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Duality versus Dual Flatness in Quantum Information Geometry

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DUALITY VERSUS DUAL FLATNESS IN QUANTUM INFORMATION GEOMETRY

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Abstract. We investigate questions in quantum information geometry which concern the existence and non-existence of dual and dually flat structures on stratified sets of density operators on finite-dimensional Hilbert spaces. We show that the set of density operators of a given rank admits dually flat connections for which one connection is complete if and only if this rank is maximal. We prove moreover that there is never a dually flat structure on the set of pure states. Thus any general theory of quantum information geometry that involves duality concepts must inevitably be based on dual structures which are non-flat.

1. Introduction

The power and strength of classical information geometry and its applications relies in many respects on the fact that in basic situations of interest the spaces under investigation are naturally endowed with the structure of a dually flat manifold (cf. [Am1], [AN], see also section 2). Of utmost importance is then that naturally associated with the dually flat structure is a distance-like *canonical divergence function*. This divergence yields a variational characterization of geodesic projections on submanifolds by a minimizing property which is crucial for applications (cf. [Am2], [AN]).

In the last years there has also been great progress in generalizing and extending fundamental concepts and results from classical information geometry to the quantum setting. One now disposes in particular of quantum analogues and versions of the Fisher metric (cf. [MC], [Pe2], [GS], [DU], [LR]) and of α-connections (cf. [Na], [Ha], [HP], [GI]). These advances allow, for example, to extend Cramér–Rao type inequalities to the information geometry of positive density operators as well as to pure state estimation theory (cf. [FN1], [FN2], [Mats]).

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On the other hand, for the fundamental spaces of study in quantum information geometry, i.e., the sets of density operators of a given rank on a finite-dimensional Hilbert space, projections and divergence functions which have properties as nice and special as in the classical setting are till now only known to exist in the special case where the operators have maximal rank. Their existence in the full rank case is a simple consequence of dual flatness, using the well-known fact that when equipped with the Boguliobov–Kobu-Mori (BKM) inner product, the complete exponential connection and the (incomplete) mixture connection define a dually flat structure on the set of positive density operators of (cf. [Na], [AN]). On the sets of density operators of a given general rank, distinguished dual structures have been constructed by Fujiwara using symmetric logarithmic derivatives (cf. [Fu]). However, none of the Fujiwara structures is dually flat.

In view of these facts and the importance of the existence of canonical divergence functions one is therefore naturally led to ask to which extent the concept of dual flatness can be carried over and put to use on the sets of density operators of a given non-maximal rank.

By employing general structure theorems for dually flat manifolds which were obtained in our previous work [AT], the present note sets out to answer this question in a systematic way. Motivated by the fact that for the above-mentioned canonical dually flat structure on the set of positive density operators, the exponential connection is complete, we also investigate the existence of dually flat structures with a complete connection. Our main results can be stated as follows:

Theorem A *There is never a dually flat structure on the set of pure states.*

Theorem B *The set $\mathcal{S}_r$ of density operators of a given rank r on a Hilbert space of complex dimension $n \in \mathbb{N}$ admits dually flat connections for which one connection is complete if and only if r is maximal, i.e., if and only if $r = n$.*

The remaining parts of the present paper are organized as follows: In section 2 we briefly recall some basic definitions and results from [AT]. Section 3 is the heart of the paper. It contains a detailed description of our approach, which emphasizes the differences and similarities between the classical and the quantum setting, and here the proofs of Theorem A and Theorem B are given. Section 4 is the final one, containing further conclusions and a discussion of other points of interest.

2. Preliminaries

Following Amari (cf. [Am1], [AN]), a *dual structure* on a Riemannian manifold (M, g) is given by a pair of affine connections ∇ and ∇^* which are dual to each other in the sense that for all vector fields X, Y, Z on M,

$$X\, g(Y, Z) \;=\; g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)\,.$$

If in addition both connections ∇ and ∇^* have vanishing torsion and curvature, the pair (∇, ∇^*) is said to define a *dually flat structure* on (M, g).

