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Quantum Riemannian Geometry

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Preface

There are many ways to extend the ideas of classical differential geometry to a noncommutative world. Our view is that there is no clear answer as to which of these is correct, given that many of them have their own rich pure mathematical theory. However, if we were to think about what *should* ideally be noncommutative differential geometry, we might identify the following considerations. There should be a broad collection of examples of interest across different branches of mathematics. Noncommutative geometry should reduce to classical geometry as a special case, though some aspects of the theory may become trivial in the classical case. Most constructions in classical differential geometry should have noncommutative geometry analogues. Last but not least, as geometry originated as a practical subject, there should be applications, which historically has meant applications in physics and applied mathematics. With this in mind, it has been one of the principles of this book to include both the pure mathematical background and applications, from categories to cosmology and from modules to Minkowski space. We shall try to explain both aspects from a relatively elementary starting point. Much of the work will be taken from our own research papers, which have been inspired by the above point of view, particularly from our experience with quantum groups (a 'quantum groups approach to noncommutative geometry'), although not limited to this. In short, we will provide one particularly constructive and computable style of noncommutative geometry, but we will also include links to other approaches where possible.

Noncommutative geometry, one way or another, arose from experience with quantum theory. By the 1920s, Dirac has already speculated about geometry with noncommuting x, p coordinates, and the great theorems of Gel'fand and Naimark for C^*-algebras and the GNS construction of Hilbert space representations of noncommutative C^*-algebras were driven by mathematical physics in the context of quantum theory. K-theory, universal differentials, vector bundles (as projective modules) and connections for noncommutative algebras, as well as Hochschild and cyclic cohomology, were a natural progression in this direction, culminating in Connes' famous notion of a spectral triple as an abstract 'Dirac operator' in the early 1980s. Meanwhile, and quite separately in the mid 1980s, large classes of noncommutative algebras appeared as part of the 'quantum groups revolution'. These objects arose on the one hand in the context of generalised symmetries in quantum integrable systems (the Drinfeld–Jimbo quantum groups $U_q(\mathfrak{g})$) and on the other hand from ideas of quantum Born reciprocity or observable-state duality in quantum gravity (the bicrossproduct quantum groups in the PhD thesis of one of the authors). These remain two main classes of quantum groups, with the first class of direct interest in many branches of mathematics, including knot theory and category theory, and the second class of particular interest as Poincaré quantum symmetries of noncommutative or 'quantum' spacetime. In either case, such quantum groups provide key examples of noncommutative algebras with a clear geometric significance which therefore *should* be foundational examples of noncommutative geometry, just as classical Lie groups were of classical differential geometry.

An outline of the book is as follows. In Chap. 1, we cover the basic theory of algebras equipped with differential structure expressed as exterior algebras of differential forms. The chapter also introduces the notion of a quantum metric and, in the case of an 'inner' calculus, an induced quantum Laplacian, as elementary layers of the theory that depend only on the differential structure. We also introduce many of the basic examples which will be further developed in subsequent chapters as we build up the different layers of noncommutative geometry in our approach. The last sections provide some applications so that, all together, Chap. 1 could be read as a self-contained first introduction to noncommutative differential geometry as we see it.

Chapter 2 provides a condensed introduction to Hopf algebras or 'quantum groups' and their representations as monoidal or (in the (co)quasitriangular case) braided monoidal categories. Much more can be found in several textbooks on quantum groups, including a text by one of the present authors, from which we borrow our notation. The chapter then develops the theory of differential structures and exterior algebras on Hopf algebras, including a braided antisymmetric algebra approach to the Woronowicz construction, and quantum/braided Lie algebras on quantum groups. The last section of Chap. 2 introduces the notion of a bar category, which is needed to formulate complex conjugation and $*$-operations in a more categorical way.

Chapter 3 introduces the basic notions of a vector bundle (as a projective module of sections) over an algebra and of a connection on such a vector bundle over a differential algebra. We also cover elements of cyclic cohomology and K-theory, including the famous Chern–Connes pairing between the two. Particularly important for in this chapter is the idea that 1-forms in noncommutative geometry are bimodules (one can multiply by the algebra from either side) with the result that one should ask for a connection to obey a Leibniz type rule from both sides. If the Leibniz rule on one side has a standard form then the other side will need to refer to a 'generalised braiding' σ and when this exists we say that we have a 'bimodule connection'. Chapters 1–3 constitute the basic foundation of the book.

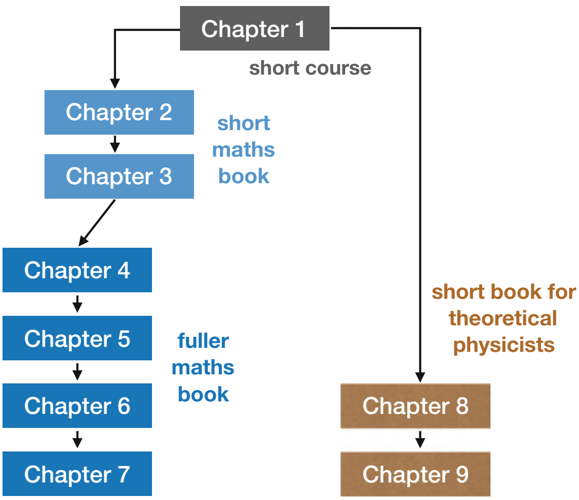
Chapter 4 proceeds to harder results about the curvature of connections such as the Bianchi identities and characteristic classes. The chapter also includes a study of the category of modules equipped with flat connections, which could be seen as playing the role of sheaves over the algebra. Various constructions with sheaves and cohomology are given which are analogous to classical constructions. Some applications of spectral sequences are given, including the Leray–Serre spectral sequence of a fibration, for a differential definition of noncommutative fibration. We also look at positive maps and Hilbert C^*-modules, and extend the idea of bimodules as generalised morphisms between algebras to a differential setting using bimodule connections on B-A bimodules for different algebras A, B.

Chapter 5 looks at quantum principal bundles, where the fibre is now a Hopf algebra or quantum group. This is a theory that goes back to Brzeziński and one of the authors, while in the case of the universal calculus it is also known in algebra as a Hopf–Galois extension. We explain the link with Galois theory and provide the general theory of associated bundles and induced bimodule connections on them when the principal bundle has a connection-form. We also study differential fibrations more generally. The last section of the chapter is an application to quantum framed spaces, i.e., differential algebras appearing as the base of a quantum principal bundle and data such that the space Ω^1 of differential 1-forms is an associated bundle. The chapter includes bundles and q-monopole bimodule connections over the standard q-deformed sphere, among other examples, which combines with our quantum framing theory to provide our first encounter with a 'quantum Levi-Civita connection' as a torsion free metric compatible bimodule connection on Ω^1. The framing approach also allows us to solve for a connection on the quantum group $\mathbb{C}_q[SU_2]$ with its 4D calculus, which turns out to be 'weak quantum Levi-Civita' in the sense of torsion free and cotorsion free.

Chapter 6 develops the theory of vector fields and the algebra of differential operators $\mathcal{D}_A$ associated to an algebra with differential calculus. As in the classical theory, modules over the algebra $\mathcal{D}_A$ are the same as A-modules with flat connection. The algebra $\mathcal{D}_A$ has extra algebraic structure, which can be expressed by saying that it is a braided-commutative algebra in the centre of the monoidal category of bimodules with bimodule connections. $\mathcal{D}_A$ for A the algebra of 2×2 matrices with a natural differential structure turns out to be generated by A and a single quantised fermion. We also introduce $T\mathfrak{X}_\bullet$ with modules the same as A-modules with connection, not necessarily flat. Chapter 7 introduces complex structures in the same manner as classical complex manifold theory. This involves a bigrading of the exterior algebra to give a double complex and allows the definition of holomorphic modules along with implications for cohomology theories. These are shorter chapters and complete the more advanced mathematical content of the book. A mathematically-minded reader should be able to use the book as Chaps. 1–3, or Chaps. 1–7, depending on how far one wishes to travel.

Chapter 8 brings together previously encountered notions of Riemannian and other structures from Chaps. 3, 4, 5 into a self-contained account of noncommutative Riemannian geometry over an algebra equipped with differential structure and choice of metric. Finding an associated torsion free and metric compatible bimodule connection (or quantum Levi-Civita connection) on Ω^1 here is a well-posed nonlinear problem and the chapter shows how it can be solved directly in a variety of models. The chapter also includes a section on Connes' spectral triples and how they can sometimes arise in a weakened form in our constructive approach, as a Dirac operator built along geometric lines from a connection and a Clifford structure. Examples include the q-sphere and the algebra of 2×2 matrices. Other topics include a wave-operator approach to quantum Riemannian geometry that short-cuts the layer-by-layer treatment by directly formulating the quantum Laplacian as a partial derivative of an extended calculus. The chapter also includes a slightly different theory of hermitian-metric compatible connections and Chern connections in noncommutative geometry.

Chapter 9 concludes the book with applications specifically to quantum spacetime. Unlike quantum phase space in quantum mechanics, there is as yet no physical evidence that the coordinates of spacetime themselves form a noncommutative differential algebra. Indeed this effect, if observed, would be a discovery on a par with the discovery of gravity itself (one can call it 'co-gravity' as it is in some sense dual to gravity). By now, this striking possibility is widely accepted in quantum gravity circles as a plausible better-than-classical model of spacetime that takes into account Planck scale or quantum gravity corrections. At the same time, physicists and applied mathematician readers should consider that just as geometry has many roles beyond gravity, so does quantum or noncommutative geometry, for example to the geometry of discrete systems as well as potentially to actual quantum-mechanics. We have written the book in such a way that such readers should be able to focus on Chaps. 1, 8, 9, with the intervening chapters dipped into as needed for further details of the underlying mathematics.



There are inevitably other topics which we have not had room to include, and many of these are of great importance and could form the basis of a future volume. We included some of these topics, at an introductory level only, in the final sections of the book. Thus, we briefly treat the semiclassical behaviour within deformation theory or ‘Poisson–Riemannian geometry’, a paradigm which includes first-order quantum gravity effects but bears the same relation to quantum gravity as does classical mechanics to quantum mechanics. We have also treated only briefly the construction of examples by ‘functorial twisting’, which is particularly interesting in the cochain case where the data on the symmetry quantum group is not a cocycle and the exterior algebra becomes nonassociative. This is a big topic in its own right and also closely tied to the semiclassical theory, where it corresponds to curvature of the Poisson-compatible connection. The reader will also notice that we have only briefly presented some areas from the C^*-algebras side of noncommutative geometry and in this case there are more comprehensive works elsewhere.

Outline of Notations and Examples

Here we outline some of the notational conventions for reference and orientation. Most constructions work over a general field $\Bbbk$ but most of the time one can keep in mind $\mathbb{R}$ or $\mathbb{C}$. We use a standard convention for expressing a tensor product of matrices as a single matrix: for the example of a 3×3 matrix A,

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & A_{13}B \\ A_{21}B & A_{22}B & A_{23}B \\ A_{31}B & A_{32}B & A_{33}B \end{pmatrix}$$

which is equivalent to $A^i{}_j B^k{}_l$ written as a matrix with rows ik and columns jl taken in order $11, 12, \cdots, 33$. A dot is typically used to emphasise a product, often to separate different types of elements. For example $\mathrm{d}a.b$ may be used to indicate $(\mathrm{d}a)b$ (rather than $\mathrm{d}(ab)$) or $e.a$ for the product of a right module by an element of the algebra. Left and right actions that are different from default module or bimodule structures will typically be denoted by $\triangleright$, $\triangleleft$ respectively. Composition of maps may for emphasis be written as $f \circ g$, meaning to apply g first.

The symbol δ_x is used for the function taking the value 1 at x and zero elsewhere. We also use the Kronecker delta $\delta_{x,y}$ with value 1 if $x = y$ and zero otherwise. We use $C(X)$ for the continuous complex-valued (unless otherwise specified) functions on a topological space X, $C^\infty(X)$ for the smooth (differentiable arbitrarily many times) continuous and typically real-valued functions on a smooth manifold X, $\Bbbk[X]$ for the coordinate algebra of an algebraic variety X and $\Bbbk_q[X]$ its q-deformation. For $\mathbb{C}_q[S^1]$, the q refers to the differential calculus as the algebra itself in this case is not deformed. (We also write $\mathbb{C}_q\mathbb{Z}$ for the same algebra but in the different context of a group Hopf algebra with coquasitriangular structure involving q.) We similarly write $\mathbb{C}_\theta[\mathbb{T}^2]$ for the algebraic noncommutative torus rather than the more common notation $\mathbb{T}^2_\theta$. The polynomial algebra in n variables is $\Bbbk[x_1, \cdots, x_n]$ and the free associative algebra in n variables is $\Bbbk\langle x_1, \cdots, x_n\rangle$. Angular brackets also denote an ideal generated in an algebra and in other contexts a duality pairing or a hermitian inner product.

We use $[x, y] = xy - yx$ for the commutator of two elements in an algebra, $\{x, y\} = xy + yx$ for the anticommutator and $[x, y\} = xy - (-1)^{|x|\,|y|}yx$ for the graded commutator when this applies. Where there is no confusion we also use [,] for Lie brackets and quantum and braided Lie brackets. In other contexts, { , } could denote a Poisson bracket or indicate a list. We use distinct notations

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (n)_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for two sorts of q-integer for any determinate or indeterminate q.

We most often use Ω^n for the n-forms on an algebra $A = \Omega^0$, with the entire collection of forms simply denoted by Ω and forming a differential graded algebra (or DGA). When we have different algebras of interest, we may write Ω_A, Ω_B etc., to avoid confusion. A roman d denotes the differential. Clearly, each Ω^n is a bimodule over A as we can multiply by degree 0 from either side. In general, for modules over an algebra A, we use ${}_A\mathrm{Hom}(E, F)$ for left A-linear maps from the left module E to the left module F. Correspondingly, $\mathrm{Hom}_A(E, F)$ is the right A-linear maps from the right module E to the right module F, and ${}_A\mathrm{Hom}_A(E, F)$ is the A-bilinear maps between bimodules. We write 'fgp' for 'finitely generated projective' as a module over an algebra. We use id for the identity map.

As noncommutative geometry often relates to complex-valued objects, the $*$ operation plays a prominent role and follows the usual rules for a complex-linear involution when extended to other contexts. It should not be confused with the Hodge operation, for which we reserve $\circledast$. Sometimes we will want to be extremely clear and use a more formal language (of bar categories). Thus, the conjugate $\overline{E}$ of a bimodule E over a $*$-algebra A has elements $\overline{e} \in \overline{E}$ denoting the same element as $e \in E$ but viewed in the conjugate module. The bimodule actions on the conjugate are $a.\overline{e} = \overline{e.a^*}$ and $\overline{e}.a = \overline{a^*.e}$ for $a \in A$. For a second bimodule F, there is a map $\Upsilon : \overline{E \otimes_A F} \to \overline{F} \otimes_A \overline{E}$ given by $\Upsilon(\overline{e \otimes f}) = \overline{f} \otimes \overline{e}$. The notation $e \mapsto e^*$ is used for the usual conjugate linear $*$ map, but we also use the more categorical notation $\star : E \to \overline{E}$ as the linear map (typically also a bimodule map) defined by $\star(e) = \overline{e^*}$. Round brackets (,) are often used for inner products linear in both factors in contrast to angular brackets $\langle\ ,\ \rangle$ for hermitian inner products. The latter could be conjugate linear on the right, or in other cases conjugate linear on the left. Often an explicit conjugate is used, for example $\langle u, \overline{v}\rangle$.

Elsewhere in algebra, the map 'flip' swaps tensor factors. A counit or 'augmentation' on an algebra means a character $A \to \Bbbk$ and when this is fixed, A^+ denotes its kernel or augmentation ideal. Left and right coactions are denoted by

$$\Delta_L v = v_{(\bar{1})} \otimes v_{(\bar{\infty})}, \quad \Delta_R v = v_{(\bar{0})} \otimes v_{(\bar{1})}.$$

Categories of modules over an algebra are denoted by ${}_A\mathcal{M}$ for left modules, $\mathcal{M}_A$ for right modules and ${}_A\mathcal{M}_A$ for bimodules. When H is a Hopf algebra, or at least a coalgebra, we use ${}^H\mathcal{M}$, $\mathcal{M}^H$, ${}^H\mathcal{M}^H$ to denote left, right and bicomodules. We also have crossed versions where the different structures do not commute, for example

$\mathcal{M}_H^H$ denotes right crossed H-modules where H both acts and coacts and there is a compatibility between these which is a linearised 'Hopf' version of the crossed G-set condition of J.C. Whitehead in algebraic topology (these objects are also called right Radford–Drinfeld–Yetter modules in the literature).

We will typically use G for a group, $\mathfrak{g}$ for a Lie algebra, not to be confused with g for a metric. We typically use ∇ for left-covariant derivatives or connections (we use these terms interchangeably) and $\tilde{\nabla}$ for right-covariant derivatives. These are frequently given subscripts such as ∇_E to indicate which module they act on. In addition, D is often used for a connection arising from the framing construction and $\heartsuit$ for a background connection on Ω^1 or on vector fields $\mathfrak{X}$ in the theory of differential operators. We use ω for principal or 'spin' connection forms on a quantum principal bundle and ϖ for the Maurer–Cartan form on a group or quantum group (which can be seen as a flat connection form). $Z(A)$ is the centre of an algebra A and $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ (as introduced independently by Drinfeld and by one of the authors as a dual $\mathcal{C}^\circ$).

We have adopted a particular compromise for the notation of indices on differential forms. To be consistent with the geometry literature, local coordinates $\{x^\mu\}$ have an upper index hence so do differential forms $\mathrm{d}x^\mu$. In Chaps. 8 and 9 an 'n-bein' basis of differential forms $\{e^i\}$, when it exists, then gets an upper index for consistency. We also adopt a geometric normalisation of the exterior derivative d so as to have a classical limit as a deformation parameter $q \to 1$ or $\lambda \to 0$. However, in the less specialised earlier chapters we do *not* adopt such strict conventions and typically keep a basis of 1-forms as $\{e_i\}$ and keep the canonical normalisations for d intrinsic to the relevant noncommutative constructions, notably in Chap. 2.

In Table 1, we give the occurrences of the most common examples used to illustrate various different aspects of the theory. The references are to example or proposition number, or by chapter and section number as indicated by §.

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Mumbles, Swansea — Edwin J. Beggs
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Table 1 Some examples by statement number, chapter-section number § or exercise number E

Algebra	Ω^1, d	Metric	Laplacian	Ω max	H_{dR}	Ω can	Integral	Projection	Connection	Bimod conn.	$*$ conn.	Line module	Regularity	Coaction	Fibration	Principal bundle	Vector fields	Complex structure	Levi-Civita	Spectral triple
$C^\infty(M)$			1.23	1.35											4.60		6.7			
$M_2(\mathbb{C})$	1.8	1.20	1.20	1.37	1.38				4.89	4.89			4.89				§6.5.3	7.6	8.13	8.46
$\mathbb{C}_\lambda[\mathbb{R}]$	1.10			1.34	1.34															
$\mathbb{C}_q[S^1]$	1.11	8.5		1.34	1.34			4.22				4.22	4.22				§6.5.2		8.5	
$\Bbbk_\Theta[V]$	1.14	1.21																		
$\Bbbk(X)$	§1.4	1.28	1.28	1.40												5.44	6.3			
$\mathbb{C}_\theta[\mathbb{T}^2]$	1.36			1.36	1.36			3.17	3.30								6.13	7.11	8.16	
$\mathbb{C}_{q,\theta}[\mathbb{T}^2]$	E1.5						E3.6							E4.3		E5.10				
$U(\mathfrak{g})$	§1.6.1	1.43																		
$U(su_2)$	1.45	1.45	1.45																8.15	8.50
$u_q(b_+)$	E2.4				E4.5	E4.5	E2.3		E4.9											
$\mathbb{C}_\lambda[S^2]$	1.46							1.46												
$\mathbb{C}S_3$	1.48	1.48	1.50		1.50				4.48					5.43		5.43	E6.2			8.49
$\Bbbk(G)$	§1.7	1.59	1.59	1.53		2.29	2.20			3.75	3.87		4.21			5.49			8.17	E8.10
$\Bbbk(S_3)$	1.60	1.60	1.60		1.60	1.60				3.76	3.88		4.18		5.64		6.30			
$U_q(sl_2)$	2.11																			
$\mathbb{C}_q[\mathbb{C}^2]$	2.79					2.79								4.33				§7.4.2		
$\mathbb{C}_q^{3D}[SU_2]$	2.32				4.68		2.21			3.77	3.89				5.63	5.51	6.4			
$\mathbb{C}_q^{4D}[SU_2]$	2.59	2.60	2.62		2.77	2.77													5.85	8.51
$\mathbb{C}_q[S^2]$	2.35	2.36			4.34		4.36	3.15	3.27	4.24		3.99			5.63		6.14	7.12	5.80	8.47
$\mathbb{C}_q[D]$	3.40			3.40	4.37		4.37	3.100				3.100		4.31		5.24	E6.3	8.57	E8.7	8.48
$\mathbb{C}(\mathbb{Z}_N)$	3.86									3.86	3.86					5.49		E7.1		
$\mathbb{C}PSL_2(\mathbb{Z})$	4.19					4.19							4.19							
$\mathbb{C}_q[SO_3]$	4.23								4.23	4.23		4.23	4.23							
$\mathbb{C}$Hg	4.62			4.62	4.67										E5.5	5.12				
$\mathbb{C}$[Klein]	5.45				5.45							5.45		5.11	5.45	5.11				

Contents

Chapter 1
Differentials on an Algebra

In differential geometry one equips a topological space with the structure of a differentiable manifold M. This means that locally we have coordinates $x^1, \ldots, x^n$ identifying an open set with a region of $\mathbb{R}^n$ (for some fixed n which is the dimension of the manifold), and that we can apply the usual methods of the calculus of several variables. Further, these local coordinates fit together so that we can talk of differentiable constructions globally over the whole manifold.

Locally, on each coordinate patch, we have vector fields $\sum_i v^i(x)\frac{\partial}{\partial x^i}$, which give a vector at every point of M. Together these vectors span the tangent bundle TM to M. The cotangent bundle T^*M is dual to this and the space of '1-forms' $\Omega^1(M)$ is spanned by elements of the form $\sum_i \omega_i(x)\mathrm{d}x^i$ in each local patch. Here the $\mathrm{d}x^i$ are a dual basis to $\frac{\partial}{\partial x^i}$ at each point. One also has an abstract map d which turns a function f into a differential 1-form

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i}\mathrm{d}x^i .$$

We denote by $C^\infty(M)$ the smooth (i.e differentiable an arbitrary number of times) real-valued functions on M. This is an algebra, so we can add and multiply such functions. In this book the role of functions on a manifold is going to be played by a 'coordinate algebra' A, except that there need not be an actual manifold or even an actual space in the picture. For example, the algebra could be noncommutative. One can still develop a theory of differential geometry over algebras in this case, and in this chapter we look its first layer, which is the differentiable structure. In most approaches to noncommutative geometry this amounts to defining a suitable space of 1-forms Ω^1 by its desired properties as an implicit definition of a 'noncommutative differentiable structure', as there are no actual open sets or local coordinates. This leads to a much cleaner development of differential geometry as

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a branch of algebra. We will look at the construction and classification of such 1-forms on a variety of algebras and also at the construction of n-forms in general as a differential graded algebra $(\Omega, \mathrm{d}, \wedge)$.

We also do not assume that differentials commute with 'functions', so this a more general conception of differential calculus of interest even when the algebra A is commutative, as we see in §1.2 for polynomial algebras in one variable and in §1.4 for functions on discrete sets. Indeed, if an algebra has a nilpotent 'fat point' element obeying $x^2 = 0$ then one needs $(\mathrm{d}x)x = -x\mathrm{d}x$ to be consistent with the Leibniz rule. Similarly, for a discrete set it is known that the only usual commutative differential calculus is the zero one, but as soon as we allow noncommutativity of differential forms then it turns out that a first-order calculus here is the same thing as a directed graph with the discrete set as vertices. Arrows $x \to y$ provide 1-forms and finite differences $f(y) - f(x)$ provide associated partial derivatives.

1.1 First-Order Differentials

The reader will likely be familiar with the idea that the smooth real-valued functions $C^\infty(M)$ on a manifold M, or the 2×2 complex matrices with complex entries $M_2(\mathbb{C})$, are examples of algebras. A formal definition on an algebra A over a field $\Bbbk$, which shall usually be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, but could in principle be, for example, a finite field, is a vector space over $\Bbbk$ equipped with an associative product which is bilinear, and so satisfies the distributive rules

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc$$

for all $a, b, c \in A$. We will assume that our algebras are unital, i.e., have a multiplicative identity or unit 1, unless otherwise stated.

A module E for an algebra A is a vector space over the same field $\Bbbk$ which has a $\Bbbk$-linear action of the algebra. The algebra can act on the left, and an example of this is the action of $M_2(\mathbb{C})$ on two-dimensional column vectors by matrix multiplication with the square matrix on the left. Similarly, the set of two-dimensional row vectors has a right action of $M_2(\mathbb{C})$ by matrix multiplication. The identity element in the algebra (in this case the 2×2 identity matrix) has the trivial action. The vital part of the definition is that the action must be compatible with the algebra product,

$$a.(b.e) = (ab).e \quad \text{(left action)}, \quad (e.a).b = e.(ab) \quad \text{(right action)}$$

for all $a, b \in A$ and $e \in E$. For our matrix example, these are just associativity of matrix multiplication. A right module means there is a right action of the algebra, and a left module a left action of the algebra. Thus we may say that two-dimensional row vectors form a right module for $M_2(\mathbb{C})$ with action just the matrix product. A bimodule has both left and right module actions such that $a.(e.b) = (a.e).b$ for

$a, b \in A$ and e in the bimodule. Any algebra is a bimodule over itself, for example $M_2(\mathbb{C})$ with the actions of matrix multiplication from the left and from the right.

Also we recall that the tensor product over a field is a way of taking products of vector spaces in such a way that it multiplies the dimension. Thus V with basis $v_1, \ldots, v_n$ and W with basis $w_1, \ldots, w_m$ have tensor product $V \otimes W$ with basis $v_i \otimes w_j$ for $1 \le i \le n$ and $1 \le j \le m$. An example is the tensor product of the space of column 2-vectors with the space of row 2-vectors to give $M_2(\mathbb{C})$ as their tensor product vector space. Tensor product is a bilinear operation and also makes sense for infinite-dimensional vector spaces, where the key defining property is the identity $v \otimes \lambda w = v\lambda \otimes w$ for all $\lambda \in \Bbbk$, $v \in V$, $w \in W$.

We will frequently need this same concept for the tensor product $E \otimes_A F$ of a right module E with a left module F over an algebra A with the key defining property $e.a \otimes f = e \otimes a.f$ for all $a \in A$, $e \in E$, $f \in F$. If E, F are bimodules then the result is again a bimodule and, moreover, the construction $\otimes_A$ is associative. We will come to this more formally in Chap. 3 as the analogue of 'pointwise tensor product' for bundles, but for now we note that if $A = C^\infty(M)$, matter fields or sections of bundles over M are examples of A-modules and the tensor product over A means their pointwise product, which is something we are very familiar with in 'tensor calculus'. In this chapter we are going to focus on the algebraic formulation of sections of the cotangent bundle and its tensor products.

Definition 1.1 A first-order 'differential calculus' (Ω^1, d) over A means

(1) Ω^1 an A-bimodule.
(2) A linear map $\mathrm{d} : A \to \Omega^1$ (the *exterior derivative*) with $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for all $a, b \in A$.
(3) $\Omega^1 = \mathrm{span}\{a\mathrm{d}b \mid a, b \in A\}$ (the *surjectivity condition*).
(4) (Optional *connectedness condition*) $\ker \mathrm{d} = \Bbbk.1$.

This is more or less the minimum that one could require for an abstract notion of 'differentials'—one should be able to multiply them from the left and from the right by 'functions' (elements of A), which is then enough to have a well-defined Leibniz rule as shown. In usual differential geometry one would assume that the left and right modules coincide, i.e., that $a.\mathrm{d}b = \mathrm{d}b.a$ for all $a, b \in A$, but this is not reasonable to impose when our algebras are noncommutative. If we did, we would deduce that $\mathrm{d}(ab - ba) = 0$ so that d would have an unusually large kernel in the noncommutative case. Thus the definition in noncommutative geometry is more general even for commutative algebras or functions on classical spaces in that we might still have different left and right actions. Note also that $\mathrm{d}1 = 0$ by the Leibniz rule applied to $1.1 = 1$. If we have occasion to drop the surjectivity condition (3) then we will say that we have a *generalised* first-order differential calculus.

An algebra map $\phi : A \to B$, where the algebras A, B are both equipped with first-order calculi, is called *differentiable* if there is a bimodule map $\phi_* : \Omega^1_A \to \Omega^1_B$ (intertwining the A-bimodule structure on Ω^1_A with the bimodule structure on Ω^1_B

pulled back via ϕ to an A-bimodule structure) such that we have a commuting square

$$\begin{tikzcd} \Omega^1_A \arrow[r, "\phi_*"] & \Omega^1_B \\ A \arrow[u, "\mathrm{d}"] \arrow[r, "\phi"] & B \arrow[u, "\mathrm{d}"] \end{tikzcd} \tag{1.1}$$

From the surjectivity assumption on differential calculi, this is the same as saying that $\phi_*(x\mathrm{d}y) := \phi(x)\mathrm{d}\phi(y)$ gives a well-defined map from Ω^1_A to Ω^1_B.

Definition 1.2 If Ω^1 is a free left (right) A-module with basis over A of cardinality n then we say that it is left (right) parallelised with *cotangent dimension n*.

Here a free left module means that there are $\omega_1, \ldots, \omega_n \in \Omega^1$ such that every element of Ω^1 can be uniquely written as $a_1.\omega_1 + \cdots + a_n.\omega_n$ for $a_i \in A$. Similarly for a free right module. This corresponds in classical differential geometry to a manifold being parallelisable (so that there is a global basis of 1-forms). The 2-dimensional torus $\mathbb{T}^2$ is parallelisable, but the 2-dimensional sphere S^2 is not. In the parallelised case the choice of basis $\{\omega_i\}$ defines *partial derivatives* $\partial_i : A \to A$ by

$$\mathrm{d}a = \sum_i (\partial_i a)\omega_i \tag{1.2}$$

for all $a \in A$. Note that these are not derivations unless our basic 1-forms commute with functions. Instead, if $\omega_i a = \sum_j C_{ij}(a)\omega_j$ for some maps $C_{ij} : A \to A$ then from the module property and the derivation property of d one easily concludes that

$$\sum_k C_{ik}(a)C_{kj}(b) = C_{ij}(ab), \quad \partial_j(ab) = \sum_i \partial_i(a)C_{ij}(b) + a\partial_j(b)$$

for all $a, b \in A$. The latter is just as well because many noncommutative algebras do not admit enough actual derivations.

We next consider a 'purely quantum' phenomenon which is not possible in classical differential geometry as it needs the left and right bimodule structures on Ω^1 to be different, i.e., 1-forms and elements of the algebra to noncommute.

Definition 1.3 A differential algebra $(A, \Omega^1, \mathrm{d})$ is said to be *inner* if there exists a $\theta \in \Omega^1$ such that for all $a \in A$,

$$\mathrm{d}a = [\theta, a].$$

Here $[\theta, a]$ is the commutator $\theta a - a\theta$. We will see later that d is a derivation on the entire exterior algebra and hence if this is sufficiently non(super)commutative then one can expect that it is inner. So this typically happens for sufficiently noncommutative calculi even if the algebra A itself is commutative.

Another ingredient of ordinary differential geometry that we need to formulate is the use of real variables. In the noncommutative case this is handled by letting A be a $*$-algebra, i.e., an algebra over $\mathbb{C}$ equipped with an antilinear map $* : A \to A$ which reverses products and which squares to the identity. In the commutative case there is a subalgebra of elements such that $a = a^*$ which plays the role of 'real-valued' functions but in the general noncommutative case this is not a subalgebra and we just work with the pair $(A, *)$ in lieu of any actual real values. When represented on a Hilbert space, the $*$ operation tells us which operators become hermitian or unitary etc., and hence plays a similar role. Furthermore, when completed to a C^*-algebra and in the commutative case, a theorem of Gel'fand and Naimark allows us to recover an actual topological space. We will say more about C^*-algebras in Chap. 3, but for the moment we need only the $*$-structure. In the same way, we want to express that our differential forms have a comparable 'reality' property as follows.

Definition 1.4 A $*$-differential calculus over a $*$-algebra means a differential calculus as above and an antilinear map $* : \Omega^1 \to \Omega^1$ that commutes with d and respects the bimodule structures in the sense that $(a.\omega)^* = \omega^*.a^*$ for all $a \in A, \omega \in \Omega^1$.

A typical application of these rules in a calculation would be $(a.\mathrm{d}b)^* = (\mathrm{d}b^*).a^*$. In the inner case this means that $\theta^* = -\theta$ up to something in Ω^1 that is central.

Proposition 1.5 (Universal Calculus)

(1) *Every algebra A has a connected differential calculus Ω^1_{uni} given by*

$$\Omega^1_{\mathrm{uni}} = \ker(\cdot) \subseteq A \otimes A, \quad \mathrm{d}_{\mathrm{uni}}a = 1 \otimes a - a \otimes 1$$

for all $a \in A$.

(2) *Any other differential calculus is isomorphic to $\Omega^1_{\mathrm{uni}}/\mathcal{N}$, for some sub-bimodule $\mathcal{N} \subseteq \Omega^1_{\mathrm{uni}}$.*

(3) *If A is finite-dimensional then Ω^1_{uni} is left and right-parallelisable with cotangent dimension* $\dim(A) - 1$.

(4) *Ω^1_{uni} is inner if and only if there exists a central $F \in A \otimes A$ with $\cdot(F) = 1$.*

(5) *In the $*$-algebra case over $\mathbb{C}$, Ω^1_{uni} is a $*$-calculus by $(a \otimes b)^* = -(b^* \otimes a^*)$.*

Proof (1) It is elementary to check that Ω^1_{uni} is indeed a differential calculus. The surjectivity axiom holds since if $\sum_i a_i \otimes b_i \in \Omega^1_{\mathrm{uni}}$, where $\sum_i a_i b_i = 0$, then $\sum_i a_i \mathrm{d}_{\mathrm{uni}} b_i = \sum_i a_i \otimes b_i$ from the definition of $\mathrm{d}_{\mathrm{uni}}$. To see that it obeys the further connectedness axiom, suppose that A^+ is any chosen complement to $\Bbbk 1$ such that $A = \Bbbk 1 \oplus A^+$. If $a = \lambda 1 + b \in \Bbbk 1 \oplus A^+$ then $\mathrm{d}_{\mathrm{uni}}a = 1 \otimes b - b \otimes 1$. Hence if $\mathrm{d}a = 0$ we see that $1 \otimes b = b \otimes 1$. By projecting the first factor onto A^+ we conclude that $b = 0$, as required. (2) The universal property follows from the surjectivity axiom: If (Ω^1, d) is another calculus then define $\pi : \Omega^1_{\mathrm{uni}} \to \Omega^1$ by $a \otimes b \to a\mathrm{d}b$, which is surjective. Indeed, an element of Ω^1 can be written as a sum of terms of the form

$a\mathrm{d}b$ which is mapped onto from $a\mathrm{d}_{\mathrm{uni}}b = a \otimes b - ab \otimes 1 \in \Omega^1_{\mathrm{uni}}$. By construction,

$$\begin{tikzcd} \Omega^1_{\mathrm{uni}} \arrow[rr, "\pi"] & & \Omega^1 \\ & A \arrow[ul, "\mathrm{d}_{\mathrm{uni}}"] \arrow[ur, "\mathrm{d}"'] & \end{tikzcd}$$

(3) If A is finite-dimensional we let $\{e_i\}$ be a basis for A^+. Then $\omega_i = \mathrm{d}_{\mathrm{uni}}e_i$ form a global basis. These span in view of the surjectivity axiom and $\mathrm{d}_{\mathrm{uni}}1 = 0$. If $\sum a_i\omega_i = 0$, this means $\sum_i (a_i \otimes e_i - a_ie_i \otimes 1) = 0$ due to the form of $\mathrm{d}_{\mathrm{uni}}$, which projecting to A^+ in the second factor gives $\sum_i a_i \otimes e_i = 0$ and hence $a_i = 0$. The argument on the other side is analogous. (4) If the condition holds then $\theta = 1 \otimes 1 - F$ lives in Ω^1_{uni} and does the job as $[F, a] = 0$ while $1 \otimes 1.a - a.1 \otimes 1 = \mathrm{d}a$. The converse clearly holds too. (5) if A is a $*$-algebra then with the stated $*$ operation, $(a.(b \otimes c))^* = (ab \otimes c)^* = -c^* \otimes b^*a^* = (b \otimes c)^*.a^*$ and $(\mathrm{d}a)^* = (1 \otimes a - a \otimes 1)^* = -(a^* \otimes 1 - 1 \otimes a^*) = \mathrm{d}a^*$ for all $a, b, c \in A$, as required. □

Universal here means with respect to the obvious notion of morphisms between calculi on a fixed algebra, namely bimodule maps π that form a commutative triangle with the exterior derivatives as in the proof of the second item (this is equivalent to saying that the identity map is differentiable from one calculus to the other). The universal property can then be stated more formally as the assertion that for every calculus Ω^1 on A there is a unique surjective morphism π such that the diagram shown commutes. The third item means that the universal calculus is generally much too big to be similar to the classical calculus on a finite-dimensional manifold, where the algebra of functions is infinite-dimensional but the cotangent dimension is finite. At the other extreme, a calculus is called *irreducible* if it has no proper quotients.

Example 1.6 $A = \Bbbk[x]/\langle x^2 \rangle$, i.e. a single Grassmann variable with the relation $x^2 = 0$. Here $A \otimes A$ has vector space basis $1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x$ and Ω^1_{uni} has vector space basis $1 \otimes x - x \otimes 1 = \mathrm{d}_{\mathrm{uni}}x$ and $x \otimes x = x\mathrm{d}_{\mathrm{uni}}x$. The latter is a free module with basis $\mathrm{d}_{\mathrm{uni}}x$ over A, i.e. has cotangent dimension 1. The right module structure is $(\mathrm{d}_{\mathrm{uni}}x)x = (1 \otimes x - x \otimes 1)x = -x \otimes x = -x\mathrm{d}_{\mathrm{uni}}x$, so the basis element is not central unless the field has characteristic 2. One can check that the calculus is not inner. ◇

In practice we may not want to go through the universal calculus, preferring to construct examples based on specific requirements. The simplest special case is the following (which contrasts with the preceding example).

Example 1.7 (Derivation Calculus) Let A be an algebra and fix the cotangent dimension 1 by $\Omega^1 = A$ as a bimodule. This means that it is a free module with a single central basis element. Differential structures of this type are in 1-1 correspondence with derivations $\tau : A \to A$ that are surjective in the sense that the map $A \otimes A \to A$ with $a \otimes b \mapsto a\tau(b)$ is surjective, namely with $\mathrm{d} = \tau$ viewed as a map $\mathrm{d} : A \to \Omega^1$. The calculus is connected if $\ker \tau = \Bbbk.1$ and inner if the derivation is inner, i.e., $\tau(a) = [\theta, a]$ for some $\theta \in A$, but the inner case can never be connected since θ regarded in A is in the kernel. In the $*$-algebra case we have a $*$-calculus if τ commutes with $*$, which in the inner case means $\theta^* + \theta \in Z(A)$, the centre of A. ◇

The proofs here are left to the reader. There are also variants of these where the action on one or both sides is twisted by an algebra automorphism and τ is a twisted derivation. Here is a concrete noncommutative example of our various considerations where pretty much everything can be computed. We let E_{ij} be the matrix with 1 at position (i, j) and zero elsewhere.

Example 1.8 Let $A = M_2(\mathbb{C})$ be the algebra of 2×2 matrices. All its calculi Ω^1 are finite-dimensional, inner and parallelisable. Up to isomorphism there are:

(i) A unique calculus of cotangent dimension 3, with $\Omega^1 = M_2 \oplus M_2 \oplus M_2$ as a bimodule and $\theta = \frac{1}{2}(E_{11} - E_{22}) \oplus E_{12} \oplus E_{21}$;
(ii) A $\mathbb{CP}^2$ moduli of calculi of cotangent dimension 2, with $\Omega^1 = M_2 \oplus M_2$ as a bimodule and $\theta = (\lambda_0 E_{12} - \frac{\lambda_+}{2}(E_{11} - E_{22})) \oplus (\lambda_0 E_{21} - \frac{\lambda_-}{2}(E_{11} - E_{22}))$ in the case $\lambda_0 \neq 0$ and $\theta = \frac{1}{2}(E_{11} - E_{22}) \oplus (\lambda_+ E_{21} - \lambda_- E_{12})$ for $\lambda_0 = 0$, where an overall scale of $(\lambda_0, \lambda_+, \lambda_-) \neq 0$ does not change the calculus up to isomorphism;
(iii) A $\mathbb{CP}^2$ moduli of calculi of cotangent dimension 1, with $\Omega^1 = M_2$ as a bimodule and $\theta = \lambda_0 \frac{1}{2}(E_{11} - E_{22}) + \lambda_+ E_{12} + \lambda_- E_{21}$, where $(\lambda_0, \lambda_+, \lambda_-) \neq 0$ as in (ii).

Proof We start with the universal calculus $\Omega_{\mathrm{uni}} A \subset M_2 \otimes M_2$, which by definition has basis elements of the form $e^+_{ij} = E_{i1} \otimes E_{2j}$, $e^-_{ij} = E_{i2} \otimes E_{1j}$ and $e^0_{ij} = E_{i1} \otimes E_{1j} - E_{i2} \otimes E_{2j}$. The universal calculus is 12-dimensional as a vector space due to the four relations that say that its elements are in the kernel of the matrix product map. Now consider the elements $\phi^\alpha = e^\alpha_{11} + e^\alpha_{22}$ where $\alpha = 0, +, -$. It follows from the $E_{ij} E_{kl} = E_{il}\delta_{jk}$ relations that these 1-forms are central and that $E_{ij}.\phi^\alpha = e^\alpha_{ij}$. Hence by dimensions, $\Omega^1_{\mathrm{uni}} A \cong M_2 \oplus M_2 \oplus M_2$ as a bimodule via $\{\phi^\alpha\}$ as a 3D basis over $M_2(\mathbb{C})$, with the standard bimodule structures on each copy since the ϕ^α are central. The e^α_{ij} are now identified with E_{ij}, viewed in the relevant copy of $M_2(\mathbb{C})$ of the direct sum as $\alpha = 0, +, -$. The calculus is inner with $\theta = \frac{1}{2}(e^0_{11} - e^0_{22}) + e^+_{12} + e^-_{21}$ since

$$[\theta, E_{11}] = [\frac{1}{2}(e^0_{11} - e^0_{22}) + e^+_{12} + e^-_{21}, E_{11}] = e^-_{21} - e^+_{12}$$

$$= E_{22} \otimes E_{11} - E_{11} \otimes E_{22} = \mathrm{d}E_{11},$$

$$[\theta, E_{12}] = [\frac{1}{2}(e^0_{11} - e^0_{22}) + e^+_{12} + e^-_{21}, E_{12}] = e^0_{12} + e^-_{22} - e^-_{11}$$

$$= E_{11} \otimes E_{12} - E_{12} \otimes E_{22} + E_{22} \otimes E_{12} - E_{12} \otimes E_{11} = \mathrm{d}E_{12},$$

and similarly for the other cases, noting that $1 = E_{11} + E_{22}$. Note also that $\theta + \frac{1}{2}\phi^0$ is an equally good inner generator to the one stated and

$$\theta + \frac{1}{2}\phi^0 = 1 \otimes 1 - E_{12} \otimes E_{21} - E_{22} \otimes E_{22}$$

in terms of the original description of $\Omega^1_{\rm uni}$, which easily generalises to prove that the universal calculus on all M_n (and over any field) is inner. We also have

$$\phi^{0*} = -\phi^0, \quad \phi^{\pm *} = -\phi^{\mp}$$

for the $*$-structure of $\Omega^1_{\rm uni}$, so that $(a \oplus b \oplus c)^* = (-a^*) \oplus (-c^*) \oplus (-b^*)$ for $a, b, c \in M_2$ for the $*$-structure of $M_2 \oplus M_2 \oplus M_2$, which means that $\theta^* = -\theta$.

Next, by the universal property, other calculi Ω^1 will be given by quotients of $\Omega^1_{\rm uni} \cong M_2 \oplus M_2 \oplus M_2$ by a nontrivial sub-bimodule N. We define $\pi^\alpha : N \to M_2$, for $\alpha \in \{0, +, -\}$, to be the projection to the M_2 summand generated by ϕ^α. For case (ii) of a 2D calculus we need that $\pi^\alpha : N \to M_2$ is a 1-1 correspondence for some α. In that case Schur's lemma tells us that the bimodule map $\pi^\beta(\pi^\alpha)^{-1} : M_2 \to M_2$ is a scalar multiple of the identity. This gives $N = \{(\lambda_0 a, \lambda_+ a, \lambda_- a) : a \in M_2\}$ for some $[(\lambda_0, \lambda_+, \lambda_-)] \in \mathbb{CP}^2$. For $\lambda_0 \neq 0$, case (iia), we have on quotient by N that $\Omega^1 \cong M_2 \oplus M_2$ by $[c \oplus d \oplus e] \mapsto (\lambda_0 d - \lambda_+ c) \oplus (\lambda_0 e - \lambda_- c)$ with

$$\mathrm{d}a = (\lambda_0[E_{12}, a] - \frac{\lambda_+}{2}[E_{11} - E_{22}, a]) \oplus (\lambda_0[E_{21}, a] - \frac{\lambda_-}{2}[E_{11} - E_{22}, a]).$$

For $\lambda_0 = 0$, case (iib), we have the quotient by N isomorphic to $M_2 \oplus M_2$ by $[c \oplus d \oplus e] \mapsto c \oplus (\lambda_+ e - \lambda_- d)$. This gives

$$\Omega^1 \cong M_2 \oplus M_2, \quad \mathrm{d}a = \frac{1}{2}[E_{11} - E_{22}, a] \oplus [\lambda_+ E_{21} - \lambda_- E_{12}, a].$$

For case (iii) of a 1-dimensional calculus we take α so that $\pi^\alpha : N \to M_2$ is surjective but not injective. Then there is a $\beta \neq \alpha$ such that $\pi^\beta : N \cap \ker \pi^\alpha \to M_2$ is a 1-1 correspondence. For the case $\alpha = 0$, on using Schur's lemma as previously we get $\{(0, \lambda a, \mu a) : a \in M_2\} = N \cap \ker \pi^\alpha$ for some λ, μ not both vanishing. For the subsequent case $\beta = +$ we get a complementary sub-bimodule $\{(\kappa a, 0, \epsilon a) : a \in M_2\} \subset N$ for some κ, ϵ not both vanishing. In any case we get unique $(\lambda_0, \lambda_+, \lambda_-)$ up to scale so that the quotient by N is isomorphic to M_2 by $[c \oplus d \oplus e] \mapsto (\lambda_0 c + \lambda_+ d + \lambda_- e)$. □

This classification is up to isomorphism of the calculi, where morphism means in the sense discussed after Proposition 1.5 of a bimodule map forming a commuting triangle with d. If the calculus is inner then it means the map connects the corresponding θ up to a central element. In the simplest of $\Omega^1 = M_2$, a bimodule isomorphism is a multiple of the identity and the ambiguity in θ is fixed by working with traceless matrices. Moreover, one can check that every nonzero traceless matrix gives a surjective calculus, which is why the moduli space is $\mathbb{CP}^2$. For the m-

dimensional case $\Omega^1 = M_2 \oplus \cdots \oplus M_2 = M_2 \otimes \mathbb{C}^m$ a bimodule map is given by a $GL_m(\mathbb{C})$ transformation acting on the second factor but the analysis of which θ obey the surjectivity condition and which are equivalent is now more complicated. The above calculations tell us in the $m = 2$ case that we can always render a calculus in the form stated as labelled by $\mathbb{CP}^2$, while for $m = 3$ we have only the one distinct calculus, namely the universal one.

By the same arguments as in the proof, 1-dimensional calculi are $*$-differential calculi for the standard hermitian adjoint $*$-operation on $A = M_2(\mathbb{C}) = \Omega^1$ precisely when θ is traceless and antihermitian as a matrix, now modulo a real scale, which is an $\mathbb{RP}^2$ moduli space. These calculi are not connected but still interesting, see Example 1.20. The 3-dimensional case is also a $*$-calculus since it is the universal calculus. The 2-dimensional calculi also include $*$-calculi, such as the following example. This will also appear in Example 7.6 as a quantum complex geometry and in Example 8.13 as part of a quantum Riemannian geometry.

Corollary 1.9 *$M_2(\mathbb{C})$ has a connected cotangent dimension 2 $*$-calculus $\Omega^1 = M_2 \oplus M_2$ as a bimodule specified by $\theta = E_{12} \oplus E_{21}$ for the two copies and $(a \oplus b)^* = (-b^*) \oplus (-a^*)$ for all $a, b \in M_2$.*

Proof We quotient the universal calculus in the form $M_2 \oplus M_2 \oplus M_2$ by the first direct summand bimodule, giving $\mathrm{d}a = [E_{12}, a] \oplus [E_{21}, a]$ and the stated $*$-structure on Ω^1 since $\phi^{+*} = -\phi^-$ in the proof above, so that θ is antihermitian. Only multiples of the identity commute with both E_{12} and E_{21}, so the calculus is connected. □

These arguments work similarly for general $M_n(\Bbbk)$. Its universal calculus is inner with $\theta = 1 \otimes 1 - \sum_i E_{in} \otimes E_{ni}$ and those 1-dimensional calculi $\Omega^1 = M_n$ with its standard bimodule structure are classified by nonzero traceless $\theta \in M_n$ modulo an overall scale. Over $\mathbb{C}$ this means that 1-dimensional calculi are classified by $\mathbb{CP}^{n^2-2}$.

1.2 Differentials on Polynomial Algebras

We start by looking at examples close to classical geometry, where A is the algebra of polynomials in some number of variables or a quotient of this by additional relations, in other words in the setting of affine algebraic geometry. In the case of the affine line, there is an additive structure and we are particularly interested in translation-invariant differentials. We will formalise this notion using Hopf algebras in Chap. 2 but here it just means with respect to translation on the underlying additive group.

Example 1.10 (Affine Line) For $A = \mathbb{C}[x]$ the algebra of polynomials in 1 variable x, irreducible translation-invariant Ω^1 are parametrised by $\lambda \in \mathbb{C}$ and take the form

$$\Omega^1 = \mathbb{C}[x].\mathrm{d}x, \quad \mathrm{d}x.f(x) = f(x+\lambda)\mathrm{d}x, \quad \mathrm{d}f = \frac{f(x+\lambda) - f(x)}{\lambda}\mathrm{d}x.$$

Only the Newton–Leibniz calculus at $\lambda = 0$ has $[\mathrm{d}x, f] = 0$. The calculus is a $*$-calculus with $x^* = x$ if and only if $\lambda \in \mathrm{i}\mathbb{R}$, which real form we denote by $\mathbb{C}_\lambda[\mathbb{R}]$. It is inner if and only if $\lambda \neq 0$, with $\theta = \lambda^{-1}\mathrm{d}x$, and is connected for all λ. ◇

Proof Here Ω^1 is defined as having a left-module basis $\mathrm{d}x$. The second equation then specifies the right module structure. In that case $\mathrm{d}x^n = \mathrm{d}x.x^{n-1} + x.\mathrm{d}x^{n-1} = (x+\lambda)^{n-1}\mathrm{d}x + x\mathrm{d}x^{n-1}$ gives the formula for d on monomials by induction and one can then check that it obeys the derivation rule. For a $*$-calculus we need $(\mathrm{d}x.x)^* = ((x+\lambda).\mathrm{d}x)^* = \mathrm{d}x.(x+\lambda^*) = (x+\lambda+\lambda^*).\mathrm{d}x$ to equal $x^*.\mathrm{d}x^* = x.\mathrm{d}x$ which forces λ to be imaginary, and one can easily check that this then works in general. Finally, if $\mathrm{d}f = 0$ we have $f(x+\lambda) = f(x)$, which for polynomials implies $f \in \mathbb{C}1$. The converse direction, that these are the only translation-invariant calculi that have no further quotients, will depend on results in Chap. 2. The inner case is clear from the commutation relations. □

Geometrically the space is the line $\mathbb{C}$ and the $*$-structure picks out the real line in it. The classification result depends on $\mathbb{C}$ being algebraically closed; over a general field $\Bbbk$ the classification is by monic irreducible polynomials $m(\lambda)$ in an indeterminate λ and $\Omega^1 = \mathbb{K}[x]$ as a vector space, where $\mathbb{K} \supseteq \Bbbk$ is the associated field extension given by adjoining λ with relation $m(\lambda) = 0$. This time the calculus is n-dimensional where n is the degree of the field extension.

We similarly have the algebraic circle which again can be classified using Hopf algebra methods in Chap. 2. Here the group structure is multiplicative.

Example 1.11 (Algebraic Circle) For $A = \mathbb{C}[t, t^{-1}]$ the algebra of polynomials in t, t^{-1}, irreducible translation-invariant Ω^1 are parametrised by $q \in \mathbb{C}^*$ and take the form

$$\Omega^1 = \mathbb{C}[t, t^{-1}].\mathrm{d}t, \quad \mathrm{d}t.f(t) = f(qt)\mathrm{d}t, \quad \mathrm{d}f = \frac{f(qt) - f(t)}{t(q-1)}\mathrm{d}t.$$

Only the Newton–Leibniz calculus at $q = 1$ has $[\mathrm{d}t, f] = 0$. The calculus is a $*$-calculus with $t^* = t^{-1}$ if and only if q is real, which real form we denote by $\mathbb{C}_q[S^1]$. It is a $*$-calculus with $t^* = t$ if and only if $|q| = 1$, which real form we denote by $\mathbb{C}_q[\mathbb{R}^\times]$. The calculus is inner if and only if $q \neq 1$, with $\theta = (q-1)^{-1}t^{-1}\mathrm{d}t$. The calculus is connected if and only q is not a nontrivial root of unity. ◇

Proof Here Ω^1 is defined as having a left-module basis $\mathrm{d}t$. The second equation gives the right module structure. In that case $\mathrm{d}t^n = \mathrm{d}t.t^{n-1} + t\mathrm{d}t.t^{n-2} + \cdots + t^{n-1}\mathrm{d}t = t^{n-1}(q^{n-1} + \cdots + q + 1)\mathrm{d}t = [n]_q t^{n-1}\mathrm{d}t$, which gives the stated result on a general 2-sided monomial. Here we see the natural appearance of the q-integers

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

One can then verify the Leibniz rule as required. For the first $*$-calculus we need $(\mathrm{d}t.t)^* = (qt.\mathrm{d}t)^* = q^*(\mathrm{d}t^{-1}).t^{-1} = q^*q^{-1}t^{-1}.\mathrm{d}t^{-1}$, which we want to equal

$t^*.\mathrm{d}t^* = t^{-1}.\mathrm{d}t^{-1}$. This happens if and only if q is real. The other case similarly gives $|q| = 1$ and can be thought of as the exponentiated version of Example 1.10. Finally, assume that q is not a nontrivial root of unity, which is equivalent to $[n]_q \neq 0$ for all $n \neq 0$. Then $\mathrm{d}(\sum a_n t^n) = \sum t^{n-1}[n]_q a_n \mathrm{d}t = 0$ implies that $a_n = 0$ for all $n \neq 0$, so our calculus is connected. Again, the proof in the converse direction follows from results in Chap. 2 and is deferred, and the inner case is clear from a short computation using the commutation relations. □

Geometrically, this is the complex line with the origin removed, i.e., the punctured plane $\mathbb{C}^\times$. The $*$-structure picks out the unit circle within it. We see from these two examples that finite differences and q-differentials arise naturally out of the classification of noncommutative differential structures on the line and the circle.

Also in both of these examples the moduli space of all differential calculi has a single 'classical point' where the translation-invariant calculus is unique and commutative. Clearly we can do a similar analysis for $A = \Bbbk[x_1, \cdots, x_n]$ in n-variables. We will see in Chap. 2 that translation-invariant calculi are classified by ideals $I \subseteq A^+$, the 'augmentation ideal' of polynomials with no constant term. We defer the general construction but suffice it to say that there is a canonical choice $I = (A^+)^2$ resulting in the classical calculus with basis $\{\mathrm{d}x^i\}$ commuting with functions. Other affine algebraic varieties are given as quotients of $\Bbbk[x_1, \cdots, x_n]$ much as we have seen for the algebraic 1-torus (where there were implicit relations $tt^{-1} = t^{-1}t = 1$) and in the algebraic group case the above remarks still apply, albeit the natural classical point would be the zero calculus in the case of a discrete group. For example, if we add the relation $t^n = 1$ then Example 1.11 descends to a calculus when $q^n = 1$, with $q = 1$ the zero calculus (cf. Exercises E1.5 and E2.5).

To put this in a larger context, we need a tiny bit of algebraic geometry for which, to keep things simple, we limit ourselves to working over $\mathbb{C}$ or some other algebraically closed field. A nonzero element of an algebra is called *nilpotent* if some power of it vanishes. An algebra is said to be *reduced* if it has no nilpotent elements. The following 'theorem' is basically the content of Hilbert's nullstellensatz.

Theorem 1.12 *There is a 1-1 correspondence between reduced commutative algebras over $\mathbb{C}$ with n generators and polynomial subsets of $\mathbb{C}^n$.*

Proof We recall that for any ideal I in an algebra, its *radical* $\mathrm{rad}(I)$ is the set of elements in the algebra which when raised to some power lie in I. It should be clear that this is an ideal and $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$, and also that the quotient algebra is reduced if and only if $I = \mathrm{rad}(I)$. By definition, a commutative algebra with generators $x_1, \cdots, x_n$, say, means a quotient of the polynomial algebra $\mathbb{C}[x_1, \cdots, x_n]$ by some ideal I (generated by polynomials in the x_i that are set to zero in defining the relations of the algebra). So, reduced algebras of this type are in correspondence with ideals $I \subseteq \mathbb{C}[x_1, \cdots, x_n]$ such that $I = \mathrm{rad}(I)$. Next, any ideal $J \subseteq \mathbb{C}[x_1, \cdots x_n]$ has a *zero set*

$$Z(J) = \{x \in \mathbb{C}^n \mid a(x) = 0, \text{ for all } a \in J\}.$$

A polynomial subset $X \subseteq \mathbb{C}^n$ means that $X = Z(J)$ of some ideal J (necessarily generated by a finite collection of polynomials). We define the corresponding algebra as $\mathbb{C}[X] = \mathbb{C}[x_1, \cdots, x_n]/I$, where $I = \mathrm{rad}(J)$. The ideal I depends only on $X = Z(J)$ rather than J itself in view of Hilbert's 'nullstellensatz', which says that

$$\mathrm{rad}(J) = \{a \in \mathbb{C}[x_1, \cdots, x_n] \mid a \text{ vanishes on } Z(J)\}$$

and $I = \mathrm{rad}(I)$, as required. Conversely, given an ideal $I = \mathrm{rad}(I) \subseteq \mathbb{C}[x_1, \cdots, x_n]$, we have a polynomial subset $Z(I)$. The two constructions are clearly inverse. □

Ideas of algebraic geometry serve as one of the motivations behind the algebraic constructions in this book but will not normally feature so directly. If a commutative algebra over $\mathbb{C}$ is not reduced then it cannot admit a nonzero commutative calculus but might admit a noncommutative one, cf. Example 1.6. If it is reduced, say $A = \mathbb{C}[x_1, \cdots, x_n]/I$ is an affine variety as above, then it might still not admit a nonzero commutative calculus as it could be zero-dimensional or singular in some way, but this is not immediately excluded by nilpotents. In this book we are more interested in noncommutative calculi, such as inner calculi. Here is an example of a construction, building on the notion of a derivation as in Example 1.7.

Example 1.13 Let A be an algebra with calculus (Ω^1, d) and $\tau : A \to A$ a derivation which is 'differentiable' in the derivation sense of extending to $\tau_* : \Omega^1 \to \Omega^1$ by

$$\tau_*(a\mathrm{d}b) = \tau(a)\mathrm{d}b + a\mathrm{d}\tau(b)$$

for $a, b \in A$. There is a cross product algebra $A \rtimes \mathbb{k}[y]$ with an adjoined generator x with $[y, a] = \lambda\tau(a)$ and a calculus $\Omega^1_{A\rtimes\mathbb{k}[y]}$ generated by Ω^1 and the calculus on $\mathbb{k}[y]$ as in Example 1.10, with relations

$$[\mathrm{d}y, \ ] = \lambda\mathrm{d}, \quad [y, \mathrm{d}a] = \lambda(\tau_* - \mathrm{id})\mathrm{d}a$$

for all $a \in A$. This calculus is by construction inner with $\theta = \lambda^{-1}\mathrm{d}y$.

Starting with $A = \mathbb{k}[x]$ itself the calculus $[\mathrm{d}x, x] = \lambda\mathrm{d}x$ as in Example 1.10 and $\tau = 0$ gives us $\mathbb{k}[x, y]$ with an inner calculus. $\mathrm{d}x, \mathrm{d}y$ are a left basis and we have

$$[\mathrm{d}x, x] = [\mathrm{d}x, y] = [\mathrm{d}y, x] = \lambda\mathrm{d}x, \quad [\mathrm{d}y, y] = \lambda\mathrm{d}y, \quad \theta = \lambda^{-1}\mathrm{d}y. \tag{1.3}$$

If we instead take $\tau = \frac{\partial}{\partial x}$ then this has $\tau(x) = 1$ and is still differentiable, with $\tau_*(\mathrm{d}x) = 0$, and the above construction gives the same bimodule relations (1.3) but now on the Heisenberg or canonical commutations relations algebra $[x, y] = \lambda$. Both cases can be iterated and are, moreover, examples of a more general construction.

Example 1.14 (Heisenberg Algebra) Let (V, Θ) be a vector space over $\Bbbk$ with antisymmetric bilinear form Θ with values in $\Bbbk$. We define the Heisenberg algebra $A = \Bbbk_\Theta[V]$ to be the unital algebra generated by a basis of V with the relations $[v, w] = \lambda\Theta(v, w)$ for all $v, w \in V$ (it is sufficient to take v, w from a basis). If V is equipped with a commutative associative algebra product $\circ$ then $\Bbbk_\Theta[V]$ has a differential calculus $\Omega^1 = A \otimes V$ with $\mathrm{d}v = 1 \otimes v$ and bimodule relations

$$[\mathrm{d}v, w] = \lambda\mathrm{d}(v \circ w)$$

which is inner when $\circ$ has an identity element $e \in V$. To prove this we simply check that the bimodule relations are compatible with d,

$$\mathrm{d}[v, w] = [\mathrm{d}v, w] + [v, \mathrm{d}w] = \lambda\mathrm{d}(v \circ w - w \circ v) = 0 = \lambda\mathrm{d}\Theta(v, w)$$

and with the algebra relations

$$\begin{aligned}[[\mathrm{d}v, w], z] &= \lambda^2\mathrm{d}((v \circ w) \circ z) = \lambda^2\mathrm{d}((v \circ z) \circ w)\\ &= [[\mathrm{d}v, z], w] = [[\mathrm{d}v, z], w] + [\mathrm{d}v, [w, z]]\end{aligned}$$

for all $v, w, z \in V$. If $\circ$ has an identity element e then set $\theta = \lambda^{-1}\mathrm{d}e$ so that $[\theta, v] = \lambda\mathrm{d}(\lambda^{-1}e\circ v) = \mathrm{d}v$, as required. Over $\mathbb{C}$, if we take λ imaginary, $\circ$ forming a $*$-algebra and $\Theta(v^*, w^*) = \overline{\Theta(v, w)}$, then we easily obtain a $*$-differential calculus.

To be concrete one can let V have basis $x_1, \cdots, x_n$, structure constants $x_i \circ x_j = c_{ij}{}^k x_k$ and $A = \Bbbk_\theta[x_1, \cdots, x_n]$ the free associative algebra with commutation relations $[x_i, x_j] = \lambda\Theta_{ij}$ for constants Θ_{ij}. The calculus has basis $\mathrm{d}x_i$ with relations

$$[\mathrm{d}x_i, x_j] = \lambda c_{ij}{}^k\mathrm{d}x_k.$$

If the product $\circ$ has a unit element, say $e = x_0$, then this is inner with $\theta = \lambda^{-1}x_0$. (The calculus could still be inner in the absence of such a unit element.) If we work over $\mathbb{C}$ and take λ imaginary, Θ_{ij} real and $x_i^* = x_i$ then we obtain a $*$-calculus. ◇

Our preceding 2-dimensional examples are given by $x \circ x = x \circ y = y \circ x = x$ and $y \circ y = y$ where $e = y$ is the identity and $\Theta(x, y) = 0, 1$ for the two cases. Taking $\Theta = 0$ in general gives us an (inner) differential structure on $\Bbbk[x_1, \cdots, x_n]$ for any (unital) algebra of dimension n over $\Bbbk$. In fact, see Corollary 1.42, every connected translation-invariant calculus of dimension n on $\Bbbk[x_1, \cdots, x_n]$ is of this form. This form of calculus, moreover, extends automatically to any choice of Θ (this arises because the generator commutation relations, being proportional to 1, are killed by d and commute with differentials). The more abstract presentation does not need V to be finite-dimensional. For example, one can take $V = C^\infty(M)$, the unital algebra of functions on a smooth manifold with $\circ$ the pointwise product. Then one has an inner noncommutative differential calculus on the symmetric algebra of functionals on $C^\infty(M)$ or its Heisenberg version for an antisymmetric bilinear Θ.

1.3 Quantum Metrics and Laplacians

One can already start to do a bit of geometry knowing only Ω^1 on an algebra A. Specifically in this book we will be very interested in the metric and the first ingredient for this is a bimodule inner product, i.e., a bimodule map

$$(\, , \,) : \Omega^1 \otimes_A \Omega^1 \to A.$$

Explicitly, this means a bilinear map such that

$$(\omega.a, \eta) = (\omega, a.\eta), \quad a(\omega, \eta) = (a.\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta.a)$$

for all $a \in A, \omega, \eta \in \Omega^1$. The first condition tells us that the map descends to a well-defined map on $\Omega^1 \otimes_A \Omega^1$ and the second two identities say that it is a bimodule map. These properties in the classical case of $A = C^\infty(M)$ just tell us that we have a 2-tensor like $g^{\mu\nu}(x)$: the first identity says that the functional-dependence on x can be associated equally well with either index while the second identities are essential to the role of metrics to contract consistently with other tensors, e.g. for an expression like $g^{\mu\nu}T_{\zeta\mu\nu}$ to make sense as a composition $(\mathrm{id} \otimes (\, , \,))(T)$ where $T \in \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$. So what we are asking for is the noncommutative version of tensoriality. In fact, Lemma 1.16 below shows that this can be quite restrictive if we also want invertibility, so we will also consider a more general approach where we drop the first condition, see §8.4.

In the $*$-algebra case it is normal to impose a compatibility condition

$$(\omega, \eta)^* = (\eta^*, \omega^*) \tag{1.4}$$

which in the case of a real manifold would be symmetry. Or in the complexified case, if we know that $(\, , \,)$ is symmetric, then the condition could be seen as a reality condition. Classically, we would normally also want $(\, , \,)$ to be nondegenerate or, in the nicest case, the associated tensor pointwise-invertible, and we would tend to call this inverse the metric or if we have not imposed symmetry then the 'generalised metric'. This leads us to focus on the following.

Definition 1.15 Let $(A, \Omega^1, \mathrm{d})$ be a differential algebra. A *(generalised) quantum metric* means $g \in \Omega^1 \otimes_A \Omega^1$ invertible in the sense that there exists a bimodule inner product $(\, , \,)$ such that for all $\omega \in \Omega^1$,

$$((\omega, \,) \otimes \mathrm{id})g = \omega = (\mathrm{id} \otimes (\, , \omega))g.$$

In the $*$-algebra case the condition on $(\, , \,)$ is then equivalent to $\mathrm{flip}(* \otimes *)g = g$.

Later on, once we have discussed Ω^2, we will impose 'quantum symmetry' by $\wedge(g) = 0$, in which case the $*$-condition should be seen as a reality condition.

Lemma 1.16 *A generalised quantum metric must be central, $ag = ga$ for all $a \in A$.*

Proof Using shorthand $g = g^1 \otimes g^2$, $(\omega, g^1)g^2 a = \omega a = (\omega a, g^1)g^2 = (\omega, a g^1)g^2$ for all $\omega \in \Omega^1$ and $a \in A$. Now take for ω the second factor of another g. $\square$

Another concept that makes sense when we have a bimodule inner product is the notion of a 2nd order 'Laplacian'. We assume that $\Bbbk$ does not have characteristic 2 (i.e., $2 \neq 0$) but this is only to fit with classical conventions and is not essential.

Definition 1.17 If $(A, \Omega^1, \mathrm{d})$ is a differential algebra, we say that $\Delta : A \to A$ is a *2nd order Laplacian* if there is a bimodule inner product $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ (which is then determined uniquely) such that

$$\Delta(ab) = (\Delta a)b + a\Delta b + 2(\mathrm{d}a, \mathrm{d}b)$$

for all $a, b \in A$. In the $*$-algebra case, we further require that Δ commutes with $*$.

Proposition 1.18 *If $(A, \Omega^1, \mathrm{d})$ is an inner differential algebra via an element θ and $(\ ,\ )$ is a bimodule map then ${}_\theta\Delta a = -2(\mathrm{d}a, \theta)$ and $\Delta_\theta a = 2(\theta, \mathrm{d}a)$ are both 2nd order Laplacians with respect to it. We call these the associated Laplacians. In the $*$-algebra case with $\theta^* = -\theta$, we have $*\Delta_\theta = {}_\theta\Delta *$.*

Proof We check that Δ_θ is a 2nd order Laplacian. Thus, $\Delta_\theta(ab) - (\Delta_\theta a)b - a\Delta_\theta b = 2(\theta, (\mathrm{d}a)b + a\mathrm{d}b) - 2(\theta, \mathrm{d}a)b - a2(\theta, \mathrm{d}b) = 2(\theta a, \mathrm{d}b) - 2(a\theta, \mathrm{d}a) = 2(\mathrm{d}a, \mathrm{d}b)$ for all $a, b \in A$. We also have $*\Delta_\theta a = (\theta, \mathrm{d}a)^* = (\mathrm{d}a^*, \theta^*) = {}_\theta\Delta(a^*)$. $\square$

It follows in the $*$-algebra case that the two Laplacians coincide if and only if either one commutes with $*$.

Example 1.19 Let A be an algebra with an inner 1-dimensional calculus $\Omega^1 = A$ as in Example 1.7. Then $\Omega^1 \otimes_A \Omega^1 = A$ and a bimodule map is necessarily given by a central element $c \in Z(A)$ so that $(\omega, \eta) = \omega\eta c$. The inner case means $\mathrm{d}a = [\theta, a]$ for some $\theta \in \Omega^1$ such that every element of A is a sum of terms of the form $a[\theta, b]$ for $a, b \in A$ (for the surjectivity condition to hold). The associated 2nd order Laplacians are $\Delta_\theta a = 2c\theta\theta a - 2c\theta a\theta$ and ${}_\theta\Delta a = 2ca\theta\theta - 2c\theta a\theta$. In the $*$-algebra case the inner product obeys our $*$ condition if and only if $c^* = c$ and in this case the two Laplacians coincide (or Δ_θ commutes with $*$) if and only if $\theta^2 \in Z(A)$. $\diamond$

For $A = M_2(\mathbb{C})$, the centre has only multiples of the identity so, without loss of generality, we take $c = 1$ and $\theta^2 = -1$. We also take θ antihermitian.

Example 1.20 Let $A = M_2(\mathbb{C}) = \Omega^1$ and let σ_i be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\sigma_i\sigma_j = \delta_{ij}1 + \mathrm{i}\epsilon_{ijk}\sigma_k$. Let $\theta = \mathrm{i}\vec{\theta}\cdot\sigma = \mathrm{i}(\theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3)$, where $\vec{\theta} \in \mathbb{R}^3$ is a unit vector, so $\theta^2 = -1$. The surjectivity condition holds for $\theta = \mathrm{i}\sigma_1$ since $\sigma_3[\sigma_1, \sigma_2] = 2\mathrm{i}$ and hence also for any direction $\vec{\theta}$ by rotational invariance. In fact this is the general form of a 1-dimensional $*$-calculus, as discussed after Corollary 1.8, with $\vec{\theta}$ modulo sign the element of $\mathbb{RP}^2$. Let $(\omega, \eta) = \frac{1}{4}\omega\eta$ and write $M_2 = \mathbb{C}1 \oplus M_2^+$, where M_2^+ denotes the traceless matrices parametrised by complex vectors $\vec{a}$ according to $a = \vec{a}\cdot\sigma$ and note that $\Delta_\theta : M_2^+ \to M_2^+$. Then $\Delta_\theta 1 = 0$ and

$$
\begin{aligned}
-\Delta_\theta a &= \frac{1}{2}(a + \theta a\theta) = \frac{1}{2}(a - \theta_i a_j \theta_k \sigma_i(\delta_{jk} + \mathrm{i}\epsilon_{jkm}\sigma_m)) \\
&= \frac{1}{2}(a - (\vec{a}\cdot\vec{\theta})\vec{\theta}\cdot\sigma - \theta_i a_j \theta_k \mathrm{i}\epsilon_{jkm}\mathrm{i}\epsilon_{imn}\sigma_n)) \\
&= \frac{1}{2}(a - (\vec{a}\cdot\vec{\theta})\vec{\theta}\cdot\sigma - \theta_i a_j \theta_k(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki})\sigma_n) \\
&= a - (\vec{a}\cdot\vec{\theta})\vec{\theta}\cdot\sigma.
\end{aligned}
$$

We see that $-\Delta_\theta$ induced by θ is a complex extension of the orthogonal projection onto the plane through the origin normal to $\vec{\theta}$. ◇

An example of a quantum metric with a 2-dimensional calculus on $M_2(\mathbb{C})$ as in Corollary 1.9 is deferred to Example 1.39 and studied further in Example 8.13.

Example 1.21 For affine planes $\Bbbk[x_1, \cdots, x_n]$ in Example 1.14 with $\Theta = 0$ and an inner calculus based on unital $\circ$, one can show that there are no central elements of $\Omega^1 \otimes_A \Omega^1$ hence no metrics unless $\Bbbk$ has characteristic 2. For $\Theta \neq 0$ there are likewise no metrics with polynomial coefficients, but there could be ones with nonpolynomial coefficients. We illustrate this on the 2-dimensional calculus (1.3), where $[\ , y]$ acts on basis elements $\mathrm{d}x, \mathrm{d}y$ as a multiple λ of the identity. (i) With $A = \Bbbk[x, y]$ and a general form for g with coefficients in A in the tensor product basis, we have $[g, y] = 2\lambda g$, which cannot then be zero for $\lambda \neq 0$. (ii) If we instead let A be the Heisenberg algebra $[x, y] = \lambda$ then we have an extra term in $[g, y]$ from differentiating the metric coefficients with respect to x. For example, if $g = a\mathrm{d}x \otimes \mathrm{d}x + \cdots$ then $[g, y] = 0$ requires $a'(x) = -2a(x)$, which has no polynomial solution. The general argument follows the same lines, looking at the polynomial degree of the coefficients to see that there is no algebraic solution even in the Heisenberg case. It is outside our algebraic scope here, but if we formally allow exponentials then we can have $a(x) \propto \mathrm{e}^{-2x}$ and find a central element, for example

$$
g = \mathrm{e}^{-2x}\left((2e^y + 1)\mathrm{d}x \otimes \mathrm{d}x - (e^y + 1)(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + \mathrm{d}y \otimes \mathrm{d}y\right).
$$

If these were classical variables then whether this is invertible would depend on the value of y. In our case, over $\mathbb{C}$, for example, would need some operator analysis. ◇

For a result going the other way, we recall that a generalised differential calculus drops the surjectivity condition and here it is natural to do so (such a calculus still contains an actual calculus as the span of elements of the form $a\mathrm{d}b$ for $a, b \in A$).

Proposition 1.22 *Let A be an algebra with calculus (Ω^1, d) and Δ a 2nd order Laplacian. Define a generalised first-order differential calculus $\tilde{\Omega}^1 = \Omega^1 \oplus A\theta'$ by*

$$\theta' \bullet a = a \bullet \theta' = a\theta', \ a \bullet \omega = a\omega, \ \omega \bullet a = \omega a + \lambda(\omega, \mathrm{d}a)\theta', \quad \tilde{\mathrm{d}}a = \mathrm{d}a + \frac{1}{2}\lambda(\Delta a)\theta',$$

where $\bullet$ denotes the bimodule product of $\tilde{\Omega}^1$ in terms of the given products and $\lambda \in \mathbb{k}^*$ is a parameter. If Ω^1 is inner then so is $\tilde{\Omega}^1$. In the $*$-differential algebra case if Δ commutes with $*$ then $\tilde{\Omega}^1$ is also a generalised $*$-differential calculus.

Proof We check that the algebra A acts from each side. Thus

$$(\omega \bullet a) \bullet b = (\omega a) \bullet b + \lambda(\omega, \mathrm{d}a)\theta' b = \omega ab + \lambda(\omega a, \mathrm{d}b)\theta' + \lambda(\omega, \mathrm{d}a)b\theta',$$
$$\omega \bullet (ab) = \omega ab + \lambda(\omega, \mathrm{d}(ab))\theta' = \omega ab + \lambda(\omega, (\mathrm{d}a)b + a\mathrm{d}b)\theta',$$

using the Leibniz rule for d. The two expressions are equal by the properties of (,), allowing us to move a and b out. We have to verify that we have a bimodule

$$(a \bullet \omega) \bullet b = (a\omega) \bullet b = a\omega b + \lambda(a\omega, \mathrm{d}b)\theta'$$
$$a \bullet (\omega \bullet b) = a \bullet (\omega b + \lambda(\omega, \mathrm{d}b)\theta') = a\omega b + \lambda a(\omega, \mathrm{d}b)\theta'$$

which are again equal by the properties of (,). Next, we verify that $\tilde{\mathrm{d}}$ is a derivation,

$$\tilde{\mathrm{d}}(ab) = \mathrm{d}(ab) + \frac{\lambda}{2}\Delta(ab)\theta' = (\mathrm{d}a)b + a\mathrm{d}b + \frac{\lambda}{2}((\Delta a)b + a\Delta b)\theta' + \lambda(\mathrm{d}a, \mathrm{d}b)\theta'$$
$$= (\mathrm{d}a)b + a\mathrm{d}b + \frac{\lambda}{2}((\Delta a)\theta' b + a\Delta b\theta') + \lambda(\mathrm{d}a, \mathrm{d}b)\theta' = \tilde{\mathrm{d}}a \bullet b + a \bullet \tilde{\mathrm{d}}b$$

from the definitions. Note that the product on the free bimodule spanned by the central element θ' is that of A and is not deformed in the construction. We may not have $a \otimes b \to a\tilde{\mathrm{d}}b$ surjective, this will depend on Δ but one can also proceed more generally without this condition. If Ω^1 is inner by θ on viewing this in $\tilde{\Omega}^1$,

$$\theta \bullet a - a \bullet \theta = \theta a - a\theta + \lambda(\theta, \mathrm{d}a)\theta' = \mathrm{d}a + \lambda(\theta, \mathrm{d}a) = \tilde{\mathrm{d}}a$$

for all $a \in A$. The $*$-structure is inherited from that of Ω^1 if we set $\lambda\theta'$ hermitian (usually we would choose $\lambda^* = -\lambda$, so that this means $\theta'^* = -\theta'$). □

Example 1.23 If $A = C^\infty(M)$ is functions on a smooth manifold, (,) is given by the metric tensor and Δ is the Laplace–Beltrami operator then we have a generalised differential calculus $(\tilde{\Omega}^1(M), \tilde{\mathrm{d}})$ in which the partial derivative in the extra θ'

direction is the Laplace–Beltrami operator. The calculus has one more dimension than classically but is noncommutative or 'quantum' at the level of differentials. ◇

This is part of a general link between Riemannian geometry and central extensions of classical differentials to quantum differentials. We will build on it in §8.3 and use it as one approach to a quantum black hole in §9.3.2.

1.4 Differentials on Finite Sets

By a directed graph we mean a set X of vertices and a collection of directed edges or arrows $x \to y$ for various $x, y \in X$. There are no arrows from a vertex to itself and at most one arrow in a given direction from one vertex to another.

Proposition 1.24 *Let X be a finite set. Differential calculi $\Omega^1(X)$ on the algebra of functions $A = \Bbbk(X)$ are inner and correspond to directed graphs on X, with*

$$\Omega^1 = \mathrm{span}_k\{\omega_{x\to y}\}, \quad f.\omega_{x\to y} = f(x)\omega_{x\to y}, \quad \omega_{x\to y}.f = \omega_{x\to y} f(y)$$

$$\mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}, \quad \theta = \sum_{x\to y}\omega_{x\to y},$$

where Ω^1 is spanned by a basis labelled by arrows. The calculus is connected if and only if the underlying (undirected) graph is connected.

Proof Ω^1 has basis labelled by the arrows while A has a basis of central projectors summing to the identity, here given by the Kronecker delta-functions δ_x with value 1 at x and zero elsewhere. Note that from the formulae stated, one necessarily has $\omega_{x\to y} = \delta_x \mathrm{d}\delta_y$ for all $x \to y$. Clearly $\Omega^1_{\mathrm{uni}} \subset A \otimes A$ has basis elements $\delta_x \otimes \delta_y = \delta_x \mathrm{d}_{\mathrm{uni}}\delta_y$ for $x \neq y$, where $\mathrm{d}_{\mathrm{uni}}\delta_x = 1 \otimes \delta_x - \delta_x \otimes 1 = \sum_{y\neq x}\delta_y \otimes \delta_x - \delta_x \otimes \delta_y$ takes the form stated for the complete directed graph (where $x \to y$ for all $x \neq y$) when applied to $f = \sum_x f(x)\delta_x$.

Now suppose that we have some other calculus defined by a sub-bimodule $\mathcal{N}$. If $n = \sum_{x\neq y} n_{x,y}\delta_x \otimes \delta_y \in \mathcal{N}$ then $\delta_x n \delta_y = n_{x,y}\delta_x \otimes \delta_y \in \mathcal{N}$. Hence either $n_{x,y} = 0$ for all elements n or $\delta_x \otimes \delta_y \in \mathcal{N}$. Hence $\mathcal{N}$ has basis $\{\delta_x \otimes \delta_y \mid (x, y) \in \bar{E}\}$ for some subset $\bar{E} \subseteq (X \times X) \setminus$ diagonal. The quotient of the universal calculus by $\mathcal{N}$ can therefore be identified with the subspace spanned by $\delta_x \otimes \delta_y$ for $(x, y) \in E$, where E is the complement of $\bar{E}$ in $(X \times X) \setminus$ diagonal. Such E are the edges of our directed graph. Clearly, $\ker \mathrm{d}$ consists of those functions for which $f(y) = f(x)$ for all $x \to y$, i.e., is a multiple of 1 if and only if the graph is connected in the weak sense of the underlying undirected graph being connected. That the calculus is inner is given by $\theta f - f\theta = \sum_{x\to y}\omega_{x\to y}(f(y) - f(x)) = \mathrm{d}f$ for all $f \in \Bbbk(X)$. □

A map $\phi : X \to Z$ between directed graphs can be defined as a set map such that if $x \to y$ is an edge of X then either $\phi(x) = \phi(y)$ or $\phi(x) \to \phi(y)$ is an edge of Z. The induced algebra map $\phi : \Bbbk(Z) \to \Bbbk(X)$ is differentiable as in (1.1) with

$$\phi(\delta_z) = \sum_{\substack{x \in X \\ \phi(x) = z}} \delta_x, \quad \phi_*(\omega_{z\to w}) = \sum_{\substack{x, y \in X \\ z = \phi(x),\, w = \phi(y)}} \omega_{x\to y}.$$

One can also show conversely that for graph calculi, this is exactly what a differentiable map entails. This provides an example of a 'contravariant functor' in the sense of §2.4 from the category of directed graphs into the category of differential algebras over a fixed field $\Bbbk$. We can also regard any undirected graph as 'bidirected' i.e., with arrows in both directions for every edge, in which case we say that the graph calculus is *symmetric*. In the above proof we had the following.

Example 1.25 $\Omega^1_{\rm uni}$ for a finite set of cardinality m is symmetric, parallelisable of cotangent dimension $m - 1$ and given by the bidirected complete graph K_m. $\diamond$

Proposition 1.26

(1) *A calculus on a finite set is left-parallelisable with left cotangent dimension n if and only if the graph is left n-regular in the sense that the number of arrows going out of a vertex is a constant n.*
(2) *It is right-parallelisable with right cotangent dimension n if and only if it is right n-regular, i.e., for incoming arrows.*
(3) *If it is both left and right-parallelisable then the cotangent dimensions coincide.*

Proof If $\{\omega_i\}$ is a global left basis then $\{\delta_x\omega_i\}$ is necessarily a linear basis of $\delta_x\Omega^1$ for each x. This is because if $\delta_x\omega \in \delta_x\Omega^1$ then we can write $\omega = \sum f_i\omega_i$ for some f_i hence $\delta_x\omega = \sum f_i(x)\delta_x\omega_i$. And if $\sum \lambda_i\delta_x\omega_i = 0$ then we let $f_i(y) = \lambda_i\delta_{x,y}$ for all $y \in X$ and conclude that $\delta_y \sum_i f_i\omega_i = 0$ for all y, hence $\sum_i f_i\omega_i = 0$. This then requires all the λ_i to vanish. Hence $\dim(\delta_x\Omega^1) = n$. But a linear basis of $\delta_x\Omega^1$ is $\{\omega_{x\to y} \mid y \in X\}$, hence there are precisely n arrows out of each vertex. Conversely, suppose the graph is outgoing n-regular and choose a colouring of the arrows out of each vertex by $i \in \{1, \cdots, n\}$ (so that the arrows are enumerated). Let

$$\omega_i = \sum_{x \xrightarrow{i} y} \omega_{x\to y},$$

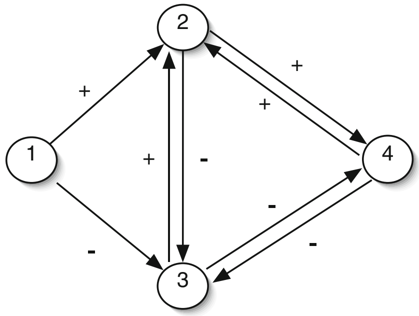
then this is a basis as follows. Every $\omega \in \Omega^1$ is a $\Bbbk$-linear combination of the $\omega_{x\to y}$ and so we can write $\omega = \sum_i f_i\omega_i$, where the f_i are defined so that $f_i(x)$ is the coefficient of $\omega_{x\xrightarrow{i}y}$. (For a fixed i, each $x \in X$ occurs just once as the source of an arrow and there is a unique y at its head.) Clearly if $\sum_i f_i\omega_i = 0$ then the f_i vanish as the $\{\omega_{x\to y}\}$ form a basis over $\Bbbk$. The right-handed result is analogous. Finally, the sum of the in-degrees and the sum of the out-degrees are equal (to the number of arrows), so if the calculus is both left and right-parallelisable then the left and right dimensions coincide. $\square$

Actually choosing a left-parallelisation means choosing a colouring of outgoing arrows from each vertex from a fixed palette of colours $i = 1, \cdots, n$, say. Then we have partial derivatives ∂_i defined by (1.2), which come out as

$$(\partial_i f)(x) = f(y) - f(x),$$

where at vertex x, y is the vertex obtained by moving along the arrow coloured i.

Example 1.27 The graph



is left-regular with left-dimension 2 but not right-regular. The edge labelling shown is an example of an edge colouring by $\{+, -\}$ with the associated left global basis

$$\omega^+ = \omega_{1\to2} + \omega_{2\to4} + \omega_{4\to2} + \omega_{3\to2}, \quad \omega^- = \omega_{1\to3} + \omega_{2\to3} + \omega_{3\to4} + \omega_{4\to3}.$$

According to (1.2) we have the associated left partial derivatives $\partial_\pm$ where

$$(\partial_+ f)(1) = f(2) - f(1), \quad (\partial_+ f)(2) = f(4) - f(2),$$
$$(\partial_+ f)(3) = f(2) - f(3), \quad (\partial_+ f)(4) = f(2) - f(4)$$

and similarly for ∂_-. By contrast there are no analogous right-handed partial derivatives associated to the reversed arrows. ◇

Proposition 1.28 *Let $\Omega^1(X)$ be a directed graph calculus on a finite set.*

(1) *Any bimodule inner product takes the form $(\omega_{x\to y}, \omega_{y'\to x'}) = \lambda_{x\to y}\delta_{x,x'}\delta_{y,y'}\delta_x$ for some arrow weights $\{\lambda_{x\to y}\}$.*
(2) *The induced 2nd order Laplacian is the standard graph Laplacian on the bidirected subgraph,*

$$(\Delta_\theta f)(x) = ({}_\theta\Delta f)(x) = 2\sum_{y|x\leftrightarrow y} \lambda_{x\to y}(f(x) - f(y)),$$

where we write $x \leftrightarrow y$ when there are arrows in both *directions.*

(3) *There is a generalised quantum metric inverse to this if and only if the calculus is symmetric and $\lambda_{x\to y}$ are all nonzero. Then*

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_A \omega_{y\to x}; \quad g_{x\to y} = \frac{1}{\lambda_{y\to x}}.$$

Proof For the bimodule inner product to be well defined on the $\otimes_A$ we need

$$(\omega_{x\to y} f, \omega_{y'\to x'}) = (\omega_{x\to y}, f\omega_{y'\to x'}),$$

i.e., $(\omega_{x\to y}, \omega_{y'\to x'})(f(y') - f(y)) = 0$ for all f. This requires a $\delta_{y,y'}$ in the evaluation. We also need

$$f(\omega_{x\to y}, \omega_{y'\to x'}) = (\omega_{x\to y}, \omega_{y'\to x'})f$$

since the algebra $A = \Bbbk(X)$ is commutative. Hence $(f(x) - f(x'))(\omega_{x\to y}, \omega_{y'\to x'}) = 0$ for all f, which requires a $\delta_{x,x'}$ in the evaluation. Moreover, $f(\omega_{x\to y}, \omega_{y'\to x'}) = f(x)(\omega_{x\to y}, \omega_{y'\to x'})$ for all f requires a value δ_x. This fixes the form of $(\ ,\ )$. Note that only $x \to y \to x$ contribute. We then compute

$$(\theta, \mathrm{d}f) = \sum_{x\to y}(\omega_{x\to y}, \sum_{y'\to x'}(f(x') - f(y'))\omega_{y'\to x'}) = \sum_{x\leftrightarrow y}\lambda_{x\to y}(f(x) - f(y))\delta_x$$

for all functions f. Similarly computing $-(\mathrm{d}f, \theta)$ gives the same. For the quantum metric one can easily check that the stated g is inverse in the required sense. $\square$

The formula for the Laplacian suggests that the inverse arrow weights $g_{x\to y}$, when they exist, should be thought of as the square of a lattice spacing or ‘length’ associated to an arrow in order to correctly scale a double finite difference (we will take this point of view in Example 8.20). Negative values are also allowed and can be thought of as the analogue of a pseudo-Riemannian structure. Also in this context one can ask for a generalised quantum metric to be *edge-symmetric* in the sense $g_{x\to y} = g_{y\to x}$ for all arrows, but note that this is not necessarily the same as the quantum symmetry imposed for a quantum metric. The canonical *Euclidean (generalised) metric* on a graph has all weights 1 and is edge-symmetric. Over $\mathbb{C}$, the algebra of functions has the usual $*$-operation given by pointwise complex conjugation.

Proposition 1.29 *A directed graph calculus $\Omega^1(X)$ is a $*$-differential calculus if and only if it is symmetric, and then $\omega_{x\to y}{}^* = -\omega_{y\to x}$. A bimodule inner product as above obeys the ‘reality’ condition (1.4) if and only if the coefficients $\lambda_{x\to y}$ are real. In this case the Laplacian commutes with $*$.*

Proof If we have a $*$-calculus then $(f.\omega_{x\to y})^* = \overline{f(x)}\omega_{x\to y}{}^* = \omega_{x\to y}{}^*.f^*$, which requires that $\omega_{x\to y}{}^*$ is a linear combination of basis elements of the form $\omega_{\cdot\to x}$. Similarly from the other side, so $\omega_{x\to y}{}^* = \epsilon_{x\to y}\omega_{y\to x}$ for some coefficients $\epsilon_{x\to y}$

with $\overline{\epsilon_{x\to y}}\epsilon_{y\to x} = 1$. This requires that the calculus is symmetric, so that $\omega_{y\to x}$ is a valid graph edge in the first place. We also need $\theta^* = \sum_{x\to y}\epsilon_{x\to y}\omega_{y\to x} = -\theta$, since there no are 1-forms that commute with all functions, which fixes $\epsilon_{x\to y} = -1$. For the inner product 'reality' we need $(\omega_{x\to y}, \omega_{y\to x})^* = \overline{\lambda_{x\to y}} = (\omega_{y\to x}{}^*, \omega_{x\to y}{}^*) = (\omega_{x\to y}, \omega_{y\to x}) = \lambda_{x\to y}$. It follows that $\Delta_\theta = 2(\theta, \mathrm{d}(\ ))$ then commutes with $*$. □

1.5 Exterior Algebra and the de Rham Complex

A key application of differential forms is to the construction of the de Rham complex

$$C^\infty(M) \to \Omega^1(M) \to \cdots \Omega^n(M) \to 0$$

with $\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M)$, where the space of i-forms $\Omega^i(M)$ consists of skew-symmetrised 1-forms. We have $\Omega^n(M) = C^\infty(M)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ in local coordinates. The space of all differential forms $\Omega(M) = \oplus_{i=1}^{i=n}\Omega^i(M)$, where $\Omega^0(M) = C^\infty(M)$, forms a graded algebra with the exterior product $\wedge$. This means that the vector space of the algebra decomposes into a direct sum over different degrees as stated, and that the degree of the product is the sum of the degrees of each element. The cohomology of this complex is the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(M)$. In this section we cover the algebraic version of this construction. We have already studied the notion of a differential structure (Ω^1, d) on an algebra A.

Definition 1.30 A *differential graded algebra* or DGA on an algebra A is

(1) A graded algebra $\Omega = \oplus_{n\geq 0}\Omega^n$ with $\Omega^0 = A$;
(2) $\mathrm{d} : \Omega^n \to \Omega^{n+1}$ such that $\mathrm{d}^2 = 0$ and

$$\mathrm{d}(\omega \wedge \rho) = (\mathrm{d}\omega) \wedge \rho + (-1)^n \omega \wedge \mathrm{d}\rho, \quad \text{for all } \omega, \rho \in \Omega,\ \omega \in \Omega^n;$$

(3) A, $\mathrm{d}A$ generate Ω (optional surjectivity condition).

When the surjectivity condition holds we shall refer to an *exterior algebra* on A. The *noncommutative de Rham cohomology* of a DGA over A is the graded algebra

$$\mathrm{H}^n_{\mathrm{dR}}(A) = \ker(\mathrm{d}|_{\Omega^n})/\mathrm{image}(\mathrm{d}|_{\Omega^{n-1}}),$$

where we understand $\mathrm{d}|_{\Omega^n} = 0$ if $n < 0$. Elements in the image here are said to be *exact*. A DGA map ϕ between DGAs Ω_A and Ω_B is an algebra map $\phi : \Omega_A \to \Omega_B$ which preserves the grade (i.e., $\phi(\Omega^n_A) \subseteq \Omega^n_B$) and commutes with d.

The first-order part of a DGA (resp. exterior algebra) is a generalised first-order calculus (resp. first-order calculus) as in §1.1. Also, if we were to omit the product from the definition of a DGA, leaving only the d operator with $\mathrm{d}^2 = 0$, then we would have a cochain complex, which is a much studied concept in algebraic topology. Several cohomology theories which we shall study later either have no obvious product, or the product requires extra structure, requiring us to work with a cochain complex. In this case, a cochain map between cochain complexes is just a linear map which preserves degree and commutes with d. One has a parallel notion of a chain complex with d decreasing index by one.

For any DGA the *volume dimension* is defined as the top degree, i.e., the largest n such that $\Omega^n \neq 0$ (infinite dimension is a possibility here). This can be different from the dimension or rank of Ω^1, if indeed that can be defined.

A DGA is called *nondegenerate* if for all $m < n$ (n the top degree, possibly infinite) the wedge products $\wedge : \Omega^m \otimes \Omega^1 \to \Omega^{m+1}$ and $\wedge : \Omega^1 \otimes \Omega^m \to \Omega^{m+1}$ are nondegenerate (this means that for any nonzero $\xi \in \Omega^m$ there are $\eta, \zeta \in \Omega^1$ such that $\xi \wedge \eta \in \Omega^{m+1}$ and $\zeta \wedge \xi \in \Omega^{m+1}$ are both nonzero). If the calculus is nondegenerate and has finite volume dimension then it follows by recursive application of nondegeneracy that for any nonzero $\xi \in \Omega^m$ there is an $\eta \in \Omega^{n-m}$ such that $\xi \wedge \eta \in \Omega^n$ is nonzero. This is one of the principles behind classical Hodge theory, which very roughly compares inner products on m-forms to the wedge product of m and $n - m$ forms. The nicest case is when the top degree Ω^n is free with one generator, i.e., 1-dimensional over the algebra, and in this case we have as basis a *volume form* Vol, defined up to normalisation.

By contrast to the classical case, we say that a DGA or exterior algebra is *inner* if there exists a $\theta \in \Omega^1$ such that for all n,

$$\mathrm{d}\omega = [\theta, \omega\} := \theta \wedge \omega - (-1)^n \omega \wedge \theta, \quad \text{for all } \omega \in \Omega^n,$$

where [, } denotes the graded commutator. As we saw in degree 1, this will be a typical feature of highly noncommutative exterior algebras related to the fact that d is a graded derivation. Also as we saw in degree 1, to do 'real' geometry we work over $\mathbb{C}$ but with a $*$-involution. This extends to all forms as follows.

Definition 1.31 A $*$-DGA (resp. $*$-exterior algebra) is a DGA (resp. exterior algebra) with an antilinear involutive operation $* : \Omega^n \to \Omega^n$ for all n such that

(1) $(\mathrm{d}\xi)^* = \mathrm{d}(\xi^*)$ for all $\xi \in \Omega$.
(2) $(\xi \wedge \eta)^* = (-1)^{nm} \eta^* \wedge \xi^*$ for all $\xi \in \Omega^n$ and $\eta \in \Omega^m$.

Here $A = \Omega^0$ is a $*$-subalgebra and Ω is said to be a $*$-DGA (resp. $*$-exterior algebra) over A.

Turning to constructions, one approach is to just construct Ω up to and including any degree of interest, just as in differential geometry one need only assume differentiability up to and including some level. One can then 'fill in' all higher degrees automatically if one wants them, as exemplified by the following result.

Lemma 1.32 *Every first-order calculus Ω^1 on A has a 'maximal prolongation' $\Omega_{\max}$ to an exterior algebra, where for every relation $\sum_i a_i.\mathrm{d}b_i = \sum_j \mathrm{d}r_j.s_j$ in Ω^1 for $a_i, b_i, r_j, s_j \in A$ we impose the relation $\sum_i \mathrm{d}a_i \wedge \mathrm{d}b_i + \sum_j \mathrm{d}r_j \wedge \mathrm{d}s_j = 0$*

in $\Omega^2_{\max}$. This is extended to higher forms, but no new relations are added. If Ω^1 is a $*$-differential structure on a $*$-algebra A then $\Omega_{\max}$ becomes a $*$-exterior algebra.

Proof We define $\Omega^2_{\max}$ as freely generated by products of elements of Ω^1 and quotient out by the minimal relations needed for an exterior algebra. If the relation shown holds in Ω^1 then applying d gives the required relations in $\Omega^2_{\max}$. We need to show that d applied to $\xi \wedge (\sum \mathrm{d}a_i \wedge \mathrm{d}b_i + \sum \mathrm{d}r_j \wedge \mathrm{d}s_j) \wedge \eta$ is another element of the ideal generated by $\sum \mathrm{d}a_i \wedge \mathrm{d}b_i + \sum \mathrm{d}r_j \wedge \mathrm{d}s_j$, and this follows from $\mathrm{d}^2 = 0$. The additional properties in the $*$-algebra case are clear. □

More formally, the maximal prolongation calculus is a quadratic algebra over A generated by Ω^1. We say *maximal* in Lemma 1.32 because we have given the smallest possible number of consistent relations, so that applying them gives the biggest consistent calculus. Thus applying d in any extension of Ω^1 to a relation $\sum_i a_i.\mathrm{d}b_i = \sum_j \mathrm{d}r_j.s_j$ requires the relation in degree 2 as stated. Similarly for higher degrees. As a result, any exterior algebra with the same Ω^1 will be a quotient of $\Omega_{\max}$ by a differential ideal. Here a differential ideal means a 2-sided ideal which is also closed under the differential d. For example, if Ω^1 is inner by an element θ then this feature need not extend to $\Omega_{\max}$ as it needs $[\theta \wedge \theta, a] = 0$ for all $a \in A$. A natural quotient in such a case is to add these as further relations.

Theorem 1.33 *Let A be an algebra. The maximal prolongation of its universal first-order calculus is its universal exterior algebra $\Omega_{\mathrm{uni}} = \bigoplus_n \Omega^n_{\mathrm{uni}}$, where $\Omega^n_{\mathrm{uni}} \subset A^{\otimes(n+1)}$ is the intersection of the kernels of all the product maps between adjacent copies of A in the tensor product. The product and differential are*

$$(a_0 \otimes \cdots \otimes a_n)(b_0 \otimes \cdots \otimes b_m) = (a_0 \otimes \cdots \otimes a_n b_0 \otimes \cdots \otimes b_m),$$

$$\mathrm{d}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n.$$

Moreover, $\mathrm{H}^i_{\mathrm{dR,uni}}(A) = \Bbbk$ if $i = 0$ and is zero otherwise (the complex is acyclic). The volume dimension is infinite. In the $*$-algebra case we have a $*$-exterior algebra with $(a_0 \otimes \cdots \otimes a_n)^* = (-1)^{\frac{n(n+1)}{2}} (a_n^* \otimes \cdots \otimes a_0^*)$.

Proof One may verify that Ω_{uni} is an exterior algebra, and since it uses only the algebra structure, it is clear that it is maximal as a prolongation of the first-order calculus. Here we just prove the cohomology as stated. Consider

$$\omega = \sum_\alpha a_0^\alpha \otimes \cdots \otimes a_n^\alpha = \sum_\alpha a_0^\alpha \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha$$
$$= \sum_\alpha \lambda^\alpha \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha + \sum_\alpha b^\alpha \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha,$$

where we replace $a_0^\alpha = \lambda^\alpha 1 \oplus b^\alpha$ according to $A = \Bbbk 1 \oplus A^+$, where A^+ is a chosen complement, as in the proof of Proposition 1.5. Then $\mathrm{d}\omega = \sum_\alpha \mathrm{d}b^\alpha \mathrm{d}a_1^\alpha \cdots \mathrm{d}a_n^\alpha$ as $\mathrm{d}1 = 0$ and $\mathrm{d}^2 = 0$. Hence if ω is closed, and writing out $\mathrm{d}b^\alpha$, we see that

$$1 \otimes \sum_\alpha b^\alpha \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha = \sum_\alpha b^\alpha \otimes \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha.$$

Projecting the first factor to A^+, we conclude that $\sum_\alpha b^\alpha \mathrm{d}a_1^\alpha \cdots \mathrm{d}a_n^\alpha = 0$ and hence

$$\omega = \sum_\alpha \lambda^\alpha \mathrm{d}a_1^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha = \mathrm{d}\Big(\sum_\alpha \lambda^\alpha a_1^\alpha \mathrm{d}a_2^\alpha \wedge \cdots \wedge \mathrm{d}a_n^\alpha\Big),$$

so ω is exact. The last step requires $n > 1$. The $n = 0$ case has already been dealt with and we computed $\ker \mathrm{d} = \Bbbk 1$ in this case. The $*$-structure in the $*$-algebra case follows from our previous results about Ω^1_{uni} and Lemma 1.32. □

We see that the universal calculus is both too big and has no nontrivial 'topology' in view of this theorem. The maximal prolongation, by contrast, can give reasonable answers for calculi that are commutative or close to commutative since in this case there is a lot of strength to the relations that we are applying d to.

Example 1.34 The maximal prolongations of $\mathbb{C}[x]$ with the calculus in Example 1.10 and $\mathbb{C}[t, t^{-1}]$ with the calculus in Example 1.11 are nondegenerate with volume dimension 1 and the cohomologies have their classical values

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{C}[x]) = \mathbb{C}.1, \qquad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}[x]) = 0,$$
$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{C}[t, t^{-1}]) = \mathbb{C}.1, \qquad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}[t, t^{-1}]) = \mathbb{C}.t^{-1}\mathrm{d}t,$$

where the latter assumes that q it not a nontrivial root of unity. These examples are $\mathbb{C}_\lambda[\mathbb{R}]$ and $\mathbb{C}_q[S^1]$ respectively for the standard real forms.

Proof We have already seen $\mathrm{H}^0_{\mathrm{dR}}$ in Examples 1.10 and 1.11. For the first case, $\Omega^i = 0$ for $i > 1$ since applying d to $\mathrm{d}x.x = (x + \lambda)\mathrm{d}x$ gives $-\mathrm{d}x \wedge \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}x$ as classically, which implies $(\mathrm{d}x)^2 = 0$. Nondegeneracy is then clear. Now suppose it is true that for all polynomials $p(x)$ of degree $< n$ we can solve $p\mathrm{d}x = \mathrm{d}f$ for f. Then we can solve $x^n \mathrm{d}x = \mathrm{d}f$ by setting $f = \frac{x^{n+1}}{n+1} + g$, where g is required to obey $\sum_{r=2}^{n+1} [{n+1 \atop r}] x^{n+1-r} \lambda^{r-1} \mathrm{d}x + \mathrm{d}g = 0$, as the first expression has degree $n-1$. Hence by induction every 1-form is exact. For the second case, $\Omega^i = 0$ for $i > 1$ since this time $0 = \mathrm{d}(\mathrm{d}t.t - qt\mathrm{d}t) = -(\mathrm{d}t)^2(1 + q) = -[2]_q(\mathrm{d}t)^2$, and hence $(\mathrm{d}t)^2 = 0$. Then we similarly have

$$\sum a_n t^n \mathrm{d}t = \mathrm{d}\Big(\sum \frac{a_n t^{n+1}}{[n+1]_q}\Big)$$

if and only if $a_{-1} = 0$, and hence $\mathrm{H}^1_{\mathrm{dR}} = \mathbb{C}t^{-1}\mathrm{d}t$. This again changes if q is a root of unity. Note that the cohomology changes drastically over other fields. $\square$

The maximal prolongation works well for the classical calculus on a manifold.

Example 1.35 The classical calculus on a manifold M of dimension n has relations $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$ and is the maximal prolongation of the classical first-order calculus, as can be seen by differentiating the relations $x^i \mathrm{d}x^j = (\mathrm{d}x^j)x^i$. It is nondegenerate as follows. Let $m < n$ and write $\xi \in \Omega^m(M)$ in terms of the basis elements $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_m}$. Choose a nonzero multiple of such an element in the expression for ξ. As $m < n$, there is a $1 \le t \le n$ which is not in the list $\{i_1, \ldots, i_m\}$. Then both $\mathrm{d}x^t \wedge \xi$ and $\xi \wedge \mathrm{d}x^t$ are nonzero. The top degree is Ω^n, which in the orientable case will be 1-dimensional, spanned by the volume form. $\diamond$

A more nontrivial example, of the same dimensions as classically, is the following.

Example 1.36 (Noncommutative Torus) The algebraic noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$ is generated as a complex algebra by two invertible generators u, v with the relation $vu = e^{i\theta}uv$ for a real parameter θ. We let $\Omega^1 = \mathbb{C}_\theta[\mathbb{T}^2].\{\mathrm{d}u, \mathrm{d}v\}$ be the left $\mathbb{C}_\theta[\mathbb{T}^2]$ module with basis $\{\mathrm{d}u, \mathrm{d}v\}$, made into a bimodule with

$$\mathrm{d}u.u = u.\mathrm{d}u, \quad \mathrm{d}v.v = v.\mathrm{d}v, \quad \mathrm{d}v.u = e^{i\theta}u.\mathrm{d}v, \quad \mathrm{d}u.v = e^{-i\theta}v.\mathrm{d}u.$$

Its maximal prolongation is nondegenerate with volume dimension 2 and natural volume form $\mathrm{Vol} = u^{-1}\mathrm{d}u \wedge v^{-1}\mathrm{d}v$. The noncommutative de Rham cohomology is the same as classically, namely

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{C}_\theta[\mathbb{T}^2]) = \mathbb{C}.1, \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}_\theta[\mathbb{T}^2]) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_\theta[\mathbb{T}^2]) = \mathbb{C}e_1 \wedge e_2,$$

where $e_1 = u^{-1}\mathrm{d}u$, $e_2 = v^{-1}\mathrm{d}v$. We have a $*$-exterior algebra structure with $u^* = u^{-1}$, $v^* = v^{-1}$ or equivalently e_1, e_2 antihermitian.

Proof Here u, u^{-1} generate an algebraic 1-torus $\mathbb{C}[u, u^{-1}]$ and Ω^1 restricts to this as classically. Similarly for v, v^{-1}. Of interest is how the two interact. From d applied to the relations we need $\mathrm{d}(vu) = \mathrm{d}v.u + v.\mathrm{d}u = e^{i\theta}\mathrm{d}(uv) = e^{i\theta}\mathrm{d}u.v + e^{i\theta}u.\mathrm{d}v$, which is provided by the relations shown. These relations also ensure that the calculus becomes the usual one on the classical algebraic torus when $\theta = 0$. We thus define $\Omega^1(\mathbb{C}_\theta[\mathbb{T}^2])$ as a free module on the left (i.e., just by the product of $\mathbb{C}_\theta[\mathbb{T}^2]$) and use the relations between 1-forms and the algebra generators to define the right module structure. Next, apply d to these relations to find

$$\mathrm{d}u \wedge \mathrm{d}u = 0, \quad \mathrm{d}v \wedge \mathrm{d}v = 0, \quad \mathrm{d}v \wedge \mathrm{d}u + e^{i\theta}\mathrm{d}u \wedge \mathrm{d}v = 0,$$

which tells us that $\Omega^2(\mathbb{C}_\theta[\mathbb{T}^2])$ is 1-dimensional over $\mathbb{C}_\theta[\mathbb{T}^2]$ while higher forms vanish. Nondegeneracy is clear. That the cohomology is the same as classically is a calculation along the same lines as when $\theta = 0$. One still has a basis

$\{u^m v^n : m, n \in \mathbb{Z}\}$ of $\mathbb{C}_\theta[\mathbb{T}^2]$ with extra $e^{i\theta}$ factors in the computations but the steps are the same. The natural basis for computations is given by the central 1-forms $e_1 = u^{-1}du$ and $e_2 = v^{-1}dv$ with $de_1 = -u^{-1}du.u^{-1}du = 0$ from the Leibniz rule and the commutation relations above, as classically. Similarly for de_2. We also have

$$\begin{aligned} d(\sum a_{mn}u^m v^n) &= \sum a_{mn}(mu^{m-1}du.v^n + u^m nv^{n-1}dv) \\ &= \sum a_{mn}(mu^m v^n e_1 + u^m nv^n e_2), \end{aligned}$$

where d on powers of u alone or v alone behaves as classically and we used that e_1, e_2 are central. We see that the partial derivatives for this basis on normal ordered monomials in the u, v have exactly the same form as classically. It follows that the cohomology has the classical form. For example, if $\lambda e_1 + \mu e_2 = d(\sum a_{mn}u^m v^n)$ then by the above we would need $a_{mn} = 0$ for all $m \neq 0$ or $n \neq 0$ for the e_1 and e_2 coefficients to match. Hence only $a_{0,0}$ can contribute, and does not as $d1 = 0$. Hence all $\lambda e_1 + \mu e_2$ are not exact. Similarly for the other facts. □

This calculus is not inner, being too close to the classical case. We close with a contrasting finite geometry example.

Example 1.37 The maximal prolongation of the 2-dimensional calculus on $M_2(\mathbb{C})$ in Corollary 1.9 is $\Omega_{\max} = M_2(\mathbb{C}).\mathbb{C}[s,t]$, where $s = 1 \oplus 0, t = 0 \oplus 1$ are central in the exterior algebra and obey $s^* = -t$. Moreover, $\Omega_{\max}$ is inner with the same $\theta = E_{12}s + E_{21}t$ as before, and it is nondegenerate with infinite volume dimension.

Proof Here $1 \in M_2(\mathbb{C})$ is the identity matrix. It follows that $[A, s] = [A, t] = 0$ for all $A \in M_2(\mathbb{C})$ and $dA = [E_{12}, A]s + [E_{21}, A]t = [\theta, A]$ in our current more algebraic notation for the first-order calculus. In particular,

$$dE_{12} = [E_{21}, E_{12}]t = (E_{22} - E_{11})t, \quad dE_{21} = [E_{12} - E_{21}]s = (E_{22} - E_{11})s$$
$$dE_{11} = [E_{12}, E_{11}]s + [E_{21}, E_{11}]t = -E_{12}s + E_{21}t = -dE_{22}.$$

The first of these tells us that

$$0 = d^2 E_{12} = 2(E_{12}s - E_{21}t)t + (E_{22} - E_{11})dt$$

which (multiplying by $E_{22} - E_{11}$) tells us that

$$dt = 2(E_{11} - E_{22})(E_{12}s - E_{21}t)t = 2E_{21}t^2 + 2E_{12}st = 2\theta t.$$

One similarly has $ds = 2\theta s$ and

$$\begin{aligned} 0 = d^2 E_{11} &= -[E_{21}, E_{12}]ts + [E_{12}, E_{21}]st + E_{21}dt - E_{12}ds \\ &= (E_{11} - E_{22})ts + (E_{11} - E_{22})st + 2E_{21}E_{12}st - 2E_{12}E_{21}ts = -ts + st, \end{aligned}$$

so that t, s commute. One can then check from the above formulae that d applied to the algebra products $E_{ij}E_{kl} = \delta_{jk}E_{il}$ introduces no more relations. Finally, $\{\theta, s\} = \theta s + s(E_{12}s + E_{21}t) = 2\theta s = \mathrm{d}s$, given the above results. It follows similarly from the form of $\Omega_{\max}$ that the whole calculus is inner. Nondegeneracy can be seen by taking the product of a form with either s or t. □

This example illustrates the fact that for many noncommutative algebras the maximal prolongation is still too big. However, it provides a useful starting point from which other calculi extending Ω^1 have to be quotients. In our case we can set $s^n = t^n = 0$ for any $n > 1$ and still have a $*$-exterior algebra, with relations $s^2 = t^2 = 0$ for the smallest extension of Corollary 1.9 in this family.

Proposition 1.38 *The exterior algebra on $M_2(\mathbb{C})$ given by $\Omega_{\max}$ in Example 1.37 modulo $s^2 = t^2 = 0$ has grade dimensions 1:2:1 (i.e., the dimensions in each degree up to the top degree) as for a classical 2-manifold, is nondegenerate, and has*

$$\mathrm{H}^0_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}.1, \quad \mathrm{H}^1_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}E_{21}s \oplus \mathbb{C}E_{12}t, \quad \mathrm{H}^2_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}s \wedge t.$$

Proof We note that from the form of $\mathrm{d}s = 2\theta s$ etc., it is clear that s^2 and t^2 generate a differential ideal. The dimensions are clear and nondegeneracy is shown as before by multiplying by s or t. We also have $\mathrm{d}(As + Bt) = (\{E_{21}, A\} + \{E_{12}, B\})st$ as an anticommutator of matrices. Then explicitly

$$\mathrm{d}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} s + \begin{pmatrix} e & f \\ g & h \end{pmatrix} t) = \begin{pmatrix} b+g & e+h \\ a+d & b+g \end{pmatrix} st.$$

A complement of the image of d is then $(E_{11} - E_{22})st$, giving $\mathrm{H}^2_{\mathrm{dR}} = \mathbb{C}$. Meanwhile,

$$\ker(\mathrm{d}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} s + \begin{pmatrix} e & f \\ -b & -e \end{pmatrix} t \right\}, \quad \mathrm{d}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d-1 \\ 0 & -c \end{pmatrix} s + \begin{pmatrix} -b & 0 \\ a-d & b \end{pmatrix} t,$$

so a complement of image(d) in ker(d) is spanned by setting $a, b, e = 0$ in the displayed kernel, leaving $\mathrm{H}^1_{\mathrm{dR}}$ as stated. □

Finally, if we have defined at least Ω^1, Ω^2 on an algebra A then we have a reasonable notion of *quantum symmetric* tensor $g \in \Omega^1 \otimes_A \Omega^1$ as $\wedge(g) = 0$. This completes the Definition 1.15 for a fully 'quantum symmetric' metric.

Example 1.39 The quantum metrics on $\Omega(M_2(\mathbb{C}))$ in Proposition 1.38 are of the form $g = cs \otimes s + \bar{c}t \otimes t + b(s \otimes t - t \otimes s)$ for constants $c \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|c|^2 \neq b^2$. This is because the s, t are central, so for g to be central the coefficients have to be in the centre of $M_2(\mathbb{C})$, hence constants. The metric should be quantum symmetric, which requires the sign in the $s \otimes t - t \otimes s$ term (because, unusually, these commute with each other). The 'reality' condition then dictates the reality properties of the coefficients. Nondegeneracy reduces to nonvanishing of a determinant. ◇

We have also seen examples of first-order calculi on finite sets and conclude with their maximal prolongation. Recall that Ω^1 is always inner by $\theta = \sum_{x\to y}\omega_{x\to y}$.

Proposition 1.40 *Let $\Omega^1(X)$ be a directed graph calculus on a finite set as in §1.4. Its maximal prolongation $\Omega_{\max}(X)$ has relations*

$$\sum_{y:p\to y\to q}\omega_{p\to y}\wedge\omega_{y\to q}$$

for all $p\neq q$ with $p\not\to q$ (the sum is over y as shown). Its quotient where we add the above relation also when $p\to q$ remains inner by θ, in which case

$$\mathrm{d}\omega_{p\to q}=\sum_{y:y\to p}\omega_{y\to p}\wedge\omega_{p\to q}+\sum_{y:q\to y}\omega_{p\to q}\wedge\omega_{q\to y}.$$

Proof In $\Omega^1(X)$ we have for all $x\in X$ that

$$\mathrm{d}\delta_x=\sum_{y\to x}\omega_{y\to x}-\sum_{x\to y}\omega_{x\to y}\tag{1.5}$$

and from this we read off all products of δ-functions and their differentials, to obtain a linear spanning set of the relations of Ω^1 which we group into cases as

(a) $p\not\to q,\ p\neq q:\qquad \delta_p\mathrm{d}\delta_q=0,\qquad (\mathrm{d}\delta_p)\delta_q=0.$
(b) $p\to q:\qquad (\mathrm{d}\delta_p)\delta_q+\delta_p\mathrm{d}\delta_q=0.$
(c) $p:\qquad \delta_p\mathrm{d}\delta_p+\sum_{p\to y}\delta_p\mathrm{d}\delta_y=0,\qquad (\mathrm{d}\delta_p)\delta_p+\sum_{y\to p}(\mathrm{d}\delta_y)\delta_p=0.$

Applying d to (a) gives $\mathrm{d}\delta_p\wedge\mathrm{d}\delta_q=0$ for all $p\not\to q$, to (b) is empty and to (c) gives

$$\mathrm{d}\delta_p\wedge\mathrm{d}\delta_p+\sum_{p\to y}\mathrm{d}\delta_p\wedge\mathrm{d}\delta_y=0,\quad \mathrm{d}\delta_p\wedge\mathrm{d}\delta_p+\sum_{y\to p}\mathrm{d}\delta_y\wedge\mathrm{d}\delta_p=0$$

for all p. To calculate these we multiply two copies of (1.5) to read off the cases

$$p\not\to q,\ p\neq q:\quad \mathrm{d}\delta_p\wedge\mathrm{d}\delta_q=-\sum_{p\to y\to q}\omega_{p\to y}\wedge\omega_{y\to q}$$

$$p\to q:\ \mathrm{d}\delta_p\wedge\mathrm{d}\delta_q=\sum_{y\to p\to q}\omega_{y\to p}\wedge\omega_{p\to q}+\sum_{p\to q\to y}\omega_{p\to q}\wedge\omega_{q\to y}-\sum_{p\to y\to q}\omega_{p\to y}\wedge\omega_{y\to q}$$

$$p:\quad \mathrm{d}\delta_p\wedge\mathrm{d}\delta_p=-\sum_{x\to p\to y}\omega_{x\to p}\wedge\omega_{p\to y}-\sum_{p\to y\to p}\omega_{p\to y}\wedge\omega_{y\to p}.$$

This then leads to the stated condition. Requiring θ^2 to commute with functions gives the stronger version for an inner extension. □

It is natural to impose further relations so that $\wedge(g) = 0$ holds and results in a quantum metric for a given class of g in Proposition 1.28. For example, asking this for the Euclidean metric requires $\sum_{y:p\to y\to p} \omega_{p\to y} \wedge \omega_{y\to p} = 0$ for all p, which is of the same form as in the proposition but now includes $p = q$. Hence one natural $\Omega(X)$ for a directed graph calculus is to adopt the stated relations for all p, q.

1.6 Exterior Algebras of Enveloping and Group Algebras

An important class of algebras are group algebras of groups G and enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. These are examples of Hopf algebras and so are covered by the general theory in Chap. 2, but in fact they are classical enough to be handled directly now. In both cases, 'translation-invariant' Ω^1 are in fact parallelisable, namely a free module over a vector space Λ^1 of left-invariant 1-forms and hence easy to describe, albeit the reason for this (and the precise definition of translation-invariance) comes out of the Hopf algebra treatment given later. The full exterior algebra Ω is also a free module over its left-invariant subalgebra Λ, which in both cases will just be the usual Grassmann or exterior algebra on Λ^1, i.e., defined by setting to zero the squares of all elements of Λ^1 in the tensor algebra. Consequently, the exterior algebras are as close to classical as one could hope, with a unique top form Vol up to scale and volume dimension equal to the cotangent dimension and equal to the dimension of Λ^1. Thus, the nontrivial part of the construction is the construction of Ω^1, after which the exterior algebra comes about automatically. The 1-dimensional cases were already covered in Example 1.34 as the algebraic line and circle, respectively. Our main result is that differential calculi on these types of algebras are classified respectively by Lie algebra and group 1-cocycles.

1.6.1 Enveloping Algebras

We recall that a Lie algebra means a vector space $\mathfrak{g}$ and a map $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ obeying $[x, x] = 0$ and the Jacobi identity $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$. Its *universal enveloping algebra* is defined by $U(\mathfrak{g}) = T\mathfrak{g}/\langle xy-yx-[x, y];\ x, y \in \mathfrak{g}\rangle$, where $T\mathfrak{g}$ is the tensor algebra on the vector space of $\mathfrak{g}$ and the angular brackets indicate the generated 2-sided ideal. To say this more explicitly, one can fix a basis $\{e_i\}$ of $\mathfrak{g}$ and then $T\mathfrak{g} = \mathbb{k}\langle e_i\rangle$ is the free associative noncommutative polynomial algebra on generators e_i (i.e., formal linear combinations of words in the alphabet $\{e_i\}$) over our field $\mathbb{k}$, and

$$U(\mathfrak{g}) = \mathbb{k}\langle e_i\rangle/\langle e_i e_j - e_j e_i - [e_i, e_j]\rangle.$$

The idea is to view this algebra when $\mathfrak{g}$ is nonabelian as a noncommutative version of $\Bbbk[\mathfrak{g}^*]$ (also denoted the symmetric algebra $S(\mathfrak{g})$) of polynomial functions on $\mathfrak{g}^*$. It is well known that $\mathfrak{g}^*$ is a Poisson manifold with the Kirillov–Kostant bracket

$$\{x, y\} = [x, y]$$

for all $x, y \in \mathfrak{g} \subset \Bbbk[\mathfrak{g}^*]$, where we view Lie algebra elements as (linear) functions on the dual and then $U(\mathfrak{g})$ is its standard quantisation. This idea goes back to the 1970s and one can equally well speak of quantising the algebra $C^\infty(\mathfrak{g}^*)$. Our new feature is to add the noncommutative differential geometry. We will have recourse to the *augmentation ideal* $U(\mathfrak{g})^+ \subset U(\mathfrak{g})$ consisting of noncommutative polynomials with no constant term. This is the kernel of the canonical algebra map $\epsilon : U(\mathfrak{g}) \to \Bbbk$ which picks out the constant term and which we think of as evaluation at the origin (in Chap. 2 it will be the 'counit'). Over $\mathbb{C}$, we recall that a *real form* means a choice of basis $\{e_i\}$ such that the structure constants defined by $[e_i, e_j] = f_{ijk}e_k$ (summation understood) are real. Then there is an associated $*$-algebra structure on $U(\mathfrak{g})$ given by $e_i^* = -e_i$. Equivalently, a real form means the specification of a $*$-Lie algebra structure on $\mathfrak{g}$ in the sense of an antilinear involution such that $[x, y]^* = -[x^*, y^*]$ for all $x, y \in \mathfrak{g}$, which we extend to $U(\mathfrak{g})$. In general the classification problem given only an algebra can be wild, but in our case we have a form of translation-invariance that we can impose and which comes from the Hopf algebra structure.

We recall that a right action $\triangleleft$ of a Lie algebra $\mathfrak{g}$ on a vector space Λ^1 means that $v\triangleleft[x, y] = (v\triangleleft x)\triangleleft y - (v\triangleleft y)\triangleleft x$ for all $x, y \in \mathfrak{g}$ and $v \in \Lambda^1$. Then a 1-cocycle $\zeta : \mathfrak{g} \to \Lambda^1$ on a Lie algebra with values in a right module Λ^1 means

$$\zeta([x, y]) = \zeta(x)\triangleleft y - \zeta(y)\triangleleft x$$

for all $x, y \in \mathfrak{g}$. Given a real form of $\mathfrak{g}$ we will say that a cocycle is *unitary* if Λ^1 is equipped with a $*$-involution such that

$$(v\triangleleft x)^* = -(v^*\triangleleft x^*), \quad \zeta(x)^* = \zeta(x^*) \tag{1.6}$$

for all $v \in \Lambda^1$, $x \in \mathfrak{g}$. Also note that $\zeta(xy) = \zeta(x)\triangleleft y$ for all $x \in U(\mathfrak{g})^+$ and $y \in \mathfrak{g}$ extends ζ inductively to all of $U(\mathfrak{g})^+$ using the cocycle relations. Then this equation also holds for all $y \in U(\mathfrak{g})^+$ when $\triangleleft$ is extended in its canonical way to a right action of the enveloping algebra. We again defer to Chap. 2 as to what precisely we mean by translation-invariance of differentials, but in line with §1.2, it means with respect to the underlying additive group of $\mathfrak{g}^*$ as a vector space.

Theorem 1.41 *Translation-invariant $\Omega^1(U(\mathfrak{g}))$ on an enveloping algebra $U(g)$ are classified by 1-cocycles $\zeta \in Z^1(\mathfrak{g}, \Lambda^1)$, where Λ^1 is a right $\mathfrak{g}$-module, such that the extension of the cocycle to $U(\mathfrak{g})^+ \to \Lambda^1$ is surjective. Such a calculus is inner precisely when the cocycle is exact. There is a canonical extension to*

$\Omega(U(\mathfrak{g})) = U(\mathfrak{g}).\Lambda$ as a free left module over Λ, the usual Grassmann algebra on Λ^1. Over $\mathbb{C}$, we have a $*$-calculus if the cocycle is unitary.

Proof We defer the full discussion to Chap. 2 and here consider only one direction: given the data (Λ^1, ζ) we construct the associated calculus. As indicated, we define Λ to be the Grassmann algebra on Λ^1 which inherits a right action $\triangleleft$ of $\mathfrak{g}$, We make $\Omega = U(\mathfrak{g}) \otimes \Lambda$ into a $U(\mathfrak{g})$-bimodule where $x \in \mathfrak{g}$ acts from the left by left multiplication in $U(\mathfrak{g})$ and from the right by both right multiplication and the given right action, so $(x \otimes v).y = xy \otimes v + x \otimes (v\triangleleft y)$ for all $x \in U(\mathfrak{g})$, $y \in \mathfrak{g} \subset U(\mathfrak{g})$. Next we define $\mathrm{d}x = 1 \otimes \zeta(x)$ for all $x \in \mathfrak{g}$, then extend this as a derivation. In a compact notation, we have bimodule relations and differential

$$[v, x] = v\triangleleft x, \quad \mathrm{d}x = \zeta(x), \quad \mathrm{d}v = 0$$

for all $v \in \Lambda^1$, $x \in \mathfrak{g} \subset U(\mathfrak{g})$. Using the cocycle identity, we verify that d is well-defined and extends as a graded derivation. Indeed $\mathrm{d}(xy - yx) = \zeta(x)y + x\zeta(y) - \zeta(y)x - y\zeta(x) = \zeta(x)\triangleleft y - \zeta(y)\triangleleft x = \zeta([x, y])$ so that extending d as a derivation is well defined. This verifies that we have a generalised first-order differential calculus. With more work one can see that to have an actual calculus we need the extension of ζ to $U(\mathfrak{g})^+$ to be surjective. This is best seen using Hopf algebras in Chap. 2, so suffice it to say that every element $\zeta(x_1 \cdots x_m)$ for $x_i \in \mathfrak{g}$ can be exhibited as a sum of elements of the form $a\mathrm{d}b$ for $a, b \in U(\mathfrak{g})$, for example $\zeta(x_1) = \mathrm{d}x_1$, $\zeta(x_1x_2) = \mathrm{d}(x_1x_2) - x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1$ etc. using the Leibniz rule and bimodule relations. If the image of ζ is all of Λ^1 then we can obtain all elements of Ω^1 also in this form since Λ^1 is a left basis. For the extension to products of 1-forms and 'functions', we check $\mathrm{d}(v.x - x.v) = -v\zeta(x) - \zeta(x).v = 0 = \mathrm{d}(v\triangleleft x)$ so that $\mathrm{d}v = 0$ is compatible with the bimodule structure.

If $\zeta(x) = \theta\triangleleft x$ for some $\theta \in \Lambda^1$ and all $x \in \mathfrak{g}$, which is the case for a Lie algebra coboundary, then clearly $\mathrm{d}x = [\theta, x]$ under the bimodule relations, so the calculus is inner. Conversely, suppose that there exists a $\tilde{\theta} = \theta_i \otimes v_i$ (summation intended) where $\theta_i \in U(\mathfrak{g})$ and $v_i \in \Lambda^1$, making the calculus inner. This entails $[\tilde{\theta}, x] = \theta_i x \otimes v_i + \theta_i \otimes v_i\triangleleft x - x\theta_i \otimes v_i = [\theta_i, x] \otimes v_i + \theta_i \otimes v_i\triangleleft x = 1 \otimes \zeta(x)$. Now projecting out $U(\mathfrak{g})^+$ we have $\epsilon(\theta_i)v_i\triangleleft x = \zeta(x)$. Then $\theta = \epsilon(\theta_i)v_i \in \Lambda^1$ also makes the calculus inner and $\zeta(x) = \theta\triangleleft x$ is its coboundary. For higher wedge products we have for example $\{\theta, xv\} = \theta xv + xv\theta = (\theta\triangleleft x)v + x\theta v + xv\theta = \mathrm{d}(xv)$. In this way the calculus becomes inner in all degrees.

Over $\mathbb{C}$, we assume a unitary cocycle in the sense of (1.6) for a map $* : \Lambda^1 \to \Lambda^1$. Then the bimodule relations are preserved by $*$ and if ζ commutes with $*$ then so does d acting on the generators $x \in \mathfrak{g}$. It follows that we have a $*$-calculus on all of $U(\mathfrak{g})$. The extension of $*$ to Λ is as a graded-$*$-algebra (so that $(vw)^* = -w^*v^*$ on $v, w \in \Lambda^1$), which is unusual but does the job. That all translation-invariant calculi come about this way is deferred to Chap. 2, where we explain what this means. □

We see that the data for an inner calculus just amounts to a right representation Λ^1 of $\mathfrak{g}$ together with a cyclic vector $\theta \in \Lambda^1$ in the sense that $\theta\triangleleft U(\mathfrak{g})^+ = \Lambda^1$.

If the element θ is not cyclic or more generally the cocycle is not surjective then we still have a DGA but without the surjectivity assumption of a usual differential calculus. On the other hand, over a field of characteristic zero (so certainly over $\mathbb{C}$) the first Whitehead lemma asserts that if $\mathfrak{g}$ is semisimple then the Lie algebra $\mathrm{H}^1(\mathfrak{g}, \Lambda^1) = 0$ and hence all differential calculi of the type above are necessarily inner in this case. Although this gives calculi on $U(\mathfrak{g})$ in the semisimple case, it does depend on the choice of θ, so is not very canonical. Also, such calculi tend to be highly non-connected, which makes them unsuitable as a quantisation of classical differential geometry. For example, if θ is a highest weight vector then $\mathrm{d}U(n_+) = 0$ by definition, where n_+ denotes the subalgebra associated to the positive roots. On the other hand, the Lie algebra does not have to be semisimple for the calculus to be inner, as our next class of examples shows. These will be based on the notion of a right *pre-Lie algebra*, namely a vector space $\mathfrak{g}$ and a product $\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ obeying

$$(x \circ y) \circ z - (x \circ z) \circ y = x \circ (y \circ z - z \circ y).$$

In this case we obtain a Lie algebra by $[x, y] = x \circ y - y \circ x$ *and* a right $\mathfrak{g}$ module by $\Lambda^1 = \mathfrak{g}$ and right action $\triangleleft = \circ$. If we start with a Lie algebra $\mathfrak{g}$ then it has a pre-Lie structure if we can recover the Lie bracket of $\mathfrak{g}$ in this way. A real form of a pre-Lie algebra means a basis for which the structure constants of $\circ$ are real and implies of course a real form of the associated Lie algebra.

Corollary 1.42 (Majid–Tao) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\Bbbk$ characteristic zero. Connected translation-invariant differential calculi $\Omega^1(U(\mathfrak{g}))$ of classical dimension are classified by pre-Lie structures $\circ$ on $\mathfrak{g}$. Here $\Lambda^1 = \mathfrak{g}$ and*

$$[\mathrm{d}x, y] = \mathrm{d}(x \circ y)$$

and $\mathrm{d}x$ is given by viewing $x \in \mathfrak{g} \subset U(\mathfrak{g})$ in Λ^1. We have a $*$-calculus whenever the pre-Lie structure has a real form with basis $\{e_i\}$, setting $e_i^* = -e_i$.

Proof We will not attempt more than few words about the proof of connectedness, and outline the rest. Firstly, pre-Lie algebras and surjective 1-cocycles with Λ^1 of classical dimension are equivalent data. Given a pre-Lie structure, we set $\zeta = \mathrm{id}$ and $\Lambda^1 = \mathfrak{g}$ with right module action $\circ$. Conversely, given a surjective (hence by dimensions, bijective) map $\zeta : \mathfrak{g} \to \Lambda^1$ forming a 1-cocycle, we define a pre-Lie structure by $x \circ y = \zeta^{-1}(\zeta(x) \triangleleft y)$. The cocycle identity tells us that $x \circ y - y \circ x = [x, y]$ and $\triangleleft$ an action gives us the pre-Lie algebra identity. If $\mathfrak{g}$ has a real form and the cocycle is unitary as in (1.6) then $(x \circ y)^* = -x^* \circ y^*$, from which we conclude that the structure constants of $\circ$ in the $\{e_i\}$ basis are real. So real forms of pre-Lie structures are the same thing as unitary surjective 1-cocycles of classical dimension.

