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Infinite-Dimensional Lie Algebras

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無限次元Lie環

INFINITE-DIMENSIONAL LIE ALGEBRAS

by Minoru Wakimoto

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Preface

Over 30 years ago, in 1967 the so-called Kac-Moody algebras were discovered as infinite-dimensional Lie algebras. The representation theories of Lie algebras can be studied without taking account of their Lie groups. Starting with a simple finite-dimensional Lie algebra, we extend it to an infinite-dimensional Lie algebra as a loop algebra over the one-dimensional torus (i.e. the circle S^1), and its central extension is an affine Lie algebra. It has structures quite similar to those of simple finite-dimensional Lie algebras. Its Dynkin diagram is the diagram obtained by adding one vertex to the Dynkin diagram of a simple finite-dimensional Lie algebra, and its Weyl group is also the affine Weyl group of a simple finite-dimensional Lie algebra. All these had already appeared within the framework of theories of simple finite-dimensional Lie algebras, so these were not new. The character formulae of affine Lie algebras have the same form as in the case of simple finite-dimensional Lie algebras. Nevertheless, as soon as character formulae of highest weight modules over affine Lie algebras were discovered, the representation theories of affine Lie algebras made great progress, which one could not imagine from the case of simple finite-dimensional Lie algebras.

The fact that an affine Lie algebra is the infinite-dimensional extension of a simple finite-dimensional Lie algebra as a loop algebra over the circle S^1 gives rise to the following remarkable features.

i. An affine Lie algebra contains an infinite-dimensional Heisenberg Lie algebra as subalgebra; namely, a Cartan subalgebra of a simple finite-dimensional Lie algebra grows up to be a Heisenberg Lie algebra by the central extension of its loop algebra. In this way, a Heisenberg Lie algebra gets into an affine Lie algebra.
ii. The vector fields on S^1 act on the derived subalgebra of an affine Lie algebra as derivations.

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Preface

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ii. The vector fields on S^1 act on the derived subalgebra of an affine Lie algebra as derivations.

(ii) is the reason why an affine Lie algebra works harmoniously with the Virasoro algebra. One might say that the above two features are the reason why representation theories of affine Lie algebras are so important.

As is well known, the irreducible representation of a Heisenberg algebra is equivalent to the Schrödinger representation on a space of functions. To construct the action of an affine Lie algebra in practice, one needs vertex operators in order to describe the action of elements not in the Heisenberg subalgebra, and by them representation theories of an affine Lie algebra become much richer than those of a Heisenberg algebra. One can see from an affine root system which sort of and how many vertex operators are necessary to construct a representation of an affine Lie algebra, and a character formula tells us how one can decompose a module over an affine Lie algebra as modules over a Heisenberg Lie algebra contained in the affine Lie algebra.

The Schrödinger representation of a Heisenberg algebra consists of creation and annihilation operators, and the former act freely. Therefore a representation of an affine Lie algebra contains free action with respect to that of the creation operators of a Heisenberg subalgebra. This will be the key of the modular invariance of characters of an affine Lie algebra. In this sense, I think that the heroes (behind the scenes) of the representation theories of affine Lie algebras are a Heisenberg Lie algebra and vertex operators.

But without such arguments, one can derive useful identities, called the Macdonald identities, by fiddling around with a character formula as an algebraic formula. When Macdonald [**Mac**] discovered the Macdonald identities by considering affine root systems, the "mysterious factors" arising as a stumbling block in the formulae are in fact the factors that have their origin in the free action of the creation operators of a Heisenberg Lie algebra.

What is interesting in an affine Lie algebra is that it may be non-trivial to apply general formulae to some concrete cases. For example, writing the denominator identity for $\hat{\mathfrak{sl}}(2,\mathbb{C})$ we obtain the Jacobi triple product identity, and writing hierarchies for $\hat{\mathfrak{sl}}(2,\mathbb{C})$ we obtain the KdV equation and the non-linear Schrödinger equation. Further, a representation theory describes not only equations but also their solutions. Namely, if one finds a solution, one can construct other solutions successively by letting vertex operators act on it. Since the action on the set of solutions is transitive, one can start from the

easiest solution, i.e. the "trivial solution". But we do not touch upon these soliton equations in this book.

In any case, many topics in affine Lie algebras are interesting enough when we specialize them to the case of $\hat{\mathfrak{sl}}(2, \mathbb{C})$ and just look at them. And in such a way, one may feel that one understands the topics. Conversely, once we find a phenomenon for one of the simplest affine Lie algebras such as $\hat{\mathfrak{sl}}(2, \mathbb{C})$, we can lift it up to general affine Lie algebras by making a fair copy of it in terms of Lie algebras, and similar phenomena will be mass-produced "at one stroke". Even if it is said that we must not do things "at one stroke" in these days, this "stroke" is quite exciting!

Thus in representation theories of affine Lie algebras, the basic tools for any purpose are character formulae, and we can say that they are the starting point of the representation theories.

So in this book, at first I will state the structure of Lie algebras (i.e. their root systems) and character formulae in Chapter 2. In this chapter, everything will be treated in a quite general setting, since one can expect that these facts will be developed and applied further. From Chapter 3 on, I will explain the modular invariance of the characters of affine Lie algebras. This is one of the most picturesque scenes, and I will sometimes pause to illuminate it further by concrete examples.

There is a book by Kac [**K4**] on Kac-Moody Lie algebras. Although the first edition was published 16 years ago, it still seems to me that this is the best book, both as an introductory book and as a side book. But I have heard that it is hard to start one's study with this book, because it begins from the theory of generalized Cartan matrices. Hence I started writing the present book as a guidebook to Kac's book. I hope this book will be an *hors d'œuvre* to the great feast of infinite-dimensional Lie algebras.

This book is not an encyclopedia. It does not contain everything that could be written, or even everything important, about infinite-dimensional Lie algebras, but only the facts needed for my development of the theory.

To ease the reader's way through this book, I will explain the derivation of each formula in full detail. The reader may mostly just follow along with his or her eyes, taking time out for occasional calculations. I can comfortably read this book like a novel. The facts appearing in mathematical nature are more beautiful and mysterious

than any fiction. I am anxious about whether I have been able to depict them successfully.

Prof. Michio Jimbo encouraged me to write this book, and moreover he advised me, politely but strongly, on the manuscript. I would like to express my deepest gratitude to him. Dr. Kenji Iohara read through the manuscript, and pointed out and corrected lots of mistakes. Dr. Hiroyuki Tagawa has constructed the TEX environment in my office from the very beginning, and moreover he not only taught me how to use TEX and to draw figures personally but also produced some of the figures in this book himself. I also would like to thank the editors, who gave me much valuable advice on the description and on the contents, and those in the Iwanami publishing house who took care of me until I could complete the manuscript of this book.

The computation of the partition numbers shown in an example in §4.2 is carried out by the mathematical system "Reduce 3.6". I first became acquainted with "Reduce" in February 1986 when I had some computations to do. Thanks to the instruction of Prof. Seigo Okamoto, "Reduce" immediately became an important supplementary tool for my research. In addition, it was Prof. Okamoto who introduced me to Kac-Moody Lie algebras. I would like to express my heartiest gratitude to him for his great care.

Finally, I would like to express my appreciation to both of my parents, who let me achieve my dream of studying mathematics.

With gratitude to my wife, Yasuko

Minoru Wakimoto

Preface to the English Edition

This book was originally written in Japanese to provide an outlook and to serve as a guide to the theory of infinite-dimensional Lie algebras, which has grown up in quite recent decades in connection with various areas in mathematics and mathematical physics.

I tried to give an exposition with much detailed explanation, so that the theory may become more familiar to readers, though the topics treated in this book are quite limited for want of space. It will greatly please me if this book may raise its readers' interest in this area and be a help for further research and development of the theory.

For the publishing of this English edition, I thank Iwanami Shoten Publishers, the editors of the AMS, the translator and, in particular, Professor Katsumi Nomizu for a lot of kind attention.

July 2000, in Fukuoka

Overview

When we consider the structure of a simple finite-dimensional Lie algebra $\mathfrak{g}$, we first choose a Cartan subalgebra and make the root space decomposition. Next we classify the roots into positive and negative ones. The minimal roots among the positive roots with respect to the decomposition (that is, the positive roots which cannot be decomposed into a sum of roots) are called **simple roots**. Denote them by $\alpha_1, \cdots, \alpha_l$. Then the matrix of inner products $\left(\frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}\right)$ is called the **Cartan matrix** of $\mathfrak{g}$. To reconstruct a Lie algebra from the Cartan matrix, we have only to impose conditions that can be extracted from the Cartan matrix on the free Lie algebra generated by $2l$ letters, say $e_1, \cdots, e_l; f_1, \cdots, f_l$. In this way, we can reproduce the original Lie algebra, and we have the one-to-one correspondence:

$$\text{Simple Finite-Dimensional Lie Algebras} \quad \longleftrightarrow \quad \text{Cartan Matrices.}$$

Namely, if two Lie algebras are isomorphic, then their Cartan matrices are the same; otherwise their Cartan matrices differ. Therefore the simple finite-dimensional Lie algebras are classified in terms of their Cartan matrices. This is the well-known Cartan-Killing theory, and it appeared about one hundred years ago. But after that, the Cartan-Killing theory on simple finite-dimensional Lie algebras fell asleep peacefully as a completed theory, i.e. as a "theory for textbooks".

But after two thirds of this century has passed, two princes suddenly approached this "Sleeping Beauty". The outrageous princes who kissed the forbidden lips were Kac and Moody. Since Cartan matrices classify the simple finite-dimensional Lie algebras,

> Relaxing conditions on Cartan matrices a bit, one might be able to obtain infinite-dimensional Lie algebras!

We call such modified Cartan matrices generalized Cartan matrices (GCM for short). A new class of infinite-dimensional Lie algebras,

i.e., **Kac-Moody Lie algebras**, was born. From its origin, one can argue similarly as in the case of finite dimension. What is necessary to explicate its structure is a representation theory of $\mathfrak{sl}(2, \mathbb{C})$ only. Furthermore, when a GCM is symmetrizable, the corresponding Lie algebra possesses an inner product, and one can define the Casimir element. As an application, character formulae can be proved. They have the same form as in the finite-dimensional case. The points that are different from the finite-dimensional cases are

i. the Weyl group becomes an infinite group, and
ii. imaginary roots appear.

Nonetheless, these two facts have great consequences.

The most lovely class of infinite-dimensional Lie algebras is the affine Lie algebras. An affine Lie algebra is the central extension of the loop algebra of a simple finite-dimensional Lie algebra, so we know the structure of its root system very well, and its Weyl group is the semi-direct product of the Weyl group of a simple finite-dimensional Lie algebra and a $\mathbb{Z}$-lattice. In the character formulae the summation over the $\mathbb{Z}$-lattice gives us theta functions, and hence the character of integrable representations over an affine Lie algebra is a modular function. I will explain this in Chapter 4, and in Chapter 5 I will explain fusion coefficients as a topic related to the transformation matrices of modular transformation of the characters.

After this brief awakening, the 'Beauty' went to sleep again. This 'Beauty' sleeps well. The next prince who disturbed her sleep was Borcherds. Anyone who is woken from a first sleep will be offended. In addition, the task of hauling Cartan matrices in is so daring that it might change the properties of simple roots. But the 'Beauty' is also sweet-tempered. She never loses her temper. Hence Borcherds built up a fine theory.

Needless to say, Borcherds started his research on infinite-dimensional Lie algebras not from the viewpoint of Cartan matrices. He discovered the vertex operator algebras from his research on representations of the Monster simple group. From the simple roots of the infinite-dimensional Lie algebra associated to a vertex operator algebra, he had an insight into the fact that the following situation is significant:

> We do not have to mind even if some strangers (i.e. simple imaginary roots) mix up within the set of simple roots.

A Cartan matrix allowing the existence of such simple roots is called a **GKM matrix** (generalized Kac-Moody matrix), and the corresponding infinite-dimensional Lie algebra is called a **GKM Lie algebra**. Even in this setting, analogous arguments work well to prove character formulae. In this case, the forms of character formulae are not entirely the same as before, but slightly more complicated, and they have a new effect. The formula obtained by specializing character formulae to the trivial representation is called the **denominator identity**. This is a formula of the form "Infinite Product = Infinite Sum". Since imaginary simple roots create a new environment in the case of GKM Lie algebras, their denominator identities show us new equalities for a much wider class of modular functions than those arising from Kac-Moody Lie algebras.

But concerning GKM algebras, the only formula that has some applications up to now is the denominator identity, and the character formulae themselves of the other integrable representations have not been studied yet. Character formulae are too good to apply only to the trivial representation. I would like to see some nice applications.

In this book, for the reason to be explained in §1.4, we use the term BKM algebra instead of GKM algebra. The character formulae for BKM Lie algebras will be important for applications. For this purpose, in Chapter 2, I will describe in detail the root system and the character formulae for BKM algebras and BKM superalgebras in a general setting.

Therefore the main theme of this book are in Chapter 2, "Character formulae for BKM Lie superalgebras", and in Chapter 4, "Modular transformation of characters of affine Lie algebras". Representation theories of affine Lie algebras have applications to many fields. One could explain each topic little by little, but I have preferred to restrict myself to the theme on modular transformations of affine Lie algebras after Chapter 4. But when I looked back after writing up till Chapter 5, I felt something was insufficient. For example, without any description of how one can use the matrix elements of modular transformations presented as examples in Chapter 4, they are simply enumerations of data and may not show the readers their beauty. After I began writing the manuscript of this book, for a while, I was going to omit the Virasoro algebra, because of the rich variety of affine Lie algebras. Representation theories of the Virasoro algebra are so ample that I cannot mention them briefly. It is impossible to fit them into this book. But representation theories of affine Lie

algebras without the Virasoro algebra became dreary, like stale beer. So Chapter 6 was added, to give a brief sketch of how representation theories of the Virasoro algebra work (the chapter entitled "In Lieu of Postscript").

Although representation theories of affine Lie algebras are related to several fields, I cannot write about them, owing to the lack of space and of my knowledge. These other fields include, in no particular order, conformal field theory and KZ equations, lattice models, soliton equations, and so on. The first two are covered in the book *Field Theory and Topology* in this series.

It seemed to me that Prof. Ryogo Hirota's use of the theory of bilinear differential operators to find the exact solutions is a marvelous method, like magic. Through investigating it, the group of Prof. Mikio Sato, Prof. Etsuro Date, Prof. Michio Jimbo, Prof. Masaki Kashiwara and Prof. Tetsuji Miwa in the Research Institute of Mathematical Science, Kyoto University, discovered that the symmetry hidden inside the non-linear differential equations called soliton equations is an affine Lie algebra. In May 1982, when the lecture course taught by Kac for the spring semester drew to a close, their theory was introduced in his lecture as a "Work of Japanese School". I got into the lecture-hall without permission or payment for this lecture, and the theory developed on the board was so beautiful that my heart burst into flames. Everyone who is alone in a foreign country will be nationalistic. In addition I am timid. It was the Kyoto school that gave me the motivation to leap at infinite-dimensional Lie algebras, forgetting everything I had been working on before. There are some explanations of the theory, e.g. in [**Sa**].

CHAPTER 1

Opening

In this chapter, we will prepare some notions to be used in the rest of this book. The simplest simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ will be an important tool to investigate the structures of infinite-dimensional Lie algebras in Chapter 2. §1.3 is the place to warm up, and we will study the structures of simple finite-dimensional Lie algebras by using the S-triple, i.e. $\mathfrak{sl}(2, \mathbb{C})$, and explain the theory which leads to the Cartan matrices and the Dynkin diagrams. The Cartan matrices set the classification of simple finite-dimensional Lie algebras. Deforming conditions on the Cartan matrices of simple finite-dimensional Lie algebras, one may be led to infinite-dimensional Lie algebras. The fact that the Cartan matrices include enough information on structures and on properties of Lie algebras was one of the backgrounds from which the theory of Kac-Moody Lie algebras emerged so beautifully in the latter half of the twentieth century. Although there are many unknown features of Lie superalgebras, in §1.5 we will briefly see some fundamental facts on them.

1.1. Basic Concepts of Lie Algebras, etc.

In this book, a vector space is always assumed to be defined over the complex number field $\mathbb{C}$, unless otherwise stated. Although exceptionally I may consider a vector space over the real number field $\mathbb{R}$, the only cases are the space spanned by roots $\sum_i \mathbb{R}\alpha_i$ and the space spanned by coroots $\sum_i \mathbb{R}\alpha_i^\vee$ when we have to treat the positivity of the inner product and so forth. When we say "infinite dimension" it always means "countably infinite dimension", and we never treat the case of dimension higher than the countably infinite dimension in this book.

DEFINITION 1.1. A vector space $\mathfrak{g}$ (possibly of infinite dimension), in addition to the structure as a vector space (the addition and

the scalar multiple), having an operation called the bracket product:

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto [x, y] \in \mathfrak{g}$$

satisfying the following properties, is called a **Lie algebra**.

1. (Bilinearity) $[\ ,\ ]$ is linear in each component. Namely,

$$[ax + bx', y] = a[x, y] + b[x', y],$$
$$[x, ay + by'] = a[x, y] + b[x, y'].$$

2. (Skew-symmetry) $[x, y] = -[y, x]$.
3. (Jacobi's Identity) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

In the above conditions, x, y, z, x', y' are elements of $\mathfrak{g}$ and a, b are complex numbers. Moreover, under the condition (2), to ensure bilinearity (1) it is enough to impose linearity on just one of the two components. If the dimension of $\mathfrak{g}$ as a vector space is n, we say that $\mathfrak{g}$ is an n-dimensional Lie algebra.

The bracket product $[x, y]$ is sometimes called the **commutator** of x and y. Let $\mathfrak{g}$ be a Lie algebra and A, B subsets of $\mathfrak{g}$. We denote the vector subspace of $\mathfrak{g}$ spanned by $\{[x, y]; x \in A, y \in B\}$ by $[A, B]$.

DEFINITION 1.2. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{p}, \mathfrak{i}$ be vector subspaces of $\mathfrak{g}$.

1. If $\mathfrak{p}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$, $\mathfrak{p}$ itself becomes a Lie algebra (with the bracket inherited from $\mathfrak{g}$). In this case, $\mathfrak{p}$ is called a **Lie subalgebra** of $\mathfrak{g}$.
2. if $\mathfrak{i}$ satisfies $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$, $\mathfrak{i}$ is called an **ideal** of $\mathfrak{g}$.

$\mathfrak{g}$ itself, and the 0-dimensional vector subspace $\{0\}$ consisting of 0 are ideals of $\mathfrak{g}$. These are called the **trivial** ideals. An ideal that is not trivial is called a non-trivial ideal, and an ideal that is not $\mathfrak{g}$ itself is called a proper ideal.

When $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, the quotient vector space $\mathfrak{g}/\mathfrak{i}$ becomes a Lie algebra by the bracket

$$[\overline{x}, \overline{y}] := \overline{[x, y]}.$$

Here $\overline{x}, \overline{y}$ are elements of $\mathfrak{g}/\mathfrak{i}$ and $x, y \in \mathfrak{g}$ are their respective representatives. This Lie algebra is called the **quotient Lie algebra** (of $\mathfrak{g}$ by $\mathfrak{i}$). In particular,

$$\mathfrak{z} := \{x \in \mathfrak{g}; [x, \mathfrak{g}] = 0\}$$

is an ideal of $\mathfrak{g}$. $\mathfrak{z}$ is called the **center** of $\mathfrak{g}$. In the case when $\mathfrak{z} = \{0\}$, we say that "$\mathfrak{g}$ has no center" or "$\mathfrak{g}$ is centerless". Notice that the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$ of $\mathfrak{g}$ by its center is not necessarily centerless.

DEFINITION 1.3. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ between two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ satisfying

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

for all $x, y \in \mathfrak{g}$ is called a **homomorphism** of these Lie algebras. Furthermore, if φ is bijective, it is called an **isomorphism** of these Lie algebras.

Here, although the brackets of the two sides in the above formulae are those of $\mathfrak{g}, \mathfrak{g}'$ respectively, I used the same notation $[\ ,\ ]$. Hereafter, I will skimp this part of my job to avoid an unnecessary flood of symbols, if no confusion seems likely to result.

DEFINITION 1.4. In particular, if $\mathfrak{g} = \mathfrak{g}'$, then a homomorphism and an isomorphism from the Lie algebra $\mathfrak{g}$ to $\mathfrak{g}$ are called an **endomorphism** and an **automorphism**, respectively.

If $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras, then

$$\mathrm{Ker}(\varphi) := \{x \in \mathfrak{g}; \varphi(x) = 0\}$$

is an ideal of $\mathfrak{g}$, and φ induces a homomorphism of Lie algebras

$$\overline{\varphi} : \mathfrak{g}/\mathrm{Ker}(\varphi) \longrightarrow \mathfrak{g}'.$$

If $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$ are Lie algebras, their direct sum as vector spaces

$$\mathfrak{g} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \{(x_1, \cdots, x_n); x_i \in \mathfrak{g}_i \ (i = 1, \cdots n)\}$$

becomes a Lie algebra via the bracket

$$[(x_1, \cdots, x_n), (y_1, \cdots, y_n)] := ([x_1, y_1], \cdots, [x_n, y_n]).$$

$\mathfrak{g}$ is called the **direct sum** of the Lie algebras $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$. In this case, each $\mathfrak{g}_i$ becomes an ideal via the embedding

$$\varphi_i : \mathfrak{g}_i \ni x \longmapsto (0, \cdots, \overset{i}{\breve{x}}, \cdots, 0) \in \mathfrak{g}.$$

DEFINITION 1.5. For a finite-dimensional Lie algebra $\mathfrak{g}$, we define the notions simple, semisimple and reductive as follows:

1. $\mathfrak{g}$ is called a **simple Lie algebra** if its dimension is greater than 1[1] and it has no proper ideal.

[1] It is easy to see that a 2-dimensional Lie algebra always has a 1-dimensional ideal. Hence this condition is equivalent to saying that its dimension is greater than or equal to 3.

2. A finite direct sum of simple Lie algebras is called a **semisimple Lie algebra**.
3. $\mathfrak{g}$ is called a **reductive Lie algebra** if it is a direct sum of its center and a semisimple Lie algebra.

Now, let us show some examples.

EXAMPLE 1.6. Any vector space $\mathfrak{g}$ becomes a Lie algebra by the bracket $[x, y] = 0$ (for all $x, y \in \mathfrak{g}$). When this happens, $\mathfrak{g}$ is called a **commutative Lie algebra** or an **abelian Lie algebra**.

The reader might feel that such a Lie algebra is worthless, but in general, for a Lie algebra $\mathfrak{g}$, an abelian Lie subalgebra of $\mathfrak{g}$ plays an important role when one studies structures and representations of $\mathfrak{g}$.

DEFINITION 1.7. The Lie algebra spanned over $2n + 1$ letters $u_1, \cdots, u_n, v_1, \cdots, v_n, z$,

$$\mathfrak{g} := \bigoplus_{i=1}^{n} \mathbb{C}u_i \oplus \bigoplus_{i=1}^{n} \mathbb{C}v_i \oplus \mathbb{C}z,$$

with the bracket defined by

$$[u_i, v_j] = -[v_j, u_i] = \delta_{i,j} z, \qquad [\mathfrak{g}, z] = 0,$$

is called the $2n + 1$-dimensional **Heisenberg algebra**. $\mathbb{C}z$ is its center.

DEFINITION 1.8. Let A be a ring with unit 1 of multiplication. Moreover, assume that the scalar multiple of $\mathbb{C}$ on A is defined and A is a vector space under this scalar multiplication and its addition. Then A is called an **associative algebra**[2] (over the complex number field $\mathbb{C}$).

For an associative algebra A, we can regard $\mathbb{C} \subset A$ by identifying $\mathbb{C}1$ with $\mathbb{C}$.

EXAMPLE 1.9. An associative algebra $\mathbb{C}$ becomes a Lie algebra via the bracket

$$[a, b] := ab - ba \quad (a, b \in A).$$

If A is commutative as an associative algebra, then it is also commutative as a Lie algebra.

[2]By the definition, we exclude the ring $\{0\}$, consisting of 0 only, from associative algebras in this book.

The following lemma (which follows immediately from the definition) is quite useful.

LEMMA 1.10. *For elements a, b, c of an associative algebra, the following formula holds:*

$$[a, bc] = [a, b]c + b[a, c].$$

EXAMPLE 1.11. For a natural number n, the set of all $n \times n$ complex matrices $M_n(\mathbb{C})$ is an associative algebra under the addition and the product of matrices, and is a Lie algebra with the bracket

$$[X, Y] := XY - YX \qquad (X, Y \in M_n(\mathbb{C})).$$

We denote this Lie algebra by $\mathfrak{gl}(n, \mathbb{C})$. The subspace

$$\mathfrak{sl}(n, \mathbb{C}) := \{X \in M_n(\mathbb{C}); \mathrm{tr}(X) = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. If we denote the unit matrix of $M_n(\mathbb{C})$ by I_n, then $\mathbb{C}I_n$ is the center of $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{C})$ decomposes as a Lie algebra as follows:

$$\mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}I_n \oplus \mathfrak{sl}(n, \mathbb{C}).$$

The fact is that $\mathfrak{sl}(n, \mathbb{C})$ for $n \geq 2$ is a simple Lie algebra, and hence $\mathfrak{gl}(n, \mathbb{C})$ is reductive.

EXAMPLE 1.12. For a vector space V (possibly of infinite dimension), the vector space $\mathrm{End}(V)$ of all linear transformations on V is an associative algebra whose multiplication is defined by the composition of maps $f \circ g$ and is a Lie algebra with the bracket

$$[f, g] := f \circ g - g \circ f.$$

If V is of finite dimension, then by choosing a basis $e_1, \cdots, e_n$ of V, each element f of $\mathrm{End}(V)$ can be expressed in matrix form as follows: For

$$f(e_j) = \sum_{i=1}^{n} a_{i,j} e_i, \tag{1.1}$$

the map

$$\mathrm{End}(V) \ni f \longmapsto (a_{i,j})_{i,j=1,\cdots,n} \in \mathfrak{gl}(n, \mathbb{C})$$

is an isomorphism of Lie algebras.

In this book, for a finite-dimensional vector space V with basis $e_1, \cdots, e_n$, we define the matrix form of a linear transformation f on V by (1.1).

DEFINITION 1.13. Let $\mathfrak{g}$ be a Lie algebra, let V be a vector space, and let $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ be a homomorphism of Lie algebras. (π, V) is called a **representation** of $\mathfrak{g}$ on V, and V is called the **representation space** of the representation (π, V).

In other words, a representation is a map

$$\mathfrak{g} \times V \ni (x, v) \longmapsto \pi(x)v \in V$$

satisfying certain conditions. The important condition is

$$\pi([x, y])v = [\pi(x), \pi(y)]v \qquad (x, y \in \mathfrak{g}, v \in V). \tag{1.2}$$

Since the right hand side of this formula is the bracket in $\mathrm{End}(V)$, this formula has the following equivalent form:

$$\pi([x, y])v = \pi(x)\pi(y)v - \pi(y)\pi(x)v \qquad (x, y \in \mathfrak{g}, v \in V). \tag{1.3}$$

If (π, V) is a representation of $\mathfrak{g}$, V is called a **$\mathfrak{g}$-module** . Abbreviating π, we may write $x \cdot v$ or xv instead of $\pi(x)v$. In this case, (1.3) can be rewritten as

$$[x, y]v = (xy - yx)v \qquad (x, y \in \mathfrak{g}, v \in V). \tag{1.4}$$

But xy and yx in the right hand side of this formula are the product not of elements of $\mathfrak{g}$, but of elements of $\mathrm{End}(V)$. In addition to the fact that (1.4) is an abbreviated form omitting π, the product xy does not *exist* in $\mathfrak{g}$.

On the whole, today is an age of 'changing one's attitude'. Although in society, it seems that usually this 'changing one's attitude' may not lead to good consequences, in mathematics it may happen that an idea freed from restraints will take a new turn. Let us grant citizenship to xy as a product of elements of $\mathfrak{g}$. That is, we rather positively change our attitude, and allow xy and yx in the right hand side of (1.4) as products of elements of $\mathfrak{g}$. But for this purpose, we have to build up a residence for these "new objects". First, take any (finite) number of elements of $\mathfrak{g}$ and consider their formal product (or, better, simply their 'arrangement')

$$x_1 \cdots x_n \quad (x_i \in \mathfrak{g}).$$

Suppose any number of elements in $x_1, \cdots, x_n$ can coincide. Assume that $n = 0$ is allowed, and we define the product of 0 elements to be 1. Consider the vector space of all finite linear combination of such

elements. Here, we allow the following operations of scalar multiplication and addition:

$$\text{(i)} \qquad (a_1x_1)\cdots(a_nx_n) = (a_1\cdots a_n)\cdot(x_1\cdots x_n),$$

$$\text{(ii)} \quad x_1\cdots x_{i-1}(x_i + x_i')x_{i+1}\cdots x_n = x_1\cdots x_i\cdots x_n + x_1\cdots x_i'\cdots x_n.$$

Here $x_j, x_i' \in \mathfrak{g}$ and a_j are complex numbers ($1 \leq j \leq n$). The condition (i) is equivalent to saying that when one wants to multiply $x_1\cdots x_n$ by a scalar, one can just multiply any one of the x_i's by the scalar. Namely, it is equivalent to the following condition (i)$'$:

$$\text{(i)}' \qquad a\cdot(x_1\cdots x_n) = x_1\cdots x_{i-1}\cdot(ax_i)x_{i+1}\cdots x_n.$$

Here i can be chosen from $1,\cdots,n$ arbitrarily. On this vector space, one can define the multiplication of two elements by

$$(x_1\cdots x_m)\cdot(y_1\cdots y_n) := x_1\cdots x_m y_1\cdots y_n.$$

In the right hand side, we simply set elements of $\mathfrak{g}$ in array. Surely, one has to check the compatibility of this operation and the above (i) and (ii), but I would like to leave it as an exercise to the reader.

By this multiplication rule, this vector space becomes an associative algebra, and moreover we assume that the following computation works:

$$\text{(iii)} \qquad uxyv - uyxv = u\cdot[x,y]\cdot v.$$

Here u, v are elements of this associative algebra and $x, y \in \mathfrak{g}$. Namely, since the left hand side of (iii) is an element of our associative algebra, we can rewrite it as $u(xy-yx)v$. The condition (iii) says that we can replace this middle term $xy - yx$ by the element $[x,y]$ of the Lie algebra $\mathfrak{g}$. Therefore, the condition (iii) is equivalent to requiring the following simple condition on a product of two elements of $\mathfrak{g}$:

$$\text{(iii)}' \qquad xy - yx = [x,y] \qquad (x, y \in \mathfrak{g})$$

without u and v.

Imposing the condition repeatedly in this unprincipled way, one may worry, "*Can we really do such things?*" One might worry whether there are some contradictions among the imposed formulae, but everything works well. The process of construction I mentioned above can be justified by first taking the universal algebraic system (in this case an associative algebra) and then taking the quotient by an appropriate ideal. The conditions (i) and (ii) mean to construct the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$, and the condition (iii) means the operation of taking its quotient by the bi-ideal generated by

$$x\otimes y - y\otimes x - [x,y] \qquad (x, y \in \mathfrak{g}).$$

The resulting algebraic system (that is, the associative algebra admitting the operations (i),(ii) and (iii)), is called the **universal enveloping algebra** of $\mathfrak{g}$ and is denoted by $\mathfrak{U}(\mathfrak{g})$. The formal product xy of elements x, y of $\mathfrak{g}$ thus acquires citizenship as an element of $\mathfrak{U}(\mathfrak{g})$. In this case, a representation (π, V) of $\mathfrak{g}$ induces a homomorphism of associative algebras $\tilde{\pi} : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$ via

$$\tilde{\pi}(x_1 \cdots x_n) := \pi(x_1) \cdots \pi(x_n).$$

In this way, it is equivalent to consider a representation of $\mathfrak{g}$ and a representation of $\mathfrak{U}(\mathfrak{g})$. Hereafter, we use the same symbol π to denote the representation of $\mathfrak{U}(\mathfrak{g})$ obtained by lifting up of π, if no confusion seems likely.

REMARK 1.14. When we regard an associative algebra A as a Lie algebra, we have to be careful not to confuse whether the product $a_1 \cdots a_n$ of elements $a_1, \cdots, a_n$ of A is a product in A or a product in the universal enveloping algebra $\mathfrak{U}(A)$ of the Lie algebra A. But since the identity map from A to A, regarded as a map from the Lie algebra A to the associative algebra A, can naturally be extended to the homomorphism p from $\mathfrak{U}(A)$ onto A as associative algebras, we can transfer an element of $\mathfrak{U}(A)$ to an element of A by this map p. Namely, we can *think* of an element of $\mathfrak{U}(A)$ as an element of A. But since $\mathfrak{U}(A)$ and A are not isomorphic, this '*thinking*' is not an *identification* but a lonely *unrequited love*.

In general, a homomorphism of Lie algebras $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ can be lifted to a homomorphism of associative algebras $\hat{\varphi} : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g}')$. If (π, V) is a representation of $\mathfrak{g}$, we construct the map $\tilde{\pi}$ as above; that is, the lift of π to $\mathfrak{U}(\mathfrak{g})$. It is the composition of

$$\mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathrm{End}(V)) \longrightarrow \mathrm{End}(V);$$

hence it is a homomorphism from $\mathfrak{U}(\mathfrak{g})$ to $\mathrm{End}(V)$ as associative algebras, and is not regarded as a map to $\mathfrak{U}(\mathrm{End}(V))$.

Note. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a commutative Lie algebra $\mathfrak{g}$ is nothing but the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. If a Lie algebra $\mathfrak{g}$ is not commutative, $S(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g})$ are isomorphic as vector spaces via **symmetrization**:

$$S(\mathfrak{g}) \ni x_1 \cdots x_n \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \in \mathfrak{U}(\mathfrak{g}),$$

but this is not an isomorphism as associative algebras. Here, $\mathfrak{S}_n$ is the group of all permutations of numbers $1, \cdots, n$ (n-th symmetric group). $\square$.

NOTATION 1.15. Let $A_1, \cdots, A_k$ be associative subalgebras of $\mathfrak{U}(\mathfrak{g})$. We denote the vector subspace of $\mathfrak{U}(\mathfrak{g})$ spanned over $\{u_1 \cdots u_k; u_i \in A_i \ (1 \leq i \leq k)\}$ by $A_1 \cdots A_k$. For an associative subalgebra A of $\mathfrak{U}(\mathfrak{g})$ and a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$, we denote the vector subspaces of $\mathfrak{U}(\mathfrak{g})$ spanned over $\{Xu; X \in \mathfrak{m}, u \in A\}$ and $\{uX; u \in A, X \in \mathfrak{m}\}$ by $\mathfrak{m}A$ and $A\mathfrak{m}$ respectively.

In $\mathfrak{U}(\mathfrak{g})$, one cannot introduce the degree of each element, since a "degree 2 element" can be a "degree 1 element" at the same time by (iii)$'$. Instead, for each $k \in \mathbb{Z}_{\geq 0}$, we introduce the vector subspace of "elements of degree less than or equal to k". Namely, let $\mathfrak{U}(\mathfrak{g})_k$ be the vector subspace of $\mathfrak{U}(\mathfrak{g})$ spanned over $\{x_1 \cdots x_j; 0 \leq j \leq k, x_1, \cdots, x_j \in \mathfrak{g}\}$. In this way, one introduces a filtration on $\mathfrak{U}(\mathfrak{g})$:

$$\mathbb{C} = \mathfrak{U}(\mathfrak{g})_0 \subset \mathfrak{U}(\mathfrak{g})_1 \subset \mathfrak{U}(\mathfrak{g})_2 \subset \cdots \subset \mathfrak{U}(\mathfrak{g})_k \subset \mathfrak{U}(\mathfrak{g})_{k+1} \subset \cdots,$$

$$\mathfrak{U}(\mathfrak{g}) = \bigcup_{k=o}^{\infty} \mathfrak{U}(\mathfrak{g})_k.$$

In particular, we have $\mathfrak{U}(\mathfrak{g})_1 = \mathbb{C} \oplus \mathfrak{g}$. When we work on problems concerning $\mathfrak{U}(\mathfrak{g})$, induction on k with respect to this filtration sometimes works effectively. The following theorem, called the **Poincaré-Birkhoff-Witt theorem** (**PBW theorem** for short), has very important applications. (1) can be easily proved by induction on k, and (2), (3) follow immediately from (1).

THEOREM 1.16. *Let $\mathfrak{g}$ be a Lie algebra (possibly of infinite dimension).*

1. *Let $x_1, x_2, \cdots$ be a basis of $\mathfrak{g}$. Then $\{x_1^{m_1} x_2^{m_2} \cdots; m_i \in \mathbb{Z}_{\geq 0}$, all but a finite number of m_i's are $0\}$ is a basis of $\mathfrak{U}(\mathfrak{g})$.*
2. *For Lie subalgebras $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$, satisfying $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ (as a vector space), we have*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_1) \cdots \mathfrak{U}(\mathfrak{g}_n).$$

In particular, this decomposition induces an isomorphism

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}_1) \otimes \cdots \otimes \mathfrak{U}(\mathfrak{g}_n)$$

as a vector space.

3. *For Lie subalgebras* $\mathfrak{h}, \mathfrak{m}, \mathfrak{n}$ *satisfying* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{n}$ *(as a vector space), we have*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{m}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}).$$

DEFINITION 1.17. Let (π, V) be a representation of $\mathfrak{g}$. If a vector subspace U of V satisfies

$$\pi(x)u \in U \qquad (x \in \mathfrak{g}, u \in U),$$

U is called an **invariant subspace** of V. In this case:

1. For each $x \in \mathfrak{g}$, if we denote $\pi(x)$ regarded as a map from U to U, i.e., as a linear transformation on U, by $\pi|_U(x)$, then $(\pi|_U, U)$ is a representation of $\mathfrak{g}$ on U. This is called a **subrepresentation** of (π, V).
2. For each $x \in \mathfrak{g}$, $\pi(x)$ induces a linear transformation $\overline{\pi}(x)$ on V/U. This representation $(\overline{\pi}, V/U)$ is called a **quotient representation**.

Let (π_i, V_i) $(1 \leq i \leq k)$ be representations of $\mathfrak{g}$ and set

$$V := V_1 \oplus \cdots \oplus V_k = \{(v_1, \cdots, v_k); v_i \in V_i \ (1 \leq i \leq k)\}.$$

By defining the action of $\mathfrak{g}$ on V by

$$\pi(x)(v_1, \cdots, v_k) := (\pi_1(x)v_1, \cdots, \pi_k(x)v_k),$$

we obtain a representation (π, V) of $\mathfrak{g}$. This representation is called the **direct sum** of the representations (π_i, V_i), and is denoted by

$$(\pi, V) = \bigoplus_{i=1}^{k} (\pi_i, V_i) = \left(\bigoplus_{i=1}^{k} \pi_i, \bigoplus_{i=1}^{k} V_i \right), \quad \text{etc.}$$

Here each V_i is an invariant subspace of V.

If (π, V) is a representation of $\mathfrak{g}$, then V itself and $\{0\}$ are invariant subspaces. They are called the **trivial invariant subspaces**. Any invariant subspace except for V itself is called a **proper invariant subspace**. If V contains no non-trivial invariant subspace, (π, V) is called an **irreducible representation** of $\mathfrak{g}$. A representation which is not irreducible is called **reducible**. If U is the maximal proper invariant subspace, $(\overline{\pi}, V/U)$ is an irreducible representation.

EXAMPLE 1.18. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. Defining the action of $\mathfrak{g}$ on V by

$$\pi(x)v = 0 \qquad (x \in \mathfrak{g}, v \in V),$$

we get a representation (π, V) of $\mathfrak{g}$. On this representation, any element x of $\mathfrak{g}$ acts on V as the 0-operator. This representation is called a **trivial representation**, and we say that $\mathfrak{g}$ acts on V trivially.

NOTATION 1.19. The **identity transformation** on a vector space V is denoted by Id_V.

EXAMPLE 1.20. Let $\mathfrak{g}$ be a $2n+1$-dimensional Heisenberg algebra as in Definition 1.7, and let $V = \mathbb{C}[x_1, \cdots, x_n]$ be the vector space of all polynomials in $x_1, \cdots, x_n$. Defining the action π of $\mathfrak{g}$ on V by

$$\begin{aligned}
&\pi(u_i) := \frac{\partial}{\partial x_i}, \\
&\pi(v_i) := x_i \times \mathrm{Id}_V \quad \text{(the multiplication of } x_i\text{)}, \\
&\pi(z) := \mathrm{Id}_V,
\end{aligned}$$

(π, V) is an (infinite-dimensional) irreducible representation of $\mathfrak{g}$. This representation of a Heisenberg algebra is called the **Schrödinger representation.**

NOTATION 1.21. Let (π, V) be a representation of $\mathfrak{g}$.

1. For each $x \in \mathfrak{g}$, set
$$V^x := \mathrm{Ker}\pi(x) = \{v \in V; \pi(x)v = 0\}.$$
2. Set $V^{\mathfrak{g}} := \bigcap_{x \in \mathfrak{g}} V^x$.

$V^{\mathfrak{g}}$ is an invariant subspace of V, and the restriction of the action of $\mathfrak{g}$ to $V^{\mathfrak{g}}$ is a trivial representation.

Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. If for every invariant subspace U there exists an invariant subspace U' satisfying $V = U \oplus U'$, the representation (π, V) is said to be **completely reducible**. As one can show easily, any subrepresentation of a completely reducible representation is again completely reducible. So if a finite-dimensional representation (π, V) of $\mathfrak{g}$ is completely reducible, one can repeat the procedure to choose the complement that is invariant under the action of $\mathfrak{g}$ and can decompose V as the direct sum of irreducible invariant subspaces

$$V = U_1 \oplus \cdots \oplus U_n.$$

For representations (π, V) and (π', V') of $\mathfrak{g}$, set

$$\mathrm{Hom}_{\mathfrak{g}}(V, V') := \{T : V \overset{linear}{\longrightarrow} V'; \pi'(x) \circ T = T \circ \pi(x) \ (x \in \mathfrak{g})\}$$

and call an element of $\mathrm{Hom}_{\mathfrak{g}}(V, V')$ an **intertwining operator** from V to V'. If T is an intertwining operator from V to V', $\mathrm{Ker}(T)$ and

$\mathrm{Im}(T)$ are invariant subspaces of V and V' respectively. In particular, for the case $V = V'$, we write $\mathrm{End}_{\mathfrak{g}}(V)$ instead of $\mathrm{Hom}_{\mathfrak{g}}(V, V)$.

DEFINITION 1.22. Let (π, V) and (π', V') be representations of a Lie algebra $\mathfrak{g}$. If there exists a bijection in $\mathrm{Hom}_{\mathfrak{g}}(V, V')$, these two representation is said to be **equivalent**.

For (π, V) a representation of $\mathfrak{g}$, one introduces a structure of the representation of $\mathfrak{g}$ on the dual space of V,

$$V^* := \{f : \overset{linear}{\longrightarrow} \mathbb{C}\},$$

as follows:

$$\langle \pi^*(x)f, v\rangle := -\langle f, \pi(x)v\rangle \qquad (x \in \mathfrak{g}, f \in V^*, v \in V). \tag{1.5}$$

Here, $\langle\ ,\ \rangle$ represents the natural pairing between V^* and V. This representation (π^*, V^*) is called the '**dual**' or the **'contragredient' representation** of (π, V).

Let us describe the above condition (1.5) in $\mathfrak{U}(\mathfrak{g})$. By applying the condition (1.5) repeatedly, we have for $X_1, \cdots, X_n \in \mathfrak{g}$:

$$\begin{aligned} \langle \pi^*(X_1 \cdots X_n)f, v\rangle &= \langle \pi^*(X_1) \cdots \pi^*(X_n)f, v\rangle \\ &= \langle f, (-\pi(X_n)) \cdots (-\pi(X_1))v\rangle \\ &= (-1)^n \langle f, \pi(X_n \cdots X_1)v\rangle. \end{aligned} \tag{1.6}$$

So, for $x = X_1 \cdots X_n$ $(X_1, \cdots, X_n \in \mathfrak{g})$, we put

$${}^t x := (-X_n) \cdots (-X_1) = (-1)^n X_n \cdots X_1. \tag{1.7}$$

The linear map from $\mathfrak{U}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g})$ defined by the above formula is called the **transposition map** of a universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. This map is characterized by the following three properties:

i. ${}^t a = a \quad$ (for $a \in \mathbb{C}$),
ii. ${}^t X = -X \quad$ (for $X \in \mathfrak{g}$),
iii. ${}^t(xy) = {}^t y {}^t x \quad$ (for all $x, y \in \mathfrak{U}(\mathfrak{g})$).

Namely, since the transposition map does not preserve the product but reverses its order, it is an anti-automorphism of an associative algebra.

Note. When an element of $\mathfrak{U}(\mathfrak{g})$ is expressed as a linear combination of products of elements of $\mathfrak{g}$, its representation is not unique. Therefore if we would like to define the transposed map by (1.7), we should check that this map is well-defined. In $\mathfrak{U}(\mathfrak{g})$, for $u, v \in \mathfrak{U}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$, we have

$$uXYv - uYXv = u[X, Y]v,$$

and this is the relation that is fundamental when we construct a universal enveloping algebra from the tensor algebra. Hence we have only to make sure that the application of the transposed map, defined by (1.7), on the two sides of the above formula is the same. This can be managed easily by the following computation:

$$\begin{aligned}
{}^t(uXYv - uYXv) &= {}^tv{}^tY{}^tX{}^tu - {}^tv{}^tX{}^tY{}^tu \\
&= {}^tv(-Y)(-X){}^tu - {}^tv(-X)(-Y){}^tu \\
&= {}^tv(YX - XY){}^tu = {}^tv[Y,X]{}^tu, \\
{}^t(u[X,Y]v) &= {}^tv{}^t[X,Y]{}^tu \\
&= {}^tv(-[X,Y]){}^tu = {}^tv[Y,X]{}^tu.
\end{aligned}$$

The transposed map is useful when one describes an invariant bilinear form, an invariant pairing, etc. □

Let (π, V) and (π', V') be representations of a Lie algebra $\mathfrak{g}$. Here we abbreviate π and π' and denote the action of $\mathfrak{g}$ on V and V' simply like xv and xv'. One can define the $\mathfrak{g}$-module structure on the tensor product $V \otimes V'$ of V and V', and on the vector space of all linear transformations $\mathrm{Hom}_{\mathbb{C}}(V, V')$ from V to V', by the following formulae:

$$\begin{aligned}
x(v \otimes v') &:= (xv) \otimes v' + v \otimes (xv'), \\
(xf)(v) &:= x(f(v)) - f(xv),
\end{aligned}$$

where $x \in \mathfrak{g}$, $f \in \mathrm{Hom}_{\mathbb{C}}(V, V')$, $v \in V$ and $v' \in V'$. Using the notation introduced in Notation 1.21, we have

$$\mathrm{Hom}_{\mathfrak{g}}(V, V') = \mathrm{Hom}_{\mathbb{C}}(V, V')^{\mathfrak{g}}.$$

If V is of finite dimension, then $\mathrm{Hom}_{\mathbb{C}}(V, V')$ and $V^* \otimes V'$ are isomorphic $\mathfrak{g}$-modules.

Let $\mathfrak{g}$ be a Lie algebra. Each element x of $\mathfrak{g}$ defines the element $\mathrm{ad}(x)$ of $\mathrm{End}(\mathfrak{g})$ as follows:

$$\mathrm{ad}(x)(y) := [x, y] \qquad (y \in \mathfrak{g}).$$

Then the map

$$\mathrm{ad} : \mathfrak{g} \ni x \longmapsto \mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$$

is a Lie algebra homomorphism by the Jacobi identity. This representation $(\mathrm{ad}, \mathfrak{g})$ is called the **adjoint representation** of $\mathfrak{g}$. In this terminology, an ideal of $\mathfrak{g}$ is an invariant subspace of the adjoint representation, and the condition on simple finite-dimensional Lie algebras can be rephrased as the requirement that the adjoint representation is irreducible.

EXAMPLE 1.23. The adjoint representation of a Heisenberg algebra is not completely reducible. Because even its center is an invariant subspace, there is no complement which is at the same time an invariant subspace.

The operator $\mathrm{ad}(x)$ $(x \in \mathfrak{g})$ also enjoys the following property:

DEFINITION 1.24. Let $\mathfrak{g}$ be a Lie algebra. D is called a **derivation** of $\mathfrak{g}$ if it satisfies

$$D([x,y]) = [Dx, y] + [x, Dy] \qquad (x, y \in \mathfrak{g}).$$

The space $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ forms a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$.

By the Jacobi identity, $\mathrm{ad}(x)$ is a derivation of $\mathfrak{g}$. $\mathrm{ad}(x)$ is called an **inner derivation** of $\mathfrak{g}$. The map

$$\mathrm{ad} : \mathfrak{g} \ni x \longmapsto \mathrm{ad}(x) \in \mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$$

is a Lie algebra homomorphism, and moreover, since

$$[D, \mathrm{ad}(x)] = \mathrm{ad}(Dx) \qquad (x \in \mathfrak{g}),$$

the image $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{g}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. Since $\mathrm{Ker}(\mathrm{ad})$ is the center $\mathfrak{z}$ of $\mathfrak{g}$, $\mathfrak{g}$ can be viewed as a subalgebra of $\mathrm{Der}(\mathfrak{g})$ if $\mathfrak{z} = \{0\}$ (i.e., $\mathfrak{g}$ has no center).

Now, if V is a finite-dimensional vector space, recall that for each $X \in \mathrm{End}(V)$, the infinite sum

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!} \tag{1.8}$$

is a well-defined regular transformation, i.e., an element of $GL(V)$. Namely, the right hand side of (1.8) converges, and e^{tX} $(t \in \mathbb{R})$ is a real analytic function.

PROPOSITION 1.25. *If D is a derivation of a finite-dimensional Lie algebra $\mathfrak{g}$, then e^D is an automorphism of $\mathfrak{g}$.*

PROOF. Since e^D is an automorphism of a vector space, we have only to check that

$$e^D[x,y] = [e^D x, e^D y].$$

Here we introduce a parameter $t \in \mathbb{R}$, and will show that

$$e^{tD}[x,y] = [e^{tD}x, e^{tD}y] \qquad (x, y \in \mathfrak{g}). \tag{1.9}$$

Let us differentiate each side of (1.9):

$$\begin{aligned}\frac{d}{dt}(\text{left hand side}) &= \frac{d}{dt}e^{tD}[x,y] = D(e^{tD}[x,y]) \\ &= D\ (\text{left hand side}), \\ \frac{d}{dt}(\text{right hand side}) &= \frac{d}{dt}[e^{tD}x, e^{tD}y] \\ &= [De^{tD}x, e^{tD}y] + [e^{tD}x, De^{tD}y] \\ &= D([e^{tD}x, e^{tD}y]) = D\ (\text{right hand side}).\end{aligned}$$

Hence each side of (1.9) (a function of t) is a solution of the differential equation

$$\frac{d}{dt}f(t) = D \cdot f(t)$$

and the initial value ($t = 0$) is $[x, y]$ for both sides. Now (1.9) follows from the uniqueness of the solution of a differential equation with given initial value. □

By this proposition,

$$\mathrm{Ad}(e^x) := e^{\mathrm{ad}(x)} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}x)^n}{n!}$$

is an automorphism of $\mathfrak{g}$.

Note. Let us make an unnecessary addition to the relation between a Lie group and a Lie algebra. The set of all automorphisms of $\mathfrak{g}$ forms a subgroup of the group of regular linear transformations $GL(\mathfrak{g})$. The subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\mathrm{Ad}(e^x)$ ($x \in \mathfrak{g}$) is denoted by $\mathrm{Int}(\mathfrak{g})$, and an element of $\mathrm{Int}(\mathfrak{g})$ is called an **inner automorphism** of the Lie algebra $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. If $\mathfrak{g}$ is of finite dimension, then both $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$ are closed subgroups of $GL(\mathfrak{g})$ and hence are linear Lie groups, whose Lie algebras are $\mathrm{Der}(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{z}$ respectively. Although these facts are not necessary to read this book further, we will use the terminology $\mathrm{Int}(\mathfrak{g})$ in §1.3 (Theorem 1.49) to describe the uniqueness of the conjugacy class of Cartan subalgebras. □

The next theorem is fundamental and very important (and its proof is simple):

THEOREM 1.26 (Schur's Lemma). *If (π, V) is an irreducible finite-dimensional representation of a Lie algebra $\mathfrak{g}$, then* $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\mathrm{Id}_V$.

PROOF. For $T \in \mathrm{End}(V)$, take an eigenvector $v_0 \neq 0$ of an eigenvalue λ of T:

$$Tv_0 = \lambda v_0.$$

Put $T' = T - \lambda \mathrm{Id}_V$. Then, since $T' \in \mathrm{End}_{\mathfrak{g}}(V)$, Ker T' is an invariant subspace of V; and since $0 \neq v_0 \in \mathrm{Ker} T'$, this invariant subspace is not $\{0\}$. By the irreducibility of V, we have Ker $T' = V$, i.e., $T' = 0$. Thus $T = \lambda \mathrm{Id}_V$. □

COROLLARY 1.27. *If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, then* $\mathrm{End}_{\mathfrak{g}}(\mathfrak{g}) = \mathbb{C}\mathrm{Id}_{\mathfrak{g}}$.

REMARK 1.28. For a Lie algebra $\mathfrak{g}$, an endomorphism of $\mathfrak{g}$ as a Lie algebra and an element of $\mathrm{End}_{\mathfrak{g}}(\mathfrak{g})$ are completely different. For example, if $\mathfrak{g}$ is a simple Lie algebra, $\mathrm{End}_{\mathfrak{g}}(\mathfrak{g})$ consists only of scalar operators by the above Corollary 1.27, while on the other hand $\mathrm{Ad}(e^x)$ $(x \in \mathfrak{g})$ is always an automorphism of $\mathfrak{g}$.

Next, let us consider an invariant symmetric bilinear form on a Lie algebra $\mathfrak{g}$:

DEFINITION 1.29. A (complex-valued) symmetric bilinear form $\langle\ ,\ \rangle$ on a Lie algebra $\mathfrak{g}$ is called **$\mathfrak{g}$-invariant** if it satisfies

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle.$$

Moreover,

$$\mathrm{Rad}\langle\ ,\ \rangle := \{x \in \mathfrak{g}; \langle x, y\rangle = 0 \text{ for all } y \in \mathfrak{g}\}$$

is called the **radical** of the bilinear form $\langle\ ,\ \rangle$. If $\mathrm{Rad}\langle\ ,\ \rangle = \{0\}$, this bilinear form $\langle\ ,\ \rangle$ is said to be **non-degenerate**. If $\langle\ ,\ \rangle$ is a symmetric invariant bilinear form, $\mathrm{Rad}\langle\ ,\ \rangle$ is an invariant subspace of $(\mathrm{ad}, \mathfrak{g})$. So, for a simple finite-dimensional Lie algebra $\mathfrak{g}$, Rad $\langle\ ,\ \rangle$ is either $\mathfrak{g}$ itself or $\{0\}$, and hence $\langle\ ,\ \rangle$ is either identically 0 or non-degenerate.

Note. Although $\mathfrak{g}$ is a vector space over the complex number field $\mathbb{C}$, an "inner product" of $\mathfrak{g}$ treated in this book is not an Hermitian symmetric bilinear form but $\mathbb{C}$-linear in both its first and its second component, and is symmetric in the usual sense (i.e., $\langle x, y\rangle = \langle y, x\rangle$). Hence we do not use the positivity of an inner product in this book (although we do frequently use the non-degeneracy of an inner product).[3] □

[3]To state the unitarity of representations, we may refer to an Hermitian product not on a Lie algebra $\mathfrak{g}$ but on a representation space V. The only places

For a finite-dimensional representation (π, V) of $\mathfrak{g}$, one defines the invariant bilinear form B_V on $\mathfrak{g}$ as follows:

$$B_V(x, y) := \mathrm{tr}_V(\pi(x)\pi(y)) \qquad (x, y \in \mathfrak{g}).$$

In particular, if $\mathfrak{g}$ is a finite-dimensional Lie algebra, the bilinear form associated to the adjoint representation of $\mathfrak{g}$ is denoted by B and is called the **Killing form**:

$$B(x, y) := \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(x)\mathrm{ad}(y)) \qquad (x, y \in \mathfrak{g}).$$

The following important theorem is called **Cartan's criterion**; it guarantees the existence of a non-degenerate symmetric invariant bilinear form on a semisimple Lie algebra. Its proof is given in standard texts on Lie algebras.

THEOREM 1.30. *For a finite-dimensional Lie algebra $\mathfrak{g}$, the following two conditions are equivalent:*

1. *$\mathfrak{g}$ is a semisimple Lie algebra.*
2. *The Killing form on $\mathfrak{g}$ is non-degenerate.*

The proof of the following theorem is extremely easy, but it is important:

THEOREM 1.31. *If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, a symmetric invariant bilinear form on $\mathfrak{g}$ is uniquely determined up to a scalar. Namely, a symmetric invariant bilinear form on a simple finite-dimensional Lie algebra is a scalar multiple of the Killing form.*

PROOF. Take two symmetric invariant bilinear forms $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ on $\mathfrak{g}$. Suppose that they are not identically 0. Then the radical of each bilinear form is $\{0\}$ (since $\mathfrak{g}$ is simple). Hence these bilinear forms are non-degenerate and induce isomorphisms of $\mathfrak{g}$-modules between $\mathfrak{g}$ and $\mathfrak{g}^*$. By composing these isomorphisms, we obtain an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}$, which must be a scalar map by Corollary 1.27. It follows that $\langle\ ,\ \rangle'$ is a scalar multiple of $\langle\ ,\ \rangle$. The rest of the theorem follows from Theorem 1.30. □

REMARK 1.32. Let me explain the terminology used in Theorem 1.31. If $\langle\ ,\ \rangle$ is an invariant bilinear form, any scalar multiple of it is also an invariant bilinear form. Hence if we have a non-zero invariant bilinear form, we may obtain infinitely many invariant bilinear form by multiplying by a scalar. But they coincide when multiplied

where this happens are in the explanation of Theorem 1.45 in §1.2 and the latter half of §3.4.

by an appropriate scalar, which means that they are essentially the same bilinear form. The phrase *unique up to a scalar* means that if two forms coincide under multiplication by an appropriate scalar, an invariant bilinear form is unique. This is one of the dialects used in the village of Mathematics. The *dialects* are completely different from speaking freely and *directly*, and often have deep contents and backgrounds.

Next we prepare some elementary formulae which will be useful to study structures and representations of a Lie algebra. First of all, the two lemmas below can be easily proved by induction and are useful.

LEMMA 1.33. *For elements x, y, z of $\mathfrak{g}$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$(\mathrm{ad}(x))^n[y,z] = \sum_{j=0}^{n} \binom{n}{j} [(\mathrm{ad}(x))^j y, (\mathrm{ad}(x))^{n-j} z].$$

LEMMA 1.34. *Let A be an associative algebra. For each element x of A, define elements L_x, R_x and $\mathrm{ad}(x)$ of $\mathrm{End}(A)$ as follows:*

$$\begin{aligned} L_x a &:= xa, \\ R_x a &:= ax, \\ \mathrm{ad}(x) &:= L_x - R_x. \end{aligned}$$

Then for $x, y \in A, \lambda, \mu \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, the following formulae hold:

$$(x-\lambda-\mu)^n a = \sum_{j=0}^{n} \binom{n}{j} ((\mathrm{ad}(x)-\lambda)^j a)(x-\mu)^{n-j}, \tag{1}$$

$$(\mathrm{ad}(x))^n a = \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{n-j} a x^j. \tag{2}$$

When a vector space V is of finite dimension, for each element X of $\mathrm{End}(V)$ we define the regular linear transformation e^X on V by (1.8). For an infinite-dimensional vector space V, even if we cannot define e^X for all X, we would like to define it for some convenient $X \in \mathrm{End}(V)$ and use it. Let us introduce the notion of local nilpotency, which is a convenient property:

DEFINITION 1.35. Let V be a vector space and $X \in \mathrm{End}(V)$. If for each element v of V there exists a natural number k satisfying $X^k v = 0$, X is said to be **locally nilpotent**.

In this case, the word "locally" explains that even if X itself is not nilpotent (that is, no power of X is 0), on each element of V, X

operates as if it is a 'nilpotent linear transformation'. In other words, if one observe the action of X from each element of V, it looks like a nilpotent operator.

In the above definition, the natural number k can depend on v. But if $X \in \mathrm{End}(V)$ is locally nilpotent, one can consider

$$e^X v := \sum_{j=0}^{\infty} \frac{1}{j!} X^j v$$

for all elements v of V. This right hand side is seemingly an infinite sum but in fact a finite sum; therefore it naturally converges and defines an element of V. Namely, we obtain a linear map $e^X : V \longrightarrow V$ by this formula. Furthermore, e^{-X} is the inverse map of e^X and e^X is a regular linear transformation on V.

In this way, for a vector space V and $X \in \mathrm{End}(V)$, one can define $e^X \in GL(V)$ whenever one of the following conditions is guaranteed:

i. V is of finite dimension.
ii. X is locally nilpotent.

Let us first consider the adjoint representation:

LEMMA 1.36. *Take a Lie algebra $\mathfrak{g}$ and its element x satisfying the following condition $(N)_{\mathrm{ad}}$:*

$$(N)_{\mathrm{ad}} \begin{cases} (i)\ \mathfrak{g} \textit{ is a finite-dimensional Lie algebra,} \\ \qquad \textit{or} \\ (ii)\ \mathrm{ad}(x) \textit{ is locally nilpotent.} \end{cases}$$

Then $g := e^{\mathrm{ad}(x)} \in GL(\mathfrak{g})$ enjoys the following properties:

1. $g[y, z] = [gy, gz] \quad (y, z \in \mathfrak{g})$,
2. $g \circ \mathrm{ad}(y) \circ g^{-1} = \mathrm{ad}(gy) \quad (y \in \mathfrak{g})$.

PROOF. (1) immediately follows from Lemma 1.33. For (2) it is enough to show that the formula $g([y, g^{-1}z]) = [gy, z]$ holds for any element z of $\mathfrak{g}$, and this follows immediately from (1). □

As was explained before, if $\mathfrak{g}$ is a finite-dimensional Lie algebra, (1) in this lemma can also be obtained as a consequence of Proposition 1.25.

NOTATION 1.37. Let (π, V) be a representation of $\mathfrak{g}$.

1. If $\mathfrak{g}$ and its element x satisfy the condition $(N)_{\mathrm{ad}}$, then set $\mathrm{Ad}(e^x) := e^{\mathrm{ad}(x)}$.

2. If we choose an element x of $\mathfrak{g}$ so that $\pi(x)$ is locally nilpotent, we set $\pi(e^x) := e^{\pi(x)}$.

The next lemma can be obtained immediately by applying Lemma 1.34 to $A := \mathrm{End}(V)$ and $\pi(x), \pi(a)$:

LEMMA 1.38. *Let (π, V) be a representation of $\mathfrak{g}$, and take an element x of $\mathfrak{g}$ satisfying*

1. *$\pi(x)$ is locally nilpotent and*
2. *$\mathfrak{g}$ and x satisfy the condition $(N)_{\mathrm{ad}}$.*

Then for any element a of $\mathfrak{g}$, we have

$$\pi(e^x)(\pi(a)v) = \pi(\mathrm{Ad}(e^x)a)\pi(e^x)v \qquad (v \in V),$$

i.e.,

$$\pi(\mathrm{Ad}(e^x)a) = \pi(e^x) \circ \pi(a) \circ \pi(e^{-x}).$$

1.2. Representation Theory of $\mathfrak{sl}(2, \mathbb{C})$

Let us take a basis $\{h, e, f\}$ of $\mathfrak{sl}(2, \mathbb{C})$ as follows:

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.10}$$

They satisfy the following relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{1.11}$$

First, let us prepare elementary formulae.

LEMMA 1.39. *In the enveloping algebra $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{C}))$ of $\mathfrak{sl}(2, \mathbb{C})$, the following formulae hold for any $k \in \mathbb{Z}_{\geq 0}$:*

1. $[h, e^k] = 2ke^k, \quad [h, f^k] = -2kf^k,$
2. $[e, f^k] = kf^{k-1}(h + 1 - k) = k(h + k - 1)f^{k-1},$
3. $[f, e^k] = -ke^{k-1}(h + k - 1) = -k(h + 1 - k)e^{k-1}.$

PROOF. It is easy to prove (1) by induction on k. (2) can also be proved with the aid of Lemma 1.10 by induction on k as follows:

$$\begin{aligned}
[e, f^k] &= [e, f \cdot f^{k-1}] \\
&= \underbrace{[e, f]}_{h} f^{k-1} + f \cdot \underbrace{[e, f^{k-1}]}_{(k-1)(h+k-2)f^{k-2}} \\
&= hf^{k-1} + (k-1) \underbrace{fh}_{hf+2f} \cdot f^{k-2} + (k-1)(k-2)f^{k-1} \\
&= khf^{k-1} + \{2(k-1) + (k-1)(k-2)\}f^{k-1} \\
&= khf^{k-1} + k(k-1)f^{k-1}.
\end{aligned}$$

Rewriting this formula by using the formula

$$hf^{k-1} = f^{k-1}h - 2(k-1)f^{k-1}$$

obtained in (1), we obtain (2). (3) can be proved similarly. □

Although it is related to Remark 1.14, one has to be careful in the above lemma that e^k is not the k-th power of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We consider the finite-dimensional representation (π, V) of $\mathfrak{sl}(2,\mathbb{C})$. To investigate the structure of V, let us study the eigenvalues and eigenspaces of the action $\pi(h)$ of h on V. Since V is of finite dimension, $\pi(h)$ is in fact diagonalizable by the complete reducibility of representations. But let me not use this fact and consider the generalized eigenspaces of $\pi(h)$. Below, for simplicity, we simply write xv instead of $\pi(x)v$ for $x \in \mathfrak{sl}(2,\mathbb{C})$ and $v \in V$. For an eigenvalue λ of $\pi(h)$, if we set

(1.12)
$$V_\lambda := \{v \in V;\ (h - \lambda \cdot \mathrm{Id}_V)^N v = 0 \text{ for some natural number } N\},$$

then V is the direct sum of these generalized eigenspaces:

$$V = \bigoplus_{\lambda \text{ is an eigenvalue of } \pi(h)} V_\lambda. \tag{1.13}$$

It is easy to see from (1.11) and (1) of Lemma 1.34 that the V_λ's satisfy

$$v \in V_\lambda \Longrightarrow ev \in V_{\lambda+2} \quad \text{and} \quad fv \in V_{\lambda-2}. \tag{1.14}$$

Now, since V is of finite dimension, $\pi(h)$ has only finitely many eigenvalues. Hence we can choose the eigenvalue λ_0 so that

$$\lambda \text{ is an eigenvalue of } \pi(h), \text{ but } \lambda_0 + 2 \text{ is not an eigenvalue of } \pi(h).$$

In this case, it follows from (1.14) that $ev = 0$ for all $v \in V_{\lambda_0}$. Let us take a 'real eigenvector' from V_{λ_0}, say v_0. Namely, v_0 ($\neq 0$) is an element which satisfies the following conditions:

$$hv_0 = \lambda_0 v_0 \quad \text{and} \quad ev_0 = 0. \tag{1.15}$$

Setting

$$v_j := f^j v_0 \quad (j \in \mathbb{Z}_{\geq 0}),$$

we obtain a sequence of points in V. By (1.14), we have $v_j \in V_{\lambda_0 - 2j}$, and since the number of eigenvalues of $\pi(h)$ is finite, this sequence should be all zeros from a certain number. Thus, there exists a natural number N which satisfies

$$v_{N-1} \neq 0 \quad \text{and} \quad v_j = 0 \ (j \geq N). \tag{1.16}$$

v_j ($j \in \mathbb{Z}_{\geq 0}$) is obtained from v_0 by applying $\pi(f)$ repeatedly, so conversely let us see what happens if we let $\pi(e)$ act on these v_j's. From (2) of Lemma 1.39, for all $j \in \mathbb{Z}_{\geq 0}$, we have:

$$\begin{aligned} ev_j &= ef^j v_0 \\ &= f^j \underbrace{ev_0}_{0} + jf^{j-1} \underbrace{(h+1-j)v_0}_{(\lambda_0+1-j)v_0} \\ &= j(\lambda_0 + 1 - j)v_{j-1}. \end{aligned} \tag{1.17}$$

So, by (1.16) and (1.17), the vector subspace $U := \bigoplus_{j=0}^{N-1} \mathbb{C}v_j$ of V is invariant with respect to the actions $\pi(h), \pi(e)$ and $\pi(f)$, and hence it is an N-dimensional invariant subspace of V. Now if we write (1.17) in particular for $j = N$, we have

$$ev_N = N(\lambda_0 + 1 - N)v_{N-1}.$$

But then, since $v_N = 0$ and $v_{N-1} \neq 0$, we must have $\lambda_0 + 1 - N = 0$. That is,

$$\lambda_0 = N - 1 \in \mathbb{Z}_{\geq 0},$$

and from these we obtain the following theorem:

THEOREM 1.40. *Let V ($\neq \{0\}$) be a finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module. Then:*

1. *The eigenvalues of h on V are all integers.*
2. *Take an element v_0 $(\neq 0)$ of V and $m \in \mathbb{Z}$ so that $ev_0 = 0$ and $hv_0 = mv_0$. Then,*
   (i) $m \geq 0$,
   (ii) $U := \bigoplus_{j=0}^{m} \mathbb{C} \cdot f^j v_0$ *is an $(m+1)$-dimensional invariant subspace of V and U is irreducible as a representation space of $\mathfrak{sl}(2,\mathbb{C})$. The actions of e, f and h on $v_j := f^j v_0$ $(0 \leq j \leq m)$ can be written as follows:*

$$\begin{aligned} hv_j &= (m-2j)v_j, \\ ev_j &= j(m+1-j)v_{j-1}, \\ fv_j &= v_{j+1}. \end{aligned} \tag{1.18}$$

   Here, if v_{-1} and v_{m+1} appear in the right hand side, we regard them as 0.

From this theorem and its proof, we obtain the following two corollaries:

COROLLARY 1.41. *Let V $(\neq \{0\})$ be an $\mathfrak{sl}(2,\mathbb{C})$-module. Then we have:*

1. *if V is of finite dimension, then (the maximal eigenvalue of h)+ (the minimal eigenvalue of h) $= 0$, and*
2. *if one of the two sets $\{$eigenvalues of $h\} \cap \mathbb{Z}_{\geq 0}$ and $\{$eigenvalues of $h\} \cap \mathbb{Z}_{\leq 0}$ is empty, then V is of infinite dimension, and*
3. *if h has a non-integral eigenvalue, then V is of infinite dimension.*

COROLLARY 1.42. *Let V $(\neq 0)$ be an irreducible finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module. If we let m be the maximal eigenvalue of h, then $m \in \mathbb{Z}_{\geq 0}$ and we have the following:*

1. *$\{$eigenvalues of $h\} = \{m - 2j; j = 0, \cdots, m\}$,*
2. *the multiplicity of each eigenvalue of $h = 1$, and*
3. $\dim V = m + 1$.

In an irreducible representation (π, V), all of the eigenvalues of $\pi(h)$ are even if the dimension of the representation space V is odd, and are odd if the dimension of V is even. Two irreducible finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$ are equivalent if their dimensions coincide. Since the formula (1.17) derived in the proof of Theorem 1.40 will be useful later, I would like to write it as a lemma:

LEMMA 1.43. *Let V be an $\mathfrak{sl}(2,\mathbb{C})$-module. Take an element v_0 ($\neq 0$) and $\lambda_0 \in \mathbb{C}$ so that*

$$ev_0 = 0 \quad \text{and} \quad hv_0 = \lambda_0 v_0, \tag{1.19}$$

and set

$$v_j := f^j v_0 \quad (j \in \mathbb{Z}_{\geq 0}).$$

Then

$$ev_j = j(\lambda_0 + 1 - j)v_{j-1}.$$

In particular, we have

$$fev_j = j(\lambda_0 + 1 - j)v_j \tag{1.20}$$

from the above lemma.

DEFINITION 1.44. A vector v_0 ($\neq 0$) and λ_0 satisfying the condition (1.19) in Lemma 1.43 are called a **singular vector** and **singular weight** of V respectively.

We now know the structure of a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module, but what happens to a (reducible) finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module? Concerning this question, the following theorem is known:

THEOREM 1.45. *Any finite-dimensional representation of a finite-dimensional simple Lie algebra is completely reducible.*

This theorem also holds for a semisimple Lie algebra, and is a criterion for semisimplicity of a Lie algebra. An outline of its proof is very simple and clear, and is given as follows:

1. There exists a compact real form of a complex finite-dimensional simple Lie algebra $\mathfrak{g}$. Here a 'compact real form' is the Lie algebra $\mathfrak{g}_u$ over the real number field $\mathbb{R}$ satisfying the following condition:

   The Lie algebra of the complexification of $\mathfrak{g}_u$ is isomorphic to $\mathfrak{g}$, and the connected and simply-connected Lie group G_u whose Lie algebra is $\mathfrak{g}_u$ is compact.
2. A finite-dimensional representation of $\mathfrak{g}$ and a finite-dimensional representation of $\mathfrak{g}_u$ on a complex vector space are equivalent.
3. Any finite-dimensional representation of $\mathfrak{g}_u$ can be lifted to a representation of G_u.

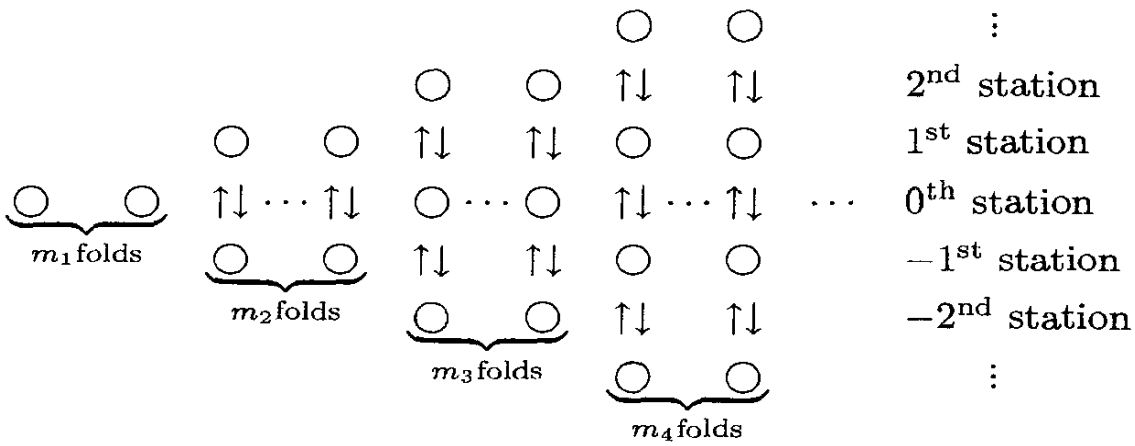


FIGURE 1.1

4. There exists a measure on G_u which is invariant under left and right translation (i.e. multiplication of a group element from the left and from the right). (Therefore there always exists an integration of a continuous function on G_u.) Using these facts, one can deduce that any finite-dimensional representation of G_u is a unitary representation, viz. there exists a positive definite Hermitian inner bilinear form on a representation space which is invariant under the action of G_u.

The keys to proving Theorem 1.45 are these four facts, and they are well-known in the representation theory of compact Lie groups. We first introduce a positive definite inner product on a representation space V (by (4)) which is invariant under the action of G_u. For any invariant subspace U of V, its orthogonal complement is G_u-invariant and V is the direct sum of these spaces. Hence, V is completely reducible.

In this way, Theorem 1.45 can easily be proved by using elementary properties of simple Lie groups and simple Lie algebras. This method of proof, using a G_u-invariant positive definite bilnear form is called the *unitary trick*[4]. (The reader who is interested in the other proof may consult [**Hum**]. Moreover, the above (1) – (4) for $\mathfrak{sl}(2,\mathbb{C})$ are explained in [**YS**] in detail.)

Let V be a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module. Then V is completely reducible, and by Corollary 1.42, its structure looks as in Figure 1.1.

[4]See e.g. [**Kn**], [**KO**]

An elegant shape of the left half of Mt. Fuji reflected in the surface of a lake, this is the proportion of the finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$. In this figure, $\bigcirc$ is an element of V, and they all together form a basis of V. They are eigenvectors of h, and the height of each $\bigcirc$ expresses its eigenvalue. The arrow $\uparrow$ shows the action of e and the arrow $\downarrow$ shows the action of f. Each column spans an irreducible invariant subspace, and each row spans an eigenspace of h. In Figure 1.1, V contains m_1 irreducible representations of dimension 1 (trivial representation), m_2 of dimension 2, m_3 of dimension 3, and so on, and V is the direct sum of these irreducible components.

Note. There are some facts that can be obviously read off from Figure 1.1. For example, let V_j be the eigenspace of V with eigenvalue j with respect to the action of the operator h. One can see that

1. $\dim V_j = \dim V_{-j}$,
2. the sequences $\{\dim V_{2j}\}_{j=0,1,2,\cdots}$ and $\{\dim V_{2j+1}\}_{j=0,1,2,\cdots}$ are both monotone decreasing,
3. $\dim V^e = \dim V_0 + \dim V_1$, where $V^e := \{v \in V; ev = 0\}$ is the vector subspace spanned by singular vectors.

etc. $\square$

$\mathfrak{sl}(2,\mathbb{C})$ is a tiny Lie algebra of dimension only 3, and (1.11) describes all of its structure as a Lie algebra. It has countably many irreducible finite-dimensional representations, and they are classified by the dimension of its representation space. Namely, for each natural number n, there exists, up to equivalence, only one n-dimensional irreducible representation, and the action of e, f, h on its representation space can be described by (1.18) by letting $m = n - 1$. Finite-dimensional representations can be decomposed into the direct sum of irreducible invariant subspaces as above. Even when a representation space is of infinite dimension and contains a finite-dimensional invariant subspace, it similarly decomposes into the direct sum of irreducible components.

A Lie algebra $\mathfrak{g}$ itself is nothing but an algebraic system with a bracket operation. But its representation (π, V) is very active. $\pi(x)$, the element that is an element x of the Lie algebra $\mathfrak{g}$ with the cover π, is an element of $\mathrm{End}(V)$. Namely, it is a linear operator on V. The linear operator acts directly on any elements of V, and we let them move here and there.

For example, in the case of $\mathfrak{sl}(2,\mathbb{C})$, as one can see from Figure 1.1, $\pi(e)$ plays the role of a pump sending $\bigcirc$ one floor above rhythmically, and $\pi(f)$ is a pump sending one floor below. Watching this

figure, one might hear a beat of ◯ moving up and down according to the work of the pumps $\pi(e)$ and $\pi(f)$. Moreover, these ◯ utter their eigenvalues in reply to the action of $\pi(h)$. When one says a 'representation', elements of the Lie algebra fit perfectly with elements of the representation, and they form a community, even though it is small.

Even if the Lie algebra under consideration is not $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$ sometimes stays nearby:

DEFINITION 1.46. Let $\mathfrak{g}$ be a Lie algebra. If elements h, e, f of $\mathfrak{g}$ satisfy the relation (1.11), this triple (h, e, f) is called an **S-triple**.

The vector subspace spanned by an **S-triple** (h, e, f) is a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In particular, if $\mathfrak{g}$ is a (finite- or infinite-dimensional) simple Lie algebra, there are many S-triples inside $\mathfrak{g}$, and an S-triple chosen properly plays quite an important role in studying the structure and the representations of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ itself is a representation space of an S-triple via its adjoint action. In the next section, we will study the structure of simple finite-dimensional Lie algebras. The only tools used there are S-triples and an invariant bilinear form (the Killing form).

Since ancient times, it has been he charm of music that has soothed the fiercest *warriors* (or *samurai*). This law seems to be universal in the physical universe, and it is also true in the world of Lie algebras. Although the theme of this book is infinite-dimensional Lie algebras, the music that plays an extremely important role in analyzing their structures and their representations is this $\mathfrak{sl}(2,\mathbb{C})$, the smallest and the loveliest algebra among the simple finite-dimensional Lie algebras. Chapter 2 of this book is the main part of representation theory of BKM algebras and BKM superalgebras, and an S-triple, i.e. $\mathfrak{sl}(2,\mathbb{C})$, always appears in an important scene. "BRING ME TO YOUR MIND!"- under this tender melody, obstructions facing us easily tumble down and the theory hidden in a veil of infinite-dimensional Lie algebras will be gradually revealed. $\mathfrak{sl}(2,\mathbb{C})$ is wonderful! Chapter 2 of this book is a paean to $\mathfrak{sl}(2,\mathbb{C})$, and the analysis of representations and the structure of infinite-dimensional Lie algebras given there is just one application of the representation theory of $\mathfrak{sl}(2,\mathbb{C})$.

1.3. Structure of Simple Finite-Dimensional Lie Algebras

In this section, let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and let us study its structure. An important role in studying representations and the structures of $\mathfrak{g}$ is played by a Cartan subalgebra.

DEFINITION 1.47. A Lie subalgebra $\mathfrak{h}$ of a simple Lie algebra $\mathfrak{g}$ satisfying the following two conditions is called a **Cartan subalgebra** :

i. $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.
ii. A linear transformation $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$ is always **diagonalizable**.

Let us see an example of Cartan subalgebras for the easiest simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:

EXAMPLE 1.48. Take a basis $\{h, e, f\}$ of $\mathfrak{sl}(2, \mathbb{C})$ as in (1.11). Then the one-dimensional vector subspace $\mathfrak{h} := \mathbb{C}h$ spanned by h is a Cartan subalgebra. In fact, arranging a basis e, h, f of $\mathfrak{sl}(2, \mathbb{C})$ in this order, the matrix representation of $\mathrm{ad}(h)$ according to (1.1) is a diagonal matrix

$$\mathrm{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

But if one takes a one-dimensional subspace $\mathbb{C}e$ spanned by e, this is a maximal abelian subalgebra but not a Cartan subalgebra. This is because the matrix representation of $\mathrm{ad}(e)$ with respect to the same basis $\{e, h, f\}$ is not a diagonal matrix but an upper triangular matrix:

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Although $\mathfrak{h}$ mentioned above is the 'canonical' Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$, a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ does not necessarily have this form. For another basis of $\mathfrak{sl}(2, \mathbb{C})$, let us adopt $\{x, y, z\}$ defined as follows:

$$x := e - f, \quad y := h + i(e + f), \quad z := h - i(e + f).$$

Then, since

$$[e - f, e + f] = 2h, \qquad [h, e \pm f] = 2(e \mp f),$$

we have

$$\begin{aligned} [x, y] &= -2(e + f) + 2ih = 2iy, \\ [x, z] &= -2(e + f) - 2ih = -2iz. \end{aligned}$$

Thus the matrix representation of $\mathrm{ad}(x)$ with respect to this basis $\{x, y, z\}$ is a diagonal matrix,

$$\mathrm{ad}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

Therefore, $\mathfrak{h}' := \mathbb{C}x = \mathbb{C} \cdot (e - f)$ is again a Cartan subalgebra.

Concerning the existence, etc., of Cartan subalgebras, the following is known:

THEOREM 1.49. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Then Cartan subalgebras exist, and they are conjugate under the group of inner automorphisms* $\mathrm{Int}(\mathfrak{g})$ *of $\mathfrak{g}$. Namely, if $\mathfrak{h}$ and $\mathfrak{h}'$ are two Cartan subalgebras of $\mathfrak{g}$, then there exists $g \in \mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{h}' = g(\mathfrak{h})$.*

By this theorem, the dimension of a Cartan subalgebra is independent of its choice and depends only on $\mathfrak{g}$.

DEFINITION 1.50. For a simple Lie algebra $\mathfrak{g}$, the dimension of a Cartan subalgebra of $\mathfrak{g}$ is called the **rank** of $\mathfrak{g}$ and is denoted by $\mathrm{rank}(\mathfrak{g})$.

Below in this section, I proceed by letting $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank l and by fixing its Cartan subalgebra, say $\mathfrak{h}$.

For any element H of $\mathfrak{h}$, $\mathfrak{g}$ decomposes into the direct sum of eigenspaces of $\mathrm{ad}(H)$, since $\mathrm{ad}(H)$ is diagonalizable by condition (ii) in the definition of a Cartan subalgebra. Let $H_1, \cdots, H_l$ be a basis of $\mathfrak{h}$. Then for each i, $\mathrm{ad}(H_i)$ is diagonalizable, and moreover $\mathrm{ad}(H_i)$ and $\mathrm{ad}(H_j)$ commute by condition (i) in the definition of a Cartan subalgebra. Hence $\mathfrak{g}$ decomposes into the direct sum of simultaneous eigenspaces of $\mathrm{ad}(H_i)$'s. These simultaneous eigenspaces are the common eigenspaces of a linear combination of $\mathrm{ad}(H_i)$'s $(1 \leq i \leq l)$, and hence they are nothing but the simultaneous eigenspaces of $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$. Namely, $\mathfrak{g}$ is the direct sum of the simultaneous eigenspaces of $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$. Here the simultaneous eigenvalues of $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$ are linear functions. In other words, they are elements of $\mathfrak{h}^*$. So let $\alpha \in \mathfrak{h}^*$ be one of the simultaneous eigenvalues of $\mathrm{ad}(H)$, and $\mathfrak{g}_\alpha$ its eigenspace with eigenvalue α:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}. \tag{1.21}$$

If α is not a simultaneous eigenvalue, then $\mathfrak{g}_\alpha = \{0\}$, and if $\alpha = 0$ then $\mathfrak{g}_\alpha = \mathfrak{h}$.

DEFINITION 1.51. If an element α ($\neq 0$) of $\mathfrak{h}^*$ satisfies $\mathfrak{g}_\alpha \neq \{0\}$, then α is called a **root** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is denoted by $\Delta(\mathfrak{g}, \mathfrak{h})$ or simply by Δ. If α is a root, then $\mathfrak{g}_\alpha$ is called a **root space** attached to the root α and an element of $\mathfrak{g}_\alpha$ is called a **root vector** of the root α.

Δ is a subset of $\mathfrak{h}^*$. For simplicity, let us set

$$\Delta_0 := \Delta \cup \{0\},$$

and call an element of Δ_0 a root 'in a wide sense'. Finally, we obtain a direct sum decomposition of $\mathfrak{g}$ (as a vector space) as follows:

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{1.22}$$

(1.22) is called the **root space decomposition** of the Lie algebra $\mathfrak{g}$ with respect to its Cartan subalgebra $\mathfrak{h}$. The following properties of $\alpha, \beta \in \Delta_0$ can be easily verified by the definition (1.21) of a root subspace $\mathfrak{g}_\alpha$:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \tag{1.23}$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}, \tag{1.24}$$

$$\text{For} \quad X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, \qquad B(X, Y) = 0 \quad \text{if } \alpha + \beta \neq 0. \tag{1.25}$$

For example, let us check (1.25). For any element H of $\mathfrak{h}$, we have

$$B([H, X], Y) + B(X, [H, Y]) = 0$$

by the invariance of B. Since X, Y are elements of root spaces, this formula can be rewritten as

$$(\alpha + \beta)(H) \cdot B(X, Y) = 0.$$

If $\alpha + \beta \neq 0$, one can choose H so that $(\alpha + \beta)(H) \neq 0$. Hence we obtain (1.25).

Since B is non-degenerate on $\mathfrak{g}$, the following can be seen easily:

LEMMA 1.52. 1. *The restriction $B|_{\mathfrak{h}\times\mathfrak{h}}$ of B on $\mathfrak{h}$ is a non-degenerate bilinear form on $\mathfrak{h}$.*

2. *For $\alpha \in \Delta$, $B|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is a non-degenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.*
3. *If α is a root, then $-\alpha$ is also a root and the dimension of their root spaces coincide. Namely,* $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$.

Let us also remark on the following:

(1.26) If $H \in \mathfrak{h}$ satisfies $\alpha(H) = 0$ for all $H \in \mathfrak{h}$, then $H = 0$.

(1.26) can be rephrased as follows:

$$\mathfrak{h}^* = \sum_{\alpha \in \Delta} \mathbb{C}\alpha. \tag{1.27}$$

Namely, any element of $\mathfrak{h}^*$ can be expressed as a linear combination of roots. But we do not claim the uniqueness of its expression. Specifically, (1.27) does not mean that Δ is a basis of $\mathfrak{h}^*$.

Now since $B|_{\mathfrak{h}\times\mathfrak{h}}$ is a non-degenerate bilinear form on $\mathfrak{h}$, one can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by this bilinear form. We denote an element of $\mathfrak{h}$ corresponding to $\alpha \in \mathfrak{h}^*$ via this identification by H_α. Namely, we define H_α by the following formula:

$$B(H_\alpha, H) = \alpha(H) \quad (H \in \mathfrak{h}). \tag{1.28}$$

In addition, the bilinear form B on $\mathfrak{h}$ defines a bilinear form on $\mathfrak{h}^*$ by this identification, and we again denote it by B:

$$B(\alpha, \beta) := B(H_\alpha, H_\beta) = \alpha(H_\beta) \quad (\alpha, \beta \in \mathfrak{h}^*). \tag{1.29}$$

Using the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$, we can rephrase (1.27) in terms of H_α as follows:

$$\mathfrak{h} = \sum_{\alpha \in \Delta} \mathbb{C}H_\alpha. \tag{1.30}$$

LEMMA 1.53. *For $\alpha \in \Delta$, we have*

1. *if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$, then $[X, Y] = B(X, Y)H_\alpha$, and*
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_\alpha$.

PROOF. (1) By (1.23), $[X, Y]$ is an element of $\mathfrak{h}$. Hence to show (1), we have only to show that the formula

$$B([X, Y], H) = B(X, Y) \cdot B(H_\alpha, H)$$

holds for any $H \in \mathfrak{h}$; and this can be checked by the following computation:

$$\begin{aligned}(\text{the left hand side}) &= B(X, \underbrace{[Y, H]}_{-[H,Y]=\alpha(H)Y}) = \underbrace{\alpha(H)}_{B(H_\alpha,H)} \cdot B(X, Y) \\ &= B(H_\alpha, H) \cdot B(X, Y) = (\text{the right hand side}).\end{aligned}$$

(2) follows from (1) and (2) of Lemma 1.52. □

Taking $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, one can take $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ satisfying

$$B(X_\alpha, X_{-\alpha}) = 1 \tag{1.31}$$

by (2) of Lemma 1.52. Then, since we have

$$[X_\alpha, X_{-\alpha}] = H_\alpha \tag{1.32}$$

by (1) of Lemma 1.53, it follows that

$$\mathfrak{s}_\alpha := \mathbb{C}H_\alpha \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$$

is a three-dimensional subalgebra of $\mathfrak{g}$. To see that this is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, let us prove the following lemma:

LEMMA 1.54. *For $\alpha, \beta \in \Delta$, we have*

1. $\alpha(H_\alpha) > 0$, *and*
2. $\beta(H_\alpha)$ *is a rational number.*

PROOF. As a preparation, let us first prove the following two 'facts'.

Fact 1: $\alpha(H_\alpha) \neq 0$,

Fact 2: Put $c_{\alpha,\beta} := \dfrac{\beta(H_\alpha)}{\alpha(H_\alpha)}$. Then $c_{\alpha,\beta}$ is a rational number.

