Squashed entanglement in infinite dimensions

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Abstract

We analyse two possible definitions of the squashed entanglement in an infinite-dimensional bipartite system: direct translation of the finite-dimensional definition and its universal extension. It is shown that the both definitions produce the same lower semicontinuous entanglement measure possessing all basis properties of the squashed entanglement on the set of states having at least one finite marginal entropy. Is also shown that the second definition gives an adequate extension of this measure to the set of all states of infinite-dimensional bipartite system.

A general condition relating continuity of the squashed entanglement to continuity of the quantum mutual information is proved and its corollaries are considered.

Continuity bound for the squashed entanglement under the energy constraint on one subsystem is obtained by using the tight continuity bound for conditional mutual information (proved in the Appendix by using Winter's technique). It is shown that the same continuity bound is valid for the entanglement of formation. As a result the asymptotic continuity of the both entanglement measures under the energy constraint on one subsystem is proved.

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1 Introduction

Entanglement is an essential feature of quantum systems which can be considered as a special quantum correlation having no classical analogue. One of the main tasks of quantum information theory consists in finding appropriate quantitative characteristics of entanglement in bi- and multipartite quantum systems and in studying their properties [7, 8, 13, 25, 30].

Among the existing entanglement measures in finite-dimensional bipartite systems, the squashed entanglement (introduced independently in [4] and in [29]) is one of the most interesting one. It possesses all the basic properties of an entanglement measure including the additivity [2, 4]. Mathematically, the squashed entanglement is interesting due to its definition which includes the infimum of the conditional mutual information over all extensions ω_{ABE} of a given state ω_{AB} with no restriction on the dimension of the system E. This leads to particular difficulties in proving continuity of the squashed entanglement and its faithfulness. Since the squashed entanglement was introduced, a lot of papers devoting to analysis of its properties appeared, see [2, 5, 15, 18, 19] and the references therein.

In this paper we try to generalize the squashed entanglement to all states of an infinite-dimensional bipartite system by using two ways: direct translation of the finite-dimensional definition and its universal extension (a construction which produces infinite-dimensional entanglement monotone starting from finite-dimensional one, it is described in Section 3 in general settings).

In Section 4 (after a short overview of properties of the squashed entanglement in finite dimensions) we analyse first a direct infinite-dimensional definition of the squashed entanglement using the extended quantum conditional mutual information considered in [28]. We show that this definition produces a function possessing all the basic properties of the squashed entanglement valid in finite dimensions (with the continuity replaced by the lower semicontinuity) on the set of states having at least one finite marginal entropy. The main problem (remained open) is to show that any separable state in an infinite-dimensional bipartite system can be extended to a short Markov chain. This problem (related to the existence of countably nondecomposable separable states [11]) prevents to prove vanishing of the directly defined version of squashed entanglement on all separable states.

Then we consider the universal extension of squashed entanglement – a lower semicontinuous function on the set of all bipartite states possessing all basis properties of the squashed entanglement valid in finite dimensions. We prove that this extension coincides with the above direct definition of the squashed entanglement on the set of states having at least one finite marginal entropy. The global coincidence is conjectured but not proved, it is shown to be equivalent to the global lower semicontinuity of the directly defined version of squashed entanglement.

Continuity properties of (the both versions of) the squashed entanglement are analysed in Section 5. We obtain a general condition relating continuity of the squashed entanglement to continuity of the quantum mutual information and consider its corollaries. In particular, we prove a weak form of the conjecture that local continuity of the squashed entanglement is preserved by local operations. We also consider several simple continuity conditions which can be used in applications.

Then the continuity bound for the squashed entanglement under the energy constraint on one subsystem is obtained by using the tight continuity bound for conditional mutual information (proved in the Appendix by using Winter's technique [32]). It is shown that the same continuity bound is

valid for the entanglement of formation. As a result, the asymptotic continuity of the both entanglement measures under the energy constraint on one subsystem is proved.

2 Preliminaries

Let \mathcal{H} be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators with the operator norm $\|\cdot\|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators in \mathcal{H} with the trace norm $\|\cdot\|_1$. Let $\mathfrak{T}_+(\mathcal{H})$ be the cone of positive operators in $\mathfrak{T}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ the set of quantum states (operators in $\mathfrak{T}_+(\mathcal{H})$ with unit trace). Note that $\mathfrak{T}_+(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ are complete separable metric space with the metric induced by the trace norm [10, 23]. Denote by $\text{ext}\,\mathfrak{S}(\mathcal{H})$ the set of all extreme points of the convex set $\mathfrak{S}(\mathcal{H})$ called *pure states*.

Trace class operators (not only states) will be denoted by the Greek letters ρ , σ , ω , ... All others linear operators (in particular, unbounded operators) will be denoted by the Latin letters A, B, F, H, ...

Denote by $I_{\mathcal{H}}$ the unit operator in a Hilbert space \mathcal{H} and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

If quantum systems A and B are described by Hilbert spaces \mathcal{H}_A and \mathcal{H}_B then the bipartite system AB is described by the tensor product of these spaces, i.e. $\mathcal{H}_{AB} \doteq \mathcal{H}_A \otimes \mathcal{H}_B$. A state in $\mathfrak{S}(\mathcal{H}_{AB})$ is denoted ω_{AB} , its marginal states $\mathrm{Tr}_{\mathcal{H}_B}\omega_{AB}$ and $\mathrm{Tr}_{\mathcal{H}_A}\omega_{AB}$ are denoted respectively ω_A and ω_B .

We will use the compactness criterion for subsets of $\mathfrak{S}(\mathcal{H}_{AB})$ (see [12, the Appendix]).

Lemma 1. A closed subset \mathfrak{S} of $\mathfrak{S}(\mathcal{H}_{AB})$ is compact if and only if

$$\mathfrak{S}_A \doteq \{\omega_A \mid \omega_{AB} \in \mathfrak{S}\} \quad and \quad \mathfrak{S}_B \doteq \{\omega_B \mid \omega_{AB} \in \mathfrak{S}\}$$

are compact subsets of $\mathfrak{S}(\mathcal{H}_A)$ and of $\mathfrak{S}(\mathcal{H}_B)$ correspondingly.

We will also use the following result of the purification theory.

Lemma 2. Let \mathcal{H} and \mathcal{K} be Hilbert spaces such that $\dim \mathcal{H} = \dim \mathcal{K}$. For an arbitrary pure state ω_0 in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ and an arbitrary sequence $\{\rho_k\}$ of states in $\mathfrak{S}(\mathcal{H})$ converging to the state $\rho_0 = \operatorname{Tr}_{\mathcal{K}}\omega_0$ there exists a sequence $\{\omega_k\}$ of pure states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ converging to the state ω_0 such that $\rho_k = \operatorname{Tr}_{\mathcal{K}}\omega_k$ for all k.

The assertion of Lemma 2 can be proved by noting that the infimum in the definition of the Bures distance (or the supremum in the definition of the Uhlmann fidelity) between two quantum states can be taken only over all purifications of one state with fixed purification of the another state and that the convergence of a sequence of states in the trace norm distance implies its convergence in the Bures distance [10, 23].

A quantum operation Φ from a system A to a system B is a completely positive trace non-increasing linear map $\mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$, where \mathcal{H}_A and \mathcal{H}_B are Hilbert spaces associated with the systems A and B. A trace preserving quantum operation is called quantum channel [10, 23].

The von Neumann entropy $H(\rho) = \text{Tr}\eta(\rho)$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$, where $\eta(x) = -x \log x$, has the natural extension to the cone $\mathfrak{T}_{+}(\mathcal{H})$ (cf.[22])¹

$$H(\rho) = [\operatorname{Tr} \rho] H\left(\frac{\rho}{\operatorname{Tr} \rho}\right) = \operatorname{Tr} \eta(\rho) - \eta(\operatorname{Tr} \rho), \quad \rho \in \mathfrak{T}_{+}(\mathcal{H}).$$

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy on the cone $\mathfrak{T}_+(\mathcal{H})$ follow from the corresponding properties of this function on the set $\mathfrak{S}(\mathcal{H})$ [22, 31]. By definition $H(\lambda \rho) = \lambda H(\rho)$, $\lambda \geq 0$.

The concavity of the von Neumann entropy is supplemented by the inequality

$$H(\lambda \rho + (1 - \lambda)\sigma) \le \lambda H(\rho) + (1 - \lambda)H(\sigma) + h_2(\lambda), \quad \lambda \in [0, 1],$$
 (1)

where $h_2(\lambda) = \eta(\lambda) + \eta(1-\lambda)$, valid for any states ρ and σ .

The quantum relative entropy for two operators ρ and σ in $\mathfrak{T}_{+}(\mathcal{H})$ is defined as follows (cf.[22])

$$H(\rho \| \sigma) = \sum_{i=1}^{+\infty} \langle i | \rho \log \rho - \rho \log \sigma + \sigma - \rho | i \rangle,$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is the orthonormal basis of eigenvectors of the operator ρ and it is assumed that $H(\rho \| \sigma) = +\infty$ if $\operatorname{supp} \rho$ is not contained in $\operatorname{supp} \sigma$. This definition implies $H(\lambda \rho \| \lambda \sigma) = \lambda H(\rho \| \sigma)$, $\lambda \geq 0$.

The quantum mutual information of a state ω_{AB} of an infinite-dimensional bipartite quantum system AB is defined as follows (cf.[21])

$$I(A:B)_{\omega} = H(\omega_{AB} \| \omega_A \otimes \omega_B).$$

¹Here and in what follows log denotes the natural logarithm.

We will use the natural extension of this quantity to positive trace-class operators

$$I(A:B)_{\omega} \doteq [\operatorname{Tr}\omega]I(A:B)_{\frac{\omega}{\mathfrak{T}_{k-1}}}, \quad \omega \in \mathfrak{T}_{+}(\mathcal{H}_{AB}).$$
 (2)

Basic properties of the relative entropy show that $\omega \mapsto I(A:B)_{\omega}$ is a lower semicontinuous function on the cone $\mathfrak{T}_{+}(\mathcal{H}_{AB})$ taking values in $[0,+\infty]$.

We will use the identity

$$I(A:B)_{\omega} + I(B:C)_{\omega} = 2H(\omega_B) \tag{3}$$

valid for any 1-rank operator $\omega \in \mathfrak{T}_{+}(\mathcal{H}_{ABC})$ (with possible value $+\infty$ in the both sides). If $H(\omega_A)$, $H(\omega_B)$ and $H(\omega_C)$ are finite then (3) is easily verified by noting that $H(\omega_A) = H(\omega_{BC})$, $H(\omega_B) = H(\omega_{AC})$ and $H(\omega_C) = H(\omega_{AB})$. In general case (3) can be proved by approximation (see the proof of Theorem 1 in [28, the Appendix]).

Identity (3) implies the upper bound (cf.[17])

$$I(A:B)_{\omega} \le 2\min\{H(\omega_A), H(\omega_B)\}. \tag{4}$$

The quantum conditional entropy

$$H(A|B)_{\omega} = H(\omega_{AB}) - H(\omega_{B})$$

can be extended to the convex set $\{\omega_{AB} | H(\omega_A) < +\infty\}$ containing states with $H(\omega_{AB}) = H(\omega_B) = +\infty$ by the formula

$$H_{e}(A|B)_{\omega} = H(\omega_{A}) - I(A:B)_{\omega} \tag{5}$$

preserving all basic properties of the conditional entropy [16] (for more detailed description of this extension see [28, Sect.5]). In what follows we will denote it by $H(A|B)_{\omega}$ omitting the subscript e.

Lemma 3. Let V_A be an operator in $\mathfrak{B}(\mathcal{H}_A)$ such that $||V_A|| \leq 1$ and ω_{AB} be a state with finite $H(\omega_A)$. Then

$$0 \le I(A:B)_{\omega} - I(A:B)_{\tilde{\omega}} \le 2\left[H(\omega_A) - H(V_A\omega_A V_A^*)\right],\tag{6}$$

or, equivalently,

$$|H(A|B)_{\omega} - H(A|B)_{\tilde{\omega}}| \le H(\omega_A) - H(V_A \omega_A V_A^*),$$

where $\tilde{\omega}_{AB} = V_A \otimes I_B \omega_{AB} V_A^* \otimes I_B$ and H(A|B) is the extended conditional entropy defined by (5).

Proof. The left inequality in (6) follows from the monotonicity of the quantum mutual information under local operations.

Let ω_{ABC} be a purification of ω_{AB} . Then identity (3) implies

$$I(A:B)_{\omega} + I(A:C)_{\omega} = 2H(\omega_A)$$

and

$$I(A:B)_{\tilde{\omega}} + I(A:C)_{\tilde{\omega}} = 2H(\tilde{\omega}_A) = 2H(V_A \omega_A V_A^*).$$

Hence

$$[I(A:B)_{\omega} - I(A:B)_{\tilde{\omega}}] + [I(A:C)_{\omega} - I(A:C)_{\tilde{\omega}}] = 2[H(\omega_A) - H(V_A\omega_A V_A^*)].$$

This implies the right inequality in (6), since $I(A:C)_{\omega} \geq I(A:C)_{\tilde{\omega}}$ by monotonicity of the quantum mutual information under local operations.