In our previous work [AT] we obtained general obstruction and structure results for dually flat manifolds. The ones we will employ in the present note may be stated as follows:

Proposition 2.1 ([AT]) *Compact manifolds with finite fundamental group do never admit dually flat structures.*

Proposition 2.2 ([AT]) *Let (M, g, ∇, ∇^*) be a dually flat manifold. If one of the two connections is complete in the sense that all of its geodesics are defined on the whole real line, then the homotopy groups $\pi_k(M)$ vanish for $2 \le k \in \mathbb{N}$.*

The completeness assumption in Proposition 2.2 is also natural in the following respect, which is, e.g., of importance in geodesic projections: it guarantees that any two points of the manifold can be joined by a geodesic of the complete connection (cf. [AT]).

3. Stratifications and Dually Flat Structures

To illustrate our results and approach we shall first discuss the following

3.1 Basic Example

Consider the set $X = \{0, 1\}$ of elementary events. The set $\bar{\mathcal{P}}$ of classical probability distributions on X consists of the disjoint union of the line segment $\mathcal{P}_2 = \{(p_0, p_1) \in \bar{\mathcal{P}} \;:\; p_0, p_1 > 0\}$ and the two Dirac measures $\mathcal{P}_1 = \{\delta_0, \delta_1\}$: $\bar{\mathcal{P}} = \mathcal{P}_1 \uplus \mathcal{P}_2$. Extending this situation to the quantum setting leads to the set $\bar{\mathcal{S}}$ of density operators on $\mathbb{C}^X \cong \mathbb{C}^2$. A concrete realization of $\bar{\mathcal{S}}$ by matrices is given as follows (cf. [Pe2]):

$$\bar{\mathcal{S}} \quad \cong \quad \left\{ \frac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \ : \ x,y,z \in \mathbb{R}, \ x^2+y^2+z^2 \leq 1 \right\}$$

The set $\bar{\mathcal{S}}$ is diffeomorphic to the closed unit ball in $\mathbb{R}^3$. It can be stratified as $\bar{\mathcal{S}} = \mathcal{S}_1 \uplus \mathcal{S}_2$, where $\mathcal{S}_2$ denotes the set of positive density operators of (maximal) rank two, and $\mathcal{S}_1$ denotes the density operators of rank one. The sets $\mathcal{S}_2$ and $\mathcal{S}_1$ are diffeomorphic to the open three-ball and the two-sphere, respectively.

For $r = 2$, both $\mathcal{P}_r$ and $\mathcal{S}_r$ admit a dually flat structure with one complete connection (compare [AT]). For $r = 1$, $\mathcal{P}_r$ admits a trivial dually flat structure. But according to our first result, Theorem A, there is no dually flat structure on the stratum $\mathcal{S}_1$.

In what follows, we shall show that the main features of the above example generalize to states of any rank and all higher dimensions.

As we shall briefly discuss in the following section, the sets $\mathcal{P}_r$ always admit a dually flat structure with one complete connection. In contrast to this fact, in section 3.3 we will prove that the quantum analogue of $\mathcal{P}_r$, the set $\mathcal{S}_r$ of density operators of a given rank r enjoys this property iff the rank r is maximal, and that the set of pure states S_1 does never admit a dually flat structure.

3.2 The Classical Setting

Consider a nonempty finite set X and the closed simplex

$$\bar{\mathcal{P}} \ = \ \bar{\mathcal{P}}(X) \ = \ \left\{ p = \left(p_x\right)_{x \in X} \in \mathbb{R}^X \ : \ p_x \geq 0 \ \text{ for all } x \in X, \ \sum_{x \in X} p_x = 1 \right\}.$$

The support set of a probability distribution $p \in \bar{\mathcal{P}}(X)$ is defined as $\operatorname{supp} p := \{x \in X \ : \ p_x > 0\}$. To each nonempty subset A of X one may associate the corresponding (open) 'face'

$$\mathcal{P}(A) := \left\{ p \in \bar{\mathcal{P}} \ : \ \operatorname{supp} p = A \right\}. \tag{1}$$

Each open face $\mathcal{P}(A)$ is a differentiable submanifold of $\mathbb{R}^X$ of dimension $|A| - 1$. It is well known that $\mathcal{P}(A)$ carries a natural dually flat structure which is given by

$$(\mathcal{P}(A), g_A, \nabla_A^{(e)}, \nabla_A^{(m)}), \tag{2}$$

where g_A denotes the Fisher metric, and $\nabla_A^{(e)}$ denotes the complete exponential connection, and $\nabla_A^{(m)}$ denotes the mixture connection. The mixture connection is not complete, whereas the exponential connection is.