To show these facts, we consider

$$V := \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\beta + j\alpha}. \tag{1.33}$$

Then V is an $\mathfrak{s}_\alpha$-module. Since $\mathrm{ad}H_\alpha = [\mathrm{ad}X_\alpha, \mathrm{ad}X_{-\alpha}]$, the trace of $\mathrm{ad}H_\alpha$ on V is

$$\mathrm{tr}_V(\mathrm{ad}H_\alpha) = 0. \tag{1.34}$$

According to the direct sum decomposition (1.33) of V, we have

$$\mathrm{tr}_V(\mathrm{ad}H_\alpha) = \sum_{j \in \mathbb{Z}} \mathrm{tr}_{\mathfrak{g}_{\beta+j\alpha}}(\mathrm{ad}H_\alpha) = \sum_{j \in \mathbb{Z}} (\beta + j\alpha)(H_\alpha) \cdot \dim \mathfrak{g}_{\beta+j\alpha}. \tag{1.35}$$

Hence, by (1.34) and (1.35), we obtain

$$\beta(H_\alpha) \underbrace{\sum_{j \in \mathbb{Z}} \dim \mathfrak{g}_{\beta+j\alpha}}_{>0} = -\alpha(H_\alpha) \sum_{j \in \mathbb{Z}} j \dim \mathfrak{g}_{\beta+j\alpha}. \tag{1.36}$$

Now if $\alpha(H_\alpha) = 0$, we get $\beta(H_\alpha) = 0$ for all $\beta \in \Delta$. But then $H_\alpha = 0$ by (1.26), and this contradicts the choice of α. Thus Fact 1 is shown, and Fact 2 follows from (1.36).

Let us turn to the proof of the lemma. For this, if we compute

$$B(H, H') = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}H)(\mathrm{ad}H') \tag{1.37}$$

by using the root space decomposition (1.22) of $\mathfrak{g}$, we obtain the following formula:

$$B(H, H') = \sum_{\beta \in \Delta} \beta(H)\beta(H') \dim \mathfrak{g}_\beta. \tag{1.38}$$

In particular, if we set $H = H' = H_\alpha$, we get

$$\alpha(H_\alpha) = B(H_\alpha, H_\alpha) = \sum_{\beta \in \Delta} \beta(H_\alpha)^2 \dim \mathfrak{g}_\beta. \tag{1.39}$$

This formula can be rewritten by using the rational numbers $c_{\alpha,\beta}$ defined in Fact 2 as follows:

$$\alpha(H_\alpha) = \alpha(H_\alpha)^2 \sum_{\beta \in \Delta} c_{\alpha,\beta}^2 \dim \mathfrak{g}_\beta. \tag{1.40}$$

By Fact 1, we can divide both sides of this formula since $\alpha(H_\alpha) \neq 0$, and we get

$$1 = \alpha(H_\alpha) \sum_{\beta \in \Delta} c_{\alpha,\beta}^2 \dim \mathfrak{g}_\beta. \tag{1.41}$$

Since $c_{\alpha,\beta}$ are rational numbers by Fact 2, it follows that $\alpha(H_\alpha)$ is a positive rational number by (1.41). From this and Fact 2, we see that $\beta(H_\alpha)$ is also a rational number, and hence the lemma is proved. □

The rational numbers $c_{\alpha,\beta}$ in the above proof have a concrete meaning. This will be made clear after Proposition 1.57.

From the above lemma and its proof, we obtain the following important proposition:

PROPOSITION 1.55. *Define the vector subspace $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$ over the real number field $\mathbb{R}$ as follows:*

$$\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha.$$

Then the following hold:

1. *the Killing form B is positive definite on $\mathfrak{h}_{\mathbb{R}}$,*
2. $\mathfrak{h}_{\mathbb{R}} \cap \sqrt{-1}\mathfrak{h}_{\mathbb{R}} = \{0\}$, *and*
3. $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \sqrt{-1}\mathfrak{h}_{\mathbb{R}}$ *(the direct sum as vector spaces over $\mathbb{R}$).*

PROOF. (1) We have only to show that $B(H,H) > 0$ for any element H ($\neq 0$). First, by (1.38), we have

$$B(H,H) = \sum_{\beta\in\Delta} \beta(H)^2 \dim \mathfrak{g}_\beta,$$

and since an element H of $\mathfrak{h}_{\mathbb{R}}$ is a linear combination of H_α ($\alpha \in \Delta$) over $\mathbb{R}$, the $\beta(H)$ are real numbers by (2) of Lemma 1.54. Hence we have $B(H,H) > 0$.

(2) follows from (1), and (3) follows from (2) and (1.30). □

Now let us look at $\mathfrak{s}_\alpha$. Since $\alpha(H_\alpha) \neq 0$ by Fact 1 in the proof of Lemma 1.54, we can put

$$h_\alpha := \frac{2}{\alpha(H_\alpha)} H_\alpha, \quad e_\alpha := X_\alpha, \quad f_\alpha := \frac{2}{\alpha(H_\alpha)} X_{-\alpha}. \tag{1.42}$$

They satisfy the following commutation relations:

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha, \quad [e_\alpha, f_\alpha] = h_\alpha. \tag{1.43}$$

Therefore $(h_\alpha, e_\alpha, f_\alpha)$ is an S-triple and $\mathfrak{s}_\alpha$ is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In this way, for each $\alpha \in \Delta$, we have an isomorphism $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$. By looking at $\mathfrak{s}_\alpha$ and its adjoint action on $\mathfrak{g}$, namely at representations of $\mathfrak{sl}(2,\mathbb{C})$, we obtain the following proposition.

PROPOSITION 1.56. 1. *If* $\alpha \in \Delta$, *then* $\dim \mathfrak{g}_\alpha = 1$.
2. *If* $\alpha, \beta, \alpha+\beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

PROOF. (1) To make things simpler, let us prove that $\dim \mathfrak{g}_{-\alpha} = 1$ here. Namely, let us assume that there is a root α satisfying $\dim \mathfrak{g}_{-\alpha} \geq 2$, and get a contradiction. For this root α, let us choose an S-triple $(h_\alpha, e_\alpha, f_\alpha)$ as above. Since $\dim \mathfrak{g}_{-\alpha} \geq 2$, there exists $Y \in \mathfrak{g}_{-\alpha}$ satisfying $B(e_\alpha, Y) = 0$. Then, by (1) of Lemma 1.53 for $\mathrm{ad}(e_\alpha)$, the action of the S-triple on Y can be computed as follows:

$$\mathrm{ad}(h_\alpha)(Y) = \frac{2}{\alpha(H_\alpha)} \underbrace{[H_\alpha, Y]}_{-\alpha(H_\alpha)Y} = -2Y,$$

$$\mathrm{ad}(e_\alpha)(Y) = \underbrace{B(e_\alpha, Y)}_{0} H_\alpha = 0.$$

In this case, since the adjoint representation of $\mathfrak{s}_\alpha$ on the representation space $\mathfrak{g}$ is finite-dimensional, this contradicts (2) (i) of Theorem 1.40. This proves (1).

(2) Since each root space has dimension one, it is enough to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$. Defining V by (1.34) as in the proof of Lemma 1.54,

this is an $\mathfrak{s}_\alpha$-module. Namely, it is a representation space of $\mathfrak{s}_\alpha$. Since there are only finitely many $j \in \mathbb{Z}$ which give $\mathfrak{g}_{\beta+j\alpha} \neq \{0\}$, putting

$$q := \max\{j; \beta + j\alpha \in \Delta_0\}, \qquad p := \min\{j; \beta + j\alpha \in \Delta_0\}, \tag{1.44}$$

we have

$$V = \sum_{p \le j \le q} \mathfrak{g}_{\beta+j\alpha}. \tag{1.45}$$

And since $\alpha + \beta$ is a root, it follows that

$$q \ge 1. \tag{1.46}$$

Let us compute the eigenvalues of $\mathrm{ad}(h_\alpha)$ on V. Since for each j ($p \le j \le q$)

$$\begin{aligned} \text{the eigenvalue of } \mathrm{ad}(h_\alpha) \text{ on } \mathfrak{g}_{\beta+j\alpha} &= \frac{2}{\alpha(H_\alpha)}(\beta + j\alpha)(H_\alpha) \\ &= \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2j, \end{aligned} \tag{1.47}$$

one proves

$$\begin{aligned} &\text{the maximal eigenvalue of } \mathrm{ad}(h_\alpha) \text{ on } V = \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2q, \\ &\text{the minimal eigenvalue of } \mathrm{ad}(h_\alpha) \text{ on } V = \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2p. \end{aligned} \tag{1.48}$$

Hence, by (1) of Corollary 1.41, we obtain

$$\left(\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2q\right) + \left(\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2p\right) = 0$$

i.e.,

$$p + q = -\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}. \tag{1.49}$$

Now suppose $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$. Then, since $\mathrm{ad}(e_\alpha)(\mathfrak{g}_\beta) = \{0\}$, a vector subspace

$$U = \sum_{p \le j \le 0} \mathfrak{g}_{\beta+j\alpha}$$

of V is invariant under the action of $\mathfrak{s}_\alpha$. That is, U is also a representation space of $\mathfrak{s}_\alpha$. And by (1.47), the eigenvalues of $\mathrm{ad}(h_\alpha)$ on U

are given as follows:

$$\text{(1.50)} \qquad \begin{aligned} &\text{the maximal eigenvalue of } \mathrm{ad}(h_\alpha) \text{ on } U = \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}, \\ &\text{the minimal eigenvalue of } \mathrm{ad}(h_\alpha) \text{ on } U = \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2p. \end{aligned}$$

So, in the same way as before, by (1) of Corollary 1.41 we get

$$\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + \left(\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} + 2p\right) = 0,$$

i.e.,

$$\text{(1.51)} \qquad p = -\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}.$$

Combining (1.49) and (1.51), we get $q = 0$, which contradicts (1.46). □

For two roots α, β, the sequence $\{\beta + j\alpha\}_{j\in\mathbb{Z}}$ of $\mathfrak{h}^*$ contains only finitely many roots. Hence one can choose two integers p, q $(p \leq q)$ as follows:

$$\text{(1.52)} \qquad \begin{aligned} &\beta + p\alpha, \beta + q\alpha \in \Delta_0 \\ &\text{but neither } \beta + (p-1)\alpha \text{ nor } \beta + (q+1)\alpha \text{ is in } \Delta_0. \end{aligned}$$

Here we remark that we do not assume that all of $\beta + j\alpha$ $(p \leq j \leq q)$ are roots (in a broad sense). Then, as in the proof of (2) of Proposition 1.56, we obtain the formula (1.49) in the same way as above by looking at the eigenvalues of $\mathrm{ad}(h_\alpha)$ on the finite-dimensional representation space of $\mathfrak{s}_\alpha$ defined by

$$V := \sum_{p\leq j\leq q} \mathfrak{g}_{\beta+j\alpha}.$$

Thus (1) of the following Proposition 1.57 is proved:

PROPOSITION 1.57. *For $\alpha, \beta \in \Delta$, choose integers p, q $(p \leq q)$ so that they satisfy (1.52). Then the following hold:*

1. $p + q = -\dfrac{2\beta(H_\alpha)}{\alpha(H_\alpha)}$,
2. $\{\beta + j\alpha; j \in \mathbb{Z}\} \cap \Delta_0 = \{\beta + j\alpha; p \leq j \leq q\}$.

Therefore, p, q are uniquely determined by α, β.

PROOF. Let us show (2). In the sequence $\{\beta + j\alpha\}_{j\in\mathbb{Z}}$ of $\mathfrak{h}^*$, suppose that the subsequence consisting of roots appears discontinuously. Namely, suppose there exist integers p', q', p, q that satisfy

$$p' \leq q' < p \leq q, \tag{1.53}$$

and

$$\cdots, \underbrace{\beta + p'\alpha, \cdots, \beta + q'\alpha}_{\text{elements of } \Delta_0}, \cdots, \underbrace{\beta + p\alpha, \cdots, \beta + q\alpha}_{\text{elements of } \Delta_0}, \cdots$$

and both neighboring sides of these subsequences consisting of elements of Δ_0 are not roots. Then by applying (1) to

$$V' := \sum_{p' \leq j \leq q'} \mathfrak{g}_{\beta + j\alpha}$$

we obtain

$$p' + q' = -\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}.$$

Hence we have $p' + q' = p + q$, and this contradicts the choice (1.53) of the integers p', q', p, q. □

DEFINITION 1.58. The sequence of roots (in a broad sense) in Proposition 1.57 is called the **α-series** through β [5] or the α-series of the root β.

Considering the case when one of p and q is zero in (1) of Proposition 1.57, we obtain the following corollary:

COROLLARY 1.59. 1. *If $\alpha, \beta \in \Delta$, then* $\dfrac{2\beta(H_\alpha)}{\alpha(H_\alpha)} \in \mathbb{Z}$.

2. *Take $\alpha, \beta \in \Delta$ so that $\alpha \pm \beta \neq 0$. Then,*
   (i) *if $\alpha + \beta \notin \Delta$, then $\beta(H_\alpha) \geq 0$,*
   (ii) *if $\alpha - \beta \notin \Delta$, then $\beta(H_\alpha) \leq 0$.*

PROPOSITION 1.60. *If $\alpha \in \Delta$, then the only scalar multiples $c\alpha$ ($c \in \mathbb{C}$) of α that are roots are $\pm\alpha$.*

PROOF. It is enough to prove the following two assertions:

i. If α and $c\alpha \in \Delta$ ($c \in \mathbb{C}$) are both roots, then c is equal to one of $\pm 1, \pm 2, \pm\frac{1}{2}$.
ii. If α is a root, then 2α is not a root.

---

[5] *Translator's note.* It is usually called the **α-string** through β.

(i) Put $\beta := c\alpha$. Then since $H_\beta = cH_\alpha$, we have

$$\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} = 2c, \quad \frac{2\alpha(H_\beta)}{\beta(H_\beta)} = \frac{2}{c},$$

but these numbers are both integers by (1) of Corollary 1.59. Thus (i) holds.

(ii) Suppose 2α is a root. Then by putting $\beta = 2\alpha$ in (2) of Proposition 1.56, one has $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$. But then one has $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{2\alpha} = 1$ by (1) of Proposition 1.56, which is a contradiction. □

Now we could prepare some fundamental properties of root systems. As a next step, let us introduce the following notion to study "fundamental root systems":

DEFINITION 1.61. If an element H of $\mathfrak{h}$ satisfies $\alpha(H) \neq 0$ for all roots α, then H is called a **regular element**.

Choose any regular element H_0 from $\mathfrak{h}_\mathbb{R}$ and fix it. Then for $\alpha \in \Delta$, $\alpha(H_0)$ is either a positive or a negative real number by (2) of Lemma 1.54. According to whether $\alpha(H_0)$ is positive or negative, α is called a positive root or a negative root respectively, and we write $\alpha > 0$ or $\alpha < 0$ respectively. We denote the subset of Δ consisting of all positive (resp. negative) roots by Δ_+ (resp. Δ_-). They enjoy the following properties:

1. If $\alpha \in \Delta_\pm$, then $-\alpha \in \Delta_\mp$.
2. $\Delta = \Delta_+ \cup \Delta_-$ and $\Delta_+ \cap \Delta_- = \emptyset$.
3. If $\alpha, \beta \in \Delta_+$ satisfy $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_+$.

DEFINITION 1.62. If a positive root α can be written in the form $\alpha = \beta + \gamma$ $(\beta, \gamma \in \Delta_+)$, α is said to be **decomposable**. If α is not decomposable, viz. one cannot express α in the form $\alpha = \beta + \gamma$ $(\beta, \gamma \in \Delta_+)$, α is called a **fundamental root** or a **simple root**.

Denote the set of all fundamental roots (a subset of Δ_+) by

$$\Pi = \{\alpha_1, \cdots, \alpha_r\}.$$

Any positive root α can be expressed as

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0}) \tag{1.54}$$

by repeating a similar decomposition as above. Since the roots span $\mathfrak{h}^*$ (1.27), it turns out that $\{\alpha_1, \cdots, \alpha_r\}$ span $\mathfrak{h}^*$. Namely, we have

$$\mathfrak{h}^* = \sum_{i=1}^{r} \mathbb{C}\alpha_i, \qquad \mathfrak{h}^*_{\mathbb{R}} = \sum_{i=1}^{r} \mathbb{R}\alpha_i. \tag{1.55}$$

Here the following naïve questions might arise.

i. Is a decomposition of a root α into a linear combination of fundamental roots as in (1.54) unique?
ii. Are $\{\alpha_1, \cdots, \alpha_r\}$ linearly independent? Namely, is the set of fundamental roots Π a basis of $\mathfrak{h}^*$?
iii. Does the number r of the fundamental roots coincide with the dimension l of $\mathfrak{h}$?

The above (i), (ii) and (iii) are pairwise equivalent. In fact this conjecture (expectation) is true. Let us prove this.

Let us first remark the following:

$$\text{If } \alpha, \beta \in \Pi, \text{ then } \alpha - \beta \notin \Delta. \tag{1.56}$$

Indeed, if $\alpha - \beta$ is a root, then it must be either positive or negative. Suppose that $\alpha - \beta$ is a positive root, and call it γ. From $\alpha - \beta = \gamma$ we get $\alpha = \beta + \gamma$. Namely, α can be expressed as a sum of two positive roots! This contradicts the fact that α is a fundamental root. The case when $\alpha - \beta$ is a negative root can also be handled in a similar manner.

So by Corollary 1.59, it follows that if $\alpha, \beta \in \Pi$ ($\alpha \neq \beta$), then $\beta(H_\alpha) \leq 0$. Describing this fact in terms of the positive definite bilinear form B induced on $\mathfrak{h}^*$ by (1.29), we have:

LEMMA 1.63. *For distinct fundamental roots α, β, the following hold:*

1. $B(\alpha, \beta) \leq 0$.
2. $B(\alpha, \beta) = 0 \Longleftrightarrow \alpha + \beta \notin \Delta$.

Now, using this fact, let us prove that $\{\alpha_1, \cdots, \alpha_r\}$ are linearly independent. For this purpose, it is enough to prove that they are linearly independent over $\mathbb{R}$ as elements of $\mathfrak{h}_{\mathbb{R}}$. Namely, assuming

$$\sum_{i=1}^{r} a_i \alpha_i = 0 \tag{1.57}$$

for some $a_i \in \mathbb{R}$ ($i = 1, \cdots, r$), it is enough to show that all the a_i's are zero. Assuming that some a_i's are non-zero, let us get a contradiction.

Classifying the a_i's into positive ones and negative ones, set

$$I := \{i; a_i > 0\}, \qquad J := \{i; a_i < 0\}. \tag{1.58}$$

Then by assumption

$$\text{at least one of } I \text{ and } J \text{ is non-empty,} \tag{1.59}$$

and one can rewrite (1.57) as follows:

$$\sum_{i\in I} a_i\alpha_i = -\sum_{j\in J} a_j\alpha_j \ (=: \alpha). \tag{1.60}$$

Let us compute $B(\alpha,\alpha)$:

$$B(\alpha,\alpha) = B\left(\sum_{i\in I} a_i\alpha_i, -\sum_{j\in J} a_j\alpha_j\right) = -\sum_{\substack{i\in I\\ j\in J}} \underbrace{a_i a_j}_{<0}\underbrace{B(\alpha_i,\alpha_j)}_{\leq 0} \leq 0.$$

On the other hand, since B is positive definite on $\mathfrak{h}^*_{\mathbb{R}}$, it follows that $\alpha = 0$. Namely,

$$\sum_{i\in I} a_i\alpha_i = \sum_{j\in J} a_j\alpha_j = 0. \tag{1.61}$$

If we compute the pairing between each side of the above formula and the previously fixed regular element H_0, it follows that

$$\sum_{i\in I} \underbrace{a_i}_{>0}\underbrace{\alpha_i(H_0)}_{>0} = \sum_{j\in J} \underbrace{a_j}_{<0}\underbrace{\alpha_j(H_0)}_{>0} = 0. \tag{1.62}$$

Thus the two sets I and J should be empty to make (1.62) consistent. But this contradicts (1.59). This contradiction occurs since we assumed the α_i's are linearly dependent. Summarizing these, we obtain the following theorem:

THEOREM 1.64. *Denote the set of all fundamental roots by* $\Pi = \{\alpha_i; 1 \leq i \leq r\}$. *Then:*

1. $r = \dim \mathfrak{h}$ $(= l)$, *i.e.* $|\Pi| = \mathrm{rank}(\mathfrak{g})$.
2. *A positive root* α *can be expressed uniquely in the form* $\alpha = \sum_{i=1}^{l} m_i\alpha_i$, *where* $m_i \in \mathbb{Z}_{\geq 0}$ *for all* i.
3. $B(\alpha_i,\alpha_j) \leq 0$ *for* $i \neq j$.

The notation $|\ |$ in (1) of the above theorem signifies the cardinality of the set (for a finite set, this is the number of elements contained in the set).

Next, let us consider the case when a positive root α is not a fundamental root. In this case, α decomposes into a sum of two

positive roots $\alpha = \beta + \gamma$, and by (2) of Proposition 1.56, an element of $\mathfrak{g}_\alpha$ can be expressed as a bracket of an element of $\mathfrak{g}_\beta$ and an element of $\mathfrak{g}_\gamma$. If one of β and γ is not a fundamental root, we may achieve the decomposition into fundamental roots

$$\alpha = \sum_{i=1}^{l} m_i \alpha_i$$

by repeating this procedure. Hence, an element of the root space $\mathfrak{g}_\alpha$ can be expressed by taking the bracket operations appropriately with an element of $\mathfrak{g}_{\alpha_1}$ m_1 times, an element of $\mathfrak{g}_{\alpha_2}$ m_2 times, $\cdots$, an element of $\mathfrak{g}_{\alpha_l}$ m_l times. Denoting the elements $h_\alpha, e_\alpha, f_\alpha$ defined in (1.42), in particular for a simple root $\alpha = \alpha_i$, by h_i, e_i, f_i, we can express an element of the subalgebras

$$\mathfrak{n}_+ := \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- := \sum_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \tag{1.63}$$

as a linear combination of brackets of $e_1, \cdots, e_l$ and $f_1, \cdots, f_l$ respectively. In other words, as subalgebras of $\mathfrak{g}$,

$$\mathfrak{n}_+ \text{ is generated by } e_1, \cdots, e_l, \tag{1.64}$$

$$\mathfrak{n}_- \text{ is generated by } f_1, \cdots, f_l. \tag{1.65}$$

Setting

$$a_{ij} := \alpha_j(H_i) = \frac{2}{\alpha_i(H_{\alpha_i})}\alpha_j(H_{\alpha_i}) = \frac{2B(\alpha_i, \alpha_j)}{B(\alpha_i, \alpha_i)}, \tag{1.66}$$

we see that h_i, e_i, f_i $(1 \le i \le l)$ satisfy the following relations:

$$\begin{aligned} &[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i, \\ &a_{ij} = 0 \Longrightarrow [e_i, e_j] = 0 \text{ and } [f_i, f_j] = 0. \end{aligned} \tag{1.67}$$

In fact the following additional relations also hold:

$$i \neq j \Longrightarrow (\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0 \text{ and } (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \tag{1.68}$$

but I will omit the detailed explanation of this relation (1.68), since I will prove these relations under a more general assumption in §2.1.

Note. We remark that the relations

$$[h_i, h_j] = 0 \quad (1 \le i, j \le l)$$

follow from (1.67). That is because by (1.67), we have

$$\begin{aligned}[h_i, h_j] &= [[e_i, f_i], h_j] = [e_i, \underbrace{[f_i, h_j]}_{a_{ji} f_i}] - [f_i, \underbrace{[e_i, h_j]}_{-a_{ji} e_i}] \\ &= a_{ji}\{[e_i, f_i] + \underbrace{[f_i, e_i]}_{-[e_i, f_i]}\} = 0.\end{aligned}$$

Therefore the relations (1.67) among the generators $e_1, \cdots, e_l$ and $f_1, \cdots, f_l$ imply that $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathbb{C}h_i$ is a Cartan subalgebra. □

DEFINITION 1.65. The matrix $A := (a_{ij})_{i,j=1,\cdots,l}$ is called the **Cartan matrix** of a simple finite-dimensional Lie algebra $\mathfrak{g}$.

The Cartan matrix constructed here enjoys the following properties:

(C0) the a_{ij} are all integers,
(C1) $a_{ii} = 2$,
(C2) $i \neq j \Longrightarrow a_{ij} \leq 0$,
(C3) $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$.

Moreover, A itself is not necessarily a symmetric matrix, but if we set

$$\begin{aligned} B &:= (B(\alpha_i, \alpha_j)), \quad \text{a positive definite symmetric matrix,} \\ D &:= \left(\frac{2}{B(\alpha_i, \alpha_i)}\delta_{ij}\right), \quad \text{a diagonal matrix,}\end{aligned}$$

we have $A = DB$. Namely,

(C4) $A =$ (a positive diagonal matrix) $\times$ (a positive definite symmetric matrix).

DEFINITION 1.66. A matrix A satisfying the above conditions (C0) – (C4) is called a **Cartan matrix of finite type**.

Given a matrix A satisfying the conditions (C0) – (C3), we construct a graph consisting of l vertices and multi-lines (with arrows)

connecting them by the following rules:

(D1) First arrange l-numbers of $\bigcirc$, and call them $\alpha_1, \cdots, \alpha_l$. (Each $\bigcirc$ is called a **vertex**.)

(D2) For each i, j $(i \neq j)$, connect two vertices α_i and α_j by $\max\{|a_{ij}|, |a_{ji}|\}$ lines. (If $a_{ij} = 0$, do not joint α_i and α_j.)

(D3) If $i \neq j$ and $|a_{ij}| \geq 2$ (i.e., $a_{ij} \leq -2$), put an arrow on the multi-line connecting α_i and α_j pointing from α_j toward α_i.

This graph is called the **Dynkin diagram** of the matrix A.

Note. One may doubt whether, conversely, one can recover a Cartan matrix from a Dynkin diagram. For example, if some vertices α_i and α_j of a Dynkin diagram are connected by more than three lines with the arrows pointing toward both of the vertices, the Cartan matrix which corresponds to this Dynkin diagram is not unique. But if we restrict ourselves to the Cartan matrices of finite type, then this sort of problem never happens. Because, if it did happen, the minor of the Cartan matrix A concerning α_i and α_j would look as follows:

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

Here, by our assumption, a, b are integers satisfying

$$a, b \geq 2 \quad \text{and} \quad \max\{a, b\} \geq 3.$$

Therefore, the determinant of this minor is negative and contradicts condition (C4), i.e., the positive definiteness of the symmetric part B of the Cartan matrix A. Thus the Cartan matrices of finite type and their Dynkin diagrams are in one-to-one correspondence. $\square$

If A is a Cartan matrix of finite type, then so is the transposed matrix ${}^tA$, and its Dynkin diagram can be obtained by reversing the direction of arrows. Let us show some examples of Cartan matrices and their Dynkin diagrams:

EXAMPLE 1.67.

1. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ corresponds to $\bigcirc\!\!-\!\!\!-\!\!\bigcirc$

2. $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ corresponds to $\bigcirc\!\Longleftarrow\!\bigcirc$

3. $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ corresponds to $\bigcirc \Rrightarrow \bigcirc$

4. $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ corresponds to $\bigcirc - \bigcirc \Rrightarrow \bigcirc$

The conditions (C0) – (C4) are strict constraints. For example, if $l = 2$ and its Dynkin diagram is connected, then it should be one of (1) – (3) in the above example (and their transposed matrices).

The Dynkin diagram of a simple Lie algebra is connected. Otherwise $\mathfrak{g}$ would be the direct sum of the subalgebras corresponding to each connected component, which contradicts the simplicity of $\mathfrak{g}$. In terms of Cartan matrices, by rearranging fundamental roots, one cannot rewrite the Cartan matrix A in the form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$. This fact is referred to as

(C5) A is **indecomposable**.

The conditions (C0) – (C5) listed above are the properties that completely characterize the Cartan matrices of simple finite-dimensional Lie algebras. The table obtained by listing all the matrices satisfying these conditions, and expressing them in terms of Dynkin diagrams, is nothing but the table of classification of simple finite-dimensional Lie algebras. And for each Dynkin diagram, the corresponding simple Lie algebra is named from A-type to G-type as follows- here the subscript in the name of each type signifies the number of vertices, i.e., rank($\mathfrak{g}$).

A_l $(l \geq 1)$

$$\overset{\alpha_1}{\bigcirc} - \overset{\alpha_2}{\bigcirc} - \cdots\cdots - \overset{\alpha_{l-2}}{\bigcirc} - \overset{\alpha_{l-1}}{\bigcirc} - \overset{\alpha_l}{\bigcirc}$$

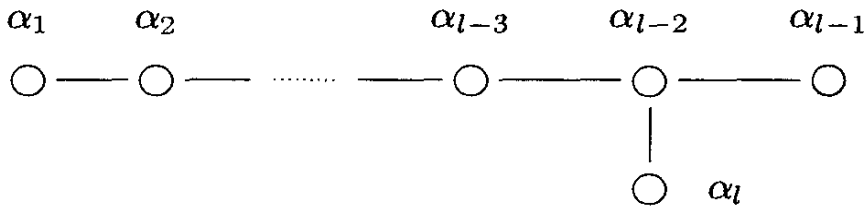
B_l $(l \geq 2)$

$$\overset{\alpha_1}{\bigcirc} - \overset{\alpha_2}{\bigcirc} - \cdots\cdots - \overset{\alpha_{l-2}}{\bigcirc} - \overset{\alpha_{l-1}}{\bigcirc} \Rrightarrow \overset{\alpha_l}{\bigcirc}$$

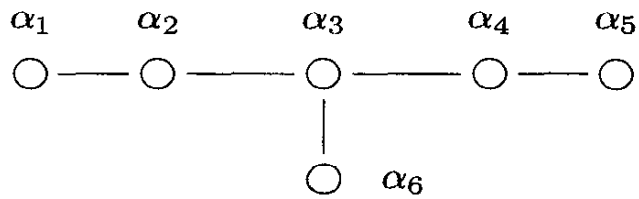
C_l $(l \geq 2)$

$$\overset{\alpha_1}{\bigcirc} - \overset{\alpha_2}{\bigcirc} - \cdots - \overset{\alpha_{l-2}}{\bigcirc} - \overset{\alpha_{l-1}}{\bigcirc} \Lleftarrow \overset{\alpha_l}{\bigcirc}$$

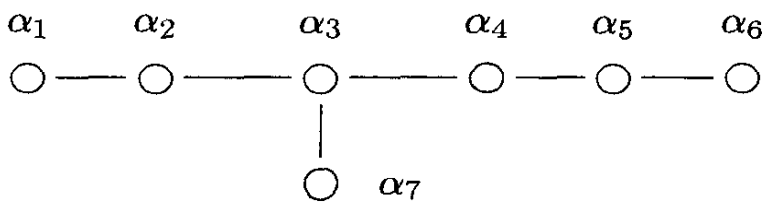
D_l $(l \geq 3)$



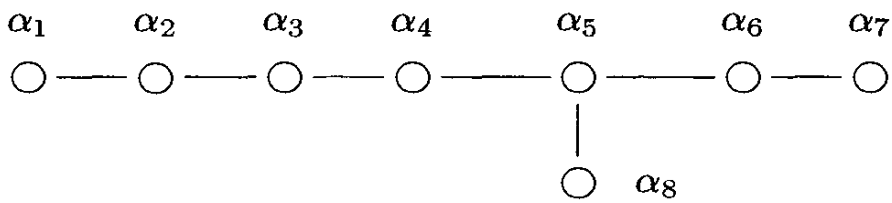
E_6



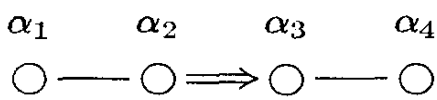
E_7



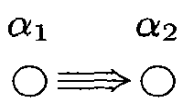
E_8



F_4



G_2



In this table, we have $B_2 \cong C_2$ and $A_3 \cong D_3$, and there are no other isomorphisms. The Lie algebras of types A, B, C, D are called of **classical type**, and those of types E, F, G are called of **exceptional type**. The classical type Lie algebras can be realized as subalgebras of $\mathfrak{gl}(n, \mathbb{C})$ as follows:

$$A_l \cong \mathfrak{sl}(l+1, \mathbb{C}),$$

$$B_l \cong \mathfrak{so}(2l+1, \mathbb{C})$$
$$:= \left\{ \begin{pmatrix} A & B & u \\ C & -{}^tA & v \\ -{}^tv & -{}^tu & 0 \end{pmatrix} ; \begin{array}{c} A, B, C \in M_l(\mathbb{C}), u, v \in \mathbb{C}^l, \\ {}^tB = -B, {}^tC = -C \end{array} \right\},$$

$$C_l \cong \mathfrak{sp}(2l, \mathbb{C})$$
$$:= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} ; \begin{matrix} A, B, C \in M_l(\mathbb{C}), \\ {}^tB = B, {}^tC = C \end{matrix} \right\}.$$

$$D_l \cong \mathfrak{so}(2l, \mathbb{C})$$
$$:= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} ; \begin{matrix} A, B, C \in M_l(\mathbb{C}), \\ {}^tB = -B, {}^tC = -C \end{matrix} \right\}.$$

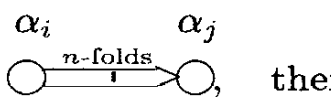
A Cartan matrix $A = (a_{i,j})_{1 \le i,j \le l}$ is in general not a symmetric matrix by its defining equation (1.66), but this implies that the matrix includes the information on the length of roots. Namely, for i, j with $a_{ij} \neq 0$ (viz. α_i and α_j are connected in its Dynkin diagram), by (1.66) we have

$$\frac{a_{ij}}{a_{ji}} = \frac{B(\alpha_j, \alpha_j)}{B(\alpha_i, \alpha_i)},$$

namely

$$B(\alpha_j, \alpha_j) = \frac{a_{ij}}{a_{ji}} B(\alpha_i, \alpha_i). \tag{1.69}$$

From this fact and the rule of making Dynkin diagrams, one obtains

(1.70) If 

, then $B(\alpha_j, \alpha_j) = \dfrac{1}{n} B(\alpha_i, \alpha_i)$.

Namely, the ratio of the lengths of fundamental roots can be extracted immediately from the Dynkin diagram. Therefore, for the Dynkin diagrams of simple finite-dimensional Lie algebras, the conditions

i. all of the roots have the same length,
ii. the Cartan matrix A of $\mathfrak{g}$ is a symmetric matrix,
iii. the Dynkin diagram of $\mathfrak{g}$ contains no multi-line,

are mutually equivalent, and simple Lie algebras with this property are of type A, D or E. In this sense, simple Lie algebras of types A, D and E are called **symmetric** simple Lie algebras or **simply-laced** simple Lie algebras, and simple Lie algebras of the other types are said to be **non-symmetric**. For a non-symmetric simple Lie algebra, as one can see from the above table of Dynkin diagrams, Π consists of roots of two different lengths. So for each fundamental root, the terms 'long root' and 'short root' make sense. As will be shown later in Lemma 3.8 in §3.1, any root of a simple finite-dimensional Lie algebra is transferred to an element of Π by the action of the Weyl

group. From this fact, it follows that any root of a symmetric simple finite-dimensional Lie algebra has the same length, and any root of a non-symmetric simple finite-dimensional Lie algebra has roots of two different lengths. We denote the set of all long roots and short roots by Δ_l and Δ_s respectively. For a symmetric Lie algebra, the lengths of all roots are the same, as I mentioned above, and hence it make no sense to say that a root is 'long' or 'short'; but usually the roots are regarded as long roots. Thus for a simple finite-dimensional Lie algebra, the set of roots has the following decomposition:

$$\Delta = \Delta_l \cup \Delta_s.$$

An invariant bilinear form on a simple finite-dimensional Lie algebra $\mathfrak{g}$ is unique up to a scalar, but is quite often normalized as follows:

$$\text{If} \quad \alpha \in \Delta_l, \quad \text{then} \quad (\alpha|\alpha) = 2. \tag{1.71}$$

This normalized bilinear form $(\ ,\ )$ is called the **standard bilinear form.**

1.4. Toward Infinite-Dimensional Lie Algebras

In the previous section, for a given simple finite-dimensional Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, we have seen that $\mathfrak{g}$ is generated by the root vectors $e_1, \cdots, e_l, f_1, \cdots, f_l$ corresponding to the $\pm$(fundamental roots) through the study of the structures of its root system. Namely, we have seen that any element of $\mathfrak{g}$ can be expressed as a linear combination of brackets of e_i's and f_i's. $e_1, \cdots, e_l, f_1, \cdots, f_l$ (or additionally, $e_1, \cdots, e_l, f_1, \cdots, f_l, h_1, \cdots, h_l$) are called **Chevalley generators** of the Lie algebra $\mathfrak{g}$. Here the bracket product $[\ ,\ ]$ satisfies not only the elementary relations imposed on a Lie algebra (= the conditions in Definition 1.1) but also the relations (1.67) and (1.68) among e_i's, f_i's and h_i's. As a matter of fact, the relations (1.67) and (1.68) are a necessary and sufficient condition to reconstruct the Lie algebra from its Chevalley generators. Namely, if the fundamental relations among $3l$ elements e_i, f_i and h_i $(1 \leq i \leq l)$ are (1.67) and (1.68) (viz., if there is no relation other than (1.67), (1.68), relations that can be derived from them and the restriction arising from the definition of a Lie algebra), then the Lie algebra constructed by linear combinations of repetitions of bracket products is isomorphic to the original Lie algebra $\mathfrak{g}$. Since the a_{ij}'s are the only information used in the fundamental relations (1.67) and (1.68), one

can reconstruct the original Lie algebra $\mathfrak{g}$ once its Cartan matrix is given.

Now, let us be free form the fetters of finite dimension, and imagine the world of infinite-dimensional simple Lie algebras. Suppose we take an infinite-dimensional simple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. Moreover, suppose that $\mathfrak{g}$ possesses a non-degenerate invariant bilinear form $\langle\ ,\ \rangle$. Let us study the structure and the properties of its root system in the same way as in §1.3. But in reality, this method will soon reach a dead end. This is because we cannot use the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

So, in the case of infinite-dimensional Lie algebras, let us consider another approach.

Thus we shall try another approach in the case of infinite Lie algebras. We begin with the same idea of listing generators and the relations among them, deleting one condition or another from (C0) – (C5), since with the same conditions (C0) – (C5) we get nothing but Cartan matrices of finite-dimensional simple Lie algebras. Another side of the coin is that if we change any one of the conditions we get an infinite Lie algebra. How do we modify any conditions to get infinite-dimensional Lie algebras that are interesting?

In the second half of the 1960s the founders of Kac-Moody Lie algebras modified condition (C4).

Removing condition (C4), a matrix A satisfying (C0) – (C3) is called a 'generalized Cartan matrix (GCM)' (a 'KM' matrix in this book). In this case, the Lie algebra $\mathfrak{g}(A)$ constructed by the fundamental relations (1.67) and (1.68) (and one more additional relation) is called a Kac-Moody Lie algebra or a Kac-Moody algebra (a 'KM algebra' in this book). But condition (C4) is the important condition for $\mathfrak{g}(A)$ to possess an invariant bilinear form, so the theory may not expand satisfactorily with only conditions (C0) – (C3).

So we will consider a slightly weakened version of condition (C4). Relaxing the constraint in (C4) that B is positive definite, let us consider afresh the following two conditions $(\widehat{\mathrm{C}}4)$ and $(\mathrm{C}4)_{aff}$:

$(\widehat{\mathrm{C}}4)$ $A = DB$, where D is a positive diagonal matrix and B is a symmetric matrix,

$(\mathrm{C}4)_{aff}$ $A = DB$, where D is a positive diagonal matrix and B is a positive semidefinite matrix and $\mathrm{corank}(B) = 1$.

Here the **corank** of a matrix is the number obtained by subtracting the rank from the size of the matrix, and the condition on the symmetric matrix $B = (b_{ij})_{1 \le i,j \le n}$ in condition $(\mathrm{C4})_{aff}$ is the same as the following:

i. $\displaystyle\sum_{i,j=1}^{n} b_{ij} u_i u_j \ge 0$ for all $(u_1, \cdots, u_n) \in \mathbb{R}^n$,

ii. the space of all $(u_1, \cdots, u_n) \in \mathbb{R}^n$ satisfying $\displaystyle\sum_{i,j=1}^{n} b_{ij} u_i u_j = 0$ forms a one-dimensional vector subspace.

If a KM matrix A satisfies condition $(\widehat{\mathrm{C}}4)$, the Lie algebra $\mathfrak{g}(A)$ is called a **symmetrizable Kac-Moody algebra**, and if A satisfies $(\mathrm{C4})_{aff}$, $\mathfrak{g}(A)$ is called an **affine Kac-Moody algebra** or an **affine Lie algebra**. (The definition of an affine Lie algebra given here looks different from that given in Chapter 2, but actually they are the same.)

Even in the case of infinite dimension, an invariant bilinear form is a powerful tool, and one can study the structure and representation theories of symmetrizable Kac-Moody algebras in the same way as simple finite-dimensional Lie algebras. In particular, in this class the affine KM algebras have connections with many fields such as modular forms, the Macdonald identities, soliton equations, fusion algebras, conformal field theories, lattice models in statistical mechanics, etc., so that they have very rich contents.

But KM matrices are not the only generalization of Cartan matrices. In the 1980's, Borcherds discovered the '(Monster) vertex algebra' by analyzing a representation space of the Monster group with a compatible ring structure, in the course of his research on the 'moonshine conjecture' based on a mysterious phenomenon (Mackey observation) which indicates a relation between the elliptic modular function $j(\tau)$ and the Monster simple group. Also he constructed infinite-dimensional Lie algebras associated to them in a natural way. Related theories are explained in [**FLM**], [**TH**]. This Lie algebra is obtained by fiddling with conditions (C0) and (C1) on Cartan matrices. Borcherds derived beautiful character formulae for highest weight representations also in this case. This Lie algebra is called a 'generalized KM algebra' (GKM algebra). Borcherds' method of generalization can also be applied to Lie superalgebras, and today the theories of 'generalized KM superalgebras' (GKM superalgebras) are producing interesting results, one after another.

But in the case of structure and representation theories of Lie superalgebras, some points of analogy to the theory of ordinary Lie algebras do not work, and phenomena peculiar to Lie superalgebras occur. Nowadays, in the above sense, what are called 'GKM superalgebras' are a generalization or an extension avoiding such strange phenomena, and most of the simple finite-dimensional Lie superalgebras are excluded. The natural conditions for Lie algebras are not necessarily natural for Lie superalgebras. Since I have a strong psychological resistance to naming such things by adding the word 'generalized', I would like to call it a 'BKM superalgebra', after Borcherds, Kac and Moody, in this book. Adjusting it to the case of Lie algebras, we speak of a 'BKM algebra' instead of a 'GKM algebra'.

I think that BKM algebras and BKM superalgebras are splendid classes of Lie (super)algebras. As we will see in Chapter 2, deriving their structures of root systems and character formulae is too easy. We have only to add a few more arguments to the case of KM algebras. I say 'simply' because we first set up the fundamental relations so that these proofs work well. This is the only reason, but I think it is marvelous that such things have 'nice applications'.

1.5. A Few Words on Lie Superalgebras

In this section, I will present the minimum facts on finite-dimen sional Lie superalgebras that are necessary in Chapter 2. Among the peculiarities of Lie superalgebras are the grading with respect to the additive group consisting of two elements $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, i.e. the '$\mathbb{Z}_2$-grading', and the 'twisting' of the bracket according to its $\mathbb{Z}_2$-grading. First, let us start from a $\mathbb{Z}_2$-graded vector space, the base of our discussion:

DEFINITION 1.68. Suppose that a vector space V and two of its vector subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are given. If V is the direct sum of these two vector subspaces, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, then V (with this decomposition) is called a $\mathbb{Z}_2$-**graded vector space**. $V_{\bar{0}}$ and $V_{\bar{1}}$ are called the **even part** and the **odd part** of V respectively.

An element v of a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ has a unique expression of the form $v = v_{\bar{0}} + v_{\bar{1}}$ ($v_i \in V_i$). In particular, if v is an element of either $V_{\bar{0}}$ or $V_{\bar{1}}$, v is said to be **homogeneous**. An element of $V_{\bar{0}}$ is called an **even element**, and an element of $V_{\bar{1}}$ is called an **odd element**. For $v \in V_i$, we set $|v| := i$, and this number (an element of $\mathbb{Z}_2$) is called the **homogeneous degree** or the **parity**

of v. We remark that for each **homogeneous element** v, $(-1)^{|v|}$ is well-defined.

DEFINITION 1.69. A $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ possessing the operation called the bracket product,

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \ni (x, y) \overset{\text{bilinear map}}{\longmapsto} [x, y] \in \mathfrak{g},$$

is called a **Lie superalgebra** if it satisfies the following conditions:

1. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad (i, j \in \mathbb{Z}_2)$,
2. $[y, x] = -(-1)^{|x||y|}[x, y]$,
3. $[x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] = [[x, y], z]$.

In the above definition, x, y, z are elements of $\mathfrak{g}$. The above conditions (2) and (3) are written for homogeneous elements, and the relations among general elements of $\mathfrak{g}$ can be computed by the bilinearity. The relation in condition (3) in the above definition is called the **super Jacobi identity**. The bracket product $[\ ,\ ]$ of a Lie superalgebra is called the **super-bracket**, and we may sometimes call $[x, y]$ the **super-commutator** of x and y. If x is an even element, we clearly have $[x, x] = 0$, but if x is an odd element, then the formula

$$[x, [x, x]] = 0 \tag{1.72}$$

follows from the super Jacobi identity.

DEFINITION 1.70. If a linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ between two Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}' = \mathfrak{g}'_{\bar{0}} \oplus \mathfrak{g}'_{\bar{1}}$ satisfies

1. $\varphi(\mathfrak{g}_i) \subset \mathfrak{g}'_i$ (preserving the grading),
2. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (preserving the bracket product),

φ is called a **homomorphism** of Lie superalgebras. In addition, if φ is bijective, then it is called an **isomorphism** of Lie superalgebras.

An endomorphism and an automorphism of a Lie superalgebra are also defined in a way analogous to the case of Lie algebras.

DEFINITION 1.71. If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space, then the linear map

$$\sigma : V_{\bar{0}} \oplus V_{\bar{1}} \ni v_{\bar{0}} + v_{\bar{1}} \longmapsto v_{\bar{0}} - v_{\bar{1}} \in V_{\bar{0}} \oplus V_{\bar{1}}$$

of V is called a **conjugation with respect to the $\mathbb{Z}_2$-grading** of V.

The following lemma can be easily checked:

LEMMA 1.72. *If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, the conjugation σ with respect to the $\mathbb{Z}_2$-grading of $\mathfrak{g}$ is an automorphism of the Lie superalgebra $\mathfrak{g}$. Namely, the formula*

$$[\sigma x, \sigma y] = \sigma[x, y]$$

holds for all $x, y \in \mathfrak{g}$.

When one consider objects associated to a Lie superalgebra, one has to be careful that the Lie superalgebra has two algebraic structures, the bracket product and the $\mathbb{Z}_2$-grading. As one sees in Definition 1.70, a homomorphism is a map preserving these two algebraic structures. Let us introduce the following concept to help us consider other objects such as Lie sub-superalgebras and ideals:

DEFINITION 1.73. If a vector subspace (in the usual sense) U of a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ can be decomposed into its even and odd parts as

$$U = (U \cap V_{\bar{0}}) \oplus (U \cap V_{\bar{1}}),$$

it is called a $\mathbb{Z}_2$-**graded vector subspace.**

A $\mathbb{Z}_2$-graded vector space is, in other words, a vector space with the action of the additive group $\mathbb{Z}_2$, i.e., a $\mathbb{Z}_2$-module, and the decomposition of it into its even and odd parts is the irreducible decomposition with multiplicities as a $\mathbb{Z}_2$-module. Namely, the irreducible representations of $\mathbb{Z}_2$ are the trivial representation and the alternating representation, and the even part of V is a sum of the trivial representations while the odd part of V is a sum of the alternating representations. From this point of view, a $\mathbb{Z}_2$-graded vector subspace of V is just an invariant subspace of V as a $\mathbb{Z}_2$-module. Moreover, in terms of the conjugation map σ, a $\mathbb{Z}_2$-graded vector subspace of V is just a vector subspace invariant under the action of σ. Under consideration of the $\mathbb{Z}_2$-structure, the definitions of a Lie sub-superalgebra and an ideal of a Lie superalgebra are analogous to the case of Lie algebras as in Definition 1.2:

DEFINITION 1.74. Let $\mathfrak{g}$ be a Lie superalgebra, and let $\mathfrak{p}, \mathfrak{i}$ be $\mathbb{Z}_2$-graded vector subspaces of $\mathfrak{g}$.

1. If $\mathfrak{p}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$, $\mathfrak{p}$ is called a **Lie sub-superalgebra** of $\mathfrak{g}$.
2. If $\mathfrak{i}$ satisfies $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$, $\mathfrak{i}$ is called an **ideal** of $\mathfrak{g}$.

Since an ideal $\mathfrak{i}$ of a Lie superalgebra $\mathfrak{g}$ is the direct sum of its even and odd parts, it also satisfies $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. For an ideal $\mathfrak{i}$ of a Lie superalgebra $\mathfrak{g}$, the quotient vector space $\mathfrak{g}/\mathfrak{i}$ has the bracket product and the $\mathbb{Z}_2$-grading; hence it becomes a Lie superalgebra. In particular, the center of $\mathfrak{g}$,

$$\mathfrak{z} := \{x \in \mathfrak{g}; [x, \mathfrak{g}] = 0\},$$

is an ideal of $\mathfrak{g}$. $\{0\}$ and $\mathfrak{g}$ itself are called the **trivial ideals**. If its dimension is greater than or equal to 3 and it has no ideal except for the trivial ideals, it is called a **simple Lie superalgebra**.

PROPOSITION 1.75. *Let $\mathfrak{g}$ be a Lie superalgebra. If a vector subspace $\mathfrak{a}$ of $\mathfrak{g}$ satisfies*

$$[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a} \quad \textit{or} \quad [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a},$$

then $\mathfrak{a}$ must be either $\{0\}$ or $\mathfrak{g}$. Namely, even forgetting its $\mathbb{Z}_2$-structure, a simple Lie superalgebra has neither a left nor a right ideal except for the trivial ideals.

PROOF. Let us consider the case when $\mathfrak{a}$ satisfies $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$. Then, since by Lemma 1.72

$$[\mathfrak{g}, \sigma\mathfrak{a}] = \sigma[\mathfrak{g}, \mathfrak{a}],$$

we have

$$[\mathfrak{g}, \mathfrak{a} + \sigma\mathfrak{a}] = [\mathfrak{g}, \mathfrak{a}] + [\mathfrak{g}, \sigma\mathfrak{a}] = [\mathfrak{g}, \mathfrak{a}] + \sigma[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a} + \sigma\mathfrak{a}.$$

Similarly,

$$[\mathfrak{g}, \mathfrak{a} \cap \sigma\mathfrak{a}] \subset \mathfrak{a} \cap \sigma\mathfrak{a}.$$

Since $\mathfrak{a} + \sigma\mathfrak{a}$ and $\mathfrak{a} \cap \sigma\mathfrak{a}$ are both σ-invariant, they are $\mathbb{Z}_2$-graded vector subspaces, and hence they are ideals of the Lie superalgebra $\mathfrak{g}$. By the simplicity of $\mathfrak{g}$, they coincide with either $\{0\}$ or $\mathfrak{g}$.

Now, to prove the proposition, let us assume that $\mathfrak{a} \neq \{0\}$ and $\mathfrak{a} \neq \mathfrak{g}$. Then we should have

$$\mathfrak{a} + \sigma\mathfrak{a} = \mathfrak{g} \quad \text{and} \quad \mathfrak{a} \cap \sigma\mathfrak{a} = \{0\}, \tag{1.73}$$

which says that there exists a unique linear isomorphism $f : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}_{\bar{1}}$ satisfying

$$u + f(u) \in \mathfrak{a} \quad (u \in \mathfrak{g}_{\bar{0}}). \tag{1.74}$$

For $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$, applying $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ to (1.74), we get

$$\underbrace{[x, u]}_{\text{even}} + \underbrace{[x, f(u)]}_{\text{odd}} \in \mathfrak{a}, \quad \underbrace{[y, u]}_{\text{odd}} + \underbrace{[y, f(u)]}_{\text{even}} \in \mathfrak{a}.$$

By the definition of f, we obtain the following relations:

$$x, u \in \mathfrak{g}_{\bar{0}} \Longrightarrow f([x,u]) = [x, f(u)], \tag{1.75}$$

$$y \in \mathfrak{g}_{\bar{1}},\ u \in \mathfrak{g}_{\bar{0}} \Longrightarrow f([y, f(u)]) = [y, u]. \tag{1.76}$$

Putting $y = f(v)$ $(v \in \mathfrak{g}_{\bar{0}})$ in (1.76), we have

$$f([f(v), f(u)]) = [f(v), u] \underset{\because(1.75)}{\underset{\uparrow}{=}} f([v,u]),$$

and since f is a linear isomorphism, we get the following formula:

$$[f(u), f(v)] = [v, u]. \tag{1.77}$$

This formula holds for all $u, v \in \mathfrak{g}_{\bar{0}}$. But since in (1.77) the bracket in the left hand side is symmetric in v, u and the right hand side is skew-symmetric in v, u, to make this formula consistent, both sides of (1.77) must be 0 for all $u, v \in \mathfrak{g}_{\bar{0}}$. Namely, we have

$$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}.$$

It follows that

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_{\bar{1}},$$

and hence $\mathfrak{g}_{\bar{1}}$ becomes an ideal. This contradicts the simplicity of $\mathfrak{g}$. Thus we have proved our propisition for the case $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$. The case $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$ can be proved similarly. □

What is especially important is a bilinear form. Concerning an inner product on a Lie superalgebra $\mathfrak{g}$, viz. a ($\mathbb{C}$-valued) bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$, we consider the following conditions:

i. (**super-invariance**) $\langle [x,y], z\rangle = \langle x, [y,z]\rangle$,
ii. (**super-symmetry**) $\langle y, x\rangle = (-1)^{|x||y|}\langle x, y\rangle$,
iii. (even) $\langle \mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}\rangle = 0$,
iv. (odd) $\langle \mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}\rangle = \langle \mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}\rangle = \{0\}$.

If a bilinear form $\langle\ ,\ \rangle$ of a Lie superalgebra is super-invariant, super-symmetric and even (or odd), then its radical

$$\mathrm{Rad}\langle\ ,\ \rangle := \{x \in \mathfrak{g}; \langle x, y\rangle = 0 \text{ for any } y \in \mathfrak{g}\}$$

is an ideal of $\mathfrak{g}$. A Lie superalgebra which has an odd inner product is also important, but we do not touch upon them owing to lack of space. If a bilinear form $\langle\ ,\ \rangle$ on a $\mathbb{Z}_2$-graded vector subspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ satisfies

$$\langle v, u\rangle = -(-1)^{|v||u|}\langle u, v\rangle,$$

it is said to be **super skew-symmetric**.

EXAMPLE 1.76. Any $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ becomes a Lie superalgebra by the bracket $[x, y] = 0$ (for all $x, y \in \mathfrak{g}$). This is called a **super-commutative** Lie superalgebra.

EXAMPLE 1.77. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space which possesses a non-degenerate super skew-symmetric bilinear form $\langle\ ,\ \rangle$. The vector space $\mathfrak{g} = V \oplus \mathbb{C}z$ spanned by V and the letter z becomes a Lie superalgebra via the bracket defined by

$$\begin{aligned} &[u, v] := \langle u, v\rangle z \quad (u, v \in V), \\ &[\mathfrak{g}, z] := \{0\}. \end{aligned}$$

This Lie superalgebra is called a **Heisenberg superalgebra**. In this case, the even and the odd part of $\mathfrak{g}$ are respectively $\mathfrak{g}_{\bar{0}} = V_{\bar{0}} \oplus \mathbb{C}z$ and $\mathfrak{g}_{\bar{1}} = V_{\bar{1}}$, and $\mathbb{C}z$ is the center of $\mathfrak{g}$.

DEFINITION 1.78. If an associative algebra A over the complex number field $\mathbb{C}$ is at the same time a $\mathbb{Z}_2$-graded vector space, and the product (as an associative algebra) in A satisfies

$$A_i \cdot A_j \subset A_{i+j} \quad (i, j \in \mathbb{Z}_2),$$

A is called a $\mathbb{Z}_2$-**graded associative algebra**. In addition, if the relation $ba = (-1)^{|a||b|}ab$ holds for all homogeneous elements a, b of A, A is said to be **super-commutative**.

If A and B are $\mathbb{Z}_2$-graded associative algebras, then their tensor product $A \otimes B$ becomes a $\mathbb{Z}_2$-graded associative algebra via the $\mathbb{Z}_2$-grading defined by

$$\begin{aligned} (A \otimes B)_{\bar{0}} &:= (A_{\bar{0}} \otimes B_{\bar{0}}) \oplus (A_{\bar{1}} \otimes B_{\bar{1}}), \\ (A \otimes B)_{\bar{1}} &:= (A_{\bar{0}} \otimes B_{\bar{1}}) \oplus (A_{\bar{1}} \otimes B_{\bar{0}}), \end{aligned} \tag{1.78}$$

and the product defined by

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'. \tag{1.79}$$

Formula (1.79) is written for homogeneous elements of A and B, and we extend it to any elements by bilinearity.

EXAMPLE 1.79. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a $\mathbb{Z}_2$-graded associative algebra. For homogeneous elements a, b of A, we set

$$[a, b] := ab - (-1)^{|a||b|} ba. \tag{1.80}$$

Then this bracket product can be uniquely extended to

$$[\ ,\ ] : A \times A \xrightarrow{\text{bilinear map}} A,$$

and A becomes a Lie superalgebra by this formula. If A is supercommutative as a $\mathbb{Z}_2$-graded associative algebra, then it is also supercommutative as a Lie superalgebra.

The following lemma is the super version of Lemma 1.10, and can be easily checked by computation:

LEMMA 1.80. *For homogeneous elements a, b, c of a $\mathbb{Z}_2$-graded associative algebra, the following formula holds:*

$$[a, bc] = [a, b]c + (-1)^{|a||b|} b[a, c].$$

EXAMPLE 1.81. For $\mathbb{Z}_2$-graded vector spaces $U = U_{\bar{0}} \oplus U_{\bar{1}}$ and $V = V_{\bar{0}} \oplus V_{\bar{1}}$, the set $\mathrm{Hom}_{\mathbb{C}}(V, U)$ of all linear maps from V to U becomes a $\mathbb{Z}_2$-graded vector space by setting

$$\mathrm{Hom}_{\mathbb{C}}(V, U)_k := \{f \in \mathrm{Hom}_{\mathbb{C}}(V, U); f(V_i) \subset U_{i+k} \ (i \in \mathbb{Z}_2)\}$$

for $k \in \mathbb{Z}_2$. In fact, the formula

$$\mathrm{Hom}_{\mathbb{C}}(V, U) = \mathrm{Hom}_{\mathbb{C}}(V, U)_{\bar{0}} \oplus \mathrm{Hom}_{\mathbb{C}}(V, U)_{\bar{1}}$$

holds since any element f of $\mathrm{Hom}_{\mathbb{C}}(V, U)$ can be decomposed as $f = f_{\bar{0}} + f_{\bar{1}}$ ($f_k \in \mathrm{Hom}_{\mathbb{C}}(V, U)_k$). To define f_k ($k = \bar{0}, \bar{1}$), it is enough to define $f_k(v)$ for each $v \in V$, and by linearity, we have only to define it for homogeneous v. So for $v \in V_i$, denoting the V_{i+k}-component of $f(v)$ by $f_k(v)$, it follows from the definition that $f_k \in \mathrm{Hom}_{\mathbb{C}}(V, U)_k$ and $f = f_{\bar{0}} + f_{\bar{1}}$.

In particular, if $U = V$, then $\mathrm{End}(V) = \mathrm{Hom}_{\mathbb{C}}(V, V)$ is an associative algebra under the product defined as the composition of maps, as can be seen in Example 1.12, and with this $\mathbb{Z}_2$-graded structure, it becomes a $\mathbb{Z}_2$-graded associative algebra. We denote this $\mathbb{Z}_2$-graded associative algebra and its associated Lie superalgebra by $\mathrm{End}(V)$. This notation is the same as that for Lie algebras, but we will distinguish them by the situation.

For simplicity, we will use the following notation below:

NOTATION 1.82. For natural numbers m, n, we denote the vector space of all $m \times n$ matrices (over $\mathbb{C}$) by $M(m, n)$.

EXAMPLE 1.83. In the above Example 1.81, let us consider the matrix representation of $\mathrm{End}(V)$ in the case when V is of finite dimension. Denote the dimension of $V_{\bar{0}}$ and $V_{\bar{1}}$ by m and n respectively, and choose the basis of V so that

$$\begin{array}{ll} e_1, \cdots, e_m & \text{(a basis of } V_{\bar{0}}), \\ e_{m+1}, \cdots, e_{m+n} & \text{(a basis of } V_{\bar{1}}). \end{array}$$

The matrix representations of elements of $\mathrm{End}(V)_k$ ($k \in \mathbb{Z}_2$) with respect to this basis look as follows:

$$(1.81) \qquad (\mathrm{End}(V))_{\bar{0}} := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}; A \in M_m(\mathbb{C}), D \in M_n(\mathbb{C}) \right\},$$

$$(1.82) \qquad (\mathrm{End}(V))_{\bar{1}} := \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}; B \in M(m,n), C \in M(n,m) \right\}.$$

For $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $y = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{End}(V)$, the bracket of these elements can be computed following the rule (1.80) as follows:

(1.83)

$$[x, y] := \begin{pmatrix} AA' - A'A + BC' + B'C & AB' + BD' - (A'B + B'D) \\ CA' + DC' - (C'A + D'C) & DD' - D'D + CB' + C'B \end{pmatrix}.$$

We denote the Lie superalgebra $M_{m+n}(\mathbb{C})$ obtained by introducing the bracket product (1.83) by $\mathfrak{gl}(m|n)$.

In the above Example 1.83, for each element $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we set

$$\mathrm{str}(x) := \mathrm{tr}A - \mathrm{tr}D$$

and call it the supertrace of x. The supertrace does not depend on the choice of a basis of V. For $x, y \in \mathrm{End}(V)$, set

$$\langle x, y \rangle := \mathrm{str}(xy).$$

Then this is a super-invariant super-symmetric even bilinear form on $\mathrm{End}(V)$. And

$$\mathfrak{sl}(m|n) := \{x \in \mathfrak{gl}(m|n); \mathrm{str}(x) = 0\}$$

is an ideal of $\mathfrak{gl}(m|n)$.

$\mathfrak{sl}(m|n)$ is a simple Lie superalgebra if $m \neq n$, but for $m = n$ this is not the case. If $m = n$, $\mathfrak{sl}(m|m)$ has a one-dimensional center $\mathfrak{z} := \mathbb{C}I_{2m}$, and its quotient $\mathfrak{sl}(m|m)/\mathbb{C}I_{2m}$ is a simple Lie superalgebra for $m \geq 2$. For $m, n \geq 1$ and $m + n \geq 3$, $\mathfrak{sl}(m|n)$ (or its quotient by the center for $m = n$) is said to be of type $A(m-1, n-1)$.

DEFINITION 1.84. Let $\mathfrak{g}$ be a Lie superalgebra. For a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and a homomorphism of Lie superalgebras $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, (π, V) is called a **representation** of $\mathfrak{g}$ on V. If (π, V) is a representation of $\mathfrak{g}$, we say that $\mathfrak{g}$ acts on V, and V is called a $\mathfrak{g}$-module.

If V is of finite dimension and $V_{\bar{0}}$ and $V_{\bar{1}}$ are of dimension m and n respectively, this representation is sometimes called a representation of dimension (m, n). If V is of finite dimension, then the quantity

$$\mathrm{sdim} V := \dim V_{\bar{0}} - \dim V_{\bar{1}}$$

is called the **super-dimension** of a representation. An invariant subspace and the irreducibility of a representation of a Lie superalgebra are the same as in the case of Lie algebras, but here we also have to be careful on the $\mathbb{Z}_2$-structure.

DEFINITION 1.85. Let (π, V) be a representation of $\mathfrak{g}$, and let U be a $\mathbb{Z}_2$-graded vector subspace of V. For all $x \in \mathfrak{g}$ and all $u \in U$, if it satisfies $\pi(x)u \in U$, U is called an **invariant subspace** of V. $\{0\}$ and V are called the trivial invariant subspaces. If it has no invariant subspace other than the trivial invariant subspaces, it is said to be **irreducible**.

A representation which is not irreducible is said to be **reducible**. For any invariant subspace U of V, if an invariant subspace W satisfying $V = U \oplus W$ exists, the representation V is said to be **completely reducible** . The definition of two representations (π, V) and (π', V') of a Lie superalgebra $\mathfrak{g}$ being equivalent is also parallel to the case of Lie algebras. Namely, we have only to replace $\mathrm{Hom}_{\mathfrak{g}}(V, V')$ in Definition 1.22 with

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g},\mathbb{Z}_2}(V, V') &:= \mathrm{Hom}_{\mathfrak{g}}(V, V')_{\bar{0}} \\ &= \left\{ T : V \xrightarrow{\text{linear}} V'; \quad \begin{array}{l} \pi'(x) \circ T = T \circ \pi(x) \ (x \in \mathfrak{g}) \\ T \text{ preserves the } \mathbb{Z}_2\text{-grading} \end{array} \right\}.\end{aligned}$$

An element of $\mathrm{Hom}_{\mathfrak{g},\mathbb{Z}_2}(V, V')$ is called an **intertwining operator** from V to V'. The other fundamental concepts on representations can be defined similarly to the case of Lie algebras, but one has to be slightly careful to define the tensor product of representations and the dual of a representation. For a Lie superalgebra, the tensor product representation of two representations (π, V) and (π', V') is defined by

$$(\pi \otimes \pi')(x)(v \otimes v') := \pi(x)v \otimes v' + (-1)^{|x||v|} v \otimes \pi'(x)v'. \tag{1.84}$$

In this case, the **switching operator**

$$S : V \otimes V' \ni v \otimes v' \longmapsto (-1)^{|v||v'|} v' \otimes v \in V' \otimes V$$

is an intertwining operator of the representations, and the representations on $V \otimes V'$ and $V' \otimes V$ are equivalent.

As in Example 1.81, $\mathrm{Hom}_{\mathbb{C}}(V, V')$ is a $\mathbb{Z}_2$-graded vector space. Defining the action of $\mathfrak{g}$ on this space by

$$(x.f)(v) := x(f(v)) - (-1)^{|x||f|} f(xv) \quad (v \in V) \tag{1.85}$$

for a homogeneous element $x \in \mathfrak{g}$ and $f \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ and extending this formula linearly in x and f, we see that $\mathrm{Hom}_{\mathbb{C}}(V, V')$ becomes a $\mathfrak{g}$-module. In particular, if $V' = \mathbb{C}$[6], then $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual space V^* of V, and the action of the Lie superalgebra $\mathfrak{g}$ on this space is given by

$$(\pi^*(x)f)(v) = -(-1)^{|x||f|} f(\pi(x)v) \quad (x \in \mathfrak{g}, f \in V^*, v \in V). \tag{1.86}$$

Just before Example 1.76, we defined a bilinear form on a Lie superalgebra $\mathfrak{g}$. Here, in the same way, we consider the following conditions on a ($\mathbb{C}$-valued) bilinear form $\langle\ ,\ \rangle$ on a representation space V:

i. (super-invariance) $\langle \pi(x)u, v\rangle + (-1)|x||u|\langle u, \pi(x)v\rangle = 0$,
ii. (super-symmetric) $\langle v, u\rangle = (-1)^{|u||v|}\langle u, v\rangle$,
iii. (even) $\langle V_{\bar{0}}, V_{\bar{1}}\rangle = \{0\}$,
iv. (odd) $\langle V_{\bar{0}}, V_{\bar{0}}\rangle = \langle V_{\bar{1}}, V_{\bar{1}}\rangle$.

Suppose there is a super-invariant super-symmetric even bilinear form $\langle\ ,\ \rangle$ on a representation space V. If we define the map

$$\varphi : V \ni u \longmapsto \varphi_u \in V^*$$

by

$$\varphi_u(v) := \langle u, v\rangle \quad (v \in V), \tag{1.87}$$

it turns out that this is an intertwining operator of $\mathfrak{g}$-modules. If the bilinear form $\langle\ ,\ \rangle$ is non-degenerate, then φ is injective, and if V is of finite dimension, then φ is an isomorphism of $\mathfrak{g}$-modules.

If (π, V) and (π', V') are representations of a Lie superalgebra $\mathfrak{g}$, then, defining the linear map from $V' \otimes V^*$ to $\mathrm{Hom}_{\mathbb{C}}(V, V')$

$$\Phi : V' \otimes V^* \ni a \longmapsto \Phi_a \in \mathrm{Hom}_{\mathbb{C}}(V, V')$$

by

$$\Phi_{v' \otimes \xi}(v) := v' \cdot \xi(v) \quad (v' \in V', \xi \in V^*, v \in V), \tag{1.88}$$

we see that Φ is an intertwining operator, and it is injective. In particular, if V is of finite dimension, Φ is an isomorphism of $\mathfrak{g}$-modules. If there is a super-invariant super-symmetric even bilinear

[6] *Translator's note.* Strictly speaking, one has to fix the parity of V', and it can be of dimension either $(1, 0)$ or $(0, 1)$. Here, $V' = \mathbb{C}$ is regarded as a $\mathbb{Z}_2$-graded vector space of dimension $(1, 0)$.

form $\langle\ ,\ \rangle$, we can do the same things by using the inner product instead of the dual space of V. Namely, defining the linear map

$$\Phi : V \otimes V' \ni a \longmapsto \Phi_a \in \mathrm{Hom}_{\mathbb{C}}(V, V')$$

by

$$\Phi_{v'\otimes u}(v) := v' \cdot \langle u, v\rangle \quad (v' \in V', u, v \in V), \tag{1.89}$$

we see that this is an intertwining operator. In particular, if the bilinear form is non-degenerate, then Φ is injective, and moreover if V is of finite dimension, then it is an isomorphism as above.

If the bilinear form is non-degenerate, then it may happen that one can identify $\mathrm{Hom}_{\mathbb{C}}(V, V')$ with an appropriate completion of the tensor product $V' \otimes V^*$. The following example shows one such method.

EXAMPLE 1.86. Let (π, V) and (π', V') be representations of a Lie superalgebra $\mathfrak{g}$, and suppose V possesses a super-invariant super-symmetric even inner product $\langle\ ,\ \rangle$. Suppose that there exists an abelian group A, and that the Lie superalgebra $\mathfrak{g}$ and the representation space V have a grading (other than the $\mathbb{Z}_2$-grading) by a set S with the action of A:

$$\mathfrak{g} = \bigoplus_{a\in A} \mathfrak{g}^a, \quad V = \bigoplus_{s\in S} V^s$$

satisfying

i. each V^s is a finite-dimensional vector subspace,
ii. $\pi(\mathfrak{g}^a)V^s \subset V^{a\cdot s}$ ($\cdot$ signifies the action of A on S), and
iii. there exists an involution[7] $S \ni s \longmapsto s^* \in S$ satisfying

$$s, t \in S \text{ and } s \neq t \text{ imply } \langle V^s, V^{t^*}\rangle = \{0\}.$$

For each $s \in S$, choose a basis $e_s^{(1)}, \cdots, e_s^{(\dim V^s)}$ of V^s, and denote the vector space spanned by all of the elements of the form

$$\sum_{s\in S} \sum_{i=1}^{\dim V^s} v'_{s,i} \otimes e_s^{(i)} \quad (v'_{s,i} \in V')$$

(allowing an infinite sum) by $V'\widehat{\otimes}V$. Then by condition (iii) the action defined in the same way as in (1.84) is well-defined, and $V'\widehat{\otimes}V$ becomes a $\mathfrak{g}$-module. Defining the linear map

$$\Phi : V'\widehat{\otimes}V \ni a \longmapsto \Phi_a \in \mathrm{Hom}_{\mathbb{C}}(V, V')$$

[7]If a map is not the identity map but its iteration is the identity map, then the map is called an involution.

by

$$\Phi_{v'\otimes u}(v) := v' \cdot \langle u, v\rangle \quad (v' \in V', u, v \in V), \tag{1.90}$$

we see that Φ is an isomorphism of $\mathfrak{g}$-modules, and one can identify $V'\widehat{\otimes}V$ with $\mathrm{Hom}_{\mathbb{C}}(V, V')$ via this map. Observe that $V'\widehat{\otimes}V$ does not depend on the choice of a basis of V^s.

In §2.1, we use the method developed in Example 1.86 to study the properties of the Casimir element.

Note. As we defined in §1.1 in this book, the dual space V^* of a vector space V is the vector space of all linear maps from V to $\mathbb{C}$. In particular, if V is of infinite dimension it is often called the **full dual space**. As in Example 1.86, if V is expressed as a (countable) sum of finite-dimensional vector spaces, we may first take the dual space of each finite-dimensional vector space and then take the direct sum of them

$$V^\dagger := \bigoplus_{s\in S}(V^s)^*. \tag{1.91}$$

$V^\dagger$ is called the **restricted dual space** of V with respect to the grading S.

It is clear, by definition, that $V^\dagger$ naturally has the grading (1.91) with respect to S, and the restricted dual space of $V^\dagger$ with respect to this grading is the original vector space V. Namely, for the restricted dual with respect to this grading, we have

$$(V^\dagger)^\dagger = V.$$

In addition, as in Example 1.86, if the action of $\mathfrak{g}$ is compatible with respect to this grading, for each $a \in A$ and $s \in S$ we have

$$\pi^*(\mathfrak{g}^a)((V^s)^*) \subset (V^{a^{-1}s})^*,$$

and $V^\dagger$ is a $\mathfrak{g}$-submodule of V^*. Therefore, if V is of infinite dimension and satisfies the above situation, the dual representation of $\mathfrak{g}$ on the full dual space is not irreducible.

Since $V^\dagger$ depends on the S-grading introduced on V, we should write S explicitly in the notation, but in concrete applications, A and S are usually chosen canonically. In this book, when we use the grading as in Example 1.86, either A is the root lattice and S is a weight lattice, or both A and S are the set $\mathbb{Z}$ of integers. □

EXAMPLE 1.87. Let us take the basis h, e, f of $\mathfrak{sl}(1|1)$ as follows:

$$h := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.92}$$

Then h is an even element, and e and f are odd elements. Their bracket products are as follows:

$$[e, f] = [f, e] = h, \quad \text{the other brackets} = 0. \tag{1.93}$$

It follows from this that $\mathbb{C}h$ is the center of $\mathfrak{sl}(1|1)$ and $\mathfrak{sl}(1|1)$ is nothing but a three-dimensional Heisenberg superalgebra. Let us consider the following representations of this Lie superalgebra. Let $V := \mathbb{C}u \oplus \mathbb{C}v$ be a two-dimensional $\mathbb{Z}_2$-graded vector space. We regard u as an even element and v as an odd element for a moment, but we can also reverse them. Let a be an arbitrary complex number and define the representation (π_a, V) of $\mathfrak{sl}(1|1)$ on V as follows:

$$\begin{aligned} &\pi_a(h)u = au, \quad \pi_a(e)u = 0, \quad \pi_a(f)u = v, \\ &\pi_a(h)v = av, \quad \pi_a(e)v = au, \quad \pi_a(f)v = 0. \end{aligned} \tag{1.94}$$

This representation is irreducible if $a \neq 0$. It is reducible if $a = 0$, and although $\mathbb{C}v$ is a non-trivial invariant subspace, it is not completely reducible.

LEMMA 1.88. *An irreducible finite-dimensional representation of $\mathfrak{sl}(1|1)$ is equivalent either to some (π_a, V) $(a \in \mathbb{C} \setminus \{0\})$ or to the one-dimensional trivial representation (shift the parity if necessary). In particular, any non-trivial irreducible finite-dimensional representation is of dimension $(1, 1)$.*

PROOF. Let (π, V) be a non-trivial irreducible finite-dimensional representation of $\mathfrak{sl}(1|1)$. For one of the eigenvalues a of $\pi(h)$, its eigenspace

$$V_a := \{v \in V; \pi(h)v = av\}$$

is invariant under the action of $\pi(e)$ and $\pi(f)$ by (1.93). Thus, by the assumption on the irreducibility of V, V should be V_a. Namely, $\pi(h)v = av$ holds for all $v \in V$. Next, since e and f are odd elements and satisfy

$$[e, e] = [f, f] = 0,$$

we see that

$$\pi(e)^2 = \pi(f)^2 = 0.$$

So one can choose $0 \neq u \in V$ so that it satisfies $\pi(e)u = 0$. Decomposing this element u as $u = u_{\bar{0}} + u_{\bar{1}}$, we see that

$$\pi(e)u_{\bar{0}} = \pi(e)u_{\bar{1}} = 0,$$

since we have

$$0 = \pi(e)u = \underbrace{\pi(e)u_{\bar{0}}}_{\text{odd}} + \underbrace{\pi(e)u_{\bar{1}}}_{\text{even}}.$$

Thus we can choose a homogeneous element for such u. Here we assume that $u \in V_{\bar{0}}$. Then setting $v := \pi(f)u$, we have

$$\pi(e)v = \pi(e)\pi(f)u = -\pi(f)\underbrace{\pi(e)u}_{0} + \underbrace{\pi(h)u}_{au} = au.$$

By this computation, it follows that the $\mathbb{Z}_2$-graded vector subspace $\mathbb{C}u \oplus \mathbb{C}v$ of V is invariant with respect to the action of h, e and f. Hence by the irreducibility of V, it follows that $V = \mathbb{C}u \oplus \mathbb{C}v$, and the lemma is proved. □

The representations (π_a, V) in Lemma 1.88 are not mutually equivalent, but if we define the automorphism φ_a (as a Lie superalgebra) of $\mathfrak{sl}(1|1)$ by

$$\varphi_a(h) := ah, \quad \varphi_a(e) := ae, \quad \varphi_a(f) = f,$$

for $0 \neq a \in \mathbb{C}$, it follows that $\pi_a = \pi_1 \circ \varphi_a$. In this sense, it might be natural to regard (π_a, V) $(a \in \mathbb{C} \setminus \{0\})$ as 'the same representations'. Namely, we obtain the following proposition:

PROPOSITION 1.89. *A non-trivial irreducible finite-dimensional representation of the three-dimensional Heisenberg Lie superalgebra* $\underbrace{\mathbb{C}h}_{\text{center}} \oplus \underbrace{\mathbb{C}e \oplus \mathbb{C}f}_{\text{odd part}}$ *is unique up to scaling of the center, and it is the following representation* (π, V)*:*

i. $V = \mathbb{C}u \oplus \mathbb{C}v$, *where u, v are even and odd respectively,*
ii. $\pi(h) = \mathrm{Id}_V$,
iii. $\pi(e)u = 0, \pi(e)v = u$,
iv. $\pi(f)u = v, \pi(f)v = 0$.

The conditions (2) and (3) (the super Jacobi identity) of a Lie superalgebra are the conditions to make a Lie superalgebra $\mathfrak{g}$ itself a representation (the adjoint representation) by assigning

$$\mathrm{ad}(x) : \mathfrak{g} \ni y \longmapsto [x, y] \in \mathfrak{g}$$

for each $x \in \mathfrak{g}$.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, its even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and its odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module via the adjoint action. From this point of view, a Lie superalgebra is a pair consisting of (an ordinary) Lie algebra $\mathfrak{g}_{\bar{0}}$ and its representation $\mathfrak{g}_{\bar{1}}$, and moreover the map

$$\varphi : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}} : \quad \text{a symmetric intertwining operator} \tag{1.95}$$

is given to satisfy

$$\varphi(u,v)w + \varphi(v,w)u + \varphi(w,u)v = 0 \tag{1.96}$$

for all $u, v, w \in \mathfrak{g}_{\bar{1}}$. Namely, considering a Lie superalgebra is nothing but considering a pair $(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}, \varphi)$ of a Lie algebra $\mathfrak{g}_0$ and its representation space $\mathfrak{g}_{\bar{1}}$ and the map satisfying (1.95) and (1.96). In this way, a Lie superalgebra can be treated as one of the objects arising in the natural framework of the representation theory of ordinary Lie algebras. If $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra and $\mathfrak{g}_{\bar{1}}$ is completely reducible as a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}$ is called a **classical Lie superalgebra** .

For a finite-dimensional Lie superalgebra $\mathfrak{g}$,

$$B(x,y) := \mathrm{str}_{\mathfrak{g}}(\mathrm{ad}(x)\mathrm{ad}(y)) \quad (x, y \in \mathfrak{g}) \tag{1.97}$$

is called the **Killing form** of $\mathfrak{g}$. Although the Killing form is a super-invariant super-symmetric even bilinear form, the analog of Cartan's criterion (Theorem 1.30) does not hold. For example, in Figure 1.2 (below), the Killing forms of the Lie superalgebras $A(n,n)$, $D(n+1,n)$ and $D(2,1,a)$ are all 0.

The complete classification of simple finite-dimensional Lie superalgebras was obtained in Kac's paper [**K1**], but its method and its result are not so easy as in the case of Lie algebras. Below in this section, we consider the only simple finite-dimensional Lie superalgebras that possess a non-degenerate super-invariant super-symmetric even bilinear form $\langle\ ,\ \rangle$. In this case, as in the case of Lie algebras, the root system is a very useful tool to describe its structure. Since the story is almost the same as in the case of Lie algebras, we only state the main points. First, take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and consider the simultaneous eigenspace of $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$. Namely, for $\alpha \in \mathfrak{h}^*$, we set

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}; [H, X] = \alpha(H)X \quad (H \in \mathfrak{h})\}.$$

If $\alpha \neq 0$ satisfies $\mathfrak{g}_\alpha \neq \{0\}$, α is called a root and $\mathfrak{g}_\alpha$ is called the root space of α, and a (non-zero) element of $\mathfrak{g}_\alpha$ is called a root vector of the root α. As we consider here, in a simple finite-dimensional Lie

superalgebra that possesses a non-degenerate super-invariant supersymmetric even bilinear form, its root spaces are known to be of dimension 1. We denote a non-zero element of $\mathfrak{g}_\alpha$ by X_α. Since $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$, each root space $\mathfrak{g}_\alpha$ is contained either in $\mathfrak{g}_{\bar{0}}$ or in $\mathfrak{g}_{\bar{1}}$. If $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}$, then α is called an 'even root', and if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}$, α is called an 'odd root'. We denote the sets of all even roots and odd roots by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ respectively, and set

$$\Delta := \Delta_{\bar{0}} \cup \Delta_{\bar{1}}.$$

We sometimes write $\Delta_{even}, \Delta_{odd}$ instead of $\Delta_{\bar{0}}, \Delta_{\bar{1}}$ respectively. Using these sets, the even and the odd part of $\mathfrak{g}$ decompose as

$$\mathfrak{g}_{\bar{0}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\bar{0}}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{\bar{1}} = \bigoplus_{\alpha \in \Delta_{\bar{1}}} \mathfrak{g}_\alpha.$$

LEMMA 1.90. *For $\alpha \in \Delta_{\bar{1}}$, the following three conditions are equivalent:*

i. *2α is not a root.*
ii. $[X_\alpha, X_\alpha] = 0$ $(X_\alpha \in \mathfrak{g}_\alpha)$.
iii. $\langle \alpha, \alpha \rangle = 0$.

Identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via the non-degenerate bilinear form $\langle\ ,\ \rangle$ instead of the Killing form, define H_α by

$$\langle H_\alpha, H \rangle = \alpha(H) \quad (H \in \mathfrak{h})$$

for each $\alpha \in \mathfrak{h}^*$. Then the reader might catch the feeling of this lemma from the following computation:

$$\begin{aligned}
[[X_\alpha, X_\alpha], X_{-\alpha}] &= 2[X_\alpha, \underbrace{[X_\alpha, X_{-\alpha}]}_{\langle X_\alpha, X_{-\alpha} \rangle H_\alpha}] \\
&= -2\langle X_\alpha, X_{-\alpha} \rangle \underbrace{[H_\alpha, X_\alpha]}_{\alpha(H_\alpha) X_\alpha} \\
&= -2\langle X_\alpha, X_{-\alpha} \rangle \langle \alpha, \alpha \rangle X_\alpha.
\end{aligned}$$

But of course, this computation is not sufficient for its proof. 'Finite-dimensionality' is essential for this lemma to be true, and if $\mathfrak{g}$ is an infinite-dimensional Lie superalgebra, the equivalence of the conditions (i), (ii) and (iii) in Lemma 1.90 does not necessarily hold (even in the case of $\dim \mathfrak{g}_\alpha = 1$).

As can be seen in Lemma 1.90, the properties of an odd root with $\langle \alpha, \alpha \rangle = 0$ and $\neq 0$ differ. So when we make a Dynkin diagram by

arranging its fundamental roots $\alpha_1, \cdots, \alpha_l$ ($l := \dim \mathfrak{h} = \mathrm{rank}(\mathfrak{g})$), we express each fundamental root α_i by the following symbols:

$$\bigcirc \iff \alpha_i \text{ is an even root,}$$
$$\bullet \iff \alpha_i \text{ is an odd root satisfying } \langle \alpha_i, \alpha_i \rangle \neq 0,$$
$$\otimes \iff \alpha_i \text{ is an odd root satisfying } \langle \alpha_i, \alpha_i \rangle = 0.$$

Figure 1.2 (below) is the list of Dynkin diagrams of simple finite-dimensional Lie superalgebras (that are not Lie algebras) which possess a non-degenerate super-invariant super-symmetric even bilinear form.

For a Lie superalgebra containing $\otimes$, its Dynkin diagram is not unique. The Dynkin diagram presented in Figure 1.2 is somewhat 'canonical'. In Figure 1.2, the number written under the lines combining roots expresses the inner product of these roots. Here the invariant bilinear form is normalized in such a way that

$$\max_i \{\langle \alpha_i, \alpha_i \rangle\} = 2$$

except for the case of $D(2,1,a)$. For example, the matrix of the inner product of fundamental roots of $B(1,2)$ reads as follows:

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq 3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \tag{1.98}$$

Note. From Figure 1.2, the Cartan matrix is computed as follows. First, the Cartan matrix of $D(2,1,a)$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -a \\ 0 & -a & 2a \end{pmatrix},$$

and this is the same as the matrix of the inner product of fundamental roots. The Cartan matrix of the other cases can be constructed as follows.

First set the diagonal component a_{ii} equal 0 to if α_i is $\otimes$, and in the case where α_i is either $\bigcirc$ or $\bullet$, depending on whether the inner product of α_i and a neighboring fundamental root is positive or

negative, we set

$$a_{ii} = \begin{cases} 2 & \text{(if the pairing of the neighboring} \\ & \qquad \text{simple roots is negative),} \\ -2 & \text{(if the pairing of the neighboring} \\ & \qquad \text{simple roots is positive).} \end{cases}$$

Next, a_{ij} $(i \neq j)$ is zero if α_i and α_j are not connected by lines, and if they are connected, we set, according to the way of connection,

$$\overset{\alpha_i}{\textcircled{?}} \xrightarrow[a]{k\text{-folds}} \overset{\alpha_j}{\textcircled{?}} \Longrightarrow \begin{cases} a_{ij} := \operatorname{sgn}(a), \\ a_{ji} := k \times \operatorname{sgn}(a). \end{cases}$$

Here $\textcircled{?}$ is one of $\bigcirc$, $\bullet$ and $\otimes$, and sgn is the signature function, defined as follows:

$$\operatorname{sgn}(a) := \begin{cases} 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

For example, the Cartan matrix of $B(1,2)$ is

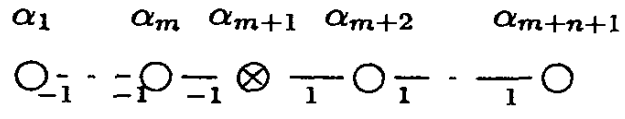
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix}.$$

In this way, to each Dynkin diagram in Figure 1.2, one can associate a Cartan matrix. Generators h_i, e_i, f_i of the Lie superalgebra satisfy the same relations as in (1.67) with this Cartan matrix. □

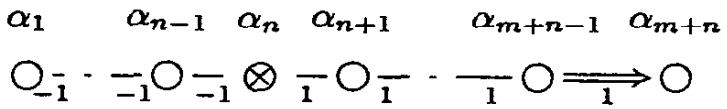
Note. When we study representations of a Lie superalgebra, it may sometimes be convenient for computations to have not -2 but 2 on the diagonal component of the Cartan matrix. On an S-triple or a super S-triple, I sometimes find it convenient to apply the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ in §1.2 or Lemmas 1.92 – 1.94 (below). For each signature of the diagonal component, one has only to choose a signature of e_i, f_i, h_i appropriately for the corresponding i. By replacing the conditions on Cartan matrices by such constraints, the Cartan matrix, for example of $B(1,2)$, becomes $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$. □

Among those in Figure 1.2, the Lie superalgebras of types A, B, C, D can be expressed by matrices. We have stated the A-type case before, so let us consider the following Lie superalgebra to describe

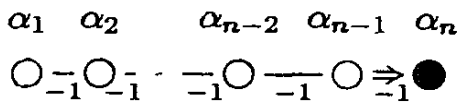
$A(m|n)$ $(m, n \geq 0)$ $(m + n \geq 1)$



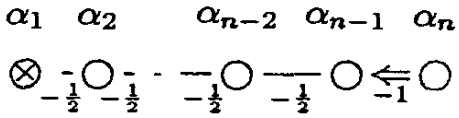
$B(m|n)$ $(m, n \geq 1)$



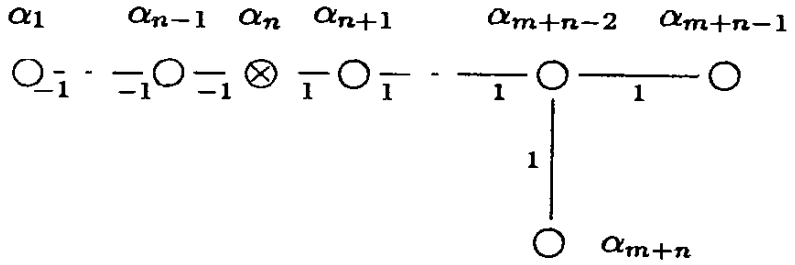
$B(0, n)$ $(n \geq 1)$



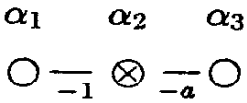
$C(n)$ $(n \geq 3)$



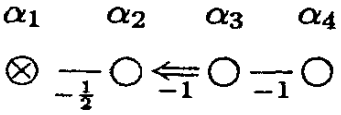
$D(m|n)$ $(m \geq 2)$ $(n \geq 1)$



$D(2, 1; a)$ $(a \in \mathbb{C})$ $(a \neq 0, -1)$



$F(4)$



$G(3)$

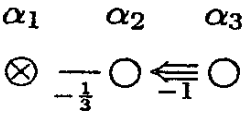


FIGURE 1.2. The Dynkin diagrams of the simple finite-dimensional Lie superalgebras (excluding the case of Lie algebras) that have a non-degenerate super-symmetric super-invariant even bilinear form

the B, C, and D type cases:

$\mathfrak{osp}(2m+1, 2n) :=$

$$\left\{ \begin{array}{c} \left(\begin{array}{ccc|cc} A & B & u & x_1 & x_2 \\ C & -{}^tA & v & y_1 & y_2 \\ -{}^tv & -{}^tu & 0 & z_1 & z_2 \\ \hline {}^ty_2 & {}^tx_2 & {}^tz_2 & D & E \\ -{}^ty_1 & -{}^tx_1 & -{}^tz_1 & F & -{}^tD \end{array}\right) \begin{array}{l} \}m \\ \}m \\ \}1 \\ \}n \\ \}n \end{array} ; \\ \begin{array}{ccccc} \underbrace{\quad}_{m} & \underbrace{\quad}_{m} & \underbrace{\quad}_{1} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} \end{array} \end{array} \begin{array}{c} B, C \text{ are} \\ \text{skew-symmetric} \\ \text{matrices,} \\ E, F \text{ are} \\ \text{symmetric} \\ \text{matrices} \end{array} \right\},$$

$\mathfrak{osp}(2m, 2n) :=$

$$\left\{ \begin{array}{c} \left(\begin{array}{cc|cc} A & B & x_1 & x_2 \\ C & -{}^tA & y_1 & y_2 \\ \hline {}^ty_2 & {}^tx_2 & D & E \\ -{}^ty_1 & -{}^tx_1 & F & -{}^tD \end{array}\right) \begin{array}{l} \}m \\ \}m \\ \}n \\ \}n \end{array} ; \\ \begin{array}{cccc} \underbrace{\quad}_{m} & \underbrace{\quad}_{m} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} \end{array} \end{array} \begin{array}{c} B, C \text{ are} \\ \text{skew-symmetric} \\ \text{matrices,} \\ E, F \text{ are} \\ \text{symmetric} \\ \text{matrices} \end{array} \right\}.$$

In the above definition of $\mathfrak{osp}(r, s)$, the column and the row line in the matrices signifies the boundary of the even part and the odd part, and the bracket product of matrices of the form $\left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$ is computed by the formula (1.83). Then we have

$$\begin{aligned} B(m,n) &\cong \mathfrak{osp}(2m+1, 2n) && (m \geq 0, n \geq 1), \\ D(m,n) &\cong \mathfrak{osp}(2m, 2n) && (m \geq 2, n \geq 1), \\ C(n) &\cong \mathfrak{osp}(2, 2n-2) && (n \geq 3). \end{aligned}$$

The structures of representation spaces of the Lie superalgebras containing $\otimes$ are very different from those of Lie algebras. As I mentioned in the previous section, the arguments concerning Lie superalgebras in Chapter 2 avoid these obstructions. By Figure 1.2, the only rank 1 Lie superalgebra that is not a Lie algebra is $B(0,1)$, and its Dynkin diagram is $\bullet$. One can choose a basis h, e', f', e, f of this Lie superalgebra $\mathfrak{g}$ in a way that

$$\begin{aligned} h, e', f' : &\quad \text{a basis of } \mathfrak{g}_{\bar{0}}, \\ e, f : &\quad \text{a basis of } \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The bracket products among these elements can be normalized as in Table 1.1.