Now, given a pre-Lie structure, the extension of ζ to $U(\mathfrak{g})^+$ is surjective already from its values on $\mathfrak{g}$, so we have a calculus by Theorem 1.41. If the product $\circ$ has a real form with basis $\{e_i\}$, we set $e_i^* = -e_i$. Then $(v \triangleleft x)^* = -v^* \triangleleft x^*$ when $\circ$ is extended complex linearly, so we have a $*$-differential calculus. Proof of connectedness is best done with Hopf algebra methods, but the idea is that d on products can be written in terms of $\zeta : U(\mathfrak{g})^+ \to \Lambda^1$ and a 'shuffle product'. Analysing the content of $\mathrm{d}f = 0$ using this and a filtration of $U(\mathfrak{g})$ into subspaces consisting of all products up to each degree n allows a proof by induction on n. Thus, if $\mathrm{d}x_1 = \zeta(x_1) = 0$ then $x_1 = 0$, and for a formal sum of terms $x_1 \otimes x_2$ (summation understood), if $\mathrm{d}(x_1x_2) = x_1\zeta(x_2) + x_2\zeta(x_1) + \zeta(x_1x_2) = 0$ then part of this $x_2\zeta(x_1) + x_1\zeta(x_2) = 0$ tells us that $x_1 \otimes x_2$ is antisymmetric, hence $x_1x_2 \in U(\mathfrak{g})$ is actually in $\mathfrak{g}$. In this case the other part $\zeta(x_1x_2) = 0$ tells us that $x_1x_2 = 0$. Similarly for higher degree. Conversely, given a connected calculus of classical dimension defined by a cocycle ζ in Theorem 1.41, if $\mathrm{d}x = 0$ for $x \in \mathfrak{g}$ then $x = 0$ since the kernel of d has to be a multiple of 1. Hence ζ must be injective and hence by finite-dimensionality it must be surjective, so we are necessarily in the pre-Lie setting. □

Here is a basic example which will feature in later chapters.

Example 1.43 (Bicrossproduct Model Quantum Spacetime) The solvable Lie algebra with generators x_i, t

$$[x_i, t] = \lambda x_i, \quad i = 1, \cdots, n-1, \quad \lambda \in \mathrm{i}\mathbb{R}$$

has a real form such that $x_i^* = x_i$ and $t^* = t$ and has two standard 1-parameter families of real pre-Lie structures:

(a) $$x_i \circ t = \lambda x_i, \quad t \circ x_i = 0, \quad x_i \circ x_j = 0, \quad t \circ t = -\lambda\alpha t,$$

which gives us the translation-invariant differential $*$-differential calculus

$$[\mathrm{d}x_i, x_j] = [\mathrm{d}t, x_i] = 0, \quad [\mathrm{d}x_i, t] = \lambda \mathrm{d}x_i, \quad [\mathrm{d}t, t] = -\lambda\alpha \mathrm{d}t$$

for α a real parameter. We focus on the case $\alpha = 1$ as the analysis for general α is not much different. Then the partial derivatives in the sense of (1.2) are the usual ones in the spatial direction and a finite difference in the time direction provided we keep our functions normal ordered, in this case meaning keeping all the x_i to the right of all t, i.e., writing $\psi(t, x) = \sum_m t^m \psi_m(x)$,

$$\mathrm{d}\psi = \partial_0\psi \mathrm{d}t + \frac{\partial\psi}{\partial x_i}\mathrm{d}x_i; \quad \partial_0\psi(t) = \frac{\psi(t) - \psi(t-\lambda)}{\lambda}. \tag{1.7}$$

Notice that this becomes the classical calculus in the limit $\lambda \to 0$. If $n = 2$ then there is a unique form of quantum metric

$$g = ax^{-2}\mathrm{d}x \otimes \mathrm{d}x + b(\mathrm{d}x \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x) + cx^2\mathrm{d}t \otimes \mathrm{d}t$$

provided we adjoin x^{-1}. The metric geometrically (after changes of variables) depends only on one free parameter $\bar{\delta} = c/(b^2 - ac)$ and in the classical limit has constant Ricci scalar curvature $S = 2\bar{\delta}$. We will see in §9.5 that if $\bar{\delta} > 0$ then this model quantises the metric of 2-dimensional de Sitter space and if $\bar{\delta} < 0$ of 2-dimensional anti-de Sitter space, i.e., we are forced to one of these geometries just from the choice of differential algebra.

(b) $$x_i \circ t = -\lambda(\beta - 1)x_i, \quad t \circ x_i = -\lambda\beta x_i, \quad x_i \circ x_j = 0, \quad t \circ t = -\lambda\beta t$$

and the translation-invariant $*$-differential calculus

$$[\mathrm{d}x_i, x_j] = 0, \quad [\mathrm{d}x_i, t] = -\lambda(\beta-1)\mathrm{d}x_i, \quad [\mathrm{d}t, x_i] = -\lambda\beta\mathrm{d}x_i, \quad [\mathrm{d}t, t] = -\lambda\beta\mathrm{d}t$$

for β a real parameter. This is inner with $\theta = -\lambda^{-1}\beta^{-1}\mathrm{d}t$. The case $\beta = 1$ is the standard differential structure for the bicrossproduct model spacetime and is a quotient of an $n+1$-dimensional calculus compatible with a Poincaré quantum group symmetry (see §9.2.2). The general case is quite similar to the $\beta = 1$ case, so we focus on the latter. If $\psi(x,t) = \sum_m \psi_m(x)t^m$ then we again have (1.7).

Next, if $n = 2$ then, up to normalisation and a shift in t by a constant, there is a unique form of quantum metric with one nonzero real parameter b, namely

$$g = \pm\mathrm{d}x\otimes\mathrm{d}x + b(v^*\otimes v + \lambda(\mathrm{d}x\otimes v - v^*\otimes\mathrm{d}x)), \quad v = x\mathrm{d}t - t\mathrm{d}x, \quad b \in \mathbb{R},\ b \neq 0,$$

where the leading sign is sign(b) if we want to have a Minkowski signature in the classical limit. The key to deriving the form of the metric is that v, $\mathrm{d}x$ are central in the algebra and can be taken as a basis. Then the coefficients of g in this basis have to be in the centre, which at least among polynomials and other reasonable extensions, means constants. The form of g is then fixed by quantum symmetry noting that $v \wedge \mathrm{d}x = -(\mathrm{d}x) \wedge v$ and $v \wedge v^* = 0$ in the exterior algebra, and can then be cast into the form shown (see also Exercise E1.9 for the general β case). In this form, the 'reality' of the quantum metric requires b real and nondegeneracy requires $b \neq 0$. The inverse metric bimodule inner product is

$$(v^*, \mathrm{d}x) = (\mathrm{d}x, v) = 0, \quad (v^*, v) = \frac{1}{b}, \quad (\mathrm{d}x, \mathrm{d}x) = \frac{1}{1 + b\lambda^2},$$

where we assume $b \neq -1/\lambda^2$. The scalar Ricci curvature in the classical limit is $S = -4b/x^2$, showing a singularity along the t axis at the origin. We will see in §9.4 that this model for $b < 0$ quantises a strong gravitational source at the origin, while for $b > 0$ it quantises an expanding universe with a Ricci singularity at an initial 'big bang'. We are forced to one or other just by the choice of differential algebra. However, $n > 2$ admits no quantum metric at a similar level of analysis.

These two families are not the only pre-Lie structures. For example, one can show for $n = 2$ that there are three other isolated possibilities at an algebraic level which, however, can be thought of as special cases of the above if we allow non-algebraic changes of variables (such as logarithms). In this way we see that the rigid requirements of noncommutative Riemannian geometry dictate relatively few translation-invariant calculi and for each of these a unique form of metric in the classical limit $\lambda \to 0$ (i.e., so as to be quantisable by our chosen differential algebra). The associated 2nd order Laplacians Δ_θ and ${}_\theta\Delta$ in the β-family are not particularly geometrical in terms of the relevant metrics. For the geometric Laplacians, we will need to construct quantum Levi-Civita connections, which we do in Chap. 9. $\diamond$

On the other hand, a simple Lie algebra cannot be a pre-Lie algebra, so this is a no-go theorem for calculi of classical dimensions on enveloping algebras of simple Lie algebras. For these we need to go to higher than classical dimension. For example, we have the following general construction.

Example 1.44 Let $\mathfrak{g}$ be a Lie algebra and V be any nonzero representation space of $\mathfrak{g}$, with $\rho : \mathfrak{g} \to \mathrm{End}(V)$ being the action viewed as a map. Extend this to an algebra map $\rho : U(\mathfrak{g})^+ \to \mathrm{End}(V)$ and let $\Lambda^1 = \rho(U(\mathfrak{g})^+)$ with right action $\omega{\triangleleft}x = \omega\rho(x)$ for all $x \in \mathfrak{g}$, $\omega \in \Lambda^1$. This is automatically a 1-cocycle with $\zeta = \rho$ as $\zeta([x, y]) = \zeta(x)\rho(y) - \zeta(y)\rho(x)$ comes down to ρ being a representation and is inner if the identity map $\mathrm{id} \in \Lambda^1$ as $[\mathrm{id}, x] = \mathrm{id}{\triangleleft}x = \mathrm{id}\rho(x) = \mathrm{d}x$. This is also typically necessary to be inner, e.g. if not all nonzero elements of Λ^1 are zero divisors. The extension to an exterior algebra Ω is by Theorem 1.41. Explicitly, $\omega.x = x.\omega + \omega\rho(x)$ and $\mathrm{d}x = \rho(x)$ for all $x \in \mathfrak{g}$ and $\omega \in \Lambda^1 \subseteq \mathrm{End}(V)$. $\diamond$

If instead we set $\Lambda^1 = \mathrm{End}(V)$ then we will always have an inner DGA but a possibly generalised first-order calculus. Also, the above construction is quite general and does not necessarily give a $*$-DGA or $*$-exterior algebra. We content ourselves with a familiar example where we do have a $*$-exterior algebra as well.

Example 1.45 (Angular Momentum Space or 'fuzzy' $\mathbb{R}^3$) For $A = U(su_2)$, we have basis $\{e_1, e_2, e_3\}$ with $e_i^* = -e_i$ and relations $[e_i, e_j] = \epsilon_{ijk}e_k$. We work with hermitian generators $x_i = 2\lambda e_i$ where $\lambda \in \mathrm{i}\mathbb{R}$ so that $[x_i, x_j] = 2\lambda\epsilon_{ijk}x_k$ as our noncommutative space. The smallest nontrivial irreducible representation is $\rho(x_i) = -\mathrm{i}\lambda\sigma_i$ in terms of Pauli matrices, so that $\mathrm{d}x_i = -\mathrm{i}\lambda\sigma_i = \rho(x_i)$. Then

$$[\mathrm{d}x_i, x_j] = \lambda\epsilon_{ijk}\mathrm{d}x_k - \lambda^2\delta_{ij}\theta, \quad [\theta, x_i] = \mathrm{d}x_i,$$

which we see has an extra dimension θ in the cotangent bundle, making the calculus inner. We see in this example that the commutation relations form a $*$-algebra with ${x_i}^* = x_i$ and $\theta^* = -\theta$. On the other hand, this is *not* usual matrix hermitian. If we write $\Lambda^1 = M_2(\mathbb{C}) = \rho(su_2) \oplus \mathbb{C}.1$ in the Pauli basis with $v = v_i\sigma_i + v_0 1$ (here 1 is the identity matrix) then $v^* = \bar{v}_i\sigma_i - \bar{v}_0 1$. We check,

$$\begin{aligned}(v{\triangleleft}x_j)^* &= (-\mathrm{i}\lambda)(v_i\sigma_i\sigma_j + v_0\sigma_j)^* = (-\mathrm{i}\lambda)(v_i(\epsilon_{ijk}\mathrm{i}\sigma_k + \delta_{ij}1) + v_0\sigma_j)^* \\ &= (-\mathrm{i}\lambda)(-\bar{v}_i(\epsilon_{ijk}\mathrm{i}\sigma_k - \delta_{ij}1) + \bar{v}_0\sigma_j) = -(-\mathrm{i}\lambda)(\bar{v}_i(\epsilon_{ijk}\mathrm{i}\sigma_k + \delta_{ij}1) - \bar{v}_0\sigma_j) \\ &= -(-\mathrm{i}\lambda)(\bar{v}_i\sigma_i\sigma_j - \bar{v}_0\sigma_j) = -v^*{\triangleleft}x_j,\end{aligned}$$

as required. To see that ζ when extended to $U(su_2)^+$ is surjective we have for example $\zeta((x_1)^2) = \zeta(x_1){\triangleleft}x_1 = (-\mathrm{i}\lambda)^2\sigma_1^2$, which is a multiple of the identity.

We also have $\zeta(x_i)^* = \zeta(x_i{}^*)$, so that Theorem 1.41 applies to give a $*$-exterior algebra. We can then compute partial derivatives according to (1.2),

$$\mathrm{d}f = \sum_i (\partial^i f)\mathrm{d}x_i - \frac{1}{2}\lambda^2(\Delta f)\theta,$$

where Δ is the partial derivative in the θ direction. One can show that this obeys $(1 - \frac{\lambda^2}{2}\Delta)^2 = 1 - \lambda^2 \sum_i (\partial^i)^2$ so that formally

$$\Delta = -\frac{2}{\lambda^2}\Big(\sqrt{1 - \lambda^2 \sum_i (\partial^i)^2} - 1\Big)$$

which tends to $\sum_i (\partial^i)^2$ and hence the classical Laplacian as $\lambda \to 0$ and $\partial^i \to \partial/\partial x_i$.

This calculus is manifestly rotationally covariant. There is a unique form of rotationally invariant quantum metric, namely, up to normalisation,

$$g = \sum_i \mathrm{d}x_i \otimes \mathrm{d}x_i + \lambda^2\theta \otimes \theta.$$

We could multiply this by an invertible central element, but at least among polynomials this central element would be a constant. The nonzero bimodule inner products are then $(\mathrm{d}x_i, \mathrm{d}x_j) = \delta_{ij}$, $(\theta, \theta) = \lambda^{-2}$. The associated 'Laplacian' is ${}_\theta\Delta = \Delta$.

This form of metric suggests to think of θ as the time direction and this as a 4D Euclidean metric emerging naturally from the noncommutative geometry of angular momentum space. To realise this, one could extend the $U(su_2)$ algebra by adding a central variable $t = t^*$ with $\theta = -\lambda^{-1}\mathrm{d}t$ and

$$[t, \mathrm{d}t] = \lambda \mathrm{d}t, \quad [x_i, \mathrm{d}t] = [t, \mathrm{d}x_i] = \lambda \mathrm{d}x_i.$$

In that case, if we have a function $\psi(x,t)$ and write $\mathrm{d}\psi = (\tilde{\partial}^t\psi)\mathrm{d}t + \sum_i (\tilde{\partial}^i\psi)\mathrm{d}x_i$ to define the associated partials, then a short calculation using the Leibniz rule on a product of functions of x_i and functions of t gives

$$\tilde{\partial}^t\psi = 0 \quad \Leftrightarrow \quad \partial_0\psi(x,t) = -\frac{1}{2}\lambda\Delta\psi(x, t-\lambda),$$

where ∂_0 is the finite time difference as in Example 1.43. Thus, a version of Schrödinger's equation arises naturally from the quantum differential calculus. $\diamond$

The 4D calculus here is connected and could not be 3D because su_2 is simple. Lower-dimensional calculi based on irreducible representations Λ^1 exist but are not connected. This obstruction to a connected 3D calculus is a kind of 'anomaly' seen above as an origin of time and quantum mechanics, in contrast to Example 1.43 where the rigidity of the differential algebra was a kind of origin of gravity.

The $U(su_2)$ example has an obvious quotient which classically corresponds to focussing on a symplectic leaf in su_2^* and which algebraically means we set the value of the central element $\sum_i x_i^2$ (the quadratic Casimir for su_2) to a constant.

Example 1.46 (Fuzzy Unit Sphere) This is defined as the algebra

$$\mathbb{C}_\lambda[S^2] = U(su_2)\Big/\langle \sum_i x_i^2 - (1-\lambda_P^2)\rangle,$$

where $\lambda = \mathrm{i}\lambda_P$. We can write the relations as

$$x^* = x, \quad [x,z] = \lambda_P z, \quad [z,z^*] = 2\lambda_P(x - \frac{1}{2}(1+\lambda_P)), \quad z^*z = x(1-x),$$

where $x = \frac{x_3+1+\lambda_P}{2}$ and $z = \frac{x_1+\mathrm{i}x_2}{2}$. Note for the usual n-dimensional irreducible representation of su_2 to descend to this algebra we will need

$$\lambda_P = n^{-1}; \quad n \in \mathbb{N}.$$

One can alternatively rescale variables so that λ_P is arbitrary and the radius of the sphere in the new variables is discretised according to the value of n.

However, the above connected 4D calculus on $U(su_2)$ above does *not* usefully descend to this quotient (one obtains the zero calculus). Instead, we start with a natural inner 3D rotationally covariant but non-connected calculus on $U(su_2)$ defined by central 1-forms s_i and $\theta = \frac{1}{2\mathrm{i}\lambda_P}x_i s_i$. From this we have

$$\begin{aligned} \mathrm{d}x_i &= \epsilon_{ijk}x_j s_k, \quad x_i\mathrm{d}x_i = \epsilon_{ijk}x_i x_j s_k = \mathrm{i}\lambda_P\epsilon_{ijk}\epsilon_{ijm}x_m s_k = (2\mathrm{i}\lambda_P)^2\theta, \\ \epsilon_{ijk}x_j\mathrm{d}x_k &= \epsilon_{ijk}x_j\epsilon_{kmn}x_m s_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})x_j x_m s_n \\ &= -x^2 s_i + 2\mathrm{i}\lambda_P(x_i\theta + \mathrm{d}x_i), \end{aligned}$$

from which it is clear that the surjectivity condition for a calculus holds provided we make x^2 invertible. The structure of this calculus is similar to Example 1.20 on $M_2(\mathbb{C})$ but the coefficients θ_i there are now central generators (and now transform under the action of su_2). We define an exterior algebra by

$$\{s_i, s_j\} = 0, \quad \mathrm{d}s_i + \frac{1}{2}\epsilon_{ijk}s_j \wedge s_k = 0,$$

which one can check is compatible with the first-order calculus relations and with $\mathrm{d}^2 = 0$. We also have a $*$-algebra with $x_i^* = x_i$ and $s_i^* = s_i$, giving a $*$-exterior algebra. The calculus is not, however, inner in higher degrees via θ, since

$$\mathrm{d}\theta = \theta \wedge \theta = \frac{1}{4\mathrm{i}\lambda_P}\epsilon_{ijk}x_i s_j \wedge s_k, \quad \{\theta, s_j\} = 0.$$

The natural top form is $\mathrm{Vol} = s_1 \wedge s_2 \wedge s_3$, so that $\theta^3 = \frac{x^2}{(2\mathrm{i}\lambda_P)^2}\mathrm{Vol}$. Moreover, it is clear that $\mathrm{d}x^2 = [\theta, x^2] = 0$, so this 3D $\Omega(U(su_2))$ is not connected. In fact, $\mathrm{H}^0_{\mathrm{dR}}$ here is the centre of $U(su_2)$, i.e., functions of x^2. This calculus descends to a rotationally covariant 3D exterior algebra on $\Omega(\mathbb{C}_\lambda[S^2])$, this time connected as we set the radius to a constant. Anything of the form $g_{ij}s_i \otimes s_j$ with an invertible symmetric matrix of constant coefficients will be a central quantum metric. $\diamond$

1.6.2 Group Algebras

There is an analogous theory for group algebras $\Bbbk G$. Here G is any group and $\Bbbk G$ is a vector space with basis G, and becomes an algebra by extending the group product bilinearly. We define a map ϵ as 1 on basis elements and $\Bbbk G^+$ will denote its kernel, i.e., where the coefficients in our basis add up to zero. Over $\mathbb{C}$ we have a $*$-algebra with $x^* = x^{-1}$ for all $x \in G$ and in this context there is a natural C^*-algebra completion $C^*(G)$ in the case when G is locally compact. When G is nonabelian, this algebra is noncommutative and should not be confused with the commutative algebra of functions on G, which is covered in the next section. Rather, it is an example of a noncommutative geometry and features, for example, in the famous Baum–Connes conjecture. In our case, we are interested in noncommutative differentials on $\Bbbk G$, and this is a Hopf algebra which gives a precise meaning to 'translation invariance' explained in Chap. 2. The reader can think of this as a discrete version of the construction for enveloping algebras in the preceding section.

We recall that a group 1-cocycle $\zeta \in Z^1(G, \Lambda^1)$ with values in a right G-module Λ^1 means a map $\zeta : G \to \Lambda^1$ such that

$$\zeta(xy) = \zeta(x){\triangleleft}y + \zeta(y)$$

for all $x, y \in G$. One can show from the cocycle condition that $\zeta(x^{-1}) = -\zeta(x){\triangleleft}x^{-1}$ for all $x \in G$ and $\zeta(1) = 0$. Over $\mathbb{C}$ we say that a group 1-cocycle is *unitary* if there is a $*$-involution on Λ^1 such that $(v{\triangleleft}x)^* = v^*{\triangleleft}x$ for all $v \in \Lambda^1$ and $x \in G$ and $\zeta(x)^* = -\zeta(x)$ for all $x \in G$. We define the augmentation ideal $\Bbbk G^+$ to be the elements of $\Bbbk G$ where the coefficients in our basis add to zero.

Theorem 1.47 *Translation-invariant $\Omega^1(\Bbbk G)$ on $A = \Bbbk G$ are classified by cocycles $\zeta \in Z^1(G, \Lambda^1)$ such that the linear extension to $\Bbbk G^+ \to \Lambda^1$ is surjective. The calculus is inner if and only if the cocycle is exact, and extends to $\Omega(\Bbbk G) = \Bbbk G.\Lambda$ as a free module over Λ, the Grassmann algebra on Λ^1, with* d *vanishing on*

Λ. The calculus is connected if and only if $\zeta(x) \neq 0$ for all $x \in G \setminus \{e\}$, and in this case

$$\mathrm{H}_{\mathrm{dR}}(\Bbbk G) = \Lambda.$$

Over $\mathbb{C}$, we have a $*$-exterior algebra if the cocycle is unitary.

Proof Here $\Omega = \Bbbk G \otimes \Lambda$ has left action the product of G and right action the tensor product: $(x \otimes v).y = xy \otimes v{\triangleleft}y$ for all $x, y \in G$. The exterior derivative is $\mathrm{d}x = x \otimes \zeta(x)$. We say this more compactly as $\Omega = \Bbbk G.\Lambda$ and

$$v.x = x(v{\triangleleft}x), \quad \mathrm{d}x = x\zeta(x), \quad \mathrm{d}v = 0$$

for all $x \in G$, $v \in \Lambda^1$. Then $\mathrm{d}(xy) = (xy)\zeta(xy) = (xy)(\zeta(x){\triangleleft}y) + (xy)\zeta(y) = x\zeta(x)y + x(y\zeta(y)) = (\mathrm{d}x)y + x\mathrm{d}y$ for all $x, y \in G \subset \Bbbk G$. By surjectivity, every element of Λ^1 is a sum of terms $\zeta(x) = x^{-1}\mathrm{d}x$. The extension to Λ is straightforward.

For connectedness, if $\zeta(x) = 0$ for some $x \neq e$ then $\mathrm{d}x = x\zeta(x) = 0$ and the calculus is not connected. Conversely, suppose the stated condition does hold and write a form as $\alpha = \sum_{x \in G\setminus\{e\}} x\alpha_x + \beta$ for some $\alpha_x, \beta \in \Lambda$ of the same degree. If $\mathrm{d}\alpha = \sum_{x\in G\setminus\{e\}} x\zeta(x) \wedge \alpha_x = 0$ then each $\zeta(x) \wedge \alpha_x$ vanishes. In degree zero we deduce that $\alpha_x = 0$, so α is a multiple of 1, hence the calculus is connected. For higher degree, given the Grassmann form of Λ, we deduce that $\alpha_x = \zeta(x) \wedge \beta_x$ for some $\beta_x \in \Lambda$ of one degree lower. Then $\mathrm{d}(x\beta_x) = x\zeta(x) \wedge \beta_x = x\alpha_x$, so all of these terms in α are exact. Meanwhile, we cannot have $\beta = \mathrm{d}\gamma$ again by the form of d (the component of γ in Λ does not contribute and any other component $x\gamma_x$ has differential in $x\Lambda$). So each degree of cohomology is spanned by that degree of Λ.

In the unitary case we have $x^{-1}v^* = (v^*{\triangleleft}x)x^{-1} = (x(v{\triangleleft}x))^* = (v.x)^*$ and $(\mathrm{d}x)^* = \zeta(x)^*x^{-1} = -\zeta(x)x^{-1} = -x^{-1}(\zeta(x){\triangleleft}x^{-1}) = x^{-1}\zeta(x^{-1}) = \mathrm{d}(x^{-1})$ for all $x \in G$. So we have a $*$-exterior algebra when we also extend the $*$ to Λ. □

Thus for an inner generalised first-order calculus we need only a right G-module Λ^1 and a 'cyclic' vector $\theta \in \Lambda^1$ in the sense that $\theta{\triangleleft}\Bbbk G^+ = \Lambda^1$. Then $\zeta(x) = \theta{\triangleleft}x - \theta$ is a 1-cocycle. Note that over $\mathbb{C}$, if Λ^1 is irreducible and nontrivial then any $\theta \in \Lambda^1$ is surjective as $\theta{\triangleleft}\mathbb{C}G^+$ would be a sub-module and cannot be zero, since that would mean θ would be G-invariant and generate a trivial 1-dimensional submodule. Changing the normalisation of θ does not change the calculus up to isomorphism, so we have a $\mathbb{CP}^{\dim \Lambda^1 - 1}$ moduli of calculi for each irreducible representation Λ^1. If θ is not cyclic or more generally the cocycle does not obey the surjectivity condition then we still have a DGA and a first-order generalised calculus.

Example 1.48 The order 6 group S_3 of permutations on 3 elements $\{1, 2, 3\}$ has generators $u = (12)$, $v = (23)$ with relations $u^2 = v^2 = e$ (the group identity) and $uvu = vuv$. The group algebra $A = \mathbb{C}S_3$ similarly has generators u, v and relations $u^2 = v^2 = 1$, $uvu = vuv$. The group has two nontrivial irreducible representations.

(1) The 1-dimensional or sign representation has $\rho(u) = \rho(v) = -1$ and gives us the 1-dimensional calculus with basis θ, say, and

$$\zeta(u) = u^{-1}\mathrm{d}u = \zeta(v) = v^{-1}\mathrm{d}v = -2\theta, \quad \theta u = -u\theta, \quad \theta v = -v\theta.$$

The calculus is inner via θ and a $*$-exterior algebra with $\theta^* = -\theta$.

(2) The other nontrivial irreducible representation is 2-dimensional and we can take it with real symmetric matrix elements

$$\rho(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(v) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Here $\Lambda^1 = \mathbb{C}^2$ has basis elements $\{f_\alpha\}$ so that $f_\alpha \triangleleft u = \rho(u)_{\alpha\beta} f_\beta$ etc. The commutation relations are then read off from the matrix entries as

$$f_1 u = u f_1, \quad f_2 = -u f_2, \quad f_1 v = \frac{1}{2} v(-f_1 + \sqrt{3} f_2), \quad f_2 v = \frac{1}{2} v(\sqrt{3} f_1 + f_2).$$

A patch in the moduli space provided by cyclic vectors in the form $\theta = \lambda f_1 + f_2$ up to overall normalisation, giving a 1-parameter family of calculi

$$\zeta(u) = u^{-1}\mathrm{d}u = -2 f_2, \quad \zeta(v) = v^{-1}\mathrm{d}v = \frac{1}{2}(1 - \sqrt{3}\lambda)(\sqrt{3} f_1 - f_2).$$

We can make this a $*$-exterior algebra with the f_i anti-hermitian and λ real, where the matrix entries for $\triangleleft$ are all real, so that the cocycle is unitary.

Finally, because all the matrices $\rho(x)$ in the representation are orthogonal, it follows that $g = \sum_i f_i \otimes f_i \in \Lambda^1 \otimes \Lambda^1$ is a central quantum metric. $\diamond$

The first calculus is not connected, while the second is connected except for specific values of λ. The following construction is more canonical and is, moreover, covariant under group conjugation.

Example 1.49 Let G be any group and V any representation, with structure map $\rho : G \to \mathrm{End}(V)$ extended linearly to $\Bbbk G$. Let $\Lambda^1 = \rho(\Bbbk G^+)$ be a right-module by right multiplication via ρ, i.e., $\omega \triangleleft x = \omega\rho(x)$ for all $\omega \in \Lambda^1$ and $x \in G$. Then $\zeta(x) = \rho(x) - \mathrm{id}$ is a cocycle and we have a calculus. This is inner if $\mathrm{id} \in \Lambda^1$ (and this is typically if and only if, for example if not every nonzero element of Λ^1 is a zero divisor). Explicitly, the bimodule relations and exterior derivative are $\omega.x = x\omega\rho(x)$ and $\mathrm{d}x = x(\rho(x) - \mathrm{id})$ for all $\omega \in \Lambda^1$ and $x \in G$. If we write $e_x = x^{-1}\mathrm{d}x = \zeta(x)$ then the commutation relations are $e_x.y = y.(e_{xy} - e_y)$ for all $x, y \in G$. Note that the $\{e_x\}$ are not typically linearly independent and that $e_e = 0$. From the relations, it is clear that the calculus is covariant under conjugation by G on the basis elements of $\Bbbk G$, and on the labels of the $\{e_x\}$ (or the action on e_x is by conjugation of matrices with G acting via ρ). Over $\mathbb{C}$, if the linear relations between the $\{e_x\}$ have real coefficients then we have a $*$-exterior algebra with $e_x^* = -e_x$, for all $x \in G$. Note that this $*$-structure is not necessarily matrix hermitian conjugation. The calculus is connected precisely when ρ is faithful. $\diamond$

If instead we let $\Lambda^1 = \mathrm{End}(V)$ then the calculus is always inner but may be a generalised calculus or DGA. Here is a concrete example of the above construction.

Example 1.50 We take the same algebra $\mathbb{C}S_3$ and 2-dimensional representation ρ as in Example 1.48, but now apply the preceding construction. We choose

$$e_u = u^{-1}\mathrm{d}u = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad e_v = v^{-1}\mathrm{d}v = \frac{1}{2}\begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$e_{uv} = (uv)^{-1}\mathrm{d}(uv) = \frac{1}{2}\begin{pmatrix} -3 & \sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix}, \quad e_{vu} = (vu)^{-1}\mathrm{d}(vu) = \frac{1}{2}\begin{pmatrix} -3 & -\sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}$$

as a basis of this 4D calculus, which is inner as $e_u + e_v + e_w = e_{uv} + e_{vu} = -3\theta$ where $w = vuv = uvu$ is defined similarly and $\theta = \mathrm{id}$. One has the relations

$$\begin{aligned} &e_u u = -u e_u, && e_u v = v(e_{uv} - e_v), && e_v u = u(e_{vu} - e_u), \\ &e_v v = -v e_v, && e_{uv} u = u(e_w - e_u), && e_{uv} v = v(e_u - e_v), \\ &e_{vu} u = u(e_v - e_u), && e_{vu} v = v(e_w - e_v). \end{aligned}$$

We also set $e_u^* = -e_u$, etc. and have a $*$-exterior algebra since the linear relations among these vectors have real coefficients. Note that this is in spite of some of the matrices not being antihermitian in the usual sense. Finally,

$$g_{u,v} := (e_{uv} - e_u) \otimes (e_{uv} - e_u) + (e_{vu} - e_v) \otimes (e_{vu} - e_v) + e_{uvu} \otimes e_{uvu} \in \Lambda^1 \otimes \Lambda^1$$

is central, quantum symmetric and obeys the reality condition. The element $g = \frac{1}{3}(g_{u,v} + g_{v,w} + g_{w,u})$ also has these features and comes out as

$$g = e_u \otimes e_u + e_v \otimes e_v + e_w \otimes e_w + e_{uv} \otimes e_{uv} + e_{vu} \otimes e_{vu} - 6\theta \otimes \theta,$$

which is nondegenerate and hence a quantum metric. It is invariant under the adjoint action of the group. The bimodule inner product in basis order $\{e_u, e_v, e_{uv}, e_{vu}\}$ is

$$(e_\cdot, e_\cdot) = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} & 1 & 1 \\ \frac{1}{3} & \frac{4}{3} & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

and from this we find that ${}_\theta\Delta$ is diagonal on the basis of $\mathbb{C}S_3$ with u, v, w having eigenvalue 4/3 and uv, vu eigenvalue 2. The partial derivatives are already diagonal,

$$\partial^u\{e, u, v, w, uv, vu\} = \{0, u, 0, -w, 0, 0\},$$

$$\partial^v\{e, u, v, w, uv, vu\} = \{0, 0, v, -w, 0, 0\},$$

$$\partial^{uv}\{e, u, v, w, uv, vu\} = \{0, 0, 0, w, uv, 0\},$$
$$\partial^{vu}\{e, u, v, uv, vu\} = \{0, 0, 0, w, 0, vu\},$$

and writing these coefficients as $\partial^i a = d^i_a a$ for some 4×6 matrix (d^i_a), we evaluate ${}_\theta\Delta a = -2ad^i_a(e_i, \theta) = \frac{2}{3}ad^i_a(e_i, e_{uv} + e_{vu})$. This ${}_\theta\Delta$ commutes with $*$ and so coincides with Δ_θ, and can also be written as $(e_i, e_j)\partial^i\partial^j$. Notice that the eigenspaces are the spans of conjugacy classes.

The calculus is connected as the representation used is faithful. The cohomology from Theorem 1.47 is then $\mathrm{H}_{\mathrm{dR}}(\mathbb{C}S_3) = \Lambda$, the exterior algebra on the $\{e_i\}$. ◇

Here S_3 acts on $\mathbb{C}S_3$ by permutation of u, v, w and the calculus is covariant, just as the calculus on $U(su_2)$ was rotationally covariant. Similarly, the analogue of the Casimirs in $U(\mathfrak{g})$ would now be elements $\theta_{\mathcal{C}} = \sum_{a\in\mathcal{C}} a$, where $\mathcal{C} \subseteq G \setminus \{e\}$ is a nontrivial conjugacy class. These generate the centre (the class functions, those constant on conjugacy classes, comprise the centre in the group convolution algebra).

Example 1.51 For S_3 we have two nontrivial conjugacy classes and these give natural candidates for 'discrete spheres' by imposing either the relation $uv + vu = \mu_1$ or the relation $u+v+w = \mu_2$, where μ_i are constants. In the first case, squaring $uv+vu$ shows that $\mu_1 = 2, -1$, while in the second case squaring $u+v+w$ shows that $uv + vu = \frac{\mu_2^2}{3} - 1$, so this is a quotient of the first case with $\mu_1 = \frac{\mu_2^2}{3} - 1$ and requires that $\mu_2 = 0, \pm 3$, with $\mu_2 = \pm 3$ implying that $\mu_1 = 2$. We also have $u+v+w = (1+uv+vu)u = (1+uv+vu)v = (1+uv+vu)w$ so that $\mu_1 = -1$ is equivalent to $\mu_2 = 0$. Altogether we obtain three distinct nonzero quotients:

$$\text{(i)}\quad \mu_1 = -1,\ \mu_2 = 0: \quad \mathbb{C}S_3\big/\langle uv + vu + 1\rangle = \mathbb{C}S_3\big/\langle u + v + uvu\rangle \cong M_2(\mathbb{C}),$$
$$\text{(ii)}\quad \mu_1 = 2: \quad \mathbb{C}S_3\big/\langle uv + vu - 2\rangle \cong \mathbb{C}(\mathbb{Z}_2),$$
$$\text{(iii)}\quad \mu_1 = 2,\ \mu_2 = \pm 3: \quad \mathbb{C}S_3\big/\langle u + v + uvu \mp 3\rangle \cong \mathbb{C},$$

where in (ii) we deduce that $u = v = w$ as the quotient of $\mathbb{C}S_3$, leaving one generator u with $u^2 = 1$, and in (iii) we further deduce that $u = \pm 1$. The commutative (iii) and (ii) are functions on 1 and 2 points, respectively.

We now focus on A, the noncommutative 'discrete sphere' (i) with one (but not the only) such isomorphism being our previous matrix representation ρ. The 4D calculus $\Omega(\mathbb{C}S_3)$ in Example 1.50 descends in this case to a 2D calculus since applying d to $u + v + w$ and to $uv + vu$ tells us that

$$ue_u + ve_v + we_w = 0, \quad uve_{uv} + vue_{vu} = 0,$$

which combined with $e_w = e_{uv} + e_{vu} - e_u - e_v$ tells us that $e_{uv} = e_v - uve_u$, $e_{vu} = e_u - vue_v$ and $e_w = -uve_u - vue_v$. Thus, we take e_u, e_v as generators and have relations and quantum metric descending from $\mathbb{C}S_3$ to

$$e_u u = -ue_u, \quad e_v v = -ve_v, \quad [e_u, v] = ue_u, \quad [e_v, u] = ve_v, \quad \Omega^2 = 0,$$

$$\theta = -\frac{1}{3}((1-uv)e_u+(1-vu)e_v), \quad g = 2e_u\otimes e_u+2e_v\otimes e_v+vue_u\otimes e_v+uve_v\otimes e_u.$$

There remains a $*$-operation whereby $e_u^* = -e_u$, etc. and an action of S_3 inherited from conjugation in $\mathbb{C}S_3$. The elements $(u + 2v)e_u$, $(2u + v)e_v$ are a central basis of Ω^1 and the four terms in g are separately central, giving more general metrics.

We can alternatively use the maximal prolongation $\Omega_{\max}$ of Ω^1, which has

$$\mathrm{d}e_u = -e_u^2, \quad \mathrm{d}e_v = -e_v^2, \quad e_v e_u = uve_u e_v,$$

with the above central basis 1-forms commuting in the wedge product. If we quotient further by $e_u^2 = e_v^2 = 0$ then a quantum-symmetric metric has the form

$$g = ae_u \otimes e_u + b(vue_u \otimes e_v - uve_v \otimes e_u) + ce_v \otimes e_v$$

for constants a, b, c. These calculations are equivalent to Example 1.37, Proposition 1.38 and Example 1.39 on $M_2(\mathbb{C})$ according to the $*$-diffeomorphism

$$\phi : A \to M_2(\mathbb{C}), \quad \phi(u) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \phi(v) = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}; \quad q = e^{\frac{\pi \mathrm{i}}{6}},$$

$$\phi((u + 2v)e_u) = -\sqrt{3}(s + t), \quad \phi((2u + v)e_v) = (q^5 - 2\mathrm{i})s + (q^5 + \mathrm{i})t,$$

which one can check forms a commuting square in the sense of (1.1). The linear combinations here are chosen so that θ maps to $E_{12} \oplus E_{21}$, putting us in the standard point $(1, 0, 0)$ of our $\mathbb{CP}^2$ moduli space for 2D calculi on $M_2(\mathbb{C})$ in Example 1.8. Note that if we had used our original identification ρ with

$$\rho((u + 2v)e_u) = t, \quad \rho((2u + v)e_v) = -\frac{3}{2}s + \frac{1}{2}t,$$

where $s = \mathrm{id} \oplus 0$, $t = 0 \oplus \mathrm{id}$, then this would similarly be diffeomorphic to a 2D calculus on $M_2(\mathbb{C})$ in Example 1.8, now at the point $(0, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$ in the $\mathbb{CP}^2$ moduli space. As A was already diffeomorphic to $(1, 0, 0)$, it means that these two points in the moduli space are diffeomorphic in the weaker sense of (1.1) but not isomorphic in the 'commuting triangle' sense for a fixed copy of $M_2(\mathbb{C})$. ◇

1.7 The Exterior Algebra of a Finite Group

We have already seen in §1.4 that a calculus on a discrete set means a directed graph. In the case of a discrete group a *Cayley graph* is a special type of graph in which the arrows have the form $x \to xa$, where $x \in G$ and $a \in \mathcal{C}$ for a fixed subset $\mathcal{C} \subseteq G \setminus \{e\}$ of the group not containing the identity. The arrows out of any point are labelled by elements of $\mathcal{C}$ and the graph is left regular. Hence by our results in §1.4, the first-order calculus at least when G is finite is parallelisable and of the form $\Omega^1 = \Bbbk(G).\Lambda^1$ as a free module over a vector space Λ^1 with basis

$$e_a = \sum_{x \in G} \omega_{x \to xa},$$

where $\omega_{x\to xa} = \delta_x \mathrm{d}\delta_{xa}$. Also, the group itself acts on the algebra of functions from the left and the right, which we can take on delta-functions as $x{\triangleright}\delta_y = \delta_{xy}$ and $\delta_x{\triangleleft}y = \delta_{xy}$. Hopf algebra covariance of the calculus as alluded to earlier comes down for $\Bbbk(G)$ to these actions extending to Ω^1 in a way that commutes with d.

Proposition 1.52 *Let G be a finite group. Left-covariant calculi $\Omega^1(G)$ on $\Bbbk(G)$ correspond to Cayley directed graphs based on subsets $\mathcal{C} \subseteq G \setminus \{e\}$, with Λ^1 the space of left-invariant 1-forms, and relations and exterior derivative*

$$e_a.f = R_a(f)e_a, \quad \mathrm{d}f = \sum_{a \in \mathcal{C}} (R_a(f) - f)e_a$$

for $f \in \Bbbk(G)$. The calculus is inner by $\theta = \sum_a e_a$, connected if and only if $\mathcal{C}$ is a generating set and right-covariant (hence 'bicovariant') if and only if $\mathcal{C}$ is stable under conjugation. Over $\mathbb{C}$, $e_a^* = -e_{a^{-1}}$ provides a $*$-calculus if $\mathcal{C}$ has inverses.

Proof Here $R_a(f) = f((\ )a)$ denotes right-translation on a general function. From the form of the e_a it is clear that the right and left bimodule products are related by R_a as shown. Clearly the left group action $\triangleright$ does not change $\sum_x \omega_{x\to xa}$ so these basis elements are left-invariant. If $\mathcal{C}$ is Ad-stable then the right action on delta-functions implies $e_a{\triangleleft}x = e_{x^{-1}ax}$. This is then a well-defined action and one can check that it commutes with d. That every left-invariant or bicovariant calculus is given by this construction is more easily done using methods of Chap. 2 and deferred to there. Finally, the $*$-structure on finite set calculi is in §1.4 and requires the graph to be symmetric. This is our case when $\mathcal{C}$ is stable under group inversion. Here $e_a^* = -\sum_x \omega_{xa\to x} = -e_{a^{-1}}$ after a change of variables. □

Using these concepts, we can ask when a finite group homomorphism $\phi : G \to H$, which determines an algebra map, also denoted by $\phi : \Bbbk(H) \to \Bbbk(G)$ and given by

$$\phi(\delta_h) = \sum_{g \in G : \phi(g) = h} \delta_g,$$

is differentiable. Suppose that G, H have calculi determined by subsets $\mathcal{C}_G \subseteq G \setminus \{e\}$ and $\mathcal{C}_H \subseteq H \setminus \{e\}$. By the comments on maps between directed graphs after Proposition 1.24, ϕ is differentiable if and only if $\phi(\mathcal{C}_G) \subseteq \mathcal{C}_H \cup \{e\}$, and then

$$\phi_*(e_a) = \sum_{b\in\mathcal{C}_G:\phi(b)=a} e_b.$$

As an example, $\mathcal{C} = G \setminus \{e\}$ gives us the universal calculus, which is always bicovariant on a group (or Hopf algebra). If G is infinite then we still have an algebra $\Bbbk(G)$ and, given a finite subset $\mathcal{C} \subseteq G \setminus \{e\}$, we define Λ^1 as having basis $\{e_a\}$ with the commutation relations and d as stated in the proposition. We would typically also be interested in analytic versions (such as smooth functions or continuous functions with compact support) and extending d to them. We now turn to the exterior algebra.

1.7.1 Left Covariant Exterior Algebras

We start with the left-covariant maximal prolongation.

Proposition 1.53 *The maximum prolongation $\Omega_{\max}$ of a left-covariant first-order calculus on a finite group G, given by a subset $\mathcal{C} \subseteq G\setminus\{e\}$, is generated by functions on G and $\{e_a\}$ with relations as for Ω^1 and the quadratic relations*

$$\sum_{a,b\in\mathcal{C}:\,ab=y} e_a \wedge e_b = 0$$

for all $y \notin \mathcal{C}$, $y \neq e$. The exterior derivative on degree 1 is given by

$$\mathrm{d}e_c = \theta \wedge e_c + e_c \wedge \theta - \sum_{a,b\in\mathcal{C}:\,ab=c} e_a \wedge e_b$$

and gives us an exterior algebra which, when $\mathcal{C}$ is stable under inversion, is a $*$-exterior algebra with $e_a^* = -e_{a^{-1}}$.

Proof For $\Omega_{\max}$ we apply the same ideas as for the maximal prolongation but assume that $\wedge$, d are left-covariant in all degrees, which potentially imposes other relations. Then calculating $\mathrm{d}^2\delta_x = 0$ gives the equation

$$0 = \sum_{a,b\in\mathcal{C}} (\delta_{xb^{-1}a^{-1}} - \delta_{xb^{-1}} - \delta_{xa^{-1}} - \delta_x)e_a \wedge e_b + \sum_{b\in\mathcal{C}}(\delta_{xb^{-1}} - \delta_x)\mathrm{d}e_b.$$

Next we use the fact that $\Omega^2_{\max}$ is freely generated by its left-invariant subspace. This is an instance of something we will see formally in Chap. 2 (the Hopf module lemma), but we have already seen the same idea in Proposition 1.52 for the construction of a left-covariant Ω^1. In this case we can separate this into a sum over δ_y times left-invariant forms, and then put each equal to zero, giving the stated quadratic relation among left-invariant forms for all $y \notin \mathcal{C}$, $y \neq e$, the stated equation for $\mathrm{d}e_c$, and $\sum_{a,b\in\mathcal{C}:\,ab=e} e_a \wedge e_b + \theta \wedge \theta - \mathrm{d}\theta = 0$, which holds automatically. Further relations of $\Omega^2_{\max}$ potentially come from differentiating $\delta_y \mathrm{d}\delta_x$ and the relation $e_c\delta_x = \delta_{xc^{-1}} e_c$, but are again implied by the relations already found. $\square$

This can also be obtained as an example of Proposition 1.40. Typically the maximal prolongation is too big to be interesting but has a natural quotient.

Corollary 1.54 *Every left-covariant calculus Ω^1 on a finite group G has a natural left-covariant exterior algebra $\Omega_L(G)$ generated by $\Bbbk(G)$ and an algebra Λ_L of left-invariant forms where the latter is generated by e_a with the quadratic relations*

$$\sum_{a,b\in\mathcal{C}:\,ab=z} e_a \wedge e_b = 0$$

for all $z \in G \setminus \{e\}$. This is inner with the same θ as before. Over $\mathcal{C}$, we have a $*$-exterior algebra if $\mathcal{C}$ contains inverses, then $e_a^* = -e_{a^{-1}}$.

Proof We construct $\Omega_L(G)$ formally by quotienting $\Omega_{\max}$ by the relations $\mathrm{d}e_c = \{\theta, e_c\}$ for all $c \in \mathcal{C}$, which ensures that the calculus is inner with the same θ as before. This is equivalent to adding the further relations $\sum_{a,b\in\mathcal{C}:\,ab=c} e_a \wedge e_b$ for all $c \in \mathcal{C}$ on top of those already in $\Omega_{\max}$ above. $\square$

These relations are still rather weak. In particular, the possibly nonsymmetric Euclidean metric on any symmetric graph in §1.4 is $g = \sum_{a\in\mathcal{C}} e_a \otimes e_{a^{-1}}$ and hence has $\wedge(g) = \sum e_a \wedge e_{a^{-1}} = \theta \wedge \theta$, which we can set to zero as an additional relation, namely the same form as the $\Omega_L(G)$ relation but now including $z = e$.

Example 1.55 Take $G = S_3$ and $\mathcal{C} = \{u, uv\}$ as a non-Ad stable generating set (using the notation of Example 1.48). The Cayley graph is

$$\begin{tikzcd}
u \arrow[rr, bend left, "uv"] \arrow[d, bend left, "u"] & uvu \arrow[l, "uv"'] \arrow[d, bend left, "u"] & v \arrow[l, "uv"'] \arrow[d, bend left, "u"] \\
e \arrow[u, bend left, "u"] \arrow[r, "uv"] & uv \arrow[r, "uv"] \arrow[u, bend left, "u"] & vu \arrow[ll, bend left, "uv"] \arrow[u, bend left, "u"]
\end{tikzcd}$$

and $\Omega_L(G)$ has the quadratic relations $e_u \wedge e_{uv} = e_{uv} \wedge e_{uv} = e_{uv} \wedge e_u = 0$. $\diamond$

1.7.2 Bicovariant Exterior Algebras

If G is a finite group and $\mathcal{C}$ is Ad-stable, so we have an inner bicovariant $\Omega^1(G)$ by Proposition 1.52, then we will see that $\Omega_L(G)$ has a canonical quotient that includes $\theta\wedge\theta=0$ and potentially other relations. To define this note that Λ^1 is a 'G-crossed module' in the sense that it has a right G-action $\triangleleft$ and a G-grading $|\ |$ such that

$$|v\triangleleft x| = x^{-1}|v|x$$

for all $v\in\Lambda^1$ of homogeneous grade and all $x\in G$. The notion here goes back to Whitehead and arises in homotopy theory. In our case $e_a\triangleleft x = e_{x^{-1}ax}$ and $|e_a|=a$, so this is clear. The general theory of crossed modules implies a canonical operator

$$\Psi:\Lambda^1\otimes\Lambda^1\to\Lambda^1\otimes\Lambda^1,\quad \Psi(e_a\otimes e_b)=e_{aba^{-1}}\otimes e_a \tag{1.8}$$

obeying the braid relations. We will explain this in Chap. 2, all we need for the moment is that this is some kind of natural 'generalised flip'.

Theorem 1.56 *A bicovariant calculus $\Omega^1(G)$ defined by $\mathcal{C}$ extends canonically to an inner exterior algebra $\Omega(G)=\Bbbk(G).\Lambda$ as a free module over an algebra Λ generated by the e_a modulo relations which in degree 2 are defined as setting to zero the kernel of $\Psi-\mathrm{id}$. Over $\mathbb{C}$, if $\mathcal{C}$ has group inverses, then we have a $*$-exterior algebra with $e_a^*=-e_{a^{-1}}$ for all $a\in\mathcal{C}$.*

Proof We only cover here the quadratic exterior algebra $\Omega_{\rm quad}(G)$, where we just impose the stated relations in degree 2, so Λ is a quadratic algebra on Λ^1. However, in many cases these are all the relations (the potentially higher degree relations of $\Omega(G)$ will be covered in Chap. 2). The bimodule relations $e_a\otimes e_b f = R_aR_b(f)e_a\otimes e_b = R_{ab}(f)e_a\otimes e_b$ specify the right bimodule product in terms of the left product on the tensor algebra in degree 2. These bimodule relations descend to Λ since $\Psi(\sum_{a,b}c_{a,b}e_a\otimes e_b)=\sum_{a,b}c_{a,b}e_{aba^{-1}}\otimes e_a=\sum_{a'b'}c_{b',b'^{-1}a'b'}e_{a'}\otimes e_{b'}$ requires $c_{a,b}=c_{b,b^{-1}ab}$ for the coefficients for such an element to be in the kernel of $\Psi-\mathrm{id}$, in which case $\Psi(\sum_{a,b}R_{ab}(f)c_{a,b}e_a\otimes e_b)=\sum_{a,b}R_{ab}(f)c_{a,b}e_{aba^{-1}}\otimes e_a=\sum_{a,b}R_{ab}(f)c_{b,b^{-1}ab}e_{aba^{-1}}\otimes e_a=\sum_{a',b'}R_{a'b'}(f)c_{a',b'}e_{a'}\otimes e_{b'}$, so this element is also in the kernel. Note that $a'b'=ab$ for the change of variables used and that $\sum_{a,b}e_a\otimes e_b$ is Ψ-invariant and hence $\theta^2=0$ (equally well, $\sum_{a,b,ab=y}e_a\otimes e_b$ for any fixed $y\in G$ is Ψ-invariant, so $\Omega(G)$ is a quotient of $\Omega_L(G)$). Setting $\mathrm{d}=[\theta,\ \}$ as the graded commutator, this obeys the graded-Leibniz rule and $\mathrm{d}^2\omega=\mathrm{d}(\theta\omega-(-1)^{|\omega|}\omega\theta)=\theta^2\omega-(-1)^{|\omega|}\theta\omega\theta-(-1)^{|\omega|+1}\theta\omega\theta-\omega\theta^2=0$ as $\theta^2=0$. We also see that $\mathrm{d}\theta=0$. Over $\mathbb{C}$ and with $\mathcal{C}$ stable under group inversion, if $\sum_{a,b}c_{a,b}e_a\otimes e_b$ is Ψ-invariant then $\Psi(-\sum_{a,b}\overline{c_{a,b}}e_{b^{-1}}\otimes e_{a^{-1}})=-\sum_{a,b}\overline{c_{a,b}}e_{b^{-1}a^{-1}b}\otimes e_{b^{-1}}=-\sum_{a,b}\overline{c_{b,b^{-1}ab}}e_{b^{-1}a^{-1}b}\otimes e_{b^{-1}}=-\sum_{a',b'}\overline{c_{a',b'}}e_{b'^{-1}}\otimes e_{a'^{-1}}$. Hence the extension of $*$ according to $(e_ae_b)^*=-e_b^*e_a^*$ is well defined. Since $\theta^*=-\theta$, we have d commuting with $*$ in all degrees. $\square$

The quadratic relations can be computed directly or by the following lemma.

Lemma 1.57 *For each $x \in G$, consider the set $\mathcal{C}_x = \mathcal{C} \cap x\mathcal{C}^{-1}$ and the automorphism $\sigma : \mathcal{C}_x \to \mathcal{C}_x$ given by $\sigma(a) = xa^{-1}$. Let $\{\lambda^{(x)\alpha}\}$ be a basis of the subspace of invariant elements of $\Bbbk\mathcal{C}_x$ under σ, and write $\lambda^{(x)\alpha} = \sum_{a\in\mathcal{C}_x} \lambda_a^{(x)\alpha} e_a$. Then the full set of quadratic relations of Λ are, for all $x \in G$,*

$$\sum_{a,b\in\mathcal{C},\, ab=x} \lambda_a^{(x)\alpha} e_a \wedge e_b = 0.$$

Proof σ corresponds to the braiding Ψ under the decomposition $\Bbbk\mathcal{C} \otimes \Bbbk\mathcal{C} = \oplus_x V_x$, where $V_x = \mathrm{span}_{\Bbbk}\{e_a \otimes e_{a^{-1}x} \mid a \in \mathcal{C}_x\}$ when we identify this with $\Bbbk\mathcal{C}_x$ by projection to the first factor. If $\lambda = \sum_a \lambda_a e_a \in \Bbbk\mathcal{C}_x$ is invariant then the corresponding relation is $\sum \lambda_a e_a \wedge e_{a^{-1}x} = 0$; a basis of such vectors for each x then gives all the relations. □

We know already the form of a generalised quantum metric and associated weighted Laplacian for a graph Laplacian from §1.4. We can now add to this the requirement of quantum symmetry (so that $\wedge(g) = 0$) in contrast to being edge-symmetric in the sense of symmetric edge-weights in Proposition 1.28.

Proposition 1.58 *Let $\Omega(G)$ be the canonical bicovariant exterior algebra on a finite group, as defined by $\mathcal{C}$. Generalised quantum metrics g exist if and only if $\mathcal{C}$ has inverses and take the form*

$$g = \sum_{a\in\mathcal{C}} c_a e_a \otimes e_{a^{-1}}, \quad (e_a, e_b) = \frac{\delta_{a^{-1},b}}{R_a(c_{a^{-1}})},$$

where $c_a \in \Bbbk(G)$ are nowhere zero. This is edge-symmetric if and only if $c_a = R_a(c_{a^{-1}})$ for all $a \in \mathcal{C}$ and a quantum metric (i.e., quantum symmetric) if and only if $c_a = c_{a^{-1}}$ for all $a \in \mathcal{C}$. The inner element Laplacians are

$$_\theta\Delta = \Delta_\theta = -2\sum_a \frac{1}{R_a(c_{a^{-1}})}\partial^a.$$

In the $*$-differential case, the metric reality condition holds if and only if the c_a are real-valued and the Laplacian then commutes with $*$.

Proof The general form of g is equivalent to Proposition 1.28 with $c_a(x) = g_{x\to xa} = 1/\lambda_{xa\to x}$, which are required to be real in the $*$-differential case. Thus, $\mathrm{flip}(*\otimes *)g = \sum_a e_a \otimes e_{a^{-1}}\bar{c}_a = \sum_a \bar{c}_a e_a \otimes e_{a^{-1}}$. We have quantum symmetry $\wedge(g) = 0$ if and only if $\Psi(g) = g$ (where Ψ is extended as a bimodule map), which happens if and only if $c_a = c_{a^{-1}}$ since $\Psi(e_a \otimes e_{a^{-1}}) = e_{a^{-1}} \otimes e_a$. By contrast, we have $g_{x\to xa} = g_{xa\to x}$ if and only if $c_a(x) = c_{a^{-1}}(xa)$. For the inverse metric, we have $\sum_b (e_a, c_b e_b) e_{b^{-1}} = (e_a, e_b) R_{b^{-1}}(c_b) e_{b^{-1}} = e_a$ and similarly on the other side. The Laplacian $_\theta\Delta f = -2(\mathrm{d}f, \theta) = -2\sum_a (\partial^a f)(e_a, \theta)$ then evaluates as stated. The other side must be the same by Proposition 1.28. Over $\mathbb{C}$, the Laplacian commutes with $*$ by Proposition 1.29. Note that a quantum metric is also edge symmetric if and only if $\partial^a c_a = 0$ for all $a \in \mathcal{C}$. □

This of course includes the canonical graph Euclidean metric where every edge has unit weight, which for $\Omega(G)$ comes out as

$$g = \sum_a e_a \otimes e_{a^{-1}}, \quad (e_a, e_b) = \delta_{a^{-1},b}, \quad \Delta_\theta = {}_\theta\Delta = -2\sum_a \partial^a. \tag{1.9}$$

Corollary 1.59 *Let G be a finite group and $\Omega(G)$ be given by $\mathcal{C}$ a conjugacy class closed under inverses. Over $\mathbb{C}$, Δ_θ for the Euclidean metric has (possibly degenerate) eigenspaces spanned by the matrix elements of each irreducible representation V of G, with eigenvalue*

$$2(1 - \chi_V(\mathcal{C}))|\mathcal{C}|,$$

where $\chi_V(\mathcal{C})$ is the normalised character evaluated on any element $a \in \mathcal{C}$. By Burnside's lemma, this fully diagonalises $\mathbb{C}(G)$.

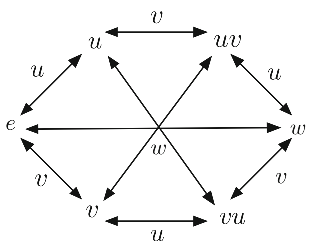
Proof The Laplacian is necessarily the same as the graph Laplacian in §1.4, and at least in the finite group case is known to diagonalise $\mathbb{C}(G)$ into (possibly degenerate) eigenspaces which are matrix elements $\rho^i{}_j$ of the irreducible representations V. We can see this easily in our approach and immediately obtain

$$-\frac{1}{2}\Delta_\theta \rho^i{}_j = \sum_a \rho^i{}_k(\rho^k{}_j(a) - \delta^k_j) = (\sum_a (\rho^k{}_j(a) - \delta^k_j))\rho^i{}_k = (\chi_V(\mathcal{C}) - 1)|\mathcal{C}|\rho^i{}_j,$$

where the value of the normalised character is $\chi_V(\mathcal{C}) = \chi(a) = \mathrm{Tr}\rho(a)/\dim(V)$ for any $a \in \mathcal{C}$. Here $\sum_a \rho(a) = \rho(\sum_a a) = \lambda\mathrm{id}$ for some constant λ by Schur's lemma as $\sum_a a$ is central. Taking traces, since all elements a then give the same value, we have $\dim(V)\chi_V(\mathcal{C})|\mathcal{C}| = \lambda \dim(V)$. □

If $\mathcal{C}$ is a sum of conjugacy classes then the eigenvalue is the sum of such terms, one for each conjugacy class in $\mathcal{C}$. It remains to compute several examples.

Example 1.60 (Permutation Group S_3) The simplest nonabelian example is the permutation group $G = S_3$ which we take with generators u, v, w and relations $u^2 = v^2 = e$, $uvu = vuv = w$ as in Example 1.48. For the calculus, we take $\mathcal{C} = \{u, v, w\}$, the set of 2-cycles, resulting in the Cayley graph



The relations between the left-invariant forms, and their differential, are

$$e_u \wedge e_v + e_v \wedge e_w + e_w \wedge e_u = 0, \quad e_v \wedge e_u + e_u \wedge e_w + e_w \wedge e_v = 0,$$
$$e_u^2 = e_v^2 = e_w^2 = 0, \qquad \mathrm{d}e_u + e_v \wedge e_w + e_w \wedge e_v = 0,$$
$$\mathrm{d}e_v + e_w \wedge e_u + e_u \wedge e_w = 0, \quad \mathrm{d}e_w + e_u \wedge e_v + e_v \wedge e_u = 0.$$

From these relations (and there are no further relations in higher degree) one can find that the dimensions of the calculus in different degrees is $1:3:4:3:1$, so volume dimension 4 like a 4-manifold but a cotangent dimension 3. Here

$$\mathrm{Vol} := e_u \wedge e_v \wedge e_u \wedge e_w = e_v \wedge e_u \wedge e_v \wedge e_w = -e_w \wedge e_u \wedge e_v \wedge e_u = -e_w \wedge e_v \wedge e_u \wedge e_v$$

and is equal to the 2 cyclic rotations of these equations. So up to normalisation there is a unique 'volume form' and it is invariant under cyclic rotations. It is also central, having trivial total G-degree. We also note that any expression of the form $e_a \wedge e_b \wedge e_a \wedge e_b$ is zero, as is any expression with a repetition in the outer (or inner) two positions, and that the basic 2-forms

$$e_u \wedge e_v, \; e_v \wedge e_u, \; e_v \wedge e_w, \; e_w \wedge e_v$$

mutually commute when multiplied in degree 4. In degree 3 we have

$$e_u \wedge e_v \wedge e_w = e_w \wedge e_v \wedge e_u = -e_w \wedge e_u \wedge e_w = -e_u \wedge e_w \wedge e_u$$

and the two cyclic rotations $u \to v \to w \to u$ of these relations, so one can take a basis given by one of these and its cyclic rotations. We define the element

$$\theta^{\circledast} = 2(e_u \wedge e_v \wedge e_w + e_v \wedge e_w \wedge e_u + e_w \wedge e_u \wedge e_v)$$

given by adding these together (the normalisation is relevant later). One can then prove by extensive linear algebra (or see Exercise E5.7 in Chap. 5) that the noncommutative de Rham cohomology over $\mathbb{C}$ is

$$\mathrm{H}^0_{\mathrm{dR}}(S_3) = \mathbb{C}.1, \qquad \mathrm{H}^1_{\mathrm{dR}}(S_3) = \mathbb{C}.\theta, \qquad \mathrm{H}^2_{\mathrm{dR}}(S_3) = 0,$$
$$\mathrm{H}^3_{\mathrm{dR}}(S_3) = \mathbb{C}.\theta^{\circledast}, \qquad \mathrm{H}^4_{\mathrm{dR}}(S_3) = \mathbb{C}.\mathrm{Vol}.$$

This exhibits Poincaré duality (and is the same cohomology as for the q-deformed 3-sphere in Example 2.77). Finally, the most general quantum metric has the form

$$g = c_u e_u \otimes e_u + c_v e_v \otimes e_v + c_w e_w \otimes e_w,$$

with nowhere zero coefficients. We take these to be real-valued and impose the $*$ structure whereby in our case the $\{e_a\}$ are antihermitian. The further symmetric-edge weight condition is that $\partial^u c_u = \partial^v c_v = \partial^w c_w = 0$, which means there are 9 nonzero free parameters for their values on the group. The canonical Laplacian is

$$\Delta_\theta = -2\Big(\frac{1}{c_u}\partial^u + \frac{1}{c_v}\partial^v + \frac{1}{c_w}\partial^w\Big).$$

In the case of the Euclidean metric there are three eigenspaces with eigenvalues 0, 6, 12 and eigenvectors respectively 1, the matrix elements of the 2D representation (see Example 1.48), and the ± 1 sign function according to the party of a permutation. This obeys $\partial^a \mathrm{sign} = -2\mathrm{sign}$ and hence has eigenvalue 12. If we write our elements of $\mathbb{C}(S_3)$ as vectors in basis order $1, u, v, w, uv, vu$ then $1 = (1,1,1,1,1,1)$, $\mathrm{sign} = (1,-1,-1,-1,1,1)$ and the four eigenvectors with eigenvalue 6 are

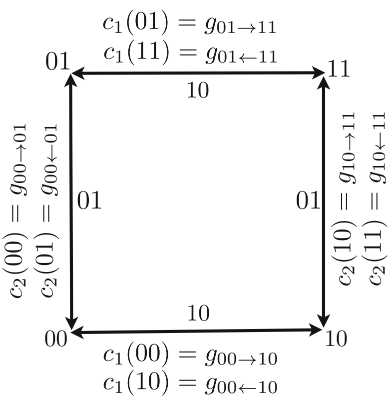
$$\rho^1{}_1 = (1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \qquad \rho^1{}_2 = \frac{\sqrt{3}}{2}(0, 0, 1, -1, 1, -1),$$
$$\rho^2{}_1 = \frac{\sqrt{3}}{2}(0, 0, 1, -1, -1, 1), \qquad \rho^2{}_2 = (1, -1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}),$$

as one may check directly. This agrees with the general analysis of Corollary 1.59 with eigenvalue computed from $\chi_V(u) = 1, 0, -1$ for V the trivial, 2D and sign representations and $|\mathcal{C}| = 3$. In practice one can also find by hand that

$$\psi_x = 2\delta_x - \delta_{xuv} - \delta_{xvu}$$

are eigenfunctions with eigenvalue 6 for each fixed $x \in G$ but obeying $\psi_e + \psi_{uv} + \psi_{vu} = 0$ and $\psi_u + \psi_v + \psi_w = 0$ so that there are four independent such eigenfunctions. This gives a more useable description of this eigenspace, which we will use later and which we think of as some kind of 'point source' at x. $\diamond$

Example 1.61 (Square Calculus) We take $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with its canonical 2D calculus given by the universal calculus (the only choice) on each copy of $\mathbb{Z}_2$. This forms a square Cayley graph with vertices 00, 01, 10, 11 in an abbreviated notation,



where $\mathcal{C} = \{10, 01\}$ in our notation labels the edges. There are correspondingly two generators e_1, e_2 with relations and differential

$$e_i f = R_i(f)e_i, \quad \mathrm{d}f = (\partial^1 f)e_1 + (\partial^2 f)e_2,$$

where as usual $R_1(f)$ shifts by 1 mod 2 (i.e., takes the other value) in the first coordinate, similarly for R_2, and $\partial^i = R_i - \mathrm{id}$. The exterior algebra is the usual Grassmann algebra on the e_i because the group is abelian so that the braiding Ψ is the usual flip. So the grade dimensions are $1 : 2 : 1$ as usual, with a top degree volume form $\mathrm{Vol} = e_1 \wedge e_2$. The cohomology over $\mathbb{C}$ is

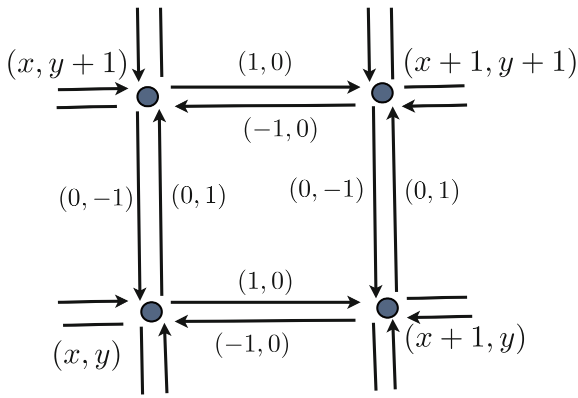
$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{C}.1, \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{C}.e_1 \oplus \mathbb{C}.e_2, \quad \mathrm{H}^2_{\mathrm{dR}}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{C}.\mathrm{Vol},$$

which is the same as for a torus. The general form of metric and Laplacian are

$$g = c_1 e_1 \otimes e_1 + c_2 e_2 \otimes e_2, \quad (e_i, e_j) = \frac{1}{R_i(c_i)}\delta_{ij}, \quad -\frac{1}{2}\Delta_\theta = \sum_i \frac{1}{R_i(c_i)}\partial^i,$$

where the c_i are functions. The symmetric edge-weight case is where $\partial^i c_i = 0$ for $i = 1, 2$, so that there are 4 real parameters in this case, one for each edge in the illustration. The Euclidean metric case is $c_i = 1$ and $\Delta_\theta = -2(\partial^1 + \partial^2)$. The representations of the group are $\psi_{n_1,n_2}(x, y) = (-1)^{n_1 x + n_2 y}$ for $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ labelled by (n_1, n_2), where $n_i = 0, 1$. Then the character values are $\psi_{n_1,n_2}(10) = (-1)^{n_1}$, $\psi_{n_1,n_2}(01) = (-1)^{n_2}$ for the two conjugacy classes that make up $\mathcal{C}$ so that the Laplacian has eigenvalue $2(2 - (-1)^{n_1} - (-1)^{n_2})$ on ψ_{n_1,n_2}. ◇

Example 1.62 (Lattice Calculus) Here the group $G = \mathbb{Z} \times \mathbb{Z}$ is infinite so any infinite sums need to be controlled. This is not an issue at the calculus level, however. We take $G = \mathbb{Z} \times \mathbb{Z}$ with $\mathcal{C} = \{(\pm 1, 0), (0, \pm 1)\}$ and let $e_{\pm 1}, e_{\pm 2}$ be the respective associated 1-forms where the index refers to a copy of $\mathbb{Z}^2$. This is a 4D calculus and the Cayley graph is a square lattice with all arrows bidirectional, namely



The relations are

$$e_{\pm i} f = R_i^{\pm 1}(f)e_{\pm i}, \quad \mathrm{d}f = (\partial_1^+ f)e_{+1} + (\partial_1^- f)e_{-1} + (\partial_2^+ f)e_{+2} + (\partial_2^- f)e_{-2},$$

where $R_1^{\pm}$ is a shift by ± 1 in the first coordinate $R_1^{\pm}(f)(x, y) = f(x \pm 1, y)$ and $R_2^{\pm}$ ditto for the second coordinate, and $\partial_i^{\pm} = R_i^{\pm} - \mathrm{id}$. We can take all functions, so $f \in \mathbb{C}(\mathbb{Z} \times \mathbb{Z})$. Because the group is abelian, the four 1-forms $e_{\pm 1}, e_{\pm 2}$ anticommute as usual to form a Grassmann algebra, and hence are all closed under d. The volume and cotangent dimensions are the same, with up to scale a unique volume form

$$\mathrm{Vol} := e_{+1} \wedge e_{-1} \wedge e_{+2} \wedge e_{-2}.$$

The doubling is needed in order to have $e_{\pm i}{}^* = -e_{\mp i}$. In fact each i can be treated separately with $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$, where

$$\mathrm{d}_i f = (\partial_i^+ f)e_{+i} + (\partial_i^- f)e_{-i}, \quad \theta_i = e_{+i} + e_{-i}, \quad \mathrm{Vol}_i = e_{+i} \wedge e_{-i}$$

and restricting to one variable gives us an inner $*$-exterior algebra $\Omega(\mathbb{Z})$ on each $\mathbb{C}(\mathbb{Z})$, with $\theta = e_+ + e_-$ is one of the θ_i in the full exterior algebra. Each exterior algebra has grade dimensions $1 : 2 : 1$ and cohomology

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{Z}) = \mathbb{C}.1, \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{Z}) = \mathbb{C}.\theta, \quad \mathrm{H}^2_{\mathrm{dR}}(\mathbb{Z}) = 0.$$

Note that we do not have Poincaré duality here since $\mathbb{Z}$ is not compact and indeed $\mathrm{Vol} = e_+ \wedge e_- = \mathrm{d}(fe_-)$ for $f(m) = m$. Next, because functions in the second copy commute with $e_\pm$ in the first copy, etc., it follows that $\Omega(\mathbb{Z} \times \mathbb{Z}) := \Omega(\mathbb{Z}) \otimes \Omega(\mathbb{Z})$ defined as a graded-tensor product of the two $*$-exterior algebras, is a sub-exterior algebra of the full exterior algebra above. Its degree zero 'coordinate algebra' is now smaller as it consists of joint functions that are tensor products of functions in each variable. At least in this restricted setting, the cohomology is the graded tensor product, so has $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{Z} \times \mathbb{Z}) = \mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2$. There are many other approaches to handling the infinite nature of the groups here, with potentially different cohomology. For example, if we want to remain algebraic then we can work with the algebra of functions of finite support and with an identity adjoined, and basic 1-forms as before. In this case, one only has a generalised calculus.

Returning to the geometry, a general quantum symmetric metric takes the form

$$g = \sum_i c_i(e_{+i} \otimes e_{-i} + e_{-i} \otimes e_{+i}), \quad (e_{\pm i}, e_{\mp i}) = \frac{1}{R_i^{\pm}(c_i)}$$

(and zero between i, j) while the case of symmetric edge-weights is $\partial^{+i} c_i = 0$ for $i = 1, 2$, i.e., $c_1(x, y)$ is independent of x and $c_2(x, y)$ is independent of y. The associated Laplacian is

$$\Delta_\theta = -2\sum_i \Big(\frac{1}{R_i(c_i)}\partial^{+i} + \frac{1}{R_i^{-1}(c_i)}\partial^{-i}\Big) = 2\sum_i \frac{1}{c_i}\partial^{+i}\partial^{-i},$$

where the second expression applies only in the symmetric edge-weight case.

The Euclidean metric here is block diagonal in our basis with each block cross diagonal so that $(e_{\pm 1}, e_{\mp 2}) = 1$ etc. The wave operator for the Euclidean metric is

$$\Delta_\theta = -2(\partial_1^+ + \partial_1^- + \partial_2^+ + \partial_2^-) = 2(\partial_1^+\partial_1^- + \partial_2^+\partial_2^-),$$

with each $\mathbb{Z}$ contributing $2\partial_i^+\partial_i^-$ to give the usual Laplacian in 2 dimensions,

$$-\frac{1}{2}\Delta_\theta f(x,y) = f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)-4f(x,y).$$

Clearly the eigenvalues of Δ_θ on $\mathbb{Z}\times\mathbb{Z}$ are $4(2-\cos\phi_1-\cos\phi_2)$ according to a pair of angles (ϕ_1,ϕ_2) with 1-dimensional eigenspaces spanned by the plane waves

$$\psi_{\phi_1,\phi_2}(x,y) = e^{i\phi_1 x + i\phi_2 y}$$

for all $x, y \in \mathbb{Z}$. This again fits the general picture where now $\mathcal{C}$ is trivially the sum of 4 conjugacy classes and the character values in the 1-dimensional representation ψ_{ϕ_1,ϕ_2} are $\psi_{\phi_1,\phi_2}((\pm 1, 0)) = e^{\pm i\phi_1}$ and $\psi_{\phi_1,\phi_2}((0,\pm 1)) = e^{\pm i\phi_2}$. This gives contributions $2(1-e^{\pm i\phi_1})$ from the first two conjugacy classes and similarly for the other two, giving the same total eigenvalue. Clearly, any linear combination of plane waves with the same fixed value of $\cos(\phi_1)+\cos(\phi_2)$ will also be an eigenvector. For example, for any $\phi\in[0,\pi]$ the choice $\phi_1 = \pi-\phi$, $\phi_2=\phi$ gives a continuum of eigenfunctions with eigenvalue 8. Integrating over ϕ with weights $\tilde f(\phi)$ we obtain

$$\psi(x,y) = (-1)^x f(x-y) = \int_0^\pi d\phi\, e^{i\pi x - i\phi(x-y)}\tilde f(\phi)$$

as an eigenfunction of Δ_θ with eigenvalue 8, for some other function f on $\mathbb{Z}$. One can check directly that any function f here will give an eigenfunction with eigenvalue 8 (one does not need to be able to expand it into Fourier modes). If one of the variables of the lattice is 'time' then this is an example of a travelling wave with, up to a phase, the same shape but shifted with each time step. ◇

We have illustrated the details for two copies of $\mathbb{Z}$ but clearly a similar analysis works for an n-dimensional lattice, with a calculus of dimension $2n$ and cohomology H^1_{dR} spanned by the θ_i at least in one algebraic setting.

1.7.3 Finite Lie Theory and Cohomology

In general very little is known about the structure of the bicovariant calculi $\Omega(G)$ for nonabelian finite groups, even the dimensions of the different degrees in most cases. Although one can define a simplicial complex (the Moore complex) associated to a Cayley graph, that has the same dimensions as the de Rham complex of a sphere of dimension $|\mathcal{C}|-1$, and that is very far from our case, as Example 1.60 shows. For some general observations, we look first at the permutation group S_n in general, with a focus on the left-invariant forms $\Lambda(S_n)$. We let $C_n(d)$ be the configuration space of ordered n-tuples in $\mathbb{R}^d$ with distinct entries, and $\mathrm{H}(C_n(d))$ its cohomology with values in $\mathbb{Z}$ and hence in any field of characteristic zero.

Proposition 1.63 *The exterior algebra $\Omega(S_n)$ with $\mathcal{C}$ the 2-cycles has invariant 1-forms $\{e_{(ij)}\}$ and at degree two the relations*

$$e_{(ij)}^2=0,\quad e_{(ij)}\wedge e_{(mn)}+e_{(mn)}\wedge e_{(ij)}=0,$$

$$e_{(ij)}\wedge e_{(jk)}+e_{(jk)}\wedge e_{(ki)}+e_{(ki)}\wedge e_{(ij)}=0,$$

where i,j,k,m,n are distinct. Moreover, if $\Bbbk$ has characteristic zero then

$$\Lambda(S_n)/\langle e_{(ij)}\wedge e_{(jk)}+e_{(jk)}\wedge e_{(ij)}\rangle\cong \mathrm{H}(C_n(2)).$$

Proof We use Lemma 1.57. There are three kinds of elements $x\in S_n$ for which $\mathcal{C}\cap x\mathcal{C}^{-1}$ is not empty: (i) $x=e$, in which case σ is trivial and $V_e=\Bbbk\mathcal{C}$. This gives the first of the relations stated; (ii) $x=(ij)(mn)$, where i,j,m,n are disjoint. In this case $\mathcal{C}\cap x\mathcal{C}^{-1}$ has two elements (ij) and (mn), interchanged by σ. The basis of $V_{(ij)(mn)}$ is 1-dimensional, namely $(ij)+(mn)$, and this gives the second type of relations stated; (iii) $x=(ij)(jk)$, where i,j,k are disjoint. Here $\mathcal{C}\cap x\mathcal{C}^{-1}$ has 3 elements $(ij),(jk),(ik)$ cyclically rotated by σ. The invariant subspace is 1-dimensional with $(ij)+(jk)+(ik)$ giving the third type of relations.

Next, the cohomology ring $\mathrm{H}(C_n(d))$ in the Arnold form is generated by E_{ij} of degree $d-1$, labelled by pairs $i\neq j$ in the range $1,\cdots,n$ with relations

$$E_{ij}=(-1)^dE_{ji},\quad E_{ij}E_{km}=(-1)^{d-1}E_{km}E_{ij},\quad E_{ij}E_{jk}+E_{jk}E_{ki}+E_{ki}E_{ij}=0$$

for all i,j,k,m and $E_{ij}^2=0$ for all i,j. Note that the 2nd relation is not restricted to disjoint indices as in $\Lambda(S_n)$ and we have rewritten the 3rd relation in a suggestive way using the first two relations. We see that $\mathrm{H}(C_n(d))$ is a graded-commutative version of the quadratic version of the algebra Λ if d is even, where we add the remaining anticommutation relations to those already in $\Lambda(S_n)$. $\square$

Some of the grade dimensions of $\Omega(S_n)$ for small n are shown in Fig. 1.1, including the volume dimensions $1,4,12,40$ respectively for $n=2,3,4,5$. The grade dimensions are shown as Hilbert polynomials $\sum_i q^i\dim(\Omega^i(S_n))$ which turn

dim	Ω^0	Ω^1	Ω^2	Ω^3	Ω^4	Hilbert polynomial(q)	Top degree
S_2	1	1				$[2]_q$	1
S_3	1	3	4	3	1	$[2]_q^2[3]_q$	4
S_4	1	6	19	42	71	$[2]_q^2[3]_q^2[4]_q^2$	12
S_5	1	10	55	220	711	$[4]_q^4[5]_q^2[6]_q^4$	40

Fig. 1.1 Dimensions and Hilbert polynomials for the exterior algebras $\Omega(S_n)$ for $n < 6$ where $[n]_q = (q^n - 1)/(q - 1)$

out to be q-integers and have a symmetric form consistent with Poincaré duality but with cotangent dimension different from the volume dimension. These numerics were obtained by computer algebra and originally for a related 'Fomin–Kirillov algebra' $\mathcal{E}_n$ that is known to have the same Hilbert series (and a similar form of relations but with generators $[ij] = -[ji]$ rather than unordered pairs and commuting for disjoint indices rather than anticommuting). Indeed, this algebra quotients to $\mathrm{H}(C_n(3))$ just by requiring all the $[ij]$ to commute, so that the Fomin–Kirillov algebra can be thought of as a quantisation of $\mathrm{H}(C_n(3))$. It also contains the cohomology of the flag variety as a certain commutative subalgebra. The composition

$$\mathrm{H}(\mathrm{Flag}_n) \hookrightarrow \mathcal{E}_n \twoheadrightarrow \mathrm{H}(C_n(3))$$

gives a natural algebra map between these two commutative cohomologies factoring through the noncommutative quadratic algebra $\mathcal{E}_n$. The latter is a braided Hopf algebra of the type that we will encounter in Chap. 2.

Returning to our noncommutative geometry, the volume dimensions of $\Omega(S_n)$ are the listed top degrees and are believed to be infinite for $n > 5$. The pattern of volume dimensions here is the same as the number of indecomposable modules of the 'preprojective algebra' of type sl_n (which in turn is related to the Lusztig–Kashiwara canonical basis of $U_q(sl_n)$). These volume dimensions are also expected to be the top degrees of $\mathrm{H}_{\mathrm{dR}}(S_n)$. In this section, we prove the following theorem about this cohomology for a class of groups that includes all permutation groups S_n with $\mathcal{C}$ the set of 2-cycles. This is due to Rietsch and one of the authors.

Theorem 1.64 *Let $\Bbbk$ not have characteristic 2. If G is the Weyl group of a simply laced complex simple Lie algebra and $\mathcal{C}$ the set of reflections then* $\mathrm{H}^1_{\mathrm{dR}}(G) = \Bbbk.\theta$.

We prove this in slightly greater generality. We start with the concept of a right quandle. This is a set $\mathcal{C}$ together with an operation $a^b \in \mathcal{C}$ for $a, b \in \mathcal{C}$, obeying

$$(a^b)^c = (a^c)^{(b^c)}$$

with $(\ )^a$ bijective and $a^a = a$ for all $a, b, c \in \mathcal{C}$. There is an obvious notion of a left quandle and ${}^a(\ )$ defined as the inverse of $(\ )^a$ is such a left quandle. We focus on the right-handed theory but the two are equivalent. By definition, a right *inverse* *property* (IP) quandle is a right quandle $\mathcal{C}$ equipped with an involutive bijection, denoted as inversion, such that $(a^{-1})^b = (a^b)^{-1}$ and $(a^b)^{b^{-1}} = a$ for all $a, b \in \mathcal{C}$.

Example 1.65 Every right quandle $\mathcal{C}$ has a 'complexification' to a right IP quandle $\mathcal{C} \sqcup \mathcal{C}$ (a disjoint union) by having a second copy and $(\ )^{-1}$ the identity map from one copy to the other. We then extend the original quandle structure to the second copy as follows. We let $a, b \in \mathcal{C}$ be in the first copy so that a^{-1}, b^{-1} are in the second copy and we define $(\ )^{a^{-1}}$ on the first copy to be the inverse of the bijective operation $(\ )^a$. We also define $(a^{-1})^b = (a^b)^{-1}$ and $(a^{-1})^{b^{-1}} = (a^{b^{-1}})^{-1}$. ◇

Given a right IP quandle $\mathcal{C}$, the map $\tilde{\Psi} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ with $\tilde{\Psi}(a, b) = (b, a^b)$ is a set-theoretic solution of the braid relations. Using this, we define the group $G_{\mathcal{C}}$ of the IP quandle as having quadratic relations given by $\tilde{\Psi}$, i.e., the free nonabelian group with generators $\mathcal{C}$ and the relations

$$ab = b(a^b), \quad aa^{-1} = e$$

for all $a, b \in \mathcal{C}$. The nice case, which we call *embeddable*, is where $\mathcal{C} \subset G_{\mathcal{C}}$.

Now start with an Ad-stable subset $\mathcal{C} \subseteq G \setminus \{e\}$ of a group G. If $\mathcal{C}$ is closed under inversion then it becomes an embeddable IP quandle with $a^b = b^{-1}ab$. The group of this IP quandle is not necessarily G but there is a group homomorphism

$$G_{\mathcal{C}} \to G \tag{1.10}$$

which is surjective if $\mathcal{C}$ generates the group. Such a $\mathcal{C}$ is exactly the data which we have seen above for a bicovariant $\Omega(G)$ admitting a metric in degree 1 or, over $\mathbb{C}$, forming a $*$-exterior algebra. In classical geometry a differential structure on a topological group makes it a Lie group and we accordingly think of $\mathcal{C}$ as a *finite Lie algebra* of G when $\mathcal{C}$ is finite, or a discrete Lie algebra otherwise. We think of a right IP quandle as a finite or discrete Lie algebra abstractly. Usually given a Lie group, one can take its Lie algebra and then construct from that its associated connected simply connected Lie group. The result need not be the group we began with, but its universal covering group. The above map $G_{\mathcal{C}} \to G$ is exactly the finite or discrete Lie theory version of this, an important difference being that whereas a classical Lie structure on a topological group, if it exists, is unique, the finite or discrete Lie algebra, and hence $G_{\mathcal{C}}$, need not be unique as there can be many such $\mathcal{C}$. This finite Lie theory is due to one of the authors and is a special case of a braided Lie theory that also applies to the quantum groups covered in Chap. 2.

This is part of a general analogy between regular Lie theory and finite Lie theory. The classical result that will concern us is that for a connected simply connected Lie group G, the de Rham cohomology is $\mathrm{H}^0(G) = \mathbb{R}.1$ and $\mathrm{H}^1(G) = 0$. We already have the former for a connected differential calculus and now Theorem 1.64 expresses the latter in so far as the 'classical part' of the cohomology vanishes, given that θ has no role classically and is a purely quantum feature of the geometry. Here θ is closed for any bicovariant $\Omega(G)$ and will be nontrivial in $\mathrm{H}^1_{\mathrm{dR}}(G_{\mathcal{C}})$ for any embeddable IP at least with finite associated group, but for this to span the whole cohomology, we will need a property motivated by the antisymmetry of the Lie bracket in regular Lie theory.

Definition 1.66 Let $\mathcal{C}$ be an IP quandle. We say that two distinct elements $a, b \in \mathcal{C}$ are 'mutually skew' if $a^b = (b^a)^{-1}$. This notion is symmetric and makes $\mathcal{C}$ an undirected graph with an edge between mutually skew elements. We say that the IP quandle is *locally skew* if this graph is connected, i.e., if any two elements are connected by a path of mutually skew elements.

This leads to various identities, including the following.

Lemma 1.67 *Let $\mathcal{C}$ be an embeddable IP quandle. Two elements $a, b \in \mathcal{C}$ are mutually skew if and only if a^{-1}, b^{-1} are. This holds if and only if the braid relation $a^{-1}ba^{-1} = ba^{-1}b$ holds in $G_{\mathcal{C}}$. In this case $(a^{-1})^{(a^b)} = b$.*

Proof If a, b are mutually skew then, using the relations of $G_{\mathcal{C}}$, $b^{-1}a^{-1}b = b^{-1}b(a^{-1})^b = (a^{-1})^b = (a^b)^{-1} = b^a = a^{-1}aa^b = a^{-1}ba$. The outer steps are reversible so a, b mutually skew is equivalent to $b^{-1}a^{-1}b = a^{-1}ba$, which is equivalent to the braid relation. Applying inverses to the braid relation, this is equivalent to $ab^{-1}a = b^{-1}ab^{-1}$, which is the same braid relation but for a^{-1}, b^{-1} in place of a, b, hence equivalent to a^{-1}, b^{-1} being mutually skew. Now given both of these, $(a^{-1})^{(a^b)} = (a^{-1})^{((b^{-1})^a)} = (((a^{-1})^{b^{-1}})^a = (b^{a^{-1}})^a = b$. □

We are now ready to state and prove our main theorem.

Theorem 1.68 *Let $\Bbbk$ not have characteristic 2 and let $\mathcal{C}$ be an embeddable IP quandle with $G_{\mathcal{C}}$ finite. The 1-form θ in the canonical bicovariant $\Omega(G_{\mathcal{C}})$ from Theorem 1.56 is a nonzero element of $\mathrm{H}^1_{\mathrm{dR}}(G_{\mathcal{C}})$. If additionally $\mathcal{C}$ is locally skew, then $\mathrm{H}^1_{\mathrm{dR}}(G_{\mathcal{C}}) = \Bbbk\theta$.*

Proof Let $\mathcal{C}$ be a right IP quandle. We note first that

$$\Psi(e_a \otimes \theta) = \theta \otimes e_a, \quad \Psi(\theta \otimes e_a) = \sum_b e_{bab^{-1}} \otimes e_b.$$

Hence if $\omega = \sum_a c_a e_a$, $c_a \in \Bbbk(G_{\mathcal{C}})$ has $\mathrm{d}\omega = 0$ in Ω^2 then by the Leibniz rule,

$$\sum_{a,b} (\partial^b c_a) e_b \otimes e_a + \sum_a c_a(\theta \otimes e_a + e_a \otimes \theta) \in \ker(\Psi - \mathrm{id}),$$

which is equivalent to

$$\sum_{b,a} R_b c_a e_{bab^{-1}} \otimes e_b - \sum_a c_a e_a \otimes \theta = \sum_{a,b} R_b c_a e_b \otimes e_a - \sum_a c_a \theta \otimes e_a$$

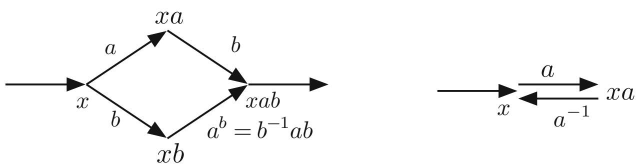
or

$$R_b c_{b^{-1}ab} + c_b = R_a c_b + c_a$$

for all $a, b \in \mathcal{C}$. This is therefore the condition for a de Rham 1-cocycle in terms of the coefficients c_a. Now consider

$$f(x) = c_{a_1}(e) + c_{a_2}(a_1) + \cdots + c_{a_n}(a_1 \cdots a_{n-1}), \quad x = a_1 a_2 \cdots a_n,$$

where we write x as obtained along a path from e in the Cayley graph by repeated multiplication from the right by generators $a_1, a_2, \cdots$. We add up the values of c at each interim vertex in the direction of the next vertex. That this is independent of the path requires invariance under application of the relations in the group $G_{\mathcal{C}}$ which are either the quadratic ones or express inverses. This means in diagram form:



at all $x \in G_{\mathcal{C}}$ and $a, b \in \mathcal{C}$. The first requires

$$c_a(x) + c_b(xa) = c_b(x) + c_{a^b}(xb), \tag{1.11}$$

which is the de Rham cocycle condition on c we found above, and therefore holds if ω is closed. The second requires $c_a(x) + c_{a^{-1}}(xa) = 0$. If both these conditions hold then f is well-defined and by construction obeys $\partial^a f = c_a$ and hence $\omega = \mathrm{d}f$. Conversely, if ω is exact then these conditions hold. We see in particular that θ is closed but not exact. More generally, if $c_a(x)+c_{a^{-1}}(xa)$ is a constant λ independent of x, a, then clearly $\bar{c}_a = c_a - \frac{\lambda}{2}$ obeys the conditions that we required for c above, i.e., $\omega - \frac{\lambda}{2}\theta$ is exact. The following lemma then completes the proof. $\square$

Lemma 1.69 *Let $\mathcal{C}$ be an embeddable IP quandle with $G_{\mathcal{C}}$ finite, $\omega = \sum_a c_a e_a$ a closed 1-form and $d_a(x) = c_a(x) + c_{a^{-1}}(xa)$. Then*

(1) *$d_a(x) = d_{a^b}(xb)$ for all $a, b \in \mathcal{C}$ and $x \in G_{\mathcal{C}}$.*
(2) *If a, b are mutually skew then $d_a(x) = d_b(x)$ for all $x \in G_{\mathcal{C}}$.*

Proof From (1.11) we also have

$$c_{a^{-1}}(xa) + c_b(x) = c_b(xa) + c_{(a^{-1})^b}(xab)$$

and adding this to (1.11) we have an equation which we use to compute

$$\begin{aligned} d_{a^b}(xb) = c_{a^b}(xb) + c_{(a^b)^{-1}}(xba^b) &= c_{a^b}(xb) + c_{(a^{-1})^b}(xab) \\ &= c_a(x) + c_{a^{-1}}(xa) = d_a(x) \end{aligned}$$

using also the definition of an IP quandle and the relations of the group. Hence $d_a(x) = d_{a^{x^{-1}}}(e)$, where we extend the action of $\mathcal{C}$ to an action of $G_{\mathcal{C}}$. In view of this we will focus on $x = e$ for simplicity. We now return to equation (1.11) with x, a, b assigned different values and overall signs as follows:

$$(x, a, b) = (e, u, v) : \; c_u(e) + c_v(u) = c_v(e) + c_{u^v}(v)$$

$$(x, a, b) = (e, u, (u^{-1})^{v^{-1}}) : \; c_u(e) + c_{(u^{-1})^{v^{-1}}}(u) = c_{(u^{-1})^{v^{-1}}}(e) + c_{u^{((u^{-1})^{v^{-1}})}}((u^{-1})^{v^{-1}})$$

$$(x, a, b) = (u, v, u^{-1}) : \; -c_v(u) - c_{u^{-1}}(uv) = -c_{u^{-1}}(u) - c_{v^{u^{-1}}}(e)$$

$$(x, a, b) = (v, u^v, v^{-1}) : \; c_{u^v}(v) + c_{v^{-1}}(uv) = c_{v^{-1}}(v) + c_u(e)$$

$$(x, a, b) = (uv, u^{-1}, v^{-1}) : \; c_{u^{-1}}(uv) + c_{v^{-1}}(v^{u^{-1}}) = c_{v^{-1}}(uv) + c_{(u^{-1})^{v^{-1}}}(u)$$

where we used the definition of an IP quandle and the relations of the group to simplify. We now add all these equations together and cancel to obtain

$$d_u(e) = d_v(e) + c_{(u^{-1})^{v^{-1}}}(e) + c_{u^{((u^{-1})^{v^{-1}})}}((u^{-1})^{v^{-1}}) - c_{v^{u^{-1}}}(e) - c_{v^{-1}}(v^{u^{-1}})$$

for all $u, v \in \mathcal{C}$. If u^{-1}, v^{-1} are mutually skew then $(u^{-1})^{v^{-1}} = v^{u^{-1}}$ and if u, v are mutually skew then $u^{((u^{-1})^{v^{-1}})} = v^{-1}$ by Lemma 1.67. Then $d_u(e) = d_v(e)$. □

This completes the proof of Theorem 1.68: when $\mathcal{C}$ is locally skew we conclude that $d_a(e)$ is independent of $a \in \mathcal{C}$ by part (2) of the lemma, then $d_a(x)$ is independent of a, x by part (1). We also get partial information if $\mathcal{C}$ is not locally skew.

Corollary 1.70 *Let $\mathcal{C}$ be an embeddable IP quandle with $G_{\mathcal{C}}$ finite. Then*

$$\dim \mathrm{H}^1_{\mathrm{dR}}(G_{\mathcal{C}}) \le |\mathcal{C}|.$$

Proof The proof of Theorem 1.68 tells us that a closed 1-form $\omega = \sum_a c_a e_a$ is exact if and only if $d_a(x) = 0$ for all a, x which, by iterating part (1) of Lemma 1.69, holds if and only if $d_a(e) = 0$ for all a. This gives $|\mathcal{C}|$ linear relations on the coefficients of ω, although not necessarily independent. □

Equality holds when the quandle structure is trivial, as in Example 1.61 where we saw that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \Bbbk^2$ was spanned by the left-invariant 1-forms associated to the elements of $\mathcal{C}$. The following now completes the proof of Theorem 1.64.

Proposition 1.71 *Every finite crystallographic reflection group (i.e., Weyl group) is of the form $G_{\mathcal{C}}$ with respect to $\mathcal{C}$, the set of reflections. Moreover, the IP quandle of reflections is locally skew* if and only if *the associated Dynkin diagram is connected and simply laced.*

Proof We choose a root system of Dynkin type appropriate to the Weyl group W, i.e., a system of roots $\{\alpha\}$ in $\mathbb{R}^n$ for some n with its standard inner product. We do not recall all of the background here, so suffice it to say that W can be realised concretely as generated by hyperplane reflections of the form $r_\alpha(v) = v - \langle\check{\alpha}, v\rangle\alpha$ for $v \in \mathbb{R}^n$, where the coroot $\check{\alpha}$ of α is a certain element of the dual space obeying $\langle\check{\alpha}, \alpha\rangle = 2$. The reflection r_α does not depend on the sign of α so we can think of it as labelled by α moduli sign. The IP quandle $\mathcal{C}$ of interest is the set of reflections as an involutive conjugacy class in W, which inherits the structure ${r_\beta}^{r_\alpha} = r_\alpha^{-1} r_\beta r_\alpha = r_{r_\alpha(\beta)}$ and $r_\alpha^{-1} = r_\alpha$ using representative roots for the reflections. We now adopt this IP quandle structure and the resulting relations of $G_{\mathcal{C}}$,

$$
{r_\beta}^{r_\alpha} = r_{r_\alpha(\beta)}, \quad r_\alpha r_\beta = r_\beta r_{r_\beta(\alpha)}, \quad r_\alpha^{-1} = r_\alpha.
$$

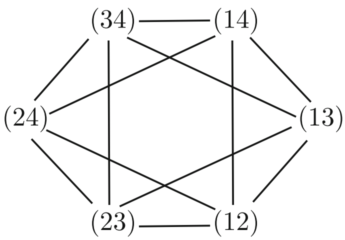
Here $G_{\mathcal{C}}$ maps onto W as in (1.10). We show that W maps onto $G_{\mathcal{C}}$ by checking that all the Coxeter relations are obtained by using only the relations of $G_{\mathcal{C}}$. We will use the notation r_i for the reflection associated to the simple root α_i.

(1) If $\langle\check{\alpha}_j, \alpha_i\rangle = 0$ then ${r_i}^{r_j} = r_i$ and hence $r_i r_j = r_j r_i$.
(2) If $\langle\check{\alpha}_j, \alpha_i\rangle = \langle\check{\alpha}_i, \alpha_j\rangle = -1$ then ${r_i}^{r_j} = r_{\alpha_i+\alpha_j} = {r_j}^{r_i}$ and hence $r_i r_j = r_j r_{\alpha_i+\alpha_j}$ and $r_j r_i = r_i r_{\alpha_i+\alpha_j}$. Hence $r_i r_j r_i = r_{\alpha_i+\alpha_j} = r_j r_i r_j$ or $(r_i r_j)^3 = 1$.
(3) If $\langle\check{\alpha}_j, \alpha_i\rangle = -2$ and $\langle\check{\alpha}_i, \alpha_j\rangle = -1$ then ${r_i}^{r_j} = r_{\alpha_i+2\alpha_j}$ and ${r_{\alpha_i+2\alpha_j}}^{r_i} = r_{\alpha_i+2\alpha_j}$, hence $r_i r_j = r_j r_{\alpha_i+2\alpha_j}$ and $r_{\alpha_i+2\alpha_j} r_i = r_i r_{\alpha_i+2\alpha_j}$. Then $r_i r_j r_i r_j = r_i r_{\alpha_i+2\alpha_j} = r_{\alpha_i+2\alpha_j} r_i = r_j r_i r_j r_i$ or $(r_i r_j)^4 = 1$.
(4) If $\langle\check{\alpha}_j, \alpha_i\rangle = -3$ and $\langle\check{\alpha}_i, \alpha_j\rangle = -1$ then ${r_i}^{r_j} = r_{\alpha_i+3\alpha_j}$, ${r_{\alpha_i+3\alpha_j}}^{r_i} = r_{2\alpha_i+3\alpha_j}$ and ${r_{2\alpha_i+3\alpha_j}}^{r_j} = r_{2\alpha_i+3\alpha_j}$, hence $r_i r_j = r_j r_{\alpha_i+3\alpha_j}$, $r_{\alpha_i+3\alpha_j} r_i = r_i r_{2\alpha_i+3\alpha_j}$ and $r_{2\alpha_i+3\alpha_j} r_j = r_j r_{2\alpha_i+3\alpha_j}$. Then we have $r_i r_j r_i r_j r_i r_j = r_i r_j r_i r_{\alpha_i+3\alpha_j} = r_i r_j r_{2\alpha_i+3\alpha_j} r_i = r_i r_{2\alpha_i+3\alpha_j} r_j r_i = r_{\alpha_i+3\alpha_j} r_i r_j r_i = r_j r_j r_{\alpha_i+3\alpha_j} r_i r_j r_i = r_j r_i r_j r_i r_j r_i$ or $(r_i r_j)^6 = 1$.

Hence $G_{\mathcal{C}}$ recovers the Weyl group. In the simply laced case, the graph Γ of mutually skew edges contains the Dynkin diagram D itself—two adjacent nodes are labeled by reflections obeying the braid relations hence by Lemma 1.67 are mutually skew. We show that any reflection is connected by a path to this Dynkin diagram. Indeed, any root can be obtained by the action of the Weyl group from a simple root and hence is of the form $r_{i_k} r_{i_{k-1}} \cdots r_{i_1}(\alpha_i)$, say. We apply these simple reflections in turn and show that the accumulated subgraph of Γ remains connected. Thus, applying r_{i_1} to D gives a translate $r_{i_1}(D)$ of D as a subgraph of Γ, because the Weyl group acts as a group automorphism by conjugation and hence two vertices are connected after the action of r_{i_1} if and only if they were before. The translate is connected to D since at least one node, α_{i_1} itself, is invariant so lies in both D and its translate. We write $\Gamma_1 = D \cup r_{i_1}(D)$ for the combined connected subgraph. Now we apply r_{i_2} to Γ_1 to obtain a subgraph $r_{i_2}(\Gamma_1)$ as its translate. This translate remains connected to D since $\alpha_{i_2} \in D$ is invariant and so also lies in $r_{i_2}(\Gamma_1)$. We write $\Gamma_2 = D \cup r_{i_2}(\Gamma_1)$ for the combined connected graph. Iterating this process with $\Gamma_{k+1} = D \cup r_{i_{k+1}}(\Gamma_k)$ we arrive at a connected subgraph of Γ which contains the element of the Weyl group that we began with as well as D. Hence the quandle is locally skew.

Next we claim that if r_α, r_β are mutually skew then α, β have the same length. Hence in the non-simply laced case the graph of mutually skew edges must have different components and the quandle is not locally skew. Indeed, r_α and r_β being mutually skew means $r_\alpha(\beta) = \pm r_\beta(\alpha)$, i.e., $\beta - \alpha\langle\check{\alpha}, \beta\rangle = \alpha - \beta\langle\check{\beta}, \alpha\rangle$ or $\beta - \alpha\langle\check{\alpha}, \beta\rangle = -\alpha + \beta\langle\check{\beta}, \alpha\rangle$. If $\alpha \neq \pm\beta$ then we conclude in the first case that $\langle\check{\alpha}, \beta\rangle = \langle\check{\beta}, \alpha\rangle = -1$ and hence that $r_\alpha(\beta) = \alpha + \beta = r_\beta(\alpha)$, which implies that $r_\beta r_\alpha(\beta) = r_\beta(\alpha) + r_\beta(\beta) = \alpha$. Hence α has the same length as β as reflections do not change length. Similarly in the other case with $r_\alpha(\beta) = \beta - \alpha$. □

For an example, S_4 has six 2-cycles forming an IP quandle with connected mutually-skewness graph



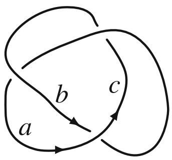
2-cycles here are mutually-skew if and only if they have a number in common.

Finally, we note that quandles arise naturally in knot theory in the course of the construction of the Wirtinger presentation of the fundamental group of the complement of a knot. Note first that for a quandle $\mathcal{C}$ one has an associated group $\mathrm{As}(\mathcal{C})$ which has the quandle elements as generators and relations $ab = ba^b$ for all $a, b \in \mathcal{C}$, along with inverses. This is the same group as $G_{\mathcal{C}\sqcup\mathcal{C}}$ in our development. Now suppose we are given a diagram of a knot K. We orient this with a flow along the knot and we label the arcs. At each crossing where an arc passes over a part of the knot we transform the label of the arc coming in under the crossing to the label of the arc coming out under the crossing according to the rules

where ${}^a(\ )$ denotes the inverse to the operation $(\ )^a$ of a right quandle. We recognise on the left the braiding $\widetilde{\Psi}$ for a right quandle, with its inverse on the left. The latter is also the form of the braiding Ψ on Λ^1 in Theorem 1.56 in a different context. Since the arcs are labelled, this defines restrictions on some of the right quandle products. The fundamental right quandle $\mathcal{C}_K$ associated to the knot is generated by all the arcs with only such restrictions as they arise from all the knot crossings.

Another way to phrase this is that we colour a knot by a quandle if we label arcs by quandle elements such that the above holds at every crossing, and the fundamental quandle of a knot is universal with this property, i.e., maps homomorphically to any of these. Colourability of a knot diagram is invariant under Reidemeister moves and hence this concept as well as $\mathcal{C}_K$ are knot invariants (in fact, the latter is a complete invariant). Moreover, $\mathrm{As}(\mathcal{C}_K)$ recovers the fundamental group $\pi_1(K^c)$ of the complement of the knot. From our point of view it defines a bicovariant exterior algebra on the fundamental group and the noncommutative de Rham cohomology of this is also a knot invariant. $\mathcal{C}_K$ is not naturally an IP quandle but we can typically address this by complexification. We limit ourselves to an example.

Example 1.72 For K the trefoil knot we have 3 arcs a, b, c with relations $a^b = c, b^c = a, c^a = b$ for the fundamental quandle as read off from



The resulting quandle can be identified with the quandle $\mathbb{ZP}^1 = \mathbb{Q} \cup \{\infty\}$. The left-hand side here has elements $\{e_{\vec{a}}\}$ labelled by 2-vectors $\vec{a}$ where the two integer entries are coprime and are taken modulo an overall sign. We let $|{\vec{a} \atop \vec{b}}|$ be the determinant of the 2×2 matrix formed by two row vectors and define the quandle operation

$$e_{\vec{a}}^{e_{\vec{b}}} = e_{\vec{a}+|{\vec{a} \atop \vec{b}}|\vec{b}}.$$

We can also think of $\vec{a} = (p, q)$ in $\mathbb{ZP}^1$ as the element p/q of $\mathbb{Q} \cup \{\infty\}$ in lowest form, which is the other description. Next, we can colour the trefoil by setting the arcs to $a = e_{(1,0)} = \infty$, $b = e_{(0,1)} = 0$, $c = e_{(1,1)} = 1$ for the trefoil arcs, or put another way we have a quandle homomorphism $\mathcal{C}_K \to \mathbb{ZP}^1$, which turns out to be an isomorphism. It is beyond our scope to prove this in detail but let

$$\frac{p}{q} = k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \cdots + \frac{1}{k_n}}}$$

be the continued fraction expansion of an element of $\mathbb{Q}$ where $k_1 \in \mathbb{Z}$ and if $n \geq 2$ then $k_2, k_3, \ldots, \geq 0$ and $k_n \geq 2$. We will use the notation $(\ )^{a^k}$ for k applications of $(\ )^a$ and $(\ )^{a^{-k}}$ for k applications of its inverse, and similarly for b. Then the corresponding element of $\mathcal{C}_K$ is

$$n \text{ even}: \quad (((a^{b^{k_n}})^{a^{-k_{n-1}}})\ldots)^{a^{-k_1}}; \quad n \text{ odd}: \quad (((b^{a^{-k_n}})^{b^{k_{n-1}}})\cdots)^{a^{-k_1}}$$

and conversely every element of $\mathcal{C}_K$ can uniquely be written as a, b, a^b, b^a or in this form. Thus to depth 3 and in the above realisation in $\mathbb{ZP}^1$,

$$b^{a^{-1}} = e_{(0,1)}{}^{e^{-1}_{(1,0)}} = e_{(0,1)-|{}^{0\,1}_{1\,0}|(1,0)} = e_{(1,1)}, \ \ldots, \quad b^{a^{-k_3}} = e_{(k_3,1)},$$
$$(b^{a^{-k_3}})^b = e_{(k_3,1)+|{}^{k_3\,1}_{0\,\,1}|(0,1)} = e_{(k_3,k_3+1)}, \ \ldots, \quad (b^{a^{-k_3}})^{b^{k_2}} = e_{(k_3,k_2k_3+1)},$$

where ... means iterating the calculation. Similarly applying $(\ )^{a^{-k_1}}$ gives

$$((b^{a^{-k_3}})^{b^{k_2}})^{a^{-k_1}} = e_{(k_3+k_1(k_2k_3+1),k_2k_3+1)} = \frac{k_3 + k_1(k_2k_3+1)}{k_2k_3+1} = k_1 + \frac{1}{k_2 + \frac{1}{k_3}}.$$

The associated group $\mathrm{As}(\mathbb{ZP}^1) = G_{\mathbb{ZP}^1 \sqcup \mathbb{ZP}^1}$ is generated by $\{a, a^{-1}, b, b^{-1}\}$ with relations $aba = bab$, which is the braid group B_3 on three strands and known to be $\pi_1(K^c)$ for the trefoil. It is also known that $\mathbb{ZP}^1 \sqcup \mathbb{ZP}^1$ is locally skew and embeddable, so our theory suitably adapted to the infinite case gives information about $\mathrm{H}^1_{\mathrm{dR}}(B_3)$ and $\Omega(B_3)$ over $\mathbb{C}$ is a $*$-exterior algebra with a canonical metric. $\diamond$

1.8 Application to Naive Electromagnetism on Discrete Groups

Given a quantum metric and our focus on differential forms, it is natural to ask for a Hodge operator $\circledast : \Omega^i \to \Omega^{n-i}$ in the case of volume dimension n. A general theory of this is missing but a naive approach, which we illustrate here, is to look for a bimodule map squaring to ± 1, as happens classically. This could be of interest to physicists and we explain it in the context of an application to electromagnetism.

We recall that the modern approach to electromagnetism is to see the electric and magnetic fields as the components of a single curvature 2-form F on spacetime. Maxwell's equations become

$$\mathrm{d}F = 0, \quad \delta F = J,$$

where $J \in \Omega^1$ is the source. The first is solved usually by viewing F as the curvature of a $U(1)$ bundle over spacetime and the second involves the divergence or Hodge codifferential $\delta = \circledast \mathrm{d} \circledast$. Noncommutative bundles are covered much later in the book and here we consider only the elementary layer of the theory which applies to trivial bundles, which we call 'Maxwell theory'. The principal quantity of interest is the electromagnetic field F but we write this as $F = \mathrm{d}\alpha$ in terms of a 'gauge potential' $\alpha \in \Omega^1$ considered modulo exact forms, since adding d of something to α does not change F. The class of α in $\mathrm{H}^1_{\mathrm{dR}}$ is also of interest as a reflection of the nontrivial topology of spacetime. There is another '$U(1)$-Yang–Mills' elementary possibility which we mention at the end of the section. In the $*$-algebra case we require $\alpha^* = -\alpha$ and $J^* = -J$. In order to have solutions it is often stated incorrectly in physics that we need $\delta J = 0$, i.e. for the source to be conserved. What we need more precisely is that J is *coexact*, by which we mean $J^\circledast$ is exact (using a super-script notation for the application of $\circledast$). If the penultimate cohomology $\mathrm{H}^{n-1}_{\mathrm{dR}}$ is trivial then being conserved is equivalent to being coexact but otherwise coexact is stronger. Indeed, we can already do this theory in nice cases armed only with an exterior algebra over an algebra A and a quantum metric, but the calculus will typically be inner, $\mathrm{H}^1_{\mathrm{dR}}(A)$ will typically contain the nonclassical element θ and the penultimate cohomology will typically contain $\theta^\circledast$. So $\mathrm{H}^1_{\mathrm{dR}}(A) = \mathbb{C}\theta$ and $\mathrm{H}^{n-1}_{\mathrm{dR}}(A) = \mathbb{C}\theta^\circledast$ will typically be the least amount of cohomology in these degrees that we will need to work with, even for elementary but noncommutative models.

We will look at this in the case where $A = \mathbb{C}(G)$ in a discrete group G and $\Omega(G)$ is a bicovariant calculus as in §1.7.2, which is inner with $\theta = \sum_a e_a$. We take the Euclidean metric (1.9) with metric coefficients in the basis denoted $\eta^{a,b} = \delta_{a,b^{-1}}$ and we assume that there is up to scale a unique top form Vol which we take as some n-fold product of the e_a basis elements. For the volume form to be central we need the degrees of these elements to multiply to e in the group. In this setting we define the 'antisymmetric tensor' ϵ by

$$e_{a_1} \wedge \cdots \wedge e_{a_n} = \epsilon_{a_1\cdots a_n}\mathrm{Vol},$$

which either is zero or has $a_1 a_2 \cdots a_n = e$ in G by centrality. From this tensor we define the Hodge star by

$$\begin{aligned}\circledast(e_{a_1} \wedge \cdots \wedge e_{a_m}) &= \sum_{b,c} d_m^{-1}\epsilon_{a_1\cdots a_m b_{m+1}\cdots b_n}\eta^{b_{m+1},c_{m+1}}\cdots\eta^{b_n c_n} e_{c_n} \wedge \cdots \wedge e_{c_{m+1}} \\ &= \sum_{a_{m+1},\cdots,a_n} d_m^{-1}\epsilon_{a_1\cdots a_n} e_{a_n^{-1}} \wedge \cdots \wedge e_{a_{m+1}^{-1}},\end{aligned}$$

for some normalisation constants d_m. One can check that this extends as a bimodule map. In nice cases we can, moreover, choose the constants so that $\circledast^2 = \epsilon_m \mathrm{id}$, where $\epsilon_m = \pm 1$ depending on the degree m, and we assume this. Note also that $\mathrm{Vol}^\circledast = 1$ by definition. One will also typically have

$$e_a^\circledast \wedge e_b = \epsilon_1 d_{n-1}\eta^{a,b}\mathrm{Vol}, \quad e_a \wedge e_b^\circledast = (-1)^{n-1}\epsilon_1 d_{n-1}\eta^{a,b}\mathrm{Vol} \tag{1.12}$$

and we assume this. In this case we find

$$\mathrm{d}e_a^\circledast = \theta e_a^\circledast - (-1)^{n-1} e_a^\circledast \theta = 0$$

so the e_a are all coclosed. Writing an $n-1$-form as $\beta = \sum_a \beta^a e_a^\circledast$, we find

$$\mathrm{d}\beta = \sum_{a,b} \partial^b \beta^a e_b \wedge e_a^\circledast = \epsilon_1 d_{n-1}(-1)^{n-1}(\sum_a \partial^{a^{-1}}\beta^a)\mathrm{Vol}$$

and from this, $\partial^{a^{-1}}\partial^a = -\partial^a - \partial^{a^{-1}}$ and $\Delta_\theta = -2\sum_a \partial^a$, it follows that

$$\circledast \mathrm{d} \circledast \mathrm{d} = \epsilon_1 d_{n-1}(-1)^{n-1}\Delta_\theta$$

on functions. Finally, when our group is finite, we define $\int f = \sum_{x\in G} f(x)$ as the analogue of integration. We extend this to n-forms by $\int f\mathrm{Vol} = \int f$ and by our assumptions we find that $\int \mathrm{d}\beta = 0$ for any $n-1$-form β.

Proposition 1.73 *Assume the above nice properties of* $\circledast$ *and assume that* $\mathrm{H}^{n-1}_{\mathrm{dR}}(G) = \mathbb{C}\theta^{\circledast}$*, and that* G *is finite. Then* $J = \sum J^a e_a \in \Omega^1$ *is coexact if and only if*

$$\sum_a \partial^{a^{-1}} J^a = 0, \quad \int \sum_a J^a = 0.$$

Any solution ψ *of the wave equation* $\Delta_\theta \psi = m^2\psi$ *generates such a source*

$$J^a = 2(\partial^a\bar\psi)\psi - \partial^a(\bar\psi\psi) + \frac{m^2}{|G||\mathcal{C}|}\int \bar\psi\psi,$$

the 'current' associated to ψ*. In the* $*$*-algebra case, this obeys* $J^* = -J$*.*

Proof By our cohomology assumption on $\mathrm{H}^{n-1}_{\mathrm{dR}}$, if $\mathrm{d}J^{\circledast} = 0$ then $J^{\circledast} = \mu\theta^{\circledast} + \mathrm{d}\beta$ for some constant μ and some $n-2$-form β. We have $\theta^{\circledast}\wedge\theta = \epsilon_1 d_{n-1}|\mathcal{C}|\mathrm{Vol}$ by our assumptions while $\int \mathrm{d}\beta\wedge\theta = \int \mathrm{d}(\beta\wedge\theta) = 0$, so imposing $\int J^{\circledast}\wedge\theta = 0$ forces $\mu = 0$ and hence $J^{\circledast}$ is exact. So

$$\mathrm{d}J^{\circledast} = 0, \quad \int J^{\circledast}\wedge\theta = 0$$

implies $J^{\circledast}$ is exact. The converse is clear, so these two conditions are equivalent to $J^{\circledast}$ being exact. We write them as stated using our formula for $\mathrm{d}J^{\circledast}$ for any $n-1$ form $J^{\circledast}$ and noting that $J^{\circledast}\wedge\theta = \epsilon_1 d_{n-1}(\sum_a J^a)\mathrm{Vol}$ under our assumptions.

For the second part, if ψ is an eigenfunction of $\Delta_\theta = -2\sum_a \partial^a$ then $\int\sum_a J^a = \int(-m^2\bar\psi\psi) + \frac{|G||\mathcal{C}|m^2}{|G||\mathcal{C}|}\int\bar\psi\psi = 0$, where we used that Δ commutes with complex conjugation and that the eigenvalues are real. We next use $\partial^{a^{-1}}\partial^a = -\partial^a - \partial^{a^{-1}}$ and $\partial^a(fg) = (\partial^a f)R_a(g) + f\partial^a g$ for any f, g, and similarly for $\partial^{a^{-1}}$, to compute

$$\begin{aligned}\partial^{a^{-1}} J^a =& -2((\partial^a + \partial^{a^{-1}})\bar\psi)R_{a^{-1}}\psi + 2(\partial^a\bar\psi)\partial^{a^{-1}}\psi + (\partial^a + \partial^{a^{-1}})(\bar\psi\psi)\\ =& -2(\partial^a\bar\psi)\psi - 2(\partial^{a^{-1}}\bar\psi)R_{a^{-1}}\psi + (\partial^a\bar\psi)R_a\psi\\ &+ (\partial^{a^{-1}}\bar\psi)R_{a^{-1}}\psi + \bar\psi(\partial^a + \partial^{a^{-1}})\psi.\end{aligned}$$

Now under a sum over a, we can rename a as a^{-1} and hence the 2nd term cancels with the 3rd and 4th so $\sum_a \partial^{a^{-1}} J^a = \sum_a(-2(\partial^a\bar{\psi})\psi + 2\bar{\psi}\partial^a\psi)$. Both parts are proportional to Δ_θ and cancel when ψ is an eigenfunction with real eigenvalue as is the case here. Hence J is coexact. In the $*$-algebra case $J^* = -J$ corresponds, given $e_a^* = -e_{a^{-1}}$, to $\overline{J^a} = R_a(J^{a^{-1}})$, which holds in our case. □

If the group is infinite, our characterisation of coexact J still applies for any well-defined $\int$ that vanishes on exact n-forms but does not vanish on Vol. Moreover, we can still use the formula for J as constructed from ψ without the last term and the relevant part of the proof still shows that this is coclosed, i.e.

$$J^a = 2(\partial^a\bar{\psi})\psi - \partial^a(\bar{\psi}\psi) = (\partial^a\bar{\psi})\psi + R_a(\bar{\psi}\partial^{a^{-1}}\psi) \tag{1.13}$$

obeys $\sum_a \partial^{a^{-1}} J^a = 0$, just not necessarily co-exact. Also, looking at the proof we can regard $\bar{\psi}$, ψ as independent eigenfunctions with the same value of m^2 but without necessarily having $J^* = -J$. And if the cohomology $\mathrm{H}^{n-1}_{\mathrm{dR}}$ has more than one generator $\theta_i^{\circledast}$ then we can use the global restriction $\int J^{\circledast} \wedge \theta_i = 0$ for each of these to characterise coexact sources provided the θ_i are closed.

The source J being coexact is necessary to be able to solve the Maxwell system and means that $J^{\circledast} = \mathrm{d}(F^{\circledast})$ for some 2-form F. If this is exact, for example if $\mathrm{H}^2_{\mathrm{dR}} = 0$, then we will have the 'gauge potential' α such that $\circledast\mathrm{d}\circledast\mathrm{d}\alpha = J$, as desired. These steps mirror the classical treatment of electromagnetism at the elementary level. In order to be more explicit, it is usual to additionally impose, for example, the 'Coulomb gauge fixing' $\delta\alpha = 0$ which, however, does not completely fix the freedom in α. Parallel to our treatment of J, it makes sense to adopt here the stronger version that α is coexact. Then

$$\circledast\mathrm{d}\circledast\mathrm{d} : \Omega^1_{\mathrm{coexact}} \to \Omega^1_{\mathrm{coexact}}$$

becomes an operator on the space of coexact 1-forms to itself and it is this which we will aim to diagonalise for the Maxwell theory at this level. For an infinite group, we should also put convergence requirements on the component functions.

Example 1.74 (Maxwell Theory on the Permutation Group S_3) From the exterior algebra in Example 1.60, we read off 12 nonzero values

$$\epsilon_{uvuw} = \epsilon_{vuvw} = 1, \quad \epsilon_{wuvu} = \epsilon_{wvuv} = -1$$

and their cyclic rotations under $u \to v \to w \to u$. We also set

$$d_0 = 12, \quad d_1 = 4, \quad d_2 = \sqrt{3}, \quad d_3 = 1, \quad d_4 = 1$$

to give the Hodge star as $\circledast 1 = -\mathrm{Vol}$, $\circledast\mathrm{Vol} = 1$ and

$$\circledast e_u = e_w \wedge e_u \wedge e_v, \quad \circledast e_v = e_u \wedge e_v \wedge e_w, \quad \circledast e_w = e_v \wedge e_w \wedge e_u,$$

$$\circledast(e_u \wedge e_v) = -3^{-\frac{1}{2}}(e_u \wedge e_v + 2e_v \wedge e_w), \quad \circledast(e_v \wedge e_w) = 3^{-\frac{1}{2}}(e_v \wedge e_w + 2e_u \wedge e_v),$$

$$\circledast(e_v \wedge e_u) = 3^{-\frac{1}{2}}(e_v \wedge e_u + 2e_w \wedge e_v), \quad \circledast(e_w \wedge e_v) = -3^{-\frac{1}{2}}(e_w \wedge e_v + 2e_v \wedge e_u),$$

$$\circledast(e_w \wedge e_u \wedge e_v) = -e_u, \quad \circledast(e_u \wedge e_v \wedge e_w) = -e_v, \quad \circledast(e_v \wedge e_w \wedge e_u) = -e_w.$$

The natural normalisations here with positive d_m are such that $\circledast\circledast = -\mathrm{id}$. We check that our assumed nice properties of $\circledast$ hold. Thus $e_u^{\circledast}e_a = -e_v e_w e_v e_a = -e_w e_v e_w e_a = 0$ unless $a = u$, in which case we obtain $-\mathrm{Vol}$ as in (1.12), while similarly $e_a e_u^{\circledast} = -e_a e_v e_w e_v = -e_a e_w e_v e_w = 0$ unless $a = u$, when we have $-e_u e_v e_w e_v = \mathrm{Vol}$, as also required. We have already studied the Laplacian Δ_θ and noting that $\mathrm{d}e_u^{\circledast} = 0$ as it must, and that $(\partial^a)^2 = -2\partial^a$, one can explicitly check that ${}^{\circledast}\mathrm{d}^{\circledast}\mathrm{d} = -\Delta_\theta$, in agreement with the general analysis above.

For the Maxwell theory we already know that the cohomology condition in Proposition 1.73 holds. We are solving $\circledast\mathrm{d}\circledast\mathrm{d}\alpha = J$ and we look for α also in the space of coexact forms. By computer calculation one finds that the eigenvalues here of ${}^{\circledast}\mathrm{d}^{\circledast}\mathrm{d}$ on $\Omega^1_{\mathrm{coexact}}$ are $-6, -12, -18$, each with 4-dimensional eigenspaces, the total space being 12-dimensional. Here 18 dimensions for the choice of the three components of α are cut down to 12 by the coexactness 'gauge fixing'.

Now let ψ obey the scalar wave equation with eigenvalue m^2. Then

$$J^a = 2(\partial^a\bar{\psi})\psi - \partial^a(\bar{\psi}\psi) + \frac{m^2}{18}\int \bar{\psi}\psi$$

obeys our source conservation conditions. The constant function and the sign function both generate zero source but the $m^2 = 6$ modes do not. In particular, the 'point source' ψ_x at any $x \in S_3$ in Example 1.60 gives

$$J_x^a = 1 - 3\delta_x - 3\delta_{xa}$$

as an element of $\Omega^1_{\mathrm{coexact}}$ and we find by direct calculation that it is an eigenmode of $\circledast\mathrm{d}\circledast\mathrm{d}$ with eigenvalue -12, and hence has gauge potential $\alpha = -\frac{1}{12}J_x$. Also,

$$J_{xu} + J_{xv} + J_{xw} = 0,$$

so that *three* symmetrically placed sources about a point x cancel out. As we vary x we account for 4 independent eigenmodes, which is a full set with this eigenvalue.

A further 'dipole' source is given by $x \in S_3$ and $b \in \mathcal{C}$ as

$$J^a_{x;b} = 1 + \frac{1}{2}(9\delta_{a.b} - 6)(\delta_x + \delta_{xa} + \delta_{xb} + \delta_{xab}),$$

which is obtained as follows. We use the polarised version of our $J(\psi)$ construction where $\bar{\psi}, \psi$ are independent eigenfunctions with the same eigenvalue. Specifically, we take a pair of point sources at ψ_x, ψ_{xb} respectively and compute the resulting joint coexact source, which we denote $J^{\mathrm{dipole}}_{x;b}$. We then define $J_{x;b} = \frac{1}{2}(J^{\mathrm{dipole}}_{x;b} + J^{\mathrm{dipole}}_{xb;b})$ to obtain the result stated. One may verify directly that this then is an eigenvector of $\circledast \mathrm{d} \circledast \mathrm{d}$ with eigenvalue -6 so that Maxwell's equations for this source is solved by $\alpha = -\frac{1}{6}J_{x;b}$. Also, there are four independent such sources as we vary x, b, with the result that we have fully diagonalised this eigenspace by means of these 'dipoles'. There are four further, more complicated, sources with eigenvalue -18. $\diamond$

We conclude with an infinite group example by way of contrast.

Example 1.75 (Maxwell Theory on a Lattice) We have already described the calculus, metric and the cohomology in Example 1.62. The tensor ϵ is the usual antisymmetric one because Λ is just the Grassmann algebra on the e_a, where $a = \pm 1, \pm 2$. We had $\mathrm{Vol} = \mathrm{Vol}_1 \wedge \mathrm{Vol}_2$, $\mathrm{Vol}_i = e_{+i} \wedge e_{-i}$ so that $\epsilon_{+1,-1,+2,-2} = 1$ and we let

$$d_0 = 24, \quad d_1 = 6, \quad d_2 = 2, \quad d_3 = 1, \quad d_4 = 1$$

so that

$$1^{\circledast} = \mathrm{Vol}, \quad e_{\pm 1}^{\circledast} = \pm \mathrm{Vol}_2 \wedge e_{\pm 1}, \quad e_{\pm 2}^{\circledast} = \pm \mathrm{Vol}_1 \wedge e_{\pm 2},$$

$$\mathrm{Vol}_1^{\circledast} = \mathrm{Vol}_2, \quad \mathrm{Vol}_2^{\circledast} = \mathrm{Vol}_1, \quad (e_{\pm 1} e_{\pm 2})^{\circledast} = e_{\pm 1} e_{\pm 2}, \quad (e_{\pm 1} e_{\mp 2})^{\circledast} = -e_{\pm 1} e_{\mp 2},$$

$$(\mathrm{Vol}_1 \wedge e_{\pm 2})^{\circledast} = \pm e_{\pm 2}, \quad (\mathrm{Vol}_2 \wedge e_{\pm 1})^{\circledast} = \pm e_{\pm 1}, \quad \mathrm{Vol}^{\circledast} = 1.$$

This has $\circledast^2 = \mathrm{id}$ if we keep all the d_i positive. The assumption (1.12) holds automatically because of the Grassmann algebra. Also because of this,

$$\circledast \mathrm{d} \circledast \mathrm{d}\alpha = -\sum_a (\Delta_\theta \alpha^a) e_a + \sum_a \partial^a \Big(\sum_b \partial^{b^{-1}} \alpha^b\Big) e_a$$

for $\alpha = \sum_a \alpha^a e_a$. If we 'gauge fix' so that $\delta\alpha = 0$ then the second term vanishes and we just need to solve $\Delta_\theta \alpha^a = -J^a$ for each component. Since we have completely diagonalised Δ_θ in Example 1.62 in terms of Fourier modes ψ_{ϕ_1,ϕ_2} we can in principle solve the system with four plane wave components of the same eigenvalue and subject to the coclosed condition.

Here we look at a more geometric subclass of solutions using the travelling waves in Example 1.62 for each component,

$$\alpha^a(x, y) = (-1)^x f^a(x - y); \quad f^{-1} = -R_-(\overline{f^{+1}}), \quad f^{-2} = R_+(\overline{f^{+2}}),$$

where f^a are functions on $\mathbb{Z}$ entering as $\underline{f^a(x - y)}$ on $\mathbb{Z} \times \mathbb{Z}$ and the displayed 'reality conditions' ensure $\alpha^* = -\alpha$ as $R_a(\overline{\alpha^{a^{-1}}}) = \alpha^a$. We solve these with $f^{+i} = f^i$ any functions on $\mathbb{Z}$ and f^{-i} as displayed. For the 'gauge fixing' we find

$$\Big(\sum_a \partial^{a^{-1}} \alpha^a\Big)(x, y) = (-1)^x \big(\partial^+(f^2 - \overline{f^2}) - (\partial^- + 2)(f^1 - \overline{f^1})\big)(x - y),$$

which needs to vanish as a condition on the f^i, for example by taking them real-valued. Since the travelling waves were eigenmodes for the scalar Laplace with eigenvalue 8, we have then solved the Maxwell system with source $J = -8\alpha$. The curvature $F = \mathrm{d}\alpha = \sum_{a,b} \partial^a \alpha^b e_a \wedge e_b$ for our real-valued solutions comes out as

$$\begin{aligned}F = &(-1)^x (2(\partial^- + 2) f^1 \mathrm{Vol}_1 - 2\partial^+ f^2 \mathrm{Vol}_2\\ &- (\partial^- f^1 + (\partial^+ + 2) f^2)(e_{-1}e_{-2} + e_{+1}e_{+2})\\ &- (\partial^+ f^1 + (\partial^+ + 2) R_+ f^2) e_{+1}e_{-2} + (\partial^- R_- f^1 - (\partial^- + 2) f^2) e_{-1}e_{+2}).\end{aligned}$$

One can also seek to solve the Maxwell system for some prescribed source and consider aspects of electromagnetism. We do not do this do this here, except to note that a travelling wave $\psi(x, y) = (-1)^x f(x - y)$ itself gives us a coclosed source

$$J(\bar{\psi}, \psi)^{\pm 1} = -(2(R_\pm \bar{f}) f + R_\pm(\bar{f} f) + \bar{f} f), \quad J(\bar{\psi}, \psi)^{\pm 2} = (\partial^\mp \bar{f}) f + R_\mp(\bar{f}\partial^\pm f)$$

for any complex function f on $\mathbb{Z}$ after a calculation from (1.13). ⋄

The theory here is Euclidean due to the signature and in spite of a 4D cotangent space is in many ways still a 2D lattice. So it is not exactly a model of physical electromagnetism; for that we could similarly use $\mathbb{Z}^4$ with an 8D calculus and a Minkowski signature of metric. We mention also that while we have covered Maxwell theory on an exterior algebra $\Omega(A)$, one can also have a '$U(1)$-Yang–Mills' theory closer to nonabelian gauge theory but still at the elementary level where $\alpha \in \Omega^1$ obeys $\alpha^* = -\alpha$ in the $*$-calculus case. Here two such 'connections' are equivalent if one can be transformed to the other by

$$\alpha \mapsto \alpha^u = u\alpha u^{-1} + u\mathrm{d}u^{-1}, \quad u \in A; \quad u^* = u^{-1}.$$

One can check that $(\alpha^u)^* = -\alpha^u$ and that $F(\alpha) = \mathrm{d}\alpha + \alpha\wedge\alpha$ transforms covariantly as $F \mapsto F(\alpha^u) = uF(\alpha)u^{-1}$ and is also anti-hermitian in the $*$-calculus case. Instead of H_{dR}, we can now ask as an indicator of 'homotopy' on an exterior algebra $\Omega(A)$ if there is a nontrivial moduli of 'flat connections' in the sense of α with $F(\alpha) = 0$ up to such gauge transformations. This can indeed be the case even in the simplest examples such as $\Omega(S_3)$ in Exercise E1.10.

1.9 Application to Stochastic Calculus

Brownian motion is the term applied to random motion of particles in a medium, originally observed by Robert Brown in 1827. The motion is probabilistic, continuous in time, but not differentiable (with probability one). Such motion, applied to a large number of particles to make the result more obvious, gives a diffusion. The Itô and Stratonovich calculi are pathwise probabilistic constructions used to describe diffusion by Brownian motion or more general stochastic effects on manifolds. They are both 'first-order calculi' in some sense. Here we shall not present any probabilistic construction, but rather an algebraic construction of a calculus in our sense of a DGA, which works to all orders and whose first order part has the same form as the stochastic differentials in Itô and Stratonovich theory.

