariant with respect to the actions of G on M and on itself (by right multiplication). Hence, according to sec. 7.19, A_z^* is a homomorphism of operations

$$A_z^*: (E, i_G, \theta_G, A(G), \delta_G) \leftarrow (E, i_M, \theta_M, A(M), \delta_M).$$

On the other hand, $\varepsilon_G\colon \wedge E^* \to A(G)$ is also a homomorphism of operations (cf. sec. 7.21). Thus it follows from sec. 7.10 that the fibre projections satisfy

$$\varrho_{A(G)} \circ (\varepsilon_G)^{\#}_{\theta=0} = \varrho_{\wedge E^\bullet} \qquad \text{and} \qquad \varrho_{A(G)} \circ (A_z^*)^{\#}_{\theta=0} = \varrho_{A(M)}.$$

Since G is connected and E is reductive, Proposition XIV, sec. 7.21, shows that $(\varepsilon_G)^{\#}_{\theta=0}$ is an isomorphism. Moreover, according to the example of sec. 7.11,

$$\varrho_{\wedge E^\bullet} = \iota\colon (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0}.$$

Thus the first equation above yields $\varrho_{A(G)} = [(\varepsilon_G)^{\#}_{\theta=0}]^{-1}$. Substituting this in the second, we find

$$[(\varepsilon_G)^{\#}_{\theta=0}]^{-1} \circ (A_z^*)^{\#}_{\theta=0} = \varrho_{A(M)}.$$

The proposition follows from this, and the commutative diagram

$$\begin{tikzcd}
H(A(M)_{\theta=0}) \arrow[r, "(A_z^*)^{\#}_{\theta=0}"] \arrow[d] & H(A(G)_{\theta=0}) \arrow[d] & (\wedge E^*)_{\theta=0} \arrow[l, "(\varepsilon_G)^{\#}_{\theta=0}"'] \arrow[dl, "\alpha_G"] \\
H(M) \arrow[r, "A_z^{\#}"'] & H(G) &
\end{tikzcd}$$

Q.E.D.

7.23. The main theorem. Let $M \times G \to M$ be a smooth action of a compact connected Lie group G on a connected manifold M. Let

$$\alpha_G \circ \varkappa_E\colon \wedge P_E \xrightarrow{\cong} H(G)$$

be the canonical isomorphism (cf. sec. 5.29 and sec. 5.32), and let

$$A_z^{\#}\colon H(M) \to H(G)$$

be the homomorphism defined above.

Then we have as an immediate consequence of Theorem I, sec. 7.13, Theorem II, sec. 7.14, and Proposition XV, sec. 7.22:

Theorem III: Let $M \times G \to M$ be a smooth action of a compact connected Lie group G on a connected manifold M. Then the isomorphism $\alpha_G \circ \varkappa_E$ identifies $\operatorname{Im} A_z^{\#}$ with $\wedge \hat{P}$ where $\hat{P}$ is a graded subspace of P_E.

Moreover, there is a graded subalgebra $A \subset H(M)$ and an isomorphism of graded algebras

$$A \otimes \wedge \hat{P} \xrightarrow{\cong} H(M)$$

which makes the diagram

$$\begin{tikzcd} A \otimes \wedge \hat{P} \arrow[r] \arrow[d, "\cong"'] & \wedge \hat{P} \arrow[d, "\alpha_G \circ \varkappa_E"] \\ H(M) \arrow[r, "A_z^*"'] & H(G) \end{tikzcd}$$

commute.

Chapter VIII

Algebraic Connections and Principal Bundles

In this chapter (R, δ_R) denotes a positively graded anticommutative differential algebra. ω_R denotes the degree involution in R.

§1. Definition and examples

8.1. Definition: An *algebraic connection* for an operation $(E, i_R, \theta_R, R, \delta_R)$ is a linear map $\chi\colon E^* \to R^1$ which satisfies the conditions

$$i_R(x)\chi(x^*) = \langle x^*, x\rangle, \qquad x^* \in E^*, \quad x \in E,$$

and

$$\theta_R(x) \circ \chi = \chi \circ \theta_E(x), \qquad x \in E.$$

(Here, as usual, θ_E denotes the representation of E in $\wedge E^*$.)

If $\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ is a homomorphism of operations, and if χ_R is an algebraic connection for the operation of E in R, then

$$\chi_S = \varphi \circ \chi_R$$

is an algebraic connection for the operation of E in S.

In fact, for $x^* \in E^*$ and $x \in E$, we have

$$i_S(x)\chi_S(x^*) = \varphi i_R(x)\chi_R(x^*) = \langle x^*, x\rangle,$$

while

$$\theta_S(x) \circ \chi_S = \varphi \circ \theta_R(x) \circ \chi_R = \chi_S \circ \theta_E(x).$$

Examples: 1. Let (P, π, B, G) be a principal bundle. Then there is a principal action $P \times G \to P$. The corresponding operation of the Lie algebra of G on the algebra of differential forms on P admits an algebraic connection (cf. article 5 at the end of this chapter).

2. The identity map $E^* \to E^*$ is an algebraic connection for the operation $(E, i_E, \theta_E, \wedge E^*, \delta_E)$ (cf. Example 2, sec. 7.4).

3. The inclusion map $\chi: E^* \to W(E)$ given by $\chi(x^*) = 1 \otimes x^*$ is the unique algebraic connection for the operation $(E, i, \theta_W, W(E), \delta_W)$ of E in the Weil algebra $W(E)$ (cf. Example 6, sec. 7.4).

4. *Subalgebras.* Let $F \subset E$ be a subalgebra of a Lie algebra E. Recall from Example 4, sec. 7.4, the definition of the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. Let $\chi: F^* \to E^*$ be an algebraic connection for this operation, with dual map $\chi^*: F \leftarrow E$. Then:

(i) $E = \ker \chi^* \oplus F$.

(ii) χ^* is the projection onto F induced by this decomposition.

(iii) $\ker \chi^*$ is stable under the operators $\mathrm{ad}_E(y)$, $y \in F$. (ad_E denotes the adjoint representation of E—cf. sec. 4.2).

Conversely, assume a direct decomposition $E = H \oplus F$, where H is a subspace stable under the operators $\mathrm{ad}_E(y)$, $y \in F$. Let $\pi: E \to F$ denote the corresponding projection. Then the dual map

$$\pi^*: F^* \to E^*$$

is an algebraic connection.

To establish these statements, let χ be any algebraic connection. Then for $y \in F$, $y^* \in F^*$

$$\langle y^*, \chi^* y\rangle = \langle \chi y^*, y\rangle = i_F(y)\chi y^* = \langle y^*, y\rangle.$$

Hence $\chi^* y = y$, $y \in F$, and so (i) and (ii) follow. Since $\chi \circ \theta_F(y) = \theta_F(y) \circ \chi$, $y \in F$, we can dualize this relation to obtain

$$\mathrm{ad}_F(y) \circ \chi^* = \chi^* \circ \mathrm{ad}_E(y).$$

It follows that $\ker \chi^*$ is stable under the operators $\mathrm{ad}_E(y)$, $y \in F$.

Conversely, assume that $E = H \oplus F$ satisfies the conditions above, and let $\pi: E \to F$ be the projection. Then for $y^* \in F^*$ and $y \in F$,

$$i_F(y)\pi^* y^* = \langle y^*, \pi y\rangle = \langle y^*, y\rangle.$$

On the other hand, since H is stable under each $\mathrm{ad}_E(y)$, $y \in F$, we have

$$\pi \circ \mathrm{ad}_E(y) = \mathrm{ad}\, y \circ \pi, \qquad y \in F.$$

Dualizing we obtain

$$\theta_F(y) \circ \pi^* = \pi^* \circ \theta_F(y), \qquad y \in F.$$

Hence, π^* is an algebraic connection for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$.

5. Let E be a reductive Lie algebra and let $(E, i, \theta, R, \delta_R)$ be an operation. Recall from sec. 7.5 the definition of the associated semisimple operation $(E, i_S, \theta_S, R_S, \delta_S)$. We shall show that an algebraic connection for $(E, i, \theta, R, \delta_R)$ is also an algebraic connection for $(E, i_S, \theta_S, R_S, \delta_S)$, and conversely.

In fact, let $\chi: E^* \to R^1$ be an algebraic connection. Since E is reductive, the representation θ_E is semisimple. Moreover, χ is an E-linear isomorphism of E^* onto $\operatorname{Im} \chi$. It follows that the restriction of θ to $\operatorname{Im} \chi$ is semisimple and so $\operatorname{Im} \chi \subset R_S^1$. Thus χ can be regarded as a linear map of E^* into R_S^1 and so it is an algebraic connection for $(E, i_S, \theta_S, R_S, \delta_S)$. The converse is trivial.

8.2. Surjective fibre projection. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra and assume that $H(R_{\theta=0})$ is connected.

Proposition I: If the fibre projection ϱ_R (cf. sec. 7.10) is surjective, then the operation admits an algebraic connection.

Proof: Choose nonzero elements $\beta \in H^n(R_{\theta=0})$ and $\Phi \in (\wedge^n E^*)_{\theta=0}$ ($n = \dim E$) so that $\varrho_R(\beta) = \Phi$. Let $z \in Z^n(R_{\theta=0})$ be a cocycle representing β, and let $a \in (\wedge^n E)_{\theta=0}$ be the unique element such that $\langle \Phi, a \rangle = 1$. Define a linear map $\chi: E^* \to R^1$ by

$$\chi(x^*) = (-1)^{n-1} i_R(i_E(x^*)a)z, \qquad x^* \in E^*.$$

Since a and z are invariant, the formulae of sec. 7.2 show that

$$\chi \circ \theta_E(x) = \theta_R(x) \circ \chi, \qquad x \in E.$$

Moreover, since $a \wedge x = 0$ $(x \in E)$ we have

$$i_R(x)\chi(x^*) = i_R(x \wedge i_E(x^*)a)z = \langle x^*, x \rangle i_R(a)z, \qquad x \in E, \quad x^* \in E^*.$$

Thus to show that χ is an algebraic connection, we must show that $i_R(a)z = 1$.

But the formulae of sec. 7.2 show that $i_R(a)z$ is an invariant cocycle of degree zero. Since $H(R_{\theta=0})$ is connected it follows that, for some $\lambda \in \Gamma$,

$$i_R(a)z = \lambda.$$

It follows that

$$\begin{aligned}\lambda = i_R(a)^{\#}(\beta) &= \varrho_R i_R(a)^{\#}(\beta) = i_E(a)\varrho_R(\beta)\\ &= i_E(a)\Phi = \langle \Phi, a\rangle = 1\end{aligned}$$

(cf. Proposition VIII, sec. 7.12). Hence $i_R(a)z = 1$.

Q.E.D.

8.3. Transformation groups. Let G be a compact Lie group acting smoothly from the right on a manifold M. Consider the induced operation $(E, i, \theta, A(M), \delta)$ of the Lie algebra E of G in the algebra of differential forms of M (cf. sec. 7.19). We shall show that this operation admits an algebraic connection if and only if the action of G is almost free (for the terminology cf. sec. 3.4, volume II).

In fact, assume that $\chi: E^* \to A^1(M)$ is an algebraic connection. Fix $h \in E$ ($h \neq 0$) and let Z_h denote the corresponding fundamental vector field on M. Choose $h^* \in E^*$ so that $\langle h^*, h\rangle \neq 0$. Then for $z \in M$, we have

$$\chi(h^*)(z; Z_h(z)) = (i(h)\chi(h^*))(z) = \langle h^*, h\rangle \neq 0,$$

whence

$$Z_h(z) \neq 0, \qquad h \in E, \quad z \in M.$$

This implies (cf. sec. 3.11, volume II) that the action of G is almost free.

Conversely, assume that the action of G is almost free. According to sec. 3.11, volume II, we can form the fundamental subbundle F_M of τ_M. Since G is compact, it follows from Example 1, sec. 3.18, volume II, that

$$\tau_M = \eta \oplus F_M,$$

where η is a subbundle of τ_M, stable under the action of G.

Now let $\varrho: \tau_M \to F_M$ be the strong bundle projection induced by the decomposition above and let

$$\varrho_*: \mathscr{X}(M) \to \operatorname{Sec} F_M$$

be the induced map of cross-sections. Since F_M and η are G-stable, it follows that

$$\varrho_*([Z_h, Z]) = [Z_h, \varrho_* Z], \qquad h \in E, \quad Z \in \mathscr{X}(M). \tag{8.1}$$

On the other hand, according to sec. 3.11, volume II, a strong bundle isomorphism

$$\alpha\colon M \times E \xrightarrow{\cong} F_M,$$

is given by

$$\alpha(z, h) = Z_h(z), \qquad z \in M, \quad h \in E.$$

Moreover,

$$\alpha(z, [h, k]) = [Z_h, Z_k](z), \qquad z \in M, \quad h, k \in E. \tag{8.2}$$

Now define a linear map $\chi\colon E^* \to A^1(M)$ by setting

$$(\chi h^*)(z; Z(z)) = \langle h^*, \alpha_z^{-1} \circ \varrho_z(Z(z))\rangle, \qquad z \in M, \quad Z \in \mathscr{X}(M).$$

Then

$$\begin{aligned} i(h)\chi(h^*)(z) &= \chi(h^*)(z; Z_h(z)) \\ &= \langle h^*, h\rangle, \qquad z \in M, \quad h \in E, \quad h^* \in E^*. \end{aligned}$$

Finally, it follows easily from formulae (8.1) and (8.2) that

$$\theta(h) \circ \chi = \chi \circ \theta_E(h), \qquad h \in E.$$

Hence χ is an algebraic connection for $(E, i, \theta, A(M), \delta)$.

§2. The decomposition of R

8.4. The decomposition of R as a tensor product. Let χ be an algebraic connection for an operation $(E, i_R, \theta_R, R, \delta_R)$. Since $\operatorname{Im} \chi \subset R^1$ and since R is anticommutative, we have

$$\chi(x^*)^2 = 0, \qquad x^* \in E^*.$$

Hence χ extends to a homomorphism of graded algebras

$$\chi_\wedge : \wedge E^* \to R.$$

It satisfies the relations

$$i_R(a) \circ \chi_\wedge = \chi_\wedge \circ i_E(a), \; a \in \wedge E, \quad \text{and} \quad \theta_R(x) \circ \chi_\wedge = \chi_\wedge \circ \theta_E(x), \; x \in E. \tag{8.3}$$

In fact since $i_R(x_1 \wedge \cdots \wedge x_p) = i_R(x_p) \circ \cdots \circ i_R(x_1)$ (cf. sec. 7.1), it is sufficient to show that

$$i_R(x) \circ \chi_\wedge = \chi_\wedge \circ i_E(x), \qquad x \in E,$$

in order to establish the first relation. Now the operators on both sides are $\chi_\wedge$-antiderivations. They coincide by definition in E^*; hence they are equal.

On the other hand, $\theta_R(x) \circ \chi_\wedge$ and $\chi_\wedge \circ \theta_E(x)$ are both $\chi_\wedge$-derivations, and they coincide by definition in E^*. Hence these operators are equal.

Theorem I: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation admitting an algebraic connection χ. Then an isomorphism of graded algebras

$$f: R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$$

is given by

$$f(z \otimes \Phi) = z \cdot \chi_\wedge(\Phi), \qquad z \in R_{i=0}, \quad \Phi \in \wedge E^*.$$

($R_{i=0} \otimes \wedge E^*$ is the skew tensor product.)

This isomorphism satisfies

$$f \circ (\omega_R^p \otimes i_E(a)) = i_R(a) \circ f, \qquad a \in \wedge^p E.$$

and

$$f \circ (\theta_R(x) \otimes \iota + \iota \otimes \theta_E(x)) = \theta_R(x) \circ f, \qquad x \in E.$$

Proof: The commutation relations are immediate consequences of the definition and the analogous formulae for $\chi_\wedge$ (cf. formula (8.3)).

To show that f is an isomorphism, we proceed as in Lemma VI, sec. 7.16.

(1) *f is injective*: Choose a nonzero element $\Omega \in R_{i=0} \otimes \wedge E^*$. Then for some $a \in \wedge^p E$ $(p \geq 0)$,

$$(\omega_R^p \otimes i_E(a))\Omega = z \otimes 1,$$

where $z \neq 0$. It follows that

$$i_R(a) f(\Omega) = f(z \otimes 1) = z.$$

Hence $f(\Omega) \neq 0$ and so f is injective.

(2) *f is surjective*: Define an operator $Y: R \to R$ by

$$Y(z) = \sum_\nu \chi(e^{*\nu}) \cdot i_R(e_\nu) z,$$

where $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E. Then

$$i_R(x) Y - Y i_R(x) = i_R(x), \qquad x \in E,$$

as follows from the relation $i_R(x)\chi(x^*) = \langle x^*, x\rangle$. Now an induction argument gives

$$i_R(a) Y - Y i_R(a) = p\, i_R(a), \qquad a \in \wedge^p E. \tag{8.4}$$

Next define subspaces $F^p \subset R$ by

$$F^p = \{z \in R \mid i_R(a) z = 0,\ a \in \wedge^p E\}.$$

Then

$$R_{i=0} = F^1 \subset F^2 \subset \cdots \subset F^{n+1} = R \qquad (n = \dim E).$$

Since, clearly, $i_R(x)\colon F^{p+1} \to F^p$, $p = 1, 2, \ldots, n$, it follows that

$$Y\colon F^{p+1} \to \operatorname{Im} f \cdot F^p.$$

On the other hand, formula (8.4) implies that

$$\left(\frac{1}{p}\, Y - \iota\right)\colon F^{p+1} \to F^p.$$

The last two equations yield

$$F^{p+1} \subset \operatorname{Im} f \cdot F^p + F^p \subset \operatorname{Im} f \cdot F^p, \qquad p = 1, \ldots, n.$$

It follows that $F^{n+1} \subset \operatorname{Im} f \cdot F^1$. Since $F^{n+1} = R$ and $F^1 = R_{i=0} \subset \operatorname{Im} f$, f must be surjective.

Q.E.D.

Corollary I: The homomorphism $\chi_\wedge\colon \wedge E^* \to R$ is injective.

Corollary II: R is generated by the subalgebras $R_{i=0}$ and $\operatorname{Im} \chi_\wedge$.

Corollary III: R is the direct sum of the subalgebra $R_{i=0}$ and the ideal generated by $\chi_\wedge(\wedge^+ E^*)$:

$$R = R_{i=0} \oplus R \cdot \chi_\wedge(\wedge^+ E^*).$$

Corollary IV: R is the direct sum of the subalgebra $R^0 \cdot \operatorname{Im} \chi_\wedge$ and the ideal generated by $R^+_{i=0}$:

$$R = R \cdot R^+_{i=0} \oplus R^0 \cdot \operatorname{Im} \chi_\wedge.$$

Corollary V: f restricts to an isomorphism

$$f_{\theta=0}\colon (R_{i=0} \otimes \wedge E^*)_{\theta=0} \xrightarrow{\cong} R_{\theta=0}.$$

Finally, let $\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations. Assume that χ_R is an algebraic connection for the first operation and let $\chi_S = \varphi \circ \chi_R$ be the induced algebraic connection in the second operation (cf. sec. 8.1). Then the induced isomorphisms

$$f_R\colon R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R \qquad \text{and} \qquad f_S\colon S_{i=0} \otimes \wedge E^* \xrightarrow{\cong} S$$

make the diagram

$$\begin{tikzcd} R_{i=0} \otimes \wedge E^* \arrow[r, "f_R", "\cong"'] \arrow[d, "\varphi_{i=0}\otimes\iota"'] & R \arrow[d, "\varphi"] \\ S_{i=0} \otimes \wedge E^* \arrow[r, "\cong", "f_S"'] & S \end{tikzcd}$$

commute, as follows from the definitions.

8.5. Covariant derivative. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation with an algebraic connection χ. Then the direct decomposition

$$R = R_{i=0} \oplus R \cdot \chi_\wedge(\wedge^+ E^*)$$

(cf. Corollary III to Theorem I, sec. 8.4) induces a linear projection

$$\pi_H \colon R \to R_{i=0}.$$

π_H is called the *horizontal projection associated with* χ.

Since $R_{i=0}$ is a subalgebra of R while $R \cdot \chi_\wedge(\wedge^+ E^*)$ is an ideal, it follows that π_H is a homomorphism of graded algebras. Evidently,

$$i_R(x) \circ \pi_H = 0 \qquad \text{and} \qquad \theta_R(x) \circ \pi_H = \pi_H \circ \theta_R(x), \qquad x \in E.$$

Now consider the operator $\nabla \colon R \to R$ given by

$$\nabla = \pi_H \circ \delta_R.$$

It is called the *covariant derivative in* R associated with χ. It follows from the definition that ∇ is homogeneous of degree 1 and that it satisfies the relations

$$i_R(x) \circ \nabla = 0 \qquad \text{and} \qquad \theta_R(x) \circ \nabla = \nabla \circ \theta_R(x), \qquad x \in E. \tag{8.5}$$

Since δ_R is an antiderivation, ∇ is a π_H-antiderivation

$$\nabla(z \cdot w) = \nabla z \cdot \pi_H w + (-1)^p \pi_H z \cdot \nabla w, \qquad z \in R^p, \quad w \in R.$$

In particular, the restriction of ∇ to $R_{i=0}$ is an antiderivation; it is denoted by $\nabla_{i=0}$.

Proposition II: The covariant derivative has the following properties:

(1) $\nabla z = \delta_R z$, $z \in R_{i=0,\theta=0}$.

(2) $\nabla z = \delta_R z - \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu)z$, $z \in R_{i=0}$.

(3) $(\nabla \circ \chi_\wedge)\Phi = 0$, $\Phi \in \sum_{j\geq 2} \wedge^j E^*$.

Proof: (1) If $z \in R_{i=0,\theta=0}$, then $\delta_R z \in R_{i=0,\theta=0}$ (cf. sec. 7.3). It follows that

$$\nabla z = \pi_H \delta_R z = \delta_R z.$$

(2) Let $z \in R_{i=0}$. Then $\theta_R(e_\nu)z \in R_{i=0}$ and so

$$i_R(x)\Big(\delta_R z - \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu)z\Big) = \theta_R(x)z - \theta_R(x)z = 0, \quad x \in E.$$

It follows that

$$\delta_R z - \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu)z \in R_{i=0}.$$

Thus

$$\begin{aligned}\delta_R(z) - \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu)z &= \pi_H\Big(\delta_R z - \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu)z\Big)\\ &= \nabla z.\end{aligned}$$

(3) Since ∇ is a π_H-antiderivation and since $\pi_H \circ \chi = 0$, it follows that for $j \geq 2$

$$\begin{aligned}&(\nabla \circ \chi_\wedge)(x_1^* \wedge \cdots \wedge x_j^*)\\ &= \sum_{\nu=1}^{j} (-1)^{\nu-1}\pi_H \chi(x_1^*) \wedge \cdots \nabla\chi(x_\nu^*) \cdots \wedge \pi_H\chi(x_j^*)\\ &= 0.\end{aligned}$$

Q.E.D.

8.6. Curvature. The *curvature* of an algebraic connection χ is the linear map

$$\varkappa: E^* \to R^2_{i=0}$$

given by

$$\varkappa = \nabla \circ \chi,$$

where ∇ is the covariant derivative associated with χ.

Thus (cf. sec. 8.1 and sec. 8.5)

$$i_R(x) \circ \not\chi = 0 \qquad \text{and} \qquad \theta_R(x) \circ \not\chi = \not\chi \circ \theta_E(x), \qquad x \in E.$$

Proposition III: The curvature of an algebraic connection χ satisfies the following identities:

(1) $\not\chi = \delta_R \circ \chi - \chi_\wedge \circ \delta_E$ (equation of Maurer and Cartan).

(2) $\nabla \not\chi = 0$ (Bianchi identity).

(3) $\nabla^2 z = -\sum_\nu \not\chi(e^{*\nu}) \cdot \theta_R(e_\nu) z,\ z \in R_{i=0}$.

(Here $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E.)

Proof: (1) Let $x \in E$, $x^* \in E^*$. Then

$$\begin{aligned} i_R(x) &\circ (\delta_R \circ \chi - \chi_\wedge \circ \delta_E)(x^*) \\ &= [\theta_R(x) \circ \chi - \delta_R \circ i_R(x) \circ \chi - \chi \circ \theta_E(x) + \chi \circ \delta_E \circ i_E(x)](x^*) \\ &= (\theta_R(x) \circ \chi - \chi \circ \theta_E(x))(x^*) \\ &= 0. \end{aligned}$$

It follows that $(\delta_R \circ \chi - \chi_\wedge \circ \delta_E)(x^*) \in R_{i=0}$. Hence

$$\begin{aligned} \delta_R \circ \chi - \chi_\wedge \circ \delta_E &= \pi_H \circ \delta_R \circ \chi - \pi_H \circ \chi_\wedge \circ \delta_E \\ &= \nabla \circ \chi = \not\chi. \end{aligned}$$

(2) Apply ∇ to both sides of (1) to obtain

$$\nabla \not\chi = \pi_H \delta_R^2 \chi - \nabla \chi_\wedge \delta_E = -\nabla \chi_\wedge \delta_E.$$

Since $\delta_E \colon E^* \to \wedge^2 E^*$, Proposition II, sec. 8.5, implies that $\nabla \chi_\wedge \delta_E = 0$.

(3) Proposition II, sec. 8.5, yields

$$\begin{aligned} \nabla^2 z &= (\nabla \circ \delta_R)(z) - \sum_\nu \nabla(\chi(e^{*\nu}) \cdot \theta_R(e_\nu) z) \\ &= -\sum_\nu (\nabla \chi(e^{*\nu})) \cdot \theta_R(e_\nu) z + \sum_\nu \pi_H \chi(e^{*\nu}) \cdot \nabla(\theta_R(e_\nu) z) \\ &= -\sum_\nu \not\chi(e^{*\nu}) \cdot \theta_R(e_\nu) z. \end{aligned}$$

Q.E.D.

8.7. The operation of E in $R_{i=0} \otimes \wedge E^*$. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation with an algebraic connection χ. Denote the curvature and

the covariant derivative by χ and ∇, respectively. We shall construct an operation $(E, i, \theta, R_{i=0} \otimes \wedge E^*, d)$ in $R_{i=0} \otimes \wedge E^*$.

First we define i and θ by

$$i(x) = \omega_R \otimes i_E(x) \qquad \text{and} \qquad \theta(x) = \theta_R(x) \otimes \iota + \iota \otimes \theta_E(x), \qquad x \in E.$$

To define d we introduce four antiderivations in $R_{i=0} \otimes \wedge E^*$, all homogeneous of degree 1.

First we have the operator

$$\omega_R \otimes \delta_E = \omega_R \otimes \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})\theta_E(e_\nu),$$

where $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E. Next, recall from sec. 5.25 that the representation θ_R of E in $R_{i=0}$ induces an antiderivation δ_θ in $R_{i=0} \otimes \wedge E^*$. It is given by

$$\delta_\theta = \sum_\nu \omega_R \theta_R(e_\nu) \otimes \mu(e^{*\nu}).$$

The third operator, denoted by h_χ, is defined by

$$h_\chi(z \otimes 1) = 0$$

and

$$\begin{aligned} h_\chi(z \otimes x_1^* \wedge \cdots \wedge x_p^*) &= (-1)^q \sum_{i=1}^p (-1)^{i-1} \chi(x_i^*) \cdot z \otimes x_1^* \wedge \cdots \widehat{x_i^*} \cdots \wedge x_p^*, \\ &\qquad z \in R_{i=0}^q, \quad x_i^* \in E^*. \end{aligned}$$

In particular,

$$h_\chi(1 \otimes x^*) = \chi(x^*) \otimes 1.$$

In terms of a pair of dual bases for E^* and E we can write

$$h_\chi = \sum_\nu \omega_R \mu(\chi e^{*\nu}) \otimes i_E(e_\nu).$$

Finally, the fourth operator δ_H is defined by

$$\delta_H = \nabla_{i=0} \otimes \iota.$$

Now set

$$d = \omega_R \otimes \delta_E + \delta_\theta + h_\chi + \delta_H.$$

Recall from sec. 5.26 that in $(R_{i=0} \otimes \wedge E^*)_{\theta=0}$

$$2\omega_R \otimes \delta_E + \delta_\theta = 0.$$

Hence, the restriction of d to this subalgebra is given by

$$d_{\theta=0} = -\omega_R \otimes \delta_E + h_\chi + \delta_H. \tag{8.6}$$

Theorem II: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation admitting an algebraic connection χ. Then (in the notation just defined):

(1) $(R_{i=0} \otimes \wedge E^*, d)$ is a graded differential algebra.

(2) $(E, i, \theta, R_{i=0} \otimes \wedge E^*, d)$ is an operation.

(3) The isomorphism $f\colon R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ of Theorem I, sec. 8.4, is an isomorphism of operations.

Proof: Since f is an isomorphism of algebras, the entire theorem will follow once it has been shown that

$$f \circ i(x) = i_R(x) \circ f, \qquad f \circ \theta(x) = \theta_R(x) \circ f, \qquad x \in E,$$

and

$$f \circ d = \delta_R \circ f.$$

The first two relations are proved in Theorem I, sec. 8.4. It remains to establish the third.

Since d and δ_R are antiderivations, we need only show that

$$(f \circ d - \delta_R \circ f)(1 \otimes x^*) = 0 = (f \circ d - \delta_R \circ f)(z \otimes 1),$$
$$x^* \in E^*, \quad z \in R_{i=0}.$$

But Proposition III, sec. 8.6, yields

$$\begin{aligned}\delta_R f(1 \otimes x^*) = \delta_R \chi(x^*) &= \bar{\chi}(x^*) + \chi_\wedge \delta_E(x^*)\\ &= f(\bar{\chi}(x^*) \otimes 1 + 1 \otimes \delta_E(x^*)).\end{aligned}$$

Since $\delta_H(1 \otimes x^*) = 0 = \delta_\theta(1 \otimes x^*)$, it follows that

$$\delta_R f(1 \otimes x^*) = f \circ (h_\chi + \omega_R \otimes \delta_E)(1 \otimes x^*) = (f \circ d)(1 \otimes x^*).$$

On the other hand, Proposition II, sec. 8.5, shows that

$$\begin{aligned}\delta_R f(z \otimes 1) = \nabla z + \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e_\nu) z &= f \circ (\delta_H + \delta_\theta)(z \otimes 1)\\ &= (f \circ d)(z \otimes 1).\end{aligned}$$

This completes the proof.

Q.E.D.

8.8. Homomorphisms. Let $\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations. Assume that χ_R is an algebraic connection for the first operation and let $\chi_S = \varphi \circ \chi_R$ be the induced connection for the second operation. Recall from sec. 8.4 the commutative diagram

$$\begin{tikzcd} R_{i=0} \otimes \wedge E^* \arrow[r, "\cong"] \arrow[d, "\varphi_{i=0}\otimes\iota"'] & R \arrow[d, "\varphi"] \\ S_{i=0} \otimes \wedge E^* \arrow[r, "\cong"'] & S. \end{tikzcd}$$

It follows from this diagram that

$$\varphi_{i=0} \circ \pi_H = \pi_H \circ \varphi,$$

where π_H denotes both horizontal projections. Hence the covariant derivatives ∇_R, ∇_S and the curvatures $\bar{\chi}_R, \bar{\chi}_S$ are related by

$$\varphi_{i=0} \circ \nabla_R = \nabla_S \circ \varphi \qquad \text{and} \qquad \varphi_{i=0} \circ \bar{\chi}_R = \bar{\chi}_S.$$

These relations imply that

$$(\varphi_{i=0} \otimes \iota) \circ (\omega_R \otimes \delta_E) = (\omega_S \otimes \delta_E) \circ (\varphi_{i=0} \otimes \iota),$$
$$(\varphi_{i=0} \otimes \iota) \circ \delta_\theta = \delta_\theta \circ (\varphi_{i=0} \otimes \iota),$$
$$(\varphi_{i=0} \otimes \iota) \circ h_\chi = h_\chi \circ (\varphi_{i=0} \otimes \iota),$$

and

$$(\varphi_{i=0} \otimes \iota) \circ \delta_H = \delta_H \circ (\varphi_{i=0} \otimes \iota).$$

(Here we have used δ_θ, h_χ, and δ_H to denote the appropriate operators both in $R_{i=0} \otimes \wedge E^*$ and $S_{i=0} \otimes \wedge E^*$, cf. sec. 8.7.)

8.9. Examples: 1. *The Weil algebra*: Consider the operation of a Lie algebra E in its Weil algebra $W(E)$ (cf. Example 6, sec. 7.4). Let χ be the algebraic connection given by $x^* \mapsto 1 \otimes x^*$, $x^* \in E^*$.

The horizontal subalgebra is $\vee E^* \otimes 1$ and the horizontal projection is induced by the direct decomposition

$$W(E) = (\vee E^* \otimes 1) \oplus (\vee E^* \otimes \wedge^+ E^*).$$

In particular, the curvature is given by

$$\bar{\chi}(x^*) = \pi_H \delta_W (1 \otimes x^*) = \pi_H(x^* \otimes 1 + 1 \otimes \delta_E x^*) = x^* \otimes 1;$$

i.e.,

$$\bar{\chi}(x^*) = x^* \otimes 1, \qquad x^* \in E^*.$$

The restriction $\nabla_{i=0}$ of ∇ to $\vee E^* \otimes 1$ is zero. Hence the operators h_χ and δ_H are given by

$$h_\chi = h \quad \text{and} \quad \delta_H = 0.$$

Thus the decomposition

$$\delta_W = \iota \otimes \delta_E + \delta_\theta + h,$$

used to define δ_W in sec. 6.4 coincides with the decomposition given in sec. 8.7.

Now Proposition III, sec. 8.6, implies that

$$0 = \nabla_{i=0}^2 = \mu(\chi(e^{*\nu})) \circ \theta_S(e_\nu).$$

Since $W(E)_{i=0} = \vee E^* \otimes 1$ and $\chi(e^{*\nu}) = e^{*\nu} \otimes 1$, we can rewrite this as

$$\sum_\nu \mu_S(e^{*\nu})\theta_S(e_\nu) = 0, \tag{8.7}$$

where $\mu_S(e^{*\nu})$ denotes multiplication by $e^{*\nu}$ in the algebra $\vee E^*$.

2. Let E be a Lie algebra and let F be a subalgebra. Assume that the operation $(E, i_F, \theta_F, \wedge E^*, \delta_E)$ admits an algebraic connection $\chi\colon F^* \to E^*$ (cf. Example 4, sec. 8.1). We shall compute the corresponding curvature χ.

Let $e_1, \ldots, e_m$ be a basis of F, and let $e_{m+1}, \ldots, e_n$ be a basis of $\ker \chi^*$. Then $e_1, \ldots, e_n$ is a basis of E. If $e^{*1}, \ldots, e^{*n}$ denotes the dual basis of E^*, then $e^{*1}, \ldots, e^{*m}$ is a basis of $\operatorname{Im} \chi$, while $e^{*m+1}, \ldots, e^{*n}$ is a basis for $F^\perp$. It will now be shown that the curvature is given by

$$\chi = \tfrac{1}{2} \sum_{\nu=m+1}^{n} \mu(e^{*\nu}) \circ \theta_E(e_\nu) \circ \chi.$$

In fact, by Proposition III, sec. 8.6, we have for $y^* \in F^*$,

$$\begin{aligned}
\chi(y^*) &= \delta_E \chi(y^*) - \chi_\wedge \delta_F(y^*) \\
&= \tfrac{1}{2}\left\{\sum_{\nu=1}^{n} e^{*\nu} \wedge \theta_E(e_\nu)\chi(y^*) - \sum_{\nu=1}^{m} e^{*\nu} \wedge \theta_E(e_\nu)\chi(y^*)\right\} \\
&= \tfrac{1}{2} \sum_{\nu=m+1}^{n} e^{*\nu} \wedge \theta_E(e_\nu)\chi(y^*).
\end{aligned}$$

8.10. The structure homomorphism. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation which admits an algebraic connection χ. Recall the definition

of the structure homomorphism

$$\gamma_R: R \to R \otimes \wedge E^*$$

(cf. sec. 7.8). On the other hand, we have the structure homomorphism

$$\gamma_{\wedge E^*}: \wedge E^* \to \wedge E^* \otimes \wedge E^*,$$

for the operation $(E, i_E, \theta_E, \wedge E^*, \delta_E)$ as well as the isomorphism $f: R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ of sec. 8.4.

Proposition IV: The isomorphism f makes the diagram

$$\begin{tikzcd} R_{i=0} \otimes \wedge E^* \arrow[rr, "f", "\cong"'] \arrow[d, "\iota\otimes\gamma_{\wedge E^*}"'] & & R \arrow[d, "\gamma_R"] \\ R_{i=0} \otimes \wedge E^* \otimes \wedge E^* \arrow[rr, "\cong", "f\otimes\iota"'] & & R \otimes \wedge E^* \end{tikzcd}$$

commute.

Proof: Since all the operators are algebra homomorphisms, we need only check the diagram for elements of the form $z \otimes 1$ $(z \in R_{i=0})$ and $1 \otimes x^*$ $(x^* \in E^*)$.

Recall from sec. 7.8 that $\gamma_R(z) = \beta(z \otimes 1)$, $z \in R$. This implies (in view of Lemma V, sec. 7.7) that

$$\gamma_R(z) = z \otimes 1, \qquad z \in R_{i=0},$$

whence

$$(\gamma_R f)(z \otimes 1) = \gamma_R(z) = z \otimes 1 = (f \otimes \iota) \circ (\iota \otimes \gamma_{\wedge E^*})(z \otimes 1), \qquad z \in R_{i=0}.$$

On the other hand, Lemma V, sec. 7.7, shows that

$$\gamma_R(\chi(x^*)) = \beta(\chi(x^*) \otimes 1) = \chi(x^*) \otimes 1 + 1 \otimes x^*, \qquad x^* \in E^*.$$

It follows that

$$\begin{aligned} (\gamma_R f)(1 \otimes x^*) &= \chi(x^*) \otimes 1 + 1 \otimes x^* \\ &= (f \otimes \iota)(1 \otimes x^* \otimes 1 + 1 \otimes 1 \otimes x^*) \\ &= (f \otimes \iota) \circ (\iota \otimes \gamma_{\wedge E^*})(1 \otimes x^*), \end{aligned}$$

(cf. Example 1, sec. 7.8).

Q.E.D.

Corollary: If $\Phi \in \wedge E^*$, then

$$(\gamma_R \circ \chi_\wedge)(\Phi) - 1 \otimes \Phi \in R^+ \otimes \wedge E^*.$$

Proof: It follows from Example 1, sec. 7.8 that

$$\gamma_{\wedge E^*}(\Phi) - 1 \otimes \Phi \in \wedge^+ E^* \otimes \wedge E^*.$$

Hence,

$$(\iota \otimes \gamma_{\wedge E^*})(1 \otimes \Phi) - 1 \otimes 1 \otimes \Phi \in R_{i=0} \otimes \wedge^+ E^* \otimes \wedge E^*.$$

Applying $f \otimes \iota$ to this relation and using the proposition, we find that

$$(\gamma_R f)(1 \otimes \Phi) - 1 \otimes \Phi \in R^+ \otimes \wedge E^*;$$

i.e., $(\gamma_R \chi_\wedge)(\Phi) - 1 \otimes \Phi \in R^+ \otimes \wedge E^*$.

Q.E.D.

§3. Geometric definition of an operation

8.11. Assume that $(E, i_R, \theta_R, R, \delta_R)$ is an operation with an algebraic connection χ_R. In articles 1 and 2 we constructed the following "geometric" objects:

(1) The horizontal subalgebra $R_{i=0}$ and the restriction of the representation of E to $R_{i=0}$.

(2) The covariant derivative (restricted to $R_{i=0}$) and the curvature

$$\nabla_{i=0}: R_{i=0} \to R_{i=0} \qquad \text{and} \qquad \chi_R: E^* \to R^2_{i=0}.$$

In sec. 8.5 and sec. 8.6 it was shown that the following properties hold:

(1) $\nabla_{i=0}$ is an antiderivation in $R_{i=0}$, homogeneous of degree 1.

(2) $\chi_R \circ \theta_E(x) = \theta_R(x) \circ \chi_R$, and $\nabla_{i=0} \circ \theta_R(x) = \theta_R(x) \circ \nabla_{i=0}$, $x \in E$.

(3) $\nabla_{i=0} \circ \chi_R = 0$.

(4) $\nabla^2_{i=0} = -\sum_\nu \mu(\chi_R(e^{*\nu}))\theta_R(e_\nu)$, where $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E.

The purpose of this article is to reverse this process; in particular we shall establish

Theorem III: Let θ_T be a representation of a Lie algebra E in an anticommutative positively graded algebra T. Assume that linear maps

$$\nabla_T: T \to T \qquad \text{and} \qquad \chi: E^* \to T^2$$

are given, subject to the following conditions:

(1) ∇_T is an antiderivation, homogeneous of degree 1.

(2) $\chi \circ \theta_E(x) = \theta_T(x) \circ \chi$, and $\nabla_T \circ \theta_T(x) = \theta_T(x) \circ \nabla_T$, $x \in E$.

(3) $\nabla_T \circ \chi = 0$.

(4) $\nabla_T^2 = -\sum_\nu \mu_T(\chi e^{*\nu})\theta_T(e_\nu)$ (where $e^{*\nu}$, e_ν is a pair of dual bases for E^*, E, and $\mu_T(z)$ denotes left multiplication by z in T).

Then there is a differential operator d in $T \otimes \wedge E^*$ and an operation $(E, i, \theta, T \otimes \wedge E^*, d)$ such that

(i) $(T \otimes \wedge E^*)_{i=0} = T \otimes 1$.

(ii) The restriction of θ to $T \otimes 1$ is θ_T.

(iii) The inclusion map $\chi\colon E^* \to 1 \otimes E^*$ is an algebraic connection for the operation.

(iv) $\mathfrak{X}$ is the curvature of the connection χ.

(v) The restriction of the covariant derivative to $T \otimes 1$ is ∇_T.

Moreover, the operators $i(x)$, $\theta(x)$ $(x \in E)$, and d are uniquely determined by conditions (i)–(v).

Remark: If $(E, i_R, \theta_R, R, \delta_R)$ is an operation with a connection χ_R and we apply Theorem III with

$$T = R_{i=0}, \qquad \nabla_T = \nabla_{i=0}, \qquad \theta_T = \theta_R, \qquad \text{and} \qquad \mathfrak{X} = \mathfrak{X}_R,$$

then the resulting operation coincides with the operation defined in sec. 8.7, as will be obvious from the definition. Hence, in view of Theorem II, sec. 8.7, the operation obtained from Theorem III in this case is isomorphic to the original operation.

8.12. The differential algebra $(T \otimes \wedge E^*, d)$. In this section we use the notation of Theorem III. It is assumed that the algebra T and the operators $\theta_T(x)$ $(x \in E)$, ∇_T, and $\mathfrak{X}$ satisfy the hypotheses of the theorem.

We wish to construct an antiderivation of degree 1,

$$d\colon T \otimes \wedge E^* \to T \otimes \wedge E^*,$$

such that $d^2 = 0$. To do so we introduce four antiderivations in $T \otimes \wedge E^*$, all homogeneous of degree 1 precisely as in sec. 8.7.

First we have the operator

$$\omega_T \otimes \delta_E = \omega_T \otimes \tfrac{1}{2} \sum_\nu \mu(e^{*\nu})\theta_E(e_\nu),$$

where $e^{*\nu}$, e_ν is a pair of dual bases for E^* and E, and ω_T denotes the degree involution in T. Next recall from sec. 5.25 that the representation θ_T of E in T induces the antiderivation δ_θ in $T \otimes \wedge E^*$ given by

$$\delta_\theta = \sum_\nu \omega_T \theta_T(e_\nu) \otimes \mu(e^{*\nu}).$$

Thirdly, define an antiderivation $h_{\mathfrak{X}}$ of degree 1 by

$$h_{\mathfrak{X}}(z \otimes 1) = 0, \qquad z \in T,$$

and

$$h_\chi(z \otimes x_1^* \wedge \cdots \wedge x_p^*)$$
$$= (-1)^q \sum_{i=1}^{p} (-1)^{i-1}\chi(x_i^*) \cdot z \otimes x_1^* \wedge \cdots \widehat{x_i^*} \cdots \wedge x_p^*,$$
$$z \in T^q, \quad x_i^* \in E^* \quad (i = 1, \ldots, p).$$

In particular,

$$h_\chi(1 \otimes x^*) = \chi(x^*) \otimes 1, \qquad x^* \in E^*.$$

If $e^{*\nu}$, e_ν is a pair of dual bases for E^*, E we can express h_χ in the form

$$h_\chi = \sum_\nu \omega_T \mu_T(\chi e^{*\nu}) \otimes i_E(e_\nu).$$

Finally, since ∇_T is an antiderivation of degree 1 in T, the operator

$$\delta_H = \nabla_T \otimes \iota$$

is an antiderivation of degree 1 in $T \otimes \wedge E^*$. Now we define d by

$$d = \omega_T \otimes \delta_E + \delta_\theta + h_\chi + \delta_H.$$

Proposition V: $(T \otimes \wedge E^*, d)$ is a graded differential algebra.

Proof: Clearly d is an antiderivation homogeneous of degree 1. Thus we have only to show that $d^2 = 0$.

According to sec. 5.25,

$$(\omega_T \otimes \delta_E + \delta_\theta)^2 = 0. \tag{8.8}$$

Moreover, it is immediate from the definitions that

$$h_\chi^2(z \otimes 1) = 0 = h_\chi^2(1 \otimes x^*), \qquad z \in T, \quad x^* \in E^*.$$

Since h_χ^2 is a derivation, it follows that

$$h_\chi^2 = 0. \tag{8.9}$$

Next we show that the equation $\nabla_T \circ \chi = 0$ implies that

$$\delta_H \circ h_\chi + h_\chi \circ \delta_H = 0. \tag{8.10}$$

In fact, $\delta_H \circ h_\chi + h_\chi \circ \delta_H$ is a derivation. Moreover,

$$(\delta_H h_\chi + h_\chi \delta_H)(z \otimes 1) = h_\chi(\nabla_T z \otimes 1) = 0, \qquad z \in T,$$

and

$$(\delta_H h_\chi + h_\chi \delta_H)(1 \otimes x^*) = \delta_H(\chi x^* \otimes 1) = (\nabla_T \chi x^*) \otimes 1 = 0, \quad x^* \in E^*.$$

Hence (8.10) is correct.

Now we shall use the relations

$$\nabla_T^2 = -\sum_\nu \mu_T(\chi e^{*\nu})\theta_T(e_\nu) \qquad \text{and} \qquad \chi \circ \theta_E(x) = \theta_T(x) \circ \chi, \qquad x \in E,$$

to prove that

$$h_\chi \circ (\omega_T \otimes \delta_E + \delta_\theta) + (\omega_T \otimes \delta_E + \delta_\theta) \circ h_\chi = -\delta_H^2. \tag{8.11}$$

In fact, denote the left-hand side by σ. Then σ is a derivation, as is $-\delta_H^2$.

Moreover, for $z \in T^q$,

$$\begin{aligned}\sigma(z \otimes 1) = h_\chi \delta_\theta(z \otimes 1) &= h_\chi\Big((-1)^q \sum_\nu \theta_T(e_\nu)z \otimes e^{*\nu}\Big)\\ &= \sum_\nu \mu_T(\chi e^{*\nu})\theta_T(e_\nu)z \otimes 1 = -\delta_H^2(z \otimes 1).\end{aligned}$$

On the other hand, for $x^* \in E^*$,

$$\begin{aligned}\sigma(1 \otimes x^*) &= h_\chi(1 \otimes \delta_E x^*) + \delta_\theta(\chi(x^*) \otimes 1)\\ &= \sum_\nu \chi(e^{*\nu}) \otimes i_E(e_\nu)\delta_E x^* + \sum_\nu \theta_T(e_\nu)\chi(x^*) \otimes e^{*\nu}\\ &= \sum_\nu \chi(e^{*\nu}) \otimes \theta_E(e_\nu)x^* + \sum_\nu \chi(\theta_E(e_\nu)x^*) \otimes e^{*\nu}\\ &= (\chi \otimes \iota)\Big(\sum_\nu e^{*\nu} \otimes \theta_E(e_\nu)x^* + \sum_\nu \theta_E(e_\nu)x^* \otimes e^{*\nu}\Big).\end{aligned}$$

But if $x, y \in E$, then

$$\begin{aligned}&\Big\langle \sum_\nu e^{*\nu} \otimes \theta_E(e_\nu)x^* + \sum_\nu \theta_E(e_\nu)x^* \otimes e^{*\nu}, x \otimes y\Big\rangle\\ &\quad = \langle \theta_E(x)x^*, y\rangle + \langle \theta_E(y)x^*, x\rangle\\ &\quad = \langle x^*, -[x, y] - [y, x]\rangle\\ &\quad = 0.\end{aligned}$$

This yields

$$\sum_\nu e^{*\nu} \otimes \theta_E(e_\nu)x^* + \sum_\nu \theta_E(e_\nu)x^* \otimes e^{*\nu} = 0,$$

and hence

$$\sigma(1 \otimes x^*) = 0 = -\delta_H^2(1 \otimes x^*).$$

Formula (8.11) follows.

Finally, we use the equations

$$\nabla_T \circ \theta_T(x) = \theta_T(x) \circ \nabla_T, \qquad x \in E,$$

to show that

$$\delta_H \circ (\omega_T \otimes \delta_E + \delta_\theta) + (\omega_T \otimes \delta_E + \delta_\theta) \circ \delta_H = 0. \tag{8.12}$$

Denote the left-hand side by τ. Then τ is a derivation. For $z \in T^q$ we have

$$\begin{aligned}\tau(z \otimes 1) &= \sum_\nu (-1)^q(\nabla_T\theta_T(e_\nu)z) \otimes e^{*\nu} + \sum_\nu (-1)^{q+1}(\theta_T(e_\nu)\nabla_T z) \otimes e^{*\nu} \\ &= 0.\end{aligned}$$

On the other hand, clearly

$$\tau(1 \otimes x^*) = 0, \qquad x^* \in E^*,$$

and so (8.12) follows.

Now adding relations (8.8)–(8.12) yields $d^2 = 0$, and so the proof of the proposition is complete.

Q.E.D.

8.13. Proof of Theorem III: We first construct the operation $(E, i, \theta, T \otimes \wedge E^*, d)$. Recall the definition of the graded differential algebra $(T \otimes \wedge E^*, d)$ in sec. 8.12.

Now define operators $i(x)$ and $\theta(x)$, $x \in E$, by

$$i(x) = \omega_T \otimes i_E(x) \qquad \text{and} \qquad \theta(x) = \theta_T(x) \otimes \iota + \iota \otimes \theta_E(x).$$

Then each $i(x)$ is an antiderivation of degree -1, while θ is a representation of E in the graded algebra $T \otimes \wedge E^*$. Moreover,

$$i(x)^2 = \iota \otimes i_E(x)^2 = 0, \qquad x \in E$$

and

$$\begin{aligned}\theta(x)i(y) - i(y)\theta(x) &= \omega_T \otimes (\theta_E(x)i_E(y) - i_E(y)\theta_E(x)) \\ &= \omega_T \otimes i_E([x, y]) \\ &= i([x, y]), \qquad x, y \in E,\end{aligned}$$

(cf. sec. 5.1).

Finally, it is evident from the definitions that

$$\begin{aligned}i(x) \circ (\omega_T \otimes \delta_E) + (\omega_T \otimes \delta_E) \circ i(x) &= \iota \otimes \theta_E(x), \\ i(x) \circ \delta_\theta + \delta_\theta \circ i(x) &= \theta_T(x) \otimes \iota, \\ i(x) \circ h_\chi + h_\chi \circ i(x) &= 0,\end{aligned}$$

and

$$i(x) \circ \delta_H + \delta_H \circ i(x) = 0, \qquad x \in E.$$

These relations imply that

$$i(x)d + di(x) = \theta(x), \qquad x \in E.$$

Hence, $(E, i, \theta, T \otimes \wedge E^*, d)$ is an operation.

Next it will be shown that this operation satisfies conditions (i)–(v) of Theorem III. The first three are immediate consequences of the definitions. To verify (iv), observe that the horizontal projection π_H corresponds to the decomposition

$$T \otimes \wedge E^* = (T \otimes 1) \oplus (T \otimes \wedge^+ E^*).$$

Hence we have

$$\begin{aligned}\nabla \chi(x^*) &= (\pi_H \circ d)(1 \otimes x^*) = (\pi_H \circ h_\chi)(1 \otimes x^*) \\ &= \bar{\chi}(x^*) \otimes 1, \qquad x^* \in E^*;\end{aligned}$$

i.e., $\bar{\chi}$ is the curvature for χ.

To prove (v) let $z \in T$. Then

$$\nabla(z \otimes 1) = \pi_H d(z \otimes 1) = \nabla_T z \otimes 1,$$

whence $\nabla_{i=0} = \nabla_T$.

It remains to be shown that the operators $i(x)$, $\theta(x)$ $(x \in E)$, and d are uniquely determined by conditions (i)–(v). Let $(E, \hat{i}, \hat{\theta}, T \otimes \wedge E^*, \hat{d})$

be a second operation which satisfies these conditions. Then

$$\hat{i}(x)(z \otimes 1) = 0, \qquad z \in T, \quad x \in E,$$

and

$$\hat{i}(x)(1 \otimes x^*) = \hat{i}(x)\chi(x^*) = \langle x^*, x \rangle, \qquad x^* \in E^*, \quad x \in E.$$

It follows that $\hat{i}(x) = i(x)$, $x \in E$.

The same argument shows that $\hat{\theta}(x) = \theta(x)$, $x \in E$.

Finally, we prove that $d = \hat{d}$. Let $z \in T$. Then Proposition II, sec. 8.5, yields

$$\begin{aligned}\hat{d}(z \otimes 1) &= \nabla_T z \otimes 1 + \sum_\nu \chi(e^{*\nu}) \cdot \theta(e_\nu)(z \otimes 1) \\ &= d(z \otimes 1).\end{aligned}$$

On the other hand, if $x^* \in E^*$, Proposition III, sec. 8.6, yields

$$\begin{aligned}\hat{d}(1 \otimes x^*) = (\hat{d} \circ \chi)(x^*) &= \chi(x^*) + \chi_\wedge \delta_E(x^*) \\ &= d(1 \otimes x^*).\end{aligned}$$

Since $\hat{d}$ and d are antiderivations, it follows that $d = \hat{d}$. This completes the proof.

Q.E.D.

8.14. The algebra $\vee \bar{E}^* \otimes R$. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation. Recall that $\bar{E}^*$ is the graded space which coincides with E^* as a vector space and all of whose elements are homogeneous of degree 2 (cf. sec. 6.1). Recall further that each $-(\operatorname{ad} x)^*$ $(x \in E)$ extends to a derivation $\theta_S(x)$ in the graded algebra $\vee \bar{E}^*$, and that the map $x \mapsto \theta_S(x)$ defines a representation of E in $\vee \bar{E}^*$.

Now consider the graded anticommutative algebra $T = \vee \bar{E}^* \otimes R$. We can define a representation θ_T of E in this algebra by setting $\theta_T(x) = \theta_S(x) \otimes \iota + \iota \otimes \theta_R(x)$, $x \in E$.

Next let h_R be the antiderivation in $\vee \bar{E}^* \otimes R$ given by

$$h_R = \sum_\nu \mu_S(e^{*\nu}) \otimes i_R(e_\nu),$$

where $e^{*\nu}$, e_ν is a pair of dual bases for E^*, E, and $\mu_S(\Psi)$ denotes multiplication by Ψ in $\vee \bar{E}^*$. Then

$$h_R \circ \theta_T(x) = \theta_T(x) \circ h_R, \qquad x \in E. \tag{8.13}$$

In fact, the derivation property of $\theta_S(x)$, together with the relations

$$\theta_R(x)i_R(y) - i_R(y)\theta_R(x) = i_R([x, y]), \qquad x, y \in E,$$

yields

$$(\theta_S(x) \otimes \iota) \circ h_R - h_R \circ (\theta_S(x) \otimes \iota) = \sum_\nu \mu_S(\theta_E(x)e^{*\nu}) \otimes i_R(e_\nu)$$

and

$$(\iota \otimes \theta_R(x)) \circ h_R - h_R \circ (\iota \otimes \theta_R(x)) = \sum_\nu \mu_S(e^{*\nu}) \otimes i_R([x, e_\nu]).$$

A computation shows that $\sum_\nu (\theta_E(x)e^{*\nu} \otimes e_\nu + e^{*\nu} \otimes [x, e_\nu]) = 0$, and so formula (8.13) follows.

Finally, define an antiderivation D_R of degree 1 in $\vee E^* \otimes R$ by setting

$$D_R = \iota \otimes \delta_R - h_R.$$

Then

$$D_R \circ \theta_T(x) = \theta_T(x) \circ D_R, \qquad x \in E, \tag{8.14}$$

and

$$D_R^2 = -\sum_\nu \mu(e^{*\nu} \otimes 1) \circ \theta_T(e_\nu). \tag{8.15}$$

In fact, the first relation follows at once from formula (8.13). To prove the second, observe that $h_R^2 = 0 = \delta_R^2$. Hence

$$\begin{aligned} D_R^2 &= -h_R \circ (\iota \otimes \delta_R) - (\iota \otimes \delta_R) \circ h_R \\ &= -\sum_\nu \mu_S(e^{*\nu}) \otimes \theta_R(e_\nu) \\ &= -\sum_\nu \mu(e^{*\nu} \otimes 1)\theta_T(e_\nu) + \sum_\nu \mu_S(e^{*\nu})\theta_S(e_\nu) \otimes \iota. \end{aligned}$$

But according to formula (8.7) in Example 1, sec. 8.9, $\sum_\nu \mu_S(e^{*\nu})\theta_S(e_\nu) = 0$. Thus formula (8.15) follows.

Now consider the graded algebra $\vee E^* \otimes R \otimes \wedge E^*$.

Proposition VI: There exists a unique differential operator d in $\vee E^* \otimes R \otimes \wedge E^*$ and a unique operation $(E, i, \theta, \vee E^* \otimes R \otimes \wedge E^*, d)$ satisfying the following conditions:

(i) $(\vee E^* \otimes R \otimes \wedge E^*)_{i=0} = \vee E^* \otimes R \otimes 1$.

(ii) The restriction of θ to $\vee E^* \otimes R \otimes 1$ is θ_T.

(iii) The inclusion map $\chi: x^* \mapsto 1 \otimes 1 \otimes x^*$, $x^* \in E^*$, is an algebraic connection for the operation.

(iv) The corresponding curvature χ is the inclusion map given by $x^* \mapsto x^* \otimes 1 \otimes 1$.

(v) The restriction of the covariant derivative to $\vee E^* \otimes R$ is given by D_R.

Proof: We shall apply Theorem III, sec. 8.11 with

$$T = \vee E^* \otimes R, \qquad \theta_T = \theta_T, \qquad \nabla_T = D_R$$

and

$$\chi(x^*) = x^* \otimes 1, \qquad x^* \in E^*.$$

To verify that the hypotheses of Theorem III are satisfied, observe that (1) is obvious, (2) follows from formula (8.14), (3) is obvious, and (4) is relation (8.15).

Q.E.D.

§4. The Weil homomorphism

8.15. Definition: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation admitting an algebraic connection χ with curvature $\chi\!\!\!/: E^* \to R_{i=0}$. Regard $\chi\!\!\!/$ as a linear map

$$\chi\!\!\!/: E^* \to R_{i=0}$$

homogeneous of degree zero.

Since $R_{i=0}$ is a graded anticommutative algebra, it follows that

$$\chi\!\!\!/(x^*) \cdot \chi\!\!\!/(y^*) = \chi\!\!\!/(y^*) \cdot \chi\!\!\!/(x^*), \qquad x^*, y^* \in E^*.$$

Hence $\chi\!\!\!/$ extends to a homomorphism of graded algebras

$$\chi\!\!\!/_\vee: \vee E^* \to R_{i=0}.$$

Recall that

$$\chi\!\!\!/ \circ \theta_E(x) = \theta_R(x) \circ \chi\!\!\!/, \qquad x \in E,$$

and that the extension of $\theta_E(x)$ from E^* to a derivation in $\vee E^*$ is denoted by $\theta_S(x)$. Evidently,

$$\chi\!\!\!/_\vee \circ \theta_S(x) = \theta_R(x) \circ \chi\!\!\!/_\vee.$$

In particular, $\chi\!\!\!/_\vee$ restricts to a homomorphism

$$(\chi\!\!\!/_\vee)_{\theta=0}: (\vee E^*)_{\theta=0} \to R_{i=0,\theta=0}.$$

Next, recall that the Bianchi identity (cf. Proposition III, sec. 8.6) states that

$$\nabla_{i=0} \circ \chi\!\!\!/ = 0,$$

where $\nabla_{i=0}$ denotes the restriction of ∇ to $R_{i=0}$. Since $\nabla_{i=0}$ is an antiderivation, it follows that

$$\nabla_{i=0} \circ \chi\!\!\!/_\vee = 0.$$

Finally, recall from sec. 8.5 that the restriction of $\nabla_{i=0}$ to $R_{i=0,\theta=0}$ coincides with δ_R. Hence the above equation restricts to

$$\delta_R \circ (\chi\!\!\!/_\vee)_{\theta=0} = 0.$$

Thus composing $(\chi_\vee)_{\theta=0}$ with the projection $Z(R_{i=0,\theta=0}) \to H(R_{i=0,\theta=0})$, we obtain a homomorphism

$$\chi^{\#}: (\vee E^*)_{\theta=0} \to H(R_{i=0,\theta=0})$$

of graded algebras. It will be shown in sec. 8.20 that $\chi^{\#}$ is independent of the connection.

The homomorphism $\chi^{\#}$ is called the *Weil homomorphism* of the operation. The image of $\chi^{\#}$, which is a subalgebra of $H(R_{i=0,\theta=0})$, is called the *characteristic subalgebra*.

In the next section we give another interpretation of $\chi_\vee$, $(\chi_\vee)_{\theta=0}$, and $\chi^{\#}$.

8.16. The classifying homomorphism. Again, let $(E, i_R, \theta_R, R, \delta_R)$ denote an operation with algebraic connection χ. Then χ determines the homomorphism of graded algebras

$$\chi_W: W(E) \to R$$

given by

$$\chi_W(\Psi \otimes \Phi) = \chi_\vee(\Psi) \cdot \chi_\wedge(\Phi), \qquad \Psi \in \vee E^*, \quad \Phi \in \wedge E^*.$$

χ_W is called the *classifying homomorphism of the operation* $(E, i_R, \theta_R, R, \delta_R)$ corresponding to the connection χ.

Proposition VII: With the notation above χ_W is a homomorphism of operations:

$$\chi_W: (E, i, \theta_W, W(E), \delta_W) \to (E, i_R, \theta_R, R, \delta_R).$$

Proof: The relations

$$i_R(x)\chi_W(1 \otimes x^*) = i_R(x)\chi(x^*) = \langle x^*, x\rangle = \chi_W i(x)(1 \otimes x^*),$$

and

$$i_R(x)\chi_W(x^* \otimes 1) = i_R(x)\chi(x^*) = 0 = \chi_W i(x)(x^* \otimes 1), \qquad x \in E, \quad x^* \in E^*,$$

show that $i_R(x)\chi_W$ coincides with $\chi_W i(x)$ in $(E^* \otimes 1) \oplus (1 \otimes E^*)$. But this space generates the algebra $W(E)$, and $\chi_W i(x)$ and $i_R(x)\chi_W$ are χ_W-antiderivations. Hence,

$$i_R(x)\chi_W = \chi_W i(x).$$

Similarly the relations

$$\theta_R(x) \circ \chi = \chi \circ \theta_E(x) \qquad \text{and} \qquad \theta_R(x) \circ \not\chi = \not\chi \circ \theta_E(x), \qquad x \in E,$$

imply that

$$\theta_R(x) \circ \chi_W = \chi_W \circ \theta_W(x), \qquad x \in E.$$

It remains to prove that $\delta_R \circ \chi_W = \chi_W \circ \delta_W$. Proposition III, sec. 8.6, yields

$$\begin{aligned} \delta_R \chi_W(1 \otimes x^*) = \delta_R \chi(x^*) &= \not\chi(x^*) + \chi_\wedge(\delta_E x^*) \\ &= \chi_W \delta_W(1 \otimes x^*). \end{aligned}$$

On the other hand, Proposition II, sec. 8.5, and Proposition III, sec. 8.6, imply that

$$\begin{aligned} \delta_R \chi_W(x^* \otimes 1) = \delta_R \not\chi(x^*) &= \nabla \not\chi(x^*) + \sum_\nu \chi(e^{*\nu}) \cdot \theta_R(e^\nu) \not\chi(x^*) \\ &= \chi_W\Big(\sum_\nu \theta_S(e_\nu) x^* \otimes e^{*\nu}\Big) = \chi_W \delta_W(x^* \otimes 1). \end{aligned}$$

Since $\delta_R \chi_W$ and $\chi_W \delta_W$ are χ_W-antiderivations, these relations yield

$$\delta_R \chi_W = \chi_W \delta_W.$$

Q.E.D.

Corollary: χ_W induces homomorphisms $(\chi_W)_{i=0}$, $(\chi_W)_{i=0,\theta=0}$, and $(\chi_W)^\#_{i=0,\theta=0}$. They are given by

$$(\chi_W)_{i=0} = \not\chi_\vee \colon \vee E^* \to R_{i=0},$$

$$(\chi_W)_{i=0,\theta=0} = (\not\chi_\vee)_{\theta=0} \colon (\vee E^*)_{\theta=0} \to R_{i=0,\theta=0},$$

and

$$(\chi_W)^\#_{i=0,\theta=0} = \not\chi^\# \colon (\vee E^*)_{\theta=0} \to H(R_{i=0,\theta=0}).$$

Proposition VIII: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation with a connection χ and let $e_R \colon R_{i=0,\theta=0} \to R_{\theta=0}$ denote the inclusion map. Then

$$e_R^\# \circ (\not\chi^\#)^+ = 0,$$

and so the ideal generated by $\operatorname{Im}(\not\chi^\#)^+$ is contained in the kernel of $e_R^\#$.

Proof: Since the classifying homomorphism χ_W is a homomorphism of operations, the corollary above yields the commutative diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "\chi^*"] \arrow[d] & H(R_{i=0,\theta=0}) \arrow[d, "\varepsilon_R^\#"] \\
H(W(E)_{\theta=0}) \arrow[r, "(\chi_W)^\#_{\theta=0}"'] & H(R_{\theta=0}).
\end{tikzcd}$$

Thus the proposition follows from the relation $H^+(W(E)_{\theta=0}) = 0$ (cf. Proposition I, sec. 6.6).

Q.E.D.

8.17. The differential algebra $(\vee E^* \otimes R)_{\theta=0}$. In this section we establish a theorem which will enable us to show that the Weil homomorphism is independent of connection.

Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation. Recall that in sec. 8.14 we introduced the graded algebra $T = \vee E^* \otimes R$, the representation θ_T, and the antiderivation $D_R = \iota \otimes \delta_R - h_R$. Moreover, it was shown that

$$D_R \circ \theta_T(x) = \theta_T(x) \circ D_R, \qquad x \in E$$

and

$$D_R^2 = -\sum_\nu \mu(e^{*\nu} \otimes 1) \circ \theta_T(e_\nu).$$

These relations imply that D_R restricts to an antiderivation D_R in $(\vee E^* \otimes R)_{\theta=0}$, and that in this algebra $D_R^2 = 0$. Hence $((\vee E^* \otimes R)_{\theta=0}, D_R)$ is a graded differential algebra.

Now consider the injection

$$\varepsilon_R \colon R_{i=0,\theta=0} \to (\vee E^* \otimes R)_{\theta=0}$$

given by

$$\varepsilon_R(z) = 1 \otimes z, \qquad z \in R_{i=0,\theta=0}.$$

Since $h_R = \sum_\nu \mu_S(e^{*\nu}) \otimes i_R(e_\nu)$ it follows that $h_R \circ \varepsilon_R = 0$. Hence,

$$D_R \circ \varepsilon_R = (\iota \otimes \delta_R) \circ \varepsilon_R = \varepsilon_R \circ \delta_R;$$

i.e., ε_R is a homomorphism of graded differential algebras.

Theorem IV: With the notation above, assume that the operation $(E, i_R, \theta_R, R, \delta_R)$ admits a connection. Then the map

$$\varepsilon_R^{\#}: H(R_{i=0,\theta=0}) \to H((\vee E^* \otimes R)_{\theta=0}, D_R),$$

is an isomorphism.

Proof: Let χ denote the connection and consider the corresponding operation $(E, i, \theta, R_{i=0} \otimes \wedge E^*, d)$ (cf. sec. 8.7). According to Theorem II, sec. 8.7, an isomorphism of operations $f: R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ is given by

$$f(z \otimes \Phi) = z \cdot (\chi_\wedge \Phi), \qquad z \in R_{i=0}, \quad \Phi \in \wedge E^*.$$

It follows that an isomorphism of graded algebras

$$g: R_{i=0} \otimes \vee E^* \otimes \wedge E^* \xrightarrow{\cong} \vee E^* \otimes R$$

is given by

$$g(z \otimes \Psi \otimes \Phi) = \Psi \otimes z \cdot (\chi_\wedge \Phi), \qquad z \in R_{i=0}, \quad \Psi \in \vee E^*, \quad \Phi \in \wedge E^*.$$

Identify these algebras via this isomorphism. Then (cf. Theorem II, sec. 8.7, and sec. 8.14)

$$\theta_T(x) = \theta_R(x) \otimes \iota \otimes \iota + \iota \otimes \theta_S(x) \otimes \iota + \iota \otimes \iota \otimes \theta_E(x),$$

$$i_R(x) = \omega_R \otimes \iota \otimes i_E(x), \qquad x \in E,$$

$$\delta_R = \omega_R \otimes \iota \otimes \delta_E + \sum_\nu \theta_R(e_\nu)\omega_R \otimes \iota \otimes \mu(e^{*\nu})$$

$$+ \sum_\nu \omega_R \circ \mu(\chi e^{*\nu}) \otimes \iota \otimes i_E(e_\nu) + \delta_H \otimes \iota \otimes \iota,$$

and

$$h_R = \sum_\nu \omega_R \otimes \mu_S(e^{*\nu}) \otimes i_E(e_\nu).$$

Moreover, ε_R is given by

$$\varepsilon_R(z) = z \otimes 1 \otimes 1, \qquad z \in R_{i=0,\theta=0}.$$

Now filter the algebra $(R_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}$ by the ideals

$$F^p = \sum_{\mu \ge p} (R^\mu_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}.$$

The formulae above show that D_R $(= \delta_R - h_R)$ is filtration preserving.

On the other hand, filter $R_{i=0,\theta=0}$ by the ideals

$$\hat{F}^p = \sum_{\mu \geq p} R^{\mu}_{i=0,\theta=0}.$$

Then ε_R is a homomorphism of graded filtered differential algebras. Hence it induces a homomorphism

$$(\varepsilon_R)_i\colon (\hat{E}_i, \hat{d}_i) \to (E_i, d_i), \qquad i \geq 0,$$

of spectral sequences (cf. sec. 1.6). We shall show that $(\varepsilon_R)_1$ is an isomorphism; in view of Theorem I, sec. 1.14, this will imply that $\varepsilon_R^\#$ is an isomorphism.

Consider the operator,

$$\omega_R \otimes \iota \otimes \delta_E + \sum_\nu \theta_R(e_\nu)\omega_R \otimes \iota \otimes \mu(e^{*\nu}) - h_R,$$

in $R \otimes \vee E^* \otimes \wedge E^*$. It commutes with $\theta_T(x)$, $x \in E$, and hence restricts to an operator δ_T in $(R \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}$. Moreover, the formulae above imply that

$$\delta_T\colon (R^p_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0} \to (R^p_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}, \qquad p = 0, 1, \ldots,$$

while $D_R - \delta_T\colon F^p \to F^{p+1}$. Thus it follows from sec. 1.7 that $\delta_T^2 = 0$, and that there is a canonical isomorphism

$$H((R_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}, \delta_T) \xrightarrow{\cong} E_1.$$

Similarly there is a canonical isomorphism

$$R_{i=0,\theta=0} \xrightarrow{\cong} \hat{E}_1.$$

Moreover, according to sec. 1.7, these isomorphisms identify $(\varepsilon_R)_1$ with the homomorphism

$$R_{i=0,\theta=0} \to H((R_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0}, \delta_T)$$

induced by ε_R. (It will be denoted by ε_R^*.) We have thus only to show that ε_R^* is an isomorphism. But this is an immediate consequence of the two lemmas in the next section.