TABLE 1.1. The table of brackets $[x, y]$ in $B(0,1)$

$x \backslash y$	h	e'	f'	e	f
h	0	$4e'$	$-4f'$	$2e$	$-2f$
e'	$-4e'$	0	$\frac{1}{2}h$	0	$-e$
f'	$4f'$	$-\frac{1}{2}h$	0	$-f$	0
e	$-2e$	0	f	$4e'$	h
f	$2f$	e	0	h	$-4f'$

DEFINITION 1.91. Let $\mathfrak{g}$ be a Lie superalgebra, h an element of $\mathfrak{g}_{\bar{0}}$ and e, f elements of $\mathfrak{g}_{\bar{1}}$. If h, e, f satisfy the same bracket products as the h, e, f in the above Table 1.1, (h, e, f) is called a **super S-triple**.

We remark that the three-dimensional vector space spanned by a super S-triple is not a Lie superalgebra. Adding $[e, e]$ and $[f, f]$ to h, e, f, we obtain a basis of the Lie superalgebra generated by the super S-triple. Namely, the Lie superalgebra generated by a super S-triple is isomorphic to $B(0,1)$. Normalizing an invariant bilinear form $\langle\ ,\ \rangle$ of $\mathfrak{g} = B(0,1)$ as $\langle h, h\rangle = 2$, one has $\langle e, f\rangle = 1$, $\langle e', f'\rangle = \frac{1}{4}$. If we put

$$h' := \frac{1}{2}h, \tag{1.99}$$

then (h', e', f') is an S-triple. The next lemma can be easily checked by induction on k and Lemma 1.80:

LEMMA 1.92. *If (h, e, f) is a super S-triple, then the following relations hold in the universal enveloping algebra for all $k \in \mathbb{Z}_{\geq 0}$:*

1. $[h, e^k] = 2ke^k, \quad [h, f^k] = -2kf^k,$
2. $[e, f^{2k}] = -2kf^{2k-1}, \ [e, f^{2k+1}] = f^{2k}(h - 2k) = (h + 2k)f^{2k},$
3. $[f, e^{2k}] = 2ke^{2k-1}, \quad [f, e^{2k+1}] = e^{2k}(h + 2k) = (h - 2k)e^{2k}.$

From (2) of this lemma, the next lemma immediately follows:

LEMMA 1.93. *Let (h, e, f) be a super S-triple of the Lie superalgebra $B(0,1)$, and let V be a representation space of $B(0,1)$. If an element v_0 $(\neq 0)$ of V and $\lambda_0 \in \mathbb{C}$ satisfy*

$$ev_0 = 0 \quad \textit{and} \quad hv_0 = \lambda v_0,$$

then the elements

$$v_j := f^j v_0 \quad (j \in \mathbb{Z}_{\geq 0})$$

satisfy the following relations for all $k \in \mathbb{Z}_{\geq 0}$*:*

$$ev_{2k} = -2kv_{2k-1}, \quad ev_{2k+1} = (\lambda_0 - 2k)v_{2k}.$$

Now let us consider an irreducible finite-dimensional representation (π, V) of $B(0,1)$. As in the case of $\mathfrak{sl}(2,\mathbb{C})$, let us decompose $V = V_{\bar{0}} \oplus V_{\bar{1}}$ into eigenspaces of $\pi(h)$. From the representation theory of an S-triple (h', e', f'), it turns out that all of the eigenvalues of $\pi(h')$ are integers, and in addition from (1.99) it follows that all of the eigenvalues of $\pi(h)$ are even integers. Let λ be the maximal eigenvalue of $\pi(h)$, and let $0 \neq v_0 \in V_\lambda$ be an eigenvector. Then

$$\pi(h)v_0 = \lambda v_0, \quad \pi(e)v_0 = 0. \tag{1.100}$$

Hence, putting

$$v_j := \pi(f)^j v_0 \quad (j = 0, 1, 2, \cdots), \tag{1.101}$$

one can prove the following formula by induction on j in the same way as in the case of $\mathfrak{sl}(2,\mathbb{C})$:

$$\pi(e)v_j := c_j v_{j-1} \quad (j \in \mathbb{Z}_{\geq 0}). \tag{1.102}$$

Here c_j is given by the following formulae:

$$c_{2k} = -2k, \tag{1.103}$$

$$c_{2k+1} = \lambda - 2k. \tag{1.104}$$

Since the v_j's belong to the eigenspaces with different eigenvalues, they are linearly independent, and since V is of finite dimension, only a finite number of them can be non-zero. Hence there exists $N \in \mathbb{Z}_{\geq 0}$ so that $c_{2N+1} = 0$. Then, by (1.104), $\lambda = 2N$ and $\{v_j; 0 \leq j \leq 2N\}$ spans an invariant subspace of V. Thus, by the irreducibility of V, it follows that

$$V = \bigoplus_{j=0}^{2N} \mathbb{C}v_j.$$

Hence we obtain the following lemma:

LEMMA 1.94. *Let* $\mathfrak{g} = B(0,1)$ *and choose an element* h *of* $\mathfrak{g}$ *as above. If* (π, V) *is an irreducible finite-dimensional representation of* $\mathfrak{g}$*, then it follows that*

1. $\dim V$ *is odd, and*

2. *putting* $\dim V = 2N+1$, *the eigenvalues of* $\pi(h)$ *are*

$$\{2N-j; 0 \le j \le 2N\} = \{j \in 2\mathbb{Z}; -2N \le j \le 2N\},$$

and the multiplicity of each eigenvalue is one.

As one can see from this lemma, in general, if $\mathfrak{g}$ is a Lie superalgebra, inside the finite-dimensional representations of the Lie superalgebra generated by a super S-triple contained in $\mathfrak{g}$, the eigenvalues of h are all even integers. The condition ($\tilde{\mathrm{C}}5$) for $i \in \Psi$ in the definition of the BKM matrices in the next chapter will reflect this fact.

Note. If $\mathfrak{g}$ is a Lie superalgebra that possesses a non-degenerate super-invariant super-symmetric even bilinear form, then the Lie (super-)subalgebra generated by $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ for a root α is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{sl}(1|1)$, or $B(0,1) \cong \mathfrak{osp}(1,2)$. □

Finally, let us state the super-version of Lemmas 1.33 – 1.38:

LEMMA 1.95. *Even if* $\mathfrak{g}$ *is a Lie superalgebra or* A *is a* $\mathbb{Z}_2$*-graded associative algebra, the formulae in Lemmas 1.33, 1.34, 1.36 and 1.38 hold for an even element* x.

This is because if x is an even element, then the properties of the brcaket product are the same as for the ordinary Lie algebras. And for the sake of the application to Chapter 2, such comments are sufficient.

CHAPTER 2

Structures and Representations of BKM (Super-) Algebras

The purpose of this chapter is to introduce the reader to the shortest course to the character formulae from BKM algebras and BKM superalgebras. In this chapter, I will take aim at the structure of the root system and the character formulae of Lie superalgebras. I will not abbreviate anything that is necessary for those goals, and will not add anything that is not necessary for them (as far as possible). I will concentrate on describing where and how which conditions and which properties work.

For example, as can be seen in §2.5, the character formulae for a symmetrizable Kac-Moody Lie algebra can be obtained, in practice, only by an inequality (2.112), if we study the invariance by Weyl group, Casimir operator and the properties of primitive weights. For BKM algebras and BKM superalgebras, it naturally requires a few more detailed arguments, but they are simply a deformation to let things work.

Concerning the Weyl group, what is important in particular is the property of 'local nilpotency'. Notice that the importance of the Weyl group (i.e., the invariance of the root system under the Weyl group, and the invariance of character under the Weyl group) does not rely only on the fact that the Weyl group is a reflection group, but also on the support of e_i's and f_i's. It may seem that the representation theory of Lie algebras is developed on the main stage called a Cartan subalgebra, but please observe that there are those sweating away behind the scenes. The property 'local nilpotency' supports their part of the task.

2. *putting* $\dim V = 2N+1$, *the eigenvalues of* $\pi(h)$ *are*

$$\{2N-j; 0 \le j \le 2N\} = \{j \in 2\mathbb{Z}; -2N \le j \le 2N\},$$

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Note. If $\mathfrak{g}$ is a Lie superalgebra that possesses a non-degenerate super-invariant super-symmetric even bilinear form, then the Lie (super-)subalgebra generated by $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ for a root α is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{sl}(1|1)$, or $B(0,1) \cong \mathfrak{osp}(1,2)$. □

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2.1. Inner Product and Cartan Matrix of BKM (Super-) Algebras

Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ real matrix. First, let us define GCM matrix, BKM matrix and BKM supermatrix:

DEFINITION 2.1. If A satisfies Conditions (C0) – (C3) just after Definition 1.46, A is called a 'generalized Cartan matrix'(GCM for short), or **KM matrix**.

DEFINITION 2.2. If A satisfies the following conditions, then A is called a **BKM matrix**:

($\widetilde{\mathrm{C}}1$) $a_{ii} = 2$ or $a_{ii} \le 0$.
($\widetilde{\mathrm{C}}2$) $i \ne j \Longrightarrow a_{ij} \le 0$.
($\widetilde{\mathrm{C}}3$) $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$.
($\widetilde{\mathrm{C}}4$) If $a_{ii} = 2$, then $a_{ij} \in \mathbb{Z}$ for all j.

DEFINITION 2.3. Let Ψ be a subset of $\{1, \cdots, n\}$. If A satisfies the following condition, in addition to ($\widetilde{\mathrm{C}}1$) – ($\widetilde{\mathrm{C}}4$),

($\widetilde{\mathrm{C}}5$) if $i \in \Psi$ and $a_{ii} = 2$, then $a_{ij} \in 2\mathbb{Z}$ for all j,

(A, Ψ) is called a **BKM supermatrix**.

Ψ in the definition of the BKM supermatrices is introduced to describe an odd simple root. If (A, Ψ) is a BKM supermatrix, we may often call A a BKM supermatrix (bearing Ψ in mind) in this book. When A is a BKM matrix or a BKM supermatrix, we define subsets Ire and I^{im} of $I := \{1, \cdots n\}$ by

$$I^{re} := \{i \in I; a_{ii} = 2\},$$
$$I^{im} := \{i \in I; a_{ii} \le 0\}.$$

Moreover, if A is a BKM supermatrix, then we set

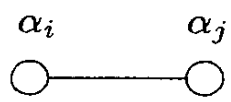
$$\Psi^{re} := \{i \in \Psi; a_{ii} = 2\} = I^{re} \cap \Psi,$$
$$\Psi_0 := \{i \in \Psi; a_{ii} = 0\},$$
$$\Psi_- := \{i \in \Psi; a_{ii} < 0\},$$
$$\Psi^{im} := \{i \in \Psi; a_{ii} \le 0\} = \Psi_0 \cup \Psi_-.$$

Note. As one can see by comparing Definition 2.2 and Definition 2.3, a BKM matrix is a BKM supermatrix with empty Ψ. At a glance, since more conditions are imposed on BKM supermatrices, the reader might feel that it is a smaller class than that of BKM matrices. But since the additional structure is introduced on a BKM supermatrix

(A, Ψ), the case where we do not consider this structure, i.e., $\Psi = \varnothing$ corresponds to a BKM matrix. $\square$

Let $A = (a_{ij})$ be a matrix satisfying one of the three definitions above. Since the definition of BKM supermatrices is the most general, we may assume that A is a BKM supermatrix, unless otherwise stated. Then, we will associate a diagram to A by the following rules:

1. Put n ○'s on a plane, call ○ a vertex, and name these vertices $\alpha_1, \cdots, \alpha_n$.
2. If $i \neq j$ and $a_{ij} < 0$, connect the two vertices α_i and α_j:



3. If $i \neq j$ and $a_{ij} = 0$, do not connect α_i and α_j.

The n vertices thus obtained and the diagram connecting them by appropriate segments is denoted by qDyn(A), and is called a **quasi-Dynkin diagram** . This is simply a diagram removing arrows and replacing multilines by a simple line from the Dynkin diagram. It is a diagram that only shows which vertices are connected.

DEFINITION 2.4. If the quasi-Dynkin diagram of A is connected, A is said to be **indecomposable**.

If the quasi-Dynkin diagram of A is not indecomposable, then by rearranging $\alpha_1, \cdots, \alpha_n$ properly (i.e., by applying the same permutation to the rows and the columns of A), A looks as follows:

$$A = \begin{pmatrix} \boxed{A_1} & & & \mathbf{0} \\ & \boxed{A_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \boxed{A_k} \end{pmatrix}.$$

In this case, the A_i's are again BKM supermatrices with small size. Later, we will construct the Lie superalgebra $\mathfrak{g}(A)$ associated to a BKM supermatrix, and then the corresponding Lie superalgebra looks like

$$\mathfrak{g}(A) = \mathfrak{g}(A_1) \oplus \cdots \oplus \mathfrak{g}(A_k).$$

DEFINITION 2.5. If a BKM supermatrix $A = (a_{ij})_{1 \leq i,j \leq n}$ decomposes as

($\widetilde{\mathrm{C}}6$) $A = DB$, where

$$D = (\varepsilon_i \delta_{ij})_{1 \le i,j \le n} : \text{ a diagonal matrix,}$$

$$B = (b_{ij})_{1 \le i,j \le n} : \text{ a symmetric matrix,}$$

A is said to be **symmetrizable.**

LEMMA 2.6. *If A is a symmetrizable BKM supermatrix, then one can choose the ε_i $(1 \le i \le n)$ in the above definition as positive numbers.*

PROOF. First, let us consider the case when A is indecomposable. Then, one can see that $\varepsilon_1, \cdots, \varepsilon_n$ are all non-zero. This is because if $\varepsilon_i = 0$ for some i, then the i-th row of A becomes all 0 by $A = DB$. Therefore, from condition ($\widetilde{\mathrm{C}}3$), it follows that the i-th column for the same i also becomes all 0, and this contradicts the assumption that A is indecomposable. Now, since

$$a_{ij} = \varepsilon_i b_{ij}, \quad a_{ji} = \varepsilon_j b_{ji}$$

and $b_{ij} = b_{ji}$, we have

$$\varepsilon_j a_{ij} = \varepsilon_i a_{ji} \tag{2.1}$$

for each i and j. Let us take any i, j $(i \ne j)$ satisfying $a_{ij} \ne 0$, and fix them. Then, since a_{ij} and a_{ji} are both negative numbers, we have

$$\frac{\varepsilon_j}{\varepsilon_i} > 0 \tag{2.2}$$

by (2.1). Namely, if α_i and α_j are connected in the quasi-Dynkin diagram, then ε_i and ε_j have the same signature. Thus, by the connectivity of the quasi-Dynkin diagram, if we take one of the ε_i's (e.g., ε_1) to be a positive number, then we get $\varepsilon_i > 0$ for all i.

If A is decomposable, then one can apply the above argument to each connected component of the quasi-Dynkin diagram of A, and one obtain $\varepsilon_i > 0$ on each connected component, which implies one can choose $\varepsilon_i > 0$ for all i. □

If A is a symmetrizable BKM supermatrix, then, taking the diagonal matrix D satisfying $\varepsilon_1, \cdots, \varepsilon_n > 0$, by $a_{ij} = \varepsilon_i b_{ij}$, we have

$$a_{ij} \gtreqless 0 \iff b_{ij} \gtreqless 0 \tag{2.3}$$

for each i and j, and hence we have

$$i \ne j \Longrightarrow b_{ij} \le 0 \tag{2.4}$$

by Condition ($\widetilde{\mathrm{C}}2$).

Note. It may seem that the assertion of Lemma 2.6 is really natural, but as we will see in the latter half of this chapter, the fact that one can take the ε_i to be all positive is the key point to proving the character formula. And, as we saw in the proof of this lemma, this property reflects condition ($\widetilde{\mathrm{C}}2$) among the conditions of BKM supermatrices. Figure 1.2 in §1.5 is the list of well-behaved simple finite-dimensional Lie superalgebras, but notice that most of them do not satisfy (2.4), hence most of them do not satisfy condition ($\widetilde{\mathrm{C}}2$). Therefore, we cannot apply the arguments developed in this chapter naïvely to simple finite-dimensional Lie superalgebras. □

Below, we assume that (A, Ψ) is a symmetrizable and indecomposable BKM supermatrix, unless otherwise stated. And, we take a diagonal matrix D satisfying $\varepsilon_1, \cdots, \varepsilon_n > 0$, and fix it. Starting from A, we will construct the Lie superalgebra $\mathfrak{g}(A)$. Let us first construct its Cartan subalgebra. Let $\mathfrak{h}$ be a vector space (over $\mathbb{C}$) whose dimension is

$$\dim \mathfrak{h} = n + \operatorname{corank}(A) = 2n - \operatorname{rank}(A). \tag{2.5}$$

And take

$$\begin{aligned} \alpha_1^\vee, \cdots, \alpha_n^\vee &: \ n \text{ linearly independent elements of } \mathfrak{h}, \\ \alpha_1, \cdots, \alpha_n &: \ n \text{ linearly independent elements of } \mathfrak{h}^* \end{aligned}$$

satisfying

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}. \tag{2.6}$$

Condition (2.5) on the dimension of $\mathfrak{h}$ is the minimal possible dimension to take such elements. Namely, if the dimension of $\mathfrak{h}$ is less than the right hand side of (2.5), then such α_i's and $\alpha_i^\vee$'s do not exist. It can be shown by a simple (but rather dull) exercise of linear algebra that under condition (2.5) on the dimension, α_i and $\alpha_i^\vee$ $(i = 1, \cdots, n)$ satisfying (2.6) exist, and they are determined up to conjugation by an element g of $GL(\mathfrak{h})$ and the corresponding element ${}^t g$ of $GL(\mathfrak{h}^*)$. In any case, we could construct a Cartan subalgebra $\mathfrak{h}$. Setting

$$\Pi^\vee := \{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}, \quad \Pi := \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*,$$

call an element Π a **fundamental root** or a **simple root**, and call an element of $\Pi^\vee$ a **fundamental coroot** or a **simple coroot**. Moreover, set

$$\Pi^{re} := \{\alpha_i; i \in I^{re}\}, \quad \Pi^{im} := \{\alpha_i; i \in I^{im}\},$$

and call an element of Π^{re} (resp. Π^{im}) a real simple root (resp. an imaginary simple root). If A is a BKM supermatrix, we also divide Π as follows:

$$\Pi_{even} := \{\alpha_i; i \in I \subset \Psi\} = \{\text{even simple roots}\},$$
$$\Pi_{odd} := \{\alpha_i; i \in \Psi\} = \{\text{odd simple roots}\}.$$

Note. For a simple finite dimensional Lie algebra, we have $|\Pi| = \dim \mathfrak{h}$ as we saw in Theorem 1.64 in §1.3, but now, in general, we have $|\Pi| \leq \dim \mathfrak{h}$. □

Next, let us introduce an inner product on $\mathfrak{h}$. For this purpose, let us set

$$\mathfrak{h}' := \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee,$$

and take a subspace $\mathfrak{h}''$ of $\mathfrak{h}$ satisfying

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'',$$

and fix it. We define the symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}$ as follows:

$$(\alpha_i|h) := \varepsilon_i\langle\alpha_i, h\rangle \quad (h \in \mathfrak{h}, 1 \leq i \leq n), \tag{2.7}$$

$$(h_1''|h_2'') := 0 \qquad (h_1'', h_2'' \in \mathfrak{h}''). \tag{2.8}$$

Then $(\ |\ )$ is non-degenerate on $\mathfrak{h}$, and this induces the linear isomorphism

$$\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*.$$

Below, we completely identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via this map ν, and omit the symbol ν. First, we have the following lemma:

LEMMA 2.7. *For $1 \leq i, j \leq n$, one has the following formulae:*

1. $\alpha_i^\vee = \varepsilon_i\alpha_i$,
2. $(\alpha_i|\alpha_j) = b_{ij}$,
3. $(\alpha_i^\vee|\alpha_j^\vee) = \varepsilon_i\varepsilon_j b_{ij}$,
4. *if $a_{ii} \neq 0$, then*

$$\frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} = \frac{2\langle\alpha_i^\vee, \lambda\rangle}{a_{ii}} \qquad (\lambda \in \mathfrak{h}^*);$$

in particular, if $a_{ii} = 2$, then one has

$$\frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} = \langle\alpha_i^\vee, \lambda\rangle \qquad (\lambda \in \mathfrak{h}^*).$$

PROOF. (1) immediately follows from (2.7). (3) can be obtained by specializing (2.7) to $h = \alpha_j^\vee$ as follows:

$$(\alpha_i^\vee|\alpha_j^\vee) = \varepsilon_i\langle\alpha_i, \alpha_j^\vee\rangle = \varepsilon_i a_{ji} = \varepsilon_i\varepsilon_j b_{ij}.$$

(2) follows from (1) and (3). (4) can be obtained by calculating the left hand side using $\alpha_i = \dfrac{\alpha_i^\vee}{\varepsilon_i}$ as follows:

$$\text{the left hand side of (4)} = \frac{2\langle\alpha_i^\vee, \lambda\rangle}{\langle\alpha_i^\vee, \alpha_i\rangle} = \frac{2\langle\alpha_i^\vee, \lambda\rangle}{a_{ii}}.$$

□

We remark that from (2) of this lemma and (2.3), one has

$$i \in I^{im}, j \in I \Longrightarrow (\alpha_i|\alpha_j) \leq 0. \tag{2.9}$$

We do not use this property in §2.1, but it is important for §2.3 when we study the structure of root systems. In terms of the inner product, one can say that

$$\Pi^{re} = \{\alpha_i; (\alpha_i|\alpha_i) > 0\}, \quad \Pi^{im} = \{\alpha_i; (\alpha_i, \alpha_i) \leq 0\}.$$

Note. When we fix the direct sum decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ of $\mathfrak{h}$, if we set

i. $\mathfrak{h}'_{\mathbb{R}} := \sum_{i=1}^n \mathbb{R}\alpha_i^\vee$,
ii. $\mathfrak{h}''_{\mathbb{R}} := \{h \in \mathfrak{h}''; \langle\alpha_i, h\rangle \in \mathbb{R}\ (1 \leq i \leq n)\}$,

then

$$\mathfrak{h}_{\mathbb{R}} := \mathfrak{h}'_{\mathbb{R}} \oplus \mathfrak{h}''_{\mathbb{R}}$$

is a vector subspace over $\mathbb{R}$ of $\mathfrak{h}$, and it satisfies $\mathfrak{h} = \mathbb{C} \otimes \mathfrak{h}_{\mathbb{R}}$. Namely, $\mathfrak{h}_{\mathbb{R}}$ is a **real form** of $\mathfrak{h}$. This $\mathfrak{h}_{\mathbb{R}}$ will be used after Chapter 3. □

When (A, Ψ) is a BKM supermatrix, to construct the Lie superalgebra $\mathfrak{g}(A)$ whose Cartan matrix is A, we first take $2n$ letters

$$e_1, \cdots, e_n, f_1, \cdots, f_n.$$

We make a Lie superalgebra by collecting all of the linear combinations of bracket products of elements of $\mathfrak{h}$ and these elements. Here, we assume that e_i, f_i $(i \in \Psi)$ are odd elements, and the other e_i, f_i's and elements of $\mathfrak{h}$ are even elements. We impose the next fundamental

relations on these bracket products:

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h}),, \tag{2.10}$$

$$[h, e_i] = \alpha_i(h)e_i \qquad (h \in \mathfrak{h}, 1 \leq i \leq n), \tag{2.11}$$

$$[h, f_i] = -\alpha_i(h)f_i \qquad (h \in \mathfrak{h}, 1 \leq i \leq n), \tag{2.12}$$

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee \qquad (1 \leq i, j \leq n). \tag{2.13}$$

We denote this Lie superalgebra by $\widehat{\mathfrak{g}}(A)$. $\widehat{\mathfrak{g}}(A)$ naturally decomposes into the direct sum of root spaces. Namely, for $\alpha \in \mathfrak{h}^*$, set

$$\widehat{\mathfrak{g}}_\alpha := \{X \in \widehat{\mathfrak{g}}(A); [h, X] = \alpha(h)X \ (h \in \mathfrak{h})\},$$

and call $\alpha \neq 0$ satisfying $\widehat{\mathfrak{g}}_\alpha \neq \{0\}$ a **root** of $\widehat{\mathfrak{g}}(A)$, denote the set of all roots by $\widehat{\Delta}$, and for $\alpha \in \widehat{\Delta}$, call $\widehat{\mathfrak{g}}_\alpha$ the **root space** of the root α. For each i, it can be easily shown that $\pm\alpha_i$ are roots of $\widehat{\mathfrak{g}}(A)$, and $\widehat{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i$, $\widehat{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i$. For $i \neq j$, it can be easily seen that

$$\alpha_i + \alpha_j, \quad \alpha_i + 2\alpha_j, \quad \alpha_i + 3\alpha_j, \quad \cdots$$

are all roots of $\widehat{\mathfrak{g}}(A)$, and each of root space is a one-dimensional vector subspace spanned by

$$[e_i, e_j], \quad [[e_i, e_j], e_j], \quad [[[e_i, e_j], e_j], e_j], \quad \cdots.$$

Here, if e_i and e_j are both odd elements, it seems that this is non-trivial, but even in this case one can prove this by (1.72) and the super Jacobi identity. Moreover, if i, j and k are all distinct, then $\alpha_i + \alpha_j + \alpha_k$ is again a root of $\widehat{\mathfrak{g}}(A)$, and the dimension of it is 2, and one can take

$$[[e_i, e_j], e_k], \quad [[e_i, e_k], e_j]$$

as a basis. The reason why it is not of dimension 3 is that $[[e_j, e_k], e_i]$ can be expressed as a linear combination of the above two elements by the (super) Jacobi identity.

By the fundamental relations (2.10) – (2.13), an element of the root space $\widehat{\mathfrak{g}}_\alpha$ can be expressed either as a linear combination of bracket products of e_i's only or as a linear combination of bracket products of f_i's only. In other words, a root α of $\widehat{\mathfrak{g}}(A)$ can be expressed as

$$\alpha = \sum_{i=1}^{n} m_i\alpha_i \quad (m_i \in \mathbb{Z}),$$

where the m_i's are either all ≥ 0 or all ≤ 0. Corresponding to whether the m_i's are all ≥ 0 or all ≤ 0, α is called a **positive root** or a

negative root respectively. And $\sum_{i=1}^{n} m_i$ is called the **height** of α, and is denoted by $\mathrm{ht}(\alpha)$. α_i $(1 \leq i \leq n)$ is a root of height 1. We denote the sets of all positive roots and negative roots by $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$ respectively. For $\alpha = \sum_{i=1}^{n} m_i\alpha_i \in \widehat{\Delta}_+$, an element of $\widehat{\mathfrak{g}}_\alpha$ can be expressed as a linear combinations of elements obtained by taking the bracket products of an element of $\mathfrak{g}_{\alpha_1}$ m_1 times, with an element of $\mathfrak{g}_{\alpha_2}$ m_2 times, $\cdots$, and with an element of $\mathfrak{g}_{\alpha_n}$ m_n times. From this, it follows that each root space $\widehat{\mathfrak{g}}_\alpha$ is of finite dimension, and by the symmetry of the conditions on e_i and f_i, one has $\dim \widehat{\mathfrak{g}}_\alpha = \dim \widehat{\mathfrak{g}}_{-\alpha}$. If we set

$$\widehat{\mathfrak{n}}_+ := \sum_{\alpha \in \widehat{\Delta}_+} \widehat{\mathfrak{g}}_\alpha, \quad \widehat{\mathfrak{n}}_- := \sum_{\alpha \in \widehat{\Delta}_-} \widehat{\mathfrak{g}}_\alpha,$$

then $\widehat{\mathfrak{n}}_+$ is a free Lie superalgebra generated by $e_1, \cdots, e_n$, and $\widehat{\mathfrak{n}}_-$ is a free Lie superalgebra generated by $f_1, \cdots, f_n$. Here, 'free' means that there is no restraining relation (other than those coming from the fact that they are Lie superalgebras). For a free Lie superalgebra, a formula on the **multiplicity** of a root, i.e., on $\dim \widehat{\mathfrak{g}}_\alpha$, is known. In §2.5 of this book, we will derive it as an application of the denominator identity. Anyway, we obtain a **triangular decomposition** of $\widehat{\mathfrak{g}}(A)$

$$\widehat{\mathfrak{g}}(A) = \mathfrak{h} \oplus \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{n}}_-$$

in this way, and we obtain the following decompositions of the universal enveloping algebra:

$$\mathfrak{U}(\widehat{\mathfrak{g}}(A)) = \mathfrak{U}(\widehat{\mathfrak{n}}_-) \cdot \mathfrak{U}(\mathfrak{h}) \cdot \mathfrak{U}(\widehat{\mathfrak{n}}_+), \tag{2.14}$$

$$\mathfrak{U}(\widehat{\mathfrak{g}}(A)) = \mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot \mathfrak{U}(\mathfrak{h}) \cdot \mathfrak{U}(\widehat{\mathfrak{n}}_-). \tag{2.15}$$

But, the Lie superalgebra $\widehat{\mathfrak{g}}(A)$ considered here is not the one we are looking for. Since $\widehat{\mathfrak{n}}_\pm$ are free Lie superalgebras, and since they are too big, $\widehat{\mathfrak{g}}(A)$ cannot be a 'simple' Lie superalgebra. Here, I blurted out the word 'simple', but for an infinite-dimensional Lie (super)algebra, we still have not defined the notion 'simple', and it is not even quite clear how to treat it. For infinite-dimensional Lie superalgebras, let us state the definition of the concept 'simple' as it is restricted to the case we are to consider here. Suppose that an indecomposable BKM supermatrix A is given. And suppose that $\mathfrak{g}$ is a Lie superalgebra generated by its Cartan subalgebra $\mathfrak{h}$ and generators $e_1, \cdots, e_n, f_1, \cdots, f_n$ which satisfy the relations (2.10) – (2.13) among generators. In this situation, we define the notion 'simple' as follows:

DEFINITION 2.8. If there does not exist an ideal $\mathfrak{i}$ of $\mathfrak{g}$ satisfying $\mathfrak{i} \cap \mathfrak{h} = \{0\}$, $\mathfrak{g}$ is called a **simple BKM Lie superalgebra**.

Since an ideal of $\mathfrak{g}$ is (naturally) invariant under the action of $\mathrm{ad}(\mathfrak{h})$, under the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \text{ is a root}} \mathfrak{g}_\alpha,$$

$\mathfrak{i}$ decomposes as

$$\mathfrak{i} = (\mathfrak{i} \cap \mathfrak{h}) \oplus \sum_{\alpha \text{ is a root}} (\mathfrak{i} \cap \mathfrak{g}_\alpha).$$

So, we have

$$\mathfrak{i} \cap \mathfrak{h} = \{0\} \Longleftrightarrow \mathfrak{i} \subset \sum_{\alpha \text{ is a root}} \mathfrak{g}_\alpha,$$

and hence one can rephrase the above definition by saying that there is no ideal $\mathfrak{i}$ of $\mathfrak{g}$ other than $\{0\}$ satisfying $\mathfrak{i} \subset \sum_{\alpha \text{ is a root}} \mathfrak{g}_\alpha$.

The reason why we define a 'simple' object as above is that we would not like to kill any part of $\mathfrak{h}$ by the ideal $\mathfrak{i}$. The assumption (2.5) on the dimension of $\mathfrak{h}$ is the minimal dimension so that one can take Π and $Pi^\vee$ satisfying (2.6). Therefore, we must not touch this part.

Why is $\widehat{\mathfrak{g}}(A)$ not a Lie superalgebra, and how can we get a simple Lie superalgebra? To see the answer, let us slightly go into the structure of $\widehat{\mathfrak{g}}(A)$. Since $\widehat{\mathfrak{g}}(A)$ is a Lie superalgebra, regarding $\widehat{\mathfrak{g}}(A)$ itself as a representation space of $\widehat{\mathfrak{g}}(A)$, one obtains the adjoint representation of $\widehat{\mathfrak{g}}$:

$$\mathrm{ad} : \widehat{\mathfrak{g}}(A) \ni x \longmapsto \mathrm{ad}(x) \in \mathrm{End}(\widehat{\mathfrak{g}}(A)).$$

We denote the homomorphism, as associative superalgebras from the universal enveloping algebra to $\mathrm{End}(\widehat{\mathfrak{g}}(A))$ induced from the above map, by

$$\varphi : \mathfrak{U}(\widehat{\mathfrak{g}}(A)) \ni u \longmapsto \varphi(u) \in \mathrm{End}(\widehat{\mathfrak{g}}(A)). \tag{2.16}$$

An element u of $\mathfrak{U}(\widehat{\mathfrak{g}}(A))$ can be written as a linear combination of elements of the form $x_1 \cdots x_n$ ($x_i \in \mathfrak{g}, n \in \mathbb{Z}_{\geq 0}$), and for $u = x_1 \cdots x_n$, the action of this element on $\widehat{\mathfrak{g}}(A)$ looks as follows:

$$\begin{aligned} \varphi(u)y &= \varphi(x_1 \cdots x_n)y \\ &= [x_1, [x_2, [\cdots [x_n, y] \cdots]]] \quad (y \in \widehat{\mathfrak{g}}(A)). \end{aligned}$$

And since

applying $\mathfrak{U}(\widehat{\mathfrak{g}}(A)) \Longleftrightarrow$ applying elements of $\widehat{\mathfrak{g}}(A)$ repeatedly,

for a given $y_0 \in \widehat{\mathfrak{g}}(A)$, it can be seen that the invariant subspace of $\widehat{\mathfrak{g}}(A)$ generated by y_0 is $\mathfrak{U}(\widehat{\mathfrak{g}}(A)) \cdot y_0$. In other words, the following, in general, holds:

LEMMA 2.9. *Let $\mathfrak{g}$ be a Lie superalgebra, and let x be an element of $\mathfrak{g}$. Then, the ideal of $\mathfrak{g}$ generated by x is $\mathfrak{U}(\mathfrak{g}) \cdot x$.*

Now, let us prove the following in $\widehat{\mathfrak{g}}(A)$:

LEMMA 2.10. *For $i \in I^{re}$ and $j \in I$ $(i \neq j)$, the following formulae hold for $k \in I$:*

1. $[f_k, (\mathrm{ad}e_i)^{1-a_{ij}} e_j] = 0$,
2. $[e_k, (\mathrm{ad}f_i)^{1-a_{ij}} f_j] = 0$.

PROOF. Since $i \in I^{re}$, we have $a_{ii} = 2$, and from (2.11) – (2.13), $(\alpha_i^\vee, e_i, f_i)$ for $i \notin \Psi$ is an S-triple, and for $i \in \Psi$ this is a super S-triple. Let us prove (1). We consider this for the following cases of k:

(i) For $k \neq i, j$, (1) is trivial by $[f_k, e_i] = [f_k, e_j] = 0$.

(ii) For $k = j$, by using $[f_k, e_i] = 0$, the left hand side of (1) looks as follows:

(2.17)

$$\begin{aligned}
\text{the left hand side of (1)} &= (-1)^{(1-a_{ij})|e_i||f_j|}(\mathrm{ad}e_i)^{1-a_{ij}} \underbrace{[f_j, e_j]}_{-(-1)^{|e_j|}\alpha_j^\vee} \\
&= -(-1)^{(1-a_{ij})|e_i||f_j|+|e_j|}(\mathrm{ad}e_i)^{-a_{ij}} \underbrace{[e_i, \alpha_j^\vee]}_{-a_{ji}e_i} \\
&= (-1)^{(1-a_{ij})|e_i||f_j|+|e_j|}(\mathrm{ad}e_i)^{-a_{ij}} e_i.
\end{aligned}$$

Let us show that the right hand side of (2.17) is 0. If $a_{ij} = 0$, then the statement is clear, so let us consider the case $a_{ij} \neq 0$. Then, by ($\widetilde{\mathrm{C}}$4), one has

$$-a_{ij} \in \mathbb{Z}_{>0}. \tag{2.18}$$

So, first when e_i is an even element, since we have $[e_i, e_i] = 0$, the right hand side of (2.17) is 0 by (2.18). And when e_i is an odd element, then by ($\widetilde{\mathrm{C}}$5), we have

$$-a_{ij} \in 2\mathbb{Z}_{>0}, \tag{2.19}$$

and since one has $[e_i, [e_i, e_i]] = 0$ by (1.72), the right hand side of (2.17) is also 0 in this case.

(iii) For $k = i$, let us prove that

$$(\mathrm{ad} f_i)(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0. \tag{2.20}$$

Using the homomorphism φ of (2.16), the left hand side of (2.20) can be regarded as the element obtained by applying the element $f_i e_i^{1-a_{ij}}$ to e_j, and one can rewrite it as follows:

$$\text{the left hand side of (2.20)} = \varphi(f_i e_i^{1-a_{ij}}) e_j. \tag{2.21}$$

We consider the cases where e_i is even or is odd, separately. When e_i is an even element, then since we have

$$f_i e_i^{1-a_{ij}} = e_i^{1-a_{ij}} f_i - (1 - a_{ij}) e_i^{-a_{ij}} (\alpha_i^\vee - a_{ij})$$

by (3) of Lemma 1.39, it follows that

$$\begin{aligned}
&\text{theright hand side of (2.21)} \\
&\quad = \varphi(e_i^{1-a_{ij}} f_i) e_j - (1 - a_{ij}) \varphi(e_i^{-a_{ij}} (\alpha_i^\vee - a_{ij})) e_j \\
&\quad = (\mathrm{ad} e_i)^{1-a_{ij}} \underbrace{\mathrm{ad} f_i(e_j)}_{0} \\
&\quad - (1 - a_{ij})(\mathrm{ad} e_i)^{-a_{ij}} \underbrace{\underbrace{[\alpha_i^\vee, e_j]}_{a_{ij} e_j} - a_{ij} e_j)}_{0} \\
&\quad = 0,
\end{aligned}$$

and we obtain (2.20).

Next, we consider the case when e_i is an odd element. In this case, since we have $-a_{ij} \in 2\mathbb{Z}_{\geq 0}$ by ($\widetilde{\mathrm{C}}$5) and ($\widetilde{\mathrm{C}}$2), it follows form (3) of Lemma 1.92 that

$$f_i e_i^{1-a_{ij}} = -e_i^{1-a_{ij}} f_i + e_i^{-a_{ij}} (\alpha_i^\vee - a_{ij}).$$

So, we have

$$\begin{aligned}
&\text{theright hand side of (2.21)}\\
&= -\varphi(e_i^{1-a_{ij}} f_i)e_j + \varphi(e_i^{-a_{ij}}(\alpha_i^\vee - a_{ij}))e_j\\
&= -(\mathrm{ad}e_i)^{1-a_{ij}} \underbrace{\mathrm{ad}f_i(e_j)}_{0} + (\mathrm{ad}e_i)^{-a_{ij}} \underbrace{\underbrace{[\alpha_i^\vee, e_j]}_{a_{ij}e_j} - a_{ij}e_j)}_{0}\\
&= 0,
\end{aligned}$$

and we also obtain (2.20) in this case. Thus (1) is proved. (2) can be proved similarly. □

In the above proof, seeing the arguments (i) and (ii) in the case when e_i is an odd element, the reader might feel that condition ($\widetilde{\mathrm{C}}5$) of the BKM supermatrices really works well. The next two lemmas are much easier than Lemma 2.10, but since they are related to the fundamental relations of Lie (super)algebras, let us check them carefully.

LEMMA 2.11. *If $i, j \in I$ $(i \neq j)$ satisfy $a_{ij} = 0$, then for all $k \in I$, the following formulae hold:*

1. $[f_k, [e_i, e_j]] = 0$,
2. $[e_k, [f_i, f_j]] = 0$.

PROOF. It is enough to show the case $k = i$ for (1). In fact, we can compute it as

$$\begin{aligned}
[f_i, [e_i, e_j]] &= [\underbrace{[f_i, e_i]}_{-(-1)^{|e_i|}\alpha_i^\vee}, e_j] + (-1)^{|e_i|}[e_i, \underbrace{[f_i, e_j]}_{0}]\\
&= -(-1)^{|e_i|} \underbrace{[\alpha_i^\vee, e_j]}_{a_{ij}e_j}\\
&= -(-1)^{|e_i|} a_{ij} e_j = 0,
\end{aligned}$$

and hence we obtain (1). (2) can be proved similarly. □

LEMMA 2.12. *For $i \in \Psi_0$, the following formulae hold for all $k \in I$:*

1. $[f_k, [e_i, e_i]] = 0$,
2. $[e_k, [f_i, f_i]] = 0$.

PROOF. It is enough to show the case $k = i$ for (1). In fact, we can compute it as

$$\begin{aligned}[f_i, [e_i, e_i]] &= [\underbrace{[f_i, e_i]}_{\alpha_i^\vee}, e_i] - [e_i, \underbrace{[f_i, e_i]}_{\alpha_i^\vee}] \\ &= 2[\alpha_i^\vee, e_i] = 2a_{ii}e_i \underset{\substack{\uparrow \\ \because a_{ii}=0}}{=} 0,\end{aligned}$$

and we obtain (1). (2) can be proved similarly. □

First, let us look at elements

$$v_{ij}^+ := (\mathrm{ad}e_i)^{1-a_{ij}} e_j, \quad v_{ij}^- := (\mathrm{ad}f_i)^{1-a_{ij}} f_j$$

in Lemma 2.10. Here, $i \in I^{re}, j \in I$ $(i \neq j)$. Taking such i and j and fixing them, we consider the ideal (of $\widehat{\mathfrak{g}}(A)$) generated by v_{ij}^+. By Lemma 2.9, the ideal is $\mathfrak{U}(\widehat{\mathfrak{g}}(A)) \cdot v_{ij}^+$, and by Lemma 2.10, any of $f_1, \cdots, f_n$ kills v_{ij}^+. Namely, any element of $\widehat{\mathfrak{n}}_-$ kills v_{ij}^+. Hence, we obtain

$$\mathfrak{U}(\widehat{\mathfrak{n}}_-) \cdot v_{ij}^+ = \mathbb{C}v_{ij}^+,$$

and by (2.15) it follows that

$$\mathfrak{U}(\widehat{\mathfrak{g}}(A)) \cdot v_{ij}^+ = \mathfrak{U}(\widehat{\mathfrak{n}}_+)\mathfrak{U}(\mathfrak{h}) \cdot v_{ij}^+. \tag{2.22}$$

By the way, since v_{ij}^+ is a root vector of the root $(1 - a_{ij})\alpha_i + \alpha_j \in \widehat{\Delta}$, the action of $\mathrm{ad}(h)$ $(h \in \mathfrak{h})$ on v_{ij}^+ is a scalar multiplication. Hence, the action of $\mathfrak{U}(\mathfrak{h})$ is also a scalar multiplication, and we obtain

$$\mathfrak{U}(\mathfrak{h}) \cdot v_{ij}^+ = \mathbb{C}v_{ij}^+. \tag{2.23}$$

Thus, by (2.22) and (2.23), we obtain

$$\mathfrak{U}(\widehat{\mathfrak{g}}(A)) \cdot v_{ij}^+ = \mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot v_{ij}^+.$$

Namely, $\mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot v_{ij}^+$ is an ideal of $\widehat{\mathfrak{g}}(A)$ lying inside $\widehat{\mathfrak{n}}_+$. Since there is such an ideal, $\widehat{\mathfrak{g}}(A)$ cannot be simple. Similar statements hold for $[e_i, e_j]$ and $[f_i, f_j]$ with $a_{ij} = 0$ appearing in Lemma 2.11, and for $[e_i, e_i]$ and $[f_i, f_i]$ with $i \in \Psi_0$ appearing in Lemma 2.12. So, taking

the ideal $\widehat{\tau}$ of $\widehat{\mathfrak{g}}(A)$ generated by these $v_{ij}^{\pm}$, $[e_i, e_j]$'s and $[f_i, f_j]$'s:

$$\widehat{\tau} := \left\{ \sum_{\substack{i \in I^{re} \\ j \in I \\ i \neq j}} \mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot v_{ij}^{+} + \sum_{\substack{i,j \in I \\ a_{ij}=0}} \mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot [e_i, e_j] + \sum_{i \in \Psi_0} \mathfrak{U}(\widehat{\mathfrak{n}}_+) \cdot [e_i, e_i] \right\}$$
$$\oplus \left\{ \sum_{\substack{i \in I^{re} \\ j \in I \\ i \neq j}} \mathfrak{U}(\widehat{\mathfrak{n}}_-) \cdot v_{ij}^{-} + \sum_{\substack{i,j \in I \\ a_{ij}=0}} \mathfrak{U}(\widehat{\mathfrak{n}}_-) \cdot [f_i, f_j] + \sum_{i \in \Psi_0} \mathfrak{U}(\widehat{\mathfrak{n}}_-) \cdot [f_i, f_i] \right\}$$
$$\subset \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{n}}_-,$$

we consider the quotient

$$\widetilde{\mathfrak{g}}(A) := \widehat{\mathfrak{g}}(A)/\widehat{\tau}.$$

Namely, we regard the above $v_{ij}^{\pm}$'s, $[e_i, e_j]$'s and $[f_i, f_j]$'s as 0. Dividing by an ideal means imposing additional fundamental relations. Namely, by imposing the additional conditions

$$\text{(2.24)} \qquad \text{if} \quad i \in I^{re}, j \in I, i \neq j, \text{ then} \begin{cases} (\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \\ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \end{cases}$$

and

$$\text{(2.25)} \qquad \text{if} \quad i, j \in I, i \neq j \ a_{ij} = 0, \text{ then} \begin{cases} [e_i, e_j] = 0, \\ [f_i, f_j] = 0, \end{cases}$$

$$\text{(2.26)} \qquad \text{if} \quad i \in \Psi_0, \text{ then} \quad [e_i, e_i] = [f_i, f_i] = 0,$$

it is the same as constructing $\widetilde{\mathfrak{g}}(A)$ as the Lie superalgebra generated by $\mathfrak{h}, e_1, \cdots, e_n, f_1, \cdots, f_n$ whose fundamental relations are (2.10) – (2.13), (2.24),(2.25) and (2.26).

By the way, is the Lie superalgebra $\widetilde{\mathfrak{g}}(A)$ obtained above a simple Lie superalgebra? Only by exterminating $v_{ij}^{\pm}$'s, $[e_i, e_j]$'s and $[f_i, f_j]$'s that we could just find out, is there no longer an ideal in $\widetilde{\mathfrak{g}}(A)$ intersecting trivially with $\mathfrak{h}$? This is okay. for an ordinary Lie algebra (not for a Lie superalgebra), ...

THEOREM 2.13. *If A is a symmetrizable BKM matrix, then $\widetilde{\mathfrak{g}}(A)$ is a simple Lie algebra.*[1]

The proof of this statement is given in §9.11 of [**K4**] in the case of symmetrizable GCM, and one can naïvely apply its proof to the case of a symmetrizable BKM matrix. Since a map between Verma modules cannot be injective for a symmetrizable BKM supermatrix, an obstruction arises in the arguments given there. Concerning the fundamental relations characterizing the simplicity of BKM Lie superalgebras, I did not know the complete answer to this problem when I was writing this book.

Including the case when the condition in Theorem 2.13 is not satisfied, in general, for a BKM supermatrix (A, Ψ), let us consider the ideal $\widetilde{\tau}$ of $\widetilde{\mathfrak{g}}(A)$ satisfying the following conditions to get the associated simple Lie superalgebra:

($\widetilde{\mathrm{R}}1$) $\widetilde{\tau}$ is an ideal of $\widetilde{\mathfrak{g}}(A)$,

($\widetilde{\mathrm{R}}2$) $\widetilde{\tau} \subset \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_-$,

($\widetilde{\mathrm{R}}3$) $\widetilde{\tau}$ is the maximum among those satisfying ($\widetilde{\mathrm{R}}1$) and ($\widetilde{\mathrm{R}}2$).

So, $\widetilde{\mathfrak{n}}_\pm$ are Lie subsuperalgebras of $\widetilde{\mathfrak{g}}(A)$ constructed in the same way as $\widehat{\mathfrak{n}}_\pm$. And set

$$\mathfrak{g}(A) := \widetilde{\mathfrak{g}}(A)/\widetilde{\tau}.$$

If A satisfies the condition of Theorem 2.13, then one has $\widetilde{\tau} = \{0\}$ and $\mathfrak{g}(A) = \widetilde{\mathfrak{g}}(A)$.

Dividing by such an ideal, we do not have to use $\widetilde{\mathfrak{g}}(A)$, and consider an ideal of $\widehat{\mathfrak{g}}(A)$ from the beginning. Namely, take the ideal $\widehat{\tau}$ satisfying

($\widehat{\mathrm{R}}1$) $\widehat{\tau}$ is an ideal of $\widehat{\mathfrak{g}}(A)$,

($\widehat{\mathrm{R}}2$) $\widehat{\tau} \subset \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{n}}_-$,

($\widehat{\mathrm{R}}3$) $\widehat{\tau}$ is the maximum among those satisfying ($\widehat{\mathrm{R}}1$) and ($\widehat{\mathrm{R}}2$),

which is the same as setting

$$\mathfrak{g}(A) := \widehat{\mathfrak{g}}(A)/\widehat{\tau}.$$

DEFINITION 2.14. For a BKM supermatrix (A, Ψ), $\mathfrak{g}(A)$ is called the **BKM Lie superalgebra** or the **BKM superalgebra** associated to (A, Ψ). When A is symmetrizable, $\mathfrak{g}(A)$ is called a **symmetrizable BKM Lie superalgebra** or a **symmetrizable BKM superalgebra**.

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[1] *Translator's note.* It has come to the translator's attention that Urmie Ray's forthcoming paper [**UR**] will contain a theorem that a Lie superalgebra $\widetilde{\mathfrak{g}}(A)$ is simple also for a symmetrizable BKM supermatrix A.

In particular, when A is a BKM matrix, the Lie algebra $\mathfrak{g}(A)$ is called a **BKM Lie algebra** (or a **BKM algebra**), and when A is a KM matrix (= a GCM), the Lie algebra $\mathfrak{g}(A)$ is called a **Kac-Moody Lie algebra** or a **Kac-Moody algebra**.

Concerning $\mathfrak{g}(A)$, we define a root α, sets of roots Δ, $\Delta_\pm$, and root space $\mathfrak{g}_\alpha$ in the same way as in the case of $\widehat{\mathfrak{g}}(A)$. We have seen multiplicities of roots of $\widehat{\mathfrak{g}}(A)$ with small height, but we cannot compute the multiplicity $\mathrm{mult}(\alpha) := \dim \mathfrak{g}_\alpha$ of the root α of $\mathfrak{g}(A)$ in the same way as $\widehat{\mathfrak{g}}(A)$. When we make bracket products of e_i's (or f_i's), they may degenerate in complicated ways by the relation (2.24), or the ideal $\widehat{\tau}$. Let us mention the following:

LEMMA 2.15. *If $\alpha \in \Delta_+ \setminus \Pi$, then there exists $i \in I$ satisfying $\alpha - \alpha_i \in \Delta_+$.*

This is because, if α satisfies $\alpha - \alpha_i \notin \Delta_+$ for all $i \in I$, then since an element x ($\neq 0$) of $\mathfrak{g}_\alpha$ satisfies

$$[f_i, x] = 0 \quad (i \in I),$$

it generates an ideal included in $\mathfrak{n}_+$.

Note. The construction of $\mathfrak{g}(A)$ given in this section is an application of the construction of a Kac-Moody Lie algebra, when A is a GCM, to the case of BKM Lie superalgebras. To describe the way of constructing $\mathfrak{g}(A)$ from its Cartan matrix A in general, it seems that this method is natural, but in some cases, strange phenomena may happen.

For example, let us consider the case where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

and $\Psi = \{1, 3\}$, viz., α_1 and α_3 are odd simple roots and α_2 is an even simple root. Since the rank of this Cartan matrix is 2, a Cartan subalgebra of $\mathfrak{g}(A)$ is of dimension four. But, on the other hand, this Cartan matrix can be obtained as a Cartan matrix of the simple finite-dimensional Lie superalgebra

$$A(1,1) = \mathfrak{sl}(2,2)/\mathfrak{z}$$

by taking an appropriate system of fundamental roots. A Cartan subalgebra of $A(1,1)$ is of dimension two, and one has

$$\mathfrak{g}(A) = (\text{two-dimensional commutative Lie algebra}) + A(1,1).$$

To be precise, one has $\mathfrak{g}(A) \cong \mathfrak{gl}(2|2)$, and the relation between $\mathfrak{g}(A)$ and $A(1|1)$ can be described as

$$\begin{aligned}\mathfrak{g}(A) =&\text{(one-dimensional center)}\\ &\oplus \{\text{(one-dimensional Lie algebra)} + A(1,1)\}.\end{aligned}$$

Here, (one-dimensional Lie algebra)$+A(1,1)$ does not mean the direct sum as Lie superalgebras, but it signifies that this one-dimensional Lie algebra acts on $A(1,1)$. Since in $A(1,1)$ we have $\alpha_1 = \alpha_3$, two odd roots are identified, and the multiplicities of odd roots becomes 2. But this 'one-dimensional Lie algebra' separates these identified odd roots. Namely, in $\mathfrak{g}(A)$, by enlarging its Cartan subalgebra, one resolves the degeneracy of roots. But instead, one lose the simplicity in the usual sense. For the representation theory, one may choose which property is suitable. In this book, we also treat such $\mathfrak{g}(A)$ as a 'simple' Lie superalgebra regarded as a BKM Lie superalgebra in the sense of Definition 2.8.

In fact, even for this Cartan matrix, the structural difference between $\mathfrak{g}(A)$ and $A(1,1)$ appear only in their Cartan subalgebras, and resolving the degeneracy of the roots of $A(1,1)$ owing to its small Cartan subalgebra, their root systems coincide. Therefore, the denominator identity (Corollary 2.79) stated later in §2.5 surely describes the root system of $A(1,1)$ correctly. But, it is related to the last **Note** in §2.5, as one can understand if one examines Definition 2.71 in §2.5, for this Cartan matrix A, the integrable representations of $\mathfrak{g}(A)$ as a BKM Lie superalgebra and finite dimensional representations of $A(1,1)$ are completely inconsistent notions (except for the trivial representation), and hence one cannot use the character formula of irreducible integrable representations of a BKM Lie superalgebra $\mathfrak{g}(A)$ to describe the character of (a non-trivial) finite-dimensional representations of $A(1,1)$. □

Concerning a bilinear form on a symmetrizable BKM Lie superalgebra $\mathfrak{g}(A)$, the following important theorem is known:

THEOREM 2.16. *For a symmetrizable BKM supermatrix (A, Ψ), there uniquely exists a non-degenerate super-symmetric super-invarian even bilinear form $(\ |\ )$, whose restriction on $\mathfrak{h}$ coincides with that described in Lemma 2.7. And this bilinear form enjoys the following properties:*

i. *If $\alpha, \beta \in \Delta$ and $\alpha + \beta \neq 0$, then one has $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = \{0\}$.*
ii. *For $\alpha \in \Delta$, $(\ |\ )$ is a non-degenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.*

iii. *For* $\alpha \in \Delta$, *one has* $[x, y] = (x|y)\alpha$ *for* $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$.

For a proof of this theorem, see e.g. §2.2 of [**K4**]. One extends the bilinear form given on $\mathfrak{h}$ to the subspace $\sum_{|\mathrm{ht}(\alpha)|\leq k} \mathfrak{g}_\alpha$ $(k \in \mathbb{Z}_{\geq 0})$ by induction on k. Looking at this method carefully, one can introduce an invariant bilinear form not only on $\mathfrak{g}(A)$ but also on $\widehat{\mathfrak{g}}(A)$ or $\widetilde{\mathfrak{g}}(A)$. A big difference between these invariant bilinear forms and an invariant bilinear form on $\mathfrak{g}(A)$ is that these inner products cannot be non-degenerate except for $\mathfrak{g}(A)$. Namely, if we denote the invariant bilinear form on $\widehat{\mathfrak{g}}(A)$ introduced in this way by the same symbol $(\ |\ )$, and define its radical by

$$\tau := \{x \in \widehat{\mathfrak{g}}(A); (x|y) = 0 \ (y \in \widehat{\mathfrak{g}}(A))\},$$

then by the invariance of this inner product, τ is an ideal of $\widehat{\mathfrak{g}}(A)$. And since $(\ |\ )$ is non-degenerate on $\mathfrak{h}$, one has $\tau \cap \mathfrak{h} = \{0\}$, i.e., $\tau \subset \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{n}}_-$. So, by the maximality of $\widehat{\tau}$, we obtain $\tau \subset \widehat{\tau}$.

Conversely, one can show that $\widehat{\tau} \subset \tau$ as follows. To see this, we have only to prove that $(\widehat{\mathfrak{g}}(A)|\widehat{\tau}) = \{0\}$. For this, it is enough to check that

i. $(\mathfrak{h}|\widehat{\tau}) = \{0\}$,
ii. $(\widehat{\mathfrak{g}}_\alpha|\widehat{\tau}) = \{0\}$ $(\alpha \in \widehat{\Delta})$.

But (i) immediately follows from $\widehat{\tau} \subset \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{n}}_-$, and (ii) can be shown by

$$(\widehat{\mathfrak{g}}_\alpha|\widehat{\tau}) = ([\mathfrak{h}, \widehat{\mathfrak{g}}_\alpha]|\widehat{\tau}) = (\mathfrak{h}|[\widehat{\mathfrak{g}}_\alpha, \widehat{\tau}]) \subset (\mathfrak{h}|\widehat{\tau}) = \{0\}.$$

Thus we could have $\tau = \widehat{\tau}$. Namely, by using the fact that one can introduce an invariant bilinear form on $\widehat{\mathfrak{g}}(A)$, the ideal $\widehat{\tau}$ used to define $\mathfrak{g}(A)$ can be characterized as the radical of this bilinear form.

Below in this book, we assume that (A, Ψ) is a symmetrizable BKM supermatrix, and we develop the representation theory of $\mathfrak{g}(A)$. The tools we are going to use are the fundamental relations (2.10) – (2.13) and (2.24) – (2.26), a non-degenerate invariant bilinear form, and the Weyl group, only. $\mathfrak{g}(A)$ differs from $\widehat{\mathfrak{g}}(A)$ and has the following two merits:

i. An invariant bilinear form is non-degenerate.
ii. One can derive important properties of the Weyl group.

In the next section, the Weyl group is defined as a subgroup of $GL(\mathfrak{h})$ by using reflections, but it does not work effectively only by this definition. By the fact that one can extend the action of each element of the Weyl group on $\mathfrak{h}$ to an automorphism of $\mathfrak{g}(A)$, and that when one

considers a representation of $\mathfrak{g}(A)$, each element of the Weyl group acts on its representation space, the Weyl group shows its power. Condition (2.24) is an important condition for this. Namely, condition (2.24) stays not only as an additional relation to get simple Lie superalgebra, but as we see in §2.2, it is an extremely important condition for deriving important properties of the Weyl group. However, since $\widehat{\mathfrak{g}}(A)$ does not enjoy the above two properties, we may meet with an obstruction if we try to study the representation space of $\widehat{\mathfrak{g}}(A)$ in the same way as we will develop below. But when we fix any element $\Lambda \in \mathfrak{h}^*$, an irreducible highest weight $\widehat{\mathfrak{g}}(A)$-module with highest weight Λ exists uniquely (up to isomorphism), as well as $\mathfrak{g}(A)$-modules. Moreover, via the natural homomorphism $\widehat{\mathfrak{g}}(A) \longrightarrow \mathfrak{g}(A)$, a highest weight $\mathfrak{g}(A)$-module itself is at the same time a highest weight $\widehat{\mathfrak{g}}(A)$-module, and the irreducibility of a module does not depend upon the choice of a Lie algebra, $\mathfrak{g}(A)$ or $\widehat{\mathfrak{g}}(A)$. Therefore, an irreducible highest weight module regarded as a $\widehat{\mathfrak{g}}(A)$-module is also an irreducible highest weight module as a $\mathfrak{g}(A)$-module, and hence the structure of an irreducible highest weight $\widehat{\mathfrak{g}}(A)$-module, e.g., the appearance of each weight and the Weyl group invariance, is the same as that regarded as a $\mathfrak{g}(A)$-module.

Summarizing these, for a symmetrizable BKM supermatrix (A, Ψ) one can take as the important properties of its associated Lie superalgebra $\mathfrak{g}(A)$ the following three facts:

1. $\mathfrak{g}(A)$ is generated by $\mathfrak{h}$ and $e_1, \cdots, e_n, f_1, \cdots, f_n$.
2. These generators satisfy (2.10) – (2.13) and (2.24) – (2.26).
3. $\mathfrak{g}(A)$ possesses a non-degenerate inner product described in Theorem 2.16.

These data on $\mathfrak{g}(A)$ are enough. Using only these data, we first study properties of the root system of $\mathfrak{g}(A)$, and next we study the representation theory of $\mathfrak{g}(A)$.

Even for a finite-dimensional Lie algebra, for example, in the concrete and important case of E_8 (of dimension 248), when we try to study some properties, we do not write down all of the elements of E_8 and do not make the multiplication table. Bracket products of each elements are not so important, and what is important is the properties of the group E_8 formed via these elements. As we saw in §1.3, to study properties of a Lie algebra, it is much more important to grasp the whole roots and their relations. By them, we will try to see the behavior of a Lie algebra.

It is the same for $\mathfrak{g}(A)$. In each section below, we are going to look at it concretely, and here we will prepare for it. Here and after, let $\mathfrak{g}(A)$ be the Lie superalgebra associated to a symmetrizable BKM supermatrix (A, Ψ), and let $(\ |\ )$ be a bilinear form in Lemma 2.7 and Theorem 2.16. Take $\rho \in \mathfrak{h}^*$ as follows:

$$(\rho|\alpha_i) = \frac{(\alpha_i|\alpha_i)}{2} \quad (1 \leq i \leq n). \tag{2.27}$$

When $n < \dim \mathfrak{h}$ (i.e., $\det A = 0$), ρ is not uniquely defined from these relations, but in this case, we may take any element satisfying (2.27), and call it ρ. If we take another element satisfying (2.27), say ρ', then it satisfies

$$(\rho - \rho'|\alpha_i) = 0 \quad (1 \leq i \leq n),$$

and regarding $\rho - \rho'$ as an element of $\mathfrak{h}$ (via the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$), it belongs to the center of $\mathfrak{g}(A)$. Namely, the choice of ρ has the freedom of a shift by a central element of $\mathfrak{g}(A)$. ρ is called a **Weyl vector**. By Lemma 2.7, for i satisfying $a_{ii} \neq 0$, we have

$$a_{ii} \neq 0 \implies (\rho|\alpha_i^\vee) = \frac{a_{ii}}{2}, \tag{2.28}$$

and in particular,

$$a_{ii} = 2 \implies (\rho|\alpha_i^\vee) = 1. \tag{2.29}$$

Let $\mathfrak{g}(A)$ be a symmetrizable BKM Lie superalgebra, and take two pairs of basis $\{x_i\}$ and $\{x^i\}$ as follows:

1. The x_i's and x^i's are elements of root spaces or elements of $\mathfrak{h}$, i.e., simultaneous eigenvectors of $\mathrm{ad}(\mathfrak{h})$.
2. $(x_i|x^j) = \delta_{ij}$.

Let us consider the element

$$\Omega' := \sum_i x^i \otimes x_i = \sum_i (-1)^{|x_i|} x_i \otimes x^i \tag{2.30}$$

of the completed space of the tensor product $\mathfrak{g}(A)\widehat{\otimes}\mathfrak{g}(A)$ in the sense of Example 1.86. As one can easily see from the choice of basis $\{x_i\}$ and $\{x^i\}$, since under the isomorphism (of $\mathfrak{g}$-modules)

$$\Phi : \mathfrak{g}(A)\widehat{\otimes}\mathfrak{g}(A) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}(A), \mathfrak{g}(A))$$

constructed in Example 1.86 we have

$$\Phi(\Omega') = \mathrm{Id}_{\mathfrak{g}(A)},$$

it follows that Ω' is invariant under the action of $\mathfrak{g}(A)$. Namely, we have

$$(\mathrm{ad} \otimes \mathrm{ad})(x)\Omega' = 0 \quad (x \in \mathfrak{g}(A)). \tag{2.31}$$

To describe this computation slightly more concretely, take two pairs of basis $\{u_i\}_{1 \le i \le \dim \mathfrak{h}}$ and $\{u^i\}_{1 \le i \le \dim \mathfrak{h}}$ of $\mathfrak{h}$ satisfying $(u_i|u^j) = \delta_{ij}$, and for each $\alpha \in \Delta_+$, take a basis $\{e_\alpha^{(i)}\}_{1 \le i \le \mathrm{mult}(\alpha)}$ of $\mathfrak{g}_\alpha$ and a basis $\{e_{-\alpha}^{(i)}\}_{1 \le i \le \mathrm{mult}(\alpha)}$ of $\mathfrak{g}_{-\alpha}$ satisfying $(e_\alpha^{(i)}|e_{-\alpha}^{(j)}) = \delta_{ij}$. Then, we have

$$\begin{aligned} \Omega' = &\sum_{i=1}^{\dim \mathfrak{h}} u^i \otimes u_i + \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} e_{-\alpha}^{(i)} \otimes e_\alpha^{(i)} \\ &+ \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} (-1)^{p(\alpha)} e_\alpha^{(i)} \otimes e_{-\alpha}^{(i)}. \end{aligned} \tag{2.32}$$

Here, $p(\alpha)$ signifies the **parity** of a root α:

$$p(\alpha) := \begin{cases} 0 & \text{if } \alpha \text{ is an even root,} \\ 1 & \text{if } \alpha \text{ is an odd root.} \end{cases} \tag{2.33}$$

Now, one can rephrase the fact that Ω' is invariant under the action of $\mathfrak{g}(A)$ by saying that it is invariant under the action of generators of $\mathfrak{g}(A)$. So, (2.31) is equivalent to the following relations:

$$\begin{aligned} &(\mathrm{ad} \otimes \mathrm{ad})(e_j)\Omega' = 0, \\ &(\mathrm{ad} \otimes \mathrm{ad})(f_j)\Omega' = 0 \quad (1 \le j \le n). \end{aligned} \tag{2.34}$$

Computing the left hand side of these formulae by using (2.32), they look as follows:

(2.35)

$$
\begin{aligned}
&(\mathrm{ad}\otimes\mathrm{ad})(e_j)\Omega' \\
&= \sum_{i=1}^{\dim\mathfrak{h}} \{[e_j, u^i]\otimes u_i + u^i \otimes [e_j, u_i]\} \\
&\quad + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[e_j, e_{-\alpha}^{(i)}]\otimes e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)} \otimes [e_j, e_\alpha^{(i)}]\} \\
&\quad + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \Big\{(-1)^{p(\alpha)}[e_j, e_\alpha^{(i)}] \\
&\qquad\qquad + (-1)^{p(\alpha)}(-1)^{p(\alpha_j)p(\alpha)} e_\alpha^{(i)} \otimes [e_j, e_{-\alpha}^{(i)}]\Big\},
\end{aligned}
$$

(2.36)

$$
\begin{aligned}
&(\mathrm{ad}\otimes\mathrm{ad})(f_j)\Omega' \\
&= \sum_{i=1}^{\dim\mathfrak{h}} \{[f_j, u^i]\otimes u_i + u^i \otimes [f_j, u_i]\} \\
&\quad + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[f_j, e_{-\alpha}^{(i)}]\otimes e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)} \otimes [f_j, e_\alpha^{(i)}]\} \\
&\quad + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \Big\{(-1)^{p(\alpha)}[f_j, e_\alpha^{(i)}] \\
&\qquad\qquad + (-1)^{p(\alpha)}(-1)^{p(\alpha_j)p(\alpha)} e_\alpha^{(i)} \otimes [f_j, e_{-\alpha}^{(i)}]\Big\}.
\end{aligned}
$$

Since they are both 0 by (2.34), in particular, the components of $\mathfrak{g}(A)\widehat{\otimes}\mathfrak{n}_+$ and $\mathfrak{g}(A)\widehat{\otimes}(\mathfrak{h}\oplus\mathfrak{n}_+)$ in these formula must be zero. So, by

looking at the $\mathfrak{g}(A)\widehat{\otimes}\mathfrak{n}_+$-component of (2.35) and the $\mathfrak{g}(A)\widehat{\otimes}(\mathfrak{n}_+ \oplus \mathfrak{h})$-component of (2.36), we obtain the next formulae:

(2.37)
$$\sum_{i=1}^{\dim \mathfrak{h}} u^i \otimes [e_j, u_i] + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[e_j, e_{-\alpha}^{(i)}] \otimes e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)} \otimes [e_j, e_\alpha^{(i)}]\} = 0,$$

(2.38)
$$\sum_{i=1}^{\dim \mathfrak{h}} [f_j, u^i] \otimes u_i + \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[f_j, e_{-\alpha}^{(i)}] \otimes e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)} \otimes [f_j, e_\alpha^{(i)}]\} = 0.$$

We consider the completion of the universal enveloping algebra of $\mathfrak{g}(A)$ following Example 1.86, and denote it by $\widehat{\mathfrak{U}}(\mathfrak{g}(A))$. Then, under the natural projection from the tensor product to the universal enveloping algebra, the above relations (2.37) and (2.38) give the following relations in $\widehat{\mathfrak{U}}(\mathfrak{g}(A))$:

(2.39)
$$\sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[e_j, e_{-\alpha}^{(i)}]e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)}[e_j, e_\alpha^{(i)}]\} = -\sum_{i=1}^{\dim \mathfrak{h}} u^i [e_j, u_i],$$

(2.40)
$$\sum_{\alpha\in\Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[f_j, e_{-\alpha}^{(i)}]e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)}[f_j, e_\alpha^{(i)}]\} = -\sum_{i=1}^{\dim \mathfrak{h}} [f_j, u^i] u_i.$$

Using these, let us prove the following important theorem:

THEOREM 2.17. *Let $\mathfrak{g}(A)$ be a symmetrizable BKM Lie superalgebra. Then the element*

$$\Omega := 2\rho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i + 2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} e_{-\alpha}^{(i)} e_{\alpha}^{(i)}$$

of the completed universal enveloping algebra $\widehat{\mathfrak{U}}(\mathfrak{g}(A))$ is invariant under the adjoint action of $\mathfrak{g}(A)$.

PROOF. For $j = 1, \cdots, n$, it enough to show that

$$[e_j, \Omega] = [f_j, \Omega] = 0. \tag{2.41}$$

First, let us compute $[e_j, \Omega]$ by using Lemma 1.80:

$$\begin{aligned}[e_j, \Omega] &= 2[e_j, \rho] + \sum_{i=1}^{\mathfrak{h}} \{[e_j, u^i]u_i + u^i[e_j, u_i]\} \\ &\quad + \underbrace{2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[e_j, e_{-\alpha}^{(i)}]e_{\alpha}^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)}[e_j, e_{\alpha}^{(i)}]\}}_{-\sum_{i=1}^{\dim \mathfrak{h}} u^i[e_j, u_i] \ (\because (2.39))}.\end{aligned}$$

Since Ω does not depend on a choice of basis, to make computations simpler, let us assume that the basis of $\mathfrak{h}$ is an orthonormal basis. Namely, here and after, we assume that $u_i = u^i$, $(u_i|u_j) = \delta_{ij}$, and we use the formulae obtained above. Let us continue the computation of $[e_j, \Omega]$:

$$\begin{aligned}[e_j, \Omega] &= 2 \underbrace{[e_j, \rho]}_{-(\alpha_j|\rho)e_j} + \sum_{i=1}^{\dim \mathfrak{h}} \left\{ \underbrace{[e_j, u_i]}_{-\alpha_j(u_i)e_j} - u_i \underbrace{[e_j, u_i]}_{-\alpha_j(u_i)e_j} \right\} \\ &= -2(\alpha_j|\rho)e_j + \sum_{i=1}^{\dim \mathfrak{h}} \alpha_j(u_i) \underbrace{[u_i, e_j]}_{\alpha_j(u_i)e_j} \\ &= 2 \underbrace{[e_j, \rho]}_{-(\alpha_j|\rho)e_j} + \underbrace{\sum_{i=1}^{\dim \mathfrak{h}} \alpha_j(u_i)^2}_{(\alpha_j|\alpha_j)} \cdot e_j \\ &= \{-2(\alpha_j|\rho) + (\alpha_j|\alpha_j)\}e_j.\end{aligned}$$

By the definition (2.27) of ρ, the right hand side of this formula is 0. Thus, we have proved that $[e_j, \Omega] = 0$. Next, for $[f_j, \Omega]$, we have

$$\begin{aligned}
[f_j, \Omega] &= 2[f_j, \Omega] + \sum_{i=1}^{\dim \mathfrak{h}} \{[f_j, u_i]u_i + u_i[f_j, u_i]\} \\
&\quad + 2 \underbrace{\sum_{\alpha \in \Delta_+} \sum_{i=1}^{\mathrm{mult}(\alpha)} \{[f_j, e_{-\alpha}^{(i)}]e_\alpha^{(i)} + (-1)^{p(\alpha_j)p(\alpha)} e_{-\alpha}^{(i)}[f_j, e_\alpha^{(i)}]\}}_{-\sum_{i=1}^{\dim \mathfrak{h}} [f_j, u_i]u_i \ (\because\ (2.40))} \\
&= 2 \underbrace{[f_j, \rho]}_{(\alpha_j|\rho)f_j} + \sum_{i=1}^{\dim \mathfrak{h}} \left\{ - \underbrace{[f_j, u_i]}_{\alpha_j(u_i)f_j} u_i + u_i \underbrace{[f_j, u_i]}_{\alpha_j(u_i)f_j} \right\} \\
&= 2(\alpha_j|\rho)f_j + \sum_{i=1}^{\dim \mathfrak{h}} \alpha_j(u_i) \underbrace{[u_i, f_j]}_{-\alpha_j(u_i)f_j} \\
&= \left\{ 2(\alpha_j|\rho) - \sum_{i=1}^{\dim \mathfrak{h}} \alpha_j(u_i)^2 \right\} \cdot f_j = \{2(\alpha_j|\rho) - (\alpha_j|\alpha_j)\} f_j \\
&= 0,
\end{aligned}$$

and hence (2.41) is shown, and the theorem is proved. □

DEFINITION 2.18. Ω is called the **Casimir element** of the BKM Lie superalgebra $\mathfrak{g}(A)$.

Since the Casimir element given here is an infinite sum of those obtained by multiplying positive root vectors from the right and negative root vectors from the left as we can see from its definition in Theorem 2.17, the "normal ordering[2]" already comes into this definition. This is because the representations considered in this book are highest weight representations. To prove Theorem 2.17, we used the merit that since there are only a few relations in tensor products, they give us detailed information when we compare corresponding terms in equalities.

When it acts on a representation space of an irreducible representation or a highest weight representation, since Ω commutes with all elements of the Lie superalgebra $\mathfrak{g}(A)$, it is an operator of scalar

[2]For a detailed explanation of the normal ordering, see e.g. Appendix A.3 of [**J2**]

multiplication. The concrete value of this scalar plays an important role.

Before closing this section, let us collect some results on the classification of GCMs (without proof). To describe the classification of GCMs, we introduce the following notation:

NOTATION 2.19. For $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, set

$$\begin{aligned} u > 0 &\iff u_i > 0 \text{ for all } i, \\ u \geq 0 &\iff u_i \geq 0 \text{ for all } i, \\ u < 0 &\iff u_i < 0 \text{ for all } i, \\ u \leq 0 &\iff u_i \leq 0 \text{ for all } i. \end{aligned}$$

Under this convention, notice that

$$u \geq 0 \not\Longrightarrow u > 0 \text{ or } u = 0.$$

On an indecomposable GCM A, we consider the following three conditions:

(Fin) There exists $u > 0$ satisfying $Au \geq 0$ and $Au \neq 0$.
(Aff) There exists $u > 0$ satisfying $Au = 0$.
(Ind) There exists $u > 0$ satisfying $Au < 0$.

At a glance, it seems that these conditions are not disjoint, but the fact is they are disjoint. Namely,

THEOREM 2.20. *If A is an indecomposable GCM, then it follows that*

1. *A satisfies only one of the above conditions, and*
2. *the transposed matrix ${}^tA$ of A is also a GCM, and A and ${}^tA$ satisfy the same condition.*

DEFINITION 2.21. For an indecomposable GCM A, we define its 'type' as follows:

$$\begin{aligned} \text{of finite type} &\iff A \text{ satisfies condition (Fin)}, \\ \text{of affine type} &\iff A \text{ satisfies condition (Aff)}, \\ \text{of indefinite type} &\iff A \text{ satisfies condition (Ind)}. \end{aligned}$$

THEOREM 2.22. *Conditions* (Fin) *and* (Aff) *on an indecomposable GCM A are equivalent to the following conditions* (Fin)$'$ *and* (Aff)$'$ *respectively:*

(Fin)′ *All principal minors of A are positive definite.*
(Aff)′ *All proper principal minors of A are positive definite, and* $\det A = 0$.

These theorems can be proved by using the theory of linear inequalities. For the method of proof and details, see e.g., Chapter 4 of [**K4**]. The next assertion follows from Theorem 2.22, but it is a part of the proof of disjointness of the conditions in Theorem 2.20.

COROLLARY 2.23. *If A is a GCM of finite type, then there does not exist $u > 0$ satisfying $Au \leq 0$.*

From (2) of Theorem 2.20, we can immediately obtain the following characterization on a GCM of finite type, and this proposition will be used in §2.3 to study properties of imaginary roots:

PROPOSITION 2.24. *Condition* (Fin) *on an indecomposable GCM A is equivalent to the following condition:*
(Fin)″ *There exists $\alpha \in \sum_{i=1}^{n} \mathbb{R}_{>0}\alpha_i$ satisfying*
i. $\langle \alpha, \alpha_j^\vee \rangle \geq 0$ *(for all j),*
ii. $\langle \alpha, \alpha_{j_0}^\vee \rangle > 0$ *(for some j_0).*

The class of GCMs of finite type coincides with the class of Cartan matrices of simple finite-dimensional Lie algebras. For an affine type A, the corresponding Kac-Moody Lie algebra $\mathfrak{g}(A)$ is called an **affine Lie algebra** or an **affine Kac-Moody algebra** .

For the next important proposition, the reader may refer to e.g., §4.6 of [**K4**]:

PROPOSITION 2.25. *A GCM of finite type or of affine type is symmetrizable.*

2.2. Weyl Groups

Let (A, Ψ) be a symmetrizable BKM supermatrix, and identify a Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via the invariant bilinear form $(\ |\ )$ on $\mathfrak{g}(A)$. When $\alpha \in \mathfrak{h}^*$ satisfies $(\alpha|\alpha) \neq 0$, we define $r_\alpha \in GL(\mathfrak{h})$ by the following formula:

$$r_\alpha(h) := h - \frac{2(\alpha|h)}{(\alpha|\alpha)}\alpha \quad (h \in \mathfrak{h}). \tag{2.42}$$

r_α is the **reflection** with respect to the hyperplane

$$\{h \in \mathfrak{h}; (\alpha|h) = 0\}.$$

Below, we refer to this fact as 'the reflection with respect to α'. From this, it can be easily checked that r_α enjoys the following properties:

$$r_\alpha^2 = \mathrm{Id}_{\mathfrak{h}}, \tag{2.43}$$

$$(r_\alpha h | r_\alpha h') = (h|h') \quad (h, h' \in \mathfrak{h}). \tag{2.44}$$

These can be also checked directly by using the definition of r_α (2.42). Moreover, if $\alpha, \beta \in \mathfrak{h}$ satisfy $(\alpha|\alpha) \neq 0$, $(\beta|\beta) \neq 0$, then one can easily check

$$r_{r_\alpha \beta} = r_\alpha \circ r_\beta \circ r_\alpha^{-1}. \tag{2.45}$$

In particular, consider the case $\alpha = \alpha_i \in \Pi^{re}$. Since $(\alpha_i|\alpha_i) > 0$ in this case, one can define the reflection r_{α_i} with respect to the simple root α_i by (2.42); we simply denote r_{α_i} by r_i, and set the subgroup of $GL(\mathfrak{h})$ generated by r_i $(i \in I^{re})$ by

$$W := \langle r_i; i \in I^{re} \rangle, \tag{2.46}$$

and call W the **Weyl group** of $\mathfrak{g}(A)$. Here and after, we denote the identity element of the Weyl group W by e. By (2.45), for $w \in W$ and $\alpha \in \mathfrak{h}$ satisfying $(\alpha|\alpha) \neq 0$, one has the following formula:

$$r_{w\alpha} = w \circ r_\alpha \circ w^{-1}. \tag{2.47}$$

Since r_α is the reflection, its determinant (as a linear transformation of $\mathfrak{h}$) is -1. So, for each element $w = r_{i_1} \cdots r_{i_k}$ of W, if we set

$$\varepsilon(w) := \det(w) = (-1)^k,$$

then the map

$$\varepsilon : W \longrightarrow \{\pm 1\}$$

is a homomorphism of groups. $\varepsilon(w)$ is called the **signature** of w, and is sometimes denoted by $\mathrm{sgn}(w)$.

Since $a_{ii} = 2$ for $i \in I^{re}$, by (4) of Lemma 2.7, the action of r_i on $\mathfrak{h}^*$

$$r_i \lambda = \lambda - \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} \alpha_i \quad (\lambda \in \mathfrak{h}^*)$$

looks as follows:

$$r_i \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i. \tag{2.48}$$

Similarly, the action of r_i on $\mathfrak{h}$ can be written as follows:

$$r_i h = h - \langle \alpha_i, h \rangle \alpha_i^\vee. \tag{2.49}$$

In particular, its action on ρ looks as follows:

$$r_i\rho = \rho - \alpha_i \quad (i \in I^{re}). \tag{2.50}$$

DEFINITION 2.26. For $i \in I^{re}$, r_i is called a **simple reflection**.

By the definition of the Weyl group and (2.44), we have the following fact:

LEMMA 2.27. *The bilinear form* $(\ |\ )$ *on* $\mathfrak{h}$ *and* $\mathfrak{h}^*$ *is invariant under the action of the Weyl group.*

The BKM Lie superalgebra $\mathfrak{g}(A)$ associated to a BKM supermatrix (A, Ψ) is the Lie superalgebra generated by its Cartan subalgebra $\mathfrak{h}$ and Chevalley generators e_i's and f_i's. And condition (2.24) implies that $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ for i satisfying $a_{ii} = 2$ act locally nilpotently on Chevalley generators. Then, how about the actions of $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ on $\mathfrak{g}(A)$? Do they also act locally nilpotently on $\mathfrak{g}(A)$? The answer is 'YES'. The following statement is extremely useful for applications:

LEMMA 2.28. *For a Lie superalgebra* $\mathfrak{g}$ *and its even element* x, *the following hold:*

1. *If one can choose generators* $x_1, x_2, \cdots$ *of* $\mathfrak{g}$ *(as a Lie superalgebra) and a sequence of natural numbers* $n_1, n_2, \cdots$ *satisfying*
$$(\mathrm{ad}(x))^{n_i} x_i = 0 \quad (\textit{for all } i),$$
then $\mathrm{ad}(x)$ *is locally nilpotent as a linear operator on* $\mathfrak{g}$.
2. *Let* (π, V) *be a representation of* $\mathfrak{g}$, *and let* $v_1, v_2 \cdots$ *be generators of* V *as a* $\mathfrak{g}$*-module. Namely, we set*
$$V = \sum_i \mathfrak{U}(\mathfrak{g}) \cdot v_i.$$
If x *satisfies the conditions*
   (i) $\mathrm{ad}(x)$ *acts locally nilpotently on* $\mathfrak{g}$ *and*
   (ii) *there exists a natural number* N_i *satisfying* $\pi(x)^{N_i}(v_i) = 0$ *for each* i,
then $\pi(x)$ *is locally nilpotent as a linear operator on* V.

PROOF. Both (1) and (2) can be easily proved by using the super-version of Lemma 1.34 (Lemma 1.95); here, let us prove (2) of the lemma. First, let us prove the following assertion:

ASSERTION A. *Take an element* v *of* V *satisfying*

$$\pi(x)^m v = 0 \quad (\textit{for some natural number } m).$$

Then, for any $a \in \mathfrak{g}$, there exists a natural number n satisfying

$$\pi(x)^n(\pi(a)v) = 0. \tag{2.51}$$

Below, for simplicity, abbreviating π, we denote $\pi(x)$ by x. Writing the formula (1) in Lemma 1.34 for $\lambda = \mu = 0$, one has

$$x^n a = \sum_{j=0}^{n} \binom{n}{j} ((\mathrm{ad}(x))^j a) x^{n-j} \tag{2.52}$$

as a formula in $A := \mathrm{End}(V)$. Applying this to v, one has

$$x^n(av) = \sum_{j=0}^{n} \binom{n}{j} ((\mathrm{ad}(x))^j a) x^{n-j} v, \tag{2.53}$$

and since we have

$$j : \text{big enough} \Longrightarrow (\mathrm{ad}(x))^j a = 0,$$
$$n - j : \text{big enough} \Longrightarrow x^{n-j} v = 0$$

by the conditions on x and v, one obtains

$$\text{the right hand side of (2.53)} = 0,$$

for big enough n, and Assertion A is proved. Next, by Assertion A and an induction on the number k of elements of $\mathfrak{g}$ acting on v, the following Assertion B can be proved:

ASSERTION B. *Take an element v of V satisfying*

$$x^m v = 0 \quad (\textit{for some natural number } m).$$

Then, for any natural number k and any k elements $a_1, \cdots, a_k$ of $\mathfrak{g}$, there exists a natural number n satisfying

$$x^n(a_1 \cdots a_k v) = 0. \tag{2.54}$$

Thus, (2) of the lemma is proved. □

From this lemma, we obtain the following proposition:

PROPOSITION 2.29. *If $\mathfrak{g}(A)$ is a BKM Lie superalgebra, then for each $i \in I^{re}$, $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ act locally nilpotently on $\mathfrak{g}(A)$. In particular, for i satisfying $i \in \Psi^{re}$, $\mathrm{ad}([e_i, e_i])$ and $\mathrm{ad}([f_i, f_i])$ also act locally nilpotently on $\mathfrak{g}(A)$.*

PROOF. By the fundamental relations (2.11) – (2.13) and (2.24), $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ for $i \in I^{re}$ act locally nilpotently on Chevalley generators. Hence, for $i \in \Psi^{re}$, $\mathrm{ad}([e_i, e_i])$ and $\mathrm{ad}(f_i, f_i])$ also act locally nilpotently on generators.

So, by the above lemma, $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ if $i \notin \Psi$, or $\mathrm{ad}([e_i, e_i])$ and $\mathrm{ad}([f_i, f_i])$ for $i \in \Psi^{re}$, act locally nilpotently on $\mathfrak{g}(A)$, and in each case the $\mathrm{ad}(e_i)$'s and $\mathrm{ad}(f_i)$'s act locally nilpotent on $\mathfrak{g}(A)$. □

PROPOSITION 2.30. *For an even root $\alpha \in \Delta$ satisfying $(\alpha|\alpha) \neq 0$, set*

$$h_\alpha := \frac{2}{(\alpha|\alpha)}\alpha, \tag{2.55}$$

and take elements of root spaces $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ so that $(h_\alpha, e_\alpha, f_\alpha)$ forms an S-triple. Moreover, assume that

(LN) *both e_α and f_α act locally nilpotent on $\mathfrak{g}(A)$,*

and set

$$\begin{aligned} \widetilde{r}_\alpha :&= e^{\mathrm{ad}(f_\alpha)} e^{-\mathrm{ad}(e_\alpha)} e^{\mathrm{ad}(f_\alpha)} \\ &= \mathrm{Ad}(e^{f_\alpha})\mathrm{Ad}(e^{-e_\alpha})\mathrm{Ad}(e^{f_\alpha}) \in \mathrm{Aut}(\mathfrak{g}). \end{aligned} \tag{2.56}$$

Then, we have

1. *$\widetilde{r}_\alpha(\mathfrak{h}) = \mathfrak{h}$, and $\widetilde{r}_\alpha|_{\mathfrak{h}} = r_\alpha$.*
2. *If $\beta \in \Delta$, then $r_\alpha\beta \in \Delta$, and*
   (i) $\mathfrak{g}_{r_\alpha\beta} = \widetilde{r}_\alpha(\mathfrak{g}_\beta)$,
   (ii) $\mathrm{mult}(r_\alpha\beta) = \mathrm{mult}(\beta)$;
   in particular, one has $r_\alpha(\Delta) = \Delta$.

PROOF. For any element $H \in \mathfrak{h}$, let us compute $\widetilde{r}_\alpha(H)$. First, one has

$$\mathrm{Ad}(e^{f_\alpha})H = H + \underbrace{[f_\alpha, H]}_{\alpha(H)f_\alpha} = H + \alpha(H)f_\alpha.$$

Next, it follows that

$$\begin{aligned}
&\mathrm{Ad}(e^{-e_\alpha})\mathrm{Ad}(e^{f_\alpha})H \\
&\quad = \mathrm{Ad}(e^{-e_\alpha})H + \alpha(H)\mathrm{Ad}(e^{-e_\alpha})f_\alpha \\
&\quad = \{H - \underbrace{[e_\alpha, H]}_{-\alpha(H)e_\alpha}\} + \alpha(H)\left\{ f_\alpha - \underbrace{[e_\alpha, f_\alpha]}_{h_\alpha} + \frac{1}{2}\underbrace{[e_\alpha, \underbrace{[e_\alpha, f_\alpha]}_{h_\alpha}]}_{-2e_\alpha} \right\} \\
&\quad = H + \alpha(H)e_\alpha + \alpha(H)(f_\alpha - h_\alpha - e_\alpha) \\
&\quad = H + \alpha(H)f_\alpha - \alpha(H)h_\alpha.
\end{aligned}$$

So, we get

$$\begin{aligned}
\widetilde{r}_\alpha(H) &= \mathrm{Ad}(e^{f_\alpha})H + \alpha(H)\mathrm{Ad}(e^{f_\alpha})f_\alpha - \alpha(H)\mathrm{Ad}(e^{f_\alpha})h_\alpha \\
&= \{H + \alpha(H)f_\alpha\} + \alpha(H)f_\alpha - \alpha(H)\{h_\alpha + \underbrace{\alpha(h_\alpha)}_{2} f_\alpha\} \\
&= H - \alpha(H)h_\alpha,
\end{aligned}$$

and (1) is proved.