We have the stratification

$$\bar{\mathcal{P}} = \biguplus_{\emptyset \neq A \subset X} \mathcal{P}(A). \tag{3}$$

Collecting all faces $\mathcal{P}(A)$ with $|A| = r$ for a fixed r, one obtains

$$\mathcal{P}_r := \left\{ p \in \bar{\mathcal{P}} \ : \ |\operatorname{supp} p| = r \right\} = \biguplus_{\substack{\emptyset \neq A \subset X \\ |A| = r}} \mathcal{P}(A). \tag{4}$$

Figure 1 below depicts the situation in the case where $X = \{1, 2, 3\}$.

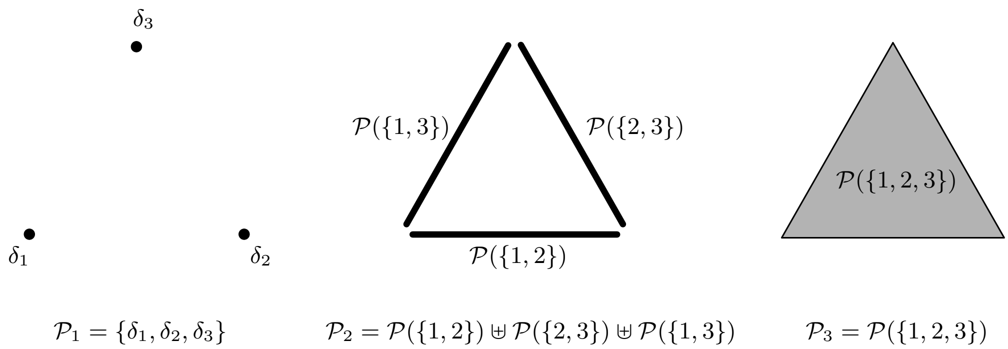

FIGURE 1.

Given a subset A of X with r elements, the set $\mathcal{P}_r$ may be considered as the orbit of $\mathcal{P}(A)$ under the permutation group of X. This gives rise to a new stratification

$$\bar{\mathcal{P}} = \biguplus_{r=1}^{|X|} \mathcal{P}_r, \tag{5}$$

which is coarser than the stratification (3). Obviously, each stratum $\mathcal{P}_r$ admits the dually flat structure given by (2).

3.3 The Quantum Setting

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space of finite (complex) dimension n, and let $\mathcal{A}$ denote the algebra of linear operators on H. A *density operator* $\rho \in \mathcal{A}$ is characterized by the properties

$$\rho = \rho^*, \qquad \rho \geq 0, \qquad \operatorname{tr}\rho = 1.$$

The set of density operators is denoted by $\bar{\mathcal{S}}$ or $\bar{\mathcal{S}}(H)$. This is a compact and convex set of real dimension

$$\dim_{\mathbb{R}} \bar{\mathcal{S}} \;=\; n^2 - 1.$$

In order to extend the definitions of $\mathcal{P}(A)$ and $\mathcal{P}_r$ to the Hilbert space setting, we have to find a generalized version of the maps $p \mapsto \operatorname{supp} p$ and $A \mapsto |A|$.

A natural candidate for the "support" of a density operator is given by its image:

$$\operatorname{im}: \;\; \bar{\mathcal{S}} \;\; \to \;\; \mathcal{G}(H) := \biguplus_{r=1}^{n} \mathcal{G}_r(H), \qquad\qquad \rho \mapsto \operatorname{im}\rho.$$

Here, $\mathcal{G}_r(H)$ denotes the Grassmann manifold of (complex) r-dimensional complex subspaces of H, which has real dimension $2r(n-r)$.