8.18. Observe that $(R_{i=0} \otimes \vee E^* \otimes \wedge E^*)_{\theta=0} = (R_{i=0} \otimes W(E))_{\theta=0}$.

Lemma I: The operator δ_T is given by

$$\delta_T = -\omega_R \otimes \delta_W .$$

Proof: In $(R_{i=0} \otimes W(E))_{\theta=0}$ we have

$$\begin{aligned}\sum_\nu \theta_R(e_\nu)\omega_R \otimes \iota \otimes \mu(e^{*\nu}) &= -\sum_\nu \omega_R \otimes \theta_S(e_\nu) \otimes \mu(e^{*\nu}) - \sum_\nu \omega_R \otimes \iota \otimes \mu(e^{*\nu})\theta_E(e_\nu) \\ &= -\sum_\nu \omega_R \otimes \theta_S(e_\nu) \otimes \mu(e^{*\nu}) - 2\omega_R \otimes \iota \otimes \delta_E ,\end{aligned}$$

(cf. formula (5.4), sec. 5.3). Thus the lemma follows at once from the definitions of δ_T and δ_W (cf. sec. 6.4).

Q.E.D.

Lemma II: The homomorphism ε_R^* (end of sec. 8.17) is an isomorphism.

Proof: Since

$$(R_{i=0} \otimes W(E))_{\theta=0} = (R_{i=0,\theta=0} \otimes 1) \oplus (R_{i=0} \otimes W^+(E))_{\theta=0} ,$$

it is sufficient to show that

$$H((R_{i=0} \otimes W^+(E))_{\theta=0} , \omega_R \otimes \delta_W) = 0. \tag{8.16}$$

In view of Lemma I, sec. 6.6, the map $\Delta = \delta_W k + k\delta_W$ is a linear automorphism of $W^+(E)$. Evidently, $\iota \otimes \Delta$ restricts to an automorphism $\psi = (\iota \otimes \Delta)_{\theta=0}$ of $(R_{i=0} \otimes W^+(E))_{\theta=0}$. It follows from the definition of Δ that ψ commutes with $\omega_R \otimes \delta_W$ and that

$$\psi(\ker(\omega_R \otimes \delta_W)) \subset \operatorname{Im}(\omega_R \otimes \delta_W).$$

Since ψ is an automorphism, these relations imply (8.16).

Q.E.D.

8.19. The inverse isomorphism. In this section we shall give an explicit expression for the isomorphism

$$H((\vee E^* \otimes R)_{\theta=0} , D_R) \xrightarrow{\cong} H(R_{i=0,\theta=0})$$

inverse to $\varepsilon_R^\sharp$ (cf. Theorem IV, sec. 8.17). Define a homomorphism of graded algebras $\alpha_\chi\colon \vee E^* \otimes R \to R_{i=0}$ by

$$\alpha_\chi(\Psi \otimes z) = \bar{\chi}_\vee(\Psi) \cdot \pi_H(z), \qquad \Psi \in \vee E^*, \quad z \in R,$$

where π_H denotes the horizontal projection onto $R_{i=0}$ (cf. sec. 8.5).

Lemma III: The homomorphism α_χ satisfies

$$\alpha_\chi \circ \theta_T(x) = \theta_R(x) \circ \alpha_\chi, \qquad x \in E$$

and

$$\alpha_\chi \circ D_R = \nabla \circ \alpha_\chi .$$

Proof: The first equation is immediate from the relations

$$\pi_H \circ \theta_R(x) = \theta_R(x) \circ \pi_H \qquad \text{and} \qquad \bar{\chi} \circ \theta_E(x) = \theta_R(x) \circ \bar{\chi}, \qquad x \in E$$

(cf. sec. 8.5 and 8.6).

To prove the second, observe first that, in view of the Bianchi identity in Proposition III, sec. 8.6,

$$\alpha_\chi \circ D_R(x^* \otimes 1) = 0 = (\nabla \circ \alpha_\chi)(x^* \otimes 1), \qquad x^* \in E^*.$$

Next, if $z \in R_{i=0}$, then

$$\begin{aligned}(\alpha_\chi \circ D_R)(1 \otimes z) &= \alpha_\chi(1 \otimes \delta_R z) = \pi_H(\delta_R z)\\ &= \nabla(z) = (\nabla \circ \alpha_\chi)(1 \otimes z).\end{aligned}$$

Finally, if $x^* \in E^*$ then the definition of $\bar{\chi}$ in sec. 8.6 yields

$$\begin{aligned}(\alpha_\chi \circ D_R)(1 \otimes \chi x^*) &= \alpha_\chi(1 \otimes \delta_R \chi(x^*) - x^* \otimes 1)\\ &= (\pi_H \delta_R \chi)(x^*) - \bar{\chi}(x^*)\\ &= (\nabla\chi - \bar{\chi})x^* = 0.\end{aligned}$$

Since

$$\alpha_\chi(1 \otimes \chi x^*) = \pi_H \chi(x^*) = 0,$$

we obtain

$$\alpha_\chi \circ D_R(1 \otimes \chi x^*) = 0 = (\nabla \circ \alpha_\chi)(1 \otimes \chi x^*).$$

Now observe that $\vee E^* \otimes R$ is generated by $E^* \otimes 1$, $1 \otimes R_{i=0}$ and $1 \otimes \chi(E^*)$. Since $\alpha_\chi \circ D_R$ and $\nabla \circ \alpha_\chi$ are α_χ-derivations, the relations

above imply that

$$\alpha_\chi \circ D_R = \nabla \circ \alpha_\chi .$$

Q.E.D.

In view of the lemma, α_χ restricts to a homomorphism of graded differential algebras

$$(\alpha_\chi)_{\theta=0}: ((\vee E^* \otimes R)_{\theta=0}, D_R) \to (R_{i=0,\theta=0}, \delta_R).$$

Proposition IX: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation admitting an algebraic connection. Then the induced homomorphism

$$(\alpha_\chi)^\#_{\theta=0}: H((\vee E^* \otimes R)_{\theta=0}) \to H(R_{i=0,\theta=0})$$

is the isomorphism inverse to $\varepsilon_R^\#$. In particular, $(\alpha_\chi)^\#_{\theta=0}$ is independent of the algebraic connection.

Proof: Clearly, $(\alpha_\chi)_{\theta=0} \circ \varepsilon_R = \iota$, whence $(\alpha_\chi)^\#_{\theta=0} \circ \varepsilon_R^\# = \iota$. By Theorem IV, sec. 8.17, $\varepsilon_R^\#$ is an isomorphism. Thus $(\alpha_\chi)^\#_{\theta=0}$ must be the inverse isomorphism.

Q.E.D.

8.20. Independence of connection. Theorem V: The Weil homomorphism $\chi_R^\#$ of an operation $(E, i_R, \theta_R, R, \delta_R)$ which admits a connection χ_R is independent of the connection.

Proof: Let

$$\xi_R: (\vee E^*)_{\theta=0} \to (\vee E^* \otimes R)_{\theta=0}$$

denote the inclusion map:

$$\xi_R(\Psi) = \Psi \otimes 1, \qquad \Psi \in (\vee E^*)_{\theta=0}.$$

In view of the definition of α_χ, we have the relation

$$(\alpha_\chi)_{\theta=0} \circ \xi_R = (\chi_R)_{\vee,\theta=0}.$$

Moreover, clearly $D_R \circ \xi_R = 0$ and so ξ_R induces a homomorphism of graded algebras

$$\xi_R^\#: (\vee E^*)_{\theta=0} \to H((\vee E^* \otimes R)_{\theta=0}, D_R).$$

It follows that

$$(\alpha_\chi)^\#_{\theta=0} \circ \xi_R^\# = \chi_R^\#.$$

Since the maps $(\alpha_\chi)^\#_{\theta=0}$ $(= (\varepsilon_R^\#)^{-1})$ and $\xi_R^\#$ are independent of the algebraic connection, so is $\chi_R^\#$.

Q.E.D.

Corollary I: Let $\eta_R \colon \vee E^* \otimes R \to R$ be the projection with kernel $\vee^+ E^* \otimes R$. Then

$$\eta_R \circ D_R = \delta_R \circ \eta_R,$$

and the diagram

$$\begin{tikzcd}
 & H((\vee E^* \otimes R)_{\theta=0}) \arrow[dd, shift right, "(\alpha_\chi)^\#_{\theta=0}"', "\cong"] \arrow[dr, "\eta_R^\#"] & \\
(\vee E^*)_{\theta=0} \arrow[ur, "\xi_R^\#"] \arrow[dr, "\chi_R^\#"'] & & H(R_{\theta=0}) \\
 & H(R_{i=0,\theta=0}) \arrow[uu, shift right, "\varepsilon_R^\#"'] \arrow[ur, "e_R^\#"'] &
\end{tikzcd}$$

commutes.

Proof: We have noted above that the left hand triangle commutes. Clearly $\eta_R \circ \varepsilon_R = e_R$, and so the right hand triangle commutes too.

Q.E.D.

Corollary II: Let $\varphi \colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations which admit algebraic connections. Then the diagram

$$\begin{tikzcd}
 & H(R_{i=0,\theta=0}) \arrow[dd, "\varphi^\#_{i=0,\theta=0}"] \\
(\vee E^*)_{\theta=0} \arrow[ur, "\chi_R^\#"] \arrow[dr, "\chi_S^\#"'] & \\
 & H(S_{i=0,\theta=0})
\end{tikzcd}$$

commutes.

Proof: Let χ_R be an algebraic connection for the first operation. Then $\chi_S = \varphi \circ \chi_R$ is an algebraic connection for the second operation

(cf. sec. 8.1). With this choice of connections we have

$$\chi_S = \varphi_{i=0} \circ \chi_R$$

(cf. sec. 8.8). It follows that $(\chi_S)_\vee = \varphi_{i=0} \circ (\chi_R)_\vee$, whence

$$\chi_S^\# = \varphi_{i=0,\theta=0}^\# \circ \chi_R^\#.$$

Since $\chi_S^\#$ is independent of the connection, the proof is complete.

Q.E.D.

8.21. Cohomology sequence of a regular operation. Definition: An operation $(E, i_R, \theta_R, R, \delta_R)$ will be called *regular* if:

(i) E is a reductive Lie algebra.
(ii) $H(R_{\theta=0})$ is connected.
(iii) The operation admits an algebraic connection.

Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation. Then there are homomorphisms:

$$\varrho_R\colon H(R_{\theta=0}) \to (\wedge E^*)_{\theta=0} \qquad \text{(fibre projection)},$$

$$e_R^\#\colon H(R_{i=0,\theta=0}) \to H(R_{\theta=0}) \qquad \text{(induced by the inclusion map)},$$

and

$$\chi_R^\#\colon (\vee E^*)_{\theta=0} \to H(R_{i=0,\theta=0}) \qquad \text{(Weil homomorphism)}.$$

The sequence

$$(\vee E^*)_{\theta=0} \xrightarrow{\chi_R^\#} H(R_{i=0,\theta=0}) \xrightarrow{e_R^\#} H(R_{\theta=0}) \xrightarrow{\varrho_R} (\wedge E^*)_{\theta=0}$$

will be called the *cohomology sequence* of the operation.

According to Proposition VIII, sec. 8.16, and sec. 7.10,

$$e_R^\# \circ (\chi_R^\#)^+ = 0 \qquad \text{and} \qquad \varrho_R \circ (e_R^\#)^+ = 0.$$

Next, let $\varphi\colon (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of regular operations. Then it follows from sec. 7.10 and Corollary

II to Theorem V, sec. 8.20, that the diagram

$$\begin{tikzcd}
 & H(R_{i=0,\theta=0}) \arrow[r, "e_R^\#"] \arrow[dd, "\varphi^\#_{i=0,\theta=0}"] & H(R_{\theta=0}) \arrow[dd, "\varphi^\#_{\theta=0}"] \arrow[dr, "\varrho_R"] & \\
(\vee E^*)_{\theta=0} \arrow[ur, "\chi_R^\#"] \arrow[dr, "\chi_S^\#"'] & & & (\wedge E^*)_{\theta=0} \\
 & H(S_{i=0,\theta=0}) \arrow[r, "e_S^\#"'] & H(S_{\theta=0}) \arrow[ur, "\varrho_S"'] &
\end{tikzcd}$$

commutes. This diagram shows that a homomorphism of regular operations induces a homomorphism of the corresponding cohomology sequences.

§5. Principal bundles

This article will make consistent reference to Chapter VI, volume II. G denotes a connected Lie group with Lie algebra E.

8.22. Principal connections. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle, with principal action, $T: P \times G \to P$, (cf. sec. 5.1, volume II). Let $Z_h \in \mathscr{X}(P)$ be the fundamental vector field generated by h ($h \in E$) (cf. sec. 7.19), and denote $i(Z_h)$ and $\theta(Z_h)$ simply by $i(h)$ and $\theta(h)$. Then $(E, i, \theta, A(P), \delta)$ is an operation of E in $A(P)$. It is called the *operation of E associated with the principal bundle $\mathscr{P}$.*

The remark at the end of sec. 6.3, volume II, shows that the homomorphism $\pi^*: A(B) \to A(P)$ can be regarded as an isomorphism

$$\pi^*: A(B) \xrightarrow{\cong} A(P)_{i=0,\theta=0}.$$

Now let V be a principal connection in $\mathscr{P}$ (cf. sec. 6.8, volume II). Thus V is a G-equivariant strong bundle map in τ_P, which projects the tangent bundle onto the vertical subbundle. The bundle map $H = \iota - V$ is the projection of τ_P onto the corresponding horizontal subbundle.

In sec. 6.10, volume II, it was shown that principal connections are in one-to-one correspondence with the E-valued 1-forms ω on P which satisfy

$$i(h)\omega = h \qquad \text{and} \qquad \theta(h)\omega = -\operatorname{ad} h(\omega), \qquad h \in E.$$

The E-valued 1-forms on P satisfying these conditions will be called *connection forms* on P. The correspondence between connections and connection forms is given, explicitly, by

$$Z_{\omega(z;\zeta)} = V(\zeta), \qquad \zeta \in T_z(P), \qquad z \in P.$$

ω is called the *connection form for V*.

On the other hand, let ω be a connection form, and define a linear map

$$\omega^*: E^* \to A^1(P)$$

by

$$\omega^*(h^*)(z;\, \zeta) = \langle h^*, \omega(z;\, \zeta)\rangle, \qquad h^* \in E^*, \quad \zeta \in T_z(P), \quad z \in P.$$

The relations above for ω imply that

$$i(h)(\omega^*(h^*)) = \langle h^*, \omega(Z_h)\rangle = \langle h^*, h\rangle$$

and

$$\begin{aligned}[\theta(h)(\omega^*(h^*))](z; \zeta) &= \langle h^*, (\theta(h)\omega)(z; \zeta)\rangle \\ &= \langle \theta_E(h)(h^*), \omega(z, \zeta)\rangle = [\omega^*(\theta_E(h)h^*)](z; \zeta), \\ &\qquad h \in E, \quad h^* \in E^*, \quad \zeta \in T_z(P), \quad z \in P.\end{aligned}$$

Hence ω^* is an algebraic connection for the operation of E in $A(P)$.

Conversely, if χ is an algebraic connection for the operation, then a connection form ω for the principal bundle is defined by

$$\langle h^*, \omega(z; \zeta)\rangle = \chi(h^*)(z; \zeta), \qquad h^* \in E^*, \quad \zeta \in T_z(P), \quad z \in P.$$

Hence we have a bijection between algebraic connections for the operation of E in $A(P)$ and principal connections in $\mathscr{P}$.

In particular, since a principal bundle always admits a principal connection (cf. sec. 6.8, volume II), the operation $(E, i, \theta, A(P), \delta)$ always admits an algebraic connection.

Finally, let V be a principal connection in $\mathscr{P}$ with corresponding algebraic connection, ω^*. Then the isomorphism

$$f\colon A(P)_{i=0} \otimes \wedge E^* \xrightarrow{\cong} A(P),$$

induced by ω^* (cf. sec. 8.4) is given by

$$\begin{aligned}&f(\Psi \otimes \Phi)(z; \zeta_1, \ldots, \zeta_{p+q}) \\ &\quad = \frac{1}{p!q!} \sum_{\sigma\in S^{p+q}} \varepsilon_\sigma \Psi(z; \zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(p)})\Phi(\omega(z; \zeta_{\sigma(p+1)}), \ldots, \omega(z; \zeta_{\sigma(p+q)})), \\ &\qquad\qquad \Psi \in A^p(P)_{i=0}, \quad \Phi \in \wedge^q E^*, \quad z \in P, \quad \zeta_i \in T_z(P).\end{aligned}$$

8.23. Horizontal projections. Fix a principal connection V in $\mathscr{P}$ with connection form ω and corresponding algebraic connection ω^*. Let $H = \iota - V$ be the projection on the horizontal subbundle. In sec. 6.11, volume II, we defined a corresponding operator

$$H^*\colon A(P) \to A(P)_{i=0}$$

by

$$\begin{aligned}(H^*\Omega)(z; \zeta_1, \ldots, \zeta_p) &= \Omega(z; H\zeta_1, \ldots, H\zeta_p), \\ &\qquad \Omega \in A^p(P), \quad \zeta_i \in T_z(P), \quad z \in P.\end{aligned}$$

H^* is called the *horizontal projection associated with* V.

On the other hand, in sec. 8.5 we defined the horizontal projection associated with the algebraic connection ω^*:

$$\pi_H: A(P) \to A(P)_{i=0}.$$

We show now that

$$\pi_H = H^*. \tag{8.17}$$

In fact, both operators reduce to the identity in $A(P)_{i=0}$. Moreover, by definition, $\pi_H \circ \omega^* = 0$, while

$$H^*(\omega^*(h^*)) = \langle h^*, H^*\omega \rangle = 0, \qquad h^* \in E^*.$$

Hence $H^* \circ \omega^* = 0$ as well. Since (cf. Corollary II to Theorem I, sec. 8.4) the algebra $A(P)$ is generated by $A(P)_{i=0}$ and Im ω^*, formula (8.17) follows.

8.24. Covariant derivative and curvature. Fix V, ω, ω^* as in sec. 8.22. In sec. 6.12, volume II, we defined the covariant exterior derivative $\nabla_{\mathscr{P}}: A(P) \to A(P)$ corresponding to V by

$$\nabla_{\mathscr{P}} = H^* \circ \delta.$$

On the other hand, in sec. 8.5 we defined the covariant derivative ∇ corresponding to ω^* by

$$\nabla = \pi_H \circ \delta.$$

Since, in view of formula (8.17), $\pi_H = H^*$, it follows that

$$\nabla = \nabla_{\mathscr{P}}. \tag{8.18}$$

Next, recall from sec. 6.14, volume II, that the curvature $\Omega \in A^2(P; E)$ for V is defined by

$$\Omega = \nabla_{\mathscr{P}}\omega,$$

where $\nabla_{\mathscr{P}}$ is regarded as an operator in $A(P; E)$. On the other hand, the "algebraic curvature" χ of ω^* is defined by

$$\chi = \nabla \circ \omega^*.$$

Since $\nabla = \nabla_{\mathscr{P}}$, it follows that

$$\langle h^*, \Omega \rangle = \nabla_{\mathscr{P}}(\langle h^*, \omega \rangle) = (\nabla \circ \omega^*)(h^*), \qquad h^* \in E^*.$$

Hence Ω and χ are related by

$$\chi(h^*) = \langle h^*, \Omega \rangle, \qquad h^* \in E^*. \tag{8.19}$$

8.25. The operator h_χ. Again fix V, ω, ω^*. Recall from sec. 8.7 the definition of the antiderivation h_χ in $A(P)_{i=0} \otimes \wedge E^*$. Under the isomorphism f induced by ω^* this operator corresponds to an antiderivation (again denoted by h_χ) in $A(P)$. It is given, explicitly, by

$$(h_\chi \Phi)(z;\, \zeta_0, \ldots, \zeta_p)$$
$$= -\sum_{i<j} (-1)^{i+j} \Phi(z;\, Z_{\Omega(z;\zeta_i,\zeta_j)}, \zeta_0, \ldots \hat{\zeta}_i \ldots \hat{\zeta}_j \ldots, \zeta_p),$$
$$\Phi \in A^p(P), \quad \zeta_i \in T_z(P), \quad z \in P.$$

In fact, let $h^{*\nu}$, h_ν be a pair of dual bases in E^* and E. Then

$$h_\chi = \sum_\nu \mu(\chi h^{*\nu}) i(h_\nu).$$

Moreover, for $z \in P$ and $\zeta_1, \zeta_2 \in T_z(P)$,

$$Z_{\Omega(z;\zeta_1,\zeta_2)} = \sum_\nu \langle h^{*\nu}, \Omega(z;\, \zeta_1, \zeta_2)\rangle Z_{h_\nu}.$$

These relations yield

$$(h_\chi \Phi)(z;\, \zeta_0, \ldots, \zeta_p)$$
$$= \frac{1}{2(p-1)!} \sum_{\sigma \in S^{p+1}} \varepsilon_\sigma (\chi h^{*\nu})(z;\, \zeta_{\sigma(0)}, \zeta_{\sigma(1)}) \Phi(z;\, Z_{h_\nu}, \zeta_{\sigma(2)}, \ldots, \zeta_{\sigma(p)})$$
$$= \frac{1}{2(p-1)!} \sum_{\sigma \in S^{p+1}} \varepsilon_\sigma \langle h^{*\nu}, \Omega(z;\, \zeta_{\sigma(0)}, \zeta_{\sigma(1)})\rangle \Phi(z;\, Z_{h_\nu}, \zeta_{\sigma(2)}, \ldots, \zeta_{\sigma(p)})$$
$$= \frac{1}{2(p-1)!} \sum_{\sigma \in S^{p+1}} \varepsilon_\sigma \Phi(z;\, Z_{\Omega(z;\zeta_{\sigma(0)},\zeta_{\sigma(1)})}, \zeta_{\sigma(2)}, \ldots, \zeta_{\sigma(p)})$$
$$= -\sum_{i<j} (-1)^{i+j} \Phi(z;\, Z_{\Omega(z;\zeta_i,\zeta_j)}, \zeta_0, \ldots \hat{\zeta}_i \ldots \hat{\zeta}_j \ldots, \zeta_p).$$

8.26. The Weil homomorphism. In sec. 6.16 through sec. 6.19, volume II, we defined the Weil homomorphism

$$h_{\mathscr{P}} \colon (\vee E^*)_I \to H(B)$$

for a principal bundle $\mathscr{P}$. (The actual definition is in sec. 6.19.) On the other hand, we have the Weil homomorphism

$$\chi^\# \colon (\vee E^*)_{\theta=0} \to H(A(P)_{i=0,\theta=0})$$

for the operation of E in $A(P)$ (cf. sec. 8.15).

Since G is connected, $(\vee E^*)_I = (\vee E^*)_{\theta=0}$. Hence, (forgetting gradations) $(\vee E^*)_I = (\vee E^*)_{\theta=0}$.

The purpose of this section is to prove

Theorem VI: The diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "\chi^*"] & H(A(P)_{i=0,\theta=0}) \\
(\vee E^*)_I \arrow[u, "="] \arrow[r, "h_{\mathscr{P}}"'] & H(B) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes.

Proof: Let ω be a connection form for $\mathscr{P}$ with algebraic connection ω^*. Recall that in sec. 6.17, volume II, we defined an algebra homomorphism

$$\gamma\colon (\vee E^*) \to A(P)_{i=0},$$

which restricted to a homomorphism

$$\gamma_I\colon (\vee E^*)_I \to A_B(P).$$

Then we set $h_{\mathscr{P}} = \gamma_B^\#$, where $\gamma_B\colon (\vee E^*)_I \to A(B)$ was defined by $\pi^* \circ \gamma_B = \gamma_I$.

Now recall that $A_B(P) = A(P)_{i=0,\theta=0}$. Hence to prove the theorem it is sufficient to show that

$$\gamma_I = (\chi_\vee)_{\theta=0}. \tag{8.20}$$

This in turn will follow from the relation

$$\gamma = \chi_\vee. \tag{8.21}$$

Since γ and $\chi_\vee$ are homomorphisms, it is enough to prove that

$$\gamma(h^*) = \chi(h^*), \qquad h^* \in E^*.$$

But in view of the definition of γ (sec. 6.17, volume II) we have

$$\gamma(h^*)(z; \zeta_1, \zeta_2) = \langle h^*, \Omega(z; \zeta_1, \zeta_2)\rangle, \qquad h^* \in E^*, \quad \zeta_i \in T_z(P), \quad z \in P.$$

Now formula (8.19), sec. 8.24, implies that $\gamma(h^*) = \chi(h^*)$.

Q.E.D.

8.27. The cohomology sequence. Assume that P is connected, and let G_x denote the fibre over $x \in B$. For each $x \in B$, the inclusion map $j_x: G_x \to P$ induces a homomorphism

$$j_x^\#: H(G_x) \leftarrow H(P).$$

Now use a local coordinate representation for $\mathscr{P}$ to identify G_x with G and $H(G_x)$ with $H(G)$. Then the resulting homomorphisms

$$j_x^\#: H(G) \leftarrow H(P)$$

all coincide (since P and G are connected).

We denote this common homomorphism by

$$\varrho_P: H(G) \leftarrow H(P).$$

It is called the *fibre projection* for the principal bundle $\mathscr{P}$. Observe that if $z \in P$ and if $A_z: G \to P$ denotes the inclusion map given by $a \mapsto z \cdot a$, $a \in G$, then

$$A_z^\# = \varrho_P.$$

Now suppose E is reductive. Then we can apply Proposition XV, sec. 7.22, to obtain the commutative diagram

$$\begin{tikzcd} H(A(P)_{\theta=0}) \arrow[r, "\varrho_{A(P)}"] \arrow[d] & (\wedge E^*)_{\theta=0} \arrow[d, "\alpha_G"] \\ H(P) \arrow[r, "\varrho_P"'] & H(G). \end{tikzcd}$$

This diagram relates the fibre projection for the operation to the fibre projection for the bundle.

Combining the diagram above with Theorem VI yields the commutative diagram

$$\begin{tikzcd} (\vee E^*)_{\theta=0} \arrow[r, "\chi^\#"] \arrow[d, "="'] & H(A(P)_{i=0,\theta=0}) \arrow[r, "e^\#"] \arrow[d, "\cong"] & H(A(P)_{\theta=0}) \arrow[r, "\varrho_{A(P)}"] \arrow[d] & (\wedge E^*)_{\theta=0} \arrow[d, "\alpha_G"] \\ (\vee E^*)_I \arrow[r, "h_{\mathscr{P}}"'] & H(B) \arrow[r, "\pi^\#"'] & H(P) \arrow[r, "\varrho_P"'] & H(G). \end{tikzcd} \tag{8.22}$$

The lower sequence is called the *cohomology sequence for the principal bundle* $\mathscr{P}$.

Thus the diagram is a homomorphism from the cohomology sequence of the operation to the cohomology sequence of the bundle. Moreover, if G is compact, then all the vertical maps are isomorphisms, so that the diagram is an isomorphism between cohomology sequences (cf. Theorem I, sec. 4.3, volume II, and sec. 5.29).

Chapter IX

Cohomology of Operations and Principal Bundles

§1. The filtration of an operation

9.1. Definition: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation. Define subspaces $F^p(R^q) \subset R^q$ $(p \leq q)$ by

$$F^p(R^q) = \{z \in R^q \mid i_R(a)z = 0,\ a \in \wedge^{q-p+1}E\},$$

and set

$$F^p(R) = \sum_{q=p}^{\infty} F^p(R^q).$$

Evidently the spaces $F^p(R)$ define a filtration of R, so that R becomes a graded filtered space.

This filtration is called the *filtration of R induced by the operation* $(E, i_R, \theta_R, R, \delta_R)$.

Proposition I: The spaces $F^p(R)$ are stable under the operators $i_R(a)$ $(a \in \wedge E)$, $\theta_R(x)$ $(x \in E)$, and δ_R. Moreover,

$$F^p(R) \cdot F^q(R) \subset F^{p+q}(R), \qquad p, q \geq 0, \tag{9.1}$$

and so the above filtration makes R into a graded filtered differential algebra.

Proof: It follows immediately from the definition that

$$i_R(x)\colon F^p(R^q) \to F^p(R^{q-1}), \qquad x \in E,$$

and so the $F^p(R)$ are stable under the operators $i_R(a)$ $(a \in \wedge E)$.

Formula (7.5), sec. 7.2, implies that the $F^p(R)$ are stable under the operators $\theta_R(x)$. On the other hand, it follows from formula (7.7), sec. 7.2, that $F^p(R)$ is stable under δ_R.

To prove formula (9.1), let $z \in F^p(R^q)$ and $w \in F^r(R^s)$. Then

$$i_R(x_1 \wedge \cdots \wedge x_{q+s-p-r+1})(z \cdot w) \qquad (x_i \in E)$$

is a sum of terms of the form

$$(i_R(a)z) \cdot (i_R(b)w), \qquad a \in \wedge^k E, \quad b \in \wedge^l E,$$

where either $k \geq q - p + 1$ or $l \geq s - r + 1$. Hence either $i_R(a)z = 0$ or $i_R(b)w = 0$, and so

$$i_R(x_1 \wedge \cdots \wedge x_{q+s-p-r-1})(z \cdot w) = 0.$$

It follows that $z \cdot w \in F^{p+r}(R^{q+s})$.

Q.E.D.

Corollary: The subspaces $F^p(R)$ are ideals in R.

Proposition II: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation. Then the basic subalgebra B_R of R with respect to the induced filtration coincides with the basic subalgebra of the operation:

$$B_R = R_{i=0,\theta=0}.$$

Proof: Recall from sec. 1.13 that an element $z \in R^p$ is contained in B_R if and only if it satisfies

$$z \in F^p(R^p) \qquad \text{and} \qquad \delta_R z \in F^{p+1}(R^{p+1}).$$

It is immediate from the definition that

$$F^p(R^p) = R^p_{i=0}.$$

Hence, $z \in B_R^p$ if and only if $z \in R^p_{i=0}$ and $\delta_R z \in R^{p+1}_{i=0}$. But for $z \in R_{i=0}$,

$$\theta_R(x)z = i_R(x)\delta_R(z), \qquad x \in E,$$

and so the proposition follows.

Q.E.D.

Suppose now that

$$\varphi: (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$$

is a homomorphism of operations. Then it follows at once from the

definition that φ preserves the induced filtrations:

$$\varphi: F^p(R^q) \to F^p(S^q).$$

Thus φ is a homomorphism of graded filtered differential algebras. The induced homomorphism φ_B of the basic subalgebras is given by $\varphi_B = \varphi_{i=0,\theta=0}$, as follows from Proposition II, above.

Finally, suppose that the operation $(E, i_R, \theta_R, R, \delta_R)$ admits an algebraic connection. Consider the operation $(E, i, \theta, R_{i=0} \otimes \wedge E^*, d)$ defined in sec. 8.7. It follows immediately from the definitions that

$$F^p(R_{i=0} \otimes \wedge E^*) = \sum_{\mu \geq p} R^\mu_{i=0} \otimes \wedge E^*.$$

Hence, since $f: R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ is an isomorphism of operations, it restricts to isomorphisms

$$\sum_{\mu \geq p} R^\mu_{i=0} \otimes \wedge E^* \xrightarrow{\cong} F^p(R).$$

This shows that $F^p(R)$ is the ideal generated by $\sum_{\mu\geq p} R^\mu_{i=0}$.

9.2. The filtration of $R_{\theta=0}$. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation. Define a filtration in $R_{\theta=0}$ by setting

$$F^p(R_{\theta=0}) = F^p(R) \cap R_{\theta=0}.$$

It follows from Proposition I, sec. 9.1, that this filtration makes $R_{\theta=0}$ into a graded filtered differential algebra. The corresponding spectral sequence will be denoted by

$$(E_i(R_{\theta=0}), \hat{d}_i), \qquad i \geq 0,$$

and called the *spectral sequence of the operation.*

Proposition II, sec. 9.1, implies that $R_{i=0,\theta=0}$ is the basic subalgebra of $R_{\theta=0}$ with respect to this filtration.

If $\varphi: R \to S$ is a homomorphism of operations, then $\varphi_{\theta=0}: R_{\theta=0} \to S_{\theta=0}$ is a homomorphism of graded filtered differential algebras.

Now assume that the operation $(E, i_R, \theta_R, R, \delta_R)$ admits a connection, and consider the associated operation $(E, i, \theta, R_{i=0} \otimes \wedge E^*, d)$ (cf. sec. 8.7). Then the corresponding filtration of $(R_{i=0} \otimes \wedge E^*)_{\theta=0}$ is given by

$$F^p((R_{i=0} \otimes \wedge E^*)_{\theta=0}) = \sum_{\mu \geq p} (R^\mu_{i=0} \otimes \wedge E^*)_{\theta=0}.$$

Hence the isomorphism f restricts to isomorphisms

$$\sum_{\mu \geq p} (R^{\mu}_{i=0} \otimes \wedge E^*)_{\theta=0} \xrightarrow{\cong} F^p(R_{\theta=0}).$$

Example: The filtration of $W(E)_{\theta=0}$ defined in sec. 6.12 is the filtration induced by the operation $(E, i, \theta_W, W(E), \delta_W)$.

§2. The fundamental theorem

In this article E denotes a reductive Lie algebra with primitive space P_E. We shall identify $\wedge P_E$ with $(\wedge E^*)_{\theta=0}$ under the isomorphism $\varkappa_E$ of Theorem III, sec. 5.18. Further,

$$\tau : P_E \rightarrow (\vee^+ E^*)_{\theta=0}$$

denotes a fixed transgression in $W(E)_{\theta=0}$ (cf. sec. 6.13), and

$$\alpha : P_E \rightarrow W(E)_{\theta=0}$$

denotes a fixed linear map, homogeneous of degree zero, such that

$$\delta_W \alpha(\Phi) = \tau(\Phi) \otimes 1 \qquad \text{and} \qquad \alpha(\Phi) - 1 \otimes \Phi \in (\vee^+ E^* \otimes \wedge E^*)_{\theta=0},$$

(cf. sec. 6.13).

9.3. The Chevalley homomorphism. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation with a fixed algebraic connection χ_R. Define a linear map, homogeneous of degree 1,

$$\tau_R : P_E \rightarrow R_{i=0,\theta=0}$$

by

$$\tau_R = (\chi_R)_{\vee,\theta=0} \circ \tau.$$

Recall from sec. 8.15 that $\delta_R \circ (\chi_R)_{\vee,\theta=0} = 0$. This implies that

$$\delta_R \circ \tau_R = 0.$$

It follows that $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ is a (P_E, δ)-algebra (cf. sec. 3.1).

Definition: The (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ is called the (P_E, δ)-*algebra associated with the operation* via the connection χ_R and the transgression τ.

Since $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ is a (P_E, δ)-algebra we can form the Koszul complex

$$(R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R)$$

(cf. sec. 3.2). ∇_R is given explicitly by

$$\nabla_R(z \otimes 1) = \delta_R z \otimes 1$$

and

$$\begin{aligned} \nabla_R(z \otimes \Phi_0 \wedge \cdots \wedge \Phi_p) &= \delta_R z \otimes \Phi_0 \wedge \cdots \wedge \Phi_p \\ &+ (-1)^q \sum_{j=0}^{p} (-1)^j \tau_R(\Phi_j) \cdot z \otimes \Phi_0 \wedge \cdots \widehat{\Phi_j} \cdots \wedge \Phi_p, \\ & \qquad z \in R^q_{i=0,\theta=0}, \quad \Phi_i \in P_E. \end{aligned}$$

Recall from sec. 3.4 that a filtration of this graded differential algebra is defined by

$$F^p(R_{i=0,\theta=0} \otimes \wedge P_E) = \sum_{\mu \geq p} R^\mu_{i=0,\theta=0} \otimes \wedge P_E.$$

The corresponding spectral sequence will be denoted by $\{E_i, d_i\}_{i\geq 0}$.

Next (cf. sec. 8.16) let $\chi_W\colon W(E) \to R$ be the classifying homomorphism for the algebraic connection χ_R. Consider the linear map

$$\vartheta_R = (\chi_W)_{\theta=0} \circ \alpha : P_E \to R_{\theta=0}.$$

Then ϑ_R is homogeneous of degree zero. Hence, since R is anticommutative and $P_E^k = 0$ for even k, ϑ_R extends to a homomorphism of graded algebras

$$\vartheta_R\colon \wedge P_E \to R_{\theta=0}.$$

Finally, extend ϑ_R to a homomorphism of graded algebras

$$\vartheta_R\colon R_{i=0,\theta=0} \otimes \wedge P_E \to R_{\theta=0},$$

by setting

$$\vartheta_R(z \otimes \Phi) = z \cdot \vartheta_R(\Phi), \qquad z \in R_{i=0,\theta=0}, \quad \Phi \in \wedge P_E.$$

ϑ_R is called the *Chevalley homomorphism* associated with the operation $(E, i_R, \theta_R, R, \delta_R)$ via the algebraic connection χ_R and the linear map α.

Since

$$(\vartheta_R \circ \nabla_R)(z \otimes 1) = \delta_R z = \delta_R \vartheta_R(z \otimes 1), \qquad z \in R_{i=0,\theta=0},$$

and

$$\begin{aligned} (\vartheta_R \circ \nabla_R)(1 \otimes \Phi) &= (\chi_R)_{\vee,\theta=0}(\tau\Phi) = (\chi_W)_{\theta=0}(\delta_W \alpha(\Phi)) \\ &= \delta_R \vartheta_R(1 \otimes \Phi), \qquad \Phi \in P_E, \end{aligned}$$

it follows that $\vartheta_R \circ \nabla_R = \delta_R \circ \vartheta_R$. Hence ϑ_R is a homomorphism of graded differential algebras:

$$\vartheta_R\colon (R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R) \to (R_{\theta=0}, \delta_R).$$

Theorem I (Fundamental theorem): Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra, which admits an algebraic connection. Then the Chevalley homomorphism ϑ_R has the following properties:

(1) The induced homomorphism

$$\vartheta_R^\#\colon H(R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R) \to H(R_{\theta=0})$$

is an isomorphism of graded algebras.

(2) ϑ_R is filtration preserving. The induced homomorphisms

$$(\vartheta_R)_i\colon E_i \to E_i(R_{\theta=0})$$

of the corresponding spectral sequences (cf. sec. 1.6) are isomorphisms for $i \geq 1$.

(3) $\vartheta_R(z \otimes 1) = z$, $z \in R_{i=0,\theta=0}$, and

$$\vartheta_R(1 \otimes \Phi) - (\chi_R)_\wedge(\Phi) \in F^1(R_{\theta=0}), \qquad \Phi \in \wedge P_E.$$

Proof: (3) The first statement is obvious. To prove the second, it is sufficient to consider the case $\Phi \in P_E$. Recall that χ_W is a homomorphism of operations, whence

$$\chi_W(F^1(W_{\theta=0})) \subset F^1(R_{\theta=0}).$$

It follows that

$$\vartheta_R(1 \otimes \Phi) - (\chi_R)_\wedge(\Phi) = \chi_W(\alpha(\Phi) - 1 \otimes \Phi) \in F^1(R_{\theta=0}).$$

(1) *and* (2) We show first that ϑ_R is filtration preserving. By definition, $R^p_{i=0,\theta=0} \subset F^p(R_{\theta=0})$. Since $F^p(R_{i=0,\theta=0} \otimes \wedge P_E)$ is the ideal generated by $\sum_{\mu \geq p} R^\mu_{i=0,\theta=0} \otimes 1$, and since $F^p(R_{\theta=0})$ is an ideal, this implies that ϑ_R preserves filtrations.

Now, in view of Theorem I, sec. 1.14, it only remains to show that the map

$$(\vartheta_R)_1\colon E_1 \to E_1(R_{\theta=0})$$

is an isomorphism. This is done in the next section.

9.4. Proposition III: With the notations and hypotheses of Theorem I,

$$(\vartheta_R)_1: E_1 \to E_1(R_{\theta=0}),$$

is an isomorphism.

Proof: Since $f: R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ is an isomorphism of operations (cf. Theorem II, sec. 8.7), we may assume that

$$R = R_{i=0} \otimes \wedge E^* \qquad \text{and} \qquad \chi_R(x^*) = 1 \otimes x^*, \qquad x^* \in E^*.$$

In this case we have $F^p(R_{\theta=0}) = \sum_{\mu \geq p}(R^\mu_{i=0} \otimes \wedge E^*)_{\theta=0}$.

Define bigradations in the algebras $R_{i=0,\theta=0} \otimes \wedge P_E$ and $R_{\theta=0}$ by setting

$$(R_{i=0,\theta=0} \otimes \wedge P_E)^{p,q} = R^p_{i=0,\theta=0} \otimes (\wedge P_E)^q,$$

and

$$R^{p,q}_{\theta=0} = (R^p_{i=0} \otimes \wedge^q E^*)_{\theta=0}.$$

Then these bigraded algebras are the bigraded algebras associated with the filtrations. Hence they coincide with E_0 and $E_0(R_{\theta=0})$ (as bigraded algebras).

Next, we show that the differential operators d_0 in E_0 and $\hat{d}_0$ in $E_0(R_{\theta=0})$ are given by

$$d_0 = 0 \qquad \text{and} \qquad \hat{d}_0 = -\omega_R \otimes \delta_E. \tag{9.2}$$

In fact, it is immediate from the definitions that

$$\nabla_R: R^p_{i=0,\theta=0} \otimes (\wedge P_E)^q \to F^{p+1}(R_{i=0,\theta=0} \otimes \wedge P_E), \qquad p, q \geq 0.$$

It follows that $d_0 = 0$.

On the other hand, recall from formula (8.6), sec. 8.7, that the restriction of δ_R to $(R_{i=0} \otimes \wedge E^*)_{\theta=0}$ is given by

$$\delta_R = -\omega_R \otimes \delta_E + h_\chi + \delta_H.$$

By definition the operators h_χ and δ_H are homogeneous of bidegrees $(2, -1)$ and $(1, 0)$ respectively, while $\omega_R \otimes \delta_E$ is homogeneous of bidegree $(0, 1)$. It follows that $\hat{d}_0 = -\omega_R \otimes \delta_E$.

Finally, we show that

$$(\vartheta_R)_0: E_0 \to E_0(R_{\theta=0})$$

is simply the inclusion map

$$j\colon R_{i=0,\theta=0} \otimes \wedge P_E \to (R_{i=0} \otimes \wedge E^*)_{\theta=0}. \tag{9.3}$$

In fact, j is homogeneous of bidegree zero. Thus we need only show that

$$(\vartheta_R - j)\colon R^p_{i=0,\theta=0} \otimes \wedge P_E \to F^{p+1}(R_{\theta=0}), \qquad p \geq 0. \tag{9.4}$$

But

$$j(z \otimes \Phi) = z \cdot (\chi_R)_\wedge(\Phi), \qquad z \in R_{i=0,\theta=0}, \quad \Phi \in \wedge P_E,$$

and so property (3) of Theorem I, sec. 9.3, yields (for $z \in R^p_{i=0,\theta=0}$, $\Phi \in \wedge P_E$)

$$\begin{aligned}(\vartheta_R - j)(z \otimes \Phi) = z \cdot (\vartheta_R(\Phi) - (\chi_R)_\wedge\Phi) &\in R^p_{i=0,\theta=0} \cdot F^1(R_{\theta=0}) \\ &\subset F^{p+1}(R_{\theta=0}).\end{aligned}$$

Thus (9.4) is established.

Since $(\vartheta_R)_i\colon (E_i, d_i) \to (E_i(R_{\theta=0}), \hat{d}_i)$ is a homomorphism of spectral sequences, we have the commutative diagram

$$\begin{array}{ccc} H(E_0, d_0) & \xrightarrow{(\vartheta_R)_0^\#} & H(E_0(R_{\theta=0}), d_0) \\ \cong \downarrow & & \downarrow \cong \\ E_1 & \xrightarrow[(\vartheta_R)_1]{} & E_1(R_{\theta=0}) \end{array}$$

(cf. sec. 1.6). Thus to prove the proposition we need only show that $(\vartheta_R)_0^\#$ is an isomorphism. In view of formulae (9.2) and (9.3) above it has to be shown that the inclusion map

$$j\colon R_{i=0,\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (R_{i=0} \otimes \wedge E^*)_{\theta=0}$$

induces an isomorphism

$$j^\#\colon R_{i=0,\theta=0} \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H((R_{i=0} \otimes \wedge E^*)_{\theta=0}, \omega_R \otimes \delta_E).$$

But since E is reductive, this follows immediately from Theorem V, sec. 4.11 (applied with $(Y, \delta_Y) = (R_{i=0}, 0)$ and $(X, \delta_X) = (\wedge E^*, \delta_E)$).

The proof of the proposition (and hence the proof of the fundamental theorem) is now complete.

Q.E.D.

9.5. Corollaries of Theorem I. Corollary I: The graded differential algebras $(R_{\theta=0}, \delta_R)$ and $(R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R)$ are c-equivalent (cf. sec. 0.10).

Corollary II: The c-equivalence class (and hence the cohomology algebra) of the differential algebra $(R_{\theta=0}, \delta_R)$ depends only on

(1) the differential algebra $(R_{i=0,\theta=0}, \delta_R)$,
(2) the Lie algebra E,
(3) the Weil homomorphism $\chi_R^\#$.

Proof: Assume that $(R_{i=0,\theta=0}, \delta_R)$, E, and $\chi_R^\#$ are given. Choose a transgression τ in $W(E)_{\theta=0}$ and let

$$\hat{\tau}: P_E \to Z(R_{i=0,\theta=0})$$

be a linear map, homogeneous of degree 1, and such that $\hat{\tau}^\# = \chi_R^\# \circ \tau$.

Then $(R_{i=0,\theta=0}, \delta_R; \hat{\tau})$ is a (P_E, δ)-algebra. Denote the corresponding Koszul complex by $(R_{i=0,\theta=0} \otimes \wedge P_E, \hat{\nabla})$. Now we have

$$\hat{\tau}^\# = \chi_R^\# \circ \tau = \tau_R^\#.$$

Hence, in view of Proposition IX, sec. 3.27, the fundamental theorem yields

$$(R_{\theta=0}, \delta_R) \underset{c}{\sim} (R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R) \underset{c}{\sim} (R_{i=0,\theta=0} \otimes \wedge P_E, \hat{\nabla}).$$

Q.E.D.

Corollary III: The first three terms of the spectral sequence for an operation are given by

$$E_0^{p,q}(R_{\theta=0}) \cong (R_{i=0}^p \otimes \wedge^q E^*)_{\theta=0}, \qquad E_1^{p,q}(R_{\theta=0}) \cong R_{i=0,\theta=0}^p \otimes (\wedge P_E)^q$$

and

$$E_2^{p,q}(R_{\theta=0}) \cong H^p(R_{i=0,\theta=0}) \otimes (\wedge P_E)^q.$$

Proof: Apply the fundamental theorem, and the observations of sec. 3.4.

Q.E.D.

Corollary IV: Assume that the graded differential algebra $(R_{i=0,\theta=0}, \delta_R)$ is c-split. Let $(H(R_{i=0,\theta=0}) \otimes \wedge P_E, \nabla_R^\#)$ denote the Koszul complex of the P_E-algebra $(H(R_{i=0,\theta=0}); \tau_R^\#)$, associated with $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ (cf. sec. 3.3). Then

$$(H(R_{i=0,\theta=0}) \otimes \wedge P_E, \nabla_R^\#) \underset{c}{\sim} (R_{\theta=0}, \delta_R).$$

In particular, the cohomology algebras are isomorphic.

Thus in this case the algebra $H(R_{\theta=0})$ depends only on E, $H(R_{i=0,\theta=0})$, and $\chi_R^\#$.

Proof: Apply the example in sec. 3.29.

Q.E.D.

Corollary V: $H(R_{\theta=0})$ is connected if and only if $H(R_{i=0,\theta=0})$ is.

Proof: In fact,

$$H^0(R_{\theta=0}) \cong H^0(R_{i=0,\theta=0} \otimes \wedge P_E) = Z^0(R_{i=0,\theta=0}) \otimes 1 = H^0(R_{i=0,\theta=0}).$$

Q.E.D.

Corollary VI: $H(R_{\theta=0})$ has finite type if and only if $H(R_{i=0,\theta=0})$ has finite type. In their case the Poincaré series are related by

$$f_{H(R_{\theta=0})} \leq f_{H(R_{i=0,\theta=0})} \cdot \prod_{i=1}^{r} (1 + t^{g_i}),$$

where $\sum_{i=1}^{r} t^{g_i}$ denotes the Poincaré polynomial for P_E.

Proof: Apply Proposition V, sec. 3.18.

Q.E.D.

Corollary VII: Suppose $H(R_{i=0,\theta=0})$ has finite dimension. Then $H(R_{\theta=0})$ has finite dimension, and

$$\dim H(R_{\theta=0}) \leq 2^r \cdot \dim H(R_{i=0,\theta=0}) \qquad (r = \dim P_E).$$

Moreover, in this case the Euler–Poincaré characteristic of $H(R_{\theta=0})$ is zero.

Proof: Apply the corollary to Proposition V, sec. 3.18.

Q.E.D.

9.6. Homomorphisms. Let $\varphi: (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations of a reductive Lie algebra E. Suppose that χ_R is an algebraic connection for the first operation, and recall that then $\chi_S = \varphi \circ \chi_R$ is an algebraic connection for the second operation. Moreover, with this choice of connections,

$$\varphi \circ (\chi_R)_\wedge = (\chi_S)_\wedge \qquad \text{and} \qquad \varphi \circ (\chi_R)_\vee = (\chi_S)_\vee$$

(cf. sec. 8.8). Hence,

$$\varphi \circ (\chi_R)_W = (\chi_S)_W .$$

These relations show that (in the notation of sec. 9.3)

$$\varphi_{i=0,\theta=0} \circ \tau_R = \varphi_{i=0,\theta=0} \circ (\chi_R)_{\vee,\theta=0} \circ \tau = \tau_S .$$

Thus

$$\varphi_{i=0,\theta=0}: (R_{i=0,\theta=0}, \delta_R; \tau_R) \to (S_{i=0,\theta=0}, \delta_S; \tau_S)$$

is a homomorphism of (P_E, δ)-algebras (cf. sec. 3.1).

On the other hand, for $\Phi \in P_E$,

$$(\varphi_{\theta=0} \circ \vartheta_R)(1 \otimes \Phi) = (\varphi_{\theta=0} \circ (\chi_R)_W \circ \alpha)\Phi = \vartheta_S(1 \otimes \Phi).$$

Thus the diagram

$$\begin{array}{ccc} R_{i=0,\theta=0} \otimes \wedge P_E & \xrightarrow{\varphi_{i=0,\theta=0} \otimes \iota} & S_{i=0,\theta=0} \otimes \wedge P_E \\ {\scriptstyle \vartheta_R} \big\downarrow & & \big\downarrow {\scriptstyle \vartheta_S} \\ R_{\theta=0} & \xrightarrow[\varphi_{\theta=0}]{} & S_{\theta=0} \end{array} \tag{9.5}$$

commutes. Hence so does the diagram

$$\begin{array}{ccc} H(R_{i=0,\theta=0} \otimes \wedge P_E) & \xrightarrow{(\varphi_{i=0,\theta=0} \otimes \iota)^\#} & H(S_{i=0,\theta=0} \otimes \wedge P_E) \\ {\scriptstyle \vartheta_R^\#} \big\downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \big\downarrow {\scriptstyle \vartheta_S^\#} \\ H(R_{\theta=0}) & \xrightarrow[\varphi_{\theta=0}^\#]{} & H(S_{\theta=0}). \end{array} \tag{9.6}$$

§3. Applications of the fundamental theorem

In this article the notation established at the start of article 2 remains in force. In particular P_E is the primitive space of a reductive Lie algebra E; $\tau\colon P_E \to (\vee E^*)_{\theta=0}$ is a transgression; and $\alpha\colon P_E \to W(E)_{\theta=0}$ satisfies

$$\delta_W \alpha(\Phi) = \tau(\Phi) \otimes 1$$

and

$$\alpha(\Phi) - 1 \otimes \Phi \in (\vee^+ E^* \otimes \wedge E^*)_{\theta=0}.$$

9.7. The cohomology sequence. Recall from sec. 8.21 that an operation $(E, i_R, \theta_R, R, \delta_R)$ is called regular, if E is reductive, $H(R_{\theta=0})$ is connected and the operation admits a connection. For such operations, the cohomology sequence was defined to be the sequence

$$(\vee E^*)_{\theta=0} \xrightarrow{\chi_R^\#} H(R_{i=0,\theta=0}) \xrightarrow{e_R^\#} H(R_{\theta=0}) \xrightarrow{\varrho_R} (\wedge E^*)_{\theta=0}.$$

On the other hand, the choice of an algebraic connection χ_R in a regular operation determines the (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$, with $\tau_R = (\chi_R)_{\vee,\theta=0} \circ \tau$ (cf. sec. 9.3). The corresponding cohomology sequence, as defined in sec. 3.14, reads

$$\vee P_E \xrightarrow{(\tau_R)_\vee^\#} H(R_{i=0,\theta=0}) \xrightarrow{l^\#} H(R_{i=0,\theta=0} \otimes \wedge P_E) \xrightarrow{\varrho^\#} \wedge P_E.$$

Finally, in Theorem III, sec. 5.18, Theorem I, sec. 6.13, and Theorem I, sec. 9.3, we established isomorphisms of graded algebras

$$\varkappa_E\colon \wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0}, \qquad \tau_\vee\colon \vee P_E \xrightarrow{\cong} (\vee E^*)_{\theta=0}$$

and

$$\vartheta_R^\#\colon H(R_{i=0,\theta=0} \otimes \wedge P_E) \xrightarrow{\cong} H(R_{\theta=0}).$$

(Note that the isomorphism $\vartheta_R^\#$ depends on the choice of an algebraic connection χ_R.)

Theorem II: Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation and let χ_R be an algebraic connection. Then the diagram

$$\begin{tikzcd}
\vee P_E \arrow[r, "(\tau_R)_\vee^\#"] \arrow[d, "\tau_\vee"', "\cong"] & H(R_{i=0,\theta=0}) \arrow[r, "l^\#"] \arrow[d, "\cong"', "\iota"] & H(R_{i=0,\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^\#"] \arrow[d, "\cong"', "\vartheta_R^\#"] & \wedge P_E \arrow[d, "\cong"', "\varkappa_E"] \\
(\vee E^*)_{\theta=0} \arrow[r, "\chi_R^\#"'] & H(R_{i=0,\theta=0}) \arrow[r, "e_R^\#"'] & H(R_{\theta=0}) \arrow[r, "\varrho_R"'] & (\wedge E^*)_{\theta=0}
\end{tikzcd}$$

commutes.

Proof: The commutativity of the left-hand square follows from the relation $\tau_R = (\chi_R)_{\vee,\theta=0} \circ \tau$. The commutativity of the centre square is a consequence of property (3) in the fundamental theorem (sec. 9.3). That the right-hand square commutes is proved in Proposition IV, below.

Remark: Theorem II also permits us to apply the Samelson and the reduction theorems (sec. 3.13, sec. 3.15) to operations. The resulting theorems, however, would coincide with Theorem I, sec. 7.13, and Theorem II, sec. 7.14. Thus we do not restate them here.

Proposition IV: Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation with algebraic connection χ_R. Then (in the notation above)

$$\varkappa_E \circ \varrho^\# = \varrho_R \circ \vartheta_R^\#.$$

Lemma I: Assume that an element $\alpha \in H(R_{\theta=0})$ is represented by a cocycle Ω of the form

$$\Omega = (\chi_R)_\wedge(\Phi) + z, \qquad \Phi \in (\wedge E^*)_{\theta=0}, \quad z \in F^1(R_{\theta=0}).$$

Then $\varrho_R(\alpha) = \Phi$.

Proof: Recall from sec. 7.8 that the structure homomorphism is a homomorphism of operations

$$\gamma_R: (E, i_R, \theta_R, R, \delta_R) \to (E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E}).$$

According to the corollary of Proposition IV, sec. 8.10,

$$\gamma_R((\chi_R)_\wedge \Phi) - 1 \otimes \Phi \in (R^+ \otimes \wedge E^*)_{\theta=0}.$$

Moreover, γ_R preserves filtrations since it is a homomorphism of operations. Evidently,

$$F^1((R \otimes \wedge E^*)_{\theta=0}) = (R^+ \otimes \wedge E^*)_{\theta=0},$$

and so $\gamma_R z \in (R^+ \otimes \wedge E^*)_{\theta=0}$. It follows that

$$\gamma_R \Omega = 1 \otimes \Phi + \hat{\Omega},$$

where $\hat{\Omega} \in (R^+ \otimes \wedge E^*)_{\theta=0}$. We also have

$$\delta_{R\otimes E} \hat{\Omega} = \delta_{R\otimes E}(\gamma_R \Omega) = \gamma_R(\delta_R \Omega) = 0.$$

Finally recall from sec. 7.10 that the fibre projection ϱ_R is defined by

$$\varrho_R = \pi_R \circ (g^\#)^{-1} \circ (\gamma_R)^\#_{\theta=0},$$

where

$$g^\#: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H((R \otimes \wedge E^*)_{\theta=0})$$

is the isomorphism induced by the inclusion map $g: R_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (R \otimes \wedge E^*)_{\theta=0}$, and

$$\pi_R: H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \to (\wedge E^*)_{\theta=0}$$

is the projection.

Since $g^\#$ is an isomorphism we can write

$$\hat{\Omega} = \Omega_1 + \delta_{R\otimes E} \Omega_2,$$

where

$$\Omega_1 \in Z(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \qquad \text{and} \qquad \Omega_2 \in (R \otimes \wedge E^*)_{\theta=0}.$$

Let Ω_1^0, Ω_2^0, and $(\delta_{R\otimes E}\Omega_2)^0$ denote the components of the elements Ω_1, Ω_2, and $\delta_{R\otimes E}(\Omega_2)$ in $(R^0 \otimes \wedge E^*)_{\theta=0}$. Then

$$\Omega_1^0 \in 1 \otimes (\wedge E^*)_{\theta=0} \qquad \text{and} \qquad \Omega_1^0 + (\delta_{R\otimes E}\Omega_2)^0 = 0.$$

Moreover, in $(R \otimes \wedge E^*)_{\theta=0}$, $\delta_{R\otimes E} = \delta_R \otimes \iota - \omega_R \otimes \delta_E$. Thus

$$(\delta_{R\otimes E}\Omega_2)^0 = -(\omega_R \otimes \delta_E)\Omega_2^0 \in (R^0 \otimes \theta(\wedge E^*))_{\theta=0}.$$

These equations imply that $\Omega_1^0 = 0$; i.e.,

$$\Omega_1 \in Z^+(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}.$$

It follows that Ω_1 represents a class α_1 in $H^+(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}$. But, clearly

$$(\gamma_R)^\#_{\theta=0}(\alpha) = g^\#(\alpha_1 + 1 \otimes \Phi).$$

Thus

$$\varrho_R(\alpha) = \pi_R(\alpha_1 + 1 \otimes \Phi) = \Phi.$$

Q.E.D.

Proof of Proposition IV: Let $\beta \in H(R_{i=0,\theta=0} \otimes \wedge P_E)$ and let Ψ be a representing cocycle. Then, by Lemma I, sec. 3.13,

$$\Psi = \Psi_1 + 1 \otimes \varrho^\#\beta,$$

where $\Psi_1 \in R^+_{i=0,\theta=0} \otimes \wedge P_E$ $(= F^1(R_{i=0,\theta=0} \otimes \wedge P_E))$.

Since ϑ_R is filtration preserving, it follows that

$$\vartheta_R\Psi - \vartheta_R(1 \otimes \varrho^\#\beta) \in F^1(R_{\theta=0}).$$

Now identify $\wedge P_E$ with $(\wedge E^*)_{\theta=0}$ via $\varkappa_E$. Then statement (3) in the fundamental theorem asserts that

$$\vartheta_R(1 \otimes \varrho^\#\beta) - (\chi_R)_\wedge(\varrho^\#\beta) \in F^1(R_{\theta=0}),$$

whence

$$\vartheta_R\Psi - (\chi_R)_\wedge(\varrho^\#\beta) \in F^1(R_{\theta=0}).$$

Thus, since $\vartheta_R(\Psi)$ represents $\vartheta_R^\#(\beta)$, Lemma I above yields

$$\varrho_R\vartheta_R^\#(\beta) = \varrho^\#(\beta).$$

Q.E.D.

Corollary: Assume that $(R_{i=0,\theta=0}, \delta_R)$ is c-split. Then

$$(R_{\theta=0}, \delta_R) \underset{c}{\sim} (H(R_{i=0,\theta=0}) \otimes \wedge P_E, \nabla_R^\#),$$

and the induced isomorphism of cohomology algebras makes the diagram

$$\begin{tikzcd}
& H(H(R_{i=0,\theta=0}) \otimes \wedge P_E) \arrow[r] \arrow[dd, "\cong"] & \wedge P_E \arrow[dd, "\cong"', "\varkappa_E"] \\
H(R_{i=0,\theta=0}) \arrow[ur] \arrow[dr, "\varrho_R^\#"'] & & \\
& H(R_{\theta=0}) \arrow[r, "\varrho_R"'] & (\wedge E^*)_{\theta=0}
\end{tikzcd}$$

commute.

Proof: Apply the example of sec. 3.29.

Q.E.D.

9.8. Homomorphisms. Recall that a homomorphism $\psi: A \to B$ of graded vector spaces is n-regular if $\psi: A^p \to B^p$ is an isomorphism for $p \leq n$ and injective for $p = n + 1$.

Theorem III: Let $\varphi: (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$ be a homomorphism of operations of a reductive Lie algebra E. Assume that the first operation admits a connection. Then

$$\varphi^{\#}_{i=0,\theta=0}: H(R_{i=0,\theta=0}) \to H(S_{i=0,\theta=0})$$

is n-regular if and only if

$$\varphi^{\#}_{\theta=0}: H(R_{\theta=0}) \to H(S_{\theta=0})$$

is n-regular.

Proof: It follows from sec. 9.6 that $\varphi^{\#}_{\theta=0}$ is n-regular if and only if

$$(\varphi_{i=0,\theta=0} \otimes \iota)^{\#}: H(R_{i=0,\theta=0} \otimes \wedge P_E) \to H(S_{i=0,\theta=0} \otimes \wedge P_E)$$

is n-regular. Now the theorem follows from Theorem I, sec. 3.10.

Q.E.D.

Corollary: $\varphi^{\#}_{\theta=0}$ is an isomorphism if and only if $\varphi^{\#}_{i=0,\theta=0}$ is.

Applying Theorem III to the classifying homomorphism (cf. sec. 8.16) yields

Theorem IV: Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra admitting a connection. Then the Weil homomorphism

$$\chi_R^{\#}: (\vee E^*)_{\theta=0} \to H(R_{i=0,\theta=0})$$

is n-regular if and only if

$$H^0(R_{\theta=0}) = \Gamma \qquad \text{and} \qquad H^p(R_{\theta=0}) = 0, \qquad 1 \leq p \leq n. \tag{9.7}$$

Proof: Recall that $H^0(W(E)_{\theta=0}) = \Gamma$ and $H^+(W(E)_{\theta=0}) = 0$ (cf. Proposition I, sec. 6.6). It follows that formula (9.7) holds if and only if $(\chi_W)^{\#}_{\theta=0}: H(W(E)_{\theta=0}) \to H(R_{\theta=0})$ is n-regular. Hence the theorem follows from Theorem III.

Q.E.D.

Corollary I: $\chi_R^\#$ is an isomorphism if and only if $H^0(R_{\theta=0}) = \Gamma$ and $H^+(R_{\theta=0}) = 0$.

Corollary II: Suppose $H^0(R_{\theta=0}) = \Gamma$ and $H^p(R_{\theta=0}) = 0$, $1 \leq p \leq n$. Then the Betti numbers $b_p = \dim H^p(R_{i=0,\theta=0})$ $(0 \leq p \leq n)$ are the coefficients of t in the series $\prod_{j=1}^r (1 - t^{g_j+1})^{-1}$, where $\sum_{j=1}^r t^{g_j}$ is the Poincaré polynomial for P_E.

Proof: This follows from the isomorphism $(\vee E^*)_{\theta=0} \cong \vee P_E$.

Q.E.D.

9.9. N.c.z. operations. Let $(E, i_R, \theta_R, R, \delta_R)$ be an operation of a reductive Lie algebra, and assume that $H(R_{\theta=0})$ is connected. Then we say $(\wedge E^*)_{\theta=0}$ is *non cohomologous to zero* in $R_{\theta=0}$ (n.c.z.) if the projection $\varrho_R \colon H(R_{\theta=0}) \to (\wedge E^*)_{\theta=0}$ is surjective. For the sake of brevity we shall often simply say that the operation is n.c.z.

According to Proposition I, sec. 8.2, an n.c.z. operation admits an algebraic connection. Thus every n.c.z. operation is regular.

Theorem V: Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation. Then the following conditions are equivalent:

(1) ϱ_R is surjective.

(2) There is an isomorphism of graded algebras

$$H(R_{i=0,\theta=0}) \otimes (\wedge E^*)_{\theta=0} \cong H(R_{\theta=0})$$

which makes the diagram

$$\begin{tikzcd}
 & H(R_{i=0,\theta=0}) \otimes (\wedge E^*)_{\theta=0} \arrow[dr] \arrow[dd, "\cong"] & \\
H(R_{i=0,\theta=0}) \arrow[ur] \arrow[dr, "e_R^*"'] & & (\wedge E^*)_{\theta=0} \\
 & H(R_{\theta=0}) \arrow[ur, "\varrho_R"'] &
\end{tikzcd} \tag{9.8}$$

commute.

(3) There is a linear isomorphism of graded vector spaces $H(R_{i=0,\theta=0}) \otimes (\wedge E^*)_{\theta=0} \xrightarrow{\cong} H(R_{\theta=0})$ which makes the diagram (9.8) commute.

(4) $e_R^\#$ is injective.

(5) $(\chi_R^\#)^+ = 0$.

(6) There is a homomorphism

$$f: (R_{i=0,\theta=0} \otimes (\wedge E^*)_{\theta=0}, \delta_R \otimes \iota) \to (R_{\theta=0}, \delta_R)$$

of graded differential algebras such that $f^\#$ is an isomorphism making the diagram (9.8) commute.

(7) The spectral sequence for $R_{\theta=0}$ collapses at the E_2-term.

Proof: In view of the fundamental theorem (cf. sec. 9.3) and Theorem II, sec. 9.7, this result is simply a translation of Theorem VII, sec. 3.17.

Q.E.D.

Theorem VI: Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation. Then

(1) $H(R_{i=0,\theta=0})$ is of finite type if and only if $H(R_{\theta=0})$ is. In this case their Poincaré series are related by

$$f_{H(R_{\theta=0})} \leq f_{H(R_{i=0,\theta=0})} \cdot \prod_{i=1}^{r} (1 + t^{g_i})$$

where $\sum_i t^{g_i}$ is the Poincaré polynomial for P_E.

Equality holds if and only if the operation is n.c.z.

(2) Suppose that $H(R_{i=0,\theta=0})$ is finite dimensional. Then so is $H(R_{\theta=0})$. In this case

$$\dim H(R_{\theta=0}) \leq 2^r \cdot \dim H(R_{i=0,\theta=0}).$$

Equality holds if and only if the operation is n.c.z.

Proof: In view of the fundamental theorem and Theorem II the theorem is a translation of Proposition V, sec. 3.18, and its corollary.

Q.E.D.

§4. The distinguished transgression

Let E be a reductive Lie algebra. Recall from sec. 6.10 the definition of the distinguished transgression $\tau_E: P_E \to (\vee E^*)_{\theta=0}$. It is immediate from the definition of τ_E that there is a linear map $\alpha: P_E \to W(E)_{\theta=0}$, homogeneous of degree zero, such that

$$\delta_W \alpha(\Phi) = \tau_E \Phi \otimes 1 \qquad \text{and} \qquad \alpha(\Phi) - 1 \otimes \Phi \in W^+(E)_{i_I=0,\theta=0}, \quad \Phi \in P_E.$$

(Recall that $W^+(E)_{i_I=0,\theta=0}$ consists of the invariant elements Ω in $W^+(E)$ which satisfy $i(a)\Omega = 0$ for $a \in (\wedge^+ E)_{\theta=0}$.)

In this article α denotes a fixed linear map, satisfying the properties listed above. In particular, the pair τ_E, α satisfies the properties listed at the beginning of article 2.

9.10. The operator $i_R(a)^\#$. Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation and let $P_*(E) \subset (\wedge^+ E)_{\theta=0}$ be the primitive subspace defined in sec. 5.14. Recall from sec. 7.12 that the operators $i_R(a)$ $(a \in P_*(E))$ restrict to operators in $R_{\theta=0}$ and induce operators $i_R(a)^\#$ in $H(R_{\theta=0})$.

On the other hand, since $P_*(E) = (P_E)^*$, the elements $a \in P_*(E)$ induce ordinary substitution operators $i_P(a)$ in the exterior algebra $\wedge P_E$. According to Lemma IX, sec. 5.22, the isomorphism $\varkappa_E: \wedge P_E \xrightarrow{\cong} (\wedge E^*)_{\theta=0}$ identifies $i_P(a)$ with $i_E(a)$; we use the latter notation.

Now extend $i_E(a)$ $(a \in P_*(E))$ to an antiderivation, $i(a)$, in the algebra $R_{i=0,\theta=0} \otimes \wedge P_E$ by setting

$$i(a) = \omega_R \otimes i_E(a).$$

Then (cf. sec. 3.2)

$$i(a)\nabla_R + \nabla_R i(a) = 0, \qquad a \in P_*(E),$$

and so we obtain antiderivations, $i(a)^\#$, in $H(R_{i=0,\theta=0} \otimes \wedge P_E)$.

The purpose of this article is to prove

Theorem VII: Let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation. Let χ_R be an algebraic connection for the operation and let

$$\vartheta_R: (R_{i=0,\theta=0} \otimes \wedge P_E, \nabla_R) \to (R_{\theta=0}, \delta_R)$$

be the corresponding Chevalley homomorphism constructed via τ_E and α. Then

$$\vartheta_R^\# \circ i(a)^\# = i_R(a)^\# \circ \vartheta_R^\#, \qquad a \in P_*(E).$$

Lemma II: Assume that the Weil homomorphism $\chi_R^\#$ of the operation $(E, i_R, \theta_R, R, \delta_R)$ is trivial: $(\chi_R^\#)^+ = 0$. Then the assertion of Theorem VII holds.

Proof: Evidently, $\tau_R^\# = \chi_R^\# \circ \tau_E = 0$. Thus we can apply the results of sec. 3.17 to the (P_E, δ)-algebra, $(R_{i=0,\theta=0}, \delta_R; \tau_R)$.

Let β and γ be elements of $H(R_{i=0,\theta=0} \otimes \wedge P_E)$ which admit representing cocycles of the form

$$z \otimes 1 \quad (z \in R_{i=0,\theta=0}) \qquad \text{and} \qquad w \otimes 1 + 1 \otimes \Phi \quad (w \in R_{i=0,\theta=0}, \Phi \in P_E)$$

respectively. (Since $\tau_R^\# = 0$, the corollary of Theorem VII, sec. 3.17, implies that $H(R_{i=0,\theta=0} \otimes \wedge P_E)$ is generated by cohomology classes with this property.)

On the other hand, according to Proposition VIII, (3), sec. 7.12, if $a \in P_*(E)$ then $i_R(a)^\#$ is an antiderivation in $H(R_{\theta=0})$. Since $i(a)^\#$ is an antiderivation in $H(R_{i=0,\theta=0} \otimes \wedge P_E)$, it is sufficient to verify that

$$i_R(a)^\# \vartheta_R^\#(\beta) = \vartheta_R^\# i(a)^\#(\beta), \qquad a \in P_*(E), \tag{9.9}$$

and

$$i_R(a)^\# \vartheta_R^\#(\gamma) = \vartheta_R^\# i(a)^\#(\gamma), \qquad a \in P_*(E). \tag{9.10}$$

Since β is represented by $z \otimes 1$, $\vartheta_R^\#(\beta)$ is represented by $z \in R_{i=0,\theta=0}$. But

$$i(a)(z \otimes 1) = 0 = i_R(a)z, \qquad a \in P_*(E),$$

and (9.9) follows.