By using (5) it is easy to show that (6) is equivalent to the second inequality of the lemma. \square

A state $\omega \in \mathfrak{S}(\mathcal{H}_{AB})$ is called *separable* if it belongs to the convex closure of the set of all product states $\rho \otimes \sigma$, where $\rho \in \mathfrak{S}(\mathcal{H}_A)$ and $\sigma \in \mathfrak{S}(\mathcal{H}_B)$. Any separable state ω_{AB} can be represented as follows

$$\omega_{AB} = \int_{X} \rho(x) \otimes \sigma(x) \mu_{\omega}(dx), \tag{7}$$

where X is a complete separable metric space, μ_{ω} is a Borel probability measure on X, $\rho(x)$ and $\sigma(x)$ are μ_{ω} -measurable functions on X taking values in ext $\mathfrak{S}(\mathcal{H}_A)$ and in ext $\mathfrak{S}(\mathcal{H}_B)$ correspondingly [11]. If the measure μ_{ω} is purely atomic then (7) is converted to the countable decomposition

$$\omega_{AB} = \sum_{i} \pi_{i} \rho_{i} \otimes \sigma_{i}, \tag{8}$$

where $\{\rho_i\} \subset \text{ext}\mathfrak{S}(\mathcal{H}_A)$ and $\{\sigma_i\} \subset \text{ext}\mathfrak{S}(\mathcal{H}_B)$ are collections of pure states and $\{\pi_i\}$ is a probability distribution.

Definition 1. If a separable state ω_{AB} has representation (8) then it is called *countably decomposable*.

An essential feature of infinite-dimensional bipartite systems consists in existence of separable states which are not countably decomposable [11]. Such states called *countably nondecomposable* play important role in this paper (see Remark 1 below).

3 On universal infinite-dimensional extension of entanglement monotones

A central role in quantitative description of entanglement in composite quantum systems is plaid by the notion of entanglement monotones [13, 25, 30]. In the case of bipartite system AB an entanglement monotone E is a non-negative function on the set $\mathfrak{S}(\mathcal{H}_{AB})$ possessing the following properties:

- EM1) $\{E(\omega_{AB}) = 0\} \Leftrightarrow \{\text{the state } \omega \text{ is separable}\};$
- EM2) monotonicity under selective unilocal operations:

$$E(\omega_{AB}) \ge \sum_{k} \pi_k E(\omega_{AB}^k), \quad \pi_k = \text{Tr}\,\Phi_k(\omega_{AB}), \quad \omega_{AB}^k = \pi_k^{-1}\Phi_k(\omega_{AB})$$

for any state ω_{AB} and any collection $\{\Phi_k\}$ of unilocal completely positive maps such that $\sum_k \Phi_k$ is a channel;

EM3) convexity:
$$E(\lambda \rho_{AB} + (1 - \lambda)\sigma_{AB}) \le \lambda E(\rho_{AB}) + (1 - \lambda)E(\sigma_{AB}).$$

The convexity of E guarantees that EM2 is equivalent to monotonicity of E under selective LOCC operations [30].

According to [25] an entanglement monotone E is called *entanglement measure* if at any pure state it coincides with the von Neumann entropy of a marginal state, i.e.

EM4)
$$E(\omega_{AB}) = H(\omega_A) = H(\omega_B)$$
 for any pure state ω_{AB} .

Other desirable properties of entanglement monotones are the following:

- EM5) additivity for product states: $E(\omega_{AB} \otimes \omega_{A'B'}) = E(\omega_{AB}) + E(\omega_{A'B'})$, where $E(\omega_{AB} \otimes \omega_{A'B'})$ corresponds to the decomposition (AA')(BB');
- EM6) subadditivity for product states: $E(\omega_{AB} \otimes \omega_{A'B'}) \leq E(\omega_{AB}) + E(\omega_{A'B'})$, where $E(\omega_{AB} \otimes \omega_{A'B'})$ corresponds to the decomposition (AA')(BB');
- EM7) strong superadditivity: $E(\omega_{(AA')(BB')}) \ge E(\omega_{AB}) + E(\omega_{A'B'});$
- EM8) monogamy: $E(\omega_{A(BC)}) \ge E(\omega_{AB}) + E(\omega_{AC})$ [15].

In the finite-dimensional case it is natural to require continuity of an entanglement monotone E on the set $\mathfrak{S}(\mathcal{H}_{AB})$ of all bipartite states. An important role is plaid by the following stronger property [13, 25]:

EM9) asymptotic continuity:

$$\lim_{n \to +\infty} \frac{E(\rho_{AB}^n) - E(\sigma_{AB}^n)}{1 + \log \dim \mathcal{H}_{AB}^n} = 0$$

for any sequences of states ρ_{AB}^n , $\sigma_{AB}^n \in \mathfrak{S}(\mathcal{H}_{AB}^n)$ such that $\|\rho_{AB}^n - \sigma_{AB}^n\|_1$ tends to zero as $n \to +\infty$.

In the infinite-dimensional case the global continuity requirement is too restrictive.² Moreover, the discontinuity of the von Neumann entropy implies discontinuity of any entanglement monotone possessing property EM4. In this case it seems reasonable to require that an entanglement monotone (measure) E must be closed or lower semicontinuous, which means that

$$\liminf_{n \to +\infty} E(\omega_{AB}^n) \ge E(\omega_{AB}^0)$$

for any sequence $\{\omega_{AB}^n\}$ converging to a state ω_{AB}^0 or, equivalently, that the set of states defined by the inequality $E(\omega_{AB}) \leq c$ is closed for any $c \geq 0$. This requirement is motivated by the natural physical observation that entanglement can not be increased by passage to a limit.

From the physical point of view it is also natural to require that entanglement monotone (measure) must be continuous on subsets of states produced in a physical experiment, for example, on the set of states with bounded mean energy. One can also consider the corresponding version of the asymptotic continuity property (see [8] and Corollary 7 below).

Assume now that E is a given entanglement monotone defined on the set of states of a bipartite system AB composed of subsystems A and B of arbitrary finite dimensions. One can construct its extension to the set of states of an infinite-dimensional bipartite system AB as follows

$$\widehat{E}(\omega_{AB}) = \sup_{P_A, P_B} E(P_A \otimes P_B \,\omega_{AB} P_A \otimes P_B), \quad \omega_{AB} \in \mathfrak{S}(\mathcal{H}_{AB}), \quad (9)$$

where the supremum is over all finite rank projectors $P_A \in \mathfrak{B}(\mathcal{H}_A)$ and $P_B \in \mathfrak{B}(\mathcal{H}_B)$ and it is assumed that E is extended to all positive trace class operators by the natural way $E(\omega_{AB}) = [\text{Tr}\omega_{AB}] E\left(\frac{\omega_{AB}}{\text{Tr}\omega_{AB}}\right)$.

²Nevertheless, in infinite-dimensional bipartite systems there exist globally continuous entanglement monotones [27, Example 5].

Reasonability of this definition is justified by the following observations.

Proposition 1. Let E be a continuous entanglement monotone on the set of states of finite-dimensional bipartite system.

- A) \widehat{E} is an unique lower semicontinuous entanglement monotone on the set $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\widehat{E}(\omega_{AB}) = E(\omega_{AB})$ for any state ω_{AB} with finite rank marginal states ω_A and ω_B .
- B) $\widehat{E}(\omega_{AB}) = \lim_{n\to\infty} E(P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n)$ for arbitrary sequences $\{P_A^n\} \subset \mathfrak{B}(\mathcal{H}_A)$ and $\{P_B^n\} \subset \mathfrak{B}(\mathcal{H}_B)$ of finite rank projectors strongly converging to the identity operators I_A and I_B .
- C) If E possesses one of the above properties EM4-EM8 then \widehat{E} possesses the same property.

Proof. A) The lower semicontinuity of \widehat{E} follows from its definition and the continuity of the function $\omega_{AB} \mapsto E(P_A \otimes P_B \omega_{AB} P_A \otimes P_B)$ for any finite rank projectors P_A and P_B .

To prove EM1 for \widehat{E} it suffices to note that separability of ω_{AB} is equivalent to separability of all states of the form $\lambda P_A \otimes P_B \omega_{AB} P_A \otimes P_B$, $\lambda \in \mathbb{C}$. Properties EM2 and EM3 for \widehat{E} will be proved later.

If ω_{AB} is a state such that $\operatorname{rank}\omega_A < +\infty$ and $\operatorname{rank}\omega_B < +\infty$ then $E(\omega_{AB}) \geq E(P_A \otimes P_B \omega_{AB} P_A \otimes P_B)$ for any finite rank projectors P_A and P_B by monotonicity of E under local operations (property EM2). This and (9) imply $\widehat{E}(\omega_{AB}) = E(\omega_{AB})$.

B) Since $E(P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n) = \widehat{E}(P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n)$ for all n, this follows from (9) and the lower semicontinuity of \widehat{E} .

Now we can prove EM2 for \widehat{E} . Let $\omega_{AB}^n = P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n$, where $\{P_A^n\}$ and $\{P_B^n\}$ are any sequences of finite rank projectors strongly converging to the identity operators I_A and I_B . Then for any finite rank projectors Q_A and Q_B the assumed validity of EM2 for E implies

$$E(\omega_{AB}^n) \ge \sum_k E(Q_A \otimes Q_B \, \Phi_k(\omega_{AB}^n) \, Q_A \otimes Q_B)$$

for any given collection $\{\Phi_k\}$ of unilocal completely positive maps such that $\sum_k \Phi_k$ is a channel. It follows from the continuity of E and assertion B that

$$\widehat{E}(\omega_{AB}) \ge \sum_{k} E(Q_A \otimes Q_B \Phi_k(\omega_{AB}) Q_A \otimes Q_B).$$

Since this holds for any projectors Q_A and Q_B , we have

$$\widehat{E}(\omega_{AB}) \ge \sum_{k} \widehat{E}(\Phi_{k}(\omega_{AB})) = \sum_{k} [\text{Tr}\Phi_{k}(\omega_{AB})] \widehat{E}\left(\frac{\Phi_{k}(\omega_{AB})}{\text{Tr}\Phi_{k}(\omega_{AB})}\right)$$

Assertion B also makes possible to derive convexity of \widehat{E} (property EM3) from the convexity of E and to prove assertion C (since the sequences $\{P_A^n\}$ and $\{P_B^n\}$ in B can be chosen arbitrarily). \square

By Proposition 1 the function \widehat{E} is an unique lower semicontinuous extension to the set $\mathfrak{S}(\mathcal{H}_{AB})$ of the function E defined on the dense subset

$$\mathfrak{S}_{\mathrm{f}}(\mathcal{H}_{AB}) \doteq \{\omega_{AB} \mid \max\{\mathrm{rank}\omega_A, \mathrm{rank}\omega_B\} < +\infty\}$$

of $\mathfrak{S}(\mathcal{H}_{AB})$. By the proof of Proposition 1 the existence and uniqueness of this extension follow from continuity of the function E and its monotonicity under local operations.

Since the above construction can be applied to arbitrary entanglement monotone E, we will call the function \widehat{E} an universal extension of E.

Example: the entanglement of formation. In the case of finite-dimensional bipartite system AB the entanglement of formation is defined as the convex roof extension to the set $\mathfrak{S}(\mathcal{H}_{AB})$ of the function $\omega_{AB} \mapsto H(\omega_A)$ on the set $\mathfrak{S}(\mathcal{H}_{AB})$ of pure states, i.e.

$$E_F(\omega_{AB}) = \inf_{\sum_i \pi_i \omega_{AB}^i = \omega_{AB}} \sum_i \pi_i H(\omega_A^i), \tag{10}$$

where the infimum is over all ensembles $\{\pi_i, \omega_{AB}^i\}$ of pure states with the average state ω_{AB} [3].

It is well known that E_F is a continuous entanglement measure on $\mathfrak{S}(\mathcal{H}_{AB})$ possessing properties EM1-EM4,EM6 and EM9 (and that EM5, EM7 and EM8 do not hold for E_F) [10, 15, 24].