Next, let us prove (2). We have only to show that

$$[H, \widetilde{r}_\alpha X] = (r_\alpha\beta)(H) \cdot \widetilde{r}_\alpha X \tag{2.57}$$

for any root vector $X \in \mathfrak{g}_\beta$. Since $\widetilde{r}_\alpha \in \mathrm{Aut}(\mathfrak{g}(A))$, it follows that

$$\begin{aligned}
[H, \widetilde{r}_\alpha X] &= \widetilde{r}_\alpha[\widetilde{r}_\alpha^{-1}H, X] = \widetilde{r}_\alpha(\beta(r_\alpha^{-1}H)X) \\
&= \langle \beta, r_\alpha^{-1}H\rangle \cdot \widetilde{r}_\alpha X = (r_\alpha\beta)(H) \cdot \widetilde{r}_\alpha X,
\end{aligned}$$

and hence we can prove (2.57). □

From the above two propositions and the definition of the Weyl group W, we obtain the following corollary:

COROLLARY 2.31. *If $w \in W$, then $w(\Delta) = \Delta$.*

In particular, from this corollary,

$$W(\Pi) := \bigcup_{w \in W} w(\Pi)$$

is a subset of Δ. Set

$$\begin{aligned}
\Delta^{re} &:= W(\Pi^{re}) \cup W(\{2\alpha_i; i \in \Psi^{re}\}), \\
\Delta^{im} &:= \Delta \setminus \Delta^{re},
\end{aligned}$$

call a root belonging to Δ^{re} or Δ^{im} a **real root** or an **imaginary root** respectively. Since $r_i(\alpha_i) = -\alpha_i$, one has

$$\alpha \in \Delta^{re} \implies -\alpha \in \Delta^{re},$$

and hence it follows that

$$\alpha \in \Delta^{im} \implies -\alpha \in \Delta^{im}.$$

For $\alpha \in \Delta^{re}$, one can take $w \in W$ and $i \in I^{re}$ so that they satisfy $\alpha = w\alpha_i$ ($i \in \Pi^{re}$) or $\alpha = w(2\alpha_i)$ ($i \in \Psi^{re}$) (by the definition of the real root). Since one has $r_\alpha = r_{\alpha/2}$, in any case, decomposing this w as

$$w = r_{j_1} \cdots r_{j_k},$$

it follows that

$$r_\alpha = w r_i w^{-1} = r_{j_1} \cdots r_{j_k} \cdot r_i \cdot r_{j_k} \cdots r_{j_1},$$

and hence r_α is an element of W. Thus, we obtain the following proposition:

PROPOSITION 2.32. *For $\alpha \in \Delta^{re}$, one has the following:*

1. $(\alpha|\alpha) > 0$,
2. $r_\alpha \in W$,
3. $\mathrm{mult}(\alpha) = 1$.

Now, let us consider $r_i : \Delta \longrightarrow \Delta$. First, by the definition of r_i, we have $r_i(\alpha_i) = -\alpha_i$. Next, if we take

$$\alpha = \sum_{j=1}^{n} m_j \alpha_j \in \Delta_+,$$

we have $m_j \geq 0$ for all j, and $r_i\alpha$ looks as follows:

$$r_i\alpha = \alpha - \langle \alpha_i^\vee, \alpha \rangle \alpha_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} m_j \alpha_j + (m_i - \langle \alpha_i^\vee, \alpha \rangle)\alpha_i. \tag{2.58}$$

Namely, expressing $r_i\alpha$ as a linear combination of simple roots, the coefficients of α_j ($j \neq i$) do not change. And since there exists j_0 ($j_0 \neq i$) satisfying $m_{j_0} > 0$ for $\alpha \in \Delta_+ \setminus \{\alpha_i\}$ ($\alpha \in \Delta_+ \setminus \{\alpha_i, 2\alpha_i\}$ for $i \in \Psi^{re}$), if we take one such j_0, then the coefficient of α_{j_0} in the expansion (2.58) of $r_i\alpha$ is positive, and hence $r_i\alpha$ is a positive root. Therefore, we obtain the following proposition:

PROPOSITION 2.33. *For $i \in I^{re}$, one has the following:*

1. $r_i(\alpha_i) = -\alpha_i$.

2. *If* $i \in I^{re} \setminus \Psi^{re}$, *then* $r_i(\Delta_+ \setminus \{\alpha_i\}) = \Delta_+ \setminus \{\alpha_i\}$.
3. *If* $i \in \Psi^{re}$, *then* $r_i(\Delta_+ \setminus \{\alpha_i, 2\alpha_i\}) = \Delta_+ \setminus \{\alpha_i, 2\alpha_i\}$.

Namely, among positive roots, those that are transferred to negative roots by r_i are only α_i if α_i is an even root, and there are two such roots, namely α_i and $2\alpha_i$, if α_i is an odd root.

Let us introduce the next important object:

$$R := \frac{\prod_{\alpha \in \Delta_{even,+}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \in \mathbb{C}[[e^{-\alpha_1}, \cdots, e^{-\alpha_n}]]. \tag{2.59}$$

Here, $\mathbb{C}[[e^{-\alpha_1}, \cdots, e^{-\alpha_n}]]$ is the ring of formal power series (that is, allowing an infinite sum) generated by $e^{-\alpha_1}, \cdots, e^{-\alpha_n}$, and the product is defined by

$$e^{-\alpha} \cdot e^{-\beta} = e^{-\alpha-\beta}.$$

Then, $r_i R$ can be computed by using Proposition 2.33 as follows. First, for $i \in I^{re} \setminus \Psi^{re}$, one has

$$\begin{aligned} r_i R &= r_i(1 - e^{-\alpha_i}) \cdot \frac{\prod_{\alpha \in \Delta_{even,+} \setminus \{\alpha_i\}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \\ &= \underbrace{(1 - e^{\alpha_i})}_{-e^{\alpha_i}(1-e^{-\alpha_i})} \cdot \frac{\prod_{\alpha \in \Delta_{even,+} \setminus \{\alpha_i\}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \\ &= -e^{\alpha_i} R, \end{aligned}$$

and next for $i \in \Psi^{re}$, one has

$$\begin{aligned} r_i R &= r_i \frac{(1 - e^{-2\alpha_i})}{(1 + e^{-\alpha_i})} \cdot \frac{\prod_{\alpha \in \Delta_{even,+} \setminus \{2\alpha_i\}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+} \setminus \{\alpha_i\}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \\ &= \underbrace{(1 - e^{\alpha_i})}_{-e^{\alpha_i}(1-e^{-\alpha_i})} \cdot \frac{\prod_{\alpha \in \Delta_{even,+} \setminus \{2\alpha_i\}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+} \setminus \{\alpha_i\}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \\ &= -e^{\alpha_i} \frac{(1 - e^{-2\alpha_i})}{(1 + e^{-\alpha_i})} \cdot \frac{\prod_{\alpha \in \Delta_{even,+} \setminus \{2\alpha_i\}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+} \setminus \{\alpha_i\}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \\ &= -e^{\alpha_i} R, \end{aligned}$$

and hence

$$r_i R = -e^{\alpha_i} R \tag{2.60}$$

irrespective of whether α_i is even or odd. To cancel out the factor e^{α_i} in the formula (2.60), let us consider the multiplication $e^\rho R$ of e^ρ and R. Then, by (2.86) and (2.60), one has

$$r_i(e^\rho R) = -e^{\rho-\alpha_i} \cdot e^{\alpha_i} R = -e^\rho R. \tag{2.61}$$

Namely, $e^\rho R$ is **skew-invariant** with respect to the action of the Weyl group W:

PROPOSITION 2.34.

$$w(e^\rho R) = \varepsilon(w) \cdot e^\rho R \quad (w \in W).$$

Here, let us present a fact which can be derived from Proposition 2.33. The set of positive imaginary roots Δ_+^{im} satisfies

$$\begin{aligned} \text{if} \quad i \in I^{re} \setminus \Psi^{re}, \text{ then} \quad & \Delta_+^{im} \subset \Delta_+ \setminus \{\alpha_i\}, \\ \text{if} \quad i \in \Psi^{re}, \text{ then} \quad & \Delta_+^{im} \subset \Delta_+ \setminus \{\alpha_i, 2\alpha_i\}, \end{aligned}$$

and if we apply r_i to these sets, we get

$$\begin{aligned} \text{if} \quad i \in I^{re} \setminus \Psi^{re}, \text{ then} \quad & r_i(\Delta_+^{im}) \subset \Delta_+ \setminus \{\alpha_i\}, \\ \text{if} \quad i \in \Psi^{re}, \text{ then} \quad & r_i(\Delta_+^{im}) \subset \Delta_+ \setminus \{\alpha_i, 2\alpha_i\}, \end{aligned}$$

by Proposition 2.33. Thus, we have $r_i(\Delta_+^{im}) = \Delta_+^{im}$, and repeated applications of r_i's also enjoy the same property. Therefore, we obtain

$$w(\Delta_+^{im}) = \Delta_+^{im} \quad (w \in W). \tag{2.62}$$

Here, for $i, j \in I^{re}$ $(i \neq j)$, the order m_{ij} of $r_i r_j$ is either a positive integer or ∞, and let us study this value. If we consider the two-dimensional vector subspace of $\mathfrak{h}$

$$\mathfrak{h}_{ij} := \mathbb{C}\alpha_i \oplus \mathbb{C}\alpha_j,$$

and the vector subspace orthogonal to this space

$$\mathfrak{h}_{ij}^\perp := \{h \in \mathfrak{h}; \alpha_i(h) = \alpha_j(h) = 0\},$$

then r_i and r_j keep $\mathfrak{h}_{ij}$ invariant, and are the identity transformation on $\mathfrak{h}_{ij}^\perp$. And, in particular, since in the case $a_{ij}a_{ji} \neq 4$, a bilinear form is non-degenerate on $\mathfrak{h}_{ij}$, and $\mathfrak{h}_{ij} \cap \mathfrak{h}_{ij}^\perp = \{0\}$, and one has

$$\mathfrak{h} = \mathfrak{h}_{ij} \oplus \mathfrak{h}_{ij}^\perp.$$

So, in any case, let us first compute the action of $r_i r_j$ on $\mathfrak{h}_{ij}$. With respect to the basis $\{\alpha_i, \alpha_j\}$ of $\mathfrak{h}_{ij}$, the matrix representations

of r_i and r_j look as follows:

$$r_i = \begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}, \quad r_j = \begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix}.$$

So, on $\mathfrak{h}_{ij}$, one has

$$r_i r_j = \begin{pmatrix} -1 + a_{ij}a_{ji} & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}, \tag{2.63}$$

and hence the characteristic polynomial of $r_i r_j|_{\mathfrak{h}_{ij}}$ looks like

$$x^2 - (a_{ij}a_{ji} - 2)x + 1 = 0. \tag{2.64}$$

Here, let us remark on the following:

(2.65) the order m_{ij} of $r_i r_j$ is finite
$\Longrightarrow$ two solutions of (2.64) are roots of unity.

Moreover, by using the discriminant of the second ordered equation, one can easily show the following:

(2.66)
two solutions of (2.64) are roots of nuity $\Longleftrightarrow$ $0 \leq a_{ij}a_{ji} \leq 4$.

And the order m_{ij} of $r_i r_j$ can be easily extracted from the solutions of the second order equations (2.64).

Here, we remark that in (2.65), the converse $\Longleftarrow$ does not necessarily hold. In fact, in the case $a_{ij}a_{ji} = 4$, the equation (2.64) becomes $(x-1)^2 = 0$, and the eigenvalues of $r_i r_j|_{\mathfrak{h}_{ij}}$ are only 1 (its multiplicity $= 2$), and in this case, the matrix

$$r_i r_j|_{\mathfrak{h}_{ij}} = \begin{pmatrix} 3 & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}$$

is not diagonalizable. Thus, in this case, we have $m_{ij} = \infty$. In this way, m_{ij} for each value of $a_{ij}a_{ji}$ can be computed as in Table 2.1.

TABLE 2.1. The order m_{ij} of $r_i r_j$

$a_{ij}a_{ji}$	0	1	2	3	no less than 4
m_{ij}	2	3	4	6	∞

Certainly, we have

$$m_{ij} = 2 \Longleftrightarrow r_i r_j = r_j r_i.$$

Thus, we see that the generators of the Weyl group $W = \langle r_i; i \in I^{re} \rangle$ satisfy the following relations:

$$r_i^2 = e, \tag{2.67}$$

$$(r_i r_j)^{m_{ij}} = e \quad (\text{where } m_{ij} \text{ is given in Table 2.1}). \tag{2.68}$$

In fact, it is known that the relations (2.67) and (2.68) exhaust all of the fundamental relations in W. Namely, W is a **Coxeter group**. This can be seen by looking at the action of W on Δ carefully, and W satisfies the 'exchange condition'. For its detailed explanation, we refer the reader to e.g., [**Bour2**].

Further, by the fundamnetal relations (2.67) and (2.68), the Weyl group of A and the Weyl group of ${}^tA$ are isomorphic, and this isomorphism is given by the natural correspondence among their generators.

2.3. Root Systems

In this section, to make the story simpler, we assume that A is a symmetrizable BKM matrix, and simply denote $\mathfrak{g}(A)$ by $\mathfrak{g}$. There is no obstruction in the case of a BKM supermatrix, but if we politely mention in a way 'in the case of an even root, we have $\cdots$, and in the case of an odd root, we have $\cdots$', then the story becomes dull, and its course becomes unclear, so in order to avoid this, we may restrict ourselves to the non-super case. The detailed arguments for BKM supermatrices are left to the reader as exercises, and we will state their results at the end of this section. If we set

$$Q := \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \quad Q_+ := \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i,$$

then we have $\Delta_+ \subset Q_+$. Q is called the **root lattice**, and Q_+ is called the **positive root lattice**. First, let us mention the following property:

$$i \in I^{im}, \alpha \in Q_+ \implies (\alpha|\alpha_i) \leq 0. \tag{2.69}$$

In fact, if we set

$$\alpha = \sum_{j=1}^{n} m_j \alpha_j \quad (m_j \geq 0),$$

then we have

$$(\alpha|\alpha_i) = \sum_{j=1}^{n} m_j (\alpha_j|\alpha_i),$$

and from (2.9), it follows that each term in the right hand side of this formula should be ≤ 0. The root lattice Q becomes a (partially) ordered set by putting

$$\alpha \geq \beta \iff \alpha - \beta \in Q_+$$

for $\alpha, \beta \in Q$. Namely, for $\alpha = \sum_i m_i \alpha_i$, $\beta = \sum_i n_i \alpha_i \in Q$, we set

$$\alpha \geq \beta \iff m_i \geq n_i \quad (\text{for all } i).$$

Now, for $\alpha = \sum_i m_i \alpha_i \in Q$, we call

$$\mathrm{supp}(\alpha) := \{i \in I; m_i \neq 0\}$$

the **support** of α. If $\mathrm{supp}(\alpha)$ is a connected subset of the quasi-Dynkin diagram of A, we say that $\mathrm{supp}(\alpha)$ is **connected**. First, the following lemma can be easily shown by induction on the height of a root:

LEMMA 2.35. *If α is a root, then* $\mathrm{supp}(\alpha)$ *is connected.*

LEMMA 2.36. *For $i \in I^{im}$ and $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, if* $\mathrm{supp}(\alpha + \alpha_i)$ *is connected, then $(\alpha|\alpha_i) < 0$.*

PROOF. If we express $\alpha \in \Delta_+$ as $\alpha = \sum_{j=1}^n m_j \alpha_j$, then by the condition $\alpha \neq \alpha_i$, there exists k $(k \neq i)$ satisfying $m_k > 0$. Here, by the assumption that $\mathrm{supp}(\alpha + \alpha_i)$ is connected, one can choose this α_k from those that are connected to α_i in the quasi-Dynkin diagram. Choosing k in such a way, one has $a_{ki} < 0$. Then, in the formula

$$(\alpha|\alpha_i) = \sum_{j=1}^n m_j (\alpha_j|\alpha_i),$$

each term $m_j(\alpha_j|\alpha_i)$ in the right hand side looks as follows:

$$m_j(\alpha_j|\alpha_i) \begin{cases} < 0 & \text{for } j = k, \\ \leq 0 & \text{for } j \neq k. \end{cases}$$

Therefore, we have $(\alpha|\alpha_i) < 0$. $\square$

Thus, we see that the generators of the Weyl group $W = \langle r_i; i \in I^{re} \rangle$ satisfy the following relations:

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$$(\alpha|\alpha_i) = \sum_{j=1}^{n} m_j (\alpha_j|\alpha_i),$$

and from (2.9), it follows that each term in the right hand side of this formula should be ≤ 0. The root lattice Q becomes a (partially) ordered set by putting

$$\alpha \geq \beta \iff \alpha - \beta \in Q_+$$

for $\alpha, \beta \in Q$. Namely, for $\alpha = \sum_i m_i \alpha_i$, $\beta = \sum_i n_i \alpha_i \in Q$, we set

$$\alpha \geq \beta \iff m_i \geq n_i \quad (\text{for all } i).$$

Now, for $\alpha = \sum_i m_i \alpha_i \in Q$, we call

$$\mathrm{supp}(\alpha) := \{i \in I; m_i \neq 0\}$$

the **support** of α. If $\mathrm{supp}(\alpha)$ is a connected subset of the quasi-Dynkin diagram of A, we say that $\mathrm{supp}(\alpha)$ is **connected**. First, the following lemma can be easily shown by induction on the height of a root:

LEMMA 2.35. *If α is a root, then* $\mathrm{supp}(\alpha)$ *is connected.*

LEMMA 2.36. *For $i \in I^{im}$ and $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, if* $\mathrm{supp}(\alpha + \alpha_i)$ *is connected, then $(\alpha|\alpha_i) < 0$.*

PROOF. If we express $\alpha \in \Delta_+$ as $\alpha = \sum_{j=1}^n m_j \alpha_j$, then by the condition $\alpha \neq \alpha_i$, there exists k $(k \neq i)$ satisfying $m_k > 0$. Here, by the assumption that $\mathrm{supp}(\alpha + \alpha_i)$ is connected, one can choose this α_k from those that are connected to α_i in the quasi-Dynkin diagram. Choosing k in such a way, one has $a_{ki} < 0$. Then, in the formula

$$(\alpha|\alpha_i) = \sum_{j=1}^n m_j (\alpha_j|\alpha_i),$$

each term $m_j(\alpha_j|\alpha_i)$ in the right hand side looks as follows:

$$m_j(\alpha_j|\alpha_i) \begin{cases} < 0 & \text{for } j = k, \\ \leq 0 & \text{for } j \neq k. \end{cases}$$

Therefore, we have $(\alpha|\alpha_i) < 0$. □

Now, let us consider the following sets:

$$K := \left\{ \alpha \in Q_+;\ \begin{matrix} \text{(i)} & \langle \alpha, \alpha_i^\vee \rangle \le 0\ (i \in I^{re}) \\ \text{(ii)} & \mathrm{supp}(\alpha) \text{ is connected} \end{matrix} \right\},$$

$$\overset{\circ}{K} := \{\alpha \in K; |\mathrm{supp}(\alpha)| \ge 2\},$$

$$\mathfrak{h}'_{\mathbb{R}} := \sum_{i=1}^{n} \mathbb{R}\alpha_i^\vee = \sum_{i=1}^{n} \mathbb{R}\alpha_i,$$

$$C^\vee := \{h \in \mathfrak{h}'_{\mathbb{R}};\ \langle \alpha_i^\vee, h \rangle \ge 0\ (i \in I^{re})\},$$

$$\overset{\circ}{C}{}^\vee := \{h \in \mathfrak{h}'_{\mathbb{R}};\ \langle \alpha_i^\vee, h \rangle > 0\ (i \in I^{re})\}.$$

DEFINITION 2.37. $C^\vee$ is called a **fundamental domain** or a "positive Weyl chamber". $\overset{\circ}{C}{}^\vee$ is the 'interior' of the fundamental domain. The set of differences $C^\vee \setminus \overset{\circ}{C}{}^\vee$ is called the **walls** of the positive Weyl chamber.

LEMMA 2.38. *If* $\alpha \in Q_+ \cap (-C^\vee)$, *then* $(\alpha|\alpha) \le 0$.

PROOF. If we write $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, then since $k_i \ge 0$ for all i, it follows that

$$\begin{aligned}(\alpha|\alpha) &= \sum_{i=1}^{n} k_i (\alpha|\alpha_i) \\ &= \sum_{i \in I^{re}} k_i \underbrace{(\alpha|\alpha_i)}_{\substack{\le 0 \\ (\because \alpha \in -C^\vee)}} + \sum_{i \in I^{im}} k_i \underbrace{(\alpha|\alpha_i)}_{\substack{\le 0 \\ (\because (2.69))}} \\ &\le 0.\end{aligned}$$

□

Let us divide imaginary roots into two kinds, i.e. an imaginary root that is conjugate to a fundamental root under the action of the Weyl group (domestic-type), and an imaginary root that is not conjugate to a fundamental root under the action of the Weyl group (alien-type) as follows:

$$\Delta^{dom-im} := W(\Pi^{im}) \cup W(-\Pi^{im}),$$
$$\Delta^{ali-im} := \Delta^{im} \setminus \Delta^{dom-im}.$$

The next formula can be checked in the same way as in Lemma 1.43, and it plays an important role in studying the distribution of imaginary roots:

LEMMA 2.39. *Let V be a $\mathfrak{g}$-module, and suppose that an element v ($\neq 0$) of V, $\lambda \in \mathfrak{h}^*$ and $i \in I^{im}$ satisfy the following three conditions:*

i. $\langle \lambda, \alpha_i^\vee \rangle < 0$,
ii. $hv = \langle \lambda, h \rangle v \quad (h \in \mathfrak{h})$,
iii. $f_i v = 0$.

Then, for all $j \in \mathbb{Z}_{\geq 0}$, the following formulae hold:

1. $f_i e_i^j v = -j \left\{ \langle \lambda, \alpha_i^\vee \rangle + \frac{a_{ii}}{2}(j-1) \right\} e_i^{j-1} v$,
2. $e_i^j v \neq 0$.

From this lemma, we obtain the following proposition:

PROPOSITION 2.40. *For $i \in I^{im}$, take $\alpha \in \Delta_+ \setminus \{\alpha_i\}$ so that* $\mathrm{supp}(\alpha + \alpha_i)$ *is connected. Then, for all $j \in \mathbb{Z}_{\geq 0}$, one has $\alpha + j\alpha_i \in \Delta_+$.*

PROOF. Consider the sequence

$$\alpha - \alpha_i, \alpha - 2\alpha_i, \alpha - 3\alpha_i, \cdots$$

in the root lattice Q. Notice that when we express $\alpha - m\alpha_i$ as a linear combination of fundamental roots, the only coefficient of α_i becomes negative if m becomes big (even the coefficients of α_j ($j \neq i$) remains ≥ 0, and in fact they do not change). Moreover, by the condition $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, there is a positive coefficient besides that of α_i. Namely, if m becomes big, then both positive numbers and negative numbers appear in the coefficients of $\alpha - m\alpha_i$ when we express it as a linear combination of fundamental roots. Thus, if m becomes big, $\alpha - m\alpha_i$ cannot be a root. So, in the proof of this proposition, we may assume

$$\alpha - \alpha_i \notin \Delta_+ \tag{2.70}$$

from the very beginning. And by Lemma 2.36, one has

$$(\alpha | \alpha_i) < 0. \tag{2.71}$$

These two equations (2.70) and (2.71) imply that if we set

$$V := \mathfrak{g}, \quad \lambda := \alpha, \quad v := x \in \mathfrak{g}_\alpha$$

(here, x is a non-zero element of the root space $\mathfrak{g}_\alpha$,), they satisfy the conditions in Lemma 2.39. So, by Lemma 2.39, we have

$$(\mathrm{ad}(e_i))^j x \neq 0$$

for all $j \in \mathbb{Z}_{\geq 0}$, and thus $\alpha + j\alpha_i$ is a root. □

The above lemma says that when α_i is an imaginary root ($i \in I^{im}$), the α_i-series through α do not truncate finitely, and they continue infinitely. On the other hand, when α_i is a real root, then $(\alpha_i^\vee, e_i, f_i)$ is an S-triple; $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ act locally nilpotent on $\mathfrak{g}$. So, the vector subspace obtained by applying $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ succesively to a root space $\mathfrak{g}_\alpha$ is a representation space of $\mathfrak{sl}(2, \mathbb{C})$, and one obtains the next lemma in the same way as Proposition 1.57 and Corollary 1.59:

LEMMA 2.41. *For $\alpha \in \Delta$ and $i \in I^{re}$, one has the following:*

1. *The set $\{j \in \mathbb{Z}; \alpha + j\alpha_i \in \Delta\}$ is a finite set. Let p be the minimum contained in this set, and let q be the maximum in this set. Then,*
   (i) $p + q = -\langle \alpha_i^\vee, \alpha \rangle$,
   (ii) $\{j \in \mathbb{Z}; \alpha + j\alpha_i \in \Delta\} = \{j \in \mathbb{Z}; p \leq j \leq q\}$,
   (iii) *the sequence $\{\mathrm{mult}(\alpha + j\alpha_i)\}_{p \leq j \leq q}$ is bilaterally symmetric, and the left half this sequence is monotone nondecreasing. Namely,*
      (a) $j + k = p + q \implies \mathrm{mult}(\alpha + j\alpha_i) = \mathrm{mult}(\alpha + k\alpha_i)$,
      (b) $p \leq j \leq k \leq \frac{p+q}{2}$
      $\implies \mathrm{mult}(\alpha + j\alpha_i) \leq \mathrm{mult}(\alpha + k\alpha_i)$.
2. $$(\alpha|\alpha_i) > 0 \implies \alpha - \alpha_i \in \Delta_+,$$
   $$(\alpha|\alpha_i) < 0 \implies \alpha + \alpha_i \in \Delta_+.$$
3. $$\alpha + \alpha_i \notin \Delta \implies (\alpha|\alpha_i) \geq 0,$$
   $$\alpha - \alpha_i \notin \Delta \implies (\alpha|\alpha_i) \leq 0.$$

(2) and (3) of the above lemma constitute a contraposition, and it is simply a rephrasing, but for later use, we state both forms. By (i) in (1) of this lemma, the next corollary immediately follows:

COROLLARY 2.42. *For $\alpha \in \Delta, i \in I^{re}$, if $\alpha + \alpha_i \notin \Delta$ and $\alpha - \alpha_i \in \Delta$, then $(\alpha|\alpha_i) > 0$.*

Later in this section, I will study the structure of Δ^{im} by using Proposition 2.40 and Lemma 2.41 depending on the cases where α_i is a real root or an imaginary root, and I want the reader to observe that Π^{re} and Π^{im} play each role quite well, and also observe the scene of their works making most of their characteristics.

LEMMA 2.43. 1. *Δ_+^{im} is invariant under the action of the Weyl group W.*
2. *If $\alpha \in \Delta_+^{im}$, then there exists $w \in W$ satisfying $w\alpha \in -C^\vee$.*
3. *For $\alpha \in \Delta_+$,*

$$\alpha \in \Delta_+^{im} \iff (\alpha|\alpha) \leq 0.$$

PROOF. (1) follows from (2.62).

(2) If we consider transformations of $\alpha \in \Delta_+^{im}$ by the Weyl group, then since we have

$$W \cdot \alpha = \{w\alpha; w \in W\} \subset Q_+$$

by (1), there exists an element whose height is minimal in $W \cdot \alpha$. Let us denote it by $\beta := w\alpha$. Applying r_i ($i \in I^{re}$) to this element β, one has

$$r_i w\alpha = r_i\beta = \beta - \langle \alpha_i^\vee, \beta\rangle \alpha_i,$$

and hence the height of $r_i w\alpha$ ($i \in I^{re}$) is

$$\mathrm{ht}(r_i w\alpha) = \mathrm{ht}(\beta) - \langle \alpha_i^\vee, \beta\rangle.$$

Thus, by the minimality of $\mathrm{ht}(\beta)$, we should have

$$-\langle \alpha_i^\vee, \beta\rangle \geq 0$$

Since this inequality holds for all $i \in I^{re}$, we have $\beta \in -C^\vee$.

(3) [The proof of $\Longrightarrow$.] For $\alpha \in \Delta_+^{im}$, take $w \in W$ as in (2), and set

$$\beta := w\alpha \in -C^\vee.$$

Then, by Lemma 2.27 and Lemma 2.38,

$$(\alpha|\alpha) = (\beta|\beta) \leq 0.$$

[The proof of $\Longleftarrow$.] If α is a real root, then it must be conjugate to one of the α_i's ($i \in I^{re}$) under the action of the Weyl group, and by Lemma 2.27 we have

$$(\alpha|\alpha) = (\alpha_i|\alpha_i) > 0.$$

□

By (2) of Lemma 2.43, we have

$$\Delta_+^{im} \subset W(K) = \bigcup_{w \in W} w(K).$$

Let us consider

$$\beta := w\alpha \in -C^\vee$$

by applying (2) of Lemma 2.43 to $\alpha \in \Delta_+^{im}$. If $\mathrm{supp}(\beta)$ consists of one point, i.e., $\beta = m\alpha_i$, then α_i must be a simple imaginary root. This is because

$$0 \geq (\alpha|\alpha) = m^2(\alpha_i|\alpha_i)$$

by (3), and hence we get $(\alpha_i\alpha_i) \leq 0$. In addition, since $\beta = m\alpha_i$ is a root, one has $m = 1$, and thus we obtain $\alpha \in \Delta_+^{dom-im}$.

Note. When A is a BKM supermatrix, and this α_i is an odd simple root, we have

$$\text{if } (\alpha_i|\alpha_i) = 0, \text{ then } m = 1, \text{ and}$$
$$\text{if } (\alpha_i|\alpha_i) \neq 0, \text{ then } m = 1, 2 \text{ are allowed.}$$

One has to be careful with this for Lie superalgebras. □

By taking the contraposition, one has $|\mathrm{supp}(\alpha)| \geq 2$ for $\alpha \in \Delta_+^{ali-im}$; thus it follows that $\beta \in \overset{\circ}{K}$. Namely, one has

$$\Delta_+^{ali-im} \subset W(\overset{\circ}{K}) = \bigcup_{w \in W} w(\overset{\circ}{K}). \tag{2.72}$$

In this formula, the '$\subset$' is in fact '$=$'. The following proposition asserts this:

PROPOSITION 2.44. 1. $\overset{\circ}{K} \subset \Delta_+^{im}$.

2. $\Delta_+^{ali-im} = W(\overset{\circ}{K})$.

PROOF. (1) Take

$$\alpha = \sum_{i=1}^{n} k_i\alpha_i \in \overset{\circ}{K}. \tag{2.73}$$

We would like to prove that α is a root. First, setting

$$\Omega_a := \{\gamma \in \Delta_+; \gamma \leq \alpha\}, \tag{2.74}$$

for i satisfying $k_i \neq 0$, we have $\alpha_i \in \Omega_a$, so Ω_a is not an empty set. So, take an element whose height is maximum in Ω_a, and denote it by

$$\beta = \sum_{i=1}^{n} m_i\alpha_i. \tag{2.75}$$

Clearly, we have

$$m_i \leq k_i$$

for all $i \in I$. We would like to prove that $\beta = \alpha$. So, assuming

$$\beta \neq \alpha,$$

we get a contradiction. Set

$$P := \{j \in \mathrm{supp}(\alpha); k_j = m_j\}.$$

(It may happen that P is an empty set, but we do not care at this stage. In fact, it will be shown in Assertion 3 that P is not an empty set.) Since we assume $\beta \neq \alpha$, we have

$$P \subsetneqq \mathrm{supp}(\alpha).$$

So, set

$$R := \mathrm{supp}(\alpha) \setminus P = \{i \in \mathrm{supp}(\alpha); m_i < k_i\},$$

and take one of the connected components of R, and denote it by R_0. First, by the maximality of $\mathrm{ht}(\beta)$, we have

$$i \in R \implies \beta + \alpha_i \notin \Delta. \tag{2.76}$$

And by Proposition 2.40, we have

$$i \in I^{im} \implies k_i = m_i, \tag{2.77}$$

in other words (i.e., its contraposition), we have

$$R \subset I^{re}. \tag{2.78}$$

By this, (2.76) and (3) of Lemma 2.41, it follows that

$$i \in R \implies (\beta|\alpha_i) \geq 0. \tag{2.79}$$

Moreover, since $\alpha \in \overset{\circ}{K}$, it follows that $|\mathrm{supp}(\alpha)| \geq 2$, and hence by the maximality of $\mathrm{ht}(\beta)$, β cannot be a fundamental root. So, by Lemma 2.15, it follows that

$$\text{there exists } i \in \mathrm{supp}(\beta) \text{ satisfying } \beta - \alpha_i \in \Delta_+. \tag{2.80}$$

Under our setting (i.e., the assumption of *reductio ad absurdum*), in addition, the followings hold:

ASSERTION 1. $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta)$,

ASSERTION 2. $\mathrm{supp}(\alpha)$ *is not of finite type,*

ASSERTION 3. $P \neq \varnothing$,

ASSERTION 4. R_0 *is a* GCM *of finite type.*

Let us check this step by step.

To prove Assertion 1, assume that $\mathrm{supp}(\beta) \subsetneqq \mathrm{supp}(\alpha)$. If we take

$$i \in \mathrm{supp}(\alpha) \setminus \mathrm{supp}(\beta),$$

then since $k_i > 0$ and $m_i = 0$, we have $i \in R$, and thus by (2.78),

$$i \in I^{re}.$$

By the connectivity of $\mathrm{supp}(\alpha)$, one can choose this i next to $\mathrm{supp}(\beta)$, but then we have

$$(\beta|\alpha_i) < 0,$$

and this contradicts (2.79).

To prove Assertion 2, note that since α ($\neq 0$) is an element of $\mathring{K}$, by Lemma 2.38, we have $(\alpha|\alpha) \leq 0$, and hence the bilinear form $(\ |\ )$ cannot be positive definite on the vector subspace of $\mathfrak{h}$ spanned over $\{\alpha_i; i \in \mathrm{supp}(\alpha)\}$.

To prove Assertion 3, assume $P = \varnothing$, and argue to a contradiction. If $P = \varnothing$, then $R = \mathrm{supp}(\alpha)$, and thus by (2.78) and Assertion 1, we have

$$\mathrm{supp}(\beta) = \mathrm{supp}(\alpha) \subset I^{re}.$$

So, by (2.76), (2.80) and Corollary 2.42, it follows that

$$\text{there exists } i_0 \in \mathrm{supp}(\alpha) \text{ saitisfying} \quad (\beta|\alpha_{i_0}) > 0.$$

And if we rewrite (2.79) for this case, then we have

$$(\beta|\alpha_i) \geq 0 \quad \text{for all } i \in \mathrm{supp}(\alpha).$$

But, if $\sum_{i\in\mathrm{supp}(\alpha)} \mathbb{Z}_{\geq 0}\alpha_i$ contains such an element β, then $\mathrm{supp}(\alpha)$ becomes a GCM of finite type by Proposition 2.24, which contradicts Assertion 2.

To prove Assertion 4, set

$$\begin{aligned} \beta' &:= \sum_{j\in R_0} m_j\alpha_j \\ &= \beta - \sum_{j\in P} m_j\alpha_j - \sum_{\substack{j\in(\text{a connected component of } R \\ \text{which is not } R_0)}} m_j\alpha_j. \end{aligned} \tag{2.81}$$

For $i \in R_0$, let us compute $(\beta'|\alpha_i)$. Since the inner product of α_i and the third term in the right hand side gives 0 (because $i \in R_0$), $(\beta'|\alpha_i)$

looks as follows:

$$(\beta'|\alpha_i) = \underbrace{(\beta|\alpha_i)}_{\substack{\geq 0 \\ (\because 2.79)}} - \sum_{j\in P} m_j(\alpha_j|\alpha_i) \tag{2.82}$$
$$\geq -\sum_{j\in P} m_j(\alpha_j|\alpha_i).$$

By the way, since we have

$$\sum_{j\in P} m_j(\alpha_j|\alpha_i) \begin{cases} = 0 & \text{if } i \text{ is not next to } P, \\ < 0 & \text{if } i \text{ is next to } P, \end{cases}$$

by (2.82), it follows that

$$(\beta'|\alpha_i) \begin{cases} \geq 0 & \text{if } i \text{ is not next to } P, \\ > 0 & \text{if } i \text{ is next to } P. \end{cases}$$

Namely, this $\beta \in \sum_{i\in R_0} \mathbb{Z}_{\geq 0}\alpha_i$ satisfies

$$\langle \beta', \alpha_i^\vee \rangle \geq 0 \quad \text{for all } i \in R_0,$$
$$\langle \beta', \alpha_i^\vee \rangle > 0 \quad \text{for some } i \in R_0,$$

and thus, by Proposition 2.24, R_0 is of finite type.

Thus, Assertions $1-4$ are proved. Now, from Assertion 4, one can derive the following contradiction. If we set

$$\gamma := \sum_{i\in R_0} (k_i - m_i)\alpha_i \in \sum_{i\in R_0} \mathbb{Z}_{>0}\alpha_i,$$

then, since R_0 is one of the connected components of $\mathrm{supp}(\alpha-\beta)$, we have

$$\begin{aligned} \langle \gamma, \alpha_i^\vee \rangle &= \langle \alpha - \beta, \alpha_i^\vee \rangle \\ &= \underbrace{\langle \alpha, \alpha_i^\vee \rangle}_{\substack{\leq 0 \\ (\because \alpha \in \mathring{K})}} - \underbrace{\langle \beta, \alpha_i^\vee \rangle}_{\substack{\geq 0 \\ (\because (2.79))}} \\ &\leq 0 \end{aligned}$$

for each $i \in R_0$. But since R_0 is of finite type, this contradicts the conclusion obtained from Corollary 2.23. Thus, we have completed a *reductio ad absurdum*, and (1) is proved.

(2) follows from (1) and (2.72). □

Finally, we obtain the following theorem:

THEOREM 2.45. *On sets of imaginary roots of a symmetrizable BKM matrix, the following hold:*

1. $\Delta_+^{ali-im} = W(\overset{\circ}{K}) = \bigcup_{w \in W} w(\overset{\circ}{K})$.
2. $\Delta_+^{im} = W(\overset{\circ}{K}) \cup W(\Pi^{im})$.

For Lie superalgebras, we have the following theorem:

THEOREM 2.46. *For a symmetrizable BKM supermatrix (A, Ψ), if we set*

$$\begin{aligned} \Delta^{re} &:= W(\Pi^{re}) \cup W(\{2\alpha_i; i \in \Psi^{re}\}), \\ \Delta^{im} &:= \Delta \setminus \Delta^{re}, \\ \Delta^{dom-im} &:= W(\pm\Pi^{im}) \cup W(\{\pm 2\alpha_i; i \in \Psi_-\}), \\ \Delta^{ali-im} &:= \Delta^{im} \setminus \Delta^{dom-im}, \end{aligned}$$

then concerning Δ^{im} and Δ^{ali-im}, we have the following:

1. $\Delta_+^{ali-im} = W(\overset{\circ}{K}) = \bigcup_{w \in W} w(\overset{\circ}{K})$.
2. $\Delta_+^{im} = W(\overset{\circ}{K}) \cup W(\Pi^{im}) \cup W(\{2\alpha_i; i \in \Psi_-\})$.

2.4. Integrable Representations

In this section, we assume that A is either a symmetrizable BKM matrix or a symmetrizable BKM supermatrix, and we simply denote $\mathfrak{g}(A)$ by $\mathfrak{g}$. Let (π, V) be a representation of $\mathfrak{g}$, and we simply denote the element $\pi(x)v$ obtained by applying an element $x \in \mathfrak{g}$ to v by xv. For $\lambda \in \mathfrak{h}^*$, set

$$V_\lambda := \{v \in V; hv = \lambda(h)v \ (h \in \mathfrak{h})\}.$$

When $V_\lambda \neq \{0\}$, λ is called a **weight** of this representation, and V_λ is called the **weight space** of weight λ. The dimension of the weight space V_λ is called the **multiplicity** of λ in this representation (π, V) (or simply V), and is denoted by $\mathrm{mult}_V(\lambda)$.

NOTATION 2.47. We denote the set of all weights of a representation (π, V), which is a subset of $\mathfrak{h}^*$, by $P(V)$.

DEFINITION 2.48. If a representation (π, V) of $\mathfrak{g}$ satisfies

1. each V_λ is of finite dimension, and
2. $V = \bigoplus_{\lambda \in P(V)} V_\lambda$,

this representation is said to be **$\mathfrak{h}$-diagonalizable**. And in this case, V is called a $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module.

Here and after, when we express a representation, we frequently abbreviate π and simply call V a representation of $\mathfrak{g}$. When V is an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module,

$$\mathrm{ch}V := \sum_{\lambda \in P(V)} (\dim V_\lambda) \cdot e^\lambda \tag{2.83}$$

is called the **character** of V. Here, we defined the character algebraically as 'an element of a ring', but the value of the character $\mathrm{ch}V$ at $h \in \mathfrak{h}$

$$(\mathrm{ch}V)(h) := \sum_{\lambda in P(V)} (\dim V_\lambda) \cdot e^{\lambda(h)} \tag{2.84}$$

may sometimes converge and make sense[3]. Since

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda$$

is the eigenspace decomposition with respect to the action of $\pi(h)$ ($h \in \mathfrak{h}$), $\pi(h)$ is the scalar operator

$$\pi(h)|_{V_\lambda} = \lambda(h) \cdot \mathrm{Id}_{V_\lambda}$$

on each V_λ, and hence we have

$$e^{\pi(h)}|_{V_\lambda} = e^{\lambda(h)} \cdot \mathrm{Id}_{V_\lambda},$$

and the right hand side of (2.4) is just the trace of the linear operator $e^{\pi(h)}$:

$$(\mathrm{ch}V)(h) = \sum_{\lambda \in P(V)} \mathrm{tr}_V(e^{\pi(h)}).$$

Note. From the viewpoint of lifting the action of a Lie algebra to the action of the corresponding Lie group, $e^{\pi(h)}$ is the action of an element e^h of a "Cartan subgroup", whose Lie algebra is $\mathfrak{h}$, on V, and its trace is the usual trace in the representation theory of groups. The defining equation (2.83) of $\mathrm{ch}V$ contains such meaning, but anyway, here we define it as an element of a ring, and compute it in the ring. Concerning 'a ring', we will soon introduce the notion 'highest weight module', and treat it accurately. □

[3]To see for which h the right hand side of this series converges, one has to study the degree of growth of $\dim V_\lambda$. For a Kac-Moody Lie algebra, there is a detailed explanation on this in §10.6 of [**K4**].

DEFINITION 2.49. Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module. If $\pi(e_i)$ and $\pi(f_i)$ act on V locally nilpotently for all $i \in I^{re}$, V is said to be **integrable**.

If V is an integrable $\mathfrak{g}$-module, then by the same argument as in §2.2, the e_α's and f_α's act locally nilpotently for all $\alpha \in \Delta^{re}$, and

$$r_\alpha^V := e^{\pi(f_\alpha)} e^{\pi(-e_\alpha)} e^{\pi(f_\alpha)} \tag{2.85}$$

makes sense as a linear operator on V. One can derive

$$v \in V_\lambda \implies r_\alpha^V v \in V_{r_\alpha \lambda} \tag{2.86}$$

in the same way as in Proposition 2.30, and the next proposition holds:

PROPOSITION 2.50. *If V is an $\mathfrak{h}$-diagonalizable integrable $\mathfrak{g}$-module then* $\mathrm{ch}(V)$ *is invariant under the action of the Weyl group.*

Next, let us introduce the important notion of a 'highest weight $\mathfrak{g}$-module':

DEFINITION 2.51. Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module. If there exist $\Lambda \in P(V)$ and an element v_Λ ($\neq 0$) of V_Λ satisfying the conditions

i. $xv_\Lambda = 0$ for all $x \in \mathfrak{n}_+$, and
ii. $\mathfrak{U}(\mathfrak{g}) \cdot v_\Lambda = V$,

then V is called a **highest weight representation** with highest weight Λ, or a highest weight $\mathfrak{g}$-module with highest weight Λ, and v_Λ is called a **highest weight vector**.

Since its name is too long, we may sometimes abbreviate 'highest weight' as 'h.w.'. Condition (i) in this definition is equivalent to

i'. $e_i v_\Lambda = 0$ for all $i \in I$.

Since we have

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{n}_-) \cdot \mathfrak{U}(\mathfrak{h}) \cdot \mathfrak{U}(\mathfrak{n}_+) \tag{2.87}$$

by the PBW theorem (Theorem 1.16), a highest weight vector v_Λ satisfies

$$\mathfrak{U}(\mathfrak{n}_-) \cdot v_\Lambda = V, \tag{2.88}$$

and the next lemma immediately follows:

LEMMA 2.52. *If V is a highest weight $\mathfrak{g}$-module with highest weight Λ, then the following hold:*

1. *Any element λ of $P(V)$ can be expressed as*

$$\lambda = \Lambda - \sum_{i=1}^{n} m_i \alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0}).$$

2. $V_\Lambda = \mathbb{C}v_\Lambda$, *namely, V_Λ is a one-dimensional vector subspace.*

So, if V is a highest weight $\mathfrak{g}$-module with highest weight Λ, then

$$\mathrm{ch}V = e^{\Lambda} \sum_{m_1, \cdots, m_n \geq 0} \left(\dim V_{\Lambda - \sum_i m_i \alpha_i}\right) e^{-\sum_{i=1}^{n} m_i \alpha_i},$$

and $e^{-\Lambda} \cdot \mathrm{ch}V$ makes sense as an element of the ring of formal series with n variables:

$$\mathbb{C}[[e^{-\alpha_1}, \cdots, e^{-\alpha_n}]].$$

Now, let us look at (2.88). By (2.88), the map

$$\mathfrak{U}(\mathfrak{n}_-) \ni x \longmapsto x v_\Lambda \in V$$

is surjective, and if it is also injective, therefore a linear isomorphism, V is called a **Verma module**, and we denote the Verma module with highest weight Λ by $M(\Lambda)$. The existence and the uniqueness (up to isomorphism) of the Verma module are not clear from this defintion, but the existence can be shown by the following construction. First, let 1_Λ be a symbol, and consider the one-dimensional vector space $\mathbb{C} \cdot 1_\Lambda$ spanned over it. This vector space becomes an $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module by setting

$$\begin{aligned} h \cdot 1_\Lambda &:= \Lambda(h) 1_\Lambda \quad (h \in \mathfrak{h}), \\ x \cdot 1_\Lambda &:= 0 \qquad (x \in \mathfrak{n}_+). \end{aligned}$$

Then, set

$$U(\Lambda) := \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{h} \oplus \mathfrak{n}_+}{\otimes} \mathbb{C} \cdot 1_\Lambda.$$

Here, $\underset{\mathfrak{h} \oplus \mathfrak{n}_+}{\otimes}$ signifies the equivalence classes of the ordinary tensor product by the following equivalence relation:

$$xy \otimes v \sim x \otimes yv \quad (x \in \mathfrak{U}(\mathfrak{g}), y \in \mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_+), v \in \mathbb{C} \cdot 1_\Lambda).$$

Then $U(\Lambda)$ naturally becomes an $\mathfrak{U}(\mathfrak{g})$-module; thus it is a $\mathfrak{g}$-module, by the multiplication of elements of $\mathfrak{U}(\mathfrak{g})$ from the left. In this case, the natural map

$$\mathfrak{U}(\mathfrak{n}_-) \ni x \longmapsto x \otimes 1_\Lambda \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{h} \oplus \mathfrak{n}_+}{\otimes} \mathbb{C} \cdot 1_\Lambda$$

does not have a kernel (since it never crushes, by the above equivalence relation), and hence $U(\Lambda)$ constructed as above is the Verma module. The uniqueness of the Verma module follows from the fact that the above definition of the Verma module $M(\Lambda)$ leads to the following properties:

LEMMA 2.53. *For $\Lambda \in \mathfrak{h}^*$, let V be a highest weight $\mathfrak{g}$-module with highest weight Λ. Then:*

1. *There exists uniquely (up to a scalar) a non-zero homomorphism of $\mathfrak{g}$-modules from $M(\Lambda)$ to V.*
2. *There exists uniquely the invariant subspace U of $M(\Lambda)$ satisfying $M(\Lambda)/U \cong V$.*

This lemma is called the **universal property of the Verma module**. Let us compute the character of the Verma module $\mathrm{ch}M(\Lambda)$. For this purpose, we arrange a basis of $\mathfrak{g}_{-\alpha}$'s ($\alpha \in \Delta_+$) and make a basis of $\mathfrak{n}_-$. To see this in detail, putting serial numbers to all of the elements of Δ_+, set

$$\Delta_+ = \{\gamma_1, \gamma_2, \cdots\}$$

and $m_i := \mathrm{mult}(\gamma_i)$. For each i, denote a basis of $\mathfrak{g}_{-\gamma_i}$ by

$$f_{\gamma_i}^{(1)}, \cdots, f_{\gamma_i}^{(m_i)},$$

and by arranging them as $i = 1, 2, \cdots$, one can take a basis of $\mathfrak{n}_-$. Call them $x_1, x_, \cdots$.

Let us first consider the case when $\mathfrak{g}$ is a Lie algebra. Then

$$\{x_1^{i_1} x_2^{i_2} \cdots ; i_k = 0 \text{ for all but a finite number of } k\text{'s}\}$$

is a basis of $\mathfrak{U}(\mathfrak{n}_-)$, and applying these to v_Λ we can form a basis of $M(\Lambda)$. By counting the weights of these elements, the character of $M(\Lambda)$ can be computed as follows:

(2.89)

$$\begin{aligned}
&\mathrm{ch}M(\Lambda) \\
&= e^\Lambda (1 + e^{-\gamma_1} + e^{-2\gamma_1} + \cdots)^{m_1} (1 + e^{-\gamma_2} + e^{-2\gamma_2} + \cdots)^{m_2} \cdots \\
&= e^\Lambda \prod_{\alpha \in \Delta_+} \frac{1}{(1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}.
\end{aligned}$$

Using R defined in (2.59) in §2.2, one can rephrase this formula as follows:

$$\mathrm{ch}M(\Lambda) = \frac{e^\Lambda}{R}. \tag{2.90}$$

In the case of a Lie superalgebra, one has to pay attention to the power of odd elements when one considers a basis of $\mathfrak{U}(\mathfrak{n}_-)$, but still one obtains the same formula (2.90) as a result. In this sense, R is called the **denominator** of characters.

Condition (ii) in Definition 2.51 of the highest weight modules does not imply that its space is irreducible. In a representation space V of a highest weight module, any element can be expressed as a linear combination of those obtained by applying elements of $\mathfrak{g}$ repeatedly to a highest weight vector v_Λ. Since only one element v_Λ generates V, one might feel that this representation space is irreducible. Surely, if an invariant subspace U contains v_Λ, then U should coincide with V. But, if there is an invariant subspace U which does not contain v_Λ, then $\cdots$. And, in fact, such things can happen.

To study this aspect, we will look at 'maximal weights' of each invariant subspace. Namely, we will study the weight which plays the role of a boss in each invariant subspace. For this purpose, let us introduce the following notion:

DEFINITION 2.54. Let V be a highest weight $\mathfrak{g}$-module, and let λ a weight of V different from its highest weight Λ. Then:

1. If an element v ($\neq 0$) of V_λ satisfies $e_i v = 0$ for all $i \in I$, then v is called a **singular vector** of weight λ.
2. Assume that $0 \neq v \in V_\lambda$. If there exists an invariant subspace U of V satisfying
   (i) $v \notin U$,
   (ii) $e_i v \in U$ for all $i \in I$,
   v is called a **primitive vector** of weight λ.
3. If there exists a singular vector of weight λ, then λ is called a **singular weight**; and if there exists a primitive vector of weight λ, then λ is called a **primitive weight**.

By this definition, a singular vector is a primitive vector (by taking $U = \{0\}$ in (2) of the above definition). From a different point of view, a primitive vector is just a singular vector in the quotient module V/U.

Let V be a highest weight module with highest weight Λ, and let us study the action of the Casimir element Ω on V. First, Ωv_Λ can be immediately computed by the definition (2.23) of Ω and the

condition of a highest weight vector:

$$\Omega v_\Lambda = \left\{ 2(\Lambda|\rho) + \underbrace{\sum_{i=1}^{n} \Lambda(u^i)\Lambda(u_i)}_{(\Lambda|\Lambda)} \right\} v_\Lambda = (\Lambda|\Lambda + 2\rho)v_\Lambda. \tag{2.91}$$

An element of V is a linear combination of elements of the form $x_1 \cdots x_n v_\Lambda$ $(x_1, \cdots, x_n \in \mathfrak{g})$. If we take an element of such a form and set $v := x_1 \cdots x_n v_\Lambda$, then since Ω commutes with the action of $\mathfrak{g}$, we have

$$\Omega v = \Omega \cdot x_1 \cdots x_n v_\Lambda = x_1 \cdots x_n \cdot \underbrace{\Omega v_\Lambda}_{(\Lambda|\Lambda+2\rho)v_\Lambda} = (\Lambda|\Lambda + 2\rho)v,$$

and thus we obtain the following lemma:

LEMMA 2.55. *If V is a highest weight $\mathfrak{g}$-module with highest weight Λ, then Ω is a scalar operator with eigenvalue $(\Lambda|\Lambda + 2\rho)$.*

Let V be a highest weight $\mathfrak{g}$-module with highest weight Λ, and let λ be a singular weight of V. Also, let $0 \neq v \in V_\lambda$ be a singular vector. Then, since we have $\mathfrak{n}_+ v = \{0\}$, we obtain

$$\Omega v = (\lambda|\lambda + 2\rho)v$$

by the same computation as (2.91). On the other hand, since Ω acts on V as a scalar operator with eigenvalue $(\Lambda|\Lambda + 2\rho)$, we also have

$$\Omega v = (\Lambda|\Lambda + 2\rho)v.$$

So, from these two formulae, we obtain

$$\text{if } \lambda \text{ is a singular weight, then} \quad (\lambda|\lambda + 2\rho) = (\Lambda|\Lambda + 2\rho). \tag{2.92}$$

In the case where λ is a primitive weight, V/U is a highest weight module with highest weight Λ as stated above; and since λ is one of the singular weights of this, we have the same formula (2.92). Thus, we obtain the next proposition:

PROPOSITION 2.56. *If V is a highest weight $\mathfrak{g}$-module with highest weight Λ, and if λ is a primitive weight of V, then*

$$(\lambda|\lambda + 2\rho) = (\Lambda|\Lambda + 2\rho),$$

i.e.,

$$|\lambda + \rho|^2 = |\Lambda + \rho|^2.$$

Below, let us consider an integrable highest weight $\mathfrak{g}$-module, say V. First, the next lemma can be easily proved:

LEMMA 2.57. *Let V be a highest weight $\mathfrak{g}$-module with highest weight Λ. If V is integrable, then Λ enjoys the following properties:*

i. $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ *for all* $i \in I^{re}$.
ii. $i \in \Psi^{re} \implies \langle \Lambda, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0}$.

PROOF. If V is integrable, then by the definition, the actions of the f_i ($i \in I^{re}$) on V are locally nilpotent, and in particular, by looking at the action on a highest weight vector v_Λ, for each $i \in I^{re}$, there exists $m_i \in \mathbb{Z}_{\geq 0}$ satisfying $f_i^{m_i+1} v_\Lambda = 0$. Suppose that we take the minimal possible integers as m_i for $i \in I^{re}$.

First, take any $i \in I^{re}$, and let us consider such an i. Then, the vector subspace of V spanned over

$$\{f_i^j v_\Lambda ; 0 \leq j \leq m_i\}$$

is the (m_i+1)-dimensional representation space of the S-triple or the super S-triple $(\alpha_i^\vee, e_i, f_i)$. So, when α_i is an even root, we obtain

$$\langle \Lambda, \alpha_i^\vee \rangle = m_i$$

by Corollary 1.42 in §1.2, and when α_i is an odd root, we obtain

$$\langle \Lambda, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0}$$

by Lemma 1.94 in §1.5. □

Here, the converse of Lemma 2.57 does not necessarily hold. This is because the only condition on V is that it be a highest weight module. For example, if we consider the Verma module $M(\Lambda)$ for V, then f_i ($i \in I^{re}$) acts 'freely' on V, and hence it cannot act locally nilpotently. Therefore, $M(\Lambda)$ cannot be integrable for any Λ. Namely, for a highest weight module V to be integrable, we have to impose a sort of conditions on V itself, not just conditions on Λ.

Verma modules are often useful when one considers highest weight modules. By Lemma 2.53, any highest weight $\mathfrak{g}$-module V with highest weight Λ can be obtained as

$$V \cong M(\Lambda)/U \quad : \quad \text{as } \mathfrak{g}\text{-modules}, \tag{2.93}$$

by choosing an appropriate invariant subspace U of $M(\Lambda)$. To avoid unnecessary confusion, here in the explanation we denote a highest weight vector of V by v_Λ and a highest weight vector of $M(\Lambda)$ by $\tilde{v}_\Lambda$.

Let V be a highest weight $\mathfrak{g}$-module, and let us consider the case where the highest weight Λ of V satisfies conditions (i) and (ii) of

Lemma 2.57. For V to be integrable, there must exist $m_i \in \mathbb{Z}_{\geq 0}$ satisfying

$$f_i^{m_i+1} v_\Lambda = 0 \tag{2.94}$$

for each $i \in I^{re}$. Here, if we choose the minimal possible integers for m_i, we have

$$\langle \Lambda, \alpha_i^\vee \rangle = m_i, \tag{2.95}$$

as we saw above. Let us consider this in the Verma module $M(\Lambda)$ by using (2.93). In $M(\Lambda)$, (2.94) is translated to the following relations:

$$f_i^{m_i+1} \tilde{v}_\Lambda \in U. \tag{2.96}$$

For $k \in I$, let us compute $e_k f_i^{m_i+1} \tilde{v}_\Lambda$. Since e_k and f_i supercommute for $k \neq i$, this element is 0. So, we consider the case $k = i$. Namely, let us compute $e_i f_i^{m_i+1} \tilde{v}_\Lambda$. Since $i \in I^{re}$, $(\alpha_i^\vee, e_i, f_i)$ is an S-triple or a super S-triple, and by (2.95) we have

$$\alpha_i^\vee \tilde{v}_\Lambda = m_i \tilde{v}_\Lambda.$$

So, by applying Lemma 1.43 for $i \in I^{re} \setminus \Psi^{re}$, and by applying Lemma 1.93 for $i \in \Psi^{re}$ ($\lambda_0 = m_i$ in any case), we obtain

$$e_i f_i^{m_i+1} \tilde{v}_\Lambda = 0$$

in each case. Thus, we have proved that

$$e_k f_i^{m_i+1} \tilde{v}_\Lambda = 0$$

for all $k \in I$. Therefore, $f_i^{m_i+1} \tilde{v}_\Lambda$ is a singular vector, and

$$\mathfrak{U}(\mathfrak{g}) \cdot f_i^{m_i+1} \tilde{v}_\Lambda = \mathfrak{U}(\mathfrak{n}_-) \cdot f_i^{m_i+1} \tilde{v}_\Lambda$$

is an invariant subspace of $M(\Lambda)$. So, summing over $i \in I^{re}$,

$$U_0 := \sum_{i \in I^{re}} \mathfrak{U}(\mathfrak{n}_-) \cdot f_i^{m_i+1} \tilde{v}_\Lambda$$

is an invariant subspace of $M(\Lambda)$, and by (2.96) we have

$$U_0 \subset U.$$

Thus, the converse of Lemma 2.57 is shown to have the following form:

PROPOSITION 2.58. *If $\Lambda \in \mathfrak{h}^*$ satisfies*

i. $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ *for all* $i \in I^{re}$,
ii. $i \in \Psi^{re} \implies \langle \Lambda, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0}$,

then we define the vector subspace U_0 of $M(\Lambda)$ as follows:

$$U_0 := \sum_{i \in I^{re}} \mathfrak{U}(\mathfrak{n}_-) \cdot f_i^{m_i+1} \tilde{v}_\Lambda.$$

Then, the following hold:

1. *U_0 is a $\mathfrak{g}$-invariant subspace of $M(\Lambda)$.*
2. *Let U be one of the invariant subspaces of $M(\Lambda)$, and set $V := M(\Lambda)/U$. Then*

$$V \text{ is integrable} \iff U_0 \subset U.$$

When we checked the local nilpotency of the action of the f_i's in the above arguments, we only checked their actions on highest weight vectors v_Λ and $\tilde{v}_\Lambda$, but (to make sure) we remark that its sufficiency is guaranteed by Lemma 2.28.

Here, we notice that there is a conspicuous difference in the relation between the 'integrability' and the 'irreducibility' of a representation for a Kac-Moody Lie algebra and those for a BKM Lie algebra (which is not a Kac-Moody Lie algebra). As we declared at the beginning of this section, everything is discussed under the assumption of symmetrizability. By the definition of a GCM and a BKM matrix, one has

$$\mathfrak{g}(A) \text{ is a Kac-Moody Lie algebra} \iff I^{re} = I$$

for a BKM matrix A. For a Kac-Moody Lie algebra, i.e., for a case $I^{re} = I$, the elements $\{f_i; i \in I^{re}\}$ exhaust generators of $\mathfrak{n}_-$. From this, it follows that U_0 in Proposition 2.58 is a maximal proper invariant subspace of $M(\Lambda)$, and an invariant proper subspace containing U_0 is U_0 itself only. But this fact is not trivial, and we will derive it as a corollary of the character formula later in this book. Namely, as is stated in the next section, it follows from Theorem 2.73 that

$$\text{(2.97)} \quad \text{an integrable h.w. module} \implies \text{an irreducible h.w. module}$$

for a symmetrizable Kac-Moody Lie algebra. In other words, for a symmetrizable Kac-Moody Lie algebra, concerning a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$ satisfying

$$\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \quad (1 \leq i \leq n),$$

the 'integrability' and the 'irreducibility' are synonyms.

But, for a BKM Lie algebra with $I^{im} \neq \varnothing$, no property of the f_i $(i \in I^{im})$ is reflected in the definition of U_0. Therefore, if there is a singular vector obtained by applying f_i $(i \in I^{im})$ several times to $\tilde{v}_\Lambda$,

then U_0 cannot be a maximal proper invariant submodule. Therefore, if $I^{im} \neq \varnothing$, then an invariant submodule U satisfying

$$U_0 \subset U \subsetneqq M(\Lambda) \tag{2.98}$$

is not necessarily unique. In this case, for any invariant subspace U satisfying (2.98), the quotient module $M(\Lambda)/U$ is integrable, and it is integrable irrespective of the irreducibility of $M(\Lambda)/U$. Namely, in a case where $I^{im} \neq \varnothing$, there might exist integrable highest weight modules which are not irreducible, and in fact this can happen. In an extreme case, for $I^{re} = \varnothing$, all highest weight modules are integrable. Namely, in general, for a BKM Lie algebra which is not a Kac-Moody Lie algebra,

$$\text{integrability} \not\Longrightarrow \text{irreducibility}.$$

Here and after, we mainly consider irreducible highest weight $\mathfrak{g}$-modules.

LEMMA 2.59. *If V is a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$, then there exists a maximal proper invariant submodule of V.*

PROOF. Set the weight space decomposition of V as

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda.$$

If we take any proper invariant subspace of V, and denote it by U_1, then since U_1 is invariant with respect to the action of $\mathfrak{h}$, it decomposes as follows:

$$U_1 = \bigoplus_{\lambda \in P(V)} (U_1 \cap V_\lambda).$$

Then, since U_1 is a proper submodule, i.e., $U_1 \subsetneqq V$, and a one-dimensional vector subspace $V_\Lambda = \mathbb{C}v_\Lambda$ generates V as a $\mathfrak{g}$-module, we should have $U_1 \cap V_\Lambda = \{0\}$. Thus, we have proved the following:

(2.99)

$$U_1 \text{ is a proper invariant subspace of } V \implies U_1 \subset \bigoplus_{\substack{\lambda \in P(V) \\ \lambda \neq \Lambda}} V_\lambda.$$

So, if we set

$$U := \sum_{U_1 : \text{a proper invariant submodule of } V} U_1,$$

then by (2.99), we have

$$U \subset \bigoplus_{\substack{\lambda \in P(V) \\ \lambda \neq \Lambda}} V_\lambda,$$

and by its construction, U is the maximal proper invariant subspace. □

NOTATION 2.60. For $\Lambda \in \mathfrak{h}^*$, we denote the maximal proper invariant subspace of the Verma module $M(\Lambda)$ by J_Λ.

By Lemma 2.53 and Lemma 2.59, we obtain the following proposition:

PROPOSITION 2.61. *For $\Lambda \in \mathfrak{h}^*$, there uniquely exists the irreducible highest weight module with highest weight Λ up to isomorphisms. If we denote it by $L(\Lambda)$, then we have $L(\Lambda) \cong M(\Lambda)/J_\Lambda$.*

NOTATION 2.62. For $\Lambda \in \mathfrak{h}^*$, we denote the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ by $L(\Lambda)$, and we also denote the character $\mathrm{ch}L(\Lambda)$ by ch_Λ.

A proper invariant subspace of a highest weight $\mathfrak{g}$-module is, in general, the sum (not necessarily the direct sum) of highest weight modules, and the space itself is not necessarily a highest weight module. Therefore, to study the structure of reducible highest weight modules, we need a systematic treatment including them. So, let us introduce the category usually denoted by $\mathcal{O}$. For detailed arguments on the category $\mathcal{O}$, we refer the reader to Chapter 9 of [**K4**]; here, we will explain only the necessary properties needed to derive the character formulae. As a preparation, for $\lambda \in \mathfrak{h}^*$, set

$$D(\lambda) := \{\mu \in \mathfrak{h}^*; \mu \leq \lambda\} = \{\lambda - \alpha; \alpha \in Q_+\}.$$

We define the objects and the morphisms of $\mathcal{O}$ as follows:

> The objects of $\mathcal{O}$ consist of $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V such that there exist $\lambda_1, \cdots, \lambda_k \in \mathfrak{h}^*$ satisfying
>
> $$P(V) \subset \bigcup_{i=1}^{k} D(\lambda_i).$$
>
> The morphisms of $\mathcal{O}$ consist of all homomorphisms of $\mathfrak{g}$-modules.

We denote the set of all objects of $\mathcal{O}$ by $\mathrm{Ob}(\mathcal{O})$. When V is an object of $\mathcal{O}$, an invariant subspace U of V and the quotient module V/U of V by U are both objects of $\mathcal{O}$. If V is an object of V, there are only finite maximal weights among the weights of V with respect to the order $\leq$. For $\lambda \in P(V)$ and $0 \neq v \in V_\lambda$, if there exists an invariant subspace U of V satisfying

$$v \notin U \quad \text{and} \quad e_i v \in U \quad (1 \leq i \leq n),$$

then λ is called a primitive weight, and v is called a primitive vector of weight λ. In particular, if one has $e_i v = 0$ for all i, then v is called a singular vector and λ is called a singular weight. The set of all primitive weights of V is denoted by $\mathrm{Prim}(V)$.

REMARK 2.63. On the definitions of primitive weights of a highest weight module, we remark that there is a gap between the definition stated here and Definition 2.54. When V is a highest weight module, its highest weight Λ is not regarded as a primitive weight in Definition 2.54, but regarding V as an object of the category $\mathcal{O}$, we may also look at its highest weight as a primitive weight. Since the highest weight weights of an object of $\mathcal{O}$, in general, are not unique, we do not look at them as special weights, but intend to treat them equally with all primitive weights and maximal weights.

To prove Proposition 2.67 and Corollary 2.68 on $V \in \mathrm{Ob}(\mathcal{O})$, let us set forth some fundamental properties of objects of $\mathcal{O}$.

LEMMA 2.64. *Let V be an object of $\mathcal{O}$, and let U be a proper invariant subspace of V.*

1. *Let v be an element of V, and denote the element of V/U defined by v by $\overline{v}$. If $\overline{v}$ is a primitive vector of V/U, then v is also a primitive vector of V.*
2. *If u is a primitive vector of U, then u is a primitive vector of V.*

PROOF. (1) Let p be the canonical projection from V to V/U. Since $\overline{v}$ is a primitive vector of V/U, one can choose a proper invariant subspace W of V/U satisfying

$$\overline{v} \notin W \quad \text{and} \quad e_i \overline{v} \in W \quad (1 \leq i \leq n).$$

Then, $p^{-1}(W)$ is a proper invariant subspace of V, and one has

$$v \notin p^{-1}(W) \quad \text{and} \quad e_i v \in p^{-1}(W) \quad (1 \leq i \leq n).$$

Thus, v is a primitive vector.

(2) follows immediately from the definition. □

LEMMA 2.65. *For $V \in \mathrm{Ob}(\mathcal{O})$, there exists a proper invariant subspace U of V such that V/U is an irreducible $\mathfrak{g}$-module.*

PROOF. Take any maximal weight λ of V, and fix it. Let r be the dimension of the vector subspace of V_λ spanned by singular vectors, and denote their basis by $v_1, \cdots, v_r$. Then, if we set

$$U' := \sum_{\substack{W:\ \text{an invariant subspace of } V \\ \text{s.t. } P(W) \subset P(V) \setminus \{\lambda\}}} W,$$

where 's.t.' stands for 'such that', and

$$U := \begin{cases} U' & \text{for } r = 1, \\ U' + \sum_{i=2}^{r} \mathfrak{U}(\mathfrak{g}) \cdot v_i & \text{for } r \geq 2, \end{cases}$$

then U is a maximal proper invariant subspace of V. □

From the proof of Lemma 2.65, we obtain the following corollary:

COROLLARY 2.66. *For $V \in \mathrm{Ob}(\mathcal{O})$, let Λ be one of its maximal weights. Then, there exists a proper invariant subspace U of V satisfying $V/U \cong L(\Lambda)$.*

PROPOSITION 2.67. *For $V \in \mathrm{Ob}(\mathcal{O})$:*

1. *There exists a series of invariant subspaces*

$$\cdots \subset W_{k+1} \subset W_k \subset \cdots \subset W_1 \subset W_0 = V$$

satisfying the following two conditions:
   (i) *W_k/W_{k+1} $(k = 0, 1, 2, \cdots)$ are irreducible $\mathfrak{g}$-modules,*
   (ii) *$\bigcap_{k=0}^{\infty} W_k = \{0\}$.*
2. *Take a series of invariant subspaces of V as in (1), and denote the highest weights of W_k/W_{k+1} $(k = 0, 1, 2, \cdots)$ which are not $\{0\}$ by μ_i $(i = 0, 1, 2, \cdots)$. Then*

$$\mathrm{ch}V = \sum_i \mathrm{ch}L(\mu_i).$$

PROOF. Since V is a vector space of at most countable dimension, the cardinality of the set of primitive weights of V with multiplicities is at most countable, and hence we denote all elements of $\mathrm{Prim}(V)$ with multiplicities by

$$\lambda_0, \lambda_1, \lambda_2, \cdots.$$

Here we arrange these numbers so that if λ_i and λ_j differ, then

$$\lambda_i \geq \lambda_j \implies i \leq j.$$

By this way of arranging, λ_0 is a maximal weight, and by Corollary 2.66, we can take an invariant subspace W_1 of V satisfying $V/W_1 \cong L(\lambda_0)$. Next, looking at W_1, since $\mathrm{Prim}(W_1)$ is a subset of $\mathrm{Prim}(V)$ by Lemma 2.64, each maximal weight of W_1 coincides with one of the λ_i $(i \leq 1)$.

Now, among the maximal weights of W_1, if we set by λ_{i_1} corresponding to the weight λ_i whose index is the minimum, then by Corollary 2.66, there exists an invariant subspace W_2 of W_1 satisfying $W_1/W_2 \cong L(\lambda_{i_1})$.

By repeating this operation, one obtains a series of invariant subspaces $\{W_k\}$, and this series satisfies (i) in (1) of this proposition.

One can show that this series satisfies (ii) of (1) in the following manner. First, if $\mathrm{Prim}(V)$ is a finite set, then one may take $\{0\}$ as W_k. Next, let us consider the case when there are infinite primitive weights of V.

If we take any $\mu \in P(V)$, then there are only finite numbers of λ_i's satisfying $\lambda \geq \mu$. There exists a natural number N satisfying

$$\lambda_k \leq \mu \implies k < N.$$

If we take N in this way, then by the contraposition of this relation, and by the choice of a sequence

$$i_1 < i_2 < i_3 < \cdots$$

appearing in the construction of W_k, we see that

$$k \geq N \implies i_k \geq N \implies W_k \cap V_\mu = \{0\},$$

and hence we have proved (ii).

(2) follows from (1) and the fact that for an exact sequence of $\mathfrak{g}$-modules

$$\{0\} \longrightarrow W_{k+1} \longrightarrow W_k \longrightarrow W_k/W_{k+1} \longrightarrow \{0\},$$

one has

$$\mathrm{ch}W_k - \mathrm{ch}W_{k+1} = \mathrm{ch}(W_k/W_{k+1}).$$

□

In the right hand side of (2) of Proposition 2.67, it may happen that $\mathrm{ch}L(\mu_i)$ appears several times. So, collecting the same one together, one can rewrite it in the following form:

COROLLARY 2.68. *For $V \in \mathrm{Ob}(\mathcal{O})$, its character has the following expression:*

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathrm{Prim}(V)} b(\lambda)\mathrm{ch}L(\lambda) \quad (b(\lambda) \in \mathbb{Z}_{\geq 0}).$$

Here, each $b(\lambda)$ is uniquely determined from V.

The latter half of this corollary says that $\mathrm{ch}L(\lambda)$ are linearly independent (since if λ differs, then the first term e^λ should differ).

In particular, for Verma modules, one obtains the following corollary from Corollary 2.68 and Proposition 2.56:

COROLLARY 2.69. *For $\Lambda \in \mathfrak{h}^*$, $\mathrm{ch}M(\Lambda)$ has the following form:*

$$\mathrm{ch}M(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ s.t.\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} b(\Lambda, \lambda)\mathrm{ch}L(\lambda) \quad (b(\Lambda, \lambda) \in \mathbb{Z}_{\geq 0}).$$

Here, in particular, $b(\Lambda, \Lambda) = 1$.

Take any $\Lambda \in \mathfrak{h}^*$, and fix it, and set

$$\begin{aligned} P'_{nice}(\Lambda) &:= \{\lambda \in \mathfrak{h}^*; \lambda \leq \Lambda; |\lambda + \rho|^2 = |\Lambda + \rho|^2\} \\ &= \{\lambda_0, \lambda_1, \lambda_2, \cdots\}. \end{aligned} \tag{2.100}$$

Here, the indices of the λ_i's are given so that they satisfy

$$\lambda_i \geq \lambda_j \implies i \leq j,$$

and we fix such an enumeration. In particular, we remark that we have $\lambda_0 = \Lambda$.

By applying Corollary 2.69 to each $\lambda_i \in P'_{nice}(\Lambda)$, we have

$$\mathrm{ch}M(\lambda_i) = \sum_{j \geq i} b(\lambda_i, \lambda_j)\mathrm{ch}L(\lambda_j). \tag{2.101}$$

Here, since $b(\lambda_i, \lambda_i) = 1$, a matrix expression of these formulae has the following form:

$$\begin{pmatrix} \mathrm{ch}M(\lambda_0) \\ \mathrm{ch}M(\lambda_1) \\ \mathrm{ch}M(\lambda_2) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ \vdots & \vdots & \cdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} \mathrm{ch}L(\lambda_0) \\ \mathrm{ch}L(\lambda_1) \\ \mathrm{ch}L(\lambda_2) \\ \vdots \end{pmatrix}.$$

Multiplying the inverse of this matrix, we obtain

$$\begin{pmatrix} \mathrm{ch}L(\lambda_0) \\ \mathrm{ch}L(\lambda_1) \\ \mathrm{ch}L(\lambda_2) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ \vdots & \vdots & \cdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} \mathrm{ch}M(\lambda_0) \\ \mathrm{ch}M(\lambda_1) \\ \mathrm{ch}M(\lambda_2) \\ \vdots \end{pmatrix}.$$

Moreover, by using Corollary 2.68 and Proposition 2.56, we obtain the next proposition:

PROPOSITION 2.70. *For $\Lambda \in \mathfrak{h}^*$, the character* $\mathrm{ch}V$ *of a highest weight module V with highest weight Λ has the following form:*

$$\mathrm{ch}V = \sum_{\substack{\lambda \leq \Lambda \\ s.t.\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c(\Lambda, \lambda)\mathrm{ch}M(\lambda) \quad (c(\Lambda, \lambda) \in \mathbb{Z}).$$

In particular, $c(\Lambda, \Lambda) = 1$.

Note. When $\mathfrak{g}$ is a BKM Lie algebra, one has $P'_{nice}(\Lambda) \subset P(M(\Lambda))$, but when $\mathfrak{g}$ is a BKM Lie superalgebra, an element of $P'_{nice}(\Lambda)$ is not necessarily a weight of the Verma module $M(\Lambda)$. The prime in $P'_{nice}(\Lambda)$ indicates this fact. Let us remark on this.

The right hand side of the formula in Corollary 2.69 contains no weight which is not a weight of the Verma module $M(\Lambda)$. Then, one may wonder why we do not define $P'_{nice}(\Lambda)$ not only by (2.100) but also with the restriction that it be a weight of $M(\Lambda)$. Of course, looking only at Corollary 2.69, this is a legitimate question. But here, we apply Corollary 2.69 to a sequence of Verma modules.

If $\mathfrak{g}$ is a BKM Lie superalgebra which has an odd root with length zero, then a map between Verma modules is not necessarily injective. Therefore, the transitivity

$$\lambda \in P(M(\Lambda)),\ \mu \in P(M(\lambda)) \implies \mu \in P(M(\Lambda))$$

does not always hold. So the weights appearing in the decomposition of $M(\lambda_k)$'s are not necessarily weights of $M(\Lambda)$. Since we also have to treat such weights, we defined $P'_{nice}(\Lambda)$ by (2.100). □

2.5. Character Formulae and Denominator Identity

Let A be a BKM matrix or a BKM supermatrix, and set

$$P_+ := \left\{ \lambda \in \mathfrak{h}^*; \begin{array}{l} \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ (i \in I^{re}) \\ \langle \lambda, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0} \ (i \in \Psi^{re}) \end{array} \right\},$$
$$P_{++} := \{\lambda \in P_+; \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0} \ (i \in I^{re})\},$$
$$P_{(+)} := \{\lambda \in P_+; \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i \in I)\}.$$

DEFINITION 2.71. An element of $P_{(+)}$ is called a **dominant integral form**.

In this section, we compute the character ch_Λ of the irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ with highest weight a dominant integral weight Λ. The starting point of this computation is the formula in Proposition 2.70 in the previous section, and since we fix Λ, we will simply denote $c(\Lambda, \lambda)$ for $V = L(\Lambda)$ by $c(\lambda)$:

$$\begin{aligned} \mathrm{ch}_\Lambda &= \sum_{\substack{\mu \leq \Lambda \\ |\mu+\rho|^2 = |\Lambda+\rho|^2}} c(\mu) \mathrm{ch} M(\mu) \\ &= \frac{1}{R} \sum_{\substack{\mu \leq \Lambda \\ |\mu+\rho|^2 = |\Lambda+\rho|^2}} c(\mu) e^\mu \\ &= \frac{1}{e^\rho R} \sum_{\substack{\mu \leq \Lambda \\ |\mu+\rho|^2 = |\Lambda+\rho|^2}} c(\mu) e^{\mu+\rho}. \end{aligned} \tag{2.102}$$

Since ch_Λ is W-invariant, and $e^\rho R$ is W-skew-invariant by Proposition 2.34, we obtain

$$\sum_{\substack{\mu \leq \Lambda \\ |\mu+\rho|^2 = |\Lambda+\rho|^2}} c(\mu) e^{\mu+\rho} \text{ is } W\text{-skew-invariant} \tag{2.103}$$

by (2.102). From this, for $\lambda, \mu \in \mathfrak{h}^*$ satisfying $\lambda, \mu \in \Lambda$, the next two facts follow:

FACT 1. *If $w(\mu+\rho) = \lambda + \rho$ and $|\mu+\rho|^2 = |\Lambda+\rho|^2$, then*

$$c(\lambda) = \varepsilon(w) c(\mu).$$

FACT 2. $c(\mu) \neq 0 \implies w(\mu+\rho) \leq \Lambda + \rho \ (\forall w \in W)$.

Here, we state the following:

LEMMA 2.72. *Suppose that* $\Lambda \in P_{(+)}$ *and* $\mu \in \mathfrak{h}^*$ *satisfy* $w(\mu + \rho) \leq \Lambda + \rho$ *for all* $w \in W$. *Then, there exists* $w_0 \in W$ *satisfying*

$$w_0(\mu + \rho) \in C^\vee,$$

i.e.,

$$\langle w_0(\mu + \rho), \alpha_i^\vee \rangle \geq 0 \quad (\forall i \in I^{re}).$$

PROOF. Writing the condition of this lemma explicitly, we have

$$\Lambda + \rho - w(\mu + \rho) \in Q_+ = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i,$$

and hence we can take $w_0 \in W$ satisfying

$$\mathrm{ht}(\Lambda + \rho - w_0(\mu + \rho)) = \min\{\mathrm{ht}(\Lambda + \rho - w(\mu + \rho)); w \in W\}.$$

Then, we have

$$w_0(\mu + \rho) \in C^\vee.$$

This is because, by the choice of w_0, for $i \in I^{re}$, we have

$$\mathrm{ht}(\Lambda + \rho - r_i w_0(\mu + \rho)) \geq \mathrm{ht}(\Lambda + \rho - w_0(\mu + \rho)),$$

i.e.,

$$\mathrm{ht}(r_i w_0(\mu + \rho) - w_0(\mu + \rho)) \geq 0.$$

From this and the equality

$$r_i w_0(\mu + \rho) = w_0(\mu + \rho) - \langle w_0(\mu + \rho), \alpha_i^\vee \rangle \alpha_i,$$

we complete the proof. □

In Lemma 2.72, for μ and $i \in I^{re}$ satisfying $\langle w_0(\mu + \rho), \alpha_i^\vee \rangle = 0$, one has

$$r_i(\mu + \rho) = \mu + \rho,$$

and hence by Fact 1, we have $c(\mu) = 0$. So, for the summation in (2.103), we only have to consider the case where $w_0(\mu + \rho)$ belongs to P_{++}. Namely, for the summation in (2.103), we only have to take the summation over μ satisfying

$$\langle w_0(\mu + \rho), \alpha_i^\vee \rangle \geq 1 \quad (\text{for all } i \in I^{re}). \tag{2.104}$$

To parametrize such μ, we set

$$\lambda + \rho := w_0(\mu + \rho). \tag{2.105}$$

Then, λ satisfies

$$\langle \lambda + \rho, \alpha_i^\vee \rangle \geq 1 \quad (i \in I^{re}),$$

i.e.,

$$\langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad (i \in I^{re}) \tag{2.106}$$

by (2.104). Moreover, since one has $\lambda + \rho \leq \Lambda + \rho$, it follows that $\lambda \leq \Lambda$, i.e., $\Lambda - \lambda \in Q_+$; and by the fact that Λ is an element of P_+ and condition ($\widetilde{\mathrm{C}}5$), λ also satisfies

$$\langle \lambda, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0} \qquad (i \in \Psi^{re}).$$

This simply means $\lambda \in P_+$. So, it follows that those μ contributing to the summation in (2.102) can be parametrized as

$$\mu + \rho = w(\lambda + \rho)$$

with $\lambda \in P_+$ and $w \in W$. Describing the summation over μ in (2.102) by using λ and w in this way, (2.102) can be rewritten as follows:

$$e^\rho R \cdot \mathrm{ch}_\Lambda = \sum_{w \in W} \varepsilon(w) \sum_{\substack{\lambda \in P_+ \\ \text{s.t. } \lambda \leq \Lambda, \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c(\lambda) e^{w(\lambda+\rho)}. \tag{2.107}$$

Namely, if we set

$$S_\Lambda := \sum_{\substack{\lambda \in P_+ \\ \text{s.t. } \lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c(\lambda) e^{w(\lambda+\rho)}, \tag{2.108}$$

then we have

$$e^\rho R \cdot \mathrm{ch}_\Lambda = \sum_{w \in W} \varepsilon(w) w(S_\Lambda). \tag{2.109}$$

So, let us study $\lambda \in P_+$ satisfying the following two conditions:

$$\lambda \leq \Lambda, \tag{2.110}$$

$$|\lambda + \rho|^2 = |\Lambda + \rho|^2, \tag{2.111}$$

and compute $c(\lambda)$ for such λ.

Up to this place, all of the arguments hold for a general BKM supermatrix A. Here and after, we will derive the character formula for BKM Lie superalgebras by studying λ satisfying (2.110) and (2.111), but since the arguments become very simple for Kac-Moody Lie algebras, let us first look that case.

[I] The case of Kac-Moody Lie algebras. When λ satisfies (2.110) and (2.111) the following formula holds:

$$
\begin{aligned}
(2.112) \qquad 0 &= |\Lambda+\rho|^2 - |\lambda+\rho|^2 \\
&= (\Lambda-\lambda|\Lambda+\lambda+2\rho) \\
&= \underbrace{(\Lambda-\lambda|\Lambda)}_{\substack{\geq 0 \\ \left(\because \ \Lambda-\lambda\in Q_+,\ \Lambda\in P_{(+)}\right)}} + \underbrace{(\Lambda-\lambda|\lambda)}_{\substack{\geq 0 \\ \left(\because \ \Lambda-\lambda\in Q_+,\ \lambda\in P_+\right)}} + 2\underbrace{(\Lambda-\lambda|\rho)}_{\substack{\geq 0 \\ (\because \ \Lambda-\lambda\in Q_+)}} .
\end{aligned}
$$

So, these three terms should all be zero, and in particular, from the third term, we have

$$(\Lambda-\lambda|\rho)=0.$$

By the way, since one has

$$\Lambda-\lambda\in Q_+=\sum_{i=1}^{n}\mathbb{Z}_{\geq 0}\alpha_i,$$

and

$$(\alpha_i|\rho)>0 \quad (\text{for all } i\in I),$$

it follows that $\Lambda-\lambda=0$, i.e.,

$$\lambda=\Lambda.$$

Thus, (2.107) looks as follows:

$$
\begin{aligned}
e^{\rho}R\cdot \mathrm{ch}_{\Lambda} &= \sum_{w\in W}\varepsilon(w)\cdot c(\Lambda)e^{w(\Lambda+\rho)} \\
&= c(\Lambda)\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}.
\end{aligned}
$$

Comparing the coefficients of $e^{\Lambda+\rho}$ in both sides, we get $c(\Lambda)=1$, and we obtain the next theorem:

THEOREM 2.73. *If $\mathfrak{g}(A)$ is a symmetrizable Kac-Moody Lie algebra, then for $\Lambda\in P_+$ we have*

$$\mathrm{ch}_{\Lambda}=\frac{1}{e^{\rho}R}\cdot\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}.$$

The formula in Theorem 2.73 is called the **Weyl-Kac character formula.**

Note. What is used in the proof of the above theorem is the Weyl group invariance of the character only, and we did not use the irreducibility of the representation. By Proposition 2.50, the Weyl group invariance of the character follows from the integrability of the representation. Therefore, the above theorem is the character formula of the integrable representations, and the fact that the character of the integrable representations are 'determined' in this way implies the irreducibility of the representations. Namely, what we stated in (2.97) in the previous section follows as a corollary of this theorem. □

[II] The case of BKM Lie algebras. In this case, we cannot say that in the right hand side of (2.112) the second and the third terms are ≥ 0, and hence some finer arguments are required. We prepare the following lemma:

LEMMA 2.74. *Let A be a symmetrizable BKM matrix or BKM supermatrix, and suppose $\Lambda \in P_{(+)}$ and $\lambda \in \mathfrak{h}^*$ satisfy*

i. $\lambda \leq \Lambda$,
ii. $(\lambda + \rho|\alpha_i) \geq 0$ *(for all $i \in I^{re}$).*

Then, the following hold:

1. *For $i \in \operatorname{supp}(\Lambda - \lambda)$,*
   (i) $i \in I^{re} \implies (\Lambda + \lambda + 2\rho|\alpha_i) > 0$,
   (ii) $i \in I^{im} \implies (\Lambda + \lambda + 2\rho|\alpha_i) \geq 0$.
2. $|\Lambda + \rho|^2 \geq |\lambda + \rho|^2$.
3. *Since $\lambda \leq \Lambda$, set*

$$\lambda = \Lambda - \sum_{i \in J} m_i \alpha_i \qquad (J \subset I, m_i \in \mathbb{Z}_{>0}). \tag{2.113}$$

Then the equality

$$|\Lambda + \rho|^2 = |\lambda + \rho|^2$$

is equivalent to J and m_i $(i \in J)$ satsifying the following conditions:

$$J \subset I^{im}, \tag{2.114}$$

$$i \in J \implies (\Lambda|\alpha_i) = 0, \tag{2.115}$$

$$i, j \in J, i \neq j \implies (\alpha_i|\alpha_j) = 0, \tag{2.116}$$

$$i \in J, a_{ii} < 0 \implies m_i = 1. \tag{2.117}$$

PROOF. (1) (i) For $i \in I^{re}$, we have

$$(\alpha_i|\Lambda+\lambda+2\rho) = \underbrace{(\alpha_i|\Lambda)}_{\geq 0} + \underbrace{(\alpha_i|\lambda+\rho)}_{\geq 0} + \underbrace{(\alpha_i|\rho)}_{>0} > 0. \tag{2.118}$$

(ii) For $i \in I^{im}$, we have

$$\begin{aligned}(\alpha_i|\Lambda+\lambda+2\rho) &= (\alpha_i|\Lambda) + (\alpha_i|\lambda) + \underbrace{2(\alpha_i|\rho)}_{(\alpha_i|\alpha_i)} \\ &= (\alpha_i|\Lambda) + (\alpha_i| \underbrace{\lambda+\alpha_i}_{\substack{=\Lambda-\beta\ (\exists\beta\in Q_+)\\ (\because i\in \mathrm{supp}(\Lambda-\lambda))}}) \\ &= 2\underbrace{(\alpha_i|\Lambda)}_{\geq 0} - \underbrace{(\alpha_i|\beta)}_{\substack{\leq 0\\ (\because (2.69))}} \geq 0.\end{aligned} \tag{2.119}$$

(2) follows immediately from (1). Namely, we have

$$|\Lambda+\rho|^2 - |\lambda+\rho|^2 = (\underbrace{\Lambda-\lambda}_{\in Q_+}|\Lambda+\lambda+2\rho) \underset{(\because (1))}{\geq} 0. \tag{2.120}$$

(3) Setting $J := \mathrm{supp}(\Lambda-\lambda)$ and

$$\Lambda - \lambda = \sum_{i\in J} m_i\alpha_i \quad (m_i > 0),$$

one has

$$\begin{aligned}|\Lambda+\rho|^2 - |\lambda+\rho|^2 &= (\Lambda-\lambda|\Lambda+\lambda+2\rho) \\ &= \sum_{i\in J} m_i(\alpha_i|\Lambda+\lambda+2\rho).\end{aligned} \tag{2.121}$$

First, by assuming

$$|\Lambda+\rho|^2 - |\lambda+\rho|^2 = 0,$$

let us show (2.114) – (2.117). Now, if $J \cap I^{re} \varnothing$, then by (i) of (1), the right hand side of this formula is > 0. So, for the right hand side of this formula being 0, we first should have $J \subset I^{im}$, and at the same time, we should also have

$$i \in J \implies (\alpha_i|\Lambda+\lambda+2\rho) = 0. \tag{2.122}$$

Now, taking any $i \in J$, fix it, and set

$$\beta := \Lambda - \lambda - \alpha_i = \sum_{\substack{j \in J \\ j \neq i}} m_j \alpha_j + (m_i - 1)\alpha_i \in Q_+. \tag{2.123}$$

Since $\lambda = \Lambda - \alpha_i - \beta$ by definition, we have

$$\Lambda + \lambda + 2\rho = 2\Lambda + 2\rho - \alpha_i - \beta. \tag{2.124}$$

For the fixed i, let us compute $(\alpha_i|\Lambda + \lambda + 2\rho)$. This is 0 by (2.122), and the computation using (2.124) looks as follows:

$$\begin{aligned} 0 &= (\alpha_i|\Lambda + \lambda + 2\rho) \\ &= \underbrace{2(\alpha_i|\Lambda)}_{\geq 0} + \underbrace{2(\alpha_i|\rho) - (\alpha_i|\alpha_i)}_{\substack{=0 \\ (\because \text{ by the definition of } \rho)}} - \underbrace{(\alpha_i|\beta)}_{\substack{\leq 0 \\ (\because (2.69))}} \\ &\geq 0. \end{aligned}$$

So, two terms in the right hand side of this formula should both be zero. Thus, we have shown that

$$i \in J \implies (\alpha_i|\Lambda) = (\alpha_i|\beta) = 0. \tag{2.125}$$

Rewriting the formula $(\alpha_i|\beta) = 0$ by using (2.123), we have

$$\begin{aligned} 0 &= (\alpha_i|\beta) \\ &= \sum_{\substack{j \in J \\ j \neq i}} \underbrace{m_j}_{>0} \underbrace{(\alpha_i|\alpha_j)}_{\substack{\leq 0 \\ (\because (2.9))}} + \underbrace{(m_i - 1)}_{\geq 0} \underbrace{(\alpha_i|\alpha_i)}_{\leq 0} \\ &\leq 0, \end{aligned}$$

and hence each term in the right hand side of this formula should be zero. Thus we have proved (2.114) – (2.117). One can prove the converse by a similar manner. □

A $\lambda \in P_+$ satisfying conditions (2.110) and (2.111) is characterized as one satisfying conditions (2.114) – (2.117) by Lemma 2.74. Next, let us study $c(\lambda)$ for such λ.