To prove (9.10), observe that since $\Phi \in P_E$,

$$i(a)(w \otimes 1 + 1 \otimes \Phi) = \langle \Phi, a \rangle, \qquad a \in P_*(E).$$

Hence $\vartheta_R^\# i(a)^\#(\gamma) = \langle \Phi, a \rangle$.

On the other hand,

$$\vartheta_R(w \otimes 1 + 1 \otimes \Phi) = w + (\chi_R)_{W,\theta=0}(\alpha(\Phi)).$$

But α was chosen so that

$$(i_E(a) \circ \alpha)(\Phi) = i_E(a)(1 \otimes \Phi) = \langle \Phi, a \rangle.$$

It follows that

$$\begin{aligned} i_R(a)\vartheta_R(w \otimes 1 + 1 \otimes \Phi) &= ((\chi_R)_{W,\theta=0} \circ i_E(a))(\alpha(\Phi)) \\ &= \langle \Phi, a \rangle. \end{aligned}$$

Hence,

$$i_R(a)^\# \vartheta_R^\#(\gamma) = \langle \Phi, a \rangle = \vartheta_R^\# i(a)^\#(\gamma), \qquad a \in P_*(E),$$

and (9.10) is proved.

Q.E.D.

9.11. Proof of Theorem VII: Recall the definition of the structure operation $(E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E})$ in sec. 7.7. The horizontal and basic subalgebras for this operation are given, respectively, by

$$(R \otimes \wedge E^*)_{i=0} = R \otimes 1 \qquad \text{and} \qquad (R \otimes \wedge E^*)_{i=0,\theta=0} = R_{\theta=0} \otimes 1.$$

Moreover, the map $\tilde{\chi}: x^* \to 1 \otimes x^*$ $(x^* \in E^*)$ is an algebraic connection for this operation. Since

$$\delta_{R\otimes E}\tilde{\chi}(x^*) = 1 \otimes \delta_E x^* = \tilde{\chi}_\wedge(\delta_E x^*), \qquad x^* \in E^*,$$

it follows that the curvature for the connection $\tilde{\chi}$ is zero (cf. Proposition III, sec. 8.6). Hence the Weil homomorphism of this operation is trivial.

On the other hand, the structure homomorphism $\gamma_R: R \to R \otimes \wedge E^*$ is a homomorphism of operations. The corresponding base homomorphism is given by

$$(\gamma_R)_{i=0,\theta=0} = e_R : R_{i=0,\theta=0} \to R_{\theta=0}.$$

Moreover, since γ_R is a homomorphism of operations, the map $\hat{\chi} = \gamma_R \circ \chi_R$ is an algebraic connection for the structure operation.

Let

$$\vartheta_{R\otimes E}: R_{\theta=0} \otimes \wedge P_E \to (R \otimes \wedge E^*)_{\theta=0}$$

be the corresponding Chevalley homomorphism. Then (cf. sec. 9.6) the diagram

$$\begin{tikzcd} R_{i=0,\theta=0} \otimes \wedge P_E \arrow[r, "e_R \otimes \iota"] \arrow[d, "\vartheta_R"] & R_{\theta=0} \otimes \wedge P_E \arrow[d, "\vartheta_{R\otimes E}"] \\ R_{\theta=0} \arrow[r, "(\gamma_R)_{\theta=0}"'] & (R \otimes \wedge E^*)_{\theta=0} \end{tikzcd}$$

commutes. Hence

$$(\gamma_R)^{\#}_{\theta=0} \circ \vartheta^{\#}_R = \vartheta^{\#}_{R\otimes E} \circ (e_R \otimes \iota)^{\#}.$$

Since the structure operation has trivial Weil homomorphism, Lemma II yields

$$\vartheta^{\#}_{R\otimes E} \circ i(a)^{\#} = i_{R\otimes E}(a)^{\#} \circ \vartheta^{\#}_{R\otimes E}, \qquad a \in P_*(E).$$

This, together with the equation above, yields

$$(\gamma_R)^{\#}_{\theta=0} \circ (\vartheta^{\#}_R \circ i(a)^{\#} - i_R(a)^{\#} \circ \vartheta^{\#}_R) = 0.$$

Since (cf. Proposition VI, sec. 7.9) $(\gamma_R)^{\#}_{\theta=0}$ is injective, the theorem is proved.

Q.E.D.

Next recall that the inclusion $g\colon R_{\theta=0} \otimes (\wedge E^*)_{\theta=0} \to (R \otimes \wedge E^*)_{\theta=0}$ induces an isomorphism in cohomology (cf. sec. 7.9) and set

$$\hat{\gamma}_R = (g^{\#})^{-1} \circ (\gamma_R)^{\#}_{\theta=0} : H(R_{\theta=0}) \to H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}.$$

Corollary: Let $\Delta_P\colon \wedge P_E \to \wedge P_E \otimes \wedge P_E$ be the homomorphism defined by $\Delta_P(\Phi) = \Phi \otimes 1 + 1 \otimes \Phi$, $\Phi \in P_E$. Then the map

$$\iota \otimes \Delta_P\colon R_{i=0,\theta=0} \otimes \wedge P_E \to R_{i=0,\theta=0} \otimes \wedge P_E \otimes \wedge P_E$$

satisfies

$$(\iota \otimes \Delta_P) \circ \nabla_R = (\nabla_R \otimes \iota) \circ (\iota \otimes \Delta_P).$$

Moreover, the homomorphism $\hat{\gamma}_R$ is given by the commutative diagram

$$\begin{array}{ccc} H(R_{i=0,\theta=0} \otimes \wedge P_E) & \xrightarrow{(\iota\otimes\Delta_P)^{\#}} & H(R_{i=0,\theta=0} \otimes \wedge P_E) \otimes \wedge P_E \\ {\scriptstyle \vartheta^{\#}_R}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \vartheta^{\#}_R\otimes\iota} \\ H(R_{\theta=0}) & \xrightarrow[\hat{\gamma}_R]{} & H(R_{\theta=0}) \otimes \wedge P_E \end{array}$$

Proof: The first equation follows from a simple computation. The commutative diagram is a consequence of Proposition IX, sec. 7.15, and the theorem.

Q.E.D.

§5. The classification theorem

In article 2 we associated with every regular operation $(E, i_R, \theta_R, R, \delta_R)$ a (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ whose associated cohomology algebra was isomorphic to the cohomology of $R_{\theta=0}$. In this article we reverse this process to show that every (P_E, δ)-algebra can be obtained in this way. In this way we obtain a "cohomology classification theorem" for operations.

In this article E again denotes a reductive Lie algebra with primitive space P_E. $(\vee E^*)_{\theta=0}$ will be identified with $\vee P_E$ via a fixed transgression τ in $W(E)_{\theta=0}$ (cf. Theorem I, sec. 6.13).

9.12. The induced operation. Theorem VIII: With the notation above let $(B, \delta_B; \tau_B)$ be a c-connected (P_E, δ)-algebra. Then there is a regular operation $(E, i_R, \theta_R, R, \delta_R)$ together with an algebraic connection χ_R with the following properties:

(1) There is an isomorphism of (P_E, δ)-algebras

$$\lambda: (B, \delta_B; \tau_B) \xrightarrow{\cong} (R_{i=0,\theta=0}, \delta_R; (\chi_R)_{\vee,\theta=0} \circ \tau).$$

(2) Let $(E, i_S, \theta_S, S, \delta_S)$ be an operation admitting a connection χ_S and let

$$\psi: (B, \delta_B; \tau_B) \to (S_{i=0,\theta=0}, \delta_S; (\chi_S)_{\vee,\theta=0} \circ \tau)$$

be a homomorphism of (P_E, δ)-algebras. Then there exists a homomorphism

$$\varphi: (E, i_R, \theta_R, R, \delta_R) \to (E, i_S, \theta_S, S, \delta_S)$$

of operations such that

$$\psi = \varphi_{i=0,\theta=0} \circ \lambda \tag{9.11}$$

and

$$\chi_S = \varphi \circ \chi_R. \tag{9.12}$$

The proof of this theorem occupies the next three sections. In sec. 9.13 below we construct the operation, the connection, and the homo-

morphism λ. In sec. 9.14 it will be shown that λ is an isomorphism. Finally, property (2) is established in sec. 9.15.

9.13. Construction of the operation. Consider first the operation $(E, i, \theta, B \otimes W(E), \delta_{B\otimes W})$ where

$$i(x) = \omega_B \otimes i_E(x), \qquad \theta(x) = \iota \otimes \theta_W(x), \qquad x \in E,$$

and

$$\delta_{B\otimes W} = \delta_B \otimes \iota + \omega_B \otimes \delta_W .$$

(ω_B denotes the degree involution in B.) Observe that

$$\hat{\chi}: x^* \mapsto 1 \otimes 1 \otimes x^*, \qquad x^* \in E^*,$$

is an algebraic connection for this operation.

Now denote by X the subspace of $B \otimes \vee E^*$ whose elements have the form

$$z = \tau_B(\Phi) \otimes 1 - 1 \otimes \tau(\Phi), \qquad \Phi \in P_E .$$

Denote by I the graded ideal in $B \otimes W(E)$ generated by $X \otimes 1$. Then

$$R = \frac{B \otimes W(E)}{I}$$

is a graded algebra. Let $\pi: B \otimes W(E) \to R$ denote the projection.

Next observe that $X \otimes 1 \subset Z(B) \otimes (\vee E^*)_{\theta=0} \otimes 1$. It follows that for $z \in X \otimes 1$, $x \in E$,

$$i(x)z = \theta(x)z = \delta_{B\otimes W}(z) = 0.$$

Since these operators are either derivations or antiderivations, the ideal I must be stable under them. Hence they induce operators

$$i_R(x), \qquad \theta_R(x), \qquad \text{and} \qquad \delta_R$$

in the factor algebra R. Since π is a surjective algebra homomorphism and $(E, i, \theta, B \otimes W(E), \delta_{B\otimes W})$ is an operation, it follows that $(E, i_R, \theta_R, R, \delta_R)$ is an operation. Moreover, π is a homomorphism of operations.

In particular, since the linear map $\hat{\chi}$ is an algebraic connection, it follows that $\chi_R = \pi \circ \hat{\chi}$ is an algebraic connection for $(E, i_R, \theta_R, R, \delta_R)$.

Next observe that the representation θ_R is semisimple. In fact, since E is reductive, the representation θ_W is semisimple and hence so

is the representation θ. Since π is surjective, it follows that θ_R is semisimple.

Finally, note that $B \otimes 1 \otimes 1 \subset (B \otimes W(E))_{i=0,\theta=0}$. Hence a homomorphism of graded differential algebras

$$\lambda: B \to R_{i=0,\theta=0}$$

is defined by $\lambda(b) = \pi(b \otimes 1 \otimes 1)$, $b \in B$.

To show that λ is a homomorphism of (P_E, δ)-algebras, we must prove that

$$\lambda \circ \tau_B = (\chi_R)_{\vee,\theta=0} \circ \tau. \tag{9.13}$$

But, since π is connection preserving,

$$\pi \circ (\hat{\chi}_\vee)_{\theta=0} = (\chi_R)_{\vee,\theta=0}.$$

Moreover, it follows easily from Example 1, sec. 8.9, that

$$(\hat{\chi}_\vee)_{\theta=0}: (\vee E^*)_{\theta=0} \to (B \otimes W(E))_{\theta=0}$$

is the inclusion map $\Psi \mapsto 1 \otimes \Psi \otimes 1$.

Since $\tau_B(\Phi) \otimes 1 - 1 \otimes \tau(\Phi) \in X$, $\Phi \in P_E$, this yields

$$\begin{aligned}\lambda \circ \tau_B(\Phi) &= \pi(\tau_B(\Phi) \otimes 1 \otimes 1)\\ &= \pi(1 \otimes \tau(\Phi) \otimes 1) = (\chi_R)_{\vee,\theta=0} \circ \tau(\Phi), \qquad \Phi \in P_E,\end{aligned}$$

whence (9.13).

9.14. Proposition V: The homomorphism $\lambda: B \to R_{i=0,\theta=0}$ is an isomorphism.

Lemma III: Let J denote the ideal in $B \otimes (\vee E^*)_{\theta=0}$ generated by X. Then the inclusion $\xi: B \to B \otimes (\vee E^*)_{\theta=0}$, induces an isomorphism,

$$\xi_1: B \xrightarrow{\cong} (B \otimes (\vee E^*)_{\theta=0})/J.$$

Proof: Let $\varrho: B \otimes (\vee E^*)_{\theta=0} \to (B \otimes (\vee E^*)_{\theta=0})/J$ be the projection. Then $\xi_1 = \varrho \circ \xi$, and

$$\varrho(1 \otimes \tau(\Phi)) = \varrho \circ \xi(\tau_B\Phi) = \xi_1(\tau_B\Phi), \qquad \Phi \in P_E.$$

Since $\tau(P_E)$ generates $(\vee E^*)_{\theta=0}$ (cf. Theorem I, sec. 6.13) this implies that

$$\varrho(B \otimes (\vee E^*)_{\theta=0}) = \xi_1(B) \cdot \varrho(1 \otimes (\vee E^*)_{\theta=0}) \subset \operatorname{Im} \xi_1.$$

Thus ξ_1 is surjective.

On the other hand, define $\eta: B \otimes (\vee E^*)_{\theta=0} \to B$ by

$$\eta(b \otimes \Psi) = b \cdot [(\tau_B)_\vee \circ (\tau_\vee)^{-1}(\Psi)], \qquad b \in B, \quad \Psi \in (\vee E^*)_{\theta=0}.$$

Then $\eta(X) = 0$ and so $\eta(J) = 0$.

Thus η induces a homomorphism

$$\eta_1: [B \otimes (\vee E^*)_{\theta=0}]/J \to B.$$

Since (clearly) $\eta_1 \circ \xi_1 = \iota$, ξ_1 is injective.

Q.E.D.

Proof of Proposition V: We show first that

$$\pi_{i=0,\theta=0}: B \otimes (\vee E^*)_{\theta=0} \to R_{i=0,\theta=0} \tag{9.14}$$

is surjective and that

$$\ker \pi_{i=0,\theta=0} = J. \tag{9.15}$$

In fact, the algebraic connection χ_R determines an isomorphism $R_{i=0} \otimes \wedge E^* \xrightarrow{\cong} R$ (cf. Theorem I, sec. 8.4). Moreover, since π is connection preserving, it follows that the diagram

$$\begin{tikzcd}
(B \otimes \vee E^*) \otimes \wedge E^* \arrow[r, "\iota", "\cong"'] \arrow[d, "\pi_{i=0} \otimes \iota"'] & B \otimes W(E) \arrow[d, "\pi"] \\
R_{i=0} \otimes \wedge E^* \arrow[r, "\cong"'] & R
\end{tikzcd} \tag{9.16}$$

commutes. Since π is surjective. this diagram implies that $\pi_{i=0}$ is surjective.

Next recall that θ and θ_R are semisimple representations. Hence

$$R_{i=0} = R_{i=0,\theta=0} \oplus \theta(R_{i=0})$$

and

$$B \otimes \vee E^* = (B \otimes (\vee E^*)_{\theta=0}) \oplus (B \otimes \theta(\vee E^*)).$$

Since $\pi_{i=0}$ is surjective, these relations imply that so is $\pi_{i=0,\theta=0}$.

Moreover, it follows from the commutative diagram (9.16) that

$$\ker \pi_{i=0} = (B \otimes \vee E^*) \cdot X.$$

Since the representations are semisimple, it follows that

$$\ker \pi_{i=0,\theta=0} = (B \otimes \vee E^*)_{\theta=0} \cdot X = J.$$

Formulae (9.14) and (9.15) show that $\pi_{i=0,\theta=0}$ induces an isomorphism

$$\pi_1 \colon [B \otimes (\vee E^*)_{\theta=0}]/J \xrightarrow{\cong} R_{i=0,\theta=0}.$$

Clearly $\lambda = \pi_1 \circ \xi_1$, and so Lemma III implies that λ is an isomorphism.
Q.E.D.

9.15. Proof of Theorem VIII, (2): Consider the operation $(E, i, \theta, B \otimes W(E), \delta_{B\otimes W})$, defined in sec. 9.13 and let $(\chi_S)_W$ denote the classifying homomorphism for χ_S.

Define a homomorphism of operations $\gamma \colon B \otimes W(E) \to S$ by

$$\gamma(b \otimes \Omega) = \psi(b) \cdot (\chi_S)_W(\Omega), \qquad b \in B, \quad \Omega \in W(E).$$

Then, for $\Phi \in P_E$,

$$\gamma(\tau_B \Phi \otimes 1 \otimes 1 - 1 \otimes \tau\Phi \otimes 1) = (\psi \circ \tau_B)\Phi - ((\chi_S)_{\vee,\theta=0} \circ \tau)\Phi = 0.$$

This implies that $\gamma(X \otimes 1) = 0$. Hence γ factors over π to yield a commutative diagram of algebra homomorphisms

$$\begin{tikzcd} B \otimes W(E) \arrow[dd, "\pi"'] \arrow[dr, "\gamma"] & \\ & S. \\ R \arrow[ur, "\varphi"'] & \end{tikzcd}$$

In particular, φ is a homomorphism of operations.

Moreover, for $b \in B$,

$$(\varphi_{i=0,\theta=0} \circ \lambda)(b) = (\varphi \circ \pi)(b \otimes 1 \otimes 1) = \psi(b).$$

Finally, clearly

$$\varphi \circ \chi_R = \varphi \circ \pi \circ \hat{\chi} = \gamma \circ \hat{\chi} = \chi_S.$$

Thus property (2) is established and the proof of Theorem VIII is complete.

Q.E.D.

9.16. Classification theorem. We shall say that an operation $(E, i_R, \theta_R, R, \delta_R)$ is *cohomologically related* to an operation $(E, i_S, \theta_S, S, \delta_S)$ if there is a homomorphism of operations $\varphi: R \to S$, which induces an isomorphism

$$\varphi_{\theta=0}^{\#}: H(R_{\theta=0}) \xrightarrow{\cong} H(S_{\theta=0}).$$

In this case we write

$$(E, i_R, \theta_R, R, \delta_R) \xrightarrow[c]{} (E, i_S, \theta_S, S, \delta_S).$$

Two operations will be called *cohomologically equivalent* (c-equivalent) if they are equivalent under the equivalence generated by the above relation.

Now let $(E, i_R, \theta_R, R, \delta_R)$ be a regular operation and let χ_R be an algebraic connection. Then a (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ is determined, where $\tau_R = (\chi_R)_{\vee,\theta=0} \circ \tau$ and τ is the transgression in $W(E)_{\theta=0}$ fixed at the start of this article. The associated P_E-algebra is given by $(H(R_{i=0,\theta=0}); \chi_R^{\#} \circ \tau)$. By Theorem V, sec. 8.20, this P_E-algebra is independent of the algebraic connection.

It follows that the c-equivalence class of the (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ is independent of the algebraic connection (cf. the corollary to Proposition XI, sec. 3.29). Hence, to every regular operation corresponds a well-defined c-equivalence class of (P_E, δ)-algebras.

Lemma IV: Suppose $(E, i_R, \theta_R, R, \delta_R)$ and $(E, i_S, \theta_S, S, \delta_S)$ are c-equivalent regular operations. Then the corresponding (P_E, δ)-algebras are c-equivalent.

Proof: It is sufficient to consider the case that there is a homomorphism of operations $\varphi: R \to S$, such that $\varphi_{\theta=0}^{\#}$ is an isomorphism. It follows that (cf. Theorem III, sec. 9.8)

$$\varphi_{i=0,\theta=0}^{\#}: H(R_{i=0,\theta=0}) \to H(S_{i=0,\theta=0})$$

is an isomorphism.

Moreover, by Corollary II to Theorem V, sec. 8.20,

$$\varphi^{\#}_{i=0,\theta=0} \circ \tau^{\#}_R = \varphi^{\#}_{i=0,\theta=0} \circ \chi^{\#}_R \circ \tau = \chi^{\#}_S \circ \tau = \tau^{\#}_S$$

Now we can apply Proposition XI, sec. 3.29, to obtain

$$(R_{i=0,\theta=0}, \delta_R; \tau_R) \underset{c}{\sim} (S_{i=0,\theta=0}, \delta_S; \tau_S).$$

Q.E.D.

In view of Lemma IV, there is a set map:

$$\alpha: \begin{Bmatrix}\text{c-equivalence classes of} \\ \text{regular operations}\end{Bmatrix} \to \begin{Bmatrix}\text{c-equivalence classes of c-connected} \\ (P_E, \delta)\text{-algebras}\end{Bmatrix}.$$

Theorem IX (Classification): With the notation above, α is a bijection.

Proof: Theorem VIII, sec. 9.12, shows that α is surjective. To show that α is injective, let $(E, i_R, \theta_R, R, \delta_R)$ and $(E, i_S, \theta_S, S, \delta_S)$ be regular operations with algebraic connections χ_R and χ_S. Assume that the corresponding (P_E, δ)-algebras, $(R_{i=0,\theta=0}, \delta_R; \tau_R)$ and $(S_{i=0,\theta=0}, \delta_S; \tau_S)$, are c-equivalent. We must show that the two operations are c-equivalent.

By hypothesis there are (P_E, δ)-algebras $(B_j, \delta_j; \tau_j)$ $j = 1, \ldots, q$ such that

(1) $(B_1, \delta_1; \tau_1) = (R_{i=0,\theta=0}, \delta_R; \tau_R)$, $(B_q, \delta_q; \tau_q) = (S_{i=0,\theta=0}, \delta_S; \tau_S)$ and

(2) For each j $(1 \leq j \leq q-1)$ either

$$(B_j, \delta_j; \tau_j) \xrightarrow[c]{} (B_{j+1}, \delta_{j+1}; \tau_{j+1}) \quad \text{or} \quad (B_{j+1}, \delta_{j+1}; \tau_{j+1}) \xrightarrow{c} (B_j, \delta_j; \tau_j).$$

Now according to Theorem VIII, sec. 9.12, there are regular operations $(E, i_j, \theta_j, T_j, d_j)$, and algebraic connections χ_j such that

$$(B_j, \delta_j, \tau_j) \cong ((T_j)_{i=0,\theta=0}, d_j; (\chi_j)_{\vee,\theta=0} \circ \tau).$$

Since it is sufficient to prove that

$$(E, i_R, \theta_R, R, \delta_R) \underset{c}{\sim} (E, i_1, \theta_1, T_1, d_1) \underset{c}{\sim} \cdots \underset{c}{\sim} (E, i_q, \theta_q, T_q, d_q) \underset{c}{\sim} (E, i_S, \theta_S, S, \delta_S),$$

we need only consider the case

$$(R_{i=0,\theta=0}, \delta_R; \tau_R) \xrightarrow[c]{} (S_{i=0,\theta=0}, \delta_S; \tau_S).$$

Thus we may assume that there is a homomorphism, $\psi: (R_{i=0,\theta=0}, \delta_R) \to (S_{i=0,\theta=0}, \delta_S)$, such that $\psi \circ \tau_R = \tau_S$, and such that $\psi^\#$ is an isomorphism.

Let $(E, \hat{i}, \hat{\theta}, \hat{R}, \hat{\delta})$ be the operation constructed in sec. 9.13 from the (P_E, δ)-algebra $(R_{i=0,\theta=0}, \delta_R; \tau_R)$. Since $\tau_R = (\chi_R)_{\vee,\theta=0} \circ \tau$, property (2) in Theorem VIII, sec. 9.12 (applied with $\psi = \iota$) yields a homomorphism of operations

$$\varphi: (E, \hat{i}, \hat{\theta}, \hat{R}, \hat{\delta}) \to (E, i_R, \theta_R, R, \delta_R),$$

such that $\varphi_{i=0,\theta=0}$ is an isomorphism.

Hence, $\varphi^\#_{i=0,\theta=0}$ is an isomorphism. Now Theorem III, sec. 9.8, implies that $\varphi^\#_{\theta=0}$ is an isomorphism; i.e., these two operations are c-equivalent.

Finally, an analogous argument yields a homomorphism of operations

$$\tilde{\varphi}: (E, \hat{i}, \hat{\theta}, \hat{R}, \hat{\delta}) \to (E, i_S, \theta_S, S, \delta_S)$$

(induced from ψ) such that $\tilde{\varphi}^\#_{\theta=0}$ is an isomorphism. Thus these two operations are also c-equivalent. It follows that

$$(E, i_R, \theta_R, R, \delta_R) \underset{c}{\sim} (E, i_S, \theta_S, S, \delta_S).$$

Q.E.D.

§6. Principal bundles

9.17. The structure of $H(P)$. Let $\mathscr{P} = (P, \pi, B, G)$ be a smooth principal bundle whose structure group G is compact and connected. Denote the Lie algebra of G by E. Since G is connected, we have $(\vee E^*)_I = (\vee E^*)_{\theta=0}$.

Let ∇ be a principal connection in $\mathscr{P}$. Recall from sec. 6.19, volume II, and sec. 8.26 that the curvature form of the connection can be used to construct a homomorphism $\gamma_B: (\vee E^*)_I \to Z(B)$, such that $\gamma_B^\# = h_{\mathscr{P}}$, where $h_{\mathscr{P}}$ denotes the Weil homomorphism for $\mathscr{P}$. Let $\tau: P_E \to (\vee E^*)_{\theta=0}$ be a fixed transgression in $W(E)_{\theta=0}$. Consider the (P_E, δ)-algebra $(A(B), \delta; \tau_B)$ given by

$$\tau_B = \gamma_B \circ \tau,$$

and let $(A(B) \otimes \wedge P_E, \nabla)$ denote the corresponding Koszul complex.

Theorem X: Let $\mathscr{P} = (P, \pi, B, G)$ be a smooth principal bundle whose structure group G is compact and connected. Let $(A(B) \otimes \wedge P_E, \nabla)$ be the Koszul complex constructed above via a principal connection. Then there is a homomorphism of graded differential algebras

$$\vartheta: (A(B) \otimes \wedge P_E, \nabla) \to (A(P), \delta)$$

such that

(1) $\vartheta^\#: H(A(B) \otimes \wedge P_E) \to H(P)$ is an isomorphism.

(2) If B is connected, the diagram

$$\begin{array}{ccccccc} \vee P_E & \xrightarrow{(\tau_B)_\vee^\#} & H(B) & \xrightarrow{l^\#} & H(A(B) \otimes \wedge P_E, \nabla) & \xrightarrow{\varrho^\#} & \wedge P_E \\ {\scriptstyle \tau_\vee}\downarrow\cong & & \cong\downarrow{\scriptstyle \iota} & & \cong\downarrow{\scriptstyle \vartheta^\#} & & \cong\downarrow{\scriptstyle \alpha_G} \\ (\vee E^*)_I & \xrightarrow[h_{\mathscr{P}}]{} & H(B) & \xrightarrow[\pi^\#]{} & H(P) & \xrightarrow[\varrho_P]{} & H(G) \end{array} \qquad (9.17)$$

commutes.

Remark: See sec. 8.27 for the lower sequence in (9.17).

Proof: Let χ be the algebraic connection for the operation $(E, i, \theta, A(P), \delta)$ obtained by dualizing the connection form ω (cf. sec. 8.22). Then according to formula (8.20), sec. 8.26, $\gamma_I = (\chi_\vee)_{\theta=0}$, whence

$$\pi^* \circ \tau_B = \pi^* \circ \gamma_B \circ \tau = \gamma_I \circ \tau = \tau_{A(P)}.$$

It follows that $\pi^*: A(B) \xrightarrow{\cong} A(P)_{i=0,\theta=0}$ may be considered as an isomorphism of (P_E, δ)-algebras. Thus we obtain an isomorphism of Koszul complexes

$$(A(B) \otimes \wedge P_E, \nabla) \xrightarrow[\cong]{\pi^* \otimes \iota} (A(P)_{i=0,\theta=0} \otimes \wedge P_E, \nabla_{A(P)}).$$

On the other hand, the Chevalley homomorphism

$$\vartheta_{A(P)}: A(P)_{i=0,\theta=0} \otimes \wedge P_E \to A(P)_{\theta=0}$$

(cf. sec. 9.3) induces an isomorphism of cohomology, as follows from the fundamental theorem (cf. sec. 9.3). Finally, since G is compact and connected, the inclusion

$$\lambda: A(P)_{\theta=0} \to A(P)$$

also induces a cohomology isomorphism (cf. Theorem I, sec. 4.3, volume II).

Thus a homomorphism of graded differential algebras

$$\vartheta: (A(B) \otimes \wedge P_E, \nabla) \to (A(P), \delta),$$

is given by $\vartheta = \lambda \circ \vartheta_{A(P)} \circ (\pi^* \otimes \iota)$, and evidently $\vartheta^\#$ is an isomorphism.

Finally, the commutativity of diagram (9.17) follows at once from diagram (8.22), sec. 8.27, together with Theorem II, sec. 9.7.

Q.E.D.

Corollary I: The graded differential algebras $(A(P), \delta)$ and $(A(B) \otimes \wedge P_E, \nabla)$ are c-equivalent.

Corollary II: If $(A(B), \delta)$ is c-split, then

$$(A(P), \delta) \underset{c}{\sim} (H(B) \otimes \wedge P_E, \nabla^\#).$$

If B is connected, the induced isomorphism of cohomology makes the diagram

$$\begin{tikzcd}
 & H(H(B) \otimes \wedge P_E, \nabla^{\#}) \arrow[r] \arrow[d, "\cong"'] & \wedge P_E \arrow[d, "\cong"', "\alpha_G"] \\
H(B) \arrow[ur] \arrow[r, "\pi^*"'] & H(P) \arrow[r, "\varrho_P"'] & H(G)
\end{tikzcd}$$

commute.

Proof: Apply the corollary to Theorem II, sec. 9.7.

Q.E.D.

Corollary III: If $H(B)$ has finite dimension (in particular if B is compact), then $H(P)$ has finite dimension and the Euler characteristic of P is zero.

Proof: Apply the corollary to Proposition V, sec. 3.18.

Q.E.D.

Theorem XI: Let $\mathscr{P} = (P, \pi, B, G)$ be a smooth principal bundle with compact connected structure group G. Then the following conditions are equivalent:

(1) The Weil homomorphism $h_{\mathscr{P}}: (\vee E^*)_I \to H(B)$ is m-regular.

(2) $H^0(P) = R$ and $H^p(P) = 0$, $1 \le p \le m$.

Proof: In view of the commutative diagram (8.22), sec. 8.27 (which identifies $h_{\mathscr{P}}$ with $\chi^{\#}$) as well as the isomorphism $H(A(P)_{\theta=0}) \cong H(P)$, the theorem follows from Theorem IV, sec. 9.8, applied to the operation $(E, i, \theta, A(P), \delta)$.

Q.E.D.

9.18. Fibres noncohomologous to zero. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with connected base. G will be called *noncohomologous to zero in* P (n.c.z.) if the fibre projection

$$\varrho_P: H(P) \to H(G)$$

is surjective (cf. sec. 8.27).

Theorem XII: Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with connected base and compact connected structure group G. Then the following conditions are equivalent:

(1) ϱ_P is surjective.

(2) There is an isomorphism of graded algebras

$$H(B) \otimes H(G) \xrightarrow{\cong} H(P)$$

which makes the diagram

$$\begin{tikzcd} & H(B) \otimes H(G) \arrow[dd, "\cong"] \arrow[dr] & \\ H(B) \arrow[ur] \arrow[dr, "\pi^\#"'] & & H(G) \\ & H(P) \arrow[ur, "\varrho_P"'] & \end{tikzcd} \tag{9.18}$$

commute.

(3) There is a linear isomorphism of graded vector spaces $H(B) \otimes H(G) \xrightarrow{\cong} H(P)$ which makes the diagram (9.18) commute.

(4) $\pi^\#$ is injective.

(5) The Weil homomorphism is trivial: $h_{\mathscr{P}}^+ = 0$.

(6) There is a c-equivalence $(A(B \times G), \delta) \underset{c}{\sim} (A(P), \delta)$ such that the induced isomorphism of cohomology makes the diagram (9.18) commute.

Moreover, if $H(B)$ has finite dimension, then these conditions are equivalent to

$$\dim H(P) = \dim H(B) \cdot \dim H(G).$$

Proof: In view of Theorem X, sec. 9.17, the theorem is a direct translation of Theorem VII, sec. 3.17, and the corollary to Proposition V, sec. 3.18.

Q.E.D.

9.19. Homomorphisms. Theorem XIII: Let $\varphi: (P, \pi, B, G) \to (\tilde{P}, \tilde{\pi}, \tilde{B}, G)$ be a homomorphism of principal bundles with compact

connected structure group G, and let $\varphi_B\colon B \to \tilde{B}$ denote the induced map between base manifolds. Then the homomorphisms ϑ and $\tilde{\vartheta}$ of Theorem X, sec. 9.17, can be chosen so that the diagram

$$\begin{tikzcd} A(B) \otimes \wedge P_E \arrow[d, "\vartheta"'] & A(\tilde{B}) \otimes \wedge P_E \arrow[l, "\varphi_B^* \otimes \iota"'] \arrow[d, "\tilde{\vartheta}"] \\ A(P) & A(\tilde{P}) \arrow[l, "\varphi^*"] \end{tikzcd}$$

commutes.

In particular,

$$\vartheta^\# \circ (\varphi_B^* \otimes \iota)^\# = \varphi^\# \circ \tilde{\vartheta}^\#.$$

Proof: Since φ is a homomorphism of principal bundles, it is equivariant with respect to the principal actions of G. It follows that the fundamental vector fields Z_h on P and $\tilde{Z}_h$ on $\tilde{P}$ are φ-related (cf. sec. 3.9, volume II). This implies that $\varphi^*\colon A(P) \leftarrow A(\tilde{P})$ is a homomorphism of operations (cf. sec. 3.14, volume II).

Moreover, since $\tilde{\pi} \circ \varphi = \varphi_B \circ \pi$, we obtain the commutative diagram

$$\begin{tikzcd} A(P)_{i=0,\theta=0} & A(\tilde{P})_{i=0,\theta=0} \arrow[l, "\varphi^*_{i=0,\theta=0}"'] \\ A(B) \arrow[u, "\pi^*", "\cong"'] & A(\tilde{B}). \arrow[l, "\varphi_B^*"] \arrow[u, "\tilde{\pi}^*"', "\cong"] \end{tikzcd}$$

The theorem follows now from sec. 9.6.

Q.E.D.

Corollary: The homomorphism $\varphi^\#\colon H(P) \leftarrow H(\tilde{P})$ is m-regular if and only if the homomorphism $\varphi_B^\#\colon H(B) \leftarrow H(\tilde{B})$ is m-regular.

Proof: Apply Theorem I, sec. 3.10.

Q.E.D.

9.20. The fibre integral. Let $\mathscr{P} = (P, \pi, B, G)$ be an oriented principal bundle with compact connected fibre. In Proposition IV, sec. 6.5,

volume II, we obtained the commutative diagram

$$\begin{tikzcd} A_I(P) \arrow[r, "\text{inclusion}"] \arrow[d, "\omega\circ i(\varepsilon)"'] & A(P) \arrow[d, "\oint_G"] \\ A_B(P) & A(B), \arrow[l, "\cong", "\pi^*"'] \end{tikzcd} \tag{9.19}$$

where $\oint_G$ denotes the fibre integral. Here $\varepsilon \in \wedge^n E$ ($n = \dim E$) is determined by $\langle \Delta(e), \varepsilon \rangle = 1$, where Δ is the unique invariant n-form on G satisfying $\int_G \Delta = 1$. Further, ω is the involution given by

$$\omega(\Phi) = (-1)^{pn}\Phi, \qquad \Phi \in A^p(P).$$

Next, recall from sec. 3.2 that $i(\varepsilon)$ is also an operator in $A(B) \otimes \wedge P_E$. Moreover, since ε is of degree n, we may regard $i(\varepsilon)$ as a linear map, homogeneous of degree $-n$, from $A(B) \otimes \wedge P_E$ to $A(B)$:

$$i(\varepsilon)(\Psi \otimes \Phi) = (-1)^{pn}\langle \Phi, \varepsilon \rangle \Psi, \qquad \Phi \in \wedge P_E, \quad \Psi \in A^p(B).$$

In particular, $i(\varepsilon)(\Psi \otimes \Phi) = 0$ if $\deg \Phi < n$. It follows from sec. 3.2 that $i(\varepsilon)$ induces an operator,

$$i(\varepsilon)^{\#}\colon H(A(B) \otimes \wedge P_E) \to H(B).$$

We show now that this operator is related to the fibre integral by the commutative diagram

$$\begin{tikzcd} H(A(B) \otimes \wedge P_E) \arrow[dd, "\vartheta^{\#}"', "\cong"] \arrow[dr, "\omega\circ i(\varepsilon)^{\#}"] & \\ & H(B) \\ H(P) \arrow[ur, "\oint_G^{\#}"'] & \end{tikzcd}$$

where $\vartheta^{\#}$ is the isomorphism of Theorem X, sec. 9.17.

In fact, recall that $\vartheta = \lambda \circ \vartheta_{A(P)} \circ (\pi^* \otimes \iota)$, where $\vartheta_{A(P)}$ is the Chevalley homomorphism determined by an algebraic connection χ. Thus,

in view of the commutative diagram (9.19), it is sufficient to show that the diagram

$$\begin{tikzcd} A(P)_{i=0,\theta=0} \otimes \wedge P_E \arrow[r, "\vartheta_{A(P)}"] \arrow[d, "i(\varepsilon)"'] & A(P)_{\theta=0} \arrow[d, "i(\varepsilon)"] \\ A(P)_{i=0,\theta=0} \arrow[r, "="'] & A_B(P) \end{tikzcd}$$

commutes.

Use the algebraic connection to write

$$A(P)_{\theta=0} = (A(P)_{i=0} \otimes \wedge E^*)_{\theta=0}.$$

Then Theorem I, (3), sec. 9.3, implies that for $\Phi \in (\wedge P_E)^q$

$$\vartheta_{A(P)}(1 \otimes \Phi) - 1 \otimes \Phi \in F^1(A^q(P)_{\theta=0}).$$

Now

$$F^1(A^q(P)_{\theta=0}) \subset (A^+(P)_{i=0} \otimes \wedge E^*)^q_{\theta=0} \subset \sum_{j<n} (A(P)_{i=0} \otimes \wedge^j E^*)_{\theta=0},$$

and so

$$i(\varepsilon)\vartheta_{A(P)}(1 \otimes \Phi) = 1 \otimes i(\varepsilon)\Phi = \langle \Phi, \varepsilon \rangle, \qquad \Phi \in \wedge P_E.$$

Finally, let $z \in A^p(P)_{i=0,\theta=0}$ and $\Phi \in \wedge P_E$. Then $\vartheta_{A(P)}(z) = z$ and so formula (7.9), sec. 7.3, yields

$$\begin{aligned} i(\varepsilon)\vartheta_{A(P)}(z \otimes \Phi) &= (-1)^{pn} z \cdot i(\varepsilon)\vartheta_{A(P)}(1 \otimes \Phi) \\ &= (-1)^{pn}\langle \Phi, \varepsilon \rangle z = i(\varepsilon)(z \otimes \Phi). \end{aligned}$$

This completes the proof.

§7. Examples

9.21. Principal $SO(2m)$-bundles. Let E denote the Lie algebra of the Lie group $SO(2m)$ (cf. Example 3, sec. 2.5, volume II). Then (cf. Theorem VII, sec. 6.23),

$$(\vee E^*)_{\theta=0} = \vee(u_4, u_8, \ldots, u_{4m-4}, v_{2m}),$$

where

$$u_{4j} = \frac{(-1)^j}{(2\pi)^{2j}} C_{2j}^{SO} \qquad \text{and} \qquad v_{2m} = \frac{(-1)^m}{(2\pi)^m} \operatorname{Pf}.$$

(Note that the subscripts of the u_i and v_{2m} denote the degrees.)

It follows (cf. Theorem II, sec. 6.14) that the elements $x_{4j-1} = \varrho_E(u_{4j})$, $j = 1, \ldots, m-1$, and $y_{2m-1} = \varrho_E(v_{2m})$ are a basis of P_E; whence

$$(\wedge E^*)_{\theta=0} = \wedge P_E = \wedge(x_3, x_7, \ldots, x_{4m-5}, y_{2m-1}).$$

In particular, a transgression $\tau: P_E \to (\vee E^*)_{\theta=0}$ is defined by

$$\tau(x_{4j-1}) = u_{4j} \qquad \text{and} \qquad \tau(y_{2m-1}) = v_{2m}.$$

Now let $\mathscr{P} = (P, \pi, B, SO(2m))$ be a principal bundle, and form the associated oriented Riemannian bundle $\xi = (P \times_{SO(2m)} \mathbb{R}^{2m}, \pi_\xi, B, \mathbb{R}^{2m})$. In sec. 9.4 and sec. 9.13 of volume II we defined the Pontrjagin classes and the Pfaffian class of ξ by

$$p_j(\xi) = k_\xi(u_{4j}) \in H^{4j}(B), \qquad j = 1, \ldots, m,$$

and

$$\operatorname{pf}(\xi) = k_\xi(v_{2m}) \in H^{2m}(B).$$

(k_ξ is the characteristic homomorphism for ξ.)

Further, according to Proposition IX, sec. 9.12, volume II, $\operatorname{pf}(\xi)^2 = (-1)^m p_m(\xi)$. On the other hand, the Gauss–Bonnet–Chern theorem (sec. 10.1, volume II) asserts that $\operatorname{pf}(\xi)$ coincides with the Euler class χ_ξ of the associated sphere bundle.

Now let $\Phi_4, \Phi_8, \ldots, \Phi_{4m-4}$, and Ψ_{2m} be any closed forms on B representing $p_1(\xi), \ldots, p_{m-1}(\xi)$ and $\operatorname{pf}(\xi)$. Let $(A(B) \otimes \wedge P_E, \nabla)$ be

the Koszul complex of the (P_E, δ)-algebra $(A(B), \delta; \sigma)$ given by

$$\sigma(x_{4j-1}) = \Phi_{4j}, \quad j = 1, \dots, m-1, \qquad \text{and} \qquad \sigma(y_{2m-1}) = \Psi_{2m}.$$

Proposition VI: There is a homomorphism of graded differential algebras

$$\vartheta\colon (A(B) \otimes \wedge P_E, \nabla) \to (A(P), \delta)$$

such that $\vartheta^\#$ is an isomorphism, and the diagram

$$\begin{tikzcd}
 & H(A(B) \otimes \wedge P_E) \arrow[r] \arrow[d, "\vartheta^\#", "\cong"'] & \wedge P_E \arrow[d, "\alpha_{SO(2m)}", "\cong"'] \\
H(B) \arrow[ur] \arrow[r, "\pi^\#"'] & H(P) \arrow[r, "\varrho_P"'] & H(SO(2m))
\end{tikzcd}$$

commutes.

Proof: Consider the (P_E, δ)-algebra $(A(B), \delta; \tau_B)$ of sec. 9.17. Since (by Theorem VII, sec. 8.24, volume II) $k_\xi = h_{\mathscr{P}}$, we have

$$\sigma^\#(x_{4j-1}) = p_j(\xi) = h_{\mathscr{P}}(u_{4j}) = h_{\mathscr{P}} \circ \tau(x_{4j-1}) = \tau_B^\#(x_{4j-1}),$$
$$j = 1, \dots, m-1.$$

Similarly $\sigma^\#(y_{2m-1}) = \tau_B^\#(y_{2m-1})$, and so $\sigma^\# = \tau_B^\#$.

Now combine Theorem X, sec. 9.17, with Proposition X, sec. 3.27, to achieve the proof.

Q.E.D.

9.22. Principal $SO(2m+1)$-bundles. Let E denote the Lie algebra of $SO(2m+1)$ (cf. Example 3, sec. 2.5, volume II, and sec. 6.21). Then (cf. Theorem VI, sec. 6.23)

$$(\vee E^*)_{\theta=0} = \vee(u_4, u_8, \dots, u_{4m}),$$

where $u_{4j} = ((-1)^j/(2\pi)^{2j}) C_{2j}^{SO}$.

It follows (cf. sec. 6.14) that the elements $x_{4j-1} = \varrho_E(u_{4j})$ ($j = 1, \dots, m$) are a basis of P_E; whence

$$(\wedge E^*)_{\theta=0} = \wedge P_E = \wedge(x_3, x_7, \dots, x_{4m-1}).$$

Thus a transgression $\tau\colon P_E \to (\vee E^*)_{\theta=0}$ is given by

$$\tau(x_{4j-1}) = u_{4j}, \qquad j = 1, \ldots, m.$$

Suppose now that $\mathscr{P} = (P, \pi, B, SO(2m+1))$ is a principal bundle with associated vector bundle $\xi = (P \times_{SO(2m+1)} \mathbb{R}^{2m+1}, \pi_\xi, B, \mathbb{R}^{2m+1})$. Let $\Phi_4, \ldots, \Phi_{4m}$ be closed forms on B representing the Pontrjagin classes $p_j(\xi)$. Let $(A(B) \otimes \wedge P_E, \nabla)$ be the Koszul complex of the (P_E, δ)-algebra $(A(B), \delta; \sigma)$, where $\sigma(x_{4j-1}) = \Phi_{4j}$. Then the argument of the preceding section also establishes

Proposition VII: There is a homomorphism

$$\vartheta\colon (A(B) \otimes \wedge P_E, \nabla) \to (A(P), \delta)$$

of graded differential algebras, such that $\vartheta^\#$ is an isomorphism, and the diagram

$$\begin{tikzcd} & H(A(B) \otimes \wedge P_E) \arrow[rr] \arrow[d, "\vartheta^\#", "\cong"'] & & \wedge P_E \arrow[d, "\alpha_{SO(2m+1)}", "\cong"'] \\ H(B) \arrow[ur] \arrow[r, "\pi^\#"'] & H(P) \arrow[rr, "\varrho_P"'] & & H(SO(2m+1)) \end{tikzcd}$$

commutes.

9.23. Principal $U(m)$-bundles. Let E be the Lie algebra of the unitary group $U(m)$. Then (cf. Theorem IX, sec. 6.27)

$$(\vee E^*)_{\theta=0} = \vee(u_2, u_4, u_6, \ldots, u_{2m}),$$

where $u_{2j} = (-1/2\pi)^j C_j^U$.

It follows that the elements $x_{2j-1} = \varrho_E(u_{2j})$ $(j = 1, \ldots, m)$ are a basis of P_E; whence

$$(\wedge E^*)_{\theta=0} = \wedge P_E = \wedge(x_1, x_3, x_5, \ldots, x_{2m-1}).$$

In particular, a transgression $\tau\colon P_E \to (\vee E^*)_{\theta=0}$ is given by

$$\tau(x_{2j-1}) = u_{2j}, \qquad j = 1, \ldots, m.$$

Now consider a principal bundle $\mathscr{P} = (P, \pi, B, U(m))$ with associated complex vector bundle $\xi = (P \times_{U(m)} C^m, \pi_\xi, B, C^m)$. In sec. 9.18, volume II, we defined the Chern classes of ξ by

$$c_j(\xi) = \tilde{l}_\xi(C_j), \qquad j = 1, \dots, m,$$

where $\tilde{l}_\xi$ is the modified characteristic homomorphism for ξ. In view of sec. 9.16, volume II,

$$c_j(\xi) = m_\xi(u_{2j}), \qquad j = 1, \dots, m,$$

where m_ξ is the characteristic homomorphism for ξ, regarded as a bundle with Hermitian inner product.

Let $\Phi_2, \Phi_4, \dots, \Phi_{2m}$ be closed forms on B representing $c_1(\xi), \dots, c_m(\xi)$, and consider the (P_E, δ)-algebra $(A(B), \delta; \sigma)$, where σ is given by

$$\sigma(x_{2j-1}) = \Phi_{2j}, \qquad j = 1, \dots, m.$$

Then

$$\sigma^\#(x_{2j-1}) = m_\xi(u_{2j}) = h_{\mathscr{P}}(u_{2j})$$

(cf. Theorem VII, sec. 8.24, volume II). Thus, exactly as in sec. 9.21, we have

Proposition VIII: Let $(A(B) \otimes \wedge P_E, \nabla)$ be the Koszul complex for $(A(B), \delta; \sigma)$. Then there is a homomorphism of graded differential algebras

$$\vartheta: (A(B) \otimes \wedge P_E, \nabla) \to (A(P), \delta)$$

such that $\vartheta^\#$ is an isomorphism, and the diagram

$$\begin{tikzcd}
 & H(A(B) \otimes \wedge P_E) \arrow[r] \arrow[d, "\cong"', "\vartheta^\#"] & \wedge P_E \arrow[d, "\cong"', "\alpha_{U(m)}"] \\
H(B) \arrow[ur] \arrow[r, "\pi^\#"'] & H(P) \arrow[r, "\varrho_P"'] & H(U(m))
\end{tikzcd}$$

commutes.

9.24. Manifolds with vanishing Pontrjagin classes. Let B be an oriented connected Riemannian n-manifold, and let $\mathscr{P} = (P, \pi, B, SO(n))$ be the frame bundle associated with the tangent bundle τ_B of B (cf. Example 3, sec. 8.20, volume II). We shall compute the cohomology algebra of P in the case that all the Pontrjagin classes of the tangent bundle vanish (examples are given below).

First notice that if B is compact and even dimensional, then the Euler class χ_{τ_B} of τ_B is given by

$$\chi_{\tau_B} = \chi_B \cdot \omega_B,$$

where ω_B is the orientation class of B, and χ_B is the Euler–Poincaré characteristic (cf. Theorem I, sec. 10.1, volume I). If B is not compact, then $H^n(B) = 0$ (cf. Proposition IX, sec. 5.15, volume I). Hence $\chi_{\tau_B} = 0$ in this case.

Now consider the following three cases.

Case I: $n = 2m + 1$. Apply Proposition VII, sec. 9.22. Since $p_k(\tau_B) = 0$ for all k, we may choose $\Phi_{4j} = 0$, $j = 1, \dots, m$. Thus $\nabla = \delta \otimes \iota$ and

$$H(P) \cong H(A(B) \otimes \wedge P_E, \delta \otimes \iota) = H(B) \otimes \wedge P_E \cong H(B) \otimes H(SO(n)).$$

In particular, $SO(n)$ is n.c.z. in P.

Case II: *$n = 2m$, and either B is not compact or $\chi_B = 0$.* In this case $\chi_{\tau_B} = 0$ and Proposition VI, sec. 9.21, gives

$$H(P) \cong H(B) \otimes H(SO(n)).$$

In particular, $SO(n)$ is n.c.z. in P.

Case III: *$n = 2m$, B is compact, and $\chi_B \neq 0$.* Apply Proposition VI, sec. 9.21. Since the Pontrjagin classes vanish we may choose $\Phi_{4j} = 0$ $(j = 1, \dots, m - 1)$. For Ψ_n, we may choose any n-form satisfying $\int_B \Psi_n = \chi_B$. Thus the Koszul complex of Proposition VI has the form

$$\{(A(B) \otimes \wedge y_{2m-1}) \otimes \wedge(x_3, \dots, x_{4m-5}), \nabla \otimes \iota\},$$

where

$$\nabla(\Phi \otimes 1 + \Psi \otimes y_{2m-1}) = \delta\Phi \otimes 1 + \delta\Psi \otimes y_{2m-1} + \Psi_n \wedge \Psi \otimes 1, \qquad \Phi, \Psi \in A(B).$$

Now apply sec. 3.6 to obtain a *linear* isomorphism

$$H(A(B) \otimes \wedge y_{2m-1}) \cong H(H(B) \otimes \wedge y_{2m-1}, \nabla^{\#}),$$

where $\nabla^{\#}(\alpha \otimes 1 + \beta \otimes y_{2m-1}) = \omega_B \cdot \beta \otimes 1$. It follows (since ω_B has top degree in $H(B)$) that

$$\ker \nabla^{\#} = (H(B) \otimes 1) \oplus (H^+(B) \otimes y_{2m-1})$$

and

$$\operatorname{Im} \nabla^{\#} = H^n(B) \otimes 1.$$

Thus we obtain linear isomorphisms of graded spaces

$$H(A(B) \otimes \wedge y_{2m-1}) \cong \sum_{j=0}^{n-1} H^j(B) \oplus \sum_{j=1}^{n} H^j(B) \otimes y_{2m-1},$$

and

$$H(P) \cong \left(\sum_{j=0}^{n-1} H^j(B) \oplus \sum_{j=1}^{n} H^j(B) \otimes y_{2m-1}\right) \otimes \wedge(x_3, x_7, \ldots, x_{4m-5}).$$

In particular, the Poincaré polynomials for $H(P)$ and $H(B)$ are related by

$$f_{H(P)} = [(1 + t^{n-1}) f_{H(B)} - t^{n-1} - t^n] \cdot \prod_{j=1}^{m-1} (1 + t^{4j-1}).$$

Examples: 1. Suppose B admits a linear connection with decomposable curvature. Then the Pontrjagin classes of B vanish (cf. sec. 9.8, volume II.)

2. If for some r, $\tau_B \oplus \varepsilon^r = \varepsilon^{n+r}$ (ε^q is the trivial bundle of rank q over B), then the Pontrjagin classes of B vanish (cf. sec. 9.4, volume II).

3. If B can be immersed into $\mathbb{R}^{n+1}$ ($n = \dim B$), then $\tau_B \oplus \varepsilon = \varepsilon^{n+1}$ and so the Pontrjagin classes are zero.

4. Let $B = G/S$, where S is a torus in a compact connected Lie group G. Then the Pontrjagin classes of B vanish. The Euler class vanishes if and only if the torus S is not maximal (cf. sec. 5.12, volume II).

9.25. The frame bundle of $\mathbb{C}P^n$ Recall from Example 3, sec. 5.14, volume II, that $\mathbb{C}P^n$ is the manifold of complex lines in $\mathbb{C}^{n+1}$, and is diffeomorphic to the homogeneous space $U(n+1)/(U(1) \times U(n))$. Moreover, if $\alpha \in H^2(\mathbb{C}P^n)$ is the Euler class of the canonical complex line

bundle ξ, then the elements $1, \alpha, \alpha^2, \ldots, \alpha^n$ are a basis of $H(CP^n)$ (cf. sec. 6.24, volume II). In particular, α^n is an orientation class for CP^n.

Next observe that $(A(CP^n), \delta)$ is c-split. Indeed, if $\Phi \in A^2(CP^n)$ is a closed form representing α, then a homomorphism $\lambda\colon (H(CP^n), 0) \to (A(CP^n), \delta)$ is defined by

$$\lambda(\alpha^p) = \Phi^p, \qquad 1 \le p \le n.$$

Clearly $\lambda^\# = \iota$ and so λ is a c-splitting.

A second graded differential algebra, c-equivalent to $(A(CP^n), \delta)$, is given as follows: Let (a) and (z_{2n+1}) be one-dimensional graded vector spaces generated by vectors a and z_{2n+1} of degrees 2 and $2n+1$, respectively. Consider the Koszul complex $(\vee(a) \otimes \wedge(z_{2n+1}), \nabla)$ defined by

$$\nabla(a \otimes 1) = 0 \qquad \text{and} \qquad \nabla(1 \otimes z_{2n+1}) = a^{n+1} \otimes 1.$$

Then a homomorphism of graded differential algebras

$$\varphi\colon (\vee(a) \otimes \wedge(z_{2n+1}), \nabla) \to (H(CP^n), 0)$$

is defined by

$$\varphi(a \otimes 1) = \alpha \qquad \text{and} \qquad \varphi(1 \otimes z_{2n+1}) = 0.$$

Evidently, $\varphi^\#$ is an isomorphism.

To determine the Pontrjagin classes of CP^n observe that

$$\underbrace{\xi^* \oplus \cdots \oplus \xi^*}_{n+1 \text{ factors}} \cong \tau_{CP^n} \oplus \varepsilon \qquad (\varepsilon = CP^n \times C)$$

as follows easily from sec. 5.16, volume II. Proposition XIII, sec. 9.20, volume II, implies that the total Chern class of ξ is simply $1 + \alpha$. Now it follows from Example 3, sec. 9.17, and Example 3, sec. 9.20, both of volume II, that

$$\begin{aligned} p(CP^n) &= c(\xi \oplus \cdots \oplus \xi) \cdot c(\xi^* \oplus \cdots \oplus \xi^*) \\ &= (1+\alpha)^{n+1}(1-\alpha)^{n+1} = (1-\alpha^2)^{n+1}. \end{aligned}$$

Thus

$$p_k(CP^n) = (-1)^k \binom{n+1}{k} \alpha^{2k}, \qquad k = 1, \ldots, n.$$

Finally, the Euler class of CP^n is given by

$$\chi = (n+1)\alpha^n.$$

Next, consider the orthonormal tangent frame bundle $\mathscr{P} = (P, \pi, CP^n, SO(2n))$ over CP^n, and assume $n \geq 2$.

Proposition IX: (1) The graded differential algebra $(A(P), \delta)$ is c-split.

(2) There is an isomorphism of graded algebras

$$H(P) \cong A \otimes \wedge(z_{2n+1}, x_7, x_{11}, \ldots, x_{4n-5}, y_{2n-1}),$$

where z_{2n+1}, x_i, y_{2n-1} are homogeneous vectors of degrees $2n + 1$, i, and $2n - 1$, and A is the truncated polynomial algebra $\vee(a)/(a^2)$. (Thus $A \cong H(CP^1) = H(S^2)$.)

(3) The Poincaré polynomial of $H(P)$ is

$$(1 + t^2)(1 + t^{2n+1})(1 + t^{2n-1}) \prod_{p=2}^{n-1} (1 + t^{4p-1}),$$

and $\dim H(P) = 2^{n+1}$.

Proof: Denote Sk$(2n)$ by E and $H(CP^n)$ by S. Then a P_E-algebra $(S; \sigma)$ is defined by

$$\sigma(x_{4k-1}) = p_k(CP^n), \quad 1 \leq k \leq n - 1, \qquad \text{and} \qquad \sigma(y_{2n-1}) = \chi.$$

It is the associated P_E-algebra of the (P_E, δ)-algebra defined in sec. 9.21 (with $B = CP^n$).

Thus, since $(A(CP^n), \delta)$ is c-split, the example in sec. 3.29 shows that the Koszul complex $(S \otimes \wedge P_E, \nabla_\sigma)$ is c-equivalent to the Koszul complex $(A(CP^n) \otimes \wedge P_E, \nabla)$ of sec. 9.21. Combining this with Proposition VI, sec. 9.21, we obtain

$$(A(P), \delta) \underset{c}{\sim} (S \otimes \wedge P_E, \nabla_\sigma). \tag{9.20}$$

Next observe that, since $n \geq 2$,

$$\sigma(x_{4p-1}) \in S^+ \cdot \sigma(x_3), \quad p \geq 2, \qquad \text{and} \qquad \sigma(y_{2n-1}) \in S^+ \cdot \sigma(x_3).$$

Since $\sigma(x_3) \neq 0$, Proposition IV, sec. 2.13, shows that the Samelson space for $(S; \sigma)$ is spanned by $x_7, x_{11}, \ldots, x_{4n-5}, y_{2n-1}$. Hence the reduction theorem of sec. 2.15 yields the relation

$$(S \otimes \wedge P_E, \nabla_\sigma) \underset{c}{\sim} (S \otimes \wedge(x_3), \nabla_\sigma) \otimes (\wedge(x_7, \ldots, x_{4n-5}, y_{2n-1}), 0). \tag{9.21}$$

Now consider the graded space $P = (z_{2n+1}, x_3)$, and the P-algebra $(\vee(a); \tau)$ defined by

$$\tau(x_3) = -(n+1)a^2, \qquad \tau(z_{2n+1}) = a^{n+1}.$$

Its Koszul complex $(\vee(a) \otimes \wedge(z_{2n+1}, x_3), \nabla_\tau)$ is also the Koszul complex of the $((x_3), \delta)$-algebra, $(\vee(a) \otimes \wedge(z_{2n+1}), \nabla; \hat{\tau})$, where $\hat{\tau}$ is given by $\hat{\tau}(x_3) = -(n+1)a^2 \otimes 1$.

But the homomorphism $\varphi\colon \vee(a) \otimes \wedge(z_{2n+1}) \to S$ defined above satisfies $\varphi(\hat{\tau}x_3) = \sigma(x_3)$. Hence

$$\psi = \varphi \otimes \iota : (\vee(a) \otimes \wedge(z_{2n+1}, x_3), \nabla_\tau) \to (S \otimes \wedge(x_3), \nabla_\sigma)$$

is a homomorphism of graded differential algebras. Moreover, because $\varphi^\#$ is an isomorphism so is $\psi^\#$ (cf. Theorem I, sec. 3.10). It follows that

$$(S \otimes \wedge(x_3), \nabla_\sigma) \underset{c}{\sim} (\vee(a) \otimes \wedge(z_{2n+1}, x_3), \nabla_\tau). \tag{9.22}$$

Finally the reduction theorem of sec. 2.15, applied to the Koszul complex on the right-hand side of (9.22) yields

$$(\vee(a) \otimes \wedge(z_{2n+1}, x_3), \nabla_\tau) \underset{c}{\sim} (\vee(a) \otimes \wedge(x_3), \nabla_\tau) \otimes (\wedge(z_{2n+1}), 0). \tag{9.23}$$

Moreover a c-equivalence from $(\vee(a) \otimes \wedge(x_3), \nabla_\tau)$ to $(A, 0)$ is given by $a \mapsto a$, $x_3 \mapsto 0$. Thus formulae (9.20), (9.21), (9.22), and (9.23) give

$$(A(P), \delta) \underset{c}{\sim} (A \otimes \wedge(z_{2n+1}, x_7, \dots, x_{4n-5}, y_{2n-1}), 0).$$

The proposition follows.

Q.E.D.

9.26. The frame bundle of $CP^n \times CP^m$. In this section we compute $H(P)$, where $\mathscr{P} = (P, \pi, CP^n \times CP^m, SO(2n + 2m))$ is the bundle of (positive) orthonormal tangent frames of $CP^n \times CP^m$.

Let $\alpha \in H^2(CP^n)$ and $\beta \in H^2(CP^m)$ be the canonical generators (as in sec. 9.25). Then the Künneth theorem (sec. 5.20, volume I) yields

$$H(CP^n \times CP^m) = H(CP^n) \otimes H(CP^m) = (1, \alpha, \dots, \alpha^n) \otimes (1, \beta, \dots, \beta^m).$$

In particular, $(A(CP^n \times CP^m), \delta)$ is c-split.

Next, note that Proposition II, (2), sec. 9.4, volume II, together with sec. 9.25 shows that the total Pontrjagin class of $CP^n \times CP^m$ is given by

$$p(CP^n \times CP^m) = (1 - \alpha^2)^{n+1}(1 - \beta^2)^{m+1},$$

whence

$$p_k(CP^n \times CP^m) = (-1)^k \sum_{i=0}^{k} \binom{n+1}{i}\binom{m+1}{k-i}\alpha^{2i}\beta^{2(k-i)}.$$

(Observe that the ith term in this sum is zero unless $k - m - 1 \leq i \leq n + 1$.) As in sec. 9.25, the Euler class is given by

$$\chi = (n + 1)(m + 1)\alpha^n\beta^m.$$

We assume throughout that $n \geq 2$ and $m \geq 2$.

Denote $\mathrm{Sk}(2n + 2m)$ by E (cf. sec. 9.21) and denote $H(CP^n \times CP^m)$ simply by S. Define a P_E-algebra $(S; \sigma)$ by

$$\sigma(x_{4k-1}) = p_k(CP^n \times CP^m),\ k = 1, \dots, n + m - 1$$

and

$$\sigma(y_{2n+2m-1}) = \chi.$$

Then, exactly as in sec. 9.25, we have

$$(A(P), \delta) \underset{c}{\sim} (S \otimes \wedge P_E, \nabla_\sigma). \tag{9.24}$$

Next observe (as follows from the formulae above for p_k and χ) that

$$\sigma(x_3) = -(n + 1)\alpha^2 - (m + 1)\beta^2$$

and

$$\sigma(x_7) = -\frac{(m + n + 2)(m + 1)}{2(n + 1)}\beta^4 + \sigma(x_3) \cdot \zeta \qquad (\text{some } \zeta \in S^+).$$

Thus since $m \geq 2$ and $n \geq 2$, $\sigma(y_{2n+2m-1})$ and $\sigma(x_{4k-1})$ $(k \geq 3)$ are in the ideal $S^+ \cdot \sigma(P_E)$. (Note that if $n = m = 2$, then $\alpha^4 = \beta^4 = 0$.)

Thus we can apply the simplification theorem of sec. 2.16 as follows: Set

$$\gamma(x_3) = -(n + 1)\alpha^2 - (m + 1)\beta^2, \qquad \gamma(x_7) = -\frac{(m+n+2)(m+1)}{2(n + 1)}\beta^4$$

and

$$\gamma(y_{2n+2m-1}) = 0 = \gamma(x_{4k-1}), \qquad k \geq 3.$$

Then

$$(S \otimes \wedge P_E, \nabla_\sigma) \underset{c}{\simeq} (S \otimes \wedge P_E, \nabla_\gamma)$$
$$= (S \otimes \wedge(x_3, x_7), \nabla_\gamma)$$
$$\otimes (\wedge(x_{11}, \ldots, x_{4n+4m-5}, y_{2n+2m-1}), 0). \tag{9.25}$$

Now let (a, b) be a two-dimensional space with basis a, b, both homogeneous of degree 2. Let P be the four-dimensional graded space $(x_3, x_7, z_{2n+1}, w_{2m+1})$ (subscripts still denote degrees). Consider the P-algebra $(\vee(a, b); \tau)$, where

$$\tau(x_3) = -(n+1)a^2 - (m+1)b^2, \qquad \tau(x_7) = b^4,$$
$$\tau(z_{2n+1}) = a^{n+1}, \qquad \text{and} \qquad \tau(w_{2m+1}) = b^{m+1}.$$

Define a homomorphism of graded differential algebras

$$\psi\colon (\vee(a, b) \otimes \wedge(x_3, x_7, z_{2n+1}, w_{2m+1}), \nabla_\tau) \to (S \otimes \wedge(x_3, x_7), \nabla_\gamma)$$

by

$$\psi(a) = \alpha, \quad \psi(b) = \beta, \quad \psi(x_3) = x_3, \quad \psi(x_7) = \frac{-2(n+1)}{(m+n+2)(m+1)}\, x_7$$

and

$$\psi(z_{2n+1}) = 0 = \psi(w_{2m+1}).$$

It follows exactly as in sec. 9.25 that $\psi^\#$ is an isomorphism, whence

$$(S \otimes \wedge(x_3, x_7), \nabla_\gamma) \underset{c}{\simeq} (\vee(a, b) \otimes \wedge(x_3, x_7, z_{2n+1}, w_{2m+1}), \nabla_\tau). \tag{9.26}$$

Now we distinguish three cases:

Case I: $n \geq 3$ *and* $m \geq 3$. Set

$$\tilde{z}_{2n+1} = \begin{cases} z_{2n+1}, & n > 3 \\ z_7 - \left(\dfrac{m+1}{4}\right)^2 x_7, & n = 3 \end{cases} \quad \text{and} \quad \tilde{w}_{2m+1} = \begin{cases} w_{2m+1}, & m > 3 \\ w_7 - x_7, & m = 3. \end{cases}$$

Then the Samelson subspace for $(\vee(a, b); \tau)$ is given by $\hat{P} = (\tilde{z}_{2n+1}, \tilde{w}_{2m+1})$. Thus the reduction theorem yields

$$(\vee(a, b) \otimes \wedge(x_3, x_7, z_{2n+1}, w_{2m+1}), \nabla_\tau)$$
$$\underset{c}{\simeq} (\vee(a, b) \otimes \wedge(x_3, x_7), \nabla_\tau) \otimes (\wedge(\tilde{z}_{2n+1}, \tilde{w}_{2m+1}), 0). \tag{9.27}$$

Moreover, it follows from (9.26) and (9.27) that $H(\vee(a, b) \otimes \wedge(x_3, x_7))$ has finite dimension. Now the corollary to Theorem VIII, sec. 2.19 $(1) \Rightarrow 5))$ implies that

$$H_+(\vee(a, b) \otimes \wedge(x_3, x_7)) = 0. \tag{9.28}$$

Finally, let $I \subset \vee(a, b)$ be the ideal generated by $(n+1)a^2 + (m+1)b^2$ and b^4. Define a homomorphism

$$\varphi: (\vee(a, b) \otimes \wedge(x_3, x_7), \nabla_\tau) \to (\vee(a, b)/I, 0)$$

by

$$\varphi(a) = \bar{a}, \qquad \varphi(b) = \bar{b}, \qquad \varphi(x_3) = 0 = \varphi(x_7).$$

In view of (9.28) it is easy to see that $\varphi^\#$ is an isomorphism. Thus

$$(\vee(a, b) \otimes \wedge(x_3, x_7), \nabla_\tau) \underset{c}{\sim} (\vee(a, b)/I, 0). \tag{9.29}$$

Proposition X: If $n \geq 3$ and $m \geq 3$, then $(A(P), \delta)$ is c-split, and

$$H(P) \cong \vee(a, b)/I \otimes \wedge(x_{11}, \ldots, x_{4n+4m-5}, y_{2n+2m-1}, \tilde{z}_{2n+1}, \tilde{w}_{2m+1})$$

(as graded algebras). The Poincaré polynomial for $H(P)$ is

$$(1 + t^2)^2(1 + t^4)(1 + t^{2n+2m-1})(1 + t^{2n+1})(1 + t^{2m+1}) \prod_{j=3}^{n+m-1} (1 + t^{4j-1}),$$

and $\dim H(P) = 2^{n+m+3}$.

Proof: The first part of the proposition follows from the c-equivalences (9.24), (9.25), (9.26), (9.27), and (9.29). The second part follows from sec. 2.20 (formula (2.16)) applied to the Koszul complex $(\vee(a, b) \otimes \wedge(x_3, x_7), \nabla_\tau)$.

Q.E.D.

Case II: $n = 2$ *and* $m \geq 3$. In this case the Samelson subspace for $(\vee(a, b); \tau)$ is given by $\hat{P} = (x_7, w_{2m+1})$. Let $J \subset \vee(a, b)$ be the ideal generated by $3a^2 + (m + 1)b^2$ and a^3. Then the same argument as given in Case I establishes

Proposition XI: If $n = 2$ and $m \geq 3$, then $(A(P), \delta)$ is c-split and

$$H(P) \cong \vee(a, b)/J \otimes \wedge(x_7, x_{11}, \ldots, x_{4m+3}, y_{2m+3}, w_{2m+1}).$$

The Poincaré polynomial of $H(P)$ is

$$(1+t^2)(1+t^2+t^4)(1+t^{2m+3})(1+t^{2m+1})\prod_{j=2}^{m+1}(1+t^{4j-1}),$$

and $\dim H(P) = 3 \cdot 2^{m+3}$.

Case III: *$n = 2$ and $m = 2$.* Let $P_1 = (x_3, x_7, x_{11}, y_7, z_5, w_5)$ and define $\tau_1 : P_1 \to \vee(a, b)$ by

$$\tau_1(x_3) = -3(a^2+b^2), \qquad \tau_1(x_7) = b^4$$
$$\tau_1(x_{11}) = \tau_1(y_7) = 0$$

and

$$\tau_1(z_5) = a^3, \qquad \tau_1(w_5) = b^3.$$

Then it follows from (9.24), (9.25), and (9.26) that

$$(A(P), \delta) \underset{c}{\simeq} (\vee(a, b) \otimes \wedge P_1, \nabla_{\tau_1}).$$

On the other hand, the Samelson subspace $\hat{P}_1$ for $(\vee(a, b); \tau_1)$ is given by $\hat{P}_1 = (x_7, y_7, x_{11})$. Hence

$$\dim \hat{P}_1 < \dim P_1 - \dim(a, b).$$

Thus Theorem XI, sec. 3.30, shows that $(\vee(a, b) \otimes \wedge P_1, \nabla_{\tau_1})$ is *not* c-split. It follows that $(A(P), \delta)$ is not c-split.