Stochastic calculus deals with the evolution in time of functions on a manifold under the twin effects of diffusion and drift. Our approach will be to model a manifold M through its exterior algebra $\Omega(M)$ and to model diffusion and drift via a homotopy deformation. In fact our construction is purely algebraic and works for any DGA $(\Omega, \mathrm{d}, \wedge)$ on an algebra $A = \Omega^0$, which need not be commutative. First we add a time variable by taking the tensor product of Ω and the classical differential calculus on the line $\mathbb{R}$, where we take t to be the coordinate on the line. This gives

$$\Omega^n_{\mathbb{R}} = \big(\Omega^n \otimes C^\infty(\mathbb{R})\big) \bigoplus \big(\Omega^{n-1} \otimes C^\infty(\mathbb{R})\big) \wedge \mathrm{d}t.$$

We have simply split the forms into those without and with a $\mathrm{d}t$. In the case of a manifold M, $\Omega^n(M) \otimes C^\infty(\mathbb{R})$ is a dense subset of n-forms on $M \times \mathbb{R}$ which have no $\mathrm{d}t$ component. Perhaps it is more illuminating to think of $\Omega^n(M) \otimes C^\infty(\mathbb{R})$ as time dependent n-forms on M, and we shall take this approach. We have the following tensor product operations for $\Omega_{\mathbb{R}}$, making it into a DGA, which we label $(\Omega_{\mathbb{R}}, \mathrm{d}_0, \wedge_0)$. For $\xi \in \Omega^n \otimes C^\infty(\mathbb{R})$ and $\eta \in \Omega^m \otimes C^\infty(\mathbb{R})$, define

$$\mathrm{d}_0(\xi) = \mathrm{d}\xi + (-1)^n \frac{\partial \xi}{\partial t} \wedge \mathrm{d}t, \quad \mathrm{d}_0(\xi \wedge \mathrm{d}t) = \mathrm{d}\xi \wedge \mathrm{d}t,$$
$$\xi \wedge_0 \eta = \xi \wedge \eta, \quad (\xi \wedge \mathrm{d}t) \wedge_0 \eta = (-1)^m (\xi \wedge \eta) \wedge \mathrm{d}t,$$
$$\xi \wedge_0 (\eta \wedge \mathrm{d}t) = (\xi \wedge \eta) \wedge \mathrm{d}t, \quad (\xi \wedge \mathrm{d}t) \wedge_0 (\eta \wedge \mathrm{d}t) = 0.$$

Here $\mathrm{d}\xi$ means applying the differential to the Ω part only, as we have explicitly written the derivative in the time direction. The subscript zero is used as we will introduce a deformation parameter, and this is the case where the parameter is zero.

A *homotopy* is a linear map $\delta : \Omega^n \to \Omega^{n-1}$ for all n (remembering that $\Omega^n = 0$ for $n < 0$). (It is called that because homotopies in algebraic topology really do generate such maps.) Let $\Delta = \delta \mathrm{d} + \mathrm{d}\delta : \Omega^n \to \Omega^n$. By construction $\mathrm{d}\Delta = \Delta \mathrm{d}$. Given δ we construct a deformation of $(\Omega_{\mathbb{R}}, \mathrm{d}_0, \wedge_0)$.

Theorem 1.76 *Given a homotopy $\delta : \Omega^n \to \Omega^{n-1}$ for all n, for a parameter $\alpha \in \mathbb{R}$, there is a DGA $(\Omega_{\mathbb{R}}, \mathrm{d}_\alpha, \wedge_\alpha)$ given by, for $\xi \in \Omega^n \otimes C^\infty(\mathbb{R})$ and $\eta \in \Omega^m \otimes C^\infty(\mathbb{R})$,*

$$\mathrm{d}_\alpha(\xi) = \mathrm{d}\xi + (-1)^n \frac{\partial \xi}{\partial t} \wedge \mathrm{d}t + (-1)^n \alpha \Delta(\xi) \wedge \mathrm{d}t, \quad \mathrm{d}_\alpha(\xi \wedge \mathrm{d}t) = \mathrm{d}\xi \wedge \mathrm{d}t,$$
$$\xi \wedge_\alpha \eta = \xi \wedge \eta - (-1)^{n+m} \alpha \big(\delta(\xi \wedge \eta) - \delta(\xi) \wedge \eta - (-1)^n \xi \wedge \delta(\eta)\big) \wedge \mathrm{d}t,$$

$$(\xi \wedge \mathrm{d}t) \wedge_\alpha \eta = (-1)^m (\xi \wedge \eta) \wedge \mathrm{d}t, \quad \xi \wedge_\alpha (\eta \wedge \mathrm{d}t) = (\xi \wedge \eta) \wedge \mathrm{d}t,$$
$$(\xi \wedge \mathrm{d}t) \wedge_\alpha (\eta \wedge \mathrm{d}t) = 0.$$

If $(\Omega, \mathrm{d}, \wedge)$ *is graded commutative, then so is* $(\Omega_{\mathbb{R}}, \mathrm{d}_\alpha, \wedge_\alpha)$.

Proof First we check that

$$\mathrm{d}_\alpha(\mathrm{d}_\alpha(\xi)) = \mathrm{d}^2\xi + (-1)^{n+1}\frac{\partial\, \mathrm{d}\xi}{\partial t} \wedge \mathrm{d}t + (-1)^{n+1}\alpha \Delta(\mathrm{d}\xi) \wedge \mathrm{d}t$$
$$+ (-1)^n \mathrm{d}\Big(\frac{\partial \xi}{\partial t}\Big) \wedge \mathrm{d}t + (-1)^n \alpha \mathrm{d}(\Delta(\xi)) \wedge \mathrm{d}t = 0.$$

Here $\mathrm{d}^2 = 0$ by definition, which also implies that $\mathrm{d}\Delta = \Delta\mathrm{d} = \mathrm{d}\delta\mathrm{d}$. The partial t-derivative and d commute as they act on different factors of $\Omega^n \otimes C^\infty(\mathbb{R})$. Next,

$$\begin{aligned}
\mathrm{d}_\alpha(\xi \wedge_\alpha \eta) =\ & \mathrm{d}(\xi \wedge \eta) + (-1)^{n+m}\frac{\partial(\xi \wedge \eta)}{\partial t} \wedge \mathrm{d}t + (-1)^{n+m}\alpha\delta\mathrm{d}(\xi \wedge \eta) \wedge \mathrm{d}t \\
& + (-1)^{n+m}\alpha \mathrm{d}\big(\delta(\xi) \wedge \eta + (-1)^n \xi \wedge \delta(\eta)\big) \wedge \mathrm{d}t, \\
\mathrm{d}_\alpha(\xi) \wedge_\alpha \eta =\ & \mathrm{d}\xi \wedge \eta + (-1)^{n+m}\alpha\big(\delta(\mathrm{d}\xi \wedge \eta) + \mathrm{d}\delta\xi \wedge \eta + (-1)^n \mathrm{d}\xi \wedge \delta\eta\big) \wedge \mathrm{d}t \\
& + (-1)^{n+m}\frac{\partial \xi}{\partial t} \wedge \eta \wedge \mathrm{d}t, \\
\xi \wedge_\alpha \mathrm{d}_\alpha(\eta) =\ & \xi \wedge \mathrm{d}\eta + (-1)^{n+m}\alpha\big(\delta(\xi \wedge \mathrm{d}\eta) - \delta\xi \wedge \mathrm{d}\eta + (-1)^n \xi \wedge \mathrm{d}\delta\eta\big) \wedge \mathrm{d}t \\
& + (-1)^m \xi \wedge \frac{\partial \eta}{\partial t} \wedge \mathrm{d}t,
\end{aligned}$$

which combine to

$$\begin{aligned}
&\mathrm{d}_\alpha(\xi \wedge_\alpha \eta) - \mathrm{d}_\alpha(\xi) \wedge_\alpha \eta - (-1)^n \xi \wedge_\alpha \mathrm{d}_\alpha(\eta) \\
&\quad = (-1)^{n+m}\alpha\Big(\mathrm{d}\big((\delta\xi) \wedge \eta + (-1)^n \xi \wedge \delta(\eta)\big) - \big(\mathrm{d}\delta\xi \wedge \eta + (-1)^n \mathrm{d}\xi \wedge \delta\eta\big) \\
&\qquad - (-1)^n\big(- \delta\xi \wedge \mathrm{d}\eta + (-1)^n \xi \wedge \mathrm{d}\delta\eta\big)\Big) \wedge \mathrm{d}t = 0
\end{aligned}$$

as d is a signed derivation. We also show that $\wedge_\alpha$ is associative,

$$\begin{aligned}
\zeta \wedge_\alpha (\xi \wedge_\alpha \eta) = \zeta \wedge \xi \wedge \eta - (-1)^{p+n+m}\alpha\big(&\delta(\zeta \wedge \xi \wedge \eta) - \delta\zeta \wedge \xi \wedge \eta \\
&- (-1)^p \zeta \wedge \delta\xi \wedge \eta - (-1)^{p+n}\zeta \wedge \xi \wedge \delta\eta\big) \wedge \mathrm{d}t,
\end{aligned}$$

for $\zeta \in \Omega^p \otimes C^\infty(\mathbb{R})$. A similar calculation of $(\zeta \wedge_\alpha \xi) \wedge_\alpha \eta$ gives the same. □

This construction is similar to the extension to higher degrees of Proposition 1.22 with $\theta' = \mathrm{d}t$ for an additional variable t, but without becoming non-graded-commutative due to the more symmetric nature of the deformation. Moreover, although the DGA is modified, it remains in fact isomorphic as we now show.

Proposition 1.77 *The map $\mathcal{I} : \Omega^n_{\mathbb{R}} \to \Omega^n_{\mathbb{R}}$ given by*

$$\mathcal{I}(\xi) = \xi - (-1)^n \alpha\delta(\xi) \wedge \mathrm{d}t, \quad \mathcal{I}(\xi \wedge \mathrm{d}t) = \xi \wedge \mathrm{d}t,$$

for $\xi \in \Omega^n \otimes C^\infty(\mathbb{R})$, is an isomorphism of DGAs from $(\Omega_{\mathbb{R}}, \mathrm{d}_0, \wedge_0)$ to $(\Omega_{\mathbb{R}}, \mathrm{d}_\alpha, \wedge_\alpha)$.

Proof The two most difficult parts to check are, for $\eta \in \Omega^m \otimes C^\infty(\mathbb{R})$,

$$\begin{aligned}
\mathcal{I}(\xi) \wedge_\alpha \mathcal{I}(\eta) &= \xi \wedge_\alpha \eta - (-1)^m \alpha\xi \wedge \delta(\eta) \wedge \mathrm{d}t - (-1)^n \alpha\delta(\xi) \wedge \mathrm{d}t \wedge \eta \\
&= \xi \wedge_0 \eta - (-1)^{n+m} \alpha\delta(\xi \wedge_0 \eta) \wedge \mathrm{d}t = \mathcal{I}(\xi \wedge_0 \eta), \\
\mathrm{d}_\alpha \mathcal{I}(\xi) &= \mathrm{d}_0\xi + (-1)^n \alpha\Delta(\xi) \wedge \mathrm{d}t - (-1)^n \alpha \mathrm{d}\delta(\xi) \wedge \mathrm{d}t \\
&= \mathrm{d}_0\xi + (-1)^n \alpha\delta\mathrm{d}(\xi) \wedge \mathrm{d}t = \mathcal{I}(\mathrm{d}_0\xi).
\end{aligned}$$

The inverse of $\mathcal{I}$ is $\mathcal{I}^{-1}(\xi) = \xi + (-1)^n \alpha\delta(\xi) \wedge \mathrm{d}t$ and $\mathcal{I}^{-1}(\xi \wedge \mathrm{d}t) = \xi \wedge \mathrm{d}t$. □

We give two natural examples of homotopies δ when the original DGA $(\Omega, \mathrm{d}, \wedge)$ is the exterior algebra on a classical manifold M with local coordinates x^μ.

Example 1.78 (Lie Derivative or Drift Homotopy) Let M be a classical manifold and i the interior product of a vector field and an n-form, giving an $n-1$ form. E.g.,

$$i_{\frac{\partial}{\partial x^1}}(\mathrm{d}x^1 \wedge \mathrm{d}x^2) = \mathrm{d}x^2, \quad i_{\frac{\partial}{\partial x^2}}(\mathrm{d}x^1 \wedge \mathrm{d}x^2) = -\mathrm{d}x^1.$$

For $i_{\frac{\partial}{\partial x^i}}\xi$ on a differential form ξ, use anticommutativity of the exterior algebra to move its $\mathrm{d}x^i$ factor to the front and cancel with $\frac{\partial}{\partial x^i}$, or get zero if there is no such factor. If $v = \sum v^a \frac{\partial}{\partial x^a}$ then we define $i_v\xi = v^a i_{\frac{\partial}{\partial x^a}}\xi$.

Now let M be equipped with a fixed 'drift' vector field v and set $\delta_v = i_v$ the interior derivative along v. Then the associated $\Delta_v = \mathrm{d}\delta_v + \delta_v\mathrm{d} = \mathcal{L}_v$, the Lie derivative along v in the Cartan form. For example, on a function f,

$$\Delta_v(f) = \delta_v \mathrm{d}f = v^a (i_{\frac{\partial}{\partial x^a}} \mathrm{d}x^b) \frac{\partial f}{\partial x^b} = v^a \frac{\partial f}{\partial x^a}$$

is the usual derivative of f in the direction of the vector field v, while on a 1-form η,

$$\Delta_v(\eta) = \Delta_v\big(\eta_b \mathrm{d}x^b\big) = \mathrm{d}\delta_v\big(\eta_b \mathrm{d}x^b\big) + \delta_v \mathrm{d}\big(\eta_b \mathrm{d}x^b\big) = \frac{\partial v^a}{\partial x^b}\eta_a \mathrm{d}x^b + v^a \frac{\partial \eta_b}{\partial x^a}\mathrm{d}x^b$$

is the Lie derivative of the 1-form η along the vector field v. The Lie derivative is used to model flows or drift on manifolds. $\diamond$

Example 1.79 (Second-Order or Diffusion Homotopy) Let M be a classical manifold equipped with a fixed bivector field with components $g^{\mu\nu}$ and a covariant derivative ∇_ν with Christoffel symbols $\Gamma^\kappa{}_{\nu\mu}$. We set a natural 'codifferential',

$$\delta_{g,\nabla}(\xi) = g^{\mu\nu} i_{\frac{\partial}{\partial x^\mu}} \nabla_\nu(\xi).$$

The associated 2nd order operator $\Delta_{g,\nabla} = \mathrm{d}\delta_{g,\nabla} + \delta_{g,\nabla}\mathrm{d}$ is, on functions,

$$\begin{aligned}\Delta_{g,\nabla}(f) = \delta_{g,\nabla}(\mathrm{d}f) \;&=\; \delta_{g,\nabla}\big(\frac{\partial f}{\partial x^\kappa}\mathrm{d}x^\kappa\big) \;=\; g^{\mu\nu} i_{\frac{\partial}{\partial x^\mu}} \nabla_\nu\big(\frac{\partial f}{\partial x^\kappa}\mathrm{d}x^\kappa\big)\\ &= g^{\mu\nu}\frac{\partial^2 f}{\partial x^\nu\,\partial x^\mu} - g^{\mu\nu}\frac{\partial f}{\partial x^\kappa}\Gamma^\kappa{}_{\nu\mu},\end{aligned}$$

where $\nabla_\mu \mathrm{d}x^\nu = -\Gamma^\nu{}_{\mu\rho}\mathrm{d}x^\rho$. This is 2nd order in the sense of Definition 1.17 with respect to $(\ ,\ )$ defined by the symmetrization of $g^{\mu\nu}$.

In particular, if M is a Riemannian manifold with metric $g^{\mu\nu}$ and ∇_ν the associated Levi-Civita connection, then $\delta_{g,\nabla} = \delta_g$, the usual Hodge codifferential. For example, on functions f,

$$\Delta_{g,\nabla} f = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\nu}\Big(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\Big) = \Delta_{\mathrm{LB}} f,$$

the Laplace–Beltrami operator featuring in the heat equation and diffusion processes on Riemannian manifolds. On higher forms, one has the Hodge–Laplacian. $\diamond$

In the Itô or Stratonovich theory we have both diffusion via a Laplacian and drift via a vector field. To model this we linearly combine the preceding two homotopies.

Example 1.80 Let M be a Riemannian manifold with metric g and equipped with a 'drift' vector field v. The Itô–Stratonovich DGA $(\Omega_{\mathbb{R}}, \mathrm{d}_I, \wedge_I)$ is defined by Theorem 1.76 applied to $(\Omega(M), \mathrm{d}, \wedge)$ with

$$\Delta = \frac{1}{2}\Delta_{\mathrm{LB}} + \Delta_v, \quad \delta = \frac{1}{2}\delta_g + \delta_v.$$

In the case of a function f and a 1-form η we have

$$\mathrm{d}_I f = \mathrm{d}f + \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\Delta_{\mathrm{LB}}\mathrm{d}t + v^a\frac{\partial f}{\partial x^a}\mathrm{d}t,$$

$$\eta \wedge_I f = f \wedge_I \eta = f\eta + \frac{1}{2}(i_{\mathrm{grad}(f)}\eta)\mathrm{d}t,$$

where grad is the usual gradient of a function. Note that $i_{\mathrm{grad}(f)}\eta = (\mathrm{d}f, \eta)$ in terms of the metric inner product notation used in Proposition 1.22, from which we see that this is a commutative version of a similar construction there. $\diamond$

For the connection with stochastic calculus, we limit ourselves to a vague account in the simplest case where $M = \mathbb{R}$. We are interested in paths B in M for motion from time 0 say to time T. Thus, we consider $B \in C([0, T], M)$, the set of continuous maps of interest, with certain boundary conditions. Here $B_t \in M$ for each $t \in [0, T]$. Now, if B is piecewise differentiable and $f \in C^\infty(\mathbb{R} \times M)$, we define

$$I_f(B) = \int_0^T f(B_t, t)\frac{\mathrm{d}B_s}{\mathrm{d}t}\mathrm{d}t$$

as the average value of f along the path B. Now when B is not differentiable we can still make sense of such $I_f(B)$ in different ways that have the same value in the differentiable case. We consider the Itô and the Stratonovich integral, which when $M = \mathbb{R}$ and t_i denotes a partition of the interval, are respectively

$$\int_0^T f(B_t, t)\mathrm{d}B_t := \lim \sum f(B_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i}),$$
$$\int_0^T f(B_t, t) \circ \partial B_t := \lim \sum \frac{1}{2}(f(B_{t_i}, t_i) + f(B_{t_{i+1}}, t_{i+1}))(B_{t_{i+1}} - B_{t_i}),$$

where $\circ\partial$ in the stochastic literature is used to distinguish a Stratonovich integral from an Itô integral.

Both expressions for $I_f(B)$ on the left are formal and defined by the right-hand sides. In either case we need to make the path space into a measure space with a particular 'Brownian' measure such that the above are defined on a randomly chosen path with probability 1. As paths that are differentiable have probability 0, we really do need to go beyond these. For each f we then interpret I_f as a random variable on this probability or measure space. An expression in the Itô or Stratonovich stochastic calculus is really shorthand for the corresponding integrals.

Stochastic processes also allow for the inclusion of a drift v and a Riemannian metric g varying from point to point on an n-manifold. We limit ourselves to the case of paths in $\mathbb{R}^n$ now described as a process $X_t = (X_t^1, \cdots, X_t^n)$. Rather than define this intrinsically, it is more convenient to formulate it as a modification of Brownian motion $B_t = (B_t^1, \cdots, B_t^n)$ defined with respect to a standard constant metric and no drift, the two being related at a formal level by

$$\mathrm{d}X_t^i = v^i(X_t)\mathrm{d}t + \sigma^i{}_j(X_t)\mathrm{d}B_t^j$$

for some matrix field σ on $\mathbb{R}^n$ related to the metric and some 'drift' vector field v. These data also enter into the appropriate probability measure on the path space. If

f is a smooth function on $\mathbb{R}^{n+1}$ then Itô's formula implies

$$\mathrm{d}f(X_t,t)=\big(\frac{\partial f}{\partial t}(X_t,t)+\frac{1}{2}g^{ij}(X_t)\frac{\partial^2 f}{\partial x^i\partial x^j}(X_t,t)\big)\mathrm{d}t+\frac{\partial f}{\partial x^i}(X_t,t)\mathrm{d}X_t^i, \tag{1.14}$$

for the behaviour of the X_i, where $g=(\sigma\sigma^T)^{-1}$. From a probabilistic point of view, the second derivative term arises from rapid variations of the paths that have probability 1 in our measure. The Itô differential of a product is given by the formula

$$\mathrm{d}(f(X_t)h(X_t))=f(X_t)\mathrm{d}h(X_t)+h(X_t)\mathrm{d}f(X_t)+g^{ij}(X_t)\frac{\partial f}{\partial x^i}\frac{\partial h}{\partial x^j}\mathrm{d}t. \tag{1.15}$$

The Stratonovich integrals can also be extended to $\mathbb{R}^n$ and are related by

$$f\circ\partial X_t^i=f\mathrm{d}X_t^i+\frac{1}{2}\frac{\partial f}{\partial x^i}\mathrm{d}t. \tag{1.16}$$

The notation indicates that unlike the Itô differential, this one has ∂ effectively behaving like a derivation with respect to a certain product $\circ$.

This completes our lightning introduction to stochastic calculus. Now we have to say how algebraic aspects of the above are encoded in the DGA in Example 1.80 for functions and 1-forms. The basic principle is that the Stratonovich calculus, with differential written ∂ and effective product $\circ$ is encoded by the DGA $(\Omega_{\mathbb{R}},\mathrm{d}_I,\wedge_I)$ in Example 1.80, whereas the Itô calculus has differential d_I but the undeformed product $\wedge_0$, which explains why the Itô differential is not a derivation as seen in (1.15), and hence not encoded as a DGA. In both cases the comparison is made by replacing the symbols X_t^i by the coordinate function x^i. To this extent, the isomorphism in Proposition 1.77 explains the observation that the Stratonovich theory is closely related in its form to the classical undeformed calculus.

We can see this in more detail. We write the differential in Example 1.80 as

$$\mathrm{d}_I f=\Big(\frac{\partial f}{\partial t}+\frac{1}{2}g^{ij}\frac{\partial^2 f}{\partial x^i\partial x^j}\Big)\mathrm{d}t+\frac{\partial f}{\partial x^i}\mathrm{d}_I x^i,$$

where $\mathrm{d}_I x^i$ includes the Christoffel symbols and the drift term part of $\mathrm{d}_I f$. This formula now has the same form as (1.14), if we remember that the Itô derivative written d in (1.14) corresponds to d_I and use the coordinate functions x_i in place of the processes X_t^i. Next, apply d_I to an ordinary product of functions f,h to get

$$\mathrm{d}_I(fh)=f\mathrm{d}_I h+h\mathrm{d}_I f+g^{ik}\frac{\partial f}{\partial x^i}\frac{\partial h}{\partial x^k}\mathrm{d}t,$$

which has the same form as (1.15). Finally we calculate

$$f^i\wedge_I \mathrm{d}_I x^i=f^i\mathrm{d}_I x^i+\frac{1}{2}\frac{\partial f^i}{\partial x^i}\mathrm{d}t,$$

which has the same form as (1.16).

Our observations also suggest that one can extend stochastic calculus to differential forms of all orders as governed by the Itô–Stratonovich DGA in Example 1.80 and that there are cohomological aspects reflected in the cohomology of the DGA which, by Proposition 1.77, is the same as the classical de Rham cohomology.

Exercises for Chap. 1

E1.1 Let A be a unital algebra. Show that any morphism $\Omega^1 \to \Omega^{1\prime}$ between differential calculi on A is surjective and unique. If $\Omega^1 = \Omega^1_{\mathrm{uni}}/N$ and $\Omega^{1\prime} = \Omega^1_{\mathrm{uni}}/N'$ then show that such a morphism exists if and only if $N \subseteq N'$. [Hence classifying Ω^1 up to isomorphism is equivalent to classifying sub-bimodules, while irreducible calculi correspond to maximal sub-bimodules.]

E1.2 If Ω^1_A, Ω^1_B are calculi on algebras A, B respectively, show that an algebra map $\phi : A \to B$ is differentiable if and only if $(\phi \otimes \phi)(N_A) \subseteq N_B$ for the corresponding sub-bimodules.

E1.3 Show that if $A \subseteq B$ is a subalgebra and B has a calculus Ω^1_B then $\Omega^1_A := \{\sum a_i \mathrm{d}a'_i \mid a_i, a'_i \in A\} \subseteq \Omega^1_B$ defines a calculus on A with the inclusion map $A \subseteq B$ differentiable. [It is the smallest in the sense that there are no proper quotients to another calculus with this property.]

E1.4 Show that if $B = A/I$ by a 2-sided ideal I and A has a calculus Ω^1_A then $\Omega^1_B := \Omega^1_A/\langle I\Omega^1_A, \Omega^1_A I, \mathrm{d}I\rangle$ (where we quotient by the sub-bimodule generated) defines a calculus on B with the canonical map $A \to B$ differentiable. [It is in some sense the largest with this property.]

E1.5 (2-Parameter and fuzzy matrix torus.) (i) Generalise the noncommutative torus in Example 1.36 to $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$ with the same algebra $vu = e^{\mathrm{i}\theta}uv$ but the calculus

$$\mathrm{d}u.u = qu.\mathrm{d}u, \quad \mathrm{d}v.v = qv.\mathrm{d}v, \quad \mathrm{d}v.u = e^{\mathrm{i}\theta}u.\mathrm{d}v, \quad \mathrm{d}u.v = e^{-\mathrm{i}\theta}v.\mathrm{d}u.$$

Show that this is a $*$-calculus when q is real, is inner when $q \neq 1$ and has an exterior algebra with $(\mathrm{d}u)^2 = (\mathrm{d}v)^2 = 0$ and $\mathrm{d}v \wedge \mathrm{d}u = -e^{\mathrm{i}\theta}\mathrm{d}u \wedge \mathrm{d}v$ (this is the maximal prolongation when $q \neq -1$).

(ii) Show when $q = e^{\mathrm{i}m\theta}$ for some integer m that $t = v^{-m}u^{-1}\mathrm{d}u$, $s = u^m v^{-1}\mathrm{d}v$ is a basis of central 1-forms.

(iii) When q is a primitive n-th root of unity, show that the general calculus descends to the 'reduced fuzzy torus' $c_{q,\theta}[\mathbb{T}^2]$ with further relations $u^n = v^n = 1$ and that if $\omega = e^{\mathrm{i}\theta}$ is also a primitive n-th root then the latter is isomorphic as an algebra to $M_n(\mathbb{C})$ by

$$\phi(u)=U=\begin{pmatrix}1&0&0&\cdots\\0&\omega&0&\cdots\\\cdots&\cdots&\cdots&\cdots\\\cdots&\cdots&0&\omega^{n-1}\end{pmatrix},\quad \phi(v)=V=\begin{pmatrix}0&1&0&\cdots\\0&0&1&\cdots\\\cdots&\cdots&\cdots&\cdots\\1&0&\cdots&0\end{pmatrix}.$$

In the case $q=\omega$, transfer the calculus to a 2D calculus on $M_n(\mathbb{C})$ such that ϕ is a diffeomorphism. For $n=2$, show that this 2D calculus on $M_2(\mathbb{C})$ is isomorphic to one given by $(0,-1,1)$ in the $\mathbb{CP}^2$ moduli space in Example 1.8.

E1.6 (Fuzzy matrix sphere.) Let $\lambda=\mathrm{i}/n$ and $n=2j+1$ where j is a half-integer. Check that the standard spin j representation $\rho: U(su_2)\to M_n(\mathbb{C})$ descends to $\mathbb{C}_\lambda[S^2]$ as claimed in Example 1.46. Define the reduced fuzzy sphere $c_\lambda[S^2]\cong M_n(\mathbb{C})$ by further relations among the x_i so as to generate the kernel of ρ and show that the calculus descends to $\Omega^1(c_\lambda[S^2])$ and hence to a diffeomorphic 3D calculus $\Omega^1(M_n)$. Show for $n=2$ that the reduced fuzzy sphere is the Pauli matrix algebra with relations $x_ix_j=\frac{1}{4}\delta_{ij}+\frac{\mathrm{i}}{2}\epsilon_{ijk}x_k$ and for $n=3$ the spin-1 algebra with relations

$$[x_i,x_j]=\frac{2\mathrm{i}}{3}\epsilon_{ijk}x_k,\quad \sum_i x_i^2=\frac{8}{9},\quad x_ix_jx_k=\frac{8\mathrm{i}}{27}\epsilon_{ijk}+\frac{4}{9}\delta_{ij}x_k+\frac{2\mathrm{i}}{3}\epsilon_{ikm}x_mx_j$$

and describe their inherited differentials and exterior algebra. For $n=2$, describe the result as a quotient of the universal exterior algebra on $M_2(\mathbb{C})$.

E1.7 In the classification of 2D calculi on $M_2(\mathbb{C})$ in Example 1.8, show (i) that the calculi for $(1,r,s)$ for $r\neq 0$ and $s=-\frac{1}{r}$ in the $\mathbb{CP}^2$ moduli space are all diffeomorphic in the sense of (1.1) to the calculus for $(0,1,0)$ by

$$\phi: M_2(\mathbb{C})\to M_2(\mathbb{C}),\quad \phi(x)=\Phi x\Phi^{-1},\quad \Phi=\begin{pmatrix}1&-\frac{1}{r}\\0&1\end{pmatrix}.$$

First show that conjugating θ for the $(1,r,-\frac{1}{r})$ calculus gives $-\frac{r}{2}(E_{11}-E_{22})\oplus(-\frac{1}{2r}(E_{11}-E_{22})+E_{21})$ for a diffeomorphic calculus. Now use the isomorphism of calculi given by the bimodule map $x\oplus y\mapsto(-\frac{1}{r}x)\oplus(y-\frac{1}{r^2}x)$ to obtain the standard form of θ for $(0,1,0)$. (ii) Show by a similar analysis that the points $(1,r,s)$ with $rs\neq-1$ are not diffeomorphic to the point $(0,1,0)$. [Note that any automorphism of $M_2(\mathbb{C})$ is given by conjugation.]

E1.8 Let $A = \mathbb{C}(X)$, where $X = \{x, y, z\}$ is the set of vertices of the graph

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[dl] \\ z & \end{tikzcd}$$

Find the maximal prolongation of the associated calculus and its H_{dR}.

E1.9 For the general β-calculus in Example 1.43 (b) on the spacetime algebra $[x, t] = \lambda x$, find a basis of central 1-forms for β real and allowing real powers of x. Hence, or otherwise, find the general central quantum metric up to normalisation and a possible shift in t so as to put this in a form extending the formula for the quantum metric for $\beta = 1$ given in the text.

E1.10 For $\Omega(S_3)$ in Example 1.60, show that the following 1-forms $\alpha \in \Omega^1$ obey $\alpha^* = -\alpha$ and $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$:

(i) $\alpha = (\mu - 1)\theta$ with $\mu \in \mathbb{R}$;
(ii) $\alpha = f e_u - \theta$ with $R_u(f) = f^* \in \mathbb{C}(S_3)$;
(iii) $\alpha = (\lambda\delta_u + \bar{\lambda}\delta_e)e_u + (\mu\delta_w + \bar{\mu}\delta_{vu})e_v + (\nu\delta_v + \bar{\nu}\delta_{uv})e_w - \theta$ with $\lambda, \mu, \nu \in \mathbb{C}$.

[These and their $u \to v \to w \to v$ rotations exhaust all real nontrivial '$U(1)$-Yang–Mills' flat connections on this calculus. With more work, one can show that the nonzero α are not $\gamma^{-1}\mathrm{d}\gamma$ for any invertible γ in the algebra.]

Notes for Chap. 1

The universal calculus on an algebra at least at first-order has a long history, starting with the module of Kähler differentials in the commutative case. Such a conception of differentials was used by Quillen and others in the 1970s, notably as an approach to super-geometry, and in the early 1980s in works of Connes on the noncommutative torus [89] and in the formal theory relating Hochschild and cyclic cohomology [84, 177]. Differential calculi here were secondary, for example being implied by a 'Dirac operator' as an element of KO-homology in Connes' approach to noncommutative geometry. Since the advent of quantum groups in the mid 1980s, a different approach to noncommutative geometry emerged, which we follow, taking an algebra equipped with a choice of calculus as the starting point and often guided by symmetry.

The analysis of differentials on matrix algebras in Example 1.8 is from [35]. The analysis of differentials on the algebraic line appeared in [205, 209] and on the circle in [225] although both have featured as elements of more complicated examples. Their cohomology in Example 1.34 is from [225]. The differentials on the Heisenberg algebra in terms of finite commutative algebras in Example 1.14

is an immediate extension of this result for the symmetric algebra of a vector space in [232, 240], where the latter studied the case over $\mathbb{F}_2$. Heisenberg algebras can be viewed as defined by a Moyal product and this in turn can be seen as an example of a 'cocycle twist'. The differential calculus similarly twists and gives $\mathrm{d}x^\mu$ central [212] or $\circ$ zero, but we saw that this is just one among many possible differential structures. Differentials on finite sets are implicit in [84] and were extensively studied in [222], from where §1.4 is taken, notably the functorial picture and remarks about parallelisation. Exterior algebra on finite groups associated to Cayley graphs follows trivially from Woronowicz's treatment for general Hopf algebras (see Chap. 2), but an early work in the physics literature is [56]. The exterior algebra for 2-cycles on S_n and the well-known conjecture about it in §1.7.3 appeared in [214] while the naive electromagnetism on S_3 in §1.8 is from [233] as is the analysis of the Laplacian for this model in Example 1.60. A similar treatment of naive electromagnetism on another finite-dimensional algebra (the reduced quantum group $c_q[SL_2]$) is in [123]. The nice thing about these finite models is that the moduli spaces of gauge fields are finite-dimensional and fully computable as precise models of continuum geometry. In another direction, the application to stochastic calculus is due to [5] and we have followed that treatment, which also potentially links up with Hodge theory. Higher order calculi for Itô calculus were first constructed by using the Moyal product in [100]. For more details on stochastic calculus, see [171].

Theorem 1.41 and Corollary 1.42 on $\Omega(U(\mathfrak{g}))$ are due to Tao and the second author in [238] and [240] respectively. The bicrossproduct model or Majid–Ruegg spacetime appeared in [235]. Its noncommutative differential geometry in the β calculus in Example 1.43(b) is due to the present authors in [32]. Its noncommutative differential geometry in the α calculus in Example 1.43(a) is due to [237]. The systematic analysis of the possible calculi on the $n = 2$ model here appeared in [240], although the α calculus was also found independently in the physics literature [248]. We will return to these models in Chap. 9. Meanwhile, the 4D calculus on the angular momentum algebra in Example 1.45 is due to [17] and explored further in [117]. The different (non-connected) 3D calculus that descends to the fuzzy sphere in Example 1.46 appears to be new. Finally, differential calculi on group algebras were part of the systematic classification of differentials on quantum groups by the second author in [204]. The detailed calculations for $\mathbb{C}S_3$ are mostly from [239].

We explored the notion of quantum metric and of 2nd order operator or Laplacian as defined for an inner calculus in §1.3. The strict notion of quantum metric we use and the centrality Lemma 1.16 are due to the present authors [32] while the inner calculus Laplacian ${}_\theta\Delta$ is due to [222]. The central extension of a calculus by a metric is due to [220]. Returning to the geometry of finite groups, the notion of braided Lie algebra and its specialisation to quandles for the geometry of finite groups is due to the second author in [200] with a follow-up in [179] and [234]. The main results in §1.7.3, including the notion of IP and locally skew quandles and the results on $\mathrm{H}^1_{\mathrm{dR}}(G)$ for crystallographic Weyl groups, are due to Rietsch and the second author and closely follow [234]. The proof that $\mathbb{ZP}^1 \sqcup \mathbb{ZP}^1$ is locally skew can be found there. The fundamental quandle of a knot goes back to Joyce [151] and Matveev [246]. Its isomorphism with $\mathbb{ZP}^1$ in the case of the trefoil is taken from [269].

Exercises E1.1–E1.4 are part of the elementary theory but are useful to clarify the notion of a differentiable map. The latter was used, for example, to define a functor from directed graphs to differential algebras in [222]. That the noncommutative torus and fuzzy sphere as algebras reduce at certain parameter values to matrix algebras is well known, but its extension to differential structures in E1.5 (where we introduce $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$) and E1.6 appear to be new. E1.9 is from [237] and E1.10 is from [233].

Chapter 2
Hopf Algebras and Their Bicovariant Calculi

Hopf algebras or 'quantum groups' can be viewed as analogues of functions on groups. Just as Lie groups played a key role in the development of classical differential geometry as a source of examples, the same will be true for quantum groups in this book. We start with a brief recap of these objects so as to be self-contained.

In fact, the axioms of a ($\mathbb{Z}_2$-graded) Hopf algebra were introduced by Hopf in 1947 but the theory lay mainly dormant for several decades, being mainly used as a tool to unify group and Lie algebra constructions. This changed in the 1980s with the emergence from mathematical physics of true 'quantum groups' or Hopf algebras that went beyond groups and Lie algebras. There emerged and remain to this day two main classes, both of which we shall need. The first class comprises the examples of Drinfeld and Jimbo associated to complex simple Lie algebras, and the second the 'bicrossproduct' class associated to local factorisations of Lie groups.

2.1 Hopf Algebras

We go back to the notion of an algebra over a field $\Bbbk$, now expressing all structures as linear maps. We have a product map $m : A \otimes A \to A$ and a unit element 1_A, which we write equivalently as a map $\eta : \Bbbk \to A$ by $\eta(1) = 1_A$. In these terms, the usual axioms of an algebra are given by the commutative diagrams in Fig. 2.1. Many algebraic constructions can similarly be expressed as commuting diagrams. On the other hand, when all premises, statements and proofs of a theorem are written out like this then reversing all arrows yields the premises, statements and proofs of a different 'dual' theorem. We apply such a duality principle.

E. J. Beggs, S. Majid, *Quantum Riemannian Geometry*,
Grundlehren der mathematischen Wissenschaften 355,
https://doi.org/10.1007/978-3-030-30294-8_2

$$
\begin{tikzcd}
& A\otimes A\otimes A \arrow[dl, "m\otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes m"] & \\
A\otimes A \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[dr, "m"] & \\
\mathbb{k}\otimes A \arrow[u, "\eta\otimes \mathrm{id}"] \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[dr, "m"] & \\
A\otimes \mathbb{k} \arrow[u, "\mathrm{id}\otimes \eta"] \arrow[r, equal] & A
\end{tikzcd}
$$

Fig. 2.1 Associativity and unit element expressed as commutative diagrams

Definition 2.1 A coalgebra C is

(1) A vector space over $\mathbb{k}$.
(2) A map $\Delta : C \to C \otimes C$ (the 'coproduct') which is coassociative in the sense that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.
(3) A map $\epsilon : C \to \mathbb{k}$ (the 'counit') obeying $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$.

These axioms are given by the commutative diagrams in Fig. 2.2, which is just Fig. 2.1 with all arrows reversed. One also has a more explicit 'Sweedler notation'

$$\Delta c =: \sum c_{(1)} \otimes c_{(2)}$$

for all $c \in C$, where we number the tensor factors in the output of Δ. In this form, (2) and (3) appear as

$$\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)},$$
$$\sum \epsilon(c_{(1)}) c_{(2)} = c = \sum c_{(1)} \epsilon(c_{(2)}).$$

$$
\begin{tikzcd}
& C\otimes C\otimes C & \\
C\otimes C \arrow[ur, "\Delta\otimes \mathrm{id}"] & & C\otimes C \arrow[ul, "\mathrm{id}\otimes \Delta"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C\otimes C \arrow[d, "\epsilon\otimes \mathrm{id}"'] & \\
\mathbb{k}\otimes C \arrow[r, equal] & C \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C\otimes C \arrow[d, "\mathrm{id}\otimes \epsilon"'] & \\
C\otimes \mathbb{k} \arrow[r, equal] & C \arrow[ul, "\Delta"']
\end{tikzcd}
$$

Fig. 2.2 Coassociativity and counit element expressed as commutative diagrams

In view of the first equality, we can renumber iterated coproducts linearly as $\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ etc., keeping the order of factors but not the 'history'.

This notion of reversing arrows has the same status as the idea, familiar in algebra, of having both left and right module versions of a construction. The theory with only left modules is equivalent to the theory with right modules, by a left-right reflection (a reversal of tensor products). But one can also consider both left and right modules interacting in some way, e.g., bimodules. Similarly, we can have constructions involving both algebras and coalgebras. This leads to the concept of a bialgebra or a Hopf algebra as a very natural 'closure' of algebra to a setting which is invariant under the arrow-reversal operation.

Definition 2.2 A bialgebra H is

(1) An algebra H, m, η.
(2) A coalgebra H, Δ, ϵ.
(3) Δ, ϵ are algebra maps, where $H \otimes H$ has the tensor product algebra structure $(h \otimes g)(h' \otimes g') = hh' \otimes gg'$ for all $h, h', g, g' \in H$.

Actually, a bialgebra is more like a quantum 'semigroup'. We still need something playing the role of group inversion, which can be done as follows.

Definition 2.3 A Hopf algebra H is

(1) A bialgebra $H, \Delta, \epsilon, m, \eta$.
(2) A map $S : H \to H$ (the 'antipode') such that $\sum (Sh_{(1)})h_{(2)} = \epsilon(h) = \sum h_{(1)} Sh_{(2)}$ for all $h \in H$.

A Hopf algebra is commutative if its algebra is. A Hopf algebra is cocommutative if $\text{flip} \circ \Delta = \Delta$, where flip is the 'flip' map $\text{flip}(h \otimes g) = g \otimes h$ for all $h, g \in H$.

The axioms that make a simultaneous algebra and coalgebra into a Hopf algebra are shown in Fig. 2.3. One can see its self-dual form under arrow-reversal. Here is a noncommutative noncocommutative example.

$$\begin{tikzcd}
H \otimes H \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"] & H \arrow[r, "\Delta"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}"] & & H \otimes H \otimes H \otimes H \arrow[u, "m \otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\epsilon"] & \Bbbk \\
H \otimes H \arrow[u, "m"] \arrow[ur, "\epsilon \otimes \epsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[dr, "\eta \otimes \eta"'] & H \arrow[d, "\Delta"] \\
& H \otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & \Bbbk \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "\mathrm{id} \otimes S,\, S \otimes \mathrm{id}"] & & H \otimes H \arrow[u, "m"']
\end{tikzcd}$$

Fig. 2.3 Additional axioms that make the algebra and coalgebra H into a Hopf algebra

Example 2.4 Let H be the 4-dimensional algebra generated by t, x with the relations $t^2 = 1$, $x^2 = 0$, $xt = -tx$ and let $\Delta 1 = 1 \otimes 1$, $\Delta t = t \otimes t$, $\Delta x = x \otimes t + 1 \otimes x$, $\Delta(tx) = tx \otimes 1 + t \otimes tx$ with $\epsilon 1 = \epsilon t = 1, \epsilon x = \epsilon(tx) = 0$. This is a Hopf algebra with $St = t$ and $Sx = tx$ (a case of the Sweedler–Taft algebra in Example 2.65).

This example and the following general result should convince the reader that these are good axioms in that the theory works as expected for some kind of 'inversion' operation. It is also provides a first exposure to the above notations and techniques.

Proposition 2.5 (Antihomomorphism Property of Antipodes)

(1) *If it exists, the antipode of a Hopf algebra is unique and obeys:*
(2) $S(hg) = S(g)S(h)$, $S(1) = 1$, *for all* $h, g \in H$ *(*S *is an* antialgebra map*);*
(3) $(S \otimes S) \circ \Delta h = \mathrm{flip} \circ \Delta \circ Sh$, $\epsilon Sh = \epsilon h$, *for all* $h \in H$ *(*S *is an* anticoalgebra map*).*

Proof During proofs, we will usually omit the $\sum$ signs, which should be understood. If S, S_1 are two antipodes on a bialgebra H then they are equal because $S_1 h = (S_1 h_{(1)})\epsilon(h_{(2)}) = (S_1 h_{(1)})h_{(2)(1)}Sh_{(2)(2)} = (S_1 h_{(1)(1)})\, h_{(1)(2)}Sh_{(2)} = \epsilon(h_{(1)})Sh_{(2)} = Sh$. Here we wrote $h = h_{(1)}\epsilon(h_{(2)})$ by the counit axioms, and then inserted $h_{(2)(1)}Sh_{(2)(2)}$ knowing that it would collapse to $\epsilon(h_{(2)})$. We then used associativity and (the more novel ingredient) coassociativity to be able to collapse $(S_1 h_{(1)(1)})h_{(1)(2)}$ to $\epsilon(h_{(1)})$. Note that the proof is not any harder than the usual one for uniqueness of group inverses, the only complication being that we are working now with parts of linear combinations and have to take care to keep the order of the coproducts. We can similarly collapse such expressions as $(S_1 h_{(1)})h_{(2)}$ or $h_{(2)}Sh_{(3)}$ wherever they occur as long as the two collapsing factors are in linear order. This is just the analogue of cancelling $h^{-1}h$ or hh^{-1} in a group. Armed with such techniques, we return now to the proof of the proposition. Consider the identity

$$\begin{aligned}(S(h_{(1)(1)}g_{(1)(1)}))h_{(1)(2)}g_{(1)(2)} \otimes g_{(2)} \otimes h_{(2)} &= (S((h_{(1)}g_{(1)})_{(1)}))(h_{(1)}g_{(1)})_{(2)} \otimes g_{(2)} \otimes h_{(2)}\\ &= \epsilon(h_{(1)}g_{(1)})1 \otimes g_{(2)} \otimes h_{(2)} = 1 \otimes g \otimes h.\end{aligned}$$

We used that Δ is an algebra homomorphism, then the antipode axiom applied to $h_{(1)}g_{(1)}$. Then we used the counity axiom. Now apply S to the middle factor of both sides and multiply the first two factors. One has the identity

$$\begin{aligned}Sg \otimes h &= (S(h_{(1)(1)}g_{(1)(1)}))h_{(1)(2)}g_{(1)(2)}Sg_{(2)} \otimes h_{(2)}\\ &= (S(h_{(1)(1)}g_{(1)}))h_{(1)(2)}g_{(2)(1)}Sg_{(2)(2)} \otimes h_{(2)} = (S(h_{(1)(1)}g))h_{(1)(2)} \otimes h_{(2)},\end{aligned}$$

where we used coassociativity applied to g. We then use the antipode axiom applied to $g_{(2)}$, and the counity axiom. We now apply S to the second factor and multiply up, to give

$$(Sg)(Sh) = (S(h_{(1)(1)}g))h_{(1)(2)}Sh_{(2)} = (S(h_{(1)}g))h_{(2)(1)}Sh_{(2)(2)} = S(hg).$$

We used coassociativity applied to h, followed by the antipode axioms applied to $h_{(2)}$ and the counity axiom. The other result, for the coproduct, is the same with reversal of all arrows and hence follows by arrow-reversal symmetry. □

Using the same methods it is a nice exercise to show that $S^2 = \mathrm{id}$ if and only if H is commutative or cocommutative. This is the classical case of a group algebra or enveloping algebra while truly 'quantum' groups are Hopf algebras that are neither commutative nor cocommutative. We will see in Theorem 2.19 that if H is finite-dimensional then S is always invertible. For our geometrical considerations, we will also need the notion of a real form, which again entails that S is invertible.

Definition 2.6 We define a real form of a Hopf algebra over $\mathbb{C}$ to be a $*$-algebra structure on the algebra of H which commutes with the coalgebra in the sense $\Delta(a^*) = (*\otimes *)\Delta a$ and $\epsilon(a^*) = \overline{\epsilon(a)}$ for all $a \in H$ and such that $(S \circ *)^2 = \mathrm{id}$. We say that H is then a Hopf $*$-algebra.

A consequence of the self-duality of the axioms of a Hopf algebra is the following.

Proposition 2.7 *If H is a finite-dimensional Hopf algebra then so is its dual H^* by adjunction of maps. The same applies to a finite-dimensional Hopf $*$-algebra.*

Proof Reversal of arrows can be realised by adjoints of maps. Thus if $\varphi : V \to W$ is a linear map then its adjoint $\varphi^* : W^* \to V^*$ goes the other way. Here

$$\langle \varphi^* \omega, v \rangle = \langle \omega, \varphi v \rangle$$

for all $v \in V,\ \omega \in W^*$, where $\langle\ ,\ \rangle : W^* \otimes W \to \Bbbk$ is the evaluation. Explicitly,

$$\langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta h\rangle = \langle \phi, h_{(1)}\rangle\langle \psi, h_{(2)}\rangle, \quad \langle \Delta\phi, h \otimes g\rangle = \langle \phi, hg\rangle,$$
$$\epsilon(\phi) = \phi(1_H), \quad \langle 1_{H^*}, h\rangle = \epsilon(h), \quad \langle S\phi, h\rangle = \langle \phi, Sh\rangle$$

for all $h, g \in H, \phi, \psi \in H^*$, where $\langle\ ,\ \rangle$ in some cases denotes the evaluation pairing extended to tensor products. In the Hopf $*$-algebra case we have

$$\langle \phi^*, h\rangle = \overline{\langle \phi, S(h^*)\rangle}$$

and check $\langle \Delta\phi^*, h \otimes g\rangle = \langle \phi^*, hg\rangle = \overline{\langle \phi, S((hg)^*)\rangle} = \overline{\langle \Delta\phi, Sh^* \otimes Sg^*\rangle} = \langle (* \otimes *)\Delta\phi, h \otimes g\rangle$, so $*$ being an involution dualises to commuting with Δ. □

Example 2.4 is isomorphic to its dual (see Exercise E2.2). In the case of an infinite-dimensional Hopf algebra, the coalgebra of H makes H^* an algebra, the product $(\phi\psi)(h) = (\phi \otimes \psi)\Delta h$ is called the *convolution product*. But going the other way, $\Delta\phi \in (H \otimes H)^* \supseteq H^* \otimes H^*$ so the dual of an algebra is not necessarily a coalgebra. More generally, we say that two Hopf algebras H, H' are *dually paired* if there is a bilinear map $\langle\ ,\ \rangle : H' \otimes H \to \Bbbk$ with the structure maps of one related to the structure maps of the other in the same way as displayed above, even if one is not exactly the algebraic dual of the other. Similarly for duality as Hopf $*$-algebras. The moral is that Hopf algebras tend to come in pairs related by duality. Later, we will see that they are also typically related by quantum Fourier transform.

Next, we want to 'polarise' the diagrams and ideas above. Thus, an action of a Hopf algebra H just means the action of the underlying algebra on a vector space i.e., a map $\triangleright : H \otimes V \to V$ (an 'action') obeying the axioms for the product but replacing two of the copies of H in the axioms with V. We refer to V as a *left module* under H. What is special about having a Hopf algebra (or at least a bialgebra) is that we can tensor product two left modules V, W, with a new action

$$h\triangleright(v \otimes w) = \sum h_{(1)}\triangleright v \otimes h_{(2)}\triangleright w$$

for all $h \in H$, $v \in V$ and $w \in W$. We will formalise the notions later, but the category ${}_H\mathcal{M}$ of H-modules is a monoidal category. The role of the antipode is to provide a dual module V^* by $(h\triangleright\phi)(v) = \phi(Sh\triangleright v)$ for all $v \in V$ and $h \in H, \phi \in H^*$. The role of the counit is to provide the trivial representation $h\triangleright\lambda = \epsilon(h)\lambda$ for all $h \in H$ and $\lambda \in \Bbbk$. So this is one sense of what a Hopf algebra *is*. An important application is that Hopf algebras can act on an algebra: we say that A is an *H-module algebra* if A is an H-module and the product map $A \otimes A \to A$ is equivariant. We also require that the identity element of A as a map $\Bbbk \to A$ is invariant. Over $\mathbb{C}$, if A is a $*$-algebra and H a Hopf $*$-algebra then we say that the action is 'unitary' if

$$(h\triangleright a)^* = S(h^*)\triangleright a^* \tag{2.1}$$

for all $a \in A$, $h \in H$. Finally, a key application of Hopf algebra actions is the construction of a cross product algebra $A{>\!\!\!\triangleleft} H$ whenever A is a left H-module algebra. This is built on the vector space $A \otimes H$ with product

$$(a \otimes h)(b \otimes g) = a(h_{(1)}\triangleright b) \otimes h_{(2)}g \tag{2.2}$$

and in the $*$-algebra case with H acting by (2.1) we have a $*$-algebra structure on $A{>\!\!\!\triangleleft} H$ with A, H $*$-subalgebras, namely $(a \otimes h)^* = (1 \otimes h^*)(a^* \otimes 1) = h^*{}_{(1)} \otimes h^*{}_{(2)}\triangleright a^*$. To balance the picture, we also have the arrow-reversal of the above, where we polarise the axioms of a coalgebra. The first step is a 'coaction'.

Definition 2.8 A *left comodule* under a Hopf algebra H is a vector space V and a map $\Delta_L : V \to H \otimes V$ obeying $(\mathrm{id} \otimes \Delta_L) \circ \Delta_L = (\Delta \otimes \mathrm{id}) \circ \Delta_L$ and $(\epsilon \otimes \mathrm{id}) \circ \Delta_L = \mathrm{id}$. One calls Δ_L a *left coaction*.

In calculations we usually use an explicit notation

$$\Delta_L v = \sum v_{(\bar{1})} \otimes v_{(\bar{\infty})}$$

for all $v \in V$ for the two tensor factors in the output of Δ_L. Then the axioms of a left coaction become

$$\sum v_{(\bar{1})} \otimes v_{(\bar{\infty})(\bar{1})} \otimes v_{(\bar{\infty})(\bar{\infty})} = \sum v_{(\bar{1})(1)} \otimes v_{(\bar{1})(2)} \otimes v_{(\bar{\infty})}, \quad \sum \epsilon(v_{(\bar{1})}) v_{(\bar{\infty})} = v$$

for all $v \in V$. Again, what is special about having a Hopf algebra (or at least a bialgebra) is that we can tensor product any two left comodules V, W to obtain another left comodule, with

$$\Delta_L(v \otimes w) = \sum v_{(\bar{1})} w_{(\bar{1})} \otimes v_{(\bar{\infty})} \otimes w_{(\bar{\infty})}$$

for all $v \in V$, $w \in W$. We will formalise the notion later as a monoidal category ${}^H\mathcal{M}$ of left H-comodules. If H coacts on an algebra A and Δ_L is an algebra map, we say that A is an *H-comodule algebra*. If A is a $*$-algebra we say that the coaction is unitary (or respects $*$) if $\Delta_L * = (* \otimes *)\Delta_L$. We also have the dual of the cross product construction above: if C is a coalgebra on which H left coacts and respects its structure then we have a left cross coproduct coalgebra $C \rtimes\!\!\blacktriangleleft H$ with coproduct

$$\Delta(c \otimes h) = c_{(1)} \otimes c_{(2)(\bar{1})} h_{(1)} \otimes c_{(2)(\bar{\infty})} \otimes h_{(2)}$$

for all $h \in H$, $c \in C$.

To complete the picture, all these notions have corresponding right-handed versions, i.e., a right action $\triangleleft : V \otimes H \to V$ and a right cross product algebra $H \triangleright\!\!< A$ when A is a right H-module algebra. We also have the notion of right coaction,

$$\Delta_R : V \to V \otimes H, \quad \Delta_R v = \sum v_{(\bar{0})} \otimes v_{(\bar{1})},$$

the notion of A a right H-comodule algebra and a right cross coproduct when $H \blacktriangleright\!\!< C$ when C is a right H-comodule coalgebra, etc. The right-handed theory is obtained by a left-right reversal of the various axioms written as diagrams and all the theory and results are analogous. We do not give details but in Hopf algebra theory one generally has *four for the price of one* on all concepts and theorems as we left-right reverse and/or arrow reverse.

If H is finite-dimensional then a left action of H is equivalent to a right coaction of H^* by

$$h \triangleright v = \sum v_{(\bar{0})} \langle v_{(\bar{1})}, h \rangle, \quad \Delta_R v = \sum_a e_a \triangleright v \otimes f^a,$$

where $\{e_a\}$ is a basis of H and $\{f^a\}$ is a dual basis. It is a nice exercise to check that in the $*$-algebra case, $\triangleright$ being unitary in the sense of (2.1) is equivalent to Δ_R commuting with $*$. One direction here, given by the first displayed formula, also works in the infinite-dimensional case to give the action of any dually paired Hopf algebra H'. For example, every Hopf algebra clearly coacts on itself from the left and the right, say $\Delta_R = \Delta$, in which case any dually paired H' acts on H by $\phi\triangleright h = h_{(1)}\langle\phi, h_{(2)}\rangle$. This is unitary in the Hopf $*$-algebra case as Δ commutes with $*$ in this case. This completes our lightning introduction to the abstract definition of a Hopf algebra and associated concepts.

Sometimes we will need the concept of a *super-Hopf algebra* H. Here $H = H_0 \oplus H_1$ is $\mathbb{Z}_2$-graded with $|h| = i$ when $h \in H_i$, and $H\underline{\otimes}H$ is the super-tensor product

$$(h \otimes g).(h' \otimes g') = (-1)^{|h'||g|}hh' \otimes gg'$$

for all h, h', g, g' with h', g homogeneous. Then $\Delta : H \to H\underline{\otimes}H$ is an algebra homomorphism with respect to this. All of the above have super versions with a similar 'supertransposition' with signs depending on the degrees, wherever there was a flip previously used in the axioms when written out as compositions of maps, for example in the notion of a super-(co)module algebra.

2.2 Basic Examples of Hopf Algebras

We start with a handful of classical examples where one hardly needs Hopf algebras but where Hopf algebras unify disparate classical concepts.

Example 2.9 Let G be a group. Its group algebra $\Bbbk G$ (consisting of finite linear combinations of elements of G) becomes an algebra with the group product of G extended linearly. Here $1 = e$ (the algebra identity is equal to the group identity). This forms a Hopf algebra with

$$\Delta x = x \otimes x, \quad \epsilon x = 1, \quad Sx = x^{-1}$$

for all $x \in G$. An element of a coalgebra with such a diagonal coproduct is said to be *grouplike*. Over $\mathbb{C}$, we have a Hopf $*$-algebra with $x^* = x^{-1}$. The reader should be able to see that a $\Bbbk G$-module algebra is an algebra on which G acts by algebra automorphisms, while a $\Bbbk G$-comodule algebra is a G-graded algebra A in the sense that

$$A = \bigoplus_{g\in G} A_g, \quad A_h.A_g \subseteq A_{hg}.$$

If G is finite then $\Bbbk(G)$, the algebra of functions on G with pointwise multiplication, is also a Hopf algebra, with

$$(\Delta f)(x, y) = f(xy), \quad \epsilon f = f(e), \quad (Sf)(x) = f(x^{-1})$$

for all $x, y \in G$, $f \in \Bbbk(G)$, where we identify $\Bbbk(G) \otimes \Bbbk(G) = \Bbbk(G \times G)$. Over $\mathbb{C}$, we have a Hopf $*$-algebra with $f^*(x) = \overline{f(x)}$. A left $\Bbbk(G)$-module algebra is a G-graded algebra A where $f \triangleright a = f(g)a$ when $a \in A_g$. A left $\Bbbk(G)$-comodule algebra means an algebra on which G right acts by algebra automorphisms, the two being related by

$$\Delta_L(a) = \sum_{g \in G} \delta_g \otimes a \triangleleft g.$$

$\diamond$

Recall the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra from §1.6.1.

Example 2.10 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. This forms a Hopf algebra with

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon x = 0, \quad Sx = -x$$

for all $x \in \mathfrak{g}$, extended to products by the homomorphism and antihomomorphism properties. An element of any bialgebra with such an additive coproduct is said to be *primitive*. Over $\mathbb{C}$, every real form of $\mathfrak{g}$ (in the sense of a basis $\{e_i\}$ for which the structure constants are real) gives a Hopf $*$-algebra with $e_i^* = -e_i$. A $U(\mathfrak{g})$-module algebra is an algebra on which $\mathfrak{g}$ acts by derivations. The notion of a $U(\mathfrak{g})$-comodule algebra makes perfect sense algebraically as a certain map $\Delta_R : A \to A \otimes U(\mathfrak{g})$ but does not have an immediate classical description (it is something like a module algebra for the associated algebraic group $\Bbbk[G]$ discussed later).

Proof We observe first that the tensor algebra $T\mathfrak{g}$ is a Hopf algebra. The product is the concatenation of words and the coproduct is defined by its action as shown on each letter of a word. Thus

$$\begin{aligned}
&\Delta((x_1 \cdots x_n)(y_1 \cdots y_m)) \\
&= (x_1 \otimes 1 + 1 \otimes x_1) \cdots (x_n \otimes 1 + 1 \otimes x_n)(y_1 \otimes 1 + 1 \otimes y_1) \cdots (y_m \otimes 1 + 1 \otimes y_m) \\
&= (\Delta(x_1 \cdots x_n))(\Delta(y_1 \cdots y_m))
\end{aligned}$$

is clearly an algebra homomorphism. Similarly for the counit and antipode, although the latter is immediately seen to be $S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1$. Having established our maps as algebra or antialgebra homomorphisms, it suffices to verify the Hopf algebra axioms on the generators of $T\mathfrak{g}$, which we leave to the reader. We now quotient our algebra by the ideal I generated by $xy - yx - [x, y]$ for all $x, y \in \mathfrak{g}$. We clearly obtain an algebra but have to check that

$$\Delta I \subseteq T\mathfrak{g} \otimes I + I \otimes T\mathfrak{g}, \quad \epsilon I = \{0\}, \quad SI \subseteq I.$$

Again it suffices to show this on the generators of the ideal, for all $x, y \in \mathfrak{g}$,

$$\begin{aligned}\Delta(xy - yx - [x, y]) &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)\\ &\quad - (y \otimes 1 + 1 \otimes y)(x \otimes 1 - 1 \otimes x) - [x, y] \otimes 1 - 1 \otimes [x, y]\\ &= (xy - yx - [x, y]) \otimes 1 + 1 \otimes (xy - yx - [x, y]),\\ S(xy - yx - [x, y]) &= (-y)(-x) - (-x)(-y) + [x, y] = -(xy - yx - [x, y])\end{aligned}$$

and more trivially for ϵ. Hence our maps descend and we obtain a Hopf algebra. □

The special case of an abelian Lie algebra of dimension n means that the polynomial algebra $\Bbbk[x_1, \cdots, x_n]$ in n variables is a Hopf algebra by the above formulae. Let us now give a key truly 'quantum group' inspired by the last example. It is part of a family $U_q(\mathfrak{g})$ associated to every complex simple Lie algebra as introduced by Jimbo and Drinfeld, as well as a reduced family $u_q(\mathfrak{g})$ when q is a root of unity.

Example 2.11 We recall that the Lie algebra sl_2 has a basis $\{H, X_\pm\}$ and relations $[H, X_\pm] = \pm 2X_\pm$ and $[X_+, X_-] = H$. The corresponding quantum group $U_q(sl_2)$ for $q^2 \neq 1$ has generators $q^{\pm\frac{H}{2}}, X_+, X_-$ and relations

$$q^{\frac{H}{2}} X_\pm q^{-\frac{H}{2}} = q^{\pm 1} X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \Delta X_\pm = X_\pm \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes X_\pm,$$

$$\Delta q^{\frac{H}{2}} = q^{\frac{H}{2}} \otimes q^{\frac{H}{2}}, \quad \epsilon q^{\frac{H}{2}} = 1, \quad \epsilon X_\pm = 0, \quad S q^{\frac{H}{2}} = q^{-\frac{H}{2}}, \quad S X_\pm = -q^{\pm 1} X_\pm.$$

Here $q^{\frac{H}{2}}$ is the actual algebraic variable, but we have used a suggestive notation such that if we write $q = e^{\frac{\lambda}{2}}$ and regard H as a generator, and expand the above expressions in orders of λ then at the lowest order we obtain the relations of $U(sl_2)$.

Over $\mathbb{C}$, we identify some real forms:

$$\begin{aligned} U_q(su_2): &\quad q^* = q, \quad X_+^* = X_-, \quad (q^{\frac{H}{2}})^* = q^{\frac{H}{2}}\\ U_q(su_{1,1}): &\quad q^* = q, \quad X_+^* = -X_-, \quad (q^{\frac{H}{2}})^* = q^{\frac{H}{2}}\\ U_q(sl_2(\mathbb{R})): &\quad q^* = q^{-1}, \quad X_\pm^* = -X_\pm, \quad (q^{\frac{H}{2}})^* = q^{\frac{H}{2}}\end{aligned}$$

which are all Hopf $*$-algebras. One can also let $g = q^H$, $E = X_+ q^{\frac{H}{2}}$, $F = q^{-\frac{H}{2}} X_-$ and regard g, E, F as the generators, then

$$g E g^{-1} = q^2 E, \quad g F g^{-1} = q^{-2} F, \quad [E, F] = \frac{g - g^{-1}}{q - q^{-1}},$$

$$\Delta E = E \otimes g + 1 \otimes E, \quad \Delta F = F \otimes 1 + g^{-1} \otimes F, \quad \Delta g = g \otimes g,$$

where our previous version has $g^{\pm\frac{1}{2}}$ adjoined. Finally, we mention a finite-dimensional reduced quantum group $u_q(sl_2)$ which applies when q is a primitive r-th root of unity, now with the additional relations

$$g^r = 1, \quad E^r = F^r = 0.$$

The proof that this gives a Hopf algebra can be done directly but more cleanly using Lemma 2.15 as the end of this section. It is not, however, a usual Hopf $*$-algebra but rather a flip Hopf $*$-algebra as explained at end of the chapter, see Example 2.113. ◇

Returning now to our classical examples, we have already explained in Theorem 1.12 the correspondence between polynomial subsets in $\mathbb{C}^n$ and reduced commutative algebras over $\mathbb{C}$. This correspondence is functorial in the sense that if $f : X \to Y$ is a polynomial map between polynomial subsets then there is an induced algebra homomorphism $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. It is also the case that $\mathbb{C}[X \times Y] = \mathbb{C}[X] \otimes \mathbb{C}[Y]$.

Example 2.12 Suppose $G \subset M_n(\mathbb{C})$ is a matrix group given by algebraic constraints. We have an algebra $\mathbb{C}[G]$ and the product map $G \times G \to G$ is clearly algebraic since it is given by matrix multiplication, so we have a corresponding map $\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$. Similarly, inversion $G \to G$ corresponds to $S : \mathbb{C}[G] \to \mathbb{C}[G]$ and $\{e\} \to G$ corresponds to $\epsilon : \mathbb{C}[G] \to \mathbb{C}$. One can verify directly that the maps fulfil the conditions on the generators, or deduce this from the arrow-reversing (contravariant) nature of the correspondence. A real form of the G would classically be a certain type of subgroup of G, but we take a more algebraic approach where a real form is specified by making $\mathbb{C}[G]$ into a Hopf $*$-algebra.

To be very concrete, we focus on

$$SL_n(\mathbb{C}) = \{x \in M_n(\mathbb{C}) \mid \det(x) = 1\}.$$

The corresponding algebra $\mathbb{C}[SL_n]$ is a quotient of the polynomial algebra $\mathbb{C}[t^i{}_j]$ for $i, j = 1, \cdots, n$, by some induced relations. Here the 'coordinate function' $t^i{}_j$ is that function which on the matrix x returns the value $x^i{}_j$ of the i, j entry. The induced relations have the same form as those defining the group but replacing the matrix elements by these coordinate functions. Thus $\mathbb{C}[SL_n] = \mathbb{C}[t^i{}_j]/\langle \det(t) - 1\rangle$, where we group all the generators together as a matrix t. For all the classical groups of Lie type, the relations can be formulated with integer coefficients and the construction then works over any field $\mathbb{k}$. The Hopf algebra structure is given by

$$\Delta t^i{}_j = \sum_k t^i{}_k \otimes t^k{}_j, \quad \epsilon t^i{}_j = \delta^i{}_j, \quad S t^i{}_j = t^{-1i}{}_j, \tag{2.3}$$

where the last expression needs some explanation. It is the formula for the matrix entries of the inverse of a matrix in terms of the matrix entries of the original matrix.

In other words, it is the usual formula for x^{-1} in terms of x but with $x^i{}_j$ replaced by $t^i{}_j$. As before, we extend Δ, ϵ, S as algebra homomorphisms on the polynomial algebra and check that the ideal defining the relations is respected in the right way. This follows from the general principles explained, but we can check it explicitly,

$$\Delta(\det(t)-1)=\det(t)\otimes\det(t)-1\otimes 1=(\det(t)-1)\otimes\det(t)+1\otimes(\det(t)-1)$$

because $\Delta\det(t) = \det(t)\otimes\det(t)$. This in turn holds by the same computation as involved in checking that $\det(xy) = \det(x)\det(y)$ for any matrices x, y, but with these replaced by $t\otimes 1$ and $1\otimes t$. Similarly, $S\det(t) = \det(t^{-1}) = \det(t)^{-1}$. For example, for $\mathbb{C}[SL_2]$ we have polynomials in four generators $t^1{}_1 = a, t^1{}_2 = b, t^2{}_1 = c, t^2{}_2 = d$, and one relation $ad - bd = 1$ for $\det(t) = 1$. We also have

$$\Delta\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad \epsilon\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\quad S\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

in a compact notation where the generators are grouped as a matrix. More explicitly,

$$\Delta a = a\otimes a + b\otimes c,\quad \epsilon a = 1,\quad Sa = d,\quad \text{etc.}$$

Similarly, a coordinate algebra $\mathbb{C}[X]$ is a $\mathbb{C}[G]$-comodule algebra when there is an algebraic action $G\times X\to X$. In our example, $\mathbb{C}[x^1,\cdots,x^n]$ or affine n-plane becomes a $\mathbb{C}[SL_n]$-comodule algebra under

$$\Delta_L x^i = \sum_j t^i{}_j\otimes x^j.$$

Finally, $\mathbb{C}[SU_n]$ means $\mathbb{C}[SL_n]$ together with $(t^j{}_i)^* = St^i{}_j$, which corresponds to a unitary matrix. We also equip two copies of the affine n-plane, $\mathbb{C}[z^1,\cdots,z^n,\bar{z}^1,\cdots,\bar{z}^n]$, with a $*$-algebra structure $z^{i*} = \bar{z}^i$ with coactions $\Delta_L z^i = \sum_j t^i{}_j\otimes z^j$ and $\Delta_L\bar{z}^i = \sum_j St^j{}_i\otimes\bar{z}^j$. Then Δ_L commutes with $*$, i.e., our action becomes unitary. ◇

Apart from the $*$-algebra part, the same formulae above give a Hopf algebra $\Bbbk[SL_n]$ for any field $\Bbbk$. We simply note that none of the relations involve anything other than integers. It turns out that one can similarly construct $\Bbbk[G]$ for all the classical groups associated to complex simple Lie algebras as well as for other algebraically defined groups. We do not necessarily have the functorial correspondence but we still have a Hopf algebra in these cases.

We now come to quantum group versions $\mathbb{C}_q[SL_n]$ of this example. In fact its structure is typified by the simplest case $\mathbb{C}_q[SL_2]$, which we focus on. We start with the *quantum plane*

$$\mathbb{C}_q[\mathbb{C}^2]=\mathbb{C}\langle x,y\rangle/\langle yx-qxy\rangle,$$

where $q \in \mathbb{C}^\times$ is a parameter. We ask what q-deformation of $\mathbb{C}[M_2]$ makes the quantum plane into a comodule algebra in a manner analogous to standard matrix operations $M_n(\mathbb{C}) \times \mathbb{C}^2 \to \mathbb{C}^2$ and $\mathbb{C}^2 \times M_n(\mathbb{C}) \to \mathbb{C}^2$.

Proposition 2.13 *Let $q \in \mathbb{C}^\times$ with $q^2 \neq -1$. There is a unique algebra on four generators a, b, c, d such that*

$$\Delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}, \quad \Delta_R \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are algebra homomorphisms. This is denoted the algebra $\mathbb{C}_q[M_2]$ of 2×2 quantum matrices *and has relations*

$$ba = qab, \;\; ca = qac, \;\; db = qbd, \;\; dc = qcd, \;\; da - ad = (q - q^{-1})bc, \;\; cb = bc.$$

It forms a a bialgebra with the matrix form of coalgebra on the generators previously displayed for $\mathbb{C}[SL_2]$ and $\mathbb{C}_q[\mathbb{C}^2]$ is a left and right comodule algebra. The quotient $\mathbb{C}_q[SL_2] = \mathbb{C}_q[M_2]/\langle \det_q - 1\rangle$ is a Hopf algebra, where

$$\det{}_q = ad - q^{-1}bc, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

We have three real forms

$$\mathbb{C}_q[SU_2]: \quad q^* = q, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix},$$

$$\mathbb{C}_q[SU_{1,1}]: \quad q^* = q, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & q^{-1}c \\ qb & a \end{pmatrix},$$

$$\mathbb{C}_q[SL_2(\mathbb{R})]: \quad q^* = q^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & q^{-1}b \\ qc & d \end{pmatrix}.$$

Proof Because Δ_L, Δ_R are supposed to be algebra maps, we define them on generators and check that these definitions respect the algebra relations when the maps are extended to products, provided $\mathbb{C}_q[M_2]$ has the algebra relations stated. Thus,

$$\begin{aligned} \Delta_L(yx) &= (c \otimes x + d \otimes y)(a \otimes x + b \otimes y) \\ &= ca \otimes x^2 + (qda + cb) \otimes xy + db \otimes y^2, \\ \Delta_L(qxy) &= q(a \otimes x + b \otimes y)(c \otimes x + d \otimes y) \\ &= qac \otimes x^2 + q(ad + qbc) \otimes xy + qbd \otimes y^2. \end{aligned}$$

The x^2, xy, y^2 are linearly independent in $\mathbb{C}_q[\mathbb{C}^2]$ so equality implies $ca = qac$, $db = qbd$ and $da - ad = qcb - q^{-1}bc$. Doing the same for $\Delta_R(yx) = \Delta_R(qxy)$ gives $ba = qab$, $dc = qcd$ and $da - ad = qbc - q^{-1}cb$. Provided $q^2 \neq -1$, this requires $bc = cb$. The matrix coproduct Δ is likewise forced and well defined. □

Note that $da - qbc = 1$ also holds in $\mathbb{C}_q[SL_2]$, given the $[d, a]$ commutation relation. There are similar *quantum group coordinate algebras* $\mathbb{C}_q[G]$ for all the classical Lie groups of complex simple Lie algebras, as well as their real forms over $\mathbb{C}$. Without worrying about real forms, we can also work over a general field $\mathbb{k}$ to have a quantum group $\mathbb{k}_q[G]$ with $q \in \mathbb{k}^\times$ or even with q an invertible indeterminate. Also we note that we can go another way and define $\mathbb{C}_q[GL_2] = \mathbb{C}_q[M_2][D, D^{-1}]$, where the central element $D = ad - q^{-1}bc = da - qbc$ and its inverse are adjoined. The antipode and $*$ for the unitary real form $\mathbb{C}_q[U_2]$ are

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = D^{-1}\begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \tag{2.4}$$

and there is a similar construction for $\mathbb{C}_q[GL_n]$ and $\mathbb{C}_q[U_n]$.

To complete the picture, our quantum groups $\mathbb{C}_q[SL_2]$ and $U_q(sl_2)$ are dually paired by a standard 2D representation ρ of the latter, whereby

$$\langle x, t^i{}_j \rangle = \rho(x)^i{}_j$$

for any $x \in U_q(sl_2)$ and the matrix of generators $t^i{}_j \in \mathbb{C}_q[SL_2]$. One can take

$$\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \rho(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \rho(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A similar story applies more generally. We also have finite-dimensional Hopf algebra versions $c_q[G]$ of these Hopf algebras when q is a root of unity and dual to $u_q(\mathfrak{g})$:

Example 2.14 When q is a primitive r-th root of unity, there is a central subalgebra

$$\mathbb{C}[SL_2] \subseteq \mathbb{C}_q[SL_2], \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}.$$

This is because the relations involve q and $q^r = 1$, for example

$$a^r d = a^{r-1}(1 + q^{-1}bc) = (1 + q^{-1-2(r-1)}bc)a^{r-1} = (1 + qbc)a^{r-1} = da^r.$$

We now quotient out by this in a manner determined by the counit. More precisely,

$$c_q[SL_2] = \mathbb{C}_q[SL_2]/\langle a^r - 1, b^r, c^r, d^r - 1 \rangle.$$

One can check that the structure maps descend and we still have a Hopf algebra. It is *not* a Hopf $*$-algebra in the usual sense. The quantum plane similarly quotients with $x^r = y^r = 0$ to an algebra $c_q[\mathbb{C}^2]$ and we have the same picture as before but now finite. One has to check that $\Delta_L(x^r) = \Delta_L(y^r) = \Delta_R(x^r) = \Delta_R(y^r) = 0$. The proof depends on the next lemma applied for example to $A = a \otimes x$, $B = b \otimes y$ as the two parts of $\Delta_L(x)$ and q^2 in the role of q, and similarly for the other cases. ◇

This finite version of the theory is somewhat analogous to working with matrix groups and linear algebra over a finite field of characteristic r. In all these and many other q-constructions one needs the following key lemma.

Lemma 2.15 (q-Binomial Formula) *For any elements A, B of an algebra obeying $BA = qAB$, we have*

$$(A+B)^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A^m B^{n-m},$$

where the q-binomial coefficients and q-factorials are

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}, \quad [m]_q! = [m]_q[m-1]_q \ldots [1]_q, \quad [m]_q = \frac{1-q^m}{1-q}.$$

Proof We proceed by induction. Assuming the result for $(A+B)^{n-1}$, we have

$$\begin{aligned}(A+B)^{n-1}(A+B) &= \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q A^m B^{n-1-m}(A+B) \\ &= \sum_{m=0}^{n-1} q^{n-1-m} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q A^{m+1} B^{n-1-m} + \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q A^m B^{n-m} \\ &= \sum_{m=1}^{n} q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q A^m B^{n-m} + \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q A^m B^{n-m} \\ &= A^n + B^n + \sum_{m=1}^{n-1} \left(q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \right) A^m B^{n-m}.\end{aligned}$$

The expression in parentheses combines to $\begin{bmatrix} n \\ m \end{bmatrix}_q$, as required, after an elementary computation using the identity $q^{n-m}[m]_q + [n-m]_q = [n]_q$. □

2.3 Translation-Invariant Integrals and Differentials

We have studied actions and coactions, but what is *really* special about Hopf algebras is that we can do constructions and 'coconstructions' at the same time.

Definition 2.16 Let H be a Hopf algebra over a field $\Bbbk$. An *H-Hopf module* is a vector space V which is both an H-module and an H-comodule and for which $\Delta_L : V \to H \otimes V$ is a left H-module map, i.e., $\Delta_L(h.v) = (\Delta h).(\Delta_L v)$ for all $h \in H, v \in V$, where the dot denotes the action of H on V and also (for the first tensor factor on the right) the product of H.

Our condition is that Δ_L is H-equivariant where H is an H-module by its product. One can write ${}^H_H\mathcal{M}$ for the category of left Hopf modules in line with our notation for other notions where the two structures mutually commute, although we will not particularly need this. Clearly, an example of a Hopf module is H itself with $\Delta_L = \Delta$ and the product of H. We also need a notion of invariant subspace under a coaction. If V is a left H-comodule, its invariant subspace is

$$ {}^H V = \{v \in V \mid \Delta_L v = 1 \otimes v\}. $$

Lemma 2.17 (Hopf Module Lemma) *Let V be a left H-Hopf module. Then $V \cong H \otimes ({}^H V)$, where the right-hand side is a Hopf module by the Hopf module structure of H alone. Conversely, every vector space defines a Hopf module by $H \otimes (\ )$, giving an equivalence between H-Hopf modules and vector spaces.*

Proof In one direction $H \otimes ({}^H V) \to V$ the map is the left action $h \otimes v \mapsto h.v$. In the other direction we provide the map

$$ v \mapsto F(v) = v_{(\bar{1})(1)} \otimes S v_{(\bar{1})(2)}.v_{(\bar{\infty})}. $$

Let us first verify that this map, which manifestly maps to $H \otimes V$, in fact maps to $H \otimes ({}^H V)$. To see this we compute

$$ \begin{aligned} (\mathrm{id}\otimes\Delta_L)F(v) = {} & v_{(\bar{1})(1)} \otimes (S v_{(\bar{1})(2)})_{(1)} v_{(\bar{\infty})(\bar{1})} \otimes (S v_{(\bar{1})(2)})_{(2)}.v_{(\bar{\infty})(\bar{\infty})} \\ = {} & v_{(\bar{1})(1)} \otimes (S v_{(\bar{1})(2)(2)}) v_{(\bar{\infty})(\bar{1})} \otimes (S v_{(\bar{1})(2)(1)}).v_{(\bar{\infty})(\bar{\infty})} \\ = {} & v_{(\bar{1})(1)(1)} \otimes (S v_{(\bar{1})(1)(2)(2)}) v_{(\bar{1})(2)} \otimes (S v_{(\bar{1})(1)(2)(1)}).v_{(\bar{\infty})} \\ = {} & v_{(\bar{1})(1)} \otimes (S v_{(\bar{1})(3)}) v_{(\bar{1})(4)} \otimes (S v_{(\bar{1})(2)}).v_{(\bar{\infty})} = v_{(\bar{1})(1)} \otimes 1 \otimes S v_{(\bar{1})(2)}.v_{(\bar{\infty})}, \end{aligned} $$

as required. The first equality uses the Hopf module property of V, the second anticomultiplicativity of S, the third that Δ_L is a coaction. We then renumber coproducts 'binary to decimal' in order to identify two consecutive coproducts to cancel by the antipode axiom. We see that the second factor of $F(v)$ lives in ${}^H V$. The map $v \mapsto F(v)$ is inverse to the first map as $v \mapsto F(v) \mapsto v_{(\bar{1})(1)}.(S v_{(\bar{1})(2)}.v_{(\bar{\infty})}) = (v_{(\bar{1})(1)} S v_{(\bar{1})(2)}).v_{(\bar{\infty})} = \epsilon(v_{(\bar{1})}) v_{(\bar{\infty})} = v$. The other way is similar. Formally, one has an equivalence of categories in the functorial sense explained in §2.4. ◻

We are going to use this concept to study integration. We recall that a Hopf algebra coacts on itself by Δ. This can be viewed as either a left or a right coaction.

Definition 2.18 A (right) invariant integral on a Hopf algebra H means a linear map $H \to \mathbb{k}$ such that $(\int \otimes \mathrm{id})\Delta = 1 \int$.

The reader may know that a classical Lie group has up to normalisation a unique left-invariant and a unique right-invariant 'Haar' integral or Haar measure.

Theorem 2.19 *A finite-dimensional Hopf algebra has a unique right-invariant integral up to normalisation and an invertible antipode.*

Proof We note that H^* is canonically a H-Hopf module by

$$h.\phi = \sum \langle \phi_{(1)}, Sh\rangle \phi_{(2)}, \quad \Delta_L \phi = \sum_a e_a \otimes \phi f^a,$$

where $\{e_a\}$ is a basis of H with dual basis $\{f^a\}$. Here Δ_L defines by evaluation a right action of H^* on itself, which is just right multiplication as $\langle \psi, e_a\rangle \phi f^a = \phi\psi$, while the action is the left coaction of H^* on itself turned into a right action of H similarly by evaluation, which the antipode S turns into a left action as stated. Hence we have only to verify that the two together provide a Hopf module, i.e., that

$$\langle \phi_{(1)}, Sh\rangle e_a \otimes \phi_{(2)} f^a = h_{(1)} e_a \otimes \langle (\phi f^a)_{(1)}, Sh_{(2)}\rangle (\phi f^a)_{(2)}$$

on inserting the definitions (the LHS is $\Delta_L(h.\phi)$ etc.). To see this, we evaluate against $\psi \in H^*$. Then the right-hand side becomes

$$\begin{aligned}\langle \psi_{(1)}, h_{(1)}\rangle \langle (\phi\psi_{(1)})_{(1)}, Sh_{(2)}\rangle (\phi\psi_{(1)})_{(2)} &= \langle \psi_{(1)}, h_{(1)}\rangle \langle S\psi_{(2)} S\phi_{(1)}, h_{(2)}\rangle \phi_{(2)}\psi_{(3)} \\ &= \langle \psi_{(1)} S\psi_{(2)}, h_{(1)}\rangle \langle h_{(2)}, S\phi_{(1)}\rangle \phi_{(2)}\psi_{(3)} = \langle \phi_{(1)}, Sh\rangle \phi_{(2)}\psi\end{aligned}$$

which is what the left-hand side of the required identity becomes.

Once we have a Hopf module, Lemma 2.17 implies $H^* \cong H \otimes ({}^H H^*)$, where

$$\begin{aligned}{}^H H^* &= \{\phi \in H^* \mid e_a \otimes \phi f^a = 1 \otimes \phi\} \\ &= \{\phi \in H^* \mid e_a \langle \phi f^a, h\rangle = \phi(h), \text{ for all } h \in H\} \\ &= \{\phi \in H^* \mid \phi(h_{(1)}) h_{(2)} = \phi(h), \text{ for all } h \in H\}\end{aligned}$$

is precisely the space of right-invariant integrals $\int : H \to \mathbb{k}$. But this space must be 1-dimensional as H and H^* have the same dimension. Hence there is a unique (right)-invariant integral up to scale. Note that fixing $\int$, the above isomorphism has the form $\phi \mapsto G(\phi) \int$ for some linear map $G : H^* \to H$, and applying the inverse isomorphism, we must recover $G(\phi).\int = \langle \int_{(1)}, SG(\phi)\rangle \int_{(2)} = \phi$ for all ϕ using the action of H on H^* in the Hopf module. It follows that S must be invertible. □

There is an analogous result for left-integrals and one can show in the finite-dimensional case that the two integrals coincide up to normalisation. In the general case, one can show similarly that the space of right-invariant integrals has dimension at most 1, i.e., an integral may not exist but if it does, it is unique, and similarly for left-invariant integrals. There are many applications of this theory. Here we mention that for any finite-dimensional Hopf algebra, one has a Fourier transform in both directions,

$$\mathcal{F} : H \to H^*, \quad \mathcal{F}(h) = \sum_a (\int e_a h) f^a, \quad \mathcal{F}^\sharp(\phi) = e_a \int^* \phi f^a,$$

where $\int, \int^*$ are respectively right-invariant integrals on H, H^*. Note that the last identity in the proof of Theorem 2.19 is that $\phi(h) = \int (SG(\phi))h = \langle \mathcal{F}(h), SG(\phi) \rangle$ for all h, which tells us that $\mathcal{F}$ is bijective, similarly for $\mathcal{F}^\sharp$. One can also show that $\mathcal{F}^\sharp \mathcal{F} = \int (\int^*) S$, which tells us that $\int (\int^*) \neq 0$ and provides $\mathcal{F}^{-1}$. Fourier transform is usually limited to abelian groups but it applies much more generally.

Example 2.20 Let G be a finite (not necessarily abelian) group. Integration on $\Bbbk(G)$ is provided by $\int f = \sum_{g \in G} f(g)$ and we take a basis δ_g for $g \in G$ of $\Bbbk(G)$ and a dual basis g of $\Bbbk G$. Hence the Fourier transform $\mathcal{F} : \Bbbk(G) \to \Bbbk G$ is

$$\mathcal{F}(f) = \sum_{g \in G} (\int \delta_g f) g = \sum_{g \in G} f(g) g.$$

Integration on $\Bbbk G$ is $\int^* g = \delta_{g,e}$ and the inverse Fourier transform is

$$\mathcal{F}^{-1}(g) = \frac{1}{\int^* \sum_{g \in G} g} \sum_{h \in G} \delta_{g^{-1}} \int^* (gh) = \delta_g.$$

Thus, Fourier transform identifies $\Bbbk G$ with the vector space of $\Bbbk(G)$ but equipped with a different (convolution) product. ◇

Fourier theory also applies for example to $\mathcal{F} : c_q[SL_2] \to u_q(sl_2)$ and in principle, with an appropriate version on the $U_q(sl_2)$ side, to $\mathcal{F} : \mathbb{C}_q[SL_2] \to \overline{U_q(sl_2)}$. There are different ways to make this precise, albeit outside the scope of the book. The main ingredient for the Fourier theory in one direction is the integral.

Example 2.21 There is a right-invariant 'Haar' integral on $\mathbb{C}_q[SL_2]$ taking the form

$$\int (bc)^n = \frac{(-1)^n q^n}{[n+1]_{q^2}}$$

using our previous q-integers and assuming that q is not a root of unity. The integral is zero on monomials not of this form in the basis $\{b^m c^n\} \cup \{a^k b^m c^n\} \cup \{d^k b^m c^n\}$, where $m, n \geq 0$, $k > 0$. In addition there is a *twisting automorphism* ς :

$\mathbb{C}_q[SL_2] \to \mathbb{C}_q[SL_2]$ such that

$$\int fg = \int \varsigma(g)f$$

for all $f, g \in \mathbb{C}_q[SL_2]$. This is an algebra map given by $\varsigma(a) = q^{-2}a$, $\varsigma(b) = b$, $\varsigma(c) = c$, $\varsigma(d) = q^2 d$.

Proof We shall outline a cohomological derivation in Example 4.36, but the formula can also be proven inductively. We illustrate this here on $\int(bc)$,

$$\begin{aligned}(\int \otimes\mathrm{id})\Delta(bc) &= (\int \otimes\mathrm{id})((a\otimes b + b\otimes d)(c\otimes a + d\otimes c))\\ &= \int ac\otimes ba + \int ad\otimes bc + \int bc\otimes da + \int bd\otimes dc\\ &= \int(1 + q^{-1}bc + qbc)\otimes bc + \int bc\otimes 1 = \int bc\otimes 1\end{aligned}$$

provided $\int bc = -1/(q+q^{-1})$. For the twisting map, it is not too difficult to check the formula for g a single generator. We take the case $g = a$. Then the only basis elements f for which $\int fa \neq 0$ are of the form $f = db^k c^k$, and now it is merely necessary to check $\int db^k c^k a = q^{-2}\int adb^k c^k$. Having checked the form of ς on the generators, it is obvious that it extends as an algebra map. □

Next we turn to differentials, the main topic of this chapter. As well as Hopf modules, we need a related concept also involving a compatible module and comodule.

Definition 2.22 (Crossed Modules) Let H be a Hopf algebra. A right H-crossed module (or Radford–Drinfeld–Yetter module) means a vector space V which is a right H-module and comodule such that

$$\Delta_R(v{\triangleleft}h) = v_{(\bar{0})}{\triangleleft}h_{(2)}\otimes(Sh_{(1)})v_{(\bar{1})}h_{(3)}$$

for all $v \in V$ and $h \in H$. In this case there is an associated map $\Psi : V\otimes V \to V\otimes V$ defined by $\Psi(v\otimes w) = \sum w_{(\bar{0})}\otimes v{\triangleleft}w_{(\bar{1})}$. Similarly, a left H-crossed module is a vector space W which is a left H module and comodule such that for $w \in W$,

$$\Delta_L(h{\triangleright}w) = h_{(1)}w_{(\bar{1})}S(h_{(3)})\otimes h_{(2)}{\triangleright}w_{(\bar{\infty})}.$$

We will see later that Ψ necessarily obeys the braid relations and is invertible when S is, so that the category $\mathcal{M}_H^H$ of right (similarly, left) crossed modules is braided. As a key example, every Hopf algebra has the following right adjoint coaction of H on itself by

$$\mathrm{Ad}_R(h) = \sum h_{(2)}\otimes(Sh_{(1)})h_{(3)} \tag{2.5}$$

for all $h \in H$. This together with right multiplication makes H a right crossed module. By restriction, so is the *augmentation ideal* $H^+ = \ker \epsilon$.

We now suppose that V is a left Hopf module as above and also a bimodule, and that Δ_L respects the right bimodule product in a similar fashion as for a Hopf module, i.e., $\Delta_L(v.h) = (\Delta_L v).\Delta h$ for all $h \in H, v \in V$. We say that such a V is a *left-covariant bimodule*. There is a similar notion that V is a right-covariant bimodule, i.e., a right Hopf module where the coaction also obeys $\Delta_R(h.v) = (\Delta h).\Delta_R v$ for all $h \in H, v \in V$. We use a dot to emphasise the bimodule structure combined, where applicable, with the product of H.

Lemma 2.23

(1) *If V is a left-covariant bimodule then $v \triangleleft h = (Sh_{(1)}).v.h_{(2)}$ restricts to a right action on ${}^H V$ and the original bimodule V is isomorphic to $H \otimes {}^H V$ with the action by the product of H from the left and the tensor product of the right action and $\triangleleft$ from the right.*

(2) *If V is also a right-covariant bimodule and the two coactions commute (we say that V is a* Hopf bimodule *or bicovariant bimodule) then the restriction of Δ_R to ${}^H V$ and $\triangleleft$ form a right crossed module. This gives an equivalence (of categories) between Hopf bimodules and crossed modules.*

Proof (1) It is easy to see that $\triangleleft$ is an action on V from the bimodule assumption and the properties of the antipode in §2.1. For $v \in {}^H V$ we check that

$$\Delta_L(Sh_{(1)}vh_{(2)}) = (Sh_{(1)})_{(1)}h_{(2)(1)} \otimes (Sh_{(1)})_{(2)}vh_{(2)(2)}$$
$$= (Sh_{(2)})h_{(3)} \otimes (Sh_{(1)})vh_{(4)} = 1 \otimes (Sh_{(1)})vh_{(2)}$$

using the assumption that we have a left-covariant bimodule for the first equality, then the antipode properties in §2.1 and coassociativity to renumber. Hence ${}^H V$ is stable under $\triangleleft$. Clearly $H \otimes {}^H V$ becomes a right module by the tensor product of $\triangleleft$ and right multiplication in H; one may check that this recovers V as a bimodule under the isomorphism in Lemma 2.17.

(2) Since the left coaction commutes with the supposed right coaction in the sense that $(\Delta_L \otimes \mathrm{id})\Delta_R = (\mathrm{id} \otimes \Delta_R)\Delta_L$, we see that Δ_R restricts to ${}^H V$. We check that $(\Delta_R, \triangleleft)$ is then a right crossed module,

$$\Delta_R(v \triangleleft h) = (Sh_{(1)})_{(1)}v_{(\bar{0})}h_{(2)(1)} \otimes (Sh_{(1)})_{(2)}v_{(\bar{1})}h_{(2)(2)}$$
$$= (Sh_{(2)})v_{(\bar{0})}h_{(3)} \otimes Sh_{(1)}v_{(\bar{1})}h_{(4)} = v_{(\bar{0})} \triangleleft h_{(2)} \otimes (Sh_{(1)})v_{(\bar{1})}h_{(3)}.$$

Moreover, $H \otimes {}^H V$ gets the structure of a right comodule by the tensor product of comultiplication on H and the restriction to ${}^H V$, which one may check recovers Δ_R on V under the isomorphism in Lemma 2.17. □

Next we recall from Chap. 1 that a first-order calculus Ω^1 over an algebra is a bimodule equipped with an exterior derivative d. In the Hopf algebra case we can ask that this is translation-invariant in a similar sense as we discussed for integrals.

Definition 2.24 A differential structure (Ω^1, d) on a Hopf algebra H is called *left-covariant* if the left coaction of H H on itself by the coproduct extends to Ω^1 by $\Delta_L \mathrm{d} = (\mathrm{id} \otimes \mathrm{d})\Delta$ and makes it a left-covariant bimodule. We say the differentials are right-covariant if there is an analogous right coaction Δ_R and *bicovariant* if both hold and the two coactions commute (so Ω^1 is a Hopf bimodule).

This seems like a lot of assumptions but the following lemma says what it amounts to in our case. It makes clear that left/bi-covariance is not in fact additional data but a property of (Ω^1, d) due to the surjectivity axiom of a standard calculus.

Lemma 2.25 *A differential structure on a Hopf algebra H is left-covariant if and only if*

$$\Delta_L(h\mathrm{d}g) = h_{(1)}g_{(1)} \otimes h_{(2)}\mathrm{d}g_{(2)}$$

for all $h, g \in H$ is well defined as a linear map $\Delta_L : \Omega^1 \to H \otimes \Omega^1$. It is right-covariant if and only if

$$\Delta_R((\mathrm{d}h)g) = (\mathrm{d}h_{(1)})g_{(1)} \otimes h_{(2)}g_{(2)}$$

for all $h, g \in H$ is well defined. When both hold then $(\Delta_L \otimes \mathrm{id})\Delta_R = (\mathrm{id} \otimes \Delta_R)\Delta_L$, making Ω^1 a Hopf bimodule. In the $*$-differential case, if the coactions exist, they are automatically unitary. In the inner left/right-covariant case we may assume without loss of generality that θ is left/right-invariant.

Proof A left-covariant bimodule means a coaction $\Delta_L : \Omega^1 \to H \otimes \Omega^1$ such that

$$\Delta_L(h\omega) = (\Delta h)\Delta_L\omega, \quad \Delta_L(\omega h) = (\Delta_L\omega)\Delta h$$

for all $h \in H$ and $\omega \in \Omega^1$, which, combined with $\Delta_L \mathrm{d} = (\mathrm{id}\otimes\mathrm{d})\Delta$, implies the first formula stated. Conversely, one checks using that d is a derivation and the axioms of a Hopf algebra that if this formula is well-defined then Δ_L is a left coaction and has the stated properties. Similarly for the right covariance. When both formulae hold then one may check that they commute (i.e., Ω^1 becomes a bicomodule). In the left-covariant inner case by $\theta \in \Omega^1$, we let $\theta' = (S\theta_{(\bar{1})})\theta_{(\bar{\infty})}$ and check that

$$\begin{aligned} \Delta_L\theta' &= (S\theta_{(\bar{1})(2)})\theta_{(\bar{\infty})(\bar{1})} \otimes (S\theta_{(\bar{1})(1)})\theta_{(\bar{\infty})(\bar{\infty})} \\ &= (S\theta_{(\bar{1})(2)})\theta_{(\bar{1})(3)} \otimes (S\theta_{(\bar{1})(1)})\theta_{(\bar{\infty})} = 1 \otimes \theta' \end{aligned}$$

by the antipode and comodule properties. Moreover, $\Delta_L(\mathrm{d}h) = h_{(1)} \otimes \mathrm{d}h_{(2)}$ means

$$\Delta_L([\theta, h]) = \theta_{(\bar{1})}h_{(1)} \otimes \theta_{(\bar{\infty})}h_{(2)} - h_{(1)}\theta_{(\bar{1})} \otimes h_{(2)}\theta_{(\bar{\infty})} = h_{(1)} \otimes [\theta, h_{(2)}].$$

We apply $S \otimes \mathrm{id}$ and multiply up to conclude $Sh_{(1)}\theta' h_{(2)} - \epsilon(h)\theta' = Sh_{(1)}[\theta, h_{(2)}]$ for all $h \in H$. Applying this to the second factor of $h_{(1)} \otimes h_{(2)}$ and multiplying, this is equivalent to $[\theta', h] = [\theta, h]$ for all $h \in H$. □

Next, we know from Lemma 2.23 that a left/bi-covariant calculus is of the form $\Omega^1 \cong H \otimes \Lambda^1$, where $\Lambda^1 = {}^H\Omega^1$ is the space of left-invariant 1-forms. By the lemma, it has a right H-action $\triangleleft$ and in the bicovariant case a right coaction $\Delta_R : \Lambda^1 \to \Lambda^1 \otimes H$ and we can recover all bimodule (co)actions from this data.

Theorem 2.26 *If* (Ω^1, d) *is a left-covariant differential structure on a Hopf algebra* H *then*

$$\Lambda^1 \cong H^+/\mathcal{I}$$

for $\mathcal{I}$ *a right ideal of* H^+*.* Ω^1 *is bicovariant if and only if* $\mathrm{Ad}_R(\mathcal{I}) \subseteq \mathcal{I} \otimes H$*. In the Hopf* $*$*-algebra case,* Ω^1 *is a* $*$*-differential structure if and only if* $*S(\mathcal{I}) \subseteq \mathcal{I}$*. Conversely, given an ideal with these properties,* $\Omega^1 = H \otimes (H^+/\mathcal{I})$ *is a left/bi-covariant calculus by the (co)actions of* H

$$h.(g \otimes v) = hg \otimes v, \quad (g \otimes v).h = \sum gh_{(1)} \otimes \pi(\tilde{v}h_{(2)}),$$
$$\mathrm{d}h = (\mathrm{id} \otimes \pi)(\Delta h - h \otimes 1), \quad (h \otimes v)^* = -h^*{}_{(1)} \otimes \pi((S^{-1}\tilde{v}^*)h^*{}_{(2)}),$$
$$\Delta_L(h \otimes v) = (\Delta \otimes \mathrm{id})(h \otimes v), \quad \Delta_R(h \otimes v) = h_{(1)} \otimes \pi v_{(2)} \otimes h_{(2)}(Sv_{(1)})v_{(3)}$$

as applicable, where $\pi : H^+ \to H^+/\mathcal{I}$ *denotes the canonical surjection and* $\tilde{v} \in H^+$ *is a representative of* $\pi(\tilde{v}) = v$*. The two constructions are inverse.*

Proof The structure of a left-covariant calculus as $\Omega^1 \cong H \otimes \Lambda^1$ is ensured by Lemma 2.23, which provides the isomorphism here as $\omega \mapsto \omega_{(\bar{1})(1)} \otimes (S\omega_{(\bar{1})(2)}).\omega_{(\bar{\infty})}$, for all $\omega \in \Omega^1$, with inverse given by the left bimodule product. The new part now is an explicit description of Λ^1 and we do this through the *quantum Maurer–Cartan form* associated to any left-covariant calculus,

$$\varpi : H^+ \to \Lambda^1, \quad \varpi(h) = (Sh_{(1)})\mathrm{d}h_{(2)} \tag{2.6}$$

for all $h \in H^+$. We can see that its image is left-invariant,

$$\begin{aligned}\Delta_L\varpi(h) &= (Sh_{(1)})_{(1)}h_{(2)(1)} \otimes (Sh_{(1)})_{(2)}\mathrm{d}h_{(2)(2)}\\ &= (Sh_{(2)})h_{(3)} \otimes Sh_{(1)}\mathrm{d}h_{(4)} = 1 \otimes \varpi(h).\end{aligned}$$

Moreover, if $\alpha = \sum a_i \mathrm{d}b_i$ is left-invariant then $a_{i\,(1)}b_{i\,(1)} \otimes a_{i\,(2)}\mathrm{d}b_{i\,(2)} = 1 \otimes \alpha$. Applying S to the first factor and multiplying tells us that $\alpha = \varpi(\epsilon(a_i)b_i)$. Hence ϖ is surjective and $\Lambda^1 \cong H^+/\mathcal{I}$, where $\mathcal{I} = \ker\varpi$. Moreover, $\varpi(g)\triangleleft h = Sh_{(1)}.\varpi(g).h_{(2)} = \varpi(gh)$, so $\mathcal{I}$ is a right ideal and that the action $\triangleleft$ is the right action on H^+ descended to $H^+/\mathcal{I}$ under the isomorphism. In the bicovariant case, we compute

$$\begin{aligned}\Delta_R(\varpi(g)) &= (Sg_{(1)})_{(1)}\mathrm{d}g_{(2)(1)} \otimes (Sg_{(1)})_{(2)}g_{(2)(2)} \\ &= Sg_{(2)(1)}\mathrm{d}g_{(2)(2)} \otimes (Sg_{(1)})g_{(3)} = \varpi(g_{(2)}) \otimes (Sg_{(1)})g_{(3)},\end{aligned}$$

which shows that $\mathcal{I}$ is Ad_R-stable and that there is a crossed module structure on Λ^1 which is isomorphic to the canonical one on H^+ descended to $H^+/\mathcal{I}$. Similarly, in the Hopf $*$-algebra case $\varpi(h)^* = (Sh_{(1)}\mathrm{d}h_{(2)})^* = (\mathrm{d}h^*{}_{(2)})S^{-1}h^*{}_{(1)} = -(SS^{-1}h^*{}_{(2)})\mathrm{d}S^{-1}h^*{}_{(1)} = -\varpi(S^{-1}h^*)$. We recover $\mathrm{d}h = h_{(1)} \otimes \varpi\pi_\epsilon h_{(2)}$ from ϖ, where the *counit projection*

$$\pi_\epsilon : H \to H^+, \quad \pi_\epsilon(h) = h - \epsilon(h) \tag{2.7}$$

projects out the unit. Conversely, given the ideal, we take this right/crossed module structure on $H^+/\mathcal{I}$ and define a left/bi-covariant differential structure on $H \otimes H^+/\mathcal{I}$ as stated. The left (co)action is just that on H by (co)multiplication and right (co)action is the tensor product of right (co)multiplication on H and the given (co)action on $H^+/\mathcal{I}$. The Maurer–Cartan form in this case is $\varpi(h) = \pi h$, where $\pi : H^+ \to H^+/\mathcal{I}$ is the canonical surjection, which gives the formula for d. It remains to check that these two constructions are inverse. For example, if we start with a calculus and construct $\mathcal{I}$ then the isomorphism $\Omega^1 \cong H \otimes \Lambda^1$ combined with the isomorphism $\Lambda^1 \cong H^+/\mathcal{I}$ gives $\Omega^1 \cong H \otimes (H^+/\mathcal{I})$. The map in the reverse direction is $h \otimes v \mapsto h \otimes \varpi(v) \mapsto h\varpi(v)$. It is then clear from the above properties of ϖ that this is an isomorphism of left/bi-covariant/$*$ calculi as applicable. □

This gives a correspondence between left/bi-covariant calculi up to isomorphism and (Ad_R-invariant) ideals $\mathcal{I}$.

Example 2.27 The universal calculus corresponds to $\mathcal{I} = \{0\}$ and is necessarily bicovariant. Here the isomorphism $\Omega^1_{\mathrm{uni}} \cong H \otimes H^+$ comes out as $h \otimes g \mapsto hSg_{(1)}\mathrm{d}_{\mathrm{uni}}g_{(2)}$ with inverse $h \otimes g \mapsto hg_{(1)} \otimes g_{(2)}$ and restricting to $\Lambda^1_{\mathrm{uni}} \cong H^+$. The left and right coactions on Ω^1_{uni} are

$$\Delta_L(h \otimes g) = h_{(1)}g_{(1)} \otimes h_{(2)} \otimes g_{(2)}, \quad \Delta_R(h \otimes g) = h_{(1)} \otimes g_{(1)} \otimes h_{(2)}g_{(2)},$$

which are manifestly well defined. The isomorphic form on $H \otimes H^+$ is provided by Theorem 2.26. The universal property that a general calculus is a quotient of Ω^1_{uni}, given that Δ_L and Δ_R, when they exist, are determined and not additional data, means that a general left/bi-covariant calculus has Λ^1 a quotient $H^+/\mathcal{I}$. ◇

A slightly better way to think of Theorem 2.26 is that bicovariant first-order differential structures on a Hopf algebra correspond to morphisms $\varpi : H^+ \to \Lambda^1$ of crossed modules, where H^+ is a crossed module by Ad_R and right multiplication and Λ^1 is a given crossed module. In the surjective case of a standard differential structure such maps are equivalent to giving the Ad_R-stable right ideal $\mathcal{I} = \ker \varpi$. We furthermore learn that a bicovariant calculus always comes with a map Ψ which we will use later to construct the full exterior algebra. In the course of the above results we have also seen that Hopf algebras with left-covariant calculi are left-parallelisable in the sense of Chap. 1. If $\{e_a\}$ is a basis of left-invariant 1-forms Λ^1 then the associated partial derivatives are defined by $\mathrm{d}h = \sum_a (\partial^a h)e_a$. In the form $H \otimes \Lambda^1$, this means $\mathrm{d}h = \sum_a \partial^a h \otimes e_a$.

Corollary 2.28 *If Ω^1 is a left-covariant calculus on a Hopf algebra and parallelised in the form $H \otimes \Lambda^1$, and $\int$ is a right-integral, then $(\int \otimes \mathrm{id})\mathrm{d}h = 0$. In particular, when Λ^1 is finite-dimensional, $\int \partial^a = 0$ for all partial derivatives.*

Proof This is immediate from the parallelised form $\mathrm{d}h = h_{(1)} \otimes \varpi(h_{(2)})$ and the definition of a right-integral and the parallelised form of Ω^1. In fact, for the universal calculus a linear map $\int$ with this property is precisely a right-integral since $\mathrm{d}_{\mathrm{uni}}h = h_{(1)} \otimes h_{(2)} - h \otimes 1$. In the case of finite-dimensional Λ^1 this tells us that $\int \partial^a = 0$, assuming the e_a form a basis. □

We can now link up with some of the examples from Chap. 1.

Example 2.29 Let $H = \Bbbk(G)$ a finite group with the calculus in Proposition 1.52. A calculation using (2.6) shows that for $x \in G \setminus \{e\}$ we have $\varpi(\delta_x) = e_x$ for $x \in \mathcal{C}$ and zero otherwise. Thus $\mathcal{I} = \ker \varpi$ is spanned by δ_x with $x \neq e, x \notin \mathcal{C}$. Conversely, defining calculi in terms of ideals, an ideal $\mathcal{I}$ of H^+ is spanned by the delta-functions on a subset of $G \setminus \{e\}$. Thus all left-covariant calculi are as described in Proposition 1.52 with this subset being the complement of some set $\mathcal{C} \subseteq G \setminus \{e\}$. The left-invariant 1-forms are spanned by $e_a = \varpi(\delta_a) = \sum_{x^{-1}y=a} S\delta_{x^{-1}}\mathrm{d}\delta_y = \delta_x \mathrm{d}\delta_{xa}$ for $a \in \mathcal{C}$. The adjoint coaction is $\mathrm{Ad}_R(\delta_a) = \sum_{xyz=a} \delta_y \otimes \delta_{x^{-1}}\delta_z = \sum_{z \in G} \delta_{zaz^{-1}} \otimes \delta_z$ so $\mathcal{I}$ being stable under Ad_R amounts to this complement of $\mathcal{C}$ being stable and hence to $\mathcal{C}$ being stable. A $*$-calculus similarly requires $\mathcal{C}$ to be stable under inversion as $S^{-1}\delta_x^* = S^{-1}\delta_x = \delta_{x^{-1}}$. Finally, the induced braiding is $\Psi(\delta_a \otimes \delta_b) = \sum \delta_{zbz^{-1}} \otimes \delta_a \delta_z = \delta_{aba^{-1}} \otimes \delta_a$, which then has the same form on the $\{e_a\}$. Thus the results of Theorem 1.56 in degree 1 are recovered correctly. ◇

One can similarly understand $\Omega(\Bbbk G)$ and $\Omega(U(\mathfrak{g}))$ in §1.7 from our Hopf algebra analysis and moreover, we now know precisely what we meant there by 'translation-invariant' differential structures, namely Definition 2.24 applied to these Hopf algebras. In these examples, Ad_R is trivial as the Hopf algebra is cocommutative and hence left and bicovariance are the same. Also Ψ here is the trivial flip map so the invariant 1-forms form a Grassmann algebra as usual.

Example 2.30 Irreducible $\Omega^1(\Bbbk[x])$ are in correspondence with monic irreducible polynomials $m \in \Bbbk[x]$ and are isomorphic to $\Omega^1(\Bbbk[x]) = \Bbbk_\mu[x]$ as a left $\Bbbk[x]$-module, where $\Bbbk_\mu = \Bbbk[\mu]/\langle m(\mu)\rangle$ is the associated field extension. The right module structure and derivative are

$$v.f(x) = f(x+\mu)v, \quad \mathrm{d}f = (f(x+\mu) - f(x))\mu^{-1}$$

for all $f \in \Bbbk[x]$, $v \in \Bbbk_\mu$. The calculus is inner if $\mu \neq 0$ with $\theta = \mu^{-1}$. It extends to a Grassmann algebra among the $\{\mu^i\}$ with $\mathrm{d}\mu^i = 0$.

Proof Note here that Ad_R is trivial so left-covariant calculi are the same as bicovariant calculi. As $H = \Bbbk[x]$ is a principal ideal domain for any field $\Bbbk$, one has $H^+ = \langle x \rangle$ (the ideal generated by x) and $\mathcal{I} = \langle xm(x) \rangle$ for some monic polynomial $m(x)$ (without loss of generality we can make the leading coefficient 1). So irreducible calculi are in 1-1 correspondence with such monic irreducible $m(x)$, and in this case $\Bbbk_\mu = \Bbbk[\mu]/\langle m(\mu) \rangle$ is a (separable) field extension of $\Bbbk$. (We use μ to avoid confusion with x generating the algebra.) We identify $\Lambda^1 = x\Bbbk[x]/\langle xm(x) \rangle \cong \Bbbk_\mu$ by $\pi(xf(x)) = f(\mu)$. Then $\Omega^1 = \Bbbk[x] \otimes \Lambda^1 \cong \Bbbk_\mu[x]$, where we write 1-forms sums of elements of the form $f(x)\mu^r$ for $r = 0, 1, \cdots, \deg(m) - 1$. This is a free module over $\Bbbk[x]$ from the left, while from the right $\mu^r.f(x) = f(x)_{(1)} \otimes \pi(x^{r+1} f(x)_{(2)}) = f(x+\mu)\mu^r$ under our identification, where we do the calculation in $\Bbbk_\mu[x]$. Similarly $\mathrm{d}f(x) = (\mathrm{id} \otimes \pi)(f(x)_{(1)} \otimes f(x)_{(2)} - f \otimes 1) = (f(x+\mu) - f(x))\mu^{-1}$ working in $\Bbbk_\mu[x]$. The exterior algebra is $\Omega = \Bbbk[x] \otimes \Lambda$ as a vector space, where Λ is the Grassmann algebra on Λ^1. This means that there is a second product $\wedge$ for Λ under which the elements μ^r anticommute (and square to zero) and for which $\mathrm{d}\mu^r = 0$. □

This includes the case of a trivial 'extension' $\Bbbk_\mu = \Bbbk$ and $m(\mu) = \mu - \lambda$ for any $\lambda \in \Bbbk$, which is the finite difference calculus in Example 1.10. More generally, if $\Bbbk \subseteq \mathbb{K}$ is any field extension and $\mu \in \mathbb{K}$ then there is a translation-invariant generalised calculus $\Omega^1(\Bbbk[x]) = \mathbb{K}[x]$ as a left $\Bbbk[x]$-module and formulae as above.

Beyond such elementary examples, we will show later in this chapter that all $\mathbb{C}_q[G]$ admit a bicovariant differential structure, but typically not of the expected dimension, and we will construct the exterior algebra in the general bicovariant case. To fully appreciate both of these constructions one needs braided categories, which we come to in the next section. On the other hand, we know from Chap. 1 that every first-order calculus has a maximal prolongation exterior algebra.

Proposition 2.31 *Let $\Omega^1 \cong H \otimes \Lambda^1$ be a left-covariant calculus in parallelised form. Its maximal prolongation is left-covariant with the form $\Omega_{\max} \cong H {\triangleright\!\!\!<} \Lambda_{\max}$ where $\Lambda_{\max}$ is generated by Λ^1 with the quadratic relations*

$$(\varpi\pi_\epsilon \wedge \varpi\pi_\epsilon)\Delta(\mathcal{I}) = 0.$$

Then d *on Λ^1 is defined by the* Maurer–Cartan equation

$$\mathrm{d}\varpi\pi_\epsilon(h) = -\varpi\pi_\epsilon(h_{(1)}) \wedge \varpi\pi_\epsilon(h_{(2)})$$

for all $h \in H$, extended as a graded-derivation to all degrees and to $\Omega_{\max}$. In the Hopf $*$-algebra case this is necessarily a $*$-exterior algebra when $*S(\mathcal{I}) \subseteq \mathcal{I}$.

Proof We use Lemma 1.32, which says that to every relation $\sum_i h_i.\mathrm{d}g_i = 0$ in Ω^1 for $h_i, g_i \in H$ there is the quadratic relation $\sum_i \mathrm{d}h_i \wedge \mathrm{d}g_i = 0$ in $\Omega^2_{\max}$. (We have used the Leibniz rule to order d to the right in the Ω^1 relation.) Applying the left coaction to $\sum_i h_i.\mathrm{d}g_i = 0$ gives $\sum_i h_{i\,(1)}g_{i\,(1)} \otimes h_{i\,(2)}.\mathrm{d}g_{i\,(2)} = 0 \in H \otimes \Omega^1$. For every $\alpha \in H^*$ we have the first-order relation $\sum_i \alpha(h_{i\,(1)}g_{i\,(1)})h_{i\,(2)}.\mathrm{d}g_{i\,(2)} = 0$, and thus the quadratic relation $\sum_i \alpha(h_{i\,(1)}g_{i\,(1)})\mathrm{d}h_{i\,(2)} \wedge \mathrm{d}g_{i\,(2)} = 0$. This means that for every quadratic relation $\sum_i \mathrm{d}h_i \wedge \mathrm{d}g_i = 0$ we have $\sum_i h_{i\,(1)}g_{i\,(1)} \otimes \mathrm{d}h_{i\,(2)} \wedge \mathrm{d}g_{i\,(2)} = 0$. From this it follows that the maximal prolongation has a well-defined left coaction.

Now we find a more explicit form for the relations of the maximal prolongation. We apply d to the equation for $\mathrm{d}h = h_{(1)}\varpi\pi_\epsilon(h_{(2)}) \in \Omega^1$ to get $0 = \mathrm{d}\mathrm{d}h = h_{(1)(1)}\varpi(\pi_\epsilon h_{(1)(2)}) \wedge \varpi(\pi_\epsilon h_{(2)}) + h_{(1)}\mathrm{d}\varpi(\pi_\epsilon h_{(2)})$ for all $h \in H$. By left-covariance, we can apply ϵ to the H factor to conclude that the Maurer–Cartan equation must hold. Now, specialising the Maurer–Cartan equation to $h \in \mathcal{I}$ we see that the quadratic relations in the statement must hold. Conversely, we show that the quadratic relations in the statement are sufficient for the maximal prolongation. We rewrite $h_i.\mathrm{d}g_i = 0$ (omitting the sum symbol) as $h_i g_{i\,(1)} \otimes \varpi\pi_\epsilon(g_{i\,(2)}) = 0 \in H \otimes \Lambda^1$. Applying $\mathrm{d} \otimes \mathrm{id}$ and then $\wedge$ to this gives

$$\mathrm{d}h_i \wedge g_{i\,(1)}\varpi\pi_\epsilon(g_{i\,(2)}) + h_i g_{i\,(1)}\varpi\pi_\epsilon(g_{i\,(2)}) \wedge \varpi\pi_\epsilon(g_{i\,(3)}) = 0$$

whereas applying $\mathrm{id} \otimes \mathrm{d}$ (using the Maurer–Cartan equation) and the product gives

$$h_i g_{i\,(1)}\varpi\pi_\epsilon(g_{i\,(2)}) \wedge \varpi\pi_\epsilon(g_{i\,(3)}) = 0.$$

We deduce that $\mathrm{d}h_i \wedge \mathrm{d}g_i = 0$, as required.

Given that $\Omega_{\max}$ is left-covariant and its higher degrees are a wedge product of Ω^1s, we may substitute $\Omega^1 \cong H \otimes \Lambda^1$ and use the commutation relations between H and Λ^1 in Theorem 2.26 to move all the H factors to the left. This gives us a left comodule and left module isomorphism $\Omega_{\max} \cong H \otimes \Lambda_{\max}$, where $\Lambda_{\max}$ in higher degree is the product of Λ^1s. The right action $\triangleleft$ on Λ^1 viewed in Ω^1 was by conjugation and the same formula in any degree similarly makes $\Lambda_{\max}$ a right H-module algebra extending $\triangleleft$ in such a way that $\Omega_{\max} \cong H \rtimes \Lambda_{\max}$ is a right module isomorphism and an algebra isomorphism with the right cross product. Moreover, $\Lambda_{\max}$ can be viewed as its left-invariant part and we note that d necessarily restricts to $\mathrm{d} : \Lambda_{\max} \to \Lambda_{\max}$ according to

$$\mathrm{d}(\varpi(h^1) \wedge \cdots \wedge \varpi(h^{n-1})) = -(\varpi\pi_\epsilon)^{\wedge n}\Big(\sum_{j=1}^{n-1}(-1)^{j+1}\Delta_j\Big)(h^1 \otimes \cdots h^{n-1})$$

for $h^1, \cdots, h^{n-1} \in H^+$, where Δ_j denotes Δ acting in the i-th position. That we have a $*$-exterior algebra is a general property of the maximal prolongation when the first-order calculus is a $*$-differential structure. □

This generalises the finite group case in Proposition 1.53. We conclude the present section with an important example: a left-covariant calculus on $\mathbb{C}_q[SU_2]$ and its application to the standard q-deformed sphere.

Example 2.32 $\mathbb{C}_q[SU_2]$ has a left-covariant 3D calculus which has basis

$$e^- = d\mathrm{d}b - qb\mathrm{d}d,\quad e^+ = q^{-1}a\mathrm{d}c - q^{-2}c\mathrm{d}a,\quad e^0 = d\mathrm{d}a - qb\mathrm{d}c$$

in our conventions for left-invariant 1-forms. It is generated by these as a left module while the right module structure is given by the bimodule relations

$$e^\pm f = q^{|f|} f e^\pm,\quad e^0 f = q^{2|f|} f e^0$$

for homogeneous f of degree $|f|$, where $|a| = |c| = 1$, $|b| = |d| = -1$, and exterior derivative

$$\mathrm{d}a = ae^0 + qbe^+,\quad \mathrm{d}b = ae^- - q^{-2}be^0,\quad \mathrm{d}c = ce^0 + qde^+,\quad \mathrm{d}d = ce^- - q^{-2}de^0.$$

The corresponding right ideal in $\mathbb{C}_q[SL_2]^+$ is

$$\mathcal{I} = \langle a + q^2 d - (1+q^2), b^2, c^2, bc, (a-1)b, (d-1)c\rangle$$

as generated by right multiplication of the listed elements by elements of the algebra. We set $e^0 = [a-1] = -q^2[d-1]$, $e^- = [b]$, $e^+ = q^{-1}[c]$, where [ ] denotes the quotient by $\mathcal{I}$. The maximal prolongation to higher forms is

$$\mathrm{d}e^0 = q^3 e^+ \wedge e^-,\quad \mathrm{d}e^\pm = \mp q^{\pm 2}[2]_{q^{-2}} e^\pm \wedge e^0,\quad e^\pm \wedge e^\pm = e^0 \wedge e^0 = 0,$$
$$q^2 e^+ \wedge e^- + e^- \wedge e^+ = 0,\quad e^0 \wedge e^\pm + q^{\pm 4} e^\pm \wedge e^0 = 0,$$

with $\mathrm{d}^2 = 0$, where $[n]_q = (1-q^n)/(1-q)$ denotes a q-integer. This means that there are the same dimensions as classically, including a unique top form $\mathrm{Vol} = e^+ \wedge e^- \wedge e^0$. Moreover, one has a $*$-exterior algebra with

$$e^{0*} = -e^0,\quad e^{+*} = -q^{-1}e^-,\quad e^{-*} = -qe^+,$$

which is required from $\mathrm{d}* = *\mathrm{d}$ and the formulae above. This also implies $\mathrm{Vol} = \mathrm{Vol}^*$. Applying d to powers of the generators gives

$$\begin{aligned}
\mathrm{d}a^n &= [n]_{q^2}(qa^{n-1}be^+ + a^n e^0),\\
\mathrm{d}b^n &= [n]_{q^2}(q^{1-n}ab^{n-1}e^- - q^{-2n}b^n e^0),\\
\mathrm{d}c^n &= [n]_{q^2}(q^{2-n}dc^{n-1}e^+ + c^n e^0),\\
\mathrm{d}d^n &= [n]_{q^2}(q^{2-2n}d^{n-1}ce^- - q^{-2n}d^n e^0).
\end{aligned}$$

From here we can write d on a linear basis, for $n > 1$ and $r, s \geq 0$,

$$\begin{aligned}
\mathrm{d}(a^n b^r c^s) &= [n+s-r]_{q^2} a^n b^r c^s e^0 + q^{1+s-r}[r]_{q^2} a^{n+1} b^{r-1} c^s e^- \\
&\quad + q^{1-s-r}\big([n+s]_{q^2} a^{n-1} b^{r+1} c^s + q[s]_{q^2} a^{n-1} b^r c^{s-1}\big) e^+, \\
\mathrm{d}(d^n b^r c^s) &= [s-r-n]_{q^2} d^n b^r c^s e^0 + q^{2-s-r}[s]_{q^2} d^{n+1} b^r c^{s-1} e^+ \\
&\quad + q^{s-r}\big(q^{2-2n}[n+r]_{q^2} d^{n-1} b^r c^{s+1} + q[r]_{q^2} d^{n-1} b^{r-1} c^s\big) e^-, \\
\mathrm{d}(b^r c^s) &= [s-r]_{q^2} b^r c^s e^0 + q^{2-s-r}[s]_{q^2} d b^r c^{s-1} e^+ + q^{1-r+s}[r]_{q^2} a b^{r-1} c^s e^-. \diamond
\end{aligned}$$

This calculus deforms the classical situation and the cohomology over $\mathbb{C}$ for generic q is the same as classically (we will show this in Example 4.68). q-deformation quantum groups also lead to natural q-deformed homogeneous spaces, as we illustrate now for the standard q-sphere in a manner analogous to the homogeneous space for the classical inclusion $S^1 \subset SU_2$. This structure will be more fully studied in terms of quantum principal bundles in Chap. 5 but for now we give a first introduction. We will also give another point of view on quantum spheres via their K-theory in the next chapter.

Proposition 2.33 *There is a Hopf $*$-algebra map*

$$\pi : \mathbb{C}_q[SU_2] \to \mathbb{C}_{q^2}[S^1], \quad \pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

which extends to the differential structures via $\pi_*(\mathrm{d}f) = \mathrm{d}\pi(f)$ for all $f \in \mathbb{C}_q[SU_2]$. Here $\mathbb{C}_{q^2}[S^1]$ denotes $\mathbb{C}[t, t^{-1}]$ with its $*$-structure $t^* = t^{-1}$ and its $*$-DGA from Example 1.11 but with q^2 and on $\mathbb{C}_q[SU_2]$ we take the 3D calculus.

Proof That we have an algebra map (and over $\mathbb{C}$ a $*$-algebra map) is easy and left to the reader. For the differential structure we require $\pi_* e^{\pm} = 0$ and $\pi_* e^0 = t^{-1}\mathrm{d}t = -q^2 t\mathrm{d}t^{-1}$ from the form of $\mathrm{d}a, \mathrm{d}b, \mathrm{d}c, \mathrm{d}d$ in the 3D calculus. The two require the relation $\mathrm{d}t.t = q^2 t\mathrm{d}t$ in $\Omega^1(\mathbb{C}[t, t^{-1}])$, which is expressed in our notation. These values of π_* are then then seen to respect the bimodule relations. For example, $\pi_*(e^0 a) = \pi_*(e^0)t = t^{-1}\mathrm{d}t.t = q^2\mathrm{d}t = \pi_*(q^2 a e^0)$, as required. $\square$

The differential structure of $\mathbb{C}_{q^2}[S^1]$ is relevant to the full Hopf fibration structure in Chap. 5, but for now we only need π as a $*$-algebra map. Specifically, just as classically SU_2 gets a right action by S^1 from the inclusion, in our case we similarly have a right coaction by push-forward of the coaction along π,

$$\Delta_R = (\mathrm{id} \otimes \pi) \circ \Delta : \mathbb{C}_q[SU_2] \to \mathbb{C}_q[SU_2] \otimes \mathbb{C}_{q^2}[S^1]$$

and we find that this extends to a coaction

$$\Delta_R : \Omega^1(\mathbb{C}_q[SU_2]) \to \Omega^1(\mathbb{C}_q[SU_2]) \otimes \mathbb{C}_{q^2}[S^1]$$

such that d is a right comodule map. With this preamble we are ready to construct the standard quantum sphere. As an algebra we define it as the fixed point $*$-subalgebra

$$\mathbb{C}_q[S^2] = \mathbb{C}_q[SU_2]^{\mathbb{C}_{q^2}[S^1]}$$

and extend this as a sub $*$-DGA

$$\Omega(\mathbb{C}_q[S^2]) \subset \Omega(\mathbb{C}_q[SU_2])^{\mathbb{C}_{q^2}[S^1]}$$

as generated by $\mathbb{C}_q[S^2]$ and d. Here the fixed subalgebra is a DGA but not surjective, so not itself an exterior algebra in the strict sense of Chap. 1. We will understand the geometry of this in Chap. 5, where a choice of connection on the above viewed as a quantum principal bundle gives a splitting

$$\Omega^1(\mathbb{C}_q[SU_2])^{\mathbb{C}_{q^2}[S^1]} = \Omega^1(\mathbb{C}_q[S^2]) \oplus \mathbb{C}_q[S^2]$$

in the present case. We now give some explicit calculations.

Lemma 2.34 $\mathbb{C}_q[S^2]$ *has generators* $x = -q^{-1}bc = -q^{-1}b_0$, $z = cd = b_+$, $z^* = -qab = -qb_-$ *(where* $b_0, b_\pm$ *are other generators in the literature) and relations*

$$zx = q^2xz, \quad z^*x = q^{-2}xz^*, \quad zz^* = q^4z^*z + q^2(1-q^2)x, \quad z^*z = x(1-x)$$

for the algebra. The middle relation can be written equally well as

$$zz^* = q^2x(1-q^2x).$$

Proof The $\mathbb{C}_{q^2}[S^1]$-coaction Δ_R on $\mathbb{C}_q[SU^2]$ comes out as $\Delta_R f = f \otimes t^{|f|}$, where this is the same degree as used in the 3D calculus above. For example, $\Delta_R a = (\mathrm{id} \otimes \pi)(a \otimes a + b \otimes c) = a \otimes t$. As a comodule algebra we know that Δ_R is multiplicative so it suffices to work it out on the generators. Then the fixed $*$-subalgebra is just the degree 0 part of $\mathbb{C}_q[SU_2]$. The relations stated are then inherited from those of $\mathbb{C}_q[SL_2]$ among the invariant elements x, z, z^*. □

We have written the relations in a form where the classical limit is clear; the first two become commutation relations and the last is the relation for a sphere in complex coordinates. More usual coordinates in the classical limit would be $x_3 = x - \frac{1}{2}$ and $z = x_1 + \mathrm{i}x_2$, giving a sphere of radius $\frac{1}{2}$ at the origin. With q real, we set $x^* = x$ and then have a $*$-algebra, in fact a $*$-subalgebra of $\mathbb{C}_q[SU_2]$. As long as $q^2 \neq 1$, one can also regard x as a function of z, z^*. The q-sphere is defined similarly over any field $\Bbbk$ regarding z, z^* as independent and $q \in \Bbbk^\times$.

Proposition 2.35 *There is an exterior algebra with* $\Omega^{1,0}$ *on* $\mathbb{C}_q[S^2]$ *spanned by* ∂z, ∂z^* *with the relation*

$$xz^*\partial z = q^{-4}(x-1)z\partial z^*$$

and with bimodule relations

$$\partial z \begin{cases} z \\ z^* \\ x \end{cases} = \begin{cases} q^2 z\partial z \\ q^2 z^*\partial z + (q^2 - q^{-2})z\partial z^* \\ q^4 x\partial z \end{cases}, \quad \partial z^* \begin{cases} z \\ z^* \\ x \end{cases} = \begin{cases} q^{-2} z\partial z^* \\ q^2 z^*\partial z^* \\ x\partial z^*. \end{cases}$$

We also have

$$\partial x = z^*\partial z - q^{-4} z\partial z^*$$

and $\Omega^{i,0} = 0$, $i > 1$. $\Omega^{0,1}$ *is similarly spanned by* $\bar\partial z$, $\bar\partial z^*$ *and* $\Omega(\mathbb{C}_q[S^2])$ *is given by the double complex*

$$\begin{tikzcd} 0 & 0 & \\ \Omega^{1,0} \arrow[u, "\partial"] \arrow[r, "\bar\partial"] & \Omega^{1,1} \arrow[u, "\partial"] \arrow[r, "\bar\partial"] & 0 \\ \Omega^{0,0} \arrow[u, "\partial"] \arrow[r, "\bar\partial"] & \Omega^{0,1} \arrow[u, "\partial"] \arrow[r, "\bar\partial"] & 0 \end{tikzcd}$$

where $\Omega^{1,1}$ *is 1-dimensional with basis* $\mathrm{Vol} = e^+ \wedge e^-$ *and* $\partial\bar\partial + \bar\partial\partial = 0$. *Here* $\Omega^1 = \Omega^{0,1} \oplus \Omega^{1,0}$ *and* $\mathrm{d} = \partial + \bar\partial$. *We have a* $*$*-DGA with*

$$* : \Omega^{0,1} \to \Omega^{1,0}, \quad *\partial = \bar\partial *$$

and similarly in the other direction, and $\mathrm{Vol}^* = -\mathrm{Vol}$ *(so* $\mathrm{i}e^+ \wedge e^-$ *is the geometric volume form).*

Proof For the differential calculus, this is not parallelisable so we do not have a free basis but $\Omega^1(\mathbb{C}_q[S^2])$ is spanned by

$$\mathrm{d}x = -q^{-1}\mathrm{d}(bc) = -bde^+ - ace^-, \quad \mathrm{d}z = \mathrm{d}(cd) = d^2e^+ + c^2e^-,$$
$$\mathrm{d}z^* = -q\mathrm{d}(ab) = -qb^2e^+ - qa^2e^-.$$

Looking at the form of these, we write $\mathrm{d} = \partial + \bar\partial$ for the parts involving e^+ and e^- respectively and these parts generate sub-bimodules $\Omega^{1,0}$, $\Omega^{0,1}$ respectively with $\Omega^1 = \Omega^{0,1} \oplus \Omega^{1,0}$. That these components actually are elements of Ω^1 follows from

$$a\mathrm{d}z + q^{-1}c\mathrm{d}x = de^+, \quad q^{-2}c\mathrm{d}z^* - qa\mathrm{d}x = be^+,$$
$$q^2b\mathrm{d}x - q^{-1}d\mathrm{d}z^* = ae^-, \quad -d\mathrm{d}x - qb.\mathrm{d}z = ce^-,$$

where multiplying the first line by b, d shows that any degree -2 element times e^+ is in $\Omega^1(\mathbb{C}_q[S^2])$ and similarly by a, c for the second line. Next we obtain the relations for these as inherited from those of $\Omega^1(\mathbb{C}_q[SU_2])$. For example $\partial x = -bde^+, \partial z = d^2e^+, \partial z^* = -qb^2e^+$ etc. implies the displayed equation for ∂x using the commutation relations in the quantum group and $ad - q^{-1}bc = 1$. Similarly,

$$\begin{aligned}(\partial z)z^* &= -qd^2e^+ab = -qd(da)be^+ = -qd(1+qbc)be^+ = -q^2bde^+ - q^3cdb^2e^+ \\ &= q^2\partial x + q^2 z\partial z^*,\end{aligned}$$

which then implies the stated commutation relation on using the formula for ∂x. The relations for $\Omega^{0,1}$ are similarly obtained and for the record they are

$$xz\bar\partial z^* = q^2(q^2x-1)z^*\bar\partial z, \quad \bar\partial x = q^{-2}z\bar\partial z^* - q^2z^*\bar\partial z,$$

$$\bar\partial z^* \begin{cases} z \\ z^* \\ x \end{cases} = \begin{cases} q^{-2}z\bar\partial z^* + (q^{-2}-q^2)z^*\bar\partial z \\ q^{-2}z^*\bar\partial z^* \\ q^{-4}x\bar\partial z^* \end{cases}, \quad \bar\partial z \begin{cases} z \\ z^* \\ x \end{cases} = \begin{cases} q^{-2}z\bar\partial z \\ q^2z^*\bar\partial z \\ x\bar\partial z. \end{cases}$$

These are exactly what one obtains by applying $*$ to the $\Omega^{1,0}$ relations while also changing ∂ into $\bar\partial$. The relations between $\Omega^{0,1}$ and $\Omega^{1,0}$ in the double complex are likewise inherited from those of $\Omega(\mathbb{C}_q[SU_2])$. Specifically, a short computation from the relations of the calculus upstairs gives

$$\partial x \wedge \bar\partial x = -q^3x(1-q^2x)\mathrm{Vol}, \quad \bar\partial x \wedge \partial x = q^{-1}x(1-x)\mathrm{Vol},$$

$$\partial z \wedge \bar\partial z^* = -q^3(1-q^3(2)_q x + q^3x^2)\mathrm{Vol}, \quad \bar\partial z \wedge \partial z^* = q^3x^2\mathrm{Vol},$$

$$\partial z^* \wedge \bar\partial z = -q^5x^2\mathrm{Vol}, \quad \bar\partial z^* \wedge \partial z = q(1-q^{-1}(2)_q x + q^{-2}x^2)\mathrm{Vol},$$

where $(2)_q = q + q^{-1}$, and from these one can obtain Vol itself as a linear combination of such expressions. This is also clear by two insertions of the q-determinant

$$e^+d^2 \wedge a^2e^- - (q+q^{-1})e^+db \wedge cde^- + q^2e^+b^2 \wedge c^2e^- = e^+ \wedge e^-.$$

In this way, $\Omega(\mathbb{C}_q[S^2])$ is constructed as a sub $*$-exterior algebra of $\Omega(\mathbb{C}_q[SU_2])$ generated by x, z, z^* and elements of the form fe^+, ge^-, where $|f| = -2$ and $|g| = 2$. Meanwhile, the above right coaction $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ on the 3D calculus restricts to Λ^1, where it takes the form

$$\Delta_R(e^0) = e^0 \otimes 1, \quad \Delta_R(e^\pm) = e^\pm \otimes t^{\pm 2}.$$

This coaction corresponds to $|e^0| = 0$ and $|e^{\pm}| = \pm 2$, so our sub-exterior algebra is invariant. The fixed DGA also includes elements of the form fe^0 where $f \in \mathbb{C}_q[S^2]$ but by the calculations above, this is not in the image of d. We shall return to this example and the theory behind it in Example 4.33 and extensively in Chap. 5. □

The above $\partial, \bar\partial$ relations imply rather more complicated commutation relations between $\mathrm{d}z, \mathrm{d}z^*, \mathrm{d}x$ and z, z^*, x which we do not list, as well as the relation

$$qz^*\mathrm{d}z + q^{-1}z\mathrm{d}z^* + q^2((2)_q x - q^{-1})\mathrm{d}x = 0 \tag{2.8}$$

by writing $\mathrm{d}z = \partial z + \bar\partial z$ etc., replacing ∂x in terms of $\partial z, \partial z^*$ and using the relation between the latter in the proposition, and a similar calculation for $\bar\partial x$. This relation (2.8) tells us that the sphere is 2-dimensional in spite of Ω^1 having three generators. We will see in Chap. 3 that it is in fact a rank 2 projective module. The deeper matter of a 'noncommutative complex structure' suggested by Proposition 2.35 will be explained in Chap. 7. The cohomology for generic q is the same as classically (see Proposition 4.34). Also, the Haar integral on $\mathbb{C}_q[SU_2]$ above restricts to the invariant subalgebra to give us an integral on $\mathbb{C}_q[S^2]$, namely

$$\int x^n = \frac{1}{[n+1]_{q^2}}. \tag{2.9}$$

This vanishes on unequal powers of z, z^* and equal powers are functions of x. We will see in Example 4.36 how this can be obtained using cohomological methods. Finally, we have a quantum metric $g \in \Omega^1 \otimes_{\mathbb{C}_q[S^2]} \Omega^1$ that is rotationally invariant under the left coaction of $\mathbb{C}_q[SL_2]$ inherited from the left-covariance of $\Omega^1(\mathbb{C}_q[SU_2])$.