In infinite dimensions there are two versions E_F^d and E_F^c of the entanglement of formation defined, respectively, by using discrete and continuous convex roof extensions, i.e.

$$E_F^d(\omega_{AB}) = \inf_{\sum_i \pi_i \omega_{AB}^i = \omega_{AB}} \sum_i \pi_i H(\omega_A^i), \quad E_F^c(\omega_{AB}) = \inf_{b(\mu) = \omega_{AB}} \int H(\omega_A) \mu(d\omega),$$

where the first infimum is over all countable convex decompositions of the state ω_{AB} into pure states and the second one is over all Borel probability measures on the set $\operatorname{ext} \mathfrak{S}(\mathcal{H}_{AB})$ with the barycenter ω_{AB} [27, Sect.5]. The discrete version seems more preferable but the assumption $E_F^d \neq E_F^c$ leads to several problems with this version, in particular, the existence of countably nondecomposable separable states prevents to prove the implication " \Leftarrow " in EM1 for E_F^d , see the end of Section 4.3.³

The coincidence of E_F^d and E_F^c on $\mathfrak{S}(\mathcal{H}_{AB})$ is an open question. In [27] it is shown that $E_F^d(\omega_{AB}) = E_F^c(\omega_{AB})$ for any state ω_{AB} such that $\min\{H(\omega_A), H(\omega_B), H(\omega_{AB})\} < +\infty$ and that the function E_F^c is lower semicontinuous on $\mathfrak{S}(\mathcal{H}_{AB})$. So, Proposition 1 implies $\widehat{E}_F = E_F^c$.

Corollary 1. The universal extension \widehat{E}_F of the entanglement of formation E_F (defined by formula (10)) coincides with the function E_F^c .

Proposition 1 and Corollary 1 show that

$$\left\{ E_F^d = E_F^c \right\} \quad \Leftrightarrow \quad \left\{ E_F^d \text{ is lower semicontinuous on } \mathfrak{S}(\mathcal{H}_{AB}) \right\}$$

and provide an alternative proof of properties EM1-EM4 and EM6 for the function E_F^c . In fact, E_F^c possesses the generalized (continuous) versions of EM2 and EM3 [27].

4 The squashed entanglement

4.1 Finite-dimensional case

The squashed entanglement of a state ω_{AB} of a finite dimensional bipartite system AB is defined as follows

$$E_{sq}(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E), \tag{11}$$

where the infimum is over all extensions ω_{ABE} of the state ω_{AB} and

$$I(A:B|E)_{\omega} = H(\omega_{AE}) + H(\omega_{BE}) - H(\omega_{E}) - H(\omega_{ABE})$$
 (12)

is the conditional mutual information of the state ω_{ABE} [4, 29]. It is essential that the dimension of the system E in (11) is assumed to be finite but not bounded (despite fixed finite dimensions of the systems A and B).

³In general, the discrete convex roof construction applied to an entropy type function (in the role of H) may give a function for which this implication is not valid [27, Rem.6].

The squashed entanglement is the only known entanglement measure possessing all properties EM1-EM9 stated in Section 3. All these properties excluding EM8,EM9 and the implication " \Rightarrow " in EM1 are proved in [4]. This implication (called the faithfulness of E_{sq}) is proved in [2]. The monogamy relation EM8 is proved in [15].

Some difficulty concerns the proof of continuity of the squashed entanglement on $\mathfrak{S}(\mathcal{H}_{AB})$. This difficulty is related to unbounded dimension of the system E in definition (11). The continuity of squashed entanglement was proved in [4] under the conjecture of validity of the Fannes type continuity bound for quantum conditional entropy H(A|B) not depending on the dimension of B which was proved later in [1]. This continuity bound also implies the asymptotic continuity of the squashed entanglement.

It is also shown in [4] that $E_D(\omega_{AB}) \leq E_{sq}(\omega_{AB}) \leq E_C(\omega_{AB})$ for any state ω_{AB} , where E_D is the distillable entanglement and E_C is the entanglement cost.

The squashed entanglement has an operational interpretation in terms of the protocol of quantum state redistribution [6]. Its interpretation as a distance to highly extendible states is given recently in [19].

4.2 Direct definition of the squashed entanglement in infinite dimensions and its properties

If A and B are infinite-dimensional systems then we may define the squashed entanglement by the same formula (11) in which it is necessary to consider that E is an infinite-dimensional system as well (see the remark after Lemma 4 below). The only problem consists in definition of I(A:B|E), since formula (12) may contain the uncertainty " $\infty - \infty$ " even for a state ω_{AB} with finite marginal entropies. This problem can be solved by using the extension of conditional mutual information defined by one of the equivalent expressions

$$I(A:B|E)_{\omega} = \sup_{P_A} [I(A:BE)_{Q_A\omega Q_A} - I(A:E)_{Q_A\omega Q_A}], \ Q_A = P_A \otimes I_{BE}, \ (13)$$

$$I(A:B|E)_{\omega} = \sup_{P_B} [I(AE:B)_{Q_B\omega Q_B} - I(E:B)_{Q_B\omega Q_B}], \ Q_B = P_B \otimes I_{AE}, \ (14)$$

where the suprema are over all finite rank projectors $P_A \in \mathfrak{B}(\mathcal{H}_A)$ and $P_B \in \mathfrak{B}(\mathcal{H}_B)$ correspondingly [28].

It is shown in [28, Th.2] that expressions (13) and (14) define a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{ABE})$ possessing all basic properties

of conditional mutual information valid in finite dimensions (including the characterization of a state ω_{ABE} such that $I(A:B|E)_{\omega}=0$ as a short Markov chain in terms of [9], i.e. as a state such that $\omega_{ABE}=\operatorname{Id}_A\otimes\Phi(\omega_{AE})$ for some channel $\Phi:E\to BE$). If one of the marginal entropies $H(\omega_A)$, $H(\omega_B)$ and $H(\omega_E)$ is finite then the above extension is given respectively by the explicit formula

$$I(A:B|E)_{\omega} = I(A:BE)_{\omega} - I(A:E)_{\omega},$$

$$I(A:B|E)_{\omega} = I(AE:B)_{\omega} - I(E:B)_{\omega},$$
(15)

and

$$I(A:B|E)_{\omega} = I(A:B)_{\omega} - I(A:E)_{\omega} - I(B:E)_{\omega} + I(AB:E)_{\omega}.$$
 (16)

We will consider in this subsection that

$$E_{sq}(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E), \quad \dim \mathcal{H}_E = +\infty$$
 (17)

where $I(A:B|E)_{\omega}$ is the extended conditional mutual information described before. Introduce the monotone sequence of functions

$$E_{sq}^{n}(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E), \quad \dim \mathcal{H}_{E} \le n$$
 (18)

pointwise converging to the function

$$E_{sq}^*(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E), \quad \dim \mathcal{H}_E < +\infty.$$
 (19)

In finite-dimensions $E_{sq}^* = E_{sq}$, but the following lemma shows that these functions do not coincide in general.

Lemma 4. A) If
$$I(A:B)_{\omega} < +\infty$$
 then $E_{sq}^*(\omega_{AB}) = E_{sq}(\omega_{AB}) < +\infty$.
B) If $I(A:B)_{\omega} = +\infty$ then $E_{sq}^*(\omega_{AB}) = +\infty$.

So, if ω_{AB} is a countably decomposable separable state such that $I(A:B)_{\omega}=+\infty$, for example, the state $\omega_{AB}=\sum_{k=1}^{+\infty}\pi_{k}|k\rangle\langle k|\otimes |k\rangle\langle k|$, where $\{|k\rangle\}$ is an orthonormal basis in $\mathcal{H}_{A}\cong\mathcal{H}_{B}$ and $\{\pi_{k}\}$ is a probability distribution with infinite Shannon entropy, then $E_{sq}^{*}(\omega_{AB})=+\infty$ while $E_{sq}(\omega_{AB})=0$ (this follows from Proposition 2A below). Lemma 4 also shows that $E_{sq}^{*}=E_{sq}$ if and only if one of the systems A and B is finite-dimensional.

Proof. A) Note first that the assumed finiteness of $I(A:B)_{\omega}$ implies finiteness of $E_{sq}^*(\omega_{AB})$ and of $E_{sq}(\omega_{AB})$.

We will use the inequality

$$\left| I(A:B|E)_{\lambda\rho + (1-\lambda)\sigma} - \lambda I(A:B|E)_{\rho} - (1-\lambda)I(A:B|E)_{\sigma} \right| \le 2h_2(\lambda) \quad (20)$$

valid for any states ρ_{ABE} and σ_{ABE} such that $I(A:B|E)_{\rho}$ and $I(A:B|E)_{\sigma}$ are finite and any $\lambda \in (0,1)$. If ρ_{ABE} and σ_{ABE} are states with finite marginal entropies then inequality (20) directly follows from formula (12), concavity of the von Neumann entropy and inequality (1). The validity of this inequality for arbitrary states with finite values of I(A:B|E) can be proved by using the approximating property for I(A:B|E) from Theorem 2 in [28].

Let $\varepsilon > 0$ be arbitrary, ω_{ABE} be an extension of ω_{AB} such that

$$I(A:B|E)_{\omega} < E_{sq}(\omega_{AB}) + \varepsilon \tag{21}$$

and ω_{ABED} be a purification of ω_{ABE} . Let $\{P_D^n\} \subset \mathfrak{B}(\mathcal{H}_D)$ be a sequence of projectors strongly converging to the operator I_D such that rank $P_D^n \leq n$ and

$$\omega_{ABED}^n = (1 - \lambda_n)^{-1} I_{ABE} \otimes P_D^n \, \omega_{ABED} I_{ABE} \otimes P_D^n, \quad \lambda_n = 1 - \text{Tr} P_D^n \omega_D,$$

for all n. Since $(1 - \lambda_n)\omega_{ABE}^n \leq \omega_{ABE}$, we have $\omega_{ABE} = (1 - \lambda_n)\omega_{ABE}^n + \lambda_n \tilde{\omega}_{ABE}^n$, where $\tilde{\omega}_{ABE}^n = \lambda_n^{-1} (\omega_{ABE} - (1 - \lambda_n)\omega_{ABE}^n)$, and hence (20) implies

$$I(A:B|E)_{\omega} \ge (1-\lambda_n)I(A:B|E)_{\omega^n} + \lambda_n I(A:B|E)_{\tilde{\omega}^n} - 2h_2(\lambda_n). \tag{22}$$

For each n the state $\hat{\omega}_{ABD}^n = (1 - \lambda_n)\omega_{ABD}^n + \lambda_n \tilde{\omega}_{AB}^n \otimes \tau_D^n$, where τ_D^n is a pure state in $\mathfrak{S}(P_D^n(\mathcal{H}_D))$, is an extension of ω_{AB} . By using (20), (22), the duality relation $I(A:B|E)_{\omega^n} = I(A:B|D)_{\omega^n}$ (cf.[6]) and nonnegativity of $I(A:B|E)_{\tilde{\omega}^n}$ we obtain

$$I(A:B|D)_{\hat{\omega}^n} \leq (1-\lambda_n)I(A:B|D)_{\omega^n} + \lambda_n I(A:B|D)_{\tilde{\omega}_{AB}^n \otimes \tau_D^n} + 2h_2(\lambda_n)$$

$$= (1-\lambda_n)I(A:B|E)_{\omega^n} + \lambda_n I(A:B)_{\tilde{\omega}^n} + 2h_2(\lambda_n)$$

$$\leq I(A:B|E)_{\omega} + \lambda_n I(A:B)_{\tilde{\omega}^n} + 4h_2(\lambda_n).$$

Since rank $\hat{\omega}_D^n \leq n$, this inequality and (21) imply

$$E_{sq}^{n}(\omega_{AB}) \leq E_{sq}(\omega_{AB}) + \varepsilon + \lambda_{n}I(A:B)_{\tilde{\omega}^{n}} + 4h_{2}(\lambda_{n}).$$

So, to prove that $\lim_{n\to+\infty} E_{sq}^n(\omega_{AB}) = E_{sq}(\omega_{AB})$ it suffices to show that

$$\lim_{n \to +\infty} \lambda_n I(A:B)_{\tilde{\omega}^n} = 0. \tag{23}$$

Since $\omega_{AB} = (1 - \lambda_n)\omega_{AB}^n + \lambda_n \tilde{\omega}_{AB}^n$, it follows from (20) that

$$I(A:B)_{\omega} \ge (1 - \lambda_n)I(A:B)_{\omega^n} + \lambda_n I(A:B)_{\tilde{\omega}^n} - 2h_2(\lambda_n). \tag{24}$$

Hence, by nonnegativity and lower semicontinuity of the quantum mutual information we have

$$\lim_{n \to +\infty} (1 - \lambda_n) I(A:B)_{\omega^n} = I(A:B)_{\omega}.$$

Since $I(A:B)_{\omega} < +\infty$, this relation and (24) imply (23).

B) If ω_{ABE} is any extension of the state ω_{AB} such that $\operatorname{rank}\omega_E < +\infty$ then formula (16) and upper bound (4) imply $I(A:B|E)_{\omega} = +\infty$. Hence $E_{sq}^*(\omega_{AB}) = +\infty$. \square

Consider now properties of the squashed entanglement E_{sq} defined by formula (17).