To compute $H(P)$ it is easiest to use (cf. equation (9.25)) the Koszul complex $(S \otimes \wedge(x_3, x_7), \nabla_\gamma)$. In this case

$$\gamma(x_7) = -3\beta^4 = 0,$$

and so the reduction theorem gives

$$H(S \otimes \wedge(x_3, x_7)) \cong H(S \otimes \wedge(x_3)) \otimes \wedge(x_7).$$

Thus formulae (9.24) and (9.25) imply that

$$H(P) \cong H(S \otimes \wedge(x_3), \nabla_\gamma) \otimes \wedge(x_7, x_{11}, y_7)$$

(as graded algebras).

A simple calculation shows that the cocycles

$$1 \otimes 1, \qquad \alpha \otimes 1, \qquad \beta \otimes 1, \qquad \alpha\beta \otimes 1, \qquad \alpha^2 \otimes 1$$

and

$$\alpha\beta \otimes x_3, \qquad (\alpha^2 - \beta^2) \otimes x_3, \qquad \alpha^2\beta \otimes x_3, \qquad \alpha\beta^2 \otimes x_3, \qquad \alpha^2\beta^2 \otimes x_3$$

represent a basis for $H(S \otimes \wedge(x_3))$.

We have thus established

Proposition XII: If $n = m = 2$, then $(A(P), \delta)$ is not c-split. The Poincaré polynomial of $H(P)$ is

$$(1 + 2t^2 + 2t^4 + 2t^7 + 2t^9 + t^{11})(1 + t^7)^2(1 + t^{11}),$$

and $\dim H(P) = 80$.

Chapter X

Subalgebras

§1. Operation of a subalgebra

10.1. Lie algebra pairs. A *Lie algebra pair* (E, F) consists of two Lie algebras E, F and an injective homomorphism $j: F \to E$. We shall identify F with its image $j(F)$ and consider F as a subalgebra of E. Recall from sec. 4.4 that a subalgebra F of a Lie algebra E is called reductive in E if the adjoint representation of F in E is semisimple. In this case the induced representations of F in $\wedge E^*$, $\vee E^*$, and $W(E)$ are semisimple (cf. Theorem III, sec. 4.4).

A Lie algebra pair (E, F) will be called a *reductive pair* if E is reductive and F is reductive in E.

Let (E, F) be a Lie algebra pair. Then the inclusion map $j: F \to E$ induces a homomorphism of graded algebras

$$j^\wedge: \wedge F^* \leftarrow \wedge E^*.$$

The kernel of $j^\wedge$ is the ideal $I_{F^\perp}$ generated by the orthogonal complement $F^\perp$ of F in E^*.

Since j is a homomorphism of Lie algebras, $j^\wedge$ restricts to a homomorphism

$$j^\wedge_{\theta=0}: (\wedge F^*)_{\theta=0} \leftarrow (\wedge E^*)_{\theta=0}$$

(cf. sec. 5.6). Moreover, $j^\wedge \circ \delta_E = \delta_F \circ j^\wedge$; i.e., $j^\wedge$ is a homomorphism of differential algebras. Hence it induces a homomorphism of cohomology algebras

$$j^\#: H^*(F) \leftarrow H^*(E),$$

and the diagram

$$\begin{tikzcd} (\wedge F^*)_{\theta=0} \arrow[d] & (\wedge E^*)_{\theta=0} \arrow[l, "j^\wedge_{\theta=0}"'] \arrow[d] \\ H^*(F) & H^*(E) \arrow[l, "j^\#"] \end{tikzcd}$$

commutes.

If E(respectively, F) is reductive, then the appropriate vertical arrow in the diagram above is an isomorphism (cf. Theorem I, sec. 5.12). If both E and F are reductive, then $j^{\wedge}_{\theta=0} = \wedge j_P$, where P_E, P_F are the primitive subspaces of $(\wedge E^*)_{\theta=0}$, $(\wedge F^*)_{\theta=0}$ (cf. sec. 5.19) and $j_P\colon P_E \to P_F$ is the restriction of $j^{\wedge}_{\theta=0}$.

Finally, j induces a homomorphism

$$j^{\vee}\colon \vee F^* \leftarrow \vee E^*$$

which restricts to a homomorphism

$$j^{\vee}_{\theta=0}\colon (\vee F^*)_{\theta=0} \leftarrow (\vee E^*)_{\theta=0}.$$

10.2. The operation associated with a pair. Let (E, F) be a Lie algebra pair. Then an operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ of F in the graded differential algebra $(\wedge E^*, \delta_E)$ is defined (cf. Example 4, sec. 7.4). The invariant, horizontal, and basic subalgebras for this operation will be denoted $(\wedge E^*)_{\theta_F=0}$, $(\wedge E^*)_{i_F=0}$, and $(\wedge E^*)_{i_F=0,\theta_F=0}$, respectively.

We shall abuse notation, and denote the cohomology algebra

$$H((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$$

by $H(E/F)$. (Observe that E/F is not equipped with a differential operator!)

Consider the inclusions

$$k\colon (\wedge E^*)_{i_F=0,\theta_F=0} \to \wedge E^*, \qquad e\colon (\wedge E^*)_{i_F=0,\theta_F=0} \to (\wedge E^*)_{\theta_F=0},$$

and

$$i\colon (\wedge E^*)_{\theta_F=0} \to \wedge E^*;$$

they are all homomorphisms of graded differential algebras. Moreover, $k = i \circ e$, and so the diagram

$$\begin{tikzcd} H(E/F) \arrow[d, "e^\#"'] \arrow[dr, "k^\#"] & \\ H((\wedge E^*)_{\theta_F=0}, \delta_E) \arrow[r, "i^\#"'] & H^*(E) \end{tikzcd}$$

commutes.

Note that if F is reductive in E, then $i^\#$ is an isomorphism, as follows from Proposition I, sec. 7.3.

10.3. Fibre projection. Let (E, F) be a Lie algebra pair with F reductive. Then (cf. sec. 7.8 and sec. 7.10) the structure homomorphism

$$\gamma_{E/F}\colon \wedge E^* \to \wedge E^* \otimes \wedge F^*,$$

and the fibre projection

$$\bar{\varrho}\colon H((\wedge E^*)_{\theta_F=0}) \to (\wedge F^*)_{\theta=0}$$

(for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$) are defined.

On the other hand, let $\alpha_\wedge\colon \wedge E^* \to \wedge E^* \otimes \wedge F^*$ be the homomorphism extending the linear map $\alpha\colon E^* \to E^* \oplus F^*$ defined by

$$\alpha(x^*) = x^* \oplus j^*x^* = x^* \otimes 1 + 1 \otimes j^*x^*, \qquad x^* \in E^*.$$

Let

$$j^\#_{\theta_F=0}\colon H((\wedge E^*)_{\theta_F=0}) \to (\wedge F^*)_{\theta=0}$$

be the homomorphism induced by j.

Proposition I: With the hypotheses and notation above,

(1) $\gamma_{E/F} = \alpha_\wedge$, and

(2) $\bar{\varrho} = j^\#_{\theta_F=0}$.

Proof: (1) It is immediate from the definition that $\alpha_\wedge$ and $\gamma_{E/F}$ agree in E^*; since they are both homomorphisms, they must coincide.

(2) Let $q\colon \wedge E^* \otimes \wedge F^* \to \wedge F^*$ be the projection. Then $q \circ \alpha_\wedge = j^\wedge$. Now Proposition VII, sec. 7.11, yields

$$\bar{\varrho} = q^\#_{\theta_F=0} \circ (\gamma_{E/F})^\#_{\theta_F=0} = j^\#_{\theta_F=0}.$$

Q.E.D.

Corollary I: The diagram

$$\begin{tikzcd}
(\wedge E^*)_{\theta=0} \arrow[d] \arrow[dr, "j^\wedge_{\theta=0}"] & \\
H((\wedge E^*)_{\theta_F=0}) \arrow[r, "\bar{\varrho}"] \arrow[d, "i^*"'] & (\wedge F^*)_{\theta=0} \arrow[d, "\cong"'] \\
H^*(E) \arrow[r, "j^*"'] & H^*(F)
\end{tikzcd}$$

commutes.

Corollary II: If (E, F) is a reductive pair, then the Samelson subspace of the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ is the image of the map $j_P\colon P_E \to P_F$.

Proof: In view of Corollary I and sec. 5.19, we have the commutative diagram

$$\begin{array}{ccc} H((\wedge E^*)_{\theta_F=0}) & \xrightarrow{\bar{\varrho}} & (\wedge F^*)_{\theta=0} \\ \downarrow \cong & & \cong \downarrow \\ \wedge P_E & \xrightarrow[\wedge j_P]{} & \wedge P_F . \end{array}$$

It follows that

$$P_F \cap \operatorname{Im} \bar{\varrho} = P_F \cap \operatorname{Im} \wedge j_P = \operatorname{Im} j_P .$$

Q.E.D.

10.4. The Samelson subspace for a pair. Let (E, F) be a Lie algebra pair with E reductive and let P_E be the primitive subspace of $(\wedge E^*)_{\theta=0}$. We shall identify $H^*(E)$ with the exterior algebra $\wedge P_E$ under the canonical isomorphism $\varkappa_E^\#$ (cf. sec. 5.18). Then $\operatorname{Im} k^\#$ is a graded subalgebra of $\wedge P_E$.

Definition: The graded subspace $\hat{P}$ of P_E given by

$$\hat{P} = \operatorname{Im} k^\# \cap P_E$$

is called the *Samelson subspace for the pair* (E, F). A *Samelson complement for* (E, F) is a graded subspace $\tilde{P} \subset P_E$ such that

$$P_E = \tilde{P} \oplus \hat{P}.$$

Observe that the Samelson subspace for *the pair* (E, F) is a subspace of P_E while (if F is reductive) the Samelson subspace *for the operation of F in $\wedge E^*$* is a subspace of P_F!

Theorem I (Samelson): Let (E, F) be a Lie algebra pair with E reductive. Then the subalgebra $\operatorname{Im} k^\# \subset H^*(E)$ is the exterior algebra over the Samelson subspace of the pair (E, F):

$$\operatorname{Im} k^\# = \wedge \hat{P}.$$

Proof: Fix $a \in (\wedge^p E)_{\theta=0}$. Formula (5.8), sec. 5.4, implies that

$$i_E(a)\delta_E + (-1)^{p-1}\delta_E i_E(a) = 0.$$

Hence, $i_E(a)$ induces an operator $i_E(a)^{\#}$ in $H^*(E)$.

Moreover, clearly $i_E(a)$ commutes or anticommutes with the operators $\theta_E(x)$ and $i_E(x)$, $x \in E$. It follows that $(\wedge E^*)_{i_F=0,\theta_F=0}$ is stable under $i_E(a)$. Hence $i_E(a)$ induces an operator $i_{E/F}(a)^{\#}$ in $H(E/F)$.

By definition, $k \circ i_E(a) = i_E(a) \circ k$, $a \in (\wedge E)_{\theta=0}$, and so

$$k^{\#} \circ i_{E/F}(a)^{\#} = i_E(a)^{\#} \circ k^{\#}, \qquad a \in (\wedge E)_{\theta=0}.$$

On the other hand, it follows from Lemma IX, sec. 5.22, that $i_E(a)^{\#} = i_P(a)$, $a \in P_*(E)$. Hence

$$k^{\#} \circ i_{E/F}(a)^{\#} = i_P(a) \circ k^{\#}, \qquad a \in P_*(E)$$

and so Im $k^{\#}$ is a subalgebra of $\wedge P_E$, stable under the operators $i_P(a)$, $a \in P_*(E)$.

Since $P_*(E)$ is dual to P_E, it follows from Proposition I, sec. 0.4 that

$$\operatorname{Im} k^{\#} = \wedge(\operatorname{Im} k^{\#} \cap P_E) = \wedge\hat{P}.$$

Q.E.D.

10.5. Algebraic connections. Let (E, F) be a Lie algebra pair and consider the associated operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. Recall from sec. 8.1 that an algebraic connection for this operation is a linear map $\chi: F^* \to E^*$ which satisfies the conditions

(1) $i_F(y) \circ \chi = \chi \circ i(y)$, and
(2) $\theta_F(y) \circ \chi = \chi \circ \theta(y)$, $y \in F$.

Let χ be an algebraic connection. Then the dual map

$$\chi^*: E \to F$$

is a projection onto F, and the kernel of χ^* is stable under the operators $\mathrm{ad}_E y$, $y \in F$ (cf. Example 4, sec. 8.1). Moreover, according to that example, the correspondence $\chi \mapsto \ker \chi^*$ is a bijection between algebraic connections for the operation, and F-stable complements for F in E.

In particular, if F is reductive in E, then the operation always admits a connection.

Proposition II: The curvature $\not\chi$ of an algebraic connection χ is given by

$$\langle \not\chi y^*, x_1 \wedge x_2 \rangle = \langle y^*, [\chi^* x_1, \chi^* x_2] - \chi^*[x_1, x_2] \rangle, \qquad y^* \in F^*, x_1, x_2 \in E.$$

In particular, $\not\chi = 0$ if and only if χ^* is a Lie algebra homomorphism.

Proof: In fact,

$$\langle \delta_E \chi y^*, x_1 \wedge x_2 \rangle = -\langle y^*, \chi^*[x_1, x_2] \rangle$$

and

$$\langle \chi_\wedge \delta_F y^*, x_1 \wedge x_2 \rangle = -\langle y^*, [\chi^* x_1, \chi^* x_2] \rangle.$$

Subtracting these relations yields the proposition.

Q.E.D.

Corollary I: Let $x_1 \in \ker \chi^*$ and $x_2 \in E$. Then

$$\langle \not\chi y^*, x_1 \wedge x_2 \rangle = -\langle \chi y^*, [x_1, x_2] \rangle.$$

Corollary II: The dual map $\not\chi^*: \wedge^2 E \to F$ is given by

$$\not\chi^*(x_1 \wedge x_2) = [\chi^* x_1, \chi^* x_2] - \chi^*[x_1, x_2].$$

Corollary III: The operation of F in $\wedge E^*$ admits a connection with zero curvature if and only if F is complemented by an ideal in E.

Again, assume χ is an algebraic connection. Set $\ker \chi^* = S$. Then the scalar product between E^* and E restricts to a scalar product between $F^\perp$ and S. Since $(\wedge E^*)_{i_F=0} = \wedge F^\perp$, it follows that the scalar product between $\wedge E^*$ and $\wedge E$ restricts to a scalar product between $(\wedge E^*)_{i_F=0}$ and $\wedge S$.

The induced (algebra) isomorphism $\wedge S^* \xrightarrow{\cong} (\wedge E^*)_{i_F=0}$ is F-linear and so restricts to an isomorphism between the invariant subalgebras. Thus the algebraic connection determines isomorphisms

$$(\wedge E^*)_{i_F=0} \xrightarrow{\cong} \wedge S^* \qquad \text{and} \qquad (\wedge E^*)_{i_F=0,\theta_F=0} \xrightarrow{\cong} (\wedge S^*)_{\theta_F=0}.$$

Under the first isomorphism the curvature $\not\chi$ of χ corresponds to the linear map

$$\not\chi_S: F^* \to \wedge^2 S^*$$

given by

$$\langle \chi_S y^*, x_1 \wedge x_2 \rangle = -\langle \chi y^*, [x_1, x_2] \rangle, \qquad y^* \in F^*, \quad x_1, x_2 \in S,$$

as follows from Corollary I to Proposition II.

Finally let $e_1, \ldots, e_m$ be a basis for F and let $e_{m+1}, \ldots, e_n$ be a basis for S. Let $e^{*1}, \ldots, e^{*n}$ be the dual basis of E^*. Then, according to Example 2, sec. 8.9, the curvature is given by

$$\chi = \tfrac{1}{2} \sum_{\varrho=m+1}^{n} \mu(e^{*\varrho}) \circ \theta_E(e_\varrho) \circ \chi.$$

10.6. The cohomology sequence. Let F be a reductive subalgebra of a Lie algebra E and assume that the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ admits an algebraic connection. Then the Weil homomorphism

$$\chi^\#: (\vee F^*)_{\theta=0} \to H(E/F)$$

is defined (cf. sec. 8.15) and is independent of the choice of connection (cf. Theorem V, sec. 8.20).

The sequence of homomorphisms

$$(\vee E^*)_{\theta=0} \xrightarrow{j^\vee_{\theta=0}} (\vee F^*)_{\theta=0} \xrightarrow{\chi^\#} H(E/F) \xrightarrow{k^\#} H^*(E) \xrightarrow{j^\#} H^*(F)$$

is called the *cohomology sequence for the pair* (E, F). If F is reductive in E, then

$$H^*(E) = H((\wedge E^*)_{\theta_F=0}), \qquad k^\# = e^\#, \qquad \text{and} \qquad j^\# = \bar{\varrho}$$

(cf. sec. 10.2 and sec. 10.3). Thus in this case the last part of the cohomology sequence of the pair coincides with the cohomology sequence for the operation of F in $(\wedge E^*, \delta_E)$ (cf. sec. 8.21).

Proposition III: The cohomology sequence of a reductive pair (E, F) has the following properties:

(1) The image of $(j^\vee_{\theta=0})^+$ generates the kernel of $\chi^\#$.

(2) The image of $(\chi^\#)^+$ is contained in the kernel of $k^\#$.

(3) The image of $k^\#$ is an exterior algebra over the Samelson subspace $\hat{P}$ and the image of $(k^\#)^+$ is contained in the kernel of $j^\#$.

(4) The image of $j^\#$ is an exterior algebra over a graded subspace of P_F.

Proof: (1) and (2) are proved in Corollary III to Theorem III, sec. 10.8. (3) is proved in Theorem I, sec. 10.4 (the last part of (3) is obvious). (4) follows from Corollary I to Theorem III, sec. 5.19.

Q.E.D.

Remark: In Theorem VII, sec. 10.16, we shall give necessary and sufficient conditions for the image of $(\chi^\#)^+$ to generate the kernel of $k^\#$. Theorem X, sec. 10.19, contains necessary and sufficient conditions for the image of $(k^\#)^+$ to generate the kernel of $j^\#$.

10.7. The bundle diagram. Let (E, F) be a Lie algebra pair with F reductive in E. Let χ be an algebraic connection for the operation of F in $(\wedge E^*, \delta_E)$, with curvature homomorphism

$$(\chi_\vee)_{\theta=0}\colon (\vee F^*)_{\theta=0} \to (\wedge E^*)_{i_F=0,\theta_F=0}$$

(cf. sec. 8.15).

Fix a transgression $\nu\colon P_F \to (\vee F^*)_{\theta=0}$ (cf. sec. 6.13) and consider the linear map

$$\nu_{E/F}\colon P_F \to (\wedge E^*)_{i_F=0,\theta_F=0}$$

given by $\nu_{E/F} = (\chi_\vee)_{\theta=0} \circ \nu$. Then the triple $((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E; \nu_{E/F})$ is the (P_F, δ)-algebra associated with the operation via the connection χ (cf. sec. 9.3).

Its Koszul complex will be denoted by $((\wedge E^*)_{i_F=0,\theta_F=0} \otimes \wedge P_F, \nabla_{E/F})$; thus

$$\begin{aligned}\nabla_{E/F}(z \otimes \Phi_0 \wedge \cdots \wedge \Phi_p) &\\ = \delta_E z \otimes \Phi_0 \wedge \cdots \wedge \Phi_p &\\ +(-1)^q \sum_{j=0}^{p} (-1)^j \nu_{E/F}(\Phi_j) \wedge z \otimes \Phi_0 \wedge \cdots \hat{\Phi}_j \cdots \wedge \Phi_p, &\\ z \in (\wedge^q E^*)_{i_F=0,\theta_F=0}, \quad \Phi_j \in P_F.&\end{aligned}$$

The cohomology sequence of this (P_F, δ)-algebra (cf. sec. 3.14) will be written

$$\vee P_F \xrightarrow{(\nu_{E/F})^\#_\vee} H(E/F) \xrightarrow{l^\#_{E/F}} H((\wedge E^*)_{i_F=0,\theta_F=0} \otimes \wedge P_F) \xrightarrow{\varrho^\#_{E/F}} \wedge P_F.$$

Next, recall that in sec. 9.3 we defined the Chevalley homomorphism

$$\vartheta\colon ((\wedge E^*)_{i_F=0,\theta_F=0} \otimes \wedge P_F, \nabla_{E/F}) \to ((\wedge E^*)_{\theta_F=0}, \delta_E).$$

Composing ϑ with the inclusion $i\colon (\wedge E^*)_{\theta_F=0} \to \wedge E^*$ yields a homomorphism

$$\vartheta_{E/F}\colon ((\wedge E^*)_{i_F=0,\,\theta_F=0} \otimes \wedge P_F, \nabla_{E/F}) \to (\wedge E^*, \delta_E)$$

of graded differential algebras.

Theorem II: Let (E, F) be a Lie algebra pair with F reductive in E. Let $\vartheta_{E/F}$ be the homomorphism above. Then

(1) $\vartheta_{E/F}^{\#}\colon H((\wedge E^*)_{i_F=0,\,\theta_F=0} \otimes \wedge P_F) \xrightarrow{\cong} H^*(E)$ is an isomorphism of graded algebras.

(2) The *bundle diagram*

$$\begin{tikzcd}
\vee P_F \arrow[r, "\nu(E/F)^{\#}_{\vee}"] \arrow[d, "\nu_{\vee}"', "\cong"] & H(E/F) \arrow[r, "l^{\#}_{E/F}"] \arrow[d, "\cong"', "\iota"] & H((\wedge E^*)_{i_F=0,\,\theta_F=0} \otimes \wedge P_F) \arrow[r, "\varrho^{\#}_{E/F}"] \arrow[d, "\cong"', "\vartheta^{\#}_{E/F}"] & \wedge P_F \arrow[d, "\cong"', "\kappa^{\#}_F"] \\
(\vee F^*)_{\theta=0} \arrow[r, "\chi^{\#}"'] & H(E/F) \arrow[r, "k^{\#}"'] & H^*(E) \arrow[r, "j^{\#}"'] & H^*(F)
\end{tikzcd}$$

commutes.

Proof: (1) Theorem I, sec. 9.3, shows that $\vartheta^{\#}$ is an isomorphism. Since F is reductive in E, $i^{\#}$ is an isomorphism. Hence so is $\vartheta_{E/F}^{\#}$.

(2) Apply Theorem II, sec. 9.7, and Proposition I, sec. 10.3.

Q.E.D.

Remark: The bundle diagram identifies the last part of the cohomology sequence for (E, F) with the cohomology sequence of the associated (P_F, δ)-algebra.

§2. The cohomology of $(\wedge E^*)_{i_F=0,\theta_F=0}$

10.8. The base diagram. Let (E, F) be a reductive Lie algebra pair with inclusion $j: F \to E$. Fix a transgression $\tau: P_E \to (\vee E^*)_{\theta=0}$ (cf. sec. 6.13). Define a linear map

$$\sigma: P_E \to (\vee F^*)_{\theta=0},$$

homogeneous of degree 1, by $\sigma = j^\vee_{\theta=0} \circ \tau$. Then $((\vee F^*)_{\theta=0}; \sigma)$ is a P_E-algebra (cf. sec. 2.4).

Definition: The pair $((\vee F^*)_{\theta=0}; \sigma)$ is called the *P_E-algebra associated with the pair (E, F) via τ.*

The Koszul complex for this P_E-algebra is given by $((\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla_\sigma)$, where

$$\nabla_\sigma(\Psi \otimes \Phi_0 \wedge \cdots \wedge \Phi_p) = \sum_{j=0}^{p} (-1)^j \Psi \vee \sigma(\Phi_j) \otimes \Phi_0 \wedge \cdots \widehat{\Phi_j} \cdots \wedge \Phi_p,$$
$$\Psi \in (\vee F^*)_{\theta=0}, \quad \Phi_j \in P_E.$$

It is called the *Koszul complex for the pair (E, F) determined by τ.* The cohomology sequence for $((\vee F^*)_{\theta=0}; \sigma)$ (cf. sec. 2.14) will be written

$$\vee P_E \xrightarrow{\sigma_\vee} (\vee F^*)_{\theta=0} \xrightarrow{l^*} H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \xrightarrow{\varrho^*} \wedge P_E.$$

Now we state the main theorem of this article, whose proof will be given in the next three sections.

Theorem III: Let (E, F) be a reductive pair and let τ be a transgression in $W(E)_{\theta=0}$. Then there is a homomorphism of graded differential algebras

$$\varphi: ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$$

with the following properties:

(1) φ induces an isomorphism,

$$\varphi^\#: H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \xrightarrow{\cong} H(E/F)$$

of graded algebras.

(2) The *base diagram*

$$\begin{tikzcd}
\vee P_E \arrow[r, "\sigma_\vee"] \arrow[d, "\tau_\vee"', "\cong"] & (\vee F^*)_{\theta=0} \arrow[r, "l^\#"] \arrow[d, "\cong"', "\iota"] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^\#"] \arrow[d, "\cong"', "\varphi^\#"] & \wedge P_E \arrow[d, "\cong"', "\varkappa_E^\#"] \\
(\vee E^*)_{\theta=0} \arrow[r, "j^\vee_{\theta=0}"'] & (\vee F^*)_{\theta=0} \arrow[r, "\chi^\#"'] & H(E/F) \arrow[r, "k^\#"'] & H^*(E)
\end{tikzcd}$$

commutes.

Remark: Theorem III shows that the base diagram identifies the first part of the cohomology sequence of (E, F) with the cohomology sequence of the associated P_E-algebra (cf. sec. 10.6).

Corollary I: The Samelson subspace $\hat{P}$ for the pair (E, F) (cf. sec. 10.4) coincides with the Samelson subspace for the P_E-algebra $((\vee F^*)_{\theta=0}; \sigma)$ (cf. sec. 2.13).

Corollary II: An element $\Phi \in P_E$ is in $\hat{P}$ if and only if

$$j^\vee_{\theta=0}(\tau\Phi) \in j^\vee_{\theta=0}(\tau P_E) \vee \vee^+(F^*)_{\theta=0}.$$

Proof: Since $\sigma = j^\vee_{\theta=0} \circ \tau$, Proposition IV, sec. 2.13, shows that the equation above characterizes the Samelson subspace for $((\vee F^*)_{\theta=0}; \sigma)$. Now apply Corollary I.

Q.E.D.

Corollary III: (1) The image of $(j^\vee_{\theta=0})^+$ generates the kernel of $\chi^\#$.
(2) The image of $(\chi^\#)^+$ is contained in the kernel of $k^\#$.

Proof: Apply Proposition V, sec. 2.14, to $((\vee F^*)_{\theta=0}; \sigma)$.

Q.E.D.

The homomorphism φ of Theorem III will be constructed as follows. In sec. 10.9 we shall apply the results of sec. 8.17, sec. 8.18 and sec. 8.19 to the operation of F in $\wedge E^*$. This will yield a graded differential algebra $((\vee F^* \otimes \wedge E^*)_{\theta_F=0}, D)$ and a homomorphism,

$$\alpha_\chi: ((\vee F^* \otimes \wedge E^*)_{\theta_F=0}, D) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$$

of graded differential algebras, such that $\alpha_\chi^\#$ is an isomorphism.

Next, in sec. 10.10, we shall construct a homomorphism of graded differential algebras

$$\vartheta\colon ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\vee F^* \otimes \wedge E^*)_{\theta_F=0}, D)$$

such that $\vartheta^\#$ is an isomorphism. Finally, φ will be defined by $\varphi = \alpha_\chi \circ \vartheta$.

10.9. The Weil algebra of a pair. The graded algebra

$$W(E, F) = \vee F^* \otimes \wedge E^*$$

will be called the *Weil algebra of the pair* (E, F). Thus in particular, $W(E, E) = W(E)$ and $W(E, 0) = \wedge E^*$.

A representation (again denoted by θ_F) of F in the algebra $W(E, F)$ is given by

$$\theta_F(y) = \theta(y) \otimes \iota + \iota \otimes \theta_F(y), \qquad y \in F.$$

The corresponding invariant subalgebra will be denoted by $W(E, F)_{\theta_F=0}$.

Now define an operator h_F in $W(E, F)$ by

$$h_F(\Psi \otimes x_0^* \wedge \cdots \wedge x_p^*) = \sum_{i=0}^{p} (-1)^i j^*(x_i^*) \vee \Psi \otimes x_0^* \wedge \cdots \widehat{x_i^*} \cdots \wedge x_p^*$$

and

$$h_F(\Psi \otimes 1) = 0, \qquad \Psi \in \vee F^*, \qquad x_i^* \in E^*.$$

Then h_F is an antiderivation, homogeneous of degree 1. It satisfies the relation

$$h_F \circ \theta_F(y) = \theta_F(y) \circ h_F, \qquad y \in F,$$

and so it restricts to an operator in $W(E, F)_{\theta_F=0}$. In terms of a pair of dual bases for E^* and E we can write $h_F = \sum_\nu \mu(j^* e^{*\nu}) \otimes i(e_\nu)$.

Denote the operator $\iota \otimes \delta_E$ in $W(E, F)$ simply by δ_E, and define an operator D in $W(E, F)$ by

$$D = \delta_E - h_F.$$

Then according to sec. 8.17 (applied to the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$) the pair $(W(E, F)_{\theta_F=0}, D)$ is a graded differential algebra.

Next, consider the inclusion map,

$$\varepsilon\colon (\wedge E^*)_{i_F=0,\theta_F=0} \to W(E, F)_{\theta_F=0}$$

given by $\varepsilon(\Phi) = 1 \otimes \Phi$. According to Theorem IV, sec. 8.17, $\varepsilon^\#$ is an isomorphism.

Finally, let χ be an algebraic connection for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. Then a homomorphism of graded differential algebras

$$\alpha_\chi\colon (W(E, F)_{\theta_F=0}, D) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$$

is given by

$$\alpha_\chi(\Psi \otimes \Phi) = (\chi_\vee \Psi) \wedge (\pi_H \Phi), \qquad \Psi \in \vee F^*, \quad \Phi \in \wedge E^*,$$

(cf. sec. 8.19). By Proposition IX, sec. 8.19, $\alpha_\chi^\#$ is the isomorphism inverse to $\varepsilon^\#$.

Moreover (cf. Corollary I to Theorem V, sec. 8.20), the diagram

$$\begin{tikzcd}
(\vee F^*)_{\theta=0} \arrow[r, "\xi^\#"] \arrow[dr, "\chi^\#"'] & H(W(E, F)_{\theta_F=0}) \arrow[r, "\eta^\#"] \arrow[d, shift left, "\alpha_\chi^\#"] & H((\wedge E^*)_{\theta_F=0}) \arrow[d, "i^\#", "\cong"'] \\
 & H(E/F) \arrow[u, shift left, "\varepsilon^\#", "\cong"'] \arrow[ur, "e^\#"] \arrow[r, "k^\#"'] & H^*(E)
\end{tikzcd} \tag{10.1}$$

commutes. (Here $\xi\colon (\vee F^*)_{\theta=0} \to W(E, F)_{\theta_F=0}$ is the inclusion and $\eta\colon W(E, F)_{\theta_F=0} \to (\wedge E^*)_{\theta_F=0}$ is the projection.)

10.10. The Chevalley homomorphism. Recall from sec. 10.8 that $\tau\colon P_E \to (\vee E^*)_{\theta=0}$ denotes a transgression in $W(E)_{\theta=0}$. Hence there is a linear map $\alpha\colon P_E \to W(E)_{\theta=0}$, homogeneous of degree zero, such that

$$\alpha(\Phi) - 1 \otimes \Phi \in (\vee^+ E^* \otimes \wedge E^*)_{\theta=0} \tag{10.2}$$

and

$$\delta_W(\alpha(\Phi)) = \tau(\Phi) \otimes 1, \qquad \Phi \in P_E. \tag{10.3}$$

Extend α to a homomorphism $\alpha_\wedge\colon \wedge P_E \to W(E)_{\theta=0}$ and observe that $j^\vee \otimes \iota$ restricts to a homomorphism

$$j^\vee \otimes \iota\colon W(E)_{\theta=0} \to W(E, F)_{\theta_F=0}.$$

Now define a homomorphism of graded algebras,

$$\vartheta\colon (\vee F^*)_{\theta=0} \otimes \wedge P_E \to W(E, F)_{\theta_F=0},$$

by

$$\vartheta(\Psi \otimes \Phi) = (\Psi \otimes 1) \cdot (j^\vee \otimes \iota)(\alpha_\wedge \Phi), \qquad \Psi \in (\vee F^*)_{\theta=0}, \quad \Phi \in \wedge P_E.$$

We show that ϑ is a homomorphism of differential algebras:

$$\vartheta \circ (-\nabla_\sigma) = D \circ \vartheta. \tag{10.4}$$

In fact, both sides give zero when applied to $\Psi \otimes 1$ ($\Psi \in (\vee F^*)_{\theta=0}$). On the other hand, if $\Phi \in P_E$, then

$$\vartheta \circ (-\nabla_\sigma)(1 \otimes \Phi) = -j^\vee_{\theta=0}(\tau\Phi) \otimes 1,$$

while

$$D \circ \vartheta(1 \otimes \Phi) = (D \circ (j^\vee \otimes \iota))(\alpha\Phi).$$

Since (cf. sec. 6.4) δ_W reduces to $h - \delta_E$ in $W(E)_{\theta=0}$, it follows that

$$D \circ (j^\vee \otimes \iota) = -(j^\vee \otimes \iota) \circ \delta_W.$$

Hence (cf. formula (10.3))

$$D \circ \vartheta(1 \otimes \Phi) = -(j^\vee \otimes \iota)(\tau\Phi \otimes 1) = \vartheta \circ (-\nabla_\sigma)(1 \otimes \Phi).$$

This shows that $D \circ \vartheta$ agrees with $\vartheta \circ (-\nabla_\sigma)$ in $1 \otimes P_E$; since these operators also agree in $(\vee F^*)_{\theta=0} \otimes 1$, and since they are ϑ-antiderivations, they must coincide. This proves formula (10.4).

The homomorphism ϑ is called *the Chevalley homomorphism for the pair* (E, F) (determined by τ and α).

Proposition IV: The Chevalley homomorphism ϑ has the following properties:

(1) $\vartheta^\#$ is an isomorphism.

(2) The diagram

$$\begin{tikzcd}
& H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^\#"] \arrow[d, "\cong"', "\vartheta^\#"] & \wedge P_E \arrow[d, "\cong"] \\
(\vee F^*)_{\theta=0} \arrow[ur, "\iota^\#"] \arrow[r, "\xi^\#"'] & H(W(E, F)_{\theta_F=0}) \arrow[r, "\eta^\#"'] & H((\wedge E^*)_{\theta_F=0})
\end{tikzcd}$$

commutes. ($\xi^\#$ and $\eta^\#$ are defined in sec. 10.9.)

Proof: (1) Filter the algebras $(\vee F^*)_{\theta=0} \otimes \wedge P_E$ and $W(E, F)_{\theta_F=0}$ respectively by the ideals

$$I^p = \sum_{j \geq p} (\vee F^*)^j_{\theta=0} \otimes \wedge P_E \qquad \text{and} \qquad \hat{I}^p = \sum_{j \geq p} ((\vee F^*)^j \otimes \wedge E^*)_{\theta_F=0}.$$

Then ϑ is filtration preserving and so induces a homomorphism $\vartheta_i: (E_i, d_i) \to (\hat{E}_i, \hat{d}_i)$ of spectral sequences.

In view of sec. 1.7, we have

$$(E_0, d_0) = ((\vee F^*)_{\theta=0} \otimes \wedge P_E, 0) \quad \text{and} \quad (\hat{E}_0, \hat{d}_0) = (W(E, F)_{\theta=0}, \delta_E).$$

Moreover, formula (10.2) implies that ϑ_0 is simply the inclusion map

$$\vartheta_0(\Psi \otimes \Phi) = \Psi \otimes \Phi, \qquad \Psi \in (\vee F^*)_{\theta=0}, \quad \Phi \in \wedge P_E.$$

(Identify $\wedge P_E$ with $(\wedge E^*)_{\theta=0}$ via the isomorphism $\varkappa_E$ of Theorem III, sec. 5.18.) Now Lemma I, below, implies that $\vartheta_0^\#$ is an isomorphism.

It follows that ϑ_1 is an isomorphism, and hence so is $\vartheta^\#$ (cf. Theorem I, sec. 1.14).

(2) Evidently $\vartheta \circ l = \xi$ and so $\vartheta^\# \circ l^\# = \xi^\#$. Moreover, if $\lambda: \wedge P_E \to (\wedge E^*)_{\theta_F=0}$ is the inclusion, then formula (10.2) shows that $\lambda \circ \varrho = \eta \circ \vartheta$. Hence $\lambda^\# \circ \varrho^\# = \eta^\# \circ \vartheta^\#$.

Q.E.D.

Lemma I: The inclusion

$$\vartheta_0: ((\vee F^*)_{\theta=0} \otimes \wedge P_E, 0) \to ((\vee F^* \otimes \wedge E^*)_{\theta_F=0}, \delta_E)$$

induces an isomorphism of cohomology.

Proof: Observe that ϑ_0 is the composite of two inclusions

$$\lambda_1: (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (\vee F^*)_{\theta=0} \otimes (\wedge E^*)_{\theta_F=0},$$

and

$$\lambda_2: (\vee F^*)_{\theta=0} \otimes (\wedge E^*)_{\theta_F=0} \to (\vee F^* \otimes \wedge E^*)_{\theta_F=0}.$$

Since the representations of F in $\vee F^*$ and $\wedge E^*$ are semisimple (because F is reductive in E) Proposition I, sec. 7.3, and Theorem V, sec. 4.11, imply, respectively, that $\lambda_1^\#$ and $\lambda_2^\#$ are isomorphisms. Hence so is $\vartheta_0^\#$.

Q.E.D.

10.11. Proof of Theorem III: Composing the Chevalley homomorphism

$$\vartheta: (\vee F^*)_{\theta=0} \otimes \wedge P_E \to W(E, F)_{\theta_F=0}$$

(cf. sec. 10.10) with the homomorphism

$$\alpha_\chi: W(F, E)_{\theta_F=0} \to (\wedge E^*)_{i_F=0, \theta_F=0}$$

(cf. sec. 10.9), we obtain a homomorphism of graded differential algebras

$$\varphi\colon ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E).$$

This homomorphism has the properties stated in Theorem III.

In fact, since $\alpha_\chi^\#$ and $\vartheta^\#$ are isomorphisms, so is $\varphi^\#$. Moreover, combining diagram (10.1) of sec. 10.9 with part (2) of Proposition IV, we see that the right-hand two squares in the diagram of Theorem III commute. The commutativity of the left-hand square follows immediately from the definition of σ.

Q.E.D.

§3. The structure of the algebra $H(E/F)$

In this article we shall study the algebra structure of $H(E/F)$, where (E, F) is a reductive pair, with Samelson subspace $\hat{P}$ and a Samelson complement $\tilde{P}$ (cf. sec. 10.4). In view of Theorem III (sec. 10.8) and the reduction theorem (sec. 2.15) there is a sequence of homomorphisms

$$\wedge\hat{P} \longrightarrow H((\vee F^*)_{\theta=0} \otimes \wedge\tilde{P}) \otimes \wedge\hat{P} \xrightarrow{\cong} H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \xrightarrow{\cong} H(E/F)$$

which imbeds $\wedge\hat{P}$ into $H(E/F)$ as a subalgebra.

On the other hand, we have the characteristic subalgebra

$$\operatorname{Im} \chi^{\#} \subset H(E/F).$$

We shall construct a graded algebra A such that $\operatorname{Im} \chi^{\#} \subset A \subset H(E/F)$ and an isomorphism,

$$A \otimes \wedge\hat{P} \xrightarrow{\cong} H(E/F).$$

10.12. Characteristic factors. A graded subalgebra $A \subset H(E/F)$ will be called a *characteristic factor* if it satisfies the following conditions:

(1) A is the direct sum of $\operatorname{Im} \chi^{\#}$ and a graded ideal I in A.

(2) There is an isomorphism of graded algebras $g: A \otimes \wedge\hat{P} \xrightarrow{\cong} H(E/F)$ ($A \otimes \wedge\hat{P}$ denotes the anticommutative tensor product) which makes the diagram

$$\begin{tikzcd}
 & A \otimes \wedge\hat{P} \arrow[r] \arrow[d, "g", "\cong"'] & \wedge\hat{P} \arrow[d, "\kappa_E^{\#}"] \\
A \arrow[ur] \arrow[dr] & & \\
 & H(E/F) \arrow[r, "k^*"'] & H^*(E)
\end{tikzcd}$$

commute.

Theorem IV: Let (E, F) be a reductive Lie algebra pair. Then $H(E/F)$ contains a characteristic factor. Moreover, if A and B are charac-

teristic factors, then there is an isomorphism $A \xrightarrow{\cong} B$ of graded algebras, which reduces to the identity in Im $\chi^{\#}$ and extends to an automorphism of the graded algebra $H(E/F)$.

Proof: *Existence*: Fix a transgression τ in $W(E)_{\theta=0}$ and consider the linear map

$$\sigma\colon P_E \to (\vee F^*)_{\theta=0}$$

given by $\sigma = j^{\vee}_{\theta=0} \circ \tau$ (cf. sec. 10.8). Let $\hat{P}$ be a Samelson complement for the pair (E, F). Corollary I to Theorem III, sec. 10.8, shows that $\hat{P}$ is also a Samelson complement for the P_E-algebra, $((\vee F^*)_{\theta=0}; \sigma)$ (cf. sec. 2.13). Set $(\vee F^*)_{\theta=0} \otimes \wedge\hat{P} = N$.

Then, by Corollary II to the reduction theorem (sec. 2.15), there is a commutative diagram of algebra homomorphisms

$$\begin{tikzcd}
 & H(N) \otimes \wedge\hat{P} \arrow[r] \arrow[d, "f^{\#}", "\cong"'] & \wedge\hat{P} \arrow[d] \\
(\vee F^*)_{\theta=0} \arrow[ur, "\bar{l}^{\#}"] \arrow[r, "l^{\#}"'] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^{\#}"'] & \wedge P_E
\end{tikzcd}$$

in which $f^{\#}$ is an isomorphism. Combining this with the commutative diagram of Theorem III, sec. 10.8, yields the commutative diagram

$$\begin{tikzcd}
 & H(N) \otimes \wedge\hat{P} \arrow[r] \arrow[d, "g", "\cong"'] & \wedge\hat{P} \arrow[d] \\
(\vee F^*)_{\theta=0} \arrow[ur, "\bar{l}^{\#}"] \arrow[r, "\chi^{\#}"'] & H(E/F) \arrow[r, "k^{\#}"'] & H^*(E)
\end{tikzcd}$$

where $g = \varphi^{\#} \circ f^{\#}$. Define a subalgebra A of $H(E/F)$ by setting $A = g(H(N) \otimes 1)$. Then g and A satisfy condition (2). Moreover, Im $\chi^{\#} \subset A$. To construct the ideal I observe that

$$H(N) = \operatorname{Im} \bar{l}^{\#} \oplus H_+(N)$$

and that $H_+(N)$ is a graded ideal in $H(N)$ (cf. sec. 2.5). Set

$$I = g(H_+(N) \otimes 1).$$

Then I is a graded ideal in A and

$$\begin{aligned} A &= g(\operatorname{Im} \chi^{\#} \otimes 1) \oplus g(H_+(N) \otimes 1) \\ &= \operatorname{Im} \chi^{\#} \oplus I. \end{aligned}$$

Thus A is a characteristic factor.

Uniqueness: Let A and B be two characteristic factors. Then the diagram

$$\begin{tikzcd}
 & & A \otimes \wedge \hat{P} \arrow[dd, "g_A"', "\cong"] \arrow[ddr] & \\
 & A \arrow[ur] \arrow[dr] & & \\
\operatorname{Im} \chi^{\#} \arrow[ur] \arrow[rr] \arrow[dr] & & H(E/F) \arrow[r, "k^{\#}"'] & \wedge \hat{P} \\
 & B \arrow[ur] \arrow[dr] & & \\
 & & B \otimes \wedge \hat{P} \arrow[uu, "g_B"', "\cong"] \arrow[uur] &
\end{tikzcd}$$

commutes. Hence so does the diagram,

$$\begin{tikzcd}
 & A \otimes \wedge \hat{P} \arrow[dd, "\cong"', "g_B^{-1} \circ g_A"] \arrow[dr] & \\
\operatorname{Im} \chi^{\#} \arrow[ur] \arrow[dr] & & \wedge \hat{P}. \\
 & B \otimes \wedge \hat{P} \arrow[ur] &
\end{tikzcd}$$

Now Proposition XI, sec. 7.18, yields an isomorphism $A \xrightarrow{\cong} B$ which reduces to the identity in $\operatorname{Im} \chi^{\#}$. It follows from property (2) for characteristic factors that this isomorphism extends to an automorphism of $H(E/F)$.

Q.E.D.

Corollary: If (E, F) is a reductive pair, then there is a homomorphism of graded algebras $\varphi: H(E/F) \to \operatorname{Im} \chi^{\#}$ which reduces to the identity in $\operatorname{Im} \chi^{\#}$.

Proof: Identify $H(E/F)$ with $A \otimes \wedge \hat{P}$ via g, and let $\pi_1: A \otimes \wedge \hat{P} \to A$ be the projection. Let $\pi_2: A \to \operatorname{Im} \chi^{\#}$ be the projection with kernel I. Then set $\varphi = \pi_2 \circ \pi_1$.

Q.E.D.

§4. Cartan pairs

Let (E, F) be a reductive pair. In view of Theorem IV, sec. 10.12, the following conditions are equivalent:

(1) There is an isomorphism of graded algebras,

$$\operatorname{Im} \chi^{\#} \otimes \wedge \hat{P} \xrightarrow{\cong} H(E/F)$$

which makes the diagram

$$\begin{tikzcd} & \operatorname{Im} \chi^{\#} \otimes \wedge \hat{P} \arrow[dd, "\cong"] \arrow[dr] & \\ \operatorname{Im} \chi^{\#} \arrow[ur] \arrow[dr] & & \wedge \hat{P} \\ & H(E/F) \arrow[ur, "k^{\#}"'] & \end{tikzcd} \tag{10.5}$$

commute.

(2) $\operatorname{Im} \chi^{\#}$ is (the unique) characteristic factor.

(3) $\dim H(E/F) = \dim \operatorname{Im} \chi^{\#} \cdot \dim \wedge \hat{P}$.

In this article we shall derive necessary and sufficient conditions for the pair (E, F) to satisfy the above properties.

10.13. Deficiency number. Let (E, F) be a reductive pair. Then the integer

$$\operatorname{def}(E, F) = \dim P_E - \dim P_F - \dim \hat{P}$$

is called the *deficiency number* for (E, F). (Here P_E and P_F are the primitive subspaces and $\hat{P}$ is the Samelson subspace for the pair (E, F).) If $\operatorname{def}(E, F) = 0$, or equivalently, if

$$\dim P_E = \dim P_F + \dim \hat{P},$$

then (E, F) is called a *Cartan pair.*

Theorem V: Let (E, F) be a reductive pair. Then $\operatorname{def}(E, F) \geq 0$; i.e.,

$$\dim P_E \geq \dim P_F + \dim \hat{P}.$$

Moreover, equality holds if and only if there is an isomorphism of graded algebras $\operatorname{Im} \chi^{\#} \otimes \wedge \hat{P} \xrightarrow{\cong} H(E/F)$ which makes the diagram (10.5) commute.

Corollary I: If (E, F) is a reductive pair, then

$$\dim P_F \leq \dim P_E .$$

Corollary II: Assume (E, F) is a Cartan pair. Then the Euler–Poincaré characteristic of $H(E/F)$ is given by

$$\chi_{H(E/F)} = \begin{cases} 0 & \text{if} \quad \chi^{\#} \text{ is not surjective,} \\ \dim H(E/F) & \text{if} \quad \chi^{\#} \text{ is surjective.} \end{cases}$$

Corollary III: If (E, F) is a Cartan pair, then the cohomology algebra $H(E/F)$ is the tensor product of an exterior algebra (over a space with odd gradation) and the factor algebra of a symmetric algebra (over an evenly graded space).

Proof: Simply observe that $\operatorname{Im} \chi^{\#}$ is a factor algebra of the symmetric algebra $(\vee F^*)_{\theta=0}$ (cf. Theorem 1, sec. 6.13).

Q.E.D.

Corollary IV: Let (E, F) be a Cartan pair. Let I_0 and I_1 denote the ideals in $H^+(E/F)$ generated respectively by the elements of even and odd degrees. Then there is a linear isomorphism of graded vector spaces

$$\hat{P} \xrightarrow{\cong} H^+(E/F)/I_0$$

and an isomorphism of graded algebras

$$\operatorname{Im} \chi^{\#} \xrightarrow{\cong} H(E/F)/I_1 .$$

Remark: In article 5, Chapter XI, it will be shown that a reductive pair is not necessarily a Cartan pair.

10.14. Proof of Theorem V: We wish to apply Theorem VII, sec. 2.17, to the P_E-algebra $((\vee F^*)_{\theta=0}; \sigma)$ defined in sec. 10.8. Observe first that, since F is reductive, Theorem I, sec. 6.13, implies that $(\vee F^*)_{\theta=0} \cong \vee P_F$. Hence $((\vee F^*)_{\theta=0}; \sigma)$ is a symmetric P_E-algebra. Moreover, Theorem III, sec. 10.8, yields the relation

$$\dim H((\vee F^*)_{\theta=0} \otimes \wedge P_E) = \dim H(E/F) < \infty .$$

Thus the hypotheses of Theorem VII are satisfied.

Since the Samelson subspace for the pair (E, F) coincides with the Samelson subspace for $((\vee F^*)_{\theta=0};\ \sigma)$ (cf. Corollary I, sec. 10.8), it follows from Theorem VII, sec. 2.17, that

$$\dim P_E \geq \dim P_F + \dim \hat{P}.$$

To prove the second part of the theorem, observe that the conditions

(1) $\dim P_E = \dim P_F + \dim \hat{P}$

and

(2) $H_+((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}) = 0$ ($\tilde{P}$ a Samelson complement)

are equivalent (cf. Theorem VII, sec. 2.17). Thus we must show that (2) holds if and only if $\operatorname{Im} \chi^\#$ is a characteristic factor for $H(E/F)$.

Write $(\vee F^*)_{\theta=0} \otimes \wedge \tilde{P} = N$. In sec. 10.12 we constructed an isomorphism,

$$g\colon H(N) \otimes \wedge \hat{P} \xrightarrow{\cong} H(E/F)$$

such that $g(H(N) \otimes 1)$ was a characteristic factor and $g(H_0(N) \otimes 1) = \operatorname{Im} \chi^\#$. Hence, if (2) holds, $\operatorname{Im} \chi^\#$ is a characteristic factor.

Conversely, assume that $\operatorname{Im} \chi^\#$ is a characteristic factor. Then

$$\dim H_0(N) = \dim \operatorname{Im} \chi^\# = \frac{\dim H(E/F)}{\dim \wedge \hat{P}} = \dim H(N),$$

and so $H_+(N) = 0$.

Q.E.D.

10.15. Poincaré polynomials. Theorem VI: Let (E, F) be a Cartan pair with Samelson space $\hat{P}$. Let

$$f_{P_E} = \sum_{i=1}^{r} t^{g_i}, \qquad f_{P_F} = \sum_{i=1}^{s} t^{l_i}, \qquad \text{and} \qquad f_{\hat{P}} = \sum_{i=s+1}^{r} t^{g_i}$$

be the Poincaré polynomials for P_E, P_F, and $\hat{P}$.

Then the Poincaré polynomials for $\operatorname{Im} \chi^\#$ and for $H(E/F)$ are given by

$$f_{\operatorname{Im}\chi^\#} = \prod_{i=1}^{s} (1 - t^{g_i+1}) \prod_{i=1}^{s} (1 - t^{l_i+1})^{-1},$$

and

$$f_{H(E/F)} = \prod_{i=1}^{s} (1 - t^{g_i+1}) \prod_{i=1}^{s} (1 - t^{l_i+1})^{-1} \prod_{i=s+1}^{r} (1 + t^{g_i}).$$

Proof: Recall again from Theorem I, sec. 6.13, that $(\vee F^*)_{\theta=0} = \vee P_F$ and observe that the Poincaré polynomial of P_F is given by $f_{P_F} = \sum_{i=1}^{s} t^{l_i+1}$.

Thus, in view of the isomorphism $H(E/F) \cong H((\vee F^*)_{\theta=0} \otimes \wedge P_E)$, (cf. Theorem III, sec. 10.8) the theorem follows at once from the results of sec. 2.20.

Q.E.D.

Corollary: The dimension and the Euler–Poincaré characteristic of $\operatorname{Im} \chi^\#$ are given by

$$\dim \operatorname{Im} \chi^\# = \chi_{\operatorname{Im} \chi^\#} = \prod_{i=1}^{s} (g_i + 1) \prod_{i=1}^{s} (l_i + 1)^{-1}.$$

10.16. In this section we translate Theorem VIII, sec. 2.19, to obtain conditions for a reductive pair to be a Cartan pair.

Let (E, F) be a reductive pair with Samelson subspace $\hat{P}$ and choose a Samelson complement $\tilde{P}$. Let $((\vee F^*)_{\theta=0}; \sigma)$ be the associated P_E-algebra.

Theorem VII: With the notation above the following conditions are equivalent:

(1) (E, F) is a Cartan pair.
(2) The kernel of $k^\#$ coincides with the ideal generated by $\operatorname{Im}(\chi^\#)^+$.
(3) The map $\tilde{l}^\#: (\vee F^*)_{\theta=0} \to H((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P})$ is surjective.
(4) There is an isomorphism of graded algebras,

$$g: \operatorname{Im} \chi^\# \otimes \wedge \hat{P} \xrightarrow{\cong} H(E/F)$$

which makes the diagram (10.5) commute.

(5) There is an isomorphism of graded $\tilde{P}$-spaces

$$h: \operatorname{Im} \chi^\# \otimes \vee \tilde{P} \xrightarrow{\cong} (\vee F^*)_{\theta=0}$$

such that the diagram

$$\begin{tikzcd}
 & \operatorname{Im} \chi^\# \otimes \vee \tilde{P} \arrow[dr] \arrow[dd, "h", "\cong"'] & \\
\vee \tilde{P} \arrow[ur] \arrow[dr, "\sigma_\vee"'] & & \operatorname{Im} \chi^\# \\
 & (\vee F^*)_{\theta=0} \arrow[ur, "\chi^\#"'] &
\end{tikzcd}$$

commutes.

(6) $H_1((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}) = 0.$

(7) $H_+((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}) = 0.$

(8) The characteristic factors of $H(E/F)$ are evenly graded.

(9) The characteristic factors of $H(E/F)$ have nonzero Euler characteristic.

(10) The restriction $\tilde{\sigma}_\vee : \vee \tilde{P} \to (\vee F^*)_{\theta=0}$ of $\sigma_\vee$ to $\vee \tilde{P}$ is injective.

Proof: This is a straightforward translation of Theorem VIII, sec. 2.19, via Theorem III, sec. 10.8, and the proof of Theorem V, sec. 10.13.

Q.E.D.

10.17. Split and c-split pairs. A reductive Lie algebra pair (E, F) will be called *split* (respectively, c-*split*), if the graded differential algebra $((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$ is split (respectively c-split) (cf. sec. 0.10).

Theorem VIII: A reductive pair (E, F) is c-split if and only if it is a Cartan pair.

Proof: In view of Theorem III, sec. 10.8, this is an immediate consequence of Theorem XI, sec. 3.30.

Q.E.D.

Now suppose (E, F) is a Cartan pair, and consider the P_F-algebra $(H(E/F); \nu_{E/F}^\#)$, where

$$\nu_{E/F}^\# = \chi^\# \circ \nu,$$

and $\nu: P_F \to (\vee F^*)_{\theta=0}$ is a transgression (cf. sec. 10.7). Denote its Koszul complex by $(H(E/F) \otimes \wedge P_F, \nabla_{E/F}^\#)$.

Since $((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$ is c-split, we can apply the example of sec. 3.29 to the results of sec. 10.7 to obtain a commutative diagram

$$\begin{tikzcd}
 & H(H(E/F) \otimes \wedge P_F, \nabla_{E/F}^\#) \arrow[r, "\varrho_{H(E/F)}^\#"] \arrow[dd, "\cong"'] & \wedge P_F \arrow[dd, "\cong"', "\kappa_F^\#"] \\
H(E/F) \arrow[ur, "l_{H(E/F)}^\#"] \arrow[dr, "k^*"'] & & \\
 & H^*(E) \arrow[r, "j^*"'] & H^*(F)
\end{tikzcd} \qquad (10.6)$$

in which the vertical arrows are isomorphisms of graded algebras.

§5. Subalgebras noncohomologous to zero

10.18. Definition: Let (E, F) be a Lie algebra pair such that F is reductive in E. Then F will be called *noncohomologous to zero* in E (or simply n.c.z. in E) if the homomorphism,

$$j^{\#}: H^*(E) \to H^*(F)$$

is surjective.

Theorem IX: Let (E, F) be a Lie algebra pair with F reductive in E. Then the following conditions are equivalent:

(1) F is n.c.z. in E.

(2) The homomorphism $k^{\#}: H(E/F) \to H^*(E)$ is injective.

(3) The characteristic homomorphism $\chi^{\#}$ for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ is trivial: $(\chi^{\#})^+ = 0$.

(4) There is an isomorphism of graded cohomology algebras $H^*(E) \cong H(E/F) \otimes H^*(F)$, which makes the diagram

$$\begin{tikzcd}
& H(E/F) \otimes H^*(F) \arrow[dr] \arrow[dd, "\cong"] & \\
H(E/F) \arrow[ur] \arrow[dr, "k^{\#}"'] & & H^*(F) \\
& H^*(E) \arrow[ur, "j^{\#}"'] &
\end{tikzcd}$$

commutative.

(5) $\dim H^*(E) = \dim H(E/F) \dim H^*(F)$.

Proof: The theorem follows immediately from Theorem II, sec. 10.7, Theorem VII, sec. 3.17, and the Corollary to Proposition V, sec. 3.18.

Q.E.D.

10.19. Theorem X: Let (E, F) be a reductive pair, with Samelson subspace $\hat{P} \subset P_E$. Then the following conditions are equivalent:

(1) F is n.c.z. in E.

(2) $j_{\theta=0}^{\wedge}$ is surjective.

(3) $j_{\theta=0}^{\vee}$ is surjective.

(4) There are transgressions ν and τ in $W(F)_{\theta=0}$ and $W(E)_{\theta=0}$ satisfying

$$j_{\theta=0}^{\vee} \circ \tau = \nu \circ j_{\theta=0}^{\wedge}.$$

(5) $\ker j_P = \hat{P}$.

(6) The kernel of $j^{\#}$ coincides with the ideal generated by $\operatorname{Im}(k^{\#})^{+}$.

(7) There is an isomorphism of graded spaces

$$P_E \cong \hat{P} \oplus P_F.$$

(8) There is an isomorphism of graded algebras

$$H(E/F) \cong \wedge \hat{P}.$$

(9) The algebra $H(E/F)$ is generated by 1 and elements of odd degree.

Proof: The equivalence of conditions (1)–(4) is established in Proposition VIII, sec. 6.15. To complete the proof we show that

$$(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9) \Rightarrow (1).$$

(1) $\Rightarrow$ (5): Since F is n.c.z. in E, it follows from Theorem IX that $(\chi^{\#})^{+} = 0$ and that $k^{\#}$ is injective. Hence the kernel of $k^{\#}$ coincides (trivially) with the ideal generated by $\operatorname{Im}(\chi^{\#})^{+}$. Thus Theorem VII, (2), sec. 10.16, implies that (E, F) is a Cartan pair:

$$\dim P_E = \dim \hat{P} + \dim P_F. \tag{10.7}$$

Now consider the commutative diagram

$$\begin{tikzcd} \wedge\hat{P} \arrow[r] \arrow[d, "\cong"'] & \wedge P_E \arrow[r, "\wedge j_P"] \arrow[d, "\cong"] & \wedge P_F \arrow[d, "\cong"] \\ \operatorname{Im} k^{\#} \arrow[r] & H^*(E) \arrow[r, "j^*"'] & H^*(F) \end{tikzcd} \tag{10.8}$$

(cf. sec. 10.4 and Corollary I, sec. 5.19). Since, by Proposition III, (3), sec. 10.6,

$$j^{\#} \circ (k^{\#})^{+} = 0,$$

it follows from this diagram that

$$\hat{P} \subset \ker j_P. \tag{10.9}$$

On the other hand (since by hypothesis $j^{\#}$ is surjective), the diagram shows that so is j_P. This implies that

$$P_E \cong \ker j_P \oplus P_F. \tag{10.10}$$

Relations (10.7), (10.9), and (10.10) show that $\ker j_P = \hat{P}$.

(5) $\Rightarrow$ (6): It is elementary algebra that $\ker \wedge j_P$ coincides with the ideal generated by $\ker j_P$. Hence, if (5) holds, $\ker \wedge j_P$ is generated by $\hat{P}$. Now the commutative diagram (10.8) shows that $\ker j^{\#}$ is generated by $(\operatorname{Im} k^{\#})^{+}$.

(6) $\Rightarrow$ (7): Assume that (6) holds. Then it follows at once from the commutative diagram (10.8) that $\ker j_P = \hat{P}$. Let $\tilde{P}$ be a Samelson complement. Then j_P restricts to an injection $\tilde{j}_P\colon \tilde{P} \to P_F$. But, by Theorem V, sec. 10.13,

$$0 \leq \dim P_E - \dim \hat{P} - \dim P_F = \dim \tilde{P} - \dim P_F.$$

Hence, since $\tilde{j}_P$ is injective, it must be an isomorphism of graded vector spaces, and (7) follows.

(7) $\Rightarrow$ (8): Assume that (7) holds. Then (E, F) is a Cartan pair, and so Theorem V, sec. 10.13, yields an isomorphism

$$\operatorname{Im} \chi^{\#} \otimes \wedge\hat{P} \cong H(E/F).$$

Now let $\tilde{P}$ be a Samelson complement for (E, F). It follows from the hypothesis that $\tilde{P}$ and P_F are isomorphic as graded vector spaces. Thus the formula in Theorem VI, sec. 10.15, for the Poincaré polynomial of $\operatorname{Im} \chi^{\#}$ reads

$$f_{\operatorname{Im} \chi^{\#}} = 1.$$

It follows that $(\chi^{\#})^{+} = 0$ and so $\wedge\hat{P} \cong H(E/F)$.

(8) $\Rightarrow$ (9): This is obvious.

(9) $\Rightarrow$ (1): The corollary to Theorem IV, sec. 10.12, yields a homomorphism $\varphi\colon H(E/F) \to \operatorname{Im} \chi^{\#}$, which restricts to the identity in $\operatorname{Im} \chi^{\#}$.

Since Im $\chi^{\#}$ is evenly graded, it follows that

$$\sum_{p \text{ odd}} H^p(E/F) \subset \ker \varphi.$$

Now assume that (9) holds. Then the relation above implies that $H^+(E/F) \subset \ker \varphi$, whence $\operatorname{Im}(\chi^{\#})^+ = 0$.

Thus $(\chi^{\#})^+ = 0$, and so Theorem IX shows that F is n.c.z. in E.

Q.E.D.

Corollary I: If (E, F) is a reductive pair, and F is n.c.z. in E, then (E, F) is a Cartan pair.

Corollary II: Let (E, F) be a reductive pair and let $\tilde{P}$ be a Samelson complement. Consider the restriction $\tilde{\sigma}: \tilde{P} \to (\vee F^*)_{\theta=0}$. Then F is n.c.z. in E if and only if the homomorphism $\tilde{\sigma}_\vee: \vee \tilde{P} \to (\vee F^*)_{\theta=0}$ is an isomorphism.

Proof: If F is n.c.z. in E, then (E, F) is a Cartan pair and, in view of Theorem IX, (3), sec. 10.18, $(\chi^{\#})^+ = 0$. Hence Theorem VII, sec. 10.16, shows that $\tilde{\sigma}_\vee$ is an isomorphism.

Conversely, if $\tilde{\sigma}_\vee$ is an isomorphism, then $\sigma_\vee$ is surjective. But $\sigma_\vee = j^\vee_{\theta=0} \circ \tau_\vee$ and so $j^\vee_{\theta=0}$ must be surjective. It follows that F is n.c.z. in E.

Q.E.D.

10.20. Semidirect sums. Let E be a Lie algebra which as a vector space is the direct sum of a subalgebra F and an ideal R: $E = F \oplus R$. Then E is called the *semidirect sum* of F and R. R is stable under the operators ad x, $x \in E$. We define the *adjoint representation* of F in R by

$$(\operatorname{ad}_R y)z = [y, z], \qquad y \in F, \quad z \in R.$$

This representation induces a representation θ_F of F in $\wedge R^*$ (cf. sec. 4.2). A simple computation shows that $\theta_F(y)\delta_R = \delta_R \theta_F(y)$ (δ_R is the differential operator corresponding to the Lie algebra R). Thus θ_F induces a representation $\theta_F^{\#}$ of F in $H^*(R)$.

Proposition V: Let E be a Lie algebra which is the semidirect sum of a subalgebra F and an ideal R with F reductive in E. Then F is n.c.z. in E and there is an isomorphism of graded algebras

$$H^*(E) \cong H^*(R)_{\theta_F^{\#}=0} \otimes H^*(F).$$

Proof: Consider the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. Corollary III to Proposition II, sec. 10.5, implies that this operation admits a connection with zero curvature. Hence $(\chi^\#)^+ = 0$. Now Theorem IX, sec. 10.18, implies that F is n.c.z. in E, and that there is an isomorphism of graded algebras,

$$H^*(E) \cong H(E/F) \otimes H^*(F).$$

A straightforward computation shows that

$$((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E) \cong ((\wedge R^*)_{\theta_F=0}, \delta_R).$$

Thus, since F is reductive in E,

$$H(E/F) \cong H((\wedge R^*)_{\theta_F=0}) \cong H^*(R)_{\theta_F^\#=0}$$

(cf. Theorem IV, sec. 4.10). The proposition follows.

Q.E.D.

10.21. Examples. 1. *Levi decomposition*: Let E be any Lie algebra. Then E is the semidirect sum of a semisimple subalgebra F and its radical R (Levi decomposition) (cf. [6; Theorem 15, p. 96]). Thus Proposition V gives the cohomology of E in terms of

(1) the cohomology of the "Levi factor" F,
(2) the cohomology of the radical R,
(3) the representation $\theta_F^\#$.

2. *The Lie algebra $L_V \oplus V$*: Let V be a vector space and define a Lie algebra structure in $L_V \oplus V$ by setting

$$[(\alpha, a), (\beta, b)] = ([\alpha, \beta], \alpha(b) - \beta(a)), \qquad \alpha, \beta \in L_V, \quad a, b \in V.$$

Then $L_V \oplus V$ is the semidirect sum of the reductive subalgebra L_V and the abelian ideal V. Moreover, L_V is reductive in $L_V \oplus V$.

The induced representation of L_V in $\wedge V^*$ is given by

$$\theta(\alpha)(v_1^* \wedge \cdots \wedge v_p^*) = -\sum_{i=1}^p v_1^* \wedge \cdots \wedge \alpha^* v_i^* \wedge \cdots \wedge v_p^*.$$

In particular, $\theta(\iota)\Phi = -p\Phi$, $\Phi \in \wedge^p V^*$.

Hence $(\wedge^p V^*)_{\theta=0} = 0, p \geq 1$. Thus by Proposition V

$$H^*(L_V \oplus V) \cong H^*(L_V).$$

3. Let (E, F) be a reductive pair with F n.c.z. in E, and let $\varphi: E \to E_1$ be a homomorphism of reductive Lie algebras such that $\operatorname{Im} \varphi$ is reductive in E_1. Define a subalgebra $F_1 \subset E \oplus E_1$ by

$$F_1 = \{(y, \varphi y) \mid y \in F\}.$$

Then $(E \oplus E_1, F_1)$ is a reductive pair (cf. Proposition III, sec. 4.7).

Moreover, F_1 is n.c.z. in $E \oplus E_1$. In fact, there is a commutative diagram of Lie algebra homomorphisms,

$$\begin{tikzcd} F_1 \arrow[r, "j_1"] \arrow[d, "\cong"'] & E \oplus E_1 \arrow[d] \\ F \arrow[r, "j"'] & E. \end{tikzcd}$$

Since the map $j^{\#}: H^*(E) \to H^*(F)$ is surjective, the diagram implies that $j_1^{\#}$ is also surjective.

§6. Equal rank pairs

10.22. Definition: A reductive Lie algebra pair (E, F) will be called an *equal rank pair* if

$$\dim P_E = \dim P_F .$$

It follows from Theorem V, sec. 10.13, that an equal rank pair is a Cartan pair, with zero Samelson subspace. In this article we establish the following

Theorem XI: Let (E, F) be a reductive Lie algebra pair. Then the following conditions are equivalent:

(1) (E, F) is an equal rank pair.

(2) The characteristic homomorphism $\chi^{\#}\colon (\vee F^*)_{\theta=0} \to H(E/F)$ is surjective.

(3) $H(E/F) \cong (\vee F^*)_{\theta=0}/I$, where I is the ideal generated by $j^{\vee}_{\theta=0}(\vee^+ E^*)_{\theta=0}$.

(4) There is an isomorphism of graded spaces

$$g\colon H(E/F) \otimes (\vee E^*)_{\theta=0} \xrightarrow{\cong} (\vee F^*)_{\theta=0},$$

which satisfies

$$g(1) = 1, \quad (\chi^{\#} \circ g)(\alpha \otimes 1) = \alpha, \quad \text{and} \quad g(\alpha \otimes \Psi) = g(\alpha \otimes 1) \cdot j^{\vee}_{\theta=0}(\Psi),$$
$$\alpha \in H(E/F),\ \Psi \in (\vee E^*)_{\theta=0}.$$

(5) $H(E/F)$ is evenly graded.

(6) $H(E/F)$ has nonzero Euler–Poincaré characteristic.

(7) $j^{\vee}_{\theta=0}\colon (\vee E^*)_{\theta=0} \to (\vee F^*)_{\theta=0}$ is injective.

Proof: Recall from Theorem III, sec. 10.8, the commutative diagram

$$\begin{tikzcd}
\vee P_E \arrow[r, "\sigma_\vee"] \arrow[d, "\tau_\vee"', "\cong"] & (\vee F^*)_{\theta=0} \arrow[r, "l^{\#}"] \arrow[d, "\iota"', "\cong"] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^{\#}"] \arrow[d, "\cong"'] & \wedge P_E \arrow[d, "\cong"] \\
(\vee E^*)_{\theta=0} \arrow[r, "j^{\vee}_{\theta=0}"'] & (\vee F^*)_{\theta=0} \arrow[r, "\chi^{\#}"'] & H(E/F) \arrow[r, "k^{\#}"'] & H^*(E).
\end{tikzcd}$$

In view of this diagram, Theorem XI is a simple translation of the Corollary to Theorem VIII, sec. 2.19.

Q.E.D.

Remark: In article 5, Chapter XI, we shall construct reductive pairs with zero Samelson space which are *not* Cartan pairs.

10.23. Cartan subalgebras. Recall (from sec. 4.5) the definition of a Cartan subalgebra, and note that a Cartan subalgebra of a reductive Lie algebra E is abelian, and reductive in E.

Theorem XII: Let F be a Cartan subalgebra of a reductive Lie algebra E. Then (E, F) is an equal rank pair. In particular, $\dim P_E = \dim F$.

Proof: Assume first that the coefficient field Γ is algebraically closed. If E is abelian, then $F = E$ and the theorem is trivial. Assume that E is not abelian, and let Δ denote the set of roots of E for the Cartan subalgebra F. Since E is not abelian and Γ is algebraically closed, we have (cf. sec. 4.5)

$$E = F \oplus \sum_{\alpha \in \Delta} E_\alpha, \qquad \Delta \neq \emptyset,$$

where E_α is the 1-dimensional root space corresponding to α. In view of [6; p. 120] there are r roots $\alpha_1, \ldots, \alpha_r$ such that every root $\alpha \in \Delta$ can be uniquely written in the form

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i, \qquad k_i \in \mathbb{Z}.$$

We shall call a root *even* (respectively, *odd*) if k_1 is even (respectively, odd). Let A (respectively, B) be the collection of even (respectively, odd) roots. Since $\alpha_1 \in B$ we have $B \neq \emptyset$. Every odd root is of the form

$$\alpha = p\alpha_1 + \sum_{i \geq 2} k_i \alpha_i, \qquad k_i \in \mathbb{Z}, \quad p \text{ odd}.$$

The odd number p will be called the *degree* of α.