Proposition 2.36 *For the q-sphere let $g = g_{+-} + g_{-+}$, where*

$$g_{+-} = q\partial z^* \otimes \bar\partial z + q^{-1}\partial z \otimes \bar\partial z^* + q^2(2)_q \partial x \otimes \bar\partial x$$

and g_{-+} is the same expression with $\bar\partial, \partial$ swapped. Then

$$g = q\mathrm{d}z^* \otimes \mathrm{d}z + q^{-1}\mathrm{d}z \otimes \mathrm{d}z^* + q^2(2)_q \mathrm{d}x \otimes \mathrm{d}x$$

is a rotationally invariant and quantum metric on $\Omega(\mathbb{C}_q[S^2])$ with inverse

$$\begin{aligned}
&(\mathrm{d}z, \mathrm{d}z^*) = q^{-1} - (2)_q zz^*, && (\mathrm{d}z^*, \mathrm{d}z) = q - (2)_q z^*z,\\
&(\mathrm{d}z, \mathrm{d}z) = -(2)_q z^2, && (\mathrm{d}z^*, \mathrm{d}z^*) = -(2)_q z^{*2},\\
&(\mathrm{d}z, \mathrm{d}x) = z(q^{-1} - (2)_q x), && (\mathrm{d}x, \mathrm{d}z) = (q^{-1} - (2)_q x)z,\\
&(\mathrm{d}z^*, \mathrm{d}x) = z^*(q^{-1} - (2)_q x), && (\mathrm{d}x, \mathrm{d}z^*) = (q^{-1} - (2)_q x)z^*,\\
&(\mathrm{d}x, \mathrm{d}x) = x(2q^{-1} - (2)_q x).
\end{aligned}$$

Proof We first check that this is quantum symmetric. Adding the identities in $\Omega^2(\mathbb{C}_q[S^2])$ displayed in the last proof, with the required weights, we find

$\wedge(g_{+-}) = -q^2\mathrm{Vol} = -\wedge(g_{-+})$. Hence $\wedge(g) = 0$. One can also see this from d applied to (2.8). Moreover, each component is invariant under flip$(*\otimes *)$ hence so is g. It remains to show that g is central. Firstly, the contributions to the commutator from g_{+-} and g_{-+} are analogous and it suffices to show that one of these is central. Indeed, working upstairs and using a different normalisation,

$$\tilde{g}_{+-} := -q^{-2}g_{+-} = e^+d^2\otimes a^2e^- + q^2e^+b^2\otimes c^2e^- - q^2(2)_q e^+bd\otimes ace^-, \tag{2.10}$$

which we can write as $e^+D_1D'_1\otimes D'_2D_2e^- = q^{-2}D_1D'_1e^+\otimes D'_2D_2e^-$ where $D=\sum D_1\otimes D_2 = d\otimes a - qb\otimes c$ and D' is another copy of D. Here $\sum D_1D_2 = 1$ as this is then the q-determinant relation of $\mathbb{C}_q[SL_2]$ and we used a similar trick already in the preceding proof for Vol. Now if f has degree 0, hence in $\mathbb{C}_q[S^2]$, we have

$$\begin{aligned}\tilde{g}_{+-}f &= e^+D_1D'_1\otimes D'_2D_2fe^- = e^+D_1D'_1\otimes D'_2D_2fD''_1D'''_1D'''_2D''_2e^-\\ &= e^+D_1D'_1D'_2D_2fD''_1D'''_1\otimes D'''_2D''_2e^- = e^+fD''_1D'''_1\otimes D'''_2D''_2e^- = f\tilde{g}_{+-}\end{aligned}$$

where D'', D'''. Finally, one can also write g as stated where the non-mixed components cancel. Working in the $\mathbb{C}_q[SL_2]$ calculus, the coaction Δ_L on the basis $\mathrm{d}z^*, \mathrm{d}x, \mathrm{d}z$ is computed from the coproduct of $\mathbb{C}_q[SL_2]$ and has the form of a q-deformed spin 1 or 'covector' transformation,

$$\Delta_L\begin{pmatrix}-q^{-1}\mathrm{d}z^*\\ -q\mathrm{d}x\\ \mathrm{d}z\end{pmatrix} = \begin{pmatrix}a^2 & (2)_qab & b^2\\ ca & 1+(2)_qbc & db\\ c^2 & (2)_qcd & d^2\end{pmatrix}\otimes\begin{pmatrix}-q^{-1}\mathrm{d}z^*\\ -q\mathrm{d}x\\ \mathrm{d}z\end{pmatrix}.$$

For generic q, there is up to normalisation a unique matrix obeying

$$\begin{pmatrix}a^2 & (2)_qab & b^2\\ ca & 1+(2)_qbc & db\\ c^2 & (2)_qcd & d^2\end{pmatrix}^T\begin{pmatrix}0&0&-q^2\\0&(2)_q&0\\-1&0&0\end{pmatrix}\begin{pmatrix}a^2 & (2)_qab & b^2\\ ca & 1+(2)_qbc & db\\ c^2 & (2)_qcd & d^2\end{pmatrix} = \begin{pmatrix}0&0&-q^2\\0&(2)_q&0\\-1&0&0\end{pmatrix}$$

on using the commutations relations of $\mathbb{C}_q[SU_2]$, where $(\ )^T$ denotes transpose. The quadratic generators here generate the even sub-Hopf algebra $\mathbb{C}_q[SO_3]\subset\mathbb{C}_q[SU_2]$ and the above characterises these as preserving the matrix shown, which then provides the coefficients of g. The inverse metric is best computed when extended to $(e_\pm, e_\pm)$ etc. and then converted back. We simply present the results and one can easily check that they are inverse, for example

$$\begin{aligned}((\ ,\ )\otimes\mathrm{id})(\mathrm{d}z\otimes g)) &= (\mathrm{d}z, q\mathrm{d}z^*)\mathrm{d}z + (\mathrm{d}z, q^{-1}\mathrm{d}z)\mathrm{d}z^* + q^2(2)_q(\mathrm{d}z,\mathrm{d}x)\mathrm{d}x\\ &= q(q^{-1}-(2)_qzz^*)\mathrm{d}z + q^{-1}(-(2)_qz^2)\mathrm{d}z^* + q^2(2)_qz(q^{-1}-(2)_qx)\mathrm{d}x = \mathrm{d}z\end{aligned}$$

using the relation (2.8). For the inverse on the other side, we use the conjugate relation

$$q(\mathrm{d}z^*)z + q^{-1}(\mathrm{d}z)z^* + q^2(\mathrm{d}x)((2)_q x - q^{-1}) = 0 \tag{2.11}$$

obtained similarly. That (,) as stated extends consistently as a bimodule map is similarly best done in terms of the values on $\partial z, \partial z^*, \bar{\partial} z, \bar{\partial} z^*$ and their relations. □

Remark 2.37 While the 3D calculus on $\mathbb{C}_q[SU_2]$ is not bicovariant, there is a vestige of this structure. The right action of $\mathbb{C}_q[SU_2]$ on Λ^1 as part of the 3D differential calculus lifts via π to an action $\triangleleft$ of $\mathbb{C}_{q^2}[S^1]$,

$$e^{\pm} \triangleleft t = q e^{\pm}, \quad e^0 \triangleleft t = q^2 e^0$$

and $(\Delta_R, \triangleleft)$ form a right $\mathbb{C}_{q^2}[S^1]$-crossed module. The cross product by this action gives the bimodule relations of Ω. The induced map $\Psi : \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ is

$$\Psi(e^0 \otimes e^0) = e^0 \otimes e^0, \quad \Psi(e^0 \otimes e^{\pm}) = q^{\pm 4} e^{\pm} \otimes e^0,$$
$$\Psi(e^{\pm} \otimes e^0) = e^0 \otimes e^{\pm}, \quad \Psi(e^{\pm'} \otimes e^{\pm}) = q^{\pm 2} e^{\pm} \otimes e^{\pm'}$$

(where $\pm'$ is an independent sign) extends to $\Omega^1 \otimes_A \Omega^1$. For generic q, $\ker(\mathrm{id} - \Psi)$ is generated by $e^0 \otimes e^0$ and $q^2 e^+ \otimes e^- + e^- \otimes e^+$, and these are contained in, but do not span, $\ker \wedge$. This gives some insight into the structure of the 3D calculus. ◇

We have focused on the compact real form $\mathbb{C}_q[SU_2]$ of $\mathbb{C}_q[SL_2]$ whereas there are two other real forms $\mathbb{C}_q[SU_{1,1}]$, $\mathbb{C}_q[SL_2(\mathbb{R})]$ listed in Proposition 2.13. Following Theorem 2.26, we find in both cases that $*S(\mathcal{I}) \subseteq \mathcal{I}$ for the ideal $\mathcal{I}$ for the 3D calculus in Example 2.32. Hence these too have $*$-differential structures, namely

$$\Omega^1(\mathbb{C}_q[SU_{1,1}]): \quad e^{-*} = q e^+, \quad e^{+*} = q^{-1} e^-, \quad e^{0*} = -e^0$$
$$\Omega^1(\mathbb{C}_q[SL_2(\mathbb{R})]): \quad e^{-*} = q^{-2} e^-, \quad e^{+*} = q^4 e^+, \quad e^{0*} = q^{-2} e^0$$

with q real and $|q| = 1$, respectively. Next, we have noted in Example 1.11 that $\mathbb{C}[t, t^{-1}]$ has a second $*$-DGA structure with $t^* = t$ and $|q| = 1$, which we called $\mathbb{C}_q[\mathbb{R}^\times]$. These are both Hopf $*$-algebras and π above becomes

$$\mathbb{C}_q[SU_{1,1}] \to \mathbb{C}_{q^2}[S^1], \quad \mathbb{C}_q[SL_2(\mathbb{R})] \to \mathbb{C}_{q^2}[\mathbb{R}^\times]$$

as maps of $*$-DGAs. For the analogue of the q-sphere, we first redo the above as an algebraic sphere $\mathbb{C}_q[S^2_{\mathbb{C}}]$ defined as the same algebra as in Lemma 2.34 but without the $*$-structure by renaming z^* in Lemma 2.34 as an independent generator $w = -qab$. We continue to use t for the generator of $\mathbb{C}_{q^2}[S^1]$ or $\mathbb{C}_{q^2}[\mathbb{C}^\times]$. For the standard real form $\mathbb{C}_q[SU_2]$, we have $\mathbb{C}_{q^2}[S^1]$ and $x^* = x$, $z^* = w$ as the standard $\mathbb{C}_q[S^2]$ above, but we now have two more real forms of $\mathbb{C}_q[S^2_{\mathbb{C}}]$, namely

$$\mathbb{C}_q[AdS_2]: \quad x^* = x, \quad z^* = -w$$
$$\mathbb{C}_q[H^2]: \quad x^* = q^2 x, \quad z^* = q^2 z, \quad w^* = q^{-2} w$$

as q-anti-de Sitter space and q-hyperbolic space for q real and $|q| = 1$, respectively.

2.4 Monoidal and Braided Categories

In order to proceed more deeply into the differential geometry of quantum groups, we will need some elements of their representation theory, best approached in terms of braided categories. We give a brief introduction focussed on concepts that we will use later, starting with the definition of a category $\mathcal{C}$. For our purposes, this is:

(1) A collection of *objects* $X, Y, V, W, \ldots$.
(2) A specification of a set $\mathrm{Mor}(X, Y)$ of *morphisms* for any objects X, Y.
(3) A composition operation $\circ : \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$ with properties analogous to the composition of maps.
(4) Every set $\mathrm{Mor}(X, X)$ should contain an identity element id_X such that $\phi \circ \mathrm{id}_X = \phi$ and $\mathrm{id}_X \circ \phi = \phi$ for any morphism for which $\circ$ is defined.

In our case, all objects will be concrete sets with structure. We are primarily going to use the language of category theory to keep our thinking clear. In particular, we indicate objects as $X \in \mathcal{C}$ by an abuse of set theory notation. Elements of $\mathrm{Mor}(X, Y)$ can be thought of as arrows $X \to Y$.

A (covariant) *functor* $F : \mathcal{C} \to \mathcal{V}$ between categories specifies an object $F(X) \in \mathcal{V}$ for every object $X \in \mathcal{C}$, *and* a morphism $F(\phi) : F(X) \to F(Y)$ for every morphism $\phi : X \to Y$, such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for morphisms. A *contravariant functor* is similar but with $F(\phi) : F(Y) \to F(X)$ and $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$.

A *natural transformation* $\Theta : F \Rightarrow G$ for $\theta \in \mathrm{Nat}(F, G)$ between two (covariant) functors $F, G : \mathcal{C} \to \mathcal{V}$ means a collection $\{\Theta_X : F(X) \to G(X) \mid X \in \mathcal{C}\}$ of morphisms in $\mathcal{V}$ which are 'functorial' in the sense that for all morphisms $\phi : X \to Y$,

$$\Theta_Y \circ F(\phi) = G(\phi) \circ \Theta_X.$$

Similarly in the contravariant case. The natural transformation Θ is called a 'natural isomorphism' if each Θ_X is an isomorphism, i.e., has an inverse which is a morphism.

Given (covariant) functors $L : \mathcal{V} \to \mathcal{C}$, $R : \mathcal{C} \to \mathcal{V}$, we say that L is *left adjoint* to R and R is *right adjoint* to L if there are isomorphism of sets of morphisms

$\mathrm{Mor}(L(V), X) \cong \mathrm{Mor}(V, R(X))$ for all objects $X \in \mathcal{C}$ and $V \in \mathcal{V}$, natural in the sense of a commutative diagram for all morphisms $\phi : W \to V, \theta : X \to Y$,

$$\begin{tikzcd}
\mathrm{Mor}(L(V), X) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\theta\circ(-)\circ L(\phi)"] & \mathrm{Mor}(V, R(X)) \arrow[d, "R(\theta)\circ(-)\circ\phi"] \\
\mathrm{Mor}(L(W), Y) \arrow[r, leftrightarrow, "\cong"] & \mathrm{Mor}(W, R(Y)).
\end{tikzcd}$$

Example 2.38 One of the easiest examples of a category is ${}_A\mathcal{M}$, the category of left A-modules for the algebra A. (For $A = \Bbbk$ we just get vector spaces.) The objects are left A-modules, and the morphisms $\mathrm{Mor}(V, W)$ are the left module maps ${}_A\mathrm{Hom}(V, W)$ from V to W. Composition of morphisms $\phi \in \mathrm{Mor}(V, W)$ and $\psi \in \mathrm{Mor}(W, U)$ is composition $\psi\circ\phi \in \mathrm{Mor}(V, U)$ of left-module maps. The morphism $\mathrm{id}_V \in \mathrm{Mor}(V, V)$ is the identity map on V.

For $\alpha : A \to B$ an algebra map, there is a covariant functor $R = \alpha^* : {}_B\mathcal{M} \to {}_A\mathcal{M}$ sending a left B-module E to a left A-module $\alpha^*(E)$ (often denoted simply as ${}_\alpha E$) with *pull back module* action $a{\triangleright}e = \alpha(a).e$ for $a \in A$ and $e \in E$. On morphisms $\theta \in \mathrm{Mor}(E, F) = {}_B\mathrm{Hom}(E, F)$, we define $\alpha^*(\theta) = \theta$. For $\varphi \in {}_B\mathrm{Hom}(F, G)$ it is immediate that $\alpha^*(\varphi\circ\theta) = \alpha^*(\varphi)\circ\alpha^*(\theta)$.

For an example of a natural transformation, let $\beta : A \to B$ be another algebra map. A natural transformation $\Xi : \alpha^* \Rightarrow \beta^*$ means a collection of morphisms $\Xi_E : \alpha^*(E) \to \beta^*(E)$ in ${}_A\mathcal{M}$ such that $\Xi_F \circ \alpha^*(\theta) = \beta^*(\theta)\circ\Xi_E$ for all $\theta : E \to F$. We set $\Xi_E(e) = \xi.e$, for some fixed $\xi \in B$, and correspondingly $\Xi_F(f) = \xi.f$ for all $f \in F$. That Ξ_E is a left A-module map is $\beta(a)\xi.e = \xi\alpha(a).e$ for all $e \in E$ and $a \in A$. This will be true when $\beta(a)\xi = \xi\alpha(a)$ for all $a \in A$. The naturality condition holds automatically for any left B-module map θ. If ξ is invertible, then Ξ is a natural isomorphism, having inverse ${\Xi_E}^{-1}(e) = \xi^{-1}.e$.

There is another covariant functor $L = \alpha_* : {}_A\mathcal{M} \to {}_B\mathcal{M}$ defined on objects by $\alpha_*(V) = B_\alpha \otimes_A V \in {}_B\mathcal{M}$. Here B_α is a B-A-bimodule, where $B_\alpha = B$ as a left B-module, but has a right action of A given by $b{\triangleleft}a = b\alpha(a)$. The functor sends a morphism $\phi : V \to W$ to the morphism $\alpha_*(\phi) = \mathrm{id}\otimes\phi : B_\alpha\otimes_A V \to B_\alpha\otimes_A W$. It is immediate that $\alpha_*(\psi\circ\phi) = \alpha_*(\psi)\circ\alpha_*(\phi)$ for $\psi : W \to U$. This is a standard *induced representation construction*, which we will use in Theorem 4.46. This time a natural transformation $\Theta : \alpha_* \Rightarrow \beta_*$ means a collection of morphisms $\Theta_V : \alpha_*(V) \to \beta_*(V)$ in ${}_B\mathcal{M}$ such that $\Theta_W\circ\alpha_*(\phi) = \beta_*(\phi)\circ\Theta_V$. This will be satisfied if we set $\Theta_V(b\otimes v) = \tau(b)\otimes v$, where $\tau : B_\alpha \to B_\beta$ is a B-A-bimodule map. If we set $\tau(b) = b\eta$ for a fixed $\eta \in B$, the bimodule map condition is satisfied if $\eta\beta(a) = \alpha(a)\eta$ for all $a \in A$, so such η give a natural transformation. If η is invertible then Θ is a natural isomorphism.

The functors α_*, α^* are adjoint. For $W \in {}_A\mathcal{M}$ and $E \in {}_B\mathcal{M}$, we define $\mathrm{Mor}(\alpha_*(W), E) \cong \mathrm{Mor}(W, \alpha^*(E))$ by sending a left A-module map $\sigma : W \to \alpha^*(E)$ to a left B-module map $\tilde\sigma(b\otimes w) = b.\sigma(w)$ (where the dot denotes the original left B action) in $\mathrm{Mor}(\alpha_*(W), E)$. Conversely, for a left B-module map $\tilde\sigma : \alpha_*(W) \to E$, there is a left A-module map $\sigma(w) = \tilde\sigma(1\otimes w)$ in $\mathrm{Mor}(W, \alpha^*(E))$.

For all $\phi \in \mathrm{Mor}(V, W)$ and $\theta \in \mathrm{Mor}(E, F)$, we have the commuting diagram

$$\begin{tikzcd}
\mathrm{Mor}(\alpha_*(W), E) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\theta\circ(-)\circ\alpha_*(\phi)"] & \mathrm{Mor}(W, \alpha^*(E)) \arrow[d, "\alpha^*(\theta)\circ(-)\circ\phi"] \\
\mathrm{Mor}(\alpha_*(V), F) \arrow[r, leftrightarrow, "\cong"] & \mathrm{Mor}(V, \alpha^*(F))
\end{tikzcd}$$

as required. The vertical maps send $\tilde{\sigma} \in \mathrm{Mor}(\alpha_*(W), E)$ to $\theta \circ \tilde{\sigma} \circ \alpha_*(\phi) \in \mathrm{Mor}(\alpha_*(V), F)$ and $\sigma \in \mathrm{Mor}(W, \alpha^*(E))$ to $\alpha^*(\theta) \circ \sigma \circ \phi \in \mathrm{Mor}(V, \alpha^*(F))$. $\diamond$

The key property of a bialgebra or Hopf algebra is that any two of its left modules have a tensor product which is another one, and likewise any two comodules. The general notion of a category with tensor products is the following.

Definition 2.39 A *monoidal category* is $(\mathcal{C}, \otimes, \underline{1}, \Phi, l, r)$, where

(1) $\mathcal{C}$ is a category;
(2) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor;
(3) $\Phi : (\ \otimes\ )\otimes \Rightarrow \otimes(\ \otimes\ )$ is a natural isomorphism (known as the *associator*), i.e., a collection of functorial isomorphisms

$$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$$

for all $V, W, Z \in \mathcal{C}$, obeying the 'pentagon condition' in Fig. 2.4;
(4) $\underline{1}$ is a unit object and $l : \mathrm{id} \Rightarrow (\ ) \otimes \underline{1}$, $r : \mathrm{id} \Rightarrow \underline{1} \otimes (\ )$ associated natural isomorphisms, i.e., collections of functorial isomorphisms, obeying the 'triangle condition' in Fig. 2.4.

In the second item, if we have morphisms $\theta : X \to Y$ and $\phi : V \to W$ in $\mathcal{C}$ then we have a morphism $\theta \otimes \phi : X \otimes V \to Y \otimes W$. For compositions of morphisms, given morphisms $\varphi : Y \to Z$ and $\psi : W \to U$, we have $(\varphi \otimes \psi) \circ (\theta \otimes \phi) = (\varphi \circ \theta) \otimes (\psi \circ \phi)$. In practice, we can omit the brackets in expressions such as $((V \otimes W) \otimes Z) \otimes U$, and write $V \otimes W \otimes Z \otimes U$ quite freely. There will be several ways to fill in the brackets and Φ in our expressions, but Mac Lane's coherence theorem says that all the different ways compose to the same. The maps l, r associated to the

$$\begin{tikzcd}
 & (V\otimes W)\otimes(Z\otimes U) \arrow[dr, "\Phi"] & \\
((V\otimes W)\otimes Z)\otimes U \arrow[ur, "\Phi"] \arrow[d, "\Phi\otimes \mathrm{id}"'] & & V\otimes(W\otimes(Z\otimes U)) \\
(V\otimes(W\otimes Z))\otimes U \arrow[rr, "\Phi"] & & V\otimes((W\otimes Z)\otimes U) \arrow[u, "\mathrm{id}\otimes\Phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(V\otimes\underline{1})\otimes W \arrow[rr, "\Phi"] & & V\otimes(\underline{1}\otimes W) \\
 & V\otimes W \arrow[ul, "l\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes r"'] &
\end{tikzcd}$$

Fig. 2.4 Pentagon and triangle identities for the associator and unity maps of a monoidal category unit object also take care of themselves once the stated consistency conditions are satisfied. We will therefore soon suppress these maps.

Example 2.40 Let H be a Hopf algebra. The category ${}_H\mathcal{M}$ of left H-modules is a monoidal category. For objects $V, W \in {}_H\mathcal{M}$ we take $V \otimes W$ to be the usual tensor product of vector spaces over the field $\Bbbk$. The H action is given by $h\triangleright(v \otimes w) = (h_{(1)}\triangleright v) \otimes (h_{(2)}\triangleright w)$. The identity object is $\underline{1} = \Bbbk$, with action $h\triangleright k = \epsilon(h)k$ for $k \in \Bbbk$. There are obvious isomorphisms $r : V \to \Bbbk \otimes V$ given by $r(v) = 1 \otimes v$ and $l : V \to V \otimes \Bbbk$ given by $l(v) = v \otimes 1$ (the inverses are just the product). The associator Φ is trivial, sending $(v \otimes w) \otimes u \in (V \otimes W) \otimes U$ to $v \otimes (w \otimes u) \in V \otimes (W \otimes U)$.

Similarly, the category $\mathcal{M}_H$ of right H-modules is a monoidal category where the right action on a tensor product is $(v \otimes w)\triangleleft h = (v\triangleleft h_{(1)}) \otimes (w\triangleleft h_{(2)})$. The monoidal category operations are the same as for ${}_H\mathcal{M}$. In fact, we also get a monoidal structure for the categories ${}^H\mathcal{M}$ of left H-comodules and $\mathcal{M}^H$ of right H-comodules, where we use the respective tensor product coactions

$$\Delta_L(v\otimes w) = v_{(\bar{1})}\otimes w_{(\bar{1})}\otimes v_{(\bar{\infty})} w_{(\bar{\infty})}, \quad \Delta_R(v\otimes w) = v_{(\bar{0})} w_{(\bar{0})}\otimes v_{(\bar{1})}\otimes w_{(\bar{1})}. \qquad \diamond$$

For an algebra A and a coalgebra C, we summarise some names and definitions of categories for convenience:

Name	Objects	Morphisms	Morphisms from E to F
${}_A\mathcal{M}$	Left A-modules	Left A-module maps	${}_A\mathrm{Hom}(E, F)$
$\mathcal{M}_A$	Right A-modules	Right A-module maps	$\mathrm{Hom}_A(E, F)$
${}_A\mathcal{M}_A$	A-bimodules	A-bimodule maps	${}_A\mathrm{Hom}_A(E, F)$
${}^C\mathcal{M}$	Left C-comodules	Left C-comodule maps	${}^C\mathrm{Hom}(E, F)$
$\mathcal{M}^C$	Right C-comodules	Right C-comodule maps	$\mathrm{Hom}^C(E, F)$

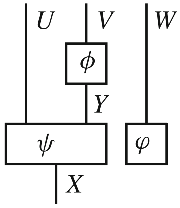
Another example is key to our approach to differential geometry on an algebra A.

Example 2.41 The A-bimodule category ${}_A\mathcal{M}_A$ is monoidal with product $\otimes_A$ and identity $\underline{1} = A$ as an A-bimodule with left and right multiplication. The bimodule $E \otimes_A F$ for objects E, F in ${}_A\mathcal{M}_A$ is defined as the quotient of the algebraic vector space tensor product $E \otimes F$ by the relation $e \otimes a.f = e.a \otimes f$ for all $e \in E$, $f \in F$ and $a \in A$. This $\otimes_A$ corresponds geometrically to taking the fibrewise tensor product of vector bundles when the algebra consists of functions on a topological space. The left and right A-actions are $a.(e\otimes f) = a.e\otimes f$ and $(e\otimes f).a = e\otimes f.a$. The associator for ${}_A\mathcal{M}_A$ is trivial, i.e., $\Phi((e \otimes f) \otimes g) = e \otimes (f \otimes g)$. $\diamond$

There is a useful diagrammatic notation for monoidal categories, writing objects as lines, tensor products of objects as parallel lines and reading down the page. Thus, given objects U, V, W, X, Y and morphisms $\phi : U \to Y$, $\psi : U \otimes Y \to X$ and

$\varphi : W \to \underline{1}$, we might have composition $U \otimes V \otimes W \to X$ drawn as

$$(\psi \otimes \varphi)(\mathrm{id} \otimes \phi \otimes \mathrm{id}) =$$



The naturality of the r and l maps means that we do not have to keep track of $\underline{1}$, so this is represented by an 'invisible' line. We have denoted morphisms by labelled rectangular boxes, but specific morphisms can be drawn with more distinctive shapes or denoted as nodes. By Mac Lane's coherence theorem, we can draw as many lines as we like without having to indicate the bracketing of tensor products.

We have the idea of a *monoidal functor* between monoidal categories, which is a way of saying that the functor preserves the product. A functor $F : \mathcal{C} \to \mathcal{V}$ between monoidal categories is (strictly) monoidal if it sends the unit object to the unit object and we have a natural isomorphism $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ which satisfies $c_{\underline{1},X} \circ l_{F(X)} = F(l_V)$, $c_{X,\underline{1}} \circ r_{F(X)} = F(r_V)$ and the consistency condition

$$\begin{tikzcd}
\big(F(X) \otimes F(Y)\big) \otimes F(Z) \arrow[r, "c\otimes\mathrm{id}"] \arrow[d, "\Phi"] & F(X \otimes Y) \otimes F(Z) \arrow[r, "c"] & F((X \otimes Y) \otimes Z) \arrow[d, "F(\Phi)"] \\
F(X) \otimes \big(F(Y) \otimes F(Z)\big) \arrow[r, "\mathrm{id}\otimes c"] & F(X) \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

One can also allow a slightly weaker notion with $c_{X,Y}$ not necessarily isomorphisms.

Example 2.42 Take the category (FinSet, $\times$, $\{*\}$) of finite sets and functions, made into a monoidal category by the cartesian product and the choice of a single element set. Also take (FinVect, $\otimes_{\Bbbk}$, $\Bbbk$) of finite-dimensional vector spaces over $\Bbbk$ and linear maps, made into a monoidal category by $\otimes_{\Bbbk}$. The functor F : FinSet $\to$ FinVect sends a finite set to the vector space with linear basis e_x for x in the set. Define $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ on the basis elements by $e_x \otimes e_y \mapsto e_{(x,y)}$, which has obvious inverse $e_{(x,y)} \mapsto e_x \otimes e_y$, making F into a monoidal functor. $\diamond$

We will also need the idea of a *left dual* $V^\sharp$ of an object V in a monodical category. This means an object $V^\sharp$, an *evaluation* map $\mathrm{ev}_V : V^\sharp \otimes V \to \underline{1}$ and a *coevaluation* map $\mathrm{coev}_V : \underline{1} \to V \otimes V^\sharp$ such that

$$(\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}) = \mathrm{id}_{V^\sharp}, \quad (\mathrm{id} \otimes \mathrm{ev})(\mathrm{coev} \otimes \mathrm{id}) = \mathrm{id}_V.$$

The existence of such a structure is not trivial. The diagrammatic notation extends to these objects as 'bend-straightening', as illustrated in Fig. 2.5. Again the identity object $\underline{1}$ is represented by an 'invisible' line. If V, W have left-duals then $W^\sharp \otimes V^\sharp$

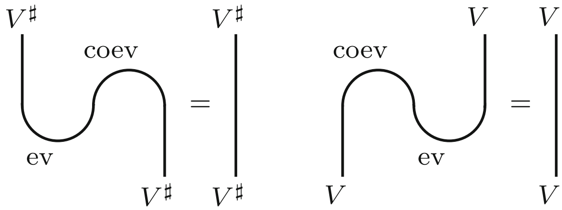
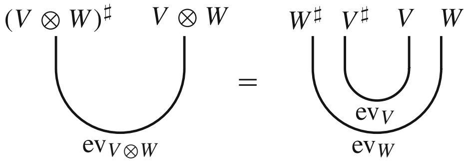
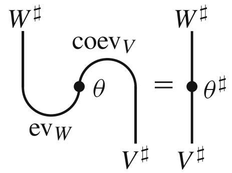


Fig. 2.5 Bend-straightening axioms for the left dual of an object in a monoidal category

becomes canonically the left-dual of $V \otimes W$ with $\mathrm{ev}_{V\otimes W} = \mathrm{ev}_W \circ (\mathrm{id} \otimes \mathrm{ev}_V \otimes \mathrm{id})$, i.e., applied in a nested fashion, with the necessary associators. This is illustrated in



Similarly for the tensor product coevaluation. Given a morphism $\theta : V \to W$ we define a dual morphism $\theta^\sharp : W^\sharp \to V^\sharp$ by



We say that a monoidal category is *left rigid* if all objects have left duals. There is a similar notion of a *right dual* $V^\flat$ with a left-right reversal of all axioms and defining diagrams; for example the evaluation map then goes $V \otimes V^\flat \to \underline{1}$.

Example 2.43 Let H be a Hopf algebra and V a finite-dimensional object in the monoidal category ${}_H\mathcal{M}$ of left H-modules. Then V has a left dual defined as $V^\sharp = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ (i.e. starting off with the usual vector space dual) together with the left H action $(h \triangleright \phi)(v) = \phi(Sh \triangleright v)$ for all $h \in H$, $v \in V$ and $\phi \in V^\sharp$. We check that the usual evaluation $\mathrm{ev}_V(\phi \otimes v) = \phi(v)$ commutes with the H action,

$$\mathrm{ev}_V(h \triangleright (\phi \otimes v)) = (h_{(1)} \triangleright \phi)(h_{(2)} \triangleright v) = \phi(S(h_{(1)})h_{(2)} \triangleright v) = \epsilon(h)\phi(v) = h \triangleright \phi(v).$$

Next, choose a vector space basis $\{f_1, \ldots, f_n\}$ of V with dual basis $\{e^1, \ldots, e^n\}$ of $V^\sharp$ so that $e^i(f_j) = \delta_{i,j}$. We define $\mathrm{coev}_V = \sum_i f_i \otimes e^i \in V \otimes V^\sharp$, which is $\mathrm{coev}_V(1)$ from the map point of view, and check the second rule in Fig. 2.5: given an element $a_j f_j \in V$ for some $a_i \in \Bbbk$ (sum over i, j to be understood), we have

$$(\mathrm{id} \otimes \mathrm{ev}_V)(\mathrm{coev}_V \otimes a_j f_j) = (\mathrm{id} \otimes \mathrm{ev}_V)(f_i \otimes e^i \otimes a_j f_j) = f_i a_j e^i(f_j) = a_j f_j.$$

$$\begin{tikzcd}[column sep=small]
 & V\otimes(W\otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
V\otimes(Z\otimes W) \arrow[d, "\Phi^{-1}"'] & & (V\otimes W)\otimes Z \arrow[d, "\Psi"] \\
(V\otimes Z)\otimes W \arrow[dr, "\Psi\otimes\mathrm{id}"'] & & Z\otimes(V\otimes W) \arrow[dl, "\Phi^{-1}"] \\
 & (Z\otimes V)\otimes W &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & (V\otimes W)\otimes Z \arrow[dl, "\Phi"'] \arrow[dr, "\Psi\otimes\mathrm{id}"] & \\
V\otimes(W\otimes Z) \arrow[d, "\Psi"'] & & (W\otimes V)\otimes Z \arrow[d, "\Phi"] \\
(W\otimes Z)\otimes V \arrow[dr, "\Phi"'] & & W\otimes(V\otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"] \\
 & W\otimes(Z\otimes V) &
\end{tikzcd}$$

Fig. 2.6 Hexagon identities for the braiding and associator of a braided monoidal category

There is a similar story for the category of comodules. For example, if $V \in \mathcal{M}^H$ then $V^\sharp$ has coaction $\Delta_R\phi = \phi_{(\bar{0})} \otimes \phi_{(\bar{1})}$ characterised by $\phi_{(\bar{0})}(v)\phi_{(\bar{1})} = \phi(v_{(\bar{0})})Sv_{(\bar{1})}$. ◇

Next we look at a feature of some monoidal categories, which group representations possess but general Hopf algebra representations do not, namely symmetry of the tensor product.

Definition 2.44 A *braided monoidal* category $(\mathcal{C}, \otimes, \underline{1}, \Phi, l, r, \Psi)$ is

(1) A monoidal category $(\mathcal{C}, \otimes, \underline{1}, \Phi, l, r)$.
(2) A natural isomorphism $\Psi : \otimes \Rightarrow \otimes^{\mathrm{op}}$, i.e., a collection of functorial isomorphisms $\Psi_{V,W} : V\otimes W \cong W\otimes V$ for all $V, W \in \mathcal{C}$, obeying the 'hexagon condition' in Fig. 2.6. Here $\otimes^{\mathrm{op}}(V, W) = W \otimes V$.

We do not assume that $\Psi_{V,W} = \Psi^{-1}_{W,V}$ (if this holds for all V, W then we have a *symmetric monoidal* category, as for example the usual flip map $V\otimes W \cong W\otimes V$ for vector spaces). The coherence theorem for braided categories (written out formally by Joyal and Street) asserts that if two composite morphisms

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \cdots V_{\sigma(n)}$$

(for some permutation σ and some bracketings on each side) that are built from $\Psi, \Psi^{-1}, \Phi, \Phi^{-1}$ correspond to the same braid, then they coincide as morphisms. Omitting the Φ for brevity, the 'hexagons' in Fig. 2.6 are

$$\Psi_{V,W\otimes Z} = \Psi_{V,Z} \circ \Psi_{V,W}, \quad \Psi_{V\otimes W,Z} = \Psi_{V,Z} \circ \Psi_{W,Z}$$

for all objects V, W, Z. From functoriality and these hexagon identities we can show the 'braid relations' between any three objects

$$\Psi_{W,Z}\Psi_{V,Z}\Psi_{V,W} = \Psi_{V,W}\Psi_{V,Z}\Psi_{W,Z}.$$

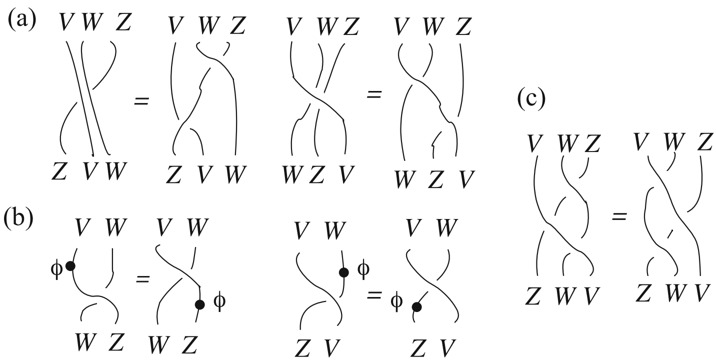
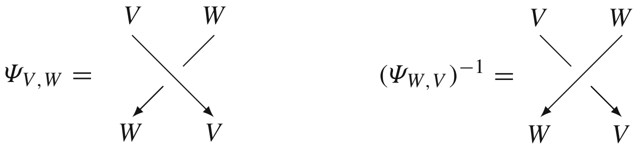


Fig. 2.7 Diagrammatic form of the braiding axioms. Doubled lines in (**a**) refer to the composite objects $V \otimes W$, $W \otimes Z$ in an extension of the notation. Functoriality of Ψ in (**b**) allows a morphism $\phi : V \to Z$ to pull through a crossing. Similarly for Ψ^{-1}. These imply the braid relations (**c**)

We accordingly extend the above diagrammatic notation for monoidal categories by



We denote any other morphisms as nodes on a string with the appropriate number of input and output legs. In this notation, the hexagon conditions and the functoriality of the 'braiding' Ψ appear in Fig. 2.7. If do not assume Ψ^{-1} then we only have the forward crossings as in the figure and we say the category is *prebraided*.

Example 2.45 (Braided Category of $\mathbb{Z}$*-Graded Vector Spaces)* Let $q \in \Bbbk^\times$. Objects are graded vector spaces $V = \oplus_i V_i$ with grade $|v| = i$ when $v \in V_i$. Morphisms are linear maps preserving the grading, $|v \otimes w| = |v| + |w|$, the associator is trivial and the braiding is $\Psi_{V,W}(v \otimes w) = q^{|v||w|} w \otimes v$ for elements of homogeneous degree.

Example 2.46 Let H be a Hopf algebra. The category $\mathcal{M}_H^H$ of right H-crossed modules in Definition 2.22 is prebraided with $\Psi_{V,W}(v \otimes w) = w_{(\bar{0})} \otimes v \triangleleft w_{(\bar{1})}$, and braided with $(\Psi_{W,V})^{-1}(v \otimes w) = w \triangleleft S^{-1} v_{(\bar{1})} \otimes v_{(\bar{0})}$ if H has invertible antipode. We tensor product co/modules and morphisms are simultaneous co/module maps. $\diamond$

Given a functor $F : \mathcal{C} \to \mathcal{V}$, there is a 'monoidal dual' $F^\circ : \mathcal{C}^\circ \to \mathcal{V}$ of 'representations' of the $\otimes$ of $\mathcal{C}$ in $\mathcal{V}$. Here we focus on the special case $\mathcal{C}^\circ \to \mathcal{C}$ of the identity functor i.e., the representations of $\mathcal{C}$ in itself, also called the 'centre' of $\mathcal{C}$. The proof of the monoidal structure for general F is very similar.

Theorem 2.47 (Drinfeld–Majid) *Let $\mathcal{C}$ be a monoidal category. The dual or* centre $\mathcal{Z}(\mathcal{C})$ *has objects (V, λ_V) consisting of $V \in \mathcal{C}$ and a natural transformation $\lambda_V : V \otimes \mathrm{id} \Rightarrow \mathrm{id} \otimes V$, i.e., a functorial collection $\{\lambda_{V,X} : V \otimes X \to X \otimes V\}$, such that*

$$(\mathrm{id} \otimes \lambda_{V,Y}) \circ (\lambda_{V,X} \otimes \mathrm{id}) = \lambda_{V,X\otimes Y}, \quad \lambda_{V,\underline{1}} = \mathrm{id}.$$

Morphisms $\phi : (V, \lambda_V) \to (W, \lambda_W)$ and the monoidal product are characterised by

$$(\mathrm{id} \otimes \phi) \circ \lambda_{V,X} = \lambda_{W,X} \circ (\phi \otimes \mathrm{id}), \quad \lambda_{V\otimes W,X} = (\lambda_{V,X} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \lambda_{W,X})$$

for all $X \in \mathcal{C}$. The centre is prebraided with $\Psi_{(V,\lambda_V),(W,\lambda_W)} = \lambda_{V,W}$ and the subcategory of objects (V, λ_V) where λ_V are natural isomorphisms, is braided.

Proof Note that functoriality of λ is $(\theta \otimes \mathrm{id}) \circ \lambda_{V,W} = \lambda_{V,Z} \circ (\mathrm{id} \otimes \theta)$ for all morphisms $\theta : W \to Z$ and $X \in \mathcal{C}$. There are a large number of axioms to check that we have a monoidal category. Many of these are routine and we give here only the heart of the proof, which is the requirement that the monoidal product of λ_V and λ_W obeys the representation condition. Omitting id morphisms for clarity, this is

$$\begin{aligned}\lambda_{V\otimes W,Z}\circ\lambda_{V\otimes W,U} &= \lambda_{V,Z} \circ \lambda_{W,Z} \circ \lambda_{V,U} \circ \lambda_{W,U}\\ &= \lambda_{V,Z} \circ \lambda_{V,U} \circ \lambda_{W,Z} \circ \lambda_{W,U} = \lambda_{V,Z\otimes U} \circ \lambda_{W,Z\otimes U} = \lambda_{V\otimes W,Z\otimes U}\end{aligned}$$

for all $U, Z \in \mathcal{C}$, as required. □

Example 2.48 Let H be a Hopf algebra. Then $\mathcal{Z}({}_H\mathcal{M})$ is the category of left H-crossed modules. If H has invertible antipode then the corresponding λ are isomorphisms and the category is braided, cf. Example 2.46.

Proof For (V, λ_V) in the centre, define $\Delta_L : V \to H\otimes V$ by $\Delta_L(v) = \lambda_{V,H}(v\otimes 1)$, where H is an H-module under left multiplication. Δ is a morphism $H \to H \otimes H$ and breaking down $\lambda_{V,H\otimes H}(v\otimes\Delta 1)$ as two applications of $\lambda_{V,H}$ tells us that Δ_L is a coaction. However, V is also an H-module and the $\lambda_{V,W}$ are required to morphisms, i.e., to commute with the action of H. This requires a compatibility condition, which we write using $\Delta_L(v) = v_{(\bar{1})} \otimes v_{(\bar{\infty})}$,

$$\begin{aligned}h_{(1)}v_{(\bar{1})} \otimes h_{(2)}\triangleright v_{(\bar{\infty})} &= h\triangleright\lambda_{V,H}(v \otimes 1) = \lambda_{V,H}(h\triangleright(v \otimes 1))\\ &= \lambda_{V,H}(h_{(1)}\triangleright v \otimes R_{h_{(2)}}(1)) = \big(\lambda_{V,H}(h_{(1)}\triangleright v \otimes 1)\big)\,(h_{(2)} \otimes 1)\\ &= (h_{(1)}\triangleright v)_{(\bar{1})}h_{(2)} \otimes (h_{(1)}\triangleright v)_{(\bar{\infty})},\end{aligned}$$

where the first equality is the definition of Δ_L and the action of H on $H \otimes V$. The second equality is that $\lambda_{V,H}$ is a morphism in $\mathcal{C}$. We then use functoriality with respect to the morphism $R_{h_{(2)}} : H \to H$ of right-multiplication to get the right-hand side of the condition for V to be a left H-crossed module.

Conversely, let V be a left H-crossed module and specify $\lambda_{V,H}(v \otimes 1) = v_{(\bar{1})} \otimes v_{(\bar{\infty})}$. This uniquely determines every $\lambda_{V,W}$ by functoriality, using the left module map $R_w : H \to W$ given by $R_w(h) = h{\triangleright}w$ for all $w \in W$,

$$\lambda_{V,W}(v \otimes w) = \lambda_{V,W}(v \otimes R_w(1)) = (R_w \otimes \mathrm{id})\lambda_{V,W}(v \otimes 1) = v_{(\bar{1})}{\triangleright}w \otimes v_{(\bar{\infty})},$$

giving the 1-1 correspondence between the centre and left H-crossed modules. If H has invertible antipode then $(\lambda_{V,W})^{-1}(w \otimes v) = v_{(\bar{\infty})} \otimes (S^{-1}v_{(\bar{1})}){\triangleright}w$, so in this case all objects have invertible λ_V. □

The $\mathcal{Z}(\mathcal{C})$ construction is very general. For example, the category of sets with Cartesian product is a perfectly good monoidal category and we can obtain further monoidal categories as sets with additional structure.

Example 2.49 (Whitehead Crossed Sets) Consider the category of G-sets, meaning sets X on which a group G acts by a left action $\triangleright$. Morphisms are equivariant maps. Its centre is the category of triples $(X, \triangleright, \partial)$ where $\partial : X \to G$ obeys $\partial(g{\triangleright}x) = g(\partial x)g^{-1}$. The braiding is $\Psi_{X,Y}(x, y) = ((\partial x){\triangleright}y, x)$ and $\Psi_{X,X}$ is a *set-theoretic solution of the braid relations* on every object X. The braiding used in the construction of $\Omega(G)$ in Proposition 1.52 and Example 2.29 is an example with $X = \mathcal{C}$, $\partial = \mathrm{id}$ and action by conjugation. Note that the linear span of a set is a vector space and the linearisation of a crossed G-set is to take $H = \Bbbk G$. Then $\mathcal{Z}({}_{\Bbbk G}\mathcal{M})$ is the category of left crossed G-modules, meaning vector spaces which are G-graded, so $V = \oplus_{g\in G} V_g$, and on which G acts such that $h{\triangleright}V_g \subseteq V_{hgh^{-1}}$. Moreover, a left action/coaction of $\Bbbk G$ means, in the case of G finite, a right coaction/action of $\Bbbk(G)$, but the category of crossed G-modules itself does not need G to be finite. ◇

Another approach is to ask what additional algebraic structure on a Hopf algebra do we need to make ${}_H\mathcal{M}$ and ${}^H\mathcal{M}$ themselves braided? We use the notation that given $\mathcal{R} \in H\otimes H$, we define elements of $H\otimes H\otimes H$ by $\mathcal{R}_{12} = \mathcal{R}\otimes 1$, $\mathcal{R}_{23} = 1\otimes\mathcal{R}$, and $\mathcal{R}_{13}$ by inserting 1 in the middle position.

Proposition 2.50 (Drinfeld) *Let H be a Hopf algebra equipped with an invertible element $\mathcal{R} \in H \otimes H$ obeying*

$$\mathrm{flip} \circ \Delta = \mathcal{R}(\Delta(\ ))\mathcal{R}^{-1}, \quad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}.$$

One says that $\mathcal{R}$ is a quasitriangular structure *or 'universal R-matrix' for H or that H is* quasitriangular. *Then*

$$\Psi_{V,W} : V \otimes W \to W \otimes V, \quad \Psi_{V,W}(v \otimes w) = \mathrm{flip}(\mathcal{R}{\triangleright}(v \otimes w))$$

makes ${}_H\mathcal{M}$ braided. Here flip *is the usual flip map and the first factor of $\mathcal{R}$ acts on $v \in V$, the second on $w \in W$.*

Proof Ψ 'functorial' says that if $\phi : V \to Z$ is equivariant (a morphism in the category ${}_H\mathcal{M}$) then $\Psi_{Z,W} \circ (\phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi) \circ \Psi_{V,W}$, and similarly on the other side. This follows by the form of Ψ as the action of an element of $H \otimes H$. We also verify that Ψ as stated is a morphism, i.e., commutes with the action of H. Thus,

$$\begin{aligned}\Psi_{V,W}(h\triangleright(v \otimes w)) &= \Psi_{V,W}((\Delta h)\triangleright(v \otimes w)) = \mathrm{flip}(\mathcal{R}(\Delta h)\triangleright(v \otimes w))\\ &= \mathrm{flip}((\mathrm{flip} \circ \Delta h)\mathcal{R}\triangleright(v \otimes w)) = h\triangleright\Psi_{V,W}(v \otimes w)\end{aligned}$$

for all $h \in H$, $v \in V$ and $w \in W$, as required. We also used $\triangleright$ to denote the action of $H \otimes H$ on $V \otimes W$ in the obvious way. The usual transposition map alone will not in general be a morphism; we need to apply $\mathcal{R}$ first. To show that Ψ obeys the 'hexagon' identities,

$$\begin{aligned}\Psi_{V,W\otimes Z}(v \otimes w \otimes z) &= \mathrm{flip}_{23} \circ \mathrm{flip}_{12}(((\mathrm{id} \otimes \Delta)\mathcal{R})\triangleright(v \otimes w \otimes z))\\ &= \mathrm{flip}_{23} \circ \mathrm{flip}_{12}(\mathcal{R}_{13}\mathcal{R}_{12}\triangleright(v \otimes w \otimes z))\\ &= \mathrm{flip}_{23} \circ \mathcal{R}_{23}\triangleright((\mathrm{flip} \circ \mathcal{R}\triangleright(v \otimes w)) \otimes z) = \Psi_{V,Z} \circ \Psi_{V,W}(v \otimes w \otimes z),\end{aligned}$$

where $\triangleright$ also denotes the action of $H \otimes H \otimes H$ on $V \otimes W \otimes Z$. Similarly for the other half of this axiom. Finally, invertibility of $\mathcal{R}$ ensures that the Ψ are invertible. □

Some facts from the theory of quasitriangular Hopf algebras are that the antipode is necessarily invertible and that

$$(\epsilon \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes \epsilon)\mathcal{R} = 1, \quad (S \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1}, \quad (\mathrm{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R}. \tag{2.12}$$

One also has that the 'quantum Killing form' $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$, where $\mathcal{R}_{21} = \mathrm{flip}(\mathcal{R})$, has the property that $(S \otimes \mathrm{id})\mathcal{Q}$ is invariant under the tensor product of left adjoint actions on H. The latter is defined by

$$h\triangleright g = h_{(1)} g S h_{(2)} \tag{2.13}$$

for all $h, g \in H$.

Example 2.51 $u_q(sl_2)$ in Example 2.11 is quasitriangular with

$$\mathcal{R} = \left(\frac{1}{r}\sum_{a,b=0}^{r-1} q^{-2ab} g^a \otimes g^b\right)\left(\sum_{i=0}^{r-1} \frac{(q - q^{-1})^i}{[i]_{q^{-2}}!} E^i \otimes F^i\right).$$

The proof is an extended computation using Lemma 2.15. A simpler example is to take just the first factor making the sub-Hopf algebra $\mathbb{C}\mathbb{Z}_r$ generated by g quasitriangular. ◇

This completes our lightning introduction to Drinfeld quantum groups based on his notion of quasitriangular Hopf algebras. However, this is not quite the theory we need. For one thing, Hopf algebras like $U_q(sl_2)$ are not quasitriangular because the required $\mathcal{R}$ would be given by an infinite power series. The series terminates on the usual finite-dimensional highest weight representations, so their braidings are well defined, but $\mathcal{R}$ itself does not live in the algebraic tensor product. There are several ways around this. One is to consider $q = e^{\lambda/2}$, where λ is a formal parameter, and work over the ring of formal power series in λ. This is not good for applications where we want q to be an actual number. Another is to switch to the quantum group coordinate algebra version of the theory given by arrow-reversal of all the axioms. We recall, see the discussion around Proposition 2.7, that the dual of a coalgebra always has a convolution product.

Proposition 2.52 *Let A be a Hopf algebra equipped with a convolution-invertible linear map $\mathcal{R} : A \otimes A \to \Bbbk$ obeying*

$$b_{(1)}a_{(1)}\mathcal{R}(a_{(2)} \otimes b_{(2)}) = \mathcal{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)},$$

$$\mathcal{R}(ab \otimes c) = \mathcal{R}(a \otimes c_{(1)})\mathcal{R}(b \otimes c_{(2)}), \quad \mathcal{R}(a \otimes bc) = \mathcal{R}(a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b)$$

for all $a, b, c \in A$. One says that $\mathcal{R}$ is a coquasitriangular structure *or that A is* coquasitriangular. *Then*

$$\Psi_{V,W} : V \otimes W \to W \otimes V, \quad \Psi_{V,W}(v \otimes w) = \sum w_{(\bar{0})} \otimes v_{(\bar{0})}\mathcal{R}(v_{(\bar{1})} \otimes w_{(\bar{1})}),$$

for all $v \in V,\ w \in W$ makes the category $\mathcal{M}^A$ of right A-comodules braided.

We omit the proof as dual to the more well-known Proposition 2.50. Here the linear maps from a coalgebra to an algebra form a 'convolution algebra' and the inverse is defined in that sense. However, here it obeys $\mathcal{R}^{-1} = \mathcal{R} \circ (S \otimes \mathrm{id})$. Again, the antipode is necessarily invertible. Also note that if $t \in M_n(A)$ is a matrix corepresentation, so $t^i{}_j \in A$, $\Delta t^i{}_j = t^i{}_k \otimes t^k{}_j$ (summation understood) and $\epsilon(t^i{}_j) = \delta^i_j$, then

$$R^i{}_a{}^j{}_b t^a{}_k t^b{}_l = t^j{}_b t^i{}_a R^a{}_k{}^b{}_l, \tag{2.14}$$

where $R^i{}_k{}^j{}_l = \mathcal{R}(t^i{}_k \otimes t^j{}_l)$ as an expression of the first axiom of a coquasitriangular Hopf algebra. The other two axioms imply that $R \in M_n \otimes M_n$ obeys the braid relations. For standard quantum groups such as $\mathbb{C}_q[G]$, one can choose a generating matrix corepresentation where A is generated by the $t^i{}_j$ with these and other relations (such as q-determinant relations). Going in the other direction, we can start with R and take the above quadratic relations as defining a bialgebra $A(R)$, which can be shown to be coquasitriangular when R obeys the braid relations.

Example 2.53 $\mathbb{C}_q[GL_2]$ has a coquasitriangular structure with

$$R = q^{\alpha}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

for any α, and if $\alpha = -\frac{1}{2}$ then this descends to a coquasitriangular structure on $\mathbb{C}_q[SL_2]$. We use the notation where $R^i{}_j{}^k{}_l$ is specified by rows ik and columns jl in order 11,12,21,22. So, for example, $\mathcal{R}(b\otimes c) = q^{\alpha}(q-q^{-1})$. Here $R \in M_2 \otimes M_2$ determines the commutation relations on the Hopf algebra by the first axiom in the definition of coquasitriangular, but not the q-determinant relation. We then extend the matrix R to products by repeated use of the 'bicharacter' coquasitriangularity axioms in the second line of Proposition 2.52. $\diamond$

We have already mentioned Ad-invariance of the quantum Killing form and we will need this particularly in the dual version

$$\mathcal{Q}(a\otimes b) = \mathcal{R}(b_{(1)}\otimes a_{(1)})\mathcal{R}(a_{(2)}\otimes b_{(2)})$$

for all $a, b \in A$ on a coquasitriangular Hopf algebra A, along with some lesser-known properties collected in the next proposition. A more obvious property is $\mathcal{Q}(Sa\otimes Sb) = \mathcal{Q}(b\otimes a)$, since $\mathcal{R}$ is invariant under $S\otimes S$.

Proposition 2.54 *With reference to the coaction* Ad_R *in (2.5), the quantum Killing form on a coquasitriangular Hopf algebra A obeys*

$$\mathcal{Q}(a_{\bar{(0)}}\otimes Sb_{\bar{(0)}})a_{\bar{(1)}}b_{\bar{(1)}} = \mathcal{Q}(a\otimes Sb), \quad \mathcal{Q}(a_{\bar{(0)}}\otimes b)a_{\bar{(1)}} = \mathcal{Q}(a\otimes b_{(2)})b_{(1)}Sb_{(3)},$$
$$\mathcal{Q}(ab, c) = \mathcal{R}(c_{(1)}, b_{(1)})\mathcal{Q}(a, c_{(2)})\mathcal{R}(b_{(2)}, c_{(3)}),$$
$$\mathcal{Q}(a\otimes bc) = \mathcal{Q}(a_{(1)}\otimes b_{(1)})\mathcal{Q}(a_{(2)\bar{(0)}}\otimes c)\mathcal{R}(a_{(2)\bar{(1)}}\otimes b_{(2)}),$$
$$\mathcal{Q}(a\otimes c_{(1)})\mathcal{Q}(b\otimes c_{(2)}) = \big((\mathcal{Q}((\ )\otimes c_{(1)})\otimes \mathcal{Q}((\ )\otimes c_{(2)})\big)\,\tilde{\Psi}(a\otimes b)$$

for all $a, b, c \in A$, where

$$\tilde{\Psi}(a\otimes b) = b_{(3)}\otimes a_{(3)}\mathcal{R}(Sb_{(2)}\otimes a_{(1)})\mathcal{R}(b_{(4)}\otimes a_{(2)})\mathcal{R}(a_{(4)}\otimes b_{(5)})\mathcal{R}(a_{(5)}\otimes Sb_{(1)})$$

is the crossed-module braiding for A as a crossed-module by Ad_R *and*

$$a\triangleleft b = a_{(2)}\mathcal{R}(b_{(1)}\otimes a_{(1)})\mathcal{R}(a_{(3)}\otimes b_{(2)})$$

for all $a, b \in A$.

Proof The first line gives two versions of Ad-invariance. The first implies

$$\begin{aligned}\mathcal{Q}(a_{\bar{(0)}} \otimes Sb_{(2)})a_{\bar{(1)}}b_{(1)} &= \mathcal{Q}(a_{\bar{(0)}} \otimes Sb_{(2)})a_{\bar{(1)}}(Sb_{(1)})b_{(3)}Sb_{(4)}\\ &= \mathcal{Q}(a_{\bar{(0)}} \otimes Sb_{(1)\bar{(0)}})a_{\bar{(1)}}b_{(1)\bar{(1)}}Sb_{(2)} = \mathcal{Q}(a \otimes Sb_{(1)})Sb_{(2)}\end{aligned}$$

for all a, b, which is equivalent to the 2nd version since S is invertible. The 2nd line follows from the multiplicativity properties of $\mathcal{R}$ to break down products, thus the left-hand side is

$$\mathcal{R}(c_{(1)}\otimes a_{(1)}b_{(1)})\mathcal{R}(a_{(2)}b_{(2)}\otimes c_{(2)}) = \mathcal{R}(c_{(2)}\otimes a_{(1)})\mathcal{R}(c_{(1)}\otimes b_{(1)})\mathcal{R}(a_{(2)}\otimes c_{(3)})\mathcal{R}(b_{(2)}\otimes c_{(4)})$$

and we then recombine the 1st and 3rd factors to give $\mathcal{Q}$ in the middle. For the 3rd line, we compute similarly, from the right-hand side

$$\begin{aligned}&\mathcal{Q}(a_{(1)} \otimes b_{(1)})\mathcal{Q}(a_{(3)} \otimes c)\mathcal{R}((Sa_{(2)})a_{(4)} \otimes b_{(2)})\\ &= \mathcal{R}(c_{(1)} \otimes a_{(4)})\mathcal{R}(a_{(5)} \otimes c_{(2)})\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes b_{(2)})\mathcal{R}^{-1}(a_{(3)} \otimes b_{(3)})\mathcal{R}(a_{(6)} \otimes b_{(4)})\\ &= \mathcal{R}(c_{(1)} \otimes a_{(2)})\mathcal{R}(a_{(3)} \otimes c_{(2)})\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(4)} \otimes b_{(2)})\\ &= \mathcal{R}((bc)_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes (bc)_{(2)}) = \mathcal{Q}(a \otimes bc).\end{aligned}$$

For the remaining result, we first check that the action and coaction stated indeed make A a right crossed module. Starting on the right-hand side of the condition in Definition 2.22 and Ad_R, we compute

$$\begin{aligned}&a_{(2)}{\triangleleft}b_{(2)} \otimes (Sb_{(1)})(Sa_{(1)})a_{(3)}b_{(3)}\\ &\quad= a_{(3)}\mathcal{R}(b_{(2)} \otimes a_{(2)})\mathcal{R}(a_{(4)} \otimes b_{(3)})(S(a_{(1)}b_{(1)}))a_{(5)}b_{(4)}\\ &\quad= a_{(3)} \otimes (S(b_{(2)}a_{(2)}))b_{(3)}a_{(4)}\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(5)} \otimes b_{(4)})\\ &\quad= a_{(3)} \otimes (Sa_{(2)})a_{(4)}\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(5)} \otimes b_{(2)})\\ &\quad= \Delta_R a_{(2)}\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes b_{(2)}) = \Delta_R(a{\triangleleft}b),\end{aligned}$$

where for the 2nd equality we use the quasicommutativity axiom of the coquasitriangular structure in two places so that we can then cancel some factors via the antipode axioms. In fact, the two copies of $\mathcal{R}$ act independently and could be two distinct coquasitriangular structures as far as having a crossed module is concerned. The crossed module braiding is then

$$\begin{aligned}\tilde{\Psi}(a \otimes b) &= b_{(2)} \otimes a_{(2)}\mathcal{R}(((Sb_{(1)})b_{(3)})_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes ((Sb_{(1)})b_{(3)})_{(2)})\\ &= b_{\bar{(0)}} \otimes a{\triangleleft}b_{\bar{(1)}} = b_{(3)} \otimes a_{(2)}\mathcal{R}((Sb_{(2)})b_{(4)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes (Sb_{(1)})b_{(5)}).\end{aligned}$$

We then expand using the multiplicativity properties to get the result stated. We now verify the 4th property of $\mathcal{Q}$. Starting with the right-hand side, we have

$$
\begin{aligned}
&\mathcal{R}(c_{(1)}\otimes b_{(3)})\mathcal{R}(b_{(4)}\otimes c_{(2)})\mathcal{R}(c_{(3)}\otimes a_{(3)})\mathcal{R}(a_{(4)}\otimes c_{(4)})\\
&\mathcal{R}(Sb_{(2)}\otimes a_{(1)})\mathcal{R}(b_{(5)}\otimes a_{(2)})\mathcal{R}(a_{(5)}\otimes b_{(6)})\mathcal{R}(a_{(6)}\otimes Sb_{(1)})\\
&\quad=\mathcal{R}(c_{(1)}\otimes b_{(3)})\mathcal{R}(b_{(5)}\otimes c_{(3)})\mathcal{R}(c_{(2)}\otimes a_{(2)})\mathcal{R}(b_{(4)}\otimes a_{(3)})\\
&\qquad\mathcal{R}(Sb_{(2)}\otimes a_{(1)})\mathcal{R}(a_{(4)}\otimes c_{(4)})\mathcal{R}(a_{(5)}\otimes b_{(6)})\mathcal{R}(a_{(6)}\otimes Sb_{(1)})\\
&\quad=\mathcal{R}(c_{(1)}\otimes b_{(3)})\mathcal{R}(b_{(6)}\otimes c_{(4)})\mathcal{R}(c_{(2)}\otimes a_{(2)})\mathcal{R}(b_{(4)}\otimes a_{(3)})\\
&\qquad\mathcal{R}(a_{(4)}\otimes b_{(5)})\mathcal{R}(a_{(5)}\otimes c_{(3)})\mathcal{R}(Sb_{(2)}\otimes a_{(1)})\mathcal{R}(a_{(6)}\otimes Sb_{(1)})\\
&\quad=\mathcal{R}(a_{(2)}\otimes (Sb_{(1)})c_{(3)}b_{(5)})\mathcal{R}((Sb_{(2)})c_{(2)}b_{(4)}\otimes a_{(1)})\mathcal{R}(c_{(1)}\otimes b_{(3)})\mathcal{R}(b_{(6)}\otimes c_{(4)})\\
&\quad=\mathcal{R}(a_{(2)}\otimes (Sb_{(1)})c_{(3)}b_{(5)})\mathcal{R}((Sb_{(2)})b_{(3)}c_{(1)}\otimes a_{(1)})\mathcal{R}(c_{(2)}\otimes b_{(4)})\mathcal{R}(b_{(6)}\otimes c_{(4)})\\
&\quad=\mathcal{R}(a_{(2)}\otimes (Sb_{(1)})c_{(3)}b_{(3)})\mathcal{R}(c_{(1)}\otimes a_{(1)})\mathcal{R}(c_{(2)}\otimes b_{(2)})\mathcal{R}(b_{(4)}\otimes c_{(4)})\\
&\quad=\mathcal{R}(a_{(2)}\otimes (Sb_{(1)})b_{(2)}c_{(2)})\mathcal{R}(c_{(1)}\otimes a_{(1)})\mathcal{R}(c_{(3)}\otimes b_{(3)})\mathcal{R}(b_{(4)}\otimes c_{(4)})\\
&\quad=\mathcal{R}(c_{(1)}\otimes a_{(1)})\mathcal{R}(a_{(2)}\otimes c_{(2)})\mathcal{R}(c_{(3)}\otimes b_{(1)})\mathcal{R}(b_{(2)}\otimes c_{(4)}),
\end{aligned}
$$

which is the required left-hand side. We repeatedly used the braid relations (obtained by gathering products via the multiplicativity axiom and then using the quasi-commutativity axioms to swap them), continuing this process until we could cancel using the antipode axioms. We will use this in the theory of braided Lie algebras. □

For a coquasitriangular Hopf $*$-algebra, we want $\mathcal{R}$ to be compatible with $*$ and we consider two main cases:

$$
\begin{aligned}
\text{real case:}\quad &\overline{\mathcal{R}(a\otimes b)}=\mathcal{R}(b^*\otimes a^*)\\
\text{antireal case:}\quad &\overline{\mathcal{R}(a\otimes b)}=\mathcal{R}^{-1}(a^*\otimes b^*)
\end{aligned}
\tag{2.15}
$$

where we use the convolution-inverse of $\mathcal{R}$ as above. In the former case, we have $\mathcal{Q}(b^*\otimes a^*)=\overline{\mathcal{Q}(a\otimes b)}$. The quantum group $\mathbb{C}_q[SU_2]$ with real q has its standard coquasitriangular structure real in the above sense.

2.5 Bicovariant Differentials on Coquasitriangular Hopf Algebras

Here we construct bicovariant differential structures on the standard quantum groups $\mathbb{C}_q[G]$ as coquasitriangular Hopf algebras. To do this we let A be a coquasitriangular Hopf algebra and construct a particular right A-crossed module Λ^1 and a right crossed module map $\varpi : A^+ \twoheadrightarrow \Lambda^1$. We will then apply Theorem 2.26 with $\mathcal{I} = \ker\varpi$ to obtain $\Omega^1 \cong A.\Lambda^1$ as a free module. Our starting point is a version of Proposition 2.54 for the quantum Killing form, but with the left A-crossed module

$$\mathrm{Ad}_L(a) = a_{(1)}Sa_{(3)} \otimes a_{(2)}, \quad a\triangleright b = b_{(2)}\mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes b_{(3)}) \tag{2.16}$$

by calculations analogous to the right-handed case previously given. Then any subcoalgebra of A becomes a left A-crossed module by restriction and its dual will provide Λ^1. Note the similarity of what follows with Theorem 1.47 for group algebras, with the quantum Killing form $\mathcal{Q}$ in a role similar to the 1-cocycle there.

Proposition 2.55 *Let A be a coquasitriangular Hopf algebra, $\mathcal{L} \subseteq A$ a nonzero finite-dimensional subcoalgebra and $\varpi : A^+ \to \mathcal{L}^*$ defined by $\varpi(a)(x) = \mathcal{Q}(a \otimes x)$ for all $a \in A^+$ and $x \in \mathcal{L}$. Then $\mathcal{L}^*$ by dualisation is a right A-crossed module, ϖ is a morphism of crossed modules and $\Lambda^1 = \mathrm{image}(\varpi)$ defines a bicovariant differential calculus Ω^1 on A. This is inner if $\theta = \epsilon|_{\mathcal{L}}$ is in the image of ϖ. Over $\mathbb{C}$, if $\mathcal{R}$ is real and $*S(\mathcal{L}) \subseteq \mathcal{L}$ then we have a $*$-differential calculus.*

Proof We do not need the dualised crossed module structure explicitly but it is

$$\Delta_R v = \sum_j e_j v(f^j{}_{(2)}) \otimes f^j{}_{(1)}Sf^j{}_{(3)},$$

$$v\triangleleft a = \sum_j e_j v(f^j{}_{(2)})\mathcal{R}(f^j{}_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes f^j{}_{(3)})$$

for all $v \in \mathcal{L}^*$, where $\{f^i\}$ is a basis of $\mathcal{L}$ and $\{e_i\}$ is a dual basis of $\mathcal{L}^*$. The first of these is the right coadjoint coaction dual to the above Ad_L when the latter is restricted to $\mathcal{L}$. Next, ϖ is a morphism of crossed modules since

$$\varpi(ab)(x) = \mathcal{Q}(ab \otimes x) = \mathcal{R}(x_{(1)} \otimes b_{(1)})\mathcal{Q}(a \otimes x_{(2)})\mathcal{R}(b_{(2)} \otimes x_{(3)}) = \mathcal{Q}(a \otimes b\triangleright x),$$

$$\varpi(a_{(\bar{0})})(x) \otimes a_{(\bar{1})} = \mathcal{Q}(a_{(\bar{0})} \otimes x) \otimes a_{(\bar{1})} = \mathcal{Q}(a \otimes x_{(2)})x_{(1)}Sx_{(3)} = (\Delta_R\mathcal{Q}(a \otimes (\ )))(x)$$

using Proposition 2.54. Hence the kernel of ϖ is an Ad-stable right ideal, giving us a bicovariant calculus. Finally, we cannot have $\mathcal{L} \subseteq A^+$ since being a subcoalgebra, this would imply $x = \epsilon(x_{(1)})x_{(2)} = 0$ for all $x \in \mathcal{L}$. Hence we can set $\theta = \epsilon|_{\mathcal{L}}$ as a nonzero element of $\mathcal{L}^*$ and find

$$(\theta\triangleleft a)(x) = \epsilon(a\triangleright x) = \epsilon(x_{(2)})\mathcal{R}(x_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes x_{(3)}) = \mathcal{Q}(a \otimes x).$$

Hence $\theta\triangleleft a = \mathcal{Q}(a \otimes (\ )) = \varpi(a)$, making the calculus inner if θ is in the image of ϖ. The ideal $\mathcal{I} \subseteq A^+$ here will be the kernel of ϖ. Over $\mathbb{C}$, we have a $*$-calculus if $*S\mathcal{I} \subseteq \mathcal{I}$ according to Theorem 2.26. In our case, if $\mathcal{Q}(a \otimes b) = \mathcal{Q}(Sb \otimes Sa) = 0$ for all $b \in \mathcal{L}$ then by our reality assumption on $\mathcal{R}$ and taking the complex conjugate, $\mathcal{Q}(*Sa \otimes *Sb) = 0$ for all $b \in \mathcal{L}$. If $*S\mathcal{L} \subseteq \mathcal{L}$ then we see that $*Sa \in \mathcal{I}$. In this case

$\varpi(a)^* := -\varpi(S^{-1}(a^*))$ translates as $v \mapsto S^2(v^*)$ when we write the $*$-operation on the left-invariant 1-forms in terms of elements of $\mathcal{L}^*$ and the $*$-operation dual to the $*$-operation on A. Note also that we have focussed on left-invariant forms but a version of Proposition 2.55 with right-invariant forms will also be used on occasion, see Example 2.87 and the discussion before it. □

If we take $\Lambda^1 = \mathcal{L}^*$ rather than the image of ϖ then we obtain an inner bicovariant generalised first-order calculus, i.e., possibly lacking the surjectivity condition. The proposition provides a powerful construction in that it is relatively easy to find subcoalgebras, but whether we can obtain all bicovariant differential calculi depends on the restricted quantum Killing form $\mathcal{Q}$; for example, if the coquasitriangular structure is trivial (a classical algebraic group) then we have no chance of obtaining anything nonzero. However, it is more or less true at the other extreme where the quantum Killing form as a map is nondegenerate. We prove this next but only in the case of A finite-dimensional. In this case it is convenient to work dually with 2-sided ideals in the dual Hopf algebra as the dual concept to that of a subcoalgebra.

Proposition 2.56 *Let A be finite-dimensional with dual H factorisable in the sense of the quantum Killing form viewed as $\mathcal{Q} : A \to H$ being a linear isomorphism. Then bicovariant Ω^1 on A are in 1-1 correspondence with 2-sided ideals $\mathcal{J} \subseteq H^+$.*

Proof Here $\mathcal{Q}(a) = (a \otimes \mathrm{id})(\mathcal{Q})$, where $\mathcal{Q}$ is the quantum Killing form of H which is now a quasitriangular Hopf algebra. This is the same as the map $\mathcal{Q}$ above from the coquasitriangular structure on A when the latter is finite-dimensional. Now, key in the previous proof was that the quantum Killing form as a map $\mathcal{Q} : A \to H$ is a morphism of right crossed A-modules, where A has Ad_R as in (2.5) and right multiplication, while H has the right coadjoint coaction of A as given at the start of the proof of Proposition 2.55 (with $\{f^j\}$ now a basis of all of A) and left action also as before but now written as

$$h \triangleleft a = \mathcal{R}_2(a_{(1)})h\mathcal{R}_1(a_{(2)}) = \mathcal{Q}(a_{(1)})\mathrm{Ad}_{S^{-1}\mathcal{R}_1(a_{(2)})}(h), \quad \text{for all } a \in A,\ h \in H$$

using the left adjoint action of H on itself. Here $\mathcal{R}_i : A \to H$ are given by evaluating on the first or second factor of $\mathcal{R}$ respectively, so $\mathcal{R}_1(a) = (a \otimes \mathrm{id})(\mathcal{R})$. By the morphism property of $\mathcal{Q}$ we see that Ad_R-stable right ideals $\mathcal{I} \subseteq A^+$ map over under $\mathcal{Q}$ to subspaces of H^+ that are stable under the right coadjoint coaction of A and right action $\triangleleft$. The former is equivalent to subspaces $\mathcal{J} \subseteq H^+$ stable under the left adjoint action of H and, given this, invariant under left multiplication under $\mathcal{Q}(A)$. In the factorisable case, this is equivalent to $\mathcal{J}$ being a 2-sided ideal. □

This result applies to the standard $c_q[G]$ at certain roots of unity, for example $c_q[SL_2]$ at an odd root has $u_q(sl_2)$ factorisable. In general the ideal structure of an algebra can be complicated but in nice cases every 2-sided ideal can be obtained as the kernel of a representation. More precisely, if $\rho : H \to \mathrm{End}(V)$ is a representation then we can set $\mathcal{J} = \ker \rho|_{H^+}$ and hence have a calculus with

$\Lambda^1 = \mathrm{End}(V)$ and $\varpi = \rho \circ \mathcal{Q} : A^+ \to \mathrm{End}(V)$ provided this map is surjective. Note the similarity with Example 1.49. If H is semisimple then it has a block decomposition into matrix blocks and if we could apply Proposition 2.56 then we would obtain one irreducible calculus for each irreducible representation V (the ideal $\mathcal{J}$ would be essentially all the other blocks in the block decomposition). We would also have that all irreducible bicovariant calculi are square-dimensional.

On the other hand, finite-dimensional quasitriangular Hopf algebras do not tend to be both semisimple and factorisable at the same time, for example $u_q(sl_2)$ is not semisimple. The standard $U_q(\mathfrak{g})$ for generic q are semisimple, or better, the standard $\mathbb{C}_q[G]$ are *cosemisimple* (they are the direct sum of their cosimple sub-coalgebras) but they are only factorisable in some formal sense working over formal power series in the deformation parameter t, where $q = e^{\frac{t}{2}}$. What this amounts to is that the conclusion we wanted essentially holds; irreducible bicovariant calculi are classified for generic q by the finite-dimensional irreducible representations provided we look only at calculi that have a classical limit as $q \to 1$. For example, the smallest nonzero bicovariant calculus for $\mathbb{C}_q[SL_2]$ is 4-dimensional, corresponding to V 2-dimensional, and there is just one of these that has a $q \to 1$ classical limit. It is also possible to approach such a result in terms of 'quasitriangular dual pairs' consisting of nondegenerately paired Hopf algebras A, H with A coquasitriangular and compatible maps $\mathcal{R}_i$ subject to axioms equivalent in the finite-dimensional case to Drinfeld's as in the preceding proof.

Returning to our constructive approach for a coquasitriangular Hopf algebra A, we focus on subcoalgebras of A that arise as the image of comodules. If $\Delta_R : V \to V \otimes A$ is a comodule then we define $\mathcal{L} = \{(\phi \otimes \mathrm{id})\Delta_R(V) \mid \phi \in V^*\}$ and check that $\Delta\phi(v_{\bar{(0)}})v_{\bar{(1)}} = \phi(v_{\bar{(0)}\bar{(0)}})v_{\bar{(0)}\bar{(1)}} \otimes v_{\bar{(1)}}$. It is clear that the first tensor factor lies in $\mathcal{L}$ and, by evaluating that against any element of $\mathcal{L}^*$, so does the second factor. If A is cosemisimple then all cosimple subcoalgebras (meaning they have no nonzero proper subcoalgebras) are of this form. The basic example is to take $\mathcal{L}$ to be a vector space with basis $\{t^i{}_j\}$, with coproduct and counit in the matrix comultiplication form that we have seen for matrix quantum groups in (2.3). If V is a finite-dimensional comodule and we choose a basis $\{e_i\}$ of it then $\mathcal{L}$ has this matrix form as defined by $\Delta_R e_j = e_i \otimes t^i{}_j$. One says that one has a matrix corepresentation (it defines a matrix representation of any dually paired Hopf algebra H by evaluation).

Corollary 2.57 *Let A be a coquasitriangular Hopf algebra, $t \in M_n(A)$ a matrix corepresentation and $\{E_\alpha{}^\beta\}$ the standard basis of $M_n(\Bbbk)$. The latter becomes a right A-crossed module with*

$$\Delta_R E_\alpha{}^\beta = E_m{}^n \otimes t^m{}_\alpha S t^\beta{}_n, \quad E_\alpha{}^\beta \triangleleft t^a{}_b = E_m{}^n R^m{}_\alpha{}^a{}_c R^c{}_b{}^\beta{}_n. \tag{2.17}$$

Then

$$\Lambda^1 = M_n(\Bbbk), \quad \varpi(a) = \mathcal{Q}(a \otimes t^\alpha{}_\beta) E_\alpha{}^\beta$$

defines a bicovariant generalised first-order calculus with $\mathcal{I} = \ker \varpi$. The $E_\alpha{}^\beta$ as left-invariant 1-forms have bimodule relations

$$E_\alpha{}^\beta t^a{}_b = t^a{}_c E_m{}^n R^m{}_\alpha{}^c{}_d R^d{}_b{}^\beta{}_n$$

and the above right covariance. The left covariance is defined as $E_\alpha{}^\beta$ invariant and the calculus is inner with

$$\mathrm{d}t^a{}_b = t^a{}_c (R_{21}R)^c{}_b{}^\alpha{}_\beta E_\alpha{}^\beta - t^a{}_b\theta = [\theta, t^a{}_b], \quad \theta = E_\alpha{}^\alpha.$$

In the $*$-algebra case for $\mathcal{R}$ of real type and a unitary corepresentation, we have a $*$-differential structure with

$$(E_\alpha{}^\beta)^* = -E_\beta{}^\alpha.$$

Proof We compute from Proposition 2.55 with $R^i{}_j{}^k{}_l = \mathcal{R}(t^i{}_j \otimes t^k{}_l)$. Note that in the general construction $\Lambda^1 \cong \mathrm{image}(\varpi)$ but since this is typically all of $M_n(\Bbbk)$, we have taken the latter as Λ^1. The rest then follows by computation in the calculus. Note that we displayed $\triangleleft$, the bimodule and d only for the $t^i{}_j$ as these typically generate the algebra, but the construction does not depend on this. Thus,

$$E_\alpha{}^\beta \triangleleft a = E_m{}^n \mathcal{R}(t^m{}_\alpha \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes t^\beta{}_n)$$

in general. If we want a usual calculus with the surjectivity condition then we can just take $\Lambda^1 = \varpi(A^+)$ and the result will be inner if this contains the identity matrix θ. Over $\mathbb{C}$ we suppose $\mathcal{R}$ is of real type and the generators are closed under $*S$, then we have a $*$-differential algebra. For example, if the matrix corepresentation is unitary in the sense that $t^\alpha{}_\beta{}^* = St^\beta{}_\alpha$, then this translates as stated, as then for all a,

$$\begin{aligned}\varpi(a)^* &= -\overline{\mathcal{Q}(a \otimes t^\alpha{}_\beta)} E_\beta{}^\alpha = -\overline{\mathcal{Q}((*S)^2 a \otimes *St^\alpha{}_\beta)} E_\alpha{}^\beta \\ &= -\mathcal{Q}(*Sa \otimes t^\alpha{}_\beta) E_\alpha{}^\beta = -\varpi(*Sa).\end{aligned}$$

□

In nice cases such as the standard $\mathbb{C}_q[G]$, q generic and $\{t^i{}_j\}$ the matrix coalgebras of the different irreducible representations, we will have a surjective inner calculus of square dimension and obtain all the irreducible calculi with classical limit as $q \to 1$, as discussed above. Before our main example, we warm up with q-deformed S^1.

Example 2.58 Let $A = \mathbb{C}_q\mathbb{Z}$ be the coquasitriangular Hopf algebra $\mathbb{C}[t, t^{-1}]$ with $\mathcal{R}(t^m \otimes t^n) = q^{mn}$. Then $M_1(\mathbb{C})$ has one basis element $\theta = E_1{}^1$ and the R-matrix is just the 1×1 matrix with entry q. Hence Ω^1 by Corollary 2.57 recovers the bicovariant calculus as in Example 1.11 but with relations $(\mathrm{d}t)t = q^2 t\mathrm{d}t$, which we have already encountered in Proposition 2.33. We have denoted this algebra with this q^2-difference calculus and the real form $t^* = t^{-1}$ as $\mathbb{C}_{q^2}[S^1]$ and we already noted that this is inner. In our present conventions, however,

$$\theta = \frac{t^{-1}\mathrm{d}t}{(q^2-1)}$$

since we have not normalised d in Corollary 2.57. $\diamond$

Example 2.59 For $A = \mathbb{C}_q[SU_2]$ the smallest nontrivial irreducible representation is 2-dimensional, giving us $\Lambda^1 = M_2(\mathbb{C})$. We write the basis as ${E_1}^1 = e_a$, ${E_1}^2 = e_b$, ${E_2}^1 = e_c$, ${E_2}^2 = e_d$ and use the R-matrix for $\mathbb{C}_q[SL_2]$ in Example 2.53 in the general formula in Corollary 2.57 to obtain the relations

$$e_a\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} qa & q^{-1}b\\ qc & q^{-1}d\end{pmatrix} e_a,\quad [e_b,\begin{pmatrix} a & b\\ c & d\end{pmatrix}] = q\lambda\begin{pmatrix} 0 & a\\ 0 & c\end{pmatrix} e_a,$$

$$[e_c,\begin{pmatrix} a & b\\ c & d\end{pmatrix}] = q\lambda\begin{pmatrix} b & 0\\ d & 0\end{pmatrix} e_a,\quad [e_d,\begin{pmatrix} a\\ c\end{pmatrix}]_{q^{-1}} = \lambda\begin{pmatrix} b\\ d\end{pmatrix} e_b,$$

$$[e_d,\begin{pmatrix} b\\ d\end{pmatrix}]_q = \lambda\begin{pmatrix} a\\ c\end{pmatrix} e_c + q\lambda^2\begin{pmatrix} b\\ d\end{pmatrix} e_a,$$

where $[x, y]_q := xy - qyx$ and $\lambda = 1 - q^{-2}$. The exterior differential is inner with $\theta = e_a + e_d$, which implies that

$$\mathrm{d}\begin{pmatrix} a\\ c\end{pmatrix} = \begin{pmatrix} a\\ c\end{pmatrix}((q-1)e_a + (q^{-1}-1)e_d) + \lambda\begin{pmatrix} b\\ d\end{pmatrix} e_b,$$

$$\mathrm{d}\begin{pmatrix} b\\ d\end{pmatrix} = \begin{pmatrix} b\\ d\end{pmatrix}((q^{-1}-1+q\lambda^2)e_a + (q-1)e_d) + \lambda\begin{pmatrix} a\\ c\end{pmatrix} e_c.$$

The map $\mathcal{Q}$ here is surjective so we obtain a standard differential algebra. This also means that we can write θ and the other basic 1-forms in terms of differentials, namely

$$\theta = \frac{q^2}{(1-q)^2[3]_q}\varpi\pi_\epsilon \mathrm{tr}_q,\quad e_b = \frac{1}{\lambda}\varpi c,\quad e_c = \frac{1}{\lambda}\varpi b,\quad e_z = \frac{q^{-1}}{\lambda}\varpi(a-d),$$

where $\mathrm{tr}_q = q^{-1}a + qd$ is the q-trace, $[n]_q = (1-q^n)/(1-q)$ and $e_z = q^{-2}e_a - e_d$. Here $\varpi\pi_\epsilon h = Sh_{(1)}\mathrm{d}h_{(2)}$ and we would need to scale d by λ^{-1} in order to have the right classical limit (we have not done this in order to have a uniform treatment according to the general construction).

In terms of ideals in $\mathbb{C}_q[SL_2]^+$, we have Ad_R-stable right ideals $\mathcal{I}_\pm$ generated by

$$(\lambda_\pm + 1 - a + q^2(1-d))x, \quad b^2, \quad b(d-a), \quad c^2,$$
$$c(d-a), \quad q^{-2}d^2 + a^2 - (1+q^{-2})(1+(2)_q bc), \tag{2.18}$$

where $x \in \{1-a, b, c, 1-d\}$ and $\lambda_\pm = -(1\mp q^{-1})(1\mp q^3)$. We used the symmetric q-integer $(2)_q = q + q^{-1}$. Only λ_+ has the correct classical limit $\mathcal{I} = (\mathbb{C}[SU_2]^+)^2$ as $q \to 1$. It gives the 4D calculus above, while the other choice is of less interest.

The right coaction on left-invariant 1-forms is

$$\Delta_R\theta = \theta \otimes 1, \quad \Delta_R(-e_b, e_z, q^{-1}e_c) = (-e_b, e_z, q^{-1}e_c) \otimes \begin{pmatrix} a^2 & (2)_q ab & b^2 \\ ca & 1+(2)_q bc & db \\ c^2 & (2)_q cd & d^2 \end{pmatrix},$$

where the calculation is from $\Delta_R e_\alpha{}^\beta$ with the relevant R-matrix.

The above works in the same way over a general field for $\Bbbk_q[SL_2]$, but in our case over $\mathbb{C}$ and with q real, we also have a $*$-differential algebra $\Omega^1(\mathbb{C}_q[SU_2])$ with

$$e_a^* = -e_a, \quad e_b^* = -e_c, \quad e_c^* = -e_b, \quad e_d^* = -e_d,$$

which implies that $e_z^* = -e_z$ and $\theta^* = -\theta$. One can verify that this works for the relations above, as it must since $\mathbb{C}_q[SU_2]$ has a real coquasitriangular structure. $\diamond$

We now study some of the geometry of this important example along the lines of our studies in Chap. 1.

Proposition 2.60 *For the above 4D calculus on $\mathbb{C}_q[SU_2]$ there is a unique bi-invariant quantum metric*

$$g = e_c \otimes e_b + q^2 e_b \otimes e_c + \frac{q^3}{(2)_q}(e_z \otimes e_z - \theta \otimes \theta)$$

of real type in the sense $\mathrm{flip}(* \otimes *)(g) = g$. *The inverse metric is*

$$(e_b, e_c) = 1, \quad (e_c, e_b) = q^{-2}, \quad (e_z, e_z) = q^{-3}(2)_q = -(\theta, \theta)$$

and the rest zero in this basis.

Proof Looking at the right coaction on Λ^1, we recognise the same $\mathbb{C}_q[SO_3]$-valued 3×3 matrix that preserved the quantum metric in Proposition 2.36. This time we have a right coaction so that invariance of $g^{ij}e_i e_j$, where we sum over $i, j = b, c, z$, requires that (g^{ij}) is the inverse of the matrix that was preserved before. This translates up to an overall normalisation as the leading part of the metric stated, where we further added a multiple of $\theta \otimes \theta$ since this is also invariant. We will see later when we cover the exterior algebra that $\wedge(g) = 0$ for any choice of this multiple, which we then fixed so that g is central. Thus for example,

$$
\begin{aligned}
&(e_c \otimes e_b + q^2 e_b \otimes e_c)a = e_c \otimes a e_b + q^3\lambda e_b \otimes b e_a + q^2 e_b \otimes a e_c \\
&\quad = a(e_c \otimes e_b + q^2 e_b \otimes e_c) + q\lambda b e_a \otimes e_b + q^3 \lambda b e_b \otimes e_a + q^4\lambda^2 a e_a \otimes e_a, \\
&(e_z \otimes e_z - \theta \otimes \theta)a = -(1+q^{-2})(\lambda e_a \otimes e_a + e_a \otimes e_d + e_d \otimes e_a)a \\
&\quad = a(e_z \otimes e_z - \theta \otimes \theta) - (1-q^{-4})(q^2\lambda a e_a \otimes e_a + q b e_b \otimes e_a + q^{-1} b e_b \otimes e_a),
\end{aligned}
$$

using the above commutation relations. Comparing these, we find $[g, a] = 0$ as it needs to be bimodule invertible. Similarly for the other generators. The reality property and the inverse are immediate. □

We will also need d on a general monomial.

Lemma 2.61 *By iterating the above relations for the 4D calculus, one has*

$$
\begin{aligned}
\mathrm{d}(c^k b^n d^m) = &\big(q^{m+n-k} - 1\big)c^k b^n d^m e_d + \lambda q^n (k)_q c^{k-1} b^n d^{m+1} e_b \\
&+ \lambda q^{-k}\left(q^{m-1}(m+n)_q c^{k+1} b^n d^{m-1} + (n)_q c^k b^{n-1} d^{m-1}\right) e_c \\
&+ \lambda^2 q\left((k+1)_q (m+n)_q c^k b^n d^m + q^{-m}(n)_q (k)_q c^{k-1} b^{n-1} d^m\right) e_a \\
&+ \big(q^{-m-n+k} - 1\big)c^k b^n d^m e_a,
\end{aligned}
$$

where $(n)_q = (q^n - q^{-n})/(q - q^{-1})$. *Here* $k, n \geq 0$ *and* $m \geq 1$.

Proof We let $u = \begin{pmatrix} a \\ c \end{pmatrix}$, $v = \begin{pmatrix} b \\ d \end{pmatrix}$ and iterate the bimodule relations to obtain

$$
\begin{aligned}
&e_a v^n = q^{-n} v^n e_a, \quad e_b v^n = v^n e_b + q\lambda(n)_q u v^{n-1} e_a, \quad e_c v^n = v^n e_c, \\
&e_d v^n = q^n v^n e_d + \lambda q^{n-1}(n)_q u v^{n-1} e_c + q\lambda^2 (n)_q v^n e_a, \\
&e_a u^n = q^n u^n e_a, \quad e_b u^n = u^n e_b, \quad e_c u^n = u^n e_c + q\lambda(n)_q v u^{n-1} e_a, \\
&e_d u^n = q^{-n} u^n e_d + \lambda q^{1-n}(n)_q v u^{n-1} e_b.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
&e_a c^k b^n d^m = q^{-m-n+k} c^k b^n d^m e_a, \\
&e_b c^k b^n d^m = c^k b^n d^m e_b + q^{1-n}\lambda\left((n)_q q^{1-m} c^k b^{n-1} d^{m-1} + (m+n)_q c^{k+1} b^n d^{m-1}\right) e_a, \\
&e_c c^k b^n d^m = c^k b^n d^m e_c + q^{k-m}\lambda(k)_q c^{k-1} b^n d^{m+1} e_a,
\end{aligned}
$$

$$
\begin{aligned}
e_d c^k b^n d^m = {} & q^{-k+n+m} c^k b^n d^m e_d + q^n \lambda (k)_q c^{k-1} b^n d^{m+1} e_b \\
& + q\lambda^2 \left((k+1)_q (m+n)_q c^k b^n d^m + q^{-m} (k)_q (n)_q c^{k-1} b^{n-1} d^m \right) e_a \\
& + q^{-k} \lambda \left((n)_q c^k b^{n-1} d^m + q^{-1} (q^{m+n} (m)_q + (n)_q) c^{k+1} b^n d^{m-1} \right) e_c,
\end{aligned}
$$

which we then use, in particular, to compute $[\theta, \,]$. Note that terms in the expression for d that might appear to involve c^{-1} or b^{-1} have zero coefficient as they would arise when $k = 0$ or $n = 0$, respectively. □

This is relevant to part of the basis of the algebra; there is another part of the basis with a in place of d for which the result is similar, and a third part with only powers of b, c which comes out as

$$
\begin{aligned}
\mathrm{d}(c^k b^n) = {} & c^k b^n ((q^{n-k} - 1) e_d + (q^{k-n} - 1) e_a) + \lambda q^n (k)_q c^{k-1} b^n d e_b \\
& + \lambda q^{-} (n)_q c^k b^{n-1} a e_c + \lambda^2 q (n)_q c^{k-1} b^{n-1} ((k)_q + (k+1)_q cb)) e_a .
\end{aligned}
$$

This approach is somewhat different from the classical case where we would allow d to be invertible in one coordinate patch and this would agree via the determinant relation with a positive power of a in another coordinate patch. In the quantum case we prefer not to invert d or a in the algebra itself (in fact, we do not think in terms of coordinate patch algebras) but our calculations do still match up using $a = (1 + q^{-1} bc) d^{-1}$ in the stated expressions for d formally extended to negative m. For example

$$
\begin{aligned}
\mathrm{d}((1 + q^{-1} bc) d^{-1}) = & (q^{-1} - 1) d^{-1} e_d - \lambda q^{-2} c d^{-2} e_c - \lambda^2 q d^{-1} e_a + (q-1) d^{-1} e_a \\
& + q^{-1} (q^{-1} - 1) cb d^{-1} e_d + \lambda b e_b + \lambda q^{-2} c d^{-2} e_c + \lambda^2 q d^{-1} e_a \\
& + q^{-1} (q-1) cb d^{-1} e_a \\
= & (q^{-1} - 1) a e_d + (q-1) a e_a + \lambda b e_b = \mathrm{d}a .
\end{aligned}
$$

From this lemma we can read off the partial derivatives in this basis defined by

$$
\mathrm{d}f = (\partial^b f) e_b + (\partial^c f) e_c + (\partial^z f) e_z + (\partial^0 f) \theta .
$$

Proposition 2.62 *For the above quantum metric on the 4D calculus on $\mathbb{C}_q[SU_2]$, we have*

$$
{}_\theta\Delta = \sum_{\mu,\nu \in \{b,c,z,0\}} (e_\mu, e_\nu) \partial^\mu \partial^\nu = 2 q^{-1} \lambda^2 \Delta_q , \quad \partial^0 = \frac{q^2 \lambda^2}{(2)_q} \Delta_q ,
$$

in terms of a q-deformation of the Laplace–Beltrami operator on SU_2 given by

$$\Delta_q(c^k b^n d^m) = q^{-m}(k)_q(n)_q c^{k-1}b^{n-1}d^m + \Big(\frac{k+n+m}{2}\Big)_q\Big(\frac{k+n+m}{2}+1\Big)_q c^k b^n d^m.$$

Proof Writing $e_a = (\theta + e_z)/(1+q^{-2})$ and $e_d = (\theta - q^2 e_z)/(1+q^2)$ we have

$$\partial^z = \frac{1}{1+q^{-2}}(\partial^a - \partial^d), \quad \partial^0 = \frac{1}{1+q^{-2}}(\partial^a + q^{-2}\partial^d),$$

in terms of the partial derivatives in our original basis read off from Lemma 2.61. The former comes out on $c^k b^n d^m$ as

$$\partial^z = \frac{\lambda}{(2)_q}(q^{m+n+2}(k)_q - q^{-k}(m+n)_q) + \frac{\lambda^2}{(2)_q}q^{2-m}(k)_q(n)_q S_c^- S_b^-,$$

where S_c^- lowers the degree of c by 1, etc. A similar computation of $\partial^a + q^{-2}\partial^d$ gives ∂^0 as stated. Then ${}_\theta\Delta = -2(\mathrm{d}(\ ),\theta) = -2(\partial^0(\ )\theta,\theta) = 2q^{-3}(2)_q\partial^0$ from the inner product $(\ ,\ )$ in Proposition 2.60, which we write in terms of Δ_q. The remaining expression for the q-Laplacian is an involved computation using all our vector fields and the inverse metric. We first compute on $c^k b^n d^m$ that

$$\begin{aligned}\partial^b\partial^c + q^{-2}\partial^c\partial^b &= \partial^b\lambda q^{-k+m-1}(m+n)_q S_c^+ S_d^- \\ &\quad + \partial^b\lambda q^{-k}(n)_q S_b^- S_d^- + q^{-2}\partial^c\lambda q^n(k)_q S_c^- S_d^+ \\ &= 2\lambda^2 q^{n-k-1}(k)_q(n)_q S_c^- S_b^- \\ &\quad + \lambda^2 q^{n-k-1+m}\left((k)_q(m+n+1)_q + (m+n)_q(k+1)_q\right).\end{aligned}$$

Next, we write for brevity

$$\begin{aligned}A &= q^{m+n+2}(k)_q - q^{-k}(m+n)_q,\\ B &= \lambda q^2\left(\frac{k+n+m}{2}\right)_q\left(\frac{k+n+m}{2}+1\right)_q,\\ C &= \lambda S_c^- S_b^- q^{2-m}(k)_q(n)_q,\end{aligned}$$

where k,n,m are now the degree operators for the powers of c,b,d respectively when acting on a monomial. Then $\partial^z = \frac{\lambda}{(2)_q}(A+C)$ and $\partial^0 = \frac{\lambda}{(2)_q}(B+C)$ and

$$(\partial^z)^2 - (\partial^0)^2 = \frac{\lambda^2}{(2)_q^2}\Big(A^2 - B^2 + 2C(A-B)\Big) = \frac{\lambda}{(2)_q}(A+B+2C)\big(1-q^{m+n-k}\big)$$

on noting that $A - B = \frac{(2)_q}{\lambda}(1-q^{m+n-k})$ commutes with C (since the latter changes both k,n equally and does not change m). Putting in these results and the value of

$A+B$ we obtain $\partial^b\partial^c+q^{-2}\partial^c\partial^b+q^{-3}(2)_q((\partial^z)^2-(\partial^0)^2)=2q^{-1}\lambda^2\Delta_q$, which is the same as our result for ${}_\theta\Delta$. This can also be used to express ∂^0 or Δ_q in terms of ∂^b, ∂^c, ∂^z.

To see that Δ_q as stated is exactly a q-deformation of the classical Laplace–Beltrami operator Δ_{LB} on SU_2, let us note that the latter is given by the action of the Casimir $x_+x_-+\frac{h^2}{4}-\frac{h}{2}$ in terms of the usual Lie algebra generators of su_2, where $[x_+,x_-]=h$. To compute the action of the vector fields for these Lie algebra generators in the part of the basis above, we let ∂^b, ∂^c, ∂^d denote partials derivatives keeping the other two generators constant but regarding $a=(1+bc)d^{-1}$ in what classically is now a coordinate patch. Then

$$\partial^b=\frac{\partial}{\partial b}+d^{-1}c\frac{\partial}{\partial a},\quad \partial^c=\frac{\partial}{\partial c}+d^{-1}b\frac{\partial}{\partial a},\quad \partial^d=\frac{\partial}{\partial d}-d^{-1}a\frac{\partial}{\partial a}$$

if one on the right regards the a,b,c,d as independent for the partial derivations. Left-invariant vector fields are usually given in the latter redundant form as $\tilde{x}=t^i{}_jx^j{}_k\frac{\partial}{\partial t^i{}_k}$ for x in the representation associated to the matrix of coordinates. Converting such formulae into our coordinate system, we find

$$\tilde{h}=c\partial^c-b\partial^b-d\partial^d,\quad \tilde{x}_+=a\partial^b+c\partial^d,\quad \tilde{x}_-=d\partial^c,$$

$$\Delta_{LB}=\partial^b\partial^c+\frac{1}{4}(c\partial^c+b\partial^b+d\partial^d)^2+\frac{1}{2}(c\partial^c+b\partial^b+d\partial^d),$$

as obtained above when $q\to 1$. The other parts of the basis are similar. □

The extra cotangent dimension here suggests adjoining a new central coordinate variable s with $\theta=\mathrm{d}s$ in additive form or $\theta=t^{-1}\mathrm{d}t$ in multiplicative form. The latter arises if we look at the calculus on $\mathbb{C}_q[GL_2]$, where we have a similar matrix of generators $s^i{}_j$ with entries $\alpha,\beta,\gamma,\delta$ say (we use different symbols from before to keep distinct from $\mathbb{C}_q[SL_2]$). Recall that now we have an additional D and D^{-1} adjoined, with antipode as given in (2.4). Also recall from Example 2.53 that this quantum group is coquasitriangular with a free parameter in the normalisation of $\mathcal{R}(s^i{}_j\otimes s^k{}_l)$, which we take to be $\alpha=1$ in the notation there. This is also the natural choice whereby the quantum plane in Proposition 2.13 becomes a Hopf algebra in the braided category of $\mathbb{C}_q[GL_2]$-comodules, as we will explain later. We then define $\Omega^1(\mathbb{C}_q[GL_2])$ by Corollary 2.57. The real form $\mathbb{C}_q[U_2]$ also has a unitary $*$-structure and hence in this case we have a $*$-differential calculus which is 4-dimensional and broadly similar to Example 2.59 but with different powers of q from the R-matrix. For example, this time

$$\theta=\frac{q}{q^6-1}S\mathrm{tr}_{q\,(1)}\mathrm{d}\mathrm{tr}_{q\,(2)},$$

where $\mathrm{tr}_q=q^{-1}\alpha+q\delta$ by a similar computation as before.

Proposition 2.63 *If we adjoin a square root of D and assume that q has a square root then*

$$\mathbb{C}_q[U_2][\sqrt{D},\sqrt{D}^{-1}]\cong\mathbb{C}_q[SU_2]\otimes\mathbb{C}_{q^3}[S^1]$$

as coquasitriangular Hopf $*$-algebras and

$$\Omega^1(\mathbb{C}_q[U_2][\sqrt{D},\sqrt{D}^{-1}])\cong\frac{\Omega^1(\mathbb{C}_q[SU_2])\rtimes\!\!\!\lhd\mathbb{C}_{q^3}[S^1]\oplus\mathbb{C}_q[SU_2]\rtimes\!\!\!\lhd\Omega^1(\mathbb{C}_{q^3}[S^1])}{\langle\theta-\frac{t^{-1}\mathrm{d}t}{q^3-1}\rangle},$$

where $\rtimes\!\!\!\lhd$ indicates the cross relations

$$t\omega=q^{-3}\omega t,\quad [\mathrm{d}t,f]=(q^3-1)t\mathrm{d}f$$

for all $f\in\mathbb{C}_q[SU_2]$, $\omega\in\Omega^1(\mathbb{C}_q[SU_2])$ and we identify the inner elements θ for the $\mathbb{C}_q[SU_2]$ and $t^{-1}\mathrm{d}t/(q^3-1)$ for the $\mathbb{C}_{q^3}[S^1]$ calculus.

Proof As algebras, $t^i{}_j\mapsto s^i{}_j\sqrt{D}^{-1}$ and $t\mapsto\sqrt{D}$ or $s^i{}_j\mapsto t^i{}_jt$ and $\sqrt{D}\mapsto t$, which we now take as an identification of two sets of generators. The coquasitriangular structure of $\mathbb{C}_q[GL_2]$ has the same form as that of $\mathbb{C}_q[SL_2]$ apart from powers of q. Thus $\mathcal{R}(s^i{}_j\otimes s^k{}_l)=\mathcal{R}(tt^i{}_j\otimes tt^k{}_l)=\mathcal{R}(t\otimes t)\mathcal{R}(t^i{}_j\otimes t^k{}_l)=q^{3/2}\mathcal{R}(t^i{}_j\otimes t^k{}_l)$, which now comes out with a factor q in front of the matrix in Example 2.53, as promised. We take the coquasitriangular structure as on $\mathbb{C}_{q^2}[S^1]$ in Example 2.58 but with q there replaced by $q^{3/2}$. These steps work over any field if we start with $q^{\frac{1}{2}}$ as the parameter. In our case, with q real and positive we have a natural choice of root and $s^i{}_j{}^*=Ss^j{}_i=t^{-1}St^j{}_i=(tt^i{}_j)^*$, as required.

The differential calculus on $\mathbb{C}_q[U_2]$ as discussed is computed via Corollary 2.57 and is 4D for the defining matrix corepresentation with extra q^3 factor from the new normalisation of the R-matrices compared to Example 2.59. Thus

$$e_a\alpha=q^4\alpha e_a,\quad \mathrm{d}\alpha=\alpha((q^4-1)e_a+(q^2-1)e_d)+\lambda q^3\beta e_b,$$

etc. In particular, we have $E_\alpha{}^\beta D=q^6DE_\alpha{}^\beta$, so in terms of t we have $E_\alpha{}^\beta t=q^3tE_\alpha{}^\beta$ and hence the first stated relation. Next, writing $\alpha=at$, $\beta=bt$,

$$\begin{aligned}(\mathrm{d}a)t+a\mathrm{d}t=\mathrm{d}(at)&=at((q^4-1)e_a+(q^2-1)e_d)+\lambda q^3bte_b\\&=a((q-q^{-3})e_a+(q^{-1}-q^{-3})e_d)t+\lambda be_bt\\&=(\mathrm{d}_{\mathbb{C}_q[SU_2]}a)t+(1-q^{-3})a(e_a+e_d)t.\end{aligned}$$

Comparing these, we see that the calculi match up provided we identify $\theta=e_a+e_d$ with $t^{-1}\mathrm{d}t/(q^3-1)$, as stated. This then implies the second stated relation. □

Alternatively, we can somewhat equivalently adjoin to $\mathbb{C}_q[SU_2]$ a central s with additive coproduct $\Delta s = s \otimes 1 + 1 \otimes s$ and $s^* = -s$ (this is then not algebraically related to $\mathbb{C}_q[U_2]$ as we would need $t = \mathrm{e}^{3s}$) and finite-difference calculus, i.e., $\mathbb{C}_\lambda[\mathbb{R}]$ in Example 1.10 with relations $[\mathrm{d}s, s] = \lambda \mathrm{d}s$. Now λ is defined by $q = e^\lambda$ (not shorthand for $1 - q^{-2}$ as above) and is therefore real, not imaginary. This time we have cross relations

$$[\omega, s] = \lambda\omega, \quad [\mathrm{d}s, f] = \lambda \mathrm{d}f$$

for $f \in \mathbb{C}_q[SU_2]$ and $\omega \in \Omega^1(\mathbb{C}_q[SU_2])$, and quotient by $\theta = \lambda^{-1}\mathrm{d}s$, where θ is the inner element for the $\mathbb{C}_q[SU_2]$ calculus. We have ∂_λ defined by $\mathrm{d}g(s) = \frac{g(s+\lambda)-g(s)}{\lambda}\mathrm{d}s = (\partial_\lambda g)\mathrm{d}s$ as usual for any $g \in \mathbb{C}_\lambda[\mathbb{R}]$ and we have $(\mathrm{d}f)g(s) = g(s+\lambda)\mathrm{d}f$ for any $f \in \mathbb{C}_q[SU_2]$. We then define the extended partials $\tilde{\partial}^i, \tilde{\partial}^0$ (where $i = b, c, z$) in the extended calculus as the coefficients of d in the basis e_i, θ and

$$\tilde{\partial}^i(fg)e_i + \tilde{\partial}^0(fg)\theta = \mathrm{d}(fg) = (\mathrm{d}f)g + f\mathrm{d}g = \partial^i(f)e_i g(s) + (\partial^0 f)\theta g(s) + \lambda f \partial_\lambda g\theta$$

tells us that $\tilde{\partial}^i(fg) = (\partial^i f)g(s+\lambda)$ and $\tilde{\partial}^0(fg) = (\partial^0 f)g(s+\lambda) + \lambda f \partial_\lambda g$ in terms of ∂^i, ∂^0 on $f \in \mathbb{C}_q[SU_2]$ as in Proposition 2.62 and for any $g \in \mathbb{C}_\lambda[\mathbb{R}]$. In these terms, Schrödinger's equation appears on normal-ordered functions $\psi(a, b, c, d, s)$, regarded as $\psi(s) \in \mathbb{C}_q[SU_2]$, as the condition

$$\tilde{\partial}^0\psi = 0 \quad \Leftrightarrow \quad \frac{\psi(s) - \psi(s-\lambda)}{\lambda} + \frac{q^2\lambda}{(2)_q}\Delta_q(\psi(s)) = 0,$$

where the first term deforms $-\mathrm{i}\frac{\partial}{\partial\tau}$ if we write $\tau = \mathrm{i}s$ so as to have a real variable. This is similar to our findings for the 4D calculus on $U(su_2)$ in Example 1.45 and again gives a point of view of quantum mechanics as emerging out of the interpretation of time as the generator of the extra cotangent dimension. In both cases one may change the normalisations so as to introduce a mass parameter.

2.6 Braided Exterior Algebras

We are almost ready to cover the construction of the exterior algebra of a bicovariant differential calculus. The proper explanation of this depends on the theory of Hopf algebras *in* braided categories, so we still need some general notions about this. In classical geometry, the exterior algebra is obtained by antisymmetrising products of 1-forms and the generalisation of this to bicovariant noncommutative differentials on Hopf algebras will be as a process of braided antisymmetrisation. This will provide a modern 'braided approach' to an exterior algebra first found by Woronowicz.

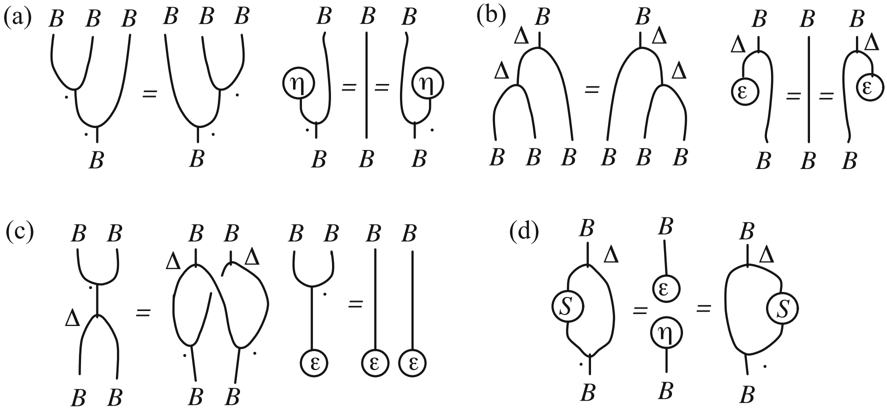


Fig. 2.8 Axioms of a 'braided group' or Hopf algebra in a braided category

First, in any monoidal category $\mathcal{C}$, we define an algebra product as a morphism $A \otimes A \to A$ where A is an object in the category and where we impose the same diagram as in Fig. 2.1 but now in the category, which means to insert any nontrivial associators Φ. An algebra 'identity' is similarly formulated as a morphism $\eta : \underline{1} \to A$. We then view the product 'flowing down the page' as a node with two legs coming in and one coming out and axioms as in Fig. 2.8a. Similarly, a coalgebra is in (b), and when $\mathcal{C}$ is braided, a bialgebra is in (c) and the antipode axiom for a 'braided group' or 'Hopf algebra B in a braided category' is in (d). Note in (c) the use of the braided tensor product algebra $B\underline{\otimes}C$, of any two algebras in the category, with product given by

$$\underline{\cdot} = (\cdot_B \otimes \cdot_C)(\mathrm{id} \otimes \Psi \otimes \mathrm{id}) : (B \otimes C) \otimes (B \otimes C) \to B \otimes C. \tag{2.19}$$

The theory of 'braided groups' or Hopf algebras in braided categories has been extensively developed since it was introduced in the early 1990s by one of the present authors. If a braided group has a left dual $B^\sharp$ object in the category then this becomes another braided group. This is shown in Fig. 2.9. The only thing to be careful about is that the dual product and coproduct in these diagrams is the opposite to those for usual Hopf algebras. This is necessary to avoid unnecessary braidings in the definitions, and to keep things clear we refer to the *categorial dual* and stick to the $B^\sharp$ notation. When B does not have a left dual, we can still talk about a dual pair B', B of braided groups and an evaluation pairing $\mathrm{ev}_B : B' \otimes B \to \underline{1}$ such that the product on one side is adjoint to the coproduct on the other, and likewise for the antipode and (co)unit. For example

$$\mathrm{ev}_B(\cdot_{B'} \otimes \mathrm{id}) = \mathrm{ev}_{B\otimes B}(\mathrm{id} \otimes \Delta_B), \quad \mathrm{ev}_B(\mathrm{id} \otimes \cdot_B) = \mathrm{ev}_{B\otimes B}(\Delta_{B'} \otimes \mathrm{id}),$$

where we recall that $\mathrm{ev}_{B\otimes B}$ is the nested iteration of ev_B as in §2.4. The theory for 'super-braided-groups' is strictly analogous and is really the same theory of braided

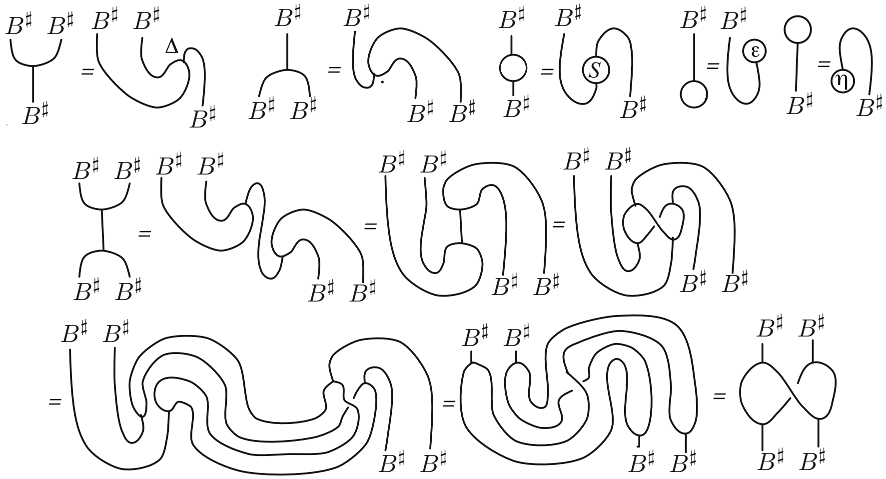


Fig. 2.9 Dual braided group product, coproduct, antipode, counit and unit. Below is the proof that $B^\sharp$ is a bialgebra in the braided category

groups but in some other super version of the original braided category. We assume that the original braided category is additive (see Definition 3.111) and we look at graded objects such as $B = B_0 \oplus B_1$. The diagrammatic theory is the same but there is a new braiding $\Psi^{\text{super}}_{V,W} = (-1)^{d_1 d_2} \Psi_{V,W}$, where d_1 is the grading operator on one factor and d_2 on the other.

Some main theorems for braided groups are transmutation, which constructs them from ordinary Hopf algebras (some formulae for this are given in (2.35)–(2.37)), and bosonisation (Theorem 2.64), which turns them into ordinary Hopf algebras. The latter theorem relates to the notion of a Hopf algebra with projection. This means a pair of Hopf algebras $A \overset{\twoheadrightarrow}{\hookleftarrow} H$ where $\pi : A \twoheadrightarrow H$ is a surjective Hopf algebra map, $i : H \hookrightarrow A$ is an injective Hopf algebra map, and $\pi \circ i = \mathrm{id}$.

Theorem 2.64 (Radford–Majid) *Let H be a Hopf algebra with bijective antipode and B a braided group in its category of right crossed modules. Then $H \bowtie B$ is an ordinary Hopf algebra. Here $\bowtie$ denotes right cross product and right cross coproduct by the action and coaction of the crossed module. There is a Hopf algebra projection of Hopf algebras $H \bowtie B \overset{\twoheadrightarrow}{\hookleftarrow} H$ and conversely every Hopf algebra projecting onto H is of this form.*

Proof For $\cdot : B \otimes B \to B$ to be a morphism in the crossed-module category means that B is a right H-module algebra and right H-comodule algebra. The former means that the cross product of $H \bowtie B$ is associative. Here $(h \otimes b)(g \otimes c) = hg_{(1)} \otimes (b \triangleleft g_{(2)})c$ for $h, g \in H$ and $b, c \in B$. Next, for brevity, we write $\Delta_B = \underline{\Delta}$, where we omit the label but underline to remind the reader that this is a braided coproduct, and we will write $\underline{\Delta} b = b_{\underline{(1)}} \otimes b_{\underline{(2)}}$ again with underlines. Then for $\underline{\Delta} : B \to B \otimes B$ to be a morphism means B is a right module coalgebra and a right comodule coalgebra. The latter means the right cross coproduct

$$\Delta(h \otimes b) = h_{(1)} \otimes b_{\underline{(1)}(\bar{0})} \otimes h_{(2)} b_{\underline{(1)}(\bar{1})} \otimes b_{\underline{(2)}}$$

is necessarily coassociative. Using the crossed-module axiom one can check that these two are compatible to form a bialgebra,

$$\begin{aligned}
\Delta((1 \otimes b)(g \otimes 1)) &= \Delta(g_{(1)} \otimes b{\triangleleft}g_{(2)}) \\
&= g_{(1)} \otimes (b{\triangleleft}g_{(3)})_{\underline{(1)}(\bar{0})} \otimes g_{(2)}(b{\triangleleft}g_{(3)})_{\underline{(1)}(\bar{1})} \otimes (b{\triangleleft}g_{(3)})_{\underline{(2)}} \\
&= g_{(1)} \otimes (b_{\underline{(1)}}{\triangleleft}g_{(3)})_{(\bar{0})} \otimes g_{(2)}(b_{\underline{(1)}}{\triangleleft}g_{(3)})_{(\bar{1})} \otimes b_{\underline{(2)}}{\triangleleft}g_{(4)} \\
&= g_{(1)} \otimes b_{\underline{(1)}(\bar{0})}{\triangleleft}g_{(4)} \otimes g_{(2)} Sg_{(3)} b_{\underline{(1)}(\bar{1})} g_{(5)} \otimes b_{\underline{(2)}}{\triangleleft}g_{(6)} \\
&= g_{(1)} \otimes b_{\underline{(1)}(\bar{0})}{\triangleleft}g_{(4)} \otimes b_{\underline{(1)}(\bar{1})} g_{(3)} \otimes b_{\underline{(2)}}{\triangleleft}g_{(4)} \\
&= (1 \otimes b_{\underline{(1)}(\bar{0})}).(g_{(1)} \otimes 1) \otimes (b_{\underline{(1)}(\bar{1})} \otimes b_{\underline{(2)}}).(g_{(2)} \otimes 1) \\
&= \Delta(1 \otimes b)\Delta(g \otimes 1).
\end{aligned}$$

The second equality uses that B is a module coalgebra. The fourth equality is the H-crossed module axiom. We also need

$$\begin{aligned}
\Delta((1 \otimes b)(1 \otimes c)) &= \Delta(1 \otimes bc) = 1 \otimes (bc)_{(1)(\bar{0})} \otimes (bc)_{(1)(\bar{1})} \otimes (bc)_{(2)} \\
&= 1 \otimes (b_{\underline{(1)}} c_{\underline{(1)}(\bar{0})})_{(\bar{0})} \otimes (b_{\underline{(1)}} c_{\underline{(1)}(\bar{0})})_{(\bar{1})} \otimes (b_{\underline{(2)}}{\triangleleft}c_{\underline{(1)}(\bar{1})}) c_{\underline{(2)}} \\
&= 1 \otimes b_{\underline{(1)}(\bar{0})} c_{\underline{(1)}(\bar{0})(\bar{0})} \otimes b_{\underline{(1)}(\bar{1})} c_{\underline{(1)}(\bar{0})(\bar{1})} \otimes (b_{\underline{(2)}}{\triangleleft}c_{\underline{(1)}(\bar{1})}) c_{\underline{(2)}} \\
&= 1 \otimes b_{\underline{(1)}(\bar{0})} c_{\underline{(1)}(\bar{0})} \otimes b_{\underline{(1)}(\bar{1})} c_{\underline{(1)}(\bar{1})(1)} \otimes (b_{\underline{(2)}}{\triangleleft}c_{\underline{(1)}(\bar{1})(2)}) c_{\underline{(2)}} \\
&= (1 \otimes b_{\underline{(1)}(\bar{0})}).(1 \otimes c_{\underline{(1)}(\bar{0})}) \otimes (b_{\underline{(1)}(\bar{1})} \otimes b_{\underline{(2)}}).(c_{\underline{(1)}(\bar{1})} \otimes c_{\underline{(2)}}) \\
&= \Delta(1 \otimes b)\Delta(1 \otimes c),
\end{aligned}$$

which uses the braided-homomorphism property of $\underline{\Delta}$ for the 3rd equality, that B is a comodule algebra for the 4th, and the comodule axiom for the 5th, and then we recognise the answer. The counit is the tensor product counit and the antipode is

$$S(1 \otimes b) = (1 \otimes \underline{S} b_{(\bar{0})})(S b_{(\bar{1})} \otimes 1) = S b_{(\bar{1})(2)} \otimes \underline{S} b_{(\bar{0})}{\triangleleft} S b_{(\bar{1})(1)},$$

where the braided-antipode $\underline{S}$ commutes with $\triangleleft$. One can check that this works using the covariance and antipode properties of B as a braided group. The full antipode is then $S(h \otimes b) = S(1 \otimes b)(Sh \otimes 1)$. Also the projection $\pi(h \otimes b) = h\epsilon(b)$ and inclusion $i(h) = h \otimes 1$ are easily seen to be Hopf algebra maps.

In the other direction, suppose we are given a Hopf algebra A and Hopf algebra maps $\pi : A \twoheadrightarrow H$ and $i : H \hookrightarrow A$ such that $\pi \circ i = \mathrm{id}_H$. We pull back left and right multiplication to make A into an H-bimodule by $h.a = i(h)a$, $a.h = ai(h)$, and push-forward left and right comultiplication to coactions $\Delta_L = (\pi \otimes \mathrm{id})\Delta_A$ and $\Delta_R = (\mathrm{id} \otimes \pi)\Delta_A$. Then A becomes a bicovariant Hopf bimodule and by Lemma 2.23 and $A \cong H \otimes B$ as vector spaces, where

$$B = {}^H A = \{a \in A \mid \Delta_L a = 1 \otimes a\}$$

is a subalgebra of A, an H-crossed module algebra by Δ_R and $b{\triangleleft}h = (Si(h_{(1)}))bi(h_{(2)})$ and a braided group by

$$\underline{\Delta} b = b_{(1)} \otimes (S \circ i \circ \pi b_{(2)})b_{(3)}, \quad \underline{S} b = (Sb_{(1)})i \circ \pi(b_{(2)})$$

and the counit from A. One can check that under the isomorphism $h \otimes b \mapsto i(h)b$, the product and coproduct of $H{\triangleright\!\!\!<}B$ map to those of A. □

When H is coquasitriangular, there is a functor of braided monoidal categories from $\mathcal{M}^H$ to H-crossed modules where a coaction induces a compatible action

$$v{\triangleleft}a = v_{(\bar{0})}\mathcal{R}(v_{(\bar{1})} \otimes a) \tag{2.20}$$

for all $a \in H$ and $v \in V \in \mathcal{M}^H$. So any braided Hopf algebra in $\mathcal{M}^H$ can be viewed in the H-crossed module category and therefore also 'bosonises' by Theorem 2.64, and similarly any braided Hopf algebra in ${}_H\mathcal{M}$ if H is quasitriangular. These cases have a more categorical origin and, at least when H is finite-dimensional, include the above theorem via a quasitriangular quantum double construction. There is an analogous super-bosonisation theorem, namely the same formulae now give $H{\triangleright\!\!\!<}B$ as a super-Hopf algebra and equivalent to the projection of a super-Hopf algebra to an ordinary (bosonic) Hopf algebra H, with B now a super-braided Hopf algebra in the category of H-crossed modules.

Example 2.65 (Braided Line) Let $\mathcal{C}$ be the category of $\mathbb{Z}$-graded vector spaces as in Example 2.45. This is also the category of right H-comodules where $H = \mathbb{C}_q\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ and the q refers to a coquasitriangular structure $\mathcal{R}(t^n \otimes t^m) = q^{nm}$. The coaction is $\Delta_R x^n = x^n \otimes t^n$. Now let $B = \mathbb{C}[x]$ be the *braided line* with $\underline{\Delta} x = x \otimes 1 + 1 \otimes x$, $\epsilon x = 0$, $\underline{S} x = -x$ but extended as a Hopf algebra in $\mathcal{C}$ by

$$\underline{\Delta} x^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \otimes x^{n-r}, \quad \underline{S} x^n = q^{\frac{n(n-1)}{2}}(-x)^n.$$

The first follows from the q-binomial theorem Lemma 2.15 since $(1 \otimes x)(x \otimes 1) = \Psi(x \otimes x) = qx \otimes x = q(x \otimes 1)(1 \otimes x)$. We can view this example as in the category of right crossed $\mathbb{C}_q\mathbb{Z}$-modules by the same coaction and induced action $x^m{\triangleleft}t = q^m x^m$ by (2.20). The bosonisation is $U_q(b_+) = k\langle x, t, t^{-1}\rangle/\langle xt - qtx\rangle$ with coproduct $\Delta x = x \otimes t + 1 \otimes x$, the Borel sub-quantum group of $U_q(sl_2)$ in Example 2.11). (There is also a full 'double-bosonisation' for $U_q(sl_2)$.)

These constructions work similarly when q is a primitive r-th root of unity. We work in the category of $\mathbb{Z}_r$-graded spaces, $H = \mathbb{C}_q\mathbb{Z}_r$ with the same formula for $\mathcal{R}$, the reduced braided line is $B = \mathbb{C}[x]/\langle x^r\rangle$ and its bosonisation is the reduced quantum group $u_q(b_+) \subset u_q(sl_2)$, the Sweedler–Taft algebra of dimension r^2, with

$$t^r = 1, \quad x^r = 0, \quad xt = qtx, \quad \Delta t = t\otimes t, \ \Delta x = x\otimes t + 1\otimes x, \quad \epsilon t = 1, \ \epsilon x = 0$$

and antipode $St = t^{-1}$, $Sx = -xt^{-1}$ of order $2r$. ◇

Example 2.66 (Braided Plane) Let $\mathcal{C}$ be the braided category of right comodules of $H = \Bbbk_q[GL_2]$ with coquasitriangular structure chosen with normalisation $\alpha = 1$ in Example 2.53. We let $B = \Bbbk\langle x, y\rangle/\langle yx - qxy\rangle = \Bbbk_q[\Bbbk^2]$ be the quantum plane, which we think of as a row vector of generators with right coaction given by matrix multiplication as in Proposition 2.13. This implies the braiding

$$\Psi(x\otimes x) = q^2 x\otimes x, \quad \Psi(x\otimes y) = qy\otimes x, \quad \Psi(y\otimes y) = q^2 y\otimes y,$$
$$\Psi(y\otimes x) = qx\otimes y + (q^2-1)y\otimes x.$$

The coproduct is again the additive one on x, y but extends as

$$\underline{\Delta}(x^m y^n) = \sum_{r=0}^{m}\sum_{s=0}^{n}\begin{bmatrix} m \\ r\end{bmatrix}_{q^2}\begin{bmatrix} n \\ s\end{bmatrix}_{q^2} x^r y^s \otimes x^{m-r}y^{n-s}q^{(m-r)s}.$$

We can also view B in the category of $\Bbbk_q[GL_2]$-crossed modules with induced action

$$(x^m y^n){\triangleleft}a = q^{2m+n}x^m y^n, \quad (x^m y^n){\triangleleft}d = q^{2n+m}x^m y^n, \quad (x^m y^n){\triangleleft}b = 0,$$
$$(x^m y^n){\triangleleft}c = q^{n+m-1}(q^2-1)[n]_{q^2}x^{m+1}y^{n-1}.$$

Its bosonisation is the quantum group 'coordinate algebra' $\Bbbk_q[GL_2]{\triangleright\!\!\!<}\Bbbk_q[\Bbbk^2]$ of translations and GL_2 transformations. ('Double cobosonisation' gives $\Bbbk_q[SL_3]$.) ◇

These examples generalise in a nice way. If V is an object in an abelian braided category then it inherits a morphism $\Psi = \Psi_{V,V} : V\otimes V \to V\otimes V$ obeying the braid relations (one says that V is 'braided' when equipped with such a morphism.)

Definition 2.67 Let (V,Ψ) be a braided object in an abelian monoidal category. The braided binomials morphisms $V^{\otimes n} \to V^{\otimes n}$ are defined recursively by

$$\begin{bmatrix} n \\ r\end{bmatrix}; \Psi\Big] = \Psi_r\Psi_{r+1}\cdots\Psi_{n-1}\Big(\Big[\begin{matrix} n-1 \\ r-1\end{matrix}; \Psi\Big]\otimes \mathrm{id}\Big) + \Big[\begin{matrix} n-1 \\ r\end{matrix}; \Psi\Big]\otimes \mathrm{id},$$
$$\Big[\begin{matrix} n \\ 0\end{matrix}; \Psi\Big] = \Big[\begin{matrix} n \\ n\end{matrix}; \Psi\Big] = \mathrm{id},$$

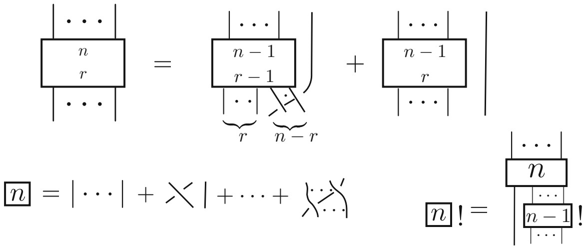


Fig. 2.10 Definitions of braided binomials, braided integers and factorials as elements of the braid group algebra reading down the page

where $0 < r < n$ and Ψ_i denotes Ψ acting in the $i, i+1$ tensor factors. We also define 'braided integers'

$$[n; \Psi] = \begin{bmatrix} n \\ 1 \end{bmatrix}; \Psi \Big] = \Psi_1\Psi_2 \cdots \Psi_{n-1} + \Big[\begin{matrix} n-1 \\ 1 \end{matrix}; \Psi \Big] \otimes \mathrm{id}$$

$$= \mathrm{id} + \Psi_1 + \Psi_1\Psi_2 + \cdots + \Psi_1\Psi_2 \cdots \Psi_{n-1}$$

and 'braided factorials' $[n, \Psi]! = (\mathrm{id} \otimes [n-1, \Psi]!)[n, \Psi]$, where $[1, \Psi]! = \mathrm{id}_V$. We take the convention $[0, \Psi]! = \mathrm{id}_{\underline{1}}$.

These are matrix versions of binomial coefficients and make sense, for example, given a vector space equipped with such an operator obeying the braid relations as depicted in Fig. 2.10. They also generalise q-binomials when applied to the category in Example 2.65 and, relevant to us, the braided factorials generalise symmetrisation and antisymmetrisation. We need the following main theorem.