Proposition 2. Let
$$\mathfrak{S}_* \doteq \{\omega_{AB} \mid \min\{H(\omega_A), H(\omega_B), H(\omega_{AB})\} < +\infty\}.$$

- A) If $E_{sq}(\omega_{AB}) = 0$ then ω_{AB} is a separable state, the converse implication holds if ω_{AB} is a state in $\operatorname{conv}(\mathfrak{S}_* \cup \mathfrak{S}_{\operatorname{cd}})$, where $\mathfrak{S}_{\operatorname{cd}}$ is the set of countably decomposable separable states (Def. 1).⁴
- B) The function E_{sq} possesses the above properties EM2-EM8.
- C) The function E_{sq} is lower-semicontinuous on the set \mathfrak{S}_* and coincides on this set with the function E_{sq}^* defined by formula (19).
- D) The function E_{sq} is continuous on any subset of $\mathfrak{S}(\mathcal{H}_{AB})$ on which **one** of the functions $\omega_{AB} \mapsto H(\omega_A)$ and $\omega_{AB} \mapsto H(\omega_B)$ is continuous.

Assertion D in Proposition 2 is essentially strengthened in Section 5.

Proof. B) Properties EM2-EM8 are proved by the same arguments as in the finite-dimensional case (see [4, 15]) with obvious modifications using properties of the extended conditional mutual information stated in [28, Th.2].

 $^{^4\}operatorname{conv}(\mathfrak{S}_* \cup \mathfrak{S}_{\operatorname{cd}})$ is the convex hull of $\mathfrak{S}_* \cup \mathfrak{S}_{\operatorname{cd}}$. Obstacles preventing to prove that $E_{sq}(\omega_{AB}) = 0$ for countably nondecomposable separable states ω_{AB} with infinite marginal entropies are considered in Remark 1 below.

C) Take increasing sequences $\{P_A^n\} \subset \mathfrak{B}(\mathcal{H}_A)$ and $\{P_B^n\} \subset \mathfrak{B}(\mathcal{H}_B)$ of finite rank projectors strongly converging to the identity operators I_A and I_B . For each n consider the functions

$$f_n(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E)_{Q_A^n \omega Q_A^n}$$
 and $g_n(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E)_{Q_B^n \omega Q_B^n}$

where $Q_A^n = P_A^n \otimes I_{BE}$, $Q_B^n = P_B^n \otimes I_{AE}$ and the infima are over all extensions ω_{ABE} of the state ω_{AB} .

We will show first that these functions are continuous on $\mathfrak{S}(\mathcal{H}_{AB})$ for all n. By symmetry it suffices to prove the continuity of f_n . Let ω_{AB}^1 and ω_{AB}^2 be states such that $\|\omega_{AB}^2 - \omega_{AB}^1\|_1 \leq \varepsilon \leq 1$. By repeating the arguments from the proof of continuity of E_{sq} in [4] we obtain

$$f_n(\omega_{AB}^k) = \frac{1}{2} \inf_{\Lambda} I(A:B|E)_{Q_A^n \operatorname{Id}_{AB} \otimes \Lambda(\omega_{ABC}^k) Q_A^n}, \quad k = 1, 2,$$
 (25)

where ω_{ABC}^1 and ω_{ABC}^2 are purifications of ω_{AB}^1 and of ω_{AB}^2 such that

$$\|\omega_{ABC}^2 - \omega_{ABC}^1\|_1 \le 2\sqrt{\varepsilon},$$

and the infimum is over all quantum channels $\Lambda: \mathfrak{T}(\mathcal{H}_C) \to \mathfrak{T}(\mathcal{H}_E)$.

For given Λ let $\omega_{ABE}^1 = \operatorname{Id}_{AB} \otimes \Lambda(\omega_{ABC}^1)$ and $\omega_{ABE}^2 = \operatorname{Id}_{AB} \otimes \Lambda(\omega_{ABC}^2)$. Then $\|Q_A^n \omega_{ABE}^2 Q_A^n - Q_A^n \omega_{ABE}^1 Q_A^n\|_1 \leq 2\sqrt{\varepsilon}$. Let $\lambda_k = \operatorname{Tr} Q_A^n \omega_{ABE}^k$, $\lambda_k = 1, 2$. Note that

$$|\lambda_2 - \lambda_1| \le \|\omega_{AB}^2 - \omega_{AB}^1\|_1 \le \varepsilon \tag{26}$$

and that

$$I(A:B|E)_{Q_A^n\omega^kQ_A^n} \le 2\lambda_k \log \operatorname{rank} P_A^n, \quad k = 1, 2.$$
(27)

Inequality (27) follows from representation (15) and the upper bound (4).

If $\lambda_1 \lambda_2 = 0$ then (26) and (27) imply

$$\Delta_{12} \doteq \left| I(A:B|E)_{Q_A^n \omega^2 Q_A^n} - I(A:B|E)_{Q_A^n \omega^1 Q_A^n} \right| \le 2\varepsilon \log \operatorname{rank} P_A^n. \tag{28}$$

If $\lambda_1 \lambda_2 \neq 0$ denote the states $\lambda_1^{-1} Q_A^n \omega_{ABE}^1 Q^n$ and $\lambda_2^{-1} Q_A^n \omega_{ABE}^2 Q^n$ respectively by $\hat{\omega}_{ABE}^2$ and $\hat{\omega}_{ABE}^1$. It follows from (26) and (27) that

$$\Delta_{12} = |\lambda_2 I(A:B|E)_{\hat{\omega}^2} - \lambda_1 I(A:B|E)_{\hat{\omega}^1}|$$

$$\leq \lambda_2 |I(A:B|E)_{\hat{\omega}^2} - I(A:B|E)_{\hat{\omega}^1}| + 2\varepsilon \log \operatorname{rank} P_A^n.$$
(29)

Since

$$\lambda_2 \|\hat{\omega}_{ABE}^2 - \hat{\omega}_{ABE}^1\|_1 = \|\lambda_2 \hat{\omega}_{ABE}^2 - \lambda_1 \hat{\omega}_{ABE}^1\|_1 + |\lambda_2 - \lambda_1| \le \tilde{\varepsilon} \doteq 2\sqrt{\varepsilon} + \varepsilon,$$

the Fannes type continuity bound for the conditional mutual information (Corollary 8 in [28]) implies

$$\lambda_2 |I(A:B|E)_{\hat{\omega}^2} - I(A:B|E)_{\hat{\omega}^1}| \le \tilde{\varepsilon} \log \operatorname{rank} P_A^n + 4\lambda_2 \theta \left(\frac{\tilde{\varepsilon}}{2\lambda_2}\right), \tag{30}$$

where $\theta(x) = (1+x)h_2\left(\frac{x}{1+x}\right)$.

It is easy to see that $\lambda_2 \theta\left(\frac{\tilde{\varepsilon}}{2\lambda_2}\right)$ tends to zero as $\tilde{\varepsilon} \to 0$ uniformly on $\lambda_2 \in (0,1]$. Hence, (28)-(30) show that $\Delta_{12} \leq \tau(\varepsilon)$ for arbitrary λ_1, λ_2 , where $\tau(\varepsilon)$ is a function tending to zero as $\varepsilon \to 0$. Since this holds for any quantum channel Λ , we conclude from (25) that $|f_n(\omega_{AB}^2) - f_n(\omega_{AB}^1)| \leq \frac{1}{2}\tau(\varepsilon)$. Thus, the function f_n is continuous.

The continuity of the functions f_n and g_n implies continuity of the function

$$h_n(\omega_{AB}) \doteq \max \{f_n(\omega_{AB}), g_n(\omega_{AB})\}.$$

So, to prove the lower semicontinuity of E_{sq} on \mathfrak{S}_* it suffices to show that

$$\sup_{n} h_n(\omega_{AB}) = E_{sq}(\omega_{AB}) \tag{31}$$

for any state ω_{AB} in \mathfrak{S}_* . If ω_{AB} is a state such that either $H(\omega_A) < +\infty$ or $H(\omega_B) < +\infty$ then (31) follows from Lemma 5 below, since it implies

$$E_{sq}(\omega_{AB}) - f_n(\omega_{AB}) \le [H(\omega_A) - H(P_A^n \omega_A P_A^n)]$$
(32)

if $H(\omega_A) < +\infty$ and

$$E_{sq}(\omega_{AB}) - g_n(\omega_{AB}) \le [H(\omega_B) - H(P_B^n \omega_B P_B^n)]$$

if $H(\omega_B) < +\infty$. Since $\lim_{n\to\infty} H(P_X^n \omega_X P_X^n) = H(\omega_X)$, X = A, B, by Simon's convergence theorem [20, the Appendix] and

$$E_{sq}(\omega_{AB}) - h_n(\omega_{AB}) = \min\{E_{sq}(\omega_{AB}) - f_n(\omega_{AB}), E_{sq}(\omega_{AB}) - g_n(\omega_{AB})\}$$

by definition of h_n , we have $\lim_{n\to\infty} h_n(\omega_{AB}) = E_{sq}(\omega_{AB})$.

If ω_{AB} is a state such that $H(\omega_A) = H(\omega_B) = +\infty$ but $H(\omega_{AB}) < +\infty$ then $I(A:B)_{\omega} = +\infty$. Since $f_n(\omega_{AB}) \leq E_{sq}(\omega_{AB})$ for all n, to prove (31) it suffices to show that

$$\lim_{n \to \infty} f_n(\omega_{AB}) = +\infty. \tag{33}$$

The lower semicontinuity of I(A:B) and Simon's convergence theorem imply respectively

$$\lim_{n \to \infty} I(A:B)_{Q_A^n \omega Q_A^n} = I(A:B)_{\omega} = +\infty$$

and

$$\lim_{n\to\infty} H(Q_A^n \omega_{AB} Q_A^n) = H(\omega_{AB}) < +\infty.$$

So, limit relation (33) follows from the inequality

$$I(A:B|E)_{\omega} \ge I(A:B)_{\omega} - 2H(\omega_{AB}),\tag{34}$$

valid for any state ω_{ABE} with finite $H(\omega_{AB})$. If $I(A:BE)_{\omega} < +\infty$ then (34) is proved by using monotonicity of the quantum mutual information under partial trace (cf.[4]):

$$I(A:B|E)_{\omega} = I(A:BE)_{\omega} - I(A:E)_{\omega}$$

$$\geq I(A:B)_{\omega} - I(AB:ED)_{\omega} \geq I(A:B)_{\omega} - 2H(\omega_{AB}),$$

where it is assumed that ω_{ABED} is a purification of ω_{ABE} . The validity of inequality (34) for arbitrary states ω_{ABE} with finite $H(\omega_{AB})$ can be proved by using the approximating property for I(A:B|E) from Theorem 2 in [28].

The coincidence of E_{sq} and E_{sq}^* on \mathfrak{S}_* follows from Lemma 4 and the above observation showing that $E_{sq}(\omega_{AB}) = +\infty$ for any state ω_{AB} such that $H(\omega_A) = H(\omega_B) = +\infty$ and $H(\omega_{AB}) < +\infty$.

- D) Assume the function $\omega_{AB} \mapsto H(\omega_A)$ is continuous on a subset \mathfrak{S}_0 of $\mathfrak{S}(\mathcal{H}_{AB})$. By Dini's lemma the increasing sequence of continuous functions $\omega_{AB} \mapsto H(P_A^n \omega_A P_A^n)$ converges to the continuous function $\omega_{AB} \mapsto H(\omega_A)$ uniformly on any compact subset of \mathfrak{S}_0 . So, inequality (32) shows that the sequence of continuous functions f_n uniformly converges to the function E_{sq} on any compact subset of \mathfrak{S}_0 . Hence the function E_{sq} is continuous on \mathfrak{S}_0 .
- A) Assume $E_{sq}(\omega_{AB}) = 0$. Take any sequences $\{P_A^n\} \subset \mathfrak{B}(\mathcal{H}_A)$ and $\{P_B^n\} \subset \mathfrak{B}(\mathcal{H}_B)$ of finite rank projectors strongly converging to the identity operators I_A and I_B . By monotonicity of the conditional mutual information under local operations we have $E_{sq}(\omega_{AB}^n) = 0$ for all n, where

 $\omega_{AB}^n = [\text{Tr}P_A^n \otimes P_B^n \omega_{AB}]^{-1} P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n$. By the faithfulness of the squashed entanglement in finite dimensions (proved in [2]) all the states ω_{AB}^n are separable. So, the state ω_{AB} is separable (as a limit of a sequence of separable states).

If ω_{AB} is a separable state in \mathfrak{S}_{cd} having representation (8) then it can be extended to a short Markov chain as follows (cf.[4])

$$\omega_{ABE} = \sum_{i} \pi_{i} \rho_{i} \otimes \sigma_{i} \otimes |i\rangle\langle i|, \qquad (35)$$

where $\{|i\rangle\}$ is an orthonormal basis in some Hilbert space \mathcal{H}_E . Hence, $E_{sq}(\omega_{AB}) = 0$

If ω_{AB} is a separable state in \mathfrak{S}_* then it can be represented as a limit of a sequence $\{\omega_{AB}^n\}$ of separable states in $\mathfrak{S}_* \cap \mathfrak{S}_{cd}$ (for examples, separable states having finite rank marginal states). Since $E_{sq}(\omega_{AB}^n) = 0$ for all n, the lower semicontinuity of E_{sq} on \mathfrak{S}_* (assertion C) implies $E_{sq}(\omega_{AB}) = 0$.