Now set

$$T = F \oplus \sum_{\alpha \in A} E_\alpha \qquad \text{and} \qquad S = \sum_{\beta \in B} E_\beta.$$

Then $E = T \oplus S$ (vector space direct sum). Since

$$[E_\alpha, E_\beta] \subset E_{\alpha+\beta}, \qquad \alpha, \beta \in \Delta,$$

it follows that T is a subalgebra, and that

$$[T, S] \subset S, \qquad [S, S] \subset T.$$

In particular, the subalgebra $\wedge S \subset \wedge E$ is stable under the operators $\theta^E(y)$, $y \in T$.

The following two lemmas will be established in the next section:

Lemma II: (E, T) is a reductive pair.

Lemma III: Let $h \in F$ be a vector such that

$$\alpha_1(h) = 1, \qquad \alpha_i(h) = 0, \qquad i = 2, \dots, r,$$

where $\alpha_1, \dots, \alpha_r$ are the roots used above to define A and B. Then

$$\ker \theta^E(h) \cap \wedge^p S = 0, \qquad p \text{ odd.}$$

Now consider the operation $(T, i_T, \theta_T, \wedge E^*, \delta_E)$. The decomposition $E = T \oplus S$ yields an isomorphism,

$$(\wedge E^*)_{i_T=0,\theta_T=0} \cong (\wedge S^*)_{\theta_T=0}$$

(cf. sec. 10.5). Since $F \subset T$, Lemma III implies that $(\wedge E^*)_{i_T=0,\theta_T=0}$ is evenly graded. Thus by Theorem XI, sec. 10.22,

$$\dim P_T = \dim P_E.$$

On the other hand, T is a proper subalgebra of E. Moreover, $F \subset T$ and so F is a Cartan subalgebra of T. It follows by induction (on $\dim E$) that

$$\dim P_F = \dim P_T,$$

whence $\dim P_E = \dim P_F = \dim F$.

Now let Γ be arbitrary and let Ω denote the algebraic closure of Γ. Then $\Omega \otimes F$ is a Cartan subalgebra of the reductive Lie algebra $\Omega \otimes E$ (over Ω). Moreover, clearly

$$\Omega \otimes (\wedge E^*)_{\theta=0} = \wedge(\Omega \otimes E)^*_{\theta=0} \qquad \text{and} \qquad \Omega \otimes P_E = P_{\Omega\otimes E}$$

(and similarly for F). It follows that

$$\dim F = \dim P_F = \dim_\Omega P_{\Omega\otimes F} = \dim_\Omega P_{\Omega\otimes E} = \dim P_E.$$

Q.E.D.

Corollary: Let F be a reductive subalgebra of a reductive Lie algebra E. Then (E, F) is an equal rank pair if and only if F contains a Cartan subalgebra of E.

Proof: Assume that (E, F) is an equal rank pair, and let H be a Cartan subalgebra of F. Then, since H is reductive in F and F is reductive in E, H must be reductive in E (cf. the corollary to Proposition III, sec. 4.7). Now Lemma II, sec. 4.5, shows that H is contained in a Cartan subalgebra H_E of E. But, in view of Theorem XII,

$$\dim H_E = \dim P_E = \dim P_F = \dim H.$$

Thus $H = H_E$; i.e., H is a Cartan subalgebra of E.

Conversely, assume that F contains a Cartan subalgebra H of E. Then $Z_F \subset H$, and so Z_F is reductive in E (cf. sec. 4.5). It follows (cf. sec. 4.4) that F is reductive in E. Moreover, by Theorem XII,

$$\dim P_F = \dim H = \dim P_E,$$

and so (E, F) is an equal rank pair.

Q.E.D.

10.24. Proof of Lemma II: Observe that T is the fixed point subalgebra of the involution $\omega: E \xrightarrow{\cong} E$ given by

$$\omega(x) = x, \quad x \in T, \qquad \text{and} \qquad \omega(x) = -x, \quad x \in S.$$

Now apply Proposition V, sec. 4.8.

Q.E.D.

Proof of Lemma III: Choose in each E_β $(\beta \in B)$ a vector $x_\beta \neq 0$. Then the vectors x_β form a basis for S. Since the x_β are eigenvectors for the transformation $\mathrm{ad}_E(h): S \to S$, where h is the vector defined in the lemma, it follows that the products, $x_{\beta_1} \wedge \cdots \wedge x_{\beta_k}$ are eigenvectors for the transformation $\theta^E(h): \wedge^k S \to \wedge^k S$. Moreover, they form a basis of $\wedge^k S$.

Now for $\beta \in B$ we have

$$(\mathrm{ad}_E\, h)x_\beta = \beta(h)x_\beta = (\deg \beta)x_\beta,$$

whence

$$\theta^E(h)(x_{\beta_1} \wedge \cdots \wedge x_{\beta_k}) = \left(\sum_{i=1}^k \deg \beta_i\right)x_{\beta_1} \wedge \cdots \wedge x_{\beta_k}.$$

In particular, since the products $x_{\beta_1} \wedge \cdots \wedge x_{\beta_k}$ span $\wedge^k S$, the eigenvalues of $\theta^E\, h)$ in $\wedge^k S$ are all of the form

$$\lambda = \sum_{i=1}^k \deg \beta_i.$$

Since $\deg \beta_i$ is an odd integer ($\beta_i \in B$), this implies that $\lambda \neq 0$ for odd k. Hence

$$\ker \theta^E(h) \cap \wedge^k S = 0, \qquad k \text{ odd}.$$

Q.E.D.

10.25. Poincaré polynomials. Let (E, F) be an equal rank pair. Let the Poincaré polynomials of $H^*(E)$, $H^*(F)$ be given by

$$f_{H(E)} = \prod_{i=1}^r (1 + t^{g_i}), \qquad f_{H(F)} = \prod_{i=1}^r (1 + t^{l_i}).$$

Then, in view of Theorem XI, sec. 10.22, and Theorem VI, sec. 10.15, the Poincaré polynomial of $H(E/F)$ is given by

$$f_{H(E/F)} = \frac{\prod_{i=1}^r (1 - t^{g_i+1})}{\prod_{i=1}^r (1 - t^{l_i+1})}.$$

Thus

$$\chi_{H(E/F)} = \dim H(E/F) = \frac{\prod_{i=1}^r (g_i + 1)}{\prod_{i=1}^r (l_i + 1)}.$$

In particular, if F is a Cartan subalgebra of E,

$$f_{H(E/F)} = \frac{1}{(1 - t^2)^r} \prod_{i=1}^r (1 - t^{g_i+1})$$

and

$$\chi_{H(E/F)} = \dim H(E/F) = \prod_{i=1}^r \left(\frac{g_i + 1}{2}\right).$$

§7. Symmetric pairs

10.26. Definition: Let E be a reductive Lie algebra and let $\omega\colon E \to E$ be an involutive Lie algebra isomorphism. Then a subalgebra F of E is given by

$$F = \{y \in E \mid \omega(y) = y\}.$$

The pair (E, F) will be called a *symmetric pair*. According to Proposition V, sec. 4.8, F is reductive in E and so a symmetric pair is reductive.

Now let (E, F) be a symmetric pair, and write

$$E = F \oplus S,$$

where $S = \{x \in E \mid \omega x = -x\}$. Then S is stable under $\operatorname{ad} y$, $y \in F$. Hence the projection $E \to F$ determined by this decomposition dualizes to an algebraic connection

$$\chi\colon F^* \to E^*$$

for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ (cf. sec. 10.5). χ will be called the *symmetric connection* for the pair (E, F).

Proposition VI: If (E, F) is a symmetric pair, then the restriction of δ_E to $(\wedge E^*)_{i_F=0,\theta_F=0}$ is zero. In particular, a symmetric pair is split (cf. sec. 10.17).

Proof: Recall from sec. 10.5 that the symmetric connection χ determines isomorphisms

$$\wedge S^* \xrightarrow{\cong} (\wedge E^*)_{i_F=0} \qquad \text{and} \qquad (\wedge S^*)_{\theta_F=0} \xrightarrow{\cong} (\wedge E^*)_{i_F=0,\theta_F=0}. \tag{10.11}$$

Next, observe that the involution ω extends to an isomorphism $\omega^\wedge$ in $\wedge E^*$. Since ω restricts to the identity in F, it follows that $\omega^\wedge$ is an automorphism of the operation of F in $\wedge E^*$. In particular, $\omega^\wedge$ restricts to isomorphisms $\omega^\wedge_{i=0}$ and $\omega^\wedge_{i=0,\theta=0}$ of the algebras $(\wedge E^*)_{i_F=0}$ and $(\wedge E^*)_{i_F=0,\theta_F=0}$.

On the other hand, ω restricts to an isomorphism ω_S of S ($\omega_S = -\iota$). Denote the induced isomorphism of $\wedge S^*$ by $\omega_S^\wedge$; then

$$\omega_S^\wedge(\Phi) = (-1)^p \Phi, \qquad \Phi \in \wedge^p S^*.$$

It follows immediately from the definitions that $\omega_S^\wedge$ corresponds to $\omega_{i=0}^\wedge$ under the isomorphism (10.11). This implies, in particular, that

$$\omega_{i=0,\theta=0}^\wedge(\Phi) = (-1)^p \Phi, \qquad \Phi \in (\wedge^p E^*)_{i_F=0,\theta_F=0}. \tag{10.12}$$

Next recall that, since ω is a homomorphism of Lie algebras $\omega^\wedge \circ \delta_E = \delta_E \circ \omega^\wedge$. Restricting this relation to $(\wedge E^*)_{i_F=0,\theta_F=0}$, and using formula (10.12), we obtain (for $\Phi \in (\wedge^p E^*)_{i_F=0,\theta_F=0}$)

$$\delta_E \Phi = (-1)^p \delta_E \omega^\wedge \Phi = (-1)^p \omega^\wedge \delta_E \Phi = -\delta_E \Phi,$$

whence $\delta_E \Phi = 0$.

Q.E.D.

Corollary: A symmetric pair is a Cartan pair.

Proof: Apply Theorem VIII, sec. 10.17.

Q.E.D.

Next, let (E, F) be a symmetric pair, with involution ω. Then, since ω is a Lie algebra homomorphism, $\omega^\wedge$ restricts to a linear involution,

$$\omega_P \colon P_E \xrightarrow{\cong} P_E.$$

Define subspaces $P_E^+ \subset P_E$ and $P_E^- \subset P_E$ by

$$P_E^+ = \{\Phi \in P_E \mid \omega_P \Phi = \Phi\} \qquad \text{and} \qquad P_E^- = \{\Phi \in P_E \mid \omega_P \Phi = -\Phi\}.$$

Then $P_E = P_E^+ \oplus P_E^-$.

Proposition VII: Let (E, F) be a symmetric pair with involution ω. Let $\tau_E \colon P_E \to (\vee E^*)_{\theta=0}$ be the distinguished transgression in $W(E)_{\theta=0}$ (cf. sec. 6.10). Then

$$\hat{P} = P_E^- = \ker \check{j}_{\theta=0} \circ \tau_E,$$

where $\hat{P}$ denotes the Samelson subspace for the pair (E, F).

Proof: It follows at once from Corollary II to Theorem III, sec. 10.8, that

$$\ker(j^{\vee}_{\theta=0} \circ \tau_E) \subset \hat{P}.$$

Let $\Phi \in \hat{P}$. Then (since $\hat{P} \subset \operatorname{Im} k^{\#}$) for some $\Psi \in \wedge E^*$,

$$\Phi + \delta_E \Psi \in (\wedge E^*)_{i_F=0,\, \theta_F=0}.$$

Since $\hat{P}^k = 0$ (k even) it follows from formula (10.12) that

$$\omega^{\wedge}\Phi + \delta_E \omega^{\wedge}\Psi = -(\Phi + \delta_E \Psi). \tag{10.13}$$

Projecting both sides of this equation into $(\wedge E^*)_{\theta=0}$ we find

$$\omega_P \Phi = -\Phi;$$

i.e., $\Phi \in P_E^-$. This shows that

$$\hat{P} \subset P_E^-.$$

Finally we prove that

$$P_E^- \subset \ker(j^{\vee}_{\theta=0} \circ \tau_E).$$

In fact, let $\Phi \in P_E^-$. Since ω is a Lie algebra automorphism it follows from Proposition VII, sec. 6.11, that $\tau_E \circ \omega_P = \omega^{\vee}_{\theta=0} \circ \tau_E$, whence

$$\omega^{\vee}_{\theta=0}(\tau_E \Phi) = -\tau_E \Phi.$$

Moreover, since ω reduces to the identity in F it follows that $j^{\vee} \circ \omega^{\vee} = j^{\vee}$ (where $j\colon F \to E$ is the inclusion). Hence

$$j^{\vee}_{\theta=0}(\tau_E \Phi) = j^{\vee}_{\theta=0}\omega^{\vee}_{\theta=0}(\tau_E \Phi) = -j^{\vee}_{\theta=0}(\tau_E \Phi).$$

This shows that $j^{\vee}_{\theta=0}(\tau_E \Phi) = 0$, and so

$$P_E^- \subset \ker j^{\vee}_{\theta=0} \circ \tau_E.$$

It has now been proved that

$$\ker j^{\vee}_{\theta=0} \circ \tau_E \subset \hat{P} \subset P_E^- \subset \ker j^{\vee}_{\theta=0} \circ \tau_E,$$

and the proposition follows.

Q.E.D.

§8. Relative Poincaré duality

10.27. The isomorphism $D_{E/F}$. Consider a Lie algebra pair (E, F) where E is unimodular and F is reductive in E (cf. sec. 5.10). Fix a basis vector e in $\wedge^n E$ ($n = \dim E$) and recall from sec. 5.11 that the Poincaré isomorphism $D: \wedge E^* \xrightarrow{\cong} \wedge E$ is defined by

$$D(\Phi) = i(\Phi)e, \qquad \Phi \in \wedge E^*.$$

It satisfies

$$(D \circ i(y))\Phi = (-1)^{p-1}(\mu(y) \circ D)\Phi, \qquad \Phi \in \wedge^p E^*, \qquad y \in F.$$

Hence it restricts to an isomorphism

$$D: (\wedge E^*)_{i_F=0} \xrightarrow{\cong} (\wedge E)_{\mu_F=0}.$$

(Here $(\wedge E)_{\mu_F=0}$ denotes the ideal in $\wedge E$ consisting of the elements a for which $y \wedge a = 0$, $y \in F$.)

Next, fix a basis vector e_F in $\wedge^m F$ ($m = \dim F$). Then multiplication from the right by e_F yields a short exact sequence

$$0 \longrightarrow I_F \xrightarrow{\lambda} \wedge E \xrightarrow{\mu_R(e_F)} (\wedge E)_{\mu_F=0} \longrightarrow 0,$$

where I_F denotes the ideal in $\wedge E$ generated by F and λ is the inclusion map.

The sequence above determines an isomorphism

$$\varphi: \wedge E/I_F \xrightarrow{\cong} (\wedge E)_{\mu_F=0}.$$

Composing φ^{-1} with D we obtain an isomorphism

$$D_{E/F}: (\wedge E^*)_{i_F=0} \xrightarrow{\cong} \wedge E/I_F.$$

It maps $(\wedge^p E^*)_{i_F=0}$ isomorphically to $(\wedge E/I_F)^{n-m-p}$.

In particular, it follows that $\dim(\wedge E/I_F)^{n-m} = 1$ and the element

$$e_{E/F} = D_{E/F}(1)$$

is a basis vector of $(\wedge E/I_F)^{n-m}$.

Definition: The isomorphism $D_{E/F}$ is called the *relative Poincaré isomorphism for the pair* (E, F).

Now observe that, with respect to the scalar products between $\wedge E^*$ and $\wedge E$,

$$(\wedge E^*)_{i_F=0} = (I_F)^{\perp}.$$

Thus there is an induced scalar product between $(\wedge E^*)_{i_F=0}$ and $\wedge E/I_F$.

Lemma IV: The isomorphism $D_{E/F}$ satisfies

$$\langle \Psi, D_{E/F}\Phi \rangle = \langle \Phi \wedge \Psi, e_{E/F} \rangle.$$

In particular, $(\wedge E^*)_{i_F=0}$ is a Poincaré algebra, and $D_{E/F}$ is the corresponding Poincaré isomorphism (cf. sec. 0.6).

Proof: It follows from the definition that, if $a \in \wedge E$ is an element satisfying $a \wedge e_F = i(\Phi)e$, then

$$\langle \Psi, D_{E/F}\Phi \rangle = \langle \Psi, a \rangle, \qquad \Phi, \Psi \in (\wedge E^*)_{i_F=0}.$$

Choose $b \in \wedge E$ so that $b \wedge e_F = e$. Then

$$i(\Phi)e = i(\Phi)(b \wedge e_F) = (i(\Phi)b) \wedge e_F.$$

Hence

$$\begin{aligned}\langle \Psi, D_{E/F}\Phi \rangle &= \langle \Psi, i(\Phi)b \rangle = \langle \Phi \wedge \Psi, b \rangle \\ &= \langle \Phi \wedge \Psi, D_{E/F}(1) \rangle = \langle \Phi \wedge \Psi, e_{E/F} \rangle.\end{aligned}$$

Q.E.D.

10.28. The representation of F in $\wedge E$ (obtained by restricting θ^E to F) will be denoted by θ^F. It induces a representation in $\wedge E/I_F$, also denoted by θ^F. Since F is reductive, it is unimodular; thus $e_F \in (\wedge^m F)_{\theta=0}$. It follows that

$$D_{E/F} \circ \theta_F(y) = \theta^F(y) \circ D_{E/F}, \qquad y \in F,$$

and so $D_{E/F}$ restricts to an isomorphism

$$(D_{E/F})_{\theta_F=0}\colon (\wedge E^*)_{i_F=0,\theta_F=0} \xrightarrow{\cong} (\wedge E/I_F)_{\theta^F=0}.$$

In particular $e_{E/F} \in (\wedge E/I_F)^{n-m}_{\theta^F=0}$.

Moreover, since F is reductive in E, the representations θ_F and θ^F are semisimple. Thus the duality between $(\wedge E^*)_{i_F=0}$ and $\wedge E/I_F$ restricts to a duality between the invariant subalgebras. Lemma IV yields

$$\langle \Psi, (D_{E/F})_{\theta_F=0}\Phi\rangle = \langle \Phi \wedge \Psi, e_{E/F}\rangle, \qquad \Phi, \Psi \in (\wedge E^*)_{i_F=0, \theta_F=0}.$$

This relation shows that $(\wedge E^*)_{i_F=0, \theta_F=0}$ is a Poincaré algebra with Poincaré isomorphism $(D_{E/F})_{\theta_F=0}$.

Finally, the equations

$$\partial_E \mu(y) + \mu(y)\partial_E = \theta^E(y), \qquad y \in F,$$

show that $(I_F)_{\theta^F=0}$ is stable under the operator ∂_E. Since F is reductive in E,

$$(\wedge E/I_F)_{\theta^F=0} = (\wedge E)_{\theta^F=0}/(I_F)_{\theta^F=0}.$$

Thus ∂_E induces a differential operator $\partial_{E/F}$ in $(\wedge E/I_F)_{\theta^F=0}$. Clearly, the restriction of δ_E to $(\wedge E^*)_{i_F=0,\theta_F=0}$ is the negative dual of $\partial_{E/F}$. Thus a scalar product is induced between $H(E/F)$ and $H((\wedge E/I_F)_{\theta^F=0}, \partial_{E/F})$.

Next observe that, since F is reductive, $\partial_F e_F = 0$ and so $\partial_E e_F = 0$. Now (the second) formula (5.8) of sec. 5.4 implies that the isomorphism

$$\varphi_{\theta=0}\colon (\wedge E/I_F)_{\theta^F=0} \xrightarrow{\cong} (\wedge E)_{\mu_F=0,\theta_F=0},$$

satisfies $\varphi_{\theta=0} \circ \partial_{E/F} = \partial_E \circ \varphi_{\theta=0}$. Since $D \circ \delta_E \circ \omega = \partial_E \circ D$ (cf. sec. 5.11), it follows that

$$(D_{E/F})_{\theta=0} \circ \delta_E \circ \omega = \partial_{E/F} \circ (D_{E/F})_{\theta=0}.$$

Thus $(D_{E/F})_{\theta=0}$ induces an isomorphism

$$D^\#_{E/F}\colon H(E/F) \xrightarrow{\cong} H((\wedge E/I_F)_{\theta^F=0}, \partial_{E/F}).$$

Evidently, $\langle \alpha \cdot \beta, \varepsilon_{E/F}\rangle = \langle \beta, D^\#_{E/F}\alpha\rangle$, where $\varepsilon_{E/F}$ denotes the class represented by $e_{E/F}$. It follows that $H(E/F)$ is a Poincaré algebra of degree $n - m$ and with Poincaré isomorphism $D^\#_{E/F}$.

In particular

$$\dim H^{n-m}(E/F) = 1.$$

On the other hand,

$$(\wedge^{n-m}E^*)_{i_F=0} \cong \wedge^{n-m}(E/F)^* \cong \Gamma.$$

It follows that

$$H^{n-m}(E/F) = (\wedge^{n-m}E^*)_{i_F=0,\theta_F=0} = (\wedge^{n-m}E^*)_{i_F=0}. \tag{10.14}$$

10.29. Reductive pairs. Proposition VIII: Let (E, F) be a reductive Lie algebra pair and let A be a characteristic factor for $H(E/F)$ (cf. sec. 10.12). Then A is a Poincaré duality algebra.

Proof: According to sec. 10.28, $H(E/F)$ is a Poincaré duality algebra. Now the relation

$$H(E/F) \cong A \otimes \wedge \hat{P}$$

together with Example 2, sec. 0.6, implies that A is a Poincaré duality algebra.

Q.E.D.

Corollary: If (E, F) is a Cartan pair, then Im $\chi^\#$ is a Poincaré duality algebra.

Proposition IX: Let (E, F) be a reductive pair and assume that $H^{n-m}(E/F) \subset \operatorname{Im} \chi^\#$ ($n = \dim E$, $m = \dim F$). Then (E, F) is an equal rank pair.

Proof: In view of Theorem XI, sec. 10.22, it is sufficient to show that $\chi^\#$ is surjective. According to the corollary to Theorem IV, sec. 10.12, there is a homomorphism

$$\varphi: H(E/F) \to \operatorname{Im} \chi^\#,$$

which restricts to the identity in Im $\chi^\#$. In particular

$$\ker \varphi \cap H^{n-m}(E/F) = 0.$$

But, since $H(E/F)$ is a Poincaré duality algebra of degree $n - m$, every nonzero ideal in $H(E/F)$ contains $H^{n-m}(E/F)$. Thus the equation above implies that $\ker \varphi = 0$. Hence φ is an isomorphism and so $\chi^\#$ is surjective.

Q.E.D.

§9. Symplectic metrics

10.30. Definition: Let (E, F) be a Lie algebra pair. Then the space $(\wedge^2 E^*)_{i_F=0}$ may be identified with the space of skew symmetric bilinear functions in E/F.

An element $\Phi \in (\wedge^2 E^*)_{i_F=0}$ is called a *symplectic metric* in E/F if Φ is nondegenerate; i.e., if the relation

$$\Phi(\bar{x}_1, \bar{x}) = 0$$

for fixed $\bar{x}_1 \in E/F$ and all $\bar{x} \in E/F$ implies that $\bar{x}_1 = 0$. Equivalently, Φ is a symplectic metric in E/F if $F = \{y \in E \mid i(y)\Phi = 0\}$.

Elementary linear algebra shows that a symplectic metric exists in E/F if and only if $\dim E/F$ is even. Now assume that this condition is satisfied: $\dim E/F = 2k$. Then an element $\Phi \in (\wedge^2 E^*)_{i_F=0}$ is a symplectic metric if and only if $\Phi^k \neq 0$ ($\Phi^k = \Phi \wedge \cdots \wedge \Phi$, k factors).

A symplectic metric Φ for E/F is called *closed* if

$$\delta_E \Phi = 0.$$

If Φ is a closed symplectic metric, then

$$\theta_F(y)\Phi = i_F(y)\delta_E\Phi + \delta_E i_F(y)\Phi = 0, \qquad y \in F,$$

and so $\Phi \in (\wedge^2 E^*)_{i_F=0,\theta_F=0}$.

Proposition X: Let (E, F) be a Lie algebra pair with E semisimple. Then E/F admits a closed symplectic metric if and only if for some $h \in E$,

$$F = \ker(\operatorname{ad} h).$$

Proof: Assume that E/F admits a closed symplectic metric Φ. Since E is semisimple, $H^2(E) = 0$ (cf. sec. 5.20). Hence, for some $h^* \in E^*$,

$$\Phi = \delta_E h^*.$$

It follows that

$$\theta_E(x)h^* = i_E(x)\delta_E h^* = i_E(x)\Phi, \qquad x \in E.$$

Now let $\alpha: E \xrightarrow{\cong} E^*$ be the isomorphism determined by the Killing form (cf. Theorem I, sec. 4.4) and set $h = \alpha^{-1}(h^*)$. Then

$$\alpha((\operatorname{ad} x)h) = \theta_E(x)h^* = i_E(x)\Phi, \qquad x \in E.$$

Since Φ is nondegenerate, this relation implies that $F = \ker(\operatorname{ad} h)$.

Conversely assume that $F = \ker(\operatorname{ad} h)$, some $h \in E$. Set $h^* = \alpha(h)$. Reversing the argument above shows that $\delta_E h^*$ is a closed symplectic metric in E/F.

Q.E.D.

10.31. Reductive pairs. Theorem XIII: Let (E, F) be a Lie algebra pair with E semisimple. Then the following conditions are equivalent:

(1) For some Cartan subalgebra H of E, and for some $h \in H$, $F = \ker(\operatorname{ad} h)$.

(2) (E, F) is a reductive pair and E/F admits a closed symplectic metric Φ.

(3) (E, F) is a reductive pair, $\dim E/F = 2k$, and, for some $\alpha \in H^2(E/F)$, $\alpha^k \neq 0$.

Moreover, if these conditions hold, then (E, F) is an equal rank pair.

Proof: First we show that (3) implies that (E, F) is an equal rank pair. Then we show that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1).$$

Assume that (3) holds. Since E is semisimple, $P_E^q = 0$, $q < 3$. This implies that $\hat{P}^q = 0$ and $\tilde{P}^q = 0$, $q < 3$, where $\hat{P}$ is the Samelson subspace and $\tilde{P}$ is a Samelson complement for (E, F).

Now according to Theorem IV, sec. 10.12,

$$H(E/F) \cong A \otimes \wedge\hat{P},$$

where

$$A = \operatorname{Im} \chi^\# \oplus I \qquad \text{and} \qquad I \cong H_+((\vee F^*)_{\theta=0} \otimes \wedge\tilde{P}).$$

This shows that $H^2(E/F) \subset \operatorname{Im} \chi^\#$.

In particular, $\alpha \in \operatorname{Im} \chi^\#$. Since $\alpha^k \neq 0$ it follows that $H^{2k}(E/F) \subset \operatorname{Im} \chi^\#$. Now Proposition IX, sec. 10.29, shows that (E, F) is an equal rank pair.

It remains to establish the equivalence of conditions (1), (2), and (3).

(1) $\Rightarrow$ (2): Since h is in a Cartan subalgebra of E, Proposition II, sec. 4.5 implies that F is reductive in E. Moreover, by Proposition X, sec. 10.30, E/F admits a closed symplectic metric.

(2) $\Rightarrow$ (3): Let $\alpha \in H^2(E/F)$ be the class represented by the closed symplectic metric Φ. Then $\dim E - \dim F = 2k$. Thus, in view of formula (10.14), sec. 10.28,

$$H^{2k}(E/F) = (\wedge^{2k} E^*)_{i_F=0,\theta_F=0},$$

and so it follows that $\alpha^k = \Phi^k \neq 0$.

(3) $\Rightarrow$ (1): Let $\Phi \in (\wedge^2 E^*)_{i_F=0,\theta_F=0}$ be a cocycle representing α. Then $\Phi^k \neq 0$ and so Φ is a closed symplectic metric. Hence, by Proposition X, sec. 10.30, for some $h \in E$,

$$F = \ker(\operatorname{ad} h).$$

Now let H be a Cartan subalgebra of F. In view of the relation above, $h \in Z_F$ and so $h \in H$. But, by the first part of the proof, (3) implies that (E, F) is an equal rank pair. Thus, in view of the corollary to Theorem XII, sec. 10.23, H is a Cartan subalgebra of E.

Q.E.D.

Chapter XI

Homogeneous Spaces

§1. The cohomology of a homogeneous space

In this chapter, the results of Chapter X will be applied to homogeneous spaces. G will always denote a connected Lie group with Lie algebra E. K is a closed connected Lie subgroup with Lie algebra F.

Recall from sec. 2.9, volume II, that the left cosets aK ($a \in G$) form a manifold G/K. It is the base of the principal bundle $\mathscr{P} = (G, \pi, G/K, K)$, where π denotes the projection $a \mapsto aK$ and the principal action of K on G is by right multiplication (cf. sec. 5.1, volume II).

Finally, note that if G is compact, then so is K. In this case both Lie algebras E and F are reductive. Moreover, Proposition XVII, sec. 1.17, volume II, implies that the adjoint representation of K in E is semisimple. Hence so is the adjoint representation of F in E. Thus, if G is compact, then (E, F) is a reductive pair.

11.1. The operation of F in $A(G)$. According to sec. 8.22 the principal bundle $\mathscr{P}$ determines an associated operation of F in $A(G)$; it will be denoted by $(F, i_K, \theta_K, A(G), \delta_G)$.

Since the principal action is right multiplication, the fundamental vector fields are simply the left invariant vector fields X_h ($h \in F$). Thus

$$i_K(h) = i(X_h) \qquad \text{and} \qquad \theta_K(h) = \theta(X_h).$$

It follows that the operation $(F, i_K, \theta_K, A(G), \delta_G)$ coincides with the restriction to F of the operation $(E, i_G, \theta_G, A(G), \delta_G)$ defined in sec. 7.21.

In sec. 7.21 we considered the left invariant operation $(E, i_L, \theta_L, A_L(G), \delta_L)$ of E in the left invariant differential forms on G. This, too, restricts to an operation $(F, i_F, \theta_F, A_L(G), \delta_L)$ of F. Further, the inclusion

$$l_G: A_L(G) \to A(G)$$

and the isomorphism

$$\tau_L\colon A_L(G) \xrightarrow{\cong} \wedge E^*$$

(cf. sec. 7.21) are homomorphisms of operations of E; hence they may be regarded as homomorphisms of operations of F.

It follows that the composite $\varepsilon_G = l_G \circ \tau_L^{-1}$ is also a homomorphism of operations of F:

$$\varepsilon_G\colon (F, i_F, \theta_F, \wedge E^*, \delta_E) \to (F, i_K, \theta_K, A(G), \delta_G).$$

Thus ε_G restricts to a homomorphism

$$(\varepsilon_G)_{i_F=0,\theta_F=0}\colon ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E) \to (A(G)_{i_K=0,\theta_K=0}, \delta_G).$$

On the other hand, since K is connected, it follows from the results of sec. 6.3, volume II (applied to the principal bundle $\mathscr{P}$) that π^* restricts to an isomorphism

$$(A(G/K), \delta_{G/K}) \xrightarrow{\cong} (A(G)_{i_K=0,\theta_K=0}, \delta_G).$$

Composing $(\varepsilon_G)_{i_F=0,\theta_F=0}$ with the inverse isomorphism yields a homomorphism

$$\varepsilon_{G/K}\colon ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E) \to (A(G/K), \delta_{G/K})$$

of graded differential algebras.

In view of sec. 6.29, volume II, $\varepsilon_{G/K}$ can be regarded as an isomorphism

$$(\wedge E^*)_{i_F=0,\theta_F=0} \xrightarrow{\cong} A_I(G/K),$$

where $A_I(G/K)$ denotes the algebra of differential forms on G/K invariant under the left action of G.

Proposition I: Assume G is compact. Then $\varepsilon_{G/K}^\#$ is an isomorphism:

$$\varepsilon_{G/K}^\#\colon H(E/F) \xrightarrow{\cong} H(G/K).$$

Proof: In view of the remarks above, it is sufficient to show that the natural homomorphism

$$H_I(G/K) \to H(G/K)$$

is an isomorphism. But this follows from Theorem I, sec. 4.3, volume II, since G is compact and connected.

Q.E.D.

Corollary: If G is compact, then

$$((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E) \underset{c}{\sim} (A(G/K), \delta_{G/K}).$$

11.2. The Samelson subspace. Assume that G is compact. Recall from sec. 4.12, volume II, (or sec. 5.32) the definition of the primitive space $P_G \subset H^+(G)$. According to sec. 4.12, volume II, $H(G) \cong \wedge P_G$.

Definition: The *Samelson subspace* $\hat{P}_G$ for the pair (G, K) is the graded space

$$\hat{P}_G = \operatorname{Im} \pi^\# \cap P_G.$$

A complementary graded space $\tilde{P}_G$ in P_G is called a *Samelson complement.*

Theorem I: Assume that G is compact. Then the image of the homomorphism $\pi^\#\colon H(G/K) \to H(G)$ is the exterior algebra over the Samelson space:

$$\operatorname{Im} \pi^\# = \wedge \hat{P}_G.$$

Proof: Identify $(\wedge E^*)_{\theta=0}$ with $H^*(E)$ via π_E (cf. sec. 5.12). Then $\varepsilon_G^\#$ coincides with α_G (cf. sec. 5.29). Thus sec. 5.32 shows that $\varepsilon_G^\#$ restricts to an isomorphism

$$\varepsilon_G^\#\colon P_E \xrightarrow{\cong} P_G.$$

Now observe that (by definition) the diagram

$$\begin{tikzcd} H(E/F) \arrow[r, "k^*"] \arrow[d, "\varepsilon_{G/K}^*"', "\cong"] & H^*(E) \arrow[d, "\cong"', "\varepsilon_G^*"] \\ H(G/K) \arrow[r, "\pi^*"'] & H(G) \end{tikzcd}$$

commutes, and apply Theorem I, sec. 10.4.

Q.E.D.

11.3. Invariant connections. Suppose χ is an algebraic connection for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$. Then $\varepsilon_G \circ \chi$ is an algebraic connection for the operation $(F, i_K, \theta_K, A(G), \delta_G)$. Let V be the corresponding principal connection in $\mathscr{P}$ (cf. sec. 8.22).

It is easy to verify that V is G-invariant. Moreover, Proposition XVIII, sec. 6.30, volume II, together with Example 4, sec. 8.1 implies that the correspondence $\chi \mapsto V$ is a bijection from algebraic connections in $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ to G-invariant principal connections in $\mathscr{P}$.

Now assume that the operation of F in $(\wedge E^*, \delta_E)$ admits an algebraic connection. Then the Weil homomorphism

$$\chi^{\#}: (\vee F^*)_{\theta=0} \to H(E/F)$$

is defined. Since ε_G is a homomorphism of operations, it follows that

$$(\varepsilon_G)^{\#}_{i_F=0,\theta_F=0} \circ \chi^{\#} = \hat{\chi}^{\#},$$

where $\hat{\chi}^{\#}$ denotes the Weil homomorphism of the operation of F in $(A(G), \delta_G)$.

Thus Theorem VI, sec. 8.26, applies to yield a commutative diagram

$$\begin{tikzcd}
(\vee F^*)_{\theta=0} \arrow[r, "\chi^*"] \arrow[d, "="'] & H(E/F) \arrow[d, "\varepsilon^*_{G/K}"] \\
(\vee F^*)_I \arrow[r, "h_{\mathscr{P}}"'] & H(G/K),
\end{tikzcd}$$

where $h_{\mathscr{P}}$ is the Weil homomorphism for the principal bundle $\mathscr{P}$.

11.4. The cohomology sequence. Let $j_K: K \to G$ and $j: F \to E$ denote the inclusions (so that $j = j_K'$). Then the sequence of homomorphisms

$$(\vee E^*)_I \xrightarrow{j_I^\vee} (\vee F^*)_I \xrightarrow{h_{\mathscr{P}}} H(G/K) \xrightarrow{\pi^*} H(G) \xrightarrow{j_K^*} H(K)$$

is called the *cohomology sequence for the pair* (G, K).

Now assume that the operation of F in E admits an algebraic connection and consider the diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "j^\vee_{\theta=0}"] \arrow[d, "="'] & (\vee F^*)_{\theta=0} \arrow[r, "\chi^*"] \arrow[d, "="'] & H(E/F) \arrow[r, "k^*"] \arrow[d, "\varepsilon^*_{G/K}"'] & H^*(E) \arrow[r, "j^*"] \arrow[d, "\varepsilon^*_G"] & H^*(F) \arrow[d, "\varepsilon^*_K"] \\
(\vee E^*)_I \arrow[r, "j_I^\vee"'] & (\vee F^*)_I \arrow[r, "h_{\mathscr{P}}"'] & H(G/K) \arrow[r, "\pi^*"'] & H(G) \arrow[r, "j_K^*"'] & H(K).
\end{tikzcd} \tag{11.1}$$

The left-hand square commutes by definition. The commutativity of the second square was shown just above, while the third square commutes, again by definition. That the fourth square commutes follows from sec. 4.7, volume II.

Thus the diagram (11.1) is commutative. It may be regarded as a homomorphism from the cohomology sequence of the pair (E, F) (cf. sec. 10.6) to the cohomology sequence of the pair (G, K). Moreover, if G is compact, then all the vertical arrows are isomorphisms, as follows from Proposition I, sec. 11.1, and sec. 5.29. Thus, in this case, the diagram is an isomorphism of cohomology sequences.

Proposition II: Assume that G is compact. Then the cohomology sequence of the pair (G, K) has the following properties:

(1) The image $(j_I^\vee)^+$ generates the kernel of $h_{\mathscr{P}}$.

(2) The image of $h_{\mathscr{P}}^+$ is contained in the kernel of $\pi^\#$.

(3) The image of $\pi^\#$ is an exterior algebra over the Samelson subspace $\hat{P}_G$, and the image of $(\pi^\#)^+$ is contained in the kernel of $j_K^\#$.

(4) The image of $j_K^\#$ is an exterior algebra over a graded subspace of the primitive space P_K.

Proof: In view of the isomorphisms

$$\varepsilon_G^\#: P_E \xrightarrow{\cong} P_G \qquad \text{and} \qquad \varepsilon_K^\#: P_F \xrightarrow{\cong} P_K$$

(cf. sec. 5.32) the proposition follows (via diagram (11.1)) from Proposition III, sec. 10.6.

Q.E.D.

§2. The structure of $H(G/K)$

11.5. The structure of $H(G/K)$. In this section G is assumed to be compact, so that (E, F) is a reductive pair. As in sec. 10.8, fix a transgression $\tau\colon P_E \to (\vee E^*)_{\theta=0}$ and define

$$\sigma\colon P_E \to (\vee F^*)_{\theta=0}$$

by $\sigma = j^{\vee}_{\theta=0} \circ \tau$.

Then the P_E-algebra $((\vee F^*)_{\theta=0}; \sigma)$ is called the *P_E-algebra associated with the pair* (G, K). (It coincides with the P_E-algebra associated with (E, F). In particular, its Koszul complex $((\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla_\sigma)$ is the one defined in sec. 10.8.)

Theorem II: Suppose G is a compact connected Lie group with compact connected subgroup K. Then there is a homomorphism of graded differential algebras

$$\varphi_{G/K}\colon ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to (A(G/K), \delta_{G/K})$$

with the following properties:

(1) $\varphi^{\#}_{G/K}$ is an isomorphism.

(2) The diagram

$$\begin{tikzcd}
\vee P_E \arrow[r, "\sigma_\vee"] \arrow[d, "\tau_\vee"', "\cong"] & (\vee F^*)_{\theta=0} \arrow[r, "l^\#"] \arrow[d, "="] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^\#"] \arrow[d, "\cong"', "\varphi^\#_{G/K}"] & \wedge P_E \arrow[d, "\cong"', "\varepsilon^\#_G"] \\
(\vee E^*)_I \arrow[r, "j^\vee_I"'] & (\vee F^*)_I \arrow[r, "h_{\mathscr{P}}"'] & H(G/K) \arrow[r, "\pi^\#"'] & H(G)
\end{tikzcd}$$

commutes.

Proof: Let $\varphi\colon (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (\wedge E^*)_{i_F=0,\theta_F=0}$ be the homomorphism in Theorem III, sec. 10.8, and set $\varphi_{G/K} = \varepsilon_{G/K} \circ \varphi$. Then $\varphi_{G/K}$ has the desired properties, as follows from Theorem III, sec. 10.8, Proposition I, sec. 11.1, and diagram (11.1), sec. 11.4.

Q.E.D.

Theorem III: Suppose G is a compact connected Lie group with compact connected subgroup K. Then there is a graded subalgebra $A \subset H(G/K)$ with the following properties:

(1) A as a vector space is the direct sum of the subalgebra $\operatorname{Im} h_{\mathscr{P}}$ and a graded ideal I in A,

$$A = \operatorname{Im} h_{\mathscr{P}} \oplus I.$$

(2) There is an isomorphism of graded algebras

$$g \colon A \otimes \wedge \hat{P}_G \xrightarrow{\cong} H(G/K),$$

which makes the diagram

$$\begin{tikzcd}
 & A \otimes \wedge \hat{P}_G \arrow[r] \arrow[dd, "g"', "\cong"] & \wedge \hat{P}_G \arrow[dd] \\
A \arrow[ur] \arrow[dr] & & \\
 & H(G/K) \arrow[r, "\pi^\#"'] & H(G)
\end{tikzcd}$$

commute. ($\hat{P}_G$ is the Samelson subspace—cf. sec. 11.2.)

Moreover, if B is a second subalgebra of $H(G/K)$ with these properties, then there is an automorphism of $H(G/K)$ which restricts to an isomorphism $A \xrightarrow{\cong} B$, and induces the identity in $\operatorname{Im} h_{\mathscr{P}}$.

Proof: In view of diagram (11.1), sec. 11.4, the theorem follows directly from Theorem IV, sec. 10.12.

Q.E.D.

Corollary: There is a homomorphism of graded algebras $\psi \colon H(G/K) \to \operatorname{Im} h_{\mathscr{P}}$ which reduces to the identity in $\operatorname{Im} h_{\mathscr{P}}$.

Theorem IV: Let G be a compact connected Lie group with compact connected subgroup K. Let $\hat{P}_G$ be the Samelson subspace for the pair (G, K) (cf. sec. 11.2). Then

$$\dim P_G \geq \dim P_K + \dim \hat{P}_G.$$

Moreover, the following conditions are equivalent:

(1) (E, F) is a Cartan pair.

(2) $\dim P_G = \dim P_K + \dim \hat{P}_G$.

(3) There is an isomorphism of graded algebras

$$\operatorname{Im} h_{\mathscr{P}} \otimes \wedge \hat{P}_G \xrightarrow{\cong} H(G/K),$$

which makes the diagram

$$\begin{tikzcd}
 & \operatorname{Im} h_{\mathscr{P}} \otimes \wedge \hat{P}_G \arrow[r] \arrow[d, "\cong"] & \wedge \hat{P}_G \arrow[d] \\
\operatorname{Im} h_{\mathscr{P}} \arrow[ur] \arrow[dr] & & \\
 & H(G/K) \arrow[r, "\pi^*"'] & H(G)
\end{tikzcd}$$

commute.

(4) $\dim H(G/K) = \dim \operatorname{Im} h_{\mathscr{P}} \cdot \dim \wedge \hat{P}_G$.

(5) The kernel of $\pi^\#$ is generated by the image of $h_{\mathscr{P}}^+$.

(6) The graded differential algebra $(A(G/K), \delta_{G/K})$ is c-split.

Proof: In view of the commutative diagram (11.1), the first statement and the equivalence of conditions (1)–(5) follows directly from the remarks at the beginning of article 4, Chapter X, together with Theorem V, sec. 10.13, and Theorem VII, sec. 10.16.

Finally, Theorem VIII, sec. 10.17, asserts that (1) holds if and only if the differential algebra $((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$ is c-split. But, in view of the corollary to Proposition I, sec. 11.1,

$$((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E) \underset{c}{\sim} (A(G/K), \delta_{G/K}),$$

and so (1) and (6) are equivalent.

Q.E.D.

Corollary I: Let G/K be a homogeneous space with G and K compact and connected. Then the validity of the conditions in Theorem IV depends only on the smooth manifold structure of G/K (and so is independent of any Lie structure).

Proof: Observe that this is correct for condition (6).

Q.E.D.

Corollary II: Assume that $K_1 \subset G_1$ and $K_2 \subset G_2$ are compact and connected Lie groups and let $\varphi\colon G_1/K_1 \to G_2/K_2$ be a smooth map such that $\varphi^\#$ is an isomorphism. Then the conditions in Theorem IV are satisfied by the pair (G_1, K_1) if and only if they hold for the pair (G_2, K_2).

In this case there is a linear isomorphism of (graded) Samelson spaces $\hat{P}_1 \xrightarrow{\cong} \hat{P}_2$, and an isomorphism of graded algebras $\operatorname{Im} h_{\mathscr{P}_1} \cong \operatorname{Im} h_{\mathscr{P}_2}$.

Proof: Since $\varphi^*\colon A(G_1/K_1, \delta_1) \leftarrow A(G_2/K_2, \delta_2)$ is a c-equivalence, the first statement follows. The second assertion is a consequence of Corollary IV to Theorem V, sec. 10.13.

Q.E.D.

Example: *Symmetric spaces*: Let ω be an involution of a compact Lie group G, and let K denote the 1-component of the subgroup left pointwise fixed by ω. Then the corollary to Proposition VI, sec. 10.26, shows that the pair (G, K) satisfies the conditions of Theorem IV.

Theorem V: Let K be a compact connected subgroup of a compact connected Lie group G. Assume that the pair (G, K) satisfies the conditions in Theorem IV. Let

$$f_{P_G} = \sum_{i=1}^{r} t^{g_i}, \qquad f_{P_K} = \sum_{i=1}^{s} t^{l_i}, \qquad \text{and} \qquad f_{\hat{P}_G} = \sum_{i=s+1}^{r} t^{g_i}$$

be the Poincaré polynomials for P_G, P_K, and $\hat{P}_G$. Then the Poincaré polynomials for $\operatorname{Im} h_{\mathscr{P}}$ and for $H(G/K)$ are given, respectively, by

$$f_{\operatorname{Im} h_{\mathscr{P}}} = \prod_{i=1}^{s} (1 - t^{g_i+1}) \prod_{i=1}^{s} (1 - t^{l_i+1})^{-1},$$

and

$$f_{H(G/K)} = \prod_{i=1}^{s} (1 - t^{g_i+1}) \prod_{i=1}^{s} (1 - t^{l_i+1})^{-1} \prod_{i=s+1}^{r} (1 + t^{g_i}).$$

Proof: In view of the commutative diagram (11.1), this follows from Theorem VI, sec. 10.15.

Q.E.D.

11.6. Subgroups noncohomologous to zero. A subgroup K of G will be called *noncohomologous to zero* (n.c.z.) in G if the homomorphism $j_K^\#\colon H(G) \to H(K)$ is surjective.

Theorem VI: Let K be a compact connected subgroup of a compact connected Lie group G. Then the following conditions are equivalent:

(1) K is n.c.z. in G.

(2) F is n.c.z. in E.

(3) The homomorphism $\pi^\#: H(G/K) \to H(G)$ is injective.

(4) The Weil homomorphism $h_{\mathscr{P}}$ is trivial; (i.e., $h_{\mathscr{P}}^+ = 0$).

(5) There is an isomorphism of graded cohomology algebras $H(G) \cong H(G/K) \otimes H(K)$, which makes the diagram

$$\begin{tikzcd}
& H(G/K) \otimes H(K) \arrow[dd, "\cong"] \arrow[dr] & \\
H(G/K) \arrow[ur] \arrow[dr, "\pi^\#"'] & & H(K) \\
& H(G) \arrow[ur, "j_K^\#"'] &
\end{tikzcd}$$

commute.

(6) $\dim H(G) = \dim H(G/K) \dim H(K)$.

(7) $j_I^\vee$ is surjective.

(8) The kernel of $j_K^\#$ coincides with the ideal generated by $\operatorname{Im}(\pi^\#)^+$.

(9) There is an isomorphism of graded spaces $P_G \cong \hat{P}_G \oplus P_K$.

(10) There is an isomorphism of graded algebras $H(G/K) \cong \wedge \hat{P}_G$.

(11) The algebra $H(G/K)$ is generated by 1 and elements of odd degree.

Proof: In view of the diagram (11.1) the theorem follows from Theorem IX, sec. 10.18, and Theorem X, sec. 10.19.

Q.E.D.

11.7. Subgroups of the same rank. Recall from sec. 4.12, volume II, that the rank of a compact connected Lie group G is the dimension of P_G. In view of sec. 5.32 we have

$$\operatorname{rank} G = \dim P_G = \dim P_E = \operatorname{rank} E,$$

where E is the Lie algebra of G.

Now let T be a maximal torus in G. Then according to Theorem IV, sec. 4.12, volume II, $\dim T = \operatorname{rank} G$. On the other hand, it is easy to

check that the Lie algebra F of T is a Cartan subalgebra of E. Thus Theorem XII, sec. 10.23, (which asserts that $\dim F = \operatorname{rank} E$) provides an algebraic proof that $\dim T = \operatorname{rank} G$.

Theorem VII: Let K be a compact connected subgroup of a compact connected Lie group G. Then the following conditions are equivalent:

(1) G and K have the same rank.

(2) The Weil homomorphism $h_{\mathscr{P}}: (\vee F^*)_I \to H(G/K)$ is surjective.

(3) $H(G/K) \cong (\vee F^*)_I/J$, where J is the ideal generated by $j_I^\vee((\vee^+ E^*)_I)$.

(4) $H(G/K)$ is evenly graded.

(5) The Euler–Poincaré characteristic of $H(G/K)$ is nonzero.

(6) $j_I^\vee$ is injective.

(7) $H^{n-m}(G/K) \subset \operatorname{Im} h_{\mathscr{P}}$ ($n = \dim G$, $m = \dim K$).

If these conditions hold, the Samelson space $\hat{P}_G = 0$ is zero and thus $(\pi^\#)^+ = 0$. Moreover if the Poincaré polynomials for P_G and P_K are given by $\sum_{i=1}^r t^{g_i}$ and $\sum_{i=1}^r t^{l_i}$, then the Poincaré polynomial for $H(G/K)$ is

$$\prod_{i=1}^{r} (1 - t^{g_i+1}) \Big/ \prod_{i=1}^{r} (1 - t^{l_i+1})$$

and

$$\dim H(G/K) = \chi_{G/K} = \frac{\prod_{i=1}^r (g_i + 1)}{\prod_{i=1}^r (l_i + 1)}.$$

Proof: In view of diagram (11.1) the equivalence of conditions (1)–(6) follows from Theorem XI, sec. 10.22. Proposition IX, sec. 10.29, shows that (2) is equivalent to (7). For the last part, apply Theorem V, sec. 11.5.

Q.E.D.

Corollary I: Suppose some Pontrjagin number of the homogeneous space G/K is nonzero. Then G and K have the same rank.

Proof: According to Proposition III, sec. 5.11, volume II, the tangent bundle of G/K has total space $G \times_K E/F$. Thus formula (8.4), sec. 8.25, volume II, shows that the characteristic classes of $\tau_{G/K}$ are in $\operatorname{Im} h_{\mathscr{P}}$. Hence, by hypothesis, $\operatorname{Im} h_{\mathscr{P}} \supset H^{n-m}(G/K)$, and so condition (7) of the theorem is satisfied.

Q.E.D.

Corollary II: Let T be a maximal torus of a compact connected Lie group G. Then $H(G/T)$ is evenly graded. Its Poincaré polynomial is given by

$$f_{H(G/T)} = \frac{\prod_{i=1}^{r} (1 - t^{g_i+1})}{(1 - t^2)^r},$$

where $\sum_{i=1}^{r} t^{g_i}$ is the Poincaré polynomial for P_E. Moreover, if $| W_G |$ is the order of the Weyl group (cf. sec. 11.8) then

$$| W_G | = \chi_{G/T} = \dim H(G/T) = \prod_{i=1}^{r} \frac{g_i + 1}{2}.$$

Proof: That $| W_G | = \chi_{G/T}$ is shown in Proposition XIII, sec. 4.21, volume II. The rest of the corollary follows from the theorem.

Q.E.D.

§3. The Weyl group

11.8. The Weyl group. Let G be a compact connected Lie group with maximal torus T. Denote the corresponding Lie algebras by E and F.

Let N_T be the normalizer of T (in G). Then the factor group

$$W_G = N_T/T$$

is a finite group (cf. sec. 2.16, volume II). It is, up to an isomorphism, independent of the choice of T and is called the *Weyl group of G*.

A smooth right action Φ of W_G on G/T is defined by

$$\Phi_{\bar{a}}(\pi x) = \Phi(\pi x, \bar{a}) = \pi(xa), \qquad \bar{a} \in W_G, \quad x \in G,$$

where $a \in N_T$ is any representative of $\bar{a}$. Hence a representation $\Phi^\#$ of W_G in $H(G/T)$ is defined by

$$\Phi^\#(\bar{a}) = \Phi_{\bar{a}}^\#, \qquad \bar{a} \in W_G.$$

The corresponding invariant subspace is denoted by $H(G/T)_{W_G=1}$.

On the other hand, the *left regular representation* of W_G is defined as follows: Let V be the real vector space whose elements are the formal sums $\sum_{\bar{a}_\nu \in W_G} \lambda^\nu \bar{a}_\nu$ with $\lambda^\nu \in \mathbb{R}$ (so that the elements of W_G are a basis of V). Set

$$\bar{a} \cdot \left(\sum_\nu \lambda^\nu \bar{a}_\nu\right) = \sum_\nu \lambda^\nu \bar{a}\bar{a}_\nu, \qquad \bar{a} \in W_G, \ \sum_\nu \lambda^\nu \bar{a}_\nu \in V.$$

Proposition III: The representation $\Phi^\#$ is equivalent to the left regular representation of W_G.

Proof: It follows from [3; (2.6), p. 12] that it is sufficient to show that

$$\operatorname{tr} \Phi_{\bar{a}}^\# = 0 \quad \text{if} \quad \bar{a} \neq \bar{e}, \quad \text{and} \quad \operatorname{tr} \Phi_{\bar{e}}^\# = |W_G|.$$

Fix $\bar{a} \in W_G$ and consider the Lefschetz number of the map $\Phi_{\bar{a}}$ (cf. sec. 10.7, volume I). It is given by

$$L(\Phi_{\bar{a}}) = \sum_p (-1)^p \operatorname{tr} \Phi_{\bar{a}}^p,$$

where $\Phi_{\bar{a}}^p$ denotes the restriction of $\Phi_{\bar{a}}^{\#}$ to $H^p(G/T)$. Since $H(G/T)$ is evenly graded (cf. Corollary II of Theorem VII), it follows that

$$L(\Phi_{\bar{a}}) = \sum_p \operatorname{tr} \Phi_{\bar{a}}^p = \operatorname{tr} \Phi_{\bar{a}}^{\#}$$

Now, if $\bar{a} \neq \bar{e}$, then $\Phi_{\bar{a}}$ has no fixed points and thus $L(\Phi_{\bar{a}}) = 0$ (cf. the Corollary of Theorem III, sec. 10.8, volume I). This shows that

$$\operatorname{tr} \Phi_{\bar{a}}^{\#} = 0 \qquad \text{if} \qquad \bar{a} \neq \bar{e}.$$

On the other hand, Proposition XIII, sec. 4.21, volume II, shows that $\chi_{G/T} = | W_G |$ and so

$$\operatorname{tr} \Phi_{\bar{e}}^{\#} = \chi_{G/T} = | W_G |.$$

Q.E.D.

Corollary: $H(G/T)_{W_G=1} = H^0(G/T)$.

Proof: Clearly, $H^0(G/T)$ is contained in the invariant subspace. But the invariant subspace of the regular representation has dimension 1.

Q.E.D.

11.9. The image of $j_{\theta=0}^{\vee}$. Observe that since T is normal in N_T, the operators Ad a ($a \in N_T$) in E restrict to operators in the Lie algebra F of T. Thus a representation Ψ of W_G in F is given by

$$\Psi(\bar{a})(y) = (\operatorname{Ad} a)y, \qquad a \in N_T, \quad y \in F.$$

Denote the induced representation in $\vee F^*$ by $\Psi^{\vee}$:

$$\Psi^{\vee}(\bar{a}) = (\Psi(\bar{a}^{-1}))^{\vee}.$$

Denote the invariant subalgebra by $(\vee F^*)_{W_G=1}$.

Now consider the inclusion $j: F \to E$ and the induced homomorphism $j_{\theta=0}^{\vee}: (\vee E^*)_{\theta=0} \to \vee F^*$. (Since F is abelian, $\vee F^* = (\vee F^*)_{\theta=0}$.)

Theorem VIII: The homomorphism $j_{\theta=0}^{\vee}$ is an isomorphism of $(\vee E^*)_{\theta=0}$ onto $(\vee F^*)_{W_G=1}$:

$$j_{\theta=0}^{\vee}: (\vee E^*)_{\theta=0} \xrightarrow{\cong} (\vee F^*)_{W_G=1}.$$

Proof: As we remarked in the beginning of sec. 11.7, (E, F) is an equal rank pair. Thus by Theorem XI, (7), sec. 10.22, $j_{\theta=0}^{\vee}$ is injective.

It remains to be shown that

$$\operatorname{Im} j^{\vee}_{\theta=0} = (\vee F^*)_{W_G=1}.$$

It follows from the definition that an element $\Omega \in \vee^p F^*$ is in $(\vee^p F^*)_{W_G=1}$ if and only if

$$\Omega(y_1, \ldots, y_p) = \Omega((\operatorname{Ad} a)y_1, \ldots, (\operatorname{Ad} a)y_p), \qquad a \in N_T, \quad y_i \in F.$$

This implies that $\operatorname{Im} j^{\vee}_{\theta=0} \subset (\vee F^*)_{W_G=1}$.

To prove the converse, consider the principal bundle $\mathscr{P} = (G, \pi, G/T, T)$. Since (G, T) is an equal rank pair, Theorem VII, sec. 11.7, implies that the Weil homomorphism $h_{\mathscr{P}}$ is surjective. Thus if $X \subset \vee F^*$ is any graded subspace satisfying $\vee F^* = X \oplus \ker h_{\mathscr{P}}$, then $h_{\mathscr{P}}$ restricts to an isomorphism

$$X \xrightarrow{\cong} H(G/T).$$

According to Lemma I, below, $h_{\mathscr{P}}$ satisfies

$$h_{\mathscr{P}} \circ \Psi^{\vee}(\bar{a}) = \Phi^{\#}_{\bar{a}} \circ h_{\mathscr{P}}.$$

Thus $\ker h_{\mathscr{P}}$ is W_G-stable. Since W_G is a finite group, the graded subspace X can be chosen to be stable under the operators $\Psi^{\vee}(\bar{a})$, $\bar{a} \in W_G$ (cf. [3; (1.1), p. 3]).

Let $\gamma\colon H(G/T) \to \vee F^*$ be the linear injection given by

$$\gamma(h_{\mathscr{P}}\Omega) = \Omega, \qquad \Omega \in X.$$

Then γ is homogeneous of degree zero, and satisfies the following properties:

(i) $\gamma(1) = 1$, (ii) $h_{\mathscr{P}} \circ \gamma = \iota$, (iii) $\gamma \circ \Phi^{\#}_{\bar{a}} = \Psi^{\vee}(\bar{a}) \circ \gamma$, $\bar{a} \in W_G$.

Now define a linear map

$$g\colon H(G/T) \otimes (\vee E^*)_{\theta=0} \to \vee F^*$$

by $g(\alpha \otimes \Omega) = \gamma(\alpha) \vee j^{\vee}_{\theta=0}(\Omega)$. According to sec. 2.9, g is an isomorphism of graded spaces. Moreover, it follows from (iii), above, that

$$g \circ (\Phi^{\#}_{\bar{a}} \otimes \iota) = \Psi^{\vee}(\bar{a}) \circ g.$$

Hence g restricts to an isomorphism

$$H(G/T)_{W_G=1} \otimes (\vee E^*)_{\theta=0} \xrightarrow{\cong} (\vee F^*)_{W_G=1}.$$

But the corollary to Proposition III, sec. 11.8, states that $H(G/T)_{W_G=1} = H^0(G/T)$; hence this isomorphism is exactly $j^\vee_{\theta=0}$.

Q.E.D.

11.10. Lemma I: The Weil homomorphism $h_{\mathscr{P}}$ satisfies

$$h_{\mathscr{P}} \circ \Psi^\vee(\bar{a}) = \Phi^\#_{\bar{a}} \circ h_{\mathscr{P}}, \qquad \bar{a} \in W_G.$$

Proof: Fix $a \in N_T$. Define a commutative diagram of smooth maps,

$$\begin{tikzcd} G \arrow[r, "\psi_a", "\cong"'] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/T \arrow[r, "\varphi_a"'] & G/T, \end{tikzcd}$$

by setting

$$\psi_a(x) = a^{-1}xa \qquad \text{and} \qquad \varphi_a(\pi x) = \pi(a^{-1}xa) = a^{-1} \cdot \pi(xa), \quad x \in G.$$

Since left translation of G/T by a^{-1} is homotopic to the identity,

$$\varphi_a^\# = \Phi_{\bar{a}}^\#.$$

Next, let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, G/T, T)$ be the pullback of $\mathscr{P}$ to G/T under φ_a. Then there is an isomorphism of principal bundles $\alpha: \hat{P} \to G$. Define a smooth fibre preserving map $\hat{\psi}_a: G \to \hat{P}$ by $\hat{\psi}_a = \psi_a \circ \alpha^{-1}$. This yields the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\hat{\psi}_a"] \arrow[d, "\pi"'] & \hat{P} \arrow[r, "\alpha", "\cong"'] \arrow[d, "\hat{\pi}"] & G \arrow[d, "\pi"] \\ G/T \arrow[r, "\iota"'] & G/T \arrow[r, "\varphi_a"'] & G/T. \end{tikzcd}$$

Since $\psi_a(xy) = \psi_a(x)\psi_a(y)$ and $\alpha(xy) = \alpha(x)y$ $(x \in G, y \in T)$, it follows that

$$\hat{\psi}_a(xy) = \hat{\psi}_a(x)\psi_a(y), \qquad x \in G, \quad y \in T.$$

Now denote the restriction of ψ_a to T by β, and apply Theorem II, sec. 6.19, volume II, to α and Theorem III, sec. 6.25, volume II, to $\hat{\psi}_a$ to obtain the relation

$$h_{\mathscr{P}} \circ (\beta')^{\vee} = h_{\hat{\mathscr{P}}} = \varphi_a^{\#} \circ h_{\mathscr{P}}.$$

Clearly $(\beta')^{\vee} = \Psi'^{\vee}(\bar{a})$. Thus, since $\varphi_a^{\#} = \Phi_{\bar{a}}^{\#}$, the lemma follows.

Q.E.D.

§4. Examples of homogeneous spaces

Recall that in article 7, Chapter VI we computed the cohomology of certain compact Lie groups. The results are contained in the following table. (E denotes the Lie algebra of G.)

G	$U(n)$	$SO(2n+1)$	$SO(2n)$	$Q(n)$
E	$\mathrm{Sk}(n; C)$	$\mathrm{Sk}(2n+1)$	$\mathrm{Sk}(2n)$	$\mathrm{Sk}(n; H)$
basis of P_E	Φ^U_{2p-1}, $1 \le p \le n$	Φ^{SO}_{4p-1}, $1 \le p \le n$	Sf, Φ^{SO}_{4p-1}, $1 \le p < n$	Φ^Q_{4p-1}, $1 \le p \le n$
$(\vee E^*)_{\theta=0}$	$\vee(C^U_1, \dots, C^U_n)$	$\vee(C^{SO}_2, \dots, C^{SO}_{2n})$	$\vee(\mathrm{Pf}, C^{SO}_2, \dots, C^{SO}_{2n-2})$	$\vee(C^Q_2, \dots, C^Q_{2n})$
rank E	n	n	n	n

In this article we consider homogeneous spaces G/K, where G is one of the groups above, and K is a product of groups, each isomorphic to one of those above.

As usual E and F denote the Lie algebras of G and K, and $j\colon F \to E$ is the inclusion.

In each case we shall determine

(1) the Samelson subspace for G/K, and

(2) the homomorphism $j^\vee_{\theta=0}\colon (\vee E^*)_{\theta=0} \to (\vee F^*)_{\theta=0}$.

Since the ranks of G and K can be read off from the table above, it will be possible for the reader to verify at once that each pair is a Cartan pair. Thus Theorem IV, sec. 11.5, and Theorem V, sec. 11.5, allow us to determine the cohomology of G/K. This information is contained in the tables at the end of this chapter.

11.11. $G = U(n)$. Let V be a complex n-dimensional vector space with Hermitian inner product $\langle\ ,\ \rangle$. Let $V = V_1 \oplus \cdots \oplus V_q \oplus W$ be a fixed orthogonal decomposition of V into complex subspaces and let

$\dim V_i = k_i$, $(i = 1, \ldots, q)$. Define an inclusion map

$$j_U: U(k_1) \times \cdots \times U(k_q) \to U(n),$$

by

$$j_U(\sigma_1, \ldots, \sigma_q) = \sigma_1 \oplus \cdots \oplus \sigma_q \oplus \iota_W, \qquad \sigma_i \in U(k_i).$$

Its derivative,

$$j: \mathrm{Sk}(k_1; C) \oplus \cdots \oplus \mathrm{Sk}(k_q; C) \to \mathrm{Sk}(n; C),$$

is given by

$$j(\varphi_1, \ldots, \varphi_q) = \varphi_1 \oplus \cdots \oplus \varphi_q \oplus 0, \qquad \varphi_i \in \mathrm{Sk}(k_i; C).$$

Examples: 1. $U(n)/U(k)$: In this case $q = 1$ and $k_1 = k$. It follows directly from the definitions that

$$j_P(\Phi_{2p-1}^{U(n)}) = \Phi_{2p-1}^{U(k)}, \qquad 1 \le p \le k.$$

This shows that j_P is surjective; hence so is $j^\#$. Thus (cf. Theorem VI, (2), sec. 11.6) $U(k)$ is n.c.z. in $U(n)$.

Now Theorem X, (5), sec. 10.19, implies that $\ker j_P = \hat{P}$. A simple degree argument shows that $\ker j_P$ is spanned by the $\Phi_{2p-1}^{U(n)}$ with $k + 1 \le p \le n$; thus

$$\hat{P} = (\Phi_{2k+1}^{U}, \ldots, \Phi_{2n-1}^{U}).$$

Finally, observe that Proposition I, sec. A.2, yields

$$\check{j}_{\theta=0}(C_p^{U(n)}) = \begin{cases} C_p^{U(k)}, & 1 \le p \le k, \\ 0, & k + 1 \le p \le n. \end{cases}$$

2. $U(n)/(U(k) \times U(n-k))$: In this case $V = V_1 \oplus V_2$, $\dim V_1 = k$ and $\dim V_2 = n - k$. Let σ_V be the isometry defined by $\sigma_V = \iota$ in V_1 and $\sigma_V = -\iota$ in V_2. Then $\sigma_V^2 = \iota$ and so an involution σ in $U(n)$ is given by $\sigma(\tau) = \sigma_V \tau \sigma_V^{-1}$. The fixed point subgroup of σ is $U(k) \times U(n-k)$.

Thus $(U(n), U(k) \times U(n-k))$ is a symmetric pair. Moreover it is clearly an equal rank pair (cf. sec. 11.7 and sec. 10.22), and hence a Cartan pair with $\hat{P} = 0$.

Finally, note that

$$(\vee F^*)_{\theta=0} = (\vee\, \mathrm{Sk}(k; C)^*)_{\theta=0} \otimes (\vee\, \mathrm{Sk}(n-k; C)^*)_{\theta=0}.$$

Hence Proposition I, sec. A.2, gives

$$\check{j}_{\theta=0}(C_p^{U(n)}) = \sum_{q+r=p} C_q^{U(k)} \otimes C_r^{U(n-k)},$$

where $C_0^U = 1$ and $C_s^{U(l)} = 0$ if $s > l$.

3. $U(n)/U(k_1) \times \cdots \times U(k_q)$: Set $k = \sum_i k_i$ and consider the inclusion $U(k_1) \times \cdots \times U(k_q) \to U(n)$, defined above. There is a commutative diagram of smooth maps

$$\begin{tikzcd} & U(n) \arrow[dl] \arrow[dr] & \\ U(n)/(U(k_1) \times \cdots \times U(k_q)) \arrow[rr] & & U(n)/U(k). \end{tikzcd}$$

Now let $\hat{P}$ and $\hat{P}_1$ denote the Samelson subspaces for the pairs $(U(n), U(k_1) \times \cdots \times U(k_q))$ and $(U(n), U(k))$. The diagram shows that $\hat{P}_1 \subset \hat{P}$. On the other hand, by Theorem IV, sec. 11.5,

$$\begin{aligned} \dim \hat{P} &\le \operatorname{rank} U(n) - \operatorname{rank}(U(k_1) \times \cdots \times U(k_q)) \\ &= n - k = \dim \hat{P}_1. \end{aligned}$$

Hence, (cf. Example 1, above)

$$\hat{P} = \hat{P}_1 = (\Phi_{k+1}^U, \ldots, \Phi_{2n-1}^U).$$

Finally, observe that

$$\check{j}_{\theta=0}(C_p^{U(n)}) = \sum_{p_1+\cdots+p_q=p} C_{p_1}^{U(k_1)} \otimes \cdots \otimes C_{p_q}^{U(k_q)};$$

this follows from the obvious generalization of Proposition I, sec. A.2, to direct decompositions into several subspaces. (Note that $C_0^U = 1$ and $C_{p_\nu}^{U(k_\nu)} = 0$ if $p_\nu > k_\nu$.)

4. $U(n)/SO(n)$: Write $V = C \otimes X$, where X is an n-dimensional Euclidean space and

$$\langle \lambda \otimes x, \mu \otimes y \rangle = \lambda\bar{\mu}\langle x, y\rangle, \qquad \lambda, \mu \in C, \quad x, y \in X.$$

Then an inclusion $SO(n) \to U(n)$ is given by $\sigma \mapsto \iota \otimes \sigma$.

Next, consider the (real) linear involution ω_V of V given by

$$\omega_V(\lambda \otimes x) = \bar{\lambda} \otimes x, \qquad \lambda \in \mathbb{C}, \quad x \in X.$$

It determines the involution ω of $U(n)$ defined by

$$\omega(\sigma) = \omega_V \sigma \omega_V^{-1}, \qquad \sigma \in U(n).$$

The 1-component of the fixed point subgroup for this involution is precisely $SO(n)$. Thus $(U(n), SO(n))$ is a symmetric pair and hence a Cartan pair (cf. sec. 10.26).

Moreover, it follows at once from the definitions that

$$(\omega')^{\wedge}_{\theta=0}(\Phi^U_{2p-1}) = (-1)^p \Phi^U_{2p-1}, \qquad p = 1, \ldots, n.$$

Thus Proposition VII, sec. 10.26, shows that $\hat{P}$ is spanned by the elements Φ^U_{2p-1} (p odd).

Finally, observe that

$$\check{j}_{\theta=0}(C^U_p) = \begin{cases} 0 & p \text{ odd}, \\ (-1)^{p/2} C^{SO}_p, & p \text{ even}, \end{cases}$$

as follows directly from the definitions.

5. $U(2m)/Q(m)$: Consider V as the underlying complex space $X_{\mathbb{C}}$ of an m-dimensional quaternionic space X as described in sec. 6.30. In particular, $Q(m) \subset U(2m)$.

Since, by definition,

$$\Phi^Q_{4p-1} = \hat{j}_{\theta=0}(\Phi^U_{4p-1}), \qquad 1 \le p \le m,$$

it follows that j_P is surjective, and so $Q(m)$ is n.c.z. in $U(2m)$. Hence $\hat{P} = \ker j_P$. A straightforward degree argument shows that $\ker j_P$ is the space spanned by Φ^U_{2p-1} (p odd). Thus $\hat{P} = (\Phi^U_1, \Phi^U_5, \Phi^U_9, \ldots)$.