Theorem 2.68 (Majid)

$$\big([r; \Psi]! \otimes [n-r; \Psi]!\big) \Big[\begin{matrix} n \\ r \end{matrix}; \Psi \Big] = [n, \Psi]!, \quad 0 \le r \le n.$$

Proof We first observe that

$$([r, \Psi] \otimes \mathrm{id})\Psi_r \cdots \Psi_{n-1} = \Psi_r \cdots \Psi_{n-1}([r-1, \Psi] \otimes \mathrm{id}) + \Psi_1 \cdots \Psi_{n-1}, \tag{2.21}$$

which follows from the definition of $[r, \Psi]$ noting that the first $r-1$ terms commute with $\Psi_r \cdots \Psi_{n-1}$. The other fact we need is

$$\Psi_1 \cdots \Psi_{n-1}(\Big[\begin{matrix} n-1 \\ r \end{matrix}; \Psi \Big] \otimes \mathrm{id}) = (\mathrm{id} \otimes \Big[\begin{matrix} n-1 \\ r \end{matrix}; \Psi \Big])\Psi_1 \cdots \Psi_{n-1}, \tag{2.22}$$

which follows in the categorical setting by functoriality (since the braided-binomial is a morphism) and $\Psi_1 \cdots \Psi_{n-1} = \Psi_{V^{\otimes n-1},V}$. It can also be proven directly by induction using Definition 2.67 and

$$\Psi_1 \cdots \Psi_{n-1}\Psi_r \cdots \Psi_{n-2} = \Psi_{r+1} \cdots \Psi_{n-1}\Psi_1 \cdots \Psi_{n-1}$$

which in turn follows from repeated use of the braid relations $\Psi_i\Psi_{i+1}\Psi_i = \Psi_{i+1}\Psi_i\Psi_{i+1}$ (if we want to work with braided integers directly at the level of a pair (V, Ψ)). Given these results, we next prove by induction on n that

$$([r, \Psi] \otimes \mathrm{id}) \begin{bmatrix} n \\ r \end{bmatrix}; \Psi \Big] = (\mathrm{id} \otimes \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}; \Psi \Big])[n, \Psi]$$

from which the stated result follows by repeated application. Thus, supposing the identity for $n-1$ in the role of n and all smaller or equal r,

$$\begin{aligned}
&([r,\Psi] \otimes \mathrm{id}) \left[\begin{matrix} n \\ r \end{matrix}; \Psi\right] \\
&= ([r, \Psi] \otimes \mathrm{id})\Psi_r \cdots \Psi_{n-1}(\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right] \otimes \mathrm{id}) + ([r, \Psi] \otimes \mathrm{id})(\left[\begin{matrix} n-1 \\ r \end{matrix}; \Psi\right] \otimes \mathrm{id}) \\
&= \Psi_r \cdots \Psi_{n-1}([r-1, \Psi] \otimes \mathrm{id})(\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right] \otimes \mathrm{id}) + \Psi_1 \cdots \Psi_{n-1}(\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right] \otimes \mathrm{id}) \\
&\quad + ([r, \Psi] \otimes \mathrm{id})(\left[\begin{matrix} n-1 \\ r \end{matrix}; \Psi\right] \otimes \mathrm{id}) \\
&= \Psi_r \cdots \Psi_{n-1}((\mathrm{id} \otimes \left[\begin{matrix} n-2 \\ r-2 \end{matrix}; \Psi\right])[n-1, \Psi] \otimes \mathrm{id}) + \Psi_1 \cdots \Psi_{n-1}(\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right] \otimes \mathrm{id}) \\
&\quad + (\mathrm{id} \otimes \left[\begin{matrix} n-2 \\ r-1 \end{matrix}; \Psi\right])[n-1, \Psi] \otimes \mathrm{id} \\
&= (\mathrm{id} \otimes \left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right])\Psi_1 \cdots \Psi_{n-1} + (\mathrm{id} \otimes \left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right])([n-1, \Psi] \otimes \mathrm{id}),
\end{aligned}$$

where the first equality is Definition 2.67, the 2nd is (2.21), the 3rd is our inductive assumption, and the 4th is (2.22) on the 2nd term and Definition 2.67 on the outer terms. The last expression then combines as required. □

The reader can and should write these definitions and proofs as manipulations of braids. They can in fact be viewed as fundamental identities in the group algebra of the braid group. The theorem extends the familiar formula for binomial coefficients to this setting and is also useful in the unbraided case. Technically, we also have two other versions by the same method, as corollaries.

Corollary 2.69 *By writing the above definitions as diagrams and then looking at them from behind (or left-right reflecting and then reversing back braid crossings), we have reversed binomial maps and reversed integers defined by*

$$\begin{aligned}
\begin{bmatrix} n \\ r \end{bmatrix}; \Psi\Big]_R &= \Psi_{n-r}\cdots\Psi_1(\mathrm{id}\otimes\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right]_R) + \mathrm{id}\otimes\left[\begin{matrix} n-1 \\ r \end{matrix}; \Psi\right]_R,\\
\left[\begin{matrix} n \\ 0 \end{matrix}; \Psi\right]_R &= \left[\begin{matrix} n \\ n \end{matrix}; \Psi\right]_R = \mathrm{id},\\
[n;\Psi]_R = \left[\begin{matrix} n \\ 1 \end{matrix}; \Psi\right]_R &= \mathrm{id} + \Psi_{n-1} + \Psi_{n-1}\Psi_{n-2} + \cdots + \Psi_{n-1}\cdots\Psi_1.
\end{aligned}$$

Moreover,

$$[n;\Psi]_R! := ([2,\Psi]_R\otimes\mathrm{id})\cdots([n-1;\Psi]_R\otimes\mathrm{id})[n;\Psi]_R = [n,\Psi]!$$

$$\left[\begin{matrix} n \\ r \end{matrix}; \Psi\right]_R = \left[\begin{matrix} n \\ n-r \end{matrix}; \Psi\right].$$

Proof The proofs for the $[\ ]_R$ theory can be done by looking at the previous proofs from behind. The new results are the relation with previous factorials and binomials. The conversion between them can be viewed as an operation (in fact a braid group isomorphism) where, given a braid expressed as a product of Ψ_i s, we can reverse all the position numbers but keep the order, thus $(\Psi_i\Psi_j)_R = \Psi_{n-i}\Psi_{n-j}$.

The recursive definition of $[n,\Psi]_R$ or this observation gives its stated form and also agrees with the expanded definition of $\left[\begin{matrix} n \\ n-1 \end{matrix}; \Psi\right]$, so these are equal. That the factorials coincide follows by induction as

$$[n;\Psi]_R! = ([n-1;\Psi]_R!\otimes\mathrm{id})[n;\Psi]_R = ([n-1;\Psi]!\otimes\mathrm{id})\left[\begin{matrix} n \\ n-1 \end{matrix}; \Psi\right] = [n;\Psi]!$$

using the braided binomial theorem. Similar arguments apply for the general equality of binomials. This can also be checked directly for small n, r. □

We will need this later, in §2.8. For the moment, we need another variant.

Corollary 2.70 *By writing the above definitions as diagrams and turning the diagrams upside down, we have co-binomial maps and co-integers defined by*

$$\begin{aligned}
\left[\begin{matrix} n \\ r \end{matrix}; \Psi\right]' &= (\mathrm{id}\otimes\left[\begin{matrix} n-1 \\ r-1 \end{matrix}; \Psi\right]')\Psi_1\cdots\Psi_{r-1} + \mathrm{id}\otimes\left[\begin{matrix} n-1 \\ r \end{matrix}; \Psi\right]',\\
\left[\begin{matrix} n \\ 0 \end{matrix}; \Psi\right]' &= \left[\begin{matrix} n \\ n \end{matrix}; \Psi\right]' = \mathrm{id},\\
[n,\Psi]' = \left[\begin{matrix} n \\ 1 \end{matrix}; \Psi\right]' &= \mathrm{id} + \Psi_{n-1} + \Psi_{n-2}\Psi_{n-1} + \cdots + \Psi_1\cdots\Psi_{n-1}.
\end{aligned}$$

Moreover,

$$[n,\Psi]'! := [n,\Psi]'([n-1,\Psi]'\otimes \mathrm{id})\cdots([2,\Psi]'\otimes\mathrm{id}) = [n,\Psi]!$$

$$\begin{bmatrix} n \\ r \end{bmatrix};\Psi\Big]'([r;\Psi]!\otimes[n-r;\Psi]!) = [n,\Psi]!,\quad 0\le r\le n.$$

Proof The proofs for the $'$ theory can be done by writing the proofs of the standard theory as diagrams and turning them upside down (a braid group order reversing isomorphism). The only new observation is that $[n,\Psi]! = [n,\Psi]'!$. This boils down to repeated use of the braid relations. For example,

$$\begin{aligned}[3,\Psi]! = (\mathrm{id}\otimes[2,\Psi])[3,\Psi] &= \mathrm{id} + \Psi_1\Psi_2 + \Psi_1\Psi_2 + \Psi_2\Psi_1\Psi_2\Psi_1\Psi_2\\ &= [3,\Psi]'([2,\Psi]'\otimes\mathrm{id})\end{aligned}$$

using $\Psi_1\Psi_2\Psi_1 = \Psi_2\Psi_1\Psi_2$ to equate the two different factorisations. It also holds because both factorials can be written as $\sum_{\sigma\in S_n}\Psi_{i_1}\cdots\Psi_{i_{l(\sigma)}}$, where $\sigma = s_{i_1}\cdots s_{i_{l(\sigma)}}$ is a reduced expression in terms of simple transpositions $s_i = (i, i+1)$. □

Next we consider the tensor algebra TV as in Example 2.10, but now in a general additive braided category (see Definition 3.111) so that we have direct sums of objects and an additive group structure on morphism sets. We can then define the tensor algebra as a direct sum of different degrees $T_nV := V^{\otimes n}$ and product given by concatenation of $\otimes$. Here $T_0V = \underline{1}$ (the unit object of the category), and the unit η of the algebra TV is the identity map from $\underline{1}\to T_0V$. In our examples $\underline{1} = \Bbbk$. The product of the tensor algebra

$$(V\otimes\cdots\otimes V)\otimes(V\otimes\cdots\otimes V)\to V\otimes\cdots\otimes V$$

is the identity map with suitable rebracketing (with Φ as necessary in the general case). We also consider the identity maps

$$\eta_n : V^{\otimes n}\to T_nV$$

with $\eta_0 = \eta$. Although all these maps are the identity, we are viewing them in different ways. We will consider two different Hopf algebra structures $T_\pm V$ on TV, as a Hopf algebra or as a super-Hopf algebra in the braided category. A super-Hopf algebra in a braided category is like a Hopf algebra but has a $\mathbb{Z}_2$ grading and an extra -1 sign according to grading in the braided tensor product.

Proposition 2.71 (Majid) *The tensor algebra has a braided Hopf algebra/super braided Hopf algebra structure $T_\pm V$ with braided coproduct*

$$\Delta|_{T_nV} = \sum_{r=0}^{n}(\eta_r\otimes\eta_{n-r})\circ\begin{bmatrix} n \\ r \end{bmatrix};\pm\Psi\Big]$$

on degree n for the two cases. The counit is $\epsilon|_{T_n V} = 0$ for all $n > 0$ and id *for $n = 0$, and the antipode S is* id *on degree* 0 *and* −id *on degree* 1.

Proof In either case we start with the linear coproduct

$$\Delta|_{T_1 V} = \mathrm{id}_V \otimes \eta + \eta \otimes \mathrm{id}_V$$

and for $T_+ V$ we extend this as a Hopf algebra in the braided category, for $T_- V$ we extend as a super-Hopf algebra in the braided category. As the tensor algebra has no relations to check we necessarily can make these extensions. There is a counit and antipode given by $\epsilon = 0$, $S = -\mathrm{id}$ on degree 1. We do the $+\Psi$ case; the $-\Psi$ case is exactly the same by replacing Ψ by $-\Psi$. We now assume the formula for $\Delta|_{T_{n-1} V}$ and note that it holds for $\Delta|_{T_1 V}$. Then,

$$\begin{aligned}\Delta|_{T_n V} &= (\Delta|_{T_{n-1} V}) \underline{\cdot} (\mathrm{id}_V \otimes \eta + \eta \otimes \mathrm{id}_V)\\ &= \Big(\sum_{r=0}^{n-1} (\eta_r \otimes \eta_{n-r-1}) \begin{bmatrix} n-1 \\ r \end{bmatrix}; \Psi \Big]\Big) \underline{\cdot} (\mathrm{id}_V \otimes \eta + \eta \otimes \mathrm{id}_V)\\ &= \sum_{r=0}^{n-1} \Big((\eta_{r+1} \otimes \eta_{n-1-r}) \Psi_{r+1} \cdots \Psi_{n-1} + (\eta_r \otimes \eta_{n-r})\Big)\Big(\begin{bmatrix} n-1 \\ r \end{bmatrix}; \Psi \Big] \otimes \mathrm{id}_V\Big),\end{aligned}$$

where $\underline{\cdot}$ is the braided tensor product in (2.19) and we use the convention that the $r = n-1$ case of $\Psi_{r+1} \cdots \Psi_{n-1}$ is the identity. Splitting the sum and reindexing, and with zero when the lower entry in a binomial coefficient is negative, we have

$$\sum_{r=0}^{n-1} (\eta_r \otimes \eta_{n-r}) \Big(\Psi_r \cdots \Psi_{n-1}\big(\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}; \Psi \Big] \otimes \mathrm{id}_V\big) + \big(\begin{bmatrix} n-1 \\ r \end{bmatrix}; \Psi \Big] \otimes \mathrm{id}_V\big)\Big) + \eta_n \otimes \eta_0$$

hence verifying the formula for Δ by Definition 2.67. The reader may find it helpful to write out the proof in the diagrammatic notation. □

For the next construction we need to take kernels and a quotient in the category in addition to our previous assumptions. The usual definition which covers all these properties is that of an abelian category (see Definition 3.113).

Corollary 2.72 *Let V be an object of an abelian braided category. There are (super)braided Hopf algebra quotients*

$$B_\pm(V) = T_\pm V / \oplus_n \ker[n, \pm\Psi]!$$

called the braided-symmetric *and* braided exterior algebras *on V respectively.*

Proof That the coproduct descends to $B_\pm(V)$ follows from the coproduct in Proposition 2.71 on showing that $\big([r; \Psi]! \otimes [n-r; \Psi]!\big) \begin{bmatrix} n \\ r \end{bmatrix}; \Psi \Big]$ applied to $\ker[n, \Psi]!$

is zero, which holds by Theorem 2.68. That $\oplus_n \ker[n, \pm\Psi]!$ is a 2-sided ideal or equivalently that the product in TV descends to the quotient follows from the last displayed equation in the statement of Corollary 2.70. The other facts are more trivial. □

These become the symmetric algebra and exterior algebra respectively on V in the unbraided case where Ψ is the flip map. One observation is that when $\phi : V \to W$ is a morphism then $\phi^{\otimes}$ (the relevant power) in each degree is a morphism $B_\pm(V) \to B_\pm(W)$ of braided groups. This is because, by functoriality of the braidings, ϕ on each strand in the diagrammatic picture can be pulled through the braidings and braided factorials. The algebra $B_+(V)$ is sometimes called the 'Nichols–Woronowicz algebra' of V when constructed in a different way, whereas we follow the original route (due to one of the authors) in which they arise naturally not only as algebras but as braided groups. Their meaning as part of this theory is clearest when V has a left dual $V^\sharp$. Recall that $V^{\sharp\otimes n}$ is left-dual to $V^{\otimes n}$ by the nested use of ev_V and we use the same nesting convention for a duality pairing $\langle\ ,\ \rangle$ on tensor products.

Corollary 2.73 *If V is an object in a braided category having a left dual $V^\sharp$ then the tensor algebras $T_\pm V^\sharp$ and $T_\pm V$ are dually paired by*

$$\langle\ ,\ \rangle|_{T_n V^\sharp \otimes T_m V} = \delta_{n,m} \mathrm{ev}_{V^{\otimes n}}(\mathrm{id} \otimes [n, \pm\Psi]!)$$

and $B_\pm(V^\sharp)$, $B_\pm(V)$ are the quotients by the kernel of the pairing.

Proof We check that the product on one side is the coproduct on the other. This again follows immediately from Proposition 2.71 and Theorem 2.68.

$$\begin{aligned}\langle\ ,\ \rangle|_{T_r V \otimes T_{n-r} V}(\mathrm{id} \otimes \Delta|_{T_n V}) &= \langle\ ,\ \rangle|_{T_r V \otimes T_{n-r} V}\left(\mathrm{id} \otimes \begin{bmatrix} n \\ r \end{bmatrix}; \Psi\right) \\ &= \mathrm{ev}_{V^{\otimes r} \otimes V^{\otimes n-r}}\left(\mathrm{id} \otimes ([r, \Psi]! \otimes [n-r, \Psi]!) \begin{bmatrix} n \\ r \end{bmatrix}; \Psi\right) \\ &= \mathrm{ev}_{V^{\otimes n}}(\mathrm{id} \otimes [n, \Psi]!) = \langle\ ,\ \rangle_{T_n V}(\cdot_{TV} \otimes \mathrm{id})\end{aligned}$$

and similarly for $-\Psi$. This can also be done diagrammatically. □

This means that $B_\pm(V^\sharp)$, $B_\pm(V)$ are nondegenerately paired (super) Hopf algebras in the braided category and the relations of $B_\pm(V)$ are the maximal relations compatible with this duality. This duality classically leads to Poincaré duality in the context of the algebra of left-invariant forms on a Lie group, which suggests in the Hopf algebra case that we take $B_-(V)$ in that role if we want a natural bicovariant construction compatible with such a duality. This leads us to the following construction of the *canonical exterior algebra* on a Hopf algebra (first found in a different way by Woronowicz). The maximal prolongation is also bicovariant and surjects to this but is generally too big for our purposes in the noncommutative case.

Theorem 2.74 (Canonical Bicovariant Exterior Algebra) *Let H be a Hopf algebra with invertible antipode and $\Omega^1 \cong H \otimes \Lambda^1$ a bicovariant differential structure. Then $B_-(\Lambda^1)$ is a DGA and Ω^1 extends to an exterior algebra $\Omega \cong H{\triangleright\!\!<} B_-(\Lambda^1)$ as its super-bosonisation. Moreover, Ω is a super-Hopf algebra and* d *is both a super-derivation and a super-coderivation*

$$\Delta \mathrm{d} = (\mathrm{d} \otimes \mathrm{id} + (-1)^{|\ |} \otimes \mathrm{d})\Delta,$$

where $|\omega|$ is the degree of homogeneous $\omega \in \Omega$. If Ω^1 is inner by $\theta \in \Lambda^1$ then so is Ω with the same θ. In the Hopf $*$-algebra case over $\mathbb{C}$, if Ω^1 is a $*$-calculus then Ω is a $*$-exterior algebra and a super-Hopf $*$-algebra.

Proof Clearly Ω^1 in the form $H \otimes \Lambda^1$ extends to $H{\triangleright\!\!<} B_-(\Lambda^1)$, which by the super version of Theorem 2.64 is a super-Hopf algebra. Note that given the braided coproduct $\Delta v = v \otimes 1 + 1 \otimes v$ on Λ^1, the cross coproduct on $H{\triangleright\!\!<}\Lambda^1$ is

$$\begin{aligned}\Delta(h \otimes v) &= (h_{(1)} \otimes v_{(\bar{0})}) \otimes (h_{(2)} v_{(\bar{1})} \otimes 1) + (h_{(1)} \otimes 1) \otimes (h_{(2)} \otimes v)\\ &= \Delta_L(h \otimes v) + \Delta_R(h \otimes v)\end{aligned}$$

when we compare with the structure of the bicovariant calculus in canonical form in Theorem 2.26. The braided antipode on $v \in \Lambda^1$ is $Sv = -v$ so that the super-antipode in the bosonisation is $S(1 \otimes v) = -(1 \otimes v_{(\bar{0})})(Sv_{(\bar{1})} \otimes 1)$ on left-invariant 1-forms. Next, we compute that

$$\Psi(\varpi\pi_\epsilon h \otimes \varpi\pi_\epsilon g) = \varpi\pi_\epsilon g_{(2)} \otimes \varpi\pi_\epsilon (h(Sg_{(1)})g_{(3)}) - \varpi\pi_\epsilon g_{(2)} \otimes \varpi\pi_\epsilon(\epsilon(h)(Sg_{(1)})g_{(3)})$$

for all $h, g \in H$, because

$$\pi_\epsilon(h)g = \pi_\epsilon(hg - \epsilon(h)g), \quad \mathrm{Ad}_R \pi_\epsilon = (\pi_\epsilon \otimes \mathrm{id})\mathrm{Ad}_R,$$

while ϖ maps the crossed module structure on Λ^1 to the canonical one given by right multiplication and Ad_R on H^+. Indeed, H with Ad_R and the action $h{\triangleleft} g = hg - \epsilon(h)g$ is an H-crossed module (then $H \to H^+ \to \Lambda^1$ is the composition of H-crossed module maps π_ϵ and ϖ) and the induced braiding $\Psi : H \otimes H \to H \otimes H$,

$$\Psi(h \otimes g) = g_{(2)} \otimes h(Sg_{(1)})g_{(3)} - \epsilon(h)g_{(2)} \otimes (Sg_{(1)})g_{(3)} \tag{2.23}$$

connects via $\varpi\pi_\epsilon$ to the one on Λ^1. We also see that

$$\Psi(\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\Delta = (\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\Delta - (\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\mathrm{Ad}_R.$$

Now, $\mathrm{Ad}_R(\mathcal{I}) \subseteq \mathcal{I} \otimes H$ because we assume that our calculus is bicovariant, and $\mathcal{I} = \ker \varpi \subseteq H^+$, so the 2nd term vanishes on elements of $\mathcal{I}$. Hence

$$(\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\Delta(\mathcal{I}) \subseteq \ker[2, -\Psi] = \ker(\mathrm{id} - \Psi),$$

meaning that the relations $(\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\Delta(\mathcal{I}) = 0$ of the maximal prolongation in Proposition 2.31 already hold among the quadratic relations in $B_-(\Lambda^1)$. The latter is therefore a (generally strict) quotient of $\Lambda_{\max}$.

We now want to show that d of the maximal prolongation descends to $B_-(\Lambda)$. Given the formula for this in the proof of Proposition 2.31 we start on the tensor algebra of H with the map $\partial_n = \sum_{j=1}^n (-1)^{j+1}\Delta_j$, where Δ_j is Δ in the i-th place of $H^{\otimes n}$. It is clear that $\partial^2 = 0$ as a general coalgebra observation for such alternating sums and Lemma 2.75 below shows that ∂ descends to $B_-(H) \to B_-(H)$ in each degree as it respects the kernels of the relevant braided-factorials. Next, π_ϵ being an H-crossed module morphism induces by $\pi_\epsilon^{\otimes n}$ in degree n a map $B_-(H) \to B_-(H^+)$. Then ∂ descends to a map $-\tilde{\mathrm{d}} : B_-(H^+) \to B_-(H^+)$ given by $-\tilde{\mathrm{d}}\pi_\epsilon^{\otimes n} = \pi_\epsilon^{\otimes(n+1)}\partial_n$ because the kernel of $\pi_\epsilon^{\otimes n}$ is spanned by elements where at least one of the tensor factors is 1. When we apply ∂_n then every term has at least one tensor factor 1 which is then killed by the final $\pi_\epsilon^{\otimes(n+1)}$. This is the first cell of

$$
\begin{array}{ccccc}
B_-(H) & \xrightarrow{\pi_\epsilon^{\otimes}} & B_-(H^+) & \xrightarrow{\varpi^{\otimes}} & B_-(\Lambda^1) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle -\tilde{\mathrm{d}}} & & \downarrow{\scriptstyle -\mathrm{d}} \\
B_-(H) & \xrightarrow{\pi_\epsilon^{\otimes}} & B_-(H^+) & \xrightarrow{\varpi^{\otimes}} & B_-(\Lambda^1)
\end{array}
$$

Similarly, $\varpi : H^+ \to \Lambda^1$ being a morphism of crossed modules induces $B_-(H^+) \to B_-(\Lambda^1)$ given by $\varpi^{\otimes n}$ in degree n, and $\tilde{\mathrm{d}}$ descends to a map $\mathrm{d} : B_-(\Lambda^1) \to B_-(\Lambda^1)$ defined by $\mathrm{d}\varpi^{\otimes n} = \varpi^{\otimes(n+1)}\tilde{\mathrm{d}}$. This is because the kernel of $\varpi^{\otimes n}$ consists of terms where at least one of the tensor factors is in $\mathcal{I}$. When we compute the $\tilde{\mathrm{d}}$ of such terms using ∂_n, either a Δ_j does not act on this tensor factor, in which case this tensor factor is present in the output of Δ_j and the whole term is killed by the action of $(\varpi\pi_\epsilon)^{\otimes(n+1)}$, or Δ_j does act on this element. But then $\cdots \otimes (\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)(\Delta\mathcal{I}) \otimes \cdots$ is in the kernel of $\mathrm{id} - \Psi$ in the relevant place as seen above, hence vanishes in $B_-(\Lambda)$. We are using the fact that the kernel in each degree contains the degree 2 relations between adjacent tensor factors. In this way, d makes $B_-(\Lambda^1)$ a DGA and as such it is clearly a quotient of $\Lambda_{\max}$ in Proposition 2.31. Finally, we extend d to $H{\triangleright\!\!<}B_-(\Lambda)$ by $\mathrm{d}h = h_{(1)} \otimes \varpi\pi_\epsilon(h_{(2)})$ and the Leibniz rule as in Proposition 2.31.

To prove the super-coderivation property we first look in degree 0. Then

$$
\begin{aligned}
\Delta\mathrm{d}h &= \Delta(h_{(1)}\varpi\pi_\epsilon(h_{(2)})) \\
&= (h_{(1)} \otimes h_{(2)}).(\varpi\pi_\epsilon h_{(3)(2)} \otimes (Sh_{(3)(1)})h_{(3)(3)} + 1 \otimes \varpi\pi_\epsilon h_{(3)}) \\
&= h_{(1)}\varpi\pi_\epsilon h_{(4)} \otimes h_{(2)}(Sh_{(3)})h_{(5)} + h_{(1)} \otimes h_{(2)}\varpi\pi_\epsilon h_{(3)} \\
&= \mathrm{d}h_{(1)} \otimes h_{(2)} + h_{(1)} \otimes \mathrm{d}h_{(2)},
\end{aligned}
$$

as required. On invariant 1-forms we have the braided coproduct

$$\begin{aligned}
&\underline{\Delta}\mathrm{d}\varpi\pi_\epsilon h = -\underline{\Delta}(\varpi\pi_\epsilon(h_{(1)})\wedge\varpi\pi_\epsilon(h_{(2)}))\\
&= -(\varpi\pi_\epsilon h_{(1)}\otimes 1 + 1\otimes\varpi\pi_\epsilon h_{(1)})\underline{.}(\varpi\pi_\epsilon h_{(2)}\otimes 1 + 1\otimes\varpi\pi_\epsilon h_{(2)})\\
&= \mathrm{d}\varpi\pi_\epsilon h\otimes 1 + 1\otimes\mathrm{d}\varpi\pi_\epsilon h - \varpi\pi_\epsilon(h_{(1)})\wedge\varpi\pi_\epsilon(h_{(2)}) + \wedge\Psi(\varpi\pi_\epsilon h_{(1)}\otimes\varpi\pi_\epsilon h_{(2)})\\
&= \mathrm{d}\varpi\pi_\epsilon h\otimes 1 + 1\otimes\mathrm{d}\varpi\pi_\epsilon h - \varpi\pi_\epsilon(h_{(2)})\wedge\varpi\pi_\epsilon((Sh_{(1)})h_{(3)}).
\end{aligned}$$

We used the super-braided tensor product for the third equality and our computation for Ψ above for the 4th. Then the coproduct of Ω is given by a right cross coproduct,

$$\begin{aligned}
&\Delta\mathrm{d}\varpi\pi_\epsilon h = (\mathrm{d}\varpi\pi_\epsilon h)_{\underline{(1)}(\bar{0})}\otimes(\mathrm{d}\varpi\pi_\epsilon h)_{\underline{(1)}(\bar{1})}(\mathrm{d}\varpi\pi_\epsilon h)_{\underline{(2)}}\\
&= (\mathrm{d}\varpi\pi_\epsilon h)_{(\bar{0})}\otimes(\mathrm{d}\varpi\pi_\epsilon h)_{(\bar{1})} + 1\otimes\mathrm{d}\varpi\pi_\epsilon h\\
&\quad - (\mathrm{d}\varpi\pi_\epsilon h_{(2)})_{(\bar{0})}\otimes(\mathrm{d}\varpi\pi_\epsilon h_{(2)})_{(\bar{1})}\varpi\pi_\epsilon((Sh_{(1)})h_{(3)})\\
&= \mathrm{d}\varpi\pi_\epsilon h_{(2)}\otimes(Sh_{(1)})h_{(3)} + 1\otimes\mathrm{d}\varpi\pi_\epsilon h - \varpi\pi_\epsilon h_{(3)}\otimes(Sh_{(2)})h_{(4)}\varpi\pi\epsilon((Sh_{(1)})h_{(5)})
\end{aligned}$$

where we computed the coaction on $\mathrm{d}\varpi\pi_\epsilon(h)$ as the tensor product coaction but after a short computation, it has same form as $\mathrm{Ad}_R h$. Meanwhile

$$\begin{aligned}
(\mathrm{d}\otimes\mathrm{id}&+(-1)^{|\ |}\otimes\mathrm{d})\Delta\varpi\pi_\epsilon h\\
&= (\mathrm{d}\otimes\mathrm{id} + (-1)^{|\ |}\otimes\mathrm{d})(1\otimes\varpi\pi_\epsilon h + \varpi\pi_\epsilon(h)_{(\bar{0})}\otimes\varpi\pi_\epsilon(h)_{(\bar{1})})\\
&= \mathrm{d}\varpi\pi_\epsilon(h)_{(\bar{0})}\otimes\varpi\pi_\epsilon(h)_{(\bar{1})} + 1\otimes\mathrm{d}\varpi\pi_\epsilon h - \varpi\pi_\epsilon(h)_{(\bar{0})}\otimes\mathrm{d}\varpi\pi_\epsilon(h)_{(\bar{1})}\\
&= \mathrm{d}\varpi\pi_\epsilon h_{(2)}\otimes(Sh_{(1)})h_{(3)} + 1\otimes\mathrm{d}\varpi\pi_\epsilon h - \varpi\pi_\epsilon h_{(2)}\otimes\mathrm{d}((Sh_{(1)})h_{(3)}),
\end{aligned}$$

which is the same expression when we compute $\mathrm{d}((Sh_{(1)})h_{(3)})$ in terms of $\varpi\pi_\epsilon$ and compare. Hence the coderivation property holds on Λ^1. The coderivation property is multiplicative, i.e., if it holds on $\omega,\eta\in\Omega$ then it holds on their product. This follows from the graded Leibniz rule for d and the graded-homomorphism property of Δ. Since our Ω is generated by H,Λ^1 we conclude the result.

In the Hopf $*$-algebra case, we first establish that $B_-(H^+)$ becomes a graded $*$-algebra under $\underline{*} = *S$. We start with T_-H^+ and

$$(h^1\otimes\cdots\otimes h^n)^{\underline{*}} := (-1)^{\frac{n(n-1)}{2}} * Sh^n\otimes\cdots\otimes *Sh^1,\quad h^i\in H^+,$$

and one can check that

$$(h^1\otimes\cdots\otimes h^n\otimes g^1\otimes\cdots\otimes g^m)^{\underline{*}} = (-1)^{nm}(g^1\otimes\cdots\otimes g^m)^{\underline{*}}\otimes(h^1\otimes\cdots\otimes h^n)^{\underline{*}}$$

as required. We also know that $\underline{*}$ is involutive as $(*S)^2 = \mathrm{id}$. Now we compute

$$(\Psi((h \otimes g)^{\underline{*}}))^{\underline{*}} = (\Psi(*Sg \otimes *Sh))^{\underline{*}} = (*Sh_{(2)} \otimes (*Sg)(S * Sh_{(3)})(*Sh_{(1)}))^{\underline{*}}$$
$$= g(S^{-1}h_{(3)})h_{(1)} \otimes h_{(2)} = \Psi^{-1}(h \otimes g)$$

from which we deduce that in the n-fold tensor power, $\underline{*}\Psi_i\underline{*} = \Psi^{-1}_{n-i}$. It follows that

$$\underline{*}[n, -\Psi]\underline{*} = \mathrm{id} - \Psi^{-1}_{n-1} + \cdots + (-1)^{n-1}\Psi^{-1}_{n-1} \cdots \Psi^{-1}_1$$
$$= (-1)^{n-1}(\Psi_1 \cdots \Psi_{n-1})^{-1}[n, -\Psi],$$

which we use to prove by induction that the factorials are also related by some braid,

$$\underline{*}[n, -\Psi]!\underline{*} = (\underline{*}[n-1, -\Psi]!\underline{*} \otimes \mathrm{id})\underline{*}[n, -\Psi]\underline{*}$$
$$= \left((-1)^{\frac{(n-1)(n-2)}{2}} f_{n-1}(\Psi)[n-1, -\Psi]! \otimes \mathrm{id}\right)(-1)^{n-1}\Psi^{-1}_{n-1} \cdots \Psi^{-1}_1[n, -\Psi]$$
$$= (-1)^{\frac{n(n-1)}{2}} f_{n-1}(\Psi)\Psi^{-1}_{n-1} \cdots \Psi^{-1}_1(\mathrm{id} \otimes [n-1, -\Psi]!)[n, -\Psi]$$
$$= (-1)^{\frac{n(n-1)}{2}} f_n(\Psi)[n, -\Psi]!$$

for some inductively defined $f_n(\Psi) = f_{n-1}(\Psi)\Psi^{-1}_{n-1} \cdots \Psi^{-1}_1$. The third equality used the monodromy relations (2.22) as in the proof of Theorem 2.68. For example,

$$\underline{*}[2, -\Psi]\underline{*} = -\Psi^{-1}_1[2, -\Psi], \quad \underline{*}[3, -\Psi]!\underline{*} = -\Psi^{-1}_1\Psi^{-1}_2\Psi^{-1}_1[3, -\Psi]!.$$

This proves that $\underline{*}$ descends to $B_-(H^+) \to B_-(H^+)$. Next we map this under $\varpi^{\otimes} : B_-(H^+) \to B_-(\Lambda^1)$ to make the latter into a graded $*$-algebra. This is well defined because terms in the kernel of $\varpi^{\otimes}$ will have a tensor factor in $\mathcal{I}$. We have $*S(\mathcal{I}) \subseteq \mathcal{I}$ if we assume that Ω^1 is a $*$-calculus, and hence $\underline{*}$ of such a term will map to zero under $\varpi^{\otimes}$. On degree 1 we already know that $\varpi(h)^* = -\varpi(*Sh)$ gives us a first-order $*$-differential calculus and we now see that this extends to all degrees as a graded $*$-algebra, being given by $(-1)^n\underline{*}$ on degree n. Moreover, we check on degree 1 that $*$ commutes with d,

$$\mathrm{d} * \varpi\pi_\epsilon h = \mathrm{d}(-\varpi\pi_\epsilon(*Sh)) = \varpi\pi_\epsilon(*Sh_{(2)}) \wedge \varpi\pi_\epsilon(*Sh_{(1)})$$
$$= -(\varpi\pi_\epsilon(h_{(1)}) \wedge \varpi\pi_\epsilon(h_{(2)}))^* = *\mathrm{d}\varpi\pi_\epsilon h.$$

The 3rd equality uses the graded-anti-algebra property of $*$. The property of d commuting with $*$ is multiplicative, i.e., it then follows on all degrees of $B_-(\Lambda^1)$ and this becomes a $*$-DGA. Together with the given $*$ on H and since d also commutes with $*$ in degree 0, Ω then becomes a $*$-DGA. Finally, the axioms of a super-Hopf $*$-algebra are the same as the usual case in that we need $(S \circ *)^2 = \mathrm{id}$ and that $*$

commutes with Δ. Here S of the super-Hopf algebra is graded-antimultiplicative, so $S \circ *$ is multiplicative and it is enough to check our properties on Λ^1 (on H we already have them as a Hopf $*$-algebra). From the definition of $*$,

$$\begin{aligned}
S * \varpi\pi_\epsilon(h) &= \varpi\pi_\epsilon(*Sh_{(2)})(S * Sh_{(3)})(*S)h_{(1)}\\
&= \varpi\pi_\epsilon(*Sh_{(2)})(*S)((S^{-1}h_{(3)})h_{(1)}),\\
(S*)^2\varpi\pi_\epsilon(h) &= \varpi\pi_\epsilon(*S(*Sh_{(2)})_{(2)})(*S)((S^{-1}(*Sh_{(2)})_{(3)})\\
&\quad (*Sh_{(2)})_{(1)})(S * *S)((S^{-1}h_{(3)})h_{(1)})\\
&= \varpi\pi_\epsilon(h_{(2)(2)})(Sh_{(2)(1)})h_{(2)(3)}S^2((S^{-1}h_{(2)})h_{(1)})\\
&= \varpi\pi_\epsilon(h)
\end{aligned}$$

on using the antipode properties and cancelling, so $(S*)^2 = \mathrm{id}$ on degree 0,1 and hence on all of Ω. Finally, we check that $*$ commutes with Δ. This is true on H and is a multiplicative property, so we need only check it on Λ^1. We have

$$\begin{aligned}
\Delta * \varpi\pi_\epsilon h &= -\Delta\varpi\pi_\epsilon(*Sh)\\
&= -1 \otimes \varpi\pi_\epsilon(*Sh) - \varpi\pi_\epsilon(*Sh_{(2)}) \otimes *S((S^{-1}h_{(3)})h_{(1)})\\
&= -1 \otimes \varpi\pi_\epsilon(*Sh) - \varpi\pi_\epsilon(*Sh_{(2)}) \otimes *((Sh_{(1)})h_{(3)})\\
&= (* \otimes *)(1 \otimes \varpi\pi_\epsilon(h) + \varpi\pi_\epsilon(h_{(2)}) \otimes (Sh_{(1)})h_{(3)}\\
&= (* \otimes *)\Delta\varpi\pi_\epsilon h,
\end{aligned}$$

as required to complete the proof of the super Hopf $*$-algebra structure.

Finally, we know for an inner left-covariant 1st order calculus that we can assume without loss of generality that $\theta \in \Lambda^1$. Clearly, $\mathrm{d}h = \theta h - h\theta = h_{(1)}\theta \triangleleft h_{(2)} - h\theta = h_{(1)}\theta \triangleleft \pi_\epsilon h_{(2)} = h_{(1)}\varpi\pi_\epsilon h_{(2)}$, which tells us that $\varpi(h) = \theta \triangleleft h$ for all $h \in H^+$. Then

$$\begin{aligned}
\mathrm{d}\varpi h &= -\varpi\pi_\epsilon(h_{(1)}) \wedge \varpi\pi_\epsilon(h_{(2)}) = -\theta \triangleleft \pi_\epsilon(h_{(1)}) \wedge \theta \triangleleft \pi_\epsilon(h_{(2)})\\
&= -\theta \triangleleft h_{(1)} \wedge \theta \triangleleft h_{(2)} + \theta \wedge (\theta \triangleleft h) + (\theta \triangleleft h) \wedge \theta\\
&= \{\theta, \varpi h\} - \theta \triangleleft h_{(1)} \wedge \theta \triangleleft h_{(2)} = \{\theta, \varpi h\}
\end{aligned}$$

for $h \in H^+$. The last equality holds in $B_-(\Lambda^1)$ since $\Psi(\theta \triangleleft h_{(1)} \otimes \theta \triangleleft h_{(2)}) = \theta \triangleleft h_{(1)} \otimes \theta \triangleleft h_{(2)}$ from Ψ a morphism. Both $[\theta, \ \}$ (the graded commutator) and d extend as graded derivations and hence coincide on all Ω. □

In practice, we will write the above super coproduct of Ω as Δ_* if we particularly want to distinguish it from Δ in degree 0. The following lemma is needed to complete the proof of the theorem.

Lemma 2.75 *Let H be a Hopf algebra, Ψ_i the braiding (2.23) acting in the $i, i+1$ position of a tensor power of H, $\mathrm{Ad}_i = \mathrm{Ad}_R$ and $\Delta_i = \Delta$ acting in the i-th position, and let $\partial_n = \sum_{j=1}^{n}(-1)^{j+1}\Delta_j$. Then*

$$[n, -\Psi]!\partial_{n-1} = \Big(\sum_{j=1}^{n-1}(-1)^{j+1}\mathrm{Ad}_1^{\otimes j}\Big)[n-1, -\Psi]!$$

Here $\mathrm{Ad}_1^{\otimes j}$ denotes the tensor product right coaction on the first j copies.

Proof Clearly

$$\Psi_i\Delta_j = \Delta_j\Psi_i, \quad \text{if } i < j-1, \quad \Psi_i\Delta_j = \Delta_j\Psi_{i-1}, \quad \text{if } i > j+1$$

since the operators act on different tensor factors, just the numbering changes in the 2nd case. We also find by direct computation in the Hopf algebra that

$$\Psi_i\Delta_i = \Delta_i - \mathrm{Ad}_i,\ \Psi_i\Delta_{i-1} = (\Delta\otimes\mathrm{Ad})_{i-1} - \mathrm{Ad}_i,\ \Psi_i(\Delta\otimes\mathrm{Ad})_i = (\Delta_{i+1} - \mathrm{Ad}_i)\Psi_i,$$

where $\mathrm{Ad} = \mathrm{Ad}_R$ and $\Delta\otimes\mathrm{Ad}$ is the tensor product right coaction. As a warm up, using these relations, we show

$$\begin{aligned}
[3, -\Psi]\partial_2 &= (\mathrm{id} - \Psi_1 + \Psi_1\Psi_2)(\Delta_1 - \Delta_2)\\
&= \Delta_1 - \Delta_2 - \Psi_1\Delta_1 + \Psi_1\Delta_2 - \Psi_1\Psi_2(\Delta_2 - \Psi_1)\\
&= \Delta_1 - \Delta_2 - \Delta_1 + \mathrm{Ad}_1 + \Psi_1\Delta_2 - \Psi_1(\Delta_2 - (\Delta\otimes\mathrm{Ad})_1)\\
&= -\Delta_2 + \mathrm{Ad}_1 + (\Delta_2 - \mathrm{Ad}_1)\Psi_1 = (\mathrm{Ad}_1 - \Delta_1 + \partial_2)[2, -\Psi].
\end{aligned}$$

Starting with this, we next prove by induction that

$$[n, -\Psi]\partial_{n-1} = (\mathrm{Ad}_1 - \Delta_1 + \partial_{n-1})[n-1, -\Psi]. \tag{2.24}$$

Assuming this for $n-1$ in the role of n, for the 2nd equality,

$$\begin{aligned}
[n, -\Psi]\partial_{n-1} &= [n-1, -\Psi]\partial_{n-2} + [n-1, -\Psi](-1)^n\Delta_{n-1}\\
&\quad + (-1)^{n-1}\Psi_1\cdots\Psi_{n-1}\partial_{n-1}\\
&= (\mathrm{Ad}_1 - \Delta_1 + \partial_{n-2})[n-2, -\Psi] + (-1)^n\Delta_{n-1}[n-2, -\Psi]\\
&\quad + \Psi_1\cdots\Psi_{n-2}\Delta_{n-1} - \Psi_1\cdots\Psi_{n-1}\Delta_{n-1}\\
&\quad + (-1)^{n-1}\Psi_1\cdots\Psi_{n-1}\partial_{n-2}\\
&= (\mathrm{Ad}_1 - \Delta_1 + \partial_{n-1})[n-2, -\Psi] + \Psi_1\cdots\Psi_{n-2}\mathrm{Ad}_{n-1}\\
&\quad + (-1)^{n-1}\Psi_1\cdots\Psi_{n-1}\partial_{n-2},
\end{aligned}$$

where we picked out and computed the $\Psi_1\cdots\Psi_{n-1}\Delta_{n-1}$ term from the sum in ∂_{n-1}. Looking now at the last expression, we compute

$$\begin{aligned}\Psi_1\cdots\Psi_{n-1}\partial_{n-2}&=\sum_{j=1}^{n-2}(-1)^{j+1}\Psi_1\cdots\Psi_{j+1}\Delta_j\Psi_{j+1}\cdots\Psi_{n-2}\\&=\sum_{j=1}^{n-2}(-1)^{j+1}\Psi_1\cdots\Psi_j((\Delta\otimes\mathrm{Ad})_j-\mathrm{Ad}_{j+1})\Psi_{j+1}\cdots\Psi_{n-2}\\&=\sum_{j=1}^{n-2}(-1)^{j+1}\Psi_1\cdots\Psi_{j-1}((\Delta_{j+1}-\mathrm{Ad}_j)\Psi_j-\Psi_j\mathrm{Ad}_{j+1})\Psi_{j+1}\cdots\Psi_{n-2}\\&=(-1)^n\Psi_1\cdots\Psi_{n-2}\mathrm{Ad}_{n-1}-\big(\mathrm{Ad}_1+\sum_{j=2}^{n-1}(-1)^{j+1}\Delta_j\big)\Psi_1\cdots\Psi_{n-2},\end{aligned}$$

where the Ad terms cancel between the sum and the displaced sum except for the top term of one sum and the bottom term of the other. In the Δ sum all the indices of Ψ are two or more smaller than the index of Δ and hence commute to the right. Combining with our previous calculation, we have

$$\begin{aligned}[][n,-\Psi]\partial_{n-1}&=(\mathrm{Ad}_1-\Delta_1+\partial_{n-1})[n-2,-\Psi]\\&\quad+(\mathrm{Ad}_1-\Delta_2+\Delta_3+\cdots(-1)^n\Delta_{n-1})(-1)^{n-2}\Psi_1\cdots\Psi_{n-2},\end{aligned}$$

proving (2.24). We use this result as the initial base for induction on i in the formula

$$\begin{aligned}[][n-i+1]\cdots[n]\partial_{n-1}&=\Big(\sum_{j=1}^{i-1}(-1)^{j+1}\mathrm{Ad}_1^{\otimes j}+\big((-1)^{i+1}\mathrm{Ad}_1^{\otimes i}\\&\quad+\sum_{j=i+1}^{n-1}(-1)^{j+1}\Delta_j\big)[n-i]\Big)[n-i+1]\cdots[n-1],\end{aligned}\tag{2.25}$$

where $[n]=[n,\Psi]$ for brevity and the nesting is rightmost as for braided factorials, so $[n-1]=\mathrm{id}\otimes[n-1,-\Psi]$. The case $i=1$ is (2.24), which we have already proven while the case $i=n-1$ or $i=n$, suitably interpreted in the sense of absent sums or products when out of range, proves the lemma. We use identities

$$\Psi_i\mathrm{Ad}_j^{\otimes i-j}=\mathrm{Ad}_j^{\otimes i}-\mathrm{Ad}_i,\quad \Psi_i\mathrm{Ad}_j^{\otimes k}=\mathrm{Ad}_j^{\otimes k}\Psi_{i-1},\quad i>j+k,$$

where the commutation relation is due to acting in different spaces, with renumbering due to the notation. The first equation is a direct computation. Also

$$\Psi_i\mathrm{Ad}_j^{\otimes k}=\mathrm{Ad}_j^{\otimes k}\Psi_i,\quad i<j-1,\quad \Psi_i\mathrm{Ad}_j^{\otimes k}=\mathrm{Ad}_j^{\otimes k}\Psi_i,\quad j\le i<j+k-1,$$

which we do not need right now, in the first case due to different tensor products and in the second case because Ψ is a morphism in the crossed module category and hence commutes with Ad applied to tensor powers that include those on which Ψ acts. Assuming (2.25) for $i-1$ in the role of i, what we need to show to prove (2.25) for i is

$$[n-i+1]\Big(\sum_{j=1}^{i-2}(-1)^{j+1}\mathrm{Ad}_1^{\otimes j}+\big((-1)^i\mathrm{Ad}_1^{\otimes i-1}+\sum_{j=i}^{n-1}(-1)^{j+1}\Delta_j\big)[n-i+1]\Big)$$

$$=\Big(\sum_{j=1}^{i-1}(-1)^{j+1}\mathrm{Ad}_1^{\otimes j}+\big((-1)^{i+1}\mathrm{Ad}_1^{\otimes i}+\sum_{j=i+1}^{n-1}(-1)^{j+1}\Delta_j\big)[n-i]\Big)[n-i+1].$$

Now, the first sum commutes with $[n-i+1]$ since on the left this is $1-\Psi_i+\Psi_i\Psi_{i+1}+\cdots+(-1)^{n-i}\Psi_i\cdots\Psi_{n-1}$ due to the rightmost embedding. These commute past the $\mathrm{Ad}_1^{\otimes j}$ getting changed to $[n-i+1]$ embedded on the right (where the numbering is reduced by one). Hence the first term on the left is $\sum_{j=1}^{i-2}(-1)^{j+1}\mathrm{Ad}_1^{\otimes j}[n-i+1]$. Next $\Psi_i\Psi_{i+1}\cdots\Psi_{n-1}\mathrm{Ad}_1^{\otimes i-1}=(\mathrm{Ad}_1^{\otimes i}-\mathrm{Ad}_i)\Psi_i\Psi_{i+1}\cdots\Psi_{n-2}$ as the Ψ_{i+1} and higher commute, reducing the index by 1, while Ψ_i computes as shown. Hence the middle terms give

$$\begin{aligned}[n-i+1](-1)^i\mathrm{Ad}_1^{\otimes i-1}[n-i+1]&=(-1)^i\mathrm{Ad}_1^{\otimes i-1}[n-i+1]\\&\quad+(-1)^{i+1}(\mathrm{Ad}_1^{\otimes i}-\mathrm{Ad}_i)[n-i][n-i+1],\end{aligned}$$

the first term of which completes our previous sum to give the first desired term. Accordingly we need only show for the remaining term that

$$[n-i+1]\sum_{j=i}^{n-1}(-1)^{j+1}\Delta_j=\Big((-1)^{i+1}\mathrm{Ad}_i+\sum_{j=i+1}^{n-1}(-1)^{j+1}\Delta_j\Big)[n-i].$$

But this is just the same identity (2.24) already proven but for $[m]\partial_{m-1}$, i.e., $m=n-i+1$ in the role of n, for the m tensor factors numbered $i,\cdots,n$. This completes our proof of (2.25) for all i. $\square$

Theorem 2.74 gives us the full structure of the bicovariant exterior algebra on a Hopf algebra and also recovers the theory for functions on groups in Theorem 1.56. We can also apply the above construction to the case of a coquasitriangular Hopf algebra A as in Proposition 2.55. We do this in the main case of a matrix coalgebra as in Corollary 2.57. The first step is to compute the crossed module braiding on Λ^1.

Lemma 2.76 *For Λ^1 constructed from a matrix subcoalgebra of a coquasitriangular Hopf algebra A, the crossed module braiding $\Psi : \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ is*

$$\Psi(E_\alpha{}^\beta \otimes E_\gamma{}^\delta) = E_{j_2}{}^{j_3} \otimes E_{k_2}{}^{k_3} \tilde{R}^{k_2}{}_{k_1}{}^{j_4}{}_{j_3} R^{k_1}{}_\alpha{}^{j_2}{}_{j_1} R^{j_1}{}_\gamma{}^\beta{}_{k_4} R^{-1}{}^\delta{}_{j_4}{}^{k_4}{}_{k_3},$$

where $\tilde{R}^i{}_j{}^k{}_l = \mathcal{R}(t^i{}_j \otimes St^k{}_l)$ is the so-called 'second inverse' of R.

Proof The braiding $\Psi(v \otimes w) = w_{(\bar{0})} \otimes v {\triangleleft} w_{(\bar{1})}$ can be computed from the right crossed module structure (2.17) but we prefer to do it from the general Proposition 2.55 with, in our case, $t^m{}_n \otimes E_m{}^n$ as the coevaluation $\sum_i e_i \otimes f^i$. We have

$$\begin{aligned}\Psi(E_\alpha{}^\beta \otimes E_\gamma{}^\delta) &= E_m{}^n \langle E_\gamma{}^\delta, t^{j_1}{}_{j_2}\rangle \otimes E_\alpha{}^\beta {\triangleleft} t^m{}_{j_1} St^{j_2}{}_n\\ &= E_m{}^n \otimes E_p{}^q \langle E_\gamma{}^\delta, t^{j_2}{}_{j_3}\rangle\langle E_\alpha{}^\beta, t^{k_2}{}_{k_3}\rangle \mathcal{R}(t^p{}_{k_1} \otimes t^m{}_{j_1} St^{j_4}{}_n)\mathcal{R}(t^{j_1}{}_{j_2} St^{j_3}{}_{j_4} \otimes t^{k_3}{}_q)\end{aligned}$$

and we then use the multiplicativity of $\mathcal{R}$. We sum over repeated indices. □

It is also possible to not start with a coquasitriangular Hopf algebra but proceed directly from a matrix R obeying the braid relations. In this case one needs to demand that R is biinvertible in the sense that the matrix $\tilde{R}$ in the lemma also exists (i.e., the matrix given by transposing the last two indices of R is also invertible).

Example 2.77 For $\mathbb{C}_q[SU_2]$ with its 4D calculus as in Example 2.59, the same R-matrix now in Lemma 2.76 gives the crossed module braiding on Λ^1 as

$$\Psi(e_a \otimes \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix}) = \begin{pmatrix} e_a & q^2 e_b \\ q^{-2} e_c & e_d \end{pmatrix} \otimes e_a,$$

$$\Psi(e_b \otimes \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix}) = \begin{pmatrix} e_a & e_b, \\ e_c & e_d \end{pmatrix} \otimes e_b + \lambda q^2 \begin{pmatrix} -e_b & 0 \\ e_z & e_b \end{pmatrix} \otimes e_a,$$

$$\Psi(e_c \otimes \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix}) = \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix} \otimes e_c + \lambda \begin{pmatrix} e_c & -q^2 e_z \\ 0 & -e_c \end{pmatrix} \otimes e_a,$$

$$\Psi(e_d \otimes e_a) = e_a \otimes e_d + \lambda^2 q^2 e_z \otimes e_a - \lambda(e_b \otimes e_c - e_c \otimes e_b),$$

$$\Psi(e_d \otimes e_b) = q^{-2} e_b \otimes e_d - \lambda e_z \otimes e_b,$$

$$\Psi(e_d \otimes e_c) = q^2 \lambda e_z \otimes e_c + (q^4 - 1 + q^{-2}) e_c \otimes e_d + \lambda(q^4 - 1) e_c \otimes e_z,$$

$$\Psi(e_d \otimes e_d) = e_d \otimes e_d + \lambda(e_b \otimes e_c - e_c \otimes e_b) - \lambda^2 q^2 e_z \otimes e_a,$$

where $\lambda = 1-q^{-2}$, from which one can see for example that $\Psi(e_i \otimes \theta) = \theta \otimes e_i$. This then gives the relations of the exterior algebra $\Lambda = B_-(\Lambda^1)$ as e_a, e_b, e_c behaving like usual forms or Grassmann variables and

$$e_z \wedge e_a + e_a \wedge e_z + \lambda e_b \wedge e_c = 0, \quad e_b \wedge e_z + q^2 e_z \wedge e_b = 0,$$
$$e_z \wedge e_c + q^2 e_c \wedge e_z = 0, \quad e_z \wedge e_z + (1-q^{-4}) e_b \wedge e_c = 0,$$
$$\mathrm{d}e_a = \lambda e_b \wedge e_c, \ \mathrm{d}e_b = \lambda q^2 e_z \wedge e_b, \ \mathrm{d}e_c = \lambda q^2 e_c \wedge e_z, \ \mathrm{d}e_z = (1-q^{-4}) e_b \wedge e_c,$$

where $e_z = q^{-2} e_a - e_d$, from which one can check that $\mathrm{d}\theta = 0$. As we saw in degree 1, it is $\lambda^{-1}\mathrm{d}$, which has the correct classical limit. Note that the relations of the exterior algebra here are quadratic, there being no further relations from the kernels of higher braided factorials in this case at least for generic q. For generic q the dimensions $\dim(\Omega^i)$ are $1:4:6:4:1$ and the cohomology in each degree is

$$\mathrm{H}^0_{\mathrm{dR}} = \mathbb{C}, \quad \mathrm{H}^1_{\mathrm{dR}} = \mathbb{C}, \quad \mathrm{H}^2_{\mathrm{dR}} = 0, \quad \mathrm{H}^3_{\mathrm{dR}} = \mathbb{C}, \quad \mathrm{H}^4_{\mathrm{dR}} = \mathbb{C},$$

with $\mathrm{H}^1_{\mathrm{dR}}$ spanned by θ. We will establish this cohomology in Example 4.27 after we show that one need only carry out the calculation among left-invariant forms. $\diamond$

We have focussed above on a specific problem in the noncommutative geometry of quantum groups, by working in a braided category that they generate. Now we close with a very different construction, this time of differential forms on a braided Hopf algebra B in a braided abelian category, such as the quantum plane Example 2.66 that we began the section with. The only data we will need is a surjective morphism $\varpi : B \to \Lambda^1$ in the category such that

$$\varpi \circ \cdot = \epsilon \otimes \varpi + \varpi \otimes \epsilon. \tag{2.26}$$

This data arises naturally as follows: suppose $B^\flat$ is a (possibly degenerately) dually paired braided Hopf algebra from the right and $\mathcal{L}$ a rigid primitive sub-object $\mathcal{L} \subset B^\flat$ (where primitive means that the coproduct restricted to $\mathcal{L}$ is the additive one). We view the duality pairing restricted to a map $B \otimes \mathcal{L} \to \underline{1}$ as a map

$$\varpi : B \to \mathcal{L}^*, \quad \varpi = (\mathrm{ev}_{\mathcal{L}} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}_{\mathcal{L}})$$

which then obeys (2.26). We let $\Lambda^1 \subseteq \mathcal{L}^*$ be its image so that $\varpi : B \to \Lambda^1$ is surjective. If B is a graded braided Hopf algebra of the form $\underline{1} \oplus V \oplus B_{>1}$ generated in degree 1 by V (for example $B_+(V)$) then we can directly take $\varpi : B \to \Lambda^1 = V$ as the projection to degree 1.

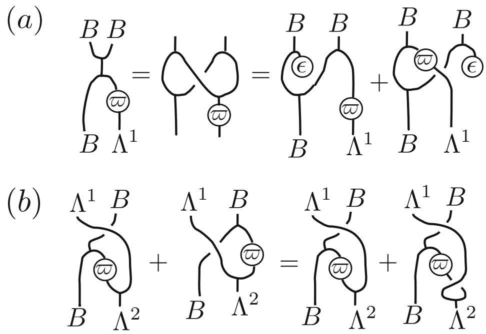


Fig. 2.11 Diagrams in the proof of Proposition 2.78 for quantum differentials on braided planes

Proposition 2.78 *Let B be a Hopf algebra in an abelian braided category and $\varpi : B \to \Lambda^1$ a surjective morphism obeying (2.26). Then*

$$\Omega = B\underline{\otimes}\Lambda, \quad \Lambda = T\Lambda^1/\langle \mathrm{image}(\mathrm{id} + \Psi_{\Lambda^1,\Lambda^1})\rangle,$$
$$\mathrm{d}|_B = (\mathrm{id} \otimes \varpi)\underline{\Delta}, \quad \mathrm{d}|_\Lambda = 0$$

is an exterior differential calculus on B in the category, with all structure maps morphisms.

Proof The proof is done diagrammatically in Fig. 2.11 and applies generally, but for convenience of exposition we also refer to concrete elements. The braided tensor product $B\underline{\otimes}\Lambda^1$ then means $(b \otimes v)(c \otimes w) = b\Psi(v \otimes c)w$ and featured already in the definition of a braided Hopf algebra. Part (a) computes $\mathrm{d}(bc)$ using the braided coproduct homomorphism property and (2.26). Using the counit axioms and ϖ a morphism, we obtain $b\mathrm{d}c$ for the first term and $(\mathrm{d}b)c$ for the second when we remember the braided tensor product. Part (b) checks that d extends as a graded derivation with respect to the braided tensor product, showing in terms of $\omega \in \Lambda^1$ that

$$\mathrm{d}(\omega b) + \omega \mathrm{d}b = (\mathrm{id} \otimes \cdot)(\mathrm{id} + \Psi^{-1})(\text{something})$$

hence vanishes in $B \otimes \Lambda^2$, agreeing with $\mathrm{d}\omega = 0$. Similarly for higher $\omega \in \Lambda$. □

This gives a differential structure on our braided-symmetric algebras $B_+(V)$ viewed as noncommutative spaces. If the category is the comodules of a coquasitriangular Hopf algebra then our construction is covariant in that all structure maps are comodule maps. Also note that if $\{e_i\}$ is a basis of V, we have explicitly

$$\mathrm{d} = \partial^i(\ )e_i,$$

where ∂^i are the (right-handed) *braided partial derivatives* defined by

$$\underline{\Delta} b = b \otimes 1 + \partial^i b \otimes e_i + \cdots,$$

where we ignore terms of degree greater than 1 in the second factor. They are given explicitly at the level of the tensor algebra by

$$\mathrm{d}(v_1 \otimes \ldots \otimes v_n) = (\eta_{n-1} \otimes \eta_1)[n; \Psi]_R(v_1 \otimes \cdots \otimes v_n),$$

where the last tensor factor of the result is viewed in Λ^1 and the right braided integers are from Corollary 2.69. We further remark that if we take the quadratic version $S(V) := B_+^{\mathrm{quad}}(V) = TV/\langle \ker(\mathrm{id} + \Psi)\rangle$ then our construction gives

$$\Omega(S(V)) = S(V)\underline{\otimes} S(V^\sharp)^!, \quad \mathrm{d}v = 1 \otimes v,$$

where ! denotes the Koszul dual. If a quadratic algebra on a vector space W has relations $R \subseteq W \otimes W$ as the subspace being set to zero then its Koszul dual is the quadratic algebra on $W^\flat$ with relations $R^\perp \subseteq W^\flat \otimes W^\flat$, where $R^\perp$ is the annihilator of R with respect to the nested pairing. This is normally done in the category of vector spaces but we do it here with the right dual so that $W^\flat = V$.

Example 2.79 Let $B = \mathbb{C}_q[\mathbb{C}^2]$ be the quantum plane in Example 2.66 in the category of $\mathbb{C}_q[GL_2]$-comodules with $q^2 \neq 1$. This is of the form $S(V)$, where $V = \mathrm{span}\{x, y\}$ is the 2D corepresentation in Example 2.66 with braiding Ψ displayed there. The kernel of $\mathrm{id}+\Psi$ gives us the relations $yx = qxy$ of the quantum plane since $(\mathrm{id} + \Psi)(y \otimes x - qx \otimes y) = 0$. The algebra $\Lambda(V) = S(V^\sharp)^!$ is the fermionic quantum plane $\mathrm{d}x \wedge \mathrm{d}x = 0$, $\mathrm{d}y \wedge \mathrm{d}y = 0$, $\mathrm{d}y \wedge \mathrm{d}x = -q^{-1}\mathrm{d}x \wedge \mathrm{d}y$, where the same basis is denoted $\{\mathrm{d}x, \mathrm{d}y\}$ as elements of $\Lambda^1 = V$. For example, $(\mathrm{id} + \Psi)(\mathrm{d}x \otimes \mathrm{d}y) = \mathrm{d}x \otimes y) = \mathrm{d}x \otimes \mathrm{d}x + q\mathrm{d}y \otimes \mathrm{d}x$ from the stated braiding. The differential is $\mathrm{d}v = 1 \otimes v$ on $v \in V \subseteq B$. The relations between $S(V)$ and $\Lambda(V)$ are the braided tensor product so $(1 \otimes v)(w \otimes 1) = \Psi(v \otimes w)$. Then from the stated braiding,

$$(\mathrm{d}x)x = q^2 x\mathrm{d}x,\ (\mathrm{d}x)y = qy\mathrm{d}x,\ (\mathrm{d}y)x = qx\mathrm{d}y + (q^2 - 1)y\mathrm{d}x,\ (\mathrm{d}y)y = q^2 y\mathrm{d}y.$$

One can check that $\mathrm{d}(yx - qxy) = 0$, as it should. By construction, this exterior algebra on the quantum plane is $\mathbb{C}_q[GL_2]$-covariant. The partial derivatives are

$$\partial^1(x^m y^n) = [m]_{q^2} x^{m-1} y^n q^n, \quad \partial^2(x^m y^n) = x^m [n]_{q^2} y^{n-1}$$

using the coproduct in Example 2.66. These are naturally 'braided right derivations' with an extra q^n in the first expression, in order that d is a left derivation, acting as braided q^2-derivatives in each variable. One can check that $\partial^2\partial^1 = q\partial^1\partial^2$ and also that we have a $*$-DGA when $|q| = 1$ with $x^* = x$ and $y^* = y$. ◇

It is beyond our scope, but for the constructions in this section the exterior algebra Λ is *itself* a (super) graded braided Hopf algebra in some category and we can therefore take *its* exterior algebra by the above constructions. The action by braided partial derivatives ∂^i (from either side) at this level provides interior products, and if a unique top form exists then this provides an integral on Λ. Fourier transform in the braided category then provides a Hodge star operator on Λ extending to Ω.

2.7 The Lie Algebra of a Quantum Group

In general in noncommutative geometry we favour 1-forms as the primary object. However, on a DGA over an algebra A we could still consider the space $\mathfrak{X}^R$ of *right vector fields* as the space of right A-module maps $\Omega^1 \to A$. The space $\mathfrak{X}^R$ is an A-bimodule by $(a\phi)(\omega) = a\phi(\omega)$ and $(\phi a)(\omega) = \phi(a\omega)$, and we have a canonical evaluation map $\mathfrak{X}^R \otimes_A \Omega^1 \to A$ making this a categorical left-dual evaluation. There is a natural notion of 'antisymmetrised vectors' $\Lambda^2\mathfrak{X}^R \subseteq \mathfrak{X}^R \otimes \mathfrak{X}^R$ compatible with the structure of Ω^2, consisting of $\sum \phi \otimes \psi$ such that

$$\sum \phi(\psi(\omega)\eta) = 0$$

for all $\sum \omega \otimes_A \eta \in \ker \wedge$. There is then a natural 'Lie bracket' defined as follows. First, let i^R be any *lifting map* $i^R : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$, not necessarily even linear, such that $\wedge \circ i^R = \mathrm{id}$ (we can always find one). We then we define

$$\sum [\![\phi, \psi]\!]_R(\omega) = \sum \phi(i^R_\psi(\mathrm{d}\omega) + \mathrm{d}\psi(\omega)) \tag{2.27}$$

for all $\sum \phi \otimes \psi \in \Lambda^2\mathfrak{X}^R$ and $\omega \in \Omega^1$, where the 'interior product' by $\psi \in \mathfrak{X}^R$ is

$$i^R_\psi = (\psi \otimes \mathrm{id})i^R : \Omega^2 \to \Omega^1.$$

Proposition 2.80 *$[\![\ ,\ ]\!]_R : \Lambda^2\mathfrak{X}^R \to \mathfrak{X}^R$ is well defined independently of the lifting map i^R. If, moreover, i^R is a right module map then the right-hand side of (2.27) defines a linear map $[\phi, \psi]_R : \Omega^1 \to A$ for any two vector fields $\phi, \psi \in \chi^R$ (but with result not necessarily an element of $\mathfrak{X}^R$) and $[\ ,\ ]_R|_{\Lambda^2\mathfrak{X}^R} = [\![\ ,\ ]\!]_R$.*

Proof We first check that $\sum [\![\phi, \psi]\!]_R$ is well defined. If we choose two lifts $\zeta', \tilde{\zeta}$ of $\zeta \in \Omega^2$ then $\zeta' - \tilde{\zeta} \in \ker \wedge$ and $\sum \phi(\psi \otimes \mathrm{id})$ of this vanishes precisely when $\sum \phi \otimes \psi \in \Lambda^2\mathfrak{X}^R$. Next, $\mathrm{d}(\omega a) = (\mathrm{d}\omega)a - \omega \wedge \mathrm{d}a$, which we choose to lift to $\tilde{\zeta}.a - \mathrm{d}\omega \otimes \mathrm{d}a$ in computing the first term of $\sum [\![\phi, \psi]\!]_R(\omega a)$. The second part of this cancels with the second part of $\mathrm{d}(\psi(\omega a)) = \mathrm{d}(\psi(\omega)a) = (\mathrm{d}\psi(\omega))a + \psi(\omega)\mathrm{d}a$ to give $\sum [\![\phi, \psi]\!]_R(\omega)a$, so the output is in $\mathfrak{X}^R$. Now suppose we are given a right module map i^R as stated. Then i^R_ψ is a right module map and our formula defines $[\ ,\ ]_R$ for all vector fields, but the result is not necessarily in $\mathfrak{X}^R$ as $[\phi, \psi]_R(\omega a) = \phi(i^R_\psi((\mathrm{d}\omega)a) - \phi(i^R_\psi(\omega \wedge \mathrm{d}a)) + \phi((\mathrm{d}\psi(\omega))a) + \phi(\psi(\omega)\mathrm{d}a) = [\phi, \psi]_R(\omega)a$ if and only if $\phi(\psi(\omega)\mathrm{d}a) - \phi(i^R_\psi(\omega \wedge \mathrm{d}a)) = 0$. This does not happen in general but since $\wedge \circ i^R = \mathrm{id}$, we have $\omega \otimes_A \eta - i^R(\omega \wedge \eta) \in \ker \wedge$ for all $\omega, \eta \in \Omega^1$ so that the identity holds on $\sum\phi\otimes\psi\in\Lambda^2\mathfrak{X}^R$. In this case we recover $[\![\ ,\ ]\!]_R$ by taking $\tilde\zeta=i^R(\zeta)$. □

Clearly, everything works just as well for *left vector fields* defined as the space $\mathfrak{X}^L$ of left-module maps $\Omega^1\to A$ where $(a\phi)(\omega)=\phi(\omega a)$ and $(\phi a)(\omega)=\phi(\omega)a$. We can view Ω^1 tautologically as contained in right module maps $\mathfrak{X}^L\to A$ by writing $\omega(\psi)=\psi(\omega)$ and we have an evaluation map $\Omega^1\otimes_A\mathfrak{X}^L\to A$. The space $\Lambda^2\mathfrak{X}^L$ consists of tensors $\sum\phi\otimes\psi$ such that $\sum\psi(\omega\phi(\eta))=0$ for all $\sum\omega\otimes_A\eta\in\ker\wedge$. Given a left module map $i^L:\Omega^2\to\Omega^1\otimes_A\Omega^1$ obeying $\wedge\circ i^L=\mathrm{id}$, we have

$$i^L_\phi=(\mathrm{id}\otimes\phi)i^L,\quad [\phi,\psi]_L(\omega)=\psi(i^L_\phi(\mathrm{d}\omega)-\mathrm{d}\phi(\omega)),\tag{2.28}$$

where the former is a left module map and the latter coincides with the canonical bracket $[\![\ ,\ ]\!]_L$ on $\Lambda^2\mathfrak{X}^L$. The proofs are similar to the previous right-handed case.

Proposition 2.81 *In the $*$-differential case we extend $*$ by $\psi^*(\omega)=\psi(\omega^*)^*$ and have the following commutative diagram*

$$\begin{tikzcd}
\Lambda^2\mathfrak{X}^R \arrow[r, "-\mathrm{flip}(*\otimes*)"] \arrow[d, "{[\![\ ,\ ]\!]_R}"'] & \Lambda^2\mathfrak{X}^L \arrow[d, "{[\![\ ,\ ]\!]_L}"] \\
\mathfrak{X}^R \arrow[r, "*"] & \mathfrak{X}^L
\end{tikzcd}$$

If i^R,i^L obey $i^L=-\mathrm{flip}(*\otimes*)\circ i^R\circ *$ then $[\phi^*,\psi^*]_L=-[\psi,\phi]_R^*$ for all $\phi,\psi\in\mathfrak{X}^R$.

Proof If $\psi\in\mathfrak{X}^R$ then $\psi^*(a\omega)=\psi((a\omega)^*)^*=(\psi(\omega^*)a^*)^*=a\psi^*(\omega)$ as required for $\psi^*\in\mathfrak{X}^L$. Similarly if $\sum\phi\otimes\psi\in\Lambda^2\mathfrak{X}^R$ and $\sum\omega\otimes_A\eta\in\ker\wedge$ then $\sum\phi^*(\omega\psi^*(\eta))=\sum\phi(\psi(\eta^*)\omega^*)=0$ as $\eta^*\wedge\omega^*=-(\omega\wedge\eta)^*=0$. So $\sum\psi^*\otimes\phi^*\in\Lambda^2\mathfrak{X}^L$. Moreover, for any lift $\tilde\zeta$ of $\zeta\in\Omega^2$ we can chose to lift ζ^* as $-\mathrm{flip}(*\otimes*)\tilde\zeta$ according to the graded-involution axiom of a $*$-DGA. Then $i^R_\psi(\zeta^*)^*=-i^L_{\psi^*}(\zeta)$ for the respective evaluations and (summations understood),

$$\begin{aligned}[\![\phi,\psi]\!]^*_R(\omega)&=\phi(i_\psi(\mathrm{d}\omega^*)+\mathrm{d}\psi(\omega^*))^*=\phi^*((i^R_\psi(\mathrm{d}\omega^*)^*)+\phi^*(\mathrm{d}\psi^*(\omega))\\&=-\phi^*(i^L_{\psi^*}(\mathrm{d}\omega)-\mathrm{d}\psi^*(\omega))=-[\![\psi^*,\phi^*]\!]_L(\omega)\end{aligned}$$

so that the diagram commutes. Similarly, if we have module maps i^R and i^L related as stated then the same computation gives the relation between $[\ ,\ ]_L$ and $[\ ,\ ]_R$. □

Classically, we have the standard antisymmetric lift $i^R(\xi\wedge\eta)=\frac{1}{2}(\xi\otimes\eta-\eta\otimes\xi)$ and the same for i^L. Both $[\ ,\ ]_L$ and $[\ ,\ ]_R$ still have outputs not necessarily a vector field, but the combination

$$[\phi,\psi]_R+[\phi,\psi]_L=\mathcal{L}_\phi(\psi)$$

is a vector field, namely the classical Lie derivative or Lie bracket of two vector fields. We study vector fields on an exterior algebra and $[\![\ ,\ ]\!]$ further in Chap. 6.

We now turn to how this theory looks in the quantum group case. In classical geometry, the tangent space at the identity of a Lie group can be identified as its Lie algebra. A tangent vector at the identity can be extended as a left-invariant vector field so the Lie algebra can also be viewed as the space of left-invariant vector fields.

We work only with the left-handed part of the general theory above. If we have a left-covariant calculus on a Hopf algebra H then we say that a map $\Omega^1 \to H$ is left-invariant if it intertwines the given left coaction on Ω^1 with the regular coaction via the coproduct on H. We let ${}^H\mathfrak{X}^L$ denote the left-invariant (left) vector fields $\mathfrak{X}^L$ and define $\Lambda^2({}^H\mathfrak{X}^L) \subseteq \Lambda^2\mathfrak{X}^L$ to be tensor products of invariant vectors that annihilate the kernel of $\wedge$ as before.

Lemma 2.82 *If Ω^1 is a left-invariant differential calculus on a Hopf algebra H with invertible antipode then $\mathfrak{X}^L \cong \Lambda^{1*} \otimes H$, ${}^H\mathfrak{X}^L \cong \Lambda^{1*}$, $\Lambda^2({}^H\mathfrak{X}^L) \cong \Lambda^{2*}$ and $[\![\ ,\ ]\!]_L$ restricts to a map $\Lambda^2({}^H\mathfrak{X}^L) \to {}^H\mathfrak{X}^L$ dual to the restriction* $\mathrm{d} : \Lambda^1 \to \Lambda^2$.

Proof We use the isomorphism $\Omega^1 \cong H \otimes \Lambda^1$ and write left-module maps $\Omega^1 \to H$ in the form $h \otimes v \mapsto \langle h \otimes v, x \otimes g\rangle = h\langle v, x\rangle g$ for all $x \in \Lambda^{1*}$ and $g \in H$, thereby identifying $\mathfrak{X}^L$ as stated. The left coaction on Ω^1 is $h \otimes v \to h_{(1)} \otimes h_{(2)} \otimes v$ so the covariant maps in $\mathfrak{X}^L$ are of the form $x \otimes 1$, i.e., we identify ${}^H\mathfrak{X}^L \cong \Lambda^{1*}$ as stated. Note that $\mathfrak{X}^L$ as a bimodule has $h.x = (h_{(2)}\triangleright x)h_{(1)}$ for the pairing with the bimodule structure on $\Omega^1 \cong H \otimes \Lambda^1$ to descend to $\otimes_H$, where $\langle v, h\triangleright x\rangle = \langle v\triangleleft h, x\rangle$ defines the adjoint left action on Λ^{1*}.

We now write elements of $\Omega^1 \otimes_H \Omega^1 \cong H \otimes \Lambda^1 \otimes \Lambda^1$ and $\mathfrak{X}^L \otimes_H \mathfrak{X}^L \cong \Lambda^{1*} \otimes \Lambda^{1*} \otimes H$ in this canonical form and take the nested pairing $\langle h \otimes v \otimes w, x \otimes y \otimes g\rangle = h\langle w, x\rangle\langle v, y\rangle g$ as dictated by the canonical nested pairing in the category of H-bimodules. Note that the action on $\Lambda^{1*} \otimes \Lambda^{1*}$ compatible with this is $h\triangleright(x \otimes y) = h_{(2)}\triangleright x \otimes h_{(1)}\triangleright y$, i.e., the tensor product one for H^{cop}. In this case $\sum x \otimes y \in \Lambda^2({}^H\mathfrak{X}^L)$ if and only if $\sum h\langle w, x\rangle\langle v, y\rangle = 0$ for all $\sum hv \wedge w = 0$. This comes down to $\sum \langle w, x\rangle\langle v, y\rangle = 0$ for all $\sum v \otimes w \in \ker\wedge : \Lambda^1 \otimes \Lambda^1 \to \Lambda^2$, i.e., $\sum x \otimes y \in \Lambda^{1*} \otimes \Lambda^{1*}$ that descend to Λ^2 as a quotient of $\Lambda^1 \otimes \Lambda^1$. In this way, we identify $\Lambda^2({}^H\mathfrak{X}^L) = \Lambda^{2*}$.

The map $[\![\ ,\ ]\!]_L$ defined by (2.28) reduces to $[\![x, y]\!]_L(v) = \langle \mathrm{d}v, y \otimes x\rangle$ on an element $x \otimes y \in \Lambda^2({}^H\mathfrak{X}^L)$ (sum understood) for any choice of lift i^L. For this, note that left-invariant elements of H itself are multiples of the identity. □

More generally, we will have some kind of 'quantum Lie bracket' $\Lambda^{1*} \otimes \Lambda^{1*} \to \Lambda^{1*}$ given by $[x, y]_L(v) = \langle i^L \circ \mathrm{d}v, y \otimes x\rangle$ for any left-covariant lift i^L (the latter necessarily restricts to $\Lambda^2 \to \Lambda^1 \otimes \Lambda^1$). This provides the general setting, which we will more or less follow in the bicovariant case below. We will also see in Chap. 6 how (in a right-handed version) the invariant vector fields generate an 'enveloping algebra' of invariant differential operators.

2.7.1 Bicovariant Quantum Lie Algebras

Here we adopt an alternative more axiomatic approach which is less geometric but which we will show works at least in the bicovariant case.

Definition 2.83 A (left) quantum Lie algebra is a vector space and linear maps $(\mathfrak{h}, \sigma, [\ ,\ ])$ where:

(1) $[\,,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ vanishes on $\ker(\mathrm{id} - \sigma)$;
(2) $\sigma : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ obeys the braid relations;
(3) $[\,,]([\,,] \otimes \mathrm{id}) = [\,,](\mathrm{id} \otimes [\,,])((\mathrm{id} - \sigma) \otimes \mathrm{id})$;
(4) $\sigma(\mathrm{id} \otimes [\,,]) = ([\,,] \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})$;
(5) $\sigma([\,,] \otimes \mathrm{id}) - (\mathrm{id} \otimes [\,,])(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = ([\,,] \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)((\mathrm{id} - \sigma^2) \otimes \mathrm{id})$.

If $\sigma = \text{flip}$ then (1),(3) are the usual axioms of a Lie algebra (in characteristic other than 2) and (2),(4),(5) are empty. The enveloping algebra of a quantum Lie algebra is defined similarly as for usual Lie algebras:

$$U(\mathfrak{h}) = T\mathfrak{h}/\langle x \otimes y - \sigma(x \otimes y) - [x, y]\rangle. \tag{2.29}$$

Similarly, a right-quantum Lie algebra has axioms as above but with a left-right reversal. If $(\mathfrak{h}, \sigma, [\ ,\ ])$ is a left quantum Lie algebra then $(\mathfrak{h}, \text{flip}\circ\sigma\circ\text{flip}, [\ ,\ ]\circ\text{flip})$ is a right enveloping algebra of a right-quantum Lie algebra is taken in the same form (2.29) but with right-handed structures $[\ ,\ ]_R$ and σ_R.

Lemma 2.84 *Let $\mathfrak{h}$ be a vector space, $\tilde{\mathfrak{h}} := \Bbbk c \oplus \mathfrak{h}$ and consider an operator $\tilde{\Psi} : \tilde{\mathfrak{h}} \otimes \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}} \otimes \tilde{\mathfrak{h}}$ of the form*

$$\tilde{\Psi}(c \otimes c) = c \otimes c, \quad \tilde{\Psi}(c \otimes x) = x \otimes c, \quad \tilde{\Psi}(x \otimes c) = c \otimes x, \quad \tilde{\Psi}(\mathfrak{h} \otimes \mathfrak{h}) \subseteq \mathfrak{h} \otimes \tilde{\mathfrak{h}}$$

for all $x \in \mathfrak{h}$. Then $\tilde{\Psi}$ obeys the braid relations if and only if $\sigma : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ and $[\ ,\ ] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ defined by $\tilde{\Psi}|_{\mathfrak{h}\otimes\mathfrak{h}} = \sigma + [\,,] \otimes c$ obey axioms (2)–(5) of a left quantum Lie-algebra.

Proof We use the notation $\tilde{\Psi}_1 = \tilde{\Psi} \otimes \mathrm{id}$ and $\tilde{\Psi}_2 = \mathrm{id} \otimes \tilde{\Psi}$ as in Definition 2.67, and first check the content of the braid relations when some of the factors are c. We find

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_1(c \otimes c \otimes x) = x \otimes c \otimes c = \tilde{\Psi}_2\tilde{\Psi}_1\tilde{\Psi}_2(c \otimes c \otimes x),$$

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_1(c \otimes x \otimes c) = c \otimes x \otimes c = \tilde{\Psi}_2\tilde{\Psi}_1\tilde{\Psi}_2(c \otimes x \otimes c),$$

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_1(c \otimes x \otimes y) = \tilde{\Psi}_1(x \otimes y \otimes c) = \sigma(x \otimes y) \otimes c + [x, y] \otimes c \otimes c$$

$$= \tilde{\Psi}_2\tilde{\Psi}_1(c \otimes \sigma(x \otimes y) + c \otimes [x, y] \otimes c) = \tilde{\Psi}_2\tilde{\Psi}_1\tilde{\Psi}_2(c \otimes x \otimes y),$$

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_1(x \otimes c \otimes y) = \tilde{\Psi}_1\tilde{\Psi}_2(c \otimes x \otimes y) = \tilde{\Psi}_1(c \otimes \sigma(x \otimes y) + c \otimes [x, y] \otimes c)$$

$$= \tilde{\Psi}_2(\sigma(x \otimes y) \otimes c + [x, y] \otimes c \otimes c) = \tilde{\Psi}_2\tilde{\Psi}_1(x \otimes y \otimes c) = \tilde{\Psi}_2\tilde{\Psi}_1\tilde{\Psi}_2(x \otimes c \otimes y),$$

so these are all empty (similarly for the 3rd cases and when all three are c). Thus the only content is when all three elements are in $\mathfrak{h}$. Then

$$\tilde{\Psi}_1\tilde{\Psi}_2\tilde{\Psi}_1(x\otimes y\otimes z)=\tilde{\Psi}_1\tilde{\Psi}_2(\sigma(x\otimes y)\otimes z+[x,y]\otimes c\otimes z)$$
$$=\tilde{\Psi}_1\big(\sigma_2\sigma_1(x\otimes y\otimes z)+(\mathrm{id}\otimes[\ ,\ ]\otimes\mathrm{id})(\sigma(x\otimes y)\otimes z\otimes c+[x,y]\otimes z\otimes c)\big)$$
$$=\sigma_1\sigma_2\sigma_1(x\otimes y\otimes z)+(\mathrm{id}\otimes\mathrm{flip})\,(([\ ,\ ]\otimes\mathrm{id})\sigma_2\sigma_1(x\otimes y\otimes z)\otimes c)$$
$$+\sigma(\mathrm{id}\otimes[\ ,\ ])(\sigma(x\otimes y)\otimes z)\otimes c+[\ ,\ ](\mathrm{id}\otimes[\ ,\ ])(\sigma(x\otimes y)\otimes z)\otimes c\otimes c$$
$$+\sigma([x,y]\otimes z)\otimes c+[[x,y],z]\otimes c\otimes c,$$
$$\tilde{\Psi}_2\tilde{\Psi}_1\tilde{\Psi}_2(x\otimes y\otimes z)=\tilde{\Psi}_2\tilde{\Psi}_1(x\otimes\sigma(y\otimes z)+x\otimes[y,z]\otimes c)=\tilde{\Psi}_2(\sigma_1\sigma_2(x\otimes y\otimes z)$$
$$+(\mathrm{id}\otimes\mathrm{flip})\,(([\ ,\ ]\otimes\mathrm{id})(x\otimes\sigma(y\otimes z))\otimes c)+\sigma(x\otimes[y,z])\otimes c+[x,[y,z]]\otimes c\otimes c)$$
$$=\sigma_2\sigma_1\sigma_2(x\otimes y\otimes z)+(\mathrm{id}\otimes[\ ,\ ])\sigma_1\sigma_2(x\otimes y\otimes z)\otimes c$$
$$+([\ ,\ ]\otimes\mathrm{id})(x\otimes\sigma(y\otimes z))\otimes c+(\mathrm{id}\otimes\mathrm{flip})(\sigma(x\otimes[y,z])\otimes c)+[x,[y,z]]\otimes c\otimes c.$$

Equating the terms in $\mathfrak{h}^{\otimes 3}$ gives (2), in $\mathfrak{h}\otimes c\otimes c$ gives (3), in $\mathfrak{h}\otimes c\otimes\mathfrak{h}$ (those with 'flip' in the final expressions, to give c in the middle) gives (4) and in $\mathfrak{h}\otimes\mathfrak{h}\otimes c$ gives

$$\sigma(\mathrm{id}\otimes[\ ,\ ])(\sigma\otimes\mathrm{id})+\sigma([\ ,\ ]\otimes\mathrm{id})=(\mathrm{id}\otimes[\ ,\ ])(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)+([\ ,\ ]\otimes\mathrm{id})(\mathrm{id}\otimes\sigma),$$

which is an alternative version of (5) given (4). This proof can also be done with diagrams and similarly works in any monoidal category with direct sums. □

Similarly, $\tilde{\Psi}(\mathfrak{h}\otimes\mathfrak{h})\subseteq\tilde{\mathfrak{h}}\otimes\mathfrak{h}$ obeys the braid relations if and only if $\sigma_R,[\ ,\ ]_R$ defined by $\tilde{\Psi}(x\otimes y)=\sigma_R(x\otimes y)+c\otimes[x,y]_R$ obey the version of axioms (2)–(5) for a right quantum Lie algebra. Now suppose that Ω is the canonical bicovariant calculus on a Hopf algebra H given by a finite-dimensional right H-crossed module Λ^1 and Maurer–Cartan form $\varpi: H^+\twoheadrightarrow\Lambda^1$ in Theorem 2.26. Then Λ^{1*} is a left H-crossed module by

$$\langle h\triangleright x,v\rangle=\langle x,v\triangleleft h\rangle,\quad x_{(\bar{1})}\langle x_{(\bar{\infty})},v\rangle=\langle x,v_{(\bar{0})}\rangle v_{(\bar{1})}\tag{2.30}$$

for all $x\in\Lambda^{1*}$, $v\in\Lambda^1$ and $h\in H$. The left-handed crossed-module braiding $\Psi(x\otimes y)=x_{(\bar{1})}\triangleright y\otimes x_{(\bar{\infty})}$ is dual to the braiding on Λ^1 as $\langle\Psi(x\otimes y),v\otimes w\rangle=\langle x\otimes y,\Psi(v\otimes w)\rangle$ with the usual (not nested) tensor product pairings.

Theorem 2.85 *Let $\Omega(H)=H.\Lambda^1$ be the canonical 'Woronowicz' exterior algebra on a Hopf algebra H. Then Λ^{1*} is a right-quantum Lie algebra by*

$$\sigma_R(x\otimes y)=x_{(\bar{1})}\triangleright y\otimes x_{(\bar{\infty})},\quad [x,y]_R=\langle y,\varpi\pi_\epsilon x_{(\bar{1})}\rangle x_{(\bar{\infty})}$$

using the left crossed module braiding.

Proof We use Lemma 2.84, extending a crossed module Λ^1 to $\tilde\Lambda^1 := \Bbbk e \oplus \Lambda^1$ by

$$e{\triangleleft}h = e\epsilon h + \varpi\pi_\epsilon h, \quad \Delta_R e = e \otimes 1$$

and the existing action and coaction on Λ^1. We check that

$$\begin{aligned}(e{\triangleleft}h){\triangleleft}g &= (e\epsilon h + \varpi\pi_\epsilon h){\triangleleft}g = e\epsilon(h)\epsilon(g) + \epsilon(h)\varpi\pi_\epsilon g + \varpi((\pi_\epsilon h)g)\\ &= e\epsilon(hg) + \varpi\pi_\epsilon(hg) = e{\triangleleft}(hg)\end{aligned}$$

so this is an action, while the crossing condition is

$$\Delta_R(e{\triangleleft}h) = e\otimes 1\epsilon h + \Delta_R\varpi\pi_\epsilon h = e\otimes 1\epsilon h + (\varpi\pi_\epsilon\otimes\mathrm{id})\mathrm{Ad}_R h = e{\triangleleft}h_{(2)}\otimes(Sh_{(1)})h_{(3)}$$

as required, using the properties of π_ϵ and ϖ in the proof of Theorem 2.74. The dual $\tilde\Lambda^1 = kc \oplus \Lambda^{1*}$ and a short calculation gives its left crossed module structure as

$$h\tilde{\triangleright}c = \epsilon(h)c, \quad h\tilde{\triangleright}x = h{\triangleright}x + \langle x, \varpi\pi_\epsilon h\rangle c, \quad \Delta_L c = c\otimes 1$$

for all $x \in \Lambda^{1*}$ and the canonical left coaction on the latter. We can now take the braiding $\tilde\Psi$ in the category of left crossed modules, technically a prebraiding as we have not assumed that S has invertible antipode, and we find that it has the required form for the right-handed version of Lemma 2.84. In particular,

$$\tilde\Psi(x\otimes y) = x_{(\bar 1)}\tilde{\triangleright}y\otimes x_{(\bar\infty)} = x_{(\bar 1)}{\triangleright}y\otimes x_{(\bar\infty)} + c\otimes\langle y, \varpi\pi_\epsilon x_{(\bar 1)}\rangle x_{(\bar\infty)}$$

for all $x, y \in \Lambda^{1*}$ allows us to read off $\sigma_R = \Psi$ (the crossed module braiding on Λ^{1*}) and $[\ ,\ ]_R$ and know that they obey the right-handed version of axioms (2)–(5).

It remains to check axiom (1). But using our previous identity

$$\Psi(\varpi\pi_\epsilon\otimes\varpi\pi_\epsilon)\Delta = (\varpi\pi_\epsilon\otimes\varpi\pi_\epsilon)(\Delta - \mathrm{Ad}_R)$$

from the proof of Theorem 2.74, and using the tensor product pairing, we have

$$\begin{aligned}[x, y]_R(\varpi\pi_\epsilon h) &= \langle x\otimes y, (\varpi\pi_\epsilon h)_{(\bar 0)}\otimes\varpi\pi_\epsilon(\varpi\pi_\epsilon h)_{(\bar 1)}\rangle\\ &= \langle x\otimes y, (\varpi\pi_\epsilon\otimes\varpi\pi_\epsilon)\mathrm{Ad}_R h\rangle\\ &= \langle x\otimes y, (\mathrm{id}-\Psi)(\varpi\pi_\epsilon\otimes\varpi\pi_\epsilon)\Delta h\rangle\\ &= \langle(\mathrm{id}-\Psi)(x\otimes y), (\varpi\pi_\epsilon\otimes\varpi\pi_\epsilon)\Delta h\rangle\end{aligned}$$

for all $h \in H$ and $x, y \in \Lambda^{1*}$. Hence $\sum[x, y]_R = 0$ if $\sum x\otimes y \in \ker(\mathrm{id}-\Psi)$. $\square$

The quantum Lie bracket on Λ^{1*} in the form at the end of the proof, using the Maurer–Cartan equations, can be written as $\langle [x, y]_R, v\rangle = -\langle x \otimes y, i \circ \mathrm{d}v\rangle$ in the spirit of the remark after Lemma 2.82, albeit with the tensor product pairing and the canonical $i = \mathrm{id} - \Psi : \Lambda^2 \to \Lambda^1 \otimes \Lambda^1$ not in general obeying $\wedge \circ i = \mathrm{id}$.

Example 2.86 For $H = \Bbbk(G)$ with its bicovariant calculus given by $\mathcal{C} \subseteq G \setminus \{e\}$, the right crossed module structure on the basic forms e_a for $a \in \mathcal{C}$ is $\Delta_R e_a = \sum_g e_{gag^{-1}} \otimes \delta_g$ and the right action $e_a \triangleright f = f(a)e_a$ (see Example 2.29). This implies for the dual basis $\{f^a\}$ the left crossed module structure $\Delta_L f^a = \sum_g \delta_g \otimes f^{g^{-1}ag}$ and $f \triangleright f^a = f(a) f^a$. The braiding and right quantum Lie bracket are $\sigma_R(f^a \otimes f^b) = f^b \otimes f^{b^{-1}ab}$ and $[f^a, f^b]_R = f^{b^{-1}ab} - f^a$, similar to the associated quandle in §1.7.3. The enveloping algebra of this right quantum Lie algebra has relations $f^a f^b - f^b f^{b^{-1}ab} = f^{b^{-1}ab} - f^a$.

We conclude with the left-quantum Lie algebra version of the above, which connects better with the conventions of the next section. These are more naturally associated to the factorisation $\Omega = \Lambda_R.H$ which equally well exists for a bicovariant calculus, where Λ_R are the right-invariant forms. Here the calculus is defined by a *left* ideal $\mathcal{I}_R$ stable under the *left* adjoint coaction $\mathrm{Ad}_L h = h_{(1)} S h_{(3)} \otimes h_{(2)}$ with right-handed Maurer–Cartan form $\varpi_R : H^+ \twoheadrightarrow \Lambda^1_R$ a morphism of left H-crossed modules and given by $\varpi_R h = (\mathrm{d}h_{(1)}) S h_{(2)}$ so that $\mathrm{d}h = (\varpi_R \pi_\epsilon h_{(1)}) h_{(2)}$. The two approaches are equivalent when S is invertible with the corresponding right ideal $\mathcal{I} = S\mathcal{I}_R$ for the same calculus and the Maurer–Cartan forms related by

$$\varpi_R \pi_\epsilon h = -h_{(1)} S h_{(3)} \varpi \pi_\epsilon S h_{(2)}. \tag{2.31}$$

In this right version, Λ^{1*}_R is a right H-crossed module by dualisation and the reflected version of Theorem 2.85 gives us a left quantum Lie algebra

$$\sigma(x \otimes y) = y_{(\bar{0})} \otimes x \triangleleft y_{(\bar{1})}, \quad [x, y] = y_{(\bar{0})} \langle x, \varpi_R \pi_\epsilon y_{(\bar{1})}\rangle. \tag{2.32}$$

Example 2.87 Let A be coquasitriangular. The reflected version of Proposition 2.55 is to start with a subcoalgebra $\mathcal{L}_R \subseteq A$ viewed as a right crossed module by Ad_R and the right action in Proposition 2.54, and $\varpi_R : A^+ \to \mathcal{L}^*_R$ given by $\varpi_R(a) = \mathcal{Q}((\ ) \otimes a)|_{\mathcal{L}_R}$. As before, we define Λ^1_R as its image or consider that we have a generalised calculus. The subcoalgebra $\mathcal{L} = S\mathcal{L}_R$ used to construct Λ^1 gives the same bicovariant calculus as we obtain now from $\mathcal{L}_R$ and Λ^1_R. This time, $\mathcal{L}_R$ inherits the braiding displayed in Proposition 2.54 (not to be confused with $\tilde{\Psi}$ above) and the reflected version of Theorem 2.85 gives a left quantum Lie algebra with this braiding as σ and quantum Lie bracket $[x, y] = y_{(\bar{0})} \mathcal{Q}(x \otimes \pi_\epsilon y_{(\bar{1})})$ defined in fact for all $x, y \in \mathcal{L}_R$.

2.7.2 Braided Lie Algebras

When our quantum group is a coquasitriangular Hopf algebra A then there is a different and more intrinsic notion of a 'braided Lie algebra' in its braided category of comodules. This will link back to quantum Lie algebras but has the benefit of being axiomatised entirely within a braided category $\mathcal{V}$. In the strictly braided case, one does not always have $\Psi^2 = \mathrm{id}$ for the braiding and hence cannot write down an immediate analogue of the familiar 3-term Jacobi identity based on cyclic rotations as for ordinary or super-Lie algebras. The solution to this is the following.

Definition 2.88 (Majid) A left braided Lie algebra is a coalgebra $\mathcal{L}$ in a braided category, together with a morphism $[\ ,\ ] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ subject to the axioms shown diagrammatically in Fig. 2.12.

The key new idea here is to start with an object $\mathcal{L} \in \mathcal{V}$ of a braided category $\mathcal{V}$ which is a coalgebra in the category, not merely an object, i.e., with equipped morphisms $\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ and counit $\epsilon : \mathcal{L} \to \underline{1}$. As we know from quantum groups, some kind of coalgebra is implicit to the notion of symmetry as it tells us how symmetries extend to tensor products, so we are picking up on this hidden structure. However, if Δ has two terms then the 'pentagonal Jacobi identity' in axiom (a) will have two terms on the left and hence a more familiar 3-term form. We are not going to develop the theory of braided Lie algebras in full, but like a usual Lie algebra, there is a notion of representation or action defined by Fig. 2.13a. Representations that obey the condition in Fig. 2.13b are called *cocommutative* and both the category ${}_{\mathcal{L}}\mathcal{M}$ of representations and its subcategory of cocommutative representations are monoidal (the tensor product is as for braided Hopf algebras, namely via Δ and the braiding). Clearly, $\mathcal{L}$ itself is a cocommutative representation of $\mathcal{L}$ via $[\ ,\ ]$.

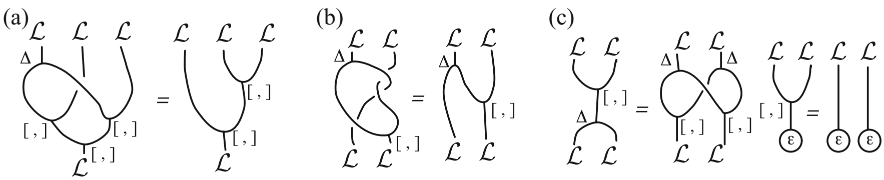


Fig. 2.12 Axioms of a left braided Lie algebra, reading maps as usual flowing down the page

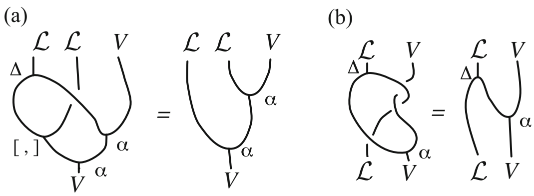


Fig. 2.13 Representation V of a braided Lie algebra and a cocommutative representation

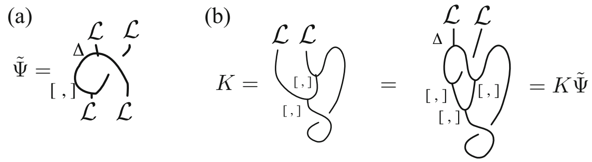


Fig. 2.14 (**a**) Fundamental braiding and (**b**) Killing form of a braided Lie algebra

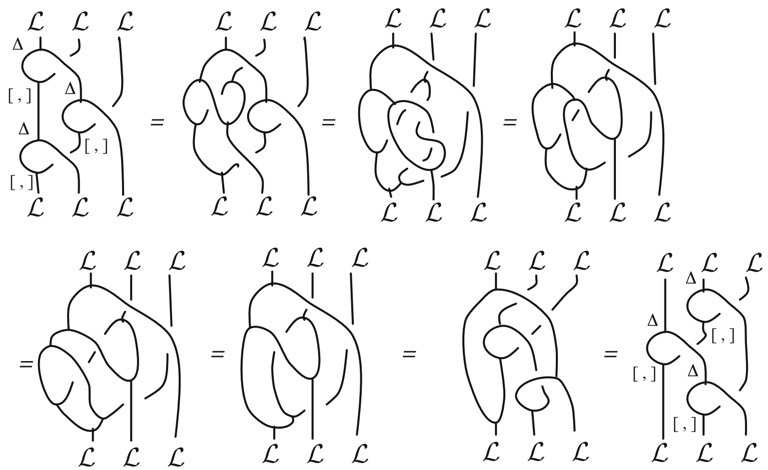


Fig. 2.15 Proof that the fundamental braiding $\tilde{\Psi}$ of a braided Lie algebra obeys the braid relations

For any braided Lie algebra $\mathcal{L} \in \mathcal{V}$, we define its *fundamental braiding* $\tilde{\Psi}$: $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ as shown in Fig. 2.14. This is shown in Fig. 2.15 to obey the braid relations on $\mathcal{L}^{\otimes 3}$. A braided Lie algebra is called *regular* if this is invertible. When $\mathcal{L}$ is rigid, there is a $\tilde{\Psi}$-symmetric *braided Killing form* shown in Fig. 2.14b. This is a braided version of the Killing form in classical Lie theory as the trace of two applications of the adjoint action.

Theorem 2.89 *Let $\mathcal{L}$ be a braided Lie algebra in a braided abelian category $\mathcal{V}$. There is a braided enveloping bialgebra $U(\mathcal{L}) \in \mathcal{V}$ defined as the tensor algebra on $\mathcal{L}$ modulo relations of commutativity with respect to $\tilde{\Psi}$ (in the concrete case this means relations $xy = \cdot\tilde{\Psi}(x \otimes y)$ for all $x, y \in \mathcal{L}$). The coproduct extends that of $\mathcal{L}$. Representations of $\mathcal{L}$ extend to representations of $U(\mathcal{L})$.*

Proof Both the product and proof can be expressed more formally in categorical terms. The key idea is in Fig. 2.16, where Δ extends to $U(\mathcal{L})$ as a bialgebra in the category, as we obtain the same answer before and after applying the relation. □

The Killing form in nice cases, e.g., when $U(\mathcal{L})$ can be localised to a braided Hopf algebra, can be shown to be invariant under the action of $\mathcal{L}$ on $\mathcal{L} \otimes \mathcal{L}$. Finally, a braided Lie algebra $\mathcal{L}$ is called *unital* if there is a morphism $\eta : \underline{1} \to \mathcal{L}$ with

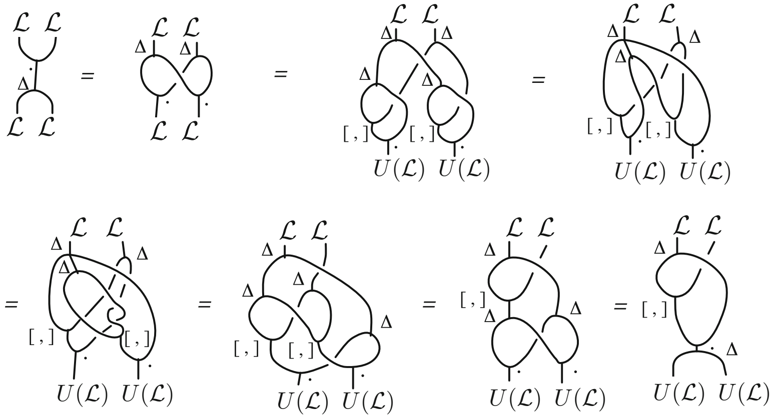


Fig. 2.16 Proof that the coproduct of the braided enveloping algebra $U(\mathcal{L})$ is well defined

certain properties which we explain for convenience in the concrete $\Bbbk$-linear setting where objects are built on $\Bbbk$-vector spaces with the inherited tensor product. This then means a bosonic element $c = \eta(1) \in \mathcal{L}$ (where bosonic means that the braiding with any other element is the flip map) such that

$$[c,\ ] = \mathrm{id}, \quad [\ , c] = c\epsilon, \quad \Delta c = c \otimes c, \quad \epsilon c = 1. \tag{2.33}$$

This implies that c is also bosonic with respect to $\tilde{\Psi}$ and hence central in $U(\mathcal{L})$. For a unital braided Lie algebra we have a projection $\pi^c_\epsilon = \mathrm{id} - c\epsilon : \mathcal{L} \to \mathcal{L}^+ = \ker \epsilon$ and $\mathcal{L} = \Bbbk c \oplus \mathcal{L}^+$ (or $\mathcal{L} = \underline{1} \oplus \mathcal{L}^+$ in general).

Lemma 2.90 *If $\mathcal{L}$ is a $\Bbbk$-linear left unital braided Lie algebra then $\mathcal{L}^+$ obeys axioms (2)–(5) of a left quantum Lie algebra in the sense of Definition 2.83 via*

$$[\ ,\ ]|_{\mathcal{L}^+}, \quad \sigma = (\mathrm{id} \otimes \pi^c_\epsilon)\tilde{\Psi}|_{\mathcal{L}^+ \otimes \mathcal{L}^+}$$

and is a left quantum Lie algebra if and only if c is not a zero divisor in degree 2 of $U(\mathcal{L})$, i.e., there is no nonzero element $x \in \mathcal{L}$ such that $cx = 0$ in $U(\mathcal{L})$.

Proof We check that $(\epsilon \otimes \mathrm{id})\tilde{\Psi}(x \otimes y) = x\epsilon(y)$ so the fundamental braiding restricts to $\tilde{\Psi}(\mathcal{L}^+ \otimes \mathcal{L}^+) \subseteq \mathcal{L}^+ \otimes \mathcal{L}$. Lemma 2.84 then gives us axioms (2)–(5) of a left quantum Lie algebra on $\mathcal{L}^+$, with σ as stated and $(\mathrm{id} \otimes \epsilon)\tilde{\Psi}(x \otimes y) = [x, y]c$, giving the quantum Lie bracket as the restriction of the extended braided bracket. Next, suppose that $\sum x \otimes y \in \mathcal{L}^{+\otimes 2}$ is in the kernel of $\mathrm{id} - \sigma$, i.e.

$$\sum x \otimes y = \sum \tilde{\Psi}(x \otimes y) - \sum [x, y] \otimes c,$$

which viewed in $U(\mathcal{L})$ implies that $(\sum [x, y])c = 0$. If c is not a zero divisor in degree 2, we conclude that $\sum [x, y] = 0$, which proves axiom (1). Conversely, suppose axiom (1) of a quantum Lie algebra holds on $\mathcal{L}^+$ and that $uc = 0$ in $U(\mathcal{L})$ for some $u \in \mathcal{L}$. Then from the definition of $U(\mathcal{L})$, we have $\sum x \otimes y \in \mathcal{L} \otimes \mathcal{L}$ with

$$u \otimes c = (\mathrm{id} - \tilde{\Psi})(\sum x \otimes y) = (\mathrm{id} - \sigma)(\pi^c_\epsilon x \otimes \pi^c_\epsilon y) - \sum [\pi^c_\epsilon x, \pi^c_\epsilon y] \otimes c$$
$$+ (\sum x\epsilon(y) - \sum \epsilon(x)y) \otimes c - c \otimes (\sum x\epsilon(y) - \sum \epsilon(x)y)$$

on making the decomposition $\mathcal{L} = \Bbbk c \oplus \mathcal{L}^+$. Comparing different terms under this decomposition we conclude that the first term and hence, by (1), the second term, vanishes and hence that $u = (\sum x\epsilon(y) - \sum \epsilon(x)y) = 0$. □

Moreover, any braided Lie algebra can have a unit adjoined by extending it to a braided Lie algebra $\tilde{\mathcal{L}}$ spanned by $c, \mathcal{L}$ with the above properties of a unit. The extended fundamental braiding is the flip when applied to c on either side. Hence $U(\tilde{\mathcal{L}}) = U(\mathcal{L}) \otimes \Bbbk[c]$, where $\Bbbk[c]$ is the (trivially braided) bialgebra with c grouplike and commutes with $U(\mathcal{L})$.

Corollary 2.91 *In the concrete $\Bbbk$-linear setting, a left braided Lie algebra $\mathcal{L}$ is a left quantum Lie algebra $\mathfrak{h} = \mathcal{L}$ in Definition 2.83 with $\sigma_{\mathfrak{h}} = \tilde{\Psi}$ and $[x, y]_{\mathfrak{h}} = [x, y] - \epsilon(x)y$ for all $x, y \in \mathcal{L}$. Moreover, $U(\mathfrak{h}) \cong U(\mathcal{L})$ as algebras by $x \mapsto x - 1\epsilon(x)$.*

Proof First consider the unital extension $\tilde{\mathcal{L}} = \Bbbk c \oplus \mathcal{L}$ and suppose in $U(\tilde{\mathcal{L}})$ that $(u + \rho c)c = 0$ for some $u \in \mathcal{L}$, $\rho \in \Bbbk$. The tensor product form of $U(\tilde{\mathcal{L}})$ tells us that $\rho = 0$ and $u = 0$, hence the no zero divisor condition holds, Lemma 2.90 applies and $\tilde{\mathcal{L}}^+$ is a quantum Lie algebra. We can also identify $\mathcal{L} \cong \tilde{\mathcal{L}}^+$ by $\pi^c_\epsilon x = x - c\epsilon(x) \in \tilde{\mathcal{L}}^+$ and compute its quantum Lie algebra structure. Working in $\tilde{\mathcal{L}}$,

$$\tilde{\Psi}(\pi^c_\epsilon x \otimes \pi^c_\epsilon y) = \tilde{\Psi}(x \otimes y) + \epsilon(x)\epsilon(y)c \otimes c - c\epsilon(y) \otimes x - \epsilon(x)y \otimes c,$$
$$\sigma(\pi^c_\epsilon x \otimes \pi^c_\epsilon y) = (\pi^c_\epsilon \otimes \pi^c_\epsilon)\tilde{\Psi}(\pi^c_\epsilon x \otimes \pi^c_\epsilon y) = (\pi^c_\epsilon \otimes \pi^c_\epsilon)\tilde{\Psi}(x \otimes y).$$

Hence if we label elements of $\tilde{\mathcal{L}}^+$ by elements $x, y \in \mathfrak{h} = \mathcal{L}$, in these terms $\sigma_{\mathfrak{h}}(x \otimes y) = \tilde{\Psi}(x \otimes y)$. Meanwhile, the bracket of $\tilde{\mathcal{L}}^+$ is given by restriction,

$$[\pi^c_\epsilon x, \pi^c_\epsilon y] = [x, y] - [\epsilon(x)c, y] - [x, \epsilon(y)c] + \epsilon(x)\epsilon(y)c = [x, y] - \epsilon(x)y$$
$$= \pi^c_\epsilon([x, y] - \epsilon(x)y),$$

giving the stated form for the bracket in terms of $\mathfrak{h} = \mathcal{L}$. We now check that $\phi : U(\mathfrak{h}) \to U(\mathcal{L})$ is well defined by extending $\phi(x) = x - 1\epsilon(x)$ as an algebra map, where $1 \in U(\mathcal{L})$ is the identity element. Evaluating in $U(\mathcal{L})$,

$$\phi(x \otimes y - \sigma_{\mathfrak{h}}(x \otimes y)) = \phi(x)\phi(y) - \cdot(\phi \otimes \phi)\tilde{\Psi}(x \otimes y)$$
$$= (x - \epsilon(x))(y - \epsilon(y)) - \cdot\tilde{\Psi}(x \otimes y) + \epsilon(y)x + [x, y] - \epsilon(x)\epsilon(y)$$
$$= [x, y] - \epsilon(x)y = \phi([x, y] - \epsilon(x)y) = \phi([x, y]_{\mathfrak{h}}),$$

as required. The inverse of ϕ on generators is $x \mapsto x + 1\epsilon(x)$ working in $U(\mathfrak{h})$ and is similarly an algebra map, giving an isomorphism of algebras. □

To complete the theory, in the concrete setting over $\mathbb{C}$, we introduce a real form of a braided Lie algebra. This requires an antilinear involution $\underline{*} : \mathcal{L} \to \mathcal{L}$ such that

$$\Delta\underline{*} = \mathrm{flip}(\underline{\star} \otimes \underline{\star})\Delta, \quad \epsilon\underline{*} = \bar{\ } \circ \epsilon, \quad \tilde{\Psi}\mathrm{flip}(\underline{*} \otimes \underline{*}) = \mathrm{flip}(\underline{\star} \otimes \underline{\star})\tilde{\Psi}^{-1} \tag{2.34}$$

plus a condition on $\underline{*}$ that it is an antilinear morphism (not merely an antilinear map) so that it can be pulled through braid crossings. What that means in examples will be clear (it means covariant) but for the general formulation we need the category to be a bar category as in the next section. Also, we assume we are in the regular case so that $\tilde{\Psi}$ is invertible. Then one can show that $\underline{*}$ extends to $U(\mathcal{L})$ to make it a $\underline{*}$-braided bialgebra where the coproduct flip-commutes with $\underline{*}$ as above, but now on $U(\mathcal{L})$, the counit commutes with $\underline{*}$ and the algebra under $\underline{*}$ is a $*$-algebra.

Example 2.92 (Braided Lie Algebra of a Leibniz Algebra) Let $\mathcal{L}$ be a vector space of the form $\mathcal{L} = \Bbbk c \oplus \mathfrak{g}$ with coalgebra structure

$$\Delta x = x \otimes c + c \otimes x, \quad \Delta c = c \otimes c, \quad \epsilon x = 0, \quad \epsilon c = 1$$

for all $x \in \mathfrak{g}$. Suppose that $[c, x] = x$, $[x, c] = 0$ and $[c, c] = c$ for all $x \in \mathfrak{g}$. Then the axioms of a braided Lie algebra in the category of vector spaces amount to

$$[[x, y], z] + [y, [x, z]] = [x, [y, z]]$$

for all $x, y, z \in \mathfrak{g}$, while regularity is automatic. We do not require antisymmetry of the bracket, which means that a braided Lie algebra of this form is the same as saying that $\mathfrak{g}$ is a *Leibniz algebra*, a slightly more general notion than that of a Lie algebra but including it. Thus every Leibniz algebra $\mathfrak{g}$ can be viewed as a unital braided Lie algebra $\mathcal{L} = \Bbbk c \oplus \mathfrak{g}$. Its fundamental braiding

$$\tilde{\Psi}(x\otimes y) = y\otimes x+[x, y]\otimes c, \ \tilde{\Psi}(c\otimes x) = x\otimes c, \ \tilde{\Psi}(x\otimes c) = c\otimes x, \ \tilde{\Psi}(c\otimes c) = c\otimes c$$

is always invertible. The Killing form restricts to the usual Killing form and in addition

$$K(c, c) = 1, \quad K(c, x) = K(x, c) = 0$$

and its invertibility is equivalent to that of the usual Killing form on $\mathfrak{g}$. Over $\mathbb{C}$, a real form is given by $c^{\underline{*}} = c$ and $e_i^{\underline{*}} = -e_i$ for a basis of $\mathfrak{g}$ in which the structure constants are real. The braided enveloping algebra $U(\mathcal{L})$ is now a quadratic ordinary bialgebra with relations $xy - yx = c[x, y]$ and c central. The zero divisor condition holds if and only if $[\ ,\ ] = 0$ on symmetric elements of the tensor product, i.e. the case when $\mathfrak{g}$ is a usual Lie algebra, and in this case the quantum Lie algebra $\mathcal{L}^+ = \mathfrak{g}$ by Lemma 2.90 just recovers $\mathfrak{g}$. Its enveloping algebra is a quotient of

$U(\mathcal{L})$ by setting $c = 1$. The fundamental braiding is still interesting in this Lie case and obeys $(\tilde{\Psi}^2 - \mathrm{id})(\tilde{\Psi} + \mathrm{id}) = 0$.

Finally, Corollary 2.91 tells us that we always have a quantum Lie algebra even in the general case, as $\mathcal{L}$ is a braided Lie algebra. This comes out as $\sigma_{\mathfrak{h}} = \tilde{\Psi}$ as above and $[x, y]_{\mathfrak{h}} = [x, y]$ that of the Leibniz algebra, with $[c, \ ]_{\mathfrak{h}} = [\ , c]_{\mathfrak{h}} = 0$. The natural enveloping algebra of the Leibniz algebra is then $U(\mathfrak{h})$ with relations $xy - yx = [x, y](1 + c)$ and c central. This is isomorphic to $U(\mathcal{L})$ by $x \mapsto x$, $c \mapsto c - 1$. Note that this c is the unit of our original braided Lie algebra $\mathcal{L}$ and should not be confused with the second unit that was adjoined during the proof of the corollary. ⋄

We can, however, construct braided Lie algebras in many other categories.

Example 2.93 A left braided Lie algebra in the category of sets and the diagonal form of coproduct $\Delta x = x \otimes x$ is a left rack, i.e., a set X and map $[x, y] = {}^x y$ obeying ${}^{^x y}({}^x z) = {}^x({}^y z)$. The counit is $\epsilon x = *$, where $\underline{1} = \{*\}$ is the set with one element. In the special case of an IP-quandle (see §1.7.3), the fundamental braiding $\tilde{\Psi} : X \times X \to X \times X$ is invertible and given by $\tilde{\Psi}(x, y) = ({}^x y, x)$. The braided enveloping algebra is the semigroup with relations $xy = {}^x y x$, which in the IP case we can make into a group G_X.

The linearised version is $\mathcal{L} = \Bbbk X$ with basis f_x labelled by $x \in X$, the set coalgebra $\Delta f_x = f_x \otimes f_x$ and $\epsilon f_x = 1$ and the bracket $[f_x, f_y] = f_{^x y}$. The ordinary bialgebra $U(\mathcal{L})$ has the $\{f_x\}$ as generators and relations $f_x f_y = f_{^x y} f_x$. If the set is finite, $\mathcal{L}$ is rigid and the Killing form $K(x, y)$ counts the number of fixed points of ${}^x({}^y(\ ))$. It is an interesting question as to when K is invertible. For example, it is known to be invertible when X is the 2-cycles conjugacy class in S_n with its IP-quandle structure, at least over $\mathbb{C}$. Also over $\mathbb{C}$, we have a real form when X is closed under inversion, with $f_x^* = -f_{x^{-1}}$. Finally, over any field, the associated quantum Lie algebra $\mathfrak{h}$ by Corollary 2.91 has the same basis as $\mathcal{L}$, the same fundamental braiding $\sigma_{\mathfrak{h}}(f_x \otimes f_y) = \tilde{\Psi}(f_x \otimes f_y) = f_{^x y} \otimes f_x$ and the bracket $[f_x, f_y]_{\mathfrak{h}} = f_{^x y} - f_y$. The enveloping algebra $U(\mathfrak{h})$ has the relations $f_x f_y - f_{^x y} f_x = f_{^x y} - f_y$ and is isomorphic to $U(\mathcal{L})$ by $f_x \mapsto f_x - 1$.

For G a group and $\mathcal{C} \subseteq G \setminus \{e\}$ Ad-stable, we have an IP-quandle with ${}^x y = xyx^{-1}$ and associated braided and quantum Lie algebras by the above. The latter by Example 2.87 is $\mathfrak{h} = \Lambda_R^{1*}$ for the associated bicovariant calculus $\Omega^1 = \Lambda_R^1.\Bbbk(G)$. Similarly, the dual of the left-invariant 1-forms gave a right quantum Lie algebra in Example 2.86 which comes from a right braided Lie algebra as in §1.7.3. ⋄

We now turn to these same ideas for the q-deformation quantum group case. Again we prefer left braided and quantum Lie algebras which will be dual to right-invariant 1-forms but one could equally well do it the other way.

Theorem 2.94

(1) *Let A be a coquasitriangular Hopf algebra and $\mathcal{L} \subseteq A$ a subcoalgebra. Then $\mathcal{L}$ with*

$$\Delta|_{\mathcal{L}}, \quad \epsilon|_{\mathcal{L}}, \quad [a, b] = b_{(\bar{0})} \mathcal{Q}(a \otimes b_{(\bar{1})})$$

for $a, b \in \mathcal{L}$ is a regular left braided Lie algebra in the category of right A-comodules.

(2) *The associated left quantum Lie algebra $(\mathfrak{h}, \sigma_{\mathfrak{h}}, [\ ,\ ]_{\mathfrak{h}})$ by Corollary 2.91 then corresponds to the right-handed version $\Omega^1 = \Lambda^1_R.A$ of the associated generalised differential calculus in Example 2.87. Moreover,*

$$U(\mathcal{L}) = (\Lambda_R^{\mathrm{quad}})^!,$$

the Koszul dual of the quadratic version of the exterior algebra.

(3) *In the real-quasitriangular case over $\mathbb{C}$, if $*S\mathcal{L} \subseteq \mathcal{L}$, we have a real form $\underline{*} = *S$ adjoint to the corresponding $*$-differential calculus.*

Proof (1) If $\mathcal{L}$ is a subcoalgebra then $\Delta(\mathcal{L}) \subseteq \mathcal{L} \otimes \mathcal{L}$ and hence $\mathrm{Ad}_R(\mathcal{L}) \subseteq \mathcal{L} \otimes A$. The right adjoint coaction on any Hopf algebra preserves the coproduct as

$$a_{(1)(2)} \otimes a_{(2)(2)} \otimes (Sa_{(1)(1)})a_{(1)(3)}(Sa_{(2)(1)})a_{(2)(3)} = \Delta a_{(2)} \otimes (Sa_{(1)})a_{(3)}$$

and this covariance also applies to $\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$. Similarly for the counit, while

$$[a_{(\bar{0})}, b_{(\bar{0})}] \otimes a_{(\bar{1})}b_{(\bar{1})} = b_{(\bar{0})}\mathcal{Q}(a_{(\bar{0})} \otimes b_{(\bar{1})(1)}) \otimes a_{(\bar{1})}b_{(\bar{1})(2)} = b_{(\bar{0})}\mathcal{Q}(a \otimes b_{(\bar{1})(2)}) \otimes b_{(\bar{1})(1)}$$

$= \mathrm{Ad}_R[a, b]$ by Proposition 2.54. So the structure maps are all morphisms.

Next we observe that the fundamental braiding $\tilde{\Psi}$ in Lemma 2.14 is

$$\begin{aligned}
\tilde{\Psi}(a \otimes b) &= [a_{(1)}, b_{(\bar{0})}] \otimes a_{(2)(\bar{0})}\mathcal{R}(a_{(2)(\bar{1})} \otimes b_{(\bar{1})}) \\
&= b_{(\bar{0})} \otimes a_{(2)(\bar{0})}\mathcal{Q}(a_{(1)} \otimes b_{(\bar{1})(1)})\mathcal{R}(a_{(2)(\bar{1})} \otimes b_{(\bar{1})(2)}) \\
&= b_{(2)} \otimes a_{(4)}\mathcal{R}(((Sb_{(1)})b_{(3)})_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes ((Sb_{(1)})b_{(3)})_{(2)}) \\
&\qquad\qquad \mathcal{R}((Sa_{(3)})a_{(5)} \otimes ((Sb_{(1)})b_{(3)})_{(3)}) \\
&= b_{(2)} \otimes a_{(2)}\mathcal{R}(((Sb_{(1)})b_{(3)})_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes ((Sb_{(1)})b_{(3)})_{(2)}) \\
&= b_{(\bar{0})} \otimes a_{(2)}\mathcal{R}(b_{(\bar{1})(1)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(2)}),
\end{aligned}$$

where the penultimate line is the same as our first expression for $\tilde{\Psi}$ in the proof of Proposition 2.54. So this is also the crossed module braiding there. We used the adjoint coaction, unpacked $\mathcal{Q}$ and then used the multiplicativity property of $\mathcal{R}$.

This allows us to verify axioms (a)–(c) of a braided Lie algebra in Fig. 2.12 since these can all be written in terms of $\tilde{\Psi}$. Here axiom (a) translates as

$$c_{(\bar{0})} \otimes \mathcal{Q}(a \otimes c_{(\bar{1})(1)})\mathcal{Q}(b \otimes c_{(\bar{1})(2)}) = c_{(\bar{0})} \otimes (\mathcal{Q}((\ ) \otimes c_{(\bar{1})(1)})\mathcal{Q}((\ ) \otimes c_{(\bar{1})(2)}))\tilde{\Psi}(a \otimes b)$$

for all $a, b, c \in \mathcal{L}$, which we have already proven as the 4th property of the quantum Killing form in Proposition 2.54. Axiom (b) can be written as

$$\Psi\tilde{\Psi} = (\mathrm{id} \otimes [\ ,\ ])(\Delta \otimes \mathrm{id}),$$

where Ψ is the braiding of $\mathcal{M}^A$. Using our above expression for $\tilde{\Psi}$, we have

$$\begin{aligned}\Psi\tilde{\Psi}(a \otimes b) &= a_{(2)(\bar{0})} \otimes b_{(\bar{0})(\bar{0})}\mathcal{R}(b_{(\bar{0})(\bar{1})} \otimes a_{(2)(\bar{1})})\mathcal{R}(b_{(\bar{1})(1)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(2)})\\
&= a_{(2)(\bar{0})} \otimes b_{(\bar{0})}\mathcal{R}(b_{(\bar{1})(1)} \otimes a_{(2)(\bar{1})})\mathcal{R}(b_{(\bar{1})(2)} \otimes a_{(1)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(3)})\\
&= a_{(2)(2)} \otimes b_{(\bar{0})}\mathcal{R}(b_{(\bar{1})(1)} \otimes a_{(1)}(Sa_{(2)(1)})a_{(2)(3)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(2)})\\
&= a_{(1)} \otimes b_{(\bar{0})}\mathcal{R}(b_{(\bar{1})(1)} \otimes a_{(2)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(2)}) = a_{(1)} \otimes b_{(\bar{0})}\mathcal{Q}(a_{(2)} \otimes b_{(\bar{1})})\\
&= a_{(1)} \otimes [a_{(2)}, b],\end{aligned}$$

where we used the coaction properties of Ad_R and the multiplicativity property of $\mathcal{R}$ to make a cancellation. Axiom (c) can also be written as in terms of $\tilde{\Psi}$ as $\Delta[\ ,\ ] = (\mathrm{id} \otimes [\ ,\ ])(\tilde{\Psi} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$ but in our case follows directly from the 3rd quantum Killing form identity in Proposition 2.54 as

$$\begin{aligned}&[a_{(1)}, b_{(1)(\bar{0})}] \otimes [a_{(2)(\bar{0})}, b_{(2)}]\mathcal{R}(a_{(2)(\bar{1})} \otimes b_{(1)(\bar{1})})\\
&\quad= b_{(1)(\bar{0})(\bar{0})}\mathcal{Q}(a_{(1)} \otimes b_{(1)(\bar{0})(\bar{1})}) \otimes b_{(2)(\bar{0})}\mathcal{Q}(a_{(2)(\bar{0})} \otimes b_{(2)(\bar{1})})\mathcal{R}(a_{(2)(\bar{1})} \otimes b_{(1)(\bar{1})})\\
&\quad= b_{(1)(\bar{0})}\mathcal{Q}(a_{(1)} \otimes b_{(1)(\bar{1})(1)}) \otimes b_{(2)(\bar{0})}\mathcal{Q}(a_{(2)(\bar{0})} \otimes b_{(2)(\bar{1})})\mathcal{R}(a_{(2)(\bar{1})} \otimes b_{(1)(\bar{1})(2)})\\
&\quad= b_{(1)(\bar{0})} \otimes b_{(2)(\bar{0})}\mathcal{Q}(a \otimes b_{(1)(\bar{1})}b_{(2)(\bar{1})}) = \Delta[a, b],\end{aligned}$$

where we used the definitions and the coaction property to obtain the second equality and applied the 3rd quantum Killing form property to $a, b = b_{(1)(\bar{1})}, c = b_{(2)(\bar{1})}$ to obtain the third. At the end we recognised the tensor product Ad_R and commuted this past Δ. The counit axioms are immediate. Our approach also means that the braided Lie algebra is regular since the crossed-module braiding is invertible on account of the antipode (necessarily) being invertible.

(2) The associated quantum Lie algebra by Corollary 2.91 is built on the same vector space $\mathcal{L}$, has the same fundamental braiding $\sigma_{\mathfrak{h}} = \tilde{\Psi}$ and, using the bracket in part (1) and $\mathcal{Q}((\ ) \otimes 1) = \epsilon$, has the bracket $[a, b]_{\mathfrak{h}} = b_{(\bar{0})}\mathcal{Q}(a \otimes b_{(\bar{1})}) - \epsilon(a)b = b_{(\bar{0})}\mathcal{Q}(a \otimes \pi_\epsilon b_{(\bar{1})})$. This is exactly as in Example 2.87 for a generalised calculus defined by $\Lambda^1_R = \mathcal{L}^*_R = \mathcal{L}^*$ in the present context (we focus on the simplest case rather than $\Lambda^1_R = \mathrm{image}\varpi_R$ for a strict calculus). Finally, the quadratic extension $\Lambda^{\mathrm{quad}}_R$ is the quadratic algebra on Λ^1_R with the relations where we set $\ker(\mathrm{id}-\Psi) = 0$ for the left crossed module braiding on $\Lambda^1_R = \mathcal{L}^*$. But this braiding is dual in the usual tensor pairing sense of the right crossed module braiding on $\mathcal{L}$ (the one in Proposition 2.54) which we have seen is the same as $\tilde{\Psi}$ for the braided Lie algebra. Hence the relations of $U(\mathcal{L})$ are to impose relations as the image$(\mathrm{id} - \Psi^*)$. This makes it an example of a Koszul dual in sense explained above Example 2.66.

(3) Over $\mathbb{C}$, the flip coalgebra properties of $\underline{*}$ are immediate from the axioms of a Hopf algebra. If $\mathcal{R}$ is of real type then $\tilde{\Psi}$ as given in Proposition 2.54 tells us

$$\mathrm{flip}(\underline{*}\otimes\underline{*})\tilde{\Psi}(a\otimes b)=S^{-1}a^*{}_{(3)}\otimes S^{-1}b^*{}_{(3)}$$
$$\mathcal{R}(Sa^*{}_{(1)}\otimes b^*{}_{(2)})\mathcal{R}(a^*{}_{(2)}\otimes b^*{}_{(4)})\mathcal{R}(b^*{}_{(5)}\otimes a^*{}_{(4)})\mathcal{R}(b^*{}_{(1)}\otimes a^*{}_{(5)})$$

using that A is a Hopf $*$-algebra. Applying $\tilde{\Psi}$ to this and cancelling via the multiplicativity properties of $\mathcal{R}$ immediately gives $S^{-1}b^*\otimes S^{-1}a^* = b^{\underline{*}}\otimes a^{\underline{*}}$, as required. One can construct $\tilde{\Psi}^{-1}$ from $\mathcal{R}$, so we are in the regular case assumed. □

In the classical limit, the Koszul dual of a classical exterior algebra is the symmetric algebra, and indeed $U(\mathcal{L})$ in the classical limit will be a symmetric algebra. This means that in examples we will need to rescale generators to identify the classical Lie algebra and its enveloping algebra in the limit. Also, the proof of Theorem 2.94 refers marginally to a construction whereby if A is coquasitriangular then there is a braided Hopf algebra version $B(A)$ of A which is the same coalgebra but has a modified unital product

$$a\bullet b=a_{(2)}b_{(3)}\mathcal{R}(a_{(3)}\otimes Sb_{(1)})\mathcal{R}(a_{(1)}\otimes b_{(2)})=a_{(\bar{0})}b_{(2)}\mathcal{R}(a_{(\bar{1})}\otimes Sb_{(1)}). \tag{2.35}$$

This is called the *transmutation* of A and lives in $\mathcal{M}^A$ by Ad_R. It is also braided-commutative, which can be written as

$$a\bullet b=b_{(3)}\bullet a_{(3)}\mathcal{R}(Sb_{(2)}\otimes a_{(1)})\mathcal{R}(b_{(4)}\otimes a_{(2)})\mathcal{R}(a_{(4)}\otimes b_{(5)})\mathcal{R}(a_{(5)}\otimes Sb_{(1)}) \tag{2.36}$$

while its braided-antipode is

$$\underline{S}a=Sa_{(2)}\mathcal{R}((S^2a_{(3)})Sa_{(1)}\otimes a_{(4)})=(Sa_{(1)})_{(\bar{0})}\mathcal{R}((Sa_{(1)})_{(\bar{1})}\otimes a_{(2)}). \tag{2.37}$$

It is beyond our scope to derive all the details of this $B(A)$ construction and we refer the reader to a main text on quantum groups for the proofs of (2.35)–(2.37). In the real-coquasitriangular case over $\mathbb{C}$, we have a braided Hopf $\underline{*}$-algebra with $\underline{*}=*S$, where we underlined the braided one. A braided Hopf $*$-algebra is like a usual Hopf $*$-algebra but flip-commutes with the braided coproduct (there is an extra flip).

Proposition 2.95 *Let A be coquasitriangular and $\mathcal{L}\subseteq A$ a subcoalgebra. The bracket of the braided Lie algebra* $([\ ,\ ],\Delta,\epsilon)$ *is the restriction to $\mathcal{L}$ of the left braided-adjoint action of $B(A)$ on itself. Moreover, there is a homomorphism*

$$U(\mathcal{L})\to B(A)$$

of braided bialgebras, commuting with the inclusion of $\mathcal{L}$.