If ω_{AB} is a convex mixture of separable states in \mathfrak{S}_* and in \mathfrak{S}_{cd} then the convexity of E_{sq} (assertion B) implies $E_{sq}(\omega_{AB}) = 0$. \square

Lemma 5. Let V_A be an operator in $\mathfrak{B}(\mathcal{H}_A)$ such that $||V_A|| \leq 1$ and ω_{ABE} be a state with finite $H(\omega_A)$. Then

$$0 \le I(A:B|E)_{\omega} - I(A:B|E)_{\tilde{\omega}} \le 2 [H(\omega_A) - H(V_A \omega_A V_A^*)], \qquad (36)$$

and hence

$$-2\delta H(V_A\omega_A V_A^*) \le I(A:B|E)_{\omega} - I(A:B|E)_{\frac{\tilde{\omega}}{\operatorname{Tr}\tilde{\omega}}} \le 2\left[H(\omega_A) - H(V_A\omega_A V_A^*)\right],$$

where $\tilde{\omega}_{ABE} = V_A \otimes I_{BE} \omega_{ABE} V_A^* \otimes I_{BE}$ and $\delta = \frac{1 - \text{Tr}\tilde{\omega}}{\text{Tr}\tilde{\omega}}$.

Proof. The left inequality in (36) follows from the monotonicity of the conditional mutual information under local operations.

To prove the right inequality in (36) it suffices to note that

$$I(A:B|E)_{\omega} = H(A|E)_{\omega} - H(A|BE)_{\omega}$$

for any $\omega \in \mathfrak{T}_{+}(\mathcal{H}_{ABE})$ with finite $H(\omega_{A})$, where H(A|X) is the extended conditional entropy defined in (5), and to apply Lemma 3 twice.

The second inequality is easily derived from the first one by noting that

$$0 \leq [\operatorname{Tr} \tilde{\omega}] I(A:B|E)_{\frac{\tilde{\omega}}{\operatorname{Tr} \tilde{\omega}}} = I(A:B|E)_{\tilde{\omega}} \leq 2H(V_A \omega_A V_A^*). \square$$

Remark 1. We cannot prove that $E_{sq}(\omega_{AB}) = 0$ for any separable state ω_{AB} , since we cannot extend a countably nondecomposable separable state to a Markov chain. It is easy to see that the integral analog of formula (35) (in which the basis $\{|i\rangle\langle i|\}$ is replaced by a basis $\{|x\rangle\langle x|\}_{x\in X}$ of nonseparable Hilbert space \mathcal{H}_E) produces a non-normal state on $\mathfrak{B}(\mathcal{H}_{ABE})$. If ω_{AB} is a countably nondecomposable separable state such that either $H(\omega_A)$ or $H(\omega_B)$ is finite then we can prove that $E_{sq}(\omega_{AB}) = 0$ by approximation but we cannot prove that the infimum in the definition of $E_{sq}(\omega_{AB})$ is attained at some state ω_{ABE} .

Thus, we have faced with the **interesting question**: Can a countably nondecomposable separable state ω_{AB} be extended to a Markov chain ω_{ABE} ?

Note that any such (hypothetical) extension would be a Markov chain having no representation described by Hayden, Jozsa, Petz and Winter in [9] characterizing Markov chains in finite-dimensional tripartite systems (it is easy to see that ω_{AB} is a countably decomposable separable state for any Markov chain ω_{ABE} having such representation).

4.3 The universal extension of squashed entanglement

According to Section 3 the universal extension of squashed entanglement is defined by the formula

$$\widehat{E}_{sq}(\omega_{AB}) = \sup_{P_A, P_B} E_{sq}(P_A \otimes P_B \,\omega_{AB} P_A \otimes P_B), \tag{37}$$

where the supremum is over all finite rank projectors P_A and P_B and E_{sq} is the natural (homogeneous) extension of the finite-dimensional squashed entanglement defined by (11) to the cone $\mathfrak{T}_+(\mathcal{H}_{AB})$.

Properties of the function \widehat{E}_{sq} and relations between this function and the infinite-dimensional squashed entanglement E_{sq} defined in Section 4.2 are presented in the following proposition.

Proposition 3. A) \widehat{E}_{sq} is an unique lower semicontinuous entanglement measure on $\mathfrak{S}(\mathcal{H}_{AB})$ coinciding with the "finite-dimensional" squashed entanglement on the set $\mathfrak{S}_f \doteq \{\omega_{AB} \mid \max\{\operatorname{rank}\omega_A, \operatorname{rank}\omega_B\} < +\infty\};$

- B) The function \widehat{E}_{sq} possesses properties EM1-EM8;
- C) $\widehat{E}_{sq}(\omega_{AB}) \leq E_{sq}(\omega_{AB})$, where $E_{sq}(\omega_{AB})$ is defined by formula (17), for any state ω_{AB} and $\widehat{E}_{sq}(\omega_{AB}) = E_{sq}(\omega_{AB})$ for a state ω_{AB} in

$$\mathfrak{S}_* \doteq \{\omega_{AB} \mid \min\{H(\omega_A), H(\omega_B), H(\omega_{AB})\} < +\infty\}. \tag{38}$$

D) $\widehat{E}_{sq} = E_{sq}$ if and only if E_{sq} is lower semicontinuous on $\mathfrak{S}(\mathcal{H}_{AB})$.

Proof. Assertions A,B,D and the inequality $\widehat{E}_{sq}(\omega_{AB}) \leq E_{sq}(\omega_{AB})$ directly follow from Propositions 1 and 2.

Let $\omega_{AB} \in \mathfrak{S}_*$ and $\omega_{AB}^n = [\operatorname{Tr} P_A^n \otimes P_B^n \omega_{AB}]^{-1} P_A^n \otimes P_B^n \omega_{AB} P_A^n \otimes P_B^n \in \mathfrak{S}_f$, where $\{P_A^n\}$ and $\{P_B^n\}$ are any sequences of finite rank projectors strongly converging to the operators I_A and I_B . By Proposition 2 the function E_{sq} is lower semicontinuous on \mathfrak{S}_* and monotone under local operations. Hence, $E_{sq}(\omega_{AB}) = \lim_{n \to \infty} E_{sq}(\omega_{AB})$. This and Proposition 1B imply $E_{sq}(\omega_{AB}) = \widehat{E}_{sq}(\omega_{AB})$. \square .

Relations between E_{sq} and \widehat{E}_{sq} are very similar to the relations between the infinite-dimensional versions E_F^d and $E_F^c = \widehat{E}_F$ of the entanglement of formation considered at the end of Section 3 (Corollary 1). Indeed (cf.[27]),

- E_F^c is a unique lower semicontinuous entanglement measure on $\mathfrak{S}(\mathcal{H}_{AB})$ coinciding with the "finite-dimensional" entanglement of formation E_F on the set \mathfrak{S}_f and inheriting all basic properties of E_F ;
- $E_F^d(\omega_{AB}) \geq E_F^c(\omega_{AB})$ for any state ω_{AB} ;
- the equality $E_F^d(\omega_{AB}) = E_F^c(\omega_{AB})$ is proved for any state ω_{AB} in the set \mathfrak{S}_* defined in (38);
- the equality $E_F^d(\omega_{AB}) = 0$ is proved for any separable state ω_{AB} in $\operatorname{conv}(\mathfrak{S}_* \cup \mathfrak{S}_{\operatorname{cd}})$, where $\mathfrak{S}_{\operatorname{cd}}$ is the set of countably decomposable separable states;
- $E_F^d = E_F^c$ if and only if E_F^d is lower semicontinuous on $\mathfrak{S}(\mathcal{H}_{AB})$.

Similar to the function E_{sq} , the equality $E_F^d(\omega_{AB}) = 0$ is not proved (as far as I know) for a countably nondecomposable separable state ω_{AB} such that $H(\omega_A) = H(\omega_B) = +\infty$.⁵

In finite dimensions $E_{sq}(\omega_{AB}) \leq E_F(\omega_{AB})$ for any state ω_{AB} [4, Pr.5]. This relation is generalized as follows.

Corollary 2. For any state ω_{AB} of an infinite-dimensional bipartite system the following relations hold

$$E_{sq}(\omega_{AB}) \le E_F^d(\omega_{AB}), \quad \widehat{E}_{sq}(\omega_{AB}) \le E_F^c(\omega_{AB}).$$

⁵So, strictly speaking, the both functions E_{sq} and E_F^d can not be considered as entanglement measures on $\mathfrak{S}(\mathcal{H}_{AB})$.

Proof. The first inequality follows from the representation (cf.[4])

$$E_F^d(\omega_{AB}) = \frac{1}{2} \inf_{\hat{\omega}_{ABE}} I(A:B|E)_{\hat{\omega}}, \tag{39}$$

where the infimum is over all extensions of the state ω_{AB} having the form

$$\hat{\omega}_{ABE} = \sum_{i} \pi_{i} \omega_{AB}^{i} \otimes |i\rangle\langle i|, \quad \operatorname{rank} \omega_{AB}^{i} = 1.$$

Since $E_F^c = \widehat{E}_F$ by Corollary 1, the second inequality follows from the definitions of the functions \widehat{E}_{sq} and \widehat{E}_F . \square

Note: we can not assert that $E_{sq}(\omega_{AB}) \leq E_F^c(\omega_{AB})$ until it is not proved that either $E_F^d(\omega_{AB}) = E_F^c(\omega_{AB})$ or $E_{sq}(\omega_{AB}) = \widehat{E}_{sq}(\omega_{AB})$.

It is shown in [27, Sect.5] that if $E_F^d \neq E_F^c$ then the function E_F^d demonstrates properties which seems non-adequate for entanglement measure. The same arguments can be repeated for the function E_{sq} if we assume that $E_{sq}(\omega_{AB}) > \widehat{E}_{sq}(\omega_{AB})$ for some state ω_{AB} . So, the function \widehat{E}_{sq} seems to be more preferable candidate on the role of infinite-dimensional squashed entanglement (until it is not proved that $E_{sq} = \widehat{E}_{sq}$).

From the physical point of view possible noncoincidences of E_{sq} with \widehat{E}_{sq} and of E_F^d with E_F^c are not too essential, since $E_{sq}(\omega_{AB}) = \widehat{E}_{sq}(\omega_{AB})$ and $E_F^d(\omega_{AB}) = E_F^c(\omega_{AB})$ for any state ω_{AB} with finite energy provided the Hamiltonian of one of the subsystems A and B satisfies some regularity condition (see Corollary 6 in Section 5.2).

5 On continuity of the squashed entanglement

5.1 General continuity condition and its corollaries

Proposition 2D states that

$$\lim_{k \to \infty} H(\omega_X^k) = H(\omega_X^0) < +\infty \implies \lim_{k \to \infty} E_{sq}(\omega_{AB}^k) = E_{sq}(\omega_{AB}^0) < +\infty, \quad (40)$$

where X is either A or B, for a sequence $\{\omega_{AB}^k\}$ converging to a state ω_{AB}^0 . In this case $E_{sq}(\omega_{AB}^k) = \widehat{E}_{sq}(\omega_{AB}^k)$ for all k (by Proposition 3). Exactly the same continuity condition holds for the entanglement of formation [27, Pr.8].

In fact, a stronger result relating continuity of the squashed entanglement with continuity of the quantum mutual information is valid (which seems more natural than condition (40) from the physical point of view).

Proposition 4. Let $\{\omega_{AB}^k\}$ be a sequence converging to a state ω_{AB}^0 .

- A) If $\min\{H(\omega_A^0), H(\omega_B^0)\} < +\infty$ then the following properties (i)-(iv) are equivalent and imply (v):
 - (i) $\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0};$
- (ii) $\lim_{k\to\infty} E_{sq}^n(\omega_{AB}^k) = E_{sq}^n(\omega_{AB}^0)$ for some $n;^6$
- (iii) $\lim_{k\to\infty} E_{sq}^n(\omega_{AB}^k) = E_{sq}^n(\omega_{AB}^0)$ for all n;
- (iv) $\lim_{k\to\infty} E_{sq}(\omega_{AB}^k) = E_{sq}(\omega_{AB}^0)$ and $\lim_{n\to\infty} \sup_{k\geq k_*} \left[E_{sq}^n(\omega_{AB}^k) E_{sq}(\omega_{AB}^k) \right] = 0$ for sufficiently large k_* ;
- (v) $\lim_{k\to\infty} \widehat{E}_{sq}(\omega_{AB}^k) = \widehat{E}_{sq}(\omega_{AB}^0).$

The condition $\min\{H(\omega_A^0), H(\omega_B^0)\} < +\infty$ implies $E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0)$ and finiteness of the above limits (but it does not imply $E_{sq}(\omega_{AB}^k) = \widehat{E}_{sq}(\omega_{AB}^k)$).