Namely, when J and m_i $(i \in J)$ satisfy (2.114) – (2.117), let us see when λ defined by (2.113) indeed appears as a weight in the left hand side of the character formula (2.108). To give the answer in advance, replacing (2.117) by the stronger condition

$$i \in J \implies m_i = 1, \tag{2.126}$$

(2.114) – (2.116) together with (2.126) are necessary and sufficient conditions for $c(\lambda) \neq 0$, and for such λ we have $c(\lambda) = (-1)^{|J|}$. This can be instantly seen by the following consideration.

Since we have $(\Lambda|\alpha_i) = 0$ for $i \in J$, it follows that

$$f_i v_\Lambda \in M(\Lambda)$$

is a singular vector, and hence this vector never appears in the irreducible representation $L(\Lambda)$. Thus, $\Lambda - \alpha_i$ is not a weight of $L(\Lambda)$, and it cannot appear in the expansion of ch_Λ.

Next, let us take some distinct elements of J, and consider a similar problem. Since the roots belonging to J are pairwise orthogonal, taking $i_1, \cdots, i_k \in J$, the quantity

$$\Lambda - \alpha_{i_1} - \cdots - \alpha_{i_k}$$

is not a weight of $L(\Lambda)$. But the term

$$e^{-\alpha_{i_1} - \cdots - \alpha_{i_k}}$$

appears in the expansion of

$$R = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}.$$

Since a sum of two such roots cannot be a root by condition (2.116), the factors of R that make such a term are only those obtained by expanding

$$(1 - e^{-\alpha_{i_1}}) \cdots (1 - e^{-\alpha_{i_k}}).$$

Thus, its coefficient is $(-1)^k$, and in the left hand side of (2.109), the multiplication of ch_Λ to R has no effect on such term, as we stated above.

Therefore, conditions (2.114) – (2.116) and (2.126) are necessary and sufficient for e^λ to appear in S_Λ, and its coefficient is then $(-1)^{|J|}$. Thus the following theorem is proved:

THEOREM 2.75. *If $\mathfrak{g}(A)$ is a symmetrizable BKM Lie superalgebra, then ch_Λ $(\Lambda \in P_{(+)})$ has the following expression:*

$$\mathrm{ch}_\Lambda = \frac{1}{e^\rho R} \sum_{w \in W} \varepsilon(w) w(S_\Lambda),$$

where we set

$$S_\Lambda := \sum_{\substack{k \in \mathbb{Z}_{\geq 0} \\ i_1, \cdots, i_k \in I^{im} \\ s.t.\ \text{(i),(ii)}}} (-1)^k e^{\Lambda + \rho - (\alpha_{i_1} + \cdots + \alpha_{i_k})},$$

and in the above formula, conditions (i) *and* (ii) *are given as follows:*

i. $(\Lambda|\alpha_{i_j}) = 0$ $(j = 1, \cdots, k)$,

ii. $i_1, \cdots, i_k$ *are all distinct, and* $\alpha_{i_1}, \cdots, \alpha_{i_k}$ *are pairwise orthogonal.*

Note. As we stated in Proposition 2.50, for a Weyl group invariant character, the representation is not necessarily irreducible but only has to be integrable. But in the (final) process of the proof of Theorem 2.75, we used more conditions than the Weyl group invariance. Namely, to compute $c(\lambda)$, we used the fact that there is no singular vector in its representation space. Thus, this character formula is the character formula describing an irreducible highest weight module. □

[III] The case of BKM Lie superalgebras. For BKM Lie superalgebras, the arguments to derive (2.114) – (2.117) are the same as in the case of BKM Lie algebras. For Lie algebras, we studied the relevant terms in the expansion of $e^{\rho}R \cdot \mathrm{ch}_{\Lambda}$ to see what $c(\lambda)$ looks like for such λ. Since, in the case of BKM Lie superalgebras, the form of R differs from those for ordinary Lie algebras, its aspect changes. Namely, for a BKM Lie algebra, we have

$$R = \frac{\prod_{\alpha \in \Delta_{even,+}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}},$$

as is given in (2.59), and for this R, we have to pick up the relevant terms in $e^{\rho}R \cdot \mathrm{ch}_{\Lambda}$. First, in the expansion of R, let us see, in particular, the terms of the form

$$e^{-(\alpha_{i_1} + \cdots + \alpha_{i_k})}. \tag{2.127}$$

Here, we assume that $\alpha_{i_1}, \cdots, \alpha_{i_k}$ satisfy (2.114) – (2.117). As is written in [II] for BKM Lie algebras, since any sum of two such fundamental roots cannot be a root, the factors generating such terms in the expansion of R are the next products:

$$\begin{aligned} &\frac{\prod_{\alpha_i \in \Pi^{im}_{even}} (1 - e^{-\alpha_i})}{\prod_{i \in \Psi_0} (1 + e^{-\alpha_i})} \cdot \prod_{i \in \Psi_-} \frac{1 - e^{-2\alpha_i}}{1 + e^{-\alpha_i}} \\ &= \frac{\prod_{\alpha_i \in \Pi^{im}_{even}} (1 - e^{-\alpha_i})}{\prod_{i \in \Psi_0} (1 + e^{-\alpha_i})} \cdot \prod_{i \in \Psi_-} (1 - e^{-\alpha_i}). \end{aligned} \tag{2.128}$$

Since the multiplication of ch_{Λ} has no effect on the terms of the form (2.127) for the same reason as was stated in [II], we only have to see

how these terms arise in the expansion of (2.128). So, by looking at the expansion of (2.128), replacing condition (2.126) by

$$i \in J \cap (I \setminus \Psi) \Longrightarrow m_i = 1, \tag{2.129}$$
$$i \in J \cap \Psi_- \Longrightarrow m_i = 1,$$

one sees that conditions (2.114) – (2.116) and (2.129) are necessary and sufficient for $c(\lambda) \neq 0$. And in this case, by using the fact that each factor in the denominator of (2.128) is expanded as

$$\frac{1}{1+e^{-\alpha_i}} = \sum_{j=0}^{\infty}(-1)^j e^{-j\alpha_i} \quad (i \in \Psi_0),$$

$c(\lambda)$ can be easily computed by looking at the coefficients of the expansion of (2.128), and the following theorem is obtained:

THEOREM 2.76. *If $\mathfrak{g}(A)$ is a symmetrizable BKM Lie superalgebra, then* ch_Λ $(\Lambda \in P_{(+)})$ *has the following expression:*

$$\mathrm{ch}_\Lambda = \frac{1}{e^\rho R} \sum_{w \in W} \varepsilon(w) w(S_\Lambda),$$

where we have

$$S_\Lambda := \sum_{\substack{k \in \mathbb{Z}_{\geq 0} \\ i_1, \cdots, i_k \in I^{im} \\ s.t.\ \text{(i),(ii),(iii)}}} (-1)^k e^{\Lambda + \rho - (\alpha_{i_1} + \cdots + \alpha_{i_k})},$$

and conditions (i),(ii) *and* (iii) *are given as follows:*

i. $(\Lambda | \alpha_{i_j}) = 0$ $(j = 1, \cdots, k)$.
ii. $\alpha_{i_1}, \cdots, \alpha_{i_k}$ *are pairwise orthogonal.*
iii. *Those $i_1, \cdots, i_k$ belonging to $I^{im} \setminus \Psi_0$ are all distinct (for those belonging to Φ_0, the same element can appear several times).*

Now, let us apply the character formulae obtained above, in particular, to the trivial representation. The trivial representaion (π, V) is that whose representation space V is of dimension one, and $\pi(X)$ for all $X \in \mathfrak{g}(A)$ are 0-operator, and is the 'pettiest' representation. Its character is surely given by

$$\mathrm{ch}L(0) = e^0 = 1.$$

Since this representation is the irreducible highest weight representation with highest weight 0, the character formulae naturally hold also for this case. So, setting $\Lambda = 0$ in Theorems 2.73 – 2.76, we obtain the following corollaries:

COROLLARY 2.77. *If $\mathfrak{g}(A)$ is a symmetrizable Kac-Moody Lie algebra, then*

$$R = e^{-\rho} \sum_{w \in W} \varepsilon(w) e^{w\rho}.$$

COROLLARY 2.78. *If $\mathfrak{g}(A)$ is a symmetrizable BKM Lie algebra, then by setting*

$$S_0 := \sum_{\substack{k \in \mathbb{Z}_{\geq 0} \\ i_1, \cdots, i_k \in I^{im} \\ s.t.\ \text{(i),(ii)}}} (-1)^k e^{\rho - (\alpha_{i_1} + \cdots + \alpha_{i_k})}$$

where conditions (i) *and* (ii) *are given by*

i. $\alpha_{i_1}, \cdots, \alpha_{i_k}$ *are pairwise orthogonal,*
ii. $i_1, \cdots, i_k$ *are all distinct,*

we have the following formula:

$$R = e^{-\rho} \sum_{w \in W} \varepsilon(w) w(S_0).$$

COROLLARY 2.79. *If $\mathfrak{g}(A)$ is a symmetrizable BKM Lie superalgebra, then by setting*

$$S_0 := \sum_{\substack{k \in \mathbb{Z}_{\geq 0} \\ i_1, \cdots, i_k \in I^{im} \\ s.t.\ \text{(i),(ii)}}} (-1)^k e^{\rho - (\alpha_{i_1} + \cdots + \alpha_{i_k})}$$

where conditions (i) *and* (ii) *are given by*

i. $\alpha_{i_1}, \cdots, \alpha_{i_k}$ *are pairwise orthogonal,*
ii. *those $i_1, \cdots, i_k$ belonging to $I^{im} \setminus \Psi_0$ are all distinct, (for those belonging to Φ_0, the same element can appear several times),*

we have the followwing formula:

$$R = e^{-\rho} \sum_{w \in W} \varepsilon(w) w(S_0).$$

Let us show one example of an application of these formulae. In a triangular decomposition $\mathfrak{g}(A) = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ of a BKM Lie (super)algebra $\mathfrak{g}(A)$, when one of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ (therefore both of them) is a free Lie (super)algebra, we may call $\mathfrak{g}(A)$ 'a Lie (super)algebra

with free nilpotent part'. In particular, when one has

$$A = \begin{pmatrix} -1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1 \end{pmatrix} \tag{2.130}$$

in Corollary 2.78, the BKM Lie algebra $\mathfrak{g}(A)$ is a Lie algebra with free nilpotent part, and since $I^{re} = \varnothing$, we have $W = \{e\} = \{$the identity element$\}$. So, the formula in Corollary 2.78 becomes

$$\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} = 1 - \sum_{i=1}^{n} e^{-\alpha_i}, \tag{2.131}$$

and by taking Taylor expansion of the log of each side of this formula with respect to $e^{-\alpha_1}, \cdots, e^{-\alpha_n}$, one obtains the formula describing the multiplicity of a root of a Lie algebra with free nilpotent part as the following proposition:

PROPOSITION 2.80. *Let $\mathfrak{g}$ be a Lie algebra with free nilpotent part, and let $\alpha_1, \cdots, \alpha_n$ be its fundamental roots. Then, the multiplicity* $\mathrm{mult}(\alpha)$ *of a root $\alpha = \sum_{i=1}^{n} m_i \alpha_i$ is given by the following formula:*

$$\mathrm{mult}(\alpha) = \sum_{d \mid \gcd(m_1, \cdots, m_n)} \frac{\mu(d)}{d} \cdot c\left(\frac{m_1}{d}, \cdots, \frac{m_n}{d}\right).$$

Here, we have

$$c(k_1, \cdots, k_n) := \frac{(k_1 + \cdots k_n - 1)!}{k_1! \cdots k_n!},$$

and μ is the Möbius function. Namely, for a natural number n, $\mu(n)$ is defined as follows:

$$\mu(n) := \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{if it is not square-free}, \\ (-1)^k & \text{for } n = p_1 \cdots p_k \ (p_1, \cdots p_k\text{: distinct primes}). \end{cases}$$

Here gcd *stands for the greatest common divisor, and $a|b$ signifies that a is a divisor of b.*

PROOF. Set

$$M(m_1, \cdots, m_n) := \begin{cases} \mathrm{mult}\left(\sum_{i=1}^{n} m_i \alpha_i\right) & \text{for } m_1, \cdots, m_n \geq 0, \sum_{i=1}^{n} m_i > 0, \\ 0 & \text{for } m_1 = \cdots = m_n = 0. \end{cases}$$

Simplifying the notations, if we set $x_i := e^{-\alpha_i}$ $(i = 1, \cdots, n)$, then (2.131) becomes

$$(2.132) \qquad 1 - \sum_{i=1}^{n} x_i = \prod_{m_1, \cdots, m_n \geq 0} (1 - x_1^{m_1} \cdots x_n^{m_n})^{M(m_1, \cdots, m_n)}.$$

Taking the logarithm of both sides of this formula, we have

(2.133)

$$\log\left(1 - \sum_{i=1}^{n} x_i\right) = \sum_{m_1, \cdots, m_n \geq 0} M(m_1, \cdots, m_n) \log(1 - x_1^{m_1} \cdots x_n^{m_n}),$$

and expanding both sides of this formula into formal power series by using

$$-\log(1-t) = \sum_{j \geq 1} \frac{t^j}{j},$$

we get

$$\sum_{j \geq 1} \frac{1}{j} \left(\sum_{i=1}^{n} x_i\right)^j = \sum_{m_1, \cdots, m_n \geq 0} \sum_{j \geq 1} \frac{M(m_1, \cdots, m_n)}{j} (x_1^{m_1} \cdots x_n^{m_n})^j,$$

and the coefficients of $x_1^{m_1} \cdots x_n^{m_n}$ in the both sides look as follows:

The coeff. of the l.h.s. of $x_1^{m_1} \cdots x_n^{m_n}$ is

$$\frac{(m_1 + \cdots + m_n - 1)!}{m_1! \cdots m_n!} = c(m_1, \cdots m_n).$$

The coeff. of the r.h.s. of $x_1^{m_1} \cdots x_n^{m_n}$ is

$$\sum_{d \mid \gcd(m_1, \cdots, m_n)} \frac{1}{d} M\left(\frac{m_1}{d}, \cdots, \frac{m_n}{d}\right).$$

Thus, we have

$$c(m_1, \cdots, m_n) = \sum_{d \mid \gcd(m_1, \cdots, m_n)} \frac{1}{d} M\left(\frac{m_1}{d}, \cdots, \frac{m_n}{d}\right),$$

and hence, by rewriting this formula using the **Möbius inversion formula**, i.e., the fact that for complex valued functions f and g defined on $\mathbb{Z}_{\geq 0}$,

$$g(n) = \sum_{d \mid n} f(d) \iff f(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) g(d),$$

we obtain the formula in the proposition. $\square$

Even if the Cartan matrix $A = (a_{ij})$ is not so beautiful as (2.130), if all of the diagonal entries a_{ii} are ≤ 0, and all of the non-diagonal entries are negative, we obtain the same conclusion.

With the same matrix A as above, and setting $\Psi := \{j \in \mathbb{Z}; k \leq j \leq n\}$, let us regard it as a BKM supermatrix. Here, k can be any integer satisfying $1 \leq k \leq n$. Then $\mathfrak{g}(A)$ is a Lie superalgebra with free nilpotent part, where $\alpha_1, \cdots, \alpha_{k-1}$ are even roots and $\alpha_k, \cdots, \alpha_n$ are odd roots, and we obtain the following formula by Corollary 2.79:

$$\frac{\prod_{\alpha \in \Delta_{even,+}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{odd,+}} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} = 1 - \sum_{i=1}^{n} e^{-\alpha_i}. \tag{2.134}$$

Observe that the right hand side of (2.131) and the right hand side of (2.134) are the same. Since the multiplicity $\mathrm{mult}(\alpha)$ of a root in the left hand side of (2.131) is obtained as in Proposition 2.80, one sees the left hand side of (2.131) concretely. Depending on whether each root α is an even root or an odd root, and if α is an odd root, then by rewriting as

$$1 - e^{-\alpha} = \frac{1 - e^{-2\alpha}}{1 + e^{-\alpha}},$$

and rewriting the left hand side of (2.131) in the form of the left hand side of (2.134), one can obtain the formula of multiplicity of a root of a Lie superalgebra with free nilpotent part. The parity of exah root $\alpha = \sum_{i-1}^{n} m_i \alpha_i$ depends upon a choice of Ψ, i.e., upon k. So, I would like to leave the problem of writing down the formula on the multiplicity of a root concretely for a given k to the reader as an exercise.

Note. The multiplicity of a root for a Lie algebra with free nilpotent part can also be derived by using the Verma module $M(0)$ with highest weight $\Lambda := 0$ as follows. Let v_0 be a highest weight vector of $M(0)$. Then $f_1 v_0, \cdots, f_n v_0$ are all singular vectors in $M(0)$, and

$$J_0 := \bigoplus_{i=1}^{n} \mathfrak{U}(\mathfrak{g}(A)) \cdot f_i v_0 \cong \bigoplus_{i=1}^{n} M(-\alpha_i)$$

is the maximal proper invariant subspace of $M(0)$. Thus, we get an exact sequence of $\mathfrak{g}(A)$-modules

$$\{0\} \longrightarrow \bigoplus_{i=1}^{n} M(-\alpha_i) \longrightarrow M(0) \longrightarrow L(0) \cong \mathbb{C} \longrightarrow \{0\},$$

and obtain the relation on the characters

$$1 = \mathrm{ch}M(0) - \sum_{i-1}^{n} \mathrm{ch}M(-\alpha_i).$$

This is the highly elementary and standard analysis using only a fundamental property of Verma modules, and in [**Bour1**], (2.131) is derived in this way, and Proposition 2.80 is obtained. □

Note. Proposition 2.80 and (2.132) can be regarded as a factorization of '$1 - \sum_{i=1}^{n} x_i$'. □

The formulae in Corollaries 2.77 – 2.79 are called the **denominator identity**. There is no longer a representation theory in the denominator identity.

Forgetting the fact that it comes from representation theory, it can be free from anything. In the next chapter, we will see an activity of the denominator identity in the case of affine Lie algebras.

Note. Before closing this section, I would like to mention the following. Let us consider the two Cartan matrices

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

For A, α_1 and α_2 are an even root and an odd root respectively, and for A', both α_1 and α_2 are odd simple roots. Then, these are both Cartan matrices of the simple finite-dimensional Lie superalgebra $\mathfrak{sl}(2|1)$. Namely, $\mathfrak{sl}(2|1)$ is a BKM Lie superalgebra, and one can apply the structure theory and the representation theory of BKM Lie superalgebras to $\mathfrak{sl}(2|1)$.

But then, the integrable representations of $\mathfrak{sl}(2|1)$ with respect to the Cartan matrix A are finite-dimensional representations. On the other hand, the integrable representations with respect to A' are all infinite-dimensional representations (except for the trivial representation). Namely, the notion 'integrable representation' considered in this section is not peculiar to the Lie superalgebra itself, but depends upon its Cartan matrix, i.e., the choice of a system of fundamental roots. The trivial representation is always integrable independently of the choice of its Cartan matrix, and its character formula, i.e., the denominator identity naturally describes the correct information of its root system for any Cartan matrix.

Here, in the example of $\mathfrak{sl}(2|1)$ presented above, the Weyl group W is the group consisting of two elements for A, and is the group consisting only of the identity element for A'.

For a simple finite-dimensional Lie superalgebra whose Dynkin diagram is given in Figure 1.2 of §1.5, or its affinization, one can define the notion 'integrable representation', in a way that is different from the one given in this book, and is independent of the choice of the system of fundamental roots, viz., as a notion peculiar to a Lie superalgebra itself. Similarly, one can also define the Weyl group, and it is rather natural and leads to significant results. But as we stated above, even when it is a BKM Lie superalgebra by accident, in many cases, its integrability condition conflicts with the integrability in this section. □

CHAPTER 3

Affine Lie Algebras

Among the Kac-Moody Lie algebras, the most important and the most lovely algebras are the affine Lie algebras. They are constructed concretely as the central extension of the loop algebra of the simple finite-dimensional Lie algebras. In this chapter, we will construct an affine Lie algebra in this way, and study its structure.

What is important in the affine Lie algebra are its root system and its Weyl group. By finding them explicitly, we can express the character formulae, especially the denominator identity, concretely.

For infinite-dimensional Lie algebras, it may happen quite often that in applying a theorem or a formula to a concrete example, we get a significant fact, even in the simplest case. Conversely, if we find an interesting phenomenon or an application, we can say that it can be immediately extended and generalized to general affine Lie algebras. In this case, to describe such things for general affine Lie algebras, we have only to rewrite them in terms of the 'roots' or the 'Weyl group'. In this chapter, we study the root system and the Weyl group of the simple finite-dimensional Lie algebras and of the affine Lie algebras.

§3.4 in this chapter is given only for Chapter 5, and it is not necessary for other sections and Chapter 4.

3.1. Properties of Weyl Groups and Roots for Simple Finite-Dimensional Lie Algebras

In this section and §3.2, let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank l. Fix a non-degenerate bilinear form $(\ |\ )$ which is a positive scalar multiple of the Killing form, and identify a Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via this bilinear form. We also confuse the pairing $\langle\ ,\ \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ and the bilinear form $(\ |\ )$. We use the same notations concerning fundamental objects such as root systems and Weyl groups. Let us recall the fundamental domain $C^\vee$:

$$C^\vee = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha_i, h\rangle \geq 0\ (1 \leq i \leq l)\}.$$

For a simple finite-dimensional Lie superalgebra whose Dynkin diagram is given in Figure 1.2 of §1.5, or its affinization, one can define the notion 'integrable representation', in a way that is different from the one given in this book, and is independent of the choice of the system of fundamental roots, viz., as a notion peculiar to a Lie superalgebra itself. Similarly, one can also define the Weyl group, and it is rather natural and leads to significant results. But as we stated above, even when it is a BKM Lie superalgebra by accident, in many cases, its integrability condition conflicts with the integrability in this section. □

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$$C^\vee = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha_i, h\rangle \geq 0\ (1 \leq i \leq l)\}.$$

LEMMA 3.1. 1. *The Weyl Group W is a finite group.*
2. *For $h \in \mathfrak{h}_{\mathbb{R}}$, there exists $w \in W$ satisfying $wh \in C^{\vee}$.*

PROOF. (1) Since each element of the Weyl group W induces a permutation of roots, one can construct a homomorphism

$$\varphi : W \longrightarrow \text{the permutation group of } \Delta.$$

And since for an element w of W

$$\begin{aligned} w\alpha = \alpha \quad & (\text{for all } \alpha \in \Delta) \\ \Longrightarrow w = \mathrm{Id}_{\mathfrak{h}} \quad & \Longrightarrow w = e \text{ (the uinty)}, \end{aligned}$$

φ is injective. Therefore we have

$$|W| \leq |\text{ the permutation group of } \Delta| < \infty.$$

(2) Fix any $h \in \mathfrak{h}_{\mathbb{R}}$. Then since

$$\{|wh - \rho|^2; w \in W\}$$

is a finite set, there exists an element of W that attains the minimum of this set, say w_0. Let us show $w_0 h \in C^{\vee}$ in this case.

For simplicity, let us set $h' := w_0 h$. Then by the choice of w_0 (i.e., the minimality of the length), it follows that

$$|r_i h' - \rho|^2 \geq |h' - \rho|^2$$

for $i = 1, \cdots, l$. Rewriting this formula, we have

$$\underbrace{|r_i h'|^2}_{|h'|^2} - 2(r_i h' | \rho) \geq |h'|^2 - 2(h' | \rho),$$

and hence the formula

$$(r_i h' - h' | \rho) \leq 0 \tag{3.1}$$

holds. Here by the definition of r_i, we have

$$r_i h' - h' = -\langle \alpha_i, h' \rangle \alpha_i^{\vee},$$

and hence (3.1) can be rewritten as

$$-\langle \alpha_i, h' \rangle \cdot \underbrace{\langle \alpha_i^{\vee}, \rho \rangle}_{1} \leq 0.$$

Hence $\langle \alpha_i, h' \rangle \geq 0$, and since this holds for all i, it follows that $h' \in C^{\vee}$. $\square$

Since $r_1, \cdots, r_l$ are generators of the Weyl group W, any element of W can be expressed in a form $w = r_{i_1} \cdots r_{i_k}$. But since $r_i^2 = e$, the way of expressing w as a product of the simple reflections is not unique. Expressing w as a product of the simple reflections, when k is the minimum possible number, this expression is called a **reduced expression** or a **minimal expression** of w. The number k in this case is called the **length** of w and is denoted by $l(w)$. Observe that a reduced expression of w is not necessarily unique. The length $l(w)$ of w has the following meaning with respect to the action of w on Δ:

PROPOSITION 3.2. *For $w \in W$, let*

$$w = r_{i_k} \cdots r_{i_1} \tag{3.2}$$

be a reduced expression. Then the following hold:

1. $l(w) = |\Delta_+ \cap w^{-1}\Delta_-| = \{\alpha \in \Delta_+; w\alpha < 0\}$.
2. $\Delta_+ \cap w^{-1}\Delta_- = \{r_{i_1} \cdots r_{i_{s-1}}(\alpha_{i_s}); 1 \leq s \leq k\}$.
3. $w(\Delta_+) \cap \Delta_- = \{-r_{i_k} r_{i_{k-1}} \cdots r_{i_{s+1}}(\alpha_{i_s}); 1 \leq s \leq k\}$.

Here in the right hand side, we regard $r_{i_1} \cdots r_{i_{s-1}}$ for $s = 1$ and $r_{i_k} r_{i_{k-1}} \cdots r_{i_{s+1}}$ for $s = k$ as the unity.

PROOF. As we have seen in §2.2, each r_i keeps $\Delta_+ \setminus \{\alpha_i\}$ invariant. Let us look at the action of w on Δ step by step according to the decomposition (3.2). First, looking at the action of r_i, we have

$$r_{i_1} : \begin{cases} \Delta_+ \setminus \{\alpha_{i_1}\} & \longrightarrow \Delta_+ \setminus \{\alpha_{i_1}\}, \\ \alpha_{i_1} & \longmapsto \alpha_{i_1}, \end{cases} \tag{3.3}$$

which means

$$r_{i_1}(\Delta_+) = (\Delta_+ \setminus \{\alpha_{i_1}\}) \cup \{-\alpha_{i_1}\}.$$

Hence

$$r_{i_1}(\Delta_+) \cap \Delta_- = \{-\alpha_{i_1}\}.$$

Next we consider the action of r_{i_2} on this. To see how many positive roots contained in Δ_+ are not contained in $r_{i_1} r_{i_2}(\Delta_+)$, let us look at the action of r_{i_1} on $\Delta_+ \setminus \{\alpha_{i_1}\}$ in slightly more detail. For this purpose, dividing $\Delta_+ \setminus \{\alpha_{i_1}\}$ into two parts as

$$\Delta_+ \setminus \{\alpha_{i_1}\} = (\Delta_+ \setminus \{\alpha_{i_1}, r_{i_1}(\alpha_{i_2})\}) \cup \{r_{i_1}(\alpha_{i_2})\},$$

the action of r_{i_1} on each set looks like

(3.4)

$$\begin{array}{ccccc}
\Delta_+ \setminus \{\alpha_{i_1}, r_{i_1}(\alpha_{i_2})\} & \xrightarrow{r_{i_1}} & \Delta_+ \setminus \{\alpha_{i_1}, \alpha_{i_2}\} & \xrightarrow{r_{i_2}} & \Delta_+ \setminus \{r_{i_2}(\alpha_{i_1}), \alpha_{i_2}\}, \\
\alpha_{i_1} & \xmapsto{r_{i_1}} & -\alpha_{i_1} & \xmapsto{r_{i_2}} & -r_{i_2}(\alpha_{i_1}), \\
r_{i_1}(\alpha_{i_2}) & \xmapsto{r_{i_1}} & \alpha_{i_2} & \xmapsto{r_{i_2}} & -\alpha_{i_2}.
\end{array}$$

Let us continue such consideration. The action of $r_{i_3}r_{i_2}r_{i_1}$ looks like

(3.5)

$$\begin{array}{ccccccc}
\alpha_{i_1} & \xmapsto{r_{i_1}} & -\alpha_{i_1} & \xmapsto{r_{i_2}} & -r_{i_2}(\alpha_{i_1}) & \xmapsto{r_{i_3}} & -r_{i_3}r_{i_2}(\alpha_{i_1}), \\
r_{i_1}(\alpha_{i_2}) & \xmapsto{r_{i_1}} & \alpha_{i_2} & \xmapsto{r_{i_2}} & -\alpha_{i_2} & \xmapsto{r_{i_3}} & -r_{i_3}(\alpha_{i_2}), \\
r_{i_1}r_{i_2}(\alpha_{i_3}) & \xmapsto{r_{i_1}} & r_{i_2}(\alpha_{i_3}) & \xmapsto{r_{i_2}} & \alpha_{i_3} & \xmapsto{r_{i_3}} & -\alpha_{i_3}.
\end{array}$$

The action of $r_{i_4}r_{i_3}r_{i_2}r_{i_1}$ looks like

(3.6)

$$\begin{array}{ccccccc}
\alpha_{i_1} & \xmapsto{r_{i_1}} & -\alpha_{i_1} & \xmapsto{r_{i_2}} & -r_{i_2}(\alpha_{i_1}) & & \\
 & & & \xmapsto{r_{i_3}} & -r_{i_3}r_{i_2}(\alpha_{i_1}) & \xmapsto{r_{i_4}} & -r_{i_4}r_{i_3}r_{i_2}(\alpha_{i_1}), \\
r_{i_1}(\alpha_{i_2}) & \xmapsto{r_{i_1}} & \alpha_{i_2} & \xmapsto{r_{i_2}} & -\alpha_{i_2} & & \\
 & & & \xmapsto{r_{i_3}} & -r_{i_3}(\alpha_{i_2}) & \xmapsto{r_{i_4}} & -r_{i_4}r_{i_3}(\alpha_{i_2}), \\
r_{i_1}r_{i_2}(\alpha_{i_3}) & \xmapsto{r_{i_1}} & r_{i_2}(\alpha_{i_3}) & \xmapsto{r_{i_2}} & \alpha_{i_3} & & \\
 & & & \xmapsto{r_{i_3}} & -\alpha_{i_3} & \xmapsto{r_{i_4}} & -r_{i_4}(\alpha_{i_3}), \\
r_{i_1}r_{i_2}r_{i_3}(\alpha_{i_4}) & \xmapsto{r_{i_1}} & r_{i_2}r_{i_3}(\alpha_{i_4}) & \xmapsto{r_{i_2}} & r_{i_3}(\alpha_{i_4}) & & \\
 & & & \xmapsto{r_{i_3}} & \alpha_{i_4} & \xmapsto{r_{i_4}} & -\alpha_{i_4}.
\end{array}$$

I would like to ask the reader to make such diagrams for a few further steps.

Watch these diagrams carefully. You may find that after each action of a simple reflection, one of the members of Δ_+ dropped out and turned into a negative root. Then you might worry: In each process of the action of a simple reflection, is it true that

(Expectation R) 'the roots that turned into a negative root will never come back to a positive root' ?

In the case of a reduced expression, we do not have to worry about that. Let us prove this. For this purpose, continuing the above diagrams further to the right and taking the simple reflections up to r_{i_p} $(1 \leq p \leq k)$, the result of the action of $r_{i_p} \cdots r_{i_1}$ looks like

$$
\begin{array}{rcl}
\alpha_{i_1} & \overset{r_{i_1}}{\longmapsto} \cdots \overset{r_{i_p}}{\longmapsto} & -r_{i_p} \cdots r_{i_2}(\alpha_{i_1}), \\
r_{i_1}(\alpha_{i_2}) & \overset{r_{i_1}}{\longmapsto} \cdots \overset{r_{i_p}}{\longmapsto} & -r_{i_p} \cdots r_{i_3}(\alpha_{i_2}), \\
r_{i_1} r_{i_2}(\alpha_{i_3}) & \overset{r_{i_1}}{\longmapsto} \cdots \overset{r_{i_p}}{\longmapsto} & -r_{i_p} \cdots r_{i_4}(\alpha_{i_3}), \\
\vdots & \overset{r_{i_1}}{\longmapsto} \cdots \overset{r_{i_p}}{\longmapsto} & \vdots \\
r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p}) & \overset{r_{i_1}}{\longmapsto} \cdots \overset{r_{i_p}}{\longmapsto} & -\alpha_{i_p}.
\end{array}
\tag{3.7}
$$

In the above diagrams (3.3) $\sim$ (3.7), α_{i_1} turned into a negative root at the first step, $r_{i_1}(\alpha_{i_2})$ turned into a negative root at the second step and $r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})$ turned into a negative root at the p-th. step. Now to show that (Expectation R) is true, let us consider what happens if we deny it. Suppose that 'there is a positive root that once turned into a negative root, but through the action of simple reflections, it again turned into a positive root', and assume that

(Assumption p) 'no such phenomenon happened up to the action of r_{i_p} but first arose when we applied $r_{i_{p+1}}$.'

Applying $w = r_{i_p} \cdots r_{i_1}$, viz., applying simple reflections from r_{i_1} to r_{i_p} in turn, the positive roots turned into negative roots are the leftmost roots in diagram (3.7), and they are mapped to the rightmost roots in diagram (3.7) by $r_{i_p} \cdots r_{i_1}$. By Assumption p, although they, the rightmost roots in diagram (3.7), are all negative roots, one of them turns into a positive root when we apply $r_{i_{p+1}}$ as a next step. Remember the following:

'the only root that turned from a positive root into a negative root (or vice versa) by the simple reflection r_i is α_i (or $-\alpha_i$).'

Therefore one of the rightmost roots in diagram (3.7) should coincide with $-\alpha_{i_{p+1}}$. Namely, there exists s ($1 \leq s \leq p$) satisfying

$$r_{i_p} r_{i_{p-1}} \cdots r_{i_{s+1}}(\alpha_{i_s}) = -\alpha_{i_{p+1}}.$$

Using the fact that in general $r_{w\alpha} = w r_\alpha w^{-1}$ for $\alpha \in \Delta$, $w \in W$ and $r_\alpha = r_{-\alpha}$ hold, we obtain the following formula from this formula:

$$r_{i_{p+1}} = r_{i_p} \cdots r_{i_{s+1}} \cdot r_{i_s} \cdot (r_{i_p} \cdots r_{i_{s+1}})^{-1},$$

i.e.,

$$r_{i_{p+1}} \cdot (r_{i_p} \cdots r_{i_{s+1}}) = r_{i_p} \cdots r_{i_{s+1}} \cdot r_{i_s}.$$

Rewriting the expression of w (3.2) by using this formula, we obtain

$$\begin{aligned} w &= r_{i_k} \cdots r_{i_{p+2}} \cdot \underbrace{r_{i_{p+1}} \cdots r_{i_{s+1}}}_{r_{i_p} \cdots r_{i_{s+1}} r_{i_s}} \cdot r_{i_s} \cdots r_{i_1} \\ &= r_{i_k} \cdots \hat{r}_{i_{p+1}} \cdots \hat{r}_{i_s} \cdots r_{i_1}. \end{aligned} \tag{3.8}$$

Here the caret means that the symbol underneath it is removed.

In (3.8), two reflections are reduced from the expression (3.2) of w. (3.8) is the formula that expresses w as the product of $k-2$ simple reflections, and it contradicts the fact that (3.2) is a reduced expression. Thus we complete our *reductio ad absurdum*, and Expectation R is proved.

Namely, when we look at the action of $w = r_{i_k} \cdots r_{i_1}$ on Δ step by step, by each action of a simple reflection only one positive root transfers to a negative root, and the roots that already have been transferred to negative roots cannot return to positive roots. Now it is easy to summarize these to complete the proof of the proposition. □

From the proof of this proposition, we have the following corollary:

COROLLARY 3.3. *If $w = r_{i_k} \cdots r_{i_1}$ is not a reduced expression, there exist natural numbers s, t $(1 \leq s < t \leq k)$ satisfying*

$$w = r_{i_k} \cdots \hat{r}_{i_t} \cdots \hat{r}_{i_s} \cdots r_{i_1}.$$

PROOF. If $w = r_{i_k} \cdots r_{i_1}$ is not a reduced expression, then by Proposition 3.2 we have

$$|w(\Delta_+) \cap \Delta_-| = l(w) < k;$$

hence Assumption p in the proof of Proposition 3.2 holds for some p $(1 \leq p \leq k)$. Repeating the argument given there, the expression of w is rewritten as (3.8). □

Note. As one can see in the proof of Proposition 3.2 and Corollary 3.3, we only counted the number of elements contained in $w(\Delta_+) \cap \Delta_-$, and we did not use the fact that Δ_+ itself is a finite set. Therefore Proposition 3.2 and Corollary 3.3 still hold for BKM Lie algebras. Since Δ^{im} is W-invariant for BKM Lie algebras, we have

$$w(\Delta_+) \cap \Delta_- = w(\Delta_+^{re}) \cap \Delta_-^{re}.$$

In the case of BKM Lie superalgebras, we have

$$r_i(\Delta_+) \cap \Delta_- = \{-\alpha_i, -2\alpha_i\}$$

for $i \in \Psi^{re}$, and hence it follows that

$$|r_i(\Delta_+) \cap \Delta_-| = 2,$$

and we need a few devices to describe Proposition 3.2. □

One also obtains the following corollary from Proposition 3.2:

COROLLARY 3.4. *For $w \in W$,*

$$w(\Delta_+) = \Delta_+ \iff w = e.$$

This is true because it follows from Proposition 3.2 that

$$w(\Delta_+) = \Delta_+ \iff l(w) = 0.$$

Note. The statements in Corollary 3.3 and Corollary 3.4 also hold for BKM Lie superalgebras. In particular, Corollary 3.3 is a property characterizing the Coxeter group; it follows that the Weyl group of a BKM Lie superalgebra is a Coxeter group. See Bourbaki [**Bour2**] for a detailed description of the Coxeter group.

But the Weyl group of a 'well-born' infinite-dimensional Lie (super-)algebra is not always a Coxeter group. For example, the Weyl group of a so-called elliptic root system is not a Coxeter group (see [**S**], [**ST**]). □

Now let us consider ρ for a simple finite-dimensional Lie algebra. Since all of the diagonal components of the Cartan matrix of a simple finite-dimensional Lie algebra are 2, we have

$$(\rho|\alpha_i^\vee) = 1 \quad (1 \le i \le l)$$

by (2.29). Thus we have

$$\alpha \in \Delta_+ \implies (\rho|\alpha) > 0, \tag{3.9}$$

and hence ρ is a regular element. Moreover, it has the following important sense:

LEMMA 3.5.

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

PROOF. Putting $\rho' := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$, for each i we have

$$\begin{aligned} r_i\rho' &= \frac{1}{2}r_i\left(\sum_{\alpha\in\Delta_+}\alpha\right) \\ &= \frac{1}{2}\left\{\underbrace{r_i\left(\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\alpha\right)}_{\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\alpha} + \underbrace{r_i\alpha_i}_{-\alpha_i}\right\} \\ &= \frac{1}{2}\left\{\left(\sum_{\alpha\in\Delta_+}\alpha-\alpha_i\right)-\alpha_i\right\} = \rho'-\alpha_i. \end{aligned}$$

On the other hand, since we have

$$r_i\rho' = \rho' - \langle\rho',\alpha_i^\vee\rangle\alpha_i$$

by the definition of r_i, comparing these two formulae, we have

$$\langle\rho',\alpha_i^\vee\rangle = 1 \quad (1\le i\le n), \tag{3.10}$$

so ρ' satisfies the same relation (2.29) as ρ does. Since an element of $\mathfrak{h}^*$ satisfying (3.10) is uniquely determined in the case of simple finite-dimensional Lie algebras, we deduce that $\rho'=\rho$. □

LEMMA 3.6. 1. *For an element w of W, the following three conditions are equivalent:*
(i) $w(\Delta_+)=\Delta_-$,
(ii) $w\rho=-\rho$,
(iii) $l(w)=|\Delta_+|$.
2. *An element satisfying the above condition in* (1) *uniquely exists, and w_0 satisfies $w_0^2=e$.*

PROOF. (1) The equivalence of (i) and (iii) follows from (1) of Proposition 3.2 and that of (i) and (ii) follows from Lemma 3.5.

(2) There exists $w\in W$ which transfers $-\rho$ in $C^\vee$. Then we have

$$(-w\rho|\alpha_i)\ge 0 \quad (1\le i\le l),$$

and since ρ is a regular element, equality in the above formula never happens. That is, we have

$$(\rho|w^{-1}(\alpha_i))<0 \quad (1\le i\le l).$$

Thus by (3.9) it follows that

$$w^{-1}\alpha_i \in \Delta_-,$$

and putting $w_0 := w^{-1}$, w_0 satisfies $w_0(\Delta_+) = \Delta_-$. The uniqueness of w_0 follows from Corollary 3.4. □

DEFINITION 3.7. The element w_0 of the Weyl group W satisfying the conditions in Lemma 3.6 is called the **longest element**.

The longest element w_0 satisfies the following conditions:

1. For type A_l, $w_0\alpha_i = -\alpha_{l+1-i}$ $(1 \le \forall i \le l)$.
2. For type D_l (l : odd),

$$w_0\alpha_i = \begin{cases} -\alpha_i & (1 \le i \le l-2), \\ -\alpha_l & (i = l-1), \\ -\alpha_{l-1} & (i = l). \end{cases}$$

3. For type E_6,

$$w_0 \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ & & \alpha_6 & & \end{pmatrix} = \begin{pmatrix} -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \\ & & -\alpha_6 & & \end{pmatrix},$$

i.e.,

$$w_0\alpha_i = \begin{cases} -\alpha_{6-i} & (1 \le i \le 5), \\ -\alpha_6 & (i = 6). \end{cases}$$

4. For the other cases, $w_0 h = -h$ $(\forall h \in \mathfrak{h})$.

LEMMA 3.8.

$$W(\Pi) = \Delta.$$

Namely, for $\alpha \in \Delta$, there exists $w \in W$ so that $w\alpha \in \Pi$.

PROOF. Consider the W-orbit of α

$$W \cdot \alpha := \{w\alpha; w \in W\},$$

and denote the element of minimal height among those belonging to $(W \cdot \alpha) \cap \Delta_+$ by $\beta = w\alpha$ $(\exists w \in W)$. Let us show that β is a fundamental root. Suppose now that β is not a fundamental root; then $r_i\beta$ are positive roots for all i. Hence by the minimality of $\mathrm{ht}(\beta)$, we should have

$$\mathrm{ht}(r_i\beta) \ge \mathrm{ht}(\beta) \quad (1 \le i \le l).$$

Thus we have

$$(\beta|\alpha_i) \leq 0 \quad (1 \leq i \leq l),$$

and since $\beta \in \Delta_+$, it follows that

$$(\beta|\beta) \leq 0.$$

But since the Killing form is positive definite on $\mathfrak{h}_{\mathbb{R}}$ by Proposition 1.55, $(\ |\ )$ is also positive definite on $\mathfrak{h}_{\mathbb{R}}$. Thus it follows that $\beta = 0$, and this contradicts the fact that β is a root. □

NOTATION 3.9. We denote the root with maximal height in Δ_+ by θ.

In other words, θ is the highest weight of the adjoint representation of $\mathfrak{g}$, and from this it follows that θ is uniquely determined. θ is called the **highest root** . Then $-\theta$ is the (lowest) root with minimal height in Δ. In §1.3, we have mentioned long roots and short roots. Let us see here that θ is a long root:

LEMMA 3.10.

$$\theta \in \Delta_l.$$

PROOF. Choose any $\alpha \in \Delta$. To show that

$$(\alpha|\alpha) \leq (\theta|\theta),$$

let us first prove the following:

$$(\alpha|\theta) \neq 0 \implies (\alpha|\alpha) \leq (\theta|\theta), \tag{3.11}$$

namely, let us show that $(\alpha|\alpha) \leq (\theta|\theta)$ when we can compare the length of α with that of θ directly. It is enough to show this for $\alpha \in \Delta_+$, replacing α with $-\alpha$ if α is a negative root.

Consider the θ-series of the roots through α,

$$\alpha + j\theta \quad (j \in \mathbb{Z}).$$

Here, even though I wrote '$j \in \mathbb{Z}$', j does not move through all the integers. It only varies as $\alpha + j\theta$ is a root. Namely, it moves inside a subset of $\mathbb{Z}$, but since it is tiresome to state that, I wrote as above. I will perform similar 'evasions' later in the proof. Since θ is the highest root, $\alpha + \theta$ is no longer a root. Moreover, since $-\theta$ is the lowest root, $\alpha - 2\theta$ is also not a root. Therefore the θ-series of the roots through α should be one of the following:

i. only α, or
ii. $\alpha - \theta, \alpha$.

The case (i), since $(\alpha|\theta) = 0$, is excluded. Hence the only possible case is (ii), and in this case we have

$$\frac{2(\alpha|\theta)}{(\theta|\theta)} = 1 \tag{3.12}$$

by Proposition 1.57.

Next, for the same α, let us consider the α-series through θ, $\theta + j\alpha$ ($j \in \mathbb{Z}$). Since $\theta + \alpha$ is not a root, this series has the following form:

$$\theta + p\alpha, \cdots, \theta - 2\alpha, \theta - \alpha, \theta.$$

Thus, again by Proposition 1.57, we have

$$-\frac{2(\alpha|\theta)}{(\alpha|\alpha)} = p, \tag{3.13}$$

and since we assume that $(\alpha|\theta) \neq 0$, p is a negative integer. Thus by (3.12) and (3.13), we have

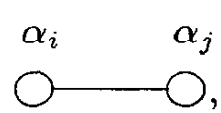
$$(\theta|\theta) = -p(\alpha|\alpha),$$

and since $p \in \mathbb{Z}_{<0}$, (3.11) is proved.

Now by Lemma 3.8, one can transfer θ in Π by the action of the Weyl group. Namely, there exist $w \in W$ and $\alpha_{i_0} \in \Pi$ satisfying $\theta = w\alpha_{i_0}$. Now if we can prove that α_{i_0} is a long root, then the proof of the lemma will be completed. Since α_{i_0} is a root which can be obtained by transferring θ by an element of the Weyl group, it satisfies the same property as (3.11). That is, the following holds:

$$\alpha \in \Delta \text{ s.t. } (\alpha|\alpha_{i_0}) \neq 0 \implies (\alpha|\alpha) \leq (\alpha_{i_0}|\alpha_{i_0}). \tag{3.14}$$

Now suppose that α_{i_0} is a short root. Then one can transfer α_{i_0} to the place next to a long fundamental root. This is because, if two fundamental roots α_i and α_j are in position as follows,



we have $r_i r_j(\alpha_i) = \alpha_j$. So by repeating this procedure, one may let α_{i_0} transfer successively, and finally one can transfer it to the place close to a lone fundamental root. Hence we may assume that α_{i_0} and α_{j_0} are adjacent. But this contradicts (3.14), if we specialize α to α_{j_0}. Thus the lemma is proved. □

Corollary 2.77 is the denominator identity for a symmetrizable Kac-Moody Lie algebra, and the next proposition, obtained by specializing the denominator identity in particular to a simple finite-dimensional Lie algebra, is very useful:

PROPOSITION 3.11. 1. *We have*

$$\sum_{w\in W} \varepsilon(w)e^{w\rho} = e^{\rho} \prod_{\alpha\in\Delta_+} (1-e^{-\alpha}) = \prod_{\alpha\in\Delta_+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}).$$

2. *For all $h \in \mathfrak{h}$, we have*

$$\sum_{w\in W} \varepsilon(w)e^{2\pi i(w\rho|h)} = (2i)^{|\Delta_+|} \prod_{\alpha\in\Delta_+} \sin \pi(\alpha|h).$$

PROOF. (1) follows from Corollary 2.77 and Proposition 3.11. (2) can be obtained by computing the quantity of (1) at $2\pi ih$. □

Next, we set

$$\begin{aligned} Q^\vee &:= \sum_{i=1}^{l} \mathbb{Z}\alpha_i^\vee, \\ P &:= (Q^\vee)^* = \{\lambda \in \mathfrak{h}^*; \langle\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}\ (1 \le i \le l)\}, \\ P_+ &:= \{\lambda \in P; \langle\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\ge 0}\ (1 \le i \le l)\}. \end{aligned}$$

The definition of P_+ is the same as was given in §2.5. The next proposition will be used to derive the Verlinde formula in Chapter 5:

PROPOSITION 3.12. *If $\lambda, \mu \in P_+$, $n \in \mathbb{Z}_{>0}$ and $w \in W$ satisfy*

$$w(\lambda+\rho) - (\mu+\rho) \in nQ^\vee,$$

then for all $\nu \in P$, the following formula holds:

$$\mathrm{ch}_\lambda\left(e^{\frac{2\pi i\nu}{n}}\right) = \varepsilon(w)\mathrm{ch}_\mu\left(e^{\frac{2\pi i\nu}{n}}\right).$$

PROOF. It is enough to show the equality in the proposition between the numerators of character formulae, i.e., the equality obtained by multiplying the character formulae by $e^\rho R$. By assumption, we can put

$$\mu+\rho = w(\lambda+\rho) + n\alpha \quad (\exists\alpha \in Q^\vee).$$

Then, we have

$$\begin{aligned}(e^{\rho}R\cdot \mathrm{ch}_{\mu})\left(e^{\frac{2\pi i\nu}{n}}\right) &= \sum_{w'\in W'}\varepsilon(w)e^{\frac{2\pi i}{n}(w'(\mu+\rho)|\nu)}\\ &= \sum_{w'\in W'}\varepsilon(w)e^{\frac{2\pi i}{n}(w'w(\lambda+\rho)|\nu)}e^{2\pi i(w'\alpha|\nu)}.\end{aligned}$$

Here $w'\alpha \in Q^{\vee}$ since $\alpha \in Q^{\vee}$; hence $(w'\alpha|\nu) \in \mathbb{Z}$. So we have $e^{2\pi i(w'\alpha|\nu)} = 1$, and the right hand side of the above formula is equal to

$$\varepsilon(w)\sum_{w'\in W'}\varepsilon(w')e^{\frac{2\pi i}{n}(w'(\lambda+\rho)|\nu)},$$

and this is just

$$\varepsilon(w)\cdot(e^{\rho}R\cdot \mathrm{ch}_{\lambda})\left(e^{\frac{2\pi i\nu}{n}}\right).$$

Thus the proposition is proved. □

In §2.4, we introduced an important concept called highest weight modules. If we replace the condition (i) in its Definition 2.51 by

(i′) $xv_{\Lambda} = 0$, for all $x \in \mathfrak{n}_{-}$,

we can define "the lowest weight module with lowest weight Λ". In this book, we denote the irreducible lowest weight module with lowest weight Λ by $L'(\Lambda)$ (not only for simple finite-dimensional Lie algebras but also for general BKM Lie superalgebras).

Note. For $\Lambda \in P_{+}$, since the character of $L(\Lambda)$ is invariant under the action of the Weyl group, then $w\Lambda$ ($w \in W$) is a weight of $L(\Lambda)$, and its multiplicity is 1. And if we let w_0 be the longest element of W, then $w_0\Lambda$ is the lowest weight of $L(\Lambda)$, and $L(\Lambda)$ can be characterized as the irreducible lowest weight module with lowest weight $w_0\Lambda$. □

For $\Lambda \in P_{+}$, the dual space $L(\Lambda)^{*}$ of $L(\Lambda)$ is an irreducible finite-dimensional $\mathfrak{g}$-module, and hence this is an irreducible highest weight module, and at the same time, is an irreducible lowest weight module. Let v_{Λ} be a highest weight vector of $L(\Lambda)$, and define $f_0 \in L(\Lambda)^{*}$ by

$$\begin{aligned}&\langle f_0, v_{\Lambda}\rangle = 1,\\ &\langle f_0, v\rangle = 0 \quad (\forall v \in \sum_{\lambda\neq\Lambda} L(\Lambda)_{\lambda}).\end{aligned}$$

Then, it can be easily checked by the definition of the action of $\mathfrak{g}$ on $L(\Lambda)^{*}$ that f_0 is a lowest weight vector of $L(\Lambda)^{*}$, and its weight is $-\Lambda$. Therefore, $L(\Lambda)^{*}$ is the irreducible lowest weight module $L'(-\Lambda)$

with lowest weight $-\Lambda$, and the highest weight of this representation is $w_0(-\Lambda) = -w_0\Lambda$.

DEFINITION 3.13. Let w_0 be the longest element of W. For each $\Lambda \in P_+$, if we set

$$^t\Lambda := -w_0\Lambda,$$

then $^t\Lambda$ is also an element of P_+. This $^t\Lambda$ is called the **transposed weight** of Λ. As I mentioned above, $^t\Lambda$ is the highest weight of the dual representation $L(\Lambda)^*$ of $L(\Lambda)$.

3.2. Invariant Form on Simple Finite-Dimensional Lie Algebras

As in the previous section, we only treat simple finite-dimensional Lie algebras in this section. We use the same notation as in the previous section. As was stated in §1.3, a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ is unique up to a scalar. In §1.3, we used the Killing form, but when we compute something, it is sometimes more useful to use the bilinear form normalized by

$$(\theta|\theta) = 2,$$

called the **standard bilinear form**. Having two bilinear forms (of course, they are mutually scalar multiples), when we convert a formula written by using one bilinear form to a formula by using another bilinear form, there is one thing that we should be careful of: the bilinear forms on the dual space induced by them. In particular, in our discussion, we have identified $\mathfrak{h}$ and $\mathfrak{h}^*$ by using a bilinear form, and confused both of them. If we change a bilinear form, then this way of identification should differ, and the induced bilinear form on $\mathfrak{h}^*$ should also differ. So it is necessary to arrange these things in order to avoid confusion.

Let V be a finite-dimensional vector space, and let $(\ |\ )$ be a non-degenerate bilinear form on V. Fix any non-zero complex number, say c, and define the new bilinear form $(\ |\ )'$ on V by

$$(\ |\ )' := c(\ |\ ).$$

Then, as two bilinear forms $(\ |\ )$ and $(\ |\ )'$ induce linear isomorphisms from V to V^*, we denote them by φ, φ' respectively. Namely,

$$\begin{aligned} \langle \varphi(u), v\rangle &= (u|v), \\ \langle \varphi'(u), v\rangle &= (u|v)' = c(u|v). \end{aligned} \tag{3.15}$$

Here $\langle\ ,\ \rangle$ signifies the pairing between V and V^*. Strictly speaking, in this pairing $\langle\ ,\ \rangle$, one of the left and right variables is a reserved seat for V, and the other is a reserved seat for V^*, but here and after we do not care about such things, and confuse them frequently.

From (3.15), it follows that

$$\langle \varphi'(u), v\rangle = c\langle \varphi(u), v\rangle,$$

and hence we have

$$\varphi'(u) = c\varphi(u). \tag{3.16}$$

Now, using the same symbol, we denote the bilinear form on V^* induced from φ and $(\ |\ )$ by $(\ |\ )$, and that induced from φ' and $(\ |\ )'$ by $(\ |\ )'$. Namely, both

$$\varphi : (V; (\ |\ )) \longrightarrow (V^*; (\ |\ ))$$

and

$$\varphi' : (V; (\ |\ )') \longrightarrow (V^*; (\ |\ )')$$

are isometries of vector spaces with metric. Let us compute $(\xi|\eta)$ and $(\xi|\eta)'$ for $\xi, \eta \in V^*$, and compare the two values. For these ξ, η, defining $u, v \in V$ by

$$\xi = \varphi'(u), \quad \eta = \varphi'(v), \tag{3.17}$$

we see from these formulae and (3.16) that

$$\xi = \varphi(cu), \quad \eta = \varphi(cv).$$

Since

$$\begin{aligned}(\xi|\eta)' &= (u|v)' = c(u|v),\\ (\xi|\eta) &= (\varphi(cu)|\varphi(cv)) = c^2(\varphi(u)|\varphi(v)) = c^2(u|v),\end{aligned}$$

it follows that

$$(\xi|\eta)' = \frac{1}{c}(\xi|\eta).$$

Namely,

$$(\ |\ )' = c(\ |\ ) \text{ on } V \Longrightarrow (\ |\ )' = \frac{1}{c}(\ |\ ) \text{ on } V^*. \tag{3.18}$$

When we rewrite a formula expressed in terms of the Killing form as a formula expressed in terms of another invariant bilinear form, we should pay attention to this fact.

Since the Killing form B is the bilinear form defined by using the adjoint representation of the Lie algebra $\mathfrak{g}$, by computing the Killing

form concretely, one can get some useful formulae. First, let us start from the following:

LEMMA 3.14.

$$B(\theta, \theta + 2\rho) = 1.$$

PROOF. Since the adjoint representation of $\mathfrak{g}$ is the irreducible representation with highest weight θ, the Casimir element Ω (made through B) is a scalar operator

$$\Omega = B(\theta, \theta + 2\rho)\mathrm{Id}_{\mathfrak{g}} \tag{3.19}$$

on $\mathfrak{g}$. Next, if we let $x_1, \cdots, x_n$ be an orthonormal basis (with respect to B) of $\mathfrak{g}$, then $\Omega = \sum_{i=1}^n x_i^2$, and since Ω acts on $\mathfrak{g}$ as $\sum_{i=1}^n (\mathrm{ad}(x_i))^2$, together with (3.19), we get

$$B(\theta, \theta + 2\rho)\mathrm{Id}_{\mathfrak{g}} = \sum_{i=1}^n (\mathrm{ad}(x_i))^2. \tag{3.20}$$

This formula is a formula as elements of $\mathrm{End}(\mathfrak{g})$. Computing the trace of both sides,

$$B(\theta, \theta + 2\rho) \cdot \dim \mathfrak{g} = \sum_{i=1}^n \underbrace{\mathrm{tr}(\mathrm{ad}(x_i))^2}_{B(x_i, x_i)} = \sum_{i=1}^n \underbrace{B(x_i, x_i)}_{1} = \dim \mathfrak{g},$$

and thus we get

$$B(\theta, \theta + 2\rho) = 1.$$

□

DEFINITION 3.15. Put

$$g := \frac{1}{B(\theta, \theta)}, \tag{3.21}$$

and call g the **dual Coxeter number** of the simple finite-dimensional Lie algebra $\mathfrak{g}$.

Since the standard bilinear form $(\ |\ )$ satisfies $(\theta|\theta) = 2$ by definition, the next relation immediately follows from (3.21):

$$(\ |\ ) = 2g \cdot B(\ ,\ ) \quad (\text{on } \mathfrak{h}^*). \tag{3.22}$$

This relation can be rephrased to a relation on $\mathfrak{h}$ by (3.18) as follows:

$$(\ |\ ) = \frac{1}{2g} \cdot B(\ ,\ ) \quad (\text{on } \mathfrak{h}). \tag{3.23}$$

By Lemma 3.14 and the definition of g, one obtains the following formula:

LEMMA 3.16.

$$(\rho|\theta) + 1 = g.$$

The following formulae are sometimes useful:

LEMMA 3.17. *For $h, h' \in \mathfrak{h}$ and $\lambda, \mu \in \mathfrak{h}^*$, the following formulae hold:*

1. $B(h, h') = 2 \sum_{\alpha \in \Delta_+} \alpha(h)\alpha(h')$.
2. (i) $g(h|h') = \sum_{\alpha \in \Delta_+} \alpha(h)\alpha(h')$,

   (ii) $g(\lambda|\mu) = \sum_{\alpha \in \Delta_+} (\alpha|\lambda)(\alpha|\mu)$.

PROOF. (1) can be obtained immediately by computing $B(h, h') = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}h\mathrm{ad}h')$ on $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Namely,

$$\begin{aligned} B(h, h') &= \sum_{\alpha \in \Delta} \mathrm{tr}_{\mathfrak{g}_\alpha}(\mathrm{ad}h\mathrm{ad}h') \\ &= \sum_{\alpha \in \Delta} \alpha(h)\alpha(h') = 2 \sum_{\alpha \in \Delta_+} \alpha(h)\alpha(h'). \end{aligned}$$

Using (3.22) and (3.23), (2) can be obtained by rewriting (1). □

Note. By Lemma 3.16, in general, it follows that for any non-degenerate invariant bilinear form (|) (not necessarily the standard one) on a simple finite-dimensional Lie algebra $\mathfrak{g}$, one has

$$g = \frac{(\theta|\theta + 2\rho)}{(\theta|\theta)} = 1 + \frac{2(\theta|\rho)}{(\theta|\theta)}. \tag{3.24}$$

Since this formula is independent of the choice of a non-degenerate invariant bilinear form, one can regard this formula as a definition of the dual Coxeter number. Then the question (with expectation) 'Can one define the dual Coxeter number for a simple finite-dimensional Lie superalgebra in Figure 1.2 by using Formula (3.24)?'might arise. Here, for $A(m|n)$, since the highest root θ defined by using the Dynkin diagram in Figure 1.2 is an odd root of length 0, switching to another Dynkin diagram (using the fact that a Dynkin diagram is not unique) in a such way that the highest root becomes an even root, we may compute formula (3.24).

But in general, the signature of g depends on a choice of a Dynkin diagram, i.e., a choice of a fundamental root system. Namely, for a Lie superalgebra in Figure 1.2, a dual Coxeter number is unique up to a signature, and a dual Coxeter number is determined when one chooses a Dynkin diagram.

As we will see in §3.5 and so forth, the dual Coxeter number is an important quantity concerning the level of a representation of an affinized Lie algebra, and it is also important in the case of Lie superalgebras. □

3.3. Structure of Affine Lie Algebras

In this section, we construct affine Lie algebras of split type by using simple finite-dimensional Lie algebras, and study their structure. First, as for the symbols used in this section, since we would like to use "ordinary" symbols for objects of constructed affine Lie algebras, we will use symbols e.g. for a simple finite-dimensional Lie algebra, adding a "bar".

Let $\overline{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra, and let $\overline{\Delta}$ be the root system associated to its Cartan subalgebra $\overline{\mathfrak{h}}$. Let $\overline{\Pi} = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental root system of $\overline{\mathfrak{g}}$, and let $\overline{h}_i, \overline{e}_i, \overline{f}_i$ $(1 \leq i \leq l)$ be its Chevalley generators. For each root $\alpha \in \overline{\Delta}$, we denote its root space by $\overline{\mathfrak{g}}_\alpha$. Based on $\overline{\mathfrak{g}}$, let us consider the next infinite-dimensional Lie algebra: Set

$$\mathfrak{g} := (\mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d, \tag{3.25}$$

and define the bracket product on $\mathfrak{g}$ as follows:

$$[t^m \otimes X, t^n \otimes Y] := t^{m+n} \otimes [X, Y] + m\delta_{m+n,0} \cdot (X|Y)c, \tag{3.26}$$

$$[c, \mathfrak{g}] := \{0\}, \tag{3.27}$$

$$[d, t^n \otimes X] := nt^n \otimes X. \tag{3.28}$$

Here $(\ |\ )$ is the standard bilinear form on $\overline{\mathfrak{g}}$. It is easy to check that this bracket product satisfies the Jacobi identity (by using the invariance of a bilinear form), and thus, by these brackets, $\mathfrak{g}$ becomes a Lie algebra. This Lie algebra $\mathfrak{g}$ is called the affine Lie algebra over $\overline{\mathfrak{g}}$, or the **affinization** of $\overline{\mathfrak{g}}$. And $\overline{\mathfrak{g}}$ is said to be the underlying simple finite-dimensional Lie algebra of $\mathfrak{g}$. We frequently use the following simplified notation:

NOTATION 3.18. We may write $X(n)$ instead of $t^n \otimes X$.

By identifying an element X of $\overline{\mathfrak{g}}$ with $X(0)$, $\overline{\mathfrak{g}}$ can be regarded as a Lie subalgebra of $\mathfrak{g}$. Set

$$\mathfrak{h} := (t^0 \otimes \overline{\mathfrak{h}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

An element λ of the dual space $\overline{\mathfrak{h}}^*$ of $\overline{\mathfrak{h}}$ can be regarded as an element of $\mathfrak{h}^*$ by putting

$$\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0.$$

So, in this way, $\overline{\Delta} \subset \overline{\mathfrak{h}}^*$ becomes a subset of $\mathfrak{h}^*$. Define the elements δ and Λ_0 of $\mathfrak{h}^*$ by

$$\langle \delta, \overline{\mathfrak{h}} \rangle := \langle \Lambda_0, \overline{\mathfrak{h}} \rangle := \{0\},$$

i.e.,

$$\langle \delta, \alpha_i^\vee \rangle := \langle \Lambda_0, \alpha_i^\vee \rangle := 0 \quad (1 \le i \le l), \tag{3.29}$$

$$\langle \delta, c \rangle := 0, \qquad \langle \delta, d \rangle := 1, \tag{3.30}$$

$$\langle \Lambda_0, c \rangle := 1, \qquad \langle \Lambda_0, d \rangle := 0; \tag{3.31}$$

then $\alpha_1, \cdots, \alpha_l, \delta, \Lambda_0$ becomes a basis of $\mathfrak{h}^*$.

Regarding $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{g}$, let us consider the root space decomposition of $\mathfrak{g}$. First, for $\alpha \in \overline{\Delta}$ and $n \in \mathbb{Z}$, $t^n \otimes \overline{\mathfrak{g}}_\alpha$ becomes a root space. In fact, for $X \in \overline{\mathfrak{g}}_\alpha$, the bracket products of $X(n)$ and elements of $\mathfrak{h}$ are

$$\begin{aligned} [h, X(n)] &= \langle \alpha, h \rangle \cdot X(n) \quad (h \in \overline{\mathfrak{h}}), \\ [c, X(n)] &= 0, \\ [d, X(n)] &= nX(n). \end{aligned}$$

These formulae can be summarized into the following form:

$$[h, X(n)] = \langle n\delta + \alpha, h \rangle \cdot X(n) \quad (h \in \mathfrak{h}).$$

Therefore, $n\delta + \alpha$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $t^n \otimes \overline{\mathfrak{g}}_\alpha$ is its root space.

Next, let us consider $t^n \otimes \overline{\mathfrak{h}}$ $(n \neq 0)$. Since $n \neq 0$, this does not intersect non-trivially with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Moreover, this is also a root space. In fact, by the same computation as above, we have

$$[h, H(n)] = \langle n\delta, h \rangle \cdot H(n) \quad (h \in \mathfrak{h}, H \in \overline{\mathfrak{h}}).$$

Therefore, $n\delta$ $(n \neq 0)$ is also a root, and its root space is $t^n \otimes \overline{\mathfrak{h}}$.

Thus, the root system Δ of $(\mathfrak{g}, \mathfrak{h})$ is

$$\Delta = \{n\delta + \alpha; n \in \mathbb{Z}, \alpha \in \overline{\Delta}\} \cup \{n\delta; n \in \mathbb{Z} \setminus \{0\}\}, \tag{3.32}$$

and one sees that the root spaces are

$$\mathfrak{g}_{n\delta+\alpha} = t^n \otimes \overline{\mathfrak{g}}_\alpha, \tag{3.33}$$

$$\mathfrak{g}_{n\delta} = t^n \otimes \overline{\mathfrak{h}}. \tag{3.34}$$

In this way, we obtain the root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

in a way similar to the case of simple finite-dimensional Lie algebras, but let us remark that, unlike the case of finite-dimensional Lie algebras, we have

$$\dim \mathfrak{g}_{n\delta} = l \geq 1.$$

As we see in slightly more detail later, $n\delta$ is an imaginary root of alien-type, namely, it is an imaginary root belonging to Δ^{ali-im}, and hence one cannot impose the constraint 'root multiplicity = 1', like Proposition 1.56 for a simple finite-dimensional Lie algebra.

We extend the standard bilinear form $(\ |\ )$ on $\overline{\mathfrak{g}}$ to the symmetric invariant bilinear form $(\ |\ )$ on $\mathfrak{g}$ as follows:

$$(X(m)|Y(n)) := \delta_{m+n,0} \cdot (X|Y), \tag{3.35}$$

$$(X(m)|c) := (X(m)|d) := 0, \tag{3.36}$$

$$(c|d) := 1, \tag{3.37}$$

$$(c|c) := (d|d) := 0. \tag{3.38}$$

It is easy to check that this is a non-degenerate invariant bilinear form on $\mathfrak{g}$.

Next, let us consider Chevalley generators of $\mathfrak{g}$. For that, it is enough to investigate generators of

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = (\mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c.$$

This is because we can get $\mathfrak{g}$ by adding d to generators of $\mathfrak{g}'$. Since, by the definition of the bracket product, we have

$$[t^m \otimes \overline{\mathfrak{g}}, t^n\overline{\mathfrak{g}}] = t^{m+n} \otimes \overline{\mathfrak{g}} + \delta_{m+n,0} \cdot \mathbb{C}c,$$

$\mathfrak{g}'$ is generated by $t^n \otimes \overline{\mathfrak{g}}$ $(n = 0, \pm 1)$. As we studied in §1.3 for $\overline{\mathfrak{g}}$, one can choose its generators as $\overline{e}_i, \overline{f}_i, \overline{h}_i$ $(1 \leq i \leq l)$. Here $(\overline{h}_i, \overline{e}_i, \overline{f}_i)$ is an S-triple. Regarding them as elements of $\mathfrak{g}$, we set

$$e_i := t^0 \otimes \overline{e}_i, \quad f_i := t^0 \otimes \overline{f}_i, \quad \alpha_i^\vee := t^0 \otimes \overline{h}_i \quad (1 \leq i \leq l). \tag{3.39}$$

Since $\overline{\mathfrak{g}}$ is a simple Lie algebra, $t^{\pm 1} \otimes \overline{\mathfrak{g}}$ are both irreducible as representation spaces of $\overline{\mathfrak{g}}$. So, if we choose a non-zero element (arbitrarily) from each of $t \otimes \overline{\mathfrak{g}}$ and $t^{-1} \otimes \overline{\mathfrak{g}}$, and call them by e_0 and f_0 respectively, then e_i, f_i $(0 \leq i \leq l)$ are generators of $\mathfrak{g}'$. If we simply need generators, e_0 and f_0 can be any non-zero elements. But we would like to make 'Chevalley generators'. For this purpose, since they should satisfy

$$i \neq j \Longrightarrow [e_i, f_j] = 0,$$

e_0 and f_0 should satisfy

$$[e_0, f_j] = [f_0, e_j] = 0 \quad (1 \leq j \leq l).$$

From this requirement, e_0 and f_0 are uniquely determined up to scalars. Namely, if we let θ be the highest root of $\overline{\Delta}$, e_0 and f_0 should be elements of $t \otimes \overline{\mathfrak{g}}_{-\theta}$ and $t^{-1} \otimes \overline{\mathfrak{g}}_{\theta}$ respectively. So we reset them as follows.

First, choosing $\overline{e}_\theta \in \overline{\mathfrak{g}}_\theta$ and $\overline{e}_{-\theta} \in \overline{\mathfrak{g}}_{-\theta}$ so that they satisfy

$$(\overline{e}_\theta | \overline{e}_{-\theta}) = 1,$$

it follows that

$$[\overline{e}_\theta, \overline{e}_{-\theta}] = \overline{h}_\theta,$$

and $(\overline{h}_\theta, \overline{e}_\theta, \overline{e}_{-\theta})$ is an S-triple. If we set

$$\begin{aligned} e_0 &:= t \otimes \overline{e}_{-\theta}, \\ f_0 &:= t^{-1} \otimes \overline{e}_\theta, \\ \alpha_0^\vee &:= [e_0, e_0] = -t^0 \otimes \overline{h}_\theta + c, \end{aligned} \tag{3.40}$$

then $e_i, f_i, \alpha_i^\vee$ $(0 \leq i \leq l)$ are generators of $\mathfrak{g}'$, and hence those with d are Chevalley generators of $\mathfrak{g}$.

Since e_0 and f_0 are root vectors of root $\delta - \theta$ and $\theta - \delta$ respectively, as one can see from their definition, if we set

$$\alpha_0 := \delta - \theta, \tag{3.41}$$

then

$$e_i \in \mathfrak{g}_{\alpha_i}, \quad f_i \in \mathfrak{g}_{-\alpha_i}$$

for all $0 \leq i \leq l$. Since θ is the highest root of $\overline{\Delta}$, let its expansion with respect to α_i $(1 \leq i \leq l)$ be

$$\theta := \sum_{i=1}^{l} \tilde{a}_i \alpha_i. \tag{3.42}$$

Then, by (3.41) and (3.42), putting $\tilde{a}_0 := 1$, we can express δ as follows:

$$\delta := \sum_{i=0}^{l} \tilde{a}_i \alpha_i. \tag{3.43}$$

The Cartan matrix $A = (a_{ij})_{0 \le i,j \le l}$ can be computed by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$, and its submatrix $\overline{A} = (a_{ij})_{1 \le i,j \le l}$ obtained by restricting its indices i and j to $1 \le i, j \le l$ is the Cartan matrix of the simple finite-dimensional Lie algebra $\overline{\mathfrak{g}}$. And since for $i = 0$ or $j = 0$ we have

$$\begin{aligned}
a_{00} &:= \langle \alpha_0^\vee, \alpha_0 \rangle = 2, \\
a_{0j} &:= \langle \alpha_0^\vee, \alpha_j \rangle = -\langle \overline{h}_\theta, \alpha_j \rangle \quad (1 \le j \le l), \\
a_{i0} &:= \langle \alpha_i^\vee, \alpha_0 \rangle = \langle \alpha_i^\vee, \delta - \theta \rangle = -\langle \alpha_i^\vee, \theta \rangle \quad (1 \le i \le l),
\end{aligned}$$

it follows that, in terms of Dynkin diagrams, the Dynkin diagram of A is that obtained by adding $-\theta$ to the Dynkin diagram of $\overline{A}$, i.e., the "extended Dynkin diagram" of $\overline{A}$. Since $-\overline{h}_\theta, \overline{h}_1^\vee, \cdots, \overline{h}_l^\vee$ is not a basis of $\overline{\mathfrak{h}}$, the matrix of the inner product of these elements loses its positive definiteness, and it becomes positive semi-definite; thus A is a GCM of affine type. And by (3.35) – (3.38), since the bilinear form naturally extended to $\mathfrak{g}$ is non-degenerate, as was stated in §2.1, $\mathfrak{g}$ itself is $\mathfrak{g}(A)$.

We call the Lie algebra thus obtained, i.e., the Lie algebra of an extended Dynkin diagram, an **affine Lie algebra of split type**. When the underlying Lie algebra $\overline{\mathfrak{g}}$ is a simple finite-dimensional Lie algebra of type X_l ($X = A \sim G$), the algebra $\mathfrak{g}$ constructed here is called an affine Lie algebra of type $X_l^{(1)}$. Although these are important class of affine Lie algebras, as the term 'of split type' indicates, the Cartan matrices of affine type are not exhausted by these. The other types of affine Lie algebras are constructed as follows.

Some of the Dynkin diagrams of simple finite-dimensional Lie algebras in §1.3 possess symmetry. For example, type A_l has a symmetry of turn-up type, and type D_l has the symmetry of exchanging α_l and α_{l-1}. Moreover, the Dynkin diagram of type D_4 has two sort of symmetries as in Figure 3.1. A symmetric transformation of turn-up type is of order 2, and a rotation is a symmetric transformation of order 3. Beside these, the Dynkin diagram of type E_6 has a symmetry of turn-up type (order 2).

Now, let $\overline{\mathfrak{g}}$ be the simple finite-dimensional Lie algebra of type X_N. Here X is one of types A, D and E. A symmetric transformation

rotation turn-up

FIGURE 3.1

of order r of the Dynkin diagram of $\overline{\mathfrak{g}}$ induces an automorphism of order r of the Lie algebra $\overline{\mathfrak{g}}$. Concerning this fact, see a textbook on Lie algebras, e.g. Chapter 3 of [**Hel**]. Let us denote it by σ. Since the eigenvalues of σ are r-th roots of unity, by setting $\varepsilon_r := e^{\frac{2\pi i}{r}}$

$$\overline{\mathfrak{g}}_j := \{x \in \overline{\mathfrak{g}}; \sigma x = \varepsilon_r^j x\},$$
$$\overline{\mathfrak{h}}_j := \{h \in \overline{\mathfrak{h}}; \sigma h = \varepsilon_r^j h\}$$

for each $j \in \mathbb{Z}/r\mathbb{Z}$, $\overline{\mathfrak{g}}$ becomes a $\mathbb{Z}/r\mathbb{Z}$-graded Lie algebra. Namely,

$$\overline{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \overline{\mathfrak{g}}_j, \quad [\overline{\mathfrak{g}}_j, \overline{\mathfrak{g}}_k] \subset \overline{\mathfrak{g}}_{j+k}. \tag{3.44}$$

Then, $\overline{\mathfrak{g}}_0$ is a simple Lie algebra and $\overline{\mathfrak{h}}_0$ is its Cartan subalgebra, and each $\overline{\mathfrak{g}}_j$ is an irreducible $\overline{\mathfrak{g}}_0$-module.

Now, let us consider the affinization of $(\overline{\mathfrak{g}}, \sigma)$: Set

$$\mathfrak{g}^{(\sigma)} := \left(\sum_{n \in \mathbb{Z}} t^n \otimes \overline{\mathfrak{g}}_{n \mod r\mathbb{Z}} \right) \oplus \mathbb{C}c \oplus \mathbb{C}d, \tag{3.45}$$

and define the bracket products of $\mathfrak{g}^{(\sigma)}$ by

$$[t^m \otimes X, t^n \otimes Y] := t^{m+n} \otimes [X, Y] + m\delta_{m+n,0} \cdot \frac{(X|Y)}{r} \cdot c, \tag{3.46}$$

$$[c, \mathfrak{g}] = \{0\}, \tag{3.47}$$

$$[d, t^n \otimes X] := nt^n \otimes X. \tag{3.48}$$

We construct Chevalley generators and the Cartan matrix of $\mathfrak{g}^{(\sigma)}$ as follows.

Put $l := \dim \overline{\mathfrak{h}}_0$, let $\overline{e}_1, \cdots, \overline{e}_l, \overline{f}_1, \cdots, \overline{f}_l, \overline{h}_1, \cdots, \overline{h}_l$ be Chevalley generators of $\overline{\mathfrak{g}}_0$, and set

$$e_i := t^0 \otimes \overline{e}_i, \quad f_i := t^0 \otimes \overline{f}_i, \quad \alpha_i^\vee := t^0 \otimes \overline{h}_i \quad (1 \le i \le l).$$

Next, take a lowest weight vector v_- of $\overline{\mathfrak{g}}_1$ and a highest weight vector v_+ of $\overline{\mathfrak{g}}_{-1}$, and set

$$e_0 := t \otimes v_-, \quad f_0 := t^{-1} \otimes v_+.$$

We normalize v_+ and v_- so that $(\alpha_0^\vee := [e_0, f_0], e_0, f_0)$ becomes an S-triple. Then, if we define a_{ij} $(0 \le i, j \le l)$ by

$$[\alpha_i^\vee, e_j] = a_{ij} e_j,$$

the matrix $A = (a_{ij})_{0 \le i,j \le l}$ is a Cartan matrix of affine type. $\mathfrak{g}^{(\sigma)}$ is called an affine Lie algebra of type $X_N^{(r)}$.

By these constructions, all of the Cartan matrices of affine type are obtained. Namely, for any Cartan matrix A of affine type, we obtained a way to construct $\mathfrak{g}(A)$ concretely. If $\mathfrak{g}$ is the affine Lie algebra of type $X_N^{(r)}$, r is called the **tier number** of $\mathfrak{g}$.

If $r \ge 2$, then $\mathfrak{g}$ is called an affine Lie algebra of 'twisted type'. When $\mathfrak{g}$ is of twisted type, we remark that the matrix $(a_{ij})_{1 \le i,j \le l}$ obtained by deleting the 0-th row and the 0-th column of the Cartan matrix $A = (a_{ij})_{0 \le i,j \le l}$ is not the Cartan matrix of $\overline{\mathfrak{g}}$ but the Cartan matrix of $\overline{\mathfrak{g}}_0$. So, in general, we denote this matrix $(a_{ij})_{1 \le i,j \le l}$ not by $\overline{A}$ but by $\overset{\circ}{A}$, and denote $\sum_{i=1}^{l} \mathbb{C}\alpha_i^\vee$ by $\overset{\circ}{\mathfrak{h}}$. When A is of split type, we often confuse the symbols $\overset{\circ}{A}$ and $\overline{A}$, $\overset{\circ}{\mathfrak{h}}$ and $\overline{\mathfrak{h}}$.

Note. The multiplicity of an imaginary root of a twisted affine Lie algebra can be computed by using the above construction of $\mathfrak{g}^{(\sigma)}$ as follows. On the eigenspace decomposition of $\overline{\mathfrak{h}}$ with respect to σ

$$\overline{\mathfrak{h}} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \overline{\mathfrak{h}}_j,$$

by the σ-invariance and the non-degeneracy of a bilinear form, one has

$$\dim \overline{\mathfrak{h}}_j = \dim \overline{\mathfrak{h}}_{-j}.$$

By the definition of N and l, we have

$$N = \dim \overline{\mathfrak{h}}, \quad l = \dim \overline{\mathfrak{h}}_0.$$

So, $\bigoplus_{\substack{j \in \mathbb{Z}/r\mathbb{Z} \\ j \ne 0}} \overline{\mathfrak{h}}_j$ is of dimension $(N - l)$, and the possible numbers for r are $r = 1, 2, 3$ by the symmetry of Dynkin diagrams, and for a

twisted type, we have $r = 2, 3$. Therefore,

$$r = 2 \Longrightarrow \dim \overline{\mathfrak{h}}_1 = N - l,$$
$$r = 3 \Longrightarrow \dim \overline{\mathfrak{h}}_1 = \dim \overline{\mathfrak{h}}_2 = \frac{N-l}{2},$$

and by this computation, the multiplicity of the imaginary root $j\delta$ ($j \in \mathbb{Z} \setminus \{0\}$) of the twisted affine Lie algebra $\mathfrak{g}^{(\sigma)}$ of type $X_N^{(r)}$ can be obtained as follows:

$$\mathrm{mult}(j\delta) = \begin{cases} l & \text{if } j \in r\mathbb{Z}, \\ \frac{N-l}{r-1} & \text{if } j \notin r\mathbb{Z}. \end{cases}$$

□

To compute several fundamental quantities of affine Lie algebras, let us restart from a Cartan matrix of affine type. If $A = (a_{ij})_{0 \le i,j \le l}$ is a GCM of affine type, then by Theorem 2.20, the transpose ${}^tA$ of A is also of affine type. So, by the definition of a GCM of affine type, there uniquely exist up to scalars positive vectors

$$u = \begin{pmatrix} u_0 \\ \vdots \\ u_l \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_0 \\ \vdots \\ v_l \end{pmatrix}$$

satisfying

$$Au = 0 \quad \text{and} \quad {}^tAv = 0. \tag{3.49}$$

Here, since the matrix components of A are all integers, the u_i and v_i can all be chosen from positive rational numbers. Next, by multiplying by the greatest common divisor of their denominators, the u_i and v_i can be all positive integers. Hence, by choosing solutions u, v of (3.49) satisfying

$$\gcd(u_0, \cdots, u_l) = \gcd(v_0, \cdots, v_l),$$

and denote them by

$$(a_o, \cdots a_l), \quad (a_0^\vee, \cdots a_l^\vee)$$

respectively. Namely, they can be uniquely determined as the minimal positive integral solutions of the equations

$$A\begin{pmatrix} a_0 \\ \vdots \\ a_l \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.50}$$
$$(a_0^\vee, \cdots, a_l^\vee)A = (0, \cdots, 0).$$

DEFINITION 3.19. $a_0, \cdots, a_l$ are called the **label** of A, and $a_0^\vee, \cdots, a_l^\vee$ are called the **colabel** of A.

LEMMA 3.20. *Set* $\varepsilon_i := \frac{a_i}{a_i^\vee}$. *If we define the diagonal matrix* D *by*

$$D := \begin{pmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_l \end{pmatrix},$$

then $D^{-1}A$ *is a symmetric matrix.*

PROOF. If we use Proposition 2.25, then it is easy to prove this lemma. Take

$$D' = \begin{pmatrix} \varepsilon_0' & & 0 \\ & \ddots & \\ 0 & & \varepsilon_l' \end{pmatrix} : \text{a positive diagonal matrix},$$
$$B = (b_{ij})_{0\le i,j\le l} : \text{ a symmetric matrix}$$

so that they satisfy $A = D'B$, and consider the equation

$$xB = 0 \quad (x \in \mathbb{R}^{l+1}). \tag{3.51}$$

Since $\mathrm{corank} B = 1$, the space of solutions of this equation is of dimension 1, and we can find its solution as follows.

First, if we set

$$u := \begin{pmatrix} a_0 \\ \vdots \\ a_l \end{pmatrix},$$

then $Bu = 0$ because $Au = 0$. Therefore, by taking the transposition of both sides, we get ${}^t uB = 0$. Namely, ${}^t u$ is one of the non-trivial solutions of (3.51).

Next, if we set $v := (a_0^\vee, \cdots, a_l^\vee)$, then, since

$$0 = vA = vD' \cdot B,$$

it follows that

$$vD' = (\varepsilon'_0 a_0^\vee, \cdots, \varepsilon'_l a_l^\vee)$$

is also a solution of (3.51).

Thus, vD' is a scalar multiple of ${}^t u$, and the lemma follows because

$$\varepsilon'_i = \frac{a_i}{a_i^\vee} \times (\text{a constant independent of } i).$$

□

If we set

$$b_{ij} := \frac{a_i^\vee}{a_i} a_{ij},$$

then B is a symmetric matrix by Lemma 3.20. Take D and B as above, and introduce a bilinear form on $\mathfrak{h}$ by Lemma 2.7. Namely, we have

$$a_i^\vee \alpha_i^\vee = a_i \alpha_i, \tag{3.52}$$

$$(\alpha_i | \alpha_j) = \frac{a_i^\vee}{a_i} a_{ij}, \tag{3.53}$$

$$(\alpha_i^\vee | \alpha_j^\vee) = \frac{a_j}{a_j^\vee} a_{ij}. \tag{3.54}$$

Also, if we set

$$c := \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee \in \mathfrak{h}, \tag{3.55}$$

$$\delta := \sum_{i=0}^{l} a_i \alpha_i \in \mathfrak{h}^*, \tag{3.56}$$

then the following formulae can be easily checked by (3.50) and (3.52) – (3.54):

$$\langle \alpha_i, c \rangle = \langle \delta, \alpha_i^\vee \rangle = 0 \quad (0 \le i \le l), \tag{3.57}$$

$$(c|c) = (\delta|\delta) = 0. \tag{3.58}$$

By (3.57), c is a central element of $\mathfrak{g}$, and we have

$$\delta \in \overset{\circ}{K} \subset K.$$

Thus, by Theorem 2.45, $j\delta$ ($j \in \mathbb{Z} \setminus \{0\}$) is an imaginary root of alien-type. c is called the **canonical central element** of $\mathfrak{g}(A)$ and δ is called the **fundamental imaginary root.**

Concerning the action of the Weyl group, we have

$$r_i\delta = \delta \quad (0 \le i \le l)$$

by (3.57). δ is fixed by all elements of W.

From this, it follows that if A is a GCM of affine type, then

$$\Delta^{im} = \Delta^{ali\text{-}im} = \{j\delta; j \in \mathbb{Z} \setminus \{0\}\}. \tag{3.59}$$

DEFINITION 3.21. Set $h := \sum_{i=0}^{l} a_i$, $h^\vee := \sum_{i=0}^{l} a_i^\vee$, and call h the **Coxeter number** and $h^\vee$ the **dual Coxeter number** of the affine Lie algebra $\mathfrak{g}(A)$.

Let us look at the relations between c and δ as a central element and as an imaginary root appearing in the concrete construction of affine Lie algebras given in the first half of this section. For simplicity, let us consider the case of split type. Since the Dynkin diagram of an affine Lie algebra of split type is the extended Dynkin diagram of a simple finite-dimensional Lie algebra, we have

$$a_0 = a_0^\vee = 1.$$

Therefore two δ's are the same, and two c's are the same, and we have

$$\delta = \alpha_0 + \theta, \quad \theta = \sum_{i=1}^{l} a_i\alpha_i. \tag{3.60}$$

Namely, the $\tilde{a}_i$'s in (3.42) and (3.43) are the same as the a_i's. Moreover, under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ via the standard bilinear form $(\ |\ )$, we have $c = \delta$.

LEMMA 3.22. *If we denote the underlying simple finite-dimensional Lie algebra of an affine Lie algebra $\mathfrak{g}$ of split type by $\overline{\mathfrak{g}}$, then the dual Coxeter number $h^\vee$ of $\mathfrak{g}$ and the dual Coxeter number g of $\overline{\mathfrak{g}}$ coincide.*

PROOF. Since $\theta = \sum_{i=1}^{l} a_i\alpha_i$, we have

$$\begin{aligned}(\overline{\rho}|\theta) &= \sum_{i=1}^{l} a_i(\overline{\rho}|\alpha_i) \underset{(\because(3.52))}{\overset{\uparrow}{=}} \sum_{i=1}^{l} a_i \cdot \frac{a_i^\vee}{a_i} \underbrace{(\overline{\rho}|\alpha_i^\vee)}_{1} \\ &= \sum_{i=1}^{l} a_i^\vee = h^\vee - a_0^\vee = h^\vee - 1,\end{aligned}$$

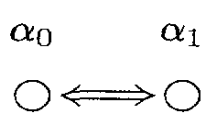
and hence we obtain $h^\vee = g$ by Lemma 3.16. □

Note. In general, if $\mathfrak{g}$ is an affine Lie algebra of twisted type $X_N^{(r)}$, then the dual Coxeter number of $\mathfrak{g}$ is not the same as that of the simple finite-dimensional Lie algebra $\mathfrak{g}(X_N)$. One should observe that the dual Coxeter number of a twisted affine Lie algebra is not the same as that of $\mathfrak{g}(\overset{\circ}{A})$. $\square$

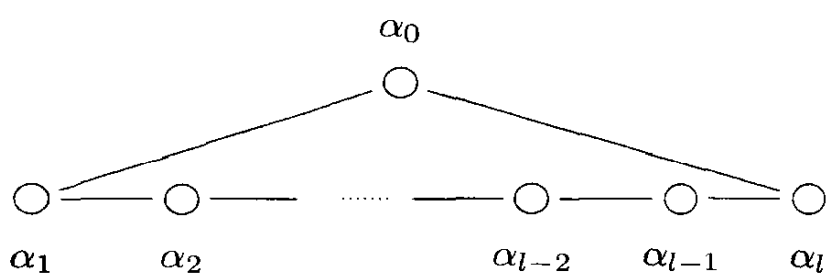
REMARK 3.23. Here, I would like to make a few remarks on the notations for affine Lie algebras. In this section, to denote an affine Lie algebra by $\mathfrak{g}$, we denoted its underlying finite-dimensional Lie algebra by $\overline{\mathfrak{g}}$. Instead, Some people denote a finite-dimensional simple Lie algebra by $\mathfrak{g}$, and the affine Lie algebra over it by $\widehat{\mathfrak{g}}$. This notation is frequently used. Those who prefer either style may denote the affine Lie algebra over $\mathfrak{sl}(n\mathbb{C})$ by $\widehat{\mathfrak{sl}}(n, \mathbb{C})$. Also, some people denote affine Lie algebras of split type, e.g., $A_l^{(1)}, B_l^{(1)}, \cdots$, by $\widehat{A}_l, \widehat{B}_l, \cdots$.