Proof We compute the left braided-adjoint action by applying $\underline{S}$ to $a_{(2)}$ and using the braiding to commute this past b before multiplying up with respect to $\bullet$:

$$\begin{aligned}
\underline{\mathrm{Ad}}_a(b) &= a_{(1)} \bullet b_{(\bar{0})} \bullet (\underline{S}a_{(2)})_{(\bar{0})}\mathcal{R}((\underline{S}a_{(2)})_{(\bar{1})} \otimes b_{(\bar{1})})\\
&= a_{(1)} \bullet \big(b_{(\bar{0})}(\underline{S}a_{(2)})_{(\bar{0})(2)}\big)\, \mathcal{R}((\underline{S}a_{(2)})_{(\bar{1})} \otimes b_{(\bar{1})(2)})\mathcal{R}(b_{(\bar{1})(1)} \otimes S(\underline{S}a_{(2)})_{(\bar{0})(1)})\\
&= a_{(1)(\bar{0})}b_{(\bar{0})(2)}(\underline{S}a_{(2)})_{(\bar{0})(3)}\mathcal{R}((\underline{S}a_{(2)})_{(\bar{1})} \otimes b_{(\bar{1})(2)})\mathcal{R}(b_{(\bar{1})(1)} \otimes S(\underline{S}a_{(2)})_{(\bar{0})(1)})\\
&\qquad\qquad \mathcal{R}(a_{(1)(\bar{1})} \otimes S(b_{(\bar{0})(1)}(\underline{S}a_{(2)})_{(\bar{0})(2)})\\
&= a_{(1)(\bar{0})}b_{(\bar{0})(2)}(Sa_{(2)})_{(\bar{0})(3)}\mathcal{R}((Sa_{(2)})_{(\bar{1})(2)} \otimes a_{(3)})\mathcal{R}((Sa_{(2)})_{(\bar{1})(1)} \otimes b_{(\bar{1})(2)})\\
&\quad \mathcal{R}(b_{(\bar{1})(1)} \otimes S(Sa_{(2)})_{(\bar{0})(1)})\mathcal{R}(a_{(1)(\bar{1})} \otimes S(b_{(\bar{0})(1)}(Sa_{(2)})_{(\bar{0})(2)}))\\
&= a_{(1)(\bar{0})}b_{(\bar{0})(2)}(Sa_{(2)})_{(\bar{0})(2)}\mathcal{R}((Sa_{(2)})_{(\bar{1})} \otimes a_{(3)}b_{(\bar{1})(2)})\\
&\quad \mathcal{R}(a_{(1)(\bar{1})(2)}b_{(\bar{1})(1)} \otimes S(Sa_{(2)})_{(\bar{0})(1)})\mathcal{R}(a_{(1)(\bar{1})(1)} \otimes Sb_{(\bar{0})(1)})
\end{aligned}$$

where we use the definitions, the coaction properties and the multiplicativity property of $\mathcal{R}$. We next unpack the adjoint coactions on a, and use multiplicativity of the last $\mathcal{R}$ to give

$$\begin{aligned}
&= a_{(3)}b_{(\bar{0})(3)}Sa_{(7)}\mathcal{R}(S^2a_{(9)}Sa_{(6)} \otimes a_{(10)}b_{(\bar{1})(2)})\mathcal{R}(Sa_{(1)}a_{(5)}b_{(\bar{1})(1)} \otimes S^2a_{(8)})\\
&\quad \mathcal{R}(a_{(2)} \otimes b_{(\bar{0})(2)})\mathcal{R}(a_{(4)} \otimes Sb_{(\bar{0})(1)})\\
&= b_{(\bar{0})(2)}a_{(2)}Sa_{(7)}\mathcal{R}(S^2a_{(9)}Sa_{(6)} \otimes a_{(10)}b_{(\bar{1})(2)})\mathcal{R}(Sa_{(1)}a_{(5)}b_{(\bar{1})(1)} \otimes S^2a_{(8)})\\
&\quad \mathcal{R}(a_{(3)} \otimes b_{(\bar{0})(3)})\mathcal{R}(a_{(4)} \otimes Sb_{(\bar{0})(1)})\\
&= b_{(\bar{0})(\bar{0})}a_{(2)}Sa_{(6)}\mathcal{R}(S^2a_{(8)}Sa_{(5)} \otimes a_{(9)}b_{(\bar{1})(2)})\mathcal{R}(Sa_{(1)}a_{(4)}b_{(\bar{1})(1)} \otimes S^2a_{(7)})\\
&\quad \mathcal{R}(a_{(3)} \otimes b_{(\bar{0})(\bar{1})})\\
&= b_{(\bar{0})}a_{(2)}Sa_{(6)}\mathcal{R}(S^2a_{(8)}Sa_{(5)} \otimes a_{(9)}b_{(\bar{1})(3)})\mathcal{R}(Sa_{(1)}a_{(4)}b_{(\bar{1})(2)} \otimes S^2a_{(7)})\\
&\quad \mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(1)})\\
&= b_{(\bar{0})}a_{(2)}Sa_{(7)}\mathcal{R}(S^2a_{(10)}Sa_{(6)} \otimes b_{(\bar{1})(3)})\nu^{-1}(a_{(11)})\mathcal{R}(Sa_{(5)} \otimes a_{(12)})\\
&\quad \mathcal{R}(Sa_{(1)}a_{(4)} \otimes S^2a_{(8)})\mathcal{R}(b_{(\bar{1})(2)} \otimes S^2a_{(9)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(1)})\\
&= b_{(\bar{0})}a_{(2)}Sa_{(7)}\mathcal{R}(S^2a_{(10)}Sa_{(6)} \otimes b_{(\bar{1})(3)})\mathcal{R}(Sa_{(5)} \otimes S^2a_{(11)})\nu^{-1}(a_{(12)})\\
&\quad \mathcal{R}(Sa_{(1)}a_{(4)} \otimes S^2a_{(8)})\mathcal{R}(b_{(\bar{1})(2)} \otimes S^2a_{(9)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(1)})\\
&= \mathcal{R}(a_{(1)} \otimes Sa_{(8)})b_{(\bar{0})}a_{(2)}Sa_{(7)}\mathcal{R}(Sa_{(5)} \otimes S^2a_{(12)})\nu^{-1}(a_{(13)})\\
&\quad \mathcal{R}(S^2a_{(11)}Sa_{(6)} \otimes b_{(\bar{1})(3)})\mathcal{R}(b_{(\bar{1})(2)} \otimes S^2a_{(10)})\mathcal{R}(a_{(4)} \otimes S^2a_{(9)})\mathcal{R}(a_{(3)} \otimes b_{(\bar{1})(1)})\\
&= \mathcal{R}(a_{(1)} \otimes Sa_{(8)})b_{(\bar{0})}a_{(2)}Sa_{(7)}\mathcal{R}(Sa_{(6)} \otimes S^2a_{(11)})\nu^{-1}(a_{(13)})\\
&\quad \mathcal{R}(Sa_{(5)}S^2a_{(12)} \otimes b_{(\bar{1})(3)})\mathcal{R}(b_{(\bar{1})(1)} \otimes S^2a_{(9)})\mathcal{R}(a_{(3)} \otimes S^2a_{(10)})\mathcal{R}(a_{(4)} \otimes b_{(\bar{1})(2)})
\end{aligned}$$

$$= \mathcal{R}(a_{(1)} \otimes Sa_{(6)}) b_{\bar{(0)}} a_{(2)} Sa_{(5)} \mathcal{R}(a_{(3)} \otimes S^2 a_{(8)}) \mathcal{R}(Sa_{(4)} \otimes S^2 a_{(9)}) \nu^{-1}(a_{(11)})$$
$$\mathcal{R}(S^2 a_{(10)} \otimes b_{\bar{(1)}(2)}) \mathcal{R}(b_{\bar{(1)}(1)} \otimes S^2 a_{(7)}) \mathcal{R}(a_{(3)} \otimes S^2 a_{(8)})$$
$$= \mathcal{R}(a_{(1)} \otimes Sa_{(4)}) b_{\bar{(0)}} a_{(2)} Sa_{(3)} \nu^{-1}(a_{(7)}) \mathcal{R}(S^2 a_{(6)} \otimes b_{\bar{(1)}(2)}) \mathcal{R}(b_{\bar{(1)}(1)} \otimes S^2 a_{(5)})$$
$$= \mathcal{R}(a_{(1)} \otimes Sa_{(2)}) b_{\bar{(0)}} \nu^{-1}(a_{(5)}) \mathcal{R}(S^2 a_{(4)} \otimes b_{\bar{(1)}(2)}) \mathcal{R}(b_{\bar{(1)}(1)} \otimes S^2 a_{(3)})$$
$$= b_{\bar{(0)}} \mathcal{R}(a_{(2)} \otimes b_{\bar{(1)}(2)}) \mathcal{R}(b_{\bar{(1)}(1)} \otimes a_{(1)}) = b_{\bar{(0)}} \mathcal{Q}(a \otimes b_{\bar{(1)}}) = [a, b],$$

where the 2nd equality is by quasicommutativity of A and the 3rd uses multiplicativity of $\mathcal{R}$ to recognise Ad_R on $b_{\bar{(0)}}$. We then expand by multiplicativity to recognise $\nu^{-1}(a) = \mathcal{R}(S^2 a_{(1)} \otimes a_{(2)})$, which is known to be convolution inverse to $\nu(a) = \mathcal{R}(a_{(1)} \otimes Sa_{(2)})$ and to obey $\nu^{-1}(a_{(1)}) a_{(2)} = S^2 a_{(1)} \nu^{-1}(a_{(2)})$, allowing us to move ν^{-1} to the right. The seventh equality uses multiplicativity of $\mathcal{R}$ so that we can use quasicommutativity on $S^2 a_{(11)} Sa_{(6)}$ and the braid relations on the last three factors to give the 8th equality. On this we use multiplicativity to cancel $a_{(4)} Sa_{(5)}$ and obtain the 9th equality and two mutually inverse copies of $\mathcal{R}$ for the 10th. We finally cancel $a_{(2)} Sa_{(3)}$ and move ν^{-1} to the left to cancel ν. We then recognise the answer.

Finally, comparing the braided-commutativity of $B(A)$ in (2.36) with the formula for $\tilde{\Psi}$ in Proposition 2.54, we see that the former is exactly $a \bullet b = \bullet \tilde{\Psi}(a \otimes b)$ or the relations of $U(\mathcal{L})$, giving our homomorphism as the identity on $\mathcal{L}$. $\square$

One can diagrammatically check that the braided-adjoint action always obeys the axiom (a) in Fig. 2.12 and by direct calculation in the case of $B(A)$ that it obeys axiom (b) (one says that $B(A)$ is 'Ad-cocommutative'). In this case one can show diagrammatically that axiom (c) also holds, so $B(A)$ itself can be regarded as a braided Lie algebra and this restricts to prove that any subcoalgebra $\mathcal{L} \subseteq A$ may be viewed as $\mathcal{L} \subseteq B(A)$ by Proposition 2.95. This is the abstract picture and an alternative line of proof behind Theorem 2.94, which in turn could be turned backwards to construct the bicovariant differential calculus in Proposition 2.55. The braided $\underline{*} = *S$ of $B(A)$ matches the requirement in Proposition 2.55 of $*S(\mathcal{L}) \subseteq \mathcal{L}$ to have a $*$-braided Lie algebra over $\mathbb{C}$. We also consider the braided Killing form for this example.

Proposition 2.96 *When the subcoalgebra $\mathcal{L} \subseteq A$ above is finite-dimensional, the braided-Killing form is*

$$K(a, b) = \sum_i u(e_{i\,\bar{(1)}(1)}) \mathcal{Q}(a, e_{i\,\bar{(1)}(2)}) \mathcal{Q}(b, e_{i\,\bar{(1)}(3)}) \langle f^i, e_{i\,\bar{(0)}} \rangle,$$

where $\{e_i\}$ is a basis of $\mathcal{L}$ and $\{f^i\}$ a dual basis and $u(a) = \mathcal{R}(a_{(2)} \otimes Sa_{(1)})$.

Proof The braided-Killing form is obtained by reading down the diagram, as (summation understood)

$$
\begin{aligned}
K(a,b) &= \mathcal{Q}(b, e_{i\,(1)})\mathcal{Q}(a, e_{i\,\bar{(0)}(1)})\langle f^i{}_{\bar{(0)}}, e_{i\,\bar{(0)}\bar{(0)}\bar{(0)}}\rangle \mathcal{R}(e_{i\,\bar{(0)}\bar{(0)}\bar{(1)}} \otimes f^i{}_{\bar{(1)}})\\
&= \mathcal{Q}(b, e_{i\,\bar{(1)}})\mathcal{Q}(a, e_{i\,\bar{(0)}\bar{(1)}})\mathcal{R}(e_{i\,\bar{(0)}\bar{(0)}\bar{(1)}} \otimes Se_{\bar{(0)}\bar{(0)}\bar{(0)}\bar{(1)}})\langle f^i, e_{i\,\bar{(0)}\bar{(0)}\bar{(0)}\bar{(0)}}\rangle\\
&= \mathcal{Q}(b, e_{i\,\bar{(1)}(4)})\mathcal{Q}(a, e_{i\,\bar{(1)}(3)})\mathcal{R}(e_{i\,\bar{(1)}(2)} \otimes Se_{\bar{(1)}(1)})\langle f^i, e_{i\,\bar{(0)}}\rangle. \qquad \square
\end{aligned}
$$

The braided Killing form can equally well be viewed as an element of $\Lambda^1_R \otimes \Lambda^1_R$ and hence a possibly noncentral metric for the corresponding $\Omega(A)$. Its braided-symmetry in Fig. 2.14b appears as quantum symmetry $\wedge(g) = 0$ in view of Theorem 2.94. We are also now well placed to see how the above construction works out in practice in the matrix case.

Corollary 2.97 *Let A be a coquasitriangular Hopf algebra and $t \in M_n(A)$ a matrix corepresentation with associated R-matrix. The braided Lie bracket structure constants are defined by $[t^i{}_j, t^k{}_l] = c^i{}_j{}^k{}_l{}^n{}_m t^m{}_n$ (summation understood).*

(1) *The matrix subcoalgebra $\mathcal{L} = \{t^i{}_j\}$ is a braided Lie algebra with*

$$
\begin{aligned}
c^i{}_j{}^k{}_l{}^n{}_m &= R^{-1k_1}{}_m{}^i{}_{i_1} R^n{}_{k_2}{}^{i_1}{}_{i_2} R^{i_2}{}_{i_3}{}^{k_2}{}_l \tilde{R}^{i_3}{}_j{}^k{}_{k_1},\\
\Psi(t^i{}_j \otimes t^k{}_l) &= t^{k_2}{}_{k_3} \otimes t^{i_2}{}_{i_3} R^i{}_{i_1}{}^{k_1}{}_{k_2} R^{-1i_1}{}_{i_2}{}^{k_3}{}_{k_4} R^{i_3}{}_{i_4}{}^{k_4}{}_l \tilde{R}^{i_4}{}_j{}^k{}_{k_1},\\
\tilde{\Psi}(t^i{}_j \otimes t^k{}_l) &= t^{k_2}{}_{k_3} \otimes t^{i_2}{}_{i_3} R^{-1k_1}{}_{k_2}{}^i{}_{i_1} R^{k_3}{}_{k_4}{}^{i_1}{}_{i_2} R^{i_3}{}_{i_4}{}^{k_4}{}_l \tilde{R}^{i_4}{}_j{}^k{}_{k_1},\\
K(t^i{}_j, t^k{}_l) &= c^i{}_j{}^n{}_m{}^{a_7}{}_{a_8}\, c^k{}_l{}^{a_3}{}_{a_4}{}^m{}_n\, R^{a_1}{}_{a_2}{}^{a_4}{}_{a_5} \nu^{a_2}{}_{a_3} \mathfrak{u}^{a_5}{}_{a_6} R^{a_6}{}_{a_7}{}^{a_8}{}_{a_1},
\end{aligned}
$$

where $\mathfrak{u}^i{}_j = \tilde{R}^a{}_j{}^i{}_a$, $\nu^i{}_j = \tilde{R}^i{}_a{}^a{}_j$. The braided enveloping algebra is $U(\mathcal{L}) = B(R)$, the braided matrix bialgebra *generated by the $\{t^i{}_j\}$ with new relations*

$$
R^k{}_{k_1}{}^i{}_{j_1} t^{j_1}{}_{j_2} \bullet R^{j_2}{}_j{}^{k_1}{}_{k_2} t^{k_2}{}_l = t^k{}_{k_1} R^{k_1}{}_{k_2}{}^i{}_{j_1} \bullet t^{j_1}{}_{j_2} R^{j_2}{}_j{}^{k_2}{}_l.
$$

Over $\mathbb{C}$, if A is of unitary type and R of real type then $U(\mathcal{L})$ is a $*$-algebra with hermitian $t^i{}_j{}^{\underline{*}} = t^j{}_i$.

(2) *The elements $x^i{}_j = t^i{}_j - \delta^i{}_j c$ in the unital extension span a left quantum Lie algebra $\tilde{\mathcal{L}}^+$ with σ the same form as $\tilde{\Psi}$ (but on the $x^i{}_j$), and*

$$
[x^i{}_j, x^k{}_l] = c^i{}_j{}^k{}_l{}^n{}_m x^m{}_n - \delta^i{}_j x^k{}_l.
$$

Proof We expand out $\mathcal{Q}$ using the properties of $\mathcal{R}$, then the above bracket can also be written explicitly as

$$
[a,b] = b_{(3)}\mathcal{R}(Sb_{(2)} \otimes a_{(1)})\mathcal{R}(b_{(4)} \otimes a_{(2)})\mathcal{R}(a_{(3)} \otimes b_{(5)})\mathcal{R}(a_{(4)} \otimes Sb_{(1)}) \qquad (2.38)
$$

and the categorical braiding in $\mathcal{M}^A$ is

$$\Psi(a\otimes b)=b_{(3)}\otimes a_{(3)}\mathcal{R}(a_{(1)}\otimes b_{(2)})\mathcal{R}(Sa_{(2)}\otimes b_{(4)})\mathcal{R}(a_{(4)}\otimes b_{(5)})\mathcal{R}(a_{(5)}\otimes Sb_{(1)}).$$

From these, we immediately read off the expressions stated, where we recall that $\tilde{R}^i{}_j{}^k{}_l=\mathcal{R}(t^i{}_j\otimes St^k{}_l)$ is the 'second inverse'. We likewise read off $\tilde{\Psi}$ from Proposition 2.54 to give the relations of $U(\mathcal{L})$ or (2.36) as stated. We moved $\tilde{R}$ and R^{-1} to the left-hand side, in the first case by the relations $\tilde{R}^i{}_j{}^k{}_l R^j{}_m{}^n{}_k=\delta^i_m\delta^n_l$. The bracket and braiding can also be written compactly as

$$R_{21}[t_1,Rt_2]=t_2R_{21}R,\quad R^{-1}\Psi(t_1\otimes Rt_2)=t_2R^{-1}\otimes t_1R,$$

where the suffixes refer to the position in $M_n\otimes M_n$. The $U(\mathcal{L})$ relations can similarly be written compactly in 'reflection form' $R_{21}t_1\bullet R_{12}t_2=t_2\bullet R_{21}t_1R$.

Over $\mathbb{C}$, if $\mathcal{R}$ is of real type for the unitary real form $t^i{}_j{}^*=St^j{}_i$ then the braided star is $t^i{}_j{}^{\underline{*}}=*St^i{}_j=S^{-1}(t^i{}_j{}^*)=t^j{}_i$ while $\overline{R^i{}_j{}^k{}_l}=R^l{}_k{}^j{}_i$. One can see that this extends to $U(\mathcal{L})$ with the quadratic relations shown, by applying $\underline{*}$ to both sides. Similarly, when we adjoin c corresponding to adding 1 to the coalgebra $\{t^i{}_j\}$, then $x^i{}_j$ form a basis of $\tilde{\mathcal{L}}^+$. Restricting the extended bracket to these gives the expression stated for the bracket. Finally, $(\mathrm{id}\otimes\pi_\epsilon)\tilde{\Psi}(x^i{}_j\otimes x^k{}_l)$ gives σ in the same form as $\tilde{\Psi}$ but with $t^i{}_j$ replaced by $x^i{}_j$. For the Killing form we compute the formula in Proposition 2.96 in our case. Indeed, letting $\{E_m{}^n\}$ be the dual basis, we have

$$\begin{aligned}K(a,b)&=u((St^m{}_pt^q{}_n)_{(1)})\mathcal{Q}(a\otimes(St^m{}_pt^q{}_n)_{(2)})\mathcal{Q}(b\otimes(St^m{}_pt^q{}_n)_{(3)})\langle E_m{}^n,t^p{}_q\rangle\\&=u(St^{m_3}{}_{m_1}t^{n_1}{}_{n_2})\mathcal{Q}(a\otimes St^{m_2}{}_{m_3}t^{n_2}{}_{n_3})\mathcal{Q}(b\otimes St^{m_1}{}_{m_2}t^{n_3}{}_{n_1}).\end{aligned}$$

Here $\mathcal{Q}(t^i{}_j\otimes St^k{}_mt^n{}_l)=c^i{}_j{}^k{}_l{}^n{}_m$ are structure constants of the braided Lie bracket as already computed, while the map u in Proposition 2.96 obeys

$$u(ab)=\mathcal{Q}^{-1}(a_{(1)}\otimes b_{(1)})u(a_{(2)})u(b_{(2)})=u(a_{(1)})u(b_{(2)})\mathcal{R}(a_{(2)}\otimes Sb_{(1)})\mathcal{R}(Sb_{(3)}\otimes a_{(3)})$$

for any coquasitriangular Hopf algebra by the properties of $\mathcal{R}$. We deduce from this that $u((Sa)b)=\mathcal{R}(a_{(2)}\otimes b_{(1)})\nu(a_{(3)})u(b_{(2)})\mathcal{R}(b_{(3)}\otimes a_{(1)})$, where $\nu(a)=u(Sa)$ as in Proposition 2.95. We now read this on the generators to obtain the remaining part of the formula for K, with $\mathrm{u}^i{}_j=u(t^i{}_j)$ and $\nu^i{}_j=\nu(t^i{}_j)$. □

The $\{t^i{}_j\}$ are normally denoted $\{u^i{}_j\}$ when used to generate $B(R)=U(\mathcal{L})$ with the above $\bullet$ product; the latter is then denoted by omission and the braided star operation, when applicable, by $*$ since there is then no confusion with $*$ on the quantum group. If $A=\mathbb{C}_q[SL_n]$ then $B(A)=B_q[SL_n]$ is the quotient of the braided-matrix bialgebra $B(R)=B_q[M_n]$ by setting a certain braided q-determinant to 1. We also have a localisation $B_q[GL_n]$, where we invert the braided q-determinant. So from our current point of view, we can both quotient and localise the above $U(\mathcal{L})$ to a braided Hopf algebra. It is also possible to view $\mathcal{L}$ as an object of a different braided category such that the same algebra $U(\mathcal{L})$ has an additive coproduct on $\mathcal{L}$, typically making it of the form $B_+(\mathcal{L})$.

Example 2.98 For $A = \mathbb{C}_q[SU_2]$ with its standard matrix coalgebra we first compute the fundamental braiding $\tilde{\Psi}$ from Corollary 2.97 using the same R-matrix as in Example 2.77. $\mathcal{L}$ as a subcoalgebra has the usual matrix of generators a, b, c, d but we now denote them by $\alpha, \beta, \gamma, \delta$ respectively since they now generate a different algebra, namely $U(\mathcal{L})$. Then

$$\tilde{\Psi}(\alpha \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = \begin{pmatrix} \alpha & q^{-2}\beta \\ q^2\gamma & \delta \end{pmatrix} \otimes \alpha,$$

$$\tilde{\Psi}(\beta \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \beta + \lambda \begin{pmatrix} \beta & 0 \\ \delta - \alpha & -q^{-2}\beta \end{pmatrix} \otimes \alpha,$$

$$\tilde{\Psi}(\gamma \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \gamma + \lambda \begin{pmatrix} -q^2\gamma & \alpha - \delta \\ 0 & \gamma \end{pmatrix} \otimes \alpha,$$

$$\tilde{\Psi}(\delta \otimes \begin{pmatrix} \beta \\ \gamma \end{pmatrix}) = \begin{pmatrix} q^2\beta \\ q^{-2}\gamma \end{pmatrix} \otimes \delta + \lambda(\delta - \alpha) \otimes \begin{pmatrix} -q^2\beta \\ \gamma \end{pmatrix} + q(2)_q\lambda^2 \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \otimes \alpha,$$

$$\tilde{\Psi}(\delta \otimes (\delta - \alpha)) = (\delta - \alpha) \otimes (\delta + \lambda^2(1+q^2)\alpha) + \lambda(1+q^2)(\gamma \otimes \beta - \beta \otimes \gamma)$$

and $\tilde{\Psi}((\ ) \otimes t) = t \otimes (\ )$ on the generators, where $\lambda = 1 - q^{-2}$ and where t and some rescaled generators that we will need are

$$t = q^{-1}\alpha + q\delta, \quad z = \lambda^{-1}(\delta - \alpha), \quad x_+ = \lambda^{-1}\beta, \quad x_- = \lambda^{-1}\gamma$$

(we use Greek symbols for the entries of ${t^i}_j$ to avoid confusion with the quantum group generators). From $\tilde{\Psi}$ we can apply $\mathrm{id} \otimes \epsilon$ to obtain that the nonzero braided Lie brackets are

$$[z, z] = q(2)_q\lambda z, \quad [t, t] = (2)_q t, \quad [t, \ ] = (q^3 + q^{-3})\mathrm{id},$$
$$[x_+, x_-] = z = -[x_-, x_+], \quad [z, x_\pm] = \pm q^{\pm 1}(2)_q x_\pm = -q^{\pm 2}[x_\pm, z].$$

The coalgebra is the matrix coalgebra but in terms of the new generators we have

$$\Delta x_\pm = (2)_q^{-1}\big(x_\pm \otimes (t \pm q^{\mp 1}\lambda z) + (t \mp q^{\pm 1}\lambda z) \otimes x_\pm\big),$$
$$\Delta z = (2)_q^{-1}\big(z \otimes (t + q^{-1}\lambda z) + (t - q\lambda z) \otimes z\big) + \lambda(x_- \otimes x_+ - x_+ \otimes x_-),$$
$$\Delta t = (2)_q^{-1}\big(t \otimes t + \lambda^2 z \otimes z) + \lambda^2(q^{-1}x_+ \otimes x_- + qx_- \otimes x_+\big),$$

so that $x_\pm, z$ are approximately primitive and $t/(2)_q$ approximately grouplike. The background braiding Ψ and braided Killing form can similarly be computed from

R; the latter coming out as an overall constant times the nonzero values

$$K(z,z)=q(2)_q,\ K(t,t)=q^3(4)_q-q^{-1}(2)_q\lambda,\ K(x_+,x_-)=1=q^{-2}K(x_-,x_+)$$

and is nondegenerate for $q\neq 1$. Also from the fundamental braiding, we find the enveloping algebra $U(\mathcal{L})=B_q[M_2]$ as generated by $\alpha,\beta,\gamma,\delta$ with relations

$$\beta\alpha=q^2\alpha\beta,\quad \gamma\alpha=q^{-2}\alpha\gamma,\quad \delta\alpha=\alpha\delta,$$
$$[\beta,\gamma]=\lambda\alpha(\delta-\alpha),\quad [\gamma,\delta]=\lambda\gamma\alpha,\quad [\delta,\beta]=\lambda\alpha\beta,$$

where [,] at this point denotes the usual commutator. Over $\mathbb{C}$ and with q real, we have the algebra of q-deformed 2×2 braided hermitian matrices in view of the $*$-structure $\alpha^*=\alpha,\beta^*=\gamma,\delta^*=\delta$, which means that geometrically it should be thought of as q-Minkowski space. There are two natural central elements, the braided determinant $\underline{\det}_q=\alpha\delta-q^2\gamma\beta$, which should be thought of as the q-Lorentzian distance from the origin, and the q-trace $t=q^{-1}\alpha+q\delta$, which should be thought of as the 'time' direction. In these variables (as opposed to the rescaled 'Lie algebra' variables), the classical limit is commutative, in keeping with thinking of this as a noncommutative geometry.

Finally, the associated left quantum Lie algebra by Corollary 2.91 has the same $z,x_\pm,t$ and a modified bracket for $[t,\ ]_{\mathfrak{h}}=[t,\ ]-(2)_q\mathrm{id}$, so

$$[t,t]_{\mathfrak{h}}=0,\quad [t,\ ]_{\mathfrak{h}}=q^2(2)_q\lambda^2\mathrm{id}$$

on $z,x_\pm$ (the others are the same as the braided Lie bracket as $\epsilon(z)=\epsilon(x_\pm)=0$, so in particular $[\ ,t]_{\mathfrak{h}}=0$ as for the braided bracket). The fundamental braiding $\sigma_{\mathfrak{h}}=\tilde{\Psi}$ for the new set of generators can be computed from previous values. Suffice it to say that $\tilde{\Psi}(\mathrm{id}\otimes t)=t\otimes\mathrm{id}$ on the other generators $z,x_\pm$ but $\tilde{\Psi}(t\otimes\mathrm{id})=\mathrm{id}\otimes t+O(\lambda)$ is braidings among the $z,x_\pm$ similarly q-deform the braiding of $\Bbbk c\oplus sl_2$ in Example 2.92 with the role of c now played by $t/(2)_q$. For example,

$$\sigma_{\mathfrak{h}}(x_\pm\otimes z)=\tilde{\Psi}(x_\pm\otimes z)=z\otimes x_\pm+[x_\pm,z]_{\mathfrak{h}}\otimes\frac{t}{(2)_q}\pm q^{\mp 1}q\lambda x_\pm\otimes z.$$

The quantum enveloping algebra $U(\mathfrak{h})$ is necessarily isomorphic to $U(\mathcal{L})$ by $t\mapsto t-(2)_q1$. This example is dual to the 4D differential calculus in Example 2.77 in a version defined in terms of right-invariant differential forms. ◇

This emergence of q-Minkowski space geometry as a quantum/braided enveloping algebra is in keeping with examples of noncommutative geometry like fuzzy $\mathbb{R}^3$ in Chap. 1 as enveloping algebras of Lie algebras. We also have a ($*$-algebra) map

$$U(\mathcal{L})\to U_q(su_2),\quad \begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\mapsto\begin{pmatrix}q^H & q^{\frac12}\lambda q^{\frac{H}{2}}X_-\\ q^{\frac12}\lambda X_+q^{\frac{H}{2}} & q^{-H}+q\lambda^2X_+X_-\end{pmatrix}$$

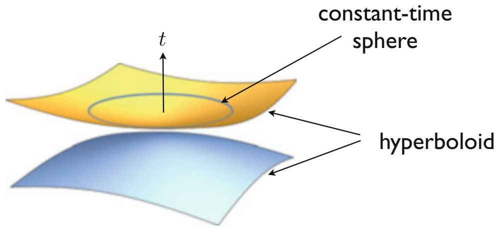


Fig. 2.17 q-Minkowski space is the braided-enveloping algebra of the 4D braided Lie algebra of $\mathbb{C}_q[SU_2]$. Setting the q-determinant to 1 gives the q-hyperboloid and setting the time or q-trace to a constant gives a family $\mathbb{C}_{q,\tau}[S^2]$ of q-spheres

again broadly q-deforming $U(\mathbb{C} \oplus su_2) \to U(su_2)$ in Example 2.92. And in $B_q[SU_2]$ as the unit q-hyperboloid in q-Minkowski space, setting the q-trace to a constant (see Fig. 2.17) gives us a 1-parameter family of q-spheres

$$\mathbb{C}_{q,\tau}[S^2] := B_q[SU_2]/\langle t - (\tau + \tau^{-1})\rangle$$

according to the time value $\tau + \tau^{-1}$ given by a real parameter. Changing variables, we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{q\tau}\begin{pmatrix} q^2 x & z \\ z^* & \tau^2 + 1 - x \end{pmatrix},$$

$$zx = q^2 xz, \quad zz^* = (q^2 x - \tau^2)(1 - q^2 x), \quad z^* z = (x - \tau^2)(1 - x).$$

These are the nonstandard q-spheres (also known as Podleś spheres) generalising the q-sphere studied in Proposition 2.33 onwards, except that usually one takes $s = \mathrm{i}\tau$ as the parameter. By construction, everything is in the category of right $\mathbb{C}_q[SU_2]$-comodules coacting by conjugation, i.e., q-rotationally invariant. The q-Minkowski space and q-hyperboloid are moreover q-Lorentz invariant.

2.8 Bar Categories

Consider $1 \otimes \mathrm{i} = \mathrm{i} \otimes 1 \in \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and observe that applying complex conjugation to the first factor would give $1 \otimes \mathrm{i} = -\mathrm{i} \otimes 1$, a contradiction. This illustrates that a certain degree of care is required around tensor products and complex conjugation, increasingly as we work with more advanced constructions on $*$-algebras and $*$-DGAs. We address this using the formal language of 'bar categories', in the first instance as a useful book-keeping device to keep track of conjugate linearity. For example conjugate linear maps such as $*$ simply become linear from an object to its conjugate, which is easier to think about. We also consider more exotic objects, for which the bar category framework is essential.

$$\begin{tikzcd}
\overline{(X\otimes Y)\otimes Z} \arrow[r, "\Upsilon_{X\otimes Y,Z}"] \arrow[d, "\overline{\Phi_{X,Y,Z}}"] & \overline{Z}\otimes\overline{X\otimes Y} \arrow[r, "\mathrm{id}\otimes\Upsilon_{X,Y}"] & \overline{Z}\otimes(\overline{Y}\otimes\overline{X}) \\
\overline{X\otimes(Y\otimes Z)} \arrow[r, "\Upsilon_{X,Y\otimes Z}"] & \overline{Y\otimes Z}\otimes\overline{X} \arrow[r, "\Upsilon_{Y,Z}\otimes\mathrm{id}"] & (\overline{Z}\otimes\overline{Y})\otimes\overline{X} \arrow[u, "\Phi_{\overline{Z},\overline{Y},\overline{X}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{X} \arrow[r, "r_{\bar{X}}"] \arrow[d, "\overline{l_X}"] & \underline{1}\otimes\overline{X} \\
\overline{X\otimes\underline{1}} \arrow[r, "\Upsilon_{X,\underline{1}}"] & \overline{\underline{1}}\otimes\overline{X} \arrow[u, "\star^{-1}\otimes\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{X} \arrow[r, "l_{\bar{X}}"] \arrow[d, "\overline{r_X}"] & \overline{X}\otimes\underline{1} \\
\overline{\underline{1}\otimes X} \arrow[r, "\Upsilon_{\underline{1},X}"] & \overline{X}\otimes\overline{\underline{1}} \arrow[u, "\mathrm{id}\otimes\star^{-1}"']
\end{tikzcd}$$

Fig. 2.18 Commuting diagrams in the definition of a bar category

A bar category is a monoidal category with extra operations. For every object X we have a conjugate object $\overline{X}$, and for every morphism $\theta : X \to Y$ we have a conjugate morphism $\overline{\theta} : \overline{X} \to \overline{Y}$ making the conjugation or bar operation into a functor. In addition we require an identification bb of the double conjugate with the original object, and an order reversing identification Υ (capital Upsilon) of the conjugate of a tensor product with the tensor product of the conjugates.

Definition 2.99 A bar category is a monoidal category $(\mathcal{C}, \otimes, \underline{1}, l, r, \Phi)$ and:

(1) A functor bar : $\mathcal{C} \to \mathcal{C}$, denoted $X \mapsto \overline{X}$ on $X \in \mathcal{C}$ and $\phi \mapsto \bar{\phi}$ on morphisms;
(2) A natural equivalence Υ (capital Upsilon): bar $\circ \otimes \Rightarrow \otimes^{\mathrm{op}} \circ$ (bar $\times$ bar), i.e., a functorial collection of isomorphisms

$$\Upsilon_{X,Y} : \overline{X\otimes Y} \cong \overline{Y}\otimes\overline{X}$$

for all $X, Y \in \mathcal{C}$, obeying the upper diagram in Fig. 2.18;
(3) A natural equivalence bb : id $\Rightarrow$ bar $\circ$ bar, i.e., a functorial collection of isomorphisms $\mathrm{bb}_X : X \cong \overline{\overline{X}}$ such that $\overline{\mathrm{bb}_X} = \mathrm{bb}_{\overline{X}} : \overline{X} \mapsto \overline{\overline{\overline{X}}}$ for all $X \in \mathcal{C}$;
(4) An invertible morphism $\star : \underline{1} \to \overline{\underline{1}}$ such that $\overline{\star}\star = \mathrm{bb}_{\underline{1}}$ and obeying the lower conditions in Fig. 2.18.

Here $\otimes^{\mathrm{op}} = \otimes\circ$flip, where the functor flip : $\mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$ is given by $(X, Y) \mapsto (Y, X)$. Functoriality of the natural equivalence Υ says that for all morphisms $\theta : X \to V$ and $\phi : Y \to W$,

$$\Upsilon_{V,W}(\overline{\theta\otimes\phi}) = (\overline{\phi}\otimes\overline{\theta})\Upsilon_{X,Y} : \overline{X\otimes Y} \to \overline{W}\otimes\overline{V}.$$

Similarly, functoriality of bb is $\mathrm{bb}_Y \circ \theta = \overline{\overline{\theta}} \circ \mathrm{bb}_X$ for all morphisms $\theta : X \to Y$. We start with a handful of examples to clarify the notions, where the bar category is not essential but provides book-keeping.

Example 2.100 (Overline Notation) The category of complex vector spaces and linear maps, with the usual tensor product is a bar category. For a vector space V, we take its conjugate $\overline{V}$ to be the same as V as a set, writing its elements as $\overline{v} \in \overline{V}$ for $v \in V$. Addition is the usual $\overline{v}+\overline{u} = \overline{v+u}$, but complex multiplication is altered by conjugation, $\lambda\overline{v} = \overline{\lambda^* v}$ for $\lambda \in \mathbb{C}$ (we shall often use $*$ for conjugate for numbers). A linear map $T : V \to W$ gives another linear map $\overline{T} : \overline{V} \to \overline{W}$ by $\overline{T}(\overline{v}) = \overline{T(v)}$. The map $\mathrm{bb}_V : V \to \overline{\overline{V}}$ is given by $\mathrm{bb}_V(v) = \overline{\overline{v}}$, and $\Upsilon_{V,W} : \overline{V \otimes W} \to \overline{W} \otimes \overline{V}$ given by $\Upsilon_{V,W}(\overline{v \otimes w}) = \overline{w} \otimes \overline{v}$ for $v \in V$ and $w \in W$. ◇

If we have a conjugate linear map, such as a $*$-operation for a $*$-algebra A with product μ, we can express it as a linear map $\star : A \to \overline{A}$ in this bar category, given by $a \mapsto \overline{a^*}$. That we have a $*$-algebra now appears as the statement that

$$\begin{tikzcd} A \otimes A \arrow[r, "\star\otimes\star"] \arrow[d, "\mu"] & \overline{A} \otimes \overline{A} \arrow[r, "\Upsilon^{-1}"] & \overline{A \otimes A} \arrow[dl, "\overline{\mu}"] \\ A \arrow[r, "\star"] & \overline{A} & \end{tikzcd} \tag{2.39}$$

commutes in the category.

Example 2.101 For a $*$-algebra A, the category ${}_A\mathcal{M}_A$ of A bimodules and bimodule maps and tensor product $\otimes_A$ is a bar category. Building on the previous example for the bar category of complex vector spaces and given an A-bimodule E, we define the left and right actions of $a \in A$ on the conjugate $\overline{E}$ by

$$a.\overline{e} = \overline{ea^*}, \quad \overline{e}.a = \overline{a^* e}.$$

On morphisms, we define $\bar{\theta}$ for a bimodule map $\theta : E \to F$ by $\overline{\theta}(\overline{e}) = \overline{\theta(e)}$ and check that is also a bimodule map,

$$\overline{\theta}(a.\overline{e}) = \overline{\theta}(\overline{e.a^*}) = \overline{\theta(e.a^*)} = \overline{\theta(e).a^*} = a.\overline{\theta(e)}$$

and similarly on the other side. The unit object is $\underline{1} = A$ with $\star$ the $*$-involution as in (2.39) while Υ and bb are inherited from the bar category of vector spaces in Example 2.100. One can check that Υ descends to bimodule maps $\Upsilon_{E,F}$: $\overline{E \otimes_A F} \to \overline{F} \otimes_A \overline{E}$ for bimodules E, F. Here

$$\Upsilon_{E,F}(a.\overline{e \otimes f}) = \Upsilon_{E,F}(\overline{e \otimes f.a^*}) = \overline{f.a^*} \otimes \overline{e} = a.\overline{f} \otimes \overline{e} = a.\Upsilon_{E,F}(\overline{e \otimes f})$$

for all $e \in E$, $f \in F$ and $a \in A$, and similarly for the other side. The remaining facts are left to the reader. ◇

The unit object of a bar category and the $*$-involution on an object on the bar category of vector spaces are all examples of a general concept.

Definition 2.102 An object X in a bar category is called a star object if equipped with a morphism $\star : X \to \overline{X}$ satisfying $\overline{\star} \circ \star = \mathrm{bb}_X$.

For a $*$-differential calculus on A as in Definition 1.4, $*$ on 1-forms can be viewed more formally as $\star : \Omega^1 \to \overline{\Omega^1}$ making Ω^1 a star object in the bimodule bar category, such that

$$\begin{tikzcd} \Omega^1 \arrow[r, "\star"] & \overline{\Omega^1} \\ A \arrow[u, "\mathrm{d}"] \arrow[r, "\star"] & \overline{A} \arrow[u, "\overline{\mathrm{d}}"'] \end{tikzcd}$$

commutes in the vector space bar category. The bimodule morphism property can also be phrased in the vector space bar category where it appears as (2.39) with one of the copies of A throughout the top line replaced by Ω^1. Note that d is not a module map and thus not a morphism in the bimodule bar category, but d and $\overline{\mathrm{d}}$ are defined in the underlying vector space bar category. In differential geometry we frequently have to use module maps ('tensorial operators') together with differentials.

Example 2.103 Let H be a Hopf $*$-algebra. The category $\mathcal{M}^H$ of right H-comodules as in Example 2.40 is a bar category. We again define $\star$, bb, Υ as for the bar category of vector spaces in Example 2.100. The coaction on $\overline{V}$ is given by

$$\Delta_{\overline{V}}(\overline{v}) = \overline{v_{\bar{(0)}}} \otimes v_{\bar{(1)}}{}^*$$

and we check that bb and Υ are right comodule maps,

$$\begin{aligned} \Delta_{\overline{\overline{V}}}\mathrm{bb}(v) = \Delta_{\overline{\overline{V}}}(\overline{\overline{v}}) &= \overline{(\overline{v})_{\bar{(0)}}} \otimes (\overline{v})_{\bar{(1)}}{}^* = \overline{\overline{v_{\bar{(0)}}}} \otimes v_{\bar{(1)}}{}^{**} = \overline{\overline{v_{\bar{(0)}}}} \otimes v_{\bar{(1)}} \\ &= (\mathrm{bb} \otimes \mathrm{id})\Delta_V v, \\ (\Upsilon \otimes \mathrm{id})\Delta_{\overline{W\otimes V}}(\overline{w \otimes v}) &= \Upsilon(\overline{(w \otimes v)_{\bar{(0)}}}) \otimes ((w \otimes v)_{\bar{(1)}})^* \\ &= \Upsilon(\overline{w_{\bar{(0)}} \otimes v_{\bar{(0)}}}) \otimes (w_{\bar{(1)}} v_{\bar{(1)}})^* \\ &= \overline{v_{\bar{(0)}}} \otimes \overline{w_{\bar{(1)}}} \otimes v_{\bar{(1)}}{}^* w_{\bar{(1)}}{}^* = \Delta_{\overline{V}\otimes\overline{W}}(\overline{v} \otimes \overline{w}) \\ &= \Delta_{\overline{V}\otimes\overline{W}} \Upsilon(\overline{w \otimes v}). \end{aligned}$$

Similarly, the categories of left comodules ${}^H\mathcal{M}$, bicomodules ${}^H\mathcal{M}^H$ and, combining with Example 2.101, Hopf bimodules as in Lemma 2.23 are all bar categories. ◇

So for a left/right/bicovariant calculus on a Hopf $*$-algebra our bar categorical view of $\star : \Omega^1 \to \overline{\Omega^1}$ is required now to hold in the appropriate one of these H-covariant bar categories. Thus, if we characterise $\star$ for a $*$-calculus by commuting diagrams in the vector space bar category then in the bicovariant theory we require these in fact in ${}^H\mathcal{M}^H$. From the bimodule bar category point of view, H itself is a Hopf bimodule by left and right (co)multiplication and provides the unit object.

Next, if the bar category $\mathcal{C}$ is also a braided category, we ask how the braiding Ψ is related to the bar structure.

Definition 2.104 If a bar category $\mathcal{C}$ is braided, we say that the braiding, or the braided category, is *real (resp. antireal)* if

$$\begin{tikzcd} \overline{X} \otimes \overline{Y} \arrow[r, "\Psi"] \arrow[d, "\Upsilon^{-1}"'] & \overline{Y} \otimes \overline{X} \arrow[d, "\Upsilon^{-1}"] \\ \overline{Y \otimes X} \arrow[r, "\overline{\Psi^{\pm 1}}"] & \overline{X \otimes Y} \end{tikzcd}$$

holds for all $X, Y \in \mathcal{C}$, with Ψ (resp. Ψ^{-1}) for the two cases.

Example 2.105 For a Hopf $*$-algebra H, the category $\mathcal{M}_H^H$ of H-crossed modules in Example 2.46 is an antireal braided bar category. We build on the bar category of vector spaces and in addition, if $V \in \mathcal{M}_H^H$ then $\overline{V}$ consists of elements $\overline{v}$ for $v \in V$ and has coaction and action

$$\Delta_R(\overline{v}) = \overline{v_{\bar{(0)}}} \otimes v_{\bar{(1)}}{}^*, \quad \overline{v}\triangleleft h = \overline{v \triangleleft S^{-1}(h^*)}.$$

The braiding $\Psi_{V,W} : V \otimes W \to W \otimes V$ given by $\Psi(v \otimes w) = w_{\bar{(0)}} \otimes v \triangleleft w_{\bar{(1)}}$ obeys

$$\begin{aligned} \Upsilon^{-1}\Psi(\overline{v} \otimes \overline{w}) &= \Upsilon^{-1}(\overline{w}_{\bar{(0)}} \otimes \overline{v}\triangleleft \overline{w}_{\bar{(1)}}) = \Upsilon^{-1}(\overline{w_{\bar{(0)}}} \otimes \overline{v}\triangleleft w^*_{\bar{(1)}}) \\ &= \Upsilon^{-1}(\overline{w_{\bar{(0)}}} \otimes \overline{v \triangleleft S^{-1}(w_{\bar{(1)}})}) = \overline{v \triangleleft S^{-1}(w_{\bar{(1)}}) \otimes w_{\bar{(0)}}} \\ &= \overline{\Psi^{-1}(w \otimes v)} = \overline{\Psi^{-1}}\Upsilon^{-1}(\overline{v} \otimes \overline{w}). \end{aligned}$$ ◇

Example 2.106 For H a coquasitriangular Hopf $*$-algebra, the braiding Ψ of $\mathcal{M}^H$ in Proposition 2.52 obeys

$$\begin{aligned} \Upsilon^{-1}\Psi\Upsilon(\overline{v \otimes w}) = \ & \Upsilon^{-1}\Psi(\overline{w} \otimes \overline{v}) = \Upsilon^{-1}(\overline{v_{\bar{(0)}}} \otimes \overline{w_{\bar{(0)}}}\, \mathcal{R}(w^*_{\bar{(1)}} \otimes v^*_{\bar{(1)}})) \\ = \ & \overline{w_{\bar{(0)}} \otimes v_{\bar{(0)}}\, \mathcal{R}(w^*_{\bar{(1)}} \otimes v^*_{\bar{(1)}})^*}. \end{aligned}$$

Hence if the coquasitriangular structure is real (see (2.15)) then

$$\Upsilon^{-1}\Psi\Upsilon(\overline{v \otimes w}) = \overline{w_{\bar{(0)}} \otimes v_{\bar{(0)}}\, \mathcal{R}(v^*_{\bar{(1)}} \otimes w^*_{\bar{(1)}})} = \overline{\Psi(v \otimes w)},$$

and the braiding is real. The converse also applies. Similarly, the braiding is antireal if and only if the coquasitriangular structure is. ◇

As a somewhat detailed illustration of the use of bar categories, we now show how one may use them to formulate the notion of real form on a left braided Lie algebra $\mathcal{L}$ as in Definition 2.88, with axioms for the coproduct and the left bracket $[\ ,\ ] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ given diagrammatically in Fig. 2.12. The axioms for a right braided Lie algebra are obtained by reflecting each of these diagrams about a vertical axis and reversing back the braid crossings so that Ψ remains Ψ.

Proposition 2.107 *Let $\mathcal{L}$ be a left braided Lie algebra in a real braided bar category $\mathcal{C}$. If there is an invertible morphism $\star : \mathcal{L} \to \overline{\mathcal{L}}$ making $\mathcal{L}$ into a star object such that*

$$\Upsilon^{-1}(\star \otimes \star)\Delta = \overline{\Delta}\star : \mathcal{L} \to \overline{\mathcal{L} \otimes \mathcal{L}}, \quad \star\epsilon = \overline{\epsilon}\star$$

then the bracket $[\ ,\ ]_R = \star^{-1}\overline{[\ ,\ ]}\,\Upsilon^{-1}(\star \otimes \star) : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ makes $\mathcal{L}$ into a right braided Lie algebra.

Proof Axiom (a) in Fig. 2.12 for a left braided Lie algebra has right version

$$[\,,]_R([\,,]_R \otimes [\,,]_R)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta) = [\,,]_R([\,,]_R \otimes \mathrm{id})$$

as morphisms $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$. We prove this in the form

$$\begin{aligned}&\star[\,,]_R([\,,]_R \otimes [\,,]_R)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\star^{-1} \otimes \star^{-1} \otimes \star^{-1})\\ &\quad = \star[\,,]_R([\,,]_R \otimes \mathrm{id})(\star^{-1} \otimes \star^{-1} \otimes \star^{-1})\end{aligned}$$

as maps $\overline{\mathcal{L}} \otimes \overline{\mathcal{L}} \otimes \overline{\mathcal{L}} \to \overline{\mathcal{L}}$. Using $[\ ,\ ]_R$ as defined in the statement, the left-hand side of our condition becomes

$$\begin{aligned}&\overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star \otimes \star)([\,,]_R \otimes [\,,]_R)(\mathrm{id} \otimes \Psi_{\mathcal{L},\mathcal{L}} \otimes \mathrm{id})(\star^{-1} \otimes \star^{-1} \otimes (\star^{-1} \otimes \star^{-1})\Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta})\\ &= \overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star \otimes \star) \otimes \overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star \otimes \star))(\mathrm{id} \otimes \Psi_{\mathcal{L},\mathcal{L}} \otimes \mathrm{id})\\ &\qquad (\star^{-1} \otimes \star^{-1} \otimes (\star^{-1} \otimes \star^{-1})\Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta})\\ &= \overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}} \otimes \overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})(\mathrm{id} \otimes (\star \otimes \star)\Psi_{\mathcal{L},\mathcal{L}}(\star^{-1} \otimes \star^{-1}) \otimes \mathrm{id})\\ &\qquad (\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta}),\end{aligned}$$

and since $\star$ is a morphism, and using naturality of Υ and Ψ, this is

$$\begin{aligned}&\overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}} \otimes \overline{[\,,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})(\mathrm{id} \otimes \Psi_{\overline{\mathcal{L}},\overline{\mathcal{L}}} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta})\\ &= \overline{[\,,]([\,,] \otimes [\,,])}\,\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L},\mathcal{L}} \otimes \Upsilon^{-1}_{\mathcal{L},\mathcal{L}})(\mathrm{id} \otimes \Psi_{\overline{\mathcal{L}},\overline{\mathcal{L}}} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta})\\ &= \overline{[\,,]([\,,] \otimes [\,,])}\,\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L},\mathcal{L}} \otimes \Upsilon^{-1}_{\mathcal{L},\mathcal{L}})(\mathrm{id} \otimes \Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Psi_{\mathcal{L},\mathcal{L}}}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}} \otimes \mathrm{id})\\ &\qquad (\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon_{\mathcal{L},\mathcal{L}}\,\overline{\Delta}),\end{aligned}$$

where we used the reality of the braiding to get the last result. We now use the properties of Υ from Definition 2.99 (b) (with the associator suppressed) to give

$$\begin{aligned}
&\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})(\mathrm{id}\otimes\Upsilon_{\mathcal{L},\mathcal{L}}\overline{\Psi_{\mathcal{L},\mathcal{L}}}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon_{\mathcal{L},\mathcal{L}})\\
&=\Upsilon^{-1}_{\mathcal{L},\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\overline{\Psi_{\mathcal{L},\mathcal{L}}}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon_{\mathcal{L},\mathcal{L}})\\
&=\Upsilon^{-1}_{\mathcal{L},\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}}(\overline{\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id}}\otimes\mathrm{id})(\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon_{\mathcal{L},\mathcal{L}})\\
&=(\overline{\mathrm{id}\otimes\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id}})\Upsilon^{-1}_{\mathcal{L},\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon_{\mathcal{L},\mathcal{L}})\\
&=(\overline{\mathrm{id}\otimes\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id}})\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L},\mathcal{L}}(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}).
\end{aligned}$$

Using this, the left-hand side of our condition becomes

$$\begin{aligned}
&=\overline{[\,,\,]([\,,\,]\otimes[\,,\,])}\,(\overline{\mathrm{id}\otimes\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id}})\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L},\mathcal{L}}(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}\,(\mathrm{id}\otimes\overline{\Delta}))\\
&=\overline{[\,,\,]([\,,\,]\otimes[\,,\,])\,(\mathrm{id}\otimes\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})}\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L},\mathcal{L}}(\mathrm{id}\otimes(\overline{\Delta\otimes\mathrm{id}})\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})\\
&=\overline{[\,,\,]([\,,\,]\otimes[\,,\,])\,(\mathrm{id}\otimes\Psi_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})(\Delta\otimes\mathrm{id}\otimes\mathrm{id})}\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})\\
&=\ \overline{[\,,\,]\,(\mathrm{id}\otimes[\,,\,])}\Upsilon^{-1}_{\mathcal{L}\otimes\mathcal{L},\mathcal{L}}(\mathrm{id}\otimes\Upsilon^{-1}_{\mathcal{L},\mathcal{L}})\\
&=\overline{[\,,\,]\,(\mathrm{id}\otimes[\,,\,])}\Upsilon^{-1}_{\mathcal{L},\mathcal{L}\otimes\mathcal{L}}(\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})=\overline{[\,,\,]}\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\overline{[\,,\,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\mathrm{id})\\
&=\star[\,,\,]_R(\star^{-1}\overline{[\,,\,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}\otimes\star^{-1})=\star[\,,\,]_R\,([\,,\,]_R\otimes\mathrm{id})(\star^{-1}\otimes\star^{-1}\otimes\star^{-1}),
\end{aligned}$$

which is the right-hand side. The 4th equality used axiom (a) in Fig. 2.12 for a left braided Lie algebra and we have proved the parallel one for $[\,,\,]_R$. The axioms (b), (c) are verified more easily. For example, axiom (c) involving the counit is

$$\begin{aligned}
&\star_{\underline{1}}\,\epsilon\,[\,,\,]_R=\ \star_{\underline{1}}\,\epsilon\,\star^{-1}\,\overline{[\,,\,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star\otimes\star)=\overline{\epsilon\,[\,,\,]}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star\otimes\star)\\
&=\overline{\epsilon\otimes\epsilon}\,\Upsilon^{-1}_{\mathcal{L},\mathcal{L}}(\star\otimes\star)=\Upsilon^{-1}_{\underline{1},\underline{1}}(\overline{\epsilon}\,\star\otimes\overline{\epsilon}\,\star)=\Upsilon^{-1}_{\underline{1},\underline{1}}(\star_{\underline{1}}\,\epsilon\otimes\star_{\underline{1}}\,\epsilon)=\star_{\underline{1}}(\epsilon\otimes\epsilon),
\end{aligned}$$

since $\underline{1}$ is the identity object. □

If the braiding were antireal, we would similarly obtain a right braided Lie algebra but in the braided bar category with inverse braiding. Note that as a right-handed braided Lie algebra, we also have an associated fundamental braiding $\tilde{\Psi}_R=(\mathrm{id}\otimes\Delta)(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes[\,,\,]_R)$ reflecting part (a) of Fig. 2.14 for the fundamental braiding $\tilde{\Psi}$ of a left braided Lie algebra.

Proposition 2.108 *Let $\mathcal{L}$ be a left braided Lie algebra in a real braided bar category, equipped with $\star$ as in Proposition 2.107. Then $\tilde{\Psi}_R = \tilde{\Psi}^{-1}$ holds if and only if*

$$\begin{array}{ccc} \mathcal{L}\otimes\mathcal{L} & \xrightarrow{\Upsilon^{-1}(\star\otimes\star)} & \overline{\mathcal{L}\otimes\mathcal{L}} \\ \downarrow{\scriptstyle \tilde{\Psi}^{-1}} & & \downarrow{\scriptstyle \overline{\tilde{\Psi}}} \\ \mathcal{L}\otimes\mathcal{L} & \xrightarrow{\Upsilon^{-1}(\star\otimes\star)} & \overline{\mathcal{L}\otimes\mathcal{L}} \end{array}$$

commutes. We say that $\star$ is then a real form *of the braided Lie algebra.*

Proof Using the formula for $[\,,\,]_R$ in Proposition 2.107 and the functoriality of Ψ,

$$\begin{aligned} \Upsilon^{-1}(\star\otimes\star)\tilde{\Psi}_R &= \Upsilon^{-1}(\star\otimes\star[\,,\,]_R)(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) \\ &= \Upsilon^{-1}(\mathrm{id}\otimes\overline{[\,,\,]}\Upsilon^{-1})(\star\otimes\star\otimes\star)(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) \\ &= \Upsilon^{-1}(\mathrm{id}\otimes\overline{[\,,\,]}\Upsilon^{-1})(\Psi\otimes\mathrm{id})(\star\otimes\star\otimes\star)(\mathrm{id}\otimes\Delta) \\ &= \overline{[\,,\,]\otimes\mathrm{id}}\;\Upsilon^{-1}(\mathrm{id}\otimes\Upsilon^{-1})(\Psi\otimes\mathrm{id})(\star\otimes\star\otimes\star)(\mathrm{id}\otimes\Delta) \\ &= \overline{[\,,\,]\otimes\mathrm{id}}\;\Upsilon^{-1}(\Upsilon^{-1}\Psi\otimes\mathrm{id})(\star\otimes\star\otimes\star)(\mathrm{id}\otimes\Delta) \\ &= \overline{[\,,\,]\otimes\mathrm{id}}\;\Upsilon^{-1}(\overline{\Psi}\,\Upsilon^{-1}\otimes\mathrm{id})(\star\otimes\star\otimes\star)(\mathrm{id}\otimes\Delta) \\ &= \overline{([\,,\,]\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)}\;\Upsilon^{-1}(\Upsilon^{-1}\otimes\mathrm{id})(\star\otimes(\star\otimes\star)\Delta) \\ &= \overline{([\,,\,]\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)}\;\Upsilon^{-1}(\star\otimes\Upsilon^{-1}(\star\otimes\star)\Delta) \\ &= \overline{([\,,\,]\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)}\;\Upsilon^{-1}(\star\otimes\overline{\Delta}\star) \\ &= \overline{([\,,\,]\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)(\Delta\otimes\mathrm{id})}\;\Upsilon^{-1}(\star\otimes\star) \\ &= \overline{\tilde{\Psi}}\;\Upsilon^{-1}(\star\otimes\star), \end{aligned}$$

where the 6th equality uses the reality of Ψ. Hence the stated diagram commutes with $\tilde{\Psi}^{-1}$ in place of $\tilde{\Psi}_R$, proving the assertion as $\Upsilon^{-1}(\star\otimes\star)$ is injective. □

This explains and properly generates the assumptions (2.34) which we had previously introduced in an ad hoc way as requirements for a real form. In the antireal case we define a real form similarly but requiring $\tilde{\Psi}_R = \tilde{\Psi}$. We next define a star braided bialgebra B as a bialgebra in a braided bar category which is a star object by $\star : B \to \overline{B}$ such that $\star\epsilon = \overline{\epsilon}\,\star : B \to \overline{1}$, $\star\eta = \overline{\eta}\,\star : 1 \to \overline{B}$ and

$$\begin{array}{ccc} B & \xrightarrow{\star} & \overline{B} \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \overline{\Delta}} \\ B\otimes B & \xrightarrow{\Upsilon^{-1}(\star\otimes\star)} & \overline{B\otimes B} \end{array} \qquad \begin{array}{ccc} B\otimes B & \xrightarrow{\Upsilon^{-1}(\star\otimes\star)} & \overline{B\otimes B} \\ \downarrow{\scriptstyle \cdot} & & \downarrow{\scriptstyle \overline{\cdot}} \\ B & \xrightarrow{\star} & \overline{B} \end{array} \tag{2.40}$$

commute. If B is a braided group or Hopf algebra in the braided category then one can show that $\star S = S\star$ holds automatically.

Theorem 2.109 *Let $(\mathcal{L}, \star)$ be a real form of a braided Lie algebra in a real braided bar category as in Proposition 2.108. Then $U(\mathcal{L})$ is a star braided bialgebra.*

Proof We define the star operation $\star : T\mathcal{L} \to \overline{T\mathcal{L}}$ recursively for $x \in \mathcal{L}^{\otimes n}$ and $y \in \mathcal{L}$ by

$$\star(x \otimes y) = \Upsilon^{-1}(\star\, x \otimes \star\, y). \tag{2.41}$$

We must check that this preserves the relations to give a well-defined map on $U(\mathcal{L})$, where those relations are the image of $\mathrm{id} - \tilde{\Psi}^{-1} : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$. Here

$$\Upsilon^{-1}(\star \otimes \star)(\mathrm{id} - \tilde{\Psi}^{-1}) = \overline{\tilde{\Psi} - \mathrm{id}}\,\Upsilon^{-1}(\star \otimes \star),$$

so we obtain a well-defined map on $x.y$ and hence by induction a map $\star : U(\mathcal{L}) \to \overline{U(\mathcal{L})}$. The product rule in (2.40) is satisfied by construction.

The coproduct rule is more difficult and also shown recursively. For x, y, we have

$$\begin{aligned}\overline{\Delta}\star(x.y) &= \overline{\Delta}\,\Upsilon^{-1}(\star\, x \otimes \star\, y)\\ &= \overline{(\cdot \otimes \cdot)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta \otimes \Delta)}\,\Upsilon^{-1}(\star\, x \otimes \star\, y)\\ &= \overline{(\cdot \otimes \cdot)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})}\,\Upsilon^{-1}(\overline{\Delta}\star x \otimes \overline{\Delta}\star y).\end{aligned}$$

Assuming the coproduct rule for $x \in \mathcal{L}^{\otimes n}$ and using brackets to indicate the input to the relevant Υ^{-1},

$$\begin{aligned}\overline{\Delta}\star(x.y) &= \overline{(\cdot \otimes \cdot)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})}\,\Upsilon^{-1}(\Upsilon^{-1}(\star \otimes \star)\Delta x \otimes \Upsilon^{-1}(\star \otimes \star)\Delta y)\\ &= \overline{(\cdot \otimes \cdot)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})}\,\Upsilon^{-1}(\mathrm{id} \otimes \Upsilon^{-1})\big(\star \otimes (\Upsilon^{-1}(\star \otimes \star) \otimes \star)\big)(\Delta x \otimes \Delta y)\\ &= \overline{(\cdot \otimes \cdot)}\,\Upsilon^{-1}(\mathrm{id} \otimes \Upsilon^{-1})\big(\star \otimes (\overline{\Psi}\,\Upsilon^{-1}(\star \otimes \star) \otimes \star)\big)(\Delta x \otimes \Delta y)\\ &= \overline{(\cdot \otimes \cdot)}\,\Upsilon^{-1}(\mathrm{id} \otimes \Upsilon^{-1})\big(\star \otimes (\Upsilon^{-1}\Psi(\star \otimes \star) \otimes \star)\big)(\Delta x \otimes \Delta y)\\ &= \overline{(\cdot \otimes \cdot)}\,\Upsilon^{-1}(\mathrm{id} \otimes \Upsilon^{-1})\big(\star \otimes (\Upsilon^{-1}(\star \otimes \star) \otimes \star)\big)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta x \otimes \Delta y)\\ &= \overline{(\cdot \otimes \cdot)}\,\Upsilon^{-1}(\Upsilon^{-1}(\star \otimes \star) \otimes \Upsilon^{-1}(\star \otimes \star))(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta x \otimes \Delta y)\\ &= \Upsilon^{-1}(\overline{\cdot}\,\Upsilon^{-1}(\star \otimes \star) \otimes \overline{\cdot}\,\Upsilon^{-1}(\star \otimes \star))(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta x \otimes \Delta y)\\ &= \Upsilon^{-1}(\star \otimes \star)(\cdot \otimes \cdot)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta x \otimes \Delta y) = \Upsilon^{-1}(\star \otimes \star)\Delta(x.y)\end{aligned}$$

on applying properties of Υ^{-1}, the reality and functoriality of Ψ, and the star property of the product at the end. $\square$

We have seen examples in §2.7.2 including the 2×2 braided matrices or q-Minkowski space. Also of interest are braided symmetric algebras.

Proposition 2.110 *If $V, \star : V \to \overline{V}$ is a rigid star object in a real braided bar category then the braided symmetric and exterior algebras $B_\pm(V)$ are star braided Hopf algebras.*

Proof We verify the result for $B_+(V)$, the other case being similar. The star operation is defined recursively on TV in the same manner as (2.41). We check that this gives a well-defined map on $B_+(V)$ by preserving its relations as defined by the kernel of $[n, \Psi]! : V^{\otimes n} \to V^{\otimes n}$. Thus, we show that if $[n, \Psi]!(\underline{v}) = 0$ for $\underline{v} \in V^{\otimes n}$ then $\overline{[n, \Psi]!}\star(\underline{v}) = \overline{0}$. By the reality of the braiding, we have $\star \circ \Psi_s = \overline{\Psi_{n-s}} \circ \star$, so in the notation of Corollary 2.69 we obtain $\star \circ [n, \Psi]! = \overline{[n, \Psi]_R!} \circ \star$. But as $[n, \Psi]_R! = [n, \Psi]!$, we see that $\star$ is well defined on $B_+(V)$.

The coproduct on $B_+(V)$ in Proposition 2.71, given the reversal of the tensor order on taking stars, is controlled by the right-handed factorials $\left[{n \atop r}\right]_R$. Then the coproduct rule for $\star$ amounts to $\left[{n \atop r}\right]_R = \left[{n \atop n-r}\right]$ in Corollary 2.69. □

Note that this does not apply as written to the construction of bicovariant exterior algebras Ω using $\Lambda = B_-(\Lambda^1)$ as in Theorem 2.74 with $\star = \underline{*} = * \circ S$ (or rather the corresponding map into the conjugate space), as the braiding $\tilde{\Psi}$ on Λ^1 there is antireal. One needs a different version of the theory which we do not cover. Likewise, the quantum plane in Example 2.66 has antireal braiding (when $|q| = 1$) but again we do have a $*$-DGA as we saw in Example 2.79 for such q. Nevertheless, there are many other situations where the basic version in Proposition 2.110 does apply.

Example 2.111 Let A be a real coquasitriangular Hopf $*$-algebra and $A \bowtie A$ two copies with $*$-algebra structure

$$(a \otimes b)(c \otimes d) = ac_{(2)} \otimes b_{(2)} d \mathcal{R}(Sb_{(1)} \otimes c_{(1)}) \mathcal{R}(b_{(3)} \otimes c_{(3)}), \quad (a \otimes b)^* = b^* \otimes a^*$$

and tensor product coalgebra (this is an example of a 'double cross product'). This can be naturally equipped with *two* coquasitriangular structures, namely a real one

$$\mathcal{R}_L(a \otimes b \otimes c \otimes d) = \mathcal{R}(Sd_{(1)} \otimes a_{(1)}) \mathcal{R}(a_{(2)} \otimes c_{(2)}) \mathcal{R}(b_{(1)} \otimes d_{(2)}) \mathcal{R}(b_{(2)} \otimes c_{(2)})$$

and an antireal one $\mathcal{R}_D$ identical to $\mathcal{R}_L$ except with $\mathcal{R}(a_{(2)} \otimes c_{(1)})$ replaced by $\mathcal{R}(Sc_{(1)} \otimes a_2)$. It is closely related to the 'quantum double' of A or more precisely of the braiding of the category of A-crossed modules. Thus, $A \bowtie A$ has two different braided categories of comodules with braidings induced by $\mathcal{R}_L$ or by $\mathcal{R}_D$ respectively.

Now consider the braided Lie algebra $\mathcal{L}$ in the construction of a bicovariant differential calculus and viewed as a subcoalgebra of A (see Proposition 2.55). As such, it is a right comodule of $A \bowtie A$ by simultaneous left and right translation,

$$\Delta_R a = a_{(2)} \otimes (Sa_{(1)} \otimes a_{(3)}).$$

Then the braiding on $\mathcal{L}$ induced by $\mathcal{R}_D$ is exactly the restriction of the crossed module braiding $\tilde{\Psi}$ in Proposition 2.54 and is the fundamental braiding of the braided Lie algebra. This defined the braided enveloping bialgebra $U(\mathcal{L})$ by setting to zero the image of $\mathrm{id} - \tilde{\Psi}$. On the other hand, the braiding induced by $\mathcal{R}_L$ comes out as

$$\Psi_L(a \otimes b) = b_{(3)} \otimes a_{(3)} \mathcal{R}(b_{(4)} \otimes a_{(2)}) \mathcal{R}(a_{(1)} \otimes b_{(2)}) \mathcal{R}(a_{(4)} \otimes b_{(5)}) \mathcal{R}(a_{(5)} \otimes Sb_{(1)})$$

and in this braided category we have a braided-symmetric Hopf algebra $B_+(\mathcal{L})$ by the construction in Corollary 2.72. In fact, the coaction is a $*$-coaction in $\mathcal{L}$ and if $\star = \underline{*}$ preserves $\mathcal{L}$ then the latter becomes a star object and $B_+(\mathcal{L})$ by Proposition 2.110 a star braided Hopf algebra with an additive coproduct on the generators. This $B_+(\mathcal{L})$ is a kind of cousin to our $U(\mathcal{L})$ construction.

For a concrete example, we take $\mathcal{L}$ to be the 4D braided Lie algebra of $\mathbb{C}_q[SU_2]$ in Example 2.98 with q real. Then $A \bowtie A = \mathbb{C}_q[SO_{1,3}]$ is the q-Lorentz group when taken with $\mathcal{R}_L$ and $B_+^{\mathrm{quad}}(\mathcal{L}) = B_q[M_2] = U(\mathcal{L})$ i.e., the same q-Minkowski spacetime algebra as before but now with both an additive coproduct with respect to the braiding Ψ_L from the q-Lorentz symmetry and a multiplicative coproduct from the point of view of 2×2 braided-hermitian matrices and fundamental braiding from $\mathcal{R}_D$ as in Example 2.98. This is a matter of showing that $\ker(\mathrm{id} + \Psi_L) = \mathrm{image}(\mathrm{id} - \tilde{\Psi})$ in this case, so that the two algebras coincide (this has its origin in the fact that the underlying R-matrix for the quantum plane is q-Hecke). $\diamond$

This provides some of the background to the remarks at the end of §2.7.2 about q-Minkowski space. Another application is the following, which includes standard q-deformation quantum groups at roots of unity.

Definition 2.112 A flip-Hopf $*$-algebra H is a Hopf algebra with a $*$-operation giving a usual $*$-algebra, but the coproduct and counit obey

$$\Delta(h^*) = (* \otimes *) \circ \mathrm{flip} \circ \Delta h, \quad \epsilon(h^*) = (\epsilon h)^*.$$

It follows that the antipode obeys $S(h^*) = (Sh)^*$.

This is nothing other than a star Hopf algebra in the bar category of vector spaces according to (2.40). In contrast to a usual Hopf $*$-algebra, its comodules separately no longer form bar categories as left and right become swapped on conjugation, but the category of bicomodules ${}^H\mathcal{M}^H$ is again a bar category. If V is a left and right comodule with our usual notations then $\overline{V}$ becomes a bicomodule with

$$\Delta_L\overline{v} = v_{(\bar{1})}{}^* \otimes \overline{v_{(\bar{0})}}, \quad \Delta_R\overline{v} = \overline{v_{(\bar{\infty})}} \otimes v_{(\bar{1})}{}^*. \tag{2.42}$$

The proof is omitted as the arrow-reversal of the proof of Example 2.101. Likewise, the category of Hopf bimodules in Lemma 2.23 becomes a bar category. Thus, if Ω is a bicovariant differential calculus on H constructed from a Hopf bimodule point of view, the construction now proceeds the same way just in the bar category version to give a $*$-DGA, now a super-flip Hopf $*$-algebra. In our case we have taken a crossed module point of view where, over any field, $\Omega = H{\triangleright\!\!<}\Lambda = \Lambda_R{>\!\!\triangleleft} H$ as two equivalent points of view when the antipode is invertible (the right Maurer–Cartan form ϖ_R is then related to the left one ϖ by (2.31)). In the flip Hopf $*$-algebra case over $\mathbb{C}$, we have

$$(\varpi h)^* = \varpi_R(h^*) = -((Sh_{(1)})h_{(3)})^*\varpi\pi_\epsilon((Sh_{(2)})^*), \tag{2.43}$$

which requires $*S(\mathcal{I}) \subseteq \mathcal{I}$ for $*$ on Λ^1 to be well defined. In bar category terms, the first equality says that we have a map $\star : \Lambda^1 \to \overline{\Lambda^1_R}$ in the vector space bar category. In fact, this is a map of right crossed modules, where Λ^1_R being a left H-crossed module makes $\overline{\Lambda^1_R}$ a right crossed module (the conversion of a left module to a right module is half of the proof of Example 2.101, and of a left comodule to a right comodule is half of (2.42)). The reason is that the actions on Λ^1, Λ^1_R come from left and right multiplication on H^+, obviously connected by the $*$ of the flip Hopf $*$-algebra, while for the coactions $\mathrm{flip}(* \otimes *)\mathrm{Ad}_R(h) = \mathrm{Ad}_L(h^*)$. Once we know that the crossed modules match up, it follows that the braidings match up via $\dagger = \Upsilon^{-1}(\star \otimes \star)$ (much as for the commutative diagram proved in the proof of Proposition 2.108) and hence that we have an order-reversing algebra map $\star : \Lambda \to \overline{\Lambda_R}$. (Super) bosonisation then combines this with $\star : H \to \overline{H}$ to obtain $\star : \Omega \to \overline{\Omega}$ in the form $H{\triangleright\!\!<}\Lambda \to \overline{\Lambda_R{>\!\!\triangleleft} H}$. Although not necessary for the $*$-DGA itself, this describes its structure in terms of the braided Hopf algebras of left and right-invariant forms.

Example 2.113 Take the Hopf algebra $H = \mathbb{C}_q[SL_2]$ in Proposition 2.13, where q is a complex number of norm 1. This forms a flip Hopf $*$-algebra, which we denote by $\mathbb{C}_q[SU_2^{fl}]$ with the hermitian $*$-structure

$$a^* = a, \quad b^* = c, \quad c^* = b, \quad d^* = d,$$

(one could also have any real factor in b^* and its inverse in c^*). Then the 4D bicovariant calculus in Example 2.59 still obeys $*S(\mathcal{I}) \subseteq \mathcal{I}$, where the ideal is given explicitly in (2.18) and one should remember that $*S$ now preserves order but is conjugate linear, so all q are turned to q^{-1}. All the generators are sent to generators. Hence this calculus becomes a $*$-DGA by the above remarks. On the other hand, the construction of the $*$ operation in terms of, say, left-invariant forms alone, is more complicated. For example, we have from (2.43) that

$$e_c^* = q^2c^2e_c - qd^2e_b - q^3cde_z$$

for the usual 4D calculus but with this flip $*$ structure. ◇

Exercises for Chap. 2

E2.1 Let H be a Hopf algebra. Show that the quantum adjoint action $\triangleright$ and the adjoint coaction Ad_R defined by

$$h\triangleright g = h_{(1)}gSh_{(2)}, \quad \mathrm{Ad}_R(h) = h_{(2)} \otimes (Sh_{(1)})h_{(3)}$$

make H into a left module algebra and a right comodule coalgebra respectively. For the Sweedler–Taft algebra $u_q(b_+)$ in Example 2.65 with q a primitive r-th root of unity, compute $t\triangleright(t^nx^m)$, $x\triangleright(t^nx^m)$, $\mathrm{Ad}_R(t)$ and $\mathrm{Ad}_R(x)$.

E2.2 Show that $u_q(b_+)$ at $q = -1$ is self dual as a Hopf algebra. [In fact, this holds for all primitive r-th roots of unity.]

E2.3 Find the unique (up to scale) right-integral on $u_q(b_+)$ for all primitive r-th roots of unity and show that $\int 1 = 0$. [Hint: Use the q-binomial formula of Lemma 2.15 and the fact that $[n]_q! \neq 0$ for $0 < n < r$.]

E2.4 Show that the proper right ideals of the augmentation ideal of $u_q(b_+)$ at $q = -1$ are the spans of

$$\{x - xt\}, \quad \{x + xt\}, \quad \{x, xt\}, \quad \{x + xt, 1 - t + \mu x\}; \quad \mu \in \mathbb{C}.$$

Which of these is Ad_R-stable? Show for the last case that

$$\omega = \varpi\pi_\epsilon(x), \quad \varpi\pi_\epsilon(t) = \mu\omega, \quad \omega\triangleleft t = -\omega, \quad \omega\triangleleft x = 0,$$

where $\omega = [x] \in \Lambda^1$ in the construction of Theorem 2.26 and hence that the corresponding 1-dimensional left-covariant calculus has

$$\mathrm{d}t = \mu t\omega, \quad \mathrm{d}x = (1 + \mu x)\omega, \quad \omega t = -t\omega, \quad \omega x = -x\omega.$$

E2.5 Let $\mathcal{C} \subseteq \{1, 2, \cdots, r-1\}$ and $\Omega(\mathbb{Z}_r)$ be the corresponding Cayley graph calculus with left-invariant forms e_a, $a \in \mathcal{C}$, by Proposition 1.52. Find the corresponding calculus on $\mathbb{C}\mathbb{Z}_r = \mathbb{C}[t]/\langle t^r - 1\rangle$ under the Fourier isomorphism $\mathbb{C}\mathbb{Z}_r \cong \mathbb{C}(\mathbb{Z}_r)$ given by $t(i) = q^i$, where q is a primitive r-th root of unity. Show that $\mathcal{C} = \{1\}$ recovers the q-circle in Example 1.11 but with the additional relation $t^r = 1$ (this reduced algebra with q-circle calculus is denoted by $c_q[S^1]$).

E2.6 For $\mathbb{C}_q\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ with coquasitriangular structure $\mathcal{R}(t^m \otimes t^n) = q^{mn}$ for $q \neq 0$, check directly from Proposition 2.55 applied to the subcoalgebra $\mathcal{L} = \mathbb{C}t$ that you obtain the q-differential calculus of $\mathbb{C}_{q^2}[S^1]$ in Example 1.11 as claimed in Example 2.58. Find $\Omega(\mathbb{C}_{q^2}[S^1])$ and its super-Hopf algebra structure from Theorem 2.74. If q is a primitive r-th root of unity and r is odd, show that the calculus is not connected but is after we reduce to $c_{q^2}[S^1]$ by adding $t^r = 1$. Also show in this case that $\mathcal{R}$ descends to make $\mathbb{C}_q\mathbb{Z}_r$ a factorisable self-dual Hopf algebra and show that the classification of its bicovariant calculi in Proposition 2.56 reduces to the subsets $\mathcal{C}$ of the preceding question.

E2.7 Show from the definition in Theorem 2.47 that the centre of the monoidal category of $\mathbb{Z}_r$-graded vector spaces (with the grade of a tensor product the sum of the grades of the factors) consists of objects V equipped with a group action of $\mathbb{Z}_r$ by morphisms on V. Show that this centre is the braided category of $\mathbb{C}_q\mathbb{Z}_r^2$-comodules where $\mathbb{C}_q\mathbb{Z}_r^2 = \mathbb{C}[s, t]/\langle s^r - 1, t^r - 1\rangle$ with the coquasitriangular structure $\mathcal{R}(s^{m_1}t^{m_2} \otimes s^{n_1}t^{n_2}) = q^{m_2 n_1}$ is a factorisable self-dual Hopf algebra. [This is a quotient in the root of unity case of the coquasitriangular Hopf algebra $\mathbb{C}_q\mathbb{Z}^2 = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ with $\mathcal{R}$ given by the same formula for any $q \neq 0$.]

E2.8 (i) Let $V = \mathbb{C}x$ be one-dimensional with basis x of grade 1 in the braided category of $\mathbb{Z}_r$-graded spaces with $r > 2$ and braiding $\Psi_{U,W}(u \otimes w) = q^{|u||w|} w \otimes u$, where q is a primitive r-th root of unity. Show that $B_+(V)$ is the reduced braided line $c_q[\mathbb{C}] = \mathbb{C}[x]/\langle x^r\rangle$ and that the braided differential calculus $\Omega(c_q[\mathbb{C}])$ by Proposition 2.78 is the 1-dimensional one with relations $(\mathrm{d}x)x = qx\mathrm{d}x$ and $(\mathrm{d}x)^2 = 0$. (ii) With respect to $\Omega(c_q[S^1])$ as a super-Hopf algebra generated by $t, \mathrm{d}t$, show that $\Omega(c_q[\mathbb{C}])$ is a super comodule algebra with coaction $\Delta_R x = x \otimes t$, $\Delta_{R*}\mathrm{d}x = \mathrm{d}x \otimes t + x \otimes \mathrm{d}t$ and likewise a super module algebra with action

$$x{\triangleleft}t = q^{-1}x, \quad \mathrm{d}x{\triangleleft}t = \mathrm{d}x, \quad x{\triangleleft}\mathrm{d}t = (q^{-1} - 1)\mathrm{d}x, \quad \mathrm{d}x{\triangleleft}\mathrm{d}t = 0.$$

Show that these fit together to 'superbosonise' $\Omega(c_q[\mathbb{C}])$ as a natural 2D (strongly bicovariant) calculus $\Omega(c_q[S^1]){\triangleright\!\!\!<}\Omega(c_q[\mathbb{C}])$ on $u_{q^{-1}}(b_+)$ defined as in Example 2.65 with $xt = q^{-1}tx$.

E2.9 Consider the 2D bicovariant calculus on $\mathbb{C}[S^1] = \mathbb{C}[t, t^{-1}]$ with two bi-invariant basic 1-forms $e_\pm$ with $e_\pm t = q^{\pm 1} t e_\pm$ and $\mathrm{d}t = t((q-1)e_+ + (q^{-1}-1)e_-)$. Prove that $\mathrm{d}t^m = t((q^m-1)e_+ + (q^{-m}-1)e_-)$, that the calculus is inner and is a $*$-calculus with $e_\pm{}^* = -e_\pm$ for q real and $e_\pm{}^* = -e_\mp$ for $|q| = 1$. Explain how the calculus arises by Proposition 2.55 and $\mathcal{L} = \mathrm{span}_{\mathbb{C}}\{t, t^{-1}\}$ if q has a square root and find the associated 2D braided and quantum Lie algebra structures by Theorem 2.85 and a right-handed version of Corollary 2.91. Explain with reference to Exercises E2.5 and E2.6 how this calculus relates to a Cayley graph calculus on $\mathbb{Z}$.

E2.10 Write out the axioms of a $*$-differential calculus in the language of bar categories. Consider $A = \mathbb{C}(S_4)$ (where S_4 is the group of permutations of $\{0, 1, 2, 3\}$) with an action of $\mathbb{Z}_4 = \{1, z, z^2, z^3\}$ by $z{\triangleright}g = (0123)g(3210)$ for $g \in S_4$, and on $\mathbb{C}(S_4)$ by e.g., $z{\triangleright}\delta_{(12)} = \delta_{(23)}$. We give A an exotic star operation $\star : A \to \overline{A}$ by $a \mapsto \overline{z{\triangleright}a^*}$, where a^* is the usual $*$. Using Definition 2.99, give ${}_A\mathcal{M}_A$ a bar category structure illustrated for bimodules E, F with $e \in E$, $f \in F$ by

$$a.\overline{e} = \overline{e.(z{\triangleright}a^*)},\ \ \overline{e}.a = \overline{(z{\triangleright}a^*).e},\ \ \Upsilon_{E,F}(\overline{e \otimes f}) = \overline{f} \otimes \overline{e},\ \ \mathrm{bb}_{\underline{1}}(a) = \overline{\overline{z^2{\triangleright}a}}.$$

Using this bar category, show that $\mathcal{C} = \{(01), (12), (23), (31)\}$ gives A a star differential calculus by taking the usual calculus for $\mathcal{C}$ with the maximal prolongation (or any other prolongation that is preserved by the action) and the above nonstandard star operation. [Hint: what is $\star : \Omega^1 \to \overline{\Omega^1}$?]

Notes for Chap. 2

There are already many texts on Hopf algebras or quantum groups and of these we continue the style and notations of [202, 209]. Some other well-known texts are [75, 155, 310] from a wide perspective and [167, 241, 257] from more specific angles. §2.1 provides a concise introduction to the basic theory and §2.2 to the basic examples including enveloping algebras $U_q(su_2)$ and coordinate algebras $\mathbb{C}_q[SU_2]$ which are respectively part of the Drinfeld–Jimbo family [103] and can be completed to C^*-algebras in the approach of Woronowicz [326]. The latter contained a definitive treatment of differentials on such coordinate algebras and the 3D and 4D calculi on $\mathbb{C}_q[SU_2]$ in particular (Examples 2.32 and 2.59). The other large class of quantum groups to arise in the 1980s was the bicrossproduct family associated to local factorisations of Lie groups, introduced by the second author in [187] and related works. An important example in the pre-quantum groups literature was the Sweedler–Taft Hopf algebra [310, 311], now seen more often as $u_q(b_+) \subseteq u_q(sl_2)$.

Our treatment of the differential geometry starts with Theorem 2.26 in §2.3 based on crossed modules and the Hopf module lemma and follows lecture notes on noncommutative geometry in [225] and recent work [226]. Crossed modules in Definition 2.22 are an obvious linearisation and Hopf version of the notion of crossed G-sets due to J.C. Whitehead [324]. They were emphasised in the Hopf case by Radford [284] and many years later by Yetter [329], while the present second author pointed out that they are essentially the same as modules of the Drinfeld double $D(H)$ (we have not covered the latter in any detail). The application of the theory to polynomial algebras in Example 2.30 is due to [205]. The application of the 3D calculus to the differential geometry of the q-sphere goes back to the q-Hopf fibration in the work of Brzeziński and the second author in [62] while a definitive application of this in [216] was the double complex in Proposition 2.35 (which we will understand further in Chap. 7) and its quantum metric in Proposition 2.36. The calculus on different quantum spheres was first obtained by Podleś [280], including nonstandard spheres beyond the q-sphere, and extended to other quantum symmetric spaces in [137]. We did not have space to treat the differential calculus on bicrossproduct quantum groups but these are classified for finite group factorisations in [26], while for the Lie case some recent results are in [240].

§2.4 is a concise introduction to category theory, braided categories [150] and Drinfeld's theory of quasitriangular Hopf algebras [103] as well as the dual theory (found in [196] among other places) of coquasitriangular Hopf algebras. Most of this is covered more fully in texts such as [202, 209] but the advanced theory of the quantum Killing form in Proposition 2.54 is recent work [226]. The use of R-matrices (obeying the braid relations) to describe q-deformation quantum groups goes back to [112]. The Drinfeld–Majid centre in Theorem 2.47 was introduced by the second author in [191] as part of a general $\mathcal{C}^\circ$ duality construction for monoidal functors $\mathcal{C} \to \mathcal{V}$. The published version of this included a private comment from Drinfeld relating the case of the identity functor to the construction to the double of a Hopf algebra. The centre construction will be needed in Chap. 6.

The clean construction of bicovariant differential calculi on coquasitriangular Hopf algebras A in Proposition 2.55 follows [226, 238], associating such a calculus to each subcoalgebra $\mathcal{L} \subseteq A$. This is a modern version of the second author's earlier result [204] classifying deformation calculi on $\mathbb{C}_q[G]$ for q-generic as being of square dimension associated to irreducible representations. That work also contained Proposition 2.56 for finite-dimensional factorisable Hopf algebras such as deformations at q a root of unity. Some similar results to [204] appeared in [18]. The matrix coalgebra associated to the quantum matrix group generators form an obvious subcoalgebra and in this case our approach reproduces 'R-matrix formulae' in Corollary 2.57 that were first found by Jurco [152]. The metric on the 4D calculus for $\mathbb{C}_q[SU_2]$ is from [211] while the q-Laplacian is from [218] and recent work [226]. The key Lemma 2.61 for the exterior derivative was first found in [123] in a study of the geometry at q a root of unity.

Our treatment in §2.6 of covariant exterior algebras on quantum groups H and quantum planes uses the theory of 'braided groups' or Hopf algebras *in* braided categories introduced by the second author at the end of the 1980s, e.g. [192, 195, 196, 208], and [198, 202, 209] for introductions. Specifically, [213, 226] used such braided groups to rework Woronowicz' construction via the Radford–Majid bosonisation theory Theorem 2.64 in [197, 284] to construct a super-braided Hopf algebra as the left-invariant part of the exterior algebra Λ and then convert this to an ordinary super-Hopf algebra $\Omega(H) = H{\triangleright\!\!<}\Lambda$ as in Theorem 2.74. That the canonical bicovariant exterior algebra forms a super-Hopf algebra is due to [59]. The notion of *strongly bicovariant* for when this happens more generally is due to [238].

That quantum planes are braided Hopf algebras is due to the second author [195, 208] and the construction of their exterior algebras as a braided tensor product in Proposition 2.78 was covered in R-matrix form in [202] and in the present form in [226]. The braided Hopf algebra $B_+(V)$ canonically associated to an object in an abelian braided category was introduced by the second author as $TV/\oplus\ker[n,\Psi]!$ in [208] with the tensor case earlier in [195]. This approach in Corollaries 2.72 and 2.73 gives both the braided Hopf algebra and its nondegenerate pairing with $B_+(V^*)$ if V is rigid and was central to the inductive construction of quantum groups in [208]. The relations can also be described as a sum over the permutation group in a construction first described in the context of [326] by Woronowicz with the result that the $B_+(V)$ are also called Nichols–Woronowicz algebras. Note that the strictly braided theory is more complicated than working in a symmetric monoidal category, which more closely follows the super case. For more on super-Hopf algebras and super-Lie algebras, a recent text is [114].

The notion of quantum Lie algebras in §2.7.1 goes back to Woronowicz in [326] while the characterisation in terms of braid relations in Lemma 2.84 is from [124]. Corollary 2.91 that every braided Lie algebra is a quantum Lie algebra is new but has its roots in that work. The theory of braided Lie algebras itself in §2.7.2 was introduced by the second author in [200] to solve the Lie problem for Drinfeld-type quantum groups. We have not had room to cover applications of the braided-Killing form to finite group theory in [179] but we included as Theorem 2.94 and Proposition 2.95 results from [226] which construct a braided Lie algebra structure directly on the subcoalgebra $\mathcal{L} \subseteq A$ underlying our construction of bicovariant calculi with the space Λ^1 of invariant 1-forms dual to $\mathcal{L}^*$.

The chapter concludes with the theory of bar categories introduced by us in [28]. The application to characterise real forms of braided Lie algebras is new. Proposition 2.110 constructing real star braided group structures on $B_+(V)$ is new but builds on experience with R-matrices in [203]. This notion is adapted to real braided bar categories and includes q-Minkowski space $U(\mathcal{L})$ as a formalisation of [192]. The full description of this as an $A \bowtie A$ comodule algebra in Example 2.111 is from [201, 202]. The construction of all 2-parameter Podleś quantum spheres [280] as constant time-slices of the unit hyperboloid in q-Minkowski space mentioned after Example 2.98 is a further result of the second author in [219] and will form our approach to these objects in Chap. 3. The antireal braided Hopf algebras including quantum planes at $|q| = 1$ do not fit into this basic theory but are covered in [28]. The application to flip Hopf $*$-algebras and their differential geometry is new.

Exercises similar to E2.1 can be found in [202, 209]. The coquasitriangular structures on $\mathbb{C}_q\mathbb{Z}$ and $\mathbb{C}_q\mathbb{Z}_r$ etc., featuring in E2.6– E2.8 are essentially in [202] with the second of these in dual form as the quasitriangular Hopf algebras $\mathbb{Z}'_r$. The construction in Exercise E2.8 is part of a general one in [13]. The exotic star calculus on S_4 in E2.10 is new.

Chapter 3
Vector Bundles and Connections

As we have seen, the standard way to approach noncommutative geometry is to take a possibly noncommutative algebra in place of the commutative algebra of real or complex-valued functions on a space. We now ask what takes the place of a vector bundle. The most obvious vector bundles are the tangent and cotangent bundles on a manifold, as in Chaps. 1 and 2, but there are many others. For example, in electrodynamics one works with sections of a complex line bundle.

A vector bundle is an assignment of a vector space to every point of a topological space in a continuous manner. A section of a vector bundle is a continuous assignment for every point in the space, of an element of the vector space at that point. For the tangent or cotangent bundles this gives the familiar ideas of vector fields and 1-forms on a manifold. Note we can multiply sections by functions, giving a module over the algebra of functions. This gives the idea in noncommutative geometry of studying the modules over a noncommutative algebra, which we think of as sections of some vector bundle over the noncommutative space.

It is usual to study locally trivial vector bundles in classical topology or differential geometry, and by the Serre–Swan theorem this translates into the idea of *finitely generated projective modules*. These have good candidates for duals and bases, and together can be thought of as a class of modules which behave in a manner not too different from vector spaces. The study of such locally trivial vector bundles is the subject of K-theory, and this is extended to the study of modules over C^*-algebras. We discuss K-theory, cyclic homology and cohomology, and the Chern–Connes pairing, which is used to get cohomological information out of K-theory. We also discuss twisted cyclic theory, which is used to extend the range of validity of the cyclic theory to more noncommutative examples. There are many excellent books on these topics so we will be brief, but will include enough by way of definitions and some constructions to be able to see how these topics fit with the rest of the book.

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Differentiation of sections of smooth bundles is of great importance in differential geometry, given that both vector fields and differential forms are examples of sections of bundles. To do this we introduce the *covariant derivative* or *connection* $\nabla_E : E \to \Omega^1 \otimes_A E$ on a left A-module E. From a connection we can form its curvature, which is fundamental to the study of manifolds in the classical case. In classical geometry, it makes no difference if we multiply a section on the left or right by a function. In noncommutative geometry we must separate these cases, and have correspondingly left and right connections. A bimodule connection on a bimodule will be based on one side but respect the product from the other side.

Given a module we can in some cases form its dual, and working over $\mathbb{C}$ we also form its conjugate in the sense of bar categories in §2.8. We study connections on these induced by connections on the original module. This will be vital in §8.4 in studying hermitian Riemannian structures. We also spend some time studying the easy but important example of line modules as a version of line bundles.

We include in this chapter two more advanced topics. Exact sequences of modules are a convenient pictorial way to discuss properties of modules which would otherwise be rather messy. Flat modules are those which preserve the exactness of sequences of modules under tensor product. Projective modules, as the name suggests, are more general than finitely generated projective modules, and are examples of flat modules. We also give a more formal treatment of abelian categories. These package a series of results, for example the ability to take kernels or add morphisms, familiar in the category of vector spaces.

3.1 Finitely Generated Projective Modules

We have already said that a vector bundle on a topological space X associates to every point $x \in X$ a vector space, which is the *fibre* of the vector bundle at $x \in X$. The problem is how the different vector spaces at different points are connected together. A trivial vector bundle is given by a direct product and projection to the second factor $\pi_2 : V \times X \to X$, where V is a vector space. Then every $\pi_2^{-1}(x) = V \times \{x\}$ is a copy of the vector space V, with its usual addition and scalar multiplication. More generally, we have the notion of a *locally trivial* vector bundle on a topological space X, where we have a cover of X by open sets $U_i \subseteq X$ and on each of these we have a trivial bundle $\pi_2 : V \times U_i \to U_i$. On the overlaps $U_i \cap U_j$ between the sets in the cover we identify the two locally trivial descriptions by a homeomorphism which is linear on the fibres. In this manner, a locally trivial fibration is made by glueing together trivial fibrations on open subsets to make one map $\pi : B \to X$, where B is the *total space* of the fibration. The *rank* of the vector bundle is the dimension of the vector space V (over $\mathbb{R}$ or $\mathbb{C}$ as appropriate). When the rank is one we have the special case of a *line bundle*. Keeping to the case where the vector bundles are finite rank to avoid complications, we may take the dual of a vector bundle, or direct sum or tensor product vector bundles together.

These constructions are applied fibrewise, i.e., at the vector space over every point in X.

A section of a vector bundle $\pi : B \to X$ is a map $s : X \to B$ such that $\pi \circ s = \mathrm{id} : X \to X$. For example, if X is a manifold then a vector field on X is a section of the tangent bundle and a 1-form is a section of the cotangent bundle. We can add sections pointwise or multiply them by constants, so they form a vector space. We can also multiply sections pointwise by continuous $\mathbb{R}$ or $\mathbb{C}$-valued functions on X, making the sections into a module for the algebra $C(X)$ of continuous functions on X. The sections of a direct sum of bundles is simply the direct sum of each module of sections. If we tensor product two bundles on X, the sections of the tensor product bundle can be identified with $\otimes_{C(X)}$ applied to the modules of sections.

In noncommutative geometry we swap the functions on a topological space X for a possibly noncommutative algebra A, and then take A-modules to stand for the sections of a hypothetical vector bundle. Because of noncommutativity, we need to distinguish between left and right modules, or bimodules with both. Our notations for these were summarised in a table in §2.4.

The left dual of a right module F is defined to be $F^\sharp = \mathrm{Hom}_A(F, A)$ with $F^\sharp$ a left module by $(a.\alpha)(f) = a\alpha(f)$ for $\alpha \in F^\sharp$, $a \in A$ and $f \in F$. For clarity, we may write $\alpha(f) = \mathrm{ev}(\alpha \otimes f)$ for the evaluation of α on its argument. The classical concept of 'finite rank' is captured in our general setting as the following.

Definition 3.1 A right A-module F is said to be finitely generated projective (fgp) if there are a finite number of $f_i \in F$ and $e^i \in F^\sharp$ (referred to together as 'dual bases') such that for all $f \in F$, $f = \sum f_i.e^i(f)$.

It follows directly that $\alpha = \sum \alpha(f_i).e^i$ for all $\alpha \in F^\sharp$ and indeed we will see shortly that $F^\sharp$ is fgp as a left A-module as might be expected by symmetry. Note that the f_i (and the e^i) span but are not separately bases except in the free case.

Example 3.2 For the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ on $\mathbb{R}^n$ with coordinates $\{x^1, \ldots, x^n\}$, the module $F = \Omega^1(\mathbb{R}^n)$ of sections of the cotangent bundle has an actual basis $\mathrm{d}x^i$ with a dual basis $\frac{\partial}{\partial x^j}$ for $F^\sharp$ of vector fields as sections of the tangent bundle. We have $\mathrm{ev}(\partial_j \otimes \mathrm{d}x^i) = \delta_{ij}$ (the Kronecker delta). These bundles are parallelisable and the modules free. By contrast, the cotangent and tangent bundle of the sphere are not parallelisable and their modules of sections are not free (for vector fields, this is the 'hairy-ball' theorem) but they are fgp as we see shortly. ⋄

The celebrated Serre–Swan theorem gives the equivalence of locally trivial vector bundles on various spaces with finitely generated projective modules over the algebra of functions on the spaces. We sketch a proof, under a simplifying assumption, to illustrate the connection between the definitions.

Proposition 3.3 *Let X be a compact Hausdorff space. There is a 1–1 correspondence between locally trivial real vector bundles over X and fgp modules over $C(X)$ (real-valued continuous functions). This is given one way by taking the module of continuous sections of the bundle.*

Proof Given a locally trivial vector bundle, take a finite open cover $U_i \subseteq X$ $(1 \leq i \leq m)$ which trivialises the bundle, so we get $\pi_2 : \mathbb{R}^n \times U_i \to U_i$. Also take a partition of unity, i.e., continuous functions $\phi_i : X \to [0, 1]$ such that the closure of the set where $\phi_i(x) > 0$ is contained in U_i and such that $\sum_i \phi_i^2(x) = 1$. For the bundle $\pi_2 : \mathbb{R}^n \times U_i \to U_i$, we take sections $e_{ij} = \phi_i v_j$ for $1 \leq j \leq n$, where v_j is the standard basis of $\mathbb{R}^n$. The e_{ij} can be extended to all of X as sections of the bundle, by setting all values outside U_i to zero. Similarly take sections of the dual bundle $\alpha_{ij} = \phi_i v^j$ for $1 \leq j \leq n$, where v^j is the dual basis to v_j in $\mathbb{R}^n$, i.e., $\mathrm{ev}(v^j \otimes v_k) = \delta_{jk}$. Now the e_{ij} and α_{ij} form dual bases of the bundle as in Definition 3.1. To see this, write a section of the bundle as $s = \sum_j s_{ij} v_j$ on the trivialisation on U_i. Then $\sum_{i,j} e_{ij}\alpha_{ij}(s) = \sum_i \phi_i^2 \sum_j s_{ij} v_j = \sum_i \phi_i^2 s = s$.

For the converse, we make things slightly less messy by restricting to connected X. Given an fgp module, take an idempotent matrix $P \in M_m(C(X))$ corresponding to the module (we shall construct this shortly). As X is connected, the rank of $P(x)$ is a constant and we take it to be n. Given $x \in X$, we get an n-dimensional subspace $V(x) = \text{image } P(x) \subseteq \mathbb{R}^m$ and have to show that this gives a locally trivial bundle. Taking any inner product on $\mathbb{R}^m$ and $x_0 \in X$, we take orthogonal projection from $V(x)$ to $V(x_0)$. This is a linear isomorphism for x in some neighbourhood of x_0, and these neighbourhoods for various $x_0 \in X$ give a trivialising cover. □

Here is an example of a module which is not fgp.

Example 3.4 Let $A = \mathrm{C}(S^1)$, the continuous complex functions on a circle, and $F_z = \mathbb{C}$ for some fixed $z \in S^1$, with right module action $\lambda{\triangleleft}f = f(z)\lambda$ for $\lambda \in F_z$ and $f \in \mathrm{C}(S^1)$. A right module map $\alpha : F_z \to A$ obeys $\alpha(\lambda{\triangleleft}f) = \alpha(\lambda)f$ so that by linearity and setting $\lambda = 1$, we have $\alpha(1)f(z) = \alpha(1)f$ for all $f \in \mathrm{C}(S^1)$. The only functions $\alpha(1) \in \mathrm{C}(S^1)$ which satisfy this would be zero at all points except $z \in S^1$, and as we are considering continuous functions, every element of $F_z^\sharp$ is zero. Thus F_z cannot be right fgp. Rather than being sections of a locally trivial vector bundle on the circle, this module is the sections of a bundle with fibre $\mathbb{C}$ at the point $z \in S^1$ and zero elsewhere. We return to a similar idea in Example 3.108. ◇

Using dual bases, we can describe module maps in terms of the dual modules.

Proposition 3.5 *Suppose that F is a right fgp A-module.*

(1) *If N is a right A-module, there is an isomorphism $N \otimes_A F^\sharp \to \mathrm{Hom}_A(F, N)$ where $n \otimes \alpha \in N \otimes_A F^\sharp$ maps to the homomorphism $e \mapsto n.\alpha(e)$.*
(2) *If G is a left A-module, there is an isomorphism $F \otimes_A G \to {}_A\mathrm{Hom}(F^\sharp, G)$ where $f \otimes g \in F \otimes_A G$ maps to the homomorphism $\alpha \mapsto \alpha(f).g$.*

Proof Let f_i, e^i be dual bases for F. The inverse map for the first case is $T \mapsto \sum T(f_i) \otimes e^i$. The inverse map for the second case is $S \mapsto \sum f_i \otimes S(e^i)$. □

A consequence of part (1) is that there is a canonical *dual bases element* $f_i \otimes e^i \in F \otimes_A F^\sharp$ corresponding to the identity id_F on F and therefore independent of the choice of dual bases. The finitely generated projective property also allows us to say more about mapping spaces and tensor products, as exemplified in the next result. We will see in §3.6.1 how to state such results in terms of exact sequences.

Corollary 3.6 *Suppose that we have a module map $T : G \to H$ between left A-modules, with kernel $K \subseteq G$. Then for a right fgp A-bimodule F, the map* $\mathrm{id} \otimes T : F \otimes_A G \to F \otimes_A H$ *has kernel $F \otimes_A K$.*

Proof We use the relevant isomorphism in Proposition 3.5 to obtain the map $T\circ : {}_A\mathrm{Hom}(F^\sharp, G) \to {}_A\mathrm{Hom}(F^\sharp, H)$, and this has kernel ${}_A\mathrm{Hom}(F^\sharp, K)$. □

We have also mentioned the concept of a free A-module, meaning freely generated by n generators, say, with no nontrivial relations. Thus we just have n copies of A in exactly the same way as an n-dimensional vector space over $\mathbb{R}$ is just $\mathbb{R}^n$. To fit with the matrix P, it is convenient to organise these free modules into row and column vectors, so we take $\mathrm{Col}^n(A)$ to be the right A-module consisting of n-dimensional column vectors with entries in A, and $\mathrm{Row}^n(A)$ to be the left A-module consisting of n-dimensional row vectors with entries in A. Now we can give an alternative characterisation of fgp modules as follows.

Theorem 3.7 *For a right A-module F, the following are equivalent:*

(1) *F satisfies the definition of fgp in Definition 3.1.*
(2) *F is isomorphic to a submodule of a right finitely generated free module, and that submodule has a complementary submodule (i.e., the intersection of the submodules is zero, and their sum is the finitely generated free module).*

In this case $P \in M_n(A)$ with entries $P_{ij} = e^i(f_j)$ is an idempotent, i.e., $P^2 = P$.

Proof If F satisfies (1), then take dual bases (f_i, e^i) and corresponding idempotent matrix $P_{ij} = e^i(f_j)$, where $P_{ij}P_{jk} = e^i(f_j)e^j(f_k) = e^i(f_j.e^j(f_k)) = e^i(f_k) = P_{ik}$ (sum over j). There is an injective map of right modules $\underline{\alpha} : F \to \mathrm{Col}^n(A)$ given by $(\underline{\alpha}(f))_i = e^i(f)$ for $f \in F$. By the usual property of dual bases, the image of this map lies in the submodule $P.\mathrm{Col}^n(A)$ of $\mathrm{Col}^n(A)$. We have the left half of

$$\underline{\alpha}(f_i) = \begin{pmatrix} P_{1i} \\ \vdots \\ P_{ni} \end{pmatrix} = P.\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \pi \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} P_{1i} \\ \vdots \\ P_{ni} \end{pmatrix}, \tag{3.1}$$

(where the single entry 1 is in the i-th position), so in fact the image of $\underline{\alpha}$ is precisely $P.\mathrm{Col}^n(A)$, giving the required isomorphism for (2). Further the submodule $P.\mathrm{Col}^n(A)$ has complement $(I_n - P).\mathrm{Col}^n(A)$.

Conversely to begin with (2), suppose that F is a complemented submodule of a right finitely generated free module, which we may as well suppose is $\mathrm{Col}^n(A)$. Denote the complementary submodule by $F^\perp$, so $F\oplus F^\perp=\mathrm{Col}^n(A)$ and $F\cap F^\perp=0$. Take the projection map $\pi:\mathrm{Col}^n(A)\to\mathrm{Col}^n(A)$ with image F and kernel $F^\perp$. Define the matrix $P\in M_n(A)$ by the right half of (3.1) (where the 1 is in the i-th position), so π is given by left multiplication by P, and since $\pi\circ\pi=\pi$ we have $P^2=P$. For the basis of F, we have f_i as given in the right formula in (3.1), and the dual has basis where e^j is projection to the j-th entry of $\mathrm{Col}^n(A)$. □

For vector bundles over topological spaces, the finitely generated free modules correspond to trivial vector bundles, so instead of looking at locally trivial vector bundles, we could (with minimal assumptions) equivalently consider sub-bundles of trivial bundles which have a complement. This is the starting point of topological K-theory, as described in Atiyah's celebrated book on K-theory. We shall consider the K-theory of C^*-algebras in §3.3.

We now turn to the important case of F an A-bimodule. Then $F^\sharp$ is also a bimodule with right A-action $(\alpha.a)(e)=\alpha(a.e)$ and evaluation descends to a bimodule map $\mathrm{ev}_F:F^\sharp\otimes_A F\to A$ given by $\mathrm{ev}_F(\alpha\otimes e)=\alpha(e)$. One can ask if there is also a 'coevaluation map' $\mathrm{coev}_F:A\to F\otimes_A F^\sharp$ such that

$$(\mathrm{id}\otimes\mathrm{ev}_F)(\mathrm{coev}_F\otimes\mathrm{id})=\mathrm{id}_F,\quad(\mathrm{ev}_F\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{coev}_F)=\mathrm{id}_{F^\sharp}\tag{3.2}$$

as identity maps on the indicated spaces. These are the conditions for $F^\sharp$ to be left dual to F in the category of A-bimodules in the categorical sense of §2.4. This map is determined by its value on 1.

Proposition 3.8 *An A-bimodule F is fgp as a right module if and only if it admits a bimodule map* $\mathrm{coev}_F:A\to F\otimes_A F^\sharp$ *making $F^\sharp$ left dual to F in the category of A-bimodules. If F has dual bases $f_i\in F$ and $e^i\in F^\sharp$ then* $\mathrm{coev}_F(1)=\sum_i f_i\otimes e^i$.

Proof When F,N in Proposition 3.5 are bimodules, (1) becomes a bijection between central elements of $N\otimes_A F^\sharp$ and bimodule maps $F\to N$. Thus if F is fgp, the canonical dual bases element corresponding to id_F is central. We take this for $\mathrm{coev}_F(1)\in F\otimes_A F^\sharp$ and $\mathrm{coev}_F(a)=a.\mathrm{coev}_F(1)=\mathrm{coev}_F(1).a$ for all $a\in A$. Then by definition of dual bases and for any $f\in F$, $\beta\in F^\sharp$,

$$(\mathrm{id}\otimes\mathrm{ev}_F)(\mathrm{coev}_F(1)\otimes f)=(\mathrm{id}\otimes\mathrm{ev}_F)(f_i\otimes e^i\otimes f)=f_i.e^i(f)=f$$
$$(\mathrm{ev}_F\otimes\mathrm{id})(\beta\otimes\mathrm{coev}_F(1))=\beta(f_i).e^i.$$

Evaluating the second line on $f\in F$ gives $\beta(f_i).e^i(f)=\beta(f_i.e^i(f))=\beta(f)$ as required. Conversely, given coev_F making $F^\sharp$ a left dual in ${}_A\mathcal{M}_A$, write $\mathrm{coev}_F(1)=\sum_i f_i\otimes_A e^i$ for some $f_i\in F$, $e^i\in F^\sharp$, which provide the dual bases. □

Finally, while we have focused on the right and bi-module theory, there is obviously an equally good left-handed version. If E is a left A-module then it has a right dual $E^\flat = {}_A\mathrm{Hom}(E, A)$, the left module maps from E to A. Then $E^\flat$ is a right A-module by $(\beta.a)(e) = \beta(e).a$ for $a \in A$, $e \in E$ and $\beta \in E^\flat$. We define E to be left finitely generated projective if there are a finite number of $e^i \in E$ and $f_i \in E^\flat$ (the 'dual bases') such that for all $e \in E$,

$$e = \sum f_i(e).e^i. \tag{3.3}$$

It follows that $\beta = \sum f_i.\beta(e^i)$ for all $\beta \in E^\flat$. Note that $e^j = \sum f_i(e^j).e^i$, and that if we set the matrix $P_{ji} = f_i(e^j)$ then $P^2 = P$. There is now an obvious left module version of Theorem 3.7, using the injective left module map $\underline{\beta} : E \to \mathrm{Row}^n(A)$ given by $(\underline{\beta}(e))_i = f_i(e)$, which gives an isomorphism of E with $\mathrm{Row}^n(A).P$.

Example 3.9 If F is a right fgp A-module we let $f_i \in F$ and $e^i \in F^\sharp$ be dual bases. Then $F^\sharp$ becomes a left fgp A-module with $(F^\sharp)^\flat = F$ and dual bases $e^i \in F^\sharp$ and $f_i \in (F^\sharp)^\flat$ defined by $f_i(\alpha) = \alpha(f_i)$ for $\alpha \in F^\sharp$. Moreover, there is an idempotent matrix P such that F is isomorphic to $P.\mathrm{Col}^n(A)$, and $F^\sharp$ is isomorphic to $\mathrm{Row}^n(A).P$. Using these isomorphisms, the evaluation map $\mathrm{ev} : F^\sharp \otimes F \to A$ is simply matrix multiplication $\mathrm{Row}^n(A).P \otimes P.\mathrm{Col}^n(A) \to A$. ◇

Similarly, if the left A-module E is also a bimodule, then its dual $E^\flat$ is a bimodule with left action $(a.\beta)(e) = \beta(e.a)$ for $a \in A$, $e \in E$ and $\beta \in E^\flat$, and there is an evaluation map $\mathrm{ev}_E : E \otimes_A E^\flat \to A$ by $\mathrm{ev}_E(e \otimes \beta) = \beta(e)$. If E is a bimodule which is fgp as a left A-module, then we also have a coevaluation map $\mathrm{coev}_E : A \to E^\flat \otimes_A E$ given in terms of the dual bases by $\mathrm{coev}_E(1) = \sum f_i \otimes e^i$, which we usually abbreviate to coev_E or just coev. These maps satisfy

$$(\mathrm{id} \otimes \mathrm{ev}_E)(\mathrm{coev}_E \otimes \mathrm{id}) = \mathrm{id}_{E^\flat}, \quad (\mathrm{ev}_E \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}_E) = \mathrm{id}_E \tag{3.4}$$

making $E^\flat$ a categorical right dual of E in ${}_A\mathcal{M}_A$. We do not distinguish between left and right in the notation for evaluation and coevaluation, but rely on context. Using the idempotent matrix P we can also study endomorphisms of E.

Proposition 3.10 *For an fgp left A-module E, the left module maps from E to itself are in 1–1 correspondence with the algebra $PM_n(A)P$, acting on* $\mathrm{Row}^n(A)P$*, the row form of E, by right multiplication.*

Proof Suppose that E has dual bases $e^i \in E$ and $f_i \in E^\flat$. Then a left module map from E to itself has to send $e^i \mapsto \sum_j b_{ij}e^j$ for some $b_{ij} \in A$, so we get $\sum_i a_i e^i \mapsto \sum_{ij} a_i b_{ij} e^j$. For this to be well defined it must preserve the relations $e^i = P_{ij}e^j$, so we must have $b_{ik}e^k = P_{ij}b_{jk}e^k$, and evaluating with f_m gives the matrix equation $BP = PBP$ for $B = (b_{ij})$. Also if B gives the zero endomorphism, then evaluating $b_{ik}e^k$ with f_m gives $BP = 0$. □

The algebra $PM_n(A)P$ (or $P.M_n(A).P$ if we stress the product in A) is called a *corner algebra*, because of the picture where P consists of a list of 1s followed by 0s down the main diagonal of the matrix. If we regard E in Proposition 3.10 as $\mathrm{Row}^n(A)P$, then the action of the corner algebra is given by matrix multiplication with $PM_n(A)P$ on the right. By definition, this action commutes with the left A-action, so we have an A-$PM_n(A)P$ bimodule. An algebra map from A to $PM_n(A)P$ (should such a thing exist) would give E the structure of an A-A bimodule.

Much is made in classical geometry of sets where sections of bundles vanish (for example, in the theory of divisors in algebraic geometry). Although the idea of a set where an element of a module vanishes may not have an obvious extension to noncommutative geometry, we can say a few things.

Definition 3.11 An element $e \in E$ of a left A-module is said to be nowhere vanishing if there is an $\alpha \in E^\flat$ such that $\mathrm{ev}(e \otimes \alpha) = 1 \in A$. The right ideal $\{\mathrm{ev}(e \otimes \alpha) : \alpha \in E^\flat\} \subseteq A$ is called the (right) vanishing ideal $\mathrm{Van}(e)$ of e, so if e is nowhere vanishing then $\mathrm{Van}(e) = A$. We extend Van to subsets of E by summing the ideals for each e in the subset. For right modules, there is a corresponding idea of left vanishing ideal using the other-sided dual.

In the classical case for a section e of a vector bundle over a space X, $\mathrm{Van}(e)$ consists of a set of functions on X which are zero on the subset where e vanishes. The vanishing ideal of a set of sections consists of a set of functions on X which are zero on the subset where all the sections simultaneously vanish.

Proposition 3.12 *Let E be a left A-module and $e \in E$ nowhere vanishing. Choose $\alpha \in E^\flat$ such that $\mathrm{ev}(e \otimes \alpha) = 1 \in A$. Then as left A-modules, $E = A.e \oplus K$, where $K = \{v \in E : \alpha(v) = 0\}$. Further, the module $A.e$ is freely generated by e.*

Proof There is a left module map $p : E \to E$ given by $p(v) = v - \alpha(v).e$ which is an idempotent, i.e., $p^2 = p$. The kernel of p is $A.e$. If the module $A.e$ was not free, there would be an equation $a.e = 0$ for some $0 \neq a \in A$. But then applying α gives $a.\alpha(e) = a = 0$. If v is in the image of p then $p(v) = v$ so $\alpha(v).e = 0$, and thus $\alpha(v) = 0$, so the image of p is K. $\square$

We now give a classical example of a line bundle on $\mathbb{CP}^1$, but in such a manner that it can be generalised to noncommutative examples. We also take the opportunity to calculate the vanishing set for an obvious section of the line bundle.

Example 3.13 The set of lines (1-dimensional complex subspaces) in $\mathbb{C}^2$ is the complex projective space $\mathbb{CP}^1$. There is a tautological bundle L on this space where L_ℓ at each $\ell \in \mathbb{CP}^1$ is the line ℓ itself in $\mathbb{C}^2$. Each $L_\ell \cong \mathbb{C}$ and we have a rank 1 vector bundle (a line bundle).

Consider $\mathbb{C}^2$ with the usual inner product $\langle (z_1, z_2), (w_1, w_2)\rangle = z_1 w_1^* + z_2 w_2^*$. By orthogonal projections, lines in $\mathbb{C}^2$ are in 1–1 correspondence with certain matrices

$$\mathbb{CP}^1 \leftrightarrow \big\{(P_{ij}) \in M_2(\mathbb{C}) \mid P^2 = P, \quad \mathrm{Tr}(P) = 1, \quad P^* = P\big\}. \tag{3.5}$$

If we specify a function from $\mathbb{CP}^1$ to the matrices in (3.5), this explicitly gives a 1-dimensional subspace for every point of $\mathbb{CP}^1$, i.e., a line bundle on $\mathbb{CP}^1$. A function from $\mathbb{CP}^1$ to $M_2(\mathbb{C})$ is just the same as taking matrices $M_2(C(\mathbb{CP}^1))$ with entries in the algebra $C(\mathbb{CP}^1)$ of continuous complex functions on $\mathbb{CP}^1$. We could write the canonical line bundle explicitly in this manner. However, we are going to take a different course, one that will eventually allow us to obtain noncommutative examples, as in Example 3.14. We choose a parameterisation for P as

$$P = \begin{pmatrix} 1-x & z \\ z^* & x \end{pmatrix}; \quad x, z \in C(\mathbb{CP}^1)$$

and ask what conditions must hold for the equations in (3.5) to be satisfied. A little calculation shows that we must have

$$x = x^*, \quad zx = xz, \quad zz^* = z^*z = x(1-x).$$

Under these conditions P is the hermitian projection with $(z^*, x).P = (z^*, x)$. A value of x, z satisfying the equations gives a line, which is simply a point of $\mathbb{CP}^1$. That line is the 1-dimensional subspace containing (z^*, x), except when $x = z = 0$, in which case the line is multiples of $(1, 0)$. Thus we have the tautological bundle (we have identified the point in $\mathbb{CP}^1$ with the line given by P), and there is a section of this bundle, given in row notation by (z^*, x), which of course vanishes at $z = x = 0$. The tautological vector bundle gives the module of sections $\kappa = \mathrm{Row}^2(\mathbb{C}[\mathbb{CP}^1]).P$. In terms of topology, the equations (3.5) correspond with functions on the sphere in $\mathbb{R}^3$ with radius $\frac{1}{2}$ by (x_1, x_2, x_3) corresponding to $z = x_1 + \mathrm{i}x_2$ and $x = x_3 + \frac{1}{2}$, which is not surprising as $\mathbb{CP}^1$ is homeomorphic to a sphere.

Now we can look at this another way: Consider a $*$-algebra generated by the entries of the matrix P, thus we have x and z satisfying the conditions in (3.5). Note that these conditions force the algebra to be commutative, we do not have to assume this from the beginning. The relations on the entries P_{ij} themselves specify the algebra, and the result is isomorphic to $\mathbb{C}[\mathbb{CP}^1]$. $\diamond$

Thus we can reconstruct the algebra of functions $\mathrm{C}[\mathbb{CP}^1]$ as the algebra generated by the entries of a matrix P subject to certain relations. Now let us impose a relation that is classically impossible to satisfy, and see what happens. For example, a hermitian projection matrix with entries in $\mathbb{C}$ must have an integer trace, whereas we could impose a non-integer trace condition.

Example 3.14 (Fuzzy Sphere) The $*$-algebra $\mathbb{C}_\lambda[\mathbb{CP}^1]$ or $\mathbb{C}_\lambda[S^2]$ is defined exactly as in Example 3.13 except with $\mathrm{Tr}(P) = 1 + \lambda_P$, where $\lambda_P \in \mathbb{R}$. We now write

$$P = \begin{pmatrix} 1+\lambda_P - x & z \\ z^* & x \end{pmatrix}$$

which solves the deformed trace condition. The conditions for being a hermitian projection this time give the relations

$$x^* = x, \quad [x, z] = \lambda_P z, \quad [z, z^*] = 2\lambda_P(x - \frac{1+\lambda_P}{2}), \quad z^* z = x(1-x)$$

The first two relations are in fact the enveloping algebra $U(su_2)$ as 'fuzzy $\mathbb{R}^3$' and the last relation sets the Casimir equal to a constant, i.e., this is the standard quantisation of S^2 which we already met in Example 1.46. This in turn is the standard quantisation of the sphere as a coadjoint orbit in su_2^* as mentioned at the end of Chap. 2. Moreover, there is a section of the bundle given by P, given in row notation by (z^*, x). As the projections to the first and second entries generate the dual, the vanishing ideal of this section is the set of all $z^*a + xb$ for $a, b \in \mathbb{C}_\lambda[\mathbb{CP}^1]$, i.e., the left ideal generated by z^*, x. Showing that this ideal is not all of $\mathbb{C}_\lambda[\mathbb{CP}^1]$, i.e., showing that the section is not nowhere vanishing, requires some work. First use the commutation relations to order the generators in any term as x's then z^*'s then z's. This must be done order by order, as the commutation relations mess up the lower orders. Next apply the last relation to replace any z^*z by xs. Doing this multiple times reduces any sum of terms to sums of the linear basis of the algebra, consisting of $x^n z^{*m}$ and $x^n z^m$ for $n, m \geq 0$. Using sums of this basis in place of a and b in $z^*a + xb = 1$ gives a contradiction, as there is no way that constants can appear in reducing $z^*a + xb$ to the linear basis elements. ◇

Example 3.15 (q-Sphere) The $*$-algebra $\mathbb{C}_q[\mathbb{CP}^1]$ or $\mathbb{C}_q[S^2]$ can be defined exactly as in Example 3.13 but with a modified 'q-trace' $\mathrm{Tr}_q(P) = P_{11} + q^2 P_{22}$, where $q \in \mathbb{R}^*$. Here we solve the $\mathrm{Tr}_q(P) = 1$ condition with

$$P = \begin{pmatrix} 1 - q^2 x & z \\ z^* & x \end{pmatrix}.$$

The result is the relations

$$zx = q^2 xz, \quad zz^* = q^4 z^* z + q^2(1-q^2)x, \quad z^* z = x(1-x),$$

which we have already met in Lemma 2.34 from a different point of view. The reason for the q-trace is that this is $\mathbb{C}_q[SU_2]$-invariant and as a result $\mathbb{C}_q[S^2]$ necessarily has a (left) coaction of $\mathbb{C}_q[SU_2]$ corresponding to the classical picture. Just after Proposition 2.33 we saw that $\mathbb{C}_q[S^2]$ is a certain invariant subalgebra of $\mathbb{C}_q[SU_2]$ by $x = bb^*$ and $z = cd$, corresponding to $S^2 = SU_2/U(1)$.

Again, there is a section of the bundle given by P, given in row notation by (z^*, x). The vanishing ideal of this section is the set of all $z^*a + xb$ for $a, b \in \mathbb{C}_q[\mathbb{CP}^1]$, i.e., the left ideal generated by z^*, x. Showing that this ideal is not all of $\mathbb{C}_q[\mathbb{CP}^1]$ is not so difficult, as there are no relations which generate constants. ◇

Example 3.16 (Fuzzy q-Sphere) One can ask what happens if one combines these ideas and asks for $\mathrm{Tr}_q(P) = 1+\lambda_P$. The result is the '$q$-fuzzy sphere' $\mathbb{C}_{q,\tau}[S^2]$, also known as the Podleś sphere, with two parameters q, λ_P as obtained from another point of view at the end of §2.7 as a 'constant time slice' of the unit hyperboloid in q-Minkowski space (with τ depending on λ_P). ◇

The algebras in Examples 3.13, 3.14, 3.15 and 3.16 can all be viewed as coming from one universal $*$-algebra $\mathbb{C}\langle \mathrm{G}_n\rangle$ defined by generators $\{p_{ij}\}$ for $i, j = 1, \cdots, n$ with just the hermitian projector relations $p_{ij}^* = p_{ji}$ and $\sum_j p_{ij} p_{jk} = p_{ik}$ or $P^\dagger = P$, $P^2 = P$ in terms of matrix P with entries p_{ij} and conjugate-transposition †. This in turn can be completed to a *universal C^*-algebra* called the *noncommutative Grassmannian* $\mathrm{G}_n^{\mathrm{nc}}$. We will cover C^*-algebras and operators on Hilbert spaces in §3.3.1 but what this comes down to here is that if we have any collection of bounded operators (labelled by the generators) on a Hilbert space obeying the above relations, then there is an extension to a representation of the universal C^*-algebra. The latter is the 'biggest' C^*-algebra satisfying the relations in the sense that it maps (with a unique C^*-algebra map) to all other C^*-algebras satisfying the relations, mapping generators to corresponding 'generators'. To pick one case, there is an algebra map $\mathrm{G}_2^{\mathrm{nc}} \to \mathbb{C}_\lambda[\mathbb{CP}^1]$ sending $p_{11} \mapsto 1 + \lambda_P - x$, $p_{12} = p_{21}^* \to z$ and $p_{22} \mapsto x$. As these elements actually generate $\mathbb{C}_\lambda[\mathbb{CP}^1]$, the map has a dense image.

For the noncommutative torus, it is convenient to use a Schwarz space version

$$S_\theta[\mathbb{T}^2] = \Big\{ \sum a_{mn} u^m v^n \mid (a_{mn}) \in S(\mathbb{Z}^2)\Big\},$$

where the Schwarz space $S(\mathbb{Z}^2)$ consists of functions on $\mathbb{Z}^2$ decaying faster than any power of n, m (more precisely $(|m| + |n|)^N |a_{mn}|$ is bounded on $\mathbb{Z}^2$ for all $N \in \mathbb{N}$). This is larger than the algebraic $\mathbb{C}_\theta[\mathbb{T}^2]$ but still workable. $S(\mathbb{R})$ is defined similarly as smooth functions on $\mathbb{R}$ all of whose derivatives decay faster than any power.

Example 3.17 For irrational θ, $S_\theta[\mathbb{T}^2]$ has finitely generated projective modules $\mathcal{E}_{p,q}$, where $p, q \in \mathbb{Z}$. As a vector space $\mathcal{E}_{p,q} = S(\mathbb{R}, \mathbb{C}^q)$ and

$$(u.s)(x) = u_0 s(x - \theta + 2\pi \frac{p}{q}), \quad (v.s)(x) = \mathrm{e}^{\mathrm{i}x} v_0 s(x),$$

where u_0, v_0 are fixed unitary matrices in $\mathbb{C}^q$ obeying $v_0 u_0 = \mathrm{e}^{2\pi\mathrm{i}\frac{p}{q}} u_0 v_0$ and $s \in S(\mathbb{R}, \mathbb{C}^q) = S(R) \otimes \mathbb{C}^q$. One may verify that these operations make $\mathcal{E}_{p,q}$ into a $S_\theta[\mathbb{T}^2]$ module. However, the projection $e_{p,q}$ (the Powers projector) and the proof that we obtain all the K-theory are a bit beyond our scope here. ◇

3.2 Covariant Derivatives

For a trivial bundle with fibre $\mathbb{R}^m$ on a subset of $\mathbb{R}^n$, we can use partial derivatives to differentiate sections of the bundle. With coordinates $(x^1, \dots, x^n)$ on $\mathbb{R}^n$, a section is just an m-tuple $(s^1, \dots, s^m)$ of real-valued functions of $(x^1, \dots, x^n)$ and the partial derivatives $\frac{\partial s^j}{\partial x^i}$ (which we typically write with a comma as $s^j{}_{,i}$) provide a new m-tuple for each i. These partial derivatives fail to be globally well-defined on a general manifold even if we have a locally trivial bundle. For that we need a more general notion of a *covariant derivative* or *connection* (we shall not attempt to distinguish between these terms) of the form

$$(\nabla_i s)^j = s^j{}_{;i} = \frac{\partial s^j}{\partial x^i} + \Gamma^j{}_{ik} s^k \tag{3.6}$$

in a particular coordinate patch, where the functions $\Gamma^j{}_{ik}$ are the *Christoffel symbols*. The reader should note the semicolon and also be careful about the positions of the indices. The summation convention is taken over a repeated index and by convention our coordinates have upstairs indices. In this case a vector field locally has the form $v = v^i \frac{\partial}{\partial x^i}$ for coefficient functions v^i and a 1-form $\xi = \xi_i \mathrm{d}x^i$ with coefficients functions ξ_i. In terms of components, a vector field is covariantly differentiated according to (3.6), and a 1-form according to

$$(\nabla_i \xi)_j = \xi_{j;i} = \frac{\partial \xi_j}{\partial x^i} - \Gamma^k{}_{ij} \xi_k. \tag{3.7}$$

The fact that (3.6) and (3.7) use the same Christoffel symbols comes from the dual pairing between vectors and 1-forms. For general vector bundles, we have to specify conventions as to whether we take upstairs or downstairs indices for bases and coefficients. Moreover, given a vector field v we define the covariant derivative of s along v as $\nabla_v s = v^i \nabla_i s$ and write this via the pairing $\mathrm{ev}(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j) = \delta_{ij}$ as

$$\nabla_v s = \mathrm{ev}\Big(v^i \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j\Big).\nabla_j s = (\mathrm{ev} \otimes \mathrm{id})(v \otimes \nabla s),$$

where $\nabla s = \mathrm{d}x^i \otimes \nabla_i s$ has values in the tensor product of 1-forms and sections. Conversely, given ∇ we recover ∇_i by $\nabla_i s = (\mathrm{ev} \otimes \mathrm{id})(\frac{\partial}{\partial x^i} \otimes \nabla s)$, so these are equivalent points of view.

Covariant derivatives have a characteristic Leibniz rule

$$\nabla_i (f.s) = \frac{\partial f}{\partial x^i} s + f \nabla_i s,$$

for any function f and section s, and any operator ∇_i on sections satisfying this can be written as (3.6) for some choice of Christoffel symbols. In terms of ∇, the Leibniz rule takes a particularly simple form

$$\nabla(f.s) = \mathrm{d}x^i \otimes \nabla_i(f.s) = \mathrm{d}x^i \frac{\partial f}{\partial x^i} \otimes s + \mathrm{d}x^i \otimes f\nabla_i s = \mathrm{d}f \otimes s + f.\nabla s.$$

In the absence of local coordinates in noncommutative geometry, it will be convenient to use this version of the Leibniz rule as a definition.

Definition 3.18 A left-covariant derivative or left connection on a left A-module E is a $\Bbbk$-linear map $\nabla_E : E \to \Omega^1 \otimes_A E$ obeying the left Leibniz rule,

$$\nabla_E(a.e) = \mathrm{d}a \otimes e + a.\nabla_E e, \quad e \in E,\ a \in A.$$

The curvature $R_E : E \to \Omega^2 \otimes_A E$ of ∇_E is defined by

$$R_E e = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\nabla_E e.$$

Note that the definition of curvature shows how we can add two terms which are not well defined, and get a well defined result. As $\nabla_E e \in \Omega^1 \otimes_A E$, we need $\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E$ to be well defined on $\Omega^1 \otimes_A E$. This means that it must give the same result when applied to $\xi.a \otimes e$ and $\xi \otimes a.e$, for $a \in A$, which follows from

$$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi.a \otimes e) = \mathrm{d}\xi.a \otimes e - \xi \wedge \mathrm{d}a \otimes e - \xi.a \wedge \nabla_E e,$$
$$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes a.e) = \mathrm{d}\xi \otimes a.e - \xi \wedge \mathrm{d}a \otimes e - \xi \wedge a.\nabla_E e.$$

The classical statement that the curvature is a tensor becomes the following.

Lemma 3.19 *The curvature $R_E : E \to \Omega^2 \otimes_A E$ is a left A-module map.*

Proof For $a \in A$ and $e \in E$, and writing $\nabla_E e = \xi \otimes f$ as a shorthand (sum of terms understood),

$$\begin{aligned} R_E(a.e) &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\nabla_E(a.e) \\ &= -(\mathrm{id} \wedge \nabla_E)(\mathrm{d}a \otimes e) + \mathrm{d}a \wedge \xi \otimes f + a.(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes f) \\ &= -\mathrm{d}a \wedge \xi \otimes f + \mathrm{d}a \wedge \xi \otimes f + a.(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes f) \\ &= a.R_E e. \end{aligned}$$

□

Example 3.20 In the classical case of a covariant derivative on a vector bundle on a manifold, we should check how the curvature we have defined corresponds to the usual commutator of covariant derivatives: If we write $\nabla e = \mathrm{d}x^i \otimes \nabla_i e$, then

$$\begin{aligned} R_\nabla e = &\ (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)(\mathrm{d}x^i \otimes \nabla_i e) \\ &= -\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \nabla_j(\nabla_i(e)) = \frac{1}{2}\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes [\nabla_i, \nabla_j]e. \end{aligned} \qquad \diamond$$

In the noncommutative case we have a parallel notion of right-covariant derivative $\tilde{\nabla}$ where we use the tilde to indicate the right-handed theory.

Definition 3.21 A right-covariant derivative on a right A-module F is a map $\tilde{\nabla}_F : F \to F \otimes_A \Omega^1$ obeying the right Leibniz rule, for $f \in F$ and $a \in A$,

$$\tilde{\nabla}_F(f.a) = f \otimes \mathrm{d}a + (\tilde{\nabla}_F f).a.$$

The curvature is now a right module map $\tilde{R}_F : F \to F \otimes_A \Omega^2$ defined by

$$\tilde{R}_F = (\mathrm{id} \otimes \mathrm{d} + \tilde{\nabla}_F \wedge \mathrm{id})\tilde{\nabla}_F.$$

Example 3.22 Suppose that A has an inner calculus, with $\theta \in \Omega^1$ such that $\mathrm{d}a = [\theta, a]$ for all $a \in A$. If E is a left A-module then it has a left connection ${}_\theta\nabla e = \theta \otimes e$ with curvature ${}_\theta Re = (\mathrm{d}\theta - \theta \wedge \theta) \otimes e$. If F is a right A-module then it has a right connection $\tilde{\nabla}_\theta f = -f \otimes \theta$ with curvature $\tilde{R}_\theta f = -f \otimes (\mathrm{d}\theta - \theta \wedge \theta)$.

Proof We check the relevant Leibniz rules, for $a \in A$, $e \in E$ and $f \in F$:

$$\begin{aligned} &{}_\theta\nabla(a.e) = \theta.a \otimes e = [\theta, a] + a.\theta \otimes e = \mathrm{d}a \otimes e + a.{}_\theta\nabla e, \\ &\tilde{\nabla}_\theta(f.a) = -f \otimes a.\theta = f \otimes [\theta, a] - f \otimes \theta.a = f \otimes \mathrm{d}a + (\tilde{\nabla}_\theta f).a. \end{aligned}$$

The curvatures are easily given by applying the formulae. □

If E is a left fgp A-module then we can describe a left-covariant derivative ∇_E by Christoffel symbols as classically. Given a dual bases $e^i \in E$ and $e_i \in E^\flat$ (using an up and down summation convention of repeated indices), these are defined by

$$\Gamma^i{}_k = -(\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes e_k) \in \Omega^1; \quad \nabla_E e^i = -\Gamma^i{}_k \otimes e^k. \tag{3.8}$$

We choose the minus sign to fit with the standard convention for the covariant derivative of 1-forms, as our basis elements of E have upstairs indices. To get the more usual three-index Christoffel symbols we would need to use a basis for the 1-forms. On a freely generated module, any choice of Christoffel symbols will give a covariant derivative. For a general dual basis on an fgp module the situation is more complicated and best expressed in terms of the associated projection matrix

$P_{ij} = e_j(e^i) = \mathrm{ev}(e^i \otimes e_j)$ with entries in A and $P^2 = P$. We fit the Christoffel symbols into matrix notation $\Gamma \in M_n(\Omega^1)$ by setting $(\Gamma)_{ij} = \Gamma^i{}_j$.