- B) If $H(\omega_A^0) = H(\omega_B^0) = +\infty$ and $H(\omega_{AB}^0) < +\infty$ then (i), (iii), the first part of (iv) and (v) hold as infinite limits.
- C) If $I(A:B)_{\omega^0} < +\infty$ and $\lambda_k \omega_{AB}^k \leq \omega_{AB}^0$ for all k, where $\{\lambda_k\}$ is a sequence of positive numbers converging to 1, then (i)-(iv) hold.

Remark 2. The main assertion of Proposition 4A is the implication

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} \quad \Rightarrow \quad \lim_{k \to \infty} E(\omega_{AB}^k) = E(\omega_{AB}^0)$$
 (41)

for $E = E_{sq}$, \widehat{E}_{sq} . It strengthens condition (40), since Theorem 1A and Example 1 in [28] show that

$$\lim_{k \to \infty} H(\omega_X^k) = H(\omega_X^0) < +\infty \quad \Rightarrow \quad \lim_{k \to \infty} I(A:B)_{\omega_{AB}^k} = I(A:B)_{\omega_{AB}^0} < +\infty,$$

for any sequence $\{\omega_{AB}^k\}$ converging to a state ω_{AB}^0 , where X is either A or B, and that the converse implication is not valid.

⁶The function E_{sq}^n is defined in (18).

The main advantage of continuity condition (41) in contrast to (40) consists in the fact that *local continuity of quantum mutual information is preserved by local operations* (in contrast to local continuity of marginal entropies), see Corollary 3 below.

It seems reasonable to conjecture that " \Leftrightarrow " holds in (41). This would give possibility to prove preserving local continuity of the squashed entanglement under local operations, see the Conjecture after Corollary 3.

Note also that the condition $\min\{H(\omega_A), H(\omega_B)\} < +\infty$ in A is used only to show that $E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0)$ and $I(A:B)_{\omega^0} < +\infty$. To prove that (iv) \Rightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (iii) it suffices to require $I(A:B)_{\omega^0} < +\infty$.

Remark 3. Proposition 4C can be treated as a dominated convergence theorem for the squashed entanglement (cf.[20]). In contrast to condition (40) and Proposition 4A it contains no assumptions concerning marginal entropies. Note that in this case we do not assert that property (v) holds (until it is not proved that $E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0)$).

The proof of Proposition 4 is based on the following two lemmas.

Lemma 6. Let $\{\omega_{AB}^k\}$ be a sequence converging to a state ω_{AB}^0 and $n \in \mathbb{N}$ A) The function E_{sq}^n is lower semicontinuous on $\mathfrak{S}(\mathcal{H}_{AB})$. If $I(A:B)_{\omega^0}$ is finite then

$$\liminf_{k\to\infty} E^n_{sq}(\omega^k_{AB}) - E^n_{sq}(\omega^0_{AB}) \ge \liminf_{k\to\infty} I(A:B)_{\omega^k} - I(A:B)_{\omega^0}.$$

B) Local continuity of I(A:B) is equivalent to local continuity of E_{sq}^n , i.e.

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} < +\infty \quad \Leftrightarrow \quad \lim_{k \to \infty} E_{sq}^n(\omega_{AB}^k) = E_{sq}^n(\omega_{AB}^0) < +\infty$$

Proof. We may assume that for any state ω_{AB} the infimum in definition (18) of $E^n_{sq}(\omega_{AB})$ is over all extensions ω_{ABE} in $\mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}^n_E)$, where \mathcal{H}^n_E is a fixed n-dimensional Hilbert space. By Lemma 1 the set of all such extensions of a given state ω_{AB} is compact. This and the lower semicontinuity of the function $\omega_{ABE} \mapsto I(A:B|E)_{\omega}$ ([28, Th.2]) imply attainability of the infimum in definition (18) of $E^n_{sq}(\omega_{AB})$.

For each k let $\widetilde{\omega}_{ABE}^k \in \mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^n)$ be an extension of the state ω_{AB}^k such that $E_{sq}^n(\omega_{AB}^k) = I(A:B|E)_{\widetilde{\omega}^k}$. By Lemma 1 the sequence $\widetilde{\omega}_{ABE}^k$ is relatively compact and hence it has a limit point $\widetilde{\omega}_{ABE}^0 \in \mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^n)$. By continuity of a partial trace $\widetilde{\omega}_{ABE}^0$ is an extension of the state ω_{AB}^0 and hence $E_{sq}^n(\omega_{AB}^0) \leq I(A:B|E)_{\widetilde{\omega}^0}$.

Assume that $I(A:B)_{\omega^0} < +\infty$ and hence $E_{sq}^n(\omega_{AB}^0) < +\infty$. Let $\epsilon > 0$ be arbitrary and $\{\omega_{AB}^{kt}\}$ be a subsequence such that

$$\lim_{t\to\infty} E_{sq}^n(\omega_{AB}^{k_t}) \le \liminf_{k\to\infty} E_{sq}^n(\omega_{AB}^k) + \epsilon \quad \text{and} \quad \lim_{t\to\infty} \tilde{\omega}_{ABE}^{k_t} = \tilde{\omega}_{ABE}^0.$$

Since the functions $\omega_{ABE} \mapsto I(X : E)_{\omega}$, X = A, B, AB, are continuous on $\mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_{E}^{n})$, formula (16) shows that

$$\liminf_{k \to \infty} I(A:B)_{\omega^k} - I(A:B)_{\omega^0} \le \liminf_{t \to \infty} I(A:B)_{\omega^{k_t}} - I(A:B)_{\omega^0}$$

$$= \lim_{t \to \infty} I(A:B|E)_{\widetilde{\omega}^{k_t}} - I(A:B|E)_{\widetilde{\omega}^0} \le \lim_{t \to \infty} E_{sq}^n(\omega_{AB}^{k_t}) - E_{sq}^n(\omega_{AB}^0).$$

So, by the lower semicontinuity of I(A:B), to prove the lower semicontinuity of E^n_{sq} on $\mathfrak{S}(\mathcal{H}_{AB})$ it suffices to show that $\lim_{k\to\infty} E^n_{sq}(\omega^k_{AB}) = +\infty$ in the case $I(A:B)_{\omega^0} = +\infty$. This can be easily done by using formula (16) and upper bound (4), since in this case the lower semicontinuity of I(A:B) implies $\lim_{k\to\infty} I(A:B)_{\omega^k} = +\infty$.

B) It suffices to show that local continuity of I(A:B) implies local upper semicontinuity of E_{sq}^n . Assume there exists a sequence $\{\omega_{AB}^k\}$ converging to a state ω_{AB}^0 such that

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega} < +\infty \quad \text{and} \quad \lim_{k \to \infty} E_{sq}^n(\omega_{AB}^k) > E_{sq}^n(\omega_{AB}^0). \tag{42}$$

Let $\widetilde{\omega}_{ABE}^0 \in \mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^n)$ be an extension of the state ω_{AB}^0 such that $E_{sq}^n(\omega_{AB}^0) = I(A:B|E)_{\widetilde{\omega}^0}$. By using Lemma 2 it is easy to show existence of a sequence $\{\widetilde{\omega}_{ABE}^k\} \subset \mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^n)$ converging to the state $\widetilde{\omega}_{ABE}^0$ such that $\widetilde{\omega}_{AB}^k = \omega_{AB}^k$ for all k.

Since the functions $\omega_{ABE} \mapsto I(X : E)_{\omega}$, X = A, B, AB, are continuous on $\mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_{E}^{n})$, formula (16) and the first relation in (42) show that

$$\lim_{k \to \infty} I(A:B|E)_{\widetilde{\omega}^k} = I(A:B|E)_{\widetilde{\omega}^0} = E_{sq}^n(\omega_{AB}^0).$$

Since $E_{sq}^n(\omega_{AB}^k) \leq I(A:B|E)_{\widetilde{\omega}^k}$ for all k, this contradicts to the second relation in (42). \square

Lemmas 4 and 6B imply the following observation.

Lemma 7. Local continuity of I(A:B) implies local upper semicontinuity of E_{sq} , i.e. for any sequence $\{\omega_{AB}^k\}$ converging to a state ω_{AB}^0 we have

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} < +\infty \quad \Rightarrow \quad \limsup_{k \to \infty} E_{sq}(\omega_{AB}^k) \le E_{sq}(\omega_{AB}^0) < +\infty.$$

Proof of Proposition 4. A) The condition $\min\{H(\omega_A^0), H(\omega_B^0)\} < +\infty$ implies, by Propositions 3, that

$$\widehat{E}_{sq}(\omega_{AB}^0) = E_{sq}(\omega_{AB}^0) \le I(A:B)_{\omega^0} < +\infty.$$
 (43)

Since $I(A:B)_{\omega^0}$ is finite, (i) \Leftrightarrow (ii) \Leftrightarrow (iii) follows from Lemma 6.

 $(i) \Rightarrow (iv) \Rightarrow (v)$. Lemma 7 shows that

$$\limsup_{k \to \infty} E_{sq}(\omega_{AB}^k) \le E_{sq}(\omega_{AB}^0). \tag{44}$$

Since \widehat{E}_{sq} is lower semicontinuous on $\mathfrak{S}(\mathcal{H}_{AB})$, we have

$$\liminf_{k \to \infty} \widehat{E}_{sq}(\omega_{AB}^k) \ge \widehat{E}_{sq}(\omega_{AB}^0). \tag{45}$$

Since $\widehat{E}_{sq}(\omega_{AB}^k) \leq E_{sq}(\omega_{AB}^k)$ for all k, (43),(44) and (45) imply

$$\lim_{k \to \infty} E_{sq}(\omega_{AB}^k) = E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0) = \lim_{k \to \infty} \widehat{E}_{sq}(\omega_{AB}^k).$$

It follows from (i) that $I(A:B)_{\omega^k} < +\infty$ for all $k \geq k_*$. Lemma 4 and (i) \Leftrightarrow (iii) show that the monotone sequence $\{E^n_{sq}\}$ of continuous functions pointwise converges to the continuous function E_{sq} on the compact set $\{\omega^0_{AB}, \omega^{k_*}_{AB}, \omega^{k_*+1}_{AB}, \ldots\}$. By Dini's lemma the sequence $\{E^n_{sq}\}$ converges uniformly on this set implying the second relation in (iv).

(iv) \Rightarrow (i). Assume that (i) is not valid. By lower semicontinuity of I(A:B) we may consider (by passing to a subsequence) that

$$\liminf_{k \to \infty} I(A:B)_{\omega^k} \ge I(A:B)_{\omega^0} + \Delta$$

for some $\Delta > 0$. So, Lemma 6A implies

$$\liminf_{k \to \infty} E_{sq}^n(\omega_{AB}^k) \ge E_{sq}^n(\omega_{AB}^0) + \Delta \quad \text{for all} \quad n.$$

Since $E_{sq}^n(\omega_{AB}^0)$ tends to $E_{sq}(\omega_{AB}^0)$ by Lemma 4, this contradicts to (iv).

B) Since in this case $I(A:B)_{\omega^0} = +\infty$, inequality (34) and Proposition 3 imply $E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0) = +\infty$. So, to prove this assertion it suffices to note that $\widehat{E}_{sq} \leq E_{sq} \leq E_{sq}^n \leq I(A:B)$ for all n and to use the lower semicontinuity of \widehat{E}_{sq} .

C) The conditions imply (cf.[28, Th.1])

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} < +\infty. \tag{46}$$

By the convexity of E_{sq} we have

$$E_{sq}(\omega_{AB}^0) \leq \lambda_k E_{sq}(\omega_{AB}^k) + (1 - \lambda_k) E_{sq}(\widetilde{\omega}_{AB}^k) \leq \lambda_k E_{sq}(\omega_{AB}^k) + (1 - \lambda_k) I(A:B)_{\widetilde{\omega}^k},$$

where $\widetilde{\omega}_{AB}^k = (1 - \lambda_k)^{-1}(\omega_{AB}^0 - \lambda_k \omega_{AB}^k)$ is bona fide state for each k. So, to prove that $\lim_{k\to\infty} E_{sq}(\omega_{AB}^k) = E_{sq}(\omega_{AB}^0)$ it suffices to show, by Lemma 7, that

$$\lim_{k \to \infty} (1 - \lambda_k) I(A:B)_{\widetilde{\omega}^k} = 0$$

This follows from (46), since inequality (20) implies

$$(1 - \lambda_k)I(A:B)_{\widetilde{\omega}^k} \le I(A:B)_{\omega^0} - \lambda_k I(A:B)_{\omega^k} + 2h_2(\lambda_k).$$

By Lemma 6 relation (46) implies (iii). So, the second part of (iv) follows from Lemma 4 and Dini's lemma. \Box

By Theorem 1B in [28] local continuity of quantum mutual information is preserved by local operations, i.e. for any sequence $\{\omega_{AB}^k\}$ of states converging to a state ω_{AB}^0 and arbitrary quantum operations $\Phi_A:A\to A$ and $\Phi_B:B\to B$ we have

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} < +\infty \quad \Rightarrow \quad \lim_{k \to \infty} I(A:B)_{\widetilde{\omega}^k} = I(A:B)_{\widetilde{\omega}^0} < +\infty,$$

where $\widetilde{\omega}_{AB}^k = \Phi_A \otimes \Phi_B(\omega_{AB}^k)$ for all k.⁷ This implication, Proposition 4A and the last sentence in Remark 2 imply the following observation.