Finally, combining Lemma XI, (1), sec. 6.24, with the definitions in sec. 6.30, we find that

$$\check{j}_{\theta=0}(C^U_p) = \begin{cases} 0, & p \text{ odd}, \\ C^Q_p, & p \text{ even}. \end{cases}$$

11.12. $G = SO(n)$. Let $(X, \langle\ ,\ \rangle)$ be an n-dimensional Euclidean space. An orthogonal decomposition of X leads (exactly as in sec. 11.11) to an inclusion

$$SO(n_1) \times \cdots \times SO(n_r) \to SO(n), \qquad \sum_{j=1}^{r} n_j \le n.$$

Examples: 1. $SO(2m+1)/SO(2k+1)$: Precisely as in Example 1, sec. 11.11, it follows that $SO(2k+1)$ is n.c.z. in $SO(2m+1)$ and that $\hat{P}$ is spanned by the elements Φ_{4p-1}^{SO} ($p = k+1, \ldots, m$). Moreover,

$$j_{\theta=0}^{\vee}(C_{2p}^{SO(2m+1)}) = \begin{cases} C_{2p}^{SO(2k+1)}, & 1 \le p \le k \\ 0, & k+1 \le p \le m. \end{cases}$$

2. $SO(2m+1)/SO(2k)$: In view of the commutative diagram

$$\begin{tikzcd} & SO(2m+1) \arrow[dl] \arrow[dr] & \\ SO(2m+1)/SO(2k) \arrow[rr] & & SO(2m+1)/SO(2k+1) \end{tikzcd}$$

the Samelson subspace $\hat{P}$ for the pair $(SO(2m+1), SO(2k+1))$ is contained in the Samelson subspace $\hat{P}$ for $(SO(2m+1), SO(2k))$. Since

$$\dim \hat{P} \le \operatorname{rank} SO(2m+1) - \operatorname{rank} SO(2k) = m - k = \dim \hat{P}_1$$

(cf. Theorem IV, sec. 11.5), it follows that $\hat{P} = \hat{P}_1$.

In particular (cf. Example 1, above), $\hat{P}$ is spanned by the elements Φ_{4p-1}^{SO} ($k+1 \le p \le m$).

The same formula for $j_{\theta=0}^{\vee}$ as in Example 1 continues to hold here. However, $C_{2k}^{SO(2k)}$ is *not* a generating element of $(\vee F^*)_{\theta=0}$, and, in fact, Proposition VI, sec. A.6, yields

$$C_{2k}^{SO(2k)} = \mathrm{Pf} \vee \mathrm{Pf}.$$

Thus

$$j_{\theta=0}^{\vee}(C_{2p}^{SO(2m+1)}) = \begin{cases} C_{2p}^{SO(2k)}, & 1 \le p < k, \\ \mathrm{Pf} \vee \mathrm{Pf}, & p = k, \\ 0, & p > k. \end{cases}$$

3. $SO(2m+1)/(SO(2k) \otimes SO(2m-2k+1))$: Consider an orthogonal decomposition $X = Y \oplus Z$, where $\dim Y = 2k$. Define involutions ω_X of X and ω of $SO(2m+1)$ by $\omega_X = \iota$ in Y, $\omega_X = -\iota$ in Z and

$$\omega(\sigma) = \omega_X \sigma \omega_X^{-1}, \qquad \sigma \in SO(2m+1).$$

Then the fixed point subgroup of ω has $SO(2k) \times SO(2m-2k+1)$ as 1-component, and so the pair is symmetric. Since it is also an equal rank pair, $\hat{P} = 0$.

Recall from Example 2, above, that $C_{2k}^{SO(2k)} = \mathrm{Pf} \vee \mathrm{Pf}$, and set $C_0^{SO} = 1$ and $C_{2s}^{SO(l)} = 0$, $2s > l$. Thus Proposition I, sec. A.2, yields

$$\check{j}_{\theta=0}(C_{2p}^{SO(2m+1)}) = \begin{cases} \sum\limits_{q+r=p} C_{2q}^{SO(2k)} \otimes C_{2r}^{SO(2m-2k+1)}, & p < k, \\ \sum\limits_{q=0}^{k-1} C_{2q}^{SO(2k)} \otimes C_{2p-2q}^{SO(2m-2k+1)} + \mathrm{Pf} \vee \mathrm{Pf} \otimes C_{2p-2k}^{SO(2m-2k+1)}, & \\ & p \geq k. \end{cases}$$

4. $SO(2m+1)/SO(2k_1) \times \cdots \times SO(2k_q) \times SO(2l+1)$, $\sum k_i \geq 1$, $l \geq 0$: Set $k = \sum_i k_i$. Exactly as in Example 2 above it follows that the Samelson subspace for this pair coincides with the Samelson subspace for $(SO(2m+1), SO(2k+2l+1))$. Hence it is spanned by the elements Φ_{4p-1}^{SO}, $k+l+1 \leq p \leq m$. The formula for $\check{j}_{\theta=0}$ is the obvious generalization of the formula in Example 3, above.

5. $SO(2m)/SO(2k)$, $k < m$: First observe that, as in Example 2, above,

$$\check{j}_{\theta=0}(C_{2p}^{SO(2m)}) = \begin{cases} C_{2p}^{SO(2k)}, & p < k, \\ \mathrm{Pf} \vee \mathrm{Pf}, & p = k, \\ 0, & p > k. \end{cases}$$

Moreover, it follows from Proposition VIII, sec. A.6, that

$$\check{j}_{\theta=0}(\mathrm{Pf}) = 0.$$

Next recall (cf. sec. 11.5) that the P_E-algebra for this pair is given by $((\vee F^*)_{\theta=0}; \sigma)$, where $\sigma = \check{j}_{\theta=0} \circ \tau$ and $\tau: P_E \to (\vee E^*)_{\theta=0}$ is a transgression. In view of formula (6.15), sec. 6.19, and formula (6.17), sec. 6.22, we may choose τ so that

$$\tau(\mathrm{Sf}) = \lambda \, \mathrm{Pf} \qquad \text{and} \qquad \tau(\Phi_{4p-1}^{SO(2m)}) = \lambda_p C_{2p}^{SO(2m)}, \qquad 1 \leq p < m,$$

where λ and λ_p are nonzero scalars.

It follows that

$$\sigma(\mathrm{Sf}) = 0 \qquad \text{and} \qquad \sigma(\Phi_{4p-1}^{SO(2m)}) = 0, \quad k+1 \leq p < m.$$

Thus these vectors are in $\hat{P}$. On the other hand,

$$\dim \hat{P} \leq \operatorname{rank} SO(2m) - \operatorname{rank} SO(2k) = m - k$$

and so the elements $\Phi_{4p-1}^{SO(2m)}$ $(p = k+1, \ldots, m-1)$ and Sf form a basis of $\hat{P}$.

6. $SO(2m)/SO(2k+1)$: In this case Proposition I, sec. A.2, and Proposition VIII, sec. A.6, yield

$$j^{\vee}_{\theta=0}(C_{2p}^{SO(2m)}) = \begin{cases} C_{2p}^{SO(2k+1)}, & 1 \le p \le k, \\ 0, & k+1 \le p \le m, \end{cases}$$

and

$$j^{\vee}_{\theta=0}(\mathrm{Pf}) = 0.$$

Thus $j^{\vee}_{\theta=0}$ is surjective and so (cf. Theorem VI, (7), sec. 11.6) $SO(2k+1)$ is n.c.z. in $SO(2m)$. Exactly as in Example 5 it follows that $\hat{P}$ is spanned by $\Phi_{4p-1}^{SO(2m)}$ $(k+1 \le p < m)$ and Sf.

7. $SO(2m)/(SO(2k) \times SO(2m-2k))$: As in Example 3, above, this is a symmetric equal rank pair and so $\hat{P} = 0$. Proposition I, sec. A.2, and Proposition VIII, sec. A.6, yield

$$j^{\vee}_{\theta=0}(C_{2p}^{SO(2m)}) = \sum_{q=0}^{p} C_{2q}^{SO(2k)} \otimes C_{2p-2q}^{SO(2m-2k)}, \qquad 1 \le p \le m$$

and

$$j^{\vee}_{\theta=0}(\mathrm{Pf}^{SO(2m)}) = \mathrm{Pf}^{SO(2k)} \otimes \mathrm{Pf}^{SO(2m-2k)}.$$

Note that in the first formula $C_0^{SO(2l)} = 1$, $C_{2l}^{SO(2l)} = \mathrm{Pf} \vee \mathrm{Pf}$, and $C_{2r}^{SO(2l)} = 0$, $r > l$.

8. $SO(2m)/(SO(2k+1) \times SO(2m-2k-1))$: As in Example 3, this is a symmetric pair (and hence a Cartan pair—cf. sec. 10.26). The involution ω_X of X reverses orientations, and hence

$$(\omega')^{\vee}_{\theta=0}(\mathrm{Pf}) = -\mathrm{Pf}$$

(cf. Proposition VII, (1), sec. A.6).

It follows (because $\mathrm{Sf} = \lambda\varrho_E(\mathrm{Pf})$—cf. sec. 6.22) that

$$(\omega')^{\wedge}_{\theta=0}(\mathrm{Sf}) = -\mathrm{Sf}.$$

Hence, by Proposition VII, sec. 10.26, $\mathrm{Sf} \in \hat{P}$.

But

$$\dim \hat{P} \le \operatorname{rank} SO(2m) - \operatorname{rank}(SO(2k+1) \times SO(2m-2k-1)) = 1,$$

and so Sf is a basis of $\hat{P}$.

Finally, as in Example 7,

$$\check{j}_{\theta=0}(C_{2p}^{SO(2m)}) = \sum_{q=0}^{p} C_{2q}^{SO(2k+1)} \otimes C_{2p-2q}^{SO(2m-2k-1)}, \qquad 1 \leq p \leq m,$$

and

$$\check{j}_{\theta=0}(\mathrm{Pf}) = 0,$$

where $C_0^{SO(2l+1)} = 1$ and $C_{2s}^{SO(2l+1)} = 0$, $s > l$.

9. $SO(2m)/U(m)$: Let X denote the underlying $2m$-dimensional Euclidean space of a complex m-dimensional Hermitian space V. Then $(SO(2m), U(m))$ is an equal rank pair, and hence $\hat{P} = 0$.

Moreover, $\check{j}_{\theta=0}$ is given by

$$\check{j}_{\theta=0}(C_{2p}^{SO}) = (-1)^p \sum_{q+r=2p} (-1)^r C_q^U \vee C_r^U \tag{11.2}$$

and

$$\check{j}_{\theta=0}(\mathrm{Pf}) = C_m^U, \tag{11.3}$$

where, as usual, $C_0^U = 1$ and $C_q^U = 0$, $q > m$.

To see this, observe first that the Hermitian inner product in V determines the $\mathbb{R}$-linear isomorphism $\alpha\colon V \xrightarrow{\cong} V^*$, given by

$$\langle \alpha x, y \rangle = \langle y, x \rangle, \qquad x, y \in V.$$

Define an isomorphism of complex spaces

$$\theta\colon \mathbb{C} \otimes X \xrightarrow{\cong} V \oplus V^*,$$

by setting

$$\theta(\lambda \otimes x) = (\lambda x, \lambda\alpha(x)), \qquad \lambda \in \mathbb{C}, \quad x \in X.$$

Now let $\varphi \in \mathrm{Sk}(m; \mathbb{C})$ $(= \mathrm{Sk}_V)$. Then $j(\varphi) \in \mathrm{Sk}(2m)$ $(= \mathrm{Sk}_X)$. Denote $j(\varphi)$ by ψ. Then $\iota \otimes \psi$ is a complex linear transformation of $\mathbb{C} \otimes X$, and, evidently,

$$\theta \circ (\iota \otimes \psi) = (\varphi \oplus -\varphi^*) \circ \theta.$$

It follows that (cf. sec. A.2)

$$\begin{aligned} C_{2p}(\psi) &= C_{2p}(\iota \otimes \psi) = C_{2p}(\varphi \oplus -\varphi^*) \\ &= \sum_{q+r=2p} (-1)^r C_q(\varphi) C_r(\varphi) \\ &= (-1)^p \sum_{q+r=2p} (-1)^r \Big(\frac{1}{i^q} C_q(\varphi)\Big)\Big(\frac{1}{i^r} C_r(\varphi)\Big), \end{aligned}$$

whence

$$(j^{\vee}_{\theta=0}C^{SO}_{2p})(\varphi, \ldots, \varphi) = (-1)^p \sum_{q+r=p} (-1)^r (C^U_q \vee C^U_r)(\varphi, \ldots, \varphi), \qquad \varphi \in \mathrm{Sk}(m; \mathbb{C}).$$

This establishes (11.2). Formula (11.3) follows from Example 3, sec. A.7.

10. $SO(4k)/Q(k)$: Regard X as the underlying $4k$-dimensional Euclidean space of a k-dimensional quaternionic space V (cf. sec. 6.30). The corresponding inclusion $Q(k) \to SO(4k)$ is the composite of the inclusions

$$Q(k) \to U(2k) \qquad \text{and} \qquad U(2k) \to SO(4k)$$

of Example 5, sec. 11.11, and Example 9, sec. 11.12.

It follows that in this case

$$j^{\vee}_{\theta=0}(C^{SO}_{2p}) = (-1)^p \sum_{q+r=p} C^Q_{2q} \vee C^Q_{2r}, \qquad 1 \le p \le 2k$$

and

$$j^{\vee}_{\theta=0}(\mathrm{Pf}) = C^Q_{2k},$$

where $C^Q_0 = 1$ and $C^Q_{2q} = 0$, $q > k$.

This implies that

$$j^{\vee}_{\theta=0}(C^{SO}_{2p}) - (-1)^p 2C^Q_{2p} \in (\vee^+ F^*)_{\theta=0} \cdot (\vee^+ F^*)_{\theta=0}, \qquad 1 \le p \le 2k.$$

Hence (cf. Theorem II, sec. 6.14)

$$\varrho_F j^{\vee}_{\theta=0}(C^{SO}_{2p}) = \lambda_p \Phi^Q_{4p-1}, \qquad 1 \le p \le k,$$

where λ_p is a nonzero scalar. This implies that there is a transgression $\tau: P_F \to (\vee F^*)_{\theta=0}$, such that $\tau(P_F)$ is spanned by the vectors $j^{\vee}_{\theta=0}(C^{SO}_{2p})$ $(1 \le p \le k)$.

But, in view of Theorem I, sec. 6.13, $\tau(P_F)$ generates $(\vee F^*)_{\theta=0}$. This shows that $j^{\vee}_{\theta=0}$ is surjective and so $Q(k)$ is n.c.z. in $SO(4k)$ (cf. Theorem VI, sec. 11.6). In particular, $\hat{P} = \ker j_P$.

A simple degree argument shows that

$$j_P(\Phi^{SO}_{4p-1}) = 0, \qquad k+1 \le p \le 2k-1.$$

Moreover,

$$\begin{aligned} j_P\varrho_E(2\mathrm{Pf} - (-1)^k C_{2k}^{SO}) &= \varrho_F(2\check{j}_{\theta=0}(\mathrm{Pf}) - (-1)^k \check{j}_{\theta=0} C_{2k}^{SO}) \\ &= 0, \end{aligned}$$

as follows from the formulae above for $\check{j}_{\theta=0}$.

Now use formula (6.15), sec. 6.19, and formula (6.17), sec. 6.22, to obtain

$$j_P(\mathrm{Sf} + (-1)^k 2^{2k-2}(2k-1)!\Phi_{4k-1}^{SO}) = 0.$$

Since $\dim \hat{P} = k$, these relations show that $\hat{P}$ is spanned by the elements

$$\Phi_{4p-1}^{SO} \quad (k+1 \leq p \leq 2k-1) \qquad \text{and} \qquad \mathrm{Sf} + (-1)^k 2^{2k-2}(2k-1)!\Phi_{4k-1}^{SO}.$$

11.13. $G = Q(n)$. Examples: 1. $Q(n)/Q(k)$: Exactly as in the unitary case (Example 1, sec. 11.11) it follows that $Q(k)$ is n.c.z. in $Q(n)$, that $\hat{P}$ is spanned by the elements Φ_{4p-1}^{Q} $(k+1 \leq p \leq n)$ and that

$$\check{j}_{\theta=0}(C_{2p}^{Q(n)}) = \begin{cases} C_{2p}^{Q(k)}, & 1 \leq p \leq k, \\ 0, & k+1 \leq p \leq n. \end{cases}$$

2. $Q(n)/(Q(k) \times Q(n-k))$: As in Example 2, sec. 11.11, this is a symmetric, equal rank pair. Thus $\hat{P} = 0$. Moreover, it follows from that example and the definition of C_{2p}^{Q} that

$$\check{j}_{\theta=0}(C_{2p}^{Q(n)}) = \sum_{r+q=p} C_{2q}^{Q(k)} \otimes C_{2r}^{Q(n-k)},$$

where $C_0^{Q(l)} = 1$ and $C_{2s}^{Q(l)} = 0$, $s > l$.

3. $Q(n)/U(n)$: Let $(Y, \langle\ ,\ \rangle_C)$ be an n-dimensional Hermitian space and consider the n-dimensional quaternionic space $Z = \mathbb{H} \otimes_C Y$ where $\mathbb{H}$ is regarded as a complex vector space via multiplication by $\mathbb{C}$ on the *right* (cf. sec. 6.30) and the quaternionic inner product is given by

$$\langle p \otimes x, q \otimes y\rangle = p\langle x, y\rangle_C \bar{q}, \qquad p, q \in \mathbb{H},\ x, y \in Y.$$

Thus we have the inclusion $U(n) \to Q(n)$ defined by $\sigma \mapsto \iota \otimes \sigma$. Evidently, $(Q(n), U(n))$ is an equal rank pair and so $\hat{P} = 0$.

To determine $\check{j}_{\theta=0}$ recall the inclusion $Q(n) \to U(2n)$ in Example 5, sec. 11.11. Let $i\colon \mathrm{Sk}(n; \mathbb{H}) \to \mathrm{Sk}(2n; \mathbb{C})$ denote the corresponding inclusion of Lie algebras and consider the composite inclusion

$$l = i \circ j\colon \mathrm{Sk}(n; \mathbb{C}) \to \mathrm{Sk}(n; \mathbb{H}) \to \mathrm{Sk}(2n; \mathbb{C}).$$

Recall from sec. 6.30 that $H = \mathbb{C} \oplus \mathbb{C}^{\perp}$ and that $\mathbb{C}^{\perp}$ is stable under multiplication from the left and from the right by $\mathbb{C}$. Let $Z_{\mathbb{C}}$ denote the $2n$-dimensional complex space underlying Z. Then

$$Z_{\mathbb{C}} = (\mathbb{C} \otimes_{\mathbb{C}} Y) \oplus (\mathbb{C}^{\perp} \otimes_{\mathbb{C}} Y) = Y \oplus (\mathbb{C}^{\perp} \otimes_{\mathbb{C}} Y).$$

Now fix a unit vector $j \in \mathbb{C}^{\perp}$, and define a $\mathbb{C}$-linear isomorphism

$$\alpha\colon \mathbb{C}^{\perp} \otimes_{\mathbb{C}} Y \xrightarrow{\cong} Y^*$$

by setting

$$\langle \alpha(j \otimes y), x\rangle = \langle x, y\rangle_{\mathbb{C}}, \qquad x, y \in Y.$$

Use α to identify $\mathbb{C}^{\perp} \otimes_{\mathbb{C}} Y$ with Y^*; then we have

$$Z_{\mathbb{C}} = Y \oplus Y^*.$$

Moreover, with this identification, l is given by

$$l(\varphi) = \varphi \oplus -\varphi^*.$$

It follows, as in Example 9, sec. 11.12, that

$$l^{\vee}_{\theta=0}(C_p^{U(2n)}) = \sum_{q+r=p} (-1)^r C_q^{U(n)} \vee C_r^{U(n)}.$$

But $l^{\vee}_{\theta=0} = j^{\vee}_{\theta=0} \circ i^{\vee}_{\theta=0}$ and $i^{\vee}_{\theta=0}$ is given by

$$i^{\vee}_{\theta=0}(C_p^{U(2n)}) = \begin{cases} 0, & p \text{ odd}, \\ C_p^{Q(n)}, & p \text{ even}, \end{cases}$$

(cf. Example 5, sec. 11.11).

This implies that

$$j^{\vee}_{\theta=0}(C_{2p}^{Q(n)}) = \sum_{q+r=2p} (-1)^r C_q^{U(n)} \vee C_r^{U(n)}, \qquad 1 \le p \le n,$$

where, as usual, $C_0^{U(n)} = 1$ and $C_q^{U(n)} = 0$, $q > n$.

4. $Q(n)/SO(n)$: Let X be an n-dimensional Euclidean space and set $Z = H \otimes X$. Then the inclusion $SO(n) \to Q(n)$, given by $\sigma \mapsto \iota \otimes \sigma$ is the composite

$$SO(n) \to U(n) \to Q(n)$$

of the inclusions in Example 4, sec. 11.11, and Example 3, above. Thus

$$j^{\vee}_{\theta=0}(C^{Q}_{2p}) = (-1)^p \sum_{q+r=p} C^{SO}_{2q} \vee C^{SO}_{2r}, \qquad 1 \leq p \leq n.$$

(Here $C^{SO}_0 = 1$, $C^{SO}_{2q} = 0$ if $q > n$; and if $n = 2m$, C^{SO}_{2m} may be replaced by $\mathrm{Pf} \vee \mathrm{Pf}$.)

Now an easy induction argument shows that $\operatorname{Im} j^{\vee}_{\theta=0}$ is the subalgebra of $(\vee F^*)_{\theta=0}$ generated by the elements C^{SO}_{2p}, $1 \leq 2p \leq n$.

In view of this, the formula above shows that

$$j^{\vee}_{\theta=0}(C^{Q}_{2p}) \in (\operatorname{Im} j^{\vee}_{\theta=0})^+ \cdot (\operatorname{Im} j^{\vee}_{\theta=0})^+, \qquad 2p > n,$$

whence $\varrho_E(C^{Q}_{2p}) \in \hat{P}$. It follows that $\Phi^{Q}_{4p-1} \in \hat{P}$, $n + 1 \leq 2p \leq 2n$. Now the standard argument on dimensions shows that these elements span $\hat{P}$.

§5. Non-Cartan pairs

11.14. The pair ($SU(6)$, $SU(3)\times SU(3)$). Recall from Example 2, sec. 11.11, that $U(3)\times U(3)$ is a subgroup of $U(6)$. The inclusion map restricts to an inclusion $SU(3)\times SU(3) \to SU(6)$. Thus we can consider the homogeneous space

$$SU(6)/(SU(3)\times SU(3)).$$

Denote the Lie algebras of $SU(6)$ and $SU(3)$ by E and F. Then, by Theorem X, sec. 6.28,

$$(\vee E^*)_{\theta=0} = \vee(C_2^{SU}, C_3^{SU}, C_4^{SU}, C_5^{SU}, C_6^{SU})$$

and

$$(\vee F^*)_{\theta=0} = \vee(C_2^{SU}, C_3^{SU}).$$

Moreover, the homomorphism $j_{\theta=0}^{\vee}\colon (\vee E^*)_{\theta=0} \to (\vee F^*)_{\theta=0} \otimes (\vee F^*)_{\theta=0}$ is given by

$$j_{\theta=0}^{\vee}(C_p^{SU}) = \sum_{q+r=p} C_q^{SU} \otimes C_r^{SU}, \tag{11.4}$$

(as follows from Example 2, sec. 11.11). Note that on the right-hand side $C_0^{SU} = 1$ and $C_q^{SU} = 0$ if $q \neq 0, 2, 3$.

Now set

$$\varrho_E(C_p^{SU}) = x_{2p-1}, \qquad 2 \le p \le 6,$$

and let Q be a graded vector space with homogeneous basis y_4, y_6, z_4, z_6 (subscripts denote degrees). Define a symmetric P_E-algebra $(\vee Q; \sigma)$ by

$$\sigma(x_{2p-1}) = \sum_{q+r=p} y_{2q} \vee z_{2r}, \qquad 2 \le p \le 6.$$

Then formula (11.4) shows that $(\vee Q; \sigma)$ is the associated P_E-algebra for the pair $(SU(6), SU(3)\times SU(3))$ (cf. sec. 11.5).

Since

$$\sigma(x_3) = y_4 + z_4, \qquad \sigma(x_5) = y_6 + z_6, \qquad \sigma(x_7) = y_4 z_4$$
$$\sigma(x_9) = y_4 z_6 + y_6 z_4, \qquad \sigma(x_{11}) = y_6 z_6,$$

it follows that the essential subspace P_1 of P_E (cf. sec. 2.22) is spanned by x_7, x_9, and x_{11}. Moreover, the subspace $Q_1 \subset Q$ (cf. sec. 2.23) may be chosen to be the subspace spanned by y_4 and y_6.

Then the associated essential P_1-algebra $(\vee Q_1; \sigma_1)$ is given by

$$\sigma_1(x_7) = -y_4^2, \qquad \sigma_1(x_9) = -2y_4y_6, \qquad \sigma_1(x_{11}) = -y_6^2.$$

It is immediate that

$$\sigma_1(x_{2p-1}) \notin \vee^+Q_1 \cdot \sigma_1(P_1), \qquad p = 4, 5, 6,$$

and so the Samelson subspace $\hat{P}_1$ is zero (cf. Proposition IV, sec. 2.13). Now Theorem X, sec. 2.23, shows that the Samelson subspace for $(\vee Q; \sigma)$ is zero.

This implies, in turn, via Theorem II, sec. 11.5, that the Samelson subspace for $(SU(6), SU(3)\times SU(3))$ is zero. Since the difference in ranks is $5 - 4$ $(= 1)$, it follows that this pair is not a Cartan pair. In particular, the differential algebra $(A(SU(6)/(SU(3)\times SU(3))), \delta)$ is *not* c-split.

Now we compute the cohomology algebra $H(SU(6)/(SU(3)\times SU(3)))$. Combining Theorem II, sec. 11.5, with Theorem X, sec. 2.23, yields an isomorphism

$$H(\vee Q_1 \otimes \wedge P_1) \xrightarrow{\cong} H(SU(6)/(SU(3) \times SU(3))).$$

Thus we have to determine the algebra $H(\vee Q_1 \otimes \wedge P_1)$.

Let α_0, α_4, and α_6 denote the cohomology classes in $H_0(\vee Q_1 \otimes \wedge P_1)$ represented by the cocycles $1 \otimes 1$, $y_4 \otimes 1$, and $y_6 \otimes 1$. Then, evidently these elements form a basis of $H_0(\vee Q_1 \otimes \wedge P_1)$.

Next observe that

$$H^p(\vee Q_1 \otimes \wedge P_1) = H^p(SU(6)/(SU(3)\times SU(3))) = 0, \qquad p > 19.$$

Thus a simple degree argument shows that

$$H_2(\vee Q_1 \otimes \wedge P_1) = 0 \qquad \text{and} \qquad H_3(\vee Q_1 \otimes \wedge P_1) = 0.$$

Finally, because $\dim P_1 > \dim Q_1$, the corollary to Theorem VIII, sec. 2.19, implies that $H(\vee Q_1 \otimes \wedge P_1)$ has zero Euler–Poincaré characteristic. Since $H_0(\vee Q_1 \otimes \wedge P_1)$ is evenly graded, while $H_1(\vee Q_1 \otimes \wedge P_1)$ is oddly graded, it follows that

$$\dim H_1(\vee Q_1 \otimes \wedge P_1) = \dim H_0(\vee Q_1 \otimes \wedge P_1).$$

Direct computation shows that the cocycles

$$-y_4 \otimes x_9 + 2y_6 \otimes x_7, \quad 2y_4 \otimes x_{11} - y_6 \otimes x_9, \quad 2y_6^2 \otimes x_7 - y_4 y_6 \otimes x_9,$$

represent nonzero cohomology classes α_{13}, α_{15}, and α_{19}. Thus these classes form a basis of $H_1(\vee Q_1 \otimes \wedge P_1)$. Hence they also form a basis of $H_+(\vee Q_1 \otimes \wedge P_1)$.

This shows that the Poincaré polynomial for

$$H(SU(6)/(SU(3) \times SU(3)))$$

is

$$1 + t^4 + t^6 + t^{13} + t^{15} + t^{19}.$$

Moreover, the algebra structure is given by

$$\alpha_i \alpha_j = \begin{cases} 0, & i + j \neq 19, \\ \alpha_{19}, & i + j = 19, \end{cases}$$

(for $\alpha_i, \alpha_j \in H^+(\vee Q_1 \otimes \wedge P_1)$).

11.15. The pair $(Q(n), SU(n))$. Recall from Example 3, sec. 11.13, that $U(n)$ is a subgroup of $Q(n)$ of the same rank n. Thus we can consider the homogeneous space

$$Q(n)/SU(n).$$

As usual, denote the Lie algebras by E and F and the inclusion by j.

According to Theorem X, sec. 6.28, and Theorem XIII, sec. 6.30,

$$(\vee E^*)_{\theta=0} = \vee(C_2^Q, \ldots, C_{2n}^Q) \quad \text{and} \quad (\vee F^*)_{\theta=0} = \vee(C_2^{SU}, C_3^{SU}, \ldots, C_n^{SU}).$$

Moreover (cf. Example 3, sec. 11.13) $j^\vee_{\theta=0}$ is given by

$$j^\vee_{\theta=0}(C_{2p}^Q) = \sum_{q+r=2p} (-1)^r C_q^{SU} \vee C_r^{SU}, \qquad 1 \le p \le n, \tag{11.5}$$

where we set $C_0^{SU} = 1$ and $C_q^{SU} = 0$ if $q = 1$ or $q > n$.

Let x_{4p-1} $(1 \le p \le n)$ be the basis of P_E given by $x_{4p-1} = \varrho_E(C_{2p}^Q)$. Let Q be a graded space with homogeneous basis $y_4, y_6, \ldots, y_{2n}$ (subscripts denote degrees) and define a P_E-algebra $(\vee Q; \sigma)$ by

$$\sigma(x_{4p-1}) = \sum_{q+r=2p} (-1)^r y_{2q} y_{2r}, \qquad p = 1, \ldots, n. \tag{11.6}$$

Then formula (11.5) shows that $(\vee Q; \sigma)$ is the associated P_E-algebra of the pair $(Q(n), SU(n))$ (cf. sec. 11.5).

Direct computation shows that if $n \leq 4$, then the Samelson subspace of this P_E-algebra has dimension 1. Since rank $Q(n)$ − rank $SU(n) = 1$, it follows that $(Q(n), SU(n))$ is a Cartan pair if $n \leq 4$.

We shall show, however, that for $n \geq 5$ the Samelson subspace is zero, and so $(Q(n), SU(n))$ is *not* a Cartan pair in this case. In particular $(A(Q(n)/SU(n)), \delta)$ is *not* c-split if $n \geq 5$.

We consider first the case $n = 5$, and then proceed by induction on n.

Case I: $n = 5$. Then $Q = (y_4, y_6, y_8, y_{10})$ and σ is given by

$$\sigma(x_3) = 2y_4, \qquad \sigma(x_7) = 2y_8 + y_4^2, \qquad \sigma(x_{11}) = 2y_4y_8 - y_6^2$$
$$\sigma(x_{15}) = -2y_6y_{10} + y_8^2, \qquad \sigma(x_{19}) = -y_{10}^2.$$

Thus the essential subspace P_1 of P_E (cf. sec. 2.22) is spanned by x_{11}, x_{15}, and x_{19}. Moreover, the essential P_1-algebra $(\vee Q_1; \sigma_1)$ (cf. sec. 2.23) may be chosen so that $Q_1 = (y_6, y_{10})$ and

$$\sigma_1(x_{11}) = -y_6^2, \qquad \sigma_1(x_{15}) = -2y_6y_{10}, \qquad \sigma_1(x_{19}) = -y_{10}^2.$$

As in sec. 11.14, if follows from Proposition IV, sec. 2.13, that the Samelson subspace $\hat{P}_1$ is zero. Now Theorem X, sec. 2.23, shows that $\hat{P} = 0$.

Case II: $n \geq 6$. Write $P_E = P_n$, $Q = Q_n$, and $\sigma = \sigma_n$. Then

$$P_n = P_{n-1} \oplus (x_{4n-1}) \qquad \text{and} \qquad Q_n = Q_{n-1} \oplus (y_{2n}).$$

Observe that

$$\sigma_n(P_{n-1}) \subset \vee^+ Q_{n-1} \otimes \vee(y_{2n}) \qquad \text{and} \qquad \sigma_n(x_{4n-1}) = (-1)^n y_{2n}^2,$$

as follows from (11.6).

This implies that

$$\sigma_n(P_{n-1}) \cdot \vee^+ Q_n \subset \vee^+ Q_{n-1} \otimes \vee(y_{2n}),$$

whence

$$\sigma_n(x_{4n-1}) \notin \sigma_n(P_{n-1}) \cdot \vee^+ Q_n.$$

This shows that $x_{4n-1} \notin \hat{P}_n$; i.e., $\hat{P}_n \subset P_{n-1}$.

Next, let $\varphi: P_n \to P_{n-1}$ and $\psi: Q_n \to Q_{n-1}$ be the projections determined by the direct decompositions above. Then

$$\begin{tikzcd} \vee Q_n \otimes \wedge P_n \arrow[r, "\psi_\vee \otimes \varphi_\wedge"] \arrow[d, "\varrho_n"'] & \vee Q_{n-1} \otimes \wedge P_{n-1} \arrow[d, "\varrho_{n-1}"] \\ \wedge P_n \arrow[r, "\varphi_\wedge"'] & \wedge P_{n-1} \end{tikzcd}$$

is a commutative diagram of graded differential algebras, as follows from formula (11.6). It follows that

$$\varphi(\hat{P}_n) \subset \hat{P}_{n-1}.$$

Now assume by induction that $\hat{P}_{n-1} = 0$ $(n \geq 6)$. Since $\hat{P}_{n-1} \subset P_{n-1}$ and the restriction of φ to P_{n-1} is injective the relation $\varphi(\hat{P}_n) \subset \hat{P}_{n-1}$ implies that $\hat{P}_n = 0$. This closes the induction.

11.16. The frame bundle over $CP^2 \times CP^2$. Let G/K be any homogeneous space with G, K compact and connected. Denote the corresponding Lie algebras by E and F. Then the adjoint representation of K in E restricts to a representation in $F^\perp$ (the orthogonal complement of F with respect to the Killing form). Thus we have a homomorphism

$$\mathrm{Ad}^\perp: K \to SO(F^\perp).$$

Now consider the inclusion

$$\psi: K \to G \times SO(F^\perp)$$

given by $\psi(y) = (y, \mathrm{Ad}^\perp(y))$, $y \in K$.

The corresponding homogeneous space is given by

$$\frac{G \times SO(F^\perp)}{K} = G \times_K SO(F^\perp).$$

It may be identified with the total space of the associated principal bundle of the vector bundle

$$\xi = (G \times_K F^\perp, \pi_\xi, G/K, F^\perp)$$

defined in sec. 5.10, volume II.

But according to Proposition III, sec. 5.11, volume II, ξ is the tangent bundle of G/K and so $(G\times SO(F^{\perp}))/K$ is the total space of the tangent orthonormal frame bundle for G/K.

In particular, consider the case

$$G = U(3) \times U(3), \qquad K = U(1) \times U(2) \times U(1) \times U(2).$$

Then $G/K = \mathbb{C}P^2 \times \mathbb{C}P^2$. According to Proposition XII, sec. 9.26, the algebra of differential forms on the manifold of frames over $\mathbb{C}P^2 \times \mathbb{C}P^2$ is *not* c-split. Now Theorem IV, sec. 11.5, implies that the pair

$$(U(3) \times U(3) \times SO(8), U(1) \times U(2) \times U(1) \times U(2))$$

is *not* a Cartan pair.

TABLE I

$G = U(n),\ E = \mathrm{Sk}(n; C),\ P_E = (\Phi_1^U, \Phi_3^U, \dots, \Phi_{2n-1}^U)$

K	$U(k),\ k < n$	$U(k) \times U(n-k)$ $(0 < k < n)$	$U(k_1) \times \cdots \times U(k_q)$ $(k_i > 0,\ \sum k_i = k \leq n)$
Basis of $\hat{P}$	$\Phi_{2p-1}^U,\ k+1 \leq p \leq n$	0	$\Phi_{2p-1}^U,\ k+1 \leq p \leq n$
$f_{\mathrm{Im}\,\chi^*}$	1	$\dfrac{\prod_{p=k+1}^n (1-t^{2p})}{\prod_{p=1}^{n-k} (1-t^{2p})^{-1}}$	$\dfrac{\prod_{p=1}^k (1-t^{2p})}{\prod_{i=1}^q \prod_{p=1}^{k_i} (1-t^{2p})}$
$f_{H(G/K)}$	$\prod_{p=k+1}^n (1+t^{2p-1})$	$\dfrac{\prod_{p=k+1}^n (1-t^{2p})}{\prod_{p=1}^{n-k} (1-t^{2p})^{-1}}$	$f_{\mathrm{Im}\,\chi^*} \cdot \prod_{p=k+1}^n (1+t^{2p-1})$
$\dim \mathrm{Im}\,\chi^*$	1	$\binom{n}{k}$	$\dfrac{k!}{k_1! \cdots k_q!}$
$\dim H(G/K)$	2^{n-k}	$\binom{n}{k}$	$\dfrac{k!}{k_1! \cdots k_q!} 2^{n-k}$
$\chi_{H(G/K)}$	0	$\binom{n}{k}$	0 if $k < n$ $\dfrac{k!}{k_1! \cdots k_q!}$ if $k = n$
n.c.z.	yes	no	yes if $q = 1$ no if $q > 1$
equal rank	no	yes	no if $k < n$ yes if $k = n$
symmetric pair	—	yes	—

TABLE I

(continued)

$SO(n)$ $(n = 2m)$	$SO(n)$ $(n = 2m + 1)$	$Q(m)$ $(n = 2m)$
Φ^U_{4p-3}, $1 \leq p \leq m$	Φ^U_{4p-3}, $1 \leq p \leq m+1$	Φ^U_{4p-3}, $1 \leq p \leq m$
$1 + t^n$	1	1
$(1 + t^n) \prod_{p=1}^{m} (1 + t^{4p-3})$	$\prod_{p=1}^{m+1} (1 + t^{4p-3})$	$\prod_{p=1}^{m} (1 + t^{4p-3})$
2	1	1
2^{m+1}	2^{m+1}	2^m
0	0	0
no	yes	yes
no	no	no
yes	yes	—

TABLE II

$G = SO(2m+1),\ E = \mathrm{Sk}(2m+1),\ P_E = (\Phi_3^{SO}, \Phi_7^{SO}, \ldots, \Phi_{4m-1}^{SO})$

K	$SO(2k+1)$	$SO(2k)$	$SO(2k) \times SO(2m-2k+1)$ $(0 < k < m)$	$SO(2k_1) \times \cdots \times SO(2k_q) \times SO(2l+1)$ $(0 < k_i,\ \sum k_i = k,\ 0 \le l)$
Basis of $\hat{P}$	$\Phi_{4p-1}^{SO},\ k+1 \le p \le m$	$\Phi_{4p-1}^{SO},\ k+1 \le p \le m$	0	$\Phi_{4p-1}^{SO},\ k+l+1 \le p \le m$
$f_{\mathrm{Im}\,\chi^*}$	1	$1 + t^{2k}$	$\dfrac{\prod_{p=m-k+1}^{m} (1 - t^{4p})}{[\prod_{p=1}^{k-1} (1-t^{4p})](1-t^{2k})}$	$\dfrac{\prod_{p=1}^{k+l} (1 - t^{4p})}{[\prod_{i=1}^{q} ((1-t^{2k_i}) \prod_{p=1}^{k_i-1} (1-t^{4p}))] \prod_{p=1}^{l} (1-t^{4p})}$
$f_{H(G/K)}$	$\prod_{p=k+1}^{m} (1 + t^{4p-1})$	$(1+t^{2k}) \prod_{p=k+1}^{m} (1+t^{4p-1})$	$f_{\mathrm{Im}\,\chi^*}$	$f_{\mathrm{Im}\,\chi^*} \cdot \prod_{p=k+l+1}^{m} (1 + t^{4p-1})$
dim Im χ^*	1	2	$2\binom{m}{k}$	$2^q \dfrac{(k+l)!}{k_1! \cdots k_q!\, l!}$
dim $H(G/K)$	2^{m-k}	2^{m-k+1}	$2\binom{m}{k}$	$2^{m+q-k-l} \cdot \dfrac{(k+l)!}{k_1! \cdots k_q!\, l!}$
$\chi_{H(G/K)}$	$0 \quad (k < m)$	$0 \quad (k < m)$ $2 \quad (k = m)$	$2\binom{m}{k}$	$0 \quad (k+l < m)$ $2^{m+q-k-l} \cdot \dfrac{(k+l)!}{k_1! \cdots k_q!\, l!} \quad (k+l = m)$
n.c.z.	yes	no	no	no
equal rank	no $(k < m)$	no $(k < m)$ yes $(k = m)$	yes	no $(k+l < m)$ yes $(k+l = m)$
symmetric pair	—	—	yes	—

TABLE III

$G = SO(2m),\ E = \mathrm{Sk}(2m),\ P_E = (\Phi_3^{SO}, \Phi_7^{SO}, \ldots, \Phi_{4m-5}^{SO}, \mathrm{Sf}_{2m-1})$

K	$SO(2k)$ $(k < m)$	$SO(2k+1)$	$SO(2k) \times SO(2m-2k)$ $(0 < k < m)$
Basis of $\hat{P}$	$\mathrm{Sf}_{2m-1},\ \Phi_{4p-1}^{SO},\ k+1 \leq p \leq m-1$	$\mathrm{Sf}_{2m-1},\ \Phi_{4p-1}^{SO},\ k+1 \leq p \leq m-1$	0
$f_{\mathrm{Im}\,\chi^*}$	$1 + t^{2k}$	1	$\dfrac{(1-t^{2m})\prod_{p=1}^{m-1}(1-t^{4p})}{(1-t^{2k})(1-t^{2m-2k})\prod_{p=1}^{k-1}(1-t^{4p})\prod_{p=1}^{m-k-1}(1-t^{4p})}$
$f_{H(G/K)}$	$(1+t^{2k})(1+t^{2m-1})\prod_{p=k+1}^{m-1}(1+t^{4p-1})$	$(1+t^{2m-1})\prod_{p=k+1}^{m-1}(1+t^{4p-1})$	$f_{\mathrm{Im}\,\chi^*}$
dim Im χ^*	2	1	$2\binom{m}{k}$
dim $H(G/K)$	2^{m-k+1}	2^{m-k}	$2\binom{m}{k}$
$\chi_{H(G/K)}$	0	0	$2\binom{m}{k}$
n.c.z.	no	yes	no
equal rank	no	no	yes
symmetric pair	—	—	yes

TABLE III (*continued*)

$G = SO(2m),\ E = \mathrm{Sk}(2m),\ P_E = (\Phi_3^{SO}, \Phi_7^{SO}, \ldots, \Phi_{4m-5}^{SO}, \mathrm{Sf}_{2m-1})$

K	$SO(2k+1) \times SO(2m-2k-1)$ $(0 < k < m-1)$	$U(m)$	$Q(k)$ $(m = 2k)$
Basis of $\hat{P}$	Sf_{2m-1}	0	$\mathrm{Sf}_{4k-1} + (-1)^k 2^{2k-2}(2k-1)!\,\Phi_{4k-1},\ \Phi_{4p-1},\ k+1 \leq p < 2k$
$f_{\mathrm{Im}\,\chi^*}$	$\dfrac{\prod_{p=1}^{m-1}(1-t^{4p})}{\prod_{p=1}^{k}(1-t^{4p})\prod_{p=1}^{m-k-1}(1-t^{4p})}$	$\prod_{p=1}^{m-1}(1+t^{2p})$	1
$f_{H(G/H)}$	$(1+t^{2m-1})f_{\mathrm{Im}\,\chi^*}$	$f_{\mathrm{Im}\,\chi^*}$	$(1+t^{2m-1})\prod_{p=k+1}^{m-1}(1+t^{4p-1})$
dim Im χ^*	$\binom{m-1}{k}$	2^{m-1}	1
dim $H(G/K)$	$2\binom{m-1}{k}$	2^{m-1}	2^{m-k}
$\chi_{H(G/K)}$	0	2^{m-1}	0
n.c.z.	no	no	yes
equal rank	no	yes	no
symmetric pair	yes	—	—

TABLE IV

$G = Q(n),\ E = \mathrm{Sk}(n; H),\ P_E = (\Phi_3^Q, \Phi_7^Q, \ldots, \Phi_{4n-1}^Q)$

K	$Q(k)$ $(k < n)$	$Q(k) \times Q(n-k)$ $(0 < k < n)$	$U(n)$	$SO(n)$ $(n = 2k)$	$SO(n)$ $(n = 2k+1)$
Basis of $\hat{P}$	$\Phi_{4p-1}^Q,\ k+1 \le p \le n$	0	0	$\Phi_{4p-1}^Q,\ k+1 \le p \le n$	$\Phi_{4p-1}^Q,\ k+1 \le p \le n$
$f_{\mathrm{Im}\,\chi^*}$	1	$\dfrac{\prod_{p=1}^{n}(1-t^{4p})}{\prod_{p=1}^{k}(1-t^{4p})\prod_{p=1}^{n-k}(1-t^{4p})}$	$\prod_{p=1}^{n}(1+t^{2p})$	$1+t^{2k}$	1
$f_{H(G/K)}$	$\prod_{p=k+1}^{n}(1+t^{4p-1})$	$f_{\mathrm{Im}\,\chi^*}$	$\prod_{p=1}^{n}(1+t^{2p})$	$(1+t^{2k})\prod_{p=k+1}^{n}(1+t^{4p-1})$	$\prod_{p=k+1}^{n}(1+t^{4p-1})$
dim Im χ^*	1	$\binom{n}{k}$	2^n	2	1
dim $H(G/K)$	2^{n-k}	$\binom{n}{k}$	2^n	2^{n-k+1}	2^{n-k}
$\chi_{H(G/K)}$	0	$\binom{n}{k}$	2^n	0	0
n.c.z.	yes	no	no	no	yes
equal rank	no	yes	yes	no	no
symmetric pair	—	yes	—	—	—

Chapter XII

Operation of a Lie Algebra Pair

§1. Basic properties

12.1. Definition: Let (E, F) be a reductive Lie algebra pair (cf. sec. 10.1) and assume that $(E, i, \theta, R, \delta_R)$ is an operation of E. This operation restricts to an operation $(F, i, \theta, R, \delta_R)$ of F (cf. Example 3, sec. 7.4). The corresponding invariant subalgebras of R will be denoted respectively by $R_{\theta_E=0}$ and $R_{\theta_F=0}$.

We shall say that $(E, F, i, \theta, R, \delta_R)$ is an *operation of the pair* (E, F) *in the graded differential algebra* (R, δ_R) if the inclusion map $R_{\theta_E=0} \to R_{\theta_F=0}$ induces an isomorphism

$$H(R_{\theta_E=0}) \xrightarrow{\cong} H(R_{\theta_F=0}).$$

Given such an operation, we adopt the following notation conventions, to remain in force for the entire chapter:

(i) ω_R denotes the degree involution of R: $\omega_R z = (-1)^p z$, $z \in R^p$.

(ii) The horizontal and invariant subalgebras for the underlying operations of E and F are denoted, respectively, by

$$R_{i_E=0}, R_{\theta_E=0} \qquad \text{and} \qquad R_{i_F=0}, R_{\theta_F=0}.$$

(iii) The basic subalgebras for the operations are written

$$B_E = R_{i_E=0,\theta_E=0} \qquad \text{and} \qquad B_F = R_{i_F=0,\theta_F=0}.$$

(iv) The obvious inclusions are denoted by

$$e_E: B_E \to R_{\theta_E=0}, \qquad e_F: B_F \to R_{\theta_F=0} \qquad \text{and} \qquad e: B_E \to B_F.$$

(v) The cohomology algebras $H(R_{\theta_E=0})$ and $H(R_{\theta_F=0})$ are identified via the isomorphism induced by the inclusion map, and are denoted

simply by $H(R_{\theta=0})$. In particular, we have the commutative diagram

$$\begin{tikzcd} H(B_E) \arrow[dd, "e^*"'] \arrow[dr, "e_E^\#"] & \\ & H(R_{\theta=0}). \\ H(B_F) \arrow[ur, "e_F^\#"'] & \end{tikzcd}$$

(vi) If the algebra $H(R_{\theta=0})$ is connected, then the fibre projections associated with the operations of E and F are denoted respectively by

$$\varrho_R\colon H(R_{\theta=0}) \to (\wedge E^*)_{\theta=0} \qquad \text{and} \qquad \varrho_R^F\colon H(R_{\theta=0}) \to (\wedge F^*)_{\theta=0}$$

(cf. sec. 7.10).

(vii) The algebras $\wedge P_E$, $(\wedge E^*)_{\theta=0}$, and $H^*(E)$ (respectively, $\wedge P_F$, $(\wedge F^*)_{\theta=0}$, and $H^*(F)$) are identified via the isomorphisms $\varkappa_E$ and $\varkappa_E^\#$ (respectively, $\varkappa_F$ and $\varkappa_F^\#$) of sec. 5.18 and sec. 5.19.

(viii) The inclusion map of F into E is written $j\colon F \to E$. It induces homomorphisms $j^\wedge$, $j_{\theta=0}^\wedge$, $j^\vee$, and $j_{\theta=0}^\vee$ as described in sec. 10.1.

(ix) The basic subalgebra for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ is denoted by $(\wedge E^*)_{i_F=0,\theta_F=0}$ and its cohomology algebra is written $H(E/F)$. The inclusion map $(\wedge E^*)_{i_F=0,\theta_F=0} \to \wedge E^*$ is denoted by k.

It is the purpose of this chapter to express $H(B_F)$ in terms of $H(B_E)$ and other invariants.

12.2. The associated semisimple operation. Consider an operation $(E, F, i, \theta, R, \delta_R)$ of a reductive pair (E, F). The operation of E in (R, δ_R) determines the associated semisimple operation $(E, i, \theta, R_S, \delta_R)$ as constructed in sec. 7.5. In particular, θ is a semisimple representation of E in R_S.

Since F is reductive in E, θ restricts to a semisimple representation of F in R_S, as follows from the definition of R_S and Proposition III, sec. 4.7. Thus the inclusions

$$(R_S)_{\theta_E=0} \to R_S \qquad \text{and} \qquad (R_S)_{\theta_F=0} \to R_S$$

induce isomorphisms of cohomology (cf. Proposition I, sec. 7.3). Hence

the same holds for the inclusion $(R_S)_{\theta_E=0} \to (R_S)_{\theta_F=0}$, and so $(E, F, i, \theta, R_S, \delta_R)$ is an operation of the pair (E, F).

Finally consider the inclusion map $R_S \to R$ as a homomorphism of operations of F.

Proposition I: Assume that the operation of F in R_S is regular. Then the inclusion induces an isomorphism

$$\begin{tikzcd}
 & H((R_S)_{i_F=0,\theta_F=0}) \arrow[r] \arrow[d, "\cong"'] & H((R_S)_{\theta_F=0}) \arrow[d, "\cong"] \arrow[dr] & \\
(\vee F^*)_{\theta=0} \arrow[ur] \arrow[dr] & & & (\wedge F^*)_{\theta=0} \\
 & H(B_F) \arrow[r] & H(R_{\theta=0}) \arrow[ur] &
\end{tikzcd}$$

between the cohomology sequences of the two operations.

Proof: In view of the corollary to Theorem III, sec. 9.8, it is sufficient to show that the inclusion $(R_S)_{\theta_F=0} \to R_{\theta_F=0}$ induces an isomorphism of cohomology. But this follows from the observation that $R_{\theta_E=0} = (R_S)_{\theta_E=0}$ and the resulting commutative diagram

$$\begin{tikzcd}
 & H((R_S)_{\theta_F=0}) \arrow[dd] \\
H(R_{\theta_E=0}) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\
 & H(R_{\theta_F=0}).
\end{tikzcd}$$

Q.E.D.

12.3. The structure operation. Let $(E, F, i, \theta, R, \delta_R)$ be an operation of a reductive pair (E, F). Recall from sec. 7.7 the definition of the structure operation

$$(E, i_{R\otimes E}, \theta_{R\otimes E}, R \otimes \wedge E^*, \delta_{R\otimes E})$$

associated with the operation of E in (R, δ_R).

Proposition II: The inclusion map

$$((R \otimes \wedge E^*)_{\theta_E=0}, \delta_{R\otimes E}) \to ((R \otimes \wedge E^*)_{\theta_F=0}, \delta_{R\otimes E})$$

induces an isomorphism of cohomology algebras; i.e., the structure operation is an operation of the pair (E, F).

Proof: Define an operator δ in $R \otimes \wedge E^*$ by setting

$$\delta = \delta_R \otimes \iota + \omega_R \otimes \delta_E .$$

In sec. 7.7 we constructed an isomorphism of graded differential algebras

$$\beta\colon (R \otimes \wedge E^*, \delta) \xrightarrow{\cong} (R \otimes \wedge E^*, \delta_{R\otimes E})$$

satisfying $\beta \circ \theta_{R\otimes E}(x) = \theta_{R\otimes E}(x) \circ \beta$, $x \in E$ (cf. Proposition V, sec. 7.7). Thus it is sufficient to show that the inclusion map induces an isomorphism

$$H((R \otimes \wedge E^*)_{\theta_E=0}, \delta) \xrightarrow{\cong} H((R \otimes \wedge E^*)_{\theta_F=0}, \delta).$$

Consider the commutative diagram

$$\begin{tikzcd}
H(R_{\theta_E=0}) \otimes (\wedge E^*)_{\theta=0} \arrow[r] \arrow[d] & H(R_{\theta_F=0}) \otimes H((\wedge E^*)_{\theta_F=0}, \delta_E) \arrow[d] \\
H((R \otimes \wedge E^*)_{\theta_E=0}, \delta) \arrow[r] & H((R \otimes \wedge E^*)_{\theta_F=0}, \delta),
\end{tikzcd}$$

which is induced by the obvious inclusions. In view of Proposition IV, sec. 7.6, the vertical arrows are isomorphisms. Our hypothesis on R implies that the upper horizontal arrow is an isomorphism. Hence so is the lower horizontal arrow.

Q.E.D.

12.4. Fibre projection. Let $(E, F, i, \theta, R, \delta_R)$ be an operation of a reductive pair and assume that $H(R_{\theta=0})$ is connected. In this section we shall define a homomorphism

$$p_R\colon H(B_F) \to H(E/F)$$

to be called the *fibre projection for the operation of the pair* (E, F).

First, consider the inclusion map

$$g\colon R_{\theta_E=0} \otimes (\wedge E^*)_{i_F=0,\theta_F=0} \to [R \otimes (\wedge E^*)_{i_F=0}]_{\theta_F=0}.$$

Since

$$\delta_{R\otimes E} = \delta_R \otimes \iota + \delta_\theta + \omega_R \otimes \delta_E$$

(cf. sec. 12.3 and sec. 7.7), and $\delta_\theta \circ g = 0$, it follows that

$$\delta_{R\otimes E} \circ g = g \circ (\delta_R \otimes \iota + \omega_R \otimes \delta_E).$$

Hence g induces a homomorphism

$$g^\#\colon H(R_{\theta=0}) \otimes H(E/F) \to H[(R \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}].$$

Lemma I: The homomorphism $g^\#$ is an isomorphism.

Proof: Filter the differential algebras by the ideals

$$F^p = \sum_{j\ge p} R_{\theta_E=0} \otimes (\wedge^j E^*)_{i_F=0,\theta_F=0}$$

and

$$\hat{F}^p = \sum_{j\ge p} [R \otimes (\wedge^j E^*)_{i_F=0}]_{\theta_F=0}.$$

Then g is filtration preserving, and so it induces homomorphisms

$$g_i\colon (E_i, d_i) \to (\hat{E}_i, \hat{d}_i)$$

between the corresponding spectral sequences.

Now consider the commutative diagram

$$\begin{tikzcd}
H(R_{\theta_E=0}) \otimes (\wedge E^*)_{i_F=0,\theta_F=0} \arrow[r, "\cong"] \arrow[d, "\lambda_1^\#"] & E_1 \arrow[dd, "g_1"] \\
H(R_{\theta_F=0}) \otimes (\wedge E^*)_{i_F=0,\theta_F=0} \arrow[d, "\lambda_2^\#"] & \\
H\{[R \otimes (\wedge E^*)_{i_F=0}]_{\theta_F=0}, \delta_R \otimes \iota\} \arrow[r, "\cong"'] & \hat{E}_1
\end{tikzcd}$$

where λ_1 and λ_2 denote the obvious inclusions (cf. formula 1.8, sec. 1.7). Our hypothesis on R asserts that $\lambda_1^\#$ is an isomorphism. Proposition IV,

sec. 7.6 (applied with $M = (\wedge E^*)_{i_F=0}$ and $\delta_M = 0$) shows that $\lambda_2^\#$ is an isomorphism. Hence g_1 is an isomorphism and so by Theorem I, sec. 1.14, $g^\#$ is an isomorphism.

Q.E.D.

Next, recall the structure homomorphism $\gamma_R: R \to R \otimes \wedge E^*$ defined in sec. 7.8. Since γ_R is a homomorphism of operations, it restricts to a homomorphism of graded differential algebras

$$(\gamma_R)_{i_F=0,\theta_F=0}: B_F \to (R \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}.$$

For the sake of brevity, this map will be denoted simply by $\gamma_{R/F}$. On the other hand, since $H(R_{\theta=0})$ is connected, we have the canonical projection

$$\pi_R: H(R_{\theta=0}) \otimes H(E/F) \to H(E/F).$$

Definition: The *fibre projection for the operation* $(E, F, i, \theta, R, \delta_R)$ is the homomorphism

$$p_R: H(B_F) \to H(E/F)$$

given by

$$p_R = \pi_R \circ (g^\#)^{-1} \circ \gamma_{R/F}^\#.$$

12.5. Projectable operations. An operation $(E, F, i, \theta, R, \delta_R)$ of a reductive pair (E, F) is called *projectable* if the underlying operation of E is projectable (cf. sec. 7.11).

Let $(E, F, i, \theta, R, \delta_R)$ be a projectable operation with projection $q: R \to \Gamma$. Then the linear map

$$q \otimes \iota : R \otimes (\wedge E^*)_{i_F=0} \to (\wedge E^*)_{i_F=0}$$

induces a map

$$(q \otimes \iota)^\#_{\theta_F=0} : H((R \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}, \delta_{R\otimes E}) \to H(E/F)$$

(cf. the relations above Proposition VII in sec. 7.11).

Proposition III: The fibre projection of a projectable operation is given by

$$p_R = (q \otimes \iota)^\#_{\theta_F=0} \circ \gamma_{R/F}^\#.$$

Proof: Consider the restriction $q_{\theta_F=0}: R_{\theta_F=0} \to \Gamma$. Then

$$\pi_R = q^{\#}_{\theta_F=0} \otimes \iota = (q \otimes \iota)^{\#}_{\theta_F=0} \circ g^{\#}.$$

The proposition follows.

Q.E.D.

Example: Consider the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$, where (E, F) is a reductive pair. Since

$$(\wedge E^*)_{\theta=0} \cong H((\wedge E^*)_{\theta_F=0}, \delta_E) \cong H^*(E),$$

it follows that this is an operation of the pair (E, F) in $(\wedge E^*, \delta_E)$. In this case

$$B_E = \Gamma \qquad \text{and} \qquad B_F = (\wedge E^*)_{i_F=0, \theta_F=0}.$$

Moreover, $\wedge E^*$ is connected and so the operation is projectable.

Now we show that the fibre projection

$$p_{\wedge E^*}: H(E/F) \to H(E/F)$$

is the identity map. In fact, according to Example 1, sec. 7.8, the structure homomorphism γ for $\wedge E^*$ satisfies

$$\gamma(\Phi) - (1 \otimes \Phi) \in (\wedge^+ E^*) \otimes \wedge E^*, \qquad \Phi \in \wedge E^*.$$

Hence,

$$(q \otimes \iota)(\gamma \Phi) = \Phi,$$

where $q: \wedge E^* \to \Gamma$ denotes the projection.

Restricting this equation to $(\wedge E^*)_{i_F=0, \theta_F=0}$ yields

$$(q \otimes \iota)_{\theta_F=0} \circ \gamma_{\wedge E^*/F} = \iota.$$

Now pass to cohomology and apply Proposition III, above.

12.6. Algebraic connections. Suppose $(E, F, i, \theta, R, \delta_R)$ is an operation of a reductive pair, such that the underlying operation of E admits an algebraic connection $\chi_E: E^* \to R^1$. Let $\chi: F^* \to E^*$ be an algebraic connection for the operation $(F, i_F, \theta_F, \wedge E^*, \delta_E)$ (cf. sec. 10.5). Then the linear map

$$\chi_F = \chi_E \circ \chi : F^* \to R^1$$

is an algebraic connection for the operation $(F, i, \theta, R, \delta_R)$.

Definition: The triple (χ, χ_E, χ_F) will be called an *algebraic connection for the operation of the pair* (E, F).

In view of Theorem I, sec. 8.4, the algebraic connections χ, χ_E, and χ_F determine isomorphisms of graded algebras

$$f = \iota \otimes \chi_\wedge : (\wedge E^*)_{i_F=0} \otimes \wedge F^* \xrightarrow{\cong} \wedge E^*,$$

$$f_E = \iota \otimes (\chi_E)_\wedge : R_{i_E=0} \otimes \wedge E^* \xrightarrow{\cong} R,$$

and

$$f_F = \iota \otimes (\chi_F)_\wedge : R_{i_F=0} \otimes \wedge F^* \xrightarrow{\cong} R.$$

Moreover, f_E restricts to an isomorphism

$$f_E : R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \xrightarrow{\cong} R_{i_F=0}$$

and, clearly, the diagram

$$\begin{tikzcd}
R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \otimes \wedge F^* \arrow[r, "\iota\otimes f", "\cong"'] \arrow[d, "f_E\otimes\iota"', "\cong"] & R_{i_E=0} \otimes \wedge E^* \arrow[d, "\cong"', "f_E"] \\
R_{i_F=0} \otimes \wedge F^* \arrow[r, "\cong", "f_F"'] & R
\end{tikzcd}$$

commutes.

Thus an isomorphism

$$f_{E,F} : R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \otimes \wedge F^* \xrightarrow{\cong} R$$

is given by $f_{E,F} = f_E \circ (\iota \otimes f) = f_F \circ (f_E \otimes \iota)$.

Lemma II: Let (χ, χ_E, χ_F) be an algebraic connection for the operation of (E, F). Then the curvatures of χ, χ_E, and χ_F are related by

$$\boldsymbol{\chi}_F(y^*) = \boldsymbol{\chi}_E(\chi y^*) + (\chi_E)_\wedge(\boldsymbol{\chi} y^*), \qquad y^* \in F^*.$$

Proof: In fact, Proposition III, sec. 8.6, yields

$$\begin{aligned}
\boldsymbol{\chi}_F(y^*) &= \delta_R \chi_F(y^*) - (\chi_F)_\wedge \delta_F(y^*) \\
&= \delta_R \chi_E \chi(y^*) - (\chi_E)_\wedge \chi_\wedge \delta_F(y^*) \\
&= (\delta_R \chi_E - (\chi_E)_\wedge \delta_E)(\chi y^*) + (\chi_E)_\wedge(\delta_E \chi - \chi_\wedge \delta_F)(y^*) \\
&= \boldsymbol{\chi}_E(\chi y^*) + (\chi_E)_\wedge(\boldsymbol{\chi} y^*).
\end{aligned}$$

Q.E.D.

If we identify $R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \otimes \wedge F^*$ with R via the isomorphism $f_{E,F}$, the formula in Lemma II reads

$$\chi_F(y^*) = \chi_E(\chi y^*) \otimes 1 \otimes 1 + 1 \otimes \chi y^* \otimes 1, \qquad y^* \in F^*. \tag{12.1}$$

An operation $(E, F, i, \theta, R, \delta_R)$ is called *regular* if the pair (E, F) is reductive, $H(R_{\theta=0})$ is connected, and the operation admits an algebraic connection. Thus an operation of a reductive pair (E, F) is regular if and only if the underlying operation of E is regular (cf. sec. 8.21).

Proposition IV: Let $(E, F, i, \theta, R, \delta_R)$ be a projectable regular operation, with algebraic connection (χ, χ_E, χ_F). Assume $\Phi \in (\wedge E^*)_{i_F=0,\theta_F=0}$ and $\Omega \in B_F$ are homogeneous elements of the same degree, such that

(1) $\delta_E\Phi = 0$ and $\delta_R(\Omega + (\chi_E)_\wedge\Phi) = 0$,

and

(2) $f_{E,F}^{-1}(\Omega) \in R^+_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \otimes 1$.

Let $\varepsilon \in H(B_F)$ be the class represented by $\Omega + (\chi_E)_\wedge\Phi$. Then Φ represents $p_R(\varepsilon)$.

Proof: It follows respectively from the definition of $f_{E,F}$ and the corollary to Proposition IV, sec. 8.10, that

$$\gamma_{R/F}\Omega \in (R^+ \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}$$

and

$$\gamma_{R/F}((\chi_E)_\wedge\Phi) - 1 \otimes \Phi \in (R^+ \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}.$$

Hence, if $q\colon R \to \Gamma$ is the projection

$$(q \otimes \iota) \circ \gamma_{R/F}(\Omega + (\chi_E)_\wedge\Phi) = \Phi.$$

Now apply Proposition III, sec. 12.5.

Q.E.D.

12.7. The cohomology diagram. Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Then the Weil homomorphisms

$$\chi_E^\#\colon (\vee E^*)_{\theta=0} \to H(B_E), \qquad \chi_F^\#\colon (\vee F^*)_{\theta=0} \to H(B_F)$$

and

$$\chi^\#\colon (\vee F^*)_{\theta=0} \to H(E/F)$$

are defined. These, together with the homomorphisms introduced in sec. 12.1 and sec. 12.4 yield the diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "\chi_E^*"] \arrow[d, "j^{\vee}_{\theta=0}"'] & H(B_E) \arrow[d, "e^*"] \arrow[dr, "e^*"] & & \\
(\vee F^*)_{\theta=0} \arrow[r, "\chi_F^*"] \arrow[dr, "\chi^*"'] & H(B_F) \arrow[r, "e_F^*"] \arrow[d, "p_R"] & H(R_{\theta=0}) \arrow[d, "\varrho_R"] \arrow[dr, "\varrho_R^F"] & \\
& H(E/F) \arrow[r, "k^*"'] & H^*(E) \arrow[r, "j^*"'] & H^*(F).
\end{tikzcd}$$

It is called the *cohomology diagram* of the regular operation $(E, F, i, \theta, R, \delta_R)$.

The cohomology diagram combines

(1) the sequence $H(B_E) \xrightarrow{e^*} H(B_F) \xrightarrow{p_R} H(E/F)$,
(2) the cohomology sequence for the operation of E in R,
(3) the cohomology sequence for the operation of F in R,
(4) the cohomology sequence for the pair (E, F).

In sec. 12.21 it will be shown that the cohomology diagram commutes.

12.8. Homomorphisms. Let $(E, F, i, \theta, R, \delta_R)$ and $(E, F, i, \theta, \hat{R}, \delta_{\hat{R}})$ denote operations of the pair (E, F). A homomorphism, $\varphi: R \to \hat{R}$, of operations of E is automatically a homomorphism of operations of F, and will be called a *homomorphism of operations of the pair* (E, F).

Such a homomorphism restricts to homomorphisms

$$\varphi_{B_E}: B_E \to \hat{B}_E \qquad \text{and} \qquad \varphi_{B_F}: B_F \to \hat{B}_F.$$

Moreover, if (E, F) is reductive and $H(R_{\theta=0})$ is connected, then the diagram

$$\begin{tikzcd}
H(B_F) \arrow[rr, "\varphi_{B_F}^*"] \arrow[dr, "p_R"'] & & H(\hat{B}_F) \arrow[dl, "p_{\hat{R}}"] \\
& H(E/F) &
\end{tikzcd}$$

commutes, as follows from the definitions.

Finally, let (χ, χ_E, χ_F) be an algebraic connection for the first operation and set $\hat{\chi}_E = \varphi \circ \chi_E$ and $\hat{\chi}_F = \varphi \circ \chi_F$. Then $(\chi, \hat{\chi}_E, \hat{\chi}_F)$ is an algebraic connection for the second operation.

In particular, a homomorphism between regular operations induces a homomorphism between the corresponding cohomology diagrams.

§2. The cohomology of B_F

12.9. Introduction. In this article (E, F) is a reductive Lie algebra pair, $\tau: P_E \to (\vee E^*)_{\theta=0}$ is a transgression (cf. sec. 6.13), and

$$\sigma = j^{\vee}_{\theta=0} \circ \tau : P_E \to (\vee F^*)_{\theta=0}.$$

Then $((\vee F^*)_{\theta=0}; \sigma)$ is a P_E-algebra, and the cohomology algebra $H((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma)$ is isomorphic to $H(E/F)$ (cf. Theorem III, sec. 10.8).

Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation with algebraic connection (χ, χ_E, χ_F) (cf. sec. 12.6). Consider the map $\tau_R: P_E \to B_E$ given by

$$\tau_R = (\chi_E)_{\vee, \theta=0} \circ \tau.$$

Then $(B_E, \delta_R; \tau_R)$ is a (P_E, δ)-algebra and the cohomology of the corresponding Koszul complex $(B_E \otimes \wedge P_E, \nabla_B)$ is isomorphic to $H(R_{\theta=0})$ (cf. Theorem I, sec. 9.3).

Now consider the tensor difference $(B_E \otimes (\vee F^*)_{\theta=0}, \delta_R; \tau_R \ominus \sigma)$, (cf. sec. 3.7). Its Koszul complex is given by

$$(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla),$$

where

$$\nabla = \delta_R \otimes \iota \otimes \iota + \nabla_{\tau_R} - \nabla_\sigma$$

with

$$\begin{aligned} &\nabla_{\tau_R}(z \otimes \Psi \otimes \Phi_0 \wedge \cdots \wedge \Phi_p) \\ &\quad = (-1)^q \sum_{i=0}^{p} \tau_R(\Phi_i) \cdot z \otimes \Psi \otimes \Phi_0 \wedge \cdots \widehat{\Phi_i} \cdots \wedge \Phi_p, \end{aligned}$$

and

$$\begin{aligned} &\nabla_\sigma(z \otimes \Psi \otimes \Phi_0 \wedge \cdots \wedge \Phi_p) \\ &\quad = (-1)^q \sum_{i=0}^{p} z \otimes \sigma(\Phi_i) \vee \Psi \otimes \Phi_0 \wedge \cdots \widehat{\Phi_i} \cdots \wedge \Phi_p, \\ &\qquad\qquad z \in B_E^q, \quad \Psi \in (\vee F^*)_{\theta=0}, \quad \Phi_i \in P_E. \end{aligned}$$

12.10. The main theorem. In this section we state the main theorem of this chapter. It contains Theorem I, sec. 9.3, and Theorem III, sec. 10.8, as special cases.

Theorem I: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation, with algebraic connection (χ, χ_E, χ_F) and let τ be a transgression in $W(E)_{\theta=0}$. Then there are homomorphisms of graded differential algebras

$$\psi\colon (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla) \to (B_F, \delta_R),$$
$$\vartheta_R\colon (B_E \otimes \wedge P_E, \nabla_B) \to (R_{\theta_E=0}, \delta_R),$$

and

$$\varphi_F\colon ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E),$$

with the following properties:

(1) The induced homomorphisms $\psi^\#$, $\vartheta_R^\#$, and $\varphi_F^\#$ are isomorphisms of graded algebras.