Proposition 3.23 *Given a left connection ∇_E on a left fgp A-module E and dual bases $e^i \in E$ and $e_i \in E^\flat$ with $P_{ij} = \mathrm{ev}(e^i \otimes e_j)$, the matrix Christoffel symbols defined in (3.8) obey*

$$\Gamma P = \Gamma, \quad \Gamma = P\Gamma - (\mathrm{d}P)P.$$

Conversely, a choice of Christoffel symbols satisfying these conditions defines a left connection on E. Its curvature is given by

$$R_E(e^i) = -((\mathrm{d}\Gamma + \Gamma \wedge \Gamma)P)_{ik} \otimes e^k.$$

Proof We have $\Gamma^i{}_k \mathrm{ev}(e^k \otimes e_j) = \Gamma^i{}_j$ from (3.8). Using the dual basis,

$$\nabla_E e^i = \nabla_E(\mathrm{ev}(e^i \otimes e_j).e^j) = \mathrm{ev}(e^i \otimes e_j).\nabla_E(e^j) + \mathrm{d}(\mathrm{ev}(e^i \otimes e_j)) \otimes e^j,$$

so in terms of the Christoffel symbols,

$$\Gamma^i{}_k \otimes e^k = \mathrm{ev}(e^i \otimes e_j).\Gamma^j{}_k \otimes e^k - \mathrm{d}(\mathrm{ev}(e^i \otimes e_k)) \otimes e^k,$$
$$\Gamma^i{}_m = \mathrm{ev}(e^i \otimes e_j).\Gamma^j{}_m - \mathrm{d}(\mathrm{ev}(e^i \otimes e_k)).\mathrm{ev}(e^k \otimes e_m)$$

where we evaluated with e_m. Conversely, if $\Gamma^i{}_k$ has the given properties then define

$$\nabla(a_i e^i) = \mathrm{d}a_i \otimes e^i - a_i \Gamma^i{}_k \otimes e^k$$

for all $a_i \in A$. For this to be well defined, we need to show that the RHS of this equation is zero if $a_i e^i = 0$, which implies $a_i P_{ik} = 0$. Indeed, then

$$\begin{aligned}
\mathrm{d}a_i \otimes e^i - a_i \Gamma^i{}_k \otimes e^k &= (\mathrm{d}a_i) P_{ik} \otimes e^k - a_i \Gamma^i{}_k \otimes e^k \\
&= -a_i \mathrm{d}P_{ik} \otimes e^k - a_i \Gamma^i{}_k \otimes e^k \\
&= -a_i (\mathrm{d}P_{im}) P_{mk} \otimes e^k - a_i \Gamma^i{}_k \otimes e^k \\
&= -a_i \big((\mathrm{d}P_{im}) P_{mk} + \Gamma^i{}_k\big) \otimes e^k \\
&= -a_i P_{ij} \Gamma^j{}_k \otimes e^k = 0.
\end{aligned}$$

The curvature is a direct calculation from its definition and (3.8). □

For a left fgp A-module, there is always at least one left-covariant derivative.

Proposition 3.24 *Given an fgp left A-module E with dual bases $e^i \in E$ and $e_i \in E^\flat$, we define the* Grassmann connection

$$\nabla_E : E \to \Omega^1 \otimes_A E, \quad \nabla_E e = \sum_i \mathrm{d}(e_i(e)) \otimes e^i$$

for all $e \in E$. Writing e^j, e_j for another set of dual bases, the curvature is

$$R_E e = -\sum_{i,j} \mathrm{d}(e_i(e)) \wedge \mathrm{d}(e_j(e^i)) \otimes e^j.$$

Writing $P_{ij} = \mathrm{ev}(e^i \otimes e_j) = e_j(e^i)$ and using $P_{ij}e^j = e^i$, we have

$$\nabla_E e^i = \sum_i (\mathrm{d}P_{ij})P_{jk} \otimes e^k, \quad R_E e^i = -\sum_{j,k,m} (\mathrm{d}P_{ij} \wedge \mathrm{d}P_{jk})P_{km} \otimes e^m.$$

Proof For the left Leibniz rule, for $a \in A$ with summation implicit

$$\begin{aligned}\nabla_E(ae) &= \mathrm{d}(e_i(ae)) \otimes e^i = \mathrm{d}(ae_i(e)) \otimes e^i = (\mathrm{d}a)e_i(e) \otimes e^i + a\mathrm{d}(e_i(e)) \otimes e^i \\ &= \mathrm{d}a \otimes e_i(e)e^i + a\mathrm{d}(e_i(e)) \otimes e^i = \mathrm{d}a \otimes e + a\nabla_E(e).\end{aligned}$$

The curvature follows from Proposition 3.23 using $\Gamma = -(\mathrm{d}P)P$ and the identity $P(\mathrm{d}P)P = 0$, which follows from applying d to the equation $P^2 = P$. (This identity can also be used to prove that Γ provides a left connection by Proposition 3.23.) □

The next theorem is a partial converse to Proposition 3.24, only using projective modules rather than fgp modules. Projective modules are defined in a similar manner to the characterisation of fgp modules in Theorem 3.7.

Definition 3.25 A left (similarly right) A-module is projective if it is isomorphic to a complemented submodule of a left (right) free A-module (i.e., having a complementary submodule).

Now we can state the Cuntz–Quillen theorem (actually the original only concerned the universal calculus, but we get a generalisation for free):

Theorem 3.26 (Cuntz and Quillen) *If a left A-module E is projective, then it has a left connection for any calculus on A. Also, if a left A-module K has a left connection for the universal calculus, then K is projective.*

Proof First, suppose that $E \subseteq F$ for some free module F, and that E has a complement G in F. Define $Q : F \to F$ be the projection map with image E

and kernel G, which is a left module map. Take free generators $f^i \in F$ for some index set $i \in \mathcal{I}$. For any calculus Ω, there is a left connection

$$\nabla_F : F \to \Omega^1 \otimes_A F, \quad \nabla_F\big(\sum a_i f^i\big) = \mathrm{d}a_i \otimes f^i.$$

Now define a connection on E by $\nabla_E = (\mathrm{id} \otimes Q)\nabla_F$.

Next, suppose that a left A-module K has a left connection $\nabla : K \to \Omega^1_{\mathrm{uni}} A \otimes_A K$. Remembering that $\Omega^1_{\mathrm{uni}} A$ is the kernel of the product in $A \otimes_{\mathbb{k}} A$, there is an isomorphism of left modules $\Omega^1_{\mathrm{uni}} A \otimes_A K$ and the kernel of the left action $\ker(\triangleright) \subset A \otimes_{\mathbb{k}} K$. This is given by $a \otimes b \otimes k \mapsto a \otimes b \triangleright k$ one way and $a \otimes 1 \otimes k - 1 \otimes a \otimes k \leftarrow\!\shortmid a \otimes k$ the other way. Using this isomorphism, define $\phi = (\mathrm{id} \otimes \triangleright)\nabla : K \to A \otimes_{\mathbb{k}} K$. Now, using $\mathrm{d}_{\mathrm{uni}} a = 1 \otimes a - a \otimes 1$ and the Leibniz rule for ∇, we get

$$\phi(a \triangleright k) = 1 \otimes a \triangleright k - a \otimes k + a\phi(k),$$

which we can rearrange to see that $\psi : K \to A \otimes_{\mathbb{k}} K$ defined by $\psi(k) = 1 \otimes k - \phi(k)$ is a left A-module map. Further $\triangleright : A \otimes_{\mathbb{k}} K \to K$ has the property $(\triangleright)\psi = \mathrm{id} : K \to K$, so K is isomorphic to a complemented submodule of $A \otimes_{\mathbb{k}} K$. Finally, $A \otimes_{\mathbb{k}} K$ is freely generated by application of $1\otimes$ to a $\mathbb{k}$-linear basis of K. $\square$

We see that the universal calculus is quite restrictive for the existence of connections in the sense that we may be interested in connections on modules which are not projective. We postpone an example of this until Example 3.108, when we have a more detailed description of projective modules. Now we return to specific examples of Grassmann connections.

Example 3.27 We consider the Grassmann connections for the versions of $\mathbb{CP}^1$ given earlier. First for the classical case in Example 3.13, its Grassmann connection is the monopole on $\mathbb{CP}^1 \cong S^2$. The projection matrix P defines a left module, which we realise as a submodule of $\mathrm{Row}^2(A)$ with generators $e^1 = (1, 0).P = (1 - x, z)$ and $e^2 = (0, 1).P = (z^*, x)$. The dual basis is given by e_i on a row vector picking out the i-th entry. Writing the result of Proposition 3.24 in matrix form,

$$\mathrm{d}P \wedge \mathrm{d}P = \begin{pmatrix} -\mathrm{d}x & \mathrm{d}z \\ \mathrm{d}z^* & \mathrm{d}x \end{pmatrix} \wedge \begin{pmatrix} -\mathrm{d}x & \mathrm{d}z \\ \mathrm{d}z^* & \mathrm{d}x \end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix} \mathrm{d}x \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}z^* & -\mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}z^* - \mathrm{d}z^* \wedge \mathrm{d}x & \mathrm{d}x \wedge \mathrm{d}x + \mathrm{d}z^* \wedge \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \mathrm{d}z \wedge \mathrm{d}z^* & -2\mathrm{d}x \wedge \mathrm{d}z \\ 2\mathrm{d}x \wedge \mathrm{d}z^* & -\mathrm{d}z \wedge \mathrm{d}z^* \end{pmatrix},$$

where we have used the classical antisymmetry of $\wedge$. Using $z^*z = x(1 - x)$ we get $\mathrm{d}z.z^* + z\mathrm{d}z^* = (1 - 2x)\mathrm{d}x$, and using this

$$\mathrm{d}P \wedge \mathrm{d}P.P = \frac{\mathrm{d}z \wedge \mathrm{d}z^*}{1 - 2x} \begin{pmatrix} 1 - 2x & 2z \\ 2z^* & -(1 - 2x) \end{pmatrix}.P = \frac{\mathrm{d}z \wedge \mathrm{d}z^*}{1 - 2x}(2P - 1).P$$

$$= \frac{\mathrm{d}z \wedge \mathrm{d}z^*}{1 - 2x} P = \mathrm{i}\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_3} P,$$

which is a constant multiple of the volume form (the rationally invariant top form) on S^2, as it should be for the monopole. P acts as the identity on the line bundle.

In the fuzzy sphere Example 3.14, the Grassmann connection defines a 'fuzzy monopole' with respect to any differential calculus. We get the middle line in (3.9), but from there the calculation depends on the relations in the calculus.

In the standard q-sphere Example 3.15, the Grassmann connection is the q-monopole with respect to any calculus. We take the 2D calculus inherited from the 3D calculus on $\mathbb{C}_q[SU_2]$, as described in Lemma 2.34. Then from the proof of Proposition 2.35 we take

$$\mathrm{d}x = -q^2e^+bd - q^{-2}e^-ac,\ \mathrm{d}z = q^2e^+d^2 + q^{-2}e^-c^2,\ \mathrm{d}z^* = -q^3e^+b^2 - q^{-1}e^-a^2.$$

Using the matrix product

$$(\mathrm{d}P)P = \begin{pmatrix} e^-ac & q^{-2}e^-c^2 \\ -q^{-1}e^-a^2 & -q^{-2}e^-ac \end{pmatrix}$$

we see that only the e^+ terms in the first $\mathrm{d}P$ factor contribute to the curvature, giving

$$(\mathrm{d}P \wedge \mathrm{d}P)P = -q^3e^+ \wedge e^- P = -q^3\mathrm{Vol}\, P,$$

where we have used the volume form given in Proposition 2.35. One can also use the 3D calculus inherited from the 4D calculus on $\mathbb{C}_q[SU_2]$ with different results. $\diamond$

As classically, given a left connection on Ω^1, we can define the torsion.

Definition 3.28 The torsion of a left A-covariant derivative ∇ on Ω^1 is the left A-module map $T_\nabla = \wedge\nabla - \mathrm{d} : \Omega^1 \to \Omega^2$.

We check that it is a module map, for $\xi \in \Omega^1$ and $a \in A$,

$$T_\nabla(a.\xi) = \wedge(a.\nabla\xi) + \wedge(\mathrm{d}a \otimes \xi) - a.\mathrm{d}\xi - \mathrm{d}a \wedge \xi = a.T_\nabla(\xi).$$

Example 3.29 Given a classical 'linear connection' on the space of 1-forms on a manifold as defined by $\nabla \mathrm{d}x^i = -\Gamma^i{}_{jk}\mathrm{d}x^j \otimes \mathrm{d}x^k$, the usual torsion and curvature are given by a torsion tensor $T^i{}_{kj} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$ and a Riemann tensor $R^i{}_{knm}$ also constructed from Γ. Our above formulations reduce in this classical case to

$$T_\nabla(\mathrm{d}x^i) = -\Gamma^i{}_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k = \frac{1}{2}(\Gamma^i{}_{kj} - \Gamma^i{}_{jk})\mathrm{d}x^j \wedge \mathrm{d}x^k = -\frac{1}{2}T^i{}_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k,$$

$$R_\nabla(\mathrm{d}x^i) = -\frac{1}{2}R^i{}_{kmn}\mathrm{d}x^m \wedge \mathrm{d}x^n \otimes \mathrm{d}x^k.$$

The proof is by direct computation in local coordinates, or in the following alternative coordinate-free way starting with the classical torsion defined as

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y] \tag{3.10}$$

for all vector fields X, Y on the manifold. Using the pairing

$$\langle\nabla_X Y,\eta\rangle=X(\langle Y,\eta\rangle)-\langle Y,\nabla_X\eta\rangle, \tag{3.11}$$

this is equivalent to

$$\langle T(X,Y),\eta\rangle=X(\langle Y,\eta\rangle)-\langle Y,\nabla_X\eta\rangle-Y(\langle X,\eta\rangle)+\langle X,\nabla_Y\eta\rangle-\langle[X,Y],\eta\rangle$$

for all 1-forms η. On the other hand, writing i_X for the interior product as a graded-derivation on forms, we have $i_Y i_X(\omega\wedge\eta)=\langle X,\omega\rangle\langle Y,\eta\rangle-\langle X,\eta\rangle\langle Y,\omega\rangle$ for all 1-forms ω, η. Also writing $\mathcal{L}_X$ for the Lie derivative and using the identities

$$\mathcal{L}_X=i_X\mathrm{d}+\mathrm{d}i_X,\quad [\mathcal{L}_X,i_Y]=i_{[X,Y]}, \tag{3.12}$$

one finds

$$\begin{aligned}
i_Y i_X T_\nabla(\eta)&=i_Y i_X(\wedge\nabla\eta-\mathrm{d}\eta)\\
&=\langle Y,\nabla_X\eta\rangle-\langle X,\nabla_Y\eta\rangle+i_Y\mathrm{d}i_X\eta-i_Y\mathcal{L}_X\eta\\
&=\langle Y,\nabla_X\eta\rangle-\langle X,\nabla_Y\eta\rangle-X(\langle Y,\eta\rangle)+Y(\langle X,\eta\rangle)+\langle[X,Y],\eta\rangle\\
&=-\langle T(X,Y),\eta\rangle.
\end{aligned}$$

Similarly, the classical Riemann curvature is usually defined in this approach as

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z \tag{3.13}$$

for all vector fields X, Y, Z. Dualising the connection to 1-forms as above, and using (3.11) a few times it is easy to see that

$$-\langle R(X,Y)Z,\eta\rangle=\langle Z,R(X,Y)\eta\rangle,\quad R(X,Y)\eta=[\nabla_X,\nabla_Y]\eta-\nabla_{[X,Y]}\eta$$

in terms of the connection on forms. By contrast, $R_\nabla=(\mathrm{d}\otimes\mathrm{id}-\wedge(\mathrm{id}\otimes\nabla))\nabla$ defined on forms evaluates on the Ω^2 factor of its output to

$$\begin{aligned}
i_Y i_X R_\nabla\eta&=(i_Y i_X\mathrm{d}\eta_{(\bar{1})})\eta_{(\bar{\infty})}-i_Y i_X(\eta_{(\bar{1})}\wedge\eta_{(\bar{\infty})(\bar{1})})\eta_{(\bar{\infty})(\bar{\infty})}\\
&=\big(X(\langle Y,\eta_{(\bar{1})}\rangle)-Y(\langle X,\eta_{(\bar{1})}\rangle)-\langle[X,Y],\eta_{(\bar{1})}\rangle\big)\,\eta_{(\bar{\infty})}\\
&\quad-\big(\langle X,\eta_{(\bar{1})}\rangle\langle Y,\eta_{(\bar{\infty})(\bar{1})}\rangle-\langle Y,\eta_{(\bar{1})}\rangle\langle X,\eta_{(\bar{\infty})(\bar{1})}\rangle\big)\,\eta_{(\bar{\infty})(\bar{\infty})}\\
&=-\nabla_{[X,Y]}\eta+[\nabla_X,\nabla_Y]\eta=R(X,Y)\eta,
\end{aligned}$$

where we wrote $\nabla\eta = \eta_{(\bar{1})} \otimes \eta_{(\bar{\infty})}$ as shorthand (with tensor product over the coordinate algebra) so that $\langle X, \eta_{(\bar{1})}\rangle \eta_{(\bar{\infty})} = \nabla_X \eta$ and $\langle X, \eta_{(\bar{1})}\rangle \langle Y, \eta_{(\bar{\infty})(\bar{1})}\rangle \eta_{(\bar{\infty})(\bar{\infty})} = \langle X, \eta_{(\bar{1})}\rangle \nabla_Y \eta_{(\bar{\infty})} = \nabla_Y \nabla_X \eta - Y(\langle X, \eta_{(\bar{1})}\rangle)\eta_{(\bar{\infty})}$, which we used. ◇

To give a noncommutative example involving Christoffel symbols, curvature and torsion, we turn to the algebraic noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$ from Example 1.36.

Example 3.30 The algebra $\mathbb{C}_\theta[\mathbb{T}^2]$ has invertible generators u, v with the relation $vu = e^{i\theta} uv$, and the 1-forms are freely generated by $e_1 = u^{-1}du$, $e_2 = v^{-1}dv$, which commute with elements of $\mathbb{C}_\theta[\mathbb{T}^2]$. We define a left connection on Ω^1 by

$$\nabla e_i = -\Gamma^i{}_{jk} e_j \otimes e_k$$

with apologies for the downstairs indices on the e_i. Here $\Gamma^i{}_{jk} \in \mathbb{C}_\theta[\mathbb{T}^2]$, and the 1-form-valued Christoffel symbols would be $\Gamma^i{}_k = \Gamma^i{}_{jk} e_j$. As the left module Ω^1 is free, we can take the idempotent matrix to be the identity. Then the conditions on Γ in Proposition 3.23 are automatically satisfied. The torsion and curvature are

$$T_\nabla e_i = \wedge\nabla e_i - de_i = -\Gamma^i{}_{jk} e_j \wedge e_k = (\Gamma^i{}_{21} - \Gamma^i{}_{12}) e_1 \wedge e_2,$$

$$R_E e_i = -(d\Gamma^i{}_k + \Gamma^i{}_j \wedge \Gamma^j{}_k) \otimes e_k$$

for the maximal prolongation calculus. ◇

Next we suppose that our 'coordinate algebra' is a Hopf algebra H and that a left H-module E is now also a left H-comodule, forming a Hopf module. According to the Hopf module lemma, Lemma 2.17, $E = H.{}^H E$, where ${}^H E \subseteq E$ is the space of invariants. If we also have a left-covariant differential calculus Ω^1 on H (see Definition 2.24) then we say that a left connection is *left-invariant* if

$$\begin{tikzcd} E \arrow[r, "\Delta_L"] \arrow[d, "\nabla"'] & H \otimes E \arrow[d, "\mathrm{id} \otimes \nabla"] \\ \Omega^1 \otimes_H E \arrow[r, "\Delta_L"] & H \otimes \Omega^1 \otimes_H E \end{tikzcd}$$

commutes, where Δ_L on the bottom line is the tensor product coaction on $\Omega^1 \otimes_H E$. From the Hopf-module lemma and $\nabla(h.e) = dh \otimes e + h.\nabla e$, it is clear that we can recover a connection from its restriction to ${}^H E$. Hence left-invariant left connections on E are in 1–1 correspondence with linear maps

$$\nabla^L : {}^H E \to \Lambda^1 \otimes ({}^H E), \tag{3.14}$$

where $\Omega^1 = H.\Lambda^1$ is itself a left H-Hopf-module when our calculus is left-covariant.

Example 3.31 Let H be a Hopf algebra with left-covariant differential calculus. The left-invariant connections are in 1–1 correspondence (in one direction by restriction) to linear maps $\nabla^L : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$. The most obvious case is to take $\nabla^L = 0$, giving the left *Maurer–Cartan connection* (compare this with the Maurer–Cartan form in §2.3). Its torsion is given by the left module map whose value on the invariant forms is $\wedge\nabla^L - \mathrm{d} = -\mathrm{d} : \Lambda^1 \to \Lambda^2$. Its curvature is given by the left module map whose value on the invariant forms is

$$(\mathrm{d} \otimes \mathrm{id})\nabla^L - (\mathrm{id} \wedge \nabla^L)\nabla^L : \Lambda^1 \to \Lambda^2 \otimes \Lambda^1,$$

which is zero. We give more examples for bicovariant calculi in Example 3.74. ◇

We now return to the general theory over an algebra A. It will be convenient to organise modules with connection into categories. The label of the category should really specify the differential calculus on A rather than just the algebra A, but as in other places, we omit some things to get a compact notation.

Name	Objects	Morphisms
${}_A\mathcal{E}$	(E, ∇_E) left modules, connections	Left module maps intertwining ∇_E
$\mathcal{E}_A$	$(F, \tilde\nabla_F)$ right modules, connections	Right module maps intertwining $\tilde\nabla_F$

The condition for a left module map $T : E \to K$ to intertwine the left-covariant derivatives (E, ∇_E) and (K, ∇_K) is that

$$\nabla_K T = (\mathrm{id} \otimes T)\nabla_E : E \to \Omega^1 \otimes_A K.$$

This is quite a strong condition, and we shall weaken it in §4.1. The condition is easier to check than it might seem, as in §4.1 we show that $\nabla_K T - (\mathrm{id} \otimes T)\nabla_E$ is always a left module map, so to check if it vanishes we would only have to check it on a set of left generators for E. The composition of morphisms is just the usual composition of module maps. To see that the condition is satisfied by a composition, take another morphism S from (K, ∇_K) to (G, ∇_G) in ${}_A\mathcal{E}$. Then

$$\nabla_G S \circ T = (\mathrm{id} \otimes S)\nabla_K T = (\mathrm{id} \otimes S)(\mathrm{id} \otimes T)\nabla_E = (\mathrm{id} \otimes S \circ T)\nabla_E.$$

For finitely generated projective modules, an obvious question is whether a connection on such a module gives a connection on its dual module. For example, classically a connection on the tangent bundle gives a dual connection on the cotangent bundle. However, remember that if E is a left fgp module, then its dual $E^\flat$ is naturally a right fgp module. We can now state the existence of a unique dual connection preserving the evaluation map $\mathrm{ev} : E \otimes E^\flat \to A$. Note that we do not have $\otimes_A$ here and indeed we have not specified any right module structure on E.

Proposition 3.32 *For the left fgp module E with left connection ∇_E, there is a unique right connection $\tilde\nabla_{E^\flat} : E^\flat \to E^\flat \otimes_A \Omega^1$ such that*

$$\mathrm{d} \circ \mathrm{ev} = (\mathrm{id} \otimes \mathrm{ev})(\nabla_E \otimes \mathrm{id}) + (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \tilde\nabla_{E^\flat}) : E \otimes E^\flat \to \Omega^1.$$

The curvatures of the connections on E and $E^\flat$ obey

$$(\mathrm{id} \otimes \mathrm{ev})(R_E \otimes \mathrm{id}) + (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \tilde R_{E^\flat}) = 0 : E \otimes E^\flat \to \Omega^2.$$

Proof Begin with dual bases $e^i \in E$ and $e_i \in E^\flat$. If $\tilde\nabla_{E^\flat}$ satisfying the statement exists then necessarily, summing over i,

$$e_i \otimes (\mathrm{ev} \otimes \mathrm{id})(e^i \otimes \tilde\nabla_{E^\flat}\gamma) = e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i \otimes \gamma) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes \gamma).$$

From (3.3) it follows that $\alpha = \sum e_i.\mathrm{ev}(e^i \otimes \alpha)$ for all $\alpha \in E^\flat$, and using this

$$\tilde\nabla_{E^\flat}\gamma = e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i \otimes \gamma) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes \gamma).$$

Now we use this to define $\tilde\nabla_{E^\flat}$ and check that this is a right connection,

$$\begin{aligned}\tilde\nabla_{E^\flat}(\gamma.a) &= e_i \otimes \mathrm{d}\,(\mathrm{ev}(e^i \otimes \gamma).a) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes \gamma).a \\ &= e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i \otimes \gamma).a + e_i \otimes \mathrm{ev}(e^i \otimes \gamma).\mathrm{d}a - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes \gamma).a \\ &= (\tilde\nabla_{E^\flat}\gamma).a + \gamma \otimes \mathrm{d}a,\end{aligned}$$

for all $a \in A$. We then verify the stated identity,

$$\begin{aligned}(\mathrm{ev} \otimes \mathrm{id})(e \otimes \tilde\nabla_{E^\flat}\gamma) &= e_i(e).\mathrm{d}\,\mathrm{ev}(e^i \otimes \gamma) - e_i(e).(\mathrm{id} \otimes \mathrm{ev})(\nabla_E(e^i) \otimes \gamma) \\ &= \mathrm{d}\,\mathrm{ev}(e_i(e).e^i \otimes \gamma) - \mathrm{d}(e_i(e)).\mathrm{ev}(e^i \otimes \gamma) - e_i(e).(\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes \gamma) \\ &= \mathrm{d}\,\mathrm{ev}(e \otimes \gamma) - (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e \otimes \gamma).\end{aligned}$$

For the curvature, we apply d to the first displayed equation in the statement. □

The dual connection in this proposition can also be characterised by

$$(\tilde\nabla_{E^\flat} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_E)(e_i \otimes e^i) = 0, \tag{3.15}$$

where $e_i \otimes e^i \in E^\flat \otimes_A E$ is the canonical dual bases element for E as left fgp. The result also holds the other way around. If F is a right fgp module with right connection $\tilde\nabla_F$ then there is a unique left connection $\nabla_{F^\sharp} : F^\sharp \to \Omega^1 \otimes_A F^\sharp$ with

$$\mathrm{d} \circ \mathrm{ev} = (\mathrm{id} \otimes \mathrm{ev})(\nabla_{F^\sharp} \otimes \mathrm{id}) + (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \tilde\nabla_F) : F^\sharp \otimes F \to \Omega^1. \tag{3.16}$$

This is defined in terms of the canonical dual bases element $f_i \otimes f^i \in F \otimes F^\sharp$ in the note after Proposition 3.5, by

$$\nabla_{F^\sharp}(\beta) = \mathrm{d}(\mathrm{ev}(\beta \otimes f_i)) \otimes f^i - (\mathrm{ev} \otimes \mathrm{id})(\beta \otimes \tilde{\nabla}_F f_i) \otimes f^i \qquad (3.17)$$

and can also be characterised as $(\tilde{\nabla}_F \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{F^\sharp})(f_i \otimes f^i) = 0$.

3.3 K-Theory and Cyclic Cohomology

Although most of this book is set at an algebraic level, to make a connection with topology we need to mention C^*-algebras, which involves analysis.

3.3.1 C-Algebras and Hilbert Spaces*

Given a compact Hausdorff topological space X, the continuous complex-valued functions $C(X)$ on X form an algebra under pointwise multiplication. The algebra is *unital* as the constant function 1 is the multiplicative identity. Taking the pointwise complex conjugate of a function defines a $*$-operation which makes $C(X)$ into a $*$-algebra. We also define the supremum norm $\|f\| \in [0, \infty)$ of $f \in C(X)$ by

$$\|f\| = \max_{x \in X} |f(x)|.$$

Recalling that a continuous real function on a compact set is bounded, and attains its maximum value, it is immediate that $\|f\|$ exists and that for $f, g \in C(X)$ we have

$$\|fg\| \le \|f\|\,\|g\|, \quad \|f^*\| = \|f\|, \quad \|f^* f\| = \|f\|^2. \qquad (3.18)$$

Finally note that any Cauchy sequence of continuous functions in $C(X)$ converges to a continuous function, i.e., the normed space $C(X)$ is complete. We now abstract these properties, and allow the product to be noncommutative.

Definition 3.33 A C^*-algebra is a $*$-algebra A over $\mathbb{C}$ with a norm, written $\|a\|$ for $a \in A$, satisfying the rules in (3.18), and such that the normed space A is complete.

$C(X)$ given above is an example of a commutative unital C^*-algebra and the 'commutative Gel'fand–Naimark theorem' says that every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff X.

The theory of C^*-algebras is very extensive, and we only mention a little which is relevant to our topic. It has been useful to define the functions on a hypothetical noncommutative compact Hausdorff topological space as a unital C^*-algebra. The fact that a C^*-algebra need not be unital is vital to the theory, but studying the resulting noncommutative 'noncompact spaces' using C^*-algebras is more complicated, and we leave this to others to describe. To explain the fundamental example of a C^*-algebra, we first need to introduce Hilbert spaces.

Example 3.34 The simplest Hilbert space is $\mathbb{C}^n$, with inner product

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\rangle = x_1 y_1^* + \cdots + x_n y_n^*.$$

This gives the usual norm $|\underline{x}|^2 = \langle \underline{x}, \underline{x}\rangle = |x_1|^2 + \cdots + |x_n|^2$. ◇

From our $\mathbb{C}^n$ example, we see that the inner product, to have the length of vectors being real, should be linear in one factor and conjugate linear in the other. The properties for a *hermitian* inner product $\langle x, y\rangle \in \mathbb{C}$ are, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} &\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, && \langle \lambda x, y\rangle = \lambda\langle x, y\rangle, \\ &\langle x, \lambda y\rangle = \lambda^*\langle x, y\rangle, && \langle x, y\rangle^* = \langle y, x\rangle, \end{aligned}$$

and finally that $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0$ if and only if $x = 0$.

Definition 3.35 A Hilbert space $\mathcal{H}$ is a complex vector space with a hermitian inner product and which is complete under the norm given by the inner product $|\underline{x}| = \sqrt{\langle \underline{x}, \underline{x}\rangle}$. The bounded operators $B(\mathcal{H})$ on $\mathcal{H}$ is the collection of linear operators $T : \mathcal{H} \to \mathcal{H}$ for which the following subset of $\mathbb{R}$ is bounded

$$\{|T(x)| : x \in \mathcal{H},\ |x| \leq 1\}.$$

The operator norm is the least $\|T\| \in [0, \infty)$ with $|T(x)| \leq \|T\|\,|x|$ for all $x \in \mathcal{H}$.

Example 3.36 The Hilbert space $\ell^2(\mathbb{Z})$ consists of square summable complex sequences (with index set $\mathbb{Z}$) with inner product

$$\begin{aligned} &\langle (\dots, a_{-1}, a_0, a_1, \dots), (\dots, b_{-1}, b_0, b_1, \dots)\rangle \\ &\qquad\qquad\qquad = \cdots + a_{-1}b_{-1}^* + a_0 b_0^* + a_1 b_1^* + \dots . \end{aligned}$$

The Hilbert space $L^2(S^1)$ of square Lebesgue integrable complex-valued functions on the unit circle has inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})g(\mathrm{e}^{\mathrm{i}\theta})^*\mathrm{d}\theta.$$

These Hilbert spaces are isomorphic by the inner product preserving linear map given by Fourier transform, where $\underline{f} \in \ell^2(\mathbb{Z})$ maps to $f(\mathrm{e}^{\mathrm{i}\theta}) = \sum f_n \mathrm{e}^{\mathrm{i}n\theta}$. ◇

Example 3.37 For any Hilbert space $\mathcal{H}$, the set of bounded linear operators $B(\mathcal{H})$ is a C^*-algebra. The product is composition of operators, the norm is the operator norm and the $*$-operation is given by the adjoint $T^* \in B(\mathcal{H})$ of $T \in B(\mathcal{H})$. The adjoint is defined by the property $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in \mathcal{H}$, and its existence and uniqueness follows from theorems of functional analysis. $\diamond$

There is a construction due to Gel'fand, Naimark & Segal (the GNS construction) which shows that every C^*-algebra is isomorphic (preserving the $*$-algebra structure and the norm) to a norm closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. See also the related KSGNS construction in Theorem 4.81. As a trivial case, if we take $\mathcal{H} = \mathbb{C}^n$ with its previously defined inner product then the matrices $M_n(\mathbb{C})$ (with the usual product and $*$-operation) form a C^*-algebra and every finite-dimensional C^*-algebra is isomorphic to a direct sum of $M_n(\mathbb{C})$ for various n. For a matrix $a \in M_n(\mathbb{C})$ the operator norm is the square root of the maximum eigenvalue of a^*a.

There are a couple of properties of C^*-algebras which we will need later. Suppose that A is a unital C^*-algebra. The spectrum of an element $a \in A$ is the nonempty compact subset of the complex numbers

$$\mathrm{spec}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}.$$

The spectral radius specradius(a) is the maximum value of $|\lambda|$ for $\lambda \in \mathrm{spec}(a)$, and we have the inequality $\mathrm{specradius}(a) \le \|a\|$.

Now suppose that $a \in A$ is *normal*, i.e., $a^*a = aa^*$ (of course this includes self-adjoint elements). If f is a continuous complex-valued function on the spectrum spec(a), there is an element $f(a) \in A$, an assignment called the *functional calculus*. This is defined such that if $f(z)$ is a polynomial in $z, z^* \in \mathbb{C}$, then $f(a)$ is the same polynomial applied to $a, a^* \in A$, and in this case the existence of $f(a)$ is trivial. The significance is that f can be any continuous function on the spectrum, and that $\|f(a)\| \le \max_{z\in\mathrm{spec}(a)}|f(z)|$. It follows that for g another such function $(f+g)(a) = f(a)+g(a)$ and $(fg)(a) = f(a)g(a)$. Often, the functional calculus is used to show the existence of the square root of certain elements. A consequence is that the spectral radius of a normal element is equal to its norm. However, there is a problem—the algebras we consider are rarely C^*-algebras, as illustrated next.

Example 3.38 A continuous complex-valued function f on the unit circle is also square integrable, so has a Fourier transform $\underline{f} \in \ell^2(\mathbb{Z})$ as in Example 3.36. The differential $\not{D}(f) = -\mathrm{i}\frac{\partial f}{\partial\theta}$ (where $\theta \in [0, 2\pi]$ is the angular coordinate on the circle) corresponds on Fourier transform to $\not{D} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by $\not{D}(\underline{f})_n = n\underline{f}_n$. This operator is *unbounded*, being only defined on a subset of $\ell^2(\mathbb{Z})$. It follows that $C^\infty(S^1)$, the smooth, or arbitrarily differentiable, functions on the circle are exactly those which have Fourier transform $\underline{f}$ obeying the following condition; for any integer $r > 0$ there is a $K_r > 0$ such that $|n^r\underline{f}_n| \le K_r$ for all $n \in \mathbb{Z}$. We say that the Fourier coefficients decay faster than polynomial. The smooth functions form a strict subalgebra of the C^*-algebra $C(S^1)$ of continuous functions. $\diamond$

If we wish to consider C^*-algebras as hypothetical compact noncommutative topological spaces then we should consider unital algebras. Differentiable functions, on the other hand, are in many cases strict subsets of the algebra of continuous functions, so how to keep many of the nice properties of a C^*-algebra if we only have a subalgebra? We take the following definition of a local C^*-algebra, and much of what follows in §3.3.2, from the book on K-theory by Blackadar.

Definition 3.39 A local C^*-algebra is a dense subalgebra of a C^*-algebra, which has the property that for any $a \in A$ (and also for any matrix algebra $M_n(A)$) and any holomorphic function f in the neighbourhood of the spectrum of a, we have $f(a) \in A$. Note that for algebras not containing an identity, we require $f(0) = 0$.

As an example, the smooth complex-valued functions on a compact smooth manifold form a local C^*-algebra with the supremum norm, and is a dense subalgebra of the C^*-algebra of continuous complex-valued functions. Central to applications of local C^*-algebras is Pedersen's result that if $\phi : A \to B$ is a $*$-algebra map from a local C^*-algebra A to C^*-algebra B, then ϕ is norm decreasing.

Now we give an example where looking at the C^*-algebra given by the generators and relations shows what is at first not at all obvious, namely that the algebra is a deformation of functions on the unit disk and not functions on the plane.

Example 3.40 We consider a deformed unit disk algebra $\mathbb{C}_q[D]$, generated by z and $\bar{z}$ with relation $z\bar{z} = q^{-2}\bar{z}z - q^{-2} + 1$ and involution $z^* = \bar{z}$. For $0 < q < 1$, we look at representations of the $*$-algebra as bounded operators on Hilbert space—these exist as we can set z to be a unitary rotation on $\mathbb{C}$ and $\bar{z}$ to be its inverse. In any Hilbert space representation, the spectrum of $q^2 z\bar{z}$ is a bounded subset of $[0, \infty)$ and, as $1 - q^2 \geq 0$, this is also the case for the spectrum of $q^2 z\bar{z} - q^2 + 1$. Thus if we set r to be the operator norm of z, the spectral radius of $q^2 z\bar{z} - q^2 + 1$ is $q^2 r^2 + 1 - q^2$, and by the relation this must be r^2. Solving the resulting equality gives $r = 1$, so we deduce that $\|z\| = 1$ in the universal C^*-algebra given by this relation. This suggests that $\mathbb{C}_q[D]$ is a deformation of the radius one disk rather than a plane. Next, the algebra $\mathbb{C}_q[D]$ is $\mathbb{Z}$-graded, by $|z| = 1$ and $|\bar{z}| = -1$. Putting $w = 1 - \bar{z}z$, we have $zw = q^{-2}wz$ and $\bar{z}w = q^2 w\bar{z}$, so for any polynomial $p(w)$

$$z.p(w) = p(q^{-2}w).z, \quad \bar{z}.p(w) = p(q^2 w).\bar{z}.$$

There is also a differential calculus generated by $\mathrm{d}z$ and $\mathrm{d}\bar{z}$, and relations given by

$$\mathrm{d}z \wedge \mathrm{d}\bar{z} = -q^{-2}\mathrm{d}\bar{z} \wedge \mathrm{d}z, \quad z.\mathrm{d}z = q^{-2}\mathrm{d}z.z, \quad z.\mathrm{d}\bar{z} = q^{-2}\mathrm{d}\bar{z}.z,$$
$$\mathrm{d}z \wedge \mathrm{d}z = \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{z} = 0, \quad \bar{z}.\mathrm{d}z = q^2\mathrm{d}z.\bar{z}, \quad \bar{z}.\mathrm{d}\bar{z} = q^2\mathrm{d}\bar{z}.\bar{z}.$$

A proof by induction on powers of w gives, for any polynomial $p(w)$,

$$\mathrm{d}p(w) = q^2 \frac{p(q^{-2}w) - p(w)}{w(1 - q^{-2})} z\mathrm{d}\bar{z} + \frac{p(q^2 w) - p(w)}{w(1 - q^2)} \bar{z}\mathrm{d}z. \tag{3.19}$$

We will use this calculus later on. Also, classically, the closed disk is a manifold with boundary. This now becomes expressed as a $*$-algebra map $\pi : \mathbb{C}_q[D] \to \mathbb{C}_{q^2}[S^1]$

where the latter is $\mathbb{C}[t, t^{-1}]$ but as the $*$-differential calculus in Example 1.11 (with q^2 instead of the q there) and $\pi(z) = t, \pi(\bar{z}) = t^* = t^{-1}$. $\diamond$

3.3.2 K-Theory and Completions

Vector bundles are one of the tools used in geometry to obtain topological invariants. From the direct sum $\oplus$ among vector bundles one can construct $K_0(X)$ as an abelian group made, loosely speaking, out of stable equivalence classes. Two vector bundles are stably equivalent if they are isomorphic after direct sum with some third vector bundle. More precisely, isomorphism classes of locally trivial vector bundles form a commutative monoid under direct sum (i.e., $\oplus$ is an associative binary operation with an identity 0). Grothendieck's construction is used to make this into a group: Given any commutative monoid S with operation $\oplus$ its associated group consists of equivalence classes of $S \times S$ where $(E, F) \sim (E', F')$ if and only if there is a $G \in S$ such that $E \oplus F' \oplus G = E' \oplus F \oplus G$. We can think of (E, F) as a formal difference $E - F$. For example, over $\mathbb{C}$, one has $K_0(S^2) = \mathbb{Z} \times \mathbb{Z}$ with generators the tautological bundle on $S^2 \cong \mathbb{CP}^1$ (see Example 3.13) and the trivial line bundle.

We remind the reader that if M, N are left A-modules then their direct sum is the set $M \times N$ with addition and action of $a \in A$,

$$(m, n) + (m', n') = (m + m', n + n'), \quad a.(m, n) = (a.m, a.n).$$

For vector spaces, this is just the usual construction which adds dimensions. We have been considering finitely generated projective modules as the noncommutative analogue of vector bundles. For a possibly noncommutative algebra A, we define $K_0(A)$ similarly, as isomorphism classes of finitely generated projective modules made into a group. The key to understanding the definition of $K_0(A)$ is to see what happens when we change the dual bases of a finitely generated projective module.

The properties of a vector space are independent of the basis used to describe it. In the same way, we wish to describe fgp modules in a manner which removes the dependence on the dual bases. Recall that dual bases for a left fgp module E over the algebra A are a collection $e^i \in E$ and $e_i \in E^\flat$ for $1 \le i \le n$ such that for all $e \in E$, $e = e_i(e).e^i$ and for all $\alpha \in E^\flat$, $\alpha = e_i.\alpha(e^i)$. Now suppose that we have another set of dual bases $c^j \in E$ and $c_j \in E^\flat$ for $1 \le j \le m$. Then

$$e^i = c_j(e^i).c^j, \quad e_i = c_j.e_i(c^j), \quad c^j = e_i(c^j).e^i, \quad c_j = e_i.c_j(e^i). \tag{3.20}$$

Now define W, an $m \times n$ matrix with entries in A, and V, an $n \times m$ matrix, by $W_{ji} = e_i(c^j)$ and $V_{ij} = c_j(e^i)$. The matrices W and V are 'change of basis' matrices. Now calculate the matrix products,

$$V_{ij} W_{ji'} = c_j(e^i).e_{i'}(c^j) = e_{i'}(c_j(e^i).c^j) = e_{i'}(e^i) = P_{ii'},$$
$$W_{ji} V_{ij'} = e_i(c^j).c_{j'}(e^i) = c_{j'}(e_i(c^j).e^i) = c_{j'}(c^j) = Q_{jj'},$$

where $P_{ii'} = e_{i'}(e^i)$ is the idempotent matrix which we associated to the dual bases e_i, e^i in §3.1, and $Q_{jj'} = c_{j'}(c^j)$ is the corresponding matrix for the dual bases c_j, c^j. To avoid the messy business of keeping track of numbers of rows and columns, we introduce 'infinite' matrices $M_\infty(A)$ by matrices with entries $M_{ij} \in A$ for all $0 \leq i, j$, but where all but finitely many M_{ij} are zero. Then, by setting all previously undefined entries to be zero, $VW = P$ and $WV = Q$ in $M_\infty(A)$.

Definition 3.41 Define $V(A)$ to be the idempotents in $M_\infty(A)$ quotiented by the relation of algebraic equivalence. This is defined by $P \sim Q$ (P is algebraically equivalent to Q) if there are $W, V \in M_\infty(A)$ with $VW = P$ and $WV = Q$. Redefining V, W if necessary (by $V \mapsto PVQ$ and $W \mapsto QWP$), we can assume $WVW = W$ and $VWV = V$.

There is a sum operation $\oplus$ on $V(A)$ given by

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P & 0 & 0 \\ 0 & P' & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

which is associative, and commutative as $P \oplus P' \sim P' \oplus P$. This corresponds to the direct sum of modules. The additive zero for this operation is simply the zero idempotent. However there are no inverses, so we do not get a group in general.

For a unital algebra, $K_0(A)$ is the abelian group given by Grothendieck's construction applied to $V(A)$. Thus $K_0(A)$ consists of formal differences $P - Q$ of elements of $V(A)$, where $P - Q = P' - Q'$ exactly when there exists an element G of $V(A)$ such that $P \oplus Q' \oplus G = P' \oplus Q \oplus G$. Nonunital algebras are a little more complicated, and we do not consider them here. There are also some useful properties for $V(A)$. First, as it is defined via $M_\infty(A)$, if we have another algebra B such that $M_\infty(A)$ is isomorphic to $M_\infty(B)$, then $V(A)$ is isomorphic to $V(B)$. The most obvious such algebra is $B = M_n(A)$. We will also need the following.

Lemma 3.42 *For A a local C^*-algebra:*

(1) *every idempotent $P \in M_\infty(A)$ is algebraically equivalent to a Hermitian idempotent Q (i.e., $Q^* = Q^2 = Q$) with a continuous path from P to Q;*
(2) *algebraically equivalent idempotents are conjugate by invertible elements (for finite matrices or adjoining the identity to $M_\infty(A)$).*

Proof The idea for (1) is to set $Z = 1+(P-P^*)(P^*-P)$ as an invertible element of $M_\infty(A)$, and then $Q = PP^*Z^{-1}$ is a Hermitian projection. The path from P to Q is a bit beyond our scope but the interested reader can find more details in Blackadar's book. For (2), if idempotents $P \sim Q$ in A then $P = VW$ and $Q = WV$, and also $WVW = W$ and $VWV = V$. We now define S as follows and find,

$$S = \begin{pmatrix} W & 1-Q \\ 1-P & V \end{pmatrix}, \; S^{-1} = \begin{pmatrix} V & 1-P \\ 1-Q & W \end{pmatrix}, \; S \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}. \quad \square$$

It can also be shown for a local C^*-algebra that $V(A)$ is isomorphic to $V(\widehat{A})$, where $\widehat{A}$ is the C^* completion of A, and in particular they have isomorphic K_0 groups. However, the differential structure is not determined by the C^* completion. This was illustrated, in the purely classical case, when Milnor showed that there were exotic differential structures on the 7-sphere, that is there are several non-diffeomorphic differential structures on the same compact topological space S^7.

Given a classical manifold M, we have a unique idea of, for example, the set of continuously differentiable functions $C^1(M)$. However suppose we are given the generators of a noncommutative algebra A with differentiable structure, which is a dense subalgebra of a C^*-algebra $\widehat{A}$. How do we define the corresponding continuously differentiable 'functions'? There is the additional problem that, just as we can change coordinates on the manifold, we could have generators for an entirely different algebra which also is a dense subalgebra of $\widehat{A}$.

Example 3.43 The simplest case would be the commutative algebra with one unitary generator z, and this should be thought of as functions on the unit circle S^1. The Laurent polynomials $A = \mathbb{C}[z, z^{-1}]$ given by generators and relations sit within the continuous functions $\widehat{A} = C(S^1)$. Taking the classical calculus where $z = \mathrm{e}^{\mathrm{i}\theta}$ commutes with $\mathrm{d}z$, then strictly placed between A and $\widehat{A}$ we have the smooth functions $C^\infty(S^1)$ and the continuously differentiable functions $C^1(S^1)$.

Since Ω^1 on A is trivial, we write it as $A.\mathrm{d}\theta$ with $\mathrm{d}a = \frac{\mathrm{d}a}{\mathrm{d}\theta}.\mathrm{d}\theta$. On completion to a C^*-algebra, we get $\widehat{\Omega}^1 = \widehat{A}.\mathrm{d}\theta$ as the module of continuous sections of the cotangent bundle. Here $\mathrm{d} : \widehat{A} \to \widehat{A}.\mathrm{d}\theta$ is originally only defined on $A \subset \widehat{A}$, but we can extend its domain by considering its graph

$$\Gamma = \{(a, \mathrm{d}a) : a \in A\} \subset \widehat{A} \times \widehat{\Omega}^1 = \widehat{A} \times \widehat{A}.\mathrm{d}\theta,$$

and taking the closure $\widehat{\Gamma}$ using the $\widehat{A}$ norm on both factors. Thus $(b, c.\mathrm{d}\theta) \in \widehat{\Gamma}$ exactly when there is a sequence $a_n \in A$ such that $a_n \to b$ and $\frac{\mathrm{d}a_n}{\mathrm{d}\theta} \to c$. Then we can extend the domain of definition of d by defining $\widehat{\mathrm{d}}(b) = c.\mathrm{d}\theta$ for all $(b, c.\mathrm{d}\theta) \in \widehat{\Gamma}$, and call the new domain $C^1(S^1)$, the functions with continuous derivative.

In order for $\widehat{\mathrm{d}}$ to be well defined, we can never have elements $(b, c_1.\mathrm{d}\theta)$ and $(b, c_2.\mathrm{d}\theta)$ in $\widehat{\Gamma}$ with $c_1 \neq c_2$. By linearity of d this reduces to showing that if $(0, c.\mathrm{d}\theta) \in \widehat{\Gamma}$ then $c = 0$. For this to fail we would have to have a sequence $a_n \in A$ with $a_n \to 0$ in norm but $\frac{\mathrm{d}a_n}{\mathrm{d}\theta} \to c \neq 0$. On the circle this is impossible, as we can integrate $c.\mathrm{d}\theta$ to get a nonconstant function and, using the supremum norm, the integral of the limit is the limit of the integral.

We show that the domain of $\widehat{\mathrm{d}}$ is all of $C^1(S^1)$. Given $f \in C^1(S^1)$, we take a Laurent polynomial $\sum r_n \mathrm{e}^{\mathrm{i}n\theta}$ (a finite sum) approximating $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ in supremum norm to within ϵ. As the integral round the circle of $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ is zero, we choose $r_0 = 0$. Then there is a constant k such that $k - \mathrm{i} \sum r_n \mathrm{e}^{\mathrm{i}n\theta}/n$ approximates f to within $2\pi\epsilon$. ◇

The argument of Example 3.43 may sound strange for such a familiar example as the circle, but it is phrased in such a way as to make sense more widely. However the reader should note that the method of working with closed graphs gives the maximal vector space on which we can sensibly extend the d operator, it does not guarantee that any of the algebraic properties which we might like still apply. The next example, which is the noncommutative calculus on the circle, illustrates some of the problems which may occur and how we can resolve them at least in this case.

Example 3.44 We consider $A = \mathbb{C}_q[S^1]$ in Example 1.11 but with $0 < q < 1$ for the calculus. As an algebra we consider $A = \mathbb{C}[t, t^{-1}]$ as Laurent polynomials on the complex plane $t \in \mathbb{C}$ and define a norm by taking the supremum norm on the unit circle $|t| = 1$. This gives for $\widehat{A} = C(S^1)$ the C^*-algebra of continuous functions on the circle. On Ω^1 we use the generator $\mathrm{d}t$ to define $\|g(t)\mathrm{d}t\| = \|g(t)\|$. As in Example 3.43, we use the closed graph method to extend the domain of d. To this end, suppose that $f_n \in A \to 0$ in norm and that $\mathrm{d}f_n$ tends to a limit. This means the Laurent polynomials $(f_n(qt) - f_n(t))/t$ tend to a limit uniformly for $t \in S^1$. However, as $f_n(t)/t$ tends to zero uniformly on the unit circle, we must have $f_n(qt)/t$ tending to a limit uniformly for $t \in S^1$, i.e., f_n tends to a limit uniformly on the circle of radius q. By the maximum modulus principle for holomorphic functions, f_n tends uniformly on $\mathcal{A}_{[q,1]}$ (the closed annulus centred on the origin between radii q and 1) to a continuous function on $\mathcal{A}_{[q,1]}$ which is holomorphic in the interior $\mathcal{A}_{(q,1)}$. As the limit is zero on the unit circle, it must be zero everywhere, hence $\mathrm{d}f_n \to 0$ and we can use the closed graph method as in Example 3.43.

Hence proceeding similarly to earlier, we define $C^1_q(S^1)$ as the domain of the linear extension of d and identify it with the functions continuous on the annulus $\mathcal{A}_{[q,1]}$ and holomorphic in the interior. We also have that the $C^1_q(S^1)$ norm (the sum of norm f and norm $\mathrm{d}f$) is equivalent to the sup norm on $\mathcal{A}_{[q,1]}$, and that $C^1_q(S^1)$ is an algebra. However, were we to try to rewrite our first-order calculus in terms of the algebra $C^1_q(S^1)$, we would discover that we no longer have the important property that Ω^1 is generated by $\mathrm{d}t$ as a left module. This is because $\mathrm{d}t.f(t) = f(qt)\mathrm{d}t$, and it is not necessarily the case that $f(qt) \in C^1_q(S^1)$ when $f(t)$ is.

One way to repair this would be to go to the smaller algebra of functions on $\mathcal{A}_{[q^2,1]}$, but of course the problem repeats and we eventually end up with the algebra of functions continuous on $\mathcal{A}_{(0,1]}$ and holomorphic on $\mathcal{A}_{(0,1)}$. This algebra is no longer normed, it is a Fréchet space, meaning it is Hausdorff and complete under a countable set of seminorms. If in addition we want a $*$-subalgebra of $\widehat{A}$ then, as $*$ swaps t and t^{-1}, we find that we have to take the holomorphic functions on $\mathcal{A}_{(0,\infty)}$, i.e., the holomorphic functions on the punctured complex plane. For this Fréchet algebra, Ω^1 is a $*$-differential calculus singly generated by $\mathrm{d}t$, as desired. $\diamond$

3.3.3 Hochschild Homology and Cyclic Homology

Now we turn to the theories which we wish to link to K-theory: cyclic homology and cohomology, for much of which we follow the exposition in Loday's book. A full account here would take a book in itself so we will be forced to progressively reduce the amount of detail in proofs, but enough for the reader to see the basic principles. We require that the field $\mathbb{k}$ contains the rational numbers, so $\mathbb{R}$ and $\mathbb{C}$ are fine, and we take all algebras to be unital. First we define Hochschild homology.

Definition 3.45 For an algebra A and an A-bimodule E, define $\mathrm{d}_j : E \otimes A^{\otimes n} \to E \otimes A^{\otimes n-1}$ for $0 \le j \le n$ by

$$\begin{aligned}
\mathrm{d}_0(e \otimes a_1 \otimes \cdots \otimes a_n) &= ea_1 \otimes \cdots\cdots \otimes a_n, \\
\mathrm{d}_j(e \otimes a_1 \otimes \cdots \otimes a_n) &= e \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n, \quad 1 \le j < n, \\
\mathrm{d}_n(e \otimes a_1 \otimes \cdots \otimes a_n) &= a_n e \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\end{aligned}$$

Then $\mathrm{d}_i \mathrm{d}_j = \mathrm{d}_{j-1} \mathrm{d}_i$ for $i < j$, so if $b = \sum_{i=0}^{n} (-1)^i \mathrm{d}_i$ then $b^2 = 0$. This is the Hochschild complex, with $E \otimes A^{\otimes n}$ in degree n,

$$\xrightarrow{b} E \otimes A^{\otimes 3} \xrightarrow{b} E \otimes A^{\otimes 2} \xrightarrow{b} E \otimes A \xrightarrow{b} E$$

and with the corresponding *Hochschild homology* denoted by $\mathrm{H}_n(A, E)$. In the case where $E = A$, the usual notation is $\mathrm{HH}_n(A) = \mathrm{H}_n(A, A)$.

The *generalised trace* map $\mathrm{Tr} : M_m(E) \otimes M_m(A)^{\otimes n} \to E \otimes A^{\otimes n}$ is

$$\mathrm{Tr}(M^0 \otimes \cdots \otimes M^n) = \sum_{i_0, \ldots, i_n} {M^0}_{i_0 i_1} \otimes {M^1}_{i_1 i_2} \otimes \cdots \otimes {M^n}_{i_n i_0}.$$

There is also an inclusion map $\mathrm{inc}_{11} : A \to M_m(A)$ where $\mathrm{inc}_{11}(a)$ is the matrix with a in the 11 position and zeros elsewhere, and similarly for $\mathrm{inc}_{11} : E \to M_m(E)$. The standard method for showing that two maps between chain complexes induce the same maps on homology is to find a *homotopy* connecting them.

Definition 3.46 For two chain complexes (C_*, d_C) and (D_*, d_D) (so $\mathrm{d}_C : C_n \to C_{n-1}$ and ${\mathrm{d}_C}^2 = 0$ and similarly for D), a map of complexes or a chain map is a map $f : C_n \to D_n$ for all n such that $f \circ \mathrm{d}_C = \mathrm{d}_D \circ f$. Two chain maps f, g from (C_*, d_C) to (D_*, d_D) are homotopic if there is a map $h : C_n \to D_{n+1}$ such that $f - g = \mathrm{d}_D \circ h + h \circ \mathrm{d}_C$. Similarly for a cochain complex with a homotopy decreasing the index, i.e., $\mathrm{d}_C : C_n \to C_{n+1}$, ${\mathrm{d}_C}^2 = 0$ and $h : C_n \to D_{n-1}$.

From this definition it is easy to see that the induced maps $f, g : \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are equal. Consider $[k] \in \mathrm{H}_n(C)$, where $k \in \ker \mathrm{d}_C$ and the square brackets denote the homology class. Then $[(f - g)k] = [\mathrm{d}_D h(k)] = [0]$. In the special case where $C = D$, $f = \mathrm{id}$ and $g = 0$, we say that h is a contracting homotopy, which then shows that the homology vanishes. Similarly for cochain complexes and cohomology. Next we show that Hochschild homology is *stably invariant*, i.e., that taking matrices over the algebra and bimodule leaves the homology unchanged. To do this, we give maps both ways between the complexes with both compositions homotopic to the identity.

Proposition 3.47 *The map* $\mathrm{Tr} : M_m(E) \otimes M_m(A)^{\otimes n} \to E \otimes A^{\otimes n}$ *induces an isomorphism in Hochschild homology* $\mathrm{H}_n(M_m(A), M_m(E)) \cong \mathrm{H}_n(A, E)$*, with inverse induced by* $\mathrm{inc}_{11} : E \otimes A^{\otimes n} \to M_m(E) \otimes M_m(A)^{\otimes n}$.

Proof Both Tr and inc_{11} commute with the differential. The composition $\mathrm{Tr} \circ \mathrm{inc}_{11} : A^{\otimes n+1} \to A^{\otimes n+1}$ is the identity. The problem is composition the other way. Define $h_j : M_m(E) \otimes M_m(A)^{\otimes n} \to M_m(E) \otimes M_m(A)^{\otimes n+1}$ for $0 \le j \le n$ by

$$h_j(M^0 \otimes \cdots \otimes M^n) = \sum_{i_0,\dots,i_{j+1}} E_{i_0 1}(M^0_{i_0 i_1}) \otimes E_{11}(M^1_{i_1 i_2}) \otimes \cdots \otimes E_{11}(M^j_{i_j i_{j+1}}) \otimes E_{1 i_{j+1}}(1) \otimes M^{j+1} \otimes \cdots \otimes M^n.$$

Here $E_{ij}(a)$ is the matrix which has a in position ij and zeros elsewhere. Then $h = \sum_j (-1)^j h_j$ obeys $bh + hb = \mathrm{id} - \mathrm{inc}_{11} \circ \mathrm{Tr}$. The parts of this calculation giving the chain maps are

$$\mathrm{d}_0 h_0(M^0 \otimes \cdots \otimes M^n) = M^0 \otimes \cdots \otimes M^n,$$

$$\begin{aligned} \mathrm{d}_{n+1} h_n(M^0 \otimes \cdots \otimes M^n) &= \sum_{i_0,\dots,i_n} E_{11}(M^0_{i_0 i_1}) \otimes E_{11}(M^1_{i_1 i_2}) \otimes \cdots \otimes E_{11}(M^0_{i_n i_0}) \\ &= \mathrm{inc}_{11} \circ \mathrm{Tr}(M^0 \otimes \cdots \otimes M^n). \end{aligned}$$

Further details can be found in Loday's book. □

It will be useful to define another differential b' in the case $E = A$.

Definition 3.48 Define $b' : A^{\otimes n+1} \to A^{\otimes n}$ by $b' = \sum_{i=0}^{n-1} (-1)^i \mathrm{d}_i$ (note we are missing d_n when compared to b), and the *bar complex* by

$$\xrightarrow{b'} A^{\otimes 4} \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0,$$

with $A^{\otimes n+2}$ in degree n and μ being the product. This is a resolution of A, i.e., it is a long exact sequence of bimodule maps (i.e., μ is surjective, at $A^{\otimes 2}$ $\ker \mu = \mathrm{image}\, b'$ and $\ker b' = \mathrm{image}\, b'$ everywhere else). This can be seen by using the following homotopy connecting the identity to zero, i.e., $b'h + hb' = \mathrm{id}$,

$$h : A^{\otimes n} \to A^{\otimes n+1}, \quad h(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n.$$

Define the cyclic shift $t_n : A^{\otimes n+1} \to A^{\otimes n+1}$ by

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \tag{3.22}$$

and define $N : A^{\otimes n+1} \to A^{\otimes n+1}$ by $N = \mathrm{id} + t_n + t_n^2 + \cdots + t_n^n$. Then we have $(1-t)b' = b(1-t)$ and $b'N = Nb$. Now we can form the *cyclic bicomplex* $CC(A)$:

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "-b'"'] & {} \arrow[d, "b"'] & {} \arrow[d, "-b'"'] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & {} \arrow[l, "N"']
\end{tikzcd}$$

This is bi-indexed by the positive integers, starting at $(0, 0)$ from A at the bottom left, giving $CC(A)_{pq} = A^{\otimes q+1}$. The total differential at any point is the left pointing arrow plus the downward arrow.

Definition 3.49 The cyclic homology $\mathrm{HC}_n(A)$ is the homology of the total bicomplex and in grade n is $\oplus_{p+q=n} CC(A)_{pq}$.

This bicomplex is useful for proofs, but there is a more concisely defined complex with the same cohomology.

Definition 3.50 Let $C_n^\lambda(A) = A^{\otimes n+1}/(1-t)$ be the quotient of $A^{\otimes n+1}$ by the image of the map $1 - t_n : A^{\otimes n+1} \to A^{\otimes n+1}$. We denote by $\mathrm{H}_n^\lambda(A)$ the homology of the *Connes complex*

$$\cdots \xrightarrow{b} C_2^\lambda(A) \xrightarrow{b} C_1^\lambda(A) \xrightarrow{b} C_0^\lambda(A).$$

Next we shall state some further facts that we need but which are beyond our scope to prove. The interested reader can find more details in Loday's excellent account. The first fact is that, assuming $\Bbbk$ contains the rationals, the map from $CC(A)$ to $C_n^\lambda(A)$ which is the quotient map from $A^{\otimes n+1}$ to $A^{\otimes n+1}/(1-t)$ on the first column of the bicomplex and zero on the rest induces an isomorphism in cohomology

$$\mathrm{HC}_n(A) \cong \mathrm{H}_n^\lambda(A). \tag{3.23}$$

The second is that the trace $\mathrm{Tr} : M_m(A)^{\otimes n+1} \to A^{\otimes n+1}$ induces an isomorphism $\mathrm{HC}_n(M_m(A)) \cong \mathrm{HC}_n(A)$ in cyclic homology, with inverse induced by

$$\mathrm{inc}_{11} : A^{\otimes n+1} \to M_m(A)^{\otimes n+1}. \tag{3.24}$$

The third fact is that, for invertible elements $g \in A$, the adjoint map $\mathrm{Ad}(g) : A \to A$ given by $\mathrm{Ad}(g)(a) = gag^{-1}$ extends to a map of $CC(A)$, and hence to $\mathrm{HC}_*(A)$ as it commutes with the differentials. Then

$$\mathrm{Ad}(g)_* = \mathrm{id} : \mathrm{HC}_n(A) \to \mathrm{HC}_n(A). \tag{3.25}$$

3.3.4 Pairing K-Theory and Cyclic Cohomology

We continue to follow Loday's exposition of the Chern–Connes pairing. Everything is already in place for the first result.

Theorem 3.51 *For a unital local C^*-algebra A, $\mathrm{ch}^{\lambda}_{0,n} : K_0(A) \to \mathrm{HC}^{\lambda}_{2n}(A)$ given by the matrix projection P maps to $(-1)^n \mathrm{Tr}(P^{\otimes 2n+1})$ is well defined and additive.*

Proof From the construction of $K_0(A)$, we need to show that algebraically equivalent idempotents are sent to the same homology class. For this we use Lemma 3.42 part (2) and the result on $\mathrm{Ad}(g)_*$. We also need to show that the map is additive with respect to direct sum of projections, as given in (3.21). For this we note that the trace map sends $(P \oplus Q)^{\otimes 2n+1}$ to $\mathrm{Tr}(P^{\otimes 2n+1}) + \mathrm{Tr}(Q^{\otimes 2n+1})$. □

We next need the dual theory, cyclic cohomology HC^n_{λ}, in order to pair with the above and obtain numerical invariants. This theory is due to Connes, who noticed that Hochschild cochains could be quotiented by the action of a cyclic group.

Definition 3.52 The cyclic cochain complex of a unital algebra A consists of vector spaces of linear maps

$$C^n_{\lambda}(A) = \left\{\phi : A^{\otimes(n+1)} \to \Bbbk \mid \phi(a_1, \cdots, a_n, a_0) = (-1)^n \phi(a_0, a_1, \cdots, a_n)\right\}$$

with the differential $b : C^n_{\lambda} \to C^{n+1}_{\lambda}$

$$(b\phi)(a_0, \cdots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \phi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \phi(a_{n+1}a_0, a_1, \cdots, a_n)$$

which obeys $b^2 = 0$. The cyclic cohomology $\mathrm{HC}^n_{\lambda}(A)$ is the kernel of b on degree n (i.e., cyclic cocycles) modulo the image of b on degree $n-1$. A cyclic n-cochain $\phi \in C^n_{\lambda}(A)$ is called *unital* if it vanishes when any of its arguments is 1 and $n \geq 1$, with no restriction when $n = 0$.

To define the pairing, we use the idempotents P giving K-theory classes.

Theorem 3.53 (Chern–Connes Pairing) *Let A be a unital local C^*-algebra. There is a pairing between $K_0(A)$ and $\mathrm{HC}^{2m}(A)$ given by*

$$\langle [P], [\phi] \rangle = \frac{1}{m!} \sum_{i_0, \cdots, i_{2m}} \phi(P_{i_0 i_1}, P_{i_1 i_2}, \cdots, P_{i_{2m} i_0}) = \frac{1}{m!} \phi \mathrm{Tr}(P, P, \cdots, P).$$

Proof We compile previous results. The critical aspect is that the pairing only depends on the K-theory class of P, which comes from Lemma 3.42 that equivalent idempotents are related by conjugation by an invertible element. We then use the fact that for invertible g, $\mathrm{Ad}(g)_* : \mathrm{HC}_n(A) \to \mathrm{HC}_n(A)$ is the identity. □

To explain the geometric meaning of this pairing, we note that Chern's original definition of the Chern class of a bundle was given in terms of the trace of powers of the curvature. To implement this we say that a linear map $\int : \Omega^n \to \Bbbk$ is *closed* if $\int \mathrm{d}\omega = 0$ for all $\omega \in \Omega^{n-1}$, which implies a well-defined map $\int : \mathrm{H}^n_{\mathrm{dR}}(A) \to \Bbbk$. We then define an n-cycle as such a closed linear map with the additional property

$$\int \omega \wedge \rho = (-1)^{|\omega||\rho|} \int \rho \wedge \omega \tag{3.26}$$

for all $\omega, \rho \in \Omega$. Classically, $\int$ is provided by integration against a submanifold of dimension n and (3.26) is empty as the classical Ω is graded-commutative.

Proposition 3.54 *From an n-cycle defined on the exterior algebra of a unital algebra A, we can construct a unital cyclic n-cocycle*

$$\phi(a_0, a_1, \cdots, a_n) = \int a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n.$$

Proof We first check cyclicity

$$\begin{aligned} \phi(a_1, \cdots a_n, a_0) &= \int a_1 \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_0 = (-1)^{n-1} \int a_1 \mathrm{d}(\mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_n . a_0) \\ &= (-1)^n \int \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n . a_0 = (-1)^n \phi(a_0, a_1, \cdots, a_n) \end{aligned}$$

using the graded Leibniz rule and that $\int$ is closed for the second line. Next,

$$\begin{aligned} (b\phi)(a_0, \cdots, a_{n+1}) = &\int a_0 a_1 \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{n+1} \\ &+ \sum_{j=1}^{n} (-1)^j \int a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}(a_j a_{j+1}) \wedge \cdots \wedge \mathrm{d}a_{n+1} \\ &+ (-1)^{n+1} \int a_{n+1} a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n \end{aligned}$$

and we expand the summed terms by the Leibniz rule to

$$\sum_{j=1}^{j=n}(-1)^j\int a_0\mathrm{d}a_1\wedge\cdots\wedge\mathrm{d}a_j.a_{j+1}\wedge\mathrm{d}a_{j+2}\wedge\cdots\wedge\mathrm{d}a_{n+1}$$
$$+\sum_{j=1}^{j=n}(-1)^j\int a_0\mathrm{d}a_1\wedge\cdots\wedge\mathrm{d}a_{j-1}\wedge a_j\mathrm{d}a_{j+1}\wedge\cdots\wedge\mathrm{d}a_{n+1}.$$

Now the interior of the first sum at j cancels with the interior of the second at $j+1$. What remains is the boundary $j=1$ of the second sum, which cancels with the first term of $b\phi$ above, and $j=n$ of the first sum giving $(-1)^n\int a_0.\mathrm{d}a_1\wedge\cdots\wedge\mathrm{d}a_n.a_{n+1}$. This cancels with the last term of $b\phi$ above due to the second requirement of $\int$ an n-cycle. Note that we only need this in the form $\int a\omega=\int\omega a$ for all $a\in A$ and $\omega\in\Omega^n$ but this implies and is therefore equivalent to the graded version for general $\rho\in\Omega$ (proof by induction on degree of ρ). Clearly ϕ is unital since $\mathrm{d}1=0$. □

To illustrate Proposition 3.54 we use a 2-cycle on the noncommutative torus.

Example 3.55 The noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$ has a 0-cycle $\int\sum a_{mn}u^m v^n = a_{00}$. Remembering that the a_{mn} are at $\theta=0$ the Fourier coefficients of a function on $S^1\times S^1$, this becomes the Haar integral on the torus. For the calculus in Example 1.36 we define a 2-cycle by $\int ae_1\wedge e_2=\int a$ for $a\in\mathbb{C}_\theta[\mathbb{T}^2]$. This defines a unital 2-cocycle on $\mathbb{C}_\theta[\mathbb{T}^2]$. To see what this looks like, it pays to use as basic 1-forms $e_1=u^{-1}\mathrm{d}u$ and $e_2=v^{-1}\mathrm{d}v$ because these commute with elements of $\mathbb{C}_\theta[\mathbb{T}^2]$, and anticommute as classically among themselves. In that case $\mathrm{d}a=(\partial_u a)e_1+(\partial_v a)e_2$ defines two derivations $\partial_u,\partial_v:\mathbb{C}_\theta[\mathbb{T}^2]\to\mathbb{C}_\theta[\mathbb{T}^2]$. Classically they look like $\partial_u=u^{-1}\frac{\partial}{\partial u}$ and similarly for ∂_v, provided we understand all expressions as 'normally ordered' with u to the left of v. We have

$$\phi(a,b,c)=\int a(\partial_u be_1+\partial_v be_2)\wedge(\partial_u ce_1+\partial_v ce_2)$$
$$=\int a(\partial_u b\partial_v c-\partial_v b\partial_u c)e_1\wedge e_2=\int a(\partial_u b\partial_v c-\partial_v b\partial_u c).\qquad\diamond$$

Remark 3.56 The converse of Proposition 3.54 is also true in the case of the universal calculus on A; in this case an n-cycle $\int$ is equivalent to a unital cyclic n-cocycle ϕ. Starting with the latter, we define $\int$ by the same formula as above but read the other way. We demonstrate this by taking the first few cases. In degree 0 an element of $\mathrm{HC}^0(A)$ and a 0-cycle both mean a 'trace', i.e., a map $\phi:A\to\Bbbk$ such that $\phi(ab)=\phi(ba)$. In degree 1 a unital 2-cocycle is a linear map $\phi:A^{\otimes 2}\to\Bbbk$ with

$$\phi(1,a)=0,\quad \phi(a_0a_1,a_2)-\phi(a_0,a_1a_2)+\phi(a_2a_0,a_1)=0$$

(which implies that ϕ is antisymmetric). Now $\int a_0 \mathrm{d}a_1 = \phi(a_0, a_1)$ is well defined because if $\sum a_0^\alpha \mathrm{d}a_1^\alpha = 0$ then $\sum a_0^\alpha \otimes a_1^\alpha = \sum a_0^\alpha a_1^\alpha \otimes 1$. Writing $a_1^\alpha = \lambda^\alpha + b^\alpha \in \Bbbk 1 \oplus A'$ (choosing a complement of $\Bbbk 1$) we find $\sum a_0^\alpha \otimes b^\alpha = \sum a_0^\alpha b^\alpha \otimes 1$ and conclude that $\sum a_0^\alpha \otimes b^\alpha = 0$ and hence $\sum a_0^\alpha \otimes a_1^\alpha \in A \otimes 1$. Hence in this case $\sum \phi(a_0^\alpha, a_1^\alpha) = 0$ as required. Once we know that $\int$ is well defined, we just push the proof of the proposition in reverse; by definition $\int \mathrm{d}a = \phi(1, a) = 0$ while the cocycle condition amounts to $\int a_0 \mathrm{d}a_1 . a_2 = \int a_2 a_0 \mathrm{d}a_1$, as required. $\diamond$

Finally, we recall that in classical geometry the Chern classes $c_n(E, \nabla_E)$ of a bundle E with connection ∇_E are defined in terms of powers of the curvature as

$$\sum c_n(E, \nabla_E) t^n = \det\big(I + \frac{\mathrm{i}t}{2\pi} R_E\big) = \exp\big(\mathrm{trace}\big(\log\big(I + \frac{\mathrm{i}t}{2\pi} R_E\big)\big)\big)$$

and are independent of the connection. We return to the independence of the trace in the noncommutative case in §4.1, while for the moment we just examine composition powers of the curvatures and their wedge product. For example, the square $(\mathrm{id} \wedge R_E) R_E$ of the curvature of the Grassmann connection, denoted R_E^2, is

$$\begin{aligned} R_E^2(e_i) &= (-1)^2 \sum_{j,k,m,p,q,r} (\mathrm{d}P_{ij} \wedge \mathrm{d}P_{jk}) P_{km} \wedge (\mathrm{d}P_{mp} \wedge \mathrm{d}P_{pq}) P_{qr} \otimes e_r \\ &= \sum_{j,k,p,q,r} \mathrm{d}P_{ij} \wedge \mathrm{d}P_{jk} \wedge \mathrm{d}P_{kp} \wedge (\mathrm{d}P_{pq}) P_{qr} \otimes e_r, \end{aligned} \tag{3.27}$$

where we have used $P\mathrm{d}P = (\mathrm{d}P)P^\perp$ and $P^\perp \mathrm{d}P = (\mathrm{d}P)P$ to simplify the expression slightly. Here $P^\perp = I - P$ and the equations come from applying d to $P^2 = P$. Hence if $\int : \Omega^{2m} \to \Bbbk$ is an $2m$-cycle and E a left fgp A-module with $R_G : E \to \Omega^2 \otimes_A E$ the curvature of the Grassmann connection then we can define

$$\mathrm{Tr}_{\int}(R_G^m) = (-1)^m \int \mathrm{Tr}(\mathrm{d}P \wedge \mathrm{d}P \wedge \cdots \wedge (\mathrm{d}P)P).$$

For a classical manifold M, this gives a map from $K_0(M)$ to $\mathrm{H}_{\mathrm{dR}}(M)$ which can be used to distinguish between different K-theory classes. In the noncommutative case we have exactly the Chern–Connes pairing in the case where the cyclic $2m$-cocycle ϕ is obtained from a $2m$-cycle.

Example 3.57 For the normalised left-invariant integral on $\mathbb{C}_q[S^2]$ in (2.9), we have

$$\int z^* z = q^2 \int z z^*.$$

In fact there is an algebra map $\varsigma : \mathbb{C}_q[S^2] \to \mathbb{C}_q[S^2]$ defined on the generators by $\varsigma(z) = q^2 z$, $\varsigma(z^*) = q^{-2} z^*$ and $\varsigma(x) = x$, such that

$$\int fg = \int (\varsigma g) f.$$

Here ς is a twisting automorphism, as we shall describe in §3.3.5, and it is quite easy to see that it is an algebra map as it respects the relations. In this case, for the purposes of discussion, we formally consider

$$\int f z x^{-1} = q^{-2} \int f x^{-1} z = \int z f x^{-1},$$

which suggests that $\phi(f) := \int f x^{-1}$ may provide a trace, but of course we can only legally apply this formula in the case where f is an expression in the generators with a strictly positive power of x on the right. If not, we could take zero. Motivated by this, one can check that there is indeed a trace defined on a linear basis of $\mathbb{C}_q[S^2]$ by

$$\phi(z^m) = \phi(z^{*m}) = 0, \quad \phi(z^m x^n) = \phi(z^{*m} x^n) = \frac{\delta_{m,0}}{1 - q^{2n}}, \quad n > 0.$$

Now we use this cyclic 0-cocycle with the tautological or monopole bundle in Example 3.15. The Chern–Connes pairing in Theorem 3.53 for $m = 0$ gives

$$\langle [e], [\phi] \rangle = \phi(\mathrm{Tr} \begin{pmatrix} 1 - q^2 x & z \\ z^* & x \end{pmatrix}) = \phi((1 - q^2) x) = 1.$$

To see if this could be a trivial bundle, we compare this result to what we would have found if P was a multiple of the identity matrix. As $\phi(1) = 0$, a multiple of the identity would give zero under the pairing, so we have showed that the q-monopole bundle is nontrivial in K_0 for q not a root of unity. ◇

The ϕ in the last example does the job of detecting the K-theory and proving that the q-Hopf fibration is nontrivial as a quantum principal bundle (see Chap. 5). However, it does not have a classical limit as $q \to 1$, suggesting that it might be better to incorporate ζ into the cyclic theory itself so as to remedy this problem.

3.3.5 Twisted Cycles

We now look at the twisted cyclic theory as first formulated by Kustermans, Murphy and Tuset. Comparing the integral in Example 3.57 with its twisting automorphism ς to the definition of an n-cycle in (3.26), the idea is that allowing a twisting automorphism in an n-cycle should allow more examples, in particular of a more geometric nature. We suppose that the algebra A has a differential calculus (Ω, d), and recall that the linear map $\int : \Omega^d \to \mathbb{C}$ is called closed if $\int \circ \mathrm{d} = 0$.

Definition 3.58 The linear map $\int : \Omega^d \to \mathbb{C}$ has right/left kernel respectively

$$R_{\int} = \left\{\xi \in \Omega^r : 0 \le r \le d \text{ and } \int \xi \wedge \eta = 0 \text{ for all } \eta \in \Omega^{d-r}\right\},$$
$$L_{\int} = \left\{\xi \in \Omega^r : 0 \le r \le d \text{ and } \int \eta \wedge \xi = 0 \text{ for all } \eta \in \Omega^{d-r}\right\}.$$

It is immediate that $R_{\int}$ is a right ideal and $L_{\int}$ a left ideal in Ω. The map is called *right faithful* if its right kernel is zero, and similarly for *left faithful*. If $R_{\int} \cap \Omega^0$ is zero the map is called *weakly right faithful*.

We can use this idea to extend the *twisting map* ς from the algebra to the calculus.

Proposition 3.59 *Suppose that the closed linear map $\int : \Omega^d \to \mathbb{C}$ is right faithful, where d is the top dimension of the calculus, and that there is an algebra map $\varsigma : A \to A$ such that $\int(\omega a) = \int(\varsigma(a)\omega)$ for all $a \in A$ and $\omega \in \Omega^d$. Then ζ extends to a map $\varsigma : \Omega \to \Omega$ of exterior algebras such that for all $\xi \in \Omega^{d-r}$, $\eta \in \Omega^r$,*

$$\int \xi \wedge \eta = (-1)^{r(d-r)} \int (\varsigma\eta) \wedge \xi.$$

Proof For $\xi \in \Omega^{d-1}$ and $a \in A$ we have

$$0 = \int \mathrm{d}(\xi a) = \int (\mathrm{d}\xi)a + (-1)^{d-1} \int \xi \wedge \mathrm{d}a = \int (\varsigma a)\mathrm{d}\xi + (-1)^{d-1} \int \xi \wedge \mathrm{d}a$$
$$= \int \mathrm{d}((\varsigma a)\xi) - \int \mathrm{d}\varsigma(a) \wedge \xi + (-1)^{d-1} \int \xi \wedge \mathrm{d}a,$$

so we have, for $b \in A$, and $r = 1$

$$\int \xi \wedge b\mathrm{d}a = (-1)^{d-1} \int (\mathrm{d}\varsigma a) \wedge \xi b = (-1)^{d-1} \int (\varsigma b)(\mathrm{d}\varsigma a) \wedge \xi.$$

If we sum this over a linear combination of $b.\mathrm{d}a$ which is zero, then we have the sum of $\varsigma(b).\mathrm{d}\varsigma(a)$ also being zero by the right faithful condition. It follows that the map $b.\mathrm{d}a \mapsto \varsigma(b).\mathrm{d}\varsigma(a)$ is well defined as $\varsigma : \Omega^1 \to \Omega^1$. Now assume that ς is well defined, and the displayed equation in the statement holds for some $r < d$, and show that it works for $r + 1$. For $\eta \in \Omega^r$ and $\kappa \in \Omega^1$,

$$\int \xi \wedge \eta \wedge \kappa = (-1)^{d-1} \int (\varsigma\kappa) \wedge \xi \wedge \eta = (-1)^{(r+1)(d-r-1)} \int (\varsigma\eta) \wedge (\varsigma\kappa) \wedge \xi.$$

By right faithfulness, the map $\eta \wedge \kappa \mapsto (\varsigma\eta) \wedge (\varsigma\kappa)$ is well defined. $\square$

A closed linear map and algebra map $(\int, d, \varsigma)$ satisfying the displayed equation in Proposition 3.59 (but not necessarily right faithful and where d is not necessarily the top dimension) is called a *twisted d-cycle* (compare with (3.26)). Suppose that we have a $*$-differential calculus, then $\int$ is said to be *self-adjoint* when $\int(\omega^*) = (\int \omega)^*$ for all $\omega \in \Omega^d$. A simple but nontrivial example is given by $\mathbb{C}_q[S^2]$.

Example 3.60 As $\Omega^3 = 0$ for the 2D calculus on $\mathbb{C}_q[S^2]$ in Lemma 2.34, for any $\xi \in \Omega^2$ we have a well-defined cohomology class $[\xi] \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_q[S^2])$. In Proposition 4.34 we shall show that $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_q[S^2]) \cong \mathbb{C}$ and that $[e^+ \wedge e^-] \neq 0$. This means that we can use the ratio to define a linear map $\int : \Omega^2 \to \mathbb{C}$ by $\xi \mapsto \mathrm{i}[\xi]/[e^+ \wedge e^-] \in \mathbb{C}$ (the i is just to make the result self-adjoint). By its definition this is a closed linear map.

In fact $-\mathrm{i}\int e^+ \wedge e^- y$ for $y \in \mathbb{C}_q[S^2]$ gives $[e^+ \wedge e^- y]/[e^+ \wedge e^-]$, and in (4.22) and (4.23) we shall see that this is actually the Haar integral from Example 2.21. There was a twisting automorphism for the Haar integral in that example, and we also get a twist in this case. Commuting the algebra elements past $e^+ \wedge e^-$ could potentially alter the formula for this but does not as the elements of $\mathbb{C}_q[S^2]$ commute with $e^+ \wedge e^-$. The twist is given in terms of the $\mathbb{C}_q[SL_2]$ generators by $\varsigma(a) = q^{-2}a$, $\varsigma(b) = b$, $\varsigma(c) = c$, $\varsigma(d) = q^2 d$ and extends to Ω by $\varsigma(e^+) = q^{-2}e^+$ and $\varsigma(e^-) = q^2 e^-$. $\diamond$

Proposition 3.61 *Suppose that $(\int, d, \varsigma)$ is a self-adjoint closed twisted d-cycle. Then its left kernel is precisely $\{\xi^* : \xi \in R_{\int}\}$. Also if $\int$ is right faithful, then ς is invertible, with $\varsigma^{-1}(\xi) = (\varsigma(\xi^*))^*$.*

Proof For the first part, we apply $*$ to the definition of kernel and obtain $R_{\int}{}^* \subseteq L_{\int}$ and $L_{\int}{}^* \subseteq R_{\int}$. Also, for all $\xi \in \Omega^p$ and $\eta \in \Omega^q$ with $p + q = d$,

$$\begin{aligned}\Big(\int \xi \wedge \eta\Big)^* &= (-1)^{pq} \int \eta^* \wedge \xi^* = \int \varsigma(\xi^*) \wedge \eta^* = (-1)^{pq}\Big(\int \eta \wedge (\varsigma(\xi^*))^*\Big)^* \\ &= \Big(\int \varsigma((\varsigma(\xi^*))^*) \wedge \eta\Big)^*\end{aligned}$$

so we deduce that $\xi = \varsigma((\varsigma(\xi^*))^*)$. $\square$

We can also use twisted cycles to construct a new differential calculus from a given differential calculus. We will need the following.

Proposition 3.62 *Suppose that $(\int, d, \varsigma)$ is a weakly right faithful closed twisted d-cycle with invertible twisting map ς. Then its right kernel plus Ω^r for all $r > d$ is a 2-sided ideal in Ω, and is closed under* d. *The quotient is a differential calculus with top dimension $\leq d$.*

Proof Using the invertible map ς to move forms, we see that the left kernel is the same as the right kernel, so we get a 2-sided ideal, and the quotient is an algebra. The closed property shows that the quotient is an exterior algebra. □

To use this, we need to start with a calculus to quotient, and the obvious choice is the universal calculus Ω_{uni} on an algebra A in Theorem 1.33. We begin with twisted cyclic homology. For an algebra map $\varsigma : A \to A$, the Hochschild homology of A with coefficients in ${}_{\varsigma}A$ (i.e., A with the left action $a{\triangleright}b = \varsigma(a)b$ and right action $b{\triangleleft}a = ba$) is given by the differential $b_{\varsigma} : A^{\otimes n+1} \to A^{\otimes n}$,

$$b_{\varsigma}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^n(\varsigma a_n)a_0 \otimes \cdots \otimes a_{n-1}.$$

If we replace the cyclic shift operator in (3.22) by

$$t_{\varsigma n}(a_0 \otimes \cdots \otimes a_n) = (-1)^n \varsigma a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

then similarly to Definition 3.50, we define $C_n^{\lambda}(A, \varsigma) = A^{\otimes n+1}/(1 - t_{\varsigma})$ with homology the twisted cyclic homology $\mathrm{H}_n^{\lambda}(A, \varsigma)$. Definition 3.52 is similarly modified:

Definition 3.63 The twisted cyclic cochain complex of (A, ς) consists of vector spaces of linear maps

$$C_{\lambda}^n(A, \varsigma) = \left\{\phi : A^{\otimes(n+1)} \to \Bbbk \mid \phi(a_1, \cdots, a_n, a_0) = (-1)^n \phi(\varsigma a_0, a_1, \cdots, a_n)\right\}$$

with the differential $b_{\varsigma} : C_{\lambda}^n \to C_{\lambda}^{n+1}$

$$(b_{\varsigma}\phi)(a_0, \cdots, a_{n+1}) = \sum_{j=0}^{n}(-1)^j \phi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1})$$
$$+ (-1)^{n+1}\phi((\varsigma a_{n+1})a_0, a_1, \cdots, a_n)$$

which obeys ${b_{\varsigma}}^2 = 0$. The twisted cyclic cohomology $\mathrm{HC}_{\lambda}^n(A, \varsigma)$ is the kernel of b_{ς} on degree n (i.e., twisted cyclic cocycles) modulo the image of b_{ς} on degree $n - 1$.

We can now give the twisted version of Proposition 3.54, which constructs an element of cyclic cohomology from a twisted d-cycle on a calculus.

Proposition 3.64 *Suppose that* $(\int, d, \varsigma)$ *is a closed twisted* d*-cycle on the calculus* $(\Omega, \mathrm{d}, \wedge)$*. Then* $\phi : A^{\otimes d+1} \to \mathbb{C}$ *defined by*

$$\phi(a_0, \ldots, a_d) = \int a_o.\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d$$

is a twisted cyclic cocycle of degree d.

Proof This is very similar to the proof of Proposition 3.54. □

The interesting point is that we can reverse this, to construct a calculus. Recall that a cochain ϕ being unital means that it vanishes if any of its arguments is 1.

Corollary 3.65 *Given an algebra automorphism* $\varsigma : A \to A$ *and* ϕ *a unital twisted cyclic cocycle in* $C^d_\lambda(A, \varsigma)$*, there is a differential calculus* $(\Omega, \mathrm{d}, \wedge)$ *of top dimension* d *with a closed twisted* d*-cycle* $(\int, d, \varsigma)$ *such that* ϕ *is the twisted cyclic cocycle associated to* $\int$ *by the construction in Proposition 3.64.*

Proof Define a linear function $\oint$ on Ω^d_{uni} of A by

$$\oint a_0.\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d = \phi(a_0, a_1, \ldots, a_d).$$

This is closed by definition. Now use Proposition 3.62. □

It is beyond our scope here, but this method was used by Kustermans, Murphy & Tuset to give an alternative construction of the 3D calculus on $\mathbb{C}_q[SU_2]$.

3.4 Bimodule Covariant Derivatives

Tensor products of vector bundles and the construction of connections on them are frequently encountered in classical differential geometry. In our case, we cannot tensor product modules over a general algebra A but we can tensor product bimodules, so these represent a nice closed class of 'vector bundles' of particular interest. In this section we let $(A, \Omega, \mathrm{d}, \wedge)$ be an algebra with a differential calculus and consider left connections on such A-bimodules E. We are not interested in giving independent covariant derivatives for the right and left module structures but rather we ask when a left-covariant derivative (say) is also compatible with the right module structure.

Definition 3.66 A left *bimodule covariant derivative* (∇_E, σ_E) or left *bimodule connection* on a bimodule E is a left connection $\nabla_E : E \to \Omega^1 \otimes_A E$ together with a bimodule map $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ obeying, for $e \in E$ and $a \in A$,

$$\nabla_E(ea) = \sigma_E(e \otimes \mathrm{d}a) + (\nabla_E e)a.$$

Note that the *generalised braiding* σ_E here is not additional data as if it exists it is uniquely determined by the stated condition. Thus, being a bimodule connection or not is a property of a left connection ∇_E on a given bimodule. In fact the above definition is a little redundant.

Lemma 3.67 *Let ∇_E be a left connection.*

(1) *If a map $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ obeys the displayed equation in Definition 3.66 then it is a bimodule map (and ∇_E is a bimodule connection).*
(2) *If $\sigma_E : E \otimes \Omega^1 \to \Omega^1 \otimes_A E$ (note the $\otimes_{\Bbbk}$ not $\otimes_A$) obeys the displayed equation in Definition 3.66 and is a right module map then it descends to $E \otimes_A \Omega^1$.*

Proof (1) If σ_E is well defined by the formula stated for all $e \in E$ and $a, b \in A$,

$$\begin{aligned}\sigma_E(e \otimes \mathrm{d}(ab)) &= \nabla_E(eab) - (\nabla_E e)ab\\ &= \nabla_E(eab) - (\nabla_E(ea))b + (\nabla_E(ea))b - (\nabla_E e)ab,\\ &= \sigma_E(ea \otimes \mathrm{d}b) + \sigma_E(e \otimes \mathrm{d}a)b.\end{aligned}$$

But the left-hand side is $\sigma_E(e\otimes a\mathrm{d}b)+\sigma_E(e\otimes(\mathrm{d}a)b) = \sigma_E(ea\otimes\mathrm{d}b)+\sigma_E(e\otimes(\mathrm{d}a)b)$ hence $\sigma_E(e \otimes (\mathrm{d}a)b) = \sigma_E(e \otimes \mathrm{d}a)b$ and so σ_E is a right module map. Also

$$\begin{aligned}\sigma_E(ae \otimes \mathrm{d}b) &= \nabla_E(aeb) - (\nabla_E(ae))b\\ &= \mathrm{d}a \otimes eb + a\nabla_E(eb) - \mathrm{d}a \otimes eb - a(\nabla_E e)b = a\sigma_E(e \otimes \mathrm{d}b).\end{aligned}$$

(2) For the converse, if σ_E obeys the formula stated and is a right module map then

$$\sigma_E(e \otimes a\mathrm{d}b) := \sigma_E(e \otimes \mathrm{d}(ab)) - \sigma_E(e \otimes \mathrm{d}a)b = \sigma_E(ea \otimes \mathrm{d}b). \qquad \square$$

If the bimodule E is not fixed but seen as part of the data then we may write (E, ∇_E, σ_E) for a bimodule connection. Classically, all connections are bimodule connections if we take the same module structure from the left and right (which is possible when A is commutative) and $\sigma_E = $ flip. More examples are as follows.

Example 3.68 If A is an algebra with differential structure then $(A, \mathrm{d}, \mathrm{id})$ is a bimodule with bimodule connection. Here $\mathrm{d} : A \to \Omega^1 \cong \Omega^1 \otimes_A A$ is viewed as a left connection and the Leibniz rule for d implies that $\sigma_A : A \otimes_A \Omega^1 \to \Omega^1 \otimes_A A$ is just the identity $\mathrm{id} : \Omega^1 \to \Omega^1$. Not surprisingly, the curvature is zero. ◇

Example 3.69 Let $A = \Bbbk[x]/\langle x^2\rangle$ be the 'fat point' or Grassmann algebra with calculus $x\mathrm{d}x = -(\mathrm{d}x)x$ (see Example 1.6). The left connections on Ω^1 are of the form $\nabla\mathrm{d}x = a\mathrm{d}x \otimes \mathrm{d}x$ for any $a \in A$ and are all bimodule connections, with

$$\begin{aligned}\sigma(\mathrm{d}x \otimes \mathrm{d}x) &= \nabla((\mathrm{d}x)x) - (\nabla\mathrm{d}x)x = -\nabla(x\mathrm{d}x) - (\nabla\mathrm{d}x)x\\ &= -(1+2xa)\mathrm{d}x \otimes \mathrm{d}x\end{aligned}$$

extending as a bimodule map $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ on noting that $\mathrm{d}x \otimes \mathrm{d}x$ is central. Here $\sigma((\mathrm{d}x)x \otimes \mathrm{d}x) = -\sigma(x\mathrm{d}x \otimes \mathrm{d}x) := -x\sigma(\mathrm{d}x \otimes \mathrm{d}x) = -\sigma(\mathrm{d}x \otimes \mathrm{d}x)x := -\sigma(\mathrm{d}x \otimes (\mathrm{d}x)x) = \sigma(\mathrm{d}x \otimes x\mathrm{d}x)$ so σ is well-defined on $\otimes_A$. ◇

In common with classical connections, affine combinations of bimodule connections are bimodule connections, so in particular the space of bimodule connections on a given bimodule is connected. If $t \in \Bbbk$ and (∇_E, σ_E) and $(\hat{\nabla}_E, \hat{\sigma}_E)$ are bimodule connections then so is $(t\nabla_E + (1-t)\hat{\nabla}_E, t\sigma_E + (1-t)\hat{\sigma}_E)$. The same applies just for left connections on a left module, omitting the σs. Also note that there is an asymmetry in the definition and we might equally be interested in a right connection (as given in Definition 3.21) that respects the left module structure. Thus a right bimodule connection $(\tilde{\nabla}_E, \tilde{\sigma}_E)$ on an A-bimodule E is defined as a right connection $\tilde{\nabla}_E : E \to E \otimes_A \Omega^1$ together with a bimodule map $\tilde{\sigma}_E : \Omega^1 \otimes_A E \to E \otimes_A \Omega^1$ obeying

$$\tilde{\nabla}_E(ae) = \tilde{\sigma}_E(\mathrm{d}a \otimes e) + a\tilde{\nabla}_E e \tag{3.28}$$

for all $e \in E$ and $a \in A$. However, it turns out that there is generically not that much difference between left and right here in view of the following lemma.

Lemma 3.70 *If ∇_E is a left bimodule connection with $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ invertible then $\sigma_E^{-1} \circ \nabla_E : E \to E \otimes_A \Omega^1$ and σ_E^{-1} form a right bimodule connection.*

Proof We check the right Leibniz rule, for $e \in E$ and $a \in A$,

$$\sigma_E^{-1} \circ \nabla_E(ea) = \sigma_E^{-1}\big(\sigma_E(e \otimes \mathrm{d}a) + (\nabla_E e)a\big) = e \otimes \mathrm{d}a + (\sigma_E^{-1} \circ \nabla_E e)a.$$

Moreover, $\sigma_E^{-1} \circ \nabla_E(ae) - a\sigma_E^{-1} \circ \nabla_E(e) = \sigma_E^{-1}(\mathrm{d}a \otimes e)$, as required. □

Since σ plays the role of 'flip' in classical geometry, it will likewise tend to be invertible in a typical noncommutative geometric context, though it need not be.

Example 3.71 Suppose that A has an inner calculus by $\theta \in \Omega^1$ and E is an A-bimodule. The left and right connections given in Example 3.22 by ${}_\theta\nabla(e) = \theta \otimes e$ and $\nabla_\theta(e) = -e \otimes \theta$ respectively are bimodule connections with $\sigma = 0$ (i.e., the connections are A-module maps from the right and left respectively). ◇

The curvature and torsion of a left connection are left-module maps by Lemma 3.19 and Definition 3.28. In the bimodule case we can ask if they are bimodule maps.

Lemma 3.72 *The curvature of a left bimodule connection* (E, ∇_E, σ_E) *obeys*

$$R_E(ea) - (R_E e)a = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}a) - (\mathrm{id} \wedge \sigma_E)(\nabla_E e \otimes \mathrm{d}a)$$

for all $e \in E, a \in A$. *The torsion of a left bimodule connection* $(\Omega^1, \nabla, \sigma)$ *obeys*

$$T_\nabla(\xi a) - (T_\nabla \xi)a = \wedge(\sigma + \mathrm{id})(\xi \otimes \mathrm{d}a)$$

for all $\xi \in \Omega^1$, $a \in A$.

Proof For $a \in A$ and $e \in E$ and writing $\nabla e = \eta \otimes f$ as a shorthand (the sum of such terms understood), we have $\nabla_E(ea) = \sigma_E(e \otimes \mathrm{d}a) + \eta \otimes fa$ and

$$\begin{aligned} R_E(ea) =& \ (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}a) + \mathrm{d}\eta \otimes fa - \eta \wedge \nabla_E(fa) \\ =& \ (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}a) + \mathrm{d}\eta \otimes fa - \eta \wedge (\nabla_E f)a - \eta \wedge \sigma_E(f \otimes \mathrm{d}a) \end{aligned}$$

and recognise the middle term as $(R_E e).a$. We also have for all $\xi \in \Omega^1$, $a \in A$,

$$\begin{aligned} T_\nabla(\xi a) &= \wedge(\nabla\xi)a + \wedge\sigma(\xi \otimes \mathrm{d}a) - (\mathrm{d}\xi)a + \xi \wedge \mathrm{d}a \\ &= (T_\nabla \xi)a + \wedge\sigma(\xi \otimes \mathrm{d}a) + \xi \wedge \mathrm{d}a. \end{aligned}$$

□

In particular, R_E is a right module (and hence bimodule) map if and only if

$$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}a) = (\mathrm{id} \wedge \sigma)(\nabla_E e \otimes \mathrm{d}a) \tag{3.29}$$

for all $e \in E$ and $a \in A$. Similarly, the torsion is a bimodule map if and only if

$$\mathrm{image}(\mathrm{id} + \sigma) \subseteq \ker \wedge. \tag{3.30}$$

3.4.1 Bimodule Connections on Hopf Algebras

We focus in this section on the case where the coordinate algebra is a Hopf algebra H with invertible antipode and $\Omega^1 \cong H \otimes \Lambda^1$ a left-covariant calculus on it as in Theorem 2.26. We recall that Λ^1 is a right H-module (we denote the action by $\triangleleft$) and there is a canonical surjective right module map $\varpi : H^+ \to \Lambda^1$ given by $\varpi h = Sh_{(1)}\mathrm{d}h_{(2)}$. We let E be an H-bimodule and also denote by $\triangleleft$ the right adjoint action of H on the bimodule by $e{\triangleleft}h = (Sh_{(1)})eh_{(2)}$ in Lemma 2.23. We similarly have a right action on $\Omega^1 \otimes_A E$ as a bimodule.

Proposition 3.73 *Let Ω^1 be a left-covariant calculus on a Hopf algebra H and E an H-bimodule. A connection $\nabla : E \to \Omega^1 \otimes_H E$ is a bimodule connection if and only if*

$$\sigma(e \otimes \varpi\pi_\epsilon h) = \varpi\pi_\epsilon(h_{(2)}) \otimes e{\triangleleft}(Sh_{(1)})h_{(3)} - (\nabla(e{\triangleleft}Sh_{(1)})){\triangleleft}h_{(2)} + (\nabla e)\epsilon h$$

for all $h \in H$ gives a well-defined map $\sigma : E \otimes \Lambda^1 \to \Omega^1 \otimes_H E$.

Proof If (E, ∇, σ) is a left bimodule connection, then

$$\begin{aligned}
\sigma(e \otimes \varpi\pi_\epsilon(h)) &= \sigma(e \otimes (Sh_{(1)})\mathrm{d}h_{(2)}) = -\sigma(e \otimes (\mathrm{d}Sh_{(1)})h_{(2)})\\
&= -\sigma(e \otimes \mathrm{d}S(h_{(1)}))h_{(2)} = (\nabla e)(Sh_{(1)})h_{(2)} - (\nabla(eSh_{(1)}))h_{(2)}\\
&= (\nabla e)\epsilon h - \nabla\big((Sh_{(2)})(e{\triangleleft}Sh_{(1)})\big)h_{(3)}\\
&= (\nabla e)\epsilon h - \mathrm{d}Sh_{(2)} \otimes (e{\triangleleft}Sh_{(1)})h_{(3)} - (Sh_{(2)})(\nabla(e{\triangleleft}Sh_{(1)}))h_{(3)}\\
&= (\nabla e)\epsilon h - (\mathrm{d}Sh_{(2)})h_{(3)} \otimes e{\triangleleft}((Sh_{(1)})h_{(4)}) - (\nabla(e{\triangleleft}Sh_{(1)})){\triangleleft}h_{(2)},
\end{aligned}$$

for $e \in E$ and $h \in H$ and the tensor product action in the last term. We used the left and right Leibniz rules for the connection and for d, and the bimodule commutation relations $eh = h_{(1)}(e{\triangleleft}h_{(2)})$. If this expression is well defined (i.e., vanishes on $h \in \ker\varpi\pi_\epsilon$) then σ has the required properties. Note that $E \otimes \Lambda^1 \cong E \otimes_H \Omega^1$, so we also have $\sigma(e \otimes \mathrm{d}h) = h_{(1)}\sigma(e{\triangleleft}h_{(2)} \otimes \varpi\pi_\epsilon h_{(3)})$, where $\pi_\epsilon = \mathrm{id} - 1\epsilon$. $\square$

We will be interested particularly in left-invariant connections in the case where E is a left-covariant H-bimodule as discussed in Lemma 2.23. We have already explained that left-covariant connections on left H-Hopf modules are determined by their restrictions $\nabla^L : {}^H E \to \Lambda^1 \otimes ({}^H E)$ as in (3.14) so such a connection is a bimodule connection if the formula above is well defined when restricted to $e \in {}^H E$ to give a well-defined map $\sigma^L : {}^H E \otimes \Lambda^1 \to \Lambda^1 \otimes ({}^H E)$. This is given by restricting the formula in Proposition 3.73 with ∇^L in place of ∇.

Example 3.74 Following Example 3.31, if Ω^1 is a bicovariant calculus on H and $E = \Omega^1$ then for all $\xi \in \Lambda^1$,

(1) $\nabla^L(\xi) = 0$, $\sigma^L = \Psi$ (left Maurer–Cartan connection);
(2) $\nabla^L(\xi) = -\varpi(\pi_\epsilon S^{-1}\xi_{(\bar 1)}) \otimes \xi_{(\bar 0)}$, $\sigma^L = \Psi^{-1}$ (right Maurer–Cartan connection);
(3) $\nabla^L(\xi) = \xi_{(\bar 0)} \otimes \varpi(\pi_\epsilon \xi_{(\bar 1)})$, $\sigma^L = \Psi + \mathrm{id} - \Psi^2$

are some examples of bimodule connections, where $\pi_\epsilon = \mathrm{id} - 1\epsilon$ and $\Psi(\eta \otimes \varpi\pi_\epsilon h) = \varpi\pi_\epsilon h_{(2)} \otimes \eta{\triangleleft}(Sh_{(1)})h_{(3)}$ is the crossed module braiding in Definition 2.22. These formulae for σ are given by substitution of the given ∇^L into

Proposition 3.73. To justify the terminology in (2), a right-invariant 1-form can be written as $h_{(1)}\mathrm{d}Sh_{(2)} = h_{(1)}Sh_{(3)}\varpi(Sh_{(2)})$ for some $h \in H^+$ and we compute

$$
\begin{aligned}
&\nabla(h_{(1)}Sh_{(3)}\varpi(Sh_{(2)}))\\
&\quad = \mathrm{d}(h_{(1)}Sh_{(3)}) \otimes \varpi(Sh_{(2)})\\
&\qquad - h_{(1)}Sh_{(3)}\varpi\pi_\epsilon S^{-1}((S(Sh_{(2)})_{(1)})(Sh_{(2)})_{(3)}) \otimes \varpi((Sh_{(2)})_{(2)})\\
&\quad = \mathrm{d}(h_{(1)}Sh_{(3)}) \otimes \varpi(Sh_{(2)}) - h_{(1)}Sh_{(5)}\varpi\pi_\epsilon(h_{(2)}Sh_{(4)}) \otimes \varpi(Sh_{(3)}) = 0
\end{aligned}
$$

on using $\mathrm{d}h = h_{(1)}\varpi(\pi_\epsilon h_{(2)})$ for all $h \in H$, applied now to $h_{(1)}Sh_{(3)}$. Note that ∇^L is adjoint to the 'quantum Lie bracket' associated to the calculus in Theorem 2.85. ◇

We now do some explicit calculations for left-invariant bimodule connections on $E = \Omega^1$ in Proposition 1.52 for functions on a finite group. We write ∇^L in terms of Christoffel symbols for the basis $\{e_a\}$ there. As Ω^1 is freely generated with this basis, we can take P to be the identity matrix in Proposition 3.23 and any choice of Christoffel symbols gives a left connection, but not always a bimodule one.

Proposition 3.75 *Let G be a finite group and $\Bbbk(G)$ have a left-covariant calculus given by a subset $\mathcal{C} \subseteq G \setminus \{e\}$. Then $\nabla^L(e_a) = -\sum_{b,c\in\mathcal{C}} \Gamma^a{}_{bc}e_b \otimes e_c$, where $\Gamma^a{}_{bc} \in \Bbbk$ gives a left-invariant bimodule covariant derivative if and only if*

$$a^{-1}cd \notin \mathcal{C} \cup \{e\} \ \Rightarrow\ \Gamma^a{}_{cd} + \delta_{d,a} = 0$$

for all $a, c, d \in \mathcal{C}$, and in this case for $b \in \mathcal{C}$,

$$\sigma^L(e_a \otimes e_b) = \sum_{c,d\in\mathcal{C}:cd=ab} (\Gamma^a{}_{cd} + \delta_{d,a})e_c \otimes e_d.$$

The torsion $T_\nabla(e_a) = -\Gamma^a{}_{bc}e_b \wedge e_c - \mathrm{d}e_a$ is a bimodule map if and only if

$$\sum_{c,d\in\mathcal{C}:cd=ab} (\Gamma^a{}_{cd} + \delta_{d,a})e_c \wedge e_d + e_a \wedge e_b = 0$$

for all $a, b \in \mathcal{C}$.

Proof We know for this calculus that $e_a{\triangleleft}\delta_x = \delta_{x,a}e_a$ and $\varpi\pi_\epsilon(\delta_a) = e_a$ for $a \in \mathcal{C}$, $\varpi\pi_\epsilon(\delta_e) = -\theta$ and zero in other cases. So $\ker\varpi\pi_\epsilon$ is spanned by 1 and δ_p for $p \notin \mathcal{C} \cup \{e\}$. The formula for σ in Proposition 3.73 gives for $v \in \Omega^1$ and $b \neq e$,

$$
\begin{aligned}
\sigma^L(v \otimes \varpi\pi_\epsilon(\delta_b)) &= \sum_{x,y,z:xyz=b} \varpi\pi_\epsilon(\delta_y) \otimes v{\triangleleft}(\delta_{x^{-1}}\delta_z) - \sum_{x,y:xy=b} (\nabla(v{\triangleleft}\delta_{x^{-1}})){\triangleleft}\delta_y\\
&= \sum_{y,z:y=zbz^{-1}} \varpi\pi_\epsilon(\delta_y) \otimes v{\triangleleft}\delta_z - \sum_{x,y:xy=b} (\nabla(v{\triangleleft}\delta_{x^{-1}})){\triangleleft}\delta_y.
\end{aligned}
$$

Now put $v = e_a$ and let $\chi_{\mathcal{C}}$ be the characteristic function of $\mathcal{C}$. Then

$$\begin{aligned}\sigma^L(e_a \otimes \varpi\pi_\epsilon(\delta_b)) &= \varpi\pi_\epsilon(\delta_{aba^{-1}}) \otimes e_a - (\nabla e_a){\triangleleft}\delta_{ab} \\ &= \chi_{\mathcal{C}}(aba^{-1})e_{aba^{-1}} \otimes e_a + (\Gamma^a{}_{cd}e_c \otimes e_d){\triangleleft}\delta_{ab} \\ &= \chi_{\mathcal{C}}(aba^{-1})e_{aba^{-1}} \otimes e_a + \delta_{ab,cd}\Gamma^a{}_{cd}e_c \otimes e_d \\ &= \sum_{c,d\in\mathcal{C}:cd=ab} (\Gamma^a{}_{cd} + \chi_{\mathcal{C}}(aba^{-1})\delta_{d,a})e_c \otimes e_d \\ &= \sum_{c,d\in\mathcal{C}:cd=ab} (\Gamma^a{}_{cd} + \delta_{d,a})e_c \otimes e_d\end{aligned}$$

as $\chi_{\mathcal{C}}(cda^{-1})\delta_{d,a} = \delta_{d,a}$ when $c \in \mathcal{C}$. This gives the stated condition for σ^L to be well defined when $b \notin \mathcal{C} \cup \{e\}$ and the formula for σ^L when $b \in \mathcal{C}$. The condition for the torsion to be a right module map is from Lemma 3.72. □

It is a natural question as to whether σ for a bimodule connection on the 1-forms satisfies the braid relation. We conclude by exploring this for two of our examples.

Example 3.76 Let $G = S_3$ be the permutations of three objects with its 3D calculus given by 2-cycles $\mathcal{C} = \{u, v, w\}$ in Example 1.60. As the product of any three elements of $\mathcal{C}$ is in $\mathcal{C}$, the condition for a bimodule connection in Proposition 3.75 always holds and any left-invariant left connection is a bimodule connection. We restrict attention to ∇ both left and right-invariant. That ∇ commutes with Δ_R is

$$\sum_{b,c} \Gamma^{gag^{-1}}{}_{b,c}e_b \otimes e_c \otimes \delta_g = \sum_{b,c} \Gamma^a{}_{b,c}e_{gbg^{-1}} \otimes e_{gcg^{-1}} \otimes \delta_g,$$

for all $a \in \mathcal{C}$ and $g \in G$, which reduces to $\Gamma^{gag^{-1}}{}_{gbg^{-1},gcg^{-1}} = \Gamma^a{}_{b,c}$ for all $a, b, c \in \mathcal{C}$ and $g \in G$. As conjugation induces all possible permutations of $\mathcal{C}$, there are only five independent values of the Christoffel symbols $\Gamma^a{}_{bc}$, which we set as

$$\Gamma^x{}_{xx} = a - 1, \quad \Gamma^x{}_{yz} = c, \quad \Gamma^x{}_{yx} = d - 1, \quad \Gamma^x{}_{xy} = e, \quad \Gamma^y{}_{xx} = b, \tag{3.31}$$

where x, y, z are all different and $a, b, c, d, e \in \Bbbk$ are parameters (not to be confused with generic elements of $\mathcal{C}$ which we no longer need). If we assume $\Bbbk$ has characteristic zero then we find

$$\sigma = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 & b \\ 0 & e & 0 & 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & e & d & 0 & 0 & 0 & c & 0 \\ 0 & 0 & c & e & 0 & 0 & 0 & d & 0 \\ b & 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ 0 & d & 0 & 0 & 0 & e & c & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & e & 0 & 0 \\ 0 & 0 & d & c & 0 & 0 & 0 & e & 0 \\ b & 0 & 0 & 0 & b & 0 & 0 & 0 & a \end{pmatrix}$$

where we use the tensor product basis $\{e_I\}$ in the lexicographical order $e_u \otimes e_u,\ e_u \otimes e_v,\ e_u \otimes e_w,\ e_v \otimes e_u, \cdots, e_w \otimes e_w$ and specify $\sigma(e_I) = \sum_J \sigma_I{}^J e_J$ in terms of the matrix $\{\sigma_I{}^J\}$ shown. For example, $\sigma(e_u \otimes e_v) = ee_u \otimes e_v + ce_v \otimes e_w + de_w \otimes e_u$ by the second row. Hence on $\Lambda^1 \otimes \Lambda^1$,

$$\det(\sigma) = (a-b)^2(a+2b)(e+c+d)^2(e^2-ce-de+c^2+d^2-cd)^2.$$

The only invertible σ obeying the braid relations are the cases:

(1) $b=c=d=0, a=e\neq 0$,
(2) $b=e=c=0, d=a\neq 0$,
(3) $b=d=e=0, c=a\neq 0$,
(4) $b\neq 0, c=d=0, e=-a^2/b=a-b$,
(5) $b,c\neq 0, c=d=-(b+e), a=-eb/(b+e), b^2+be+e^2=0$,
(6) $b,c\neq 0, a=c=e, d=b^2/e, b^2+be+e^2=0$,
(7) $b,c\neq 0, a=e, c=-b-e, d=-b^2/(b+e), b^2+be+e^2=0$.

Calculating the eigenvectors of σ and using the result of Lemma 3.72 shows that the torsion of the connection given by (3.31) is a right and hence bimodule map exactly when both $d=c$ and $e=c-1$. The torsion here is

$$T_\nabla(e_u) = (1-d)(e_v+e_w)\wedge e_u - ee_u\wedge(e_v+e_w) + (1-c)(e_v\wedge e_w + e_w\wedge e_v)$$

and similarly with permutations $u\to v\to w\to u$. So these are just the conditions for the torsion to vanish. Similarly, for the curvature

$$R_\nabla(e_a) = -\left(\Gamma^a{}_{bc}\mathrm{d}e_b + \Gamma^a{}_{bs}\Gamma^s{}_{rc}e_b\wedge e_r\right)\otimes e_c$$

to be a right-module and hence bimodule map, we need, using the relations $e_a f = R_a(f)e_a$ where R_a is right translation by a, that

$$\left(\Gamma^a{}_{bc}\mathrm{d}e_b + \Gamma^a{}_{bs}\Gamma^s_{rc}e_b\wedge e_r\right)f = R_{ac^{-1}}(f)\left(\Gamma^a{}_{bc}\mathrm{d}e_b + \Gamma^a{}_{bs}\Gamma^s{}_{rc}e_b\wedge e_r\right).$$

A calculation for our 5-parameter moduli space gives the following 1-parameter and 2-parameter moduli for the curvature to be a right and hence bimodule map:

(i) $b=c=d=e=0$,
(ii) $a=-b(b+c)/c, c\neq 0, d=c, e=-b-c$,
(iii) $d=a, e=c=b$,
(iv) $a=c, e=d=b$,
(v) $a=b+c-b^2/c, c\neq 0, d=c, e=b$.

The curvature here is given by

$$\begin{aligned}R_\nabla(e_u) = {}& e_w\wedge e_u\otimes\left(Ae_u+Be_v+Ce_w\right) + e_u\wedge e_v\otimes\left(De_u+Ee_v+Fe_w\right)\\ &+e_u\wedge e_w\otimes\left(De_u+Fe_v+Ee_w\right) + e_v\wedge e_u\otimes\left(Ae_u+Ce_v+Be_w\right);\end{aligned}$$

$$A = d^2 + ce - b^2 - ad, \quad B = (d-c)(b-e),$$
$$C = b^2 + ac - c^2 - de, \quad D = bc - ad + d^2 - e^2,$$
$$E = bd - ab + cd - ae - be + ce, \quad F = (b-e)(b+d+e)$$

and similarly with $u \to v \to w \to u$. We see that (i)–(iv) have zero curvature and the only possibly nonzero case of the curvature being a bimodule map is (v), where

$$R_\nabla(e_u) = \frac{(c-e)^2(c+2e)}{c}(e_u \wedge e_v \otimes e_v + e_u \wedge e_w \otimes e_w). \tag{3.32}$$

Here $e = c$ falls back to an instance of case (iv) and $e = -c/2$ to case (ii). $\diamond$

Example 3.77 We let $H = \mathbb{C}_q[SU_2]$ with its 3D calculus in Example 2.32. Let ∇ be a left-invariant left connection and note that the first term for σ^L in Proposition 3.73 is still well defined as discussed in Remark 2.37. If we denote the rest of σ^L by $\hat\sigma$ and use the actions $e^\pm{\triangleleft}t = qe^\pm$ and $e^0{\triangleleft}t = q^2e^0$ from Remark 2.37 then we find different values for $\hat\sigma$ when evaluated on $h = a-1, b, c, d-1$, namely

$$\hat\sigma(\xi \otimes e^0) = -q^2\nabla^L\xi + q^2(\nabla^L(\xi{\triangleleft}t)){\triangleleft}t^{-1}, \quad \hat\sigma(\xi \otimes e^-) = 0, \quad \hat\sigma(\xi \otimes e^+) = 0$$

and another result for $\hat\sigma(\xi \otimes e^0)$ requiring for consistency that

$$-q^2\nabla^L\xi + q^2(\nabla^L(\xi{\triangleleft}t)){\triangleleft}t^{-1} = \nabla^L\xi - (\nabla^L(\xi{\triangleleft}t^{-1})){\triangleleft}t$$

for any $\xi \in \Omega^1$. Now taking $\xi = e^\pm, e^0$ we find that ∇ is a bimodule covariant derivative if and only if

(1) $\nabla^L e^\pm$ is a linear combination of $e^0 \otimes e^+$, $e^0 \otimes e^-$, $e^+ \otimes e^0$ and $e^- \otimes e^0$,
(2) $\nabla^L e^0$ is a linear combination of $e^0 \otimes e^0$, $e^\pm \otimes e^\pm$ and $e^\mp \otimes e^\pm$,

and in this case σ is

$$\sigma(\xi \otimes e^0) = e^0 \otimes \xi + \nabla^L\xi - (\nabla^L(\xi{\triangleleft}t^{-1})){\triangleleft}t, \ \sigma(\xi \otimes e^\pm) = e^\pm \otimes \xi{\triangleleft}t^{\pm 2}.$$

We cut down the possibilities by supposing that ∇ is also invariant under the right $\mathbb{C}_{q^2}[S^1]$ coaction of the q-Hopf fibration (i.e., preserves the relevant $\mathbb{Z}$-grading on the calculus). Then the bimodule connections are

$$\nabla^L(e^0) = \gamma e^0 \otimes e^0 + \nu e^+ \otimes e^- + \mu e^- \otimes e^+,$$
$$\nabla^L(e^\pm) = \alpha_\pm e^0 \otimes e^\pm + \beta_\pm e^\pm \otimes e^0$$

for parameters $\alpha_\pm, \beta_\pm, \mu, \nu, \gamma \in \mathbb{C}$ or, equivalently, the matrix of Christoffel symbols using basis order e_+, e_0, e_- is

$$\Gamma = -\begin{pmatrix} \alpha_+ e^0 & \beta_+ e^+ & 0 \\ \mu e^- & \gamma e^0 & \nu e^+ \\ 0 & \beta_- e^- & \alpha_- e^0 \end{pmatrix}.$$

Using the same conventions as in Example 3.76 and basis order $e^+ \otimes e^+, e^+ \otimes e^0, \cdots, e^- \otimes e^-$ we then obtain

$$\sigma = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_+(1-q^2) & 0 & 1+\alpha_+(1-q^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 \\ 0 & q^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\gamma(1-q^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-4} & 0 \\ 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\alpha_-(1-q^2) & 0 & \beta_-(1-q^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} \end{pmatrix}$$

so that, for example, $\sigma(e^0 \otimes e^+) = q^4 e^+ \otimes e^0$ from the fourth row. We then find that σ obeys the braid relations if and only if either:

(i) $\beta_+ = \beta_- = 0$,
(ii) $\alpha_+(1-q^2) = -1 - q^{-2}p,\ \ \alpha_- = -(1+q^2),\ \ \beta_+ = q^2\alpha_+,\ \ \beta_- = 0$,
(iii) $\alpha_+ = q^{-4}(1+q^2),\ \ \alpha_-(1-q^2) = -1 - q^2 p,\ \ \beta_+ = 0,\ \ \beta_- = q^{-2}\alpha_-$,

where we set $p := 1 + \gamma(1-q^2)$. For case (i), σ has eigensystem

eigenvector	eigenvalue	eigenvector	eigenvalue
$q^2 e^+ \otimes e^- \pm e^- \otimes e^+$	± 1	$e^0 \otimes e^0$	p
$q^2 e^+ \otimes e^0 \pm e^0 \otimes e^+ q_+$	$\pm q^2 q_+$	$e^+ \otimes e^+$	q^2
$q^2 q_- e^0 \otimes e^- \pm e^- \otimes e^0$	$\pm q^{-2} q_-$	$e^- \otimes e^-$	q^{-2}

where $q_\pm := \sqrt{1 + \alpha_\pm(1-q^2)}$.

Returning to the general parameter space, the torsion is given by

$$T_\nabla(e^\pm) = (\beta_\pm - q^{\pm 4}\alpha_\pm \pm q^{\pm 2}(1+q^{-2}))e^\pm \wedge e^0,$$
$$T_\nabla(e^0) = (\nu - q^2\mu - q^3)e^+ \wedge e^-,$$

which is easily seen in view of the relations of the 3D differential calculus to be a right module map if and only if $T_\nabla(e^\pm) = 0$. Finally the matrix $-(\mathrm{d}\Gamma + \Gamma \wedge \Gamma).P$ which describes the curvature R_∇ in the notation of Proposition 3.23 is just given by $P = \mathrm{id}$ and Γ the matrix already displayed above. ◇

3.4.2 The Monoidal Category of Bimodule Connections

It will be convenient to organise bimodules with bimodule connections into a category, for which we need a notion of morphisms. As for the categories of left modules with left connections, the most restrictive sensible choice is bimodule maps which commute with, or *intertwine*, the bimodule covariant derivatives. The latter for $\theta : (E, \nabla_E, \sigma_E) \to (F, \nabla_F, \sigma_F)$ is $\nabla_F \circ \theta = (\mathrm{id} \otimes \theta)\nabla_E$ and easily implies

$$(\mathrm{id} \otimes \theta)\sigma_E = \sigma_F(\theta \otimes \mathrm{id}). \tag{3.33}$$

We define the following categories:

Name	Objects	Morphisms
${}_A\mathcal{E}_A$	(E, ∇_E, σ_E) bimodules, left bimodule connections	Bimodule maps intertwining ∇_E
${}_A\mathcal{E}\mathcal{I}_A$	(E, ∇_E, σ_E) bimodules, left bimodule connections, σ_E invertible	Bimodule maps intertwining ∇_E

and are now ready to generalise the classical notion of the tensor product of bundles with connection, using the notion of monoidal categories in §2.4.

Theorem 3.78 *${}_A\mathcal{E}_A$ and ${}_A\mathcal{E}\mathcal{I}_A$ are monoidal categories. The tensor product of objects (E, ∇_E, σ_E) and (F, ∇_F, σ_F) is $(E \otimes_A F, \nabla_{E\otimes F}, \sigma_{E\otimes F})$, where*

$$\nabla_{E\otimes F} = \nabla_E \otimes \mathrm{id} + (\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \nabla_F) : E \otimes_A F \to \Omega^1 \otimes_A E \otimes_A F,$$

$$\sigma_{E\otimes F} = (\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_F) : E \otimes_A F \otimes_A \Omega^1 \to \Omega^1 \otimes_A E \otimes_A F.$$

The unit object is $(A, \mathrm{d}, \mathrm{id})$ as given in Example 3.68, and we take the usual tensor product of bimodule maps.

Proof We have already covered the monoidal category structure on the A-bimodule category ${}_A\mathcal{M}_A$ in §2.4. Thus we only have to extend the connections in a consistent way to tensor products. First we need to check that the given formula is well defined on $E \otimes_A F$. Applying $\nabla_{E\otimes F}$ to $e \otimes af$, for $a \in A$, we get

$$\nabla_E e \otimes af + (\sigma_E \otimes \mathrm{id})(e \otimes \mathrm{d}a \otimes f + e \otimes a\nabla f).$$

Applying the formula to $ea\otimes f$ we get $\nabla_E(ea)\otimes f + (\sigma_E\otimes\mathrm{id})(ea\otimes\nabla_F f)$, and these are the same by definition of σ_E. The left Leibniz rule for $\nabla_{E\otimes F}$ is true because σ_E is a left A-module map. To check that $\nabla_{E\otimes F}$ is a bimodule connection,

$$\begin{aligned}\nabla_{E\otimes F}(e \otimes f.a) - \nabla_{E\otimes F}(e \otimes f).a =&\ (\sigma_E \otimes \mathrm{id})(e \otimes (\nabla_F(f.a) - \nabla_F(f).a))\\ =&\ \sigma_{E\otimes F}(e \otimes f \otimes \mathrm{d}a).\end{aligned}$$

Now suppose that $\theta : (E, \nabla_E, \sigma_E) \to (V, \nabla_V, \sigma_V)$ and $\phi : (F, \nabla_F, \sigma_F) \to (W, \nabla_W, \sigma_W)$ are morphisms. We show that

$$\begin{aligned}\nabla_{V\otimes W}(\theta\otimes\phi) &= \nabla_V\theta\otimes\phi + (\sigma_V\otimes\mathrm{id})(\theta\otimes\nabla_W\phi)\\ &= (\mathrm{id}\otimes\theta\otimes\phi)(\nabla_E\otimes\mathrm{id}) + (\sigma_V(\theta\otimes\mathrm{id})\otimes\phi)(\mathrm{id}\otimes\nabla_F)\\ &= (\mathrm{id}\otimes\theta\otimes\phi)\nabla_{E\otimes F}.\end{aligned}$$

Finally we note for ${}_A\mathcal{E}\mathcal{I}_A$ that if σ_E and σ_F are both invertible then so is $\sigma_{E\otimes F}$. □

Note that if we only want a left connection on $E \otimes_A F$ rather than a bimodule connection then it is enough to assume that $(E, \nabla_E, \sigma_E) \in {}_A\mathcal{E}_A$ and $(F, \nabla_F) \in {}_A\mathcal{E}$. Next, in the monoidal category ${}_A\mathcal{E}_A$, it is reasonable to ask if an object has a dual (or is 'rigid') in the categorical sense of ev and coev maps explained in §2.4.

Proposition 3.79 *Let F be a right fgp A-bimodule F with coevaluation element $f_i \otimes f^i \in F \otimes F^\sharp$ (sum understood) and ∇_F is a left bimodule connection with σ_F invertible. Then there is a unique left bimodule connection $(F^\sharp, \nabla_{F^\sharp}, \sigma_{F^\sharp})$ such that $\mathrm{ev}_F, \mathrm{coev}_F$ in Proposition 3.8 are morphisms in ${}_A\mathcal{E}_A$, namely*

$$\nabla_{F^\sharp}\alpha = \sum \mathrm{d}\alpha(f_i)\otimes f^i - \sum(\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f_i)\otimes f^i,$$

$$\sigma_{F^\sharp}(\alpha\otimes\xi) = \sum(\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\alpha\otimes\xi\otimes f_i)\otimes f^i,$$

where $\alpha \in F^\sharp$ and $\xi \in \Omega^1$. This is left dual to (F, ∇_F, σ_F) in ${}_A\mathcal{E}_A$.

Proof From Lemma 3.70, ${\sigma_F}^{-1}\nabla_F$ is a right connection on F. Its dual left connection on $F^\sharp$ from (3.17) comes out as stated. Next, it is easy to see that

$$\begin{aligned}&\mathrm{d}(\alpha(fa)) - (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes fa)\\ &\qquad = (\mathrm{d}\alpha(f))a - (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f)a\end{aligned}$$

for all $a \in A$, $f \in F$ and $\alpha \in F^\sharp$. Hence

$$\begin{aligned}\nabla_{F^\sharp}(\alpha a) &= \mathrm{d}(\alpha(af_i))\otimes f^i - (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes af_i)\otimes f^i\\ &\quad + (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\alpha\otimes\mathrm{d}a\otimes f_i)\otimes f^i\\ &= \mathrm{d}\alpha(f_i)\otimes f^i a - (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f_i)\otimes f^i a\\ &\quad + (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\alpha\otimes\mathrm{d}a\otimes f_i)\otimes f^i\\ &= (\nabla_{F^\sharp}\alpha)a + \sigma_{F^\sharp}(\alpha\otimes\mathrm{d}a),\end{aligned}$$

as required, where the first expression depends only on the value of $af_i \otimes f^i = \mathrm{coev}_F(a) = f_i \otimes f^i a \in F \otimes_A F^\sharp$ (see Proposition 3.8). Also, $\sigma_{F^\sharp}$ is a composition of bimodule maps. That $\nabla_{F^\sharp\otimes F}$ preserves the evaluation is

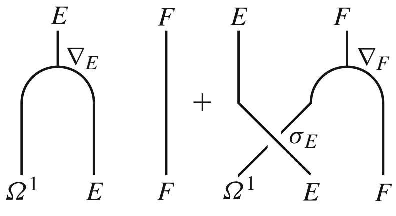
$$\begin{aligned}(\mathrm{id}\otimes\mathrm{ev}_F)(\nabla_{F^\sharp}(\alpha)\otimes f) &= (\mathrm{d}\alpha(f_i))f^i(f)\\ &\quad -((\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f_i))f^i(f)\\ &= \mathrm{d}(\alpha(f)) - (\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F^{-1})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f)\\ &\quad -\alpha(f_i)\mathrm{d}(f^i(f)) + (\mathrm{ev}_F\otimes\mathrm{id})(\alpha\otimes f_i\otimes\mathrm{d}(f^i(f)))\\ &= \mathrm{d}(\alpha(f)) - (\mathrm{id}\otimes\mathrm{ev}_F)(\sigma_{F^\sharp}\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)(\alpha\otimes f).\end{aligned}$$

For $\mathrm{coev}_F : A \to F\otimes_A F^\sharp$ a morphism, we show that $\nabla_{F\otimes F^\sharp}\mathrm{coev}_F(1)=0$. But

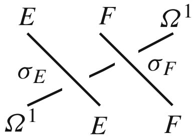
$$\begin{aligned}&(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_F)(\nabla_{F\otimes F^\sharp}\mathrm{coev}_F(1)\otimes f)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_F)\big(\nabla_F f_i\otimes f^i\otimes f + (\sigma_F\otimes\mathrm{id}\otimes\mathrm{id})(f_i\otimes\nabla_{F^\sharp}f^i\otimes f)\big)\\ &= (\nabla_F f_i)f^i(f) + \sigma_F(f_i\otimes(\mathrm{id}\otimes\mathrm{ev}_F)(\nabla_{F^\sharp}f^i\otimes f))\\ &= (\nabla_F f_i)f^i(f) + \sigma_F(f_i\otimes\mathrm{d}f^i(f)) - \sigma_F(f_i\otimes(\mathrm{id}\otimes\mathrm{ev}_F)(\sigma_{F^\sharp}\otimes\mathrm{id})(f^i\otimes\nabla_F f))\\ &= \nabla_F(f) - \sigma_F(f_i\otimes(\mathrm{id}\otimes\mathrm{ev}_F)(\sigma_{F^\sharp}\otimes\mathrm{id})(f^i\otimes\nabla_F f)).\end{aligned}$$

Using the formula for $\sigma_{F^\sharp}$ (requiring separate dual bases $f_j\otimes f^j$) and two uses of the dual bases property shows that this expression vanishes. □

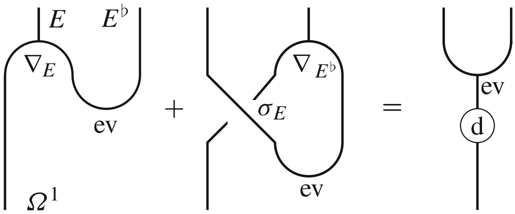
To close this section we point out that it can be useful for proofs such as those above to use a pictorial representation somewhat in the manner of braided algebra in §2.6. For example, we can illustrate the formula for $\nabla_{E\otimes F}$ in Theorem 3.78 as



where we read down the page. Here $\sigma_E : E\otimes_A\Omega^1 \to \Omega^1\otimes_A E$ has been drawn as a braid crossing in order to distinguish it from its inverse $\sigma_E^{-1} : \Omega^1\otimes_A E \to E\otimes_A\Omega^1$ (if it exists), which would be drawn with the crossing the other way. We have already observed in (3.33) that morphisms in ${}_A\mathcal{E}_A$ commute with the relevant generalised braidings σ, which means that they can be pulled through such crossings. Note, however, that many of the operations which we draw (such as ∇_E, d etc.) are not morphisms, so our pictorial representation is less of a computational tool and more of a shorthand. In particular, we do *not* suppose that any form of braid relation necessarily holds for σ, though it can do as we saw in Examples 3.76 and 3.77. The reader should also be warned that neither term in the formula for $\nabla_{E\otimes F}$ in isolation is well defined on $E \otimes_A F$, only their sum is. In general it will be convenient to think of the operators that are not module maps as defined on $\otimes_{\Bbbk}$ rather than on $\otimes_A$, with a separate check that the required sum of operations is well defined on the given domain. Similarly, to represent $\sigma_{E\otimes F}$ in Theorem 3.78 we use



We illustrate this method on a different version of the above duality result, this time starting with E a *left* fgp module. We have already seen in Proposition 3.32 how a left connection ∇_E dualises to get a right connection $\tilde\nabla_{E^\flat}$ on $E^\flat$, but now in our bimodule connection case we want to convert this to another left connection $\nabla_{E^\flat}$ in such a way that $\mathrm{ev}_E : E \otimes_A E^\flat \to A$ and $\mathrm{coev}_E : A \to E^\flat \otimes_A E$ are morphisms, where we have seen that $(A, \mathrm{d}, \mathrm{id})$ is the unit object. This requirement of ev_B to intertwine the connections has the pictorial form



The requirement for coev_E is that $\mathrm{coev} = e_i \otimes e^i \in E^\flat \otimes_A E$ (sum understood) is covariantly constant.

Proposition 3.80 *Let (E, ∇_E, σ_E) be a left bimodule connection with E fgp as a left A-module. Then $\tilde\nabla_{E^\flat} : E^\flat \to E^\flat \otimes_A \Omega^1$ from Proposition 3.32 is a right bimodule connection with*

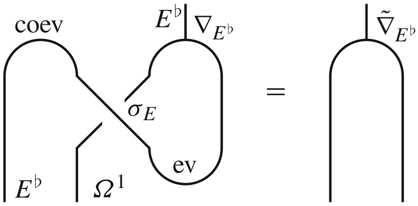
$$\tilde\sigma_{E^\flat} : \Omega^1 \otimes_A E^\flat \to E^\flat \otimes_A \Omega^1, \quad \tilde\sigma_{E^\flat}(\xi \otimes \gamma) = e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\sigma_E(e^i \otimes \xi) \otimes \gamma)$$

and if $\tilde\sigma_{E^\flat}$ is invertible then $\nabla_{E^\flat} = \tilde\sigma_{E^\flat}{}^{-1}\tilde\nabla_{E^\flat}$ and $\sigma_{E^\flat} = \tilde\sigma_{E^\flat}{}^{-1}$ make $(E^\flat, \nabla_{E^\flat}, \sigma_{E^\flat})$ a left bimodule connection right dual to (E, ∇_E, σ_E) in ${}_A\mathcal{E}_A$.