Corollary 3. If one of properties (i)-(iv) in Proposition 4A holds for a sequence $\{\omega_{AB}^k\}$ converging to a state ω_{AB}^0 such that $I(A:B)_{\omega^0} < +\infty$ then all the properties (i)-(v) hold for the sequence $\{\widetilde{\omega}_{AB}^k\}$, where

$$\widetilde{\omega}_{AB}^k = [\operatorname{Tr} \Phi_A \otimes \Phi_B(\omega_{AB}^k)]^{-1} \Phi_A \otimes \Phi_B(\omega_{AB}^k),$$

for any local quantum operations $\Phi_A : A \to A$ and $\Phi_B : B \to B$ such that $\Phi_A \otimes \Phi_B(\omega_{AB}^0) \neq 0$ and min $\{H(\Phi_A(\omega_A^0)), H(\Phi_B(\omega_B^0))\} < +\infty$. In particular,

$$\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0} < +\infty \quad \Rightarrow \quad \lim_{k \to \infty} E(\widetilde{\omega}_{AB}^k) = E(\widetilde{\omega}_{AB}^0) < +\infty,$$

⁷If Φ_A and Φ_B are not channels then $I(A:B)_{\tilde{\omega}^k}$ in the above relation is defined by formula (2).

for
$$E = E_{sq}$$
, \widehat{E}_{sq} .

The main assertion of Corollary 3 is a weak form of the following

Conjecture: Local continuity of the squashed entanglement is preserved by local operations.

To prove this conjecture it suffices to show that " \Leftrightarrow " holds in (41).

Theorem 1A in [28] and Proposition 4A imply the following continuity condition.

Corollary 4. Let ω_{AB}^0 be a state such that $\min \{H(\omega_A^0), H(\omega_B^0)\} < +\infty$ and $\{\omega_{AB}^k\}$ be a sequence of states converging to the state ω_{AB}^0 such that $\lambda_k \omega^k \leq \Phi_A^k \otimes \Phi_B^k(\omega^0)$ for some local quantum operations Φ_A^k and Φ_B^k , where $\{\lambda_k\}$ is a sequence converging to 1. Then

$$\lim_{k \to \infty} E_{sq}(\omega_{AB}^k) = \lim_{k \to \infty} \widehat{E}_{sq}(\omega_{AB}^k) = E_{sq}(\omega_{AB}^0) = \widehat{E}_{sq}(\omega_{AB}^0) < +\infty. \tag{47}$$

Corollary 4 shows in particular that (47) holds if $\{\omega_{AB}^k\}$ is a sequence of states proportional to the operators

$$\operatorname{Tr}_C \Phi_A^k \otimes \Phi_B^k \otimes \Phi_C^k(\omega_{ABC}^0),$$

where ω_{ABC}^0 is a state such that min $\{H(\omega_A^0), H(\omega_B^0)\} < +\infty$ and $\{\Phi_A^k\}, \{\Phi_B^k\}, \{\Phi_C^k\}$ are sequences of local quantum operations strongly converging to the identity channels $\mathrm{Id}_A, \mathrm{Id}_B, \mathrm{Id}_C$ (this means that $\lim_k \Phi_X^k(\rho) = \rho$ for arbitrary states $\rho \in \mathfrak{S}(\mathcal{H}_X), X = A, B, C$ [12]).

Corollary 5. If one of the systems A and B, say A, is finite-dimensional then $E_{sq} = \tilde{E}_{sq} = E_{sq}^*$ (the function E_{sq}^* is defined by (19)) is a continuous entanglement measure on $\mathfrak{S}(\mathcal{H}_{AB})$ and

$$|E_{sq}(\omega_{AB}^2) - E_{sq}(\omega_{AB}^1)| \le \sqrt{\varepsilon} \log \dim \mathcal{H}_A + 2(1+\sqrt{\varepsilon})h_2\left(\frac{\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}\right),$$

for any ω_{AB}^1 , ω_{AB}^2 such that $\varepsilon = \|\omega_{AB}^2 - \omega_{AB}^1\|_1 < 1$.

Proof. Lemma 4 and Proposition 3 imply $E_{sq} = \widehat{E}_{sq} = E_{sq}^*$. To prove the continuity bound it suffices to note that in this case E_{sq} coincides with the function f_n in the proof of assertion C of Proposition 2 for some finite n and to repeat the arguments from that proof. \square

5.2 Continuity bounds for E_{sq} and for E_F under the energy constraint on one subsystem

In this subsection we restrict attention to subsets of bipartite states ω_{AB} with bounded energy of ω_A , i.e. subsets of the form

$$\mathfrak{S}_E \doteq \{ \omega_{AB} \mid \text{Tr} H_A \omega_A \le E \}, \tag{48}$$

where H_A is a Hamiltonian of the system A and E > 0.

It is well known that the von Neumann entropy is bounded on the set $\{\rho_A \mid \text{Tr} H_A \rho_A \leq E\}$ if and only if $\text{Tr} e^{-\beta H_A} < +\infty$ for some $\beta > 0$ and it is continuous on this set if and only if $\text{Tr} e^{-\beta H_A} < +\infty$ for all $\beta > 0$ [26, 31]. So, Propositions 2D and 3C imply the following assertions.

Corollary 6. A) If $\operatorname{Tr} e^{-\beta H_A}$ is finite for some $\beta > 0$ then the functions E_{sq} and \widehat{E}_{sq} coincide and are lower semicontinuous on \mathfrak{S}_E for any E > 0.

- B) If $\operatorname{Tr} e^{-\beta H_A}$ is finite for all $\beta > 0$ then the functions E_{sq} and \widehat{E}_{sq} coincide and are continuous on \mathfrak{S}_E for any E > 0.
- C) Assertions A and B also hold for the infinite-dimensional versions E_F^d and E_F^c of the entanglement of formation (considered at the end of Sect.3).

The last assertion of Corollary 6 follows from the results in [27, Sect.5].

Corollary 6 implies coincidence and continuity of the functions E_{sq} and \widehat{E}_{sq} (and of the functions E_F^d and E_F^c) on the set of states of a bipartite finite-mode Bosonic system with bounded mean energy [10, Ch.12].

Corollary 6 can be strengthened by using the approach recently proposed by Winter in [32], where the tight continuity bounds for the von Neumann entropy and for the conditional entropy under the energy constraint are obtained. Combining this approach, Lemma 5 in Section 4 and the Fannes type continuity bound for the conditional mutual information one can obtain tight continuity bound for the conditional mutual information under the energy constraint on one subsystem (Lemma 8 in the Appendix), which makes possible to derive continuity bounds for the squashed entanglement and for the entanglement of formation under the same constraint.

Let H_A be a Hamiltonian of system A such that

$$Z(\beta) \doteq \text{Tr}e^{-\beta H_A} < +\infty \quad \text{for all} \quad \beta > 0.$$
 (49)

Then for any E > 0 the von Neumann entropy attains its maximum on the set $\{\rho_A \mid \text{Tr} H_A \rho_A \leq E\}$ at the Gibbs state $\gamma(E) = [Z(\beta(E))]^{-1} e^{-\beta(E)H_A}$, where

 $\beta(E)$ is the solution of the equation $\text{Tr}H_A e^{-\beta H_A} = EZ(\beta)$ [31]. Condition (49) implies that H_A has discrete spectrum $\{E_1, E_2, ...\}$ of finite multiplicity. We will assume that $E_1 = 0$.

We will obtain continuity bound for the squashed entanglement on the set \mathfrak{S}_E defined by (48) and will show that the same continuity bound is valid for the entanglement of formation. Corollary 6 shows that $E_{sq}(\omega_{AB}) = \widehat{E}_{sq}(\omega_{AB})$ and $E_F^d(\omega_{AB}) = E_F^c(\omega_{AB})$ for any state $\omega_{AB} \in \mathfrak{S}_E$. So, in what follows we will forget about possible noncoincidence of E_{sq} with \widehat{E}_{sq} and of E_F^d with E_F^c and will denote these functions respectively by E_{sq} and by E_F .

Proposition 5. Let ω_{AB}^1 and ω_{AB}^2 be arbitrary states in \mathfrak{S}_E such that $\|\omega_{AB}^2 - \omega_{AB}^1\|_1 = \varepsilon < 1$. Let $\varepsilon' \in (\sqrt{\varepsilon}, 1]$ and $\delta = \frac{\varepsilon' - \sqrt{\varepsilon}}{1 + \varepsilon'}$. Then

$$\left| E_{sq}(\omega_{AB}^2) - E_{sq}(\omega_{AB}^1) \right| \le (\varepsilon' + 2\delta) H(\gamma(E/\delta)) + 2(1+\varepsilon') h_2\left(\frac{\varepsilon'}{1+\varepsilon'}\right) + 2h_2(\delta)$$

The same continuity bound is valid for the entanglement of formation E_F .

The coincidence of the continuity bounds for E_{sq} and for E_F is not surprising due to representation (39).

Remark 4. By Proposition 1 in [26] condition (49) implies

$$\lim_{\delta \to +0} \delta H(\gamma(E/\delta)) = 0. \tag{50}$$

So, the continuity bounds in Proposition 5 show uniform continuity of the squashed entanglement and of the entanglement of formation on the set \mathfrak{S}_E .

Proof. By repeating the arguments from the proof of continuity of E_{sq} in [4] we obtain

$$E_{sq}(\omega_{AB}^k) = \frac{1}{2} \inf_{\Lambda} I(A:B|E)_{\mathrm{Id}_{AB} \otimes \Lambda(\omega_{ABC}^k)}, \quad k = 1, 2,$$
 (51)

where ω_{ABC}^1 and ω_{ABC}^2 are purifications of ω_{AB}^1 and of ω_{AB}^2 such that

$$\|\omega_{ABC}^2 - \omega_{ABC}^1\|_1 \le 2\sqrt{\varepsilon},$$

and the infimum is over all quantum channels $\Lambda : \mathfrak{T}(\mathcal{H}_C) \to \mathfrak{T}(\mathcal{H}_E)$.

For given Λ let $\omega_{ABE}^1 = \operatorname{Id}_{AB} \otimes \Lambda(\omega_{ABC}^1)$ and $\omega_{ABE}^2 = \operatorname{Id}_{AB} \otimes \Lambda(\omega_{ABC}^2)$. Then $\|\omega_{ABE}^2 - \omega_{ABE}^1\|_1 \leq 2\sqrt{\varepsilon}$, and Lemma 8 in the Appendix implies

$$|I(A:B|E)_{\omega^2} - I(A:B|E)_{\omega^1}| \le (2\varepsilon' + 4\delta)H(\gamma(E/\delta)) + 4\theta(\varepsilon') + 4h_2(\delta),$$

where $\theta(x) = (1+x)h_2(\frac{x}{1+x})$. Since this estimate holds for any quantum channel Λ , expression (51) implies the above continuity bound for E_{sq} .

To obtain the continuity bound for E_F we will use the modification of Nielsen's technique [24]. Let $\epsilon > 0$ be arbitrary and $\{\pi_i, \varpi_{AB}^i\}$ be an ensemble (finite or countable) of pure states such that

$$\omega_{AB}^1 = \sum_i \pi_i \varpi_{AB}^i \quad \text{and} \quad E_F(\omega_{AB}^1) \ge \sum_i \pi_i H(\varpi_A^i) - \epsilon.$$

Let R be an infinite-dimensional reference system, ω_{ABR}^1 and ω_{ABR}^2 be purifications of ω_{AB}^1 and of ω_{AB}^2 such that

$$\|\omega_{ABR}^2 - \omega_{ABR}^1\|_1 \le 2\sqrt{\varepsilon}.$$

By the arguments from [14] we can choose such basic $\{|i\rangle\}$ in \mathcal{H}_R that

$$\tilde{\omega}_{ABR}^1 = \sum_i I_{AB} \otimes |i\rangle\langle i| \,\omega_{ABR}^1 I_{AB} \otimes |i\rangle\langle i| = \sum_i \pi_i \varpi_{AB}^i \otimes |i\rangle\langle i|.$$

Thus,

$$\sum_{i} \pi_{i} H(\varpi_{A}^{i}) = \frac{1}{2} I(A:B|R)_{\tilde{\omega}^{1}} \text{ and hence } E_{F}(\omega_{AB}^{1}) \geq \frac{1}{2} I(A:B|R)_{\tilde{\omega}^{1}} - \epsilon.$$

On the other hand, since

$$\tilde{\omega}_{ABR}^2 = \sum_{i} I_{AB} \otimes |i\rangle\langle i| \,\omega_{ABR}^2 \,I_{AB} \otimes |i\rangle\langle i| = \sum_{i} |\psi_i\rangle\langle\psi_i| \otimes |i\rangle\langle i|,$$

where $\{|\psi_i\rangle\}$ is a collection of vectors such that $\sum_i |\psi_i\rangle\langle\psi_i| = \tilde{\omega}_{AB}^2 = \omega_{AB}^2$, it is easy to see that $\frac{1}{2}I(A:B|R)_{\tilde{\omega}^2} \geq E_F(\omega_{AB}^2)$.