Let us show the Dynkin diagrams of affine Lie algebras in the following table. In this book, we follow the enumeration of the vertices of Dynkin diagrams by the next table:

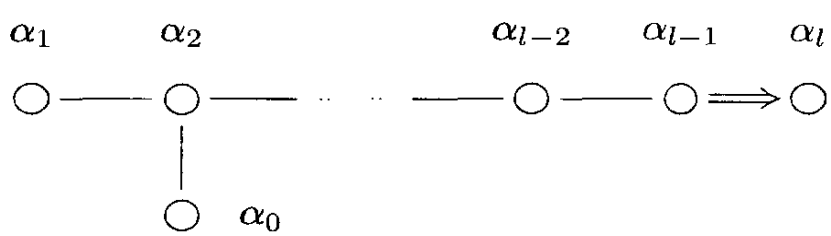
$A_1^{(1)}$



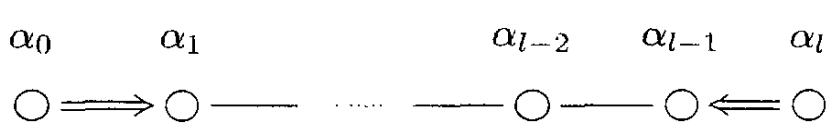
$A_l^{(1)}$ $(l \geq 2)$



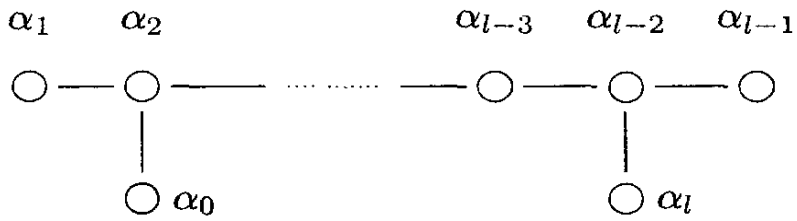
$B_l^{(1)}$ $(l \geq 3)$



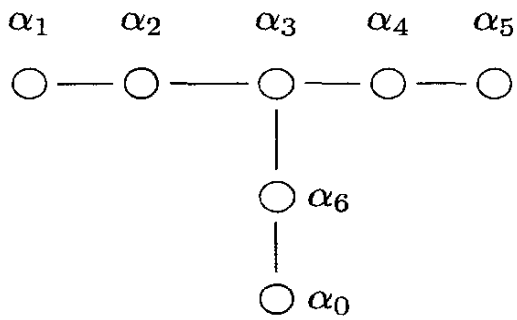
$C_l^{(1)}$ $(l \geq 2)$



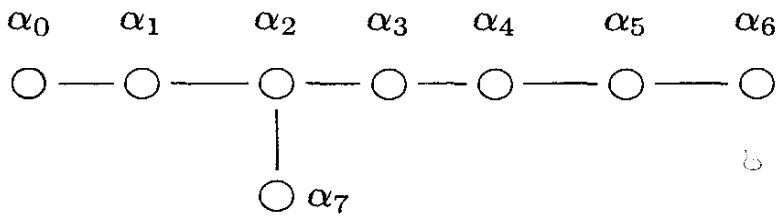
$D_l^{(1)}$ $(l \geq 4)$



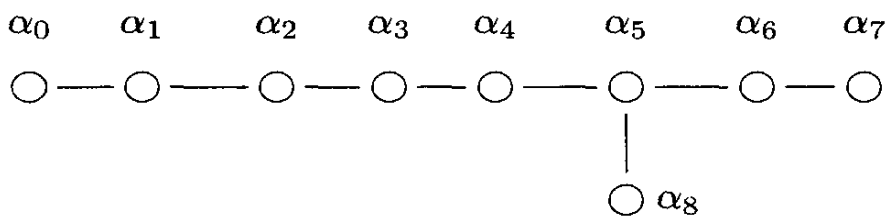
$E_6^{(1)}$



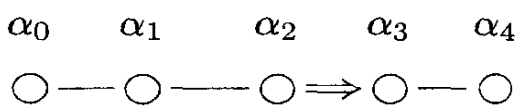
$E_7^{(1)}$



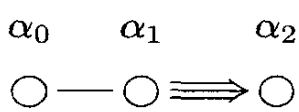
$E_8^{(1)}$



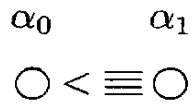
$F_4^{(1)}$



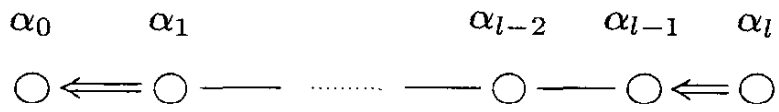
$G_2^{(1)}$



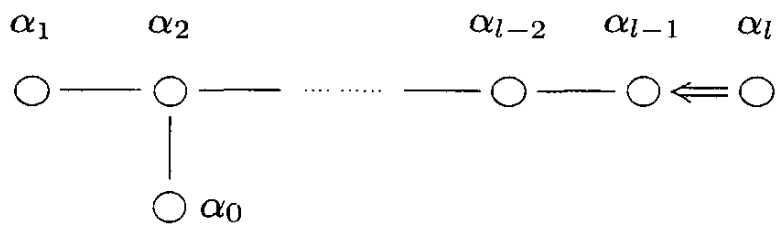
$A_2^{(2)}$



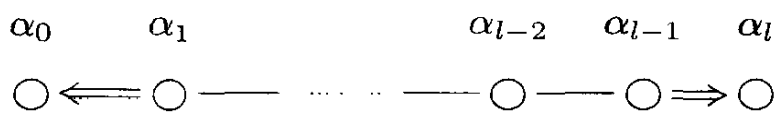
$A_{2l}^{(2)}$ $(l \geq 2)$



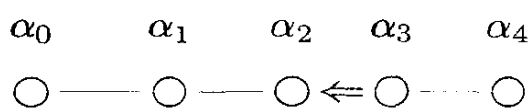
$A_{2l-1}^{(2)}$ $(l \geq 3)$



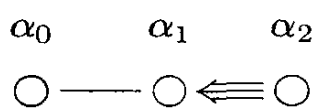
$D_{l+1}^{(2)}$ $(l \geq 2)$



$E_6^{(2)}$



$D_4^{(3)}$



3.4. Pairing and Contravariant Form on Verma Modules

Since the arguments in this section hold for general BKM algebras, we let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a BKM matrix, and $\mathfrak{g} = \mathfrak{g}(A)$ a BKM algebra. For Lie superalgebras, we need a modification due to the parity, and we would like to leave the arguments for this case to the reader as exercises. We cannot generalize the positivity of the Hermitian inner product, stated at the end of this section, to Lie superalgebras.

Let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition, and set

$$\mathfrak{n}_+ := \sum_{\alpha > 0} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- := \sum_{\alpha < 0} \mathfrak{g}_\alpha.$$

Then, by (3) of Theorem 1.16, we have

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}_- \mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g}) \mathfrak{n}_+). \tag{3.61}$$

We denote the image of $x \in \mathfrak{U}(\mathfrak{g})$ under the projection $\mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{h})$ with respect to this decomposition by $\langle x \rangle \in \mathfrak{U}(\mathfrak{h})$, and call it the expectation value of x. Next, when an element Λ of $\mathfrak{h}^*$ is given, we denote the homomorphism from $\mathfrak{U}(\mathfrak{h})$ to $\mathbb{C}$ obtained by naturally extending linear map $\Lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ by

$$\mathfrak{U}(\mathfrak{h}) \ni u \longmapsto u_\Lambda \in \mathbb{C}. \tag{3.62}$$

Then, by considering the composition of two maps

$$\mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{h}) \longrightarrow \mathbb{C},$$

one may assign $\langle x \rangle_\Lambda$ for each $x \in \mathfrak{U}(\mathfrak{g})$ and $\Lambda \in \mathfrak{h}^*$. This complex number $\langle x \rangle_\Lambda$ is called the 'expectation value of x at Λ'.

Note. Since $\mathfrak{h}$ is a commutative Lie algebra, $\mathfrak{U}(\mathfrak{h})$ can be identified with $S(\mathfrak{h})$, i.e., the polynomial ring over $\mathfrak{h}^*$. Via this identification, the above mentioned map (3.62) is just the map that associate a polynomial u on $\mathfrak{h}^*$ to the value $u(\Lambda)$ at Λ, i.e., the 'evaluation map at the point Λ'. □

There exists a unique automorphism ω of $\mathfrak{U}(\mathfrak{g})$ (as an associative algebra) satisfying

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega|_{\mathfrak{h}} = -\mathrm{Id}_{\mathfrak{h}}. \tag{3.63}$$

In other words, since the transformation (3.63) on generators e_i, f_i and $\mathfrak{h}$ preserves the fundamental relations (2.10) – (2.13), it defines an automorphism of the Lie algebra $\mathfrak{g}$, and hence it lifts to an automorphism of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. As one can see immediately, ω enjoys the following properties:

$$\omega(\mathfrak{n}_+) = \mathfrak{n}_-, \quad \omega(\mathfrak{n}_-) = \mathfrak{n}_+, \tag{3.64}$$

$$\omega^2 = \mathrm{Id}_{\mathfrak{U}(\mathfrak{g})}. \tag{3.65}$$

ω is called the **Chevalley involution** of $\mathfrak{U}(\mathfrak{g})$. ω commutes with the transposition map ${}^t(\cdot)$ of $\mathfrak{U}(\mathfrak{g})$ defined in §1.1. Namely, for $x \in \mathfrak{U}(\mathfrak{g})$, we have

$${}^t(\omega(x)) = \omega({}^t x). \tag{3.66}$$

This is because both ${}^t(\cdot) \circ \omega$ and $\omega \circ {}^t(\cdot)$ are anti-automorphisms of $\mathfrak{U}(\mathfrak{g})$, and their actions coincide on generators. We denote the composition of ${}^t(\cdot)$ and ω by ${}^t\omega$.

LEMMA 3.24. *For $x \in \mathfrak{U}(\mathfrak{g})$ and $u \in \mathfrak{U}(\mathfrak{h})$, we have the following equalities on an expectation value:*

1. $\langle {}^t\omega(x) \rangle = \langle x \rangle$,

2. $\langle xu \rangle = \langle ux \rangle = u\langle x \rangle$.

PROOF. By the definition of the expectation value, we have

$$x - \langle x \rangle \in \mathfrak{n}_-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}_+,$$

and applying ω, we get

$$\omega(x) - \omega(\langle x \rangle) \in \mathfrak{n}_+\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}_-.$$

Moreover, if we apply the transposition map ${}^t(\cdot)$, we get

$${}^t(\omega(x)) - {}^t(\omega(\langle x \rangle)) \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_-\mathfrak{U}(\mathfrak{g}),$$

and this formula says that the expectation value of ${}^t\omega(x))$ is ${}^t(\omega(\langle x \rangle))$. Namely, we have

$$\langle {}^t(\omega(x)) \rangle = {}^t(\omega(\langle x \rangle)). \tag{3.67}$$

Let us see the right hand side of (3.67). For $u = H_1 \cdots H_k \in \mathfrak{U}(\mathfrak{h})$, we have

$$\omega(u) = (-H_1) \cdots (-H_k),$$

and applying the transposition map, we get

$$\begin{aligned} {}^t(\omega(u)) &= {}^t(-H_k) \cdots {}^t(-H_1) = H_k \cdots H_1 \\ &= H_1 \cdots H_k = u. \end{aligned}$$

Namely, the composition map ${}^t\omega$ of ω and the transposition map is the identity map on $\mathfrak{U}(\mathfrak{h})$. Therefore, the right hand side of (3.67) is $\langle x \rangle$, and (1) is proved. (2) easily follows from the definition of the expectation value and

$$\mathfrak{n}_\pm\mathfrak{U}(\mathfrak{h}) = \mathfrak{U}(\mathfrak{h})\mathfrak{n}_\pm.$$

□

For $\Lambda \in \mathfrak{h}^*$, we define the lowest weight Verma module $M'(-\Lambda)$ with lowest weight $-\Lambda$ as follows. Let $1'_{-\Lambda}$ be a symbol, and regard the one-dimensional vector space $\mathbb{C} \cdot 1'_{-\Lambda}$ as an $(\mathfrak{h} \oplus \mathfrak{n}_-)$-module by

$$\begin{aligned} H \cdot 1'_{-\Lambda} &:= -\Lambda(H) \cdot 1'_{-\Lambda} \quad (\forall H \in \mathfrak{h}), \\ X \cdot 1'_{-\Lambda} &:= 0 \qquad (\forall X \in \mathfrak{n}_-), \end{aligned}$$

and set

$$M'(-\Lambda) := \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{h}\oplus\mathfrak{n}_-}{\otimes} \mathbb{C} \cdot 1'_{-\Lambda}.$$

Here, $\otimes_{\mathfrak{h}\oplus\mathfrak{n}_-}$ expresses the equivalence classes of the following equivalence relation introduced on the ordinary tensor product:

$$xy \otimes v \sim x \otimes yv \quad (x \in \mathfrak{U}(\mathfrak{g}), y \in \mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_-), v \in \mathbb{C} \cdot 1'_{-\Lambda}).$$

$M'(-\Lambda)$ naturally becomes a $\mathfrak{U}(\mathfrak{g})$-module by the multiplication of elements of $\mathfrak{U}(\mathfrak{g})$ from the left; therefore it is a $\mathfrak{g}$-module. Then, the natural map

$$\mathfrak{U}(\mathfrak{n}_+) \ni x \longmapsto x \otimes 1'_{-\Lambda} \in \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{h}\oplus\mathfrak{n}_-}{\otimes} \mathbb{C} \cdot 1'_{-\Lambda} = M'(-\Lambda)$$

is a linear isomorphism, and an element of $M'(-\Lambda)$ can be uniquely expressed as $x \cdot 1'_{-\Lambda}$ $(x \in \mathfrak{U}(\mathfrak{n}_+))$. $M'(-\Lambda)$ is called the **lowest weight Verma module** with lowest weight $-\Lambda$. For $\Lambda \in \mathfrak{h}^*$, we define the pairing between $M(\Lambda)$ and $M'(-\Lambda)$,

$$M(\Lambda) \otimes M'(-\Lambda) \ni (u, v) \longmapsto \langle u, v \rangle \in \mathbb{C},$$

as follows: For $M(\Lambda) \times M'(-\Lambda) \ni (u, v)$, take $x, y \in \mathfrak{U}(\mathfrak{g})$ satisfying $u = x \cdot 1'_{-\Lambda}$ and $v = y \cdot 1'_{-\Lambda}$, and set

$$\langle u, v \rangle := \langle {}^t yx \rangle_\Lambda. \tag{3.68}$$

To see that this is well-defined, we have only to prove that for other $x', y' \in \mathfrak{U}(\mathfrak{g})$ satisfying $u = x' \cdot 1'_{-\Lambda}$ and $v = y' \cdot 1'_{-\Lambda}$, we have

$$\langle {}^t yx \rangle_\Lambda = \langle {}^t y'x' \rangle_\Lambda.$$

This can be easily checked as follows. Since we have, under the change of representatives of u and v,

$$\begin{aligned} x' - x &\in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{U}(\mathfrak{g}) \cdot \{H - \Lambda(H); H \in \mathfrak{h}\}, \\ y' - y &\in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_- + \mathfrak{U}(\mathfrak{g}) \cdot \{H + \Lambda(H); H \in \mathfrak{h}\}, \end{aligned}$$

we can take these elements as follows:

$$\begin{aligned} x' - x &= \sum_{i=1}^{r} z_i (H_i - \langle \Lambda, H_i \rangle) + \xi_+, \\ y' - y &= \sum_{j=1}^{s} z'_j (H'_j + \langle \Lambda, H'_j \rangle) + \xi_-. \end{aligned}$$

Here, we have $z_i, z'_j \in \mathfrak{U}(\mathfrak{g})$, $H_i, H'_j \in \mathfrak{h}$ and $\xi_\pm \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_\pm$. Then, since

$$\begin{aligned} {}^t y'x' - {}^t yx &= {}^t y'(x' - x) + {}^t(y' - y)x \\ &= \sum_{i=1}^{r} {}^t y' z_i (H_i - \langle \Lambda, H_i \rangle) + \sum_{j=1}^{s} (-H'_j + \langle \Lambda, H'_j \rangle) {}^t z'_j x \\ &\quad + \underbrace{{}^t y' \xi_+}_{\in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_+} + \underbrace{{}^t \xi_- x}_{\in \mathfrak{n}_- \mathfrak{U}(\mathfrak{g})}, \end{aligned}$$

it follows from the definition of the expectation value and (2) of Lemma 3.24 that

$$\begin{aligned} &\langle {}^t y'x' \rangle - \langle {}^t yx \rangle \\ &= \sum_{i=1}^{r} \langle {}^t y' z_i \rangle \cdot (H_i - \langle \Lambda, H_i \rangle) - \sum_{j=1}^{s} (H'_j - \langle \Lambda, H'_j) \cdot \langle {}^t z'_j x \rangle. \end{aligned}$$

Hence, in particular, for its value at Λ, each term in the right hand side becomes 0, and we get

$$\langle {}^t y'x' \rangle_\Lambda = \langle {}^t yx \rangle_\Lambda.$$

Thus the well-definedness is proved. It can be immediately shown by definition that this pairing satisfies

$$\langle xu, v \rangle = \langle u, {}^t xv \rangle \quad (u \in M(\Lambda), v \in M'(-\Lambda), x \in \mathfrak{U}(\mathfrak{g})). \tag{3.69}$$

In particular, if we choose x from the elements of $\mathfrak{g}$, we have

$$\langle Xu, v \rangle + \langle u, Xv \rangle = 0 \quad (u \in M(\Lambda), v \in M'(-\Lambda), X \in \mathfrak{g}). \tag{3.70}$$

Thus this pairing is $\mathfrak{g}$-invariant.

This pairing can be rewritten to a bilinear form on $M(\Lambda)$ as follows. For this purpose, let us remark that the linear isomorphism between $M(\Lambda)$ and $M'(-\Lambda)$ defined as follows is well-defined:

$$\varphi : M(\Lambda) \ni x \cdot 1_\Lambda \longmapsto \omega(x) \cdot 1'_{-\Lambda} \quad (x \in \mathfrak{U}(\mathfrak{g})).$$

Via this map φ, $M(\Lambda)$ and $M'(-\Lambda)$ can be naturally identified as vector spaces, but this is not an isomorphism of $\mathfrak{g}$-modules. As for the action of $\mathfrak{g}$, the following formula holds:

$$\varphi(Xu) = \omega(X)\varphi(u) \quad (u \in M(\Lambda),\ X \in \mathfrak{g}).$$

In any case, a pairing on $M(\Lambda)$ and $M'(-\Lambda)$ defines a bilinear form on $M(\Lambda)$ via this linear isomorphism. Let us use the same symbol for this pairing

$$M(\Lambda) \times M(\Lambda) \ni (u, v) \longmapsto \langle u, v \rangle \in \mathbb{C}.$$

When we express $u, v \in M(\Lambda)$ as $u = x \cdot 1_\Lambda, v = y \cdot 1_\Lambda$ $(x, y \in \mathfrak{U}(\mathfrak{g}))$, it follows that

$$\langle u, v \rangle = \langle {}^t\omega(y)x \rangle_\Lambda. \tag{3.71}$$

Since for $a \in \mathfrak{U}(\mathfrak{g})$ we have

$$\langle u, av \rangle = \langle {}^t\omega(ay)x \rangle_\Lambda = \langle {}^t\omega(y)\,{}^t\omega(a)x \rangle_\Lambda = \langle {}^t\omega(a)u, v \rangle,$$

in other words,

$$\langle au, v \rangle = \langle u, {}^t\omega(a)v \rangle \quad (a \in \mathfrak{U}(\mathfrak{g})). \tag{3.72}$$

In particular, if we choose a from $\mathfrak{g}$, the following equality holds:

$$\langle Xu, v \rangle + \langle u, \omega(X)v \rangle = 0 \quad (X \in \mathfrak{g}, u, v \in M(\Lambda)). \tag{3.73}$$

Moreover, by Lemma 3.24, this bilinear form is symmetric.

DEFINITION 3.25. The bilinear form $\langle\ ,\ \rangle$ on $M(\Lambda)$ obtained as above is called a **contravariant bilinear form** on $M(\Lambda)$.

Denote the weight space decomposition of $M(\Lambda)$ by

$$M(\Lambda) = \bigoplus_\lambda M(\Lambda)_\lambda.$$

Here, the condition on λ to be a weight is

$$\lambda \in \Lambda - \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i.$$

Let λ and μ be weights of $M(\Lambda)$, and take elements of weight spaces $u \in M(\Lambda)_\lambda$ and $v \in M(\Lambda)_\mu$. Then, for $H \in \mathfrak{h}$, we obtain the following relation from (3.73):

$$\begin{aligned} 0 &= \langle Hu, v \rangle - \langle u, Hv \rangle \\ &= \langle \lambda(H)u, v \rangle - \langle u, \mu(H)v \rangle \\ &= (\lambda - \mu)(H) \cdot \langle u, v \rangle. \end{aligned} \tag{3.74}$$

Since we can choose $H \in \mathfrak{h}$ satisfying $(\lambda - \mu)(H) \neq 0$ provided that $\lambda \neq \mu$, we obtain

$$\lambda \neq \mu \Longrightarrow \langle M(\Lambda)_\lambda, M(\Lambda)_\mu \rangle = \{0\}. \tag{3.75}$$

Now, let U be a proper invariant subspace of $M(\Lambda)$. Then, since

$$U \subset \bigoplus_{\lambda \neq \Lambda} M(\Lambda)_\lambda$$

by (2.99), by (3.75), the pairing of 1_Λ and U becomes

$$\langle U, 1_\Lambda \rangle = \{0\}.$$

Next, let us compute the pairing of $u \in U$ and each element of $M(\Lambda)$. If we take $v \in M(\Lambda)$ and $x \in \mathfrak{U}(\mathfrak{g})$ satisfying $v = x \cdot 1_\Lambda$, then it follows that

$$\langle u, v \rangle = \langle u, x \cdot 1_\Lambda \rangle = \langle \underbrace{{}^t\omega(x)u}_{\in U}, 1_\Lambda \rangle = 0,$$

and hence

$$\langle U, M(\Lambda) \rangle = \{0\}.$$

Conversely, by the $\mathfrak{g}$-invariance of this pairing, the radical of this pairing

$$\mathrm{Rad} := \{u \in M(\Lambda); \langle u, v \rangle = 0 \ (\forall v \in M(\Lambda))\}$$

is a proper invariant subspace of $M(\Lambda)$. Hence, if we let J_Λ be the maximal proper invariant subspace of $M(\Lambda)$, then a non-degenerate contravariant bilinear form is induced on the quotient space, i.e., on the irreducible representation $M(\Lambda)/J_\Lambda$. Let us summarize these facts as a proposition:

PROPOSITION 3.26. *For $\Lambda \in \mathfrak{h}^*$, the contravariant symmetric bilinear form $\langle u, v \rangle$ on $M(\Lambda)$ is well-defined by*

$$\langle x \cdot 1_\lambda, y \cdot 1_\Lambda \rangle := \langle {}^t\omega(y)x \rangle_\Lambda \quad (x, y \in \mathfrak{U}(\mathfrak{g})),$$

and this pairing enjoys the following properties:

1. $\langle au, v \rangle = \langle u, {}^t\omega(a)v \rangle \quad (\forall a \in \mathfrak{U}(\mathfrak{g}))$.
2. $\langle Xu, v \rangle + \langle u, \omega(X)v \rangle = 0 \qquad (\forall X \in \mathfrak{g})$.
3. *For $u \in M(\Lambda)_\lambda$ and $v \in M(\Lambda)_\mu$,*
$$\lambda \neq \mu \Longrightarrow \langle u, v \rangle = 0.$$
4. *If U is a proper invariant subspace of $M(\Lambda)$, then $\langle U, M(\Lambda) \rangle = \{0\}$.*
5. *This pairing induces a non-degenerate contravariant symmetric bilinear form on the irreducible representation $L(\Lambda)$.*

For $\lambda = \Lambda - \gamma$ ($\gamma \in \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$), we consider this inner product restricted to the weight space $M(\Lambda)_\lambda$. Choose one basis of $M(\Lambda)_\lambda$, and denote the matrix of the inner product of them by $\mathfrak{M}_\gamma(\Lambda)$, and its determinant by $D_\gamma(\Lambda)$:

$$D_\gamma(\Lambda) := \det \mathfrak{M}_\gamma(\Lambda).$$

Here, we can choose a basis of a weight space independently of Λ. This can be done as follows. For each $\gamma \in \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i$, we set

$$\mathfrak{U}(\mathfrak{n}_-)_{-\gamma} := \{x \in \mathfrak{U}(\mathfrak{n}_-); [H, x] = -\gamma(H)x \ (\forall H \in \mathfrak{h})\},$$

and choose and fix one of their bases, and make a basis of $M(\Lambda)_{\Lambda-\gamma}$ by applying them to 1_Λ. Here, since there is a freedom in a choice of a basis of $\mathfrak{U}(\mathfrak{n}_-)$, the determinant $D_\gamma(\Lambda)$ has a freedom of the multiplication by a (non-zero) scalar; but since we have fixed a basis of $\mathfrak{U}(\mathfrak{n}_-)$, $D_\gamma(\Lambda)$ is determined as a function of Λ up to multiplication by a (non-zero) scalar. The determinant $D_\gamma(\Lambda)$ (= a function of Λ) is called the **Shapovalov determinant**. This was obtained by Shapovalov [**Sh**] when $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, and extended to the case of symmetrizable Kac-Moody algebras by Kac and Kazhdan [**KK**] as follows:

PROPOSITION 3.27. *If $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra, $D_\gamma(\Lambda)$ ($\gamma \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i$) looks as follows:*

$$D_\gamma(\Lambda) = \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left((\Lambda + \rho|\alpha) - \frac{n(\alpha|\alpha)}{2} \right)^{K(\gamma - n\alpha)\cdot \mathrm{mult}(\alpha)}.$$

*Here, $K(\alpha)$ ($\alpha \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i$) is **Kostant's partition function** defined as follows:*

$$\frac{1}{R} = \sum_{\alpha \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i} K(\alpha) e^{-\alpha}.$$

For its proof, see the above original articles or §§2.8, 2.9 and 6.6 in [**MP**].

Note. As can be seen from the above argument, the matrix $\mathfrak{M}_\gamma(\Lambda)$ contains information on the maximal proper invariant subspace of $M(\Lambda)$. In fact, we have

$$\dim L(\Lambda)_{\Lambda-\gamma} = \mathrm{rank}\ \mathfrak{M}_\gamma(\Lambda),$$

and hence the character of the irreducible representation can be expressed as

$$\mathrm{ch}_\Lambda = e^\Lambda \sum_{\gamma \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i} \mathrm{rank}\ \mathfrak{M}_\gamma(\Lambda) e^{-\gamma}.$$

But, as we see from the simple example

$$\det \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \det \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix} = x^2,$$

since we cannot see the rank of a matrix even if we know its determinant, it is, in general, a difficult problem to derive detailed information on the rank of the matrix $\mathfrak{M}_\gamma(\Lambda)$ from the Shapovalov determinant $D_\gamma(\Lambda)$. □

As for an Hermitian inner product on $M(\Lambda)$, we can develop a similar consideration as above. In general, an **Hermitian inner product** on a $\mathbb{C}$-vector space V is an $\mathbb{R}$-bilinear map

$$\langle\ ,\ \rangle_0 : V \times V \ni (u, v) \mapsto \langle u, v\rangle_0 \in \mathbb{C}$$

satisfying the following two conditions:
(i) $\langle u, v\rangle_0$ is $\mathbb{C}$-linear in u, and $\mathbb{C}$-conjugate linear in v.
(ii) $\langle u, v\rangle_0 = \overline{\langle v, u\rangle}_0$.
Here, $\mathbb{C}$-conjugate linear (or in other words $\overline{\mathbb{C}}$-linear) in v means it satisfies

$$\langle u, v + v'\rangle_0 = \langle u, v\rangle_0 + \langle u, v'\rangle_0,$$
$$\langle u, av\rangle_0 = \overline{a}\langle u, v\rangle_0$$

for all $u, v, v' \in V$ and $a \in \mathbb{C}$. An Hermitian inner product is said to be positive definite if it satisfies:
(iii) For all $u \in V$, it follows that $\langle u, u\rangle_0 \geq 0$. Here, the equality holds if and only if $u = 0$.

If we define a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$ as the Note just after (2.9), then there uniquely exists a $\overline{\mathbb{C}}$-linear automorphism ω_0 satisfying

$$\omega_0(e_i) = -f_i, \quad \omega_0(f_i) = -e_i, \quad \omega_0|_{\mathfrak{h}_\mathbb{R}} = -\mathrm{Id}_{\mathfrak{h}_\mathbb{R}}. \tag{3.76}$$

As in the case of ω, ω_0 commutes with the transposition map ${}^t(\cdot)$, and satisfies the following properties:

$$\omega_0(\mathfrak{n}_+) = \mathfrak{n}_-, \quad \omega_0(\mathfrak{n}_-) = \mathfrak{n}_+, \tag{3.77}$$

$$\omega_0^2 = \mathrm{Id}_{\mathfrak{U}(\mathfrak{g})}. \tag{3.78}$$

ω_0 is called the **Cartan involution** of $\mathfrak{U}(\mathfrak{g})$ (or $\mathfrak{g}$).
Note. ω_0 transfers $\mathfrak{g}$ to $\mathfrak{g}$, and its eigenspace with eigenvalue 1

$$\mathfrak{g}_\mathbb{R} = \{X \in \mathfrak{g}; \omega_0(X) = X\}$$

is a Lie algebra over $\mathbb{R}$, and satisfies $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_\mathbb{R}$. $\mathfrak{g}_\mathbb{R}$ is called a **compact real form** of $\mathfrak{g}$. □.

If we choose $\Lambda \in \mathfrak{h}^*$ satisfying $\langle \Lambda, h\rangle \in \mathbb{R}$ $(\forall h \in \mathfrak{h})$, an Hermitian inner product $\langle\ ,\ \rangle_0$ on $M(\Lambda)$ defined by

$$\langle x \cdot 1_\lambda, y \cdot 1_\Lambda\rangle_0 := \langle {}^t\omega_0(y)x\rangle_\Lambda \tag{3.79}$$

is well-defined. It can be shown similarly to (3.72) that this inner product satisfies

$$\langle xu, v\rangle_0 = \langle u, {}^t\omega_0(x)v\rangle_0 \quad (x \in \mathfrak{U}(\mathfrak{g}), u, v \in M(\Lambda)), \tag{3.80}$$

and hence it satisfies

$$\langle Xu, v\rangle_0 + \langle u, \omega_0(X)v\rangle_0 = 0 \quad (X \in \mathfrak{g}, u, v \in M(\Lambda)). \tag{3.81}$$

Therefore, by the same argument as for a symmetric inner product given above, it induces a non-degenerate Hermitian inner product on the irreducible representation space $L(\Lambda)$. By using the Casimir operator, one can investigate when this inner product becomes positive definite. The proof of the next proposition is given in §11.7 of [**K4**]:

PROPOSITION 3.28 (cf. Theorem 11.7 in [**K4**]). *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. This Hermitian inner product $\langle\ ,\ \rangle_0$ on $L(\Lambda)$ is positive definite if and only if $\Lambda \in P_+$.*

Note. By (3.81), it follows that

$$\langle Xu, v\rangle_0 + \langle u, Xv\rangle_0 = 0$$

for $X \in \mathfrak{g}_{\mathbb{R}}$. Namely, this Hermitian inner product is $\mathfrak{g}_{\mathbb{R}}$-invariant, and $L(\Lambda)$ ($\Lambda \in P_+$) is a 'unitary representation' of $\mathfrak{g}_{\mathbb{R}}$. So, when $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra, $L(\Lambda)$ ($\Lambda \in P_+$) is called a **unitarizable representation** of $\mathfrak{g}$. □

3.5. Weyl Group of Affine Lie Algebras and Characters of Irreducible Representations

In this section, we let $A = (a_{ij})_{0\le i,j\le l}$ be a GCM of affine type, and we study its Weyl group.

DEFINITION 3.29. Let i_0 be an integer satisfying $0 \le i_0 \le l$. i_0 is called a **special index** if a proper scalar multiple of $\delta - a_{i_0}\alpha_{i_0}$ is a root. In this book, we denote the set of special indices by J.

Below in this section, we fix a special index i_0. In the above definition, we allow 'a scalar multiple' in order to treat $i_0 := l$ for $A_{2l}^{(2)}$ as a special index. But here in this section, for simplicity, we assume that $\delta - a_{i_0}\alpha_{i_0}$ itself is a root, and we denote it by θ_{i_0}. If its scalar multiple is a root, we have only to do some simple modification of the arguments given below. First, observe that since

$$|\theta_{i_0}|^2 = a_{i_0}^2|\alpha_{i_0}|^2,$$

it follows that θ_{i_0} is a real root, and hence the reflection $r_{\theta_{i_0}}$ with respect to θ_{i_0} is an element of W.

The $l \times l$ matrix $A^{(i_0)}$ obtained by deleting the i_0-th column and the i_0-th row from A is a Cartan matrix of finite type, and the Weyl group of $A^{(i_0)}$,

$$W^{(i_0)} := \langle r_i; i \neq i_0 \rangle,$$

is a finite group. And W is the group obtained by adding r_{i_0} to $W^{(i_0)}$. Namely, we have

$$W := \langle W^{(i_0)}, r_{i_0} \rangle. \tag{3.82}$$

Let us consider θ_{i_0}. θ_{i_0} is a root of the affine Lie algebra $\mathfrak{g}(A)$, and, as can be immediately seen from

$$\theta_{i_0} = \delta - a_{i_0}\alpha_{i_0} = \sum_{i \neq i_0} a_i \alpha_i, \tag{3.83}$$

its support lies in that obtained by deleting α_{i_0} from the Dynkin diagram, and hence is a root of the simple finite-dimensional Lie algebra $\mathfrak{g}(A^{(i_0)})$. Therefore, the reflection $r_{\theta_{i_0}}$ with respect to θ_{i_0} is an element of $W^{(i_0)}$:

$$r_{\theta_{i_0}} \in W^{(i_0)}. \tag{3.84}$$

By (3.82) and (3.84), one can rearrange generators of W as follows:

$$W = \langle W^{(i_0)}, r_{i_0} r_{\theta_{i_0}} \rangle. \tag{3.85}$$

So, let us compute the action of $r_{i_0} r_{\theta_{i_0}}$ on $\mathfrak{h}^*$ concretely. By computing the composition of reflections naïvely, we obtain the following formula for $\lambda \in \mathfrak{h}^*$:

$$r_{i_0} r_{\theta_{i_0}}(\lambda) = \lambda + (\lambda|\delta)\theta_{i_0}^\vee - \left\{ \frac{|\theta_{i_0}^\vee|^2}{2}(\lambda|\delta) + (\lambda|\theta_{i_0}^\vee) \right\} \delta. \tag{3.86}$$

Here, we set

$$\theta_{i_0}^\vee := \frac{2\theta_{i_0}}{(\theta_{i_0}|\theta_{i_0})}. \tag{3.87}$$

To obtain (3.86), the following method may simplify its computation. First, we compute it mod $\mathbb{C}\delta$, and derive

$$r_{i_0} r_{\theta_{i_0}}(\lambda) \equiv \lambda + (\lambda|\delta)\theta_{i_0}^\vee \qquad \mathrm{mod}\ \mathbb{C}\delta.$$

Next, we compute the coefficient of δ by using

$$|r_{i_0} r_{\theta_{i_0}}(\lambda)|^2 = |\lambda|^2.$$

In any case, (3.86) can be obtained by a simple computation.

According to (3.85), we have only to add $r_{i_0}r_{\theta_{i_0}}$ to $W^{(i_0)}$ to obtain W, and $r_{i_0}r_{\theta_{i_0}}$ is the linear transformation on $\mathfrak{h}$ whose concrete action is given by (3.86). Now, by replacing $\theta_{i_0}^\vee$ in (3.86) with a general element α of $\sum_{i\neq i_0}\mathbb{C}\alpha_i$, we introduce the following linear transformation t_α:

$$t_\alpha\lambda := \lambda + (\lambda|\delta)\alpha - \left\{\frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha)\right\}\delta \quad (\lambda \in \mathfrak{h}^*). \tag{3.88}$$

Then, it can be easily checked (by computation) that the t_α's ($\alpha \in \sum_{i\neq i_0}\mathbb{C}\alpha_i$) satisfy the following properties:

$$(t_\alpha\lambda|t_\alpha\mu) = (\lambda|\mu), \tag{3.89}$$

$$t_\alpha t_\beta = t_{\alpha+\beta}, \quad t_0 = \mathrm{Id}_{\mathfrak{h}^*}, \tag{3.90}$$

$$t_{w\alpha} = wt_\alpha w^{-1} \quad (w \in W^{(i_0)}, \alpha \in \sum_{i\neq i_0}\mathbb{C}\alpha_i). \tag{3.91}$$

Using this symbol, (3.85) can be rephrased as follows:

$$W = \langle W^{(i_0)}, t_{\theta_{i_0}^\vee}\rangle, \tag{3.92}$$

and by (3.91) we have

$$t_{w\theta_{i_0}^\vee} \in W \quad (\forall w \in W^{(i_0)}).$$

Namely,

$$\alpha \in W^{(i_0)}\cdot\theta_{i_0}^\vee \Longrightarrow t_\alpha \in W. \tag{3.93}$$

Moreover, by using (3.90), one can take α from the $\mathbb{Z}$-lattice spanned over $W^{(i_0)}\cdot\theta_{i_0}^\vee$:

$$M^{(i_0)} := \sum_{\alpha\in W^{(i_0)}\cdot\theta_{i_0}^\vee}\mathbb{Z}\alpha. \tag{3.94}$$

By (3.90) and (3.93), it follows that

$$\alpha \in M^{(i_0)} \Longrightarrow t_\alpha \in W. \tag{3.95}$$

Now, let us set

$$T^{(i_0)} := \{t_\alpha; \alpha \in M^{(i_0)}\}.$$

By (3.90) and (3.95), it follows that $T^{(i_0)}$ is a subgroup of W, and $W^{(i_0)}$ normalizes $T^{(i_0)}$ by (3.91). Thus

$$W^{(i_0)}\cdot T^{(i_0)} := \{wt; w \in W^{(i_0)}, t \in T^{(i_0)}\}$$

is a subgroup of W, and since this subgroup contains a system of generators of W by (3.92), it coincides with W. Namely, we have

$$W = W^{(i_0)} \cdot T^{(i_0)}.$$

Since $W^{(i_0)}$ normalizes $T^{(i_0)}$ and

$$W^{(i_0)} \cap T^{(i_0)} = \{e\}, \tag{3.96}$$

this product is a semi direct product. Here (3.96) follows from the fact that $T^{(i_0)}$ is a free abelian group by (3.90) and $W^{(i_0)}$ is a finite group. Namely, any positive power of an element of $T^{(i_0)}$ (except for the identity element e) cannot be e, but each element of $W^{(i_0)}$ can be e by an appropriate power. Thus, we obtain the next lemma:

LEMMA 3.30. *For a special index i_0, we have*

$$W = W^{i_0} \ltimes T^{(i_0)}.$$

Note. In general, the direct product of two groups G_1 and G_2 is denoted by $G_1 \times G_2$. Then G_1 and G_2 are both normal subgroups of $G_1 \times G_2$. In the above lemma, $T^{(i_0)}$ is a normal subgroup of W, but $W^{(i_0)}$ is not normal but 'simply' a subgroup. Namely, W is a semi direct product of $W^{(i_0)}$ and $T^{(i_0)}$, and $\ltimes$ is a symbol signifying the relation of being a normal subgroup. □

By the above lemma, any element w of W can be uniquely decomposed as

$$w = \overline{w} t_\alpha \quad (\overline{w} \in W^{(i_0)}, \alpha \in M^{(i_0)}),$$

and since this formula can be rewritten as $w = t_{\overline{w}\alpha}\overline{w}$ by (3.91), we can write this decomposition as

$$w = t_\beta \overline{w} \quad (\overline{w} \in W^{(i_0)}, \beta \in M^{(i_0)}).$$

Below, we take $i_0 := 0$ as a special index i_0, and then denote $W^{(i_0)}$, $M^{(i_0)}$,$T^{(i_0)}$ and θ_{i_0} by $\overset{\circ}{W}, M, T, \theta$ respectively. In this case, $\theta^\vee$ and its $\overset{\circ}{W}$-orbit can be computed concretely, and M has the following expression:

LEMMA 3.31. *For $A = X_N^{(r)}$,*

1. *if $r = 1$ or $A = A_{2l}^{(2)}$, then $M = \overset{\circ}{Q}^\vee = \sum_{i=1}^l \mathbb{Z}\alpha_i$, and*
2. *if $r \geq 2$ and $A \neq A_{2l}^{(2)}$, then $M = \overset{\circ}{Q} = \sum_{i=1}^l \mathbb{Z}\alpha_i$.*

Note. If we consider an extension of the Weyl group

$$\widetilde{W} := \overset{\circ}{W} \ltimes \overline{P},$$

then $\widetilde{W}$ keeps the set of roots Δ invariant. Now, if we set

$$\widetilde{W}_+ := \{w \in \widetilde{W}; w(\Delta_+) = \Delta_+\},$$

then we have

$$\widetilde{W}_+ \cdot (\{\alpha_i; i \in J\}) = \{\alpha_i; i \in J\},$$

and the action of $\widetilde{W}_+$ on $\{\alpha_i; i \in J\}$ is simply transitive. Namely, for any two elements i, j of J, there exists a unique element w of $\widetilde{W}_+$ satisfying $w(\alpha_i) = \alpha_j$. Thus in particular, it follows that

$$|\widetilde{W}_+| = |J|.$$

Since (by definition) $\widetilde{W}_+$ keeps the set of positive roots Δ_+ invariant, it transfers the set of fundamental roots Π to Π. Therefore, an element of $\widetilde{W}_+$ induces an automorphism of the Dynkin diagram. □

Now, let us study the action of an affine Weyl group. For this purpose, we consider the next fundamental domain $\mathfrak{C}^\vee$ and its interior $\overset{\circ}{\mathfrak{C}}{}^\vee$ defined as follows:

$$\mathfrak{C}^\vee := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}}; \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (\forall i)\},$$

$$\overset{\circ}{\mathfrak{C}}{}^\vee := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}}; \langle \lambda, \alpha_i^\vee \rangle > 0 \ (\forall i)\}.$$

Note. Observe that $\mathfrak{C}^\vee$ and $\overset{\circ}{\mathfrak{C}}{}^\vee$ are not the same as $C^\vee$ and $\overset{\circ}{C}{}^\vee$ defined in §2.3. Even if we call it a 'fundamental domain', we consider it the convenient one depending on a situation. □

Can we transfer any element of $\mathfrak{h}^*_{\mathbb{R}}$ into $\mathfrak{C}^\vee$ by an action of W? For a finite-dimensional Lie algebra, this is possible, but for an affine Lie algebra, this is not necessarily possible. Since W fixes δ, one cannot transfer λ satisfying $(\lambda|\delta) < 0$ into $\mathfrak{C}^\vee$. Then, how about $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ satisfying $(m :=)\ (\lambda|\delta) > 0$? In this case, it is probably possible[1], but we need a slightly tedious argument to prove it by the same way as in the proof of Lemma 3.32. Since $|W| = \infty$ for an affine Lie algebra, it is not obvious whether there exists w attaining the

[1] *Translator's note.* It is, in fact, possible to choose such $w \in W$. See, e.g. Proposition 5.8 in [**K4**].

minimum of $\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2$. Since the bilinear form is positive semi-definite, this value is always ≥ 0, but we should check the possibility that it accumulates to their infimum.

If λ is an integral weight, then by multiplying by an integer (independent of w), we may assume that the $\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2$ take their value in integers, and the argument becomes simple. Even if we weaken the integrality conditions on a weight, we can still apply this method and obtain the following lemma:

LEMMA 3.32. *Suppose $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ satisfies the following two conditions:*

1. $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Q}$ $(\forall i)$.
2. $(\lambda|\delta) = m > 0$.

Then there exists an element w of W satisfying $w\lambda \in \mathfrak{C}^\vee$. If λ is a regular element, then such a w is unique.

PROOF. For each element w of W, let us consider $\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2$. The reason why we consider $w\lambda - \frac{m}{h^\vee}\rho$ is that the level of this weight is 0, and hence even if we add a scalar multiple of δ to λ, its value does not change. Then, for all $w \in W$, we have

$$\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2 = \left|\overline{w\lambda} - \frac{m}{h^\vee}\overline{\rho}\right|^2 \geq 0.$$

Now, by assumption, if we choose a positive integer N satisfying

$$N\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \quad (\forall i),$$

then since $w\lambda - \lambda \in \frac{1}{N}Q$, one can choose a positive integer N' satisfying

$$\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2 \in \frac{1}{N'} \cdot \mathbb{Z}_{\geq 0} \quad (\forall w \in W).$$

So, there exists the minimum of

$$\left\{\left|w\lambda - \frac{m}{h^\vee}\rho\right|^2 ; w \in W\right\}.$$

If we take a w attaining the minimum, and set $\mu := w\lambda$, then it follows that $\mu \in \mathfrak{C}^\vee$. In fact, by the minimality of μ, the following

inequalities hold for all i:

$$|r_i\mu|^2 - \frac{2m}{h^\vee}(r_i\mu|\rho) + \frac{m^2}{h^{\vee 2}}|\rho|^2 = \left|r_i\mu - \frac{m}{h^\vee}\rho\right|^2$$
$$\geq \left|\mu - \frac{m}{h^\vee}\rho\right|^2 = |\mu|^2 - \frac{2m}{h^\vee}(\mu|\rho) + \frac{m^2}{h^{\vee 2}}|\rho|^2.$$

Since $m > 0$, it follows that

$$\begin{array}{c}(r_i\mu|\rho) \leq (\mu|\rho)\\ \|\\ (\mu|\rho) - \langle\mu, \alpha_i^\vee\rangle(\alpha_i|\rho),\end{array}$$

and hence we have

$$\langle\mu, \alpha_i^\vee\rangle(\alpha_i|\rho) \geq 0.$$

Since $(\alpha_i|\rho) > 0$, we obtain $\langle\mu, \alpha_i^\vee\rangle \geq 0$.

To prove the latter half of the lemma, we only have to prove that if $w\lambda \in \mathring{\mathfrak{C}}^\vee$ for an element λ of $\mathring{\mathfrak{C}}^\vee$ and an element w of W, then the only possible w is $w = e$. To prove this, we observe that the set of positive roots Δ_+ can be characterized by an (arbitrarily chosen and fixed) element λ of $\mathring{\mathfrak{C}}^\vee$ as follows:

$$\Delta_+ := \{\alpha \in \Delta; \langle\lambda, \alpha\rangle > 0\}.$$

If both λ and $w\lambda$ belong to $\mathring{\mathfrak{C}}^\vee$, then it follows that $w(\Delta_+) = \Delta_+$, since $\langle\ ,\ \rangle$ is W-invariant, and by Corollary 3.4 and the Note after it, we obtain $w = e$. $\square$

Now, if we set

$$\mathring{\mathfrak{h}} := \sum_{i=1}^{l} \mathbb{C}\alpha_i^\vee \quad \text{and} \quad \mathring{\mathfrak{h}}^* := \sum_{i=1}^{l} \mathbb{C}\alpha_i,$$

it follows that

$$\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\delta;$$

we denote the $\mathring{\mathfrak{h}}^*$-component of $\lambda \in \mathfrak{h}^{ast}$ with respect to this decomposition by $\overline{\lambda}$. Moreover, by identifying $\mathfrak{h}$ and $\mathfrak{h}^*$, we frequently express $h \in \mathfrak{h}$ as

$$h = 2\pi i(-\tau\Lambda_0 + z + t\delta) \quad (\tau, t \in \mathbb{C}, z \in \mathring{\mathfrak{h}}).$$

We may also express this element simply by

$$h = (\tau, z, t).$$

In §3.1, we defined the transposed weight for a dominant integral weight, and here we define the **transposed weight** ${}^t\Lambda \in P_+^m$ of an integral weight $\Lambda = m\Lambda_0 + \overline{\Lambda} + b\delta \in P_+^m$ ($m \in \mathbb{Z}_{\geq 0}$) of an affine Lie algebra as follows:

$$(3.97) \qquad {}^t\Lambda := m\Lambda_0 + {}^t(\overline{\Lambda}) + b\delta = m\Lambda_0 - w_0\overline{\Lambda} + b\delta.$$

Here, w_0 is the **longest element** of $\mathring{W}$ (the element satisfying $w_0(\mathring{\Delta}_+) = \mathring{\Delta}_-$). Notice that

$$(3.98) \qquad {}^t\rho = \rho$$

because $w_0\overline{\rho} = -\overline{\rho}$.

We first prepare the following formula:

LEMMA 3.33. *For* $\lambda = a\Lambda_0 + \overline{\lambda} + b\delta \in \mathfrak{h}^*, \alpha \in M, w \in \mathring{W}$,

$$(t_\alpha w(a\Lambda_0 - \overline{\lambda} + b\delta)|(\tau, z, t)) = (t_{-\alpha} w\lambda|(\tau, -z, t)).$$

PROOF. Let us first prove the formula for $w = e$:

$$(3.99) \qquad (t_\alpha(a\Lambda_0 - \overline{\lambda} + b\delta)|(\tau, z, t)) = (t_{-\alpha}\lambda|(\tau, -z, t)).$$

Since we have

$$t_\alpha(a\Lambda_0 - \overline{\lambda} + b\delta) = a\Lambda_0 + a\alpha - \overline{\lambda} - \left\{\frac{a}{2}|\alpha|^2 - (\overline{\lambda}|\alpha)\right\}\delta + b\delta,$$

$$t_{-\alpha}\lambda = a\Lambda_0 - a\alpha + \overline{\lambda} - \left\{\frac{a}{2}|\alpha|^2 - (\overline{\lambda}|\alpha)\right\}\delta + b\delta,$$

both sides of (3.99) become

$$\left(a\Lambda_0 + \left\{b - \frac{a}{2}|\alpha|^2 - (\overline{\lambda}|\alpha)\right\}\delta\Big|(\tau, 0, t)\right) + 2\pi i(a\alpha - \overline{\lambda}|z),$$

and we obtain (3.99). The lemma is obtained by replacing λ in (3.99) by $w\lambda$. □

By Lemma 3.33 and the definition of ${}^t\Lambda$, we obtain the next lemma:

LEMMA 3.34. *For* $\lambda \in P_+, \alpha \in M, w \in \mathring{W}$,

$$(t_\alpha w({}^t\lambda)|(\tau, z, t)) = (t_{-\alpha} w w_0(\lambda)|(\tau, -z, t)).$$

By using this, we obtain the following proposition:

PROPOSITION 3.35. *For* $\Lambda \in P_+$,

$$\mathrm{ch}_{t\Lambda}(\tau, z, t) = \mathrm{ch}_{\Lambda}(\tau, -z, t).$$

PROOF. It follows from the character formula that

$$\mathrm{ch}_{t\Lambda}(\tau, z, t) = \frac{\sum_{w\in W} \varepsilon(w) e^{(w({}^t\Lambda+\rho)|(\tau,z,t))}}{\sum_{w\in W} \varepsilon(w) e^{(w\rho|(\tau,z,t))}}.$$

Here, both in the denominator and in the numerator, decomposing w as $w = t_\alpha \overline{w}$ ($\alpha \in M, \overline{w} \in \overset{\circ}{W}$), and applying Lemma 3.34 together with (3.98), we obtain the proposition. □

3.6. The Jacobi Triple Product Identity

We will derive the Jacobi triple product identity as one of the important applications of the denominator identity (Corollary 2.77) for affine Lie algebras. This is an equality obtained by computing two sides of the denominator identity for $A_1^{(1)}$. First, let us see the root system of $A_1^{(1)}$ to compute the left hand side. Since $A_1^{(1)}$ is the central extension of the loop algebra of $\mathfrak{sl}(2, \mathbb{C})$, its root system can be described by using $\delta = \alpha_0 + \alpha_1$ as follows:

$$\Delta^{re} = \{n\delta \pm \alpha_1; n \in \mathbb{Z}\}, \quad \Delta^{im} = \{n\delta; n \in \mathbb{Z} \setminus \{0\}\}.$$

In terms of fundamental roots α_0 and α_1, the set Δ_+ of all positive roots looks like

$$\Delta_+ = \{n(\alpha_0 + \alpha_1), (n-1)\alpha_0 + n\alpha_1, n\alpha_0 + (n-1)\alpha_1; n \in \mathbb{N}\},$$

and the multiplicity of each root is 1. So, putting $e^{-\alpha_0} =: u$ and $e^{-\alpha_1} =: v$, we have

(3.100)

$$R = \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha}) = \prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^{n-1} v^n)(1 - u^n v^{n-1}).$$

Next, let us compute the right hand side of the denominator identity (Corollary 2.77). The standard bilinear form of $A_1^{(1)}$ is given by

$$(\alpha_0|\alpha_0) = (\alpha_1|\alpha_1) = 2, \quad (\alpha_0|\alpha_1) = -2,$$

and $M = \mathbb{Z}\alpha_1$. And its Weyl group W is

$$W = \{t_{n\alpha_1}, t_{n\alpha_1} r_1; n \in \mathbb{Z}\}.$$

Let us compute the action of the Weyl group on $\rho = \Lambda_0 + \Lambda_1 = 2\Lambda_0 + \frac{1}{2}\alpha_1$. We have

$$r_1\rho = 2\Lambda_0 - \frac{1}{2}\alpha_1,$$

and since

$$\begin{aligned} t_{n\alpha_1}\Lambda_0 &= \Lambda_0 + n\alpha_1 - n^2\delta, \\ t_{n\alpha_1}\alpha_1 &= \alpha_1 - 2n\delta, \end{aligned}$$

it follows that

$$\begin{aligned} t_{n\alpha_1}\rho &= 2(\Lambda_0 + n\alpha_1 - n^2\delta) + \left(\frac{1}{2}\alpha_1 - n\delta\right) \\ &= \rho + 2n\alpha_1 - (2n\ 2 + n)\delta, \\ t_{n\alpha_1}r_1\rho &= 2(\Lambda_0 + n\alpha_1 - n^2\delta) - \left(\frac{1}{2}\alpha_1 - n\delta\right) \\ &= \rho + (2n-1)\alpha_1 - (2n^2 - n)\delta, \end{aligned}$$

and the right hand side (r.h.s. for short) of the denominator identity (d.i. for short) becomes

$$\begin{aligned} &\text{the r.h.s. of the d.i.} \\ &= \sum_{w\in W} \varepsilon(w)e^{w\rho-\rho} = \sum_{n\in\mathbb{Z}} e^{t_{n\alpha_1}\rho-\rho} - \sum_{n\in\mathbb{Z}} e^{t_{n\alpha_1}r_1\rho-\rho} \\ &= \sum_{n\in\mathbb{Z}} e^{2n\alpha_1}e^{-(2n^2+n)\delta} - \sum_{n\in\mathbb{Z}} e^{(2n-1)\alpha_1}e^{-(2n^2-n)\delta}. \end{aligned} \tag{3.101}$$

Since in the right hand side of this formula the coefficients of δ on e in the first and the second term can be written as

$$\begin{aligned} 2n^2 + n &= \frac{(2n)^2 + (2n)}{2}, \\ 2n^2 - n &= \frac{(2n-1)^2 + (2n-1)}{2}, \end{aligned}$$

the right hand side of (3.101) is

$$\text{the r.h.s of the d.i.} = \sum_{n\in\mathbb{Z}} (-1)^n e^{n\alpha_1} e^{-\frac{n\ 2+n}{2}\delta},$$

and this can be written in terms of u and v as follows:

$$\begin{aligned}\text{the r.h.s. of the d.i.} &= \sum_{n\in\mathbb{Z}}(-1)^n v^{-n}(uv)^{\frac{n^2+n}{2}} \\ &= \sum_{n\in\mathbb{Z}}(-1)^n u^{\frac{n^2+n}{2}} v^{\frac{n^2-n}{2}}.\end{aligned}$$

Thus, we obtain the following formula:

LEMMA 3.36 (The Jacobi triple product identity).

(3.102)
$$\prod_{n=1}^{\infty}(1-u^n v^n)(1-u^{n-1}v^n)(1-u^n v^{n-1}) = \sum_{n\in\mathbb{Z}}(-1)^n u^{\frac{n^2+n}{2}} v^{\frac{n^2-n}{2}}.$$

There is no longer a form of the 'roots' in this formula, This is just an algebraic equality. The Jacobi triple product identity which is important for the theory of automorphic functions can be thus obtained as a denominator identity of an affine Lie algebra. If we set $u = zq$ and $v = z^{-1}q$ in (3.102), we get

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-z^{-1}q^{2n-1})(1-zq^{2n-1}) = \sum_{n\in\mathbb{Z}}(-1)^n z^n q^{n^2}. \tag{3.103}$$

The Jacobi triple product identity is often written in this form.

Let us fiddle with the triple product identity. For example, if we put $u = q^2$ and $v = q$, the two sides of (3.102) looks like

$$\begin{aligned}\text{the l.h.s.} &= \prod_{n=1}^{\infty}(1-q^{3n})(1-q^{3n-2})(1-q^{3n-1}) = \prod_{n=1}^{\infty}(1-q^n), \\ \text{the r.h.s.} &= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2+n} q^{\frac{n^2-n}{2}} = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}}.\end{aligned}$$

By this formula, we obtain the formal series expansion of the **Euler function**

$$\varphi(q) := \prod_{n=1}^{\infty}(1-q^n) \tag{3.104}$$

and of the **Dedekind eta function**

$$\eta(\tau) := q^{\frac{1}{24}}\varphi(q) \quad (q = e^{2\pi i\tau}) \tag{3.105}$$

as follows:

$$\varphi(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}}, \tag{3.106}$$

$$\eta(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}\left(n+\frac{1}{6}\right)^2}. \tag{3.107}$$

Moreover, if we put $z = 1$ in (3.103), we get

$$\frac{\varphi(q)^2}{\varphi(q^2)} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}. \tag{3.108}$$

This is one of the formulae called the **Gauss formulae** and is one of the important formulae in the theory of automorphic functions.

Next, we differentiate (3.103) with respect to z, and set $z = q$. We obtain

$$\varphi(q^2)^3 = \sum_{n\in\mathbb{Z}} nq^{n^2+n}. \tag{3.109}$$

Separating the right hand side of this formula into $\sum_{n\geq 0}$ and $\sum_{n<0}$, we replace n by $-m-1$ for $\sum_{n<0}$. Also replacing q^2 by q, (3.109) can be rephrased as follows:

$$\varphi(q)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n^2+n}{2}}. \tag{3.110}$$

Although the reader might feel it like a 'distortion', the quantities appearing in the right hand side of this formula have the following meanings. First, its hometown is the affine Lie algebra $A_1^{(1)}$, and it was born from the affinization of $\mathfrak{sl}(2,\mathbb{C})$. "3" of the cubic power in the left hand side coincides with the dimension of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. $2n+1$ in the right hand side is the dimension of an irreducible representation (or, to be precise, $(-1)^n(2n+1)$ is the special value of a character on an irreducible representation space), and the number put on q is the value of the Casimir operator of $\mathfrak{sl}(2,\mathbb{C})$ on the $(2n+1)$-dimensional irreducible representation.

Note. To derive (3.110) directly from (3.102), we first divide both sides of (3.102) by $1-v$, and then set $u = q$ and $v \to 1$. This operation is just the 'specialization'

$$e^{-\alpha_0} \longrightarrow q, \quad e^{-\alpha_1} \longrightarrow 1$$

in the denominator identity. □

For a general affine Lie algebra, by specializing its denominator identity by such a method, namely by setting

$$e^{-\alpha_0} \longrightarrow q, e^{-\alpha_i} \longrightarrow 1 \quad (1 \leq i \leq l),$$

we obtain a formal series expansion of $\varphi(q)^{\dim \mathfrak{g}}$ described in terms of quantities of the Lie algebra $\mathfrak{g}$. This is one of the beautiful facts discovered in representation theory in the 1970's. Every formula derived from the denominator identity is called a 'Macdonald identity'.

CHAPTER 4

Modular Transformations of Characters of Affine Lie Algebras

One of the important properties of the character of an integrable representation of an affine Lie algebra is its modular invariance. This property does not exist for a finite-dimensional Lie algebra and is one of the features of infinite-dimensional Lie algebras. The modular invariance of the characters is significant not only because it is interesting in its own, but because of the existence of the modular invariance. Even when we restrict ourselves to representation theory, it becomes a very effective method to study problems such as 'Regarding a representation of an affine Lie algebra as a representation of its Lie subalgebra, how does it decompose?' or 'How does the tensor product of representations decompose?' But, to explain them, we need the representation theory of the "Virasoro algebra", and unfortunately, there is no place to explain it in this book. So, some topics concerning the Virasoro algebra will be just briefly reviewed in Chapter 6.

In this sense, what is going to be described in this chapter is just a preface to modular transformations of characters of an affine Lie algebra. As a preparation, let us start from theories of theta functions.

4.1. Classical Theta Functions

Topics in theta functions (in particular, relating them to theories of Lie algebras) are explained in §§13.1 – 13.5 of [**K4**] in detail, and this section is just a précis. See, e.g., [**K4**], for details such as proofs. The symbols used here are the same as those for affine Lie algebras; for example, we denote a vector space by $\mathfrak{h}$, and we also use δ or t_α ($\alpha \in M$), etc. This is because the purpose of this chapter is to consider the modular transformations of characters by directly applying the story developed here to an affine Lie algebra. But, as one

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can see from the set-up in this section, topics treated in this section hold, independently of an affine Lie algebra, for a general integral lattice with Lorentzian type inner product.

Let l be a positive integer, and let $\mathfrak{h}_{\mathbb{R}}$ be an $(l+2)$-dimensional real vector space. Let $(\ |\ )$ be a non-degenerate symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}$, and suppose that its index is $(l+1,1)$. Here, the index of a bilinear form signifies the pair (the number of positive eigenvalues, the number of negative eigenvalues).

Assume that M is a lattice in $\mathfrak{h}_{\mathbb{R}}$ of rank l, and that the bilinear form $(\ |\ )$ takes its value in integers on M and is positive definite. We choose M in this way, and fix it. In addition, if M satisfies the condition

$$\alpha \in M \Longrightarrow (\alpha|\alpha) \in 2\mathbb{Z},$$

M is called an "even lattice"; otherwise it is called an "odd lattice". Take $\delta \in \mathfrak{h}_{\mathbb{R}}$ so that it satisfies

$$(\delta|\delta) = (\delta|\alpha) = 0 \quad (\forall \alpha \in M),$$

and, associated to this element δ, we define $\Lambda_0 \in \mathfrak{h}_{\mathbb{R}}$ by

$$(\Lambda_0|\delta) = 1, \quad (\Lambda_0|\Lambda_0) = (\Lambda_0|\alpha) = 0 \quad (\forall \alpha \in M). \tag{4.1}$$

Let us denote the complexification of $\mathfrak{h}_{\mathbb{R}}$ by $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$, and the subspaces of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}$ spanned over M by $\mathring{\mathfrak{h}}_{\mathbb{R}}$ and $\mathring{\mathfrak{h}}$ respectively. Moreover, we denote the dual lattice of M in $\mathring{\mathfrak{h}}_{\mathbb{R}}$ by M^*:

$$M^* := \{\alpha \in \mathring{\mathfrak{h}}_{\mathbb{R}}; (\alpha|\beta) \in \mathbb{Z}\ (\forall \beta \in M)\}.$$

Since the symmetric bilinear form on $\mathfrak{h}$ that is an extension of $(\ |\ )$ is non-degenerate, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$, or $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$, via this bilinear form. We have

$$\mathfrak{h} = \mathbb{C}\Lambda_0 \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}\delta, \tag{4.2}$$

and we denote an element h of $\mathfrak{h}$ according to this direct sum decomposition

$$h = 2\pi i(-\tau\Lambda_0 + z + t\delta) \quad (\tau, t \in \mathbb{Z}, z \in \mathring{\mathfrak{h}}) \tag{4.3}$$

simply by

$$h = (\tau, z, t).$$

We also denote the image of $\lambda \in \mathfrak{h}$ under the projection $\mathfrak{h} \longrightarrow \mathring{\mathfrak{h}}$ with respect to the direct sum decomposition (4.3) by $\overline{\lambda}$ $(\in \mathring{\mathfrak{h}})$. Namely,

for $h = 2\pi i(-\tau\Lambda_0 + z + t\delta)$, we set $\overline{\lambda} := 2\pi i z$. (The reason why we denote a general element of $\mathfrak{h}$ by h or λ is that we are identifying $\mathfrak{h}$ and $\mathfrak{h}^*$.)

Now, by using the coordinate of $\mathfrak{h}$ introduced above, we define the subset Y of $\mathfrak{h}$ as follows:

$$Y := \{(\tau, z, t); \mathrm{Im}\tau > 0\}.$$

And, for $\alpha \in \overset{\circ}{\mathfrak{h}}_{\mathbb{R}}$, we define $t_\alpha, p_\alpha \in GL(\mathfrak{h})$ by the following formulae:

$$t_\alpha(\lambda) := \lambda + (\lambda|\delta)\alpha - \left\{\frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha)\right\}\delta, \tag{4.4}$$

$$p_\alpha(\lambda) := \lambda + 2\pi i\alpha \quad (\forall\lambda \in \mathfrak{h}). \tag{4.5}$$

Then, it is easy to see that

$$t_\alpha \circ t_\beta = t_{\alpha+\beta}, \quad p_\alpha \circ p_\beta = p_{\alpha+\beta},$$

for all $\alpha, \beta \in \overset{\circ}{\mathfrak{h}}_{\mathbb{R}}$. Moreover, t_α keeps the bilinear form. Next, if we set

$$N := \overset{\circ}{\mathfrak{h}}_{\mathbb{R}} \times \overset{\circ}{\mathfrak{h}}_{\mathbb{R}} \times i\mathbb{R},$$

N becomes a group by the following product:

(4.6)

$$(\alpha, \beta, u) \cdot (\alpha', \beta', u') := (\alpha + \alpha', \beta + \beta', u + u' + \pi i\{(\alpha|\beta') - (\alpha'|\beta)\}).$$

If we define the action of N on $\mathfrak{h}$ by

$$(\alpha, \beta, u) \cdot \lambda := t_\beta\lambda + 2\pi i\alpha + \{u - \pi i(\alpha|\beta)\}\delta, \tag{4.7}$$

then Y is N-invariant. Let us consider the subgroup

$$N_{\mathbb{Z}} := \{(\alpha, \beta, u) \in N; \alpha, \beta \in M, u + \pi i(\alpha|\beta) \in 2\pi i\mathbb{Z}\}$$

of N. Let $O(Y)$ be the vector space formed by all holomorphic functions on Y, and define the action of N and $N_{\mathbb{Z}}$ on $O(Y)$ by the following formula:

$$F|_{(\alpha,\beta,u)}(\lambda) := F((\alpha, \beta, u) \cdot \lambda). \tag{4.8}$$

Then, if we set

$$\widetilde{Th}_m := \left\{F \in O(Y); \begin{array}{l}\text{(i) } F|_{(\alpha,\beta,u)} = F \quad (\forall(\alpha, \beta, u) \in N_{\mathbb{Z}}), \\ \text{(ii) } F(\lambda + a\delta) = e^{ma}F(\lambda) \quad (\forall a \in \mathbb{C})\end{array}\right\}$$

for each $m \in \mathbb{Z}_{\geq 0}$ and

$$\widetilde{Th} := \bigoplus_{m=0}^{\infty} \widetilde{Th}_m,$$

then $\widetilde{Th}$ becomes a graded associative algebra by the ordinary product. An element of $\widetilde{Th}_m$ is called a **theta function** of degree m. Here, the condition (i) can be replaced by the condition

(i′) $\quad F(p_\alpha\lambda) = F(t_\alpha\lambda) = F(\lambda) \quad (\forall\alpha \in M, \forall\lambda \in Y)$

under the condition (ii). Namely, we only have to require the invariance under $(\alpha, 0, 0)$ and $(0, \alpha, 0)$ among elements of $N_{\mathbb{Z}}$.

For a positive integer m, set

$$P_m := \{\lambda \in \mathfrak{h}; (\lambda|\delta) = m, \overline{\lambda} \in M^*\},$$

and for each $\lambda \in P_m$,

$$\Theta_\lambda := e^{-\frac{(\lambda|\lambda)}{2m}\delta} \sum_{\alpha\in M} e^{t_\alpha\lambda} \tag{4.9}$$

is called a **classical theta function** of degree m. By a simple computation, one obtains

$$\Theta_\lambda(\tau, z, t) = e^{2\pi imt} \sum_{\gamma\in M+\frac{\overline{\lambda}}{m}} q^{\frac{m}{2}|\gamma|^2} e^{2\pi im(\gamma|z)}. \tag{4.10}$$

Here, as usual, we set $q = e^{2\pi i\tau}$. This formula can be rewritten as follows:

$$\Theta_\lambda := e^{m\Lambda_0} \sum_{\gamma\in M+\frac{\overline{\lambda}}{m}} q^{\frac{m}{2}|\gamma|^2} e^{m\gamma}. \tag{4.11}$$

Since by the definition (4.9) of Θ_λ we have

$$\Theta_{\lambda+a\delta} = \Theta_\lambda \tag{4.12}$$

for all $a \in \mathbb{C}$, Θ_λ can be regarded as is defined for $\lambda \mod \mathbb{C}\delta$. Moreover, since the inner product $(\ |\ )$ is invariant with respect to the action of t_α, it follows that

$$\Theta_{\lambda+m\alpha} = \Theta_\lambda \quad (\forall\alpha \in M), \tag{4.13}$$

and hence Θ_λ is defined for $\lambda \mod (mM + \mathbb{C}\delta)$. So, for each $m \in \mathbb{N}$, the set of all classical theta functions of degree m

$$\{\Theta_\lambda; \lambda \in P_m \mod (mM + \mathbb{C}\delta)\}$$

is a finite set.

One has $\Theta_\lambda \in \widetilde{Th}_m$, but let us consider the vector subspace of $\widetilde{Th}_m$ spanned by Θ_λ's. For this purpose, let us introduce the following

differential operator D. First, let $u_1, \cdots, u_l$ be an orthonormal basis of $\mathring{\mathfrak{h}}_{\mathbb{R}}$, and introduce a coordinate on Y by this basis:

$$Y \ni 2\pi i\left(-\tau\Lambda_0 + \sum_{i=1}^{l} z_i u_i + t\delta\right) \longleftrightarrow (\tau, z_1, \cdots, z_l, t). \tag{4.14}$$

Set

$$D := \frac{1}{4\pi^2}\left\{2\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \tau}\right) - \sum_{i=1}^{l}\left(\frac{\partial}{\partial z_i}\right)^2\right\}. \tag{4.15}$$

If $F \in \widetilde{Th}_m$, it follows from the condition (ii) that

$$\frac{\partial F}{\partial t} = 2\pi i m F,$$

and the action of D on $\widetilde{Th}_m$ is, in practice, a linear combination of the first order derivative of τ and the second order derivatives of z_i's. So regarding τ as the time variable, this is a heat (or a diffusion) differential operator. And for each $\lambda \in P_m$, it follows that

$$D\Theta_\lambda = 0.$$

Now, let us set

$$Th_m := \{F \in \widetilde{Th}_m; DF = 0\}.$$

LEMMA 4.1 (Proposition 13.3 in [**K4**]). *For a positive integer m,*

$$\{\Theta_\lambda; \lambda \in P_m \mod (mM + \mathbb{C}\delta)\}$$

is a basis of the vector space Th_m.

The proof of this lemma is not so difficult. We compute the relations among the coefficients of the Fourier series expansion of $F \in Th_m$ by using the condition (i). By using $DF = 0$, we see that each of its coefficients can be expressed as a linear combination of Fourier coefficients of Θ_λ's.

Next, let $O(\mathbb{C}_+)$ be the set of holomorphic functions on the upper half plane $\mathbb{C}_+ := \{\tau \in \mathbb{C}; \mathrm{Im}\tau > 0\}$, and consider the double cover of $SL_2(\mathbb{Z})$

$$Mp_2(\mathbb{Z}) := \left\{(A, j); \begin{array}{c} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad j \in O(\mathbb{C}_+), \\ j(\tau)^2 = c\tau + d. \end{array}\right\},$$

and define the action of $Mp_2(\mathbb{Z})$ on Y and on $N_{\mathbb{Z}}$ as follows: For $(A,j) \in Mp_2(\mathbb{Z})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(\tau, z, t) \in Y$ and $(\alpha, \beta, u) \in N_{\mathbb{Z}}$, set

$$(A,j)\cdot(\tau,z,t) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, t - \frac{c|z|^2}{2(c\tau+d)}\right), \tag{4.16}$$

$$(A,j)\cdot(\alpha,\beta,u)\cdot(A,j)^{-1} := (a\alpha+b\beta, c\alpha+d\beta, u). \tag{4.17}$$

Define the right action of $Mp_2(\mathbb{Z})$ on $O(Y)$ by

$$F|_{(A,j)}(\tau,z,t) := \frac{1}{j(\tau)^l} F((A,j)\cdot(\tau,z,t)). \tag{4.18}$$

From this formula and (4.8), it follows that $N_{\mathbb{Z}}$ and $Mp_2(\mathbb{Z})$ act on $O(Y)$. $Mp_2(\mathbb{Z})$ acts on $N_{\mathbb{Z}}$ by (4.17), and the compatibility of these actions, i.e., the following formula, can be checked by a simple computation:

$$F|_{(A,j)}|_{(\alpha,\beta,u)}|_{(A,j)^{-1}} = F|_{(A,j)\cdot(\alpha,\beta,u)\cdot(A,j)^{-1}}. \tag{4.19}$$

Let us consider a subgroup of $SL_2(\mathbb{Z})$,

$$\Gamma_\theta := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); ac \text{ and } bc \text{ are both even} \right\}.$$

As is well-known, $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$, and S and T^2 generate Γ_θ. (For these facts, see e.g., [**Ser**].)

LEMMA 4.2 (Proposition 13.4 in [**K4**]). 1. *If M is an even lattice or m is even, then the vector space Th_m is invariant under the action of $Mp_2(\mathbb{Z})$.*
2. *If M is an odd lattice and m is odd, then Th_m is invariant under the action of $\{(A,j) \in Mp_2(\mathbb{Z}); A \in \Gamma_\theta\}$.*

The action of T can be easily computed by using (4.10):

(4.20)

$$\begin{aligned}
&\Theta_\lambda(\tau+1,z,t)\\
&= e^{2\pi imt}\sum_{\gamma\in M+\frac{\overline{\lambda}}{m}} e^{\pi im|\gamma|^2(\tau+1)}e^{2\pi im(\gamma|z)}\\
&= e^{2\pi imt}\sum_{\gamma\in M} e^{\pi im\left|\gamma+\frac{\overline{\lambda}}{m}\right|^2} q^{\frac{m}{2}\left|\gamma+\frac{\overline{\lambda}}{m}\right|^2} e^{2\pi im\left(\gamma+\frac{\overline{\lambda}}{m}\middle|z\right)}\\
&= e^{\frac{\pi i|\overline{\lambda}|^2}{m}} e^{2\pi imt}\sum_{\gamma\in M} e^{\pi im|\gamma|^2}\underbrace{e^{2\pi im(\gamma|\overline{\lambda})}}_{1} q^{\frac{m}{2}\left|\gamma+\frac{\overline{\lambda}}{m}\right|^2} e^{2\pi im\left(\gamma+\frac{\overline{\lambda}}{m}\middle|z\right)}\\
&= e^{\frac{\pi i|\overline{\lambda}|^2}{m}} e^{2\pi imt}\sum_{\gamma\in M} e^{\pi im|\gamma|^2} q^{\frac{m}{2}\left|\gamma+\frac{\overline{\lambda}}{m}\right|^2} e^{2\pi im\left(\gamma+\frac{\overline{\lambda}}{m}\middle|z\right)}.
\end{aligned}$$

In the right hand side of this formula, $|\gamma|^2$ is an integer, and if M is an even lattice or m is even, then it follows that $e^{\pi im|\gamma|^2}=1$, and the right hand side becomes

$$e^{\frac{\pi i|\overline{\lambda}|^2}{m}}\Theta_\lambda(\tau,z,t).$$

If M is an odd lattice and m is odd, then since $e^{\pi im|\gamma|^2}(=\pm 1)$ depends upon γ, it does not have such a form, but if we apply T once more, then $\Theta_\lambda(\tau+2,z,t)$ becomes a scalar multiple of $\Theta_\lambda(\tau,z,t)$. The reason why the conclusion of Lemma 4.2 depends on whether M is an even lattice or m is even, or M is an odd lattice and m is odd, is that the action of T differs in a way explained above.

In any case, Th_m is invariant under the action of (S,j). The following theorem describes how to transform the basis Θ_λ of Th_m under the action of (S,j):

THEOREM 4.3 (Theorem 13.5 in [**K4**]).

$$\Theta_\lambda|_{(S,j)} = (-i)^{\frac{l}{2}}|M^*/mM|^{-\frac{1}{2}}\sum_{\mu\in P_m \bmod (mM+\mathbb{C}\delta)} e^{-\frac{2\pi i(\overline{\lambda}|\overline{\mu})}{m}}\Theta_\mu.$$

By using the coordinate (τ, z, t), this formula can be written as follows:

(4.21)

$$\Theta_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right) = (-i)^{\frac{l}{2}} |M^*/mM|^{-\frac{1}{2}} \times \sum_{\mu \in P_m \bmod (mM + \mathbb{C}\delta)} e^{-\frac{2\pi i(\overline{\lambda}|\overline{\mu})}{m}} \Theta_\mu(\tau, z, t).$$

Replacing $\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right)$ by (τ, z, t), this formula becomes

(4.22)

$$\Theta_\lambda(\tau, z, t) = \left(\frac{i}{\tau}\right)^{\frac{l}{2}} |M^*/mM|^{-\frac{1}{2}} \times \sum_{\mu \in P_m \bmod (mM + \mathbb{C}\delta)} e^{-\frac{2\pi i(\overline{\lambda}|\overline{\mu})}{m}} \Theta_\mu\left(-\frac{1}{\tau}, -\frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right).$$

The action of T can be obtained immediately from (4.20):

THEOREM 4.4. *If M is an even lattice or m is even, then*

$$\Theta_\lambda(\tau + 1, z, t) = e^{\frac{\pi i|\overline{\lambda}|^2}{m}} \Theta_\lambda(\tau, z, t).$$

4.2. Jacobi's Theta Function – Modular Transformations and Asymptotic Behavior

In the previous section, the phrase 'Lie algebra' did not appear, and the theory was developed only in terms of a vector space with Lorentzian metric, and a positive definite lattice in it. Here and after, we apply the theory in the previous section to a Cartan subalgebra of an affine Lie algebra and the (co)root lattice of the underlying simple finite-dimensional Lie algebra. In this section, as a warmup, let us compute the Jacobi theta functions in the case of the simplest affine Lie algebra $A_1^{(1)}$, and let us see their properties.

If we let $\mathfrak{h}$ a Cartan subalgebra of $A_1^{(1)}$, then it follows that

$$\mathfrak{h} = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta,$$

and the pairing $(\ |\ )$ is given by

$$(\alpha_1|\alpha_1) = 2, \quad (\Lambda_0|\delta) = 1,$$

and this pairing gives 0 otherwise. If we set $M = \mathbb{Z}\alpha_1$, then we can choose $m\Lambda_0 + \frac{j}{2}\alpha_1$ $(j \in \mathbb{Z})$ as representatives of $P_m \bmod \mathbb{C}\delta$. Let us

denote Θ_λ for $\lambda = m\Lambda_0 + \frac{j}{2}\alpha_1$ by $\Theta_{j,m}$. Then, for $\alpha = k\alpha_1 \in \mathbb{Z}\alpha_1$, since we have

$$(4.23) \qquad t_\alpha\lambda = \lambda + \underbrace{m\alpha}_{mk\alpha_1} - \left(\underbrace{\frac{|\alpha|^2}{2}}_{k^2} m + (\alpha|\lambda)\right)\delta$$
$$= m\Lambda_0 + \left(\frac{j}{2} + km\right)\alpha_1 - (mk^2 + jk)\delta,$$

it follows that

$$(4.24) \qquad \Theta_{j,m} = e^{-\frac{j^2}{4m}\delta} \sum_{k\in\mathbb{Z}} e^{m\Lambda_0 + \left(\frac{j}{2}+km\right)\alpha_1 - (mk^2+jk)\delta}.$$

Therefore, its value at $(\tau, z, t) := 2\pi i(-\tau\Lambda_0 + z\frac{\alpha_1}{2} + t\delta)$ is

$$(4.25) \qquad \Theta_{j,m}(\tau, z, t) = q^{\frac{j^2}{4m}} e^{2\pi i t m} \sum_{k\in\mathbb{Z}} q^{mk^2+jk} e^{\pi i z(j+2km)}$$
$$= e^{2\pi i t m} \sum_{k\in\mathbb{Z}} q^{m\left(k+\frac{j}{2m}\right)^2} e^{2\pi i m z\left(k+\frac{j}{2m}\right)}$$
$$= e^{2\pi i t m} \sum_{k\in\frac{j}{2m}+\mathbb{Z}} q^{mk^2} e^{2\pi i m k z}.$$

Since Θ_λ is defined for $\lambda \mod (mM + \mathbb{C}\delta)$, for fixed m, $\Theta_{j,m}$ is also defined for $j \mod 2m\mathbb{Z}$, and this is obvious from (4.25).

Now, let us compute how $\Theta_{j,m}$'s transform under the action of $SL_2(\mathbb{Z})$. First, the action of T can be immediately obtained from (4.25):

(4.26)
$$\Theta_{j,m}(\tau+1, z, t) = e^{2\pi i t m} \sum_{k\in\frac{j}{2m}+\mathbb{Z}} \underbrace{e^{2\pi i m k^2}}_{=e^{2\pi i m\left(\frac{j}{2m}\right)^2} = e^{\frac{\pi i j^2}{2m}}} q^{mk^2} e^{2\pi i m k z}$$
$$= e^{\frac{\pi i j^2}{2m}} \Theta_{j,m}(\tau, z, t).$$

Next, the action of S can be computed by using Theorem 4.3 as follows. For $A_1^{(1)}$, since $M^* = \frac{1}{2}\mathbb{Z}\alpha_1$, it follows that $|M^*/M| = 2$.

Thus, by (4.21), we have

$$\begin{aligned}(4.27)\qquad &\Theta_{j,m}\left(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{|z\alpha_1/2|^2}{2\tau}\right)\\ &=\frac{(-i\tau)^{\frac12}}{\sqrt{2m}}\sum_{k\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{2\pi i\left(\frac{j}{2}\alpha_1\middle|\frac{k}{2}\alpha_1\right)}{m}}\Theta_{k,m}(\tau,z,t)\\ &=\frac{(-i\tau)^{\frac12}}{\sqrt{2m}}\sum_{k\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi ijk}{m}}\Theta_{k,m}(\tau,z,t).\end{aligned}$$

We define $\theta_{j,m}$ by specializing $\Theta_{j,m}$ at $t=0$ and regarding it as a function of (τ,z) only:

$$\begin{aligned}(4.28)\qquad \theta_{j,m}(\tau,z)&:=\Theta_{j,m}(\tau,z,0)\\ &=\sum_{k\in\frac{j}{2m}+\mathbb{Z}} q^{mk^2}e^{2\pi imkz}\quad (m\in\mathbb{N},j\in\mathbb{Z}/2m\mathbb{Z}).\end{aligned}$$

$\theta_{j,m}$ is the **Jacobi theta function**, and its properties are explained in textbooks on automorphic functions (e.g., see Mumford [**Mum**]). Since we have

$$(4.29)\qquad \Theta_{j,m}(\tau,z,t)=e^{2\pi imt}\theta_{j,m}(\tau,z)$$

by this definition and (4.25), we can rephrase (4.27) in terms of Jacobi theta functions as follows:

$$e^{2\pi im\left(t-\frac{z^2}{4\tau}\right)}\theta_{j,m}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\frac{(-i\tau)^{\frac12}}{\sqrt{2m}}\sum_{k\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi ijk}{m}}e^{2\pi imt}\theta_{k,m}(\tau,z).$$

Thus, formulae on transformations of $\theta_{j,m}$ are obtained:

PROPOSITION 4.5.

1. $\theta_{j,m}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\frac{(-i\tau)^{\frac12}}{\sqrt{2m}}e^{\frac{\pi imz^2}{2\tau}}\sum_{k\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi ijk}{m}}\theta_{k,m}(\tau,z)$,
2. $\theta_{j,m}(\tau+1,z,t)=e^{\frac{\pi ij^2}{2m}}\theta_{j,m}(\tau,z)$.

In particular, if we set $z=0$, and put

$$f_{j,m}(\tau):=\theta_{j,m}(\tau,0)=\sum_{k\in\frac{j}{2m}+\mathbb{Z}} q^{mk^2}=\sum_{k\in\mathbb{Z}} q^{m\left(k+\frac{j}{2m}\right)^2},$$

it follows that

$$f_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{k\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi i j k}{m}} f_{k,m}(\tau). \tag{4.30}$$

Here, we defined $f_{j,m}$ for $m \in \mathbb{N}$ and $j \in \mathbb{Z}$, but let us extend these m and j to half integers. Namely, for $m \in \frac{1}{2}\mathbb{N}, j \in \frac{1}{2}\mathbb{Z}$, let us consider the functions $f_{j,m}$ and $g_{j,m}$ as follows:

$$f_{j,m}(\tau) := \sum_{k\in\mathbb{Z}} q^{m\left(k+\frac{j}{2m}\right)^2}, \tag{4.31}$$

$$g_{j,m}(\tau) := \sum_{k\in\mathbb{Z}} (-1)^k q^{m\left(k+\frac{j}{2m}\right)^2}. \tag{4.32}$$

By definition, it can be easily checked that these $f_{j,m}$ and $g_{j,m}$ satisfy the following relations:

$$f_{j,m} = f_{-j,m} = f_{j+2m,m}, \tag{4.33}$$

$$g_{j,m} = g_{-j,m} = -g_{j+2m,m}. \tag{4.34}$$

Thus, in particular, the $g_{j,m}$'s enjoy the following properties:

$$g_{j,m} = -g_{2m-j,m}, \tag{4.35}$$

$$g_{m,m} = 0. \tag{4.36}$$

Now, for $m \in \frac{1}{2}\mathbb{N}$, by dividing the summation $\sum_{k\in\mathbb{Z}}$ in the right hand side of the definitions (4.31) and (4.32) of $f_{j,m}$ and $g_{j,m}$ respectively into $\sum_{k=\text{even}} + \sum_{k=\text{odd}}$, these definitions can be rewritten in the following form:

$$\begin{aligned} f_{j,m}(\tau) &= \sum_{k\in\mathbb{Z}} q^{m\left(2k+\frac{j}{2m}\right)^2} + \sum_{k\in\mathbb{Z}} q^{m\left(2k-1+\frac{j}{2m}\right)^2} \\ &= \sum_{k\in\mathbb{Z}} q^{4m\left(k+\frac{j}{4m}\right)^2} + \sum_{k\in\mathbb{Z}} q^{4m\left(k+\frac{j-2m}{4m}\right)^2} \\ &= f_{2j,4m}(\tau) + f_{2j-4m,4m}(\tau) \\ &= f_{2j,4m}(\tau) + f_{4m-2j,4m}(\tau). \end{aligned} \tag{4.37}$$

Similarly, for $g_{j,m}$, we have

$$g_{j,m}(\tau) = f_{2j,4m}(\tau) - f_{4m-2j,4m}(\tau). \tag{4.38}$$

And by (4.33) – (4.30) the transformation formulae for the $f_{j,m}$'s and $g_{j,m}$'s for $m \in \frac{1}{2}\mathbb{N}$, $j \in \frac{1}{2}\mathbb{Z}$ looks as follows:

THEOREM 4.6. *For $m \in \frac{1}{2}\mathbb{N}, j \in \frac{1}{2}\mathbb{Z}$, the following formulae hold:*

1. *For* $j \in \mathbb{Z}$,

$$f_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{0 \le k \le 2m-1} e^{\frac{\pi i j k}{m}} f_{k,m}(\tau),$$

$$g_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{\substack{0 \le k \le 4m-1 \\ k = odd}} e^{\frac{\pi i j k}{2m}} f_{\frac{k}{2},m}(\tau).$$

2. *For* $j \in \frac{1}{2} + \mathbb{Z}$,

$$f_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{0 \le k \le 2m-1} e^{\frac{\pi i j k}{m}} g_{k,m}(\tau),$$

$$g_{j,m}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} \sum_{\substack{0 \le k \le 4m-1 \\ k = odd}} e^{\frac{\pi i j k}{2m}} g_{\frac{k}{2},m}(\tau).$$

Here, $k \in \mathbb{Z}$ *in each of the right hand sides.*

THEOREM 4.7. *For* $m \in \frac{1}{2}\mathbb{N}, j \in \frac{1}{2}\mathbb{Z}$, *the following formulae hold:*

1. *For* $j + m \in \mathbb{Z}$,

$$f_{j,m}(\tau + 1) = e^{\frac{\pi i j^2}{2m}} f_{j,m}(\tau),$$

$$g_{j,m}(\tau + 1) = e^{\frac{\pi i j^2}{2m}} g_{j,m}(\tau).$$

2. *For* $j + m \in \frac{1}{2} + \mathbb{Z}$,

$$f_{j,m}(\tau + 1) = e^{\frac{\pi i j^2}{2m}} g_{j,m}(\tau),$$

$$g_{j,m}(\tau + 1) = e^{\frac{\pi i j^2}{2m}} f_{j,m}(\tau).$$

In particular, we have

$$g_{\frac{1}{2},\frac{3}{2}}\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} g_{\frac{1}{2},\frac{3}{2}}(\tau), \tag{4.39}$$

$$g_{\frac{1}{2},\frac{3}{2}}(\tau + 1) = e^{\frac{\pi i}{12}} g_{\frac{1}{2},\frac{3}{2}}(\tau). \tag{4.40}$$

In fact, by (2) of Theorem 4.6, (4.39) can be shown by the following computation:

$$\begin{aligned} g_{\frac{1}{2},\frac{3}{2}}\left(-\frac{1}{\tau}\right) &= \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{3}} \sum_{k=1,3,5} e^{\frac{\pi i k}{6}} g_{\frac{1}{2},\frac{3}{2}}(\tau) \\ &= \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{3}} \left\{ e^{\frac{\pi i}{6}} g_{\frac{1}{2},\frac{3}{2}}(\tau) + e^{\frac{5\pi i}{6}} \underbrace{g_{\frac{5}{2},\frac{3}{2}}(\tau)}_{-g_{\frac{1}{2},\frac{3}{2}}(\tau)} \right\} \\ &= (-i\tau)^{\frac{1}{2}} g_{\frac{1}{2},\frac{3}{2}}(\tau). \end{aligned}$$

Now, by (3.107) in §3.5, we have $\eta(\tau) = g_{\frac{1}{2},\frac{3}{2}}(\tau)$. Therefore, the formulae (4.39) and (4.40) imply the following formulae on the Dedekind eta function $\eta(\tau)$:

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau), \tag{4.41}$$

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau). \tag{4.42}$$

Let us consider the following condition on a holomorphic function $f(\tau)$ on the upper half plane $\mathbb{C}_+$:

Condition (H). $f(\tau)$ can be expanded into the following form of a formal series:

$$f(\tau) = q^r \sum_{n=0}^{\infty} a_n q^n \quad (r \in \mathbb{Q}, a_n \in \mathbb{C}).$$

Here r for $a_0 \neq 0$ is called the leading power of $f(\tau)$. When we expand $f(\tau)$ in this form, the part of f consisting of minus the 0-th power of q is called the **polar part** of f, and is denoted by $\mathrm{Polar}(f)$. (Here, the constant term is included in the polar part.) The following lemma is very useful when we compare two systems of functions:

LEMMA 4.8. *Let us consider two families of holomorphic functions on the upper half plane $\mathbb{C}_+$, denoted by $f_1, \cdots, f_n; g_1, \cdots, g_n$. Suppose they satisfy the following conditions* (i) – (v)*:*

i. *The f_i's and g_i's all satisfy condition* (H).
ii. *The vector subspaces of $O(\mathbb{C}_+)$ spanned by $\{f_i\}_{1\le i\le n}$ and $\{g_i\}_{1\le i\le n}$ are invariant under the action of $SL_2(\mathbb{Z})$.*

iii. *There exist* c_{ij}*'s satisfying*

$$f_i\left(-\frac{1}{\tau}\right) = \sum_{j=1}^{n} c_{ij} f_j(\tau), \quad g_i\left(-\frac{1}{\tau}\right) = \sum_{j=1}^{n} c_{ij} g_j(\tau).$$

iv. $\mathrm{Polar}(f_i) = \mathrm{Polar}(g_i)$ *for any* i.

v. *For all* i, *the leading power of* $f_i \equiv$ *the leading power of* g_i mod $\mathbb{Z}$.