Considering the "cardinality" of a subspace A of H to be given by the complex dimension of A, by making use of an orthonormal basis $X = \{x_1, \dots, x_n\}$ of H we obtain the following commutative diagram:

$$\begin{tikzcd}
\bar{\mathcal{P}}(X) \arrow[r, "\iota"] \arrow[d, "\mathrm{supp}"'] & \bar{\mathcal{S}}(H) \arrow[d, "\mathrm{im}"] \\
2^X \arrow[r, "\mathrm{span}"] \arrow[d, "|\cdot|"'] & \mathcal{G}(H) \arrow[d, "\dim"] \\
\{1, \dots, n\} \arrow[r, "\mathrm{id}"] & \{1, \dots, n\}
\end{tikzcd}$$

Here ι is the inclusion map

$$(p_x)_{x\in X} \;\mapsto\; \sum_{x\in X} p_x\, \pi_x\,,$$

where for $x \in X$ the symbol π_x denotes the orthogonal projection onto the subspace $\mathbb{C} \cdot x$, and where 'span' assigns to each subset A of X the linear hull of A.

As in (1), to each subspace $A \subset H$ we now associate the (open) A-face of $\bar{\mathcal{S}}$, defined by

$$\mathcal{S}(A) \ := \ \left\{\rho \in \bar{\mathcal{S}} \ : \ \operatorname{im}\rho = A\right\}.$$

This is a convex subset of $\bar{\mathcal{S}}$ of real dimension

$$\dim_{\mathbb{R}} \mathcal{S}(A) = (\dim_{\mathbb{C}} A)^2 - 1,$$

which can be identified with the set of all positive density operators on the Hilbert space A.

It is well known that $\mathcal{S}(A)$ carries a dually flat structure with one complete connection. Moreover, this structure can be chosen in such a way that for each orthonormal basis $X = \{x_1, \ldots, x_r\}$ of A, the ι-pullback of this structure coincides with the structure on $\mathcal{P}(X)$ discussed in Section 3.2 (cf. [AN]).

We obtain the following analogue to the stratification (3):

$$\bar{\mathcal{S}} = \biguplus_{A \in \mathcal{G}(H)} \mathcal{S}(A). \tag{6}$$

Collecting all faces $\mathcal{S}(A)$ with $\dim A = r$ for a fixed r yields now the set of density operators of rank r:

$$\mathcal{S}_r \ := \ \left\{\rho \in \bar{\mathcal{S}} \ : \ \operatorname{rank}\rho = r\right\} \ = \ \biguplus_{A \in \mathcal{G}_r(H)} \mathcal{S}(A).$$

The sets $\mathcal{S}_r$ are differentiable manifolds of real dimension $2nr - r^2 - 1$. Notice also that the manifold $\mathcal{S}_1$ is diffeomorphic to $\mathbb{CP}^{n-1}$, the complex projective space of real dimension $2(n-1)$. Its elements are also known as the *pure states*.

Given a subspace $A \in \mathcal{G}_r$, the set $\mathcal{S}_r$ may be considered as the orbit of $\mathcal{S}(A)$ under the unitary group on H. This leads to the following stratification:

$$\bar{\mathcal{S}} \ = \ \biguplus_{r=1}^{n} \mathcal{S}_r. \tag{7}$$

The stratification (7) is obviously coarser than the stratification (6). The natural question is therefore if there is also in this case a dually flat structure on the individual strata $\mathcal{S}_r$. Notice also that from $\mathcal{S}_n = \mathcal{S}(H)$ we know that the largest stratum admits a natural dually flat structure.

We first treat the case of pure states (i.e., $r = 1$) which is of special importance in the statistical estimation theory of pure state models (cf. [FN1], [FN2], [Mats]). The quantum analogue of the Fisher metric is here given by the Fubini-Study metric on $\mathcal{S}_1 \cong \mathcal{G}_1$, and it is clear that $\mathcal{S}_1$ admits a multitude of dual structures. However, in contrast to this fact one has:

Theorem 3.1 *There exists no dually flat structure on the set $\mathcal{S}_1$ of pure states.*

Proof. The stratum $\mathcal{S}_1$ is diffeomorphic to a complex projective space so that, in particular, its fundamental group is trivial. Proposition 2.1 implies therefore that $\mathcal{S}_1$ does not admit any dually flat structure. □

To deal with the existence problem for dually flat structures on sets of mixed states, we shall employ the following proposition.