(2) The isomorphisms $\psi^\#$, $\vartheta_R^\#$, $\varphi_F^\#$, and the isomorphism

$$\tau_\vee\colon \vee P_E \xrightarrow{\cong} (\vee E^*)_{\theta=0}$$

determine an isomorphism between the cohomology diagram

$$\begin{tikzcd}
\vee P_E \arrow[r, "(\tau_R)_\vee^\#"] \arrow[d, "\sigma_\vee"'] & H(B_E) \arrow[d, "m_B^\#"] \arrow[drr, "l_B^\#"] & & \\
(\vee F^*)_{\theta=0} \arrow[r, "m_F^\#"] \arrow[dr, "l^*"'] & H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[rr, "p_F^\#"] \arrow[d, "p_B^\#"] & & H(B_E \otimes \wedge P_E) \arrow[d, "\varrho_B^\#"] \\
& H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[rr, "\varrho^*"'] & & \wedge P_E
\end{tikzcd}$$

of the tensor difference, and the subdiagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "\chi_E^\#"] \arrow[d, "\check{j}_{\theta=0}"'] & H(B_E) \arrow[d, "e^*"] \arrow[dr, "e_E^\#"] & \\
(\vee F^*)_{\theta=0} \arrow[r, "\chi_F^\#"] \arrow[dr, "\chi^*"'] & H(B_F) \arrow[r, "e_F^\#"] \arrow[d, "p_R"] & H(R_{\theta=0}) \arrow[d, "\varrho_R"] \\
& H(E/F) \arrow[r, "k^*"'] & H^*(E)
\end{tikzcd}$$

of the cohomology diagram of the operation.

Remark: The rest of this article is devoted to the construction of ϑ_R and ψ and the proof that $\vartheta_R^\#$ and $\psi^\#$ are isomorphisms. The rest of (1), and (2), will be established in article 3. In article 4, we shall derive corollaries and less immediate consequences of the theorem. In particular, the reader may skip directly to article 4, without going through the proof of Theorem I.

The proof of the first part of the theorem is organized as follows. We construct a (noncommutative) diagram

$$\begin{tikzcd}
 & B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \arrow[r, "p_F"] \arrow[d, "\vartheta"] & B_E \otimes \wedge P_E \arrow[d, "\vartheta_R"] \\
(\vee F^*)_{\theta=0} \arrow[ur, "m_F"] \arrow[r, "\xi_F"] \arrow[dr, "(\chi_F)_\vee"'] & (\vee F^* \otimes R)_{\theta_F=0} \arrow[r, "\eta_F"] \arrow[d, shift left, "\alpha_F"] & R_{\theta_F=0} \\
 & B_F \arrow[u, shift left, "\varepsilon_F"] \arrow[ur, "e_F"'] &
\end{tikzcd} \tag{12.2}$$

of graded differential algebras, which yields a commutative diagram of cohomology algebras. In particular, the lower half of the cohomology diagram is the commutative diagram in Corollary I, sec. 8.20.

The difficulty is in the construction of ϑ and ϑ_R; this is done in sec. 12.13 and sec. 12.14. The homomorphism ψ is defined by $\psi = \alpha_F \circ \vartheta$, and the article concludes with the proof that $\psi^\#$ is an isomorphism (sec. 12.16).

12.11. The algebra $((\vee F^* \otimes R)_{\theta_F=0}, D_F)$. We apply the results of secs. 8.17–8.20 to the operation of F in R. The antiderivation denoted there by D_R will here be denoted by D_F:

$$D_F = \iota \otimes \delta_R - \sum_\varrho \mu_S(f^{*\varrho}) \otimes i(f_\varrho),$$

where $f^{*\varrho}, f_\varrho$ is a pair of dual bases for F^* and F. According to sec. 8.17, $((\vee F^* \otimes R)_{\theta_F=0}, D_F)$ is a graded differential algebra.

Let $\varepsilon_F: B_F \to (\vee F^* \otimes R)_{\theta_F=0}$ be the inclusion. In sec. 8.19 we constructed a homomorphism (here denoted by α_F)

$$\alpha_F: ((\vee F^* \otimes R)_{\theta_F=0}, D_F) \to (B_F, \delta_R)$$

of graded differential algebras such that $\alpha_F \circ \varepsilon_F = \iota$. Moreover, by Theo-

rem IV, sec. 8.17, and Proposition IX, sec. 8.19, $\alpha_F^\#$ and $\varepsilon_F^\#$ are inverse isomorphisms.

Finally, let

$$\xi_F\colon (\vee F^*)_{\theta=0} \to (\vee F^* \otimes R)_{\theta_F=0} \qquad \text{and} \qquad \eta_F\colon (\vee F^* \otimes R)_{\theta_F=0} \to R_{\theta_F=0}$$

be the obvious inclusion and projection. Then Corollary I of sec. 8.20 yields the commutative diagram

$$\begin{tikzcd}[column sep=large]
(\vee F^*)_{\theta=0} \arrow[r, "\xi_F^\#"] \arrow[dr, "\chi_F^\#"'] & H((\vee F^* \otimes R)_{\theta_F=0}) \arrow[r, "\eta_F^\#"] \arrow[d, shift left, "\alpha_F^\#"] \arrow[d, phantom, "\cong"] & H(R_{\theta_F=0}) \\
& H(B_F) \arrow[u, shift left, "\varepsilon_F^\#"] \arrow[ur, "e_F^\#"'] &
\end{tikzcd} \tag{12.3}$$

12.12. The algebra $\vee E^* \otimes W(E)$. Consider the operation $(E, i, \theta_W, W(E), \delta_W)$ (cf. Example 6 of sec. 7.4 and Chapter VI). Recall the decomposition

$$\delta_W = \delta_E + \delta_\theta + h$$

and the projection $\pi_E\colon W(E) \to \wedge E^*$. Finally recall the canonical map (cf. sec. 6.7)

$$\varrho_E\colon (\vee^+ E^*)_{\theta=0} \to (\wedge^+ E^*)_{\theta=0}.$$

Extend π_E to a homomorphism

$$\pi_E\colon \vee E^* \otimes W(E) \to \wedge E^*$$

by setting $\pi_E(\vee^+ E^* \otimes W(E)) = 0$. Similarly define homomorphisms

$$\pi_j\colon \vee E^* \otimes W(E) \to W(E), \qquad j = 1, 2,$$

by

$$\pi_1(\Psi_1 \otimes \Psi_2 \otimes \Phi) = \begin{cases} \Psi_2 \otimes \Phi & \text{if } \Psi_1 = 1, \\ 0 & \text{if } \Psi_1 \in \vee^+ E^*; \end{cases}$$

and

$$\pi_2(\Psi_1 \otimes \Psi_2 \otimes \Phi) = \begin{cases} \Psi_1 \otimes \Phi & \text{if } \Psi_2 = 1, \\ 0 & \text{if } \Psi_2 \in \vee^+ E^*. \end{cases}$$

Then $\pi_E \circ \pi_1 = \pi_E$ and $\pi_E \circ \pi_2 = \pi_E$.

The homomorphisms π_E, π_1, and π_2 restrict to homomorphisms

$$\pi_E\colon (\vee E^* \otimes W(E))_{\theta=0} \to (\wedge E^*)_{\theta=0}$$

and

$$\pi_j\colon (\vee E^* \otimes W(E))_{\theta=0} \to W(E)_{\theta=0}, \qquad j = 1, 2.$$

Now apply the results of secs. 8.17–8.20 to the operation $(E, i, \theta_W, W(E), \delta_E)$. Consider the operator

$$D_W = \iota \otimes \delta_W - \sum_\nu \mu_S(e^{*\nu}) \otimes i(e_\nu)$$

in $\vee E^* \otimes W(E)$ ($e^{*\nu}$, e_ν a pair of dual bases for E^* and E). According to sec. 8.17, D_W restricts to a differential operator in $(\vee E^* \otimes W(E))_{\theta=0}$.

Moreover, combining Theorem IV, sec. 8.17, with Corollary I to Theorem V, sec. 8.20 yields the commutative diagram

$$\begin{tikzcd}
 & H((\vee E^* \otimes W(E))_{\theta=0}) \\
(\vee E^*)_{\theta=0} \arrow[ur, "\xi^*"] \arrow[dr, "\iota"', "\cong"] & \\
 & (\vee E^*)_{\theta=0}. \arrow[uu, "\varepsilon^*"', "\cong"]
\end{tikzcd}$$

(Here ξ and ε are the inclusion maps given by

$$\xi(\Psi) = \Psi \otimes 1 \otimes 1 \qquad \text{and} \qquad \varepsilon(\Psi) = 1 \otimes \Psi \otimes 1, \qquad \Psi \in (\vee E^*)_{\theta=0}.)$$

Recall from Theorem I, sec. 12.10 that τ denotes a transgression in $W(E)_{\theta=0}$.

Lemma III: There is a linear map, homogeneous of degree zero,

$$s\colon P_E \to (\vee E^* \otimes W(E))_{\theta=0}$$

with the following properties: For $\Phi \in P_E$,

(1) $D_W(s\Phi) = 1 \otimes \tau\Phi \otimes 1 - \tau\Phi \otimes 1 \otimes 1$.

(2) $\delta_W \pi_j(s\Phi) = \tau\Phi \otimes 1$, $j = 1, 2$.

(3) $\pi_E(s\Phi) = \Phi$.

Proof: The diagram above shows that for $\Phi \in P_E$,

$$1 \otimes \tau\Phi \otimes 1 - \tau\Phi \otimes 1 \otimes 1 \in \operatorname{Im} D_W.$$

Hence there is a linear map, homogeneous of degree zero,

$$s: P_E \to (\vee E^* \otimes W(E))_{\theta=0},$$

such that

$$D_W(s\Phi) = 1 \otimes \tau\Phi \otimes 1 - \tau\Phi \otimes 1 \otimes 1, \qquad \Phi \in P_E.$$

This establishes (1).

A simple calculation shows that in $(\vee E^* \otimes W(E))_{\theta=0}$

$$\pi_1 D_W = \delta_W \pi_1 \qquad \text{and} \qquad \pi_2 D_W = -\delta_W \pi_2.$$

Substituting these relations in (1) we obtain (2).

Finally, recall from Lemma VII, sec. 6.13, that $\varrho_E \circ \tau = \iota$. In view of the definition of ϱ_E it follows from (2) that (for $\Phi \in P_E$)

$$\Phi = \varrho_E \tau(\Phi) = \pi_E(\pi_j(s\Phi)) = \pi_E(s\Phi), \qquad j = 1, 2.$$

Q.E.D.

12.13. The homomorphism ϑ. In this section we define a homomorphism of graded differential algebras

$$\vartheta: (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla) \to ((\vee F^* \otimes R)_{\theta_F=0}, D_F).$$

First, consider the homomorphism

$$j^\vee \otimes (\chi_E)_W: \vee E^* \otimes W(E) \to \vee F^* \otimes R,$$

where $(\chi_E)_W$ is the classifying homomorphism of the algebraic connection χ_E (cf. sec. 8.16). This homomorphism is F-linear (with respect to the obvious representations of F) and satisfies

$$D_F \circ (j^\vee \otimes (\chi_E)_W) = (j^\vee \otimes (\chi_E)_W) \circ D_W.$$

Hence it restricts to a homomorphism

$$I: ((\vee E^* \otimes W(E))_{\theta_E=0}, D_W) \to ((\vee F^* \otimes R)_{\theta_F=0}, D_F)$$

of graded differential algebras. Lemma III, (1) shows that

$$(D_F \circ I)(s\Phi) = 1 \otimes \tau_R\Phi - \sigma\Phi \otimes 1, \qquad \Phi \in P_E. \tag{12.4}$$

Now extend s to a homomorphism

$$s_\wedge : \wedge P_E \to (\vee E^* \otimes W(E))_{\theta_E=0},$$

and define ϑ by

$$\vartheta(z \otimes \Psi \otimes \Phi) = (\Psi \otimes z) \cdot I(s_\wedge \Phi), \qquad z \in B_E, \quad \Psi \in (\vee F^*)_{\theta=0}, \quad \Phi \in \wedge P_E.$$

Then it follows at once from formula (12.4) that

$$\vartheta \circ \nabla = D_F \circ \vartheta,$$

and so ϑ is a homomorphism of graded differential algebras.

In sec. 12.15 it will be shown that $\vartheta^\#$ is an isomorphism.

12.14. The homomorphism ϑ_R. Let $s_1 : P_E \to W(E)_{\theta=0}$ be the linear map given by $s_1 = \pi_1 \circ s$. In view of Lemma III, sec. 12.12, we have

$$\delta_W(s_1 \Phi) = \tau\Phi \otimes 1 \quad \text{and} \quad s_1\Phi - 1 \otimes \Phi \in (\vee^+ E^* \otimes \wedge E^*)_{\theta=0}, \quad \Phi \in P_E.$$

Now define a homomorphism

$$\vartheta_R : B_E \otimes \wedge P_E \to R_{\theta_E=0}$$

by setting

$$\vartheta_R(z \otimes \Phi) = z \cdot [(\chi_E)_W (s_1)_\wedge(\Phi)], \qquad z \in B_E, \quad \Phi \in \wedge P_E.$$

Then ϑ_R is the Chevalley homomorphism associated with the algebraic connection χ_E and the linear map s_1 (cf. sec. 9.3).

In particular, ϑ_R is a homomorphism of graded differential algebras. Moreover, by Theorem I, sec. 9.3, $\vartheta_R^\#$ is an isomorphism.

12.15. Proposition V: The homomorphism

$$\vartheta^\# : H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla) \to H((\vee F^* \otimes R)_{\theta_F=0}, D_F)$$

induced by ϑ is an isomorphism (cf. sec. 12.13).

Proof: Filter the algebras $B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E$ and $(\vee F^* \otimes R)_{\theta_F=0}$, respectively, by the ideals

$$F^p = \sum_{j \geq p} B_E \otimes (\vee F^*)^j_{\theta=0} \otimes \wedge P_E \quad \text{and} \quad \hat{F}^p = \sum_{j \geq p} [(\vee F^*)^j \otimes R]_{\theta_F=0}.$$

Then ϑ is filtration preserving and so we have an induced homomorphism of spectral sequences

$$\vartheta_i\colon (E_i, d_i) \to (\hat{E}_i, \hat{d}_i), \qquad i \geq 0.$$

In view of the comparison theorem (sec. 1.14) it is sufficient to show that

$$\vartheta_0^\#\colon H(E_0, d_0) \to H(\hat{E}_0, \hat{d}_0)$$

is an isomorphism.

First observe that

$$(E_0, d_0) = (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \delta_R \otimes \iota \otimes \iota + \nabla_{\tau_R})$$

(cf. sec. 12.10), and that

$$(\hat{E}_0, \hat{d}_0) = ((\vee F^* \otimes R)_{\theta_F=0}, \iota \otimes \delta_R).$$

Lemma IV, below, implies that the diagram

$$\begin{tikzcd}
B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \arrow[r, "\lambda"] \arrow[rrdd, "\vartheta_0"'] & (\vee F^*)_{\theta=0} \otimes B_E \otimes \wedge P_E \arrow[r, "\iota\otimes\vartheta_R"] & (\vee F^*)_{\theta=0} \otimes R_{\theta_E=0} \arrow[d, "\iota\otimes\varepsilon_1"] \\
 & & (\vee F^*)_{\theta=0} \otimes R_{\theta_F=0} \arrow[d, "\iota\otimes\varepsilon_2"] \\
 & & (\vee F^* \otimes R)_{\theta_F=0},
\end{tikzcd} \tag{12.5}$$

commutes, where λ is given by

$$\lambda(z \otimes \Psi \otimes \Phi) = \Psi \otimes z \otimes \Phi,$$

and $\varepsilon_1, \varepsilon_2$ are the obvious inclusions.

Since λ is an isomorphism, so is

$$\lambda^\#\colon H(E_0, d_0) \xrightarrow{\cong} (\vee F^*)_{\theta=0} \otimes H(B_E \otimes \wedge P_E, \nabla_B).$$

According to sec. 12.14, $\vartheta_R^\#$ is an isomorphism. By hypothesis

$$\varepsilon_1^\#\colon H(R_{\theta_E=0}) \to H(R_{\theta_F=0})$$

is an isomorphism. Finally, Proposition IV, sec. 7.6, (applied with

$M = \vee F^*$, $\delta_M = 0$, and the Lie algebra F) shows that

$$(\iota \otimes \varepsilon_2)^{\#}: (\vee F^*)_{\theta=0} \otimes H(R_{\theta_F=0}) \to H((\vee F^* \otimes R)_{\theta_F=0}, \iota \otimes \delta_R)$$

is an isomorphism.

It follows that $\vartheta_0^{\#}$ (and hence $\vartheta^{\#}$) is an isomorphism.

Q.E.D.

Lemma IV: The diagram (12.5) commutes.

Proof: Let

$$\hat{\vartheta}: B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (\vee F^* \otimes R)_{\theta_F=0}$$

be the homomorphism given by

$$\hat{\vartheta}(z \otimes \Psi \otimes \Phi) = \Psi \otimes \vartheta_R(z \otimes \Phi).$$

Then diagram (12.5) commutes (obviously) if ϑ_0 is replaced by $\hat{\vartheta}$. Hence it is sufficient to show that $\vartheta_0 = \hat{\vartheta}$; equivalently, we must prove that

$$\hat{\vartheta} - \vartheta : B_E \otimes (\vee F^*)^p_{\theta=0} \otimes \wedge P_E \to \sum_{j \geq p+1} ((\vee F^*)^j \otimes R)_{\theta_F=0}.$$

Since $\hat{\vartheta}$ and ϑ agree in $B_E \otimes (\vee F^*)_{\theta=0} \otimes 1$, and since both are homomorphisms, it is sufficient to check that for $\Phi \in P_E$

$$(\hat{\vartheta} - \vartheta)(1 \otimes 1 \otimes \Phi) \in (\vee^+ F^* \otimes R)_{\theta_F=0}. \tag{12.6}$$

But

$$\begin{aligned}(\hat{\vartheta} - \vartheta)(1 \otimes 1 \otimes \Phi) &= 1 \otimes (\chi_E)_W(s_1\Phi) - (j^\vee \otimes (\chi_E)_W)(s\Phi) \\ &= (j^\vee \otimes (\chi_E)_W)(1 \otimes \pi_1 s\Phi - s\Phi).\end{aligned}$$

In view of the definition of π_1

$$1 \otimes \pi_1 \Omega - \Omega \in \vee^+ E^* \otimes W(E), \qquad \Omega \in \vee E^* \otimes W(E).$$

Hence, in particular,

$$\begin{aligned}(j^\vee \otimes (\chi_E)_W)(1 \otimes \pi_1 s\Phi - s\Phi) &\in [j^\vee \otimes (\chi_E)_W](\vee^+ E^* \otimes W(E))_{\theta_E=0} \\ &\subset (\vee^+ F^* \otimes R)_{\theta_F=0}.\end{aligned}$$

This establishes formula (12.6).

Q.E.D.

12.16. The homomorphism ψ. Recall from sec. 12.11 and sec. 12.13 the homomorphisms

$$\alpha_F\colon (\vee F^* \otimes R)_{\theta_F=0} \to B_F$$

and

$$\vartheta\colon B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (\vee F^* \otimes R)_{\theta_F=0}.$$

Their composite will be denoted by

$$\psi\colon (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla) \to (B_F, \delta_R).$$

It follows from sec. 12.11 and Proposition V, sec. 12.15, that $\psi^\#$ is an isomorphism.

Next, consider the diagram (12.2) of sec. 12.10. It is immediate from the definitions and formula (12.6), sec. 12.15, that the upper half commutes. Thus in view of the commutative diagram (12.3) of sec. 12.11 the diagram of cohomology algebras induced by (12.2) is commutative. In particular we obtain the commutative diagram

$$\begin{tikzcd} & H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "p_F^\#"] \arrow[d, "\psi^\#", "\cong"'] & H(B_E \otimes \wedge P_E) \arrow[d, "\vartheta_R^\#", "\cong"'] \\ (\vee F^*)_{\theta=0} \arrow[ur, "m_F^\#"] \arrow[r, "\chi_F^\#"'] & H(B_F) \arrow[r, "e_F^\#"'] & H(R_{\theta=0}). \end{tikzcd} \tag{12.7}$$

§3. Isomorphism of the cohomology diagrams

12.17. The purpose of this article is to prove the rest of Theorem I, sec. 12.10. We carry over all the notation developed in article 2.

Recall from sec. 12.14 and sec. 12.16 the isomorphisms

$$\vartheta_R^\#: H(B_E \otimes \wedge P_E) \xrightarrow{\cong} H(R_{\theta=0})$$

and

$$\psi^\#: H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \xrightarrow{\cong} H(B_F).$$

To finish the proof of Theorem I, we have to construct a homomorphism of graded differential algebras

$$\varphi_F: ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E),$$

with the following properties:

(1) $\varphi_F^\#$ is an isomorphism.

(2) The isomorphisms $\vartheta_R^\#$, $\varphi_F^\#$, $\psi^\#$, and $\tau_\vee$ define an isomorphism from the cohomology diagram of the tensor difference to the cohomology diagram of the operation $(E, F, i, \theta, R, \delta_R)$. In other words, the following diagrams commute:

$$\begin{tikzcd}
\vee P_E \arrow[r, "(\tau_R)_\vee^\#"] \arrow[d, "\tau_\vee"', "\cong"] & H(B_E) \arrow[r, "l_B^\#"] \arrow[d, "\iota"', "\cong"] & H(B_E \otimes \wedge P_E) \arrow[r, "\varrho_B^\#"] \arrow[d, "\cong"', "\vartheta_R^\#"] & \wedge P_E \arrow[d, "\cong"] \\
(\vee E^*)_{\theta=0} \arrow[r, "\chi_E^\#"'] & H(B_E) \arrow[r, "e_E^\#"'] & H(R_{\theta=0}) \arrow[r, "\varrho_R"'] & (\wedge E^*)_{\theta=0},
\end{tikzcd} \tag{12.8}$$

$$\begin{tikzcd}
\vee P_E \arrow[r, "\sigma_\vee"] \arrow[d, "\tau_\vee"', "\cong"] & (\vee F^*)_{\theta=0} \arrow[r, "l^*"] \arrow[d, "\iota"', "\cong"] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "\varrho^*"] \arrow[d, "\cong"', "\varphi_F^\#"] & \wedge P_E \arrow[d, "\cong"'] \\
(\vee E^*)_{\theta=0} \arrow[r, "j_{\theta=0}^\vee"'] & (\vee F^*)_{\theta=0} \arrow[r, "\chi^*"'] & H(E/F) \arrow[r, "k^*"'] & H^*(E),
\end{tikzcd} \tag{12.9}$$

$$\begin{tikzcd}
& H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "p_F^*"] \arrow[d, "\cong"', "\psi^*"] & H(B_E \otimes \wedge P_E) \arrow[d, "\cong"', "\vartheta_R^*"] \\
(\vee F^*)_{\theta=0} \arrow[ur, "m_F^*"] \arrow[r, "\chi_F^*"'] & H(B_F) \arrow[r, "e_F^*"'] & H(R_{\theta=0}),
\end{tikzcd} \tag{12.10}$$

$$\begin{tikzcd}
& H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[d, "\cong"', "\psi^*"] \\
H(B_E) \arrow[ur, "m_B^*"] \arrow[r, "e^*"'] & H(B_F),
\end{tikzcd} \tag{12.11}$$

and

$$\begin{tikzcd}
H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r, "p_B^*"] \arrow[d, "\psi^*"', "\cong"] & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[d, "\cong"', "\varphi_F^*"] \\
H(B_F) \arrow[r, "p_R"'] & H(E/F).
\end{tikzcd} \tag{12.12}$$

First observe that, in view of the definition of ϑ_R (sec. 12.14), it follows from Theorem II, sec. 9.7, that (12.8) commutes. That (12.10) commutes is established in sec. 12.16. Moreover, clearly $\psi \circ m_B = e$, and so diagram (12.11) commutes.

Finally, in sec. 12.18 we shall construct φ_F, prove that $\varphi_F^\#$ is an isomorphism, and show that (12.9) commutes. In sec. 12.19 we show that diagram (12.12) commutes.

12.18. The homomorphism φ_F. Define $s_2\colon P_E \to W(E)_{\theta=0}$ by

$$s_2 = \pi_2 \circ s$$

(cf. sec. 12.12). Lemma III of sec. 12.12 implies that for $\Phi \in P_E$:

$$\delta_W s_2(\Phi) = \tau\Phi \otimes 1 \qquad \text{and} \qquad s_2\Phi - 1 \otimes \Phi \in (\vee^+ E^* \otimes \wedge E^*)_{\theta=0}.$$

The corresponding Chevalley homomorphism for the pair (E, F) (as defined in sec. 10.10) is the homomorphism

$$\vartheta_F: (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (\vee F^* \otimes \wedge E^*)_{\theta_F=0},$$

given by

$$\vartheta_F(\Psi \otimes \Phi) = (\Psi \otimes 1) \cdot (j^\vee \otimes \iota)((s_2)_\wedge \Phi), \qquad \Psi \in (\vee F^*)_{\theta=0}, \quad \Phi \in \wedge P_E.$$

Recall that (χ, χ_E, χ_F) is a fixed algebraic connection for the operation of (E, F) in R. In sec. 10.9 we constructed a homomorphism

$$\alpha_\chi: (\vee F^* \otimes \wedge E^*)_{\theta_F=0} \to (\wedge E^*)_{i_F=0, \theta_F=0},$$

induced by the algebraic connection χ.

Now set

$$\varphi_F = \vartheta_F \circ \alpha_\chi : ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) \to ((\wedge E^*)_{i_F=0, \theta_F=0}, \delta_E).$$

It follows from Theorem III, sec. 10.8 (as proved in sec. 10.11) that φ_F is a homomorphism of graded differential algebras, and that $\varphi_F^\#$ is an isomorphism making (12.9) commute.

12.19. Proposition VI: The diagram (12.12) commutes; i.e.,

$$p_R \circ \psi^\# = \varphi_F^\# \circ p_B^\#.$$

Proof: Use the algebraic connection (χ, χ_E, χ_F) to write

$$\wedge E^* = (\wedge E^*)_{i_F=0} \otimes \wedge F^*, \qquad R = R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \otimes \wedge F^*$$

and

$$B_F = (R_{i_E=0} \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0},$$

as described in sec. 12.6. Define a homomorphism

$$\hat{\psi}: B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (R_{i_E=0} \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0},$$

by setting

$$\hat{\psi}(w \otimes \Psi \otimes \Phi) = w \otimes \varphi_F(\Psi \otimes \Phi).$$

Now let $\zeta \in H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E)$. According to sec. 3.19, ζ can be represented by a cocycle of the form

$$z = 1 \otimes \tilde{\Omega} + \hat{\Omega},$$

where $\tilde{\Omega} \in (\vee F^*)_{\theta=0} \otimes \wedge P_E$ is a cocycle representing $p_B^\# \zeta$, and

$$\hat{\Omega} \in B_E^+ \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E.$$

Next, write

$$\psi(z) = \psi(\hat{\Omega}) + (\psi - \hat{\varphi})(1 \otimes \tilde{\Omega}) + 1 \otimes \varphi_F \tilde{\Omega}.$$

Then (by the definition of ψ)

$$\psi(\hat{\Omega}) \in [R^+_{i_E=0} \otimes (\wedge E^*)_{i_F=0}]_{\theta_F=0}.$$

Moreover, Lemma V, sec. 11.20, (below) shows that

$$(\psi - \hat{\varphi})(1 \otimes \tilde{\Omega}) \in [R^+_{i_E=0} \otimes (\wedge E^*)_{i_F=0}]_{\theta_F=0}.$$

Thus, *if the operation is projectable,* Proposition IV, sec. 12.6 (applied with $\Phi = \varphi_F(\tilde{\Omega})$) shows that $\varphi_F(\tilde{\Omega})$ represents $p_R(\psi^\# \zeta)$. Hence

$$\varphi_F^\#(p_B^\# \zeta) = p_R(\psi^\# \zeta),$$

and the proposition is proved in this case.

Finally, suppose $(E, F, i, \theta, R, \delta_R)$ is *any* regular operation. Let $(E, F, i, \theta, R_S, \delta_R)$ denote the associated semisimple operation (cf. sec. 12.2). Note that

$$(R_S)_{\theta_E=0} = R_{\theta_E=0} \qquad \text{and} \qquad (R_S)_{i_E=0,\theta_E=0} = B_E.$$

Since $\chi_E(E^*) \subset R_S^1$, χ_E may be regarded as an algebraic connection $\tilde{\chi}_E$ for the operation of E in R_S. Set

$$\tilde{\chi}_F = \tilde{\chi}_E \circ \chi.$$

Next, observe that the inclusion map $\lambda\colon R_S \to R$ is a homomorphism of operations. Thus it restricts to a homomorphism

$$\lambda_{i_F=0,\theta_F=0}\colon (R_S)_{i_F=0,\theta_F=0} \to B_F.$$

Since the operation of (E, F) in R_S is regular, the construction of articles 2 and 3 may be carried out with R_S replacing R and with $(\chi, \tilde{\chi}_E, \tilde{\chi}_F)$ replacing (χ, χ_E, χ_F), but with the same linear map $s\colon P_E \to (\vee E^* \otimes W(E))_{\theta=0}$. This yields a homomorphism

$$\tilde{\psi}\colon B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E \to (R_S)_{i_F=0,\theta_F=0}.$$

Clearly,

$$\lambda_{i_F=0,\theta_F=0} \circ \tilde{\psi} = \psi.$$

Let

$$p_{R_S}: H((R_S)_{i_F=0,\theta_F=0}) \to H(E/F)$$

denote the fibre projection for the operation of (E, F) in R_S. Then (cf. sec. 12.8)

$$p_{R_S} = p_R \circ \lambda^{\#}_{i_F=0,\theta_F=0}.$$

Moreover, since the representation of E in R_S is semisimple, the operation of (E, F) in R_S is projectable. Thus, by the first part of the proof,

$$p_{R_S} \circ \tilde{\psi}^{\#} = \varphi_F^{\#} \circ p_B^{\#}.$$

It follows that

$$p_R \circ \psi^{\#} = p_R \circ \lambda^{\#}_{i_F=0,\theta_F=0} \circ \tilde{\psi}^{\#} = \varphi_F^{\#} \circ p_B^{\#}.$$

This completes the proof of Proposition VI and hence the proof of Theorem I.

Q.E.D.

12.20. Lemma V: Let I be the ideal in B_F given by

$$I = \sum_{j \geq 2} (R^j_{i_E=0} \otimes (\wedge E^*)_{i_F=0})_{\theta_F=0}.$$

Then

$$\operatorname{Im}(\psi - \hat{\varphi}) \subset I.$$

Proof: It is sufficient to establish the relations:

$$(\psi - \hat{\varphi})(z \otimes 1 \otimes 1) \in I, \qquad z \in B_E. \tag{12.13}$$

$$(\psi - \hat{\varphi})(1 \otimes \Psi \otimes 1) \in I, \qquad \Psi \in (\vee F^*)_{\theta=0}. \tag{12.14}$$

$$(\psi - \hat{\varphi})(1 \otimes 1 \otimes \Phi) \in I, \qquad \Phi \in P_E. \tag{12.15}$$

It follows from the definitions that ψ and $\hat{\varphi}$ agree in $B_E \otimes 1 \otimes 1$, whence formula (12.13). Moreover if $\Psi \in (\vee F^*)_{\theta=0}$, then

$$(\psi - \hat{\varphi})(1 \otimes \Psi \otimes 1) = \alpha_F(\Psi \otimes 1) - 1 \otimes \alpha_\chi(\Psi \otimes 1) \tag{12.16}$$

(cf. sec. 12.11 and sec. 12.18). Further, α_F and α_χ extend to the homomorphisms

$$\alpha_F: \vee F^* \to R_{i_E=0} \otimes (\wedge E^*)_{i_F=0} \qquad \text{and} \qquad \alpha_\chi: \vee F^* \to (\wedge E^*)_{i_F=0}$$

given by

$$\alpha_F(y^*) = \chi_F(y^*) \qquad \text{and} \qquad \alpha_\chi(y^*) = \chi(y^*), \qquad y^* \in F^*.$$

Now formula (12.1), sec. 12.6, shows that

$$\alpha_F(y^*) - 1 \otimes \alpha_\chi(y^*) = \chi_E(\chi y^*) \otimes 1 \in R^2_{i_E=0} \otimes 1.$$

In view of this, relation (12.16) implies formula (12.14).

To prove (12.15) observe that

$$s\Phi = \sum_i \Psi_i \otimes 1 \otimes \Phi_i + \Omega_1,$$

where $\sum_i \Psi_i \otimes \Phi_i = s_2\Phi$ and $\Omega_1 \in \vee E^* \otimes \vee^+ E^* \otimes \wedge E^*$. Since

$$(\chi_E)_W: \vee^+ E^* \to \sum_{j\geq 2} R^j_{i_E=0},$$

it follows that

$$\vartheta(1 \otimes 1 \otimes \Phi) - \sum_i j^\vee \Psi_i \otimes 1 \otimes \Phi_i \in \vee F^* \otimes \sum_{j\geq 2} R^j_{i_E=0} \otimes \wedge E^*. \quad (12.17)$$

Next, write $\Phi_i = \hat{\Phi}_i + \tilde{\Phi}_i$, where

$$\hat{\Phi}_i \in (\wedge E^*)_{i_F=0} \otimes 1 \qquad \text{and} \qquad \tilde{\Phi}_i \in (\wedge E^*)_{i_F=0} \otimes \wedge^+ F^*.$$

Then

$$\alpha_F\Big(\sum_i j^\vee \Psi_i \otimes 1 \otimes \Phi_i\Big) = \sum_i [(\chi_F)_\vee(j^\vee \Psi_i)] \cdot (1 \otimes \hat{\Phi}_i).$$

Thus applying formula (12.17) we find

$$\psi(1 \otimes 1 \otimes \Phi) - \sum_i [(\chi_F)_\vee(j^\vee \Psi_i)] \cdot (1 \otimes \hat{\Phi}_i) \in I. \qquad (12.18)$$

Now it follows from formula (12.1), sec. 12.6, that

$$((\chi_F)_\vee - \chi_\vee): \vee F^* \to \sum_{j\geq 2} R^j_{i_E=0} \otimes (\wedge E^*)_{i_F=0}.$$

In view of formula (12.18), this implies that

$$\psi(1 \otimes \mathbf{1} \otimes \Phi) - 1 \otimes \sum_i (\chi_\vee j^\vee(\Psi_i)) \cdot \hat{\Phi}_i \in I.$$

But

$$\begin{aligned}\hat{\psi}(1 \otimes 1 \otimes \Phi) &= 1 \otimes [\alpha_\chi(j^\vee \otimes \iota)(s_2\Phi)] \\ &= 1 \otimes \alpha_\chi\Big(\sum_i j^\vee \Psi_i \otimes \Phi_i\Big) \\ &= 1 \otimes \sum_i \chi_\vee(j^\vee \Psi_i) \cdot \hat{\Phi}_i,\end{aligned}$$

and so (12.15) is established.

Q.E.D.

§4. Applications of the fundamental theorem

12.21. Immediate consequences. Corollary I: The cohomology diagram of a regular operation of a reductive pair commutes.

Proof: According to Proposition VI, sec. 3.20, the cohomology diagram of a tensor difference commutes. Thus, in view of Theorem I, (2) we have only to show that the diagram

$$\begin{tikzcd} H(R_{\theta=0}) \arrow[d, "\varrho_R"'] \arrow[dr, "\varrho_R^F"] & \\ H^*(E) \arrow[r, "j^\#"'] & H^*(F) \end{tikzcd}$$

commutes.

Let $\gamma_R^F\colon R \to R \otimes \wedge F^*$ be the structure homomorphism for the operation $(F, i, \theta, R, \delta_R)$. Then, it follows from the definitions of γ_R (sec. 7.8) and β (sec. 7.7) that the diagram

$$\begin{tikzcd} R \arrow[d, "\gamma_R"'] \arrow[dr, "\gamma_R^F"] & \\ R \otimes \wedge E^* \arrow[r, "\iota\otimes j^\wedge"'] & R \otimes \wedge F^* \end{tikzcd}$$

commutes. This yields the commutative diagram

$$\begin{tikzcd} & H((R \otimes \wedge E^*)_{\theta_E=0}) \arrow[d, "(\iota\otimes j^\wedge)^\#_{\theta=0}"] & H(R_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \arrow[l, "\cong"'] \arrow[d, "\iota\otimes j^\wedge_{\theta=0}"] \\ H(R_{\theta=0}) \arrow[ur, "(\gamma_R)^\#_{\theta=0}"] \arrow[dr, "(\gamma_R^F)^\#_{\theta_F=0}"'] & & \\ & H((R \otimes \wedge F^*)_{\theta_F=0}) & H(R_{\theta=0}) \otimes (\wedge F^*)_{\theta=0}, \arrow[l, "\cong"] \end{tikzcd}$$

whence $\varrho_R^F = j^\# \circ \varrho_R$.

Q.E.D.

Corollary II: There is a spectral sequence converging to $H(B_F)$ whose E_2-term is given by

$$E_2^{p,q} \cong H^p(B_E) \otimes H^q(E/F).$$

Proof: Apply the results of sec. 3.9, recalling from sec. 10.8 that

$$H(E/F) \cong H((\vee F^*)_{\theta=0} \otimes \wedge P_E).$$

Q.E.D.

Corollary III: There is a spectral sequence converging to $H(B_F)$ whose E_2-term is given by

$$E_2^{p,q} \cong (\vee F^*)^p_{\theta=0} \otimes H^q(R_{\theta=0}).$$

Proof: Apply the results of sec. 3.9, recalling from sec. 9.3 that

$$H(R_{\theta=0}) \cong H(B_E \otimes \wedge P_E).$$

Q.E.D.

Corollary IV: Assume (B_E, δ_R) is c-split. Then there are c-equivalences

$$(H(B_E) \otimes \wedge P_E, \nabla^{\#}_{\tau_R}) \underset{c}{\sim} (R_{\theta_E=0}, \delta_R)$$

and

$$(H(B_E) \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla^{\#}_{\tau_R} - \nabla_\sigma) \underset{c}{\sim} (B_F, \delta_R).$$

They can be chosen so that the induced cohomology isomorphisms, together with the identity map of $H(B_E)$, define an isomorphism from the cohomology diagram of the operation to the diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r] \arrow[d] & H(B_E) \arrow[d] \arrow[dr] & \\
(\vee F^*)_{\theta=0} \arrow[r] \arrow[dr] & H(H(B_E) \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r] \arrow[d] & H(H(B_E) \otimes \wedge P_E) \arrow[d] \\
 & H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \arrow[r] & \wedge P_E .
\end{tikzcd}$$

Proof: This follows from the fundamental theorem, together with Proposition XI, sec. 3.29, applied to a c-splitting. (In applying Proposi-

tion XI, copy the example of sec. 3.29, replacing $B_E \otimes \wedge P_E$ by $B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E$, as in sec. 3.28.)

Q.E.D.

12.22. N.c.z. operations of a pair. Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Then $(\wedge E^*)_{i_F=0,\theta_F=0}$ is called *noncohomologous to zero* (*n.c.z.*) *in* B_F if the fibre projection

$$p_R\colon H(B_F) \to H(E/F)$$

is surjective.

Theorem II: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Then $(\wedge E^*)_{i_F=0,\theta_F=0}$ is n.c.z. in B_F if and only if there is a linear isomorphism of graded vector spaces

$$f\colon H(B_E) \otimes H(E/F) \xrightarrow{\cong} H(B_F)$$

such that $f(\alpha \otimes \beta) = e^\#(\alpha) \cdot f(1 \otimes \beta)$ and the diagram

$$\begin{tikzcd}
 & H(B_E) \otimes H(E/F) \arrow[dd, "f"', "\cong"] \arrow[dr] & \\
H(B_E) \arrow[ur] \arrow[dr, "e^*"'] & & H(E/F) \\
 & H(B_F) \arrow[ur, "p_R"'] &
\end{tikzcd}$$

commutes.

Proof: Apply Theorem VIII, sec. 3.21.

Q.E.D.

Theorem III: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Then:

(1) If $H(B_E)$ has finite type, then so does $H(B_F)$ and the corresponding Poincaré series satisfy

$$f_{H(B_F)} \le f_{H(B_E)} f_{H(E/F)}.$$

Equality holds if and only if $(\wedge E^*)_{i_F=0,\theta_F=0}$ is n.c.z. in B_F.

(2) If $H(B_E)$ has finite dimension then so does $H(B_F)$ and

$$\dim H(B_F) \le \dim H(B_E) \dim H(E/F).$$

Equality holds if and only if $(\wedge E^*)_{i_F=0,\theta_F=0}$ is n.c.z. in B_F.

(3) If $H(B_E)$ has finite dimension, then

$$\chi_{H(B_F)} = \chi_{H(B_E)}\chi_{H(E/F)}.$$

In particular, the Euler–Poincaré characteristic of $H(B_F)$ is zero unless (E, F) is an equal rank pair.

Proof: Apply Corollaries IV, V, and VI to Theorem VIII, sec. 3.21, together with Theorem XI, sec. 10.22.

Q.E.D.

12.23. Algebra isomorphisms. Proposition VII: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation, and assume that $(\chi_E^\#)^+ = 0$. Then there are c-equivalences

$$(B_E \otimes (\wedge E^*)_{i_F=0,\theta_F=0},\ \delta_R \otimes \iota + \omega_R \otimes \delta_E) \underset{c}{\simeq} (B_F,\ \delta_R)$$

and

$$(B_E \otimes \wedge P_E,\ \delta_R \otimes \iota) \underset{c}{\simeq} (R_{\theta_E=0},\ \delta_R).$$

Moreover, these can be chosen so the induced isomorphisms of cohomology make the diagrams

$$\begin{tikzcd}
 & H(B_E) \otimes H(E/F) \arrow[dr] \arrow[dd, "\cong"] & \\
H(B_E) \arrow[ur] \arrow[dr, "e^*"'] & & H(E/F) \\
 & H(B_F) \arrow[ur, "p_R"'] &
\end{tikzcd} \tag{12.19}$$

and

$$\begin{tikzcd}
 & H(B_E) \otimes H(E/F) \arrow[r, "\iota \otimes k^*"] \arrow[dd, "\cong"] & H(B_E) \otimes \wedge P_E \arrow[dr] \arrow[dd, "\cong"'] & \\
(\vee F^*)_{\theta=0} \arrow[ur, "1 \otimes \chi^*"] \arrow[dr, "\chi_F^*"'] & & & \wedge P_E \\
 & H(B_F) \arrow[r, "e_F^*"'] & H(R_{\theta=0}) \arrow[ur, "\varrho_R"'] &
\end{tikzcd} \tag{12.20}$$

commute.

Proof: Since $(\chi_E^\#)^+ = 0$, the (P_E, δ)-algebras $(B_E, \delta_R; 0)$ and $(B_E, \delta_R; (\chi_E)_\vee \circ \tau)$ are equivalent (cf. sec. 3.27). Hence (cf. the corollary to Proposition XI, sec. 3.29) they are c-equivalent.

Now it follows from the remarks at the end of sec. 3.28 that there are c-equivalences

$$(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla)$$
$$\underset{c}{\sim} (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \delta_R \otimes \iota \otimes \iota - \nabla_\sigma)$$

and

$$(B_E \otimes \wedge P_E, \nabla_B) \underset{c}{\sim} (B_E \otimes \wedge P_E, \delta_R \otimes \iota)$$

such that the induced isomorphisms of cohomology define an isomorphism between the cohomology diagrams.

The proposition follows now from Theorem I, sec. 12.10.

Q.E.D.

Theorem IV: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Then the following conditions are equivalent:

(1) There is a homomorphism $\lambda: (\vee F^*)_{\theta=0} \to H(B_E) \otimes \operatorname{Im} \chi^\#$ of graded algebras, which makes the diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "\chi_E^\#"] \arrow[d, "j^\vee_{\theta=0}"'] & H(B_E) \arrow[d] \\
(\vee F^*)_{\theta=0} \arrow[r, "\lambda"] \arrow[dr, "\chi^\#"'] & H(B_E) \otimes \operatorname{Im} \chi^\# \arrow[d] \\
& \operatorname{Im} \chi^\#
\end{tikzcd}$$

commute.

(2) There is a c-equivalence

$$(B_E \otimes (\wedge E^*)_{i_F=0, \theta_F=0}, \delta_R \otimes \iota + \omega_R \otimes \delta_E) \underset{c}{\sim} (B_F, \delta_R)$$

such that the induced isomorphism of cohomology makes the diagram (12.19) commute.

(3) There is an isomorphism of graded algebras

$$H(B_E) \otimes H(E/F) \cong H(B_F),$$

which makes the diagram (12.19) commute.

Proof: Apply Theorem IX, sec. 3.23, together with Theorem I, (2), sec. 12.10.

Q.E.D.

Corollary I: If $(\chi_E^\#)^+ = 0$, then the conditions of the theorem hold.

Corollary II: If (B_E, δ_R) and $((\wedge E^*)_{i_F=0,\theta_F=0}, \delta_E)$ are c-split, and if the conditions of the theorem hold, then (B_F, δ_R) is c-split.

Theorem V: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation. Assume F is abelian and E is semisimple. Then the following conditions are equivalent:

(1) $(\chi_E^\#)^+ = 0$.

(2) There is an isomorphism $H(B_E) \otimes H(E/F) \xrightarrow{\cong} H(B_F)$ of graded algebras such that the diagram (12.19) commutes.

Proof: Since F is abelian, $(\vee F^*)_{\theta=0} = \vee F^*$ and so $(\vee F^*)_{\theta=0}$ is generated by elements of degree 2. Since E is semisimple, $P_E^k = 0$ for $k < 3$. Hence $(\vee E^*)_{\theta=0}^k = 0$ for $k < 4$, and so

$$j_{\theta=0}^\vee(\vee E^*)_{\theta=0} \subset (\vee^+ F^*)_{\theta=0} \cdot (\vee^+ F^*)_{\theta=0}.$$

This shows that $((\vee F^*)_{\theta=0}; \sigma)$ is an essential symmetric P_E-algebra. Now apply the corollary to Proposition VIII, sec. 3.25.

Q.E.D.

Theorem VI: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation which satisfies the conditions of Theorem IV. Let τ be a transgression in $W(E)_{\theta=0}$ and define a subspace P_1 of P_E by

$$P_1 = \{\Phi \in P_E \mid j_{\theta=0}^\vee(\tau\Phi) \subset (\vee^+ F^*)_{\theta=0} \cdot (\vee^+ F^*)_{\theta=0}\}.$$

Then P_1 is contained in the Samelson subspace for the operation $(E, i, \theta, R, \delta_R)$.

Proof: Apply Theorem X, sec. 3.26.

Q.E.D.

12.24. Special Cartan pairs. Let (E, F) be a Cartan pair with Samelson subspace $\hat{P} \subset P_E$. Let τ be a transgression in $W(E)_{\theta=0}$ and consider the linear map

$$\sigma = j^{\vee}_{\theta=0} \circ \tau : P_E \to (\vee F^*)_{\theta=0},$$

(cf. sec. 6.13 and sec. 10.8).

The pair (E, F) will be called a *special Cartan pair* if

$$\sigma(\hat{P}) \subset (\operatorname{Im} j^{\vee}_{\theta=0})^+ \cdot (\operatorname{Im} j^{\vee}_{\theta=0})^+. \tag{12.21}$$

Observe that if τ' is a second transgression, then $\varrho_E \circ (\tau - \tau') = 0$ and so

$$(\tau - \tau')(P_E) \subset (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0}$$

(cf. Lemma VII, sec. 6.13, and Theorem II, sec. 6.14). This shows that the condition (12.21) is independent of the choice of τ.

Clearly, an equal rank pair is a special Cartan pair. Moreover, Theorem X, (4) and (5), sec. 10.19, show that if F is n.c.z. in E, then (E, F) is a special Cartan pair. Finally, Proposition VII, sec. 10.26, implies that a symmetric pair is a special Cartan pair.

Lemma VI: Let (E, F) be a special Cartan pair. Then there is a transgression τ in $W(E)_{\theta=0}$ such that

$$(j^{\vee}_{\theta=0} \circ \tau)(\hat{P}) = 0.$$

Proof: Let τ_1 be any transgression in $W(E)_{\theta=0}$. Then it follows from relation (12.21) that there is a linear map

$$\alpha : \hat{P} \to (\vee^+ E^*)_{\theta=0} \cdot (\vee^+ E^*)_{\theta=0},$$

homogeneous of degree 1, such that

$$(j^{\vee}_{\theta=0} \circ \alpha)(\Phi) = (j^{\vee}_{\theta=0} \circ \tau_1)(\Phi), \qquad \Phi \in \hat{P}.$$

Write $P_E = \tilde{P} \oplus \hat{P}$ and define $\tau : P_E \to (\vee E^*)_{\theta=0}$ by

$$\tau(\Phi) = \begin{cases} \tau_1(\Phi), & \Phi \in \tilde{P}, \\ \tau_1(\Phi) - \alpha(\Phi), & \Phi \in \hat{P}. \end{cases}$$

Then Theorem II, sec. 6.14, yields $\varrho_E \circ \tau = \iota$; thus, by Lemma VII, sec. 6.13, τ is a transgression. Clearly,

$$(\check{j}_{\theta=0} \circ \tau)(\hat{P}) = 0.$$

Q.E.D.

A transgression satisfying the condition of Lemma VI will be called *adapted* to the special Cartan pair (E, F).

Next, consider the Koszul complex, $((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma)$, where $\sigma = \check{j}_{\theta=0} \circ \tau$ and τ is adapted. Let $\tilde{P}$ be a Samelson complement and let $\tilde{\sigma}$ denote the restriction of σ to $\tilde{P}$. Then, since $\sigma(\hat{P}) = 0$,

$$((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma) = ((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}, -\nabla_{\tilde{\sigma}}) \otimes (\wedge \hat{P}, 0).$$

Moreover, because (E, F) is a Cartan pair, we have

$$(\vee F^*)_{\theta=0} = \vee P_F \qquad \text{and} \qquad \dim \tilde{P} = \dim P_F.$$

Thus Theorem VII, sec. 2.17 yields

$$\begin{aligned} H((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}) &= H_0((\vee F^*)_{\theta=0} \otimes \wedge \tilde{P}) \\ &= (\vee F^*)_{\theta=0}/(\vee F^*)_{\theta=0} \circ P_E \cong \operatorname{Im} l^\#, \end{aligned}$$

where

$$l^\#\colon (\vee F^*)_{\theta=0} \to H((\vee F^*)_{\theta=0} \otimes \wedge P_E)$$

is the homomorphism induced by the inclusion map. It follows that

$$H((\vee F^*)_{\theta=0} \otimes \wedge P_E) = \operatorname{Im} l^\# \otimes \wedge \hat{P}.$$

Finally, consider the isomorphism

$$\varphi_F^\#\colon H((\vee F^*)_{\theta=0} \otimes \wedge P_E) \xrightarrow{\cong} H(E/F)$$

(cf. sec. 12.18). It determines (via the equation above) the isomorphism

$$\psi_F\colon \wedge \hat{P} \otimes \operatorname{Im} \chi^\# \xrightarrow{\cong} H(E/F)$$

given by

$$\psi_F(\Phi \otimes \chi^\# \Psi) = \varphi_F^\#(l^\# \Psi \otimes \Phi), \qquad \Phi \in \wedge \hat{P}, \quad \Psi \in (\vee F^*)_{\theta=0}.$$

Evidently, the diagram

$$\begin{tikzcd}
 & \wedge\hat{P} \otimes \operatorname{Im} \chi^{\#} \arrow[dd, "\psi_F"', "\cong"] \arrow[dr] & \\
\operatorname{Im} \chi^{\#} \arrow[ur] \arrow[dr] & & \wedge\hat{P} \\
 & H(E/F) \arrow[ur, "k^{\#}"'] &
\end{tikzcd}$$

commutes (cf. diagram (12.9) in sec. 12.17).

12.25. Operation of a special Cartan pair. Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation of a special Cartan pair. Assume that the (P_E, δ)-algebra $(B_E, \delta_R; \tau_R)$ and the P_E-algebra $((\vee F^*)_{\theta=0}; \sigma)$ are defined via an adapted transgression τ (cf. sec. 12.9). Let $\hat{P}$ be the Samelson subspace for (E, F) and let $\tilde{P}$ be a Samelson complement.

Observe that ∇_B restricts to a differential operator $\hat{\nabla}_B$ in $B_E \otimes \wedge\hat{P}$. Moreover, the inclusion

$$i\colon (B_E \otimes \wedge\hat{P}, \hat{\nabla}_B) \to (B_E \otimes \wedge P_E, \nabla_B)$$

is a homomorphism of graded differential algebras.

On the other hand, since $\sigma(\hat{P}) = 0$, the inclusion

$$i_F\colon (B_E \otimes \wedge\hat{P}, \hat{\nabla}_B) \to (B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla)$$

is also a homomorphism of graded differential algebras. Hence so is

$$\psi \circ i_F\colon (B_E \otimes \wedge\hat{P}, \hat{\nabla}_B) \to (B_F, \delta_R),$$

where ψ is the homomorphism in Theorem I, sec. 12.10.

Now let

$$\alpha\colon \operatorname{Im} \chi^{\#} \to (\vee F^*)_{\theta=0}$$

be a linear map, homogeneous of degree zero, such that $\chi^{\#} \circ \alpha = \iota$. Then $p_R \circ \chi_F^{\#} \circ \alpha = \iota$ as follows from the cohomology diagram in sec. 12.7.

Theorem VII: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation of a special Cartan pair. Then a linear isomorphism of graded spaces

$$f\colon H(B_E \otimes \wedge\hat{P}) \otimes \operatorname{Im} \chi^{\#} \xrightarrow{\cong} H(B_F)$$

is defined by

$$f(\zeta \otimes \eta) = (\psi \circ i_F)^\#(\zeta) \cdot (\chi_F^\# \circ \alpha)(\eta), \qquad \zeta \in H(B_E \otimes \wedge \hat{P}), \quad \eta \in \operatorname{Im} \chi^\#.$$

Moreover, this isomorphism makes the diagrams

$$\begin{tikzcd}
 & H(B_E \otimes \wedge \hat{P}) \otimes \operatorname{Im} \chi^\# \arrow[r, "\hat{\varrho}_B^\# \otimes \iota"] \arrow[d, "f"', "\cong"] & \wedge \hat{P} \otimes \operatorname{Im} \chi^\# \arrow[d, "\cong"', "\psi_F"] \\
H(B_E) \arrow[ur] \arrow[r, "e^*"'] & H(B_F) \arrow[r, "p_R"'] & H(E/F)
\end{tikzcd} \tag{12.22}$$

and

$$\begin{tikzcd}
H(B_E \otimes \wedge \hat{P}) \otimes \operatorname{Im} \chi^\# \arrow[r] \arrow[d, "f"', "\cong"] & H(B_E \otimes \wedge \hat{P}) \arrow[r, "i^*"] & H(B_E \otimes \wedge P_E) \arrow[d, "\cong"', "\vartheta_R^\#"] \\
H(B_E) \arrow[rr, "e_F^\#"'] & & H(R_{\theta=0})
\end{tikzcd} \tag{12.23}$$

commute (cf. sec. 12.24 for ψ_F and sec. 12.14 for $\vartheta_R^\#$).

Proof: The commutativity of the diagrams (12.22) and (12.23) is an immediate consequence of the definition of f and Theorem I, sec. 12.10. It remains to be shown that f is a linear isomorphism.

Identify $\operatorname{Im} \chi^\#$ with $\operatorname{Im} l^\#$ via $\varphi_F^\#$ (cf. sec. 12.24 and sec. 12.17). Then α becomes a linear map

$$\alpha \colon \operatorname{Im} l^\# \to (\vee F^*)_{\theta=0}$$

satisfying $l^\# \circ \alpha = \iota$.

Now define a linear map

$$g \colon H(B_E \otimes \wedge \hat{P}) \otimes \operatorname{Im} l^\# \to H(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E)$$

by

$$g(\zeta \otimes \eta) = i_F^\#(\zeta) \cdot (m_F^\# \circ \alpha)(\eta), \qquad \zeta \in H(B_E \otimes \wedge \hat{P}), \quad \eta \in \operatorname{Im} l^\#,$$

(cf. sec. 12.10 for $m_F^\#$). In view of Theorem I, sec. 12.10, we have $f = \psi^\# \circ g$ and so it is sufficient to show that g is a linear isomorphism.

Consider the $(\hat{P}, \delta)$-algebra $(B_E \otimes \wedge \hat{P}, \hat{V}_B; \tilde{\tau}_R)$ and the $\hat{P}$-algebra $((\vee F^*)_{\theta=0}; \tilde{\sigma})$ given by

$$\tilde{\tau}_R(\Phi) = \tau_R(\Phi) \otimes 1 \qquad \text{and} \qquad \tilde{\sigma}(\Phi) = \sigma(\Phi), \qquad \Phi \in \hat{P}.$$

The Koszul complex of their tensor difference

$$((B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0} \otimes \wedge \hat{P}, \tilde{V})$$

coincides with the Koszul complex $(B_E \otimes (\vee F^*)_{\theta=0} \otimes \wedge \hat{P} \otimes \wedge \hat{P}, V)$ under the obvious identification (since $\sigma(\hat{P}) = 0$).

With this identification, i_F becomes the base inclusion for the tensor difference. Since

$$(\vee F^*)_{\theta=0} = \vee P_F \qquad \text{and} \qquad \dim P_F = \dim \hat{P}$$

it follows (as in sec. 12.24) that

$$H((\vee F^*)_{\theta=0} \otimes \wedge \hat{P}) = \operatorname{Im} \tilde{l}^{\#} = \operatorname{Im} l^{\#}.$$

Now Corollary II to Theorem VIII, sec. 3.21, shows that g is an isomorphism.

Q.E.D.

Again let $(E, F, i, \theta, R, \delta_R)$ be a regular operation of a special Cartan pair. Let I be the ideal in $H(B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0}$ generated by the elements of the form

$$\tilde{\tau}_R^{\#}\Phi \otimes 1 - 1 \otimes \tilde{\sigma}\Phi, \qquad \Phi \in \hat{P},$$

and let

$$\pi\colon H(B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0} \to (H(B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0})/I$$

be the corresponding projection.

Consider the homomorphism

$$\gamma\colon H(B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0} \to H(B_F)$$

given by

$$\gamma(\zeta \otimes \Psi) = (\psi \circ i_F)^{\#}(\zeta) \cdot \chi_F^{\#}(\Psi), \qquad \zeta \in H(B_E \otimes \wedge \hat{P}), \;\; \Psi \in (\vee F^*)_{\theta=0}.$$

Proposition VIII: With the hypotheses above, γ factors over π to yield an isomorphism of graded algebras

$$(H(B_E \otimes \wedge \hat{P}) \otimes (\vee F^*)_{\theta=0})/I \xrightarrow{\cong} H(B_F).$$

Proof: This follows from Corollary III to Theorem VIII, sec. 3.21, in exactly the same way as Theorem VII followed from Corollary II. Q.E.D.

12.26. Examples. 1. *Equal rank pairs*: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation of an equal rank pair. Then according to Theorem XI sec. 10.22, $\chi^\#$ is surjective. But

$$p_R \circ \chi_F^\# = \chi^\#.$$

Thus p_R is surjective, and so $(\wedge E^*)_{i_F=0,\theta_F=0}$ is n.c.z. in B_F. Moreover $\hat{P} = 0$, and Theorem VII, sec. 12.25, provides a linear isomorphism

$$H(B_E) \otimes H(E/F) \xrightarrow{\cong} H(B_F)$$

(cf. also Theorem II, sec. 12.22).

Note that diagram (12.22) reduces to the diagram of Theorem II in this case. On the other hand, diagram (12.23) yields the commutative diagram

$$\begin{array}{ccc} H(B_E) \otimes H(E/F) & \longrightarrow & H(B_E) \\ \cong \big\downarrow & & \big\downarrow e_E^\# \\ H(B_F) & \xrightarrow[e_F^\#]{} & H(R_{\theta=0}). \end{array}$$

This shows that

$$\operatorname{Im} e_F^\# = \operatorname{Im} e_E^\#. \tag{12.24}$$

Finally, Proposition VIII, sec. 12.25, yields an algebra isomorphism

$$(H(B_E) \otimes (\vee F^*)_{\theta=0})/I \xrightarrow{\cong} H(B_F),$$

where I denotes the ideal which is generated by the elements of the form $\tau_R^\# \Phi \otimes 1 - 1 \otimes \sigma\Phi$, $\Phi \in P_E$.

2. *N.c.z. pairs*: Let $(E, F, i, \theta, R, \delta_R)$ be a regular operation of a reductive pair such that F is n.c.z. in E. Then, by Theorem IX, sec. 10.18, $(\chi^\#)^+ = 0$.

Thus the isomorphism f of Theorem VII is given by

$$f = (\psi \circ i_F)^\#: H(B_E \otimes \wedge \hat{P}) \xrightarrow{\cong} H(B_F),$$

and so in this case f is an isomorphism of graded algebras induced from an isomorphism of graded differential algebras.

In particular,

$$(B_E \otimes \wedge\hat{P}, \hat{V}_B) \underset{c}{\sim} (B_F, \delta_R).$$

Moreover, diagrams (12.22) and (12.23) reduce to the commutative diagrams

$$\begin{tikzcd}
 & H(B_E \otimes \wedge\hat{P}) \arrow[r, "\hat{\varrho}_B^*"] \arrow[d, "f"', "\cong"] & \wedge\hat{P} \arrow[d, "\cong"'] \\
H(B_E) \arrow[ur] \arrow[dr, "e^*"'] & & \\
 & H(B_F) \arrow[r, "p_R"'] & H(E/F)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H(B_E \otimes \wedge\hat{P}) \arrow[r, "i^*"] \arrow[d, "f"', "\cong"] & H(B_E \otimes \wedge P_E) \arrow[d, "\cong"', "\vartheta_R^*"] \\
H(B_F) \arrow[r, "e_F^*"'] & H(R_{\theta=0}).
\end{tikzcd}$$

12.27. Lie algebra triples. A *reductive Lie algebra triple* (L, E, F) is a sequence of Lie algebras $L \supset E \supset F$ such that

(1) L is reductive.
(2) E is reductive in L.
(3) F is reductive in E.

Note that then F is reductive in L (cf. Proposition III, sec. 4.7).

Moreover, the commutative diagram

$$\begin{tikzcd}
H((\wedge L^*)_{\theta_E=0}) \arrow[dd] \arrow[dr, "\cong"] & \\
 & H^*(L) \\
H((\wedge L^*)_{\theta_F=0}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

implies that the vertical arrow is an isomorphism, and so $(E, F, i_E, \theta_E, \wedge L^*, \delta_L)$ is a regular operation of the pair (E, F).

With the terminology of sec. 12.1 we have

$$B_E = (\wedge L^*)_{i_E=0, \theta_E=0} \qquad \text{and} \qquad B_F = (\wedge L^*)_{i_F=0, \theta_F=0}.$$

Thus, Theorem I, sec. 12.10, expresses $H(L/F)$ in terms of $H(L/E)$ and other invariants.

Now assume that (E, F) is a special Cartan pair, and let τ be an adapted transgression in $W(E)_{\theta=0}$. Then Theorem VII, sec. 12.25, yields a linear isomorphism

$$f\colon H((\wedge L^*)_{i_E=0, \theta_E=0} \otimes \wedge \hat{P}) \otimes \operatorname{Im} \chi^\# \xrightarrow{\cong} H(L/F).$$

Theorem VIII: Let (L, E, F) be a reductive triple and assume that (E, F) is an equal rank pair. Then

$$\operatorname{def}(L, E) = \operatorname{def}(L, F).$$

In particular, (L, E) is a Cartan pair if and only if (L, F) is.

Proof: In view of formula (12.24) in sec. 12.26, the inclusions

$$k_E\colon (\wedge L^*)_{i_E=0, \theta_E=0} \to \wedge L^* \qquad \text{and} \qquad k_F\colon (\wedge L^*)_{i_F=0, \theta_F=0} \to \wedge L^*$$

satisfy

$$\operatorname{Im} k_E^\# = \operatorname{Im} k_F^\#.$$

It follows that the Samelson subspaces $\hat{P}_E$ and $\hat{P}_F$ for the pairs (L, E) and (L, F) coincide.

Since by hypothesis $\dim P_E = \dim P_F$, it follows that

$$\operatorname{def}(L, E) = \dim P_L - \dim \hat{P}_E - \dim P_E = \operatorname{def}(L, F).$$

Q.E.D.

Corollary: Let H be a Cartan subalgebra of E. Then (L, E) is a Cartan pair if and only if (L, H) is.

Proof: Apply Theorem XII, sec. 10.23.

Q.E.D.

§5. Bundles with fibre a homogeneous space

12.28. The cohomology diagram. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with compact connected structure group G and assume that B is connected. Let K be a closed connected subgroup of G. Restricting the principal action of G on P to K yields an action of K.

Let $\pi_1 : P \to P/K$ be the canonical projection. Then $\mathscr{P}_1 = (P_1, \pi_1, P/K, K)$ is again a principal bundle (cf. sec. 5.7, volume II). Moreover, π factors over π_1 to yield a smooth map $\pi_\xi : P/K \to B$, and $\xi = (P/K, \pi_\xi, B, G/K)$ is a fibre bundle. Finally, we have the principal bundle $\mathscr{P}_K = (G, \pi_K, G/K, K)$.

These bundles are combined in the commutative diagram

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d, "j_K"'] & K \arrow[d, "\hat{A}_z"] & \\
G \arrow[r, "A_z"] \arrow[d, "\pi"'] & P \arrow[r, "\pi"] \arrow[d, "\pi_1"] & B \arrow[d, equal] \\
G/K \arrow[r, "\bar{A}_z"'] & P/K \arrow[r, "\pi_\xi"'] & B,
\end{tikzcd} \tag{12.25}$$

where, for $z \in P$,

(1) A_z is the inclusion map $a \mapsto z \cdot a$, $a \in G$.
(2) $\hat{A}_z$ is the restriction of A_z to K.
(3) $\bar{A}_z$ is the induced map.

Note that A_z, $\hat{A}_z$, and $\bar{A}_z$ are the fibre inclusions for the bundles $\mathscr{P}$, $\mathscr{P}_1$, and ξ. Observe also that A_z is a homomorphism of K-principal bundles.

The homomorphism $\bar{A}_z^\# : H(P/K) \to H(G/K)$ is independent of the choice of z in P. In fact, if $z_0 \in P$ and $z_1 \in P$, choose a smooth path z_t connecting z_0 and z_1. Then the maps $\bar{A}_{z_t}$ define a homotopy from $\bar{A}_{z_0}$ to $\bar{A}_{z_1}$, and thus $\bar{A}_{z_0}^\# = \bar{A}_{z_1}^\#$. We denote this common homomorphism by

$$\varrho_\xi : H(P/K) \to H(G/K)$$

and call it the *fibre projection* for the bundle ξ.

The diagram

$$
\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "h_{\mathscr{P}}"] \arrow[d, "\overset{\vee}{j}_{\theta=0}"'] & H(B) \arrow[d, "\pi_\xi^\#"] \arrow[dr, "\pi^\#"] & & \\
(\vee F^*)_{\theta=0} \arrow[r, "h_{\mathscr{P}_1}"] \arrow[dr, "h_{\mathscr{P}_K}"'] & H(P/K) \arrow[r, "(\pi_1)^\#"] \arrow[d, "\varrho_\xi"] & H(P) \arrow[d, "\varrho_P"] \arrow[dr, "\hat{\varrho}_P"] & \\
& H(G/K) \arrow[r, "\pi_K^\#"'] & H(G) \arrow[r, "j_K^\#"'] & H(K)
\end{tikzcd}
\tag{12.26}
$$

is called the *cohomology diagram* corresponding to the diagram (12.25). Here $\varrho_P = A_z^\#$ and $\hat{\varrho}_P = \hat{A}_z^\#$ are the fibre projections for the bundles $\mathscr{P}$ and $\mathscr{P}_1$.

The cohomology diagram commutes. In fact, it was shown in sec. 6.27, volume II, that the upper square commutes. Since A_z is a homomorphism of principal bundles, and since the Weil homomorphism is natural it follows that $\varrho_\xi \circ h_{\mathscr{P}_1} = h_{\mathscr{P}_K}$. Finally, the commutativity of the rest of the diagram follows directly from diagram (12.25).

12.29. Induced operation of a pair. Let E and F denote the Lie algebras of G and K. Then we have the associated operations $(E, i, \theta, A(P), \delta)$ and $(F, i, \theta, A(P), \delta)$ (cf. sec. 8.22). Since the principal action of K on P is the restriction to K of the principal action of G, it follows that the second operation is the restriction of the first operation.

In view of Theorem I, sec. 4.3, volume II, the inclusion maps

$$A(P)_{\theta_E=0} \to A(P) \qquad \text{and} \qquad A(P)_{\theta_F=0} \to A(P)$$

induce isomorphisms of cohomology.

Thus the inclusion map

$$A(P)_{\theta_E=0} \to A(P)_{\theta_F=0}$$

induces an isomorphism of cohomology, and so $(E, F, i, \theta, A(P), \delta)$ is an operation of the pair (E, F).

Now since G is compact, the pair (E, F) is reductive. Thus the "algebraic" fibre projection

$$p_{A(P)}\colon H(A(P)_{i_F=0,\theta_F=0}) \to H(E/F)$$

is defined (cf. sec. 12.4). On the other hand, π_1^* can be regarded as an isomorphism

$$\pi_1^*: A(P/K) \xrightarrow{\cong} A(P)_{i_F=0,\theta_F=0},$$

while in sec. 11.1 we defined an isomorphism

$$\varepsilon_{G/K}^{\#}: H(E/F) \xrightarrow{\cong} H(G/K).$$

Proposition IX: With the hypotheses and notation above, the diagram

$$\begin{tikzcd} H(A(P)_{i_F=0,\theta_F=0}) \arrow[r, "p_{A(P)}"] \arrow[d, "((\pi_1^*)^{-1})^*"', "\cong"] & H(E/F) \arrow[d, "\cong"', "\varepsilon_{G/K}^{\#}"] \\ H(P/K) \arrow[r, "\varrho_\xi"'] & H(G/K) \end{tikzcd}$$

commutes.

Proof: Consider the operation $(E, F, i, \theta, A(G), \delta_G)$, where $i(h) = i(X_h)$ and $\theta(h) = \theta(X_h)$ (X_h is the left invariant vector field generated by h) (cf. sec. 7.21). Since the map $A_z: G \to P$ is G-equivariant, $A_z^*: A(G) \leftarrow A(P)$ is a homomorphism of operations.

On the other hand, in sec. 7.21 we defined a homomorphism of operations

$$\varepsilon_G: (E, F, i, \theta, \wedge E^*, \delta_E) \to (E, F, i, \theta, A(G), \delta_G)$$

inducing an isomorphism in cohomology.

Recall from the example of sec. 12.5 that the fibre projection

$$p_{\wedge E^*}: H(E/F) \to H(E/F)$$

is just the identity map. Now the naturality of the algebraic fibre projection gives the commutative diagram

$$\begin{tikzcd} H(E/F) \arrow[r, "(\varepsilon_G)^{\#}_{i_F=0,\theta_F=0}"] \arrow[dr, "\iota"'] & H(A(G)_{i_F=0,\theta_F=0}) \arrow[d, "p_{A(G)}"] & H(A(P)_{i_F=0,\theta_F=0}) \arrow[l, "(A_z^*)^{\#}_{i_F=0,\theta_F=0}"'] \arrow[dl, "p_{A(P)}"] \\ & H(E/F) & \end{tikzcd}$$

connecting the various fibre projections.

It follows that

$$\begin{aligned}\varepsilon^{\#}_{G/K} \circ p_{A(P)} &= \varepsilon^{\#}_{G/K} \circ \left((\varepsilon_G)^{\#}_{i_F=0,\theta_F=0}\right)^{-1} \circ (A_z^*)^{\#}_{i_F=0,\theta_F=0} \\ &= \bar{A}_z^{\#} \circ ((\pi_1^*)^{-1})^{\#} = \varrho_\xi \circ ((\pi_1^*)^{-1})^{\#}.\end{aligned}$$

Q.E.D.

12.30. The cohomology of P/K. Theorem IX: Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with compact connected structure group G and connected base. Let K be a closed connected subgroup of G. Then there are c-equivalences

$$(A(P/K), \delta) \underset{c}{\sim} (A(B) \otimes (\vee F^*)_{\theta=0} \otimes \wedge P_E, \nabla),$$

$$(A(P), \delta) \underset{c}{\sim} (A(B) \otimes \wedge P_E, \nabla_B),$$

and

$$(A(G/K), \delta) \underset{c}{\sim} ((\vee F^*)_{\theta=0} \otimes \wedge P_E, -\nabla_\sigma).$$

The induced isomorphism of cohomology algebras determines an isomorphism from the cohomology diagram of sec. 12.10 to the cohomology diagram (12.26) (with B_E replaced by $A(B)$ and $H(B_E)$ replaced by $H(B)$).