So, we have

$$E_F(\omega_{AB}^2) - E_F(\omega_{AB}^1) \le \frac{1}{2}I(A:B|R)_{\tilde{\omega}^2} - \frac{1}{2}I(A:B|R)_{\tilde{\omega}^1} + \epsilon.$$

Since $\|\tilde{\omega}_{ABR}^2 - \tilde{\omega}_{ABR}^1\|_1 \leq 2\sqrt{\varepsilon}$, by applying Lemma 8 in the Appendix we obtain

$$E_F(\omega_{AB}^2) - E_F(\omega_{AB}^1) \le (\varepsilon' + 2\delta)H(\gamma(E/\delta)) + 2(1+\varepsilon')h_2\left(\frac{\varepsilon'}{1+\varepsilon'}\right) + 2h_2(\delta) + \epsilon.$$

By permuting ω_{AB}^2 and ω_{AB}^1 in the above argumentation we obtain the same upper bound for $E_F(\omega_{AB}^1) - E_F(\omega_{AB}^2)$. Since ϵ is arbitrary, this gives the continuity bound for E_F coinciding with the continuity bound for E_{sq} . \square

Proposition 5 implies the following asymptotic continuity property of the squashed entanglement and of the entanglement of formation (cf. [8]).

Corollary 7. Let $H_{A^n} = H_A \otimes I_A \otimes ... \otimes I_A + ... + I_A \otimes ... \otimes I_A \otimes H_A$ be a Hamiltonian of the system A^n and $\{\omega_n^1\}$, $\{\omega_n^2\}$ be sequences of states such that

$$\omega_n^k \in \mathfrak{S}(\mathcal{H}_{AB}^{\otimes n}), \quad \text{Tr} H_{A^n}[\omega_n^k]_{A^n} \le nE, \quad k = 1, 2, \qquad \lim_{n \to \infty} \|\omega_n^2 - \omega_n^1\|_1 = 0.$$

If H_A satisfies condition (49) then

$$\lim_{n\to\infty} \frac{|E_{sq}(\omega_n^2) - E_{sq}(\omega_n^1)|}{n} = 0 \quad and \quad \lim_{n\to\infty} \frac{|E_F(\omega_n^2) - E_F(\omega_n^1)|}{n} = 0.$$

Proof. Note that H_{A^n} satisfies condition (49) and that $\gamma(E)^{\otimes n}$ is the Gibbs state of the system A^n corresponding to the energy nE. So, the above limit relations directly follow from the continuity bounds in Proposition 5 and relation (50). \square

6 Conclusion

We have presented a detailed analysis of properties of the squashed entanglement E_{sq} obtained by direct translation (17) of the finite-dimensional definition in comparison with the universal extension \hat{E}_{sq} defined by formula (37). The function \hat{E}_{sq} is a result of application to the case of squashed entanglement of the general method for construction of infinite-dimensional entanglement monotones starting from finite-dimensional ones described in Section 3.

It is shown that the functions E_{sq} and \widehat{E}_{sq} produce the same entanglement measure on the set

$$\mathfrak{S}_* \doteq \{\omega_{AB} \mid \min\{H(\omega_A), H(\omega_B), H(\omega_{AB})\} < +\infty\}$$

possessing all basis properties of the squashed entanglement valid in finite dimensions (with continuity replaced by lower semicontinuity). It is proved

that the function \widehat{E}_{sq} is an adequate lower semicontinuous extension of this measure to the set $\mathfrak{S}(\mathcal{H}_{AB})$ of all bipartite states. Coincidence of E_{sq} and \widehat{E}_{sq} on $\mathfrak{S}(\mathcal{H}_{AB})$ is a conjecture equivalent to lower semicontinuity of E_{sq} on $\mathfrak{S}(\mathcal{H}_{AB})$.

It is shown that local continuity of the quantum mutual information implies local continuity of the squashed entanglement (Proposition 4, Remark 2). A weak form of the conjecture that local continuity of the squashed entanglement is preserved by local operations is proved (Corollary 3).

The common continuity bound for the squashed entanglement and for the entanglement of formation under the energy constraint on one subsystem is obtained and used to prove the asymptotic continuity of these entanglement measures under the same constraint (Proposition 5, Corollary 7).

Some open questions are formulated. One of them concerns possibility to extend a countably nondecomposable separable state to a short Markov chain (Remark 1).

7 Appendix: Tight continuity bound for conditional mutual information under the energy constraint on one subsystem

Since for any state ω_{ABC} with finite $H(\omega_A)$ we have

$$I(A:C|B)_{\omega} = H(A|B)_{\omega} - H(A|BC)_{\omega}$$

where H(A|X) is the extended conditional entropy defined in (5), continuity bound for $I(A:C|B)_{\omega}$ under the energy constraint on ω_A can be directly obtained from Meta-Lemma 14 in [32] (by proving that this lemma remans valid for the extended conditional entropy).

But more sharp continuity bound for $I(A:C|B)_{\omega}$ can be obtained by using Winter's technique (rather than his final results) and some estimates for conditional mutual information.

Let H_A be a Hamiltonian of system A such that $Z(\beta) \doteq \operatorname{Tr} e^{-\beta H_A}$ is finite for all $\beta > 0$. This implies that H_A has discrete spectrum of finite multiplicity, i.e. $H_A = \sum_{n=1}^{+\infty} E_n |n\rangle\langle n|$, where $\{|n\rangle\}_{n=1}^{+\infty}$ is an orthonormal basis of eigenvectors of H_A corresponding to the nondecreasing sequence $\{E_n\}_{n=1}^{+\infty}$ of eigenvalues (energy levels of H_A) such that $\sum_{n=1}^{+\infty} e^{-\beta E_n}$ is finite for all $\beta > 0$. We will assume that $E_1 = 0$ for simplicity.

For any E > 0 the von Neumann entropy attains its maximum under the constraint $\text{Tr}\rho H_A \leq E$ at the Gibbs state $\gamma(E) = [Z(\beta(E))]^{-1}e^{-\beta(E)H_A}$, where $\beta(E)$ is the solution of the equation $\text{Tr}H_Ae^{-\beta H_A} = EZ(\beta)$ [31].

Lemma 8. Let ρ and σ be states of the tripartite system ABC such that $\operatorname{Tr} \rho_A H_A, \operatorname{Tr} \sigma_A H_A \leq E, \frac{1}{2} \|\rho - \sigma\|_1 \leq \varepsilon < \varepsilon' \leq 1$ and $\delta = \frac{\varepsilon' - \varepsilon}{1 + \varepsilon'}$. Then

$$|I(A:C|B)_{\rho} - I(A:C|B)_{\sigma}| \le (2\varepsilon' + 4\delta)H(\gamma(E/\delta)) + 4(1+\varepsilon')h_2\left(\frac{\varepsilon'}{1+\varepsilon'}\right) + 4h_2(\delta).$$

Remark 5. Since $\lim_{\delta \to +0} \delta H(\gamma(E/\delta)) = 0$ (see Remark 4), Lemma 8 shows that the function $\omega_{ABC} \mapsto I(A:C|B)_{\omega}$ is uniformly continuous on the set $\{\omega_{ABC} \mid \text{Tr} H_A \omega_A \leq E\}$ for any E > 0.

Proof. Following the proofs of Lemmas 13,14 in [32] define the projectors

$$P_{\leq} \doteq \sum_{0 \leq E_n \leq E/\delta} |n\rangle\langle n|$$

and consider the states

$$\rho \le = \frac{P_\le \rho P_\le}{\operatorname{Tr} P_\le \rho}, \quad \sigma_\le = \frac{P_\le \sigma P_\le}{\operatorname{Tr} P_\le \sigma}.$$

In the proof of Lemma 13 in [32] it is shown that

$$H(\omega) - [\operatorname{Tr} P_{\leq} \omega] H(\omega_{\leq}) \leq \delta H(\gamma(E/\delta)) + h_2(\operatorname{Tr} P_{\leq} \omega),$$
 (52)

$$H(\omega_{<}) \le H(\gamma(E/\delta)), \quad \text{Tr} P_{<}\omega \ge 1 - \delta,$$
 (53)

where $\omega = \rho, \sigma$, and that

$$\log \operatorname{Tr} P_{\leq} \leq H(\gamma(E/\delta)), \quad \frac{1}{2} \|\rho_{\leq} - \sigma_{\leq}\| \leq \varepsilon'. \tag{54}$$

By using (52) and (53) it is easy to derive from the second inequality of Lemma 5 that

$$\left| I(A:C|B)_{\omega} - I(A:C|B)_{\omega_{<}} \right| \le 2\delta H(\gamma(E/\delta)) + 2h_2(\delta), \quad \omega = \rho, \sigma. \quad (55)$$

By using (54) and applying Corollary 8 in [28] (Fannes type continuity bound for conditional mutual information) we obtain

$$\left| I(A:C|B)_{\rho_{\leq}} - I(A:C|B)_{\sigma_{\leq}} \right| \leq 2\varepsilon' \log \operatorname{Tr} P_{\leq} + 4(1+\varepsilon')h_{2}\left(\frac{\varepsilon'}{1+\varepsilon'}\right)
\leq 2\varepsilon' H(\gamma(E/\delta)) + 4(1+\varepsilon')h_{2}\left(\frac{\varepsilon'}{1+\varepsilon'}\right).$$
(56)

Since

$$|I(A:C|B)_{\rho} - I(A:C|B)_{\sigma}| \le |I(A:C|B)_{\rho \le \sigma} - I(A:C|B)_{\sigma \le \sigma}|$$

$$+ |I(A:C|B)_{\rho} - I(A:C|B)_{\rho \le \sigma}| + |I(A:C|B)_{\sigma} - I(A:C|B)_{\sigma \le \sigma}|,$$

the required continuity bound follows from (55) and (56). \square

Remark 6. By using Lemma 8 one can obtain analog of Lemma 15 in [32] for conditional mutual information, i.e. a continuity bound for conditional mutual information I(A:C|B) under the energy constraint on the system A composed of ℓ oscillators (assuming that B and C are arbitrary systems). Since the main terms in Lemma 8 and in Meta-Lemma 14 in [32] coincide, the main term in this continuity bound coincides with the main term

$$2\varepsilon \left(\frac{1+\alpha}{1-\alpha}+2\alpha\right) \sum_{i=1}^{\ell} \log \left(\frac{\bar{E}}{\hbar\omega_i}+1\right), \quad \bar{E}=E/\ell,$$

in the continuity bound for conditional entropy in Lemma 15 in [32], where $\omega_1, ..., \omega_\ell$ are frequencies of the oscillators and $\alpha \in (0, 1/2)$ is a free parameter. By choosing arbitrarily small α and large energy E we see (as in Remark 16 in [32]) that this term is approximately equal to $2\varepsilon H(\gamma(E))$.

Let $\rho = \gamma(E) \otimes \tau_B \otimes \tau_C$ and $\sigma = (1 - \varepsilon)\rho + \varepsilon |\phi_{AC}\rangle \langle \phi_{AC}| \otimes \tau_B'$, where $|\phi_{AC}\rangle \langle \phi_{AC}| \in \mathfrak{S}(\mathcal{H}_{AC})$ is a purification of the Gibbs state $\gamma(E) \in \mathfrak{S}(\mathcal{H}_A)$, τ_B and τ_B' are orthogonal pure states in $\mathfrak{S}(\mathcal{H}_B)$, τ_C is a pure state in $\mathfrak{S}(\mathcal{H}_C)$. Then $I(A:C|B)_{\rho} = 0$ and $I(A:C|B)_{\sigma} = 2\varepsilon H(\gamma(E))$, so that

$$I(A:C|B)_{\sigma} - I(A:C|B)_{\rho} = 2\varepsilon H(\gamma(E)).$$

Since $\frac{1}{2} \|\sigma - \rho\|_1 \leq \varepsilon$, we conclude that Lemma 8 gives asymptotically tight continuity bound for conditional mutual information.

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