Then, $f_i = g_i$ *for all* i.

PROOF. If we set $h_i := f_i - g_i$ for each i, then h_i is a holomorphic function on $\mathbb{C}_+$, and it satisfies

$$\lim_{\tau \to i\infty} h_i(\tau) = 0 \tag{4.43}$$

by condition (iv). The vector subspace V of $O(\mathbb{C}_+)$ spanned by the h_i $(1 \le i \le n)$ is S-invariant by condition (iii), and is also T-invariant by conditions (ii) and (v). By (4.43), it follows that

$$\lim_{\tau \to a} h_i(\tau) = 0 \tag{4.44}$$

for all $a \in \mathbb{Q}$. So, if we consider the compact Riemann surface R obtained by adding all of the rational points $\mathbb{Q}$ of the real axis and $i\infty$ to the upper half plane, and in addition pinch all of the added rational points and $i\infty$ to one point (cusp), then each h_i is a holomorphic function on R and its value at the cusp is 0. Hence $h_i = 0$ $(\forall i)$, and the lemma is proved. □

Here, let us consider the following functions:

$$\begin{aligned}
\phi_1(\tau) &:= \frac{\eta(\tau)^2}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1 + q^{n-\frac{1}{2}}),\\
\phi_2(\tau) &:= \frac{\eta(\frac{\tau}{2})}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1 - q^{n-\frac{1}{2}}),\\
\phi_3(\tau) &:= \sqrt{2} \cdot \frac{\eta(2\tau)}{\eta(\tau)} = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 + q^{n}).
\end{aligned} \tag{4.45}$$

Modular transformations of these functions can be immediately computed from (4.41) and (4.42), and they look as follows:

$$\phi_1\left(-\frac{1}{\tau}\right) = \phi_1(\tau),\ \phi_2\left(-\frac{1}{\tau}\right) = \phi_3(\tau),\ \phi_3\left(-\frac{1}{\tau}\right) = \phi_2(\tau), \tag{4.46}$$

$$\begin{aligned} &\phi_1(\tau+1) = e^{-\frac{\pi i}{24}}\phi_2(\tau), \quad \phi_2(\tau+1) = e^{-\frac{\pi i}{24}}\phi_1(\tau), \\ &\phi_3(\tau+1) = e^{\frac{\pi i}{12}}\phi_3(\tau). \end{aligned} \tag{4.47}$$

Therefore, ϕ_1, ϕ_2 and ϕ_3 span a three-dimensional $SL_2(\mathbb{Z})$-invariant subspace inside $O(\mathbb{C}_+)$. If we consider ϕ_1^n, ϕ_2^n and ϕ_3^n for any non-zero integer n (not necessarily positive), then since they also satisfy the same relations as (4.46) and (4.47) by modifying their scalar multiple slightly, they also span an $SL_2(\mathbb{Z})$-invariant subspace. These ϕ_1, ϕ_2 and ϕ_3 are important functions for describing characters of affine Lie algebras. The next lemma is sometimes useful for concrete computations:

LEMMA 4.9. *Suppose holomorphic functions f_1, f_2 and f_3 satisfy the following conditions:*

i. *f_1, f_2 and f_3 satisfy condition* (H).

ii. *f_1, f_2 and f_3 can be written in the following forms:*

$$f_1(\tau) = q^{-\frac{a}{48}} \sum_{n=0}^{\infty} a_n q^{\frac{n}{2}},$$

$$f_2(\tau) = q^{-\frac{a}{48}} \sum_{n=0}^{\infty} (-1)^n a_n q^{\frac{n}{2}},$$

$$f_3(\tau) = q^{\frac{a}{24}} \sum_{n=0}^{\infty} b_n q^n,$$

where a is an integer, and a_n, b_n are complex numbers, and $a_0 = 1$.

iii. $f_1\left(-\frac{1}{\tau}\right) = f_1(\tau), \quad f_2\left(-\frac{1}{\tau}\right) = f_3(\tau), \quad f_3\left(-\frac{1}{\tau}\right) = f_2(\tau).$

Then, $f_i(\tau) = \phi_i(\tau)^a$ $(i = 1, 2, 3)$.

PROOF. If we set $h_i := f_i - \phi_i^a$, the formal series expansion of h_i has the following form:

$$h_1(\tau) = c_1 q^{\frac{-a}{48}+\frac{1}{2}} + \cdots,$$

$$h_2(\tau) = c_2 q^{\frac{-a}{48}+\frac{1}{2}} + \cdots,$$

$$h_3(\tau) = c_3 q^{\frac{a}{24}} + \cdots.$$

Thus, the function obtained by multiplying these three functions is

$$h(\tau) := h_1(\tau)h_2(\tau)h_3(\tau) = c_1c_2c_3q + \cdots .$$

Namely, it is a formal series consisting of positive powers of q, and hence we have

$$\lim_{\tau \to i\infty} h(\tau) = 0. \tag{4.48}$$

h is S-invariant by condition (iii), and is also T-invariant by condition (ii). Thus, the holomorphic function h on the upper half plane is invariant under the action of $SL_2(\mathbb{Z})$, and together with (4.48), we obtain $h = 0$. So, one of the h_i $(i = 1, 2, 3)$ should be 0, but since we have

$$h_2\left(-\frac{1}{\tau}\right) = h_3(\tau), \quad h_1(\tau + 1) = e^{-\frac{\pi i a}{24}} h_2(\tau),$$

it follows that if one of the h_i's is zero, they all are. □

By using modular transformations, one can study the **asymptotic behavior** of a function. Let us explain this by taking the Dedekind eta function $\eta(\tau)$ as an example. By (3.105), we have

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n \tau}) \quad (q = e^{2\pi i \tau}),$$

and this is a holomorphic function on the upper half plane. As one can immediately see from this formula, the limit when τ approaches a rational point of the real axis is 0. Namely, we have

$$\lim_{\tau \to a} \eta(\tau) = 0$$

for all rational numbers a. Observe here that under the relation $q = e^{2\pi i \tau}$, one has

$$\mathrm{Im}\tau > 0 \Longleftrightarrow |q| < 1, \tag{4.49}$$

$$\tau \longrightarrow 0 \Longleftrightarrow q \longrightarrow 1. \tag{4.50}$$

The coefficients of the formal series of the reciprocal of Euler's function $\varphi(q)$ in q (in $|q| < 1$) are the partition numbers $p(n)$:

$$\frac{1}{\varphi(q)} = \sum_{n=0}^{\infty} p(n)q^n. \tag{4.51}$$

Here, we set the partition number of $n = 0$ to be 1 (i.e., $p(0) = 1$). If we let $q \longrightarrow 1$ in this formula, then clearly we get

$$\lim_{q \longrightarrow 1} \sum_{n=0}^{\infty} p(n) q^n = \sum_{n=0}^{\infty} p(n) = \infty,$$

so, this formula itself does not make sense. But suppose we would like to know 'How fast does this diverge?' Modular transformations easily lead us to the answer.

To study the behavior of $\varphi(q)$ (or $\frac{1}{\varphi(q)}$) as $q \longrightarrow 1$, we only have to study the behavior of $\eta(\tau)$ (or $\frac{1}{\eta(\tau)}$) as $\tau \to 0$. By the way,

$$\tau \longrightarrow 0 \Longleftrightarrow -\frac{1}{\tau} \longrightarrow \text{infinity}, \tag{4.52}$$

and the formula (4.41) on a modular transformation is a formula describing the relation of the value of the function near 0 and that near infinity.

We signify τ approaching to the origin 0 along the imaginary axis by $\tau \downarrow 0$. Similarly, we signify τ approaching infinity along the imaginary axis by $\tau \uparrow i\infty$ or by $\tau \longrightarrow i\infty$. Namely, we set

$$\tau \downarrow 0 \underset{\text{Def.}}{\Longleftrightarrow} \tau = iT \ (T > 0) \quad \text{and} \quad T \longrightarrow 0,$$
$$\tau \longrightarrow i\infty \underset{\text{Def.}}{\Longleftrightarrow} \tau = iT \ (T > 0) \quad \text{and} \quad T \longrightarrow \infty.$$

Let us study $\eta(\tau)$ at $\tau \downarrow 0$. By the formula (4.41) on a modular transformation, we have

$$\begin{aligned} \eta(\tau) &= (-i\tau)^{-\frac{1}{2}} \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{-\frac{1}{2}} g_{\frac{1}{2},\frac{3}{2}}\left(-\frac{1}{\tau}\right) \\ &= (-i\tau)^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} (-1)^k e^{2\pi i \left(-\frac{1}{\tau}\right) \cdot \frac{3}{2} \left(k + \frac{1}{6}\right)^2}. \end{aligned} \tag{4.53}$$

If we set $\tau = iT$ in the summation in the right hand side of this formula, it becomes

$$\eta(\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} (-1)^k e^{-\frac{3\pi}{T}\left(k + \frac{1}{6}\right)^2}. \tag{4.54}$$

For $T \longrightarrow 0$, one has $e^{-\frac{3\pi}{T}} \longrightarrow 0$, and the term that contributes the most in the summation in the right hand side is the term for $k = 0$; the contribution from the other terms becomes much smaller (as T approaches 0). Needless to say, we have to estimate appropriately to check this, but this can be done easily, as follows. We estimate the series in the right hand side except for the $k = 0$ term, i.e., the

rest of the terms, it is enough to estimate the sum of their absolute values. Since the exponent of e is a polynomial in k of degree two, it is dominated by those obtained by keeping the degree 1 term in k as it is, and replacing k^2 by $|k|$. So, by using the summation formula on geometric series, we obtain an estimation of the rest of the terms, and the above assertion can be shown.

For $\tau \downarrow 0$, by approximating the $k = 0$ term in the right hand side of (4.53), we get the following formula:

$$\eta(\tau) \underset{\tau\downarrow 0}{\sim} (-i\tau)^{-\frac{1}{2}} e^{-\frac{3\pi i}{\tau}\cdot\frac{1}{36}} = (-i\tau)^{-\frac{1}{2}} e^{-\frac{\pi i}{12\tau}}. \tag{4.55}$$

Since $\eta(\tau)$ is $q^{\frac{1}{24}}$ times $\varphi(q)$, their asymptotic behaviors at $\tau \downarrow 0$ (i.e., $q \longrightarrow 1$) are the same. Therefore, the asymptotic behavior of $\varphi(q)$ can also be given by the same formula as above:

$$\varphi(q) \underset{\tau\downarrow 0}{\sim} (-i\tau)^{-\frac{1}{2}} e^{-\frac{\pi i}{12\tau}} \quad (q = e^{2\pi i\tau}). \tag{4.56}$$

This formula shows which function one can use to 'approximate' $\sum_{n=0}^{\infty} p(n)q^n$ as $\tau \downarrow 0$. Now, when we find an asymptotic of $\frac{1}{\varphi(q)}$ in this way, can we find the increasing order of each coefficient $p(n)$ of this formal series? There is a theorem that provides us with an answer to this question. It is a " Tauber type theorem".[1]

THEOREM 4.10 (Ingham, cf. Lemma 4.22 of [**KP**]). *Suppose that*

$$f(\tau) = q^{\lambda} \sum_{n=0}^{\infty} a_n q^n \quad (q = e^{2\pi i\tau})$$

is a holomorphic function on $\mathbb{C}_+$, *and satisfies the following two conditions:*

i. a_n *are real numbers, and they satisfies* $a_n \leq a_{n+1}$ *for all* n.
ii. *There exist* $c \in \mathbb{C}, d \in \mathbb{R}$ *and* $N > 0$ *satisfying* $f(\tau) \underset{\tau\downarrow 0}{\sim} c \cdot (-i\tau)^{-d} e^{\frac{2\pi i N}{\tau}}$.

Then

$$a_n \underset{n\to\infty}{\sim} \frac{c}{\sqrt{2}} N^{-\frac{1}{2}\left(d-\frac{1}{2}\right)} n^{\frac{1}{2}\left(d-\frac{3}{2}\right)} e^{4\pi\sqrt{Nn}}.$$

In the above theorem, the symbol "$a_n \underset{n\to\infty}{\sim} b_n$" signifies

$$\lim_{n\longrightarrow\infty} \frac{b_n}{a_n} = 1.$$

[1]It seems that Tauber type theorems are also used frequently in the theory of probability. See e.g., Chapter XIII of [**Fel**]. See also §8.3 of [**Ta**].

A proof of this theorem is not so difficult, and can be obtained by computing a few Riemann-Stieltjes integrations.

Now, let us apply this theorem to

$$f(\tau) := \frac{1}{\varphi(\tau)} = \sum_{n=0}^{\infty} p(n) q^n.$$

Since we have

$$c = 1, \quad d = -\frac{1}{2}, \quad N = \frac{1}{24}$$

by (4.56), it immediately follows that

$$p(n) \underset{n\to\infty}{\sim} \frac{1}{4\sqrt{3}} \cdot \frac{1}{n} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{4.57}$$

How well does this approximation work? For reference, let us show some results obtained by using the mathematical system "Reduce 3.6". Programming by 'Reduce' is very simple, and to obtain the partition number $p(n)$ and the right hand side of (4.57), we have only to proceed as follows:

(1) A sample of the program of computing $p(n)$ by using its generating function:

```
let  x^1001 = 0;
operator  f,p;
forall  j  let  f(j) =for  k := 0 : 1000/j  sum  x^(j*k);
ff := for  j := 1 : 1000  product  f(j)$
for  n := 1 : 1000  do  write  p(n) := coeffn(ff,x,n);
```

(2) A sample of the program of computing the right hand side of (4.57) (here, we denote it by $a(n)$):

```
operator  a;
forall  n  let  a(n) = (1/(4*n))*((1/3)^(1/2))*
e^(PI*(2*n/3)^(1/2));
for  n := 1 : 1000  do  write  a(n) := a(n);
```

When I execute this, my computer replies

$$p(1) := 1, \quad p(2) := 2, \quad \cdots$$
$$a(1) := 1.87667042261, \quad a(2) := 2.71516043156, \quad \cdots$$

in an instant. Try it yourself if you are interested in it. Since in this book we cannot waste pages to show concrete values of both sides of (4.57), instead, let us show its error,

$$\text{error} := \frac{(\text{the r.h.s. of } (4.57)) - p(n)}{p(n)},$$

only for $n = 100, 200, \cdots, 1000$:

n	error	n	error
100	0.045713563075	600	0.0183221480923
200	0.03202930629	700	0.016947300342
300	0.026046656731	800	0.0158409292204
400	0.0225032776482	900	0.0149257599658
500	0.0200949029227	1000	0.0141524388575

In the next section, we will consider the modular transformations of characters, and by studying their asymptotic behaviors, we can see the increasing order of the multiplicities of weights (= the dimensions of weight subspaces), and also define the quantity called "asymptotic dimension" of a representation space. Although a representation of an affine Lie algebra is of infinite dimension (except for the trivial representation), its "asymptotic dimension" makes it possible to treat an infinite-dimensional representation like a finite-dimensional representation of a simple finite-dimensional Lie algebra.

4.3. Modular Transformations of Characters

In this scetion, let A be an affine Cartan matrix of split type or of type $A_{2l}^{(2)}$. Since $M = \overset{\circ}{Q}{}^{\vee}$ in this case, it follows that $M^* = P$. Moreover, since P_+ and $P_{(+)}$ in §2.5 coincide in the case of affine Lie algebras, we do not have to distiguish them. For $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{N}$, set

$$\begin{aligned} P^m &:= \{\lambda \in P; (\lambda|\delta) = m\}, \\ P_+^m &:= \{\lambda \in P_+; (\lambda|\delta) = m\}, \\ P_{++}^n &:= \{\lambda \in P_+^n; \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}\ (\forall i)\}. \end{aligned}$$

For $\lambda \in P^n$ $(n \in \mathbb{N})$, set

$$A_\lambda := e^{-\frac{|\lambda|^2}{2n}\delta} \sum_{w \in W} \varepsilon(w) e^{w\lambda}. \tag{4.58}$$

By this definition, it immediately follows that

$$A_{\lambda + a\delta} = A_\lambda \quad (\forall a \in \mathbb{C}). \tag{4.59}$$

Therefore A_λ is defined for $\lambda \mod \mathbb{C}\delta$. A_λ is the denominator and the numerator of characters. Namely, by using A_λ, the character formula (Theorem 2.73) for ch_Λ $(\Lambda \in P_+^m)$ can be expressed as follows:

$$\mathrm{ch}_\Lambda = e^{\left(\frac{|\Lambda+\rho|^2}{2(m+h^\vee)} - \frac{|\rho|^2}{2h^\vee}\right)\delta} \cdot \frac{A_{\Lambda+\rho}}{A_\rho}. \tag{4.60}$$

By using $W = \overset{\circ}{W} \ltimes M$, A_λ $(\lambda \in P^n)$ becomes

$$A_\lambda = e^{-\frac{|\lambda|^2}{2n}\delta} \sum_{w \in \overset{\circ}{W}} \sum_{\alpha \in M} \varepsilon(w) e^{t_\alpha(w\lambda)} = \sum_{w \in \overset{\circ}{W}} \varepsilon(w) \Theta_{w\lambda}. \tag{4.61}$$

To study modular transformations of characters, let us study that of A_λ. For $\lambda \in P^m$, one can choose $w \in W$ and $\mu \in P_+^m$ satisfying $\lambda = w\mu$ by Lemma 3.32. Since we have $A_\lambda = \varepsilon(w) A_\mu$ in this case, we have only to consider A_Λ for $\lambda \in P_+$. If $\lambda \in P_+$, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all i, but if $\langle \lambda, \alpha_i^\vee \rangle = 0$ for some i, then $r_i \lambda = \lambda$ for such i, and hence

$$A_\lambda = A_{r_i\lambda} = \varepsilon(r_i) A_\lambda = -A_\lambda,$$

which implies $A_\lambda = 0$. Thus, A_λ $(\lambda \in P_+)$ can fail to vanish only when $\lambda \in P_{++}$. Therefore, the following holds:

$$\text{For } \lambda \in P, \lambda \neq \text{a regular element} \Longrightarrow A_\lambda = 0. \tag{4.62}$$

Now, take any positive integer n and fix it, and let us study the modular transformations of A_λ $(\lambda \in P_{++})$. This can be easily computed by Theorem 4.3:

$$\begin{aligned} A_\lambda|_{(S,j)} &= \sum_{w \in \overset{\circ}{W}} \varepsilon(w) \Theta_{w\lambda}|_{(S,j)} \\ &= (-i)^{\frac{l}{2}} |M^*/nM|^{-\frac{1}{2}} \sum_{w \in \overset{\circ}{W}} \sum_{\substack{\mu \in P^n \\ \mathrm{mod}\ (nM + \mathbb{C}\delta)}} \varepsilon(w) e^{-\frac{2\pi i}{n}(w\overline{\lambda}|\overline{\mu})} \Theta_\mu \\ &= (-i)^{\frac{l}{2}} |M^*/nM|^{-\frac{1}{2}} \sum_{w \in \overset{\circ}{W}} \sum_{\substack{\mu \in P^n \\ \mathrm{mod}\ (nM + \mathbb{C}\delta)}} \varepsilon(w) e^{-\frac{2\pi i}{n}(\overline{\lambda}|w^{-1}\overline{\mu})} \Theta_\mu. \end{aligned}$$

If we replace $w^{-1}\mu$ by μ in the summation in the right hand side, this formula becomes

$$\begin{aligned} A_\lambda|_{(S,j)} &= (-i)^{\frac{l}{2}}|M^*/nM|^{-\frac{1}{2}} \\ &\quad \times \sum_{\mu\in P^n \bmod (nM+\mathbb{C}\delta)} e^{-\frac{2\pi i}{n}(\overline{\lambda}|\overline{\mu})} \sum_{w\in \mathring{W}} \varepsilon(w)\Theta_{w\mu} \\ &= (-i)^{\frac{l}{2}}|M^*/nM|^{-\frac{1}{2}} \sum_{\mu\in P^n \bmod (nM+\mathbb{C}\delta)} e^{-\frac{2\pi i}{n}(\overline{\lambda}|\overline{\mu})} A_\mu. \end{aligned}$$

Here, we only have to take the summation in the right hand side over regular elements by (4.62). Since there uniquely exist $w \in \mathring{W}$ and $\mu' \in P^n_{++}$ satisfying

$$\mu \equiv w\mu' \mod (nM + \mathbb{C}\delta)$$

if $\mu \in P^n$ is regular, the right hand side of the above formula can be rewritten as follows:

$$A_\lambda|_{(S,j)} = (-i)^{\frac{l}{2}}|M^*/nM|^{-\frac{1}{2}} \sum_{\mu\in P^n_{++} \bmod \mathbb{C}\delta} \sum_{w\in \mathring{W}} e^{-\frac{2\pi i}{n}(\overline{\lambda}|w\overline{\mu})} \underbrace{A_{w\mu}}_{\varepsilon(w)A_\mu}$$

$$= (-i)^{\frac{l}{2}}|M^*/nM|^{-\frac{1}{2}} \sum_{\mu\in P^n_{++} \bmod \mathbb{C}\delta} \left(\sum_{w\in \mathring{W}} \varepsilon(w) e^{-\frac{2\pi i}{n}(\overline{\lambda}|w\overline{\mu})} \right) A_\mu.$$

In particular, let us consider the case $\lambda = \rho$. Since we have $n = (\rho|\delta) = h^\vee$, and $P^{h^\vee}_{++} \bmod \mathbb{C}\delta = \{\rho\}$ (i.e., it consists of only one element) in this case, it follows that

$$A_\rho|_{(S,j)} = (-i)^{\frac{l}{2}}|M^*/h^\vee M|^{-\frac{1}{2}} \left(\sum_{w\in \mathring{W}} \varepsilon(w) e^{-\frac{2\pi i}{h^\vee}(\overline{\rho}|w\overline{\rho})} \right) A_\rho.$$

Then, since one has

$$\sum_{w\in \mathring{W}} \varepsilon(w) e^{-\frac{2\pi i}{h^\vee}(\overline{\rho}|w\overline{\rho})} = (-2i)^{|\mathring{\Delta}_+|} \prod_{\alpha\in \mathring{\Delta}_+} \sin \frac{\pi(\overline{\rho}|\alpha)}{h^\vee}$$

by the denominator identity of a simple finite-dimensional Lie algebra (Proposition 3.11), the above formula becomes

(4.63)

$$A_\rho|_{(S,j)} = (-i)^{\frac{l}{2}+|\mathring{\Delta}_+|}\left(2^{|\mathring{\Delta}_+|}|M^*/h^\vee M|^{-\frac{1}{2}}\prod_{\alpha\in\mathring{\Delta}_+}\sin\frac{\pi(\overline{\rho}|\alpha)}{h^\vee}\right)A_\rho.$$

Here, let us remark that the quantity in parentheses on the right hand side of (4.63) is a positive real number. Moreover, since $S^4 = \mathrm{Id}$, notice that $(S,j)^8 = \mathrm{Id}$. Namely, the eighth power of the factor in the right hand side of (4.63) should be 1. Thus, the quantity in parentheses in the right hand side of (4.63) is 1. Namely, we have

$$2^{|\mathring{\Delta}_+|}|M^*/h^\vee M|^{-\frac{1}{2}}\prod_{\alpha\in\mathring{\Delta}_+}\sin\frac{\pi(\overline{\rho}|\alpha)}{h^\vee} = 1.$$

Summarizing these, we obtained the following theorem:

THEOREM 4.11. *If A is an affine Cartan matrix of split type or of type $A^{(2)}_{2l}$, then the following formulae hold for $\lambda \in P^n_{++}$ ($n \in \mathbb{N}$) and ρ:*

1.

$$A_\lambda|_{(S,j)} = (-i)^{\frac{l}{2}}|M^*/nM|^{-\frac{1}{2}} \times \sum_{\mu\in P^n_{++} \bmod \mathbb{C}\delta}\left(\sum_{w\in\mathring{W}}\varepsilon(w)e^{-\frac{2\pi i}{n}(\overline{\lambda}|w\overline{\mu})}\right)A_\mu,$$

2. $A_\rho|_{(S,j)} = (-i)^{\frac{l}{2}+|\mathring{\Delta}_+|}A_\rho$,
3. $\displaystyle\prod_{\alpha\in\mathring{\Delta}_+} 2\sin\frac{\pi(\overline{\rho}|\alpha)}{h^\vee} = |M^*/h^\vee M|^{\frac{1}{2}}$. .

Next, let us consider the modular transformations of characters. For $\Lambda \in P_+^m$ ($m \in \mathbb{Z}_{\geq 0}$), set

$$z_\Lambda := z_m := \frac{m}{m+h^\vee} \cdot \frac{12|\overline{\rho}|^2}{h^\vee}, \tag{4.64}$$

$$s_\Lambda := \frac{|\Lambda+\rho|^2}{2(m+h^\vee)} - \frac{|\rho|^2}{2h^\vee}, \tag{4.65}$$

$$h_\Lambda := \frac{(\Lambda|\Lambda+2\rho)}{2(m+h^\vee)}. \tag{4.66}$$

Here ρ is chosen so that $\rho = h^\vee\Lambda_0 + \overline{\rho}$, i.e., it satisfies $\langle \rho, d\rangle = 0$. This choice is the same as $|\rho|^2 = |\overline{\rho}|^2$. Moreover, set

$$s_{\overline{\Lambda};m} := \frac{|\overline{\Lambda}+\overline{\rho}|^2}{2(m+h^\vee)} - \frac{|\overline{\rho}|^2}{2h^\vee}, \tag{4.67}$$

$$h_{\overline{\Lambda};m} := \frac{(\overline{\Lambda}|\overline{\Lambda}+2\overline{\rho})}{2(m+h^\vee)}. \tag{4.68}$$

By simple computations, s_Λ can be rewritten as follows:

$$\begin{aligned} s_\Lambda &= h_\Lambda + \frac{|\overline{\rho}|^2}{2}\left(\frac{1}{m+h^\vee} - \frac{1}{h^\vee}\right) \\ &= h_\Lambda - \frac{m}{m+h^\vee} \cdot \frac{|\overline{\rho}|^2}{2h^\vee} = h_\Lambda - \frac{1}{24} z_m, \end{aligned} \tag{4.69}$$

$$s_{\overline{\Lambda};m} = h_{\overline{\Lambda};m} - \frac{1}{24} z_m. \tag{4.70}$$

Here, we do not have any place to explain, but for a split type A, the formula

$$\frac{|\overline{\rho}|^2}{2g} = \frac{1}{24} \dim \overline{\mathfrak{g}} \tag{4.71}$$

can be easily derived from the denominator identity of Weyl, and (4.71) is called the Freudenthal-de Vries 'strange formula'. For a split type, since we have $h^\vee = g$ by Lemma 3.22, it follows that

$$z_m = \frac{m}{m+h^\vee} \cdot \dim \overline{\mathfrak{g}}.$$

s_Λ and h_Λ are not the quantities defined for $\Lambda \mod \mathbb{C}\delta$, but may change by shifting a scalar multiple of δ to Λ. But, if we slightly modify ch_Λ and consider

$$\mathrm{ch}'_\Lambda := e^{-s_\Lambda \delta}\mathrm{ch}_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}, \tag{4.72}$$

i.e.,

$$\mathrm{ch}'_\Lambda(\tau, z, t) := q^{s_\Lambda}\mathrm{ch}_\Lambda(\tau, z, t), \tag{4.73}$$

then this ch'_Λ is defined for $\Lambda \mod \mathbb{C}\delta$, and its modular transformations can be immediately obtained from Theorem 4.11:

THEOREM 4.12. *If A is an affine Cartan matrix of split type or of type $A^{(2)}_{2l}$, then for $\lambda \in P^m_+$ $(m \in \mathbb{Z}_{\geq 0})$, the following formula holds:*

$$\mathrm{ch}'_\lambda|_{(S,j)} = \sum_{\mu \in P^m_+} a(\lambda, \mu)\mathrm{ch}'_\mu.$$

Here, $a(\lambda, \mu)$ is defined by

(4.74)

$$a(\lambda, mu) := i^{|\mathring{\Delta}_+|}|M^*/(m+h^\vee)M|^{-\frac{1}{2}} \sum_{w \in \mathring{W}} \varepsilon(w) e^{-\frac{2\pi i}{m+h^\vee}(\overline{\lambda+\rho}|w(\overline{\mu+\rho}))}.$$

By the definition (4.74), it immediately follows that

$$a(\lambda, \mu) = a(\mu, \lambda). \tag{4.75}$$

In the above computation, we have studied the transformation $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The transformation of ch'_Λ under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be easily computed. If A is of split type, then M is an even lattice, and the transformation of ch_Λ under the action of T follows from (4.61) and Theorem 4.6 as follows:

$$A_\Lambda(\tau + 1, z, t) = e^{\pi i \frac{|\overline{\lambda}|^2}{n}} A_\Lambda(\tau, z, t). \tag{4.76}$$

Thus, $\mathrm{ch}'_\Lambda(\tau+1, z, t)$ can be written by the definition (4.61) as follows:

$$\mathrm{ch}'_\Lambda(\tau + 1, z, t) = e^{2\pi i\left(\frac{|\overline{\Lambda}+\overline{\rho}|^2}{2(m+h^\vee)} - \frac{|\overline{\rho}|^2}{2h^\vee}\right)} \cdot \mathrm{ch}'_\Lambda(\tau, z, t).$$

Rewriting the exponent of e in the right hand side by (4.65), the following theorem is obtained:

THEOREM 4.13. *If A is an affine Cartan matrix of split type, then for $\Lambda \in P^m_+$ $(m \in \mathbb{Z}_{\geq 0})$, the following formula holds:*

$$\mathrm{ch}'_\Lambda(\tau + 1, z, t) = e^{2\pi i s_\Lambda}\mathrm{ch}'_\Lambda(\tau, z, t).$$

If we write the formula in Theorem 4.12 in the coordinates (τ, z, t), it looks like

$$\mathrm{ch}'_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right) = \sum_{\mu \in P_+^m} a(\lambda, \mu)\mathrm{ch}'_\mu(\tau, z, t), \tag{4.77}$$

$$\mathrm{ch}'_\lambda(\tau, z, t) = \sum_{\mu \in P_+^m} a(\lambda, \mu)\mathrm{ch}'_\mu\left(-\frac{1}{\tau}, -\frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right). \tag{4.78}$$

The right hand side of (4.78) can be rewritten by Proposition 3.35 as follows:

$$\begin{aligned}\mathrm{ch}'_\lambda(\tau, z, t) &= \sum_{\mu \in P_+^m} a(\lambda, \mu)\mathrm{ch}'_{{}^t\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right) \\ &= \sum_{\mu \in P_+^m} a(\lambda, {}^t\mu)\mathrm{ch}'_\mu\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{|z|^2}{2\tau}\right).\end{aligned} \tag{4.79}$$

Rewriting the right hand side of (4.79) by the left hand side of (4.77), we get

$$\mathrm{ch}'_\lambda(\tau, z, t) = \sum_{\mu \in P_+^m} \sum_{\nu \in P_+^m} a(\lambda, {}^t\mu)a(\mu, \nu)\mathrm{ch}'_\nu(\tau, z, t). \tag{4.80}$$

By the way, if we fix $m \in \mathbb{Z}_{\geq 0}$, then ch'_Λ $(\Lambda \in P_+^m)$ are linearly independent (as one can see from the exponent of their top terms). Therefore, from this formula, we obtain

$$\sum_{\mu \in P_+^m} a(\lambda, {}^t\mu)a(\mu, \nu) = \delta_{\lambda,\nu}. \tag{4.81}$$

Next, let us compute $a(\lambda, {}^t\mu)$. By (3.97), we have

$${}^t\mu = m\Lambda_0 + {}^t\overline{\mu} = m\Lambda_0 - w_0\overline{\mu}.$$

Here, w_0 is the longest element of $\mathring{W}$. By the definition (4.74) of $a(\lambda, \mu)$, it follows that

$$\begin{aligned}&a(\lambda, {}^t\mu) \\ &= i^{|\mathring{\Delta}|}|M^*/(m+h^\vee)M|^{-\frac{1}{2}} \sum_{w \in \mathring{W}} \varepsilon(w)e^{-\frac{2\pi i}{m+h^\vee}(\overline{\lambda+\rho}|w(-w_0\overline{\mu}-w_0\overline{\rho}))} \\ &= i^{|\mathring{\Delta}|}|M^*/(m+h^\vee)M|^{-\frac{1}{2}} \sum_{w \in \mathring{W}} \varepsilon(w)e^{\frac{2\pi i}{m+h^\vee}(\overline{\lambda+\rho}|w(w_0\overline{\mu}+w_0\overline{\rho}))}.\end{aligned}$$

Replacing ww_0 by w here, we get

$$a(\lambda, {}^t\mu)$$
$$= i^{|\mathring{\Delta}_+|}|M^*/(m+h^\vee)M|^{-\frac{1}{2}} \sum_{w\in\mathring{W}} \varepsilon(w_0)\varepsilon(w)e^{\frac{2\pi i}{m+h^\vee}(\overline{\lambda+\rho}|w(\overline{\mu+\rho}))}.$$

Since w_0 is the longest element of $\mathring{W}$, it follows that $\varepsilon(w_0) = (-1)^{|\mathring{\Delta}_+|}$. Rewriting the above formula by using this fact, one obtains

$$a(\lambda, {}^t\mu) = \overline{a(\lambda, \mu)}. \tag{4.82}$$

Next, set

$$a(\lambda) := a(\lambda, m\Lambda_0)$$
$$= i^{|\mathring{\Delta}_+|}|M^*/(m+h^\vee)M|^{-\frac{1}{2}} \sum_{w\in\mathring{W}} \varepsilon(w)e^{-\frac{2\pi i}{m+h^\vee}(\overline{\lambda+\rho}|w(\overline{\rho}))}.$$

Rewriting the right hand side of this formula by using the denominator identity for a simple finite-dimensional Lie algebra (Proposition 3.11), we get

$$a(\lambda) = |M^*/(m+h^\vee)M|^{-\frac{1}{2}} \prod_{\alpha\in\mathring{\Delta}_+} 2\sin\frac{\pi(\overline{\lambda+\rho}|\alpha)}{m+h^\vee}. \tag{4.83}$$

By the way, since $\lambda \in P_+^m$, for each $\alpha \in \mathring{\Delta}_+$, one has

$$(\overline{\lambda+\rho}|\alpha) = (\overline{\lambda}|\alpha) + (\overline{\rho}|\alpha) < m + h^\vee,$$

and any multiplicative factor of the right hand side of (4.83) is a positive number. Therefore, one has

$$a(\lambda) > 0 \quad (\forall\lambda \in P_+^m). \tag{4.84}$$

$a(\lambda)$ is called the amplitude of the representation $L(\lambda)$.

For $\lambda, \mu \in P_+^m$, let us compute

$$\frac{a(\lambda,\mu)}{a(\lambda)} = \frac{a(\lambda,\mu)}{a(\lambda, m\Lambda_0)}.$$

If we compute this by using the definition (4.74) of $a(\lambda,\mu)$, the first factor in the right hand side of (4.74) cancels by the denominator and the numerator, and the next formula follows:

$$\frac{a(\lambda,\mu)}{a(\lambda)} = \frac{\sum_{w\in\mathring{W}} \varepsilon(w)e^{-\frac{2\pi i}{m+\vee}(\overline{\lambda+\rho}|w(\overline{\mu+\rho}))}}{\sum_{w\in\mathring{W}} \varepsilon(w)e^{-\frac{2\pi i}{m+\vee}(\overline{\lambda+\rho}|w(\overline{\rho}))}}.$$

The right hand side of this formula is just the character formula of the simple finite-dimensional Lie algebra $\mathfrak{g}(\mathring{A})$. Namely, this is the value of the character of the irreducible finite-dimensional representation $\mathring{L}(\overline{\mu})$ with highest weight $\overline{\mu}$ at $e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}$. Therefore, we obtain

$$\frac{a(\lambda,\mu)}{a(\lambda)} = \mathrm{ch}_{\mathring{L}(\overline{\mu})}\left(e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}\right). \tag{4.85}$$

Note. Since the right hand side of (4.85) is the trace of en element $e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}$ (this is an element of the compact Lie group correspoding to $\mathfrak{g}(\mathring{A})$) on the irreducible finite-dimensional representation $\mathring{L}(\overline{\mu})$ of the simple finite-dimensional Lie algebra $\mathfrak{g}(\mathring{A})$, we may sometimes denote the right hand side of (4.85) by

$$\mathrm{tr}_{\mathring{L}(\overline{\mu})}\left(e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}\right) \quad \text{or} \quad \mathrm{tr}_{\overline{\mu}}\left(e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}\right). \qquad \square$$

Here, we summarize the properties of $a(\lambda,\mu)$ obtained above:

THEOREM 4.14. *For $m \in \mathbb{Z}_{\geq 0}$, the following hold:*

1. $a(\lambda) > 0$ *for all* $\lambda \in P_+^m$.
2. *For* $\lambda, \mu \in P_+^m$,
   (i) $a(\lambda,\mu) = a(\lambda)\cdot \mathrm{ch}_{\mathring{L}(\overline{\mu})}\left(e^{-\frac{2\pi i}{m+h^\vee}\overline{\lambda+\rho}}\right)$,
   (ii) $a(\lambda, {}^t\mu) = \overline{a(\lambda,\mu)}$.
3. $(a(\lambda,\mu))_{\lambda,\mu\in P_+^m}$ *is a unitary matrix.*

In particular, from (3) of this theorem, we obtain the following corollary:

COROLLARY 4.15. *For any $m \in \mathbb{Z}_{\geq 0}$,*

$$\sum_{\Lambda\in P_+^m} a(\Lambda)^2 = 1.$$

Note. In (2) of Proposition 6.11 in Chapter 6, we will derive a 'grade-up' version of this corollary by using the decomposition of the tensor product of representations. $\square$

As we saw above, $a(\lambda,\mu)$ is the 'normalized' number obtained by multiplying a certain positive number to the special value of the character $\mathrm{ch}_{\mathring{L}(\overline{\mu})}$ of the simple finite-dimensional Lie algebra $\mathfrak{g}(\mathring{A})$, and one can regard the unitarity of the matrix $(a(\lambda,\mu))$ just as the

orthogonality relation of the characters. Of course, there already exists the theory describing the orthogonality relation of the characters of a simple finite-dimensional Lie algebra as representations of the compact Lie group, and what we called 'the orthogonality relation' here is used in a different sense. To describe this in a slightly clearer way we need a fusion algebra, and I will explain this in the next chapter.

Now, let us consider

$$\chi_\lambda(\tau) := \mathrm{ch}'_\lambda(\tau, 0, 0).$$

This function has the following sense. For simplicity, we may choose $\lambda \in P_+^m$ so that $\langle \lambda, d \rangle = 0$. The eigenvalues of d on $L(\lambda)$ are 0 or negative integers, and for each $j \in \mathbb{Z}_{\geq 0}$,

$$L(\lambda)_{-j} := \{v \in L(\lambda); dv = -jv\}$$

is a $\overline{\mathfrak{g}}$-module, and $L(\lambda)$ decomposes into the direct sum of them. In this case we have

$$\chi_\lambda(\tau) = q^{s_{\overline{\lambda};m}} \sum_{j=0}^{\infty} \dim L(\lambda)_{-j} \cdot q^j \quad (q = e^{2\pi i \tau}). \tag{4.86}$$

Let us study a property of the leading exponent $s_{\overline{\lambda};m}$ of $\chi_\lambda(\tau)$. For $\lambda \in P_+^m$, we have

$$(\overline{\lambda}|\overline{\lambda} + 2\overline{\rho}) = (\overline{\lambda}|\overline{\lambda}) + 2(\overline{\lambda}|\overline{\rho}) = (\overline{\lambda}|\overline{\lambda}) + \sum_{\alpha \in \mathring{\Delta}_+} (\overline{\lambda}|\alpha),$$

and since $\overline{\lambda} \in \overline{P}_+$, each term in the right hand side of this formula is ≥ 0, and hence $(\overline{\lambda}|\overline{\lambda} + 2\overline{\rho}) \geq 0$. Namely,

$$\lambda \in P_+^m \Longrightarrow h_{\overline{\lambda};m} \geq 0, \tag{4.87}$$

and the equality holds only if $\lambda = m\Lambda_0$. Rephrasing this fact in terms of $s_{\overline{\lambda};m}$, we obtain the following lemma:

LEMMA 4.16.

$$\lambda \in P_+^m \Longrightarrow s_{\overline{\lambda};m} \geq -\frac{1}{24} z_m.$$

Let us study the asymptotic behavior of $\chi_\lambda(\tau)$ as $\tau \downarrow 0$. First, the modular transformations of these functions can be immediately obtained from (4.78) by specializing $z = t = 0$:

$$\chi_\lambda(\tau) = \sum_{\mu \in P_+^m} a(\lambda, \mu) \chi_\mu \left(-\frac{1}{\tau}\right). \tag{4.88}$$

The asymptotic behavior of these functions can be obtained in the same way as was done for $\eta(\tau)$. But here, the functions appearing in the right hand side are not single but a linear combination of χ_μ. Since the term among them that contribute most to the behavior as $\tau \downarrow 0$ is $\mu = m\Lambda_0$ by Lemma 4.16, it follows that

$$\chi_\lambda(\tau) \underset{\tau\downarrow 0}{\sim} a(\lambda, m\Lambda_0)e^{2\pi i(-\frac{1}{\tau})(-\frac{1}{24}z_m)}.$$

Thus, we obtain the following theorem:

THEOREM 4.17. *If A is an affine Carta matrix of split type or of type $A_{2l}^{(2)}$, then the asymptotic behavior of χ_Λ ($\Lambda \in P_+^m$) looks like*

$$\chi_\Lambda(\tau) \underset{\tau\downarrow 0}{\sim} a(\Lambda)e^{\frac{\pi i}{12\tau}z_m}.$$

Since $q = e^{2\pi i\tau}$, $q \longrightarrow 1$ when $\tau \downarrow 0$. So, by (4.86), the limit of χ_Λ as $\tau \downarrow 0$ coincide with the limit of $\sum_{j=0}^{\infty} \dim L(\Lambda)_{-j} \cdot q^j$ as $q \longrightarrow 1$, and if we compute it, we get

$$\sum_{j=0}^{\infty} \dim L(\Lambda)_{-j} = \dim L(\Lambda) = \infty.$$

Therefore, Theorem 4.17 can be interpreted as the 'asymptotic' quantity of $\dim L(\Lambda) = \infty$. In this sense,

$$a(\Lambda)e^{\frac{\pi i}{12\tau}z_m}$$

is called the **asymptotic dimension** of the representation $L(\Lambda)$. $a(\Lambda)$ and z_m are called its amplitude and its growth, respectively. The growth depends only on its level m, and not on each Λ.

When we study representations of an affine Lie algebra, for example, when we study their decompositions regarded as modules over a sub (affine) Lie algebra, their asymptotic dimensions are sometimes effective. In some cases, one can get exact information on the irreducible decomposition by simple arithmetic operations from of their growth and amplitude. For such applications, the concrete value of $a(\Lambda)$ is important.

Note. Here, we have explained things only for an affine Lie algebra of split type or of type $A_{2l}^{(2)}$, but we can also compute the modular transformations of characters and their asymptotic behavoir for an affine Lie algebra of twisted type which is not of type $A_{2l}^{(2)}$. But, for an affine Lie algebra of twisted type, since the dual lattice of M is not

P but $P^\vee$, it is necessary to introduce its "adjacent root system" to describe modular transformations of its characters (see, e.g. [**KP**]). Such an argument is explained in §13.9 of [**K4**], but since enough of the necessary tools for considering such problem were prepared in §4.1, I recommend that you try it for yourself. □

$a(\lambda, \mu)$ enjoys important properties other than the unitarity, but here we cannot help skipping it owing to the lack of space. For small m, one can compute the concrete values of $a(\lambda, \mu)$ (by using several techniques). Here, we will just list them, since they are important for applications:

1. For $A_1^{(1)}$, putting

$$\lambda_{m,j} := (m-j)\Lambda_0 + j\Lambda_1 \quad (0 \le j \le m), \tag{4.89}$$

we have

$$a(\lambda_{m,j}, \lambda_{m,k}) = \frac{2}{\sqrt{m+2}} \sin \frac{\pi(j+1)(k+1)}{m+2}. \tag{4.90}$$

2. For $A_l^{(1)}$,

$$a(\Lambda_j, \Lambda_k) = \frac{1}{\sqrt{l+1}} e^{\frac{2\pi i j k}{l+1}}.$$

3. For $B_l^{(1)}$,

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_1	Λ_l
Λ_0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
Λ_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{2}}$
Λ_l	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0

(ii) The list of $a(\lambda, \mu)$ for $m = 2$:

	$2\Lambda_0$	$2\Lambda_1$	$\Lambda_0 + \Lambda_l$	$\Lambda_1 + \Lambda_l$	λ_k
$2\Lambda_0$	$\frac{1}{2\sqrt{2l+1}}$	$\frac{1}{2\sqrt{2l+1}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2l+1}}$
$2\Lambda_1$	$\frac{1}{2\sqrt{2l+1}}$	$\frac{1}{2\sqrt{2l+1}}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{1}{\sqrt{2l+1}}$
$\Lambda_0 + \Lambda_l$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	0
$\Lambda_1 + \Lambda_l$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	0
λ_j	$\frac{1}{2\sqrt{2l+1}}$	$\frac{1}{2\sqrt{2l+1}}$	0	0	$\frac{2}{\sqrt{2l+1}} \cos \frac{2\pi ijk}{2l+1}$

Here the λ_j $(1 \leq j \leq l)$ are defined as follows:

$$\lambda_j := \begin{cases} \Lambda_0 + \Lambda_1 & (\text{for } j = 1), \\ \Lambda_j & (\text{for } 2 \leq j \leq l-1), \\ 2\Lambda_l & (\text{for } j = l). \end{cases}$$

4. For $C_l^{(1)}$,

$$a(\Lambda_j, \Lambda_k) = \frac{2}{\sqrt{l+2}} \sin \frac{\pi(j+1)(k+1)}{l+2}.$$

5. For $D_l^{(1)}$,

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_1	Λ_{l-1}	Λ_l
Λ_0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Λ_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$
Λ_{l-1}	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{(-i)^l}{2}$	$\frac{-(-i)^l}{2}$
Λ_l	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-(-i)^l}{2}$	$\frac{(-i)^l}{2}$

(ii) The list of $a(\lambda, \mu)$ for $m = 2$:

	$2\Lambda_0$	$2\Lambda_1$	$2\Lambda_{l-1}$	$2\Lambda_l$	λ_k
$2\Lambda_0$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{\sqrt{2l}}$
$2\Lambda_1$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{\sqrt{2l}}$
$2\Lambda_{l-1}$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{(-1)^l}{2\sqrt{2l}}$	$\frac{(-1)^l}{2\sqrt{2l}}$	$\frac{(-1)^k}{2\sqrt{2l}}$
$2\Lambda_l$	$\frac{1}{2\sqrt{2l}}$	$\frac{1}{2\sqrt{2l}}$	$\frac{(-1)^l}{2\sqrt{2l}}$	$\frac{(-1)^l}{2\sqrt{2l}}$	$\frac{(-1)^k}{2\sqrt{2l}}$
λ_j	$\frac{1}{\sqrt{2l}}$	$\frac{1}{\sqrt{2l}}$	$\frac{(-i)^j}{\sqrt{2l}}$	$\frac{(-i)^j}{\sqrt{2l}}$	$\frac{2}{\sqrt{2l}} \times \cos\frac{\pi jk}{l}$
$\Lambda_0 + \Lambda_{l-1}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	0
$\Lambda_0 + \Lambda_l$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	0
$\Lambda_1 + \Lambda_{l-1}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	0
$\Lambda_1 + \Lambda_l$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	0

	$\Lambda_0 + \Lambda_{l-1}$	$\Lambda_0 + \Lambda_l$	$\Lambda_1 + \Lambda_{l-1}$	$\Lambda_1 + \Lambda_l$
$2\Lambda_0$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$2\Lambda_1$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$
$2\Lambda_{l-1}$	$\frac{(-i)^l}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$
$2\Lambda_l$	$\frac{-(-i)^l}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	$\frac{(-i)^l}{2\sqrt{2}}$	$\frac{-(-i)^l}{2\sqrt{2}}$
λ_j	0	0	0	0
$\Lambda_0 + \Lambda_{l-1}$	$\frac{1}{2}$	0	0	$\frac{-1}{2}$
$\Lambda_0 + \Lambda_l$	0	$\frac{1}{2}$	$\frac{-1}{2}$	0
$\Lambda_1 + \Lambda_{l-1}$	0	$\frac{-1}{2}$	$\frac{1}{2}$	0
$\Lambda_1 + \Lambda_l$	$\frac{-1}{2}$	0	0	$\frac{1}{2}$

Here the λ_j $(1 \le j \le l-1)$ are defined as follows:

$$\lambda_j := \begin{cases} \Lambda_0 + \Lambda_1 & (\text{for } j = 1), \\ \Lambda_j & (\text{for } 2 \le j \le l-2), \\ \Lambda_{l-1} + \Lambda_l & (\text{for } j = l). \end{cases}$$

6. For $E_6^{(1)}$, let σ be an automorphism of the Dynkin diagram of order 3 (a rotation of the Dynkin diagram in Figure 4.1).

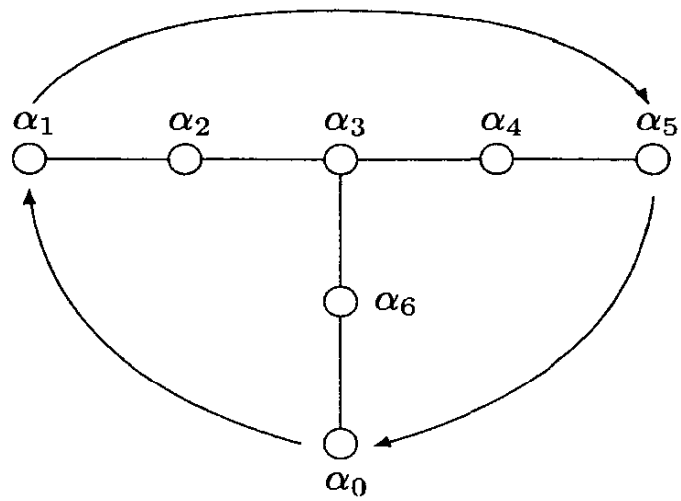

FIGURE 4.1

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_1	Λ_5
Λ_0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
Λ_1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}e^{-\frac{2\pi i}{3}}$	$\frac{1}{\sqrt{3}}e^{\frac{2\pi i}{3}}$
Λ_5	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}e^{\frac{2\pi i}{3}}$	$\frac{1}{\sqrt{3}}e^{-\frac{2\pi i}{3}}$

i.e.,

$$a(\sigma^j\Lambda_0, \sigma^k\Lambda_0) = \frac{1}{\sqrt{3}}e^{-\frac{2\pi ijk}{3}}.$$

(ii) For $m = 2$, putting

$$\lambda_1 := 2\Lambda_0, \quad \lambda_3 := \Lambda_1 + \Lambda_5, \quad \lambda_5 := \Lambda_6,$$

we have

$$a(\sigma^j\lambda_r, \sigma^k\lambda_s) = \frac{2}{\sqrt{21}}e^{\frac{2\pi ijk}{3}} \sin\frac{\pi rs}{7}.$$

7. For $E_7^{(1)}$,

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_6
Λ_0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
Λ_6	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

(ii) For $m = 2$, putting

$$\lambda_1 := 2\Lambda_0, \quad \lambda_3 := \Lambda_5, \quad \mu_1 := \Lambda_7, \quad \mu_3 := \Lambda_0 + \Lambda_6,$$

the list of $a(\lambda, \mu)$ is as follows:

	λ_k	$\sigma\lambda_k$	μ_k
λ_j	$\frac{1}{\sqrt{5}} \sin \frac{\pi jk}{5}$	$\frac{1}{\sqrt{5}} \sin \frac{\pi jk}{5}$	$\frac{\sqrt{2}}{\sqrt{5}} \sin \frac{\pi jk}{5}$
$\sigma\lambda_j$	$\frac{1}{\sqrt{5}} \sin \frac{\pi jk}{5}$	$\frac{1}{\sqrt{5}} \sin \frac{\pi jk}{5}$	$\frac{-\sqrt{2}}{\sqrt{5}} \sin \frac{\pi jk}{5}$
μ_j	$\frac{\sqrt{2}}{\sqrt{5}} \sin \frac{\pi jk}{5}$	$\frac{-\sqrt{2}}{\sqrt{5}} \sin \frac{\pi jk}{5}$	0

Here, σ is an automorphism of order 2 given as follows:

$$\sigma\lambda_1 = 2\Lambda_6, \quad \sigma\lambda_3 = \Lambda_2.$$

8. For $E_8^{(1)}$,

(i) For $m = 1$, we have $a(\Lambda_0, \Lambda_0) = 1$,

(ii) The list of $a(\lambda, \mu)$ for $m = 2$:

	$2\Lambda_0$	Λ_7	Λ_1
$2\Lambda_0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
Λ_7	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{2}}$
Λ_1	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0

(iii) For $m = 3$, putting

$$\lambda_1 := 3\Lambda_0,\ \lambda_3 := \Lambda_2,\ \lambda_5 := \Lambda_0 + \Lambda_7,\ \lambda_7 := \Lambda_0 + \Lambda_1,\ \lambda_9 := \Lambda_8,$$

we have

$$a(\lambda_j, \lambda_k) = \frac{2}{\sqrt{11}} \sin \frac{\pi jk}{11}.$$

9. For $F_4^{(1)}$,

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_4
Λ_0	$\frac{2}{\sqrt{5}} \sin \frac{\pi}{5}$	$\frac{2}{\sqrt{5}} \sin \frac{2\pi}{5}$
Λ_4	$\frac{2}{\sqrt{5}} \sin \frac{2\pi}{5}$	$\frac{-2}{\sqrt{5}} \sin \frac{\pi}{5}$

(ii) For $m = 2$, putting

$$\lambda_1 := 2\Lambda_0,\ \lambda_3 := 2\Lambda_4,\ \lambda_5 := \Lambda_0 + \Lambda_4,\ \lambda_7 := \Lambda_3,\ \lambda_9 := \Lambda_1,$$

we have

$$a(\lambda_j, \lambda_k) := \frac{2}{\sqrt{11}} \sin \frac{\pi jk}{11}.$$

10. For $G_2^{(1)}$,

(i) The list of $a(\lambda, \mu)$ for $m = 1$:

	Λ_0	Λ_2
Λ_0	$\frac{2}{\sqrt{5}} \sin \frac{\pi}{5}$	$\frac{2}{\sqrt{5}} \sin \frac{2\pi}{5}$
Λ_2	$\frac{2}{\sqrt{5}} \sin \frac{2\pi}{5}$	$\frac{-2}{\sqrt{5}} \sin \frac{\pi}{5}$

(ii) For $m = 2$, putting

$$\lambda_1 := 2\Lambda_0, \quad \lambda_3 := 2\Lambda_2, \quad \lambda_5 := \Lambda_0 + \Lambda_2, \quad \lambda_7 := \Lambda_1,$$

we have

$$a(\lambda_j, \lambda_k) = \frac{2}{3} \sin \frac{\pi jk}{9}.$$

For types $A - D$, one can rewrite a summation over the Weyl group in the form of a determinant, and using this fact one can derive the following formulae on $a(\lambda, \mu)$. In Propositions 4.18 – 4.20, λ and μ are elements of P_+^m, and we set

$$\lambda = \sum_{i=0}^{l} m_i \Lambda_i, \quad \mu = \sum_{i=0}^{l} n_i \Lambda_i.$$

PROPOSITION 4.18. *Let A be the Cartan matrix of type $A_l^{(1)}$ or $C_l^{(1)}$, and for $1 \le i \le l$ set*

$$s_i := \sum_{j=i}^{l} m_j + (l+1-i), \quad t_i := \sum_{j=i}^{l} n_j + (l+1-i).$$

Then $a(\lambda, \mu)$ has the following form:

1. *For $A = A_l^{(1)}$,*

$$a(\lambda, \mu) = i^{\frac{l(l+1)}{2}} (l+1)^{-\frac{1}{2}} (m+l+1)^{-\frac{l}{2}} \det \left(e^{-\frac{\pi i}{m+l+1} s_j t_k} \right)_{1 \le j,k \le l},$$

2. *For $A = C_l^{(1)}$,*

$$a(\lambda, \mu) = (-1)^l i^{l(l+1)} \left(\frac{2}{m+l+1} \right)^{\frac{l}{2}} \det \left(\sin \frac{\pi s_j t_k}{m+l+1} \right)_{1 \le j,k \le l}.$$

PROPOSITION 4.19. *Let A be the Cartan matrix of type $B_l^{(1)}$ and set*

$$(m_1', \cdots, m_l') := \left(m_1, \cdots, m_{l-1}, \frac{m_l}{2} \right),$$
$$(n_1', \cdots, n_l') := \left(n_1, \cdots, n_{l-1}, \frac{n_l}{2} \right).$$

For $1 \le i \le l$, put

$$s_i := \sum_{j=i}^{l} m_j' + \left(l + \frac{1}{2} - i \right), \quad t_i := \sum_{j=i}^{l} n_j' + \left(l + \frac{1}{2} - i \right).$$

Then

$$a(\lambda, \mu) = i^{l(l-1)} 2^{l-1} (m+2l-1)^{-\frac{l}{2}} \det \left(\sin \frac{2\pi s_j t_k}{m+2l-1} \right)_{1 \le j,k \le l}.$$

PROPOSITION 4.20. *Let A be the Cartan matrix of type $D_l^{(1)}$, and set*

$$(m_1', \cdots, m_l') := \left(m_1, \cdots, m_{l-2}, \frac{m_{l-1}}{2} \frac{m_l}{2}\right),$$
$$(n_1', \cdots, n_l') := \left(n_1, \cdots, n_{l-2}, \frac{n_{l-1}}{2} \frac{n_l}{2}\right),$$

and

$$s_i := \begin{cases} \sum_{j=i}^{l} m_j' + (l-1-i) & \text{for } 1 \le i \le l-1, \\ -m_{l-1}' + m_l' + \frac{1}{2} & \text{for } i = l, \end{cases}$$
$$t_i := \begin{cases} \sum_{j=i}^{l} n_j' + (l-1-i) & \text{for } 1 \le i \le l-1, \\ -n_{l-1}' + n_l' + \frac{1}{2} & \text{for } i = l. \end{cases}$$

Then

$$\begin{aligned} a(\lambda, \mu) =& i^{l(l+1)} 2^{l-2} (m+2l-2)^{-\frac{l}{2}} \\ & \times \left\{ \det \left(\sin \frac{2\pi s_j t_k}{m+2l-2} \right)_{1 \le j,k \le l} \right. \\ & \left. + i^l \cdot \det \left(\cos \frac{2\pi s_j t_k}{m+2l-2} \right)_{1 \le j,k \le l} \right\}. \end{aligned}$$

PROPOSITION 4.21. *Let A be the Cartan matrix of type $G_2^{(1)}$, and set*

$$(s_1, s_2, s_3) := (2m_1 + m_2 + 3, -(m_1+1), -(m_1+m_2+2)),$$
$$(t_1, t_2, t_3) := (2n_1 + n_2 + 3, -(n_1+1), -(n_1+n_2+2)).$$

Then we have

$$a(\lambda, \mu) = \frac{-2}{(m+3)\sqrt{3}} \cdot \mathrm{Re} \left\{ \det \left(e^{\frac{2\pi i}{3(m+4)} s_j t_k} \right)_{1 \le j,k \le 3} \right\}.$$

By using these values of $a(\lambda, \mu)$ and Lemma 4.9, one can compute some $\chi_\Lambda(\tau)$'s. We will present a few examples below. Here ϕ_1, ϕ_2, ϕ_3 are the functions defined in (4.45).

EXAMPLE 4.22. For $B_l^{(1)}$,

1. The characters for $m = 1$:

$$\chi_{\Lambda_0} + \chi_{\Lambda_1} = \phi_1^{2l+1}, \quad \chi_{\Lambda_0} - \chi_{\Lambda_1} = \phi_2^{2l+1}, \quad \chi_{\Lambda_l} = \frac{1}{\sqrt{2}} \phi_3^{2l+1}.$$

2. The characters for $m = 2$:

$$\chi_{\Lambda_0+\Lambda_l} - \chi_{\Lambda_1+\Lambda_l} = 2^l \phi_1^{-2l},$$
$$\chi_{\Lambda_0+\Lambda_l} + \chi_{\Lambda_1+\Lambda_l} = 2^l \phi_2^{-2l},$$
$$\chi_{2\Lambda_0} - \chi_{2\Lambda_1} = 2^l \phi_3^{-2l}.$$

EXAMPLE 4.23. For $D_l^{(1)}$,

1. The characters for $m = 1$:

$$\chi_{\Lambda_0} + \chi_{\Lambda_1} = \phi_1^{2l}, \quad \chi_{\Lambda_0} - \chi_{\Lambda_1} = \phi_2^{2l}, \quad \chi_{\Lambda_l} = \frac{1}{2}\phi_3^{2l}.$$

2. The chracters for $m = 2$:

$$\chi_{\Lambda_0+\Lambda_l} - \chi_{\Lambda_1+\Lambda_{l-1}} = 2^{l-1} \phi_1^{1-2l},$$
$$\chi_{\Lambda_0+\Lambda_l} + \chi_{\Lambda_1+\Lambda_{l-1}} = 2^{l-1} \phi_2^{1-2l},$$
$$\chi_{2\Lambda_0} - \chi_{2\Lambda_1} = 2^{l-\frac{1}{2}} \phi_3^{1-2l}.$$

Next, let us consider the characters of the level 1 representations of $E_8^{(1)}$ as a beautiful example of a relation between characters of an affine Lie algebra and modular functions. Since the only level 1 dominant integral weight is Λ_0, it follows that

$$\chi_{\Lambda_0}\left(-\frac{1}{\tau}\right) = \chi_{\Lambda_0}(\tau), \tag{4.91}$$

because $a(\Lambda_0, \Lambda_0) = 1$. Moreover, since

$$z_1 = \frac{1}{1+30} \dim E_8 = \frac{1}{1+30} \cdot 248 = 8,$$

it follows that

$$s_{\overline{\Lambda}_0;1} = -\frac{8}{24} = -\frac{1}{3}, \tag{4.92}$$

and hence we have

$$\chi_{\Lambda_0}(\tau+1) = e^{-\frac{2\pi i}{3}} \chi_{\Lambda_0}(\tau). \tag{4.93}$$

By (4.92), the exponent of the leading term of $\chi_{\Lambda_0}(\tau)$ is $-\frac{1}{3}$, and it can be seen that the formal series expansion of $\chi_{\Lambda_0}(\tau)$ in q has the following form:

$$\chi_{\Lambda_0}(\tau) = q^{-\frac{1}{3}}\{1 + (\text{positive powers in } q)\}. \tag{4.94}$$

On the other hand, since the cubic root of the **elliptic modular function**

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

i.e.,

$$j(\tau)^{\frac{1}{3}} = q^{-\frac{1}{3}}(1 + 248q + 4124q^2 + 34752q^3 + \cdots),$$

also satisfies the same conditions (4.91), (4.93) and (4.94), it follows from Lemma 4.8 that

$$j(\tau)^{\frac{1}{3}} = \chi_{\Lambda_0}(\tau) = q^{-\frac{1}{3}} \sum_{j=0}^{\infty} \dim L(\Lambda_0)_{-j} \cdot q^j.$$

Since $L(\Lambda_0)_{-j}$ is a (not necessarily irreducible) representation space of E_8, this formula signifies that each coefficient of the formal series expansion of $j(\tau)^{\frac{1}{3}}$ has the meaning of the sum of the dimensions of finite-dimensional irreducible representations of E_8. This fact was conjectured by McKay in 1979, and was proved by Kac [**K3**] in 1980 in the way we explained here. A similar sort of "McKay observation" on the relation between $j(\tau)$ and the Monster simple group was starting point for hatching out the vertex operator algebras and the BKM algebras.

CHAPTER 5

Fusion Algebras

In this chapter, we consider the fusion algebras of an affine Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ of split type. Let $\overline{\mathfrak{g}}$ be the underlying finite-dimensional simple Lie algebra of $\mathfrak{g}$, and put a bar over the objects concerning $\overline{\mathfrak{g}}$. For example, for a dominant integral weight $\overline{\Lambda}$ of $\overline{\mathfrak{g}}$, we denote the irreducible representation space with highest weight $\overline{\Lambda}$ by $\overline{L}(\overline{\Lambda})$. Since the affine lie algebras treated in this chapter are all of split type, we have $\overset{\circ}{A} = \overline{A}$.

Fix $m \in \mathbb{Z}_{\geq 0}$, and for $\lambda, \mu, \nu \in P_+^m$, set

$$N_{\mu,\nu}^{\lambda} := \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) \tag{5.1}$$

$$N_{\mu,\nu,\lambda} := N_{\mu,\nu}^{{}^t\lambda}, \tag{5.2}$$

and call $N_{\mu,\nu}^{\lambda}$ and $N_{\mu,\nu,\lambda}$ **fusion coefficients**. Here

$$\mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) := \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}), \overline{L}(\gamma))$$

signifies the multiplicity of $\overline{L}(\gamma)$ in the tensor product representation $\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu})$. To distinguish this from the multiplicity of the weight γ in $\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu})$,

$$\mathrm{mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) = \dim(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}))_{\gamma},$$

we capitalize the Mult in this book. The following lemma is the key to studying fusion coefficients:

i.e.,

$$j(\tau)^{\frac{1}{3}} = q^{-\frac{1}{3}}(1 + 248q + 4124q^2 + 34752q^3 + \cdots),$$

also satisfies the same conditions (4.91), (4.93) and (4.94), it follows from Lemma 4.8 that

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Fix $m \in \mathbb{Z}_{\geq 0}$, and for $\lambda, \mu, \nu \in P_+^m$, set

$$N_{\mu,\nu}^{\lambda} := \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) \tag{5.1}$$

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and call $N_{\mu,\nu}^{\lambda}$ and $N_{\mu,\nu,\lambda}$ **fusion coefficients**. Here

$$\mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) := \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}), \overline{L}(\gamma))$$

signifies the multiplicity of $\overline{L}(\gamma)$ in the tensor product representation $\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu})$. To distinguish this from the multiplicity of the weight γ in $\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu})$,

$$\mathrm{mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma) = \dim(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}))_{\gamma},$$

we capitalize the Mult in this book. The following lemma is the key to studying fusion coefficients:

LEMMA 5.1. *For $\mu, \nu \in P_+^m$ and $\overline{x} \in \overline{P}$, the following formula holds:*

$$\begin{aligned}&\mathrm{ch}_{\overline{L}(\overline{\mu})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\cdot \mathrm{ch}_{\overline{L}(\overline{\nu})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\\ &\quad=\sum_{\lambda\in P_+^m} N_{\mu,\nu}^{\lambda}\mathrm{ch}_{\overline{L}(\overline{\lambda})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right).\end{aligned}$$

PROOF. Let us compute the right hand side of the formula in this lemma. By (5.1), we have the following:

$$\begin{aligned}(\text{the right hand side}) &= \sum_{\lambda\in P_+^m}\ \sum_{\substack{\gamma\in\overline{P}_+ \text{ s.t.}\\ \gamma+\overline{\rho}\equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M\\ (\text{for some } \exists w\in\overline{W})}} \varepsilon(w)\\ &\quad\times \mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma)\cdot \underbrace{\mathrm{tr}_{\overline{L}(\overline{\lambda})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)}_{\varepsilon(w)\mathrm{tr}_{\overline{L}(\overline{\lambda})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)}\\ &= \sum_{\gamma\in\overline{P}_+}\mathrm{Mult}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}(\gamma)\cdot \mathrm{tr}_{\overline{L}(\overline{\lambda})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\\ &=\mathrm{ch}_{\overline{L}(\overline{\mu})\otimes\overline{L}(\overline{\nu})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\\ &=\mathrm{ch}_{\overline{L}(\overline{\mu})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\cdot \mathrm{ch}_{\overline{L}(\overline{\nu})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)\\ &=(\text{the left hand side}).\end{aligned}$$

□

In particular for $x \in P_+^m$, applying (4.85)

$$\frac{a(\lambda,x)}{a(x)} = \mathrm{ch}_{\overline{L}(\overline{\lambda})}\left(e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}\right)$$

to λ, μ and ν, the formula in Lemma 5.1 becomes

$$\frac{a(\mu,x)}{a(x)}\cdot\frac{a(\nu,x)}{a(x)} = \sum_{x\in P_+^m} N_{\mu,\nu}^{\lambda}\cdot\frac{a(\lambda,x)}{a(x)}.$$

By multiplying $a(x)$ to both sides and using the fact that $(a(x,y))_{x,y\in P_+^m}$ is a unitary matrix, we obtain the following theorem:

THEOREM 5.2 (Verlinde's Formula). *For $\lambda, \mu, \nu \in P_+^m$, the following formula holds:*

$$N_{\lambda,\mu,\nu} = \sum_{x \in P_+^m} \frac{a(\lambda, x)a(\mu, x)a(\nu, x)}{a(x)}.$$

It follows from this formula that $N_{\lambda,\mu,\nu}$ is symmetric in λ, μ, ν. Moreover, by Theorem 4.12,

$$N^{\lambda}_{\mu, m\Lambda_0} = \delta_{\lambda,\mu}, \tag{5.3}$$

$$N^{m\Lambda_0}_{\mu,\nu} = \delta_{\mu, {}^t\nu}. \tag{5.4}$$

LEMMA 5.3. *For $\lambda, \mu, \nu \in P_+^m$, the following formula on the fusion coefficients $N_{\lambda,\mu,\nu}$ holds:*

$$N_{\lambda,\mu,\nu} = \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}) \otimes \overline{L}(\gamma), \mathbb{C}).$$

PROOF. By (5.1) and (5.2),

$$N_{\mu,\nu,\lambda} = N^{{}^t\lambda}_{\mu,\nu} = \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}), \overline{L}(\gamma))$$

$$= \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}) \otimes \overline{L}(\gamma)^*, \mathbb{C}).$$

By the way, since as we stated in §3.1 we have

$$\overline{L}(\gamma)^* = \overline{L}({}^t\gamma) = \overline{L}(-w_0\gamma) \quad (w_0 \text{ is the longest element of } \overline{W}),$$

the above formula has the following form:

$$N_{\mu,\nu,\lambda} = \sum_{\substack{\gamma \in \overline{P}_+ \text{ s.t.} \\ \gamma+\overline{\rho} \equiv w(\overline{\lambda}+\overline{\rho}) \bmod (m+g)M \\ (\text{for some } \exists w \in \overline{W})}} \varepsilon(w) \dim \mathrm{Hom}_{\overline{\mathfrak{g}}}(\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\nu}) \otimes \overline{L}(-w_0\gamma), \mathbb{C}). \tag{5.5}$$

Now let us investigate the condition on γ in the summation in the right hand side of this formula. Applying w_0 to both sides of

$$\gamma + \overline{\rho} \equiv w({}^t\overline{\lambda} + \overline{\rho}) \qquad \mathrm{mod}\ (m+g)M,$$

we have

$$w_0\gamma + w_0\overline{\rho} \equiv w_0 w({}^t\overline{\lambda} + \overline{\rho}) \qquad \mathrm{mod}\ (m+g)M. \tag{5.6}$$

On the other hand, since $w_0\overline{\rho} = -\overline{\rho}$, the two sides of this formula are

$$\begin{aligned} w_0\gamma + w_0\overline{\rho} &= -{}^t\gamma - \overline{\rho} = -({}^t\gamma + \overline{\rho}), \\ w_0 w({}^t\overline{\lambda} + \overline{\rho}) &= w_0 w w_0^{-1}(w_0\,{}^t\overline{\lambda} + w_0\overline{\rho}) \\ &= w_0 w w_0^{-1}(-\overline{\lambda} - \overline{\rho}) = -w_0 w w_0^{-1}(\overline{\lambda} + \overline{\rho}). \end{aligned}$$

Hence (5.6) can be rewritten as follows:

$${}^t\gamma + \overline{\rho} \equiv w_0 w w_0^{-1}(\overline{\lambda} + \overline{\rho}) \qquad \mathrm{mod}\ (m+g)M. \tag{5.7}$$

Replacing $t\gamma = -w_0\gamma$ with γ in (5.5) and $w_0 w w_0^{-1}$ with w in (5.7), we obtain the lemma. □

Since $N_{\mu,\nu,\lambda}$ is symmetric in λ, μ and ν, it follows from Lemma 5.3 that $N^{\lambda}_{\mu,\nu}$ also has the following expression:

LEMMA 5.4. *For $\lambda, \mu, \nu \in P^m_+$, the following formula on fusion coefficients $N^{\lambda}_{\mu,\nu}$ holds:*

$$N^{\lambda}_{\mu,\nu} = \sum_{\substack{\gamma \in \overline{P}_+ \ s.t. \\ \gamma + \overline{\rho} \equiv w(\overline{\nu} + \overline{\rho}) \ \mathrm{mod}\ (m+g)M \\ (for\ some\ \exists w \in \overline{W})}} \varepsilon(w) \mathrm{Mult}_{\overline{L}(\overline{\mu}) \otimes \overline{L}(\overline{\gamma})}(\overline{\lambda}).$$

Now, Lemma 5.1 is relations among the characters of irreducible finite-dimensional representations over a simple finite-dimensional Lie algebra $\overline{\mathfrak{g}}$ at the point $e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}$ (i.e., special values of characters). These relations do not imply the decomposition of the tensor product of representations, but imply that on the discrete set

$$\left\{ e^{-\frac{2\pi i}{m+g}(\overline{x}+\overline{\rho})}; x \in \overline{P} \right\},$$

the relations

$$\mathrm{ch}_{\overline{L}(\overline{\mu})} \cdot \mathrm{ch}_{\overline{L}(\overline{\nu})} = \sum_{\lambda \in P^m_+} N^{\lambda}_{\mu,\nu} \mathrm{ch}_{\overline{L}(\overline{\lambda})}$$

hold. So with each $\lambda \in P_+^m$ we associate the letter x_λ and consider the set of all linear combinations of the x_λ ($\lambda \in P_+^m$) over integers

$$\mathfrak{F}_m := \bigoplus_{\lambda \in P_+^m} \mathbb{Z} x_\lambda.$$

Besides the natural addition on $\mathfrak{F}_m$, defining the multiplication by

$$x_\mu \cdot x_\nu := \sum_{\lambda \in P_+^m} N_{\mu,\nu}^\lambda x_\lambda,$$

we see that this product is associative by Lemma 5.1, and hence $\mathfrak{F}_m$ becomes a commutative associative algebra by these rules. Moreover, by (5.3), the (multiplicative) inverse of x_λ is $x_{{}^t\lambda}$. This commutative associative algebra $\mathfrak{F}_m$[1] is called the **fusion algebra** of level m.

As one can see from the above discussion, $\mathfrak{F}_m$ is one modification of the "character ring", and one can interpret the unitarity of the matrix $(a(\lambda, \mu)$ as the orthogonality relations among characters.

Now let us show some examples of fusion algebras. These are computed by using the definition (5.1) of $N_{\mu,\nu}^\lambda$, or by the Verlinde formula and the numbers $a(\lambda, \mu)$ computed in §4.3.

1. For $A_1^{(1)}$, defining $\lambda_{m,j}$ by (4.89) in §4.3, we have

$$N_{\lambda_{m,i},\lambda_{m,j},\lambda_{m,k}} = \begin{cases} 1 & (i+j+k \in 2\mathbb{Z},\ |j-k| \le i \le j+k, \\ & \text{and } i+j+k \le 2m), \\ 0 & (\text{otherwise}). \end{cases}$$

2. The fusion algebra for $(A_l^{(1)}, m = 1)$ is given as follows:

$$\Lambda_i \cdot \Lambda_j = \Lambda_{i+j \bmod (l+1)}.$$

3. The fusion algebra for $(B_l^{(1)}, m = 1)$, if we put $x_i = \Lambda_i$ ($i = 0, 1, l$), is given by the following table:

	x_0	x_1	x_l
x_0	x_0	x_1	x_l
x_1	x_1	x_0	x_l
x_l	x_l	x_l	$x_0 + x_1$

[1] *Translator's note.* The structure constants $N_{\lambda,\mu}^\nu$ are believed to be non-negative, but there is no proof in the literature.

4. For $(C_l^{(1)}, m = 1)$,

$$N_{\Lambda_i,\Lambda_j,\Lambda_k} = \begin{cases} 1 & (i+j+k \in 2\mathbb{Z},\ |j-k| \le i \le j+k, \\ & \text{and } i+j+k \le 2l), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence the product of the fusion algebra is given as follows:

$$\Lambda_j \cdot \Lambda_k = \sum_{\substack{i \equiv j+k \bmod 2 \\ |j-k| \le i \le j+k \\ i+j+k \le 2l}} \Lambda_i.$$

5. The fusion algebra for $(D_l^{(1)}, m = 1)$ is given as follows:
   (a) For an odd l,

	Λ_0	Λ_1	Λ_{l-1}	Λ_l
Λ_0	Λ_0	Λ_1	Λ_{l-1}	Λ_l
Λ_1	Λ_1	Λ_0	Λ_l	Λ_{l-1}
Λ_{l-1}	Λ_{l-1}	Λ_l	Λ_1	Λ_0
Λ_l	Λ_l	Λ_{l-1}	Λ_0	Λ_1

   (b) For an even l,

	Λ_0	Λ_1	Λ_{l-1}	Λ_l
Λ_0	Λ_0	Λ_1	Λ_{l-1}	Λ_l
Λ_1	Λ_1	Λ_0	Λ_l	Λ_{l-1}
Λ_{l-1}	Λ_{l-1}	Λ_l	Λ_0	Λ_1
Λ_l	Λ_l	Λ_{l-1}	Λ_1	Λ_0

6. For $(E_6^{(1)}, m = 1)$,

	Λ_0	Λ_1	Λ_5
Λ_0	Λ_0	Λ_1	Λ_5
Λ_1	Λ_1	Λ_5	Λ_0
Λ_5	Λ_5	Λ_0	Λ_1

7. For $(E_7^{(1)}, m = 1)$,

	Λ_0	Λ_6
Λ_0	Λ_0	Λ_6
Λ_6	Λ_6	Λ_0

8. For $(E_8^{(1)}, m = 2)$, putting

$$x_0 := 2\Lambda_0, x_1 := \Lambda_1, x_7 := \Lambda_7,$$

the fusion algebra is given as follows:

	x_0	x_1	x_7
x_0	x_0	x_1	x_7
x_1	x_1	$x_0 + x_7$	x_1
x_7	x_7	x_1	x_0

9. For $(F_4^{(1)}, m = 1)$,

	Λ_0	Λ_4
Λ_0	Λ_0	Λ_4
Λ_4	Λ_4	$\Lambda_0 + \Lambda_4$

10. For $(G_2^{(1)}, m = 1)$,

	Λ_0	Λ_2
Λ_0	Λ_0	Λ_2
Λ_2	Λ_2	$\Lambda_0 + \Lambda_2$

Here for the affine Lie algebra of type $A_l^{(1)}$, one can also compute fusion coefficients from the computation the decomposition of the tensor product of representations by the Young diagrams and Richardson rule.

Note. If the level of representations rises, then the fusion algebra suddenly becomes complicated. Fusion coefficients do not always take just the values 0 and 1. For example, when $\mathfrak{g}$ is of type $C_2^{(1)}$ and $m = 4$, the rank of the fusion algebra is 15, and although we cannot show the complete list of the multiplications, concerning elements

$$\lambda := \Lambda_0 + 2\Lambda_1 + \Lambda_2, \quad \mu := 2\Lambda_0 + 2\Lambda_1,$$

we have

$$N_{\lambda,\lambda,\lambda} = 3, \quad N_{\lambda,\lambda,\mu} = 2, \quad N_{\lambda,\mu,\mu} = N_{\mu,\mu,\mu} = 1.$$

This sort of thing also happens even for an affine Lie algebra of type A. If $\mathfrak{g}$ is of type $A_2^{(1)}$ and $m = 3$, we have $N_{\rho,\rho,\rho} = 2$.

See a book on conformal field theory, e.g. [**TK**], for a more detailed discussion of fusion algebras. It seems to me that [**FMS**] contains many topics and is suitable for the next step in this area. There are also expositions of conformal field theory and related topics in [**Koh**] in this series. □

CHAPTER 6

In Lieu of Postscript – Virasoro Algebra –

In Chapter 4 and 5, I have explained the representation theory of affine Lie algebras and some of its applications. Even if we restrict ourselves to affine Lie algebras, there still are a lot of topics that have to be mentioned, for example, the theory of the branching functions of representations, soliton equations, and so on. Concerning soliton equations, an excellent expository book has already been published by those who contributed a lot to the development of the theory, so if the reader is interested in it, see, e.g., [**Hir**], [**MJD**].

The branching functions of representations are important theories, and one can naturally develop them as a continuation of Chapter 4. Although this problem itself lies in the framework of affine Lie algebras, the representation of the Virasoro algebra plays a great role when we consider this problem and develop its theory.

The Virasoro algebra is also one of the infinite-dimensional Lie algebras, but belongs to a different class from Kac-Moody algebras or BKM algebras. The Virasoro algebra is also still young. Her figure appeared and disappeared in physics in the latter half of the 1960's, but its representation theory began to be studied only in the 1980's. The fact that it coincides with the period when the theory of affine Lie algebras developed, is not an accident.

In the mid 1980's, The Virasoro algebra fell in passionate love with affine Lie algebras. The love of an affine Lie algebra for the Virasoro algebra, and the love of the Virasoro algebra for affine Lie algebras $\cdots$ each representation theory has cooperated, elevated each other, and grown up beautifully. Namely, the development of the theory of affine Lie algebras and that of the Virasoro algebra is the same event. One evidence of their affection has already appeared in this book, namely, the list of $a(\lambda, \mu)$ in §4.3.

10. For $(G_2^{(1)}, m = 1)$,

	Λ_0	Λ_2
Λ_0	Λ_0	Λ_2
Λ_2	Λ_2	$\Lambda_0 + \Lambda_2$

Here for the affine Lie algebra of type $A_l^{(1)}$, one can also compute fusion coefficients from the computation the decomposition of the tensor product of representations by the Young diagrams and Richardson rule.

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$$\lambda := \Lambda_0 + 2\Lambda_1 + \Lambda_2, \quad \mu := 2\Lambda_0 + 2\Lambda_1,$$

we have

$$N_{\lambda,\lambda,\lambda} = 3, \quad N_{\lambda,\lambda,\mu} = 2, \quad N_{\lambda,\mu,\mu} = N_{\mu,\mu,\mu} = 1.$$

This sort of thing also happens even for an affine Lie algebra of type A. If $\mathfrak{g}$ is of type $A_2^{(1)}$ and $m = 3$, we have $N_{\rho,\rho,\rho} = 2$.

See a book on conformal field theory, e.g. [**TK**], for a more detailed discussion of fusion algebras. It seems to me that [**FMS**] contains many topics and is suitable for the next step in this area. There are also expositions of conformal field theory and related topics in [**Koh**] in this series. □

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The branching functions of representations are important theories, and one can naturally develop them as a continuation of Chapter 4. Although this problem itself lies in the framework of affine Lie algebras, the representation of the Virasoro algebra plays a great role when we consider this problem and develop its theory.

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In the mid 1980's, The Virasoro algebra fell in passionate love with affine Lie algebras. The love of an affine Lie algebra for the Virasoro algebra, and the love of the Virasoro algebra for affine Lie algebras $\cdots$ each representation theory has cooperated, elevated each other, and grown up beautifully. Namely, the development of the theory of affine Lie algebras and that of the Virasoro algebra is the same event. One evidence of their affection has already appeared in this book, namely, the list of $a(\lambda, \mu)$ in §4.3.

$a(\lambda, \mu)$ itself is the summation over the Weyl group, and is defined by (4.74). One can derive many properties of $a(\lambda, \mu)$ from this formula, but this is not always suitable for a practical computation of its value. For example, the number of elements of the Weyl group of the exceptional Lie algebra E_8 is

$$\prod_{\substack{1 \le p \le 29 \\ \gcd(p,30)=1}} (p+1) = 2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30 = 696729600.$$

No one would try to compute (4.74) directly when one wants to know the values of $a(\lambda, \mu)$ for level 2 or 3 in the case of the affine Lie algebra of type $E_8^{(1)}$. Much of the numerical data in this list is obtained through the analysis of branching functions using the Virasoro algebra.

In this way, the Virasoro algebra is so important that the representation theory of an affine Lie algebra without the Virasoro algebra becomes too dreary. So, in lieu of postscript, I will briefly introduce a profile of the Virasoro algebra.

The **Virasoro algebra** is the vector space

$$\mathrm{Vir} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}L_j \oplus \mathbb{C}Z$$

with basis L_j ($j \in \mathbb{Z}$) and Z, and is the Lie algebra satisfying the following bracket products:

$$[L_j, L_k] = (j-k)L_{j+k} + \frac{j^3 - j}{12}\delta_{j+k,0}Z, \tag{6.1}$$

$$[L_j, Z] = 0. \tag{6.2}$$

By setting

$$\mathrm{Vir}_+ := \bigoplus_{j>0} \mathbb{C}L_j, \quad \mathrm{Vir}_- := \bigoplus_{j<0} \mathbb{C}L_j, \quad \mathrm{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}Z,$$

Vir decomposes as

$$\mathrm{Vir} = \mathrm{Vir}_- \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_+, \tag{6.3}$$

and by the Poincaré-Birkhoff-Witt Theorem 1.16, it follows that

$$\mathfrak{U}(\mathrm{Vir}) = \mathfrak{U}(\mathrm{Vir}_-) \cdot \mathfrak{U}(\mathrm{Vir}_0) \cdot \mathfrak{U}(\mathrm{Vir}_+). \tag{6.4}$$

Vir_0 plays the role of a Cartan subalgebra of Vir, and one can regard $\mathbb{C}L_j$ as a root space for each j, but there is no root which plays the role of the fundamental root.

By using the triangular decomposition (6.3), one can define a highest weight representation of the Virasoro algebra in the same way as before:

DEFINITION 6.1. Let (π, V) be a representation of Vir.

1. If the linear transformation $\pi(L_0)$ on V is diagonalizable, and each eigenspace is of finite dimension, then the representation (π, V) is said to be **L_0-diagonalizable.**
2. If (π, V) is L_0-diagonalizable, and one can choose complex numbers h, z and $0 \neq v_0 \in V$ satisfying the following three conditions
   (i) $\pi(Z)v_0 = zv_0$,
   (ii) $\pi(L_0)v_0 = hv_0$, $\pi(\mathrm{Vir}_+)v_0 = \{0\}$,
   (iii) $\pi(\mathfrak{U}(\mathrm{Vir}))v_0 = V$,
   then (π, V) is called a **highest weight representation**, (h, z) is called its **highest weight**, and v_0 is called a **highest weight vector**. z is called its **central charge**, and h is called its **conformal weight**.

If (π, V) is a highest weight representation, then its highest weight is uniquely determined, but its highest weight vector is determined only up to a scalar. We denote the irreducible highest weight module with highest weight (h, z) by $\mathfrak{V}(h, z)$.

We define the character of an L_0-diagonalizable representation (π, V) by

$$\mathrm{ch}_V(q) := \sum_{\lambda \in \mathbb{C}} \dim V_\lambda \cdot q^\lambda.$$

Here, V_λ $(\lambda \in \mathbb{C})$ is the eigenspace of $\pi(L_0)$ with eigenvalue λ:

$$V_\lambda := \{v \in V; \pi(L_0)v = \lambda v\}.$$

If V is the irreducible highest weight module $\mathfrak{V}(h, z)$, then we have

$$V = \bigoplus_{\lambda \in h + \mathbb{Z}_{\geq 0}} V_\lambda,$$

and hence it follows that

$$\mathrm{ch}_{\mathfrak{V}(h,z)}(q) = q^h \sum_{n=0}^{\infty} \dim V_{h+n} \cdot q^n.$$

We may simply denote the character $\mathrm{ch}_{\mathfrak{V}(h,z)}$ of an irreducible representation by $\mathrm{ch}_{(h,z)}$.

Below, we may omit π, and simply write $L_j v$ instead of $\pi(L_j)v$.

DEFINITION 6.2. Let V be a Vir-module. If there exists a positive definite Hermitian inner product $\langle\ ,\ \rangle$ satisfying the two conditions

i. $\langle L_j u, v\rangle = \langle u, L_{-j} v\rangle \quad (u, v \in V, j \in \mathbb{Z})$,

ii. $\langle Zu, v\rangle = \langle u, Zv\rangle \quad (u, v \in V)$,

then this representation is said to be **unitarizable**.

It is easy to see that for the irreducible highest weight representation $\mathfrak{V}(h, z)$, one has

$$\text{unitarizable} \Longrightarrow h, z \geq 0. \tag{6.5}$$

Moreover, one can show that

$$h \geq 0, z \geq 1 \Longrightarrow \mathfrak{V}(h, z) \text{ is unitarizable} \tag{6.6}$$

by constructing concretely a "Fock module" of the Virasoro algebra on a space of functions $\mathbb{C}[x_1, x_2, \cdots]$. From the above two assertions (6.5) and (6.6), the following question might arise:

What happens when $h \geq 0$ and $0 \leq z < 1$?

Those who considered this problem and opened a door to solve it were physicists. The Shapovalov determinant of the Virasoro algebra is called the Kac determinant. By analyzing the Kac determinant [**K2**], Friedan, Qiu and Shenker showed in [**FQS1**] and [**FQS2**] that the only possible places in this domain where a unitarizable representation exists are

$$z = z^{(m)} := 1 - \frac{6}{(m+2)(m+3)} \quad (m \in \mathbb{Z}_{\geq 0}), \tag{6.7}$$

$$h = h_{r,s}^{(m)} := \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)} \tag{6.8}$$

$$(r, s \in \mathbb{N} \quad \text{s.t.} \quad 1 \leq r \leq m+1,\ 1 \leq s \leq m+2),$$

and the fact that they are, in fact, unitarizable was shown by using a **Goddard-Kent-Olive construction** [**GKO1**],[**GKO2**] (GKO construction for short) of integrable representations of an affine Lie algebra and the Virasoro operators. A GKO construction is also called a coset construction. With this opportunity, remarkable applications of the representation theory of the Virasoro algebra started.