Proof: In view of the canonical isomorphisms

$$A(P/K) \xrightarrow{\cong} A(P)_{i_F=0,\theta_F=0} \qquad \text{and} \qquad A(B) \xrightarrow{\cong} A(P)_{i_E=0,\theta_E=0},$$

the theorem follows from Theorem I, sec. 12.10, together with diagram 8.22, sec. 8.27, diagram 11.1, sec. 11.4, and Proposition IX, sec. 12.29.

Q.E.D.

12.31. N.c.z. fibres. The fibre of ξ, G/K, is called *noncohomologous to zero in* P/K if the map ϱ_ξ is surjective. In this case, Theorem IX, together with Theorem VIII, sec. 3.21, yield the commutative diagram

$$\begin{tikzcd} & H(B) \otimes H(G/K) \arrow[dr] \arrow[dd, "\cong"', "f"] & \\ H(B) \arrow[ur] \arrow[dr, "\pi_\xi^*"'] & & H(G/K) \\ & H(P/K) \arrow[ur, "\varrho_\xi"'] & \end{tikzcd} \tag{12.27}$$

where f is an isomorphism of graded vector spaces satisfying

$$f(\alpha \otimes \beta) = (\pi_\xi^\# \alpha) \cdot f(1 \otimes \beta), \qquad \alpha \in H(B), \quad \beta \in H(G/K).$$

In particular, if G and K have the same rank, then the map

$$h_{\mathscr{P}_K}: (\vee F^*)_{\theta=0} \to H(G/K)$$

is surjective (cf. sec. 11.7). The cohomology diagram (sec. 12.28) shows that in this case G/K is n.c.z. in P/K.

On the other hand, we can apply Theorem IX, sec. 3.23, to obtain

Theorem X: Let $\mathscr{P} = (P, \pi, B, G)$, K, be as in Theorem IX. Then the following conditions are equivalent:

(1) There is a homomorphism of graded algebras

$$\psi: (\vee F^*)_{\theta=0} \to H(B) \otimes \operatorname{Im} h_{\mathscr{P}_K}$$

which makes the diagram

$$\begin{tikzcd}
(\vee E^*)_{\theta=0} \arrow[r, "h_{\mathscr{P}}"] \arrow[d, "j^\vee_{\theta=0}"'] & H(B) \arrow[d, "\pi_\xi^\#"] \\
(\vee F^*)_{\theta=0} \arrow[r, "\psi"] \arrow[dr, "h_{\mathscr{P}_K}"'] & H(B) \otimes \operatorname{Im} h_{\mathscr{P}_K} \arrow[d] \\
& H(G/K)
\end{tikzcd}$$

commute.

(2) There is a c-equivalence

$$(A(B \times G/K), \delta) \underset{c}{\sim} A(P/K)$$

such that the induced isomorphism of cohomology makes the diagram (12.27) commute.

(3) There is an isomorphism of graded algebras

$$f: H(B) \otimes H(G/K) \xrightarrow{\cong} H(P/K)$$

which makes the diagram (12.27) commute.

Finally, as in Theorem V, sec. 12.23, we have

Theorem XI: Let $\mathscr{P} = (P, \pi, B, G)$, K be as in Theorem IX. Assume the Lie algebra of G is semisimple and that K is a torus in G. Then the following conditions are equivalent:

(1) $h_{\mathscr{P}}^{+} = 0$.

(2) The conditions of Theorem X hold.

Example: Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle as in Theorem XI, and assume $h_{\mathscr{P}}^{+} \neq 0$. Let T be a maximal torus in G. Then rank T $=$ rank G. Hence, as we have just seen, there is a linear isomorphism

$$f\colon H(B) \otimes H(G/T) \xrightarrow{\cong} H(P/T)$$

such that $f(\alpha \otimes \beta) = (\pi_\xi^\# \alpha) \cdot f(1 \otimes \beta)$, $\alpha \in H(B)$, $\beta \in H(G/T)$, and the diagram (12.27) commutes.

On the other hand, since $h_{\mathscr{P}}^{+} \neq 0$, Theorem XI, together with Theorem X, (3), shows that *f cannot be an algebra isomorphism.*

Appendix A

Characteristic Coefficients and the Pfaffian

In this chapter all vector spaces are defined over a field Γ of characteristic zero.

A.0. The algebra of homogeneous functions. Given a vector space F a function $f: F \to \Gamma$ is called *homogeneous of degree p* if

$$f(\lambda x) = \lambda^p f(x), \qquad x \in F, \quad \lambda \in \Gamma.$$

The functions homogeneous of degree p form a vector space, $\mathscr{H}^p(F)$. Multiplication of functions makes the direct sum,

$$\mathscr{H}(F) = \sum_{p=0}^{\infty} \mathscr{H}^p(F),$$

into a graded commutative algebra.

Consider the inclusion map $\alpha: F^* \to \mathscr{H}(F)$. Since $\mathscr{H}(F)$ is a commutative algebra, α extends to a homomorphism,

$$\alpha: \vee F^* \to \mathscr{H}(F),$$

of graded algebras. For simplicity, we usually denote $\alpha(\Psi)(x)$ by $\Psi(x)$.

On the other hand, a homomorphism of graded algebras $\beta: \otimes F^* \to \mathscr{H}(F)$ is given by

$$\beta(\Phi)(x) = \Phi(x, \ldots, x), \qquad \Phi \in \otimes^p F^*, \quad x \in F.$$

Let $\pi_S: \otimes F^* \to \vee F^*$ be the projection (cf. sec. 6.17, volume II); then

$$(\alpha \circ \pi_S)(x^*) = x^* = \beta(x^*) \qquad x^* \in F^*.$$

It follows that

$$\alpha \circ \pi_S = \beta.$$

This shows that

$$\Psi(x) = \frac{1}{p!} \Psi(x, \ldots, x), \qquad \Psi \in \vee^p F^*, \quad x \in F.$$

In particular, α is injective.

§1. Characteristic and trace coefficients

A.1. The characteristic algebra of a vector space. Let F be an n-dimensional vector space. Define bilinear maps,

$$\Box : L_{\wedge^p F} \times L_{\wedge^q F} \to L_{\wedge^{p+q} F},$$

by setting

$$(\Phi \,\Box\, \Psi)(x_1 \wedge \cdots \wedge x_{p+q})$$
$$= \frac{1}{p!q!} \sum_{\sigma \in S^{p+q}} \varepsilon_\sigma \Phi(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) \wedge \Psi(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}),$$
$$\Phi \in L_{\wedge^p F}, \quad \Psi \in L_{\wedge^q F}, \quad x_i \in F.$$

These bilinear maps make the space $\sum_{p=0}^n L_{\wedge^p F}$ into a graded algebra, $C(F)$. It is called the *characteristic algebra for F*.

On the other hand, make the direct sum $\Delta(F) = \sum_{p=0}^n (\wedge^p F^* \otimes \wedge^p F)$ into a commutative and associative algebra by setting

$$(u^* \otimes u) \cdot (v^* \otimes v) = (u^* \wedge v^*) \otimes (u \wedge v), \quad u^*, v^* \in \wedge F^*, \quad u, v \in \wedge F.$$

Then the canonical linear isomorphisms $\wedge^p F^* \otimes \wedge^p F \xrightarrow{\cong} L_{\wedge^p F}$ define an algebra isomorphism

$$\Delta(F) \xrightarrow{\cong} C(F).$$

In particular, it follows that $C(F)$ is commutative and associative. Henceforth we shall identify the algebras $\Delta(F)$ and $C(F)$ under the isomorphism above.

The pth power of an element $\Phi \in C(F)$ will be denoted by $\Phi^{\boxed{p}}$,

$$\Phi^{\boxed{p}} = \underbrace{\Phi \,\Box \cdots \Box\, \Phi}_{(p \text{ factors})}.$$

In particular,

$$\varphi^{\boxed{p}} = p!\wedge^p \varphi, \qquad \varphi \in L_F.$$

More particularly, if ι denotes the identity map of F and ι_p denotes the identity map of $\wedge^p F$, this formula becomes

$$\iota^{\boxed{p}} = p!\iota_p.$$

It follows that

$$\iota_p \square \iota_q = \frac{(p+q)!}{p!q!} \iota_{p+q}.$$

Next, recall the substitution operators $i(x)\colon \wedge F^* \to \wedge F^*$ and $i(x^*)\colon \wedge F \to \wedge F$ determined by vectors $x \in F$ and $x^* \in F^*$. They are the unique antiderivations that satisfy

$$i(x)y^* = \langle y^*, x\rangle \quad \text{and} \quad i(x^*)y = \langle x^*, y\rangle, \quad y^* \in F^*, \ y \in F.$$

An algebra homomorphism $i\colon \Delta(F) \to L_{\Delta(F)}$ is defined by

$$i(x^{*1} \wedge \cdots \wedge x^{*p} \otimes x_1 \wedge \cdots \wedge x_p) = i(x_p) \circ \cdots \circ i(x_1) \otimes i(x^{*p}) \circ \cdots \circ i(x^{*1}).$$

With the aid of the identification above we may regard i as a homomorphism

$$i\colon C(F) \to L_{C(F)}.$$

Finally, note that the spaces $L_{\wedge^p F}$ are self-dual with respect to the inner product given by

$$\langle \Phi, \Psi\rangle = \operatorname{tr}(\Phi \circ \Psi) = \langle \iota_p, \Phi \circ \Psi\rangle = i(\Phi)\Psi.$$

It satisfies

$$\langle u^* \otimes u, v^* \otimes v\rangle = \langle u^*, v\rangle\langle v^*, u\rangle, \qquad u^*, v^* \in \wedge^p F^*, \quad u, v \in \wedge^p F.$$

Moreover $i(\Phi)$ is dual to multiplication by Φ, $\Phi \in L_{\wedge^p F}$.

A.2. Characteristic coefficients. The pth *characteristic coefficient* for an n-dimensional vector space F is the element $C_p^F \in \vee^p L_F^*$ given by $C_0^F = 1$ and

$$C_p^F(\varphi_1, \ldots, \varphi_p) = \operatorname{tr}(\varphi_1 \square \cdots \square \varphi_p) = \langle \iota_p, \varphi_1 \square \cdots \square \varphi_p\rangle, \qquad p \geq 1, \quad \varphi_i \in L_F.$$

Note that $C_n^F = 0$ if $p > n$. C_p^F will be denoted by Det^F.

The homogeneous functions, C_p^F, corresponding to C_p^F are given by

$$C_p^F(\varphi) = \operatorname{tr} \wedge^p \varphi, \qquad \varphi \in L_F$$

(cf. sec. A.0). We shall show that

$$\det(\varphi + \lambda\iota) = \sum_{p=0}^{n} C_p^F(\varphi)\lambda^{n-p}, \qquad \lambda \in \Gamma, \quad \varphi \in L_F. \tag{A.1}$$

In particular,

$$\det \varphi = \frac{1}{n!} \operatorname{Det}^F(\varphi, \ldots, \varphi).$$

To prove formula (A.1) we argue as follows. Let $e_1, \ldots, e_n$ be a basis of F. Then

$$\begin{aligned}
&\det(\varphi + \lambda\iota)\, e_1 \wedge \cdots \wedge e_n \\
&\quad = (\varphi + \lambda\iota)e_1 \wedge \cdots \wedge (\varphi + \lambda\iota)e_n \\
&\quad = \sum_{p=0}^{n} \lambda^{n-p} \sum_{i_1 < \cdots < i_p} e_1 \wedge \cdots \wedge \varphi e_{i_1} \wedge \cdots \wedge \varphi e_{i_p} \wedge \cdots \wedge e_n .
\end{aligned}$$

The elements $e_{i_1} \wedge \cdots \wedge e_{i_p}$ $(i_1 < \cdots < i_p)$ are a basis for $\wedge^p F$. Moreover, writing

$$\wedge^p \varphi(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j_1 < \cdots < j_p} \lambda_{i_1 \cdots i_p}^{j_1 \cdots j_p} e_{j_1} \wedge \cdots \wedge e_{j_p},$$

we see that $e_1 \wedge \cdots \wedge \varphi e_{i_1} \wedge \cdots \wedge \varphi e_{i_p} \wedge \cdots \wedge e_n = \lambda_{i_1 \cdots i_p}^{i_1 \cdots i_p} e_1 \wedge \cdots \wedge e_n$.

It follows that

$$\begin{aligned}
\det(\varphi + \lambda\iota)\, e_1 \wedge \cdots \wedge e_n &= \sum_{p=0}^{n} \lambda^{n-p} \sum_{i_1 < \cdots < i_p} \lambda_{i_1 \cdots i_p}^{i_1 \cdots i_p} e_1 \wedge \cdots \wedge e_n \\
&= \left(\sum_{p=0}^{n} \operatorname{tr} \wedge^p \varphi \cdot \lambda^{n-p} \right) e_1 \wedge \cdots \wedge e_n .
\end{aligned}$$

Relation (A.1) is now established.

Relation (A.1) implies that $C_p^F \in (\vee^p L_F^*)_I$; i.e.,

$$C_p^F(\sigma \circ \varphi_1 \circ \sigma^{-1}, \ldots, \sigma \circ \varphi_p \circ \sigma^{-1}) = C_p^F(\varphi_1, \ldots, \varphi_p),$$

or, equivalently

$$C_p^F(\sigma \circ \varphi_1, \ldots, \sigma \circ \varphi_p) = C_p^F(\varphi_1 \circ \sigma, \ldots, \varphi_p \circ \sigma),$$
$$\varphi_i \in L_F, \quad \sigma \in GL(F).$$

Setting $\sigma = \psi + \lambda\iota$ ($\psi \in L_F$, $-\lambda$ not an eigenvalue of ψ) and comparing the coefficients of λ^{p-1} we obtain

$$\sum_{i=1}^{p} C_p^F(\varphi_1, \ldots, [\psi, \varphi_i], \ldots, \varphi_p) = 0, \qquad \varphi_i, \psi \in L_F .$$

The nonhomogeneous element $C^F \in (\vee L_F^*)_I$, given by

$$C^F = \sum_{p=0}^{n} C_p^F,$$

is called the *characteristic element for F*.

Next, let H be a second finite-dimensional vector space. The inclusion map $j: L_F \oplus L_H \to L_{F\oplus H}$, given by $j(\varphi \oplus \psi) = \varphi \oplus \psi$, induces a homomorphism

$$j^\vee: \vee L_F^* \otimes \vee L_H^* \leftarrow \vee L_{F\oplus H}^*.$$

(Recall that multiplication induces a canonical isomorphism

$$\vee L_F^* \otimes \vee L_H^* \xrightarrow{\cong} \vee(L_F^* \oplus L_H^*).\ )$$

Proposition I: The characteristic elements of F, H, and $F \oplus H$ are connected by the relation

$$j^\vee(C^{F\oplus H}) = C^F \otimes C^H.$$

Proof: The equation

$$\det(\varphi \oplus \psi + \lambda\iota_{F\oplus H}) = \det(\varphi + \lambda\iota) \cdot \det(\psi + \lambda\iota), \qquad \varphi \in L_F, \quad \psi \in L_H,$$

shows that

$$C_r^{F\oplus H}(\varphi \oplus \psi) = \sum_{p=0}^{r} C_p^F(\varphi) C_{r-p}^H(\psi).$$

Let $\alpha: \vee(L_F^* \oplus L_H^*) \to \mathscr{H}(L_F \oplus L_H)$ be the homomorphism of sec. A.0.

The relation above yields

$$\alpha(j^\vee(C^{F\oplus H})) = \alpha(C^F \otimes C^H).$$

Since α is injective, the proposition follows.

Q.E.D.

A.3. Trace coefficients. Let F be a finite-dimensional vector space. The *trace coefficients* of F are the elements $\mathrm{Tr}_p^F \in \vee^p L_F^*$, given by

$$\mathrm{Tr}_0^F = \dim F$$

and

$$\mathrm{Tr}_p^F(\varphi_1, \ldots, \varphi_p) = \sum_{\sigma\in S^p} \mathrm{tr}(\varphi_{\sigma(1)} \circ \cdots \circ \varphi_{\sigma(p)}), \qquad \varphi_\nu \in L_F, \quad p \geq 1.$$

Evidently,

$$\mathrm{Tr}_p^F(\sigma\circ\varphi_1\circ\sigma^{-1},\ldots,\sigma\circ\varphi_p\circ\sigma^{-1})=\mathrm{Tr}_p^F(\varphi_1,\ldots,\varphi_p),\qquad \sigma\in GL(F),$$

and

$$\sum_{j=1}^{p}\mathrm{Tr}_p^F(\varphi_1,\ldots,[\psi,\varphi_j],\ldots,\varphi_p)=0,\qquad \psi\in L_F.$$

Let H be a second finite-dimensional vector space. Consider the inclusion maps,

$$j\colon L_F\oplus L_H\to L_{F\oplus H}\qquad\text{and}\qquad i\colon L_F\oplus L_H\to L_{F\otimes H},$$

given by

$$j(\varphi\oplus\psi)=\varphi\oplus\psi\qquad\text{and}\qquad i(\varphi\oplus\psi)=\varphi\otimes\iota+\iota\otimes\psi.$$

A straightforward computation establishes (cf. sec. A.2)

Proposition II: The trace coefficients of $F, H, F\oplus H, F\otimes H$ are connected by the relations

$$j^\vee(\mathrm{Tr}_p^{F\oplus H})=\mathrm{Tr}_p^F\otimes 1+1\otimes\mathrm{Tr}_p^H$$

and

$$i^\vee(\mathrm{Tr}_p^{F\otimes H})=\sum_{i+j=p}\binom{p}{i}\mathrm{Tr}_i^F\otimes\mathrm{Tr}_j^H.$$

Next, consider the commutative algebra,

$$\vee^{**}L_F^*=\prod_{p=0}^{\infty}(\vee^pL_F^*),$$

whose elements are the infinite sequences

$$\Phi=(\Phi_0,\Phi_1,\ldots,\Phi_p,\ldots),\qquad \Phi_p\in\vee^pL_F^*.$$

Addition is defined componentwise, while the product is given by

$$(\Phi\cdot\Psi)_k=\sum_{i+j=k}\Phi_i\vee\Psi_j$$

(cf. sec. 6.21, volume II). Clearly $\vee^{**}L_F^*$ contains $\vee L_F^*$ as a subalgebra.

The *trace series* of F is the element $\mathrm{Tr}^F \in \vee^{**} L_F^*$, given by

$$\mathrm{Tr}^F = \left(\mathrm{Tr}_0^F, \ldots, \frac{1}{p!}\mathrm{Tr}_p^F, \ldots\right).$$

Proposition II implies that

$$j^\vee(\mathrm{Tr}^{F\oplus H}) = \mathrm{Tr}^F \otimes 1 + 1 \otimes \mathrm{Tr}^H$$

and

$$i^\vee(\mathrm{Tr}^{F\oplus H}) = \mathrm{Tr}^F \otimes \mathrm{Tr}^H.$$

Proposition III: The trace and characteristic coefficients of a finite-dimensional vector space F are related by

$$C_p^F = -\frac{1}{p}\sum_{j=0}^{p-1}(-1)^{p-j} C_j^F \vee \mathrm{Tr}_{p-j}^F, \qquad p \geq 1.$$

Lemma I: The operator, d, in $\vee L_F^*$ given by

$$\begin{aligned}(d\Phi)(\varphi_0, \ldots, \varphi_p) &= \sum_{i<j} \Phi(\varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i, \varphi_0, \ldots, \hat{\varphi}_i, \ldots, \hat{\varphi}_j, \ldots, \varphi_p), \\ &\qquad\qquad \Phi \in \vee^p L_F^*, \varphi_i \in L_F,\end{aligned}$$

is a derivation, homogeneous of degree 1. It satisfies

$$d\,\mathrm{Tr}_p^F = p\,\mathrm{Tr}_{p+1}^F$$

and

$$dC_p^F = -(p+1)C_{p+1}^F + C_p^F \vee \mathrm{Tr}_1^F, \qquad p \geq 0.$$

Proof: A simple calculation yields the formula

$$d(\Phi \vee \Psi)(\varphi, \ldots, \varphi) = (d\Phi \vee \Psi + \Phi \vee d\Psi)(\varphi, \ldots, \varphi), \qquad \varphi \in L_F.$$

This implies that d is a derivation. Clearly d is homogeneous of degree 1.

The first formula follows at once from the definition of d. To establish the second formula note that (cf. sec. A.1)

$$\begin{aligned}(p+1)C_{p+1}^F(\varphi_0, \ldots, \varphi_p) &= \langle \iota \,\square\, \iota_p, \varphi_0 \,\square\, \cdots \,\square\, \varphi_p \rangle \\ &= \langle \iota_p, i(\iota)(\varphi_0 \,\square\, \cdots \,\square\, \varphi_p)\rangle\end{aligned}$$

and

$$
\begin{aligned}
&i(\iota)(\varphi_0 \square \cdots \square \varphi_p) \\
&\quad = \sum_{j=0}^{p} \langle \iota, \varphi_j \rangle \varphi_0 \square \cdots \hat{\varphi}_j \cdots \square \varphi_p \\
&\qquad - \sum_{j<k} (\varphi_j \circ \varphi_k + \varphi_k \circ \varphi_j) \square \varphi_1 \square \cdots \hat{\varphi}_j \cdots \square \hat{\varphi}_k \cdots \varphi_p .
\end{aligned}
$$

Combining these relations yields the second formula.

Q.E.D.

Proof of the proposition: The proposition is trivial for $p = 1$. In the general case, it follows by induction via the formulae in the lemma and the derivation property of d.

Q.E.D.

Corollary I:

$$\sum_{j=0}^{p-1} (-1)^{p-j} C_j^F \vee \mathrm{Tr}_{p-j}^F = 0, \qquad p > n.$$

Proof: Apply the proposition and observe that $C_p^F = 0$, $p > n$.

Q.E.D.

Corollary II: The subalgebras of $\vee L_F^*$ generated respectively, by $C_0^F, \ldots, C_n^F$ and by $\mathrm{Tr}_0^F, \ldots, \mathrm{Tr}_n^F$, coincide and contain all the trace coefficients and characteristic coefficients.

Q.E.D.

§2. Inner product spaces

In this article F denotes an n-dimensional vector space and $\langle\ ,\ \rangle$ denotes an inner product in F. It induces a linear isomorphism $F \xrightarrow{\cong} F^*$ which we use to identify F with F^*. Further, $\langle\ ,\ \rangle$ extends to an inner product in each space $\wedge^p F$.

Sk_F denotes the Lie subalgebra of L_F consisting of the linear transformations which are skew with respect to $\langle\ ,\ \rangle$.

A.4. Multiplications in $\wedge F \otimes \wedge F$. In the vector space $\wedge F \otimes \wedge F$ we introduce *two* algebra structures: the first is the *canonical* tensor product of the algebras $\wedge F$ and $\wedge F$; the second is the *anticommutative* tensor product of $\wedge F$ and $\wedge F$.

The first algebra contains $\Delta(F)$ as a subalgebra (cf. sec. A.1) and so its multiplication is denoted by $\Box$:

$$(u \otimes v) \,\Box\, (u_1 \otimes v_1) = (u \wedge u_1) \otimes (v \wedge v_1).$$

The second algebra is canonically isomorphic to $\wedge(F \oplus F)$, and so multiplication is denoted by $\wedge$:

$$(u \otimes v) \wedge (u_1 \otimes v_1) = (-1)^{qr}(u \wedge u_1) \otimes v \wedge v_1, \qquad v \in \wedge^q F, \quad u_1 \in \wedge^r F.$$

The two products are connected by the relation

$$\Phi \,\Box\, \Psi = (-1)^{qr} \Phi \wedge \Psi, \qquad \Phi \in \wedge F \otimes \wedge^q F, \quad \Psi \in \wedge^r F \otimes \wedge F.$$

This implies that

$$\varphi_1 \,\Box\, \cdots \,\Box\, \varphi_p = (-1)^{p(p-1)/2} \varphi_1 \wedge \cdots \wedge \varphi_p, \qquad \varphi_j \in F \otimes F. \tag{A.2}$$

In particular,

$$\iota_p = \frac{1}{p!}\, \iota \,\Box\, \cdots \,\Box\, \iota = \frac{1}{p!}\, (-1)^{p(p-1)/2} \iota \wedge \cdots \wedge \iota,$$

where ι_p is regarded as an element of $\wedge^p F \otimes \wedge^p F$.

Now define an inner product in $F \oplus F$ by

$$\langle x \oplus y, x_1 \oplus y_1 \rangle = \langle x, y_1 \rangle + \langle y, x_1 \rangle.$$

(This is *not* the usual inner product!) Extend it to an inner product in $\wedge(F \oplus F)$. The induced inner product in $\wedge F \otimes \wedge F$ (via the standard algebra isomorphism $(\wedge F \otimes \wedge F, \wedge) \cong \wedge(F \oplus F)$) is given by

$$\langle \wedge^p F \otimes \wedge^q F, \wedge^r F \otimes \wedge^s F \rangle = 0, \qquad \text{unless} \quad p = s \quad \text{and} \quad q = r,$$

and

$$\langle a \otimes b, u \otimes v \rangle = (-1)^{pq} \langle a, v \rangle \langle b, u \rangle, \qquad a, v \in \wedge^p F, \quad b, u \in \wedge^q F.$$

Remark: Up to sign, this inner product agrees with the inner product in $C(F)$ defined in sec. A.1.

Next, identify $F \oplus F$ with $(F \oplus F)^*$ under the above inner product. Let $\tau: F \oplus F \to F \oplus F$ be the linear isomorphism given by

$$\tau(x, y) = (x + y, x - y), \qquad x, y \in F.$$

Its dual, τ^*, is given by

$$\tau^*(x, y) = (y - x, y + x), \qquad x, y \in F.$$

τ and τ^* extend to algebra automorphisms $\tau_\wedge$ and $\tau^\wedge$ of $(\wedge F \otimes \wedge F, \wedge)$ which are dual with respect to the inner product defined above.

Observe that

$$\begin{aligned} \tau_\wedge(x \wedge y \otimes 1) &= (x \wedge y) \otimes 1 + x \otimes y - y \otimes x + 1 \otimes (x \wedge y), \\ \tau_\wedge(x \otimes y) &= (x \wedge y) \otimes 1 - x \otimes y - y \otimes x - 1 \otimes (x \wedge y) \end{aligned}$$

and

$$\tau_\wedge(1 \otimes x \wedge y) = (x \wedge y) \otimes 1 - x \otimes y + y \otimes x + 1 \otimes (x \wedge y).$$

Lemma II: τ has the following properties:

(1) $\tau_\wedge(x \otimes y - y \otimes x) = 2((x \wedge y) \otimes 1 - 1 \otimes (x \wedge y))$.

(2) $\tau^\wedge(\iota_p) = 2^p \iota_p$.

Proof: (1) is immediate from the formula above as is (2) in the case $p = 1$. To obtain (2) in general observe that

$$\begin{aligned} \tau^\wedge(\iota_p) &= (-1)^{p(p-1)/2} \frac{1}{p!} \tau^\wedge(\iota \wedge \cdots \wedge \iota) \\ &= (-1)^{p(p-1)/2} \frac{1}{p!} (\tau^\wedge \iota \wedge \cdots \wedge \tau^\wedge \iota) = 2^p \iota_p. \end{aligned}$$

Q.E.D.

A.5. Characteristic coefficients for F. Let $\beta\colon \wedge^2 F \xrightarrow{\cong} \mathrm{Sk}_F$ be the canonical isomorphism given by

$$\beta(x \wedge y)(z) = \langle x, z\rangle y - \langle y, z\rangle x.$$

Proposition IV: Let $\varphi \in \mathrm{Sk}_F$. Then the characteristic coefficients $C_p^F(\varphi)$ are given by

$$C_p^F(\varphi) = 0, \qquad p \text{ odd},$$

and

$$C_{2k}^F(\varphi) = \frac{1}{(k!)^2}\langle \underbrace{\beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)}_{(k \text{ factors})}, \underbrace{\beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)}_{(k \text{ factors})}\rangle.$$

Proof: Let $\varphi \in \mathrm{Sk}_F$. Then

$$\det(\varphi + \lambda\iota) = \det(\varphi^* + \lambda\iota) = \det(-\varphi + \lambda\iota).$$

It follows that $C_{2k+1}^F(\varphi) = -C_{2k+1}^F(\varphi)$, whence $C_{2k+1}^F(\varphi) = 0$.

To establish the second formula, regard the inclusion $j\colon \mathrm{Sk}_F \to L_F$ as a linear map from Sk_F into $F \otimes F$. Then

$$j\beta(x \wedge y) = x \otimes y - y \otimes x.$$

Thus Lemma II, (1) shows that

$$\tau_\wedge j(\varphi) = 2(\beta^{-1}(\varphi) \otimes 1 - 1 \otimes \beta^{-1}(\varphi)), \qquad \varphi \in \mathrm{Sk}_F.$$

Now let $\langle\, ,\rangle$ be the inner product in $\wedge F \otimes \wedge F$ defined in sec. A.4. Then, for $\varphi \in \mathrm{Sk}_F$ (cf. Lemma II and formula (A.2), sec. A.4),

$$\begin{aligned} C_{2k}^F(\varphi) &= \frac{1}{(2k)!}\langle \iota_{2k}, j(\varphi) \square \cdots \square j(\varphi)\rangle \\ &= \frac{(-1)^k}{(2k)!2^{2k}}\langle \tau^\wedge(\iota_{2k}), j(\varphi) \wedge \cdots \wedge j(\varphi)\rangle \\ &= \frac{1}{(k!)^2}\langle \iota_{2k}, \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi) \otimes \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)\rangle \\ &= \frac{1}{(k!)^2}\langle \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi), \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)\rangle. \end{aligned}$$

Q.E.D.

Next, define elements $B_k \in \vee^{2k}\mathrm{Sk}_F^*$ by

$$B_k(\varphi_1, \ldots, \varphi_{2k})$$
$$= \frac{1}{(k!)^2} \sum_{\sigma \in S^{2k}} \langle \beta^{-1}(\varphi_{\sigma(1)}) \wedge \cdots \wedge \beta^{-1}(\varphi_{\sigma(k)}), \beta^{-1}(\varphi_{\sigma(k+1)}) \wedge \cdots \wedge \beta^{-1}(\varphi_{\sigma(2k)}) \rangle.$$

Then, as an immediate consequence of Proposition IV, we have

Proposition V: Let $j\colon \mathrm{Sk}_F \to L_F$ be the inclusion. Then

$$j^\vee(C^F_{2k+1}) = 0 \qquad \text{and} \qquad j^\vee(C^F_{2k}) = B_k.$$

A.6. Pfaffian. Suppose F has even dimension $n = 2m$ and let $a \in \wedge^n F$. Then the *Pfaffian of the pair* (F, a) is the element, $\mathrm{Pf}^F_a \in \vee^m \mathrm{Sk}_F^*$, given by

$$\mathrm{Pf}^F_a(\varphi_1, \ldots, \varphi_m) = \langle a, \beta^{-1}(\varphi_1) \wedge \cdots \wedge \beta^{-1}(\varphi_m) \rangle, \qquad \varphi_\mu \in \mathrm{Sk}_F.$$

It determines the homogeneous function Pf^F_a given by

$$\mathrm{Pf}^F_a(\varphi) = \frac{1}{m!} \mathrm{Pf}^F_a(\varphi, \ldots, \varphi), \qquad \varphi \in \mathrm{Sk}_F.$$

The scalar $\mathrm{Pf}^F_a(\varphi)$ is called the *Pfaffian* of φ with respect to a.

We extend the definition to odd-dimensional spaces by setting the Pfaffian equal to zero in this case.

Proposition VI: Let $a \in \wedge^n F$ and $b \in \wedge^n F$. Then

$$\mathrm{Pf}^F_a \vee \mathrm{Pf}^F_b = \langle a, b \rangle j^\vee(\mathrm{Det}),$$

where $j\colon \mathrm{Sk}_F \to L_F$ denotes the inclusion. In particular,

$$(\mathrm{Pf}^F_a(\varphi))^2 = \langle a, a \rangle \det \varphi, \qquad \varphi \in \mathrm{Sk}_F.$$

Proof: In fact,

$$(\mathrm{Pf}^F_a \vee \mathrm{Pf}^F_b)(\varphi_1, \ldots, \varphi_{2m})$$
$$= \frac{1}{(m!)^2} \sum_\sigma \langle a, \beta^{-1}\varphi_{\sigma(1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(m)} \rangle \langle b, \beta^{-1}\varphi_{\sigma(m+1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(2m)} \rangle$$
$$= \frac{1}{(m!)^2} \sum_\sigma \langle a, b \rangle \langle \beta^{-1}\varphi_{\sigma(1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(m)}, \beta^{-1}\varphi_{\sigma(m+1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(2m)} \rangle$$

(since $a \in \wedge^n F$ and $b \in \wedge^n F$). This shows that

$$\mathrm{Pf}_a^F \vee \mathrm{Pf}_b^F = \langle a, b\rangle B_m.$$

Now apply Proposition V, with $k = m$.

Q.E.D.

Next, let $\tau: F \to F$ be an isometry; i.e.,

$$\langle \tau x, \tau y\rangle = \langle x, y\rangle, \qquad x, y \in F.$$

If $\det \tau = 1$, τ is called *proper*.

Proposition VII: (1) If τ is an isometry of F, then

$$\mathrm{Pf}_a^F(\tau \circ \varphi_1 \circ \tau^{-1}, \ldots, \tau \circ \varphi_m \circ \tau^{-1}) = \det \tau \, \mathrm{Pf}_a^F(\varphi_1, \ldots, \varphi_m), \quad \varphi_i \in \mathrm{Sk}_F.$$

(2) If $\psi \in \mathrm{Sk}_F$, then

$$\sum_{i=1}^m \mathrm{Pf}_a^F(\varphi_1, \ldots, [\psi, \varphi_i], \ldots, \varphi_m) = 0, \qquad \varphi_i \in \mathrm{Sk}_F.$$

Proof: In fact, since

$$\beta(\tau x \wedge \tau y) = \tau \circ \beta(x \wedge y) \circ \tau^{-1}, \qquad x, y \in F,$$

it follows that

$$\mathrm{Pf}_a^F(\tau \circ \varphi_1 \circ \tau^{-1}, \ldots, \tau \circ \varphi_m \circ \tau^{-1}) = \det \tau \, \mathrm{Pf}_a^F(\varphi_1, \ldots, \varphi_m),$$

which establishes (1).

Similarly, for $\psi \in \mathrm{Sk}_F$

$$\beta(\psi x \wedge y + x \wedge \psi y) = [\psi, \beta(x \wedge y)],$$

whence

$$\sum_{i=1}^m \mathrm{Pf}_a^F(\varphi_1, \ldots, [\psi, \varphi_i], \ldots, \varphi_m) = \mathrm{tr}\, \psi \cdot \mathrm{Pf}_a^F(\varphi_1, \ldots, \varphi_m) = 0.$$

Q.E.D.

Let H be a second inner product space and give $F \oplus H$ the induced inner product; i.e.,

$$\langle x \oplus y, x_1 \oplus y_1\rangle = \langle x, x_1\rangle + \langle y, y_1\rangle.$$

The inclusion map $j\colon \mathrm{Sk}_F \oplus \mathrm{Sk}_H \to \mathrm{Sk}_{F\oplus H}$ induces a homomorphism

$$j^{\vee}\colon \vee\, \mathrm{Sk}_F^* \otimes \vee\, \mathrm{Sk}_H^* \leftarrow \vee\, \mathrm{Sk}_{F\oplus H}^*.$$

Moreover, multiplication defines a canonical algebra isomorphism,

$$\wedge F \otimes \wedge H \xrightarrow{\cong} \wedge(F \oplus H),$$

which preserves the inner products. We shall identify the algebras $\wedge F \otimes \wedge H$ and $\wedge(F \oplus H)$ under this isomorphism.

Proposition VIII: Let $a \in \wedge^n F$ and $b \in \wedge^r H$, where $n = \dim F$ and $r = \dim H$. Then, with the identification above,

$$j^{\vee}(\mathrm{Pf}_{a\otimes b}^{F\oplus H}) = \mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H.$$

Proof: If $n + r$ is odd both sides are zero. Now assume that $n + r = 2k$. Then we have, for $\varphi \in \mathrm{Sk}_F$ and $\psi \in \mathrm{Sk}_H$,

$$\begin{aligned}(j^{\vee}\mathrm{Pf}_{a\otimes b}^{F\oplus H})(\varphi \oplus \psi, \ldots, \varphi \oplus \psi) &= \langle a \otimes b, (\varphi \otimes \iota + \iota \otimes \psi)^k\rangle \\ &= \sum_{i+j=k} \binom{k}{i} \langle a, \varphi^i\rangle\langle b, \psi^j\rangle.\end{aligned}$$

If n and r are odd, it follows that

$$j^{\vee}\, \mathrm{Pf}_{a\otimes b}^{F\oplus H} = 0 = \mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H.$$

If $n = 2m$ and $r = 2s$, we obtain

$$\begin{aligned}(j^{\vee}\, \mathrm{Pf}_{a\otimes b}^{F\oplus H})(\varphi \oplus \psi, \ldots, \varphi \oplus \psi) &= \binom{k}{m} \mathrm{Pf}_a^F(\varphi)\, \mathrm{Pf}_b^H(\psi) \\ &= (\mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H)(\varphi \oplus \psi, \ldots, \varphi \oplus \psi).\end{aligned}$$

Q.E.D.

Corollary: $\mathrm{Pf}_{a\otimes b}^{F\oplus H}(\varphi \oplus \psi) = \mathrm{Pf}_a^F(\varphi)\, \mathrm{Pf}_b^H(\psi), \qquad \varphi \in \mathrm{Sk}_F, \psi \in \mathrm{Sk}_H.$

A.7. Examples: 1. *Oriented inner product spaces*: Let F be a real inner product space of dimension $n = 2m$ (note that we do not require the inner product to be positive definite). Let $e \in \wedge^n F$ be the unique element which represents the orientation and satisfies $|\langle e, e\rangle| = 1$.

Then Pf_e^F is called the *Pfaffian of the oriented inner product space* F, and is denoted by Pf^F. Reversing the orientation changes the sign of the

Pfaffian. Proposition VI implies that

$$\det \varphi = \langle e, e\rangle(\mathrm{Pf}^F \varphi)^2, \qquad \varphi \in \mathrm{Sk}_F.$$

Next let $F = F^+ \oplus F^-$ be an orthogonal decomposition of F such that the restriction of the inner product to F^+ (respectively, F^-) is positive (respectively, negative) definite. Define a positive definite inner product $(\,,\,)$ in F by setting

$$(x^+ + x^-, y^+ + y^-) = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle, \qquad x^+, y^+ \in F^+, \quad x^-, y^- \in F^-.$$

Let φ be a skew linear transformation of F that stabilizes F^+ and F^-,

$$\varphi = \varphi^+ \oplus \varphi^-, \qquad \varphi^+ \colon F^+ \to F^+, \qquad \varphi^- \colon F^- \to F^-.$$

Then φ is skew with respect to both of the inner products $\langle\,,\,\rangle$ and $(\,,\,)$ and so the Pfaffians $\mathrm{Pf}^F_{\langle,\rangle}(\varphi)$ and $\mathrm{Pf}^F_{(,)}(\varphi)$ are defined.

Proposition IX: Suppose φ satisfies the conditions above. Then:

(1) If $\dim F^-$ is odd,

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = 0, \qquad \mathrm{Pf}^F_{(,)}(\varphi) = 0.$$

(2) If $\dim F^- = 2q$. Then

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = (-1)^q\, \mathrm{Pf}^F_{(,)}(\varphi).$$

Proof: The corollary to Proposition VIII, sec. A.6, shows that, for suitable orientations of F^+ and F^-,

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = \mathrm{Pf}^{F^+}_{\langle,\rangle}(\varphi^+) \cdot \mathrm{Pf}^{F^-}_{\langle,\rangle}(\varphi^-)$$

and

$$\mathrm{Pf}^F_{(,)}(\varphi) = \mathrm{Pf}^{F^+}_{(,)}(\varphi^+) \cdot \mathrm{Pf}^{F^-}_{(,)}(\varphi^-).$$

Since $\langle\,,\,\rangle$ and $(\,,\,)$ coincide in F^+, it follows that

$$\mathrm{Pf}^{F^+}_{\langle,\rangle}(\varphi^+) = \mathrm{Pf}^{F^+}_{(,)}(\varphi^+)$$

We are thus reduced to the case that $\langle\,,\,\rangle$ is negative definite; i.e., $F = F^-$ and $\varphi = \varphi^-$.

In this case, $\langle\,,\,\rangle = -(\,,\,)$ and so the linear isomorphisms $\beta_{\langle,\rangle}$ and $\beta_{(,)}$ are related by

$$\beta_{\langle,\rangle} = -\beta_{(,)}.$$

If $\dim F$ is odd, then, by definition

$$\mathrm{Pf}^F_{\langle,\rangle} = \mathrm{Pf}^F_{(,)} = 0.$$

On the other hand, if $\dim F = 2q$, then

$$\begin{aligned}\mathrm{Pf}^F_{\langle,\rangle}(\varphi) &= \langle e, \beta^{-1}_{\langle,\rangle}(\varphi) \wedge \cdots \wedge \beta^{-1}_{\langle,\rangle}(\varphi)\rangle \\ &= (-1)^q \langle e, \beta^{-1}_{(,)}(\varphi) \wedge \cdots \wedge \beta^{-1}_{(,)}(\varphi)\rangle = (-1)^q\, \mathrm{Pf}^F_{(,)}(\varphi).\end{aligned}$$

Q.E.D.

2. *Oriented Euclidean spaces*: Let F be an oriented $2m$-dimensional Euclidean space. Fix $\varphi \in \mathrm{Sk}_F$ and choose a positive orthonormal basis $x_1, \ldots, x_{2m}$ of F so that

$$\varphi(x_{2i-1}) = \lambda_i x_{2i},$$

and

$$\varphi(x_{2i}) = -\lambda_i x_{2i-1}, \qquad i = 1, \ldots, m.$$

Then $\mathrm{Pf}^F(\varphi) = \lambda_1 \cdots \lambda_m$.

On the other hand, the characteristic coefficients of φ are given by

$$C^F_p(\varphi) = \sum_{i_1 < \cdots < i_p} \lambda^2_{i_1} \cdots \lambda^2_{i_p}.$$

3. *Complex spaces*: Let F be an m-dimensional complex space with a Hermitian inner product. Orient the underlying real vector space F_R as described in Example 2, sec. 9.17, volume II, and define a positive definite inner product in F_R by

$$\langle\, , \rangle_R = \mathrm{Re}\langle\, , \rangle.$$

Then a skew Hermitian linear transformation φ of F may be considered as a skew linear transformation φ_R of F_R. We shall show that

$$i^m\, \mathrm{Pf}^{F_R}(\varphi_R) = \det \varphi, \qquad \varphi \in \mathrm{Sk}_F.$$

In fact, let $z_1, \ldots, z_m$ be an orthonormal basis of F and let $\lambda_\mu \in R$ be scalars, such that

$$\varphi z_\mu = i\lambda_\mu z_\mu, \qquad \mu = 1, \ldots, m.$$

Then $\det \varphi = i^m \lambda_1 \cdots \lambda_m$.

On the other hand, the vectors $z_1, iz_1, \ldots, z_m, iz_m$ form a positive orthonormal basis of F_R. Moreover,

$$\varphi_R(z_\mu) = \lambda_\mu(iz_\mu) \qquad \text{and} \qquad \varphi_R(iz_\mu) = -\lambda_\mu(z_\mu), \qquad \mu = 1, \ldots, m.$$

It follows that (cf. Example 2)

$$i^m\, \mathrm{Pf}^{F_R}(\varphi_R) = i^m \lambda_1 \cdots \lambda_m = \det \varphi.$$

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Notes

In these notes we attempt to give the original sources for the theorems of this volume, as well as some of the recent applications which have been made. There have been two effectively different techniques applied to the study of the cohomology of principal bundles and homogeneous spaces: E. Cartan's method of invariant differential forms, as extended in [53] by H. Cartan, and the classic methods of algebraic topology.

The latter techniques were used by Pontrjagin, Ehresmann, Hopf, Samelson, Leray, and Borel and depend in part on the construction of a topological universal bundle which plays a role analogous to that of the Weil algebra in the first method.

Both methods frequently produced the same results at the same time, although the topological method often produced results with coefficients Z or Z_p. Nonetheless this book is an exposition of the first method, and the notes will therefore tend to concentrate on proofs achieved that way.

For more details the reader is referred to the two excellent survey articles by Samelson [239] and Borel [25], as well as the Springer Lectures Notes [28] by Borel.

Finally, we should like to apologize for any omissions or mistakes in crediting discoveries. We should also like to thank J.–L. Koszul, who gave a series of lectures at Toronto, on which note 18 is based.

1. *C*-equivalence and minimum models. A *minimum model* is a graded differential algebra (R, d) such that (i) R is the tensor product of an exterior algebra (over an oddly graded space) with a symmetric algebra (over an evenly graded space) and (ii) $d(R) \subset R^+ \cdot R^+$. They were introduced by Sullivan [265], who shows that each c-equivalence class of anticommutative graded differential algebras contains exactly one isomorphism class of minimum models. For example, the Koszul complex of the "associated essential P_1-algebra" of a P-algebra is the minimum model for the Koszul complex of the P-algebra (cf. sec. 2.23).

Sullivan also shows that if the ground field is Q then there is a natural bijection between isomorphism classes of minimum models and rational homotopy types of C. W. complexes (at least in the simply connected case.)

2. Koszul complexes of *P*-algebras and *P*-differential algebras. These were introduced by Koszul in [168], and have become an important homological tool in commutative algebra. They are also used by Borel in his thesis [19]. The cohomology of the Koszul complex of a (P, δ) algebra is a function now known as *differential tor* introduced by Eilenberg and Moore (cf. [87], [209], and [140]). Differential tor is defined for two graded differential modules over a graded differential algebra; the notion has been further extended by Stasheff and Halperin (cf. [253], [212]) to "strongly homotopy associative modules."

Many of the results of Chapter II, including the use of the Poincaré–Koszul series, are either explicit or implicit in Koszul [168] and explicit in André [9]. A major exception is the proof of Theorem VII, sec. 2.17, which is due to J. C. Moore.

3. Spectral sequences. In chapter IX we introduce the spectral sequence of an operation and show it is isomorphic with the spectral sequence of the associated (P_E, δ)-algebra. If the operation arises from a principal bundle with compact Lie group, this spectral sequence coincides with the classical Leray–Serre sequence.

On the other hand, the (P_E, δ)-algebra determines the lower spectral sequence (cf. sec. 3.5); this was introduced by Koszul in [168]. It coincides with the Eilenberg–Moore sequence for the bundle (cf. [87], [249]) in singular theory as has been observed by So.

Further, it has been shown (Halperin, unpublished) that this is the same sequence as that arising from the filtration of the invariant differential forms $A(P)_{\theta=0}$ by the subspaces

$$F^{-k} = \bigcap_{a_{i_\nu} \in (\wedge^+ E)_{\theta=0}} \ker i(a_{i_1} \wedge \cdots \wedge a_{i_k}).$$

4. Cohomology of Lie groups and Lie algebras. E. Cartan [47] showed that biinvariant forms on a connected compact Lie group coincided with the de Rham cohomology of the group; this was used by Brauer [45] to calculate the Poincaré polynomials of the classical groups.

Pontrjagin [221], [222], and [223] constructed the primitive cycles in the classical groups, introduced the Pontrjagin multiplication in homology, and showed that the homology of each group was an exterior algebra over the primitive homology classes. He also obtained the Poincaré polynomials for the Grassmann manifolds and some results on torsion. Ehresmann [82] and [84] calculates the homology of the Grassmann manifolds and classical groups using cell complexes.

Hopf [131] introduced finite-dimensional H-spaces and showed their cohomology was an exterior algebra. Then Samelson [236] gave a topological proof of all the main results of Chapter V for compact Lie groups (the cohomology and homology algebras are dual exterior algebras over the

primitive subspaces) for coefficients a field of characteristic zero. Leray [178] and [187] obtained similar results using fibre bundle methods.

Koszul [167] approaches and solves all these problems by working only with the Lie algebra; articles 1 through 6 of Chapter V are an exposition of his work. His starting point is the graded differential algebra $(\wedge E^*, \delta_E)$ of E. Cartan (see also Chevalley and Eilenberg [68]). It is E. Cartan's theorem [47] that $H(E) = H(G)$ when E is the Lie algebra of the compact connected group G.

Stiefel [264] uses the action of the Weyl group on a maximal torus to calculate the Poincaré polynomials of the classical groups.

The Poincaré polynomials for the exceptional groups were first written down by Yen [294] and Chevalley [65], although they are implicit in Racah [226]; for proofs see Borel and Chevalley [29].

5. Weil algebra and transgression. The Weil algebra was invented by Weil (unpublished) and is introduced by H. Cartan in [53], where most of the major results of Chapter VI are announced; this includes the equalities Im $\rho_E = P_E$ and ker $\rho_E = (\vee^+ E^*)^2_{\theta=0}$ of Theorem II, sec. 6.14 (also proved by Leray [187]). These had been conjectured by Weil, and the first one was partially established by Chevalley, using the methods of Koszul [167].

The map ρ_E has reappeared recently in the work of Chern and Simons (cf. [61], [62], and [120]) under the name "transgression map."

The fundamental notion of transgression is first defined explicitly by Koszul in his thesis [167] (for the operation in $\wedge L^*$ of a reductive subalgebra E of a Lie algebra L), although the idea is clearly present in Hirsch [123] and one example is implicit in Chern [56]. By constructing a specific transgression Koszul shows that primitive elements are transgressive. That same construction in the Weil algebra (used by Chevalley to prove Im $\rho_E \supset P_E$) gives rise to the canonical transgression as described in sec. 6.10.

The notion of transgression plays a major role in Borel's thesis [19], where he uses it to obtain topological proofs of many of the results in this volume, often with coefficients $\boldsymbol{Z}$, $\boldsymbol{Z}_p$, or $\boldsymbol{Q}$ rather than $\boldsymbol{R}$.

6. The structure of $(\vee E^*)_{\theta=0}$ and $\vee E^*$. Let T be a Cartan subalgebra of a reductive Lie algebra E. Then there are isomorphisms

$$\vee Q \overset{(1)}{\cong} (\vee E^*)_{\theta=0} \overset{(2)}{\cong} (\vee T^*)_{W_G=1} \overset{(3)}{\cong} \vee Q$$

as shown in Theorem I, sec. 6.13 and Theorem VIII, sec. 11.9.

The first isomorphism is given by Koszul [168], the second by Borel [19] and Leray [187], and the third independently by Leray [187] and Chevalley [63]. Chevalley's proof uses only the fact that the action of W_G in T^* is generated by reflections. In many cases these proofs also

give the Poincaré series for $(\vee E^*)_{\theta=0}$ in terms of the Poincaré polynomial for $H(E)$.

The isomorphism (of W_G modules) $(\vee E^*)_{\theta=0} \otimes L \cong \vee T^*$, where L is the regular representation of W_G, is also established by Leray [187] and Chevalley [63].

Borel [19] also obtains analogous results for coefficients $\mathbf{Z}$ or $\mathbf{Z}_p$ under suitable hypotheses; in this case $(\vee E^*)_{\theta=0}$ must be replaced by $H^*(B_G)$ (E the Lie algebra of a compact connected group G with classifying space B_G). In the process he obtains an isomorphism $H^*(B_G\,;\,\mathbf{R}) \cong (\vee E^*)_{\theta=0}$. This isomorphism is also constructed in [30].

Suppose E is a reductive Lie algebra. As in the case of $\wedge E^*$ the elements $a \in (\vee^+ E)_{\theta=0}$ determine operators $i_S(a)$ in $\vee E^*$. Set

$$\vee E^*_{i_I=0} = \bigcap_{a \in (\vee^+ E)_{\theta=0}} \ker i_S(a).$$

Then Kostant [161] shows that multiplication defines a linear isomorphism $(\vee E^*)_{\theta=0} \otimes (\vee E^*)_{i_I=0} \xrightarrow{\cong} \vee E^*$. A major simplification in the proof is given by Johnson [143].

7. Operations and algebraic connections. The notions of an operation of a Lie algebra and of an algebraic connection were introduced by H. Cartan in [53], together with the example of differential forms on a principal bundle. The results of the latter part of Chapter VIII (construction of the classifying homomorphism, construction of the Weil homomorphism, independence of connection) as well as the idea of the proofs are all in [53]; details can be found in [9].

These ideas derive from the earlier work of Chern [56], who first expressed characteristic classes as polynomials in the curvature; Ehresmann, who introduced the notion of connection in a principal bundle (C.R. Acad. Sci. Paris, 1938, p. 1433–1434); and Koszul [167], who analyzed the special case of the principal bundle associated with a homogeneous space.

In the subsequent development of the general theory Weil (unpublished) played a major role; in particular, he constructed the Weil algebra and the Weil homomorphism and proved the latter independent of connection.

8. Cohomology of an operation. The fundamental theorem of Chapter IX, which identifies the cohomology of an operation with the cohomology of a Koszul complex, is due to Chevalley (cf. [53]) as an application of the theory of Hirsch–Koszul (exposited in [9]). The proof we give is essentially the same.

A topological proof for bundles is given by Borel [19]. The simpler Koszul complex $(H(B) \otimes P_E, \nabla)$ for the case $A(B)$ is c-split is announced by Koszul [168] and established by Borel [19].

The specific application to n-connected operations (Theorem IV, sec. 9.8.) is also given by H. Cartan [53].

The definition of the filtration of an operation, its spectral sequence, and the first three terms of the spectral sequence are all given earlier by Kozul in [167] for the operation of a subalgebra E of L in ΛL^*; this is apparently the first time Leray's spectral sequence is interpreted as a sequence of differential spaces (E_i, d_i) with $E_{i+1} \cong H(E_i)$.

9. Cohomology of homogeneous spaces. Most of the results of Chapters X and XI are due to H. Cartan [53] (although many of the proofs are only given in André [9] or Rashevskii [227]). These include Theorems II through V and Theorem VII of Chapter X and the corresponding results in Chapter XI; at least part of Theorem V is apparently also due to Weil.

Theorem VI is due to Koszul [168]; it generalizes a conjecture of Hirsch (see below) and is frequently referred to as a "Hirsch formula."

Most of the results in article 1 (operation of a subalgebra), article 5 (n.c.z. subalgebras), and article 8 (relative Poincaré duality) in Chapter X are in Koszul [167].

In particular the identification of the algebra $(\wedge E^*_{i_F=0,\,\theta_F=0}, \delta_E)$ as the differential algebra of invariant forms on G/K is made by Chevalley and Eilenberg in [68] and is a starting point for Koszul; the cohomology is sometimes called the *relative cohomology* and written $H^*(E, F)$ rather than our notation of $H(E/F)$.

The proofs given in this book in Chapter X are based on the ideas discovered by H. Cartan and Koszul.

N.c.z. subgroups were studied earlier by Samelson [236] and by Kudo [173]. The formula for the Poincaré polynomial of $H(G/K)$ (Theorem VI) was originally conjectured by Hirsch for equal rank pairs, and it is established by Leray [187] in that case.

Homogeneous spaces in which the subgroup has the same rank were studied extensively by Leray [185], [187] and Borel [19], who establish all of the results of article 6, Chapter X as well as those in article 3, Chapter XI. Many of the results of article 6 are also at least implicit in H. Cartan [53].

Theorem XII, sec. 10.23, is equivalent to dim $H(E) = 2^l$, l the dimension of a Cartan subalgebra, and in this form it was known to E. Cartan, at least for compact Lie groups. As stated, it is proved explicitly by Hopf [131] again for compact groups. The algebraic proof given here is a slight modification of the argument due to Borel [19]. The main idea is to prove that if $K \subset G$ has the same rank (G, K compact and connected), then $H(G/K)$ is evenly graded; this result remains true for integer coefficients, as was shown by Bott and Samelson [43].

The example in sec. 11.15 is due to Borel (unpublished).

H. Cartan's main theorem, $H(G/K) = H(\vee F^*_{\theta=0} \otimes \wedge P_E)$, is also in Borel [19], again for coefficients a field of characteristic zero and with $H(B_K)$ replacing $(\vee F^*)_{\theta=0}$. The same theorem has recently been established for coefficients $\boldsymbol{Z}$ or $\boldsymbol{Z}_p$ (under suitable homological restrictions on G and K) independently by Husemoller *et al.* [140], May [202], Munkholm [213], and Wolf [290]; partial results had been obtained earlier by Baum [12].

10. Symmetric spaces. If G/K is a symmetric space with G compact and K connected, then $H(G/K) = (\wedge E^*)_{i_F=0,\, \theta_F=0}$, as was proved by E. Cartan [47]. This enabled Ehresmann, and Iwamoto [141], to calculate the cohomology in certain cases.

This same fact is used by Koszul [168] to show that symmetric spaces satisfy the Cartan condition (cf. sec. 10.13). Theorem VIII, sec. 10.17, is a simple generalization of Koszul's result.

11. The Samelson and reduction theorems. The "Samelson" theorem (Theorem IV, sec. 2.13; Theorem IV, sec. 3.13; Theorem I, sec. 7.13; Theorem III, sec. 7.23; Theorem I, sec. 10.4; Theorem I, sec. 11.2) is in each case effectively the same theorem proved the same way. The original version is due to Samelson [236]; the algebraic form (and proof) which we give is due to Koszul [167].

The reduction theorem (Theorem V, sec. 2.15; Theorem V, sec. 3.15; Theorem II, sec. 7.14; Theorem 3, sec. 7.23; Theorem IV, sec. 10.12; Theorem III, sec. 11.5) is again effectively a single theorem. The version in sec. 2.15 is established in André [9] (and hence also those in sec. 10.12 and sec. 11.5). The one in sec. 10.12, however, is announced earlier by H. Cartan in [53].

Dynkin [76] gives a topological proof of the Samelson theorem; Leray [183] and [184] does the same for both theorems. Borel [28] extends these results to actions of a Hopf space. These proofs work in any characteristic with suitable homological restrictions on the group or Hopf space.

12. Operation of a Lie algebra pair. The main theorem of Chapter XII (sec. 12.10) was inspired by a theorem of Baum and Smith [13], which is essentially an additive version of Corollary IV, sec. 12.21. A weaker version of part (1) of the main theorem was established independently by Kamber and Tondeur [151]. They also considered special Cartan pairs and obtained many of the results of sec. 12.24.

Diagram (12.27) was established by Leray [187] for homogeneous spaces G/K with rank K = rank G.

13. Tensor difference. Suppose $\{A(B_i), \delta_i; \tau_i\}$ $(i = 1, 2)$ are the (P_E, δ) algebras arising from principal bundles $P_i \xrightarrow{\pi} B_i$. Then the Koszul

complex of the tensor difference is c-equivalent to the algebra of differential forms on $P_1 \times_G P_2$ if G is compact and connected (Halperin, unpublished); this follows easily from the main theorem of Chapter XII given the identification of $P_1 \times_G P_2$ as a bundle over $B_1 \times B_2$ with fibre $(G \times G)/G$.

14. Associated bundles. Let G be a compact connected Lie group with Lie algebra E. The canonical transgression $\tau : P_E \to (\vee E^*)_{\theta=0}$ extends to a canonical linear map $\tau : \wedge P_E \to (\vee E^*)_{\theta=0}$, homogeneous of degree 1.

Moreover, let (P, π, B, G) be a smooth principal bundle and suppose G acts smoothly on a manifold F. Then $H(P \times_G F) \cong H(A(B) \otimes A_I(F), D)$, where

$$D = \delta_{B \times F} - \Sigma\mu(\Phi^\nu) \otimes i(a_\nu).$$

(Here a_ν is a basis for $\wedge P_E$ with dual basis Ψ^ν, and Φ^ν is a closed form representing $\chi^*(\tau\Psi^\nu)$.)

This result (Halperin, unpublished) uses the result on the tensor difference stated above

15. Cohomology of an operation with coefficients in a module. A linear connection in a vector bundle ξ determines a covariant exterior derivative ∇ in the space $A(B; \xi)$ of bundle-valued forms over the base (cf. Chapter VII, volume II). If R is the curvature of ∇, then $\nabla^2(\Phi) = R(\Phi)$ and so $(A(B; \xi)/R(A(B; \xi)), \bar{\nabla})$ is a chain complex.

The cohomology of this chain complex was introduced by Vaisman [272], who used a different complex to calculate it. Later it was observed by Halperin and Lehmann [115] that the chain complex described above could be constructed abstractly from the following data: an operation of a Lie algebra E, a finite dimensional representation of E, and an algebraic connection for the operation. Moreover the cohomology contains a *characteristic submodule* which is a finitely generated submodule over the characteristic subalgebra of $H(B)$.

16. Foliations. Let ξ be an involutive distribution on a manifold M and let $(P, \pi, M, GL(q))$ be the principal bundle associated with the quotient bundle τ_M/ξ (τ_M the tangent bundle). Every linear connection ∇ in τ_M/ξ determines a principal connection and hence an algebraic connection in $A(P)$.

Bott [37] shows that for suitable linear connections χ, $\chi_\vee(\Sigma_{j>q} \vee^j E^*) = 0$ (E the Lie algebra of $GL(q)$). Thus the classifying homomorphism becomes a homomorphism

$$\tilde{\chi}_W : (\vee E^*/I) \otimes \wedge E^* \to A(P), \qquad I = \sum_{j>q} \vee^j E^*.$$

Moreover, it is clear that the operators $\theta_W(x)$, $i(x)$, and d_W induce operators in $(\vee E^*/I) \otimes \wedge E^*$. The resulting operation of E is called the ***truncated Weil algebra***. The fundamental theorem of Chapter IX shows that $H\{(\vee E^*/I) \otimes \wedge E^*\} \cong H(W_q, \nabla)$, where $W_q = \{\vee(c_1, \ldots, c_q)/J\} \otimes \wedge(x_1, x_2, x_3, x_4, \ldots, x_q)$, $\deg c_i = 2i$, $\deg x_i = 2i - 1$, J is the ideal of elements of degree $> 2q$, and ∇ is given by $\nabla x_i = \bar{c}_i$. Thus $\bar{\chi}_W$ determines a homomorphism $H(W_q, \nabla) \to H(P)$.

Now let F be the Lie algebra of $O(q)$. Write

$$\begin{aligned}\{(\vee E^*/I) \otimes \wedge E^*\}_{O(q)\text{-basic}} \\ = \{(\vee E^*/I) \otimes \wedge E^*\}_{i_F = 0} \cap \{(\vee E^*/I) \otimes \wedge E^*\}_{O(q)\text{-invariant}}\end{aligned}$$

and

$$A(P)_{O(q)\text{-basic}} = A(P)_{i_F = 0} \cap A(P)_{O(q)\text{-invariant}}$$

Then $\bar{\chi}_W$ restricts to a homomorphism

$$\chi_O : \{(\vee E^*/I) \otimes \wedge E^*\}_{O(q)\text{-basic}} \to A(P)_{O(q)\text{-basic}}$$

Applying the main theorem of Chapter XII (slightly modified because $O(q)$ is not connected), we find that $H((\vee E^*/I) \otimes \wedge E^*)_{O(q)\text{-basic}} = H(WO_q, \nabla)$, where WO_q is the sub-differential algebra of W_q given by

$$WO_q = (\vee(c_1, \ldots, c_q)/J) \otimes \wedge(x_1, x_3, x_5, \ldots).$$

(Explicit bases for $H(W_q)$ and $H(WO_q)$ have been given by Vey; cf. [119].)

On the other hand $A(P)_{O(q)\text{-basic}} = A(P/O(q))$ and the projection $P/O(q) \to M$ induces a cohomology isomorphism. Thus χ_O induces a homomorphism

$$H(WO_q) \to H(M).$$

This clearly extends the classic characteristic homomorphism of P; the new elements are called secondary characteristic classes. The first construction of such a class is due to Godbillon and Vey [100]; Roussarie [100] gave an example of a foliation for which the class did not vanish.

The main results listed above seem to have been discovered independently by a number of people (see for example Bott [39], Bott–Haefliger [41], Bernstein–Rosenfeld [14], Kamber–Tondeur [151]).

17. Gelf'and–Fuks cohomology. Let E be the Lie algebra of compactly supported vector fields on a manifold M. Let $C^p(E)$ denote the p-linear skew symmetric functions $E \times \cdots \times E \to R$, which are continuous with respect to the C^∞ topology of E. Then formula (5.7), sec. 5.3, defines an antiderivation δ_E in $C^*(E)$; the cohomology algebra $H^*(E)$ is called the Gelf'and–Fuks cohomology of M (cf. [95]). They show that if $H(M)$ is finite dimensional, then each $H^p(E)$ is finite dimensional.

18. Formal vector fields. An important step in the study of Gelf'and–Fuks cohomology is the study of the Lie algebra of formal vector fields in R^q. An account is given in Godbillon [99]. A *polynomial vector field in* R^q of degree $\le p$ is a vector field of the form $\sum_{i=1}^q f_i\, \partial/\partial x_i$, where each f_i is a polynomial of degree $\le p+1 (p=-1, 0,\dots)$; these form a Lie algebra L_p. There are obvious projections $\to L_{p+1} \to L_p \to \cdots \to L_{-1}$, and the inverse limit $L = \varprojlim L_p$ is called the *Lie algebra of formal vector fields on* R^q. Note that L_0 is the Lie algebra of the Lie group of affine isomorphisms of R^q.

Now consider the induced directed system $\to \wedge L_p^* \to \wedge L_{p+1}^* \to \cdots$ of graded differential algebras and set $\{C^*(L), \delta\} = \varinjlim (\wedge L_p^*, \delta)$. Its cohomology is called the *cohomology algebra of* L and is written $H(L)$. Next set $E = L(R^q; R^q)$. Observe that $L_p = \sum_{j \le p+1} \vee^j (R^q)^* \otimes R^q$, and the inclusions

$$E = (R^q)^* \otimes R^q \to L_p$$

are Lie algebra homomorphisms and define a Lie algebra homomorphism of E into L.

This defines an operation of E in $\{C^*(L), \delta\}$, which admits a natural algebraic connection χ. Moreover the homomorphism $\chi_W : W(E)_{\theta_E=0} \to C^*(L)_{\theta_E=0}$ maps the ideal $(\mathrm{I} \otimes \wedge E^*)_{\theta_E=0}$ into zero $(\mathrm{I} = \sum_{j>q} \vee^j E^*)$. Thus χ_W induces a homomorphism

$$\{(\vee E^*/\mathrm{I}) \otimes \wedge E^*)_{\theta_E=0} \to C^*(L)_{\theta_E=0}.$$

It has been shown that the induced homomorphism (cf. note 16) $H(W_q) \to H(L)$ is an isomorphism; this is a main step in the proof of the Gelf'and–Fuks theorem (cf. note 17).

19. Bundles over any space. Let (P, π, B, G) be a principal bundle over any topological space B, with G a compact connected Lie group. Then the methods of Chapter IX can be applied as follows to calculate $H(P)$, as has been shown by Watkiss ([283]).

Case I: B is a simplicial complex. For each simplex σ of B, set $P_\sigma = \pi^{-1}(\sigma)$. Then a smooth structure can be assigned to each P_σ so that the inclusions $P_\tau \hookrightarrow P_\sigma$ (when $\tau < \sigma$) are smooth. Let $A(P)$ be the subalgebra of $\Pi_\sigma A(P_\sigma)$ defined by $(\Phi_\sigma) \in A(P)$ if and only if $\Phi_\sigma|_{P_\tau} = \Phi_\tau$ whenever $\tau < \sigma$. Then $\{A(P), \delta\}$ is a graded differential algebra in which the Lie algebra of G operates and which admits an algebraic connection. Moreover, $H(A(P)) = H(P)$.

Case II: B is arbitrary. Replace B by the associated singular complex (subdivided twice).

20. The bicomplex of an open cover. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover of a manifold B. For each ordered $(p+1)$-tuple $\sigma = \langle i_0, \ldots, i_p \rangle$ $(i_\nu \in I)$ write $p = |\sigma|$ and $\langle i_0 \ldots \hat{i}_\nu \ldots i_p \rangle = \partial_\nu \sigma$. Then set $U_\sigma = \bigcap_{i_\nu} U_{i_\nu}$ and define $A^q(\mathscr{U}^p) = \Pi_{|\sigma|=p} A^q(U_\sigma)$ {note that $A(\phi) = 0$!}. Set $A^{p,q}(\mathscr{U}) = A^q(\mathscr{U}^p)$ and $A(\mathscr{U}) = \sum_{p,q} A^{p,q}(\mathscr{U})$. Operators d and δ of bidegrees $(1, 0)$ and $(0, 1)$ are defined in $A(\mathscr{U})$ by

$$(d\Phi)_\sigma = \sum_{\nu=0}^{p+1} (-1)^\nu \Phi_{\partial_\nu \sigma}|_{U_\sigma} \qquad \text{and} \qquad (\delta\Phi)_\sigma = (-1)^p \delta(\Phi_\sigma), \quad \Phi \in A^{p,q}(\mathscr{U}).$$

Set $D = d + \delta$; then $D^2 = 0$ and $H(A, D) \cong H(B)$. $(A(\mathscr{U}), D)$ is called the bicomplex of the open cover.

Next observe that a "piecewise smooth" space $|\mathscr{U}|$ can be constructed by glueing together the pieces $\sigma \times U_\sigma$ via the inclusions $\partial_\nu\sigma \times U_\sigma \to \partial_\nu\sigma \times U_{\partial_\nu\sigma}$. There is an algebra of piecewise differential forms on $|\mathscr{U}|$ (defined as described above in note 19) and a chain equivalence $J: \{A(|\mathscr{U}|), \delta\} \to \{A(\mathscr{U}), D\}$ obtained by "fibre integration over the simplices."

Suppose now that B is the base of a principal bundle and that connections ω_i are given for the restriction of the bundle to each U_i. Then a piecewise smooth bundle is determined over $|\mathscr{U}|$, and the ω_i determine a canonical connection for the corresponding operation (which is constructed as in note 19). Thus we obtain the sequence

$$(\vee E^*)_{\theta=0} \xrightarrow{\chi_{\vee,\theta=0}} A(|\mathscr{U}|) \xrightarrow{J} A(\mathscr{U}),$$

which provides representatives in the bicomplex for the characteristic classes.

The composite $J \circ \chi_{\vee,\theta=0}$ was first constructed by Bott [39]. Shulman [247] provides a "universal construction" of this composite map in the case that the bundle is trivial over each U_i. The approach described above is due to Watkiss [283].

Another approach is given by Kamber and Tondeur [150]. They construct a semisimplicial Weil algebra W_1 and show that $H\{(W_1)_{\text{basic}}\} \cong (\vee E^*)_{\theta=0}$. The ω_i then determine a chain map $(W_1)_{\text{basic}} \to A(\mathscr{U})$.

Given representatives of the characteristic classes in $A(\mathscr{U})$, one would like to write down a Koszul complex $\{A(\mathscr{U}) \otimes \wedge P_E, \nabla)$ whose cohomology is isomorphic with the cohomology of the total space of the bundle (in analogy with the fundamental theorem of Chapter IX). This problem is completely solved by Watkiss [283]. The operator $\nabla = \nabla_0 + \nabla_1 + \nabla_2 + \cdots$; ∇_i carries $A(\mathscr{U}) \otimes \wedge^p P_E$ into $A(\mathscr{U}) \otimes \wedge^{p-i} P_E$. ∇_0 and ∇_1 are the operators defined in Chapter III, while the other ∇_i are needed because $A(\mathscr{U})$ is not anticommutative.

These constructions apply generally to the bicomplex of a simplicial graded commutative differential algebra, and most of the results of Chap-

ter III carry over to this case. Applied to bundles over a simplicial complex K, it gives an operator in $C^*(K) \otimes \wedge P_E$ whose cohomology is isomorphic with that of the total space ($C^*(K)$ is the algebra of simplicial cochains). It is conjectured that this remains true when the coefficients are $\mathbb{Z}$ or $\mathbb{Z}_p$.

Finally, an earlier construction of Toledo and Tong (cf. [270] and [270a]), in some ways analogous to that of Watkiss, makes use of local Koszul complexes to prove the Riemann–Roch theorem.

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