Proposition 3.2 *The stratum $\mathcal{S}_r$ and the Grassmann manifold $\mathcal{G}_r$ are homotopy equivalent and therefore have isomorphic homotopy groups.*

Proof. Given a complex subspace $A \subset H$ of complex dimension r, define the center of the face $\mathcal{S}(A)$ as

$$\rho_A := \frac{1}{\dim A} \text{Orthogonal projection onto } A.$$

We will now construct a homotopy equivalence between $\mathcal{G}_r$ and $\mathcal{S}_r$ as follows (compare Figure 2):

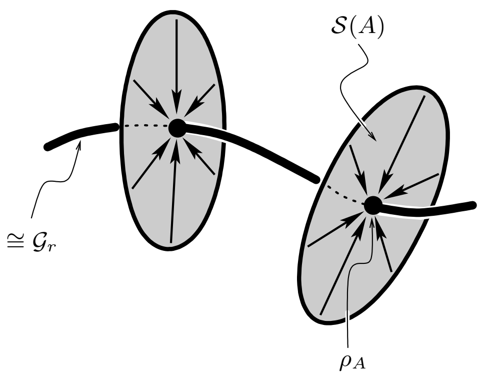

FIGURE 2.

Consider now the following maps:

$$f : \mathcal{G}_r \to \mathcal{S}_r, \qquad A \mapsto \rho_A,$$

$$g : \mathcal{S}_r \to \mathcal{G}_r, \qquad \rho \mapsto \operatorname{im} \rho.$$

Obviously, the composition $g \circ f$ is equal to the identity on $\mathcal{G}_r$. Furthermore, the homotopy

$$F : [0, 1] \times \mathcal{S}_r \to \mathcal{S}_r, \qquad (t, \rho) \mapsto (1 - t)\, \rho + t\, \rho_{\operatorname{im} \rho}$$

which satisfies

$$F(0, \cdot) = \mathrm{id}_{\mathcal{S}_r}, \qquad F(1, \cdot) = f \circ g,$$

provides a homotopic deformation of the composition $f \circ g$ to the identity on $\mathcal{S}_r$. □

Theorem 3.3 *The set $\mathcal{S}_r$ of density operators of a given rank r on a Hilbert space of complex dimension $n \in \mathbb{N}$ admits dually flat connections for which one connection is complete if and only if r is maximal, i.e., if and only if $r = n$.*

Proof. Notice first that when equipped with the Boguliobov–Kobu-Mori inner product, the exponential and mixture connection define a dually flat structure on the set of density operators of full rank, and that the exponential connection is complete.

Assume now that for $r < n$ there is a dually flat structure with one complete connection on $\mathcal{S}_r$.

According to Proposition 3.2, the manifold $\mathcal{S}_r$ and the Grassmann manifold $\mathcal{G}_r$ have isomorphic homotopy groups.

Using the coset representation $\mathcal{G}_r = SU(n)/(\,SU(n) \cap (U(r) \times U(n - r))\,)$ of the Grassmannian as a symmetric space (cf., e.g., [He]), one easily sees that $\mathcal{G}_r$ is simply connected. The Hurewicz Isomorphism Theorem in Algebraic Topology (cf. [Sp]) implies therefore that $\mathcal{G}_r$ and $\mathcal{S}_r$ possess at least one higher-dimensional homotopy group which is non-trivial. This, however, contradicts Proposition 2.2. □

4. Conclusions and Further Remarks

Any general theory of quantum information geometry must include as a special case a theory of density operators on finite-dimensional Hilbert spaces and, in particular, the estimation theory of quantum pure state models. An important consequence of our results is therefore that any such general theory that involves duality concepts must inevitably be based on dual structures which are non-flat.

As we mentioned in the introduction, for dually flat manifolds there is a natural as well as important variational characterization of geodesic projections by distance-like divergence functions. We feel that a general theory of dual but not necessarily dually flat structures must extend this divergence concept appropiately. In this regard, special emphasis should be placed on the investigation of dual structures for which the associated connections have vanishing torsion. In fact, for torsion-free dual structures one knows from work of Matumoto (cf. [Matu]) that any such structure can—though, however, not in a canonical way—be obtained from divergence functions in the sense of Eguchi (cf. [Eg]).

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