As one can easily see, since $h_{r,s}^{(m)}$ satisfies

$$h_{r,s}^{(m)} = h_{(m+2)-r,(m+3)-s}^{(m)},$$

one can restrict the range of r and s to

$$1 \leq s \leq r \leq m+1, \tag{6.9}$$

and also to

$$1 \le r \le m+2,\ 1 \le s \le m+3 \text{ and } r \equiv s \mod 2. \tag{6.10}$$

By imposing such a condition, one can establish the one-to-one correspondence between the range of the parameters (r, s) and representations.

In other words, one can say as follows. For $m \in \mathbb{Z}_{\ge 0}$, set

$$\widetilde{\mathrm{VIR}}^{(m)} := \{(r,s) \in \mathbb{N} \times \mathbb{N}; 1 \le r \le m+1, \quad 1 \le s \le m+2\},$$

and denote the equivalence classes of this set, where the equivalence relation is defined by

$$(r,s) \sim (m+2-r, m+3-s),$$

by $\mathrm{VIR}^{(m)}$. We denote the element of $\mathrm{VIR}^{(m)}$ defined by $(r,s) \in \widetilde{\mathrm{VIR}}^{(m)}$, i.e., the equivalence class of (r,s), by the same symbol (r,s). (6.9) and (6.10) are just choices of representatives of $\mathrm{VIR}^{(m)}$.

A unitary representation with $0 \le z < 1$ is called the **discrete series** representation of the Virasoro algebra. Its character $\mathrm{ch}_{r,s}^{(m)} := \mathrm{ch}_{\mathfrak{V}(h_{r,s}^{(m)}, z^{(m)})}$ can be obtained as an application of results of Feigin and Fuchs [**FeFu1**],[**FeFu2**], and is given by the following formula:

$$\mathrm{ch}_{r,s}^{(m)}(q) = \frac{1}{\varphi(q)} \sum_{n \in \mathbb{Z}} \left\{ q^{\frac{[2(m+2)(m+3)n+(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}} - q^{\frac{[2(m+2)(m+3)n+(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}} \right\}.$$

So, if we multiply $q^{-\frac{1}{24}z^{(m)}}$ by the character and denote it by $\Xi_{r,s}^{(m)}(\tau)$ ($q = e^{2\pi i \tau}$), then $\Xi_{r,s}^{(m)}(\tau)$ can be expressed by using the modular functions defined in (4.31) in §4.2 as follows:

$$\begin{aligned} &\Xi_{r,s}^{(m)}(\tau) = \\ &\frac{1}{\eta(\tau)} \{ f_{(m+3)r-(m+2)s,(m+2)(m+3)}(\tau) \\ &\qquad - f_{(m+3)r+(m+2)s,(m+2)(m+3)}(\tau) \}. \end{aligned} \tag{6.11}$$

Therefore, the (normalized) character of a discrete series representation of the Virasoro algebra is a modular function, and its modular transformations can be computed by Theorem 4.6. Moreover, the leading power of $\Xi_{r,s}^{(m)}(\tau)$ is $h_{r,s}^{(m)} - \frac{1}{24}z^{(m)}$, and it has the following

form:

$$\Xi_{r,s}^{(m)}(\tau) = q^{h_{r,s}^{(m)} - \frac{1}{24}z^{(m)}}\{1 + (\text{formal series in positive powers of } q)\}.$$

Suppose that a representation of the Virasoro algebra appears by accident, and its central charge is less than 1. In addition, we assume that we may realize from the situation of its birth that it is a unitarizable representation. In such a case, z and h of the representation should coincide with the list of the values (6.7) and (6.8). This is called the "conformal invariance constraint" .

What connects the Virasoro algebra and an affine Lie algebra are the differential operators

$$d_j := t^{j+1}\frac{d}{dt} \quad (j \in \mathbb{Z})$$

on $\mathbb{R}$. Since d_j becomes

$$d_j = -ie^{ij\theta}\frac{d}{d\theta}$$

by the transformation $t = e^{i\theta}$, d_j can be also regarded as a vector field on the one-dimensional torus S^1. The commutation relations as differential operators are

$$[d_j, d_k] = (k - j)d_{j+k},$$

i.e.,

$$[-d_j, -d_k] = (j - k) \cdot (-d_{j+k}),$$

and the bracket products of elements L_j which form a basis of the Virasoro algebra are those obtained by adding a central element to the bracket products of $-d_j$'s. Therefore, the Virasoro algebra can be regarded as the central extension of the Lie algebra of polynomial vector fields on the one-dimensional torus S^1, and this point of view is sometimes important.

Now, let $\mathfrak{g}$ be the affine Lie algebra over $\overline{\mathfrak{g}}$. What is going to be stated in this section can be applied for any affine Lie algebra, but I will restrict myself to a split type for simplicity. Then, first, we have $h^\vee = g$ for the dual Coxeter number. As is stated in §3.3, $\mathfrak{g}$ can be realized as

$$\mathfrak{g} = \sum_{n\in\mathbb{Z}}(t^n \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The **derived algebra** $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is the one obtained by slicing off the part $\mathbb{C}d$:

$$\mathfrak{g}' = \sum_{n \in \mathbb{Z}} (t^n \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c.$$

If we define the action of d_j on $\mathfrak{g}'$ by

$$\begin{aligned} [d_j, X(n)] &:= t^{j+1} \frac{d}{dt}(t^n \otimes X) = nX(n+j) \quad (X \in \overline{\mathfrak{g}}, n \in \mathbb{Z}), \\ [d_j, c] &:= 0, \end{aligned}$$

then the d_j $(j \in \mathbb{Z})$ act on $\mathfrak{g}'$ as derivations. Notice that as an action on $\mathfrak{g}'$, $d = d_0$ holds. Namely, the element d sliced off from $\mathfrak{g}$ simply changed its seat.

Note. d is an important element and is also called the **energy operator**. If we let d_j act on d, then it becomes a scalar multiple of d_j, and goes outside of $\mathfrak{g}$. The reason why we considered the derived subalgebra $\mathfrak{g}'$ is to avoid such problems. As it may slightly wander from the subject, we may sometimes regard a representation of $\mathfrak{g}$ as a representation of $\mathfrak{g}'$ below. These two points of view are equivalent for highest weight modules, and one can naturally define the action of d on a highest weight $\mathfrak{g}'$-module, and make it a $\mathfrak{g}$-module. And the irreducibility regarded either as a $\mathfrak{g}$-module or as a $\mathfrak{g}'$-module is equivalent. □

We consider a highest weight representation (π, V) of $\mathfrak{g}$. We do not require it to be irreducible, nor that it have non-dominant integral highest weight. For this representation, we consider the following question:

Do d_j's act on this representation space?

This means 'Can we define the element $\pi(d_j)$ of $\mathrm{End}(V)$ for each j satisfying the following three conditions?':

i. $[\pi(d_j), \pi(X(n))] = n\pi(X(n+j)) \quad (\forall X \in \overline{\mathfrak{g}},\ \forall n \in \mathbb{Z})$,
ii. $\pi(d_0) = \pi(d)$,
iii. $[\pi(d_j), \pi(d_k)] = (k-j)\pi(d_{j+k})$.

The answer to this question is 'Yes?' The reason why we added the '?' is that if we add some terms (the term coming from the central extension and shifting d by a scalar operator) to the right hand side of condition (iii) viz., if we allow some concessions or modifications to condition (iii), then the answer can be 'Yes'. Physicists call this 'additional term' a conformal anomaly. It is not an obstructive term,

but a meaningful and important term. Representations of the central extension of d_j's, i.e., the Virasoro algebra, naturally appear here.

Concerning these facts, let us proceed more concretely. For this purpose, if we choose two sets $\{u_i\}$ and $\{u^i\}$ of bases of $\overline{\mathfrak{g}}$ satisfying $(u_i, u^j) = \delta_{ij}$, then the Casimir element of $\mathfrak{g}$ can be expressed as follows:

$$\Omega = 2(c+g)d + 2\overline{\rho} + \sum_i u^i u_i + 2\sum_{n=1}^{\infty}\sum_i u^i(-n)u_i(n). \tag{6.12}$$

As was stated in §2.1, any element of $\mathfrak{g}$ commutes with Ω, but since d_j $(0 \neq j \in \mathbb{Z})$ is not an element of $\mathfrak{g}$, one cannot say that $[d_j, \Omega]$ is 0. For $j \in \mathbb{Z}$, let us study their properties. For each $j \in \mathbb{Z}$, set

$$S_j := \left[d_j, \sum_{n=1}^{\infty}\sum_i u^i(-n)u_i(n)\right]. \tag{6.13}$$

As we saw in §2.1, this is an element of the completion $\widehat{\mathfrak{U}}(\mathfrak{g})$ of the universal enveloping algebra. By (6.12) and (6.13), it follows that

$$\begin{aligned}[d_j, \Omega] &= 2(c+g)\underbrace{[d_j, d]}_{-jd_j} + 2\underbrace{[d_j, \overline{\rho}]}_{0} + \underbrace{[d_j, \sum_i u^i u_i]}_{0} + 2S_j \\ &= 2\{S_j - j(c+d)d_j\}.\end{aligned} \tag{6.14}$$

And for $X \in \overline{\mathfrak{g}}$ and $n \in \mathbb{Z}$, by simple computations using the Jacobi identity, one obtains

$$[[d_j, \Omega], X(n)] = [d_j, \underbrace{[\Omega, X(n)]}_{0}] - \underbrace{[\Omega, \underbrace{[d_j, X(n)]}_{\in \mathfrak{g}}]}_{0} = 0, \tag{6.15}$$

$$[[d_j, \Omega], d] = [d_j, \underbrace{[\Omega, d]}_{0}] - [\Omega, \underbrace{[d_j, d]}_{-jd_j}] = j[\Omega, d_j] = -j[d_j, \Omega]. \tag{6.16}$$

So, by applying (6.14) to (6.15) and (6.16), it follows that

$$\begin{aligned}&[S_j - j(c+g)d_j, X(n)] = 0, \\ &[S_j - j(c+g)d_j, d] = -j\{S_j - j(c+g)d_j\},\end{aligned}$$

and hence we obtain

$$[S_j, X(n)] = j(c+g)[d_j, X(n)] = jn(c+g)X(n+j), \tag{6.17}$$

$$[S_j, d] = -j^2(c+g)d_j - j\{S_j - j(c+g)d_j\} = -jS_j. \tag{6.18}$$

Since S_j is an element of $\widehat{\mathfrak{U}}(\mathfrak{g})$ and acts on a highest weight module as stated above, one can put π on S_j, and obtain formulae as a linear operator on V as follows:

$$[\pi(S_j), \pi(X(n))] = jn(\pi(c) + g)\pi(X(n+j)), \tag{6.19}$$

$$[\pi(S_j), \pi(d)] = -j\pi(S_j). \tag{6.20}$$

If we set the level of the representation (π, V) to be m, then $\pi(c)$ is the scalar operator $m \cdot \mathrm{Id}_V$. So for $m \neq -g$, if we set

$$\pi(d_j) := \frac{1}{j(m+g)}\pi(S_j) \tag{6.21}$$

for each j $(0 \neq j \in \mathbb{Z})$, then condition (i) is satisfied. And for $m \neq g$, by computing the commutation relations among the $\pi(S_j)$'s using (6.17) and (6.18), the commutation relations among $\pi(d_j)$'s can be computed as follows:

$$\begin{aligned}[\pi(d_j), \pi(d_k)] = (k-j)&\left\{\pi(d_{j+k}) - \frac{\pi(\Omega)}{2(m+g)}\delta_{j+k,0}\right\} \\ &+ \frac{j^3 - j}{12}\delta_{j+k,0} \cdot \frac{m}{m+g}\dim\overline{\mathfrak{g}} \cdot \mathrm{Id}_V.\end{aligned} \tag{6.22}$$

So, if we set

$$\pi(L_j) := -\pi(d_j) + \frac{\pi(\Omega)}{2(m+g)}\delta_{j,0} \quad (j \in \mathbb{Z}), \tag{6.23}$$

$$\pi(Z) := \frac{m}{m+g}\dim\overline{\mathfrak{g}} \cdot \mathrm{Id}_V, \tag{6.24}$$

then a representation space V of a highest weight representation of an affine Lie algebra becomes a representation space of the Virasoro algebra, and in this case, the central charge z and the conformal weight h of the representation of the Virasoro algebra are given by

$$z = z_\Lambda = \frac{m}{m+g}\dim\overline{\mathfrak{g}}, \tag{6.25}$$

$$h = h_{\overline{\Lambda};m} = h_\Lambda - (\Lambda|d) = \frac{(\Lambda|\Lambda + 2\rho)}{2(m+g)} - (\Lambda|d), \tag{6.26}$$

where $\Lambda \in \mathfrak{h}^*$ is the highest weight of V. Here z_Λ, $h_{\overline{\Lambda};m}$ and h_Λ are defined by the same formulae as (4.64), (4.68) and (4.67) in §4.3 (even in the case of generic $\Lambda \in \mathfrak{h}^*$). If it is necessary to indicate that a quantity is associated to $\mathfrak{g}$, then we use $\mathfrak{g}$ as a supercript, and, for example, denote z_Λ by $z_\Lambda^{\mathfrak{g}}$. Moreover, we set

$$h_\Lambda^{\mathfrak{g}} = h_\Lambda - (\Lambda|d). \tag{6.27}$$

z_Λ depends only on the level of Λ, and h does not depend on a choice of representative Λ of $\Lambda \mod \mathbb{C}\delta$. It is interesting that the quantities appearing in the topic of modular transformations reappear in relation to representations of the Virasoro algebra. In the paper that discovered these facts (appendix 3 of §2.5 in [**KP**]), it was said 'The material of this section has no apparent relevance.....the unexpected mysterious coincidence of a constant...'. It was the 'predawn' of the analysis of representations using the modular invariance and the conformal invariance.

DEFINITION 6.3. $\pi(L_j)$ (or regarded it as an element of $\widehat{\mathfrak{U}}(\mathfrak{g})$) is called a **Virasoro operator** of the affine Lie algebra $\mathfrak{g}$, and is denoted by $L_j^{\mathfrak{g}}$.

In particular, for $\Lambda \in P_+$, $L(\Lambda)$ is unitarizable as a representation of the affine Lie algebra, and it can be easily seen from the form of $L_j^{\mathfrak{g}}$ that it is also unitarizable as a representation of the Virasoro algebra. But its central charge (except for the trivial representation) is always greater than 1, and we cannot use the technique of conformal invariance constraint. So, let us consider the following situation.

Let us take a reductive subalgebra $\overline{\mathfrak{p}}$ of $\overline{\mathfrak{g}}$ so that it satisfies

$$\text{a Cartan subalgebra of } \overline{\mathfrak{p}} \subset \text{a Cartan subalgebra of } \overline{\mathfrak{g}},$$

and let $\mathfrak{p}$ be the affinization of $\overline{\mathfrak{p}}$. Then, since the Virasoro operators $L_j^{\mathfrak{g}}$ and $L_j^{\mathfrak{p}}$ satisfy

$$[L_j^{\mathfrak{g}}, X(n)] = -[d_j, X(n)] = -nX(n+j) \quad (X \in \overline{\mathfrak{g}}, n \in \mathbb{Z}), \tag{6.28}$$

$$[L_j^{\mathfrak{p}}, X(n)] = -[d_j, X(n)] = -nX(n+j) \quad (X \in \overline{\mathfrak{p}}, n \in \mathbb{Z}), \tag{6.29}$$

it follows that in particular for $X \in \overline{\mathfrak{p}}$, one has

$$[L_j^{\mathfrak{g}} - L_j^{\mathfrak{p}}, X(n)] = 0, \tag{6.30}$$

and thus

$$L_j^{\mathfrak{g};\mathfrak{p}} := L_j^{\mathfrak{g}} - L_j^{\mathfrak{p}}$$

commute with any element of the derived subalgebra $\mathfrak{p}' := [\mathfrak{p}, \mathfrak{p}]$ of $\mathfrak{p}$. The commutation relation between $L_j^{\mathfrak{g}}$ and $L_k^{\mathfrak{p}}$ can be computed in the same way as $[L_j^{\mathfrak{g}}, L_k^{\mathfrak{g}}]$ using (6.28) instead, and it is

$$[L_j^{\mathfrak{g}}, L_k^{\mathfrak{p}}] = (j-k)L_{j+k}^{\mathfrak{p}} + \frac{j^3 - j}{12}\delta_{j+k,0} \cdot \frac{\dot{c}}{\dot{c} + \dot{g}} \dim \overline{\mathfrak{p}}.$$

Here, a dot over a symbol signifies a quantity associated to $\mathfrak{p}$. Namely, $\dot{g}$ is the dual Coxeter number of $\mathfrak{p}$, and $\dot{c}$ signifies a central element

of $\mathfrak{p}$. Thus, one can immediately obtain the commutation relations among the $L_j^{\mathfrak{g};\mathfrak{p}}$:

$$[L_j^{\mathfrak{g};\mathfrak{p}}, L_k^{\mathfrak{g};\mathfrak{p}}] = (j-k)L_{j+k}^{\mathfrak{p}} + \frac{j^3-j}{12}\delta_{j+k,0}\cdot(z_m - \dot{z}_{\dot{m}}). \tag{6.31}$$

Here, m is the level of the representation (π, V) of $\mathfrak{g}$, and $\dot{m}$ is the level of the same representation regarded as a $\mathfrak{p}$-module. $L_j^{\mathfrak{g};\mathfrak{p}}$ is called a coset Virasoro operator. Its central charge is $z = z_m - \dot{z}_{\dot{m}}$.

Let us consider decomposing a representation (π, V) of $\mathfrak{g}$ regarded as $\mathfrak{p}$-module. For this purpose, we denote the space of singular vectors of V with respect to $\mathfrak{p}$ by

$$V^{\mathfrak{p}_+} := \{v \in V; xv = 0 \ (\forall x \in \mathfrak{p}_+)\}.$$

Here, $\mathfrak{p}_+$ is the positive part of $\mathfrak{p}$ with respect to a triangular decomposition. Since $L_j^{\mathfrak{g};\mathfrak{p}}$ commutes with any element of $\mathfrak{p}'$ by (6.30), $V^{\mathfrak{p}_+}$ is invariant under the action of $L_j^{\mathfrak{g};\mathfrak{p}}$. Namely, $V^{\mathfrak{p}_+}$ is a representation space of the Virasoro algebra! This representation is called a **coset representation** of the Virasoro algebra, and its representation space $V^{\mathfrak{p}_+}$ is called a **coset Virasoro module**.

When (π, V) is an integrable representation of $\mathfrak{g}$, i.e., $V = L(\Lambda)$ ($\Lambda \in P_+^m, m \in \mathbb{Z}_{\geq 0}$), it is unitarizable as a representation of $\mathfrak{g}$, and $V^{\mathfrak{p}_+}$ becomes a unitarizable representation of the Virasoro algebra. If its central charge

$$z_m^{\mathfrak{g};\mathfrak{p}} := z_m - \dot{z}_{\dot{m}} = z_m^{\mathfrak{g}} - z_{\dot{m}}^{\mathfrak{p}}$$

is smaller than 1, then a trick of the conformal invariance constraint works. Such a lucky thing, i.e, $z_m^{\mathfrak{g};\mathfrak{p}} < 1$, really happens. We refer the reader to [**KW1**] for detailed arguments, and here we show some simple examples.

EXAMPLE 6.4. Since by assumption that $\overline{\mathfrak{p}}$ is a reductive Lie subalgebra, we can take the Cartan subalgebra $\overline{\mathfrak{h}}$ of $\overline{\mathfrak{g}}$ as $\overline{\mathfrak{p}}$. Then, the "string functions" of $L(\Lambda)$ can be computed by using the conformal invariance as follows. Since we cannot denote the affinization of $\overline{\mathfrak{g}}$ by $\mathfrak{h}$ ($\because$ it is the same symbol as a Cartan subalgebra of $\mathfrak{g}$), here we denote it by

$$\widetilde{\mathfrak{h}} := \sum_{n\in\mathbb{Z}}(t^n \otimes \overline{\mathfrak{h}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

In this case, the dual Coxeter number of $\widetilde{\mathfrak{h}}$ is $\dot{g} = 0$, and being independent of the level, one has

$$z_{\dot{m}}^{\widetilde{\mathfrak{h}}} = \dim \overline{\mathfrak{h}} = l.$$

And since the dimension of a simple finite-dimensional Lie algebra can be expressed in terms of the Coxeter number h as

$$\dim \overline{\mathfrak{g}} = l(h+1)$$

(cf. [**Kos1**]), the central charge of this coset Virasoro module is

$$z_m^{\mathfrak{g};\widetilde{\mathfrak{h}}} = z_m^{\mathfrak{g}} - l = \frac{m}{m+g} \cdot l(h+1) - l.$$

For simplicity, let us consider the case where $\overline{\mathfrak{g}}$ is a symmetric simple finite-dimensional Lie algebra, i.e., of type A, D or E. Since we have $g = h$ in this case, it follows that

$$z_m^{\mathfrak{g}} = \frac{m}{m+g} \cdot l(g+1), \tag{6.32}$$

and in particular, if $m = 1$, then

$$z_{\dot{m}}^{\widetilde{\mathfrak{h}}} = l - l = 0.$$

Namely, for $\Lambda \in P_+^1$, the representation of the Virasoro algebra on $V^{\mathfrak{p}_+}$ is the direct sum of the trivial representations. Combining this fact, the distribution of the weights of $L(\Lambda)$ and the Weyl group invariance of ch_Λ, we obtain

$$\mathrm{ch}_\Lambda = \left(\sum_{\alpha \in M} e^{t_\alpha \Lambda}\right) \times \left(\text{the character of an irreducible } \widetilde{\mathfrak{h}}\text{-module}\right).$$

By the way, since the action of the negative part of $\widetilde{\mathfrak{h}}$ on $L(\Lambda)$ is free by Proposition 11.9 in Chapter 11 of [**K4**], we can summarize the result obtained in this example as a proposition:

PROPOSITION 6.5. *If $\mathfrak{g}$ is a symmetric affine Lie algebra (viz., it is an affine Lie algebra of split type of type A, D or E), and $\Lambda \in P_+^1$, the character of $L(\Lambda)$ can be expressed as follows:*

$$\mathrm{ch}_\Lambda = \frac{1}{\varphi(q)^l} \sum_{\alpha \in M} e^{t_\alpha \Lambda}.$$

Here, l is the rank of $\overline{\mathfrak{g}}$, and $\varphi(q)$ is the Euler function defined in (3.104).

Note. If λ is a weight of $L(\Lambda)$, then $\lambda + n\delta$ is a weight for $n \in \mathbb{Z}_{\geq 0}$ by Proposition 11.9 of [**K4**], but since it is a highest weight module, $\lambda + n\delta$ cannot always be a weight for $n \in \mathbb{Z}_{>0}$. So, there exists a weight λ of $L(\Lambda)$ such that $\lambda + \delta$ is not a weight of $L(\Lambda)$. Such a weight of $L(\Lambda)$ is called a **maximal weight**. When λ is a maximal weight of $L(\Lambda)$, the generating series of the dimension of the weight spaces $L(\Lambda)_{\lambda - n\delta}$ $(n \in \mathbb{Z}_{\geq 0})$

$$c_\lambda^\Lambda(q) := \sum_{n=0}^{\infty} \dim L(\Lambda)_{\lambda - n\delta} \cdot q^n$$

is called the **string function** of $L(\Lambda)$ through λ. Proposition 6.5 implies that if $\mathfrak{g}$ is a symmetric affine Lie algebra and $\Lambda \in P_+^1$, then

1. $c_\lambda^\Lambda(q) = \dfrac{1}{\varphi(q)^l}$, and
2. the set of maximal weights of $L(\Lambda) = \{t_\alpha \Lambda; \alpha \in M\}$.

□

EXAMPLE 6.6. Let us consider the decomposition of the tensor product representations $L(\Lambda') \otimes L(\Lambda'')$ of an affine Lie algebra. For $\Lambda' \in P_+^{m'}$ and $\Lambda'' \in P_+^{m''}$, the central charge of the tensor product representation

$$\pi(L_j) \otimes \mathrm{Id} + \mathrm{Id} \otimes \pi(L_j) \quad \text{and} \quad \pi(Z) \otimes \mathrm{Id} + \mathrm{Id} \otimes \pi(Z)$$

of the Virasoro algebra is $z_{m'} + z_{m''}$. On the other hand, if we take one of the irreducible components, say $L(\Lambda)$ $(\Lambda \in P_+^{m'+m''})$, then the central charge is $z_{m'+m''}$ on it. So, the central charge of the coset representation of the Virasoro algebra is

$$\begin{aligned} z = z_{m' \otimes m''} &= z_{m'} + z_{m''} - z_{m'+m''} \\ &= \left\{ \frac{m'}{m'+g} + \frac{m''}{m''+g} - \frac{m'+m''}{m'+m''+g} \right\} \dim \bar{\mathfrak{g}} \end{aligned} \tag{6.33}$$

Let us consider the simplest case, i.e., $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C})$ and $m'' = 1$. Since $\dim \bar{\mathfrak{g}} = 3$ and $g = 2$ in this case, the above formula becomes

$$z_{m' \otimes 1} = \left\{ \frac{m'}{m'+2} + \frac{1}{3} - \frac{m'+1}{m'+3} \right\} \times 3 = 1 - \frac{6}{(m'+2)(m'+3)}.$$

This is just the central charge of discrete series representations of the Virasoro algebra! In fact, as we see in Example 6.12, if Λ' varies all of the elements of P_+, the discrete series representations of the Virasoro algebra all appear in this decomposition of the tensor product. Since they are realized on the space of unitary representations of an affine

Lie algebra, they are naturally unitarizable as representations of the Virasoro algebra. This was one used method to solve the unitarity problem when $0 \leq z < 1$.

Proposition 6.5 is the formula expressing the character ch_Λ in terms of string functions. By using this expression, one can derive the formula decomposing the tensor product of a level 1 representation and a level m representation. Here, for simplicity, we state the result restricting ourselves to the 'basic representation' $L(\Lambda_0)$ for a level 1 representation:

PROPOSITION 6.7. *If* $\mathfrak{g}$ *is a symmetric affine Lie algebra, and* $\Lambda \in P_+^m$ $(m \in \mathbb{Z}_{\geq 0})$, *then*

$$\mathrm{ch}'_{\Lambda_0} \cdot \mathrm{ch}'_\Lambda = \sum_{\substack{\Lambda' \in P_+^{m+1} \ s.t. \\ \Lambda' - (\Lambda_0 + \Lambda) \in Q}} b_{\Lambda'}^{\Lambda_0 \otimes \Lambda}(\tau) \cdot \mathrm{ch}'_{\Lambda'},$$

where

(6.34)

$$\begin{aligned} b_{\Lambda'}^{\Lambda_0 \otimes \Lambda}(\tau) &= \frac{1}{\eta(\tau)^l} \sum_{w \in W} \varepsilon(w) q^{\frac{(m+g)(m+g+1)}{2} \left| \frac{w(\Lambda'+\rho)}{m+g+1} - \frac{\Lambda+\rho}{m+g} \right|^2} \\ &= \frac{1}{\eta(\tau)^l} \sum_{\alpha \in M} \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+g)(m+g+1)}{2} \left| \alpha + \frac{\overline{w}(\Lambda'+\rho)}{m+g+1} - \frac{\Lambda+\rho}{m+g} \right|^2}. \end{aligned}$$

PROOF. Since

$$\mathrm{ch}'_{\Lambda_0} = \frac{1}{\eta(\tau)^l} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}.$$

by Proposition 6.5, and

$$\mathrm{ch}'_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}$$

by (4.72), multiplying these, one obtains

$$\mathrm{ch}'_{\Lambda_0} \cdot \mathrm{ch}'_\Lambda = \frac{1}{\eta(\tau)^l} \cdot \frac{1}{A_\rho} \cdot A_{\Lambda+\rho} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}. \tag{6.35}$$

So, set

$$(\mathrm{I}) := A_{\Lambda+\rho} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}, \tag{6.36}$$

and let us compute this. Since

$$A_{\Lambda+\rho} = e^{-\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho)}$$

by (4.58), it follows that

$$\begin{aligned} e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) &= \sum_{w\in W}\sum_{\gamma\in M} \varepsilon(w) e^{w(\Lambda+\rho)+t_\gamma(\Lambda_0)} \\ &= \sum_{\overline{w}\in \overline{W}}\sum_{\beta\in M}\sum_{\gamma\in M} \varepsilon(\overline{w}) e^{t_\beta \overline{w}(\Lambda+\rho)+t_\gamma(\Lambda_0)}. \end{aligned}$$

Here if we substitute $\gamma = \beta + \gamma'$, this formula becomes

$$e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\overline{w}\in \overline{W}}\sum_{\beta\in M}\sum_{\gamma'\in M} \varepsilon(\overline{w}) e^{t_\beta(\overline{w}(\Lambda+\rho)+t_{\gamma'}(\Lambda_0))}.$$

By the way, since we have

$$t_{\gamma'}(\Lambda_0) = \overline{w} t_{\overline{w}^{-1}\gamma'} \cdot \underbrace{\overline{w}^{-1}(\Lambda_0)}_{\Lambda_0} = \overline{w} t_{\overline{w}^{-1}\gamma'}(\Lambda_0),$$

the above formula is

$$e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\overline{w}\in \overline{W}}\sum_{\beta\in M}\sum_{\gamma'\in M} \varepsilon(\overline{w}) e^{t_\beta \overline{w}(\Lambda+\rho+t_{\overline{w}^{-1}\gamma'}(\Lambda_0))}.$$

Here, if we set $t_\beta \overline{w} = w$, and substitute $\gamma := \overline{w}^{-1}\gamma'$, then, since

$$t_\gamma(\Lambda_0) = \Lambda_0 + \gamma - \frac{1}{2}|\gamma|^2\delta,$$

the above formula takes the form

$$e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\gamma\in M}\sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho+\Lambda_0+\gamma)} e^{-\frac{1}{2}|\gamma|^2\delta}. \tag{6.37}$$

Since the signature function $\varepsilon(w)$ is in the right hand side of this formula, if $\Lambda+\rho+\Lambda_0+\gamma$ is not a regular element, then the summation of this term over $w \in W$ vanishes. Thus, we have only to take the summation over γ so that $\Lambda + \rho + \Lambda_0 + \gamma$ becomes regular. In this case, i.e., when $\Lambda + \rho + \Lambda_0 + \gamma$ is a regular element, one can transfer it to the interior of the fundamental chamber via the action of the Weyl group. Namely, by Lemma 3.32, there uniquely exist $\sigma \in W$ and $\Lambda' \in P_+^{m+1}$ (mod $\mathbb{C}\delta$) satisfying

$$\begin{aligned} &\Lambda + \rho + \Lambda_0 + \gamma \equiv \sigma(\Lambda' + \rho) \qquad \mathrm{mod}\ \mathbb{C}\delta, \\ &\Lambda' - (\Lambda + \Lambda_0) \in Q. \end{aligned} \tag{6.38}$$

If we write this σ as $\sigma = t_\alpha \overline{\sigma}$ ($\alpha \in M, \overline{\sigma} \in \overline{W}$), then $\sigma(\Lambda' + \rho)$ looks like

$$\begin{aligned}(6.39)\qquad \sigma(\Lambda' + \rho) &= t_\alpha \cdot \overline{\sigma}(\Lambda' + \rho) \\ &= \overline{\sigma}(\Lambda' + \rho) + (m + g + 1)\alpha \\ &\quad - \left\{ (\overline{\sigma}(\Lambda' + \rho)|\alpha) + \frac{1}{2}|\alpha|^2(m + g + 1) \right\} \delta.\end{aligned}$$

So, from (6.38) and (6.39), we obtain

$$(6.40)\qquad \Lambda + \rho + \Lambda_0 + \gamma = \overline{\sigma}(\Lambda' + \rho) + (m + g + 1)\alpha,$$

i.e.,

$$(6.41)\qquad \gamma = \overline{\sigma}(\Lambda' + \rho) + (m + g + 1)\alpha - (\Lambda + \rho + \Lambda_0).$$

Let us rewrite (6.37) by using (6.40) and (6.41). By setting

$$\begin{aligned}F(\Lambda', \overline{\sigma}, \alpha) :=& (\overline{\sigma}(\Lambda' + \rho)|\alpha) + \frac{1}{2}|\alpha|^2(m + g + 1) \\ &- \frac{1}{2}|\overline{\sigma}(\Lambda' + \rho) + (m + g + 1)\alpha - (\Lambda + \Lambda_0 + \rho)|^2,\end{aligned}$$

we get (6.37) in the form

$$\begin{aligned}& e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) \\ &= \sum_{w \in W} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \varepsilon(w) e^{w \cdot t_\alpha \cdot \overline{\sigma}(\Lambda' + \rho)} e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \\ &= \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \varepsilon(\overline{\sigma}) \underbrace{\left\{ \sum_{w \in W} \varepsilon(w) e^{w(\Lambda' + \rho)} \right\}}_{e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta} \cdot A_{\Lambda'+\rho}} e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \\ &= \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \left\{ e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta} \cdot A_{\Lambda'+\rho} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \varepsilon(\overline{\sigma}) e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \right\}.\end{aligned}$$

Thus, (I) of (6.36) becomes

(6.42)

$$\sum_{\substack{\Lambda'\in P_+^{m+1} \text{ s.t.} \\ \Lambda'-(\Lambda+\Lambda_0)\in Q}} A_{\Lambda'+\rho}\left\{\sum_{\overline{\sigma}\in\overline{W}}\sum_{\alpha\in M}\varepsilon(\overline{\sigma})e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta}e^{-\frac{|\Lambda+\rho|^2}{2(m+g)}\delta}e^{F(\Lambda',\overline{\sigma},\alpha)\delta}\right\}.$$

Therefore, putting the coefficients of δ in the exponent of e in this formula by $G(\Lambda',\overline{\sigma},\alpha)$, we get

(6.43)

$$\begin{aligned}
&G(\Lambda',\overline{\sigma},\alpha)\\
&:= F(\Lambda',\overline{\sigma},\alpha)+\frac{|\Lambda'+\rho|^2}{2(m+g+1)}-\frac{|\Lambda+\rho|^2}{2(m+g)}\\
&= -\frac{1}{2}|\overline{\sigma}(\Lambda'+\rho)+(m+g+1)\alpha-(\Lambda+\Lambda_0+\rho)|^2\\
&\quad+(\overline{\sigma}(\Lambda'+\rho)|\alpha)+\frac{1}{2}|\alpha|^2(m+g+1)+\frac{|\Lambda'+\rho|^2}{2(m+g+1)}-\frac{|\Lambda+\rho|^2}{2(m+g)}\\
&= -\frac{1}{2}\Big\{|\overline{\sigma}(\Lambda'+\rho)+(m+g+1)\alpha-(\Lambda+\Lambda_0+\rho)|^2\\
&\qquad -2(\overline{\sigma}(\Lambda'+\rho)|\alpha)-(m+g+1)|\alpha|^2-\frac{|\Lambda'+\rho|^2}{m+g+1}+\frac{|\Lambda+\rho|^2}{m+g}\Big\}\\
&= -\frac{(m+g)(m+g+1)}{2}\cdot\left|\frac{\overline{\sigma}(\Lambda'+\rho)}{m+g+1}+\alpha-\frac{\Lambda+\rho}{m+g}\right|^2.
\end{aligned}$$

Rewriting (6.42) by using (6.43), we obtain the proposition by (6.35) and (6.36). □

Note. The level of the element of $\mathfrak{h}^*$ put on e in (6.34) of this proposition,

$$\alpha+\frac{\overline{w}(\Lambda'+\rho)}{m+g+1}-\frac{\Lambda+\rho}{m+g},$$

is 0, and hence its norm is invariant even if we add a scalar multiple of δ to this element. Therefore, we can replace this part by its $\overline{\mathfrak{h}}^*$ as follows:

(6.44)

$$b_{\Lambda'}^{\Lambda_0\otimes\Lambda}(\tau)=\frac{1}{\eta(\tau)^l}\sum_{\alpha\in M}\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})q^{\frac{(m+g)(m+g+1)}{2}\left|\alpha+\frac{\overline{w}(\overline{\Lambda'+\rho})}{m+g+1}-\frac{\overline{\Lambda+\rho}}{m+g}\right|^2}. \quad \square$$

Note. In Proposition 6.7, we impose the condition that $\mathfrak{g}$ be symmetric, and that one of the representations be of level 1, but the decomposition theorem of the tensor product of representations in this form is proved in some other cases. In any case, if the character of one of the representations is described in terms of its string functions, one can apply the above method of proof. There is no place to explain in this book, but when one of the representations is an admissible representation that is of a wider class than the integrable representations, one can also compute the decomposition of the character of the tensor product of representations. $b_{\Lambda'}^{\Lambda_0\otimes\Lambda}(\tau)$ also has a meaning as a character of the W-algebra associated to the affine Lie algebra. □

$b_{\Lambda'}^{\Lambda_0\otimes\Lambda}(\tau)$ arising in the above decomposition of the tensor product of the representations is a function only of $q = e^{2\pi i\tau}$. This is because, in the decomposition of $L(\Lambda_0)\otimes L(\Lambda)$, for each $\Lambda' \in P_+^{m+1}$, we are looking at the singular vectors whose weights are the same as $\Lambda' \in P_+^{m+1}$ modulo $\mathbb{C}\delta$, i.e., elements of $\{\Lambda' + n\delta; n \in \mathbb{Z}\}$. In general, when we write the decomposition of $L(\Lambda')\otimes L(\Lambda'')$ as

$$\mathrm{ch}'_{\Lambda'}\cdot\mathrm{ch}'_{\Lambda''} = \sum_{\Lambda\in P_+^{m'+m''}} b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)\cdot\mathrm{ch}'_{\Lambda}, \tag{6.45}$$

$b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ is called a **branching function** of the decomposition of the tensor product of representations. Since ch'_{Λ}'s are modular functions, $b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ is also a modular function, and its modular transformations can be obtained as follows. First, if we apply S to (6.45), we have

$$\mathrm{ch}'_{\Lambda'}|_S\cdot\mathrm{ch}'_{\Lambda''}|_S = \sum_{\Lambda\in P_+^{m'+m''}} b_{\Lambda}^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)\cdot\mathrm{ch}'_{\Lambda}|_S,$$

and if we rewrite this formula by using the formula for a modular transformation of ch'_{λ},

$$\mathrm{ch}'_{\lambda}|_S = \sum_{\mu} a(\lambda,\mu)\mathrm{ch}'_{\mu},$$

for Λ', Λ'' and Λ, we obtain

$$\begin{aligned}\sum_{\mu'}\sum_{\mu''} a(\Lambda',\mu')a(\Lambda'',\mu'')\cdot\mathrm{ch}'_{\mu'}\cdot\mathrm{ch}'_{\mu''} \\ = \sum_{\Lambda}\sum_{\mu} b_{\Lambda}^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)\cdot a(\Lambda,\mu)\mathrm{ch}'_{\mu}.\end{aligned} \tag{6.46}$$

By the way, since we have

$$\mathrm{ch}'_{\mu'} \cdot \mathrm{ch}'_{\mu''} = \sum_{\mu} b_{\mu}^{\mu' \otimes \mu''}(\tau) \cdot \mathrm{ch}'_{\mu}$$

from (6.45), we can rewrite (6.46) as follows:

$$\begin{aligned}\sum_{\mu'} \sum_{\mu''} \sum_{\mu} & a(\Lambda', \mu') a(\Lambda'', \mu'') b_{\mu}^{\mu' \otimes \mu''}(\tau) \cdot \mathrm{ch}'_{\mu} \\ &= \sum_{\Lambda} \sum_{\mu} b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) \cdot a(\Lambda, \mu) \mathrm{ch}'_{\mu}.\end{aligned}$$

ch'_{μ} is a function of (τ, z, t), and by the linearly independence of these functions, we obtain

$$\begin{aligned}\sum_{\mu'} \sum_{\mu''} & a(\Lambda', \mu') a(\Lambda'', \mu'') b_{\mu}^{\mu' \otimes \mu''}(\tau) \\ &= \sum_{\Lambda} b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) a(\Lambda, \mu)\end{aligned} \tag{6.47}$$

for each $\mu \in P_{+}^{m'+m''}$. To pick up the single $b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right)$ in the right hand side of this formula, we use the fact that the matrix $(a(\lambda, \mu))$ is unitary. Namely, since we have

$$\sum_{\mu} a(\Lambda, \mu) a({}^t\mu, \nu) = \delta_{\Lambda, \nu}$$

by (4.81), multiplying by both sides of (6.47) $a({}^t\mu, \nu)$ and summing over μ, we obtain the next formula:

$$b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) = \sum_{\mu} \sum_{\mu'} \sum_{\mu''} a(\Lambda', \mu') a(\Lambda'', \mu'') a(\Lambda, {}^t\mu) b_{\mu}^{\mu' \otimes \mu''}(\tau).$$

Sine we have $a({}^t\mu, \nu) = \overline{a(\mu, \nu)}$ by (4.82), this formula can be rewritten as follows:

$$b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) = \sum_{\mu} \sum_{\mu'} \sum_{\mu''} a(\Lambda', \mu') a(\Lambda'', \mu'') \overline{a(\Lambda, \mu)} b_{\mu}^{\mu' \otimes \mu''}(\tau).$$

Moreover, it is easy to compute $b_{\Lambda}^{\Lambda' \otimes \Lambda''}(\tau + 1)$, and this can be computed by Theorem 4.13 as follows. First, by letting $\tau \longrightarrow \tau + 1$ in (6.45), we have

$$e^{2\pi i s_{\Lambda'}} e^{2\pi i s_{\Lambda''}} \cdot \mathrm{ch}'_{\Lambda'} \cdot \mathrm{ch}'_{\Lambda''} = \sum_{\Lambda \in P_{+}^{m'+m''}} b_{\Lambda}^{\Lambda' \otimes \Lambda''}(\tau + 1) \cdot e^{2\pi i s_{\Lambda}} \mathrm{ch}'_{\Lambda},$$

and comparing this formula with (6.45), we obtain

$$b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau+1) = e^{2\pi i(s_{\Lambda'}+s_{\Lambda''}-s_\Lambda)} b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau).$$

Let us summarize these as a proposition:

PROPOSITION 6.8. *If $\mathfrak{g}$ is an affine Lie algebra of split type, and $\Lambda' \in P_+^{m'}, \Lambda'' \in P_+^{m''}$ and $\Lambda \in P_+^{m'+m''}$, modular transformations of the branching function $b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau)$ of the decomposition of the tensor product of the representations $L(\Lambda')\otimes L(\Lambda'')$ are given as follows:*

1. $b_\Lambda^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)$
   $= \sum_\mu \sum_{\mu'} \sum_{\mu''} a(\Lambda',\mu')a(\Lambda'',\mu'')\overline{a(\Lambda,\mu)} b_\mu^{\mu'\otimes\mu''}(\tau)$,
2. $b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau+1) = e^{2\pi i(s_{\Lambda'}+s_{\Lambda''}-s_\Lambda)} b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau)$.

REMARK 6.9. The decomposition of a tensor product has the form

$$L(\Lambda')\otimes L(\Lambda'') = \bigoplus_{\Lambda\in P_+^{m'+m''}} (V_\Lambda^{\Lambda'\otimes\Lambda''}\otimes L(\Lambda)), \tag{6.48}$$

and it is (6.45) that expresses this in terms of their characters. Here, $V_\Lambda^{\Lambda'\otimes\Lambda''}$ is a Vir-module. Notice that the right hand side of (6.48) is the decomposition regarded not as $\mathfrak{g}$-modules but as $\mathfrak{g}'$-modules. This is because, $\mathfrak{g}$ contains d, and the values of this element d depend on the weight spaces where the singular vectors, that give $L(\Lambda)$ in $L(\Lambda')\otimes L(\Lambda'')$, lie, and they are the branching functions that measure them. Therefore, the right hand side of (6.48) expresses the decomposition regarded as $(\mathrm{Vir}\oplus\mathfrak{g}')$-module. In general, the Vir-module $V_\Lambda^{\Lambda'\otimes\Lambda''}$ is not necessarily irreducible.

If we replace τ by $-\frac{1}{\tau}$ in (1) of the above proposition, we get

(6.49)

$$b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau) = \sum_\mu \sum_{\mu'} \sum_{\mu''} a(\Lambda',\mu')a(\Lambda'',\mu'')\overline{a(\Lambda,\mu)} b_\mu^{\mu'\otimes\mu''}\left(-\frac{1}{\tau}\right).$$

Using this formula, we can study the asymptotic behavior of $b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau)$ as $\tau\downarrow 0$. In the right hand side of (6.49), the terms that contribute to the principal part of the asymptotic behavior as $\tau\downarrow 0$ are those (μ',μ'',μ) when it is $(m'\Lambda_0, m''\Lambda_0, (m'+m'')\Lambda_0)$ or its image of the action of $\widetilde{W}_+$, i.e., the terms

$$(\mu',\mu'',\mu) = (m'\sigma\Lambda_0, m''\sigma\Lambda_0, (m'+m'')\sigma\Lambda_0) \quad (\sigma\in\widetilde{W}_+).$$

And for $\lambda \in P_+^m$ and $\sigma \in \widetilde{W}_+$, the formula

$$a(\lambda, m\Lambda_0) = a(\lambda, \sigma(m\Lambda_0))$$

does not necessarily hold, but in general, one can prove the following:

LEMMA 6.10. *Let $\mathfrak{g}$ be an affine Lie algebra of split type. If $\lambda', \mu' \in P_+^{m'}$; $\lambda'', \mu'' \in P_+^{m''}$; $\lambda, \mu \in P_+^{m'+m''}$ satisfy*

$$\lambda' + \lambda'' - \lambda, \mu' + \mu'' - \mu \in Q,$$

then the next formula holds for any $\sigma \in \widetilde{W}_+$:

$$a(\lambda', \mu')a(\lambda'', \mu'')a({}^t\lambda, \mu) = a(\lambda', \sigma(\mu'))a(\lambda'', \sigma(\mu''))a({}^t\lambda, \sigma(\mu)).$$

By this lemma, it follows that

$$\begin{aligned} &a(\Lambda', \sigma(m'\Lambda_0))a(\Lambda'', \sigma(m''\Lambda_0))\overline{a(\Lambda, \sigma((m'+m'')\Lambda_0))} \\ &= a(\Lambda', m'\Lambda_0)a(\Lambda'', m''\Lambda_0)\overline{a(\Lambda, (m'+m'')\Lambda_0)} \\ &= a(\Lambda')a(\Lambda'')a(\Lambda), \end{aligned}$$

and hence we obtain

$$b_\Lambda^{\Lambda' \otimes \Lambda''}(\tau) \sim |J| \cdot a(\Lambda')a(\Lambda'')a(\Lambda)e^{\frac{\pi i}{12\tau}(z_{m'} + z_{m''} - z_{m'+m''})} \tag{6.50}$$

as $\tau \downarrow 0$.

Next, if we specialize (6.45), in particular, to $z = t = 0$, then we have

(6.51)

$$\mathrm{ch}'_{\Lambda'}(\tau, 0, 0) \cdot \mathrm{ch}'_{\Lambda''}(\tau, 0, 0) = \sum_{\Lambda \in P_+^{m'+m''}} b_\Lambda^{\Lambda' \otimes \Lambda''}(\tau) \cdot \mathrm{ch}'_\Lambda(\tau, 0, 0),$$

but since the asymptotic behavior of $\mathrm{ch}'_\lambda(\tau, 0, 0) = \chi_\Lambda(\tau)$ $(\lambda \in P_+^m)$ as $\tau \downarrow 0$ is given by

$$\chi_\lambda(\tau) \sim a(\lambda)e^{\frac{\pi i}{12\tau} z_m} \quad (\lambda \in P_+^m)$$

by Theorem 4.17, it follows from this, (6.50) and (6.51) that

$$a(\Lambda')a(\Lambda'') = \sum_{\substack{\Lambda \in P_+^{m'+m''} \\ \text{s.t.}\, \Lambda - (\Lambda' + \Lambda'') \in Q}} |J| a(\Lambda')a(\Lambda'')a(\Lambda) \cdot a(\Lambda),$$

and hence we obtain

$$\sum_{\substack{\Lambda \in P_+^{m'+m''} \\ \text{s.t.}\, \Lambda - (\Lambda' + \Lambda'') \in Q}} a(\Lambda)^2 = \frac{1}{|J|}.$$

Summarizing these, we obtain the following proposition:

PROPOSITION 6.11. *Let $\mathfrak{g}$ be an affine Le algebra of split type.*

1. *For $\Lambda' \in P_+^{m'}, \Lambda'' \in P_+^{m''}$ and $\Lambda \in P_+^{m'+m''}$, the asymptotic behavior as $\tau \downarrow 0$ of the branching function $b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau)$ of the decomposition of the tensor product of representations $L(\Lambda')\otimes L(\Lambda'')$ is given by*

$$b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau) \sim |J| \cdot a(\Lambda')a(\Lambda'')a(\Lambda)e^{\frac{\pi i}{12\tau}(z_{m'}+z_{m''}-z_{m'+m''})}$$

2. *For $m \in \mathbb{Z}_{\geq 0}$, the following formula holds for any $\Lambda \in P_+^m$:*

$$\sum_{\substack{\lambda\in P_+^m \\ s.t.\lambda-\Lambda\in Q}} a(\lambda)^2 = \frac{1}{|J|}.$$

Note. From (1) of this proposition, we can conclude that for $\Lambda' \in P_+^{m'}$ and $\Lambda'' \in P_+^{m''}$, a $\Lambda \in P_+^{m'+m''}$ satisfying $\Lambda - (\Lambda' + \Lambda'') \in Q$ necessarily appears in the decomposition of the tensor product of representations $L(\Lambda')\otimes L(\Lambda'')$. □

Note. From (2) of this proposition and Corollary 4.15 in §4.3, it follows that $|J|$ coincides with the number of conjugacy classes of P_+^m with respect to Q for each fixed $m \in \mathbb{N}$. Moreover, it is known (cf. [**Kos2**]) that $|J|$ is equal to the determinant of the Cartan matrix of the simple finite-dimensional Lie algebra $\overline{\mathfrak{g}}$. □

EXAMPLE 6.12. For $A_1^{(1)} = \hat{\mathfrak{sl}}(2,\mathbb{C})$, since $\dim\overline{\mathfrak{g}} = \dim\mathfrak{sl}(2,\mathbb{C}) = 3$ and $g = 2$, the central charge of the Virasoro algebra on an integrable representation of level m is

$$z_m = \frac{m}{m+g}\dim\overline{\mathfrak{g}} = \frac{3m}{m+2}.$$

Let us consider the decomposition of the tensor product of representations $L(\Lambda_0)\otimes L(\Lambda)$. If we set

$$\lambda_{m,j} := (m-j)\Lambda_0 + j\Lambda_1 = m\Lambda_0 + \frac{j}{2}\alpha_1,$$

in the same way as (4.89) in §4.3, then we have

$$\lambda_{m,j} + 2\rho = (m+4)\Lambda_0 + \left(\frac{j}{2}+1\right)\alpha_1,$$

and

$$(\lambda_{m,j}|\lambda_{m,j}+2\rho) = j\left(\frac{j}{2}+1\right) = \frac{j(j+2)}{2},$$

which implies

$$h^{\mathfrak{g}}_{\lambda_{m,j}} = \frac{1}{2(m+2)} \cdot \frac{j(j+2)}{2} = \frac{j(j+2)}{4(m+2)}.$$

Since the level of a representation of $A_1^{(1)}$ appearing in the decomposition $L(\Lambda_0) \otimes L(\lambda_{m,j})$ is $m+1$, if we put its weight as $\lambda_{m+1,k}$, then we have

$$h^{\mathfrak{g}}_{\lambda_{m+1,k}} = \frac{k(k+2)}{4(m+3)},$$

and hence their difference is

$$\begin{aligned}
&h^{\mathfrak{g}}_{\Lambda_0} + h^{\mathfrak{g}}_{\lambda_{m,j}} - h^{\mathfrak{g}}_{\lambda_{m+1,k}} \\
&= O + \frac{j(j+2)}{4(m+2)} - \frac{k(k+2)}{4(m+3)} \\
&= \frac{j(j+2)(m+3) - k(k+2)(m+2)}{4(m+2)(m+3)} \\
&= \frac{[(j+1)(m+3) - (k+1)(m+2)]^2 - 1}{4(m+2)(m+3)} - \frac{(k-j)^2}{4} \\
&= h^{(m)}_{j+1,k+1} - \frac{(k-j)^2}{4}.
\end{aligned}$$

Namely, we have

$$h^{(m)}_{j+1,k+1} = h^{\mathfrak{g}}_{\Lambda_0} + h^{\mathfrak{g}}_{\lambda_{m,j}} - h^{\mathfrak{g}}_{\lambda_{m+1,k}} + \frac{(k-j)^2}{4}. \tag{6.52}$$

This formula shows that the first place where the singular vector of $L(\lambda_{m+1,k})$ appears in the decomposition of $L(\Lambda_0) \otimes L(\lambda_{m,j})$ is the weight space of weight

$$\Lambda_0 + \lambda_{m,j} - \frac{(k-j)^2}{4}\delta + \frac{k-j}{2}\alpha_1.$$

Here, if $L(\lambda_{m+1,k})$ appears in the decomposition of $L(\Lambda_0) \otimes L(\lambda_{m,j})$, then since it should satisfy

$$\lambda_{m+1,k} - (\Lambda_0 + \lambda_{m,j}) \in \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1,$$

it follows that $k \equiv j \mod 2$.

Next, let us compute the branching function

$$b^{\Lambda_0 \otimes \lambda_{m,j}}_{\lambda_{m+1,k}}(\tau) = \frac{1}{\eta(\tau)} \sum_{\alpha \in M} \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+2)(m+3)}{2}\left|\alpha + \frac{\overline{w}(\overline{\lambda}_{m+1,k} + \overline{\rho})}{m+3} - \frac{\overline{\lambda}_{m,j} + \overline{\rho}}{m+2}\right|^2}$$

by using (6.44). Since we have

$$M = \mathbb{Z}\alpha_1, \quad \overline{\lambda}_{m,j} + \overline{\rho} = \frac{j+1}{2}\alpha_1, \quad \overline{\lambda}_{m+1,k} + \overline{\rho} = \frac{k+1}{2}\alpha_1,$$

by putting $\alpha = n\alpha_1$ ($n \in \mathbb{Z}$), the above formula becomes

$$\begin{aligned}
(6.53)\quad & b^{\Lambda_0 \otimes \lambda_{m,j}}_{\lambda_{m+1,k}}(\tau) \\
&= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+2)(m+3)}{2}\left|n\alpha_1 + \frac{k+1}{2(m+3)}\overline{w}(\alpha_1) - \frac{j+1}{2(m+2)}\alpha_1\right|^2} \\
&= \frac{1}{\eta(\tau)} \left\{ \sum_{n\in\mathbb{Z}} q^{(m+2)(m+3)\left[n + \frac{k+1}{2(m+3)} - \frac{j+1}{2(m+2)}\right]^2} \right. \\
&\qquad \left. - \sum_{n\in\mathbb{Z}} q^{(m+2)(m+3)\left[n - \frac{k+1}{2(m+3)} - \frac{j+1}{2(m+2)}\right]^2} \right\} \\
&= \Xi^{(m)}_{j+1,k+1}(\tau).
\end{aligned}$$

Therefore, we obtain the following decomposition for $A_1^{(1)}$:

$$L(\Lambda_0) \otimes L(\lambda_{m,j}) = \sum_{\substack{0\le k\le m+1 \\ \text{s.t.} \\ j\equiv k \bmod 2}} \mathfrak{V}(h^{(m)}_{j+1,k+1}, z^{(m)}) \otimes L(\lambda_{m+1,k}).$$

(6.53) is a formula expressing characters of the Virasoro algebra in terms of the Weyl group of $A_1^{(1)}$. There is a restriction $j \equiv k \bmod 2$ in the decomposition of the tensor product, but by using the relation $h^{(m)}_{j+1,k+1} = h^{(m)}_{m+1-j,m+2-k}$, one can remove this restriction. And for all j, k, we obtain the following formula:

PROPOSITION 6.13. *For $m \in \mathbb{N}$ and $j, k \in \mathbb{Z}_{\ge 0}$ satisfying $0 \le j \le m$ and $0 \le k \le m+1$, $\Xi^{(m)}_{j+1,k+1}(\tau)$ can be expressed in terms of $A_1^{(1)}$ as follows:*

$$\Xi^{(m)}_{j+1,k+1}(\tau) = \frac{1}{\eta(\tau)} \sum_{w\in W} \varepsilon(w) q^{\frac{(m+2)(m+3)}{2}\left|\frac{w(\lambda_{m+1,k}+\rho)}{m+3} - \frac{\lambda_{m,j}+\rho}{m+2}\right|^2}.$$

Note. The characters of the discrete series of the Virasoro algebra enjoy the modular invariance properties, but there is a wider class called "minimal series" (or, BPZ series) known (thanks to [**BPZ**]) to be a class of representations with the modular invariance properties. The characters of the minimal series representations can also be expressed by using the affine Lie algebra $A_1^{(1)}$ as follows (here, we set

$l = 1, g = 2$):

$$\varphi_{\lambda,\mu}(q) := \frac{1}{\eta(\tau)^l} \sum_{w \in W} \varepsilon(w) q^{\frac{(m+g)(m'+g)}{2}\left|\frac{w(\lambda+\rho)}{m+g} - \frac{\mu+\rho}{m'+g}\right|^2} \tag{6.54}$$

$$(\lambda \in P_+^m,\ \mu \in P_+^{m'}).$$

For a minimal series representation, m and m' are non-negative integers satisfying $\gcd(m+2, m'+2) = 1$, and in particular, it is a discrete series if $m - m' = \pm 1$. When $\mathfrak{g}$ is a symmetric affine Lie algebra, the functions $\varphi_{\lambda,\mu}$ defined by (6.54) are the characters of minimal series representations of the W-algebra, and their modular transformations can be described by the matrices of the modular transformations of the affine Lie algebra. (cf. [**FKW**], [**KW2**]) □

The modular transformations and the asymptotic behavior of the characters of the Virasoro algebra can be computed very easily by using Proposition 6.8, Proposition 6.11, (6.53), (4.90) and the relation associated to the symmetry of the Dynkin diagram of $A_1^{(1)}$,

$$b_{\lambda_{m+1,k}}^{\Lambda_1 \otimes \lambda_{m,j}}(\tau) = b_{\lambda_{m+1,m+1-k}}^{\Lambda_0 \otimes \lambda_{m,m-j}}(\tau) = \Xi_{m+1-j,m+2-k}^{(m)}(\tau),$$

and we obtain the following proposition:

PROPOSITION 6.14.

1. $$\Xi_{r,s}^{(m)}\left(-\frac{1}{\tau}\right) = \sum_{\substack{r',s' \in \mathbb{N} s.t. \\ 1 \le s' \le r' \le m+1}} S_{(r,s),(r',s')}^{(m)} \Xi_{r',s'}^{(m)}(\tau).$$

   Here, $S_{(r,s),(r',s')}^{(m)}$ *are defined as follows:*

(6.55)

$$S_{(r,s),(r',s')}^{(m)} := (-1)^{(r+s)(r'+s')} \sqrt{\frac{8}{(m+2)(m+3)}} \sin\frac{\pi r r'}{m+2} \sin\frac{\pi s s'}{m+3}.$$

2. $$\Xi_{r,s}^{(m)}(\tau) \underset{\tau \downarrow 0}{\sim} \sqrt{\frac{8}{(m+2)(m+3)}} \sin\frac{\pi r}{m+2} \sin\frac{\pi s}{m+3} \cdot e^{\frac{\pi i}{12\tau} z^{(m)}}.$$

EXAMPLE 6.15. Let us compute the characters of the Virasoro algebra for $m = 1$. In this case, its central charge is $z^{(1)} = \frac{1}{2}$, and the possible values of (r, s) are

$$(r, s) = (1, 1),\ (1, 2),\ (2, 2);$$

correspondingly, $h^{(1)}_{r,s}$ is

$$h^{(1)}_{1,1} = 0, \quad h^{(1)}_{2,1} = \frac{1}{2}, \quad h^{(1)}_{2,2} = \frac{1}{16}.$$

Thus, the characters of the irreducible representations have the forms

$$\begin{aligned}
\Xi^{(1)}_{1,1}(\tau) &= q^{-\frac{1}{48}}(1 + \cdots), \\
\Xi^{(1)}_{2,1}(\tau) &= q^{-\frac{1}{48}+\frac{1}{2}}(1 + \cdots), \\
\Xi^{(1)}_{2,2}(\tau) &= q^{-\frac{1}{48}+\frac{1}{16}}(1 + \cdots) = q^{\frac{1}{24}}(1 + \cdots).
\end{aligned}$$

Arranging (r, s) in this order, the matrix of the transformation of the characters becomes

$$\left(S^{(1)}_{(r,s),(r',s')}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}.$$

So, if we set

$$\begin{aligned}
f_1(\tau) &:= \Xi^{(1)}_{1,1}(\tau) + \Xi^{(1)}_{2,1}(\tau), \\
f_2(\tau) &:= \Xi^{(1)}_{1,1}(\tau) - \Xi^{(1)}_{2,1}(\tau), \quad f_3(\tau) := \sqrt{2} \cdot \Xi^{(1)}_{2,2}(\tau),
\end{aligned}$$

then these functions have the following forms:

$$\begin{aligned}
f_1(\tau) &= q^{-\frac{1}{48}} \sum_{n=0}^{\infty} a_n q^{\frac{n}{2}} \quad (a_0 = 1), \\
f_2(\tau) &= q^{-\frac{1}{48}} \sum_{n=0}^{\infty} (-1)^n a_n q^{\frac{n}{2}}, \\
f_3(\tau) &= q^{\frac{1}{24}} \sum_{n=0}^{\infty} b_n q^n,
\end{aligned}$$

and their modular transformations are

$$f_1\left(-\frac{1}{\tau}\right) = f_1(\tau), \quad f_2\left(-\frac{1}{\tau}\right) = f_3(\tau), \quad f_3\left(-\frac{1}{\tau}\right) = f_2(\tau).$$

Thus, by Lemma 4.9, it follows that $f_i(\tau) = \phi_i(\tau)$ $(i = 1, 2, 3)$. Here, the $\phi_i(\tau)$'s are the functions defined in (4.45) in §4.2. In this way, the characters of the irreducible representations of the Virasoro algebra

for $m = 1$ are obtained as follows:

$$\begin{aligned}
\Xi^{(1)}_{1,1}(\tau) + \Xi^{(1)}_{2,1}(\tau) &= \frac{\eta(\tau)^2}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1 + q^{n-\frac{1}{2}}), \\
\Xi^{(1)}_{1,1}(\tau) - \Xi^{(1)}_{2,1}(\tau) &= \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1 - q^{n-\frac{1}{2}}), \qquad (6.56) \\
\Xi^{(1)}_{2,2}(\tau) &= \frac{\eta(2\tau)}{\eta(\tau)} = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 + q^{n}).
\end{aligned}$$

Moreover, by (2) of Proposition 6.14, the asymptotic behavior of $\Xi^{(1)}_{r,s}(\tau)$ as $\tau \downarrow 0$ is

$$\begin{aligned}
\Xi^{(1)}_{1,1}(\tau) &\sim \frac{1}{2} \cdot e^{\frac{\pi i}{24\tau}}, \\
\Xi^{(1)}_{2,1}(\tau) &\sim \frac{1}{2} \cdot e^{\frac{\pi i}{24\tau}}, \qquad (6.57) \\
\Xi^{(1)}_{2,2}(\tau) &\sim \frac{1}{\sqrt{2}} \cdot e^{\frac{\pi i}{24\tau}}.
\end{aligned}$$

EXAMPLE 6.16. Let us consider the tensor product of level 1 representations of $E_8^{(1)}$. Since the colabels $a_i^\vee$ $(0 \le i \le 8)$ of $E_8^{(1)}$ are given by

$$\begin{pmatrix} a_0^\vee & a_1^\vee & a_2^\vee & a_3^\vee & a_4^\vee & a_5^\vee & a_6^\vee & a_7^\vee \\ & & & & & a_8^\vee & & \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & 3 & & \end{pmatrix},$$

the only weight of level 1 is Λ_0, and there are three weights $2\Lambda_0, \Lambda_1, \Lambda_7$ of level 2. Moreover, since we have $h = g = \sum_{i=0}^{8} a_i^\vee = 30$, it follows that $\dim E_8 = 8 \cdot (30 + 1) = 248$. So, for $m = 1, 2$, each z_m takes

$$z_1 = \frac{1}{1+30} \cdot 248 = 8, \quad z_2 = \frac{2}{2+30} \cdot 248 = 15 + \frac{1}{2},$$

and the central charge of the Virasoro algebra appearing in the decomposition of the tensor product of representations $L(\Lambda_0) \otimes L(\Lambda_0)$ is

$$2z_1 - z_2 = \frac{1}{2}.$$

h_Λ for weights of level 1 and 2 is

$$h_{\Lambda_0} = h_{2\Lambda_0} = 0, \quad h_{\Lambda_1} = \frac{15}{16}, \quad h_{\Lambda_7} = \frac{3}{2}.$$

Let us study the decomposition of this tensor product of representations. If we let v_0 be a highest weight vector of $L(\Lambda_0)$, then $v_0 \otimes v_0$ and $f_0 v_0 \otimes v_0 - v_0 \otimes f_0 v_0$ are singular vectors, and their weights are $2\Lambda_0$ and $2\Lambda_0 - \alpha_0 = \Lambda_1$, respectively. We have

$$\begin{aligned} &2h_{\Lambda_0} - h_{2\Lambda_0} = 0 = h^{(1)}_{1,1}, \\ &2h_{\Lambda_0} - h_{\lambda_1} = -\frac{15}{16} = \frac{1}{16} - 1 = h^{(1)}_{2,2} - 1, \end{aligned} \tag{6.58}$$

and since there are no other values of h of the discrete series for $z = \frac{1}{2}$ of the Virasoro algebra than those in (6.58), which differ from $2h_{\Lambda_0} - h_{2\Lambda_0}$ or $2h_{\Lambda_0} - h_{\lambda_1}$ by an integer, $L(2\Lambda_0)$ and $L(\Lambda_1)$ can appear only in the forms $L(2\Lambda_0) \otimes \mathfrak{V}(0, \frac{1}{2})$ and $L(\Lambda_1) \otimes \mathfrak{V}(\frac{1}{16}, \frac{1}{2})$. Here, the discrepancy -1 in (6.58) comes from that the fact that the first place where a singular vector of this weight appear is $2\Lambda_0 - \alpha_0$, and that $\langle d, \alpha_0 \rangle = 1$.

Let us see what $L(\Lambda_7)$ looks like. Since

$$2h_{\Lambda_0} - h_{\Lambda_7} = -\frac{3}{2} = \frac{1}{2} - 2 = h^{(1)}_{2,1} - 2,$$

if we denote the first place where a singular vector of this weight appears by $2\Lambda_0 - \alpha$, then $\alpha \in \sum_{i=0}^{8} \mathbb{Z}_{\geq 0}\alpha_i$ should satisfy the following conditions:

i. $\Lambda_7 = 2\Lambda_0 - \alpha$,

ii. $\langle \alpha, d \rangle = 2$, i.e., $\alpha = 2\alpha_0 + \sum_{i=1}^{8} m_i \alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0})$.

For $E_8^{(1)}$, if we express $\sum_{i=0}^{8} m_i \alpha_i$ as

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 \\ & & & & & m_8 & & \end{pmatrix},$$

then by the above conditions, α looks like

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ & & & & & 1 & & \end{pmatrix}.$$

We denote the multiplicity of singular vectors in this weight space $(L(\Lambda_0) \otimes L(\Lambda_0))_{2\Lambda_0 - \Lambda_7}$ by a. What we can say about a at this stage is $a \geq 1$ only. So, we can write the decomposition of $L(\Lambda_0) \otimes L(\Lambda_0)$ as follows:

(6.59)

$$\begin{aligned} L(\Lambda_0) \otimes L(\Lambda_0) = &\left(L(2\Lambda_0) \otimes \mathfrak{V}(h^{(1)}_{1,1}, z^{(1)})\right) \oplus \left(L(\Lambda_1) \otimes \mathfrak{V}(h^{(1)}_{2,2}, z^{(1)})\right) \\ &\oplus \left(a \cdot L(\Lambda_7) \otimes \mathfrak{V}(h^{(1)}_{2,1}, z^{(1)})\right). \end{aligned}$$

Let us prove that $a = 1$.

From (6.59) and the asymptotic behavior (6.57) of the characters of the Virasoro algebra for $m = 1$, the asymptotic behavior of the branching functions is

$$
\begin{aligned}
b_{2\Lambda_0}^{\Lambda_0\otimes\Lambda_0}(\tau) &= \Xi_{1,1}^{(1)}(\tau) \sim \frac{1}{2}\cdot e^{\frac{\pi i}{24\tau}},\\
b_{\Lambda_1}^{\Lambda_0\otimes\Lambda_0}(\tau) &= \Xi_{2,2}^{(1)}(\tau) \sim \frac{1}{\sqrt{2}}\cdot e^{\frac{\pi i}{24\tau}},\\
b_{\Lambda_7}^{\Lambda_0\otimes\Lambda_0}(\tau) &= a\cdot \Xi_{2,1}^{(1)}(\tau) \sim \frac{a}{2}\cdot e^{\frac{\pi i}{24\tau}}.
\end{aligned}
\tag{6.60}
$$

On the other hand, since the asymptotic behavior of $b_{\Lambda}^{\Lambda_0\otimes\Lambda_0}(\tau)$ ($\Lambda \in P_+^2$) is given by

$$
b_{\Lambda}^{\Lambda_0\otimes\Lambda_0}(\tau) \sim a(\Lambda)e^{\frac{\pi i}{24\tau}}
\tag{6.61}
$$

by (1) of Proposition 6.11, one can see from (6.60) and (6.61) that $a(\Lambda)$ ($\Lambda \in P_+^2$) is given by

$$
a(2\Lambda_0) = \frac{1}{2}, \quad a(\Lambda_1) = \frac{1}{\sqrt{2}}, \quad a(\Lambda_7) = \frac{a}{2}.
$$

So, by using $\sum_{\Lambda\in P_+^2} a(\Lambda)^2 = 1$, we conclude that $a = 1$. Thus we obtain the decomposition of $L(\Lambda_0)\otimes L(\Lambda_0)$ as follows:

(6.62)

$$
\begin{aligned}
L(\Lambda_0)\otimes L(\Lambda_0) = &\left(L(2\Lambda_0)\otimes \mathfrak{V}(h_{1,1}^{(1)}, z^{(1)})\right) \oplus \left(L(\Lambda_1)\otimes \mathfrak{V}(h_{2,2}^{(1)}, z^{(1)})\right)\\
&\oplus \left(L(\Lambda_7)\otimes \mathfrak{V}(h_{2,1}^{(1)}, z^{(1)})\right).
\end{aligned}
$$

In terms of their characters, it has the following form:

$$
(\mathrm{ch}'_{\Lambda_0})^2 = \mathrm{ch}'_{2\Lambda_0}\cdot \Xi_{1,1}^{(1)}(\tau) + \mathrm{ch}'_{\Lambda_7}\cdot \Xi_{2,1}^{(1)}(\tau) + \mathrm{ch}'_{\Lambda_1}\cdot \Xi_{2,2}^{(1)}(\tau).
\tag{6.63}
$$

Since ch'_{Λ_0} is invariant under the modular transformations, and the modular transformations of the characters of the Virasoro algebra are known, it is quite easy to compute the modular transformations of ch'_{Λ} ($\Lambda \in P_+^2$) by using (6.63). We showed their modular transformations in §4.3. Some of the concrete values of $a(\lambda, \mu)$ shown in §4.3 are computed in this way.

We define the fusion coefficients $N^{(m)}_{\xi,\xi',\xi''}$ $(\xi,\xi',\xi'' \in \mathrm{VIR}^{(m)})$ of the Virasoro algebra by using the Verlinde formula as follows:

$$N^{(m)}_{\xi,\xi',\xi''} := \sum_{x\in\mathrm{VIR}^{(m)}} \frac{S^{(m)}_{\xi,x} S^{(m)}_{\xi',x} S^{(m)}_{\xi'',x}}{S^{(m)}_{(1,1),x}}.$$

Concerning this **fusion coefficient**, we have the next proposition:

PROPOSITION 6.17 (cf. Theorem 4.3 of [**FKW**]). *Set representatives of* ξ,ξ',ξ'' *in* $\widetilde{\mathrm{VIR}}^{(m)}$ *as* $(j+1,k+1),(j'+1,k'+1),(j''+1,k''+1)$. *The fusion coefficient* $N^{(m)}_{\xi,\xi',\xi''}$ *can be expressed by using that of the affine Lie algebra* $A^{(1)}_1$ *as follows:*

$$N^{(m)}_{\xi,\xi',\xi''} = \begin{cases} N_{\lambda_{m,j},\lambda_{m,j'},\lambda_{m,j''}} \cdot N_{\lambda_{m+1,k},\lambda_{m+1,k'},\lambda_{m+1,k''}} \\ \quad (\textit{if one can choose representatives of} \\ \quad\ \xi,\xi',\xi'' \textit{ satisfying} \\ \quad j+j'+j''\in 2\mathbb{Z},\ k+k'+k''\in 2\mathbb{Z}), \\ 0 \quad (\textit{otherwise}). \end{cases}.$$

Since we know the fusion coefficients for $A^{(1)}_1$ from §5.1, by using them, the fusion coefficients of the Virasoro algebra can be concretely described as follows:

COROLLARY 6.18. *If we set representatives of* ξ,ξ',ξ'' *in* $\widetilde{\mathrm{VIR}}^{(m)}$ *as* $(j+1,k+1),(j'+1,k'+1),(j''+1,k''+1)$, *then the fusion coefficient* $N^{(m)}_{\xi,\xi',\xi''}$ *is given by the following formula:*

$$N^{(m)}_{\xi,\xi',\xi''} = \begin{cases} 1 \quad (\textit{if one can choose representatives of} \\ \qquad \xi,\xi',\xi'' \textit{ satisfying} \\ \qquad j+j'+j'', k+k'+k''\in 2\mathbb{Z}, \\ \qquad j+j'+j''\le 2m, \\ \qquad k+k'+k''\le 2(m+1), \\ \qquad |j'-j''|\le j\le j'+j'', \\ \qquad |k'-k''|\le k\le k'+k''), \\ 0 \quad (\textit{otherwise}). \end{cases}$$

Note. There exists a similar formula for a fusion coefficient of the minimal series of the W-algebra associated to a symmetric affine Lie algebra (cf. [**FKW**]). □

The Virasoro algebra plays an important role in conformal field theories and vertex operator algebras.

Further Developments

Those who scrape together a living day by day clear have no prospect of the future. All day and every day, they only manage to finish their jobs.

There are too many fields related to infinite-dimensional Lie algebras for me to be able to cover them all. Automorphic forms, combinatorial identities, link invariants, soliton equations, quantum groups, solvable lattice models in statistical physics, theory of superstrings in particle physics, conformal field theory, gravitational field theory, etc.- the related branches in both mathematics and physics are broad, and the course of the development of the representation theory of infinite-dimensional Lie algebras has been affected by all of them. Because of these systematic and intricately intertwined connections, the representation theory of infinite-dimensional Lie algebras has grown to include even deeper contents, has absorbed new lives constantly, and has maintained its freshness.

In fact, there are more researchers who are studying the representation theory of infinite-dimensional Lie algebras for such applications than there are those studying it as a representation theory. Such things were a motive for extending the representation theory of Lie algebras in the right direction. For example, it seems that conformal field theories cannot be discovered without affects from physics. Those who told us the importance of Virasoro algebra, Neveu-Schwarz algebra, Ramond algebra and vertex operator algebras are physicists. Even now, the researchers who are studying superalgebras 'practically' are physicists. The fact that physicists are interested in those things guarantees that it is a nice 'object' to study, and the author is delighted. We mathematicians can concentrate on inquiring deeply, on pursuing beauty, and on searching for deep significance without sticking to its applications. It is good for research on the representation theory of Lie algebras to be in such a favored environment. Without retiring into oneself, nature tells us what we should do.

There always is a symmetry underlying a physical phenomenon, and if we accept that non-artificial 'good mathematics' possesses a symmetry, as the universal principle, then the fundamental algebraic systems describing its symmetry might be groups or Lie algebras. Of course, they are not all of the symmetry. But even quantum groups are born in the framework of Lie algebras, and it seems that Lie algebras are still the mother of quantum groups. Other "mothers" of vertex operator algebras include the Monster simple group and infinite-dimensional Lie algebras.

In this way, the influence on groups and Lie algebras as algebraic systems describing symmetry must continue as long as the sciences try to elucidate nature, whether mathematical or physical. Thus, I think that the representation theory of Lie algebras will enlarge its contents and make sound development without bursting like bubbles or going bankrupt.

So, I cannot imagine the direction after now. Who on earth could have predicted the present situation 30 years ago, i.e., when Kac-Moody algebras appeared? Even 20 years ago?

Therefore, I will guess about the research subjects for only a few years from now, restricting myself to the framework of the representation theory of Lie algebras. It will be a highly biased point of view, owing to my narrow outlook. Concerning the representation theory, I feel that the following directions are possible:

1. The representation theory of simple finite-dimensional Lie superalgebras and of affine Lie superalgebras.
2. The structure and the representation theory of the W-algebra associated to an affine Lie algebra and an affine Lie superalgebra.
3. The representation theory of super-conformal algebras, and of their associated conformal superalgebras. Also the relations among them.
4. The representation theory of toroidal Lie algebras.
5. Their extension to quantum groups, and their applications.
6. Et cetera $\cdots$.

The representation theory of affine Lie superalgebras is, surprisingly, an unexplored field. Even the general cases of finite-dimensional representations of simple finite-dimensional Lie superalgebras are still open. The character formulae of finite-dimensional representations of

$\mathfrak{sl}(m|n)$ were completed only recently (cf. [**Serg1**]).[1] The representation theory of the affinization of this algebra is almost unknown, but it should be a nice object.

One may feel that for a Lie superalgebra, the only difference is that the bracket product becomes the super-bracket product, and many theories work as well as Lie algebras, but this is not true. There are big obstructions. Certainly, there are some technical difficulties, but once we overcome them, we can find completely different aspects. Therefore, this originates essentially in its interior.

First, an obstruction arises in the analysis of Verma modules, but this is not the only problem. For an affine Lie algebra, there is a modular invariance as we saw in this book, and the discrete series of Virasoro algebra played a role. But, for a Lie superalgebra, the unitarity does not hold even for an integrable representation. So we cannot apply the discrete series of Virasoro algebra. It seems that the modular invariance is also broken. Therefore, its representation theory eventually differs from the representation theory of affine Lie algebras.

Even for affine Lie superalgebras, there is a case when modular invariance necessarily appears. That is, the character formula of the trivial representation, i.e., the denominator identity. One side of the denominator identity is infinite products on roots, and it is a modular function that can be expressed as a product and a quotient of the Jacobi theta functions. The denominator identity is a formula expressing it by an infinite sum. If we do this in practice, we obtain something completely different from those for affine Lie algebras.

Let us see this by a simple example. Let us consider the denominator identities of affine Lie superalgebras of type $\widehat{A}(1,0) = \widehat{\mathfrak{sl}}(2,1)$ and $\widehat{B}(1,1) = \widehat{\mathfrak{osp}}(3,2)$ that are affinizations of $A(1,0) = \mathfrak{sl}(2,1)$ and $B(1,1) = \mathfrak{osp}(3,2)$ in Figure 1.2. We name the simple roots of $A(1,0)$ and $B(1,1)$ as in Figure 1.2, and denote a new root appearing by the affinization by α_0. Then, for each case, the fundamental imaginary root looks as follows:

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 & \text{for } \widehat{A}(1,0), \\ \alpha_0 + 2\alpha_1 + \alpha_2 & \text{for } \widehat{B}(1,1). \end{cases}$$

δ is an even root in each case.

[1] *Translator's note.* See [**Serg2**] for further developments.

First, when we set $u := e^{-\alpha_0}, v := e^{-\alpha_2}, q := e^{-\delta}$, the denominator identity of $\widehat{A}(1,0)$ becomes

$$(1) \qquad \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^{n-1})(1-(uv)^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)} = \sum_{m,n\geq 0} (-1)^{m+n} u^m v^n q^{mn} - \sum_{m,n<0} (-1)^{m+n} u^m v^n q^{mn}.$$

One can also find this formula in the old analysis text of Whittaker and Watson [**WW**]. The denominator identity for $\widehat{B}(1,1)$ is slightly more complicated, and by setting $u := e^{-\alpha_1}, v := e^{-\alpha_2}, q := e^{-\delta}$, it becomes

(2)

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-vq^{n-1})(1-v^{-1}q^n)(1-(uv)^2q^{n-1})(1-(uv)^{-2}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+uvq^{n-1})(1+(uv)^{-1}q^n)(1+uv^2q^{n-1})(1+u^{-1}v^{-2}q^n)}$$

$$= \left\{ \sum_{\substack{m,n\geq 0 \\ \text{s.t.} \\ m\equiv n+1 \bmod 2}} - \sum_{\substack{m,n<0 \\ \text{s.t.} \\ m\equiv n+1 \bmod 2}} \right\} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}}.$$

According to [**Hic**], these are the Ramanujan mock theta functions! Then, one could call the denominator identity of affine Lie superalgebras a generalization of the mock theta functions.

As an application of (1), one can derive the modular transformations of the characters of the minimal series representations of the $N = 2$ super-conformal algebra (cf. [**KW3**]). And by using their modular transformations, one can compute the fusion algebra of the $N = 2$ super-conformal algebra. The reason why the denominator identity of $\widehat{A}(1,0)$ has such an application is that the $N = 2$ super-conformal algebra is the W-algebra of $\widehat{A}(1,0)$. The relations between an affine Lie superalgebra and its W-algebra, in particular between their representations, are very interesting research subjects after this. In the same way as I described the characters and their modular transformations of representations of Virasoro algebra by using $\widehat{\mathfrak{sl}}(2,\mathbb{C})$, one may possibly describe representations and their properties of the W-algebra by using the associated affine Lie superalgebra. For this purpose, one has to study representations and their properties for affine Lie superalgebras.

Similar to the case of Lie algebras and Lie superalgebras, there are some series of super-conformal algebras, and they are classified by Kac [**K1**]. Among them, the simplest are $N = 0$ (Virasoro), $N = 1$ (Neveu-Schwarz and Ramond), $N = 2$, $N = 3$, $N = 4$ super-conformal algebras,[2] and $N = 2$ super-conformal algebras related with the mirror symmetry, and they are mainly studied by algebraic geometers and physicists. From the viewpoint of the structure of $N = 2$ super-conformal algebra, this symmetry is an exchange of two odd fields. For $N = 3$ and $N = 4$, the even part of the super-conformal algebra is $\widehat{\mathfrak{sl}}(2, \mathbb{C})$ and is non-abelian. So their representation theory becomes more interesting and difficult, since they are related to the representation theory of the affine Lie algebra $\widehat{\mathfrak{sl}}(2, \mathbb{C})$. Including these, it is interesting to study the representation theory of super-conformal algebras, the modular transformations of their characters and the associated fusion algebras.

Moreover, the conformal superalgebra (CSA for short) has recently been discovered by Kac, and its definition is given in §2.7 of [**K5**]. Their representation theory has been started in [**CK**]. It is like a *matsutake* mushroom derived from a big tree called a vertex operator algebra, and it is a portable version of a super-conformal algebra and a vertex operator algebra. There is an experimental report saying that it is more delicious to munch a *matsutake* mushroom than its landlord i.e., a Japanese red pine.

Let us also munch it a bit. A portable version of Virasoro algebra is called Virasoro conformal algebra. Using the symbol L, this is the $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]L$ generated by L with the operation

$$L \underset{\lambda}{\cdot} L = (\partial + 2\lambda)L,$$

and this operation is an algebraic expression of the operator products in the corresponding vertex operator algebra. In the above formula, λ is a parameter, and the symbol $L \underset{\lambda}{\cdot} L$ is the generating function of $L_{(n)}L \in \mathbb{C}[\partial]L$ ($n \in \mathbb{Z}_{\geq 0}$). The representation theory of this algebra as a CSA, in particular the problem of extensions, can be very easily obtained (cf. [**CKW**]) by simply computing polynomials in two variables (regarding ∂ and λ as indeterminates). In other words, this is the cohomology of Virasoro algebra, and one can compare this with

[2]These names were not used yet in [**K1**]. $K_1, K_2(\cong W_2), K_3$ in [**K1**] are $N = 1, N = 2, N = 3$ respectively; S_2' is $N = 4$; and W_0 is $N = 0$. N represents the number of odd fields whose conformal dimension is $\frac{3}{2}$.

the result stated in item F in Section 3 of Chapter 2 of [**Fu**]. Furthermore, the classification of differential Lie algebras (cf. [**Cas**]) can be very easily obtained by using CSA's.

The representation theory of CSA's is in the process of developing as a new branch of representation theory, under the name 'algebraic conformal field theory'. . It will be quite interesting to see further developments such as which sort of physical or mathematical applications they have and what are the relations with the so-called topological conformal field theory, etc.

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the result stated in item F in Section 3 of Chapter 2 of [**Fu**]. Furthermore, the classification of differential Lie algebras (cf. [**Cas**]) can be very easily obtained by using CSA's.

The representation theory of CSA's is in the process of developing as a new branch of representation theory, under the name 'algebraic conformal field theory'. . It will be quite interesting to see further developments such as which sort of physical or mathematical applications they have and what are the relations with the so-called topological conformal field theory, etc.

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Among these references, let us give some remarks related to each chapter.

Chapter 1:

In [**YS**], representation theories of a group of rotations $SO(3)$ and its universal covering group $SU(2)$ are explained in detail. Going back to some years ago, when I was a student of physics, I was bothered by the words 'a representation of a group' or 'an infinitesimal transformation', which are not defined in a lecture for junior year, and by this book, I felt that the relation between a Lie group and the corresponding Lie algebra or representation theories of Lie algebras, at least for $SU(2)$ and $\mathfrak{sl}(2,\mathbb{C})$, became clear to me. It was a very impressive book in my student days.

When I could understand $\mathfrak{sl}(2,\mathbb{C})$, I wanted to know further theories. Then, the book that I found in a library which seemed easy even for me was [**Iw1**]. After explaining fundamental facts on semi-simple Lie algebras and their general properties, in II, for example, the Killing form for each simple Lie algebra is computed case by case, and it was my great pleasure to compute them along with the book. I was taken prisoner by the Lie algebras. But, even if it was interesting, there was nothing left to make a living in the world of simple finite-dimensional Lie algebras. In 1982, when I met with Kac-Moody Lie algebras, it reminded me of my deep emotion on reading these two books in my school days, and I burned with ardor. Even if it was simply a useless game at a stage, it might turn out to be useful. In this sense, although it is my personal impression, [**YS**], [**Iw1**] and [**Iw2**] might be the best introductory books to Lie algebras.[3] Even though they are old publications, their contents have never lost vitality. [**Po**] is also famous as an introductory book to a relevant topic.

Since I had a hard time in my seminar for senior year and master course with an old version of [**Hel**], it was especially memoried, and I think its Chapter 3 (Topics in Cartan subalgebras and root systems) is well-organized.

[**Bour2**] is a detailed exposition of root systems and Coxeter groups, and probably (if the reader does not dislike linear algebra) the reader may learn enough by reading only this volume. The list of root systems in its appendix is also useful.

There is an exposition of Lie superalgebras and related geometry in [**Man**].

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[3] *Translator's note.* [**Iw2**] and [**GW**] treat similar topics.

Concerning Kac-Moody Lie algebras, [**K4**] seems to be the best book both as an introductory book and as a side book. As an introductory book, the first edition of this book (Birkhäuser 1983) is easy to follow, and is rather suitable since it does not explain so many things. I was crazy about reading the manuscript of the first edition of this book. Kac-Moody Lie algebras are similar to and, at the same time, slightly different from finite-dimensional Lie algebras, but because of this small difference, we have such a new world, and I was greatly touched by several beautiful scenes developed in this book. I thought I would like to go on further than this book. This book was really impressive.

Chapter 2:

A proof of the character formula and the structure of the root system of a BKM Lie algebra is explained in §11.13 of [**K4**]. Superizing it, one may find the character formula and the root system of a Lie superalgebra given in an appendix of [**GN1**], but the theory itself can be developed in analogy to the case of Lie algebras, as I explained in Chapter 2 of this book. In particular, it seems that relations with modular forms will be developed further (see, e.g., [**Bor2**], [**Bor3**], [**Bor4**], [**GN1**] and [**GN2**]).

Chapter 3:

Concerning the representation theory of simple finite-dimensional Lie algebras, [**Hum**] is said to be a standard textbook, but it might depend upon the reader's taste. Is seems that [**FH**] is an accessible book. [**CSM**] consists of three chapters Lie algebras written by Carter, Lie groups by Segal and linear algebraic group by Macdonald and it is an well-arranged book with 190 pages. One can find a detailed explanation of the Shapovalov determinant in [**MP**].

Chapter 4:

[**Ser**] and [**Mum**] are readable introductory books on theta functions. [**Z**] is also a well-summarized exposition. It seems to me that [**KP**] is a monument to the representation theory of affine Lie algebras. It is in this paper where the modular invariance of the characters was discovered. For a continuation of this chapter, the reader may refer to Chapter 13 of [**K4**], or [**KW1**].

Chapter 5:

There are many papers on conformal field theories and on fusion algebras, and the published monographs, at the moment, are almost all written by physicists for physicists, and there are too many hard

obstructions for me to understand their symbols and the way of descriptions. In such situations it seems that [**FMS**] is accessible also to mathematicians and explains conformal field theories and fusion algebras in detail. An outline of conformal field theories and their related topics is also given in [**Koh**] in this series of courses.

Chapter 6:

Let me introduce some references treating soliton equations which are not touched upon in this book. The articles where one discovered how to describe soliton equations by means of symmetry of affine Lie algebras are [**DJKM1**], [**DJKM2**] and [**DJKM3**]. In connection with that, [**I.Fr**] was also impressive. As an exposition in Japanese, I shall first recommend [**MJ**]. This is a vivid exposition which explains the theory discovered just then. In [**MJ**], after a report by Miwa and Jimbo, they unrolled a round-table talk with Mikio Sato; the excitement and the heated atmosphere when they discovered this magnificent theory are recorded as they were. Now, it has already been a decade, without growing old as we listen to Beethoven's music, it still makes a fresh impression on me when I reread it even now.

[**MJD**] is an exposition by those who discovered the transformation group of soliton equations. On the contrary, the author of [**J1**] did not touch upon affine Lie algebras, but stated the history and the background of soliton equations and related topics in other fields of mathematics. So, it seems more recondite to me.

The bridges that connect soliton equations and affine Lie algebras are bilinear differential operators of Hirota. In [**Hir**], I am pulled in by the skillful technique of treating several soliton equations used by the author. The phrase 'affine Lie algebras' does not appear in this book; it is quite interesting to turn over in the reader's mind through looking at several examples treated in this book.

Studies on hierarchies for toroidal Lie algebras have just started, and it seems they provide us with new types of soliton equations that are different from those for affine Lie algebras (cf. [**Bil**], [**ISW**]).

Representation theories of the Virasoro algebra are well-organized in [**KR**]. This is a document of twelve lectures given by Kac in December 1985 and January 1986 at the Tata Institute in India. It was exactly the period when research on the representation theory of the Virasoro algebra was begun from both the physical and the mathematical side, and was accelerating. When these lectures were given, Kac himself was also getting more and more excited about the

relations between Virasoro algebra and affine Lie algebras. Although this book is small, it well depicts the heated atmosphere of that time.

There are some textbooks on vertex operator algebras, but since the ways to treat δ-function are different and hence the ways to describe real calculus, i.e. computation, differ, and they seem to be different at least at a glance, the reader may follow his or her own taste. I personally prefer [**FLM**] as a pioneer of vertex operator algebras. In [**K5**], it says that the essence of a vertex operator algebra is in its "locality", and the author constructs the theory from this point of view. Its description is clear, and it may tell us what a 'pleasure' it is to play with vertex operator algebras.

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