Proof From Proposition 3.32, we have

$$\begin{aligned}
\tilde\nabla_{E^\flat}(a\gamma) &= e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i \otimes a\gamma) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i \otimes a.\gamma) \\
&= e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i a \otimes \gamma) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E e^i)a \otimes \gamma) \\
&= e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i a \otimes \gamma) - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E(e^i a) \otimes \gamma) \\
&\quad + e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\sigma_E(e^i \otimes \mathrm{d}a) \otimes \gamma)
\end{aligned}$$

for all $a \in A$ and $\gamma \in E^\flat$. Since the first two terms depend only on the value $e_i \otimes e^i a = a e_i \otimes e^i \in E^\flat \otimes_A E$ (similarly to the proof of Proposition 3.79), we obtain the formula for $\tilde{\sigma}_{E^\flat}$. Next, to satisfy the diagram for ev a morphism we need



which is just $\tilde{\sigma}_{E^\flat}\nabla_{E^\flat} = \tilde{\nabla}_{E^\flat}$, and a left-right reversed version of Lemma 3.70 gives the result. Finally, for the derivative of the coevaluation element, we have

$$
\begin{aligned}
&\tilde{\nabla}_{E^\flat}(e_j) \otimes e^j + e_i \otimes \nabla_E(e^i) \\
&\qquad = e_i \otimes \mathrm{d}\,\mathrm{ev}(e^i \otimes e_j) \otimes e^j - e_i \otimes (\mathrm{id} \otimes \mathrm{ev})(\nabla_E(e^i) \otimes e_j) \otimes e^j \\
&\qquad\quad + e_i \otimes \mathrm{ev}(e^i \otimes e_j)\nabla_E(e^j) \\
&\qquad = e_i \otimes \nabla_E(\mathrm{ev}(e^i \otimes e_j)e^j) - e_i \otimes \nabla_E(e^i) = 0. \qquad \square
\end{aligned}
$$

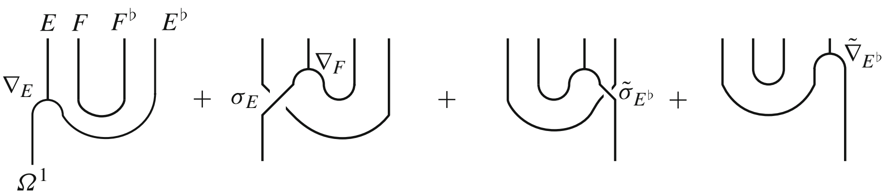
Similar diagrammatic methods may be used to see that duality and the tensor product of bimodule connections are compatible. Thus we suppose (E, ∇_E, σ_E) and (F, ∇_F, σ_F) are left bimodule connections with E, F fgp as left A-modules. It is clear that $E \otimes_A F$ is also fgp as a left A-module using $(E \otimes_A F)^\flat = F^\flat \otimes_A E^\flat$ and the nested form of the evaluation and coevaluation maps as discussed in §2.4. A version of this will be applied in Proposition 6.6.

Corollary 3.81 *Let (E, ∇_E, σ_E) and (F, ∇_F, σ_F) be bimodules with left bimodule connections and fgp as left A-modules. Then, by Proposition 3.80, the dual of the tensor product $\nabla_{E\otimes F}$ of left bimodule connections and the tensor product $\tilde{\nabla}_{F^\flat \otimes E^\flat}$ of right bimodule connections coincide.*

Proof The usual tensor product left bimodule connection and the tensor product of the dual right bimodule connections are, respectively,

$$\nabla_{E\otimes F} = \nabla_E \otimes \mathrm{id} + (\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \nabla_F)\,,\ \ \tilde{\nabla}_{F^\flat \otimes E^\flat} = (\mathrm{id} \otimes \tilde{\sigma}_{E^\flat})(\tilde{\nabla}_{F^\flat} \otimes \mathrm{id}) + \mathrm{id} \otimes \tilde{\nabla}_{E^\flat}.$$

Applying d to the nested evaluation map $\mathrm{ev}_E(\mathrm{id} \otimes \mathrm{ev}_F \otimes \mathrm{id})$ gives



as required. $\square$

3.4.3 Conjugates of Bimodule Connections

Previously in §2.8, we explained a 'bar category' approach to complex conjugation, including the conjugate right module $\overline{E}$ to a left module E of a $*$-algebra A. We suppose further that $(\Omega, \mathrm{d}, \wedge)$ is a $*$-exterior algebra on A and now look at how to conjugate connections defined with respect to this data.

Lemma 3.82 *Given a left connection ∇_E on the left A-module E, there is a right connection $\tilde{\nabla}_{\overline{E}}$ on $\overline{E}$ given by*

$$\tilde{\nabla}_{\overline{E}}\overline{e} = (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\nabla_E e}).$$

More explicitly, if $\nabla_E e = \xi \otimes f$ (sum understood) then $\tilde{\nabla}_{\overline{E}}\overline{e} = \overline{f} \otimes \xi^*$. Likewise if its curvature is $R_E e = \omega \otimes h$ then $\tilde{\nabla}_{\overline{E}}$ has curvature $\tilde{R}_{\overline{E}}\overline{e} = \overline{h} \otimes \omega^*$.

Proof We check the right Leibniz property to see that we have a right connection.

$$\begin{aligned}\tilde{\nabla}_{\overline{E}}(\overline{e}.a) &= \tilde{\nabla}_{\overline{E}}(\overline{a^*.e}) = (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\nabla_E(a^*.e)})\\ &= (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\mathrm{d}a^* \otimes e + a^*.\nabla_E e}) = \overline{e} \otimes \mathrm{d}a + (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{a^*.\nabla_E e})\\ &= \overline{e} \otimes \mathrm{d}a + (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\nabla_E e}).a.\end{aligned}$$

Its curvature is given by Definition 3.21 and if we write $\nabla_E e = \xi \otimes f$ and $\nabla_E f = \kappa \otimes g$ (sums understood) then

$$\tilde{R}_{\overline{E}}\overline{e} = \overline{f} \otimes \mathrm{d}\xi^* + \overline{g} \otimes \kappa^* \wedge \xi^* = \overline{f} \otimes \mathrm{d}\xi^* - \overline{g} \otimes (\xi \wedge \kappa)^*,$$

which can be compared with the direct computation of $R_E e$. □

How do we get from this right connection back to a left connection on $\overline{E}$? We shall begin with a left bimodule covariant derivative on E, and then use a similar argument to that in Lemma 3.70 to switch the sides of the connection.

Proposition 3.83 *Suppose that (E, ∇_E, σ_E) is a left bimodule connection with σ_E invertible. Then we have a left bimodule connection $(\overline{E}, \nabla_{\overline{E}}, \sigma_{\overline{E}})$ defined by*

$$\nabla_{\overline{E}}\overline{e} = (\star^{-1} \otimes \mathrm{id})\Upsilon\, \overline{\sigma_E^{-1}\nabla_E e}, \quad \sigma_{\overline{E}} = (\star^{-1} \otimes \mathrm{id})\Upsilon\, \overline{\sigma_E^{-1}}\, \Upsilon^{-1}(\mathrm{id} \otimes \star).$$

Proof Using the right connection from Lemma 3.82,

$$\begin{aligned}\tilde{\nabla}_{\overline{E}}(a\overline{e}) &= \tilde{\nabla}_{\overline{E}}(\overline{ea^*}) = (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\nabla_E(ea^*)})\\ &= (\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{(\nabla_E e)a^*}) + (\mathrm{id} \otimes \star^{-1})\Upsilon\, \overline{\sigma_E(e \otimes \mathrm{d}a^*)}\\ &= a(\mathrm{id} \otimes \star^{-1})\Upsilon(\overline{\nabla_E e}) + (\mathrm{id} \otimes \star^{-1})\Upsilon\, \overline{\sigma_E(e \otimes \mathrm{d}a^*)},\end{aligned}$$

so $\tilde\nabla_{\overline{E}}$ is a right bimodule connection with $\tilde\sigma_{\overline{E}} = (\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\sigma_E}\,\Upsilon^{-1}(\star\otimes\mathrm{id})$ and applying $\tilde\sigma_{\overline{E}}^{-1}$ to $\tilde\nabla_{\overline{E}}$ gives the result. $\square$

As we need invertibility of σ_E in Proposition 3.83, ${}_A\mathcal{E}\mathcal{I}_A$ as opposed to ${}_A\mathcal{E}_A$ is better placed to be a bar category.

Theorem 3.84 *Suppose the $*$-algebra A has a $*$-differential calculus. Then ${}_A\mathcal{E}\mathcal{I}_A$ is a bar category with $\overline{(E,\nabla_E,\sigma_E)} = (\overline{E},\nabla_{\overline{E}},\sigma_{\overline{E}})$ as in Proposition 3.83.*

Proof This is mostly inherited from the bar category structure of ${}_A\mathcal{M}_A$. We still need to check that if θ is a morphism then so is $\overline{\theta}$, and also that Υ and bb are morphisms in ${}_A\mathcal{E}\mathcal{I}_A$. First, for a morphism $\theta : (E,\nabla_E,\sigma_E)\to(F,\nabla_F,\sigma_F)$ in ${}_A\mathcal{E}\mathcal{I}_A$, we check that $\overline{\theta} : (\overline{E},\nabla_{\overline{E}},\sigma_{\overline{E}})\to(\overline{F},\nabla_{\overline{F}},\sigma_{\overline{F}})$ intertwines the connections:

$$
\begin{aligned}
\nabla_{\overline{F}}(\overline{\theta}\,\overline{e}) &= (\star^{-1}\otimes\mathrm{id})\Upsilon_{F,\Omega^1}(\overline{\sigma_F^{-1}\nabla_E(\theta e)}) = (\star^{-1}\otimes\mathrm{id})\Upsilon_{F,\Omega^1}(\overline{\sigma_F^{-1}(\mathrm{id}\otimes\theta)\nabla_E e})\\
&= (\star^{-1}\otimes\mathrm{id})\Upsilon_{F,\Omega^1}(\overline{(\theta\otimes\mathrm{id})\sigma_E^{-1}\nabla_E e})\\
&= (\star^{-1}\otimes\overline{\theta})\Upsilon_{E,\Omega^1}(\overline{\sigma_E^{-1}\nabla_E e}) = (\mathrm{id}\otimes\overline{\theta})\nabla_{\overline{E}}\overline{e}.
\end{aligned}
$$

Next we check that the bimodule map $\Upsilon_{F,E} : \overline{F\otimes_A E}\to\overline{E}\otimes_A\overline{F}$, defined by $\Upsilon_{F,E}(\overline{f\otimes e}) = \overline{e}\otimes\overline{f}$ intertwines the connections. We begin with

$$
\begin{aligned}
(\mathrm{id}\otimes\Upsilon_{F,E})\nabla_{\overline{F\otimes E}}(\overline{f\otimes e}) &= (\star^{-1}\otimes\Upsilon_{F,E})\Upsilon_{F\otimes E,\Omega^1}(\overline{\sigma_{F\otimes E}^{-1}\nabla_{F\otimes E}(f\otimes e)}),\\
\nabla_{\overline{E}\otimes\overline{F}}\Upsilon_{F,E}(\overline{f\otimes e}) &= \nabla_{\overline{E}\otimes\overline{F}}(\overline{e}\otimes\overline{f}) = \nabla_{\overline{E}}\overline{e}\otimes_A\overline{f} + (\sigma_{\overline{E}}\otimes\mathrm{id})(\overline{e}\otimes\nabla_{\overline{F}}\overline{f})\\
&= (\star^{-1}\otimes\mathrm{id})\Upsilon_{E,\Omega^1}(\overline{\sigma_E^{-1}\nabla_E(e)})\otimes\overline{f}\\
&\quad + \big((\star^{-1}\otimes\mathrm{id})\Upsilon_{E,\Omega^1}\overline{\sigma_E^{-1}}\Upsilon^{-1}_{\Omega^1,E}(\mathrm{id}\otimes\star)\otimes\mathrm{id}\big)(\overline{e}\otimes\nabla_{\overline{F}}\overline{f})
\end{aligned}
$$

for $e\in E$ and $f\in F$. Now, as $(\Upsilon^{-1}_{E,\Omega^1}\otimes\mathrm{id})(\mathrm{id}\otimes\Upsilon_{F,E})\Upsilon_{F\otimes E,\Omega^1} = \Upsilon_{F,E\otimes\Omega^1}$, the equation we need to be intertwining becomes

$$
\begin{aligned}
&\Upsilon_{F,E\otimes\Omega^1}(\overline{\sigma_{F\otimes E}^{-1}\nabla_{F\otimes E}(f\otimes e)})\\
&\quad = (\overline{\sigma_E^{-1}}\otimes\mathrm{id})\Big(\overline{\nabla_E(e)}\otimes\overline{f} + \big(\Upsilon^{-1}_{\Omega^1,E}\otimes\mathrm{id}\big)(\overline{e}\otimes(\star\otimes\mathrm{id})\nabla_{\overline{F}}\overline{f})\Big).
\end{aligned}
$$

We can simplify this equation using

$$
\begin{aligned}
\sigma_{F\otimes E}^{-1}\nabla_{F\otimes E}(f\otimes e) &= (f\otimes\sigma_E^{-1}\nabla_E e) + (\mathrm{id}\otimes\sigma_E^{-1})(\sigma_F^{-1}\nabla_F f\otimes e),\\
\Upsilon_{F,E\otimes\Omega^1}(\overline{f\otimes\sigma_E^{-1}\nabla_E e}) &= (\overline{\sigma_E^{-1}}\otimes\mathrm{id})\left(\overline{\nabla_E e}\otimes\overline{f}\right),
\end{aligned}
$$

leaving us with having to verify that

$$\Upsilon_{F,E\otimes\Omega^1}\overline{((\mathrm{id}\otimes\sigma_E^{-1})(\sigma_F^{-1}\nabla_F f\otimes e))}$$
$$=(\overline{\sigma_E^{-1}}\otimes\mathrm{id})\left(\left(\Upsilon^{-1}_{\Omega^1,E}\otimes\mathrm{id}\right)(\overline{e}\otimes(*\otimes\mathrm{id})\nabla_{\overline{F}}\overline{f})\right).$$

Simplifying this and using the definition of $\nabla_{\overline{F}}$ gives

$$\Upsilon_{F,\Omega^1\otimes E}(\overline{\sigma_F^{-1}\nabla_F f\otimes e})=\left(\Upsilon^{-1}_{\Omega^1,E}\otimes\mathrm{id}\right)(\overline{e}\otimes\Upsilon_{F,\Omega^1}\overline{\sigma_F^{-1}\nabla_F f}),$$

and on using the formula $(\mathrm{id}\otimes\Upsilon^{-1}_{F,\Omega^1})(\Upsilon_{\Omega^1,E}\otimes\mathrm{id})\Upsilon_{F,\Omega^1\otimes E}=\Upsilon_{F\otimes\Omega^1,E}$ the equation we have to verify simply becomes the definition of Υ.

Finally we have to show that the bimodule map $\mathrm{bb}_E: E\to\overline{\overline{E}}$ defined by $\mathrm{bb}_E(e)=\overline{\overline{e}}$ intertwines the connections. Begin with

$$\nabla_{\overline{\overline{E}}}(\overline{\overline{e}})=(\star^{-1}\otimes\mathrm{id})\Upsilon_{\overline{E},\Omega^1}(\overline{\sigma_{\overline{E}}^{-1}\nabla_{\overline{E}}\overline{e}})$$
$$=(\star^{-1}\otimes\mathrm{id})\Upsilon_{\overline{E},\Omega^1}(\overline{\sigma_{\overline{E}}^{-1}(\star^{-1}\otimes\mathrm{id})\Upsilon_{E,\Omega^1}(\overline{\sigma_E^{-1}\nabla_E e})}),$$

and we have to check that this is $(\mathrm{id}\otimes\mathrm{bb}_E)\nabla_E e=(\mathrm{id}\otimes\mathrm{bb}_E)\sigma_E\sigma_E^{-1}\nabla_E e$. Put $\sigma_E^{-1}\nabla_E e=f\otimes\xi$, and then we need to check if

$$(\mathrm{id}\otimes\mathrm{bb}_E)\sigma_E(f\otimes\xi)=(\star^{-1}\otimes\mathrm{id})\Upsilon_{\overline{E},\Omega^1}(\overline{\sigma_{\overline{E}}^{-1}(\star^{-1}\otimes\mathrm{id})\Upsilon_{E,\Omega^1}(\overline{f\otimes\xi})}),$$

which can be rewritten as

$$\sigma_E(f\otimes\xi)=(\star^{-1}\otimes\mathrm{bb}_E^{-1})\Upsilon_{\overline{E},\Omega^1}(\overline{\sigma_{\overline{E}}^{-1}(\xi^*\otimes\overline{f}})).$$

If we substitute $\sigma_{\overline{E}}^{-1}=(\mathrm{id}\otimes\star^{-1})\Upsilon_{\Omega^1,E}\overline{\sigma_E}\Upsilon^{-1}_{E,\Omega^1}(\star\otimes\mathrm{id})$ then it becomes

$$\sigma_E(f\otimes\xi)=(\star^{-1}\otimes\mathrm{bb}_E^{-1})\Upsilon_{\overline{E},\Omega^1}(\overline{(\mathrm{id}\otimes\star^{-1})\Upsilon_{\Omega^1,E}\overline{\sigma_E(f\otimes\xi)}}),$$

and on substituting $\sigma_E(f\otimes\xi)=\eta\otimes h$, this becomes

$$\eta\otimes h=(\star^{-1}\otimes\mathrm{bb}_E^{-1})\Upsilon_{\overline{E},\Omega^1}(\overline{(\mathrm{id}\otimes\star^{-1})\Upsilon_{\Omega^1,E}\overline{\eta\otimes h}})$$
$$=(\star^{-1}\otimes\mathrm{bb}_E^{-1})\Upsilon_{\overline{E},\Omega^1}(\overline{\bar{h}\otimes\eta^*})=(\star^{-1}\otimes\mathrm{bb}_E^{-1})(\overline{\eta^*}\otimes\bar{\bar{h}}),$$

verifying that $\mathrm{bb}_E: E\to\overline{\overline{E}}$ intertwines the connections □

To deal with modules which have a real structure, we need an idea of a star operation on modules. Recall that Definition 2.102 provided the required notion of a star object in a general bar category while in the category of bimodules over a $*$-algebra A, the bimodule E is a star object if there is a bimodule map $\star : E \to \overline{E}$, which we shall write as $e \mapsto \overline{e^*}$, with the property $e^{**} = e$. The definition of the conjugate module immediately required that $e \mapsto e^*$ is conjugate linear, and that $(a.e)^* = e^*.a^*$ and $(e.a)^* = a^*.e^*$ for all $a \in A$. As we have now developed the idea of connections on conjugate modules, we can ask for the star operation on E to preserve the connection, making it into a morphism in ${}_A\mathcal{E}\mathcal{I}_A$. It will also be useful to have a weaker notion '$*$-compatible' where we just keep the σ part (3.33) of the morphism condition (the full meaning of which will be explained in Theorem 4.11, where it will be incorporated generally into a less restrictive class of morphisms).

Definition 3.85 If (E, ∇_E, σ_E) is an object in ${}_A\mathcal{I}\mathcal{E}_A$, and $\star : E \to \overline{E}$ makes E into a star object in ${}_A\mathcal{M}_A$, we say that ∇_E is $*$-preserving if

$$\overline{\sigma_E}\, \Upsilon^{-1}\, (\star \otimes \star)\, \nabla_E(e) = \overline{\nabla_E(e^*)}$$

and is $*$-*compatible* if

$$\sigma_E = (\star^{-1} \otimes \star^{-1}) \Upsilon\, \overline{\sigma_E^{-1}}\, \Upsilon^{-1} (\star \otimes \star).$$

These are the content of $\nabla_{\overline{E}}\star = (\mathrm{id}\otimes\star)\nabla_E$ and $(\mathrm{id}\otimes\star)\sigma_E = \sigma_{\overline{E}}(\star\otimes\mathrm{id}) : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A \overline{E}$ respectively using the definition of $\nabla_{\overline{E}}$ and $\sigma_{\overline{E}}$ in Proposition 3.83. It remains to unravel the categorical notation and state these definitions in more conventional terms by noting that

$$\dagger : \Omega^1 \otimes_A E \to E \otimes_A \Omega^1, \quad \dagger(\xi \otimes e) = e^* \otimes \xi^* \tag{3.34}$$

is a well-defined antilinear operation. Then the condition for $\star$ to be a morphism (i.e for the connection to be $*$-preserving) is the first of

$$\nabla_E \circ * = \sigma_E \circ \dagger \circ \nabla_E, \quad \dagger \circ \sigma_E = \sigma_E^{-1} \circ \dagger. \tag{3.35}$$

The second is implied by the first and by itself is the weaker notion of $*$-compatible.

Recall that the Möbius bundle is a real vector bundle on the circle in which a constant fibre $\mathbb{R}$ in one semicircular patch is glued with a minus sign on one of the overlaps to constant fibres in the other patch, as depicted in Fig. 3.1. Such a local trivialisation comes with a flat connection in each patch relating to the constant

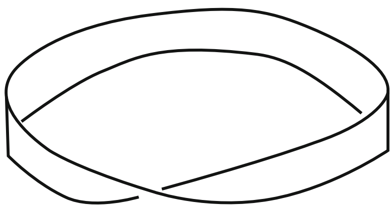


Fig. 3.1 Sketch of the Möbius bundle of which we give a discrete version in Example 3.86

sections, which glue to a global flat connection on the vector bundle. The figure actually sketches the interval $[-1, 1]$ as part of the fibre but its end points form a nontrivial bundle with fibre consisting of two points. If we have a flat metric on the Möbius bundle then these are points of distance 1 from the zero section.

In our formulation, the real Möbius bundle will appear via a nontrivial star operation on a trivial complex bundle and will be equipped with a $*$-preserving connection as in Definition 3.85. At the same time we consider a discretisation of the circle as the finite cyclic group $(\mathbb{Z}_N, +)$, which we consider as the subgroup of the unit circle, with $0 \in \mathbb{Z}_N$ corresponding to $1 \in S^1$ and $1 \in \mathbb{Z}_N$ corresponding to $e^{2i\pi/N}$. Classically of course there is no separate idea of Möbius bundle over the set $\mathbb{Z}_N$ as there is no notion of continuity on a discrete space, but our connection $\nabla_{\mathrm{Möb}_N}$ will capture some of the above geometry in the behaviour of its flat sections.

Example 3.86 We equip the group $\mathbb{Z}_N$ of integers mod N under addition with differential structure given $\mathcal{C} = \{-1, 1\}$ and left-invariant basis e_+, e_- of Ω^1. Now consider $\mathrm{Möb}_N = \mathbb{C}(\mathbb{Z}_N)$ as a $\mathbb{C}(\mathbb{Z}_N)$-bimodule with a single basis element 1 and consider a connection with a particular form of constant Christoffel symbols,

$$\nabla_{\mathrm{Möb}_N}(1) = (\lambda_+ e_+ + \lambda_- e_-) \otimes 1, \quad \lambda_\pm = 1 - e^{\pm i\pi/N}.$$

On a general section $s = \sum_{x\in\mathbb{Z}_N} s_x \delta_x \in \mathrm{Möb}_N$, we have

$$\begin{aligned}\nabla_{\mathrm{Möb}_N} s &= \sum_{x\in\mathbb{Z}_N} s_x \Big((\delta_{x-1} - \delta_x).e_+ + \delta_x.\lambda_+ e_+ + (\delta_{x+1} - \delta_x).e_- + \delta_x.\lambda_- e_- \Big) \otimes 1 \\ &= \sum_{x\in\mathbb{Z}_N} \Big((s_{x+1} - e^{i\pi/N} s_x)\delta_x.e_+ + (s_{x-1} - e^{-i\pi/N} s_x)\delta_x.e_- \Big) \otimes 1\end{aligned}$$

which for our particular choice of $\lambda_\pm$ means that if we seek flat sections in the sense $\nabla_{\mathrm{Möb}_N} s = 0$ then $s_{x+1} = e^{i\pi/N} s_x$ and hence $s_N = -s_0$. As $s_N = s_0$, we have no nonzero flat sections with respect to $\nabla_{\mathrm{Möb}_N}$ as is the case in the classical limit. The reader might think of this multiple of minus one as the 'holonomy' around the loop $0, 1, \ldots, N = 0$. Note also that our connection is a bimodule connection with

$$\sigma_{\mathrm{Möb}_N}(\delta_m \otimes e_+) = e^{i\pi/N} e_+ \otimes \delta_{m+1}, \quad \sigma_{\mathrm{Möb}_N}(\delta_m \otimes e_-) = e^{-i\pi/N} e_- \otimes \delta_{m-1}$$

and that the curvature is zero since we are using $\mathbb{Z}_N$ to model the circle and can reasonably define $\Omega^2 = 0$. The alternative is to use the canonical exterior algebra from §1.7.2, which has the basic one-forms anticommuting and hence Ω^2 1-dimensional with volume form $e_+ \wedge e_-$; the curvature is also zero in this case.

We now express a 'real form' by defining a $*$-structure on $\mathrm{M\ddot{o}b}_N$, namely $\star(\delta_m) = \overline{\mu^m\delta_m}$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. We compute

$$\begin{aligned}
&\tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}\star(\delta_m) = \tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}(\overline{\mu^m\delta_m}) = \mu^{-m}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_{\mathrm{M\ddot{o}b}_N}(\delta_m)}\\
&= \mu^{-m}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{(\delta_{m-1}e_+ - \mathrm{e}^{\mathrm{i}\pi/N}\delta_m e_+ + \delta_{m+1}e_- - \mathrm{e}^{-\mathrm{i}\pi/N}\delta_m e_-)\otimes 1}\\
&= \mu^{-m}\overline{1}\otimes(\delta_{m-1}e_+ - \mathrm{e}^{\mathrm{i}\pi/N}\delta_m e_+ + \delta_{m+1}e_- - \mathrm{e}^{-\mathrm{i}\pi/N}\delta_m e_-)^*\\
&= -\mu^{-m}\overline{1}\otimes(\delta_m e_- - \mathrm{e}^{-\mathrm{i}\pi/N}\delta_{m+1}e_- + \delta_m e_+ - \mathrm{e}^{\mathrm{i}\pi/N}\delta_{m-1}e_+),
\end{aligned}$$

and from this

$$\begin{aligned}
&(\star^{-1}\otimes\mathrm{id})\tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}\star(\delta_m)\\
&\quad = -\delta_m\otimes e_- + \delta_{m+1}\otimes\mu\mathrm{e}^{-\mathrm{i}\pi/N}e_- - \delta_m\otimes e_+ + \delta_{m-1}\otimes\mu^{-1}\mathrm{e}^{\mathrm{i}\pi/N}e_+.
\end{aligned}$$

For the connection to be $*$-preserving we require

$$\begin{aligned}
\nabla_{\mathrm{M\ddot{o}b}_N}(\delta_m) &= \sigma_{\mathrm{M\ddot{o}b}_N}(\star^{-1}\otimes\mathrm{id})\tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}\star(\delta_m)\\
&= \mu\mathrm{e}^{-2\mathrm{i}\pi/N}e_-\otimes\delta_m - \mathrm{e}^{-\mathrm{i}\pi/N}e_-\otimes\delta_{m-1}\\
&\quad + \mu^{-1}\mathrm{e}^{2\mathrm{i}\pi/N}e_+\otimes\delta_m - \mathrm{e}^{\mathrm{i}\pi/N}e_+\otimes\delta_{m+1}
\end{aligned}$$

which forces us to set $\mu = \mathrm{e}^{2\mathrm{i}\pi/N}$. We can also focus on 'real' sections with respect to $*$, which means $s = \sum s_m e^{m\mathrm{i}\pi/N}\delta_m$ with real s_m. Note that this space of 'real' sections is not invariant under translation on $\mathbb{Z}_N$ but under a modified shift $S(\delta_m) = \mathrm{e}^{\mathrm{i}\pi/N}\delta_{m+1}$, generating an action of $\mathbb{Z}_{2N}$ as a double cover of $\mathbb{Z}_N$. Our sections change sign under N shifts in keeping with the classical picture.

We will study the discrete Möbius bundle further in Example 4.47 and Example 8.37, where we show that the classical fact that there are no nonvanishing continuous sections of bundle becomes the statement that there are nonvanishing global sections but they have infinite 'Sobolev norm' (or 'energy') as $N \to \infty$. ◇

We next analyse the important case of a connection on a finite group calculus.

Proposition 3.87 *Let G be a finite group and $\mathbb{C}(G)$ have $*$-calculus given by $\mathcal{C} \subseteq G\setminus\{e\}$ closed under inverse as in Proposition 1.52. Bimodule connections $(\Omega^1, \nabla, \sigma)$ as in Proposition 3.75 are $\star$-compatible and $\star$-preserving (these conditions coincide in this case) if and only if*

$$\sum_{c,d\in\mathcal{C};cd=pq}(\Gamma^r{}_{cd} + \delta_{d,r})^*(\Gamma^{d^{-1}}{}_{q^{-1}p^{-1}} + \delta_{p^{-1},d^{-1}}) = \delta_{r,p}$$

holds for all $p, q, r \in \mathcal{C}$.

Proof Set $\sigma^L(e_a\otimes e_b)=\sigma_{ab}{}^{cd}e_c\otimes e_d$ (summed over $c,d\in\mathcal{C}$). Then the condition for $\star$ compatibility in Definition 3.85 applied to $e_a\otimes e_b$ gives

$$(\sigma_{ab}{}^{cd})^* e_{d^{-1}}\otimes e_{c^{-1}}=(\sigma^L)^{-1}(e_{b^{-1}}\otimes e_{a^{-1}}),$$

and applying σ^L to both sides gives $(\sigma_{ab}{}^{cd})^*\sigma_{d^{-1}c^{-1}}{}^{q^{-1}p^{-1}}=\delta_{b,q}\delta_{a,p}$, and substituting $\sigma_{ab}{}^{cd}=(\Gamma^a{}_{cd}+\delta_{d,a})\delta_{ab,cd}$ from Proposition 3.75 gives the result. Similarly checking $\star$-preservation from (3.35) yields no further restriction. $\square$

Example 3.88 Left-invariant bimodule connections on the standard $\Omega^1(S_3)$ as in Example 3.76 are $*$-compatible (and $*$-preserving) if and only if

$$N_e=\begin{pmatrix}a&b&b\\b&a&b\\b&b&a\end{pmatrix},\quad N_{uv}=\begin{pmatrix}d&e&c\\c&d&e\\e&c&d\end{pmatrix}$$

are unitary. Here $(N_{cd})_{a,d}=\Gamma^a{}_{cd}+\delta_{a,d}$ is the combination appearing in Proposition 3.87 and in our case the elements of $\mathcal{C}$ are all involutive. $\diamond$

In general, $*$-preserving is a stronger condition.

Example 3.89 The bimodule connections on $\Omega^1(\mathbb{C}_q[SU_2])$ with its 3-dimensional calculus given in Example 3.77 with q real are $\star$-compatible if and only if

$$\alpha_-^*=\frac{-\alpha_+}{1+\alpha_+(1-q^2)},\quad \beta_-^*=\frac{-q^{-4}\beta_+}{1+\alpha_+(1-q^2)},\quad \gamma^*+\gamma+|\gamma|^2(1-q^2)=0$$

and $\star$-preserving if and only if in addition $\nu=-\mu^*q^2$. $\diamond$

3.5 Line Modules and Morita Theory

Line bundles appear in many places, from algebraic geometry to theoretical physics. Classically they have rank one—a one-dimensional fibre—but even more, they are invertible with respect to tensor products. We now formulate this in general.

Definition 3.90 Let A be a unital algebra. A *line A-module* is a left fgp A-bimodule L for which the bimodule maps of coevaluation $\mathrm{coev}:A\to L^\flat\otimes_A L$ and evaluation $\mathrm{ev}:L\otimes_A L^\flat\to A$ are isomorphisms.

We may just say 'line module' when the context is clear, and we can also just say fgp rather than left fgp due to the following result.

Lemma 3.91 *If L is a left line A-module then* $\mathrm{coev}^{-1}:L^\flat\otimes_A L\to A$ *and* $\mathrm{ev}^{-1}:A\to L\otimes_A L^\flat$ *make L into a right fgp module with dual $L^\flat$. Moreover, if L and M are line modules then so is $L\otimes_A M$.*

Proof Recall from (3.4) that L left fgp means $L^\flat$ is its categorical right dual in the sense that $(\mathrm{ev}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{coev})=\mathrm{id}_L$ and $(\mathrm{id}\otimes\mathrm{ev})(\mathrm{coev}\otimes\mathrm{id})=\mathrm{id}_{L^\flat}$. Taking the inverse of these gives the required

$$(\mathrm{id}\otimes\mathrm{coev}^{-1})(\mathrm{ev}^{-1}\otimes\mathrm{id})=\mathrm{id}_L,\quad (\mathrm{coev}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{ev}^{-1})=\mathrm{id}_{L^\flat}.$$

For the tensor product, define the dual $(L\otimes_A M)^\flat=M^\flat\otimes_A L^\flat$, with

$$\begin{aligned}\mathrm{ev}_{L\otimes M}&=\mathrm{ev}_L(\mathrm{id}\otimes\mathrm{ev}_M\otimes\mathrm{id}):L\otimes_A M\otimes_A M^\flat\otimes_A L^\flat\to A,\\ \mathrm{coev}_{L\otimes M}&=(\mathrm{id}\otimes\mathrm{coev}_L\otimes\mathrm{id})\mathrm{coev}_M:A\to M^\flat\otimes_A L^\flat\otimes_A L\otimes_A M.\end{aligned}\qquad\square$$

Now we give one of the most useful results about line modules.

Lemma 3.92 *Let L be a line A-module. If $T:L\to L\otimes_A F$ is a left A-module map for some left A-module F then there is a unique $f\in F$ such that $T(e)=e\otimes f$. If $S:L\to E\otimes_A L$ is a right A-module map for some right A-module E then there is a unique $g\in E$ such that $S(e)=g\otimes e$.*

Proof For the first part we define f as the image of $1\in A$ under

$$A\xrightarrow{\mathrm{coev}}L^\flat\otimes_A L\xrightarrow{\mathrm{id}\otimes T}L^\flat\otimes_A L\otimes_A F\xrightarrow{\mathrm{coev}^{-1}\otimes\mathrm{id}}F.$$

Then as $(\mathrm{coev}\otimes\mathrm{id})(f)=(\mathrm{id}\otimes T)(\mathrm{coev})$, we have

$$\begin{aligned}e\otimes f&=(\mathrm{ev}\otimes\mathrm{id})(e\otimes(\mathrm{coev}\otimes\mathrm{id})f)=(\mathrm{ev}\otimes\mathrm{id})(e\otimes(\mathrm{id}\otimes T)(\mathrm{coev}))\\ &=T((\mathrm{ev}\otimes\mathrm{id})(e\otimes\mathrm{coev})=T(e).\end{aligned}$$

The proof for the second part is similar with g the image of $1\in A$ under

$$A\xrightarrow{\mathrm{ev}^{-1}}L\otimes_A L^\flat\xrightarrow{S\otimes\mathrm{id}}E\otimes_A L\otimes_A L^\flat\xrightarrow{\mathrm{id}\otimes\mathrm{ev}}E.\qquad\square$$

There is an immediate application of this proposition in the case $F=A$.

Corollary 3.93 *Let L be a line A-module. There is a 1–1 correspondence between left module maps ${}_A\mathrm{Hom}(L,L)$ and A, given by $T(x)=x.t$ for some $t\in A$. There is a 1–1 correspondence between bimodule maps ${}_A\mathrm{Hom}_A(L,L)$ and the centre $Z(A)$ of A, given by some t in the centre.*

Proof Write a left A-module map $T : L \to L$ as $T : L \to L \otimes_A A$, and from Lemma 3.92 we see that there is a unique element $t \in A$ such that $T(x) = x.t$. If T is a bimodule map, and $a \in A$, then the left module map $x \mapsto T(x.a) - T(x).a$ corresponds to the element $at - ta \in A$, but it also must be zero by uniqueness. □

This in turn has an immediate application.

Corollary 3.94 (Frölich Map) *For L a line A-module there is a unital invertible algebra map $\Phi_L : Z(A) \to Z(A)$ such that $z.e = e.\Phi_L(z)$ for all $e \in L$ and $z \in Z(A)$. This map depends only on the isomorphism class of the bimodule and obeys $\Phi_{L\otimes M} = \Phi_M \circ \Phi_L$.*

Proof Given $z \in Z(A)$ the map $e \mapsto z.e$ is a bimodule map from L to L, so by Corollary 3.93 there is a unique $\Phi_L(z) \in Z(A)$ such that $z.e = e.\Phi_L(z)$ for all $e \in L$. Most of the results follow quite simply. The invertibility comes from the tensor product result, which has the special case $\Phi_{L^\flat} = \Phi_L{}^{-1}$. □

The Frölich map Φ_L also behaves well for conjugates.

Proposition 3.95 *If A is a $*$-algebra then the conjugate $\overline{L}$ of a line A-module L is also a line A-module and $\Phi_{\overline{L}}(z) = (\Phi_L{}^{-1}(z^*))^*$.*

Proof Taking the dual of $\overline{L}$ to be $\overline{L^\flat}$, the evaluation and coevaluation maps are

$$\overline{L} \otimes_A \overline{L^\flat} \xrightarrow{\Upsilon^{-1}} \overline{L^\flat \otimes_A L} \xrightarrow{\mathrm{coev}^{-1}} \overline{A} \xrightarrow{\star^{-1}} A, \quad A \xrightarrow{\star} \overline{A} \xrightarrow{\mathrm{ev}^{-1}} \overline{L \otimes_A L^\flat} \xrightarrow{\Upsilon} \overline{L^\flat} \otimes_A \overline{L}$$

which are compositions of isomorphisms. Finally, we have, for $\overline{e} \in \overline{L}$ and $z \in Z(A)$

$$z.\overline{e} = \overline{e.z^*} = \overline{\Phi_L{}^{-1}(z^*).e} = \overline{e}.(\Phi_L{}^{-1}(z^*))^*.$$ □

We extend the Frölich map to forms in Corollary 4.20. Next, there is a standard idea of Morita contexts in algebraic K-theory, which generalise line modules.

Definition 3.96 A *Morita context* for unital algebras A, B consists of $E \in {}_A\mathcal{M}_B$, $F \in {}_B\mathcal{M}_A$ with bimodule maps $\mu_1 : E \otimes_B F \to A$, $\mu_2 : F \otimes_A E \to B$ such that

$$\mu_1 \otimes \mathrm{id} = \mathrm{id} \otimes \mu_2 : E \otimes_B F \otimes_A E \to E,$$
$$\mu_2 \otimes \mathrm{id} = \mathrm{id} \otimes \mu_1 : F \otimes_A E \otimes_B F \to F.$$

This is called a *strict Morita context* in the case where μ_1 and μ_2 are surjective.

A strict Morita context $(A, B, E, F, \mu_1, \mu_2)$ has several properties which are beyond our scope to prove here but for which the interested reader can find more details in the book on algebraic K-theory by Bass. It can be shown in particular that:

(a) μ_1 and μ_2 are isomorphisms.
(b) E and F are finitely generated projective left A- and B-modules, respectively.
(c) E and F are finitely generated projective right B- and A-modules, respectively.

The purpose of Morita contexts is to implement equivalences between categories of modules over algebras. Two unital algebras A and B are said to be *Morita equivalent* if the categories of modules ${}_A\mathcal{M}$ and ${}_B\mathcal{M}$ are equivalent. By an equivalence of categories we mean that there are functors $P : {}_A\mathcal{M} \to {}_B\mathcal{M}$ and $Q : {}_B\mathcal{M} \to {}_A\mathcal{M}$ together with invertible natural transformations from $P \circ Q$ to the identity on ${}_B\mathcal{M}$, and from $Q \circ P$ to the identity on ${}_A\mathcal{M}$. It is a basic fact from algebraic K-theory that such equivalences between ${}_A\mathcal{M}$ and ${}_B\mathcal{M}$ are in 1–1 correspondence with strict Morita contexts $(A, B, E, F, \mu_1, \mu_2)$. Given a strict Morita context, the functors implementing the equivalence are given by

$$E \otimes_B - : {}_B\mathcal{M} \to {}_A\mathcal{M}, \quad F \otimes_A - : {}_A\mathcal{M} \to {}_B\mathcal{M}.$$

The following relates line modules to Morita contexts as promised.

Proposition 3.97 *Suppose that L is a line A-module. Then*

$$\mathrm{ev} \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{coev}^{-1} : L \otimes_A L^\flat \otimes_A L \to L,$$
$$\mathrm{id} \otimes \mathrm{ev} = \mathrm{coev}^{-1} \otimes \mathrm{id} : L^\flat \otimes_A L \otimes_A L^\flat \to L^\flat.$$

Thus there is a 1–1 correspondence between line A-module L and strict Morita contexts $(A, A, L, L^\flat, \mathrm{ev}, \mathrm{coev}^{-1})$ from A to itself.

Proof The map $\mathrm{id} \otimes \mathrm{coev} : L \to L \otimes_A L^\flat \otimes_A L$ is invertible, and we have

$$(\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}) = (\mathrm{id} \otimes \mathrm{coev}^{-1})(\mathrm{id} \otimes \mathrm{coev}) : L \to L,$$

as both sides are the identity, proving the first equation. Likewise $\mathrm{coev} \otimes \mathrm{id} : L^\flat \to L^\flat \otimes_A L \otimes_A L^\flat$ is invertible, and we deduce the second equation from

$$(\mathrm{id} \otimes \mathrm{ev})(\mathrm{coev} \otimes \mathrm{id}) = (\mathrm{coev}^{-1} \otimes \mathrm{id})(\mathrm{coev} \otimes \mathrm{id}) : L^\flat \to L^\flat. \qquad \square$$

In §5.2.3 we will give another characterisation of a line module as part of a Hopf–Galois extension. For the moment we give a related construction. Recall that an algebra B being graded by a group G means that B is the direct sum of vector subspaces B_g for all $g \in G$, and that the grading is preserved by the product, i.e., $B_g.B_{g'} \subseteq B_{gg'}$. Clearly if B is unital then $1 \in B_e$ for $e \in G$ the identity element.

Proposition 3.98 *Let B be a group G-graded unital algebra, $A = B_e$ and for some fixed $g \in G$ suppose that we have*

$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathrm{Col}^n(B_g), \quad \underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathrm{Col}^n(B_{g^{-1}}),$$

such that $\underline{w}^T \underline{v} = \sum_i w_i v_i = 1$. If the product : $B_g \otimes_A B_{g^{-1}} \to A$ is surjective then

$$L = \{b.\underline{w}^T \subseteq \mathrm{Row}^n(A) \mid b \in B_g\}, \quad L^\flat = \{\underline{v}.b \subseteq \mathrm{Col}^n(A) \mid b \in B_{g^{-1}}\}$$

defines a line A-module L and its dual $L^\flat$. The projector is $P = \underline{v}\underline{w}^T \in M_n(A)$.

Proof It is clear that the stated P satisfies $P^2 = P$. We make L and $L^\flat$ into A-bimodules with actions

$$a{\triangleright}(b.\underline{w}^T) = ab.\underline{w}^T, \quad (b.\underline{w}^T){\triangleleft}a = b.\underline{w}^T.P(a) = ba.\underline{w}^T,$$
$$a{\triangleright}(\underline{v}.b) = P(a).\underline{v}.b = \underline{v}.ab, \quad (\underline{v}.b){\triangleleft}a = \underline{v}.ba,$$

using the notation $P(a) = \underline{v}a\underline{w}^T$. The evaluation map and coevaulation maps are

$$\mathrm{ev}: L \otimes_A L^\flat \to A, \quad \mathrm{ev}(b.\underline{w}^T \otimes \underline{v}.b') = b.\underline{w}^T \underline{v}.b' = bb',$$
$$\mathrm{coev}: A \to L^\flat \otimes_A L, \quad \mathrm{coev}(a) = \sum_i \underline{v}aw_i \otimes v_i\underline{w}^T,$$

where the former uses matrix multiplication. Here coev is clearly a left module map and a right module map by

$$\underline{v}aw_i \otimes v_i\underline{w}^T = \underline{v}w_jv_jaw_i \otimes v_i\underline{w}^T = \underline{v}w_j \otimes v_jaw_iv_i\underline{w}^T = \underline{v}w_j \otimes v_ja\underline{w}^T.$$

For evaluation and coevaluation, summing over i,

$$(\mathrm{ev} \otimes \mathrm{id})(b.\underline{w}^T \otimes \mathrm{coev}(1)) = b.\underline{w}^T.\underline{v}w_i{\triangleright}v_i\underline{w}^T = bw_iv_i\underline{w}^T = b.\underline{w}^T,$$
$$(\mathrm{id} \otimes \mathrm{ev})(\mathrm{coev} \otimes \underline{v}b) = \underline{v}w_i{\triangleleft}v_i\underline{w}^T\underline{v}.b = \underline{v}w_iv_ib = \underline{v}b.$$

Also, $\mathrm{coev}^{-1}(\underline{x} \otimes \underline{y}^T) = \underline{w}^T\underline{x}\underline{y}^T\underline{v}$ is the inverse of the coevaluation map,

$$\mathrm{coev}^{-1} \circ \mathrm{coev}(1) = \mathrm{coev}^{-1}(\underline{v}w_i \otimes v_i\underline{w}^T) = \underline{w}^T\underline{v}w_iv_i\underline{w}^T\underline{v} = 1_B,$$
$$\mathrm{coev} \circ \mathrm{coev}^{-1}(\underline{v}b \otimes b'\underline{w}^T) = \mathrm{coev}(bb') = \underline{v}bb'w_i \otimes v_i\underline{w}^T$$
$$= \underline{v}b \otimes b'w_iv_i\underline{w}^T = \underline{v}b \otimes b'\underline{w}^T.$$

As the product $B_g \otimes B_{g^{-1}} \to B_0 = A$ is surjective, choose $r_j \otimes s_j \in B_g \otimes B_{g^{-1}}$ such that $\sum_j r_j s_j = 1_B$, and define $\mathrm{ev}^{-1}(a) = \sum_j a r_j \underline{w}^T \otimes \underline{v} s_j$. Then

$$\begin{aligned}
&\mathrm{ev} \circ \mathrm{ev}^{-1}(a) = \mathrm{ev}(a r_j \underline{w}^T \otimes \underline{v} s_j) = a r_j s_j = a,\\
&\mathrm{ev}^{-1} \circ \mathrm{ev}(b' \underline{w}^T \otimes \underline{v}.b) = \mathrm{ev}^{-1}(b'b) = b' b r_j \underline{w}^T \otimes \underline{v} s_j = b' \underline{w}^T \underline{v} b r_j \underline{w}^T \otimes \underline{v} s_j\\
&\quad = b' \underline{w}^T \otimes \underline{v} b r_j \underline{w}^T \underline{v} s_j = b' \underline{w}^T \otimes \underline{v} b r_j s_j = b' \underline{w}^T \otimes \underline{v} b. \qquad \square
\end{aligned}$$

In fact the construction of L in Proposition 3.98 is simply an explicit dual basis. The bimodule L is isomorphic to B_g and $L^\flat$ is isomorphic to $B_{g^{-1}}$, independently of the choice of $\underline{v}$ and $\underline{w}$, as can be seen by the formulae for the left and right actions in the proof of the proposition. One of the simplest interesting examples of a line module is given by the q-Hopf fibration.

Example 3.99 From Example 2.32 there is a $\mathbb{Z}$ grading on $B = \mathbb{C}_q[SL_2]$ for which a, c have degree $+1$ and b, d have degree -1. We choose $g = -1 \in \mathbb{Z}$ and

$$\underline{w} = \begin{pmatrix} a \\ q^{-1}c \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} d \\ -b \end{pmatrix},$$

which obey $\underline{w}^T \underline{v} = 1$. The element $d \otimes a - qb \otimes c$ in degree $-1 \otimes$ degree 1 maps to 1 under the product map, so Proposition 3.98 holds and we get a line module over the degree zero algebra, which is $\mathbb{C}_q[S^2]$. In terms of the usual $\mathbb{C}_q[S^2]$ coordinates,

$$P = \underline{v}\underline{w}^T = \begin{pmatrix} da & q^{-1}dc \\ -ba & -q^{-1}bc \end{pmatrix} = \begin{pmatrix} 1 - q^2 x & z \\ w & x \end{pmatrix},$$

which is the projection from Example 3.15 except that we consider $w = z^*$ as an independent generator so that we are not tied to the standard $*$-algebra structure. Elements of L have the form $\underline{u} = (fa, f q^{-1} c)$, where f has degree -1.

Next we look at the three different star operations or 'real forms' in Proposition 2.13. By Proposition 3.95 in each case we have a conjugate line A-module:

(1) The $\mathbb{C}_q[SU_2]$ $*$-operation with q real is the simplest and results in $w = z^*$ on the sphere $A = \mathbb{C}_q[S^2]$. We have $\underline{w}^{T*} = \underline{v}$, and the map $G : \overline{L} \to L^\flat$ defined by $\overline{\underline{u}} \mapsto \underline{u}^*$ is an invertible bimodule map. This can be viewed as corresponding to a hermitian inner product $\langle\ ,\ \rangle$ on L by the composition

$$L \otimes_A \overline{L} \xrightarrow{\mathrm{id} \otimes G} L \otimes_A L^\flat \xrightarrow{\mathrm{ev}} A.$$

This is positive as $\langle (r, s), \overline{(r, s)} \rangle = rr^* + ss^*$. We will return to this in §8.4.

(2) The $\mathbb{C}_q[SU_{1,1}]$ $*$-operation with q real is a little more complicated as we have

$$\underline{w}^{T*} = U\underline{v}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $UG : \overline{L} \to L^\flat$ (where U operates by left multiplication) is an invertible bimodule map, and we similarly get a hermitian inner product on L, except this time it may not be positive. For both of these $*$-algebra structures P is a hermitian projection. In terms of the additional real forms on $\mathbb{C}_q[S^2_{\mathbb{C}}]$ at the end of §2.3 this gives a line bundle with hermitian metric (not necessarily positive) on $\mathbb{C}_q[AdS_2]$.

(3) The $\mathbb{C}_q[SL_2(\mathbb{R})]$ $*$-operation with $|q| = 1$ is completely different. In this case there is an invertible bimodule map $\star^{-1} : \overline{L} \to L$ given by

$$\star^{-1}(\overline{\underline{u}}) = \underline{u}^{*T} V, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Thus L becomes a star object, for which we need to check that

$$\overline{\star^{-1}}(\overline{\overline{\underline{u}}}) = \overline{\underline{u}^{*T} V} = \star(\underline{u} V^{*T} V) = \star\underline{u}.$$

This also tells us that the $*$-operation on L is $\underline{u}^* = \underline{u}^{*T} V$. In terms of the additional real forms on $\mathbb{C}_q[S^2_{\mathbb{C}}]$ at the end of §2.3, this gives a line bundle with $*$-structure on $\mathbb{C}_q[H^2]$. We explain after Proposition 5.54 that the connection on L as in Example 5.51 is $*$-preserving at least for the standard real form. $\diamond$

Our last example of the construction of line modules in Proposition 3.98 illustrates an interesting point: a group action on a manifold often gives a quotient which is not a manifold as it has singular points, but this problem can disappear on quantum deformation. A typical example is the group $\mathbb{Z}_N$ acting on the classical disk by multiples of $360°/N$ rotations about the centre. The quotient is a cone, with the singular point at the vertex. The quotient of the cotangent plane to the disk is singular—any 1-form which is invariant under the $360°/N$ rotation must vanish at the centre. Most obviously, the invariant 1-forms do not form a locally trivial bundle over the cone. The following nonsingular noncommutative deformation of this construction was found by Brzeziński.

Example 3.100 (q-Cone) Consider the $*$-algebra of functions on a 'deformed disk' $\mathbb{C}_q[D]$, generated by z and z^*, and its star differential calculus as given in Example 3.40. We take q generic but real, so in particular $q^2 \neq 1$. We take a coaction of $\mathbb{C}(\mathbb{Z}_N)$ on $\mathbb{C}_q[D]$,

$$\Delta_R z = z \otimes \chi, \quad \Delta_R \bar{z} = \bar{z} \otimes \chi^*, \quad \chi = \sum_{0 \le j < N} \exp(2\pi \mathrm{i} j/N)\delta_j.$$

We will see in Example 4.31 that this coaction is not differentiable for any nontrivial calculus on $\mathbb{Z}_N$. The invariant subalgebra $A=\mathbb{C}_q[C]=\mathbb{C}_q[D]^{\mathbb{Z}_N}$ (or coinvariants under $\mathbb{C}(\mathbb{Z}_N)$) is generated by $a=z\bar{z}$, z^N and $\bar{z}^N$. We define $\Omega^1=A\mathrm{d}A$ as the subalgebra inside the calculus on $\mathbb{C}_q[D]$ generated by elements of $\mathbb{C}_q[C]$ and their differentials, i.e. with generators $z^{N-1}\mathrm{d}z$, $\bar{z}\mathrm{d}z$, $\bar{z}^{N-1}\mathrm{d}\bar{z}$, $z\mathrm{d}\bar{z}$. This splits as $\Omega^{1,0}\oplus\Omega^{0,1}$, where $\Omega^{1,0}$ has the first two on our list of generators. (One can equally well do a splitting of the calculus on $\mathbb{C}_q[D]$ into holomorphic and antiholomorphic parts, which we do in Chap. 7, and then construct $\Omega^{1,0}$ on the cone from there.)

We show how $\Omega^{1,0}$ can be obtained from Proposition 3.98. We view the coaction above as grading by $\hat{\mathbb{Z}}_N$, the character group of $\mathbb{Z}_N$, with generator χ (so we have degrees $|z|=1$, $|\bar{z}|=-1$). We take $B=\mathbb{C}_q[D]$ viewed as $\hat{\mathbb{Z}}_N$-graded with degree zero part $A=\mathbb{C}_q[C]$. Define $a_m=q^{2m}z\bar{z}-[m]_{q^2}(q^2-1)$, and then $\bar{z}a_m=a_{m+1}\bar{z}$. From this we can show that $\bar{z}^m z=a_m\bar{z}^{m-1}$ and hence that $\bar{z}^{N-1}z^{N-1}=a_{N-1}\dots a_1$. Now $a_{N-1}\dots a_1$ is a polynomial in $a=z\bar{z}$ with a nonzero constant term, which we label γ_N. Then $\gamma_N^{-1}\bar{z}^{N-1}z^{N-1}+p_N(a)a=1$, where $p_N(a)$ is a polynomial in a. Then we have $w_1v_1+w_2v_2=1$, where $w_1=\gamma_N^{-1}\bar{z}^{N-1}$, $w_2=p_N(a)z$, $v_1=z^{N-1}$ and $v_2=\bar{z}$. Then the $|w_i|=1\in\hat{\mathbb{Z}}_N$ and $|v_i|=-1\in\hat{\mathbb{Z}}_N$ and by Proposition 3.98 with $g=-1$ we have a projection

$$P=\begin{pmatrix} v_1w_1 & v_1w_2\\ v_2w_1 & v_2w_2\end{pmatrix}$$

for a line bundle $L=\{b(w_1,w_2)\}$, where $|b|=-1$. Identifying $\Omega^{1,0}=L$ by $v_i\mathrm{d}z=v_i(w_1,w_2)$, this becomes fgp as a left $\mathbb{C}_q[C]$-module. Taking the q-deformation has in some sense 'de-singularised' the point of the cone. Where our method goes wrong in the classical case is obvious: if $q=1$ then $\gamma_N=0$. ⋄

3.6 Exact Sequences and Abelian Categories

Here we consider more technical details about modules and categories which the reader may avoid at the cost of taking certain things on trust later. We take A to be a unital algebra over the field $\Bbbk$. In this section we use im for the image of a map.

3.6.1 Exact Sequences, Flat and Projective Modules

In the theory of modules, many arguments are simplified by considering exact sequences of the underlying vector spaces. Linear maps

$$D\xrightarrow{\theta}E\xrightarrow{\phi}F$$

are said to be *exact at* E if the composition $\phi \circ \theta = 0$ and if the kernel of ϕ is equal to the image of θ. A sequence of linear maps is exact if it is exact at every object with an incoming and an outgoing map. Of particular importance is a *short exact sequence*, which is of the form

$$0 \longrightarrow D \xrightarrow{\theta} E \xrightarrow{\phi} F \longrightarrow 0. \tag{3.36}$$

The maps to and from the zero space are zero, so there is no point labelling them. To be exact, the sequence must be exact at E, which we have already given the conditions for, and also exact at D and F. The image of the map into D is zero, so exactness at D means that θ has zero kernel, i.e., is injective or one to one. The kernel of the map out of F is F, so exactness at F means that ϕ is onto or surjective.

The relevance of this to us is when our short exact sequence is one of (say) left A-modules and we want to tensor it with a right A-module M to get a sequence of maps of vector spaces

$$0 \longrightarrow M \otimes_A D \xrightarrow{\mathrm{id}\otimes\theta} M \otimes_A E \xrightarrow{\mathrm{id}\otimes\phi} M \otimes_A F \longrightarrow 0. \tag{3.37}$$

The end points are still 0 as tensoring 0 with something gives 0. We have no algebra actions left, as we did not specify that any of the modules was actually a bimodule. However, the objects are still vector spaces and we can ask whether the sequence is exact as such. (In fact, it was misleading to write (3.37) in that way as it would often be taken that something in that format was necessarily a short exact sequence.)

Proposition 3.101 *Tensoring the short exact sequence (3.36) of left A-modules with a right A-module M gives an exact sequence*

$$M \otimes_A D \xrightarrow{\mathrm{id}\otimes\theta} M \otimes_A E \xrightarrow{\mathrm{id}\otimes\phi} M \otimes_A F \longrightarrow 0.$$

Proof To show that $\mathrm{id} \otimes \phi$ is onto, take $m \otimes f \in M \otimes_A F$. As ϕ is onto, there is an $e \in E$ such that $\phi(e) = f$, so then $(\mathrm{id} \otimes \phi)(m \otimes e) = m \otimes f$. To show exactness at $M \otimes_A E$, first note that the composition $(\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \theta) = 0$. Now take $m_i \otimes e_i \in \ker(\mathrm{id} \otimes \phi)$ (summing over i). Now consider $\mathrm{id} \otimes \phi : M \otimes E \to M \otimes F$, where we take $\otimes$ over the field rather than $\otimes_A$. Now $M \otimes_A F$ is obtained from $M \otimes F$ by adding extra relations, so in $M \otimes F$ (not $M \otimes_A F$) we have

$$m_i \otimes \phi(e_i) = m'_j \otimes a_j.f_j - m'_j.a_j \otimes f_j$$

(summing over j) for some $a_j \in A$, $m'_j \in M$ and $f_j \in F$. As ϕ is onto, there exist $e'_j \in E$ such that $\phi(e'_j) = f_j$, hence

$$m_i \otimes \phi(e_i) - m'_j \otimes \phi(a_j.e'_j) + m'_j.a_j \otimes \phi(e'_j) = 0 \in M \otimes F.$$

We see that $m_i \otimes e_i - m'_j \otimes a_j.e'_j + m'_j.a_j \otimes e'_j$ is in the kernel of $\mathrm{id} \otimes \phi$: $M \otimes E \to M \otimes F$, which is the image of $\mathrm{id} \otimes \theta : M \otimes D \to M \otimes F$ since our initial sequence was exact and we merely tensor over the field at this stage. Thus there exist $m''_k \otimes d_k \in M \otimes D$ (summing over k) such that

$$m''_k \otimes \theta(d_k) = m_i \otimes e_i - m'_j \otimes a_j.e'_j + m'_j.a_j \otimes e'_j \in M \otimes E.$$

Quotienting this by the further relations of $M \otimes_A E$ gives the answer. □

In general the map $\mathrm{id} \otimes \theta$ in (3.37) will not be injective, see Example 3.112 later. However, there is a useful type of module for which we do get injectivity:

Definition 3.102 A right A-module M is called *flat* if for every short exact sequence (3.36) of left A-modules, (3.37) is also a short exact sequence. It is called *faithfully flat* if it is flat and the converse also holds, i.e., if (3.37) is exact then (3.36) is exact.

In fact, by splitting up longer exact sequences into several short exact sequences, we see that tensoring with a flat module preserves exactness of any exact sequence. We could similarly define flat left modules by swapping all the sides. A basic result about flat modules is the following:

Proposition 3.103 *An arbitrary direct sum of right modules is flat if and only if each direct summand is flat. (Similarly for left modules.)*

Proof Take a direct sum $\bigoplus_{i \in I} M_i$, and note that

$$0 \longrightarrow (\bigoplus_{i \in I} M_i) \otimes_A D \stackrel{\mathrm{id}\otimes\theta}{\longrightarrow} (\bigoplus_{i \in I} M_i) \otimes_A E \stackrel{\mathrm{id}\otimes\phi}{\longrightarrow} (\bigoplus_{i \in I} M_i) \otimes_A F \longrightarrow 0$$

is exact if and only if the following is exact for every $i \in I$:

$$0 \longrightarrow M_i \otimes_A D \stackrel{\mathrm{id}\otimes\theta}{\longrightarrow} M_i \otimes_A E \stackrel{\mathrm{id}\otimes\phi}{\longrightarrow} M_i \otimes_A F \longrightarrow 0.$$ □

We now recall the definition of a projective module as a complemented submodule of a free module from Definition 3.25. In §3.1 it was shown that every finitely generated projective module is a complemented submodule of $A^{\oplus n}$ for some n, and is therefore projective. (A submodule $N \subseteq M$ is *complemented* or a *direct summand* if there is another submodule $N^c \subseteq M$ with $N \cap N^c = 0$ and $N + N^c = M$.)

Corollary 3.104 *Every free module over a unital algebra is flat, and every projective module is flat.*

Proof Taking right modules, we use $M \otimes_A A \cong M$ for every right A-module M. A free module is isomorphic to a direct sum of some number (possibly infinitely many) of copies of A, and is then also flat. A projective module is a direct summand of a free module and is then also flat. □

As a consequence, every vector space is a flat module over the field. As an application of flatness, we have the following result about invariant subspaces of coactions for a Hopf algebra H. This is implicitly used in several places in Chap. 5.

Example 3.105 Suppose that E is a left A-module and a right H-comodule such that the coaction $\Delta_R : E \to E \otimes H$ is an A-module map. By definition, we have an exact sequence

$$0 \longrightarrow E^H \xrightarrow{\text{inc}} E \xrightarrow{\otimes 1 - \Delta_R} E \otimes H.$$

(We could make a short exact sequence by using the image of $\otimes 1 - \Delta_R$ instead of $E \otimes H$, but it would make no difference to the result.) If M is a flat right A-module, on applying $M \otimes_A$ to the exact sequence we get another exact sequence

$$0 \longrightarrow M \otimes_A E^H \xrightarrow{\text{inc}} M \otimes_A E \xrightarrow{\text{id}\otimes(\otimes 1 - \Delta_R)} M \otimes_A E \otimes H.$$

As a result we see that $(M \otimes_A E)^H = M \otimes_A E^H$. ◇

Not every flat module is faithfully flat. The simplest example is the trivial zero module $M = 0$, which is fgp (as we can decompose the free module A as $A = 0 \oplus A$) and therefore flat. However, it is not faithfully flat since any sequence of maps (3.36) becomes the trivial zero short exact sequence on tensoring by zero. For examples of modules which are faithfully flat, the obvious one is the algebra A itself, as $A \otimes_A$ is the identity operation, since A is unital. More examples of faithfully flat modules can be given by the following construction.

Proposition 3.106 *A direct sum of flat modules is faithfully flat if at least one of the summands is faithfully flat.*

Proof If M is a direct sum of flat modules then we already know from Proposition 3.103 that is its flat. Suppose that $M = N \oplus C$, where N is faithfully flat. Then tensoring the not-necessarily exact sequence (3.36) by M gives the following, which we assume to be exact:

$$0 \longrightarrow (N \oplus C) \otimes_A D \xrightarrow{\text{id}\otimes\theta} (N \oplus C) \otimes_A E \xrightarrow{\text{id}\otimes\phi} (N \oplus C) \otimes_A F \longrightarrow 0.$$

This implies that the following sequence is also exact

$$0 \longrightarrow N \otimes_A D \xrightarrow{\text{id}\otimes\theta} N \otimes_A E \xrightarrow{\text{id}\otimes\phi} N \otimes_A F \longrightarrow 0,$$

and as N is faithfully flat, we deduce that the original sequence was exact. □

It will be useful to have an alternative definition of projective modules from that given in Definition 3.25, which appears as the first characterisation in the following.

Proposition 3.107 *The following properties are equivalent for a left (resp. right) A-module M:*

(1) *M is isomorphic to a complemented submodule of a left (right) free A-module.*
(2) *Given modules and module maps (the solid arrows) with π surjective,*

$$\begin{tikzcd} & M \arrow[dl, dotted, "\theta"'] \arrow[d, "\psi"] & \\ E \arrow[r, "\pi"] & K \arrow[r] & 0 \end{tikzcd}$$

we can always fill in the dotted module map θ to make the diagram commute.

Proof Suppose that M is isomorphic to a complemented submodule of the left free A-module F via $\iota : M \to F$, and that we have the solid arrows in the diagram

$$\begin{tikzcd} F \arrow[r, dotted, "\alpha"'] \arrow[d, dotted, "\kappa"'] & M \arrow[l, bend right, "\iota"'] \arrow[r] \arrow[dl, dotted, "\theta"'] \arrow[d, "\psi"] & 0 \\ E \arrow[r, "\pi"] & K \arrow[r] & 0 \end{tikzcd}$$

As we have a complemented submodule, we may take a module map $\alpha : F \to M$ such that $\alpha \circ \iota = \mathrm{id} : M \to M$. Now take a set of free generators f_i for F, and define $\kappa(f_i) \in \pi^{-1}(\phi \circ \alpha(f_i))$ where we take any element of the inverse image. As F is freely generated by the f_i, this extends to a module map $\kappa : F \to E$. Now define $\theta = \kappa \circ \iota : M \to E$.

Conversely, suppose that M obeys (2) and fix a set of generators g_i for M (these will not generate M freely in general). Now consider a free module F with free generators f_i (same index set as for the g_i) and define $\pi : F \to M$ by $\pi(f_i) = g_i$ such that we have the diagram

$$\begin{tikzcd} & M \arrow[dl, dotted, "\theta"'] \arrow[d, "\mathrm{id}"] & \\ F \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

providing a module map $\theta : M \to F$ making the diagram commute. Hence $\theta \circ \pi : F \to F$ is an idempotent splitting F into a copy of M and a complement. $\square$

This alternative definition is useful for showing that modules are not projective.

Example 3.108 Let $A = C^\infty(\mathbb{R})$ be the smooth $\mathbb{R}$-valued functions on $\mathbb{R}$ and define

$$D = \{f \in C^\infty(\mathbb{R}) : f^{(n)}(0) = 0 \text{ all } n \geq 0\},$$

the ideal in A where the function and all derivatives vanish at $0 \in \mathbb{R}$. Now display the quotient A-module $F = C^\infty(\mathbb{R})/D$ as a short exact sequence

$$0 \longrightarrow D \xrightarrow{i} C^\infty(\mathbb{R}) \xrightarrow{\pi} F \longrightarrow 0.$$

If F was projective then we could fill in the dotted module map in the diagram

$$\begin{tikzcd} & F \arrow[dl, dotted, "\theta"'] \arrow[d, "\mathrm{id}"] & \\ C^\infty(\mathbb{R}) \arrow[r, "\pi"] & F \arrow[r] & 0. \end{tikzcd}$$

This implies a direct sum decomposition of $C^\infty(\mathbb{R})$ into submodules $D \oplus \operatorname{im}\theta$, where the restriction $\pi : \operatorname{im}\theta \to F$ is an isomorphism. Now take the unique $g \in \operatorname{im}\theta$ such that $g(0) = 1$ and all derivatives $g^{(n)}(0) = 0$ for $n > 0$ (i.e., $\pi(g) = [1]$). By continuity there is an $x_0 \neq 0 \in \mathbb{R}$ such that $g(x_0) \neq 0$. Consider $f \in C^\infty(\mathbb{R})$ identically 1 in a neighbourhood of $0 \in \mathbb{R}$ but with $f(x_0) = 0$. Then $f.g \in \operatorname{im}\theta$ and $\pi(f.g - g) = 0$, but $f.g \neq g$, giving a contradiction. So F is not projective.

We can now deliver the example promised in §3.2 of a connection on a module which is not projective. Define a connection on F by $\nabla(\pi(f)) = \mathrm{d}x \otimes \pi(\frac{\mathrm{d}f}{\mathrm{d}x})$. This is well-defined because if $\pi(f) = 0$ then $f \in D$, hence $\frac{\mathrm{d}f}{\mathrm{d}x} \in D$ and so $\mathrm{d}x \otimes \pi(\frac{\mathrm{d}f}{\mathrm{d}x}) = 0$. One could call F a 'smooth skyscraper sheaf' (a skyscraper sheaf is a sheaf which is only nonzero at one point, here $0 \in \mathbb{R}$). This construction of a connection on a quotient module is an example of Theorem 3.114. ◇

We close with a couple of definitions. Firstly, in addition to some of the module theory above, one can also work with comodules. This will prove useful in Chap. 5.

Definition 3.109 Let C be a coalgebra as in Chap. 2, V a right C-comodule and F a left C-comodule. We define the cotensor product $V \otimes^C F$ to be the kernel of

$$\Delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_L : V \otimes F \to V \otimes C \otimes F.$$

Here V is called *coflat* if for every short exact sequence

$$0 \longrightarrow D \xrightarrow{\theta} E \xrightarrow{\phi} F \longrightarrow 0$$

of left C-comodules, we get a short exact sequence

$$0 \longrightarrow V \otimes^C D \xrightarrow{\mathrm{id}\otimes\theta} V \otimes^C E \xrightarrow{\mathrm{id}\otimes\phi} V \otimes^C F \longrightarrow 0.$$

There is also a property of exact sequences which we shall find useful.

Definition 3.110 A short exact sequence as follows (in solid arrows) is called split if we have a dotted arrow ψ, the splitting map, such that $\phi \circ \psi = \mathrm{id} : F \to F$

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r, "\theta"] & E \arrow[r, "\phi"] & F \arrow[r] \arrow[l, dotted, bend left, "\psi"] & 0. \end{tikzcd}$$

The word *split* is used because in that case $\psi \circ \phi : E \to E$ is an idempotent (i.e., $(\psi \circ \phi)^2 = \psi \circ \phi$) and E splits as a direct sum $E = \mathrm{im}\,\theta \bigoplus \mathrm{im}\,(\psi \circ \phi)$. There is a useful case where splitting is guaranteed, namely if F in the short exact sequence of (say) left A-modules and left module maps in Definition 3.110 (in solid arrows) is projective then the exact sequence is split. This is because Proposition 3.107 gives the existence of the splitting map ψ according to the diagram

$$\begin{tikzcd} & & & F \arrow[dl, dotted, "\psi"'] \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & D \arrow[r, "\theta"] & E \arrow[r, "\phi"] & F \arrow[r] & 0. \end{tikzcd}$$

3.6.2 Abelian Categories

There are several constructions that we are used to performing on vector spaces or on modules, such as taking direct sums, adding morphisms, and taking kernels or quotients. To formulate the idea of an exact sequence we needed kernels and images. Do these ideas continue to apply as we add more structure, such as connections? The appropriate formal construction to consider here is that of an *abelian category*. The name comes from the basic example of the category of abelian groups, and the theory goes back to Buchsbaum and Grothendieck. We outline some of the theory and refer the interested reader to the excellent lecture notes by Keller for more details.

Definition 3.111 A category $\mathcal{C}$ is *additive* if for all objects x, y, z:

(1) There is a binary operation making $(\mathrm{Mor}(x, y), +)$ into an abelian group.
(2) The composition $\circ : \mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z)$ is bi-additive, i.e., $(\theta + \phi) \circ \psi = \theta \circ \psi + \phi \circ \psi$ and $\theta \circ (\phi + \psi) = \theta \circ \phi + \theta \circ \psi$.
(3) There is a direct sum object $x \oplus y$ with given morphisms $\pi_1 : x \oplus y \to x$ and $\pi_2 : x \oplus y \to y$, which induce isomorphisms $\mathrm{Mor}(z, x \oplus y) \cong \mathrm{Mor}(z, x) \times \mathrm{Mor}(z, y)$ and $\mathrm{Mor}(x \oplus y, z) \cong \mathrm{Mor}(x, z) \times \mathrm{Mor}(y, z)$.
(4) There is a zero object $\underline{0} \in \mathcal{C}$ such that $\mathrm{Mor}(\underline{0}, x) = 0$ and $\mathrm{Mor}(x, \underline{0}) = 0$.

To explain Definition 3.111 of an additive category we will use a standard example of vector spaces over a field $\Bbbk$, though a little thought will show that modules ${}_A\mathcal{M}$ over an algebra A will work in the same way. (We shall not use abelian groups, as our focus on algebras here means that vector spaces are a more relevant example.) For any vector spaces V, W, the linear maps $L(V, W)$ (the morphisms in the category of vector spaces) form an additive group and composition of linear maps obeys the distributive law over $+$. We can also form the direct sum $V \oplus W$. The zero vector space is the zero object in the category.

Now we have to explain the idea of kernel and cokernel. Continuing with vector spaces, take a morphism (linear map) $T : V \to W$. Then $\ker T = \{v \in V : T(v) = 0\}$ is a vector subspace of V. To emphasise the subspace part, we take a morphism $i_T : \ker T \to V$ to imbed the kernel in V. The image $\operatorname{im} T = \{T(v) : v \in W\}$ is a subspace of W, and the cokernel is $\operatorname{coker} T = W/\operatorname{im} T$, a quotient vector space. To emphasise the quotient part, we take the quotient map $\pi_T : W \to \operatorname{coker} T$. This is all we need for vector spaces and linear maps, or even for modules over algebras and module maps. However, for more general categories we have to define kernel and cokernel purely in terms of the category and as we do this the reader can see how the vector space case is being generalised.

Given objects $x, y \in \mathcal{C}$ and a morphism $\phi : x \to y$, the kernel consists of an object $\ker \phi \in \mathcal{C}$ and a morphism $i_\phi : \ker \phi \to x$ such that the composition $\phi \circ i_\phi = 0 : \ker \phi \to y$ and also for any morphism $\theta : k \to x$ with $\phi \circ \theta = 0 : k \to y$ we have a unique morphism $\alpha : k \to \ker \phi$ such that $i_\phi \circ \alpha = \theta$. This is equivalent to saying that if we have the morphisms given by solid arrows in the left picture in (3.38), then we can uniquely fill in the dotted arrow.

Given a morphism $\phi : x \to y$, the cokernel consists of an object $\operatorname{coker} \phi \in \mathcal{C}$ and a morphism $\pi_\phi : y \to \operatorname{coker} \phi$ such that the composition $\pi_\phi \circ \phi = 0 : x \to \operatorname{coker} \phi$ and also for any morphism $\psi : y \to c$ such that $\psi \circ \phi = 0 : x \to c$ we have a unique morphism $\beta : \operatorname{coker} \phi \to c$ such that $\beta \circ \pi_\phi = \psi$. This is equivalent to saying that if we have the morphisms given by solid arrows in the right picture in (3.38) then we can uniquely fill in the dotted arrow.

$$
\begin{tikzcd}
k \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\theta"] \arrow[ddr, bend right, "0"'] & & \\
& \ker\phi \arrow[r, "i_\phi"'] \arrow[d, "0"] & x \arrow[dl, "\phi"] \\
& y &
\end{tikzcd}
\qquad
\begin{tikzcd}
c & & \\
& \operatorname{coker}\phi \arrow[ul, dotted, "\beta"'] & x \arrow[l, "0"] \arrow[ull, bend right, "0"'] \arrow[dl, "\phi"] \\
& y \arrow[u, "\pi_\phi"'] \arrow[uul, bend left, "\psi"] &
\end{tikzcd}
\tag{3.38}
$$

Having done this, we can define the image $\operatorname{im}\phi$ of $\phi : x \to y$ as the kernel of $\pi_\phi : y \to \operatorname{coker}\phi$ (which really is the usual image in the case of vector spaces and linear maps). Symmetrically, we can define the coimage $\operatorname{coim}\phi$ as the cokernel of $i_\phi : \ker\phi \to x$. We have the following commuting diagram for a unique $\tilde{\phi}$.

$$
\begin{tikzcd}
 & x \arrow[rrr, "\phi"] \arrow[dr, "\pi_{i_\phi}"] & & & y \arrow[dr, "\pi_\phi"] & \\
\ker\phi \arrow[ur, "i_\phi"] & & \mathrm{coim}\phi \arrow[r, "\tilde{\phi}"] & \mathrm{im}\,\phi \arrow[ur, "i_{\pi_\phi}"] & & \mathrm{coker}\,\phi
\end{tikzcd}
\tag{3.39}
$$

We construct $\tilde{\phi}$ in two stages, first a morphism γ from the left diagram of

$$
\begin{tikzcd}
y & & \\
 & \mathrm{coim}\,\phi \arrow[ul, dotted, "\gamma"] & \ker\phi \arrow[l, "0"] \arrow[ull, bend right, "0"'] \arrow[dl, "i_\phi"] \\
 & x \arrow[u, "\pi_{i_\phi}"] \arrow[uul, bend left, "\phi"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{coim}\,\phi \arrow[dr, dotted, "\tilde{\phi}"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "0"'] & & \\
 & \mathrm{im}\,\phi \arrow[r, "i_{\pi_\phi}"] \arrow[d, "0"] & y \arrow[dl, "\pi_\phi"] \\
 & \mathrm{coker}\,\phi &
\end{tikzcd}
$$

and then $\tilde{\phi}$ from the right. That $\pi_\phi \circ \gamma = 0$ uses uniqueness of the induced maps. Of course, the kernels and cokernels may not exist in a general additive category. As we have said, these constructions do work for modules over an algebra. However, if we are more specific and take a category of fgp modules over an algebra then we find that not all cokernels of maps between fgp modules are fgp.

Example 3.112 Consider the semigroup $(\mathbb{Z}, \times)$ with identity (i.e., monoid) of integers under multiplication. If we write the elements as e_n for $n \in \mathbb{Z}$ then $e_n.e_m = e_{nm}$. Using the e_n as a vector space basis and extending the product bilinearly gives the monoid algebra $\mathbb{C}(\mathbb{Z}, \times)$. For example, in the algebra we have

$$2e_3(4e_2 - 5\,e_{-1}) = 8e_6 - 10e_{-3}.$$

Now $\mathbb{C}(\mathbb{Z}, \times)$ is a module over itself (everything here is commutative, so we will not care about sides). We have an injective module map $e_2\times : \mathbb{C}(\mathbb{Z}, \times) \to \mathbb{C}(\mathbb{Z}, \times)$ given by $(e_2\times)(e_n) = e_{2n}$. The image of this map is a submodule of $\mathbb{C}(\mathbb{Z}, \times)$, which has basis $\{e_{2n} : n \in \mathbb{Z}\}$ (call this submodule Even). Now both $\mathbb{C}(\mathbb{Z}, \times)$ and Even are fgp (in fact rank 1 free) modules over $\mathbb{C}(\mathbb{Z}, \times)$, with generators e_1 and e_2 respectively. Their quotient module, which we call Odd, has linear basis $\{e_n : \text{odd } n \in \mathbb{Z}\}$. Now tensoring our module map $e_2\times$ with Odd gives

$$\mathrm{Odd} \cong \mathrm{Odd} \otimes_{\mathbb{C}(\mathbb{Z},\times)} \mathbb{C}(\mathbb{Z}, \times) \xrightarrow{\mathrm{id}\otimes(e_2\times)} \mathrm{Odd} \otimes_{\mathbb{C}(\mathbb{Z},\times)} \mathbb{C}(\mathbb{Z}, \times) \cong \mathrm{Odd},$$

and this map is zero. Thus Odd is a cokernel, and the quotient of a rank one free module by a rank one free submodule, which is not flat, let alone fgp. ◇

Definition 3.113 An abelian category is an additive category $\mathcal{C}$ such that:

(1) All morphisms in $\mathcal{C}$ have a kernel and a cokernel.
(2) For all morphisms $\phi : x \to y$, the morphism $\tilde{\phi}$ in (3.39) is invertible.

Whereas (1) in Definition 3.113 may seem justified as kernels, images and such things are expected to be useful, (2) takes more explanation. It allows us to say that if $\ker\phi = 0$ (e.g., ϕ is injective for vector spaces) and $\mathrm{coker}\,\phi = 0$ (e.g., ϕ is surjective for vector spaces), then ϕ is invertible. For vector spaces and linear maps, (2) simply becomes the statement that for a linear map $T : V \to W$ the induced map $\tilde{T} : V/\ker T \to \mathrm{im}\, T$ is an isomorphism. Indeed it might be said that the purpose of abelian categories is to allow people to do similar manipulations to those that they might do naturally for vector spaces and linear maps.

We now apply the idea of abelian categories to the category ${}_A\mathcal{E}$ of left modules equipped with left connections as defined in §3.2. We assume for this that our algebra A is equipped with a first-order differential calculus $(\Omega^1, \mathrm{d}, \wedge)$.

Theorem 3.114 *If Ω^1 is a flat right A-module then ${}_A\mathcal{E}$ is an abelian category.*

Proof Going through Definition 3.111 of an additive category, adding two morphisms gives another morphism because the sum of module maps is a module map and the sum of maps intertwining the connections also intertwines the connections. There is a zero morphism, and composition preserves sums. Define the direct sum $(E, \nabla_E) \oplus (F, \nabla_F)$ by $E \oplus F$ as a module, with elements $(e, f) \in E \oplus F$ where $e \in E$ and $f \in F$. The rules for addition and left action are, for $a \in A$,

$$(e, f) + (e', f') = (e + e', f + f'), \quad a.(e, f) = (a.e, a.f).$$

If $\nabla_E e = \xi \otimes e'$ and $\nabla_F f = \eta \otimes f'$ (summation implicit) then we can define a connection on the direct sum by

$$\nabla_{E\oplus F}(e, f) = \xi \otimes (e', 0) + \eta \otimes (0, f').$$

The zero object is the zero module with the zero connection (there is no other connection on the zero module).

Let $\phi : (E, \nabla_E) \to (F, \nabla_F)$ be a morphism in ${}_A\mathcal{E}$. Let K be the kernel and C the cokernel of ϕ in the category of left A-modules. Since Ω^1 is flat, the bottom row in the following commuting diagram with solid arrows is exact:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"'] \arrow[d, dotted, "\nabla_K"'] & E \arrow[r, "\phi"'] \arrow[d, "\nabla_E"'] & F \arrow[r, "\pi"'] \arrow[d, "\nabla_F"] & C \arrow[r] \arrow[d, dotted, "\nabla_C"] & 0 \\
0 \arrow[r] & \Omega^1 \otimes_A K \arrow[r, "\mathrm{id}\otimes i"] & \Omega^1 \otimes_A E \arrow[r, "\mathrm{id}\otimes\phi"] & \Omega^1 \otimes_A F \arrow[r, "\mathrm{id}\otimes\pi"] & \Omega^1 \otimes_A C \arrow[r] & 0
\end{tikzcd}$$

We want to define ∇_K such that the diagram commutes. To do this we need to show that $\nabla_E \circ i : K \to \Omega^1 \otimes_A E$ has image contained in the image of $\mathrm{id}\otimes i$. As the bottom row is exact, this means that we have to show that $(\mathrm{id}\otimes\phi) \circ \nabla_E \circ i : K \to \Omega^1 \otimes_A F$ is zero. But this map is $\nabla_F \circ \phi \circ i : K \to \Omega^1 \otimes_A F$, which is zero. Now to show that $\nabla_K(a.k) = \mathrm{d}a \otimes k + a.\nabla_K(k)$ we apply the injective map $\mathrm{id} \otimes i$ to this and use the Leibniz property of ∇_E.

We also want to define ∇_C such that the diagram commutes. We do this by $\nabla_C(c) = (\mathrm{id} \otimes \pi)\nabla_F(\pi^{-1}(c))$, where in standard homological algebra shorthand, $\pi^{-1}(c)$ is a choice of the inverse image of c. We need to check that the value does not depend on the choice of this inverse image, i.e., by exactness of the top line that $(\mathrm{id} \otimes \pi)\nabla_F \circ \phi$ is zero. But this is just $(\mathrm{id} \otimes \pi)(\mathrm{id} \otimes \phi)\nabla_E$, which is zero. Hence

$$\begin{aligned}\nabla_C(a.c) &= (\mathrm{id} \otimes \pi)\nabla_F(a.\pi^{-1}(c)) \\ &= (\mathrm{id} \otimes \pi)(\mathrm{d}a \otimes \pi^{-1}(c)) + a.(\mathrm{id} \otimes \pi)\nabla_F(\pi^{-1}(c)) \\ &= \mathrm{d}a \otimes c + a.\nabla_C(c).\end{aligned}$$

□

Why are we making a fuss about abelian categories? The answer is that with modules it is very easy to think just about fgp modules, which classically is the same as thinking about locally trivial vector bundles. However, this is not enough for sheaf theory. It is very common to consider sheaves which are nowhere near locally trivial, for example 'skyscraper' sheaves which are zero except at a single point. Even if we start with fgp objects in ${}_A\mathcal{E}$, the kernel and cokernel constructions allow us to make examples in ${}_A\mathcal{E}$ which may not be fgp. The following quotient construction contains an annoying left/right flip, but is an interesting application of the vanishing ideal (see Definition 3.11). Sheaves themselves are covered in Chap. 4.

Example 3.115 Let (F, ∇_F) be a left A-module connection on a left fgp module F. Suppose that $f \in F$ obeys $\nabla_F f = 0$ and that $f \in F$ has a nontrivial vanishing right ideal $\mathrm{Van}(f) \subseteq A$. Now take the dual right connection from Proposition 3.32 $\tilde{\nabla}_{F^\flat} : F^\flat \to F^\flat \otimes_A \Omega^1$ so that

$$\mathrm{d}\,\mathrm{ev}(f \otimes \alpha) = (\mathrm{id} \otimes \mathrm{ev})(\nabla_F f \otimes \alpha) + (\mathrm{ev} \otimes \mathrm{id})(f \otimes \tilde{\nabla}_{F^\flat}\alpha)$$

for $\alpha \in F^\flat$. The first term on the right vanishes and as a result $\mathrm{dVan}(f) \subseteq \mathrm{Van}(f) \otimes_A \Omega^1$. Thus $(\mathrm{Van}(f), \tilde{\nabla}_{\mathrm{Van}(f)} = \mathrm{d})$ is a right connection as the restriction of the right connection $\mathrm{d} : A \to A \otimes_A \Omega^1$ and has zero curvature since $\mathrm{d}^2 = 0$. If Ω^1 is flat, we deduce that the quotient $E = A/\mathrm{Van}(f)$ also has a right connection $\tilde{\nabla}_E[a] = [1] \otimes \mathrm{d}a \in A/\mathrm{Van}(f) \otimes_A \Omega^1$, where [ ] denotes equivalence class. ◇

The motivating example for abelian categories was the category of abelian groups, thus the name abelian category. However our categories have all been about vector spaces over a field $\Bbbk$, with various extra structures, and so we have used the category of $\Bbbk$-vector spaces as our motivating example. The following definition formalises the difference between these two examples of abelian categories:

Definition 3.116 A $\Bbbk$-category is an additive category with a ring homomorphism from $\Bbbk$ to the natural transformations from the identity functor to itself, i.e., a morphism $\lambda_X : X \to X$ for every $\lambda \in \Bbbk$ and object X such that for every morphism $\phi : X \to Y$ we have a commuting diagram

$$\begin{tikzcd} X \arrow[r, "\lambda_X"'] \arrow[d, "\phi"] & X \arrow[d, "\phi"'] \\ Y \arrow[r, "\lambda_Y"] & Y \end{tikzcd}$$

and where $\lambda_X \circ \mu_X = (\lambda\mu)_X$ and $\lambda_X + \mu_X = (\lambda + \mu)_X$.

The obvious example of a $\Bbbk$-category is the category of vector spaces and linear maps over $\Bbbk$, where $\Bbbk$ acts by multiplication on each object.

Exercises for Chap. 3

E3.1 Let $A = \mathbb{C}\mathbb{Z}_{6,\times}$ be the monoid algebra with basis a_i for $i \in \mathbb{Z}_6$, product $a_i a_j = a_{ij}$ and identity $1 = a_1$ (the monoid $\mathbb{Z}_{6,\times}$ under multiplication of integers mod 6, extended linearly). Let $E = \mathrm{span}_{\mathbb{C}}\{a_0, a_3\}$, $F = \mathrm{span}_{\mathbb{C}}\{a_0, a_2, a_4\} \subset A$ regarded as left A-modules by left multiplication. Show that $E^\flat \cong E$ and $F^\flat \cong F$ with the evaluation map under this identification given by product in A. By finding explicit dual bases for E and F, show that they are fgp and write down the corresponding projection matrices associated to E and F. Find a complementary module $F^\perp \subset A$ such that $F \oplus F^\perp = A$.

E3.2 We define a calculus on $A = \mathbb{C}\mathbb{Z}_{6,\times}$ in the preceding question by a similar method as for group algebras in Theorem 1.47 and the inner case after it. Namely, let ρ be the right action of A on $V = \mathrm{Row}^2(\mathbb{C})$ by right multiplication by the matrices $\rho(a_0) = 0$, $\rho(a_1) = \mathrm{id}$ and

$$\rho(a_2) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(a_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(a_5) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then for a fixed $\theta \in V$ define $\zeta(x) = \theta.\rho(x) - \theta \in V$ and $\Omega^1 = A \otimes V$ with

$$\mathrm{d}x = x \otimes \zeta(x), \quad x.(y \otimes v) = xy \otimes v, \quad (y \otimes v).x = yx \otimes v.\rho(x)$$

for $x, y \in \mathbb{Z}_6$. Use Proposition 3.23 to find all left connections on the fgp module E.

E3.3 Verify the formula for σ^L claimed in case (2) of Example 3.74. You will require the formula $\varpi\pi_\epsilon(hg) = \epsilon(h)\varpi\pi_\epsilon(g) + \varpi\pi_\epsilon(h)\triangleleft g$ for $h, g \in H$ and the formulae for a right crossed module.

E3.4 Consider an element w of a $*$-algebra obeying $ww^* - q^2w^*w = q^2 - 1$, where $q \in \mathbb{R}$. Show that if $q^2 \neq 1$ then it is impossible to represent w as a bounded operator and w^* as its adjoint on a Hilbert space. [Hint: Rearrange the equation slightly, and then use the idea of spectral radius (assuming that w was bounded) as in Example 3.40. This will be relevant to Example 7.12, where classically w would be the standard complex coordinate on $\mathbb{C}$ and unbounded as an operator by multiplication.]

E3.5 For the 3D differential calculus on $A = U(su_2)$ in Example 1.46 and any trace $\mathrm{Tr} : A \to \mathbb{C}$, define a map $\int : \Omega^3 \to \mathbb{C}$ by

$$\int as_1 \wedge s_2 \wedge s_3 = \mathrm{Tr}(a).$$

Show that $\int$ is a 3-cycle according to the definition in (3.26). In addition, define a map $\int_\theta : \Omega^2 \to \mathbb{C}$ by $\int_\theta \eta = \int(\theta \wedge \eta)$ for $\eta \in \Omega^2$ but show that this is *not* a closed map if Tr is the matrix trace in a standard irreducible matrix representation. Also find an expression for $\int_\theta(\eta.a - a.\eta)$ in terms of $\int$, $a \in A$ and η. [These results also apply to the fuzzy sphere $\mathbb{C}_\lambda[S^2]$, where the matrix trace in the $n \times n$ matrix representation applies if $\lambda_P = 1/n$.]

E3.6 Let $A = \mathbb{C}_{q,\theta}[\mathbb{T}^2]$ be the noncommutative torus with its q-deformed calculus as in Exercise E1.5. Show that $\int : \Omega^2 \to A$ defined in the same way as in Example 3.55 is a twisted 2-cycle and find its associated twisted cyclic cocycle.

E3.7 Let $\Omega(X)$ be the maximal prolongation of the directed graph calculus on $A = \mathbb{C}(X)$ with $X = \{x, y, z\}$ in Exercise E1.8. Find all left connections on Ω^1, together with their curvatures and torsions. Determine which of these left connections are bimodule connections and find the associated σ.

E3.8 Let $\Omega(\mathbb{Z}_3)$ be the canonical exterior algebra on $A = \Bbbk(\mathbb{Z}_3)$ with its (universal) Cayley graph calculus Ω^1 with basis e_i associated to $\mathcal{C} = \{1, 2\} \subset \mathbb{Z}_3$ (so $e_i f = R_i(f)e_i$ and $\partial_i = R_i - \mathrm{id}$ in Proposition 1.52, and the e_i anticommute). Show that all connections on Ω^1 are bimodule connections and give formulae for the generalised braiding, curvature and torsion in terms of the coefficients of ∇e_i in the basis. [See also Example 8.19 and Exercise E8.5.]

E3.9 Let A be an algebra with trivial centre and an inner calculus Ω^1 freely generated by a finite central basis $\{s^i\}$. Write $\theta = \theta_i s^i$ (summation understood) for the inner element. Let E similarly be an A-bimodule freely generated by a finite central basis $\{e^\mu\}$. We specify a left bimodule connection ∇ on E in terms of Christoffel symbols $\nabla e^\mu = -\Gamma^\mu{}_{i\nu} s^i \otimes e^\nu$ and write the associated map σ as $\sigma(e^\mu \otimes s^i) = \sigma^{\mu i}{}_{j\nu} s^j \otimes e^\nu$. Show under our assumptions that

$$\sigma^{\mu i}{}_{j\nu} \in \Bbbk, \quad \Gamma^\mu{}_{i\nu} = \theta_j(\sigma^{\mu j}{}_{i\nu} - \delta_{\mu\nu}\delta_{ij}) + \alpha^\mu{}_{i\nu}$$

for some $\alpha^\mu{}_{i\nu} \in \Bbbk$. Assuming that all $\mathrm{d}s^i = 0$, show that the curvature is

$$R_\nabla(e^\mu) = -([\theta_i, \Gamma^\mu{}_{j\nu}] + \Gamma^\mu{}_{i\rho}\Gamma^\rho{}_{j\nu}) s^i \wedge s^j \otimes e^\nu.$$

E3.10 Apply the preceding exercise to the standard $\Omega(M_2(\Bbbk))$ as in Proposition 1.38 with central basis $s^1 = s$, $s^2 = t$ and $s \wedge s = t \wedge t = 0$. Show that if E is a free module over $M_2(\Bbbk)$ with a single central generator e^1 then a bimodule connection on it has zero curvature if and only if

$$c_2{}^2 = c_1{}^1, \quad \gamma_2 c_1{}^1 + \gamma_1 c_2{}^1 = \gamma_2 c_1{}^2 + \gamma_1 c_2{}^2 = 0,$$
$$c_1{}^1 c_2{}^2 + c_1{}^2 c_2{}^1 + 2\gamma_1\gamma_2 = 0,$$

where $\gamma_i = \alpha^1{}_{i1}$ and $c_i{}^j = \sigma^{1j}{}_{i1} - \delta_{ij}$. Find all such flat connections on E.

Notes for Chap. 3

The study of locally trivial vector bundles over topological spaces in §3.1 is the starting point of topological K-theory, see for example Atiyah's book [12]. The Serre–Swan theorem on classical vector bundles as finitely generated projective modules is in [309]. The algebraic treatment of fgp modules using idempotents is completely standard, see e.g., [39]. The idea of corner algebra, mentioned after Proposition 3.10, goes back to [277].

Concerning examples, the characterisation of the $*$-coordinate algebra of $\mathbb{CP}^1$ as generated by a hermitian projector of trace 1 in Example 3.13 is from [53] and extended to fuzzy-spheres as in Example 3.14, q-spheres as in Example 3.15 and to q-fuzzy spheres by the second author in [219]. The latter are shown in [219] to be equivalent to the 2-parameter 'Podleś spheres' in the classification of noncommutative quantum-group covariant spheres due to Podleś [280]. Some results on the standard q-sphere and in particular the cyclic 0-cocycle or trace ϕ used later in Example 3.57 are due to [245]. Universal C^*-algebras, including the example Gr_n, are in [41]. The matter of what relations between generators actually give C^*-algebras is quite involved, see [181, 278]. Example 3.17 on modules for the noncommutative torus is taken from the pathbreaking paper by Connes & Rieffel [89]. Among many works on the irrational rotation noncommutative torus, the structure theory from the point of view of C^*-algebras appeared in [110].

Covariant derivatives or connections (we do not attempt to differentiate between these terms) in §3.2 have a long history in classical differential and Riemannian geometry, going back to Christoffel [80]. Connections in noncommutative geometry are directly analogous to the commutative case, via the left/right Leibniz rules, as is the curvature and featured notably in the works of Quillen in the 1980s. The Cuntz–Quillen theorem about covariant derivatives for the universal calculus can be found in [91]. The matrix Christoffel symbols are taken from [31]. Covariant derivatives on the noncommutative torus were already in [89].

§3.3 covers background from an enormous literature, of which we have only scratched the surface. To the expert we apologise for the limited coverage and to the beginner we point to some of the many excellent expositions available for further reading. For the properties of Hilbert spaces in §3.3.1, including the functional calculus of operators, and much more, see [291]. Excellent references on C^*-algebras are too numerous to list, but some we have found useful here are [8, 84, 111, 153, 263, 276]. An elegant account of the commutative Gel'fand–Naimark theorem that every commutative unital C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff X is in [8]. The Gel'fand–Naimark–Segal construction that every C^*-algebra is isomorphic (preserving the $*$-algebra structure and the norm) to a norm closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is in [297]. Example 3.40 follows the account of [81] for the deformed disc algebra $\mathbb{C}_q[D]$, and [319] for its differential calculus. Klimek & Leśniewski in [164] consider a two-parameter deformation of the unit disc algebra, but for the purposes of this chapter it is convenient to only take one of these parameters. For K-theory itself in §3.3.2, one of standard reference books discussing the foundations of the subject, namely Blackadar [40], is actually based on a more general notion of local C^*-algebras, and we use his approach. The norm decreasing result for local C^*-algebras is from [276]. In [255] Milnor showed that there are exotic differential structures on the 7-sphere. The idea of completing the graph of unbounded operators is standard, see e.g., [291].

Hochschild constructed his theory of algebra cohomology in [140]. Cyclic cohomology was introduced by Connes [83] and independently cyclic homology by Tsygan [314], with other independent work by Loday & Quillen [178]. Our introduction to these three topics in §3.3.3 and §3.3.4 follows in part Loday's excellent exposition [177] and the reader should turn to this for more details. We also used the excellent text of Connes [84] to which we refer the reader for the role of these ideas in noncommutative geometry more generally. We also note the survey article [90] by Cuntz covering bivariant cyclic and K-theories on more general topological algebras. Chern's original definition of the Chern class of a bundle in classical geometry was given in terms of the trace of powers of the curvature [77] and more details of this can be found in various places, including [254]. The noncommutative Chern–Connes pairing is taken from [84], while Example 3.57, which provided the first application to quantum groups (to prove nontriviality of the q-Hopf fibration), is due to Hajac and the second author in [134].

§3.3.5 on twisted cyclic theory is largely based on the work [172] of Kustermans, Murphy & Tuset who developed a generalisation of n-cycles based on the idea of KMS states. The Kubo, Martin & Schwinger or KMS condition on a state on an algebra comes from the idea of thermal equilibrium in thermodynamics [170, 243]. Specifically, [172] introduced twisted cyclic and twisted Hochschild cohomology so as to give an alternative derivation of the 3D calculus on $\mathbb{C}_q[SL_2]$ from a twisted cycle. The twisted Hochschild and cyclic homology itself on this quantum group was obtained by Hadfield & Krähmer [132]. Both cyclic cohomology and K-theory can incorporate an action of a Hopf algebra to give equivariant versions, and there is a pairing between equivariant cyclic cohomology and equivariant K-theory given by Neshveyev & Tuset in [266]. Some cases of twisted cyclic cohomology fit into this framework [266]. Twisted cyclic theory is linked to equivariant KK-theory in [70]. Wagner's construction of the twisted Chern–Connes pairing can be found in [321].

Bimodule covariant derivatives in §3.4 have their origins in [105, 106, 259], and were later used in [113, 185]. They took on an extensive role for noncommutative Riemannian geometry in work of the authors, such as [31, 32] among others. Tensoring bimodule connections was noted in [56, Appendix A]. Invariant bimodule connections in Propositions 3.73 are taken from the authors' work [31], as are Example 3.74, Proposition 3.75 and Examples 3.76 for bimodule connections on S_3 and 3.77 for bimodule connections on $\mathbb{C}_q[SU_2]$. The theorem in §3.4.3 that ${}_A\mathcal{E}\mathcal{I}_A$ is a bar category is due to the authors in [31] as are the notion of $*$-preserving bimodule connection and the examples covered in Proposition 3.87 and Example 3.89.

The description of Morita contexts in §3.5 can be found in more detail in the book by Bass on algebraic K-theory [16], and Definition 3.96 and Proposition 3.97 are quoted from there. The consequences of Morita equivalence for C^*-algebras are studied in [286]. The map Φ_L in Corollary 3.94 is from A. Fröhlich's paper [120], and we have taken the liberty of calling it the Fröhlich map. Much of this section is based on the work of Brzeziński and the first author in [25] on line modules and the Thom construction. The group-graded algebra construction of line modules was used in [175] to explain the classical Dirac monopole bundle, and was subsequently used to construct other nontrivial noncommutative bundles, for example in [65] or [176]. Example 3.100 showing that noncommutative deformation removes the singularity in the cone is due to Brzeziński in [60] and a follow-up with the same phenomenon for some other deformations is in [67].

In §3.6.1, we gave an introduction to exact sequences, flat and faithfully flat modules, for which more details can be found in many textbooks, for example [173]. In particular, Proposition 3.103 on the direct sum of modules and flatness can be found in [173, Prop 4.2]. We have made extensive use of the lecture notes [157] by Keller in §3.6.2 on abelian categories. Original references for the theory of abelian categories are [68, 129] while some interesting recent results can be found in [154]. One difference is that we have shifted emphasis away from abelian groups to vector spaces as our basic example. The idea of $\Bbbk$-category is taken from [256].

The exercises mostly extend examples from previous exercises and are mostly new calculations, including cycles for the fuzzy sphere in E3.5 and for the q-noncommutative torus in E3.6. Exercise E3.9 is a general result for the inner case with central basis and trivial centre and can be seen as a special case of Proposition 8.11, which is from [222].

Chapter 4
Curvature, Cohomology and Sheaves

In this chapter we explore the curvature in greater depth, including an approach to characteristic classes in which traces of the powers of the curvature are independent of the choice of connection. This is needed for a geometric approach to Chern classes and requires us in the noncommutative case to look at the implications of the curvature being a bimodule map as well as the differentiation of morphisms. Previously we considered morphisms between modules with connection which necessarily intertwined the connections, but this category ${}_A\mathcal{E}$ will be too strict for our current purposes. Rather, we define the derivative of a left module map $\phi : (E, \nabla_E) \to (F, \nabla_F)$ to be the commutator with the covariant derivatives,

$$\mathbb{W}(\phi) = \nabla_F \phi - (\mathrm{id} \otimes \phi)\nabla_E : E \to \Omega^1 \otimes_A F \tag{4.1}$$

which vanishes precisely when the map intertwines the connections. In classical Riemannian geometry, for example, this is just viewing a map as a tensor and taking its covariant derivative as such. It would also be sensible for the derivative of a morphism to be another morphism, which leads us to the definition of a category ${}_A\mathcal{G}$ with morphisms graded by $\mathbb{N} \cup \{0\}$, where ϕ above has grade 0 and $\mathbb{W}(\phi)$ grade 1. This will be a DG category where the morphisms are graded and have differentials.

We then consider cohomology in more detail, including extending the results of Chevalley & Eilenberg on Lie group actions on manifolds to Hopf algebra coactions on an algebra with differential structure. This uses an integral on the Hopf algebra to relate the cohomology calculation to the forms invariant under the coaction. In particular, the cohomology for a left-covariant calculus on a Hopf algebra in some cases reduces to the cohomology of the left-invariant forms, which classically would be the cohomology of the Lie algebra. We compute $\mathrm{H}_{\mathrm{dR}}(\mathbb{C}_q[SL_2])$ with its 4D calculus among the examples.

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Classically, it was found for some purposes that one needs cohomology with coefficients that vary from point to point (the Leray–Serre spectral sequence being one case). This idea was formalised in *sheaf theory*. We consider similarly adding coefficients to noncommutative de Rham cohomology. These take the form of a module with a zero curvature connection and we make a case for this being an analogue of classical sheaf theory by considering the long exact sequences of cohomology resulting from short exact sequences of such noncommutative sheaves. We will return to this sheaf theory in the complex geometry case in Chap. 7. The biggest apparent limitation of our approach to noncommutative sheaf cohomology theory is that we only have a differential definition. Classically, there are both differential geometric (de Rham) and purely topological definitions of cohomology. In many cases these agree as they are based on different resolutions of the same sheaf, so for example singular cohomology and de Rham cohomology (both with real coefficients) agree at least for compact manifolds. This then provides a means of transferring information between differential geometry and topology. In our case we have instead freedom in the choice of differential calculus, with the universal calculus at one end.

Spectral sequences meanwhile are a calculational machine from homological algebra and we turn in §4.4, which could easily be skipped on a first reading, to two mechanisms for constructing a spectral sequence in the noncommutative case. Classically, the van Est spectral sequence uses the spectral sequence of a particular double complex on a Lie group to connect the Lie algebra cohomology, the cohomology of the group as a topological space, and a version of the group cohomology. We present a version of the van Est spectral sequence for Hopf algebras with differential calculus. The Hopf algebra cochains here are given by the invariant part of the universal calculus with coefficients. In a different direction, the classical Leray–Serre spectral sequence starts from the cohomology of the base of a fibration with coefficients the cohomology of the fibre, and converges to the cohomology of the total space of the fibration. We define a noncommutative fibration in a differential setting and derive a noncommutative version of the Leray–Serre spectral sequence, providing supporting evidence that our definition of noncommutative sheaves is reasonable. As an example, we compute $\mathrm{H}(\mathbb{C}_q[SL_2])$ with its $3D$ calculus. We will return to differential fibrations in Chap. 5 on quantum principal bundles.

Another recurring problem in noncommutative geometry which we address is that there are often not enough algebra maps. There are times when classically we would say ‘take a continuous map with the property…’, but in the noncommutative world there may be no obvious way to construct a corresponding algebra map. One popular way of generalising the idea of algebra map is to use bimodules. This is the established notion of *correspondences* in topology and C^*-algebra theory, and links to completely positive maps by the Kasparov, Stinespring, Gel’fand, Naĭmark & Segal (KSGNS) construction. We look at replacing algebra maps by bimodules for differential structures and sheaves. In particular, in Proposition 4.86 we give an extension of the KSGNS construction to differential calculi as a cochain map.

We conclude the chapter with a definition of relative cohomology and a discussion of some exact sequences associated to that. This could be seen as a first step towards a model category for homotopy theory in the sense of Quillen. As with sheaf theory, we will only be able to give half of the picture as we lack a theory of cofibrations to pair with the fibrations that we have already discussed.

4.1 Differentiating Module Maps and Curvature

So far we have constructed categories of modules with connections and morphisms which are module maps and intertwine the relevant connections. This was our category ${}_A\mathcal{E}$ in §3.2 in the case of left A-modules with connection. Intertwining connection is, however, quite a strong condition and we will often need to consider other module maps. Once we do this, we naturally have the idea of a differential of a morphism and hence of a *differential graded category* or DG category. It is customary to speak in this context of a cochain complex on each morphism space, but in fact this will just mean a graded space and a map increasing degree by 1 without actually requiring it to square to zero.

Definition 4.1 A differential graded category (or DG category) is a $\mathbb{k}$-category (see Definition 3.116) where each $\mathrm{Mor}(X, Y)$ is a cochain complex (in the weaker sense explained above) of $\mathbb{k}$-vector spaces, and composition $\mathrm{Mor}(Y, Z) \otimes \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$ is a map of cochain complexes.

The notion of a map of cochain complexes here is also a little nonstandard, being the standard one if we were to write the composition of morphisms oppositely. We gave a formula for the derivative of a morphism $\phi : (E, \nabla_E) \to (F, \nabla_F)$ in (4.1). From this, taking the left module map $\phi : E \to F$ as a grade zero morphism, we would have $\mathbb{W}(\phi) : E \to \Omega^1 \otimes_A F$ a grade one morphism. Based on this, we give our candidate for a DG category as follows.

Definition 4.2 The category ${}_A\mathcal{G}$ is defined as having objects (E, ∇_E) consisting of left A-modules E with a left-covariant derivative ∇_E and $\mathbb{N}\cup\{0\}$-graded morphisms, where $\psi \in \mathrm{Mor}_n((E, \nabla_E), (F, \nabla_F))$ of grade n is a left module map $\psi : E \to \Omega^n \otimes_A F$. Composition of morphisms is given by the formula

$$\phi \circ \psi = (\mathrm{id} \wedge \phi)\psi : E \to \Omega^{n+m} \otimes_A G,$$

where $\phi \in \mathrm{Mor}_m((F, \nabla_F), (G, \nabla_G))$.

To give the formula for derivatives of morphisms in all grades, it will be convenient to extend the left connection $\nabla_E : E \to \Omega^1 \otimes_A E$ to

$$\nabla_E{}^{[n]} : \Omega^n \otimes_A E \to \Omega^{n+1} \otimes_A E, \quad \nabla_E{}^{[n]}(\xi \otimes e) = \mathrm{d}\xi \otimes e + (-1)^n \xi \wedge \nabla_E e \tag{4.2}$$

for $n \geq 0$. The $n = 0$ case is the usual connection $\nabla_E{}^{[0]} = \nabla_E$. A brief check shows that $\nabla_E{}^{[n]}$ is well defined and satisfies the Leibniz rule

$$\nabla_E{}^{[n]}(a\xi \otimes e) = \mathrm{d}a \wedge \xi \otimes e + a\nabla_E{}^{[n]}(\xi \otimes e)$$

for all $a \in A$, $\xi \in \Omega^n$ and $e \in E$. The curvature R_E in Definition 3.18 can be rewritten now as $R_E = \nabla_E{}^{[1]}\nabla_E{}^{[0]} \in \mathrm{Mor}_2((E, \nabla_E), (E, \nabla_E))$.

Theorem 4.3 *The category ${}_A\mathcal{G}$ is a DG category. If $\psi : E \to \Omega^n \otimes_A F$ is a morphism then*

$$\mathbb{W}(\psi) = \nabla_F{}^{[n]} \circ \psi - (\mathrm{id} \wedge \psi)\nabla_E : E \to \Omega^{n+1} \otimes_A F$$

is a morphism of one degree higher. If $\phi : F \to \Omega^m \otimes_A G$ is another morphism then

$$\mathbb{W}(\phi \circ \psi) = \phi \circ \mathbb{W}(\psi) + (-1)^n \mathbb{W}(\phi) \circ \psi.$$

Moreover,

(1) *(2nd Bianchi) For all objects E, the curvature R_E has $\mathbb{W}(R_E) = 0$;*
(2) *For all morphisms $\psi : E \to F$, $\mathbb{W}(\mathbb{W}(\psi)) = R_F \circ \psi - \psi \circ R_E$.*

Proof We first show that $\mathbb{W}(\psi)$ is a left module map. For $a \in A$, $e \in E$, and $\psi(e) = \xi \otimes f$ (summation implicit),

$$\begin{aligned}
\nabla_F^{[n]} \circ \psi(a.e) - (\mathrm{id} \wedge \psi)\nabla_E(a.e) &= \nabla_F^{[n]}(a.\xi \otimes f) - (\mathrm{id} \wedge \psi)(\mathrm{d}a \otimes e + a.\nabla_E(e)) \\
&= \mathrm{d}a \wedge \xi \otimes f + a.\nabla_F^{[n]}(\xi \otimes f) - \mathrm{d}a \wedge \psi(e) + a.(\mathrm{id} \wedge \psi)\nabla_E(e) \\
&= a.(\nabla_F^{[n]} \circ \psi(e) - (\mathrm{id} \wedge \psi)\nabla_E(e)).
\end{aligned}$$

Next setting $\phi(f) = \kappa \otimes g$ (summation implicit),

$$\begin{aligned}
&\nabla_G{}^{[n+m]}(\mathrm{id} \wedge \phi)\psi(e) = \nabla_G{}^{[n+m]}(\xi \wedge \phi(f)) = \nabla_G{}^{[n+m]}(\xi \wedge \kappa \otimes g) \\
&= \mathrm{d}(\xi \wedge \kappa) \otimes g + (-1)^{n+m}\xi \wedge \kappa \wedge \nabla_G g = \mathrm{d}\xi \wedge \phi(f) + (-1)^n \xi \wedge \nabla_G{}^{[m]}(\phi(f)) \\
&= (\mathrm{id} \wedge \phi)\nabla_F{}^{[n]}\psi(e) + (-1)^n \xi \wedge \mathbb{W}(\phi) f \\
&= (\mathrm{id} \wedge \phi)\mathbb{W}(\psi)e + (\mathrm{id} \wedge (\mathrm{id} \wedge \phi)\psi)\nabla_E e + (-1)^n \xi \wedge \mathbb{W}(\phi) f.
\end{aligned}$$

For (1), the curvature is a left module map by Lemma 3.19. Also

$$\begin{aligned}
\mathbb{W}(R_E) &= \nabla_E{}^{[2]} \circ R_E - (\mathrm{id} \wedge R_E)\nabla_E \\
&= \nabla_E{}^{[2]} \circ \nabla_E{}^{[1]} \circ \nabla_E - (\mathrm{id} \wedge (\nabla_E{}^{[1]} \circ \nabla_E))\nabla_E.
\end{aligned}$$

Now set $\nabla_E e = \xi \otimes h$ and $\nabla_E h = \eta \otimes f$ (summation implicit),

$$
\begin{aligned}
&(\mathbb{W}(R_E))(e) = \nabla_E{}^{[2]}(\nabla_E{}^{[1]}(\xi \otimes h)) - \xi \wedge (\nabla_E{}^{[1]} \circ \nabla_E)h \\
&= \nabla_E{}^{[2]}(\mathrm{d}\xi \otimes h - \xi \wedge \nabla_E h) - \xi \wedge (\nabla_E{}^{[1]} \circ \nabla_E)h \\
&= -\mathrm{d}(\xi \wedge \eta) \otimes f + \mathrm{d}\xi \wedge \nabla_E h - \xi \wedge \eta \wedge \nabla_E f - \xi \wedge \mathrm{d}\eta \otimes f + \xi \wedge \eta \wedge \nabla_E f = 0.
\end{aligned}
$$

To prove (2) we begin by showing that for all $n \geq 0$,

$$
\nabla_E{}^{[n+1]} \circ \nabla_E{}^{[n]} = \mathrm{id} \wedge R_E : \Omega^n \otimes_A E \to \Omega^{n+2} \otimes_A E. \tag{4.3}
$$

Putting $\nabla_E e = \xi \otimes h$ again for convenience, and for all $\omega \in \Omega^n$,

$$
\begin{aligned}
\nabla_E{}^{[n+1]}(\nabla_E{}^{[n]}(\omega \otimes e)) &= \nabla_E{}^{[n+1]}(\mathrm{d}\omega \otimes e + (-1)^n \omega \wedge \nabla_E e) \\
&= \nabla_E{}^{[n+1]}(\mathrm{d}\omega \otimes e + (-1)^n \omega \wedge \xi \otimes h) \\
&= (-1)^{n+1}\mathrm{d}\omega \wedge \nabla_E e + (-1)^n \mathrm{d}\omega \wedge \xi \otimes e + \omega \wedge \mathrm{d}\xi \otimes h - \omega \wedge \xi \wedge \nabla_E h \\
&= \omega \wedge (\mathrm{d}\xi \otimes h - \xi \wedge \nabla_E h) = \omega \wedge R_E(e).
\end{aligned}
$$

To complete the proof of (2),

$$
\begin{aligned}
\mathbb{W}(\mathbb{W}(\psi)) &= \nabla_F{}^{[n+1]} \circ \mathbb{W}(\psi) - (\mathrm{id} \wedge \mathbb{W}(\psi))\nabla_E \\
&= \nabla_F{}^{[n+1]} \circ \nabla_F{}^{[n]} \circ \psi - \nabla_F{}^{[n+1]} \circ (\mathrm{id} \wedge \psi)\nabla_E \\
&- (\mathrm{id} \wedge (\nabla_F{}^{[n]} \circ \psi))\nabla_E + (\mathrm{id} \wedge \psi)(\mathrm{id} \wedge \nabla_E)\nabla_E \\
&= (\mathrm{id} \wedge R_F)\psi - \nabla_F{}^{[n+1]} \circ (\mathrm{id} \wedge \psi)\nabla_E - (\mathrm{id} \wedge (\nabla_F{}^{[n]} \circ \psi))\nabla_E \\
&- (\mathrm{id} \wedge \psi)R_E + (\mathrm{id} \wedge \psi)(\mathrm{d} \otimes \mathrm{id})\nabla_E.
\end{aligned}
$$

And if $\nabla_E e = \xi \otimes h$ and $\psi(h) = \eta \otimes f$ then

$$
\begin{aligned}
\nabla_F{}^{[n+1]} \circ (\mathrm{id} \wedge \psi)\nabla_E(e) &= \nabla_F{}^{[n+1]}(\xi \wedge \eta \otimes f) \\
&= \mathrm{d}\xi \wedge \eta \otimes f - \xi \wedge \mathrm{d}\eta \otimes f - (-1)^n \xi \wedge \eta \wedge \nabla_F f, \\
(\mathrm{id} \wedge (\nabla_F{}^{[n]} \circ \psi))\nabla_E(e) &= \xi \wedge \nabla_F{}^{[n]}(\eta \otimes f) \\
&= \xi \wedge \mathrm{d}\eta \otimes f + (-1)^n \xi \wedge \eta \wedge \nabla_F f. \qquad \square
\end{aligned}
$$

The content of (1) becomes the usual 2nd Bianchi identity in the classical case when this is viewed in Ω^3, while (2) tells us that the curvatures form an obstruction to the cochain complex on each morphism space having differential $\mathbb{W}(\ )$ that squares to zero. The $(-1)^n$ rather than $(-1)^m$ in the Leibniz rule is not a typographical error and reflects the fact that $\circ$ for historical reasons is in reverse order, i.e., $\mathbb{W}(\ )$ is a graded derivation of degree 1 if we take the opposite product. We summarise the definition of the category ${}_A\mathcal{G}$ in a table:

Name	Objects	$\mathrm{Mor}_n((E,\nabla_E),(F,\nabla_F))$	Derivative
${}_A\mathcal{G}$	(E,∇_E) Left modules & left connections	$\phi : E \to \Omega^n \otimes_A F$ Left module maps	$\mathbb{W} : \mathrm{Mor}_n(E,F) \to \mathrm{Mor}_{n+1}(E,F)$ $\mathbb{W}(\phi) = \nabla_F{}^{[n]} \circ \phi - (\mathrm{id} \wedge \phi)\nabla_E$

The next examples show how some almost tautological morphisms in ${}_A\mathcal{G}$ can have a nice geometric interpretation, including a version of the first Bianchi identity.

Example 4.4 Let A be an algebra with differential structure and consider (A, d) as a bimodule with connection. Suppose that Ω^1 has a left connection ∇ and consider the morphism $\tau \in \mathrm{Mor}_1(\Omega^1, A)$ given by $\xi \mapsto \xi \otimes_A 1$. Then $\mathbb{W}(\tau)(\xi) = \mathrm{d}\xi \otimes 1 - (\mathrm{id} \wedge \tau)\nabla_{\Omega^1}\xi = (\mathrm{d}\xi - \wedge\nabla_{\Omega^1}\xi) \otimes 1$ giving

$$\mathbb{W}(\tau) = -T_\nabla \in \mathrm{Mor}_2(\Omega^1, A),$$

i.e., the torsion tensor (see Definition 3.28). ◇

Example 4.5 Let A be an algebra with differential structure and suppose that Ω^1 is right fgp with dual $\mathfrak{X}^R$, the right vector fields as described in §2.7, and suppose $\mathfrak{X}^R$ has a left connection with $(\mathfrak{X}^R, \nabla_{\mathfrak{X}^R}) \in {}_A\mathcal{G}$. Then the coevaluation map $\mathrm{coev} : A \to \Omega^1 \otimes_A \mathfrak{X}^R$ is in $\mathrm{Mor}_1(A, \mathfrak{X}^R)$ and from Theorem 4.3 we have

$$\mathbb{W}(\mathbb{W}(\mathrm{coev})) = (\mathrm{id} \wedge R_{\mathfrak{X}^R})\mathrm{coev} \in \mathrm{Mor}_3(A, \mathfrak{X}^R), \tag{4.4}$$

which is our version of the first Bianchi identity. We used $R_{\mathfrak{X}^R} \circ \mathrm{coev} = (\mathrm{id} \wedge R_{\mathfrak{X}^R})\mathrm{coev}$ from the definition of composition. We have not assumed that the connection is torsion free so the first Bianchi identity has various additional terms which we see as part of the left-hand side here. To explain this further, we suppose that there is a left bimodule connection $(\Omega^1, \nabla_{\Omega^1}, \sigma_{\Omega^1})$ with σ_{Ω^1} invertible and that ∇_χ is its dual bimodule connection provided by Proposition 3.79. Recall that this is such that $\nabla_{\Omega^1 \otimes \mathfrak{X}^R} \circ \mathrm{coev}(1) = 0$, which can be written

$$(\sigma_{\Omega^1}^{-1}\nabla_{\Omega^1} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\mathfrak{X}^R})\mathrm{coev}(1) = 0.$$

Applying $\wedge$ and assuming that ∇_{Ω^1} is torsion free, this implies by Lemma 3.72 that

$$(- \wedge \nabla_{\Omega^1} \otimes \mathrm{id} + \mathrm{id} \wedge \nabla_{\mathfrak{X}^R})\mathrm{coev}(1) = (-\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \wedge \nabla_{\mathfrak{X}^R})\mathrm{coev}(1) = 0,$$

which is the content of $\mathbb{W}(\mathrm{coev}) = 0$. Hence (4.4) reduces to the more familiar

$$(\mathrm{id} \wedge R_{\mathfrak{X}^R})\mathrm{coev}(1) = 0$$

when the torsion is zero. In the classical case, using the conventions for the Riemann tensor in Example 3.29, our identity reads $R^a{}_{bcd}\mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d = 0$, which is the classical first Bianchi identity. There will also be a more elementary version of the 1st Bianchi identity for any connection on Ω^1, in Lemma 8.1. ◇

The powers of the curvature which appeared in an ad hoc fashion in §3.3.4 are quite natural in ${}_A\mathcal{G}$, as we simply define ${R_E}^2 = R_E \circ R_E$, ${R_E}^3 = R_E \circ R_E \circ R_E$ etc. Then the following is an immediate corollary of Theorem 4.3.

Corollary 4.6 *For (E, ∇_E) in ${}_A\mathcal{G}$, ${R_E}^n : E \to \Omega^{2n} \otimes_A E$ obeys $\mathbb{V}({R_E}^n) = 0$.*

This is part of what we need if we want a noncommutative version of the classical construction of Chern classes in which powers of the curvature of a connection are viewed as bundle endomorphisms and traced to obtain a de Rham cohomology class. We now turn to the noncommutative trace. As happens frequently in noncommutative geometry, the trace represents a problem because if E is (say) a left fgp A-module with dual $E^\flat$ with evaluation $\mathrm{ev} : E \otimes_A E^\flat \to A$ then the natural way to 'trace' a left module map $\theta : E \to F \otimes_A E$ would be to apply $\mathrm{id} \otimes \mathrm{ev}$ to $(\theta \otimes \mathrm{id})\mathrm{coev}^R \in F \otimes_A E \otimes_A E^\flat$ for some $\mathrm{coev}^R \in E \otimes_A E^\flat$. Unfortunately, our canonical coevaluation element $\mathrm{coev} \in E^\flat \otimes_A E$ is the wrong way around for this. One approach, which is used in a braided category, is to insert a flip of tensor factors via a braiding (a 'braided trace'), but in our case we do not have this luxury and take a different approach. This is to choose an actual basis e^i and dual basis f_i such that $\mathrm{coev} = f_i \otimes_A e^i$ (sum understood), i.e. we effectively lift this to $\otimes_{\Bbbk}$. Then we are more free to assemble expressions how we wish, but have to evaluate F further in such a way that the evaluated trace does not see the basis dependence.

Lemma 4.7 *Let F be an A-bimodule, E a left fgp module and $\phi : F \to \Bbbk$ have the trace property $\phi(a.f) = \phi(f.a)$ for all $a \in A$, $f \in F$. Then a left module map $\theta : E \to F \otimes_A E$ has a well-defined trace*

$$\mathrm{Tr}_\phi(\theta) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta e^i \otimes f_i)) \in \Bbbk$$

for any choice of basis e^i of E as a left fpg module with dual f_i of $E^\flat$.

Proof We need to show that the formula does not depend on the choice of dual bases. If we take different dual bases d_j, c^j of $E^\flat, E$ then

$$\begin{aligned}
&\phi((\mathrm{id} \otimes \mathrm{ev})(\theta(c^j) \otimes d_j)) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(f_i(c^j)e^i) \otimes f_k d_j(e^k)))\\
&= \phi(f_i(c^j)(\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes f_k d_j(e^k))) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes f_k d_j(e^k) f_i(c^j)))\\
&= \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes f_k f_i(d_j(e^k)c^j))) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes f_k f_i(e^k)))\\
&= \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes f_i))
\end{aligned}$$

on using the change of basis formulae in (3.20) and summing over i, j, k. □

Using the definition of n-cycle in (3.26), we can relate the derivative operation in ${}_A\mathcal{G}$ to de Rham classes with the help of the following result.

Proposition 4.8 *Let E be an A-bimodule which is fgp as a left module, let ∇_E be a left connection and $\int$ an $n+1$-cycle. If we view $(E, \nabla_E) \in {}_A\mathcal{G}$ then any $\theta \in \mathrm{Mor}_n(E, E)$ has $\mathrm{Tr}_{\int}(\mathbb{W}(\theta)) = 0$ in the sense of Lemma 4.7.*

Proof Take dual bases $e^i \in E$ and $f_i \in E^\flat$, and another copy $e^j \in E$ and $f_j \in E^\flat$. From Proposition 3.32 we recall the dual right connection $\tilde{\nabla}_{E^\flat}$ given by

$$\tilde{\nabla}_{E^\flat}(\alpha) = f_j \otimes \mathrm{d}\,\mathrm{ev}(e^j \otimes \alpha) - f_j \otimes (\mathrm{id} \otimes \mathrm{ev}_E)(\nabla_E(e^j) \otimes \alpha)$$

with sum over j understood. Using this dual connection, we have (by Lemma 4.7 the result is independent of the choice of the dual basis)

$$\begin{aligned}
\mathrm{Tr}_{\int}(\mathbb{W}(\theta)) &= \int (\mathrm{id} \otimes \mathrm{ev}_E)(\mathbb{W}(\theta)(e^i) \otimes f_i) \\
&= \int (\mathrm{id} \otimes \mathrm{ev}_E)\big((\mathrm{d} \otimes \mathrm{id} + (-1)^n(\mathrm{id} \wedge \nabla_E))\theta(e^i) \otimes f_i - (\mathrm{id} \wedge \theta)\nabla_E(e^i) \otimes f_i\big) \\
&= \int \mathrm{d}(\mathrm{id} \otimes \mathrm{ev}_E)(\theta(e^i) \otimes f_i) - (-1)^n \int (\mathrm{id} \wedge \mathrm{ev}_E \wedge \mathrm{id})(\theta(e^i) \otimes \tilde{\nabla}_{E^\flat}(f_i)) \\
&\quad - \int (\mathrm{id} \otimes \mathrm{ev}_E)((\mathrm{id} \wedge \theta)\nabla_E(e^i) \otimes f_i).
\end{aligned}$$

Using that $\int$ is an $n+1$ cycle and the shorthand $\tilde{\nabla}_{E^\flat}(f_i) = f_j \otimes \eta_{ji}$, we have

$$(-1)^n \int (\mathrm{id} \wedge \mathrm{ev}_E \wedge \mathrm{id})(\theta(e^i) \otimes \tilde{\nabla}_{E^\flat}(f_i)) = \int (\mathrm{id} \otimes \mathrm{ev}_E)(\eta_{ji} \wedge \theta(e^i) \otimes f_j)$$

while using the explicit formula for η_{ji} in Proposition 3.32 and the shorthand $\nabla_E(e^j) = \kappa \otimes f$ (summation implicit),

$$\begin{aligned}
\eta_{ji} \wedge \theta(e^i) &= \mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) - (\mathrm{id} \otimes \mathrm{ev}_E)(\nabla_E(e^j) \otimes f_i) \wedge \theta(e^i) \\
&= \mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) - \kappa\ \mathrm{ev}_E(f \otimes f_i) \wedge \theta(e^i) \\
&= \mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) - \kappa\ \wedge \theta(\mathrm{ev}_E(f \otimes f_i)e^i) \\
&= \mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) - \kappa \wedge \theta(f) \\
&= \mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) - (\mathrm{id} \wedge \theta)\nabla_E(e^j).
\end{aligned}$$

Also setting $P_{ji} = \mathrm{ev}(e^j \otimes f_i)$ so that $e^i = P_{ik} e^k$ and $f_j = f_m P_{mj}$, we arrive at

$$\begin{aligned}\mathrm{Tr}_{\int}(\mathbb{W}(\theta)) &= -\int (\mathrm{id} \otimes \mathrm{ev}_E)(\mathrm{d}\,\mathrm{ev}(e^j \otimes f_i) \wedge \theta(e^i) \otimes f_j) \\ &= -\int (\mathrm{id} \otimes \mathrm{ev}_E)(P_{mj}(\mathrm{d}P_{ji})P_{ik} \wedge \theta(e^k) \otimes f_m).\end{aligned}$$

But applying d to $P^2 = P$ tells us that $P(\mathrm{d}P)P = 0$. □

Now we give a noncommutative version of the standard classical result that the cohomology class of the trace of the curvature is independent of the connection.

Theorem 4.9 *Suppose that E is an fgp left A-module, and that $\int$ is a $2n$-cycle on the differential calculus Ω. Then for any left-covariant derivative ∇_E and any n, the value of $\mathrm{Tr}_{\int}(R_E{}^n)$ is independent of the choice of ∇_E.*

Proof The set of left connections on E is an affine space, so any two left connections can be connected by a straight line segment. Take such a line segment, with left connection $\nabla_E{}^t$ parameterised by $t \in \mathbb{R}$, and define $\dot{\nabla}_E : E \to \Omega^1 \otimes_A E$ by

$$\dot{\nabla}_E(e) = \frac{\mathrm{d}\nabla_E{}^t(e)}{\mathrm{d}t}.$$

(As we are moving along a straight line interpolating two connections, $\dot{\nabla}_E$ does not depend on t.) For all t, we have $\nabla_E{}^t(a.e) = \mathrm{d}a \otimes e + a.\nabla_E{}^t(e)$ from which it follows that $\dot{\nabla}_E$ is a left module map. Differentiating $R_E{}^t = (\nabla_E{}^t)^{[1]} \circ \nabla_E{}^t$, we obtain

$$\dot{R}_E{}^t = -(\mathrm{id} \wedge \dot{\nabla}_E)\nabla_E{}^t + (\nabla_E{}^t)^{[1]}\dot{\nabla}_E = \mathbb{W}^t(\dot{\nabla}_E).$$

Now omitting the explicit t dependence for clarity, by Theorem 4.3,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} R_E{}^n &= \dot{R}_E \circ R_E{}^{n-1} + R_E \circ \dot{R}_E \circ R_E{}^{n-2} + \cdots + R_E{}^{n-1} \circ \dot{R}_E \\ &= \mathbb{W}(\dot{\nabla}_E) \circ R_E{}^{n-1} + R_E \circ \mathbb{W}(\dot{\nabla}_E) \circ R_E{}^{n-2} + \cdots + R_E{}^{n-1} \circ \mathbb{W}(\dot{\nabla}_E) \\ &= \mathbb{W}(\dot{\nabla}_E \circ R_E{}^{n-1} + R_E \circ \dot{\nabla}_E \circ R_E{}^{n-2} + \cdots + R_E{}^{n-1} \circ \dot{\nabla}_E).\end{aligned}$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}_{\int}(R_E{}^n) = 0$ by Proposition 4.8. □

We should compare this geometric approach to Chern classes to the cyclic cohomology approach in §3.3.4. They are both about associating numbers to fgp modules which depend only on some form of isomorphism class of the module, and in fact by substituting the Grassmann connection into the formula in the last approach we see that the numbers obtained are the same. However, the construction and motivation behind the approaches are different. Connes' cyclic cohomology approach depends on showing that there is a well-defined pairing with K-theory, so that we recover information about the K-theory classes. Chern's approach, in which classically a trace of the curvature gives a de Rham cohomology class, depends on showing that the class is independent of the connection.

Next, following our general strategy we consider the same ideas as above but with bimodules. Thus we consider A-bimodules with left bimodule connections, i.e., objects $(E, \nabla_E, \sigma_E), (F, \nabla_F, \sigma_F) \in {}_A\mathcal{E}_A$ and suppose that $\phi : E \to F$ is a bimodule map. We already know that $\mathbb{W}(\phi) : E \to \Omega^1 \otimes_A F$ is a left module map, but from the right we have from the elementary definitions that

$$\begin{aligned}\mathbb{W}(\phi)(ea) &= \nabla_F\phi(ea) - (\mathrm{id} \otimes \phi)\nabla_E(ea) = \nabla_F((\phi e)a) - (\mathrm{id} \otimes \phi)\nabla_E(ea)\\ &= (\nabla_F\phi e)a - ((\mathrm{id} \otimes \phi)\nabla_E e)a + \sigma_F(\phi(e) \otimes \mathrm{d}a) - (\mathrm{id} \otimes \phi)\sigma_E(e \otimes \mathrm{d}a)\\ &= (\mathbb{W}(\phi)(e))a + \sigma_F(\phi e \otimes \mathrm{d}a) - (\mathrm{id} \otimes \phi)\sigma_E(e \otimes \mathrm{d}a)\end{aligned}$$

for $a \in A$ and $e \in E$. So $\mathbb{W}(\phi)$ is a bimodule map if and only if $(\mathrm{id} \otimes \phi)\sigma_E = \sigma_F(\phi \otimes \mathrm{id})$, which is precisely (3.33) discussed in §3.4.2 as a weaker version of the morphism condition in ${}_A\mathcal{E}_A$. Hence if we want to have a DG category version of ${}_A\mathcal{E}_A$ with a weaker notion of morphism such that $\mathbb{W}(\phi)$ need not be zero but is another morphism then (3.33) is part of the answer, for a 0-morphism ϕ, as it ensures that $\mathbb{W}(\phi)$ remains a bimodule map. We still need to generalise (3.33) to every grade but in such a way that is obeyed by $\mathbb{W}(\phi) \in \mathrm{Mor}_1(E, F)$ when ϕ is grade zero, and so on for every grade. The required generalisation for a morphism $\phi : E \to \Omega^n \otimes_A F$ of grade n is a bimodule map obeying the condition

$$(\mathrm{id} \wedge \phi)\sigma_E = (-1)^n(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}) : E \otimes_A \Omega^1 \to \Omega^{n+1} \otimes_A F \tag{4.5}$$

and we will show in Lemma 4.13 that then $\mathbb{W}(\phi)$ is indeed an $n+1$-morphism. This essentially gives us our category ${}_A\mathcal{G}_A$ as the bimodule version of ${}_A\mathcal{G}$ except that we need a restriction on the objects (E, ∇_E, σ_E) that we consider.

Definition 4.10 A bimodule with left connection (E, ∇_E, σ_E) is called *extendable* if $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ extends for all $n \geq 1$ to $\sigma_E : E \otimes_A \Omega^n \to \Omega^n \otimes_A E$ such that

$$(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E)(\sigma_E \otimes \mathrm{id}) = \sigma_E(\mathrm{id} \otimes \wedge) : E \otimes_A \Omega^n \otimes_A \Omega^m \to \Omega^{n+m} \otimes_A E$$

for all $m \geq 1$. If Ω is an exterior algebra then the extension, if it exists, is unique. We include $m, n = 0$ in the above with $\sigma_E(e \otimes 1) = 1 \otimes e$.

Recall that an exterior algebra as in Definition 1.30 means that Ω is generated by A, $\mathrm{d}A$ and we assume this from now on. In this case uniqueness means that the higher degree σ_E are not additional data and their existence or not is a property of ∇_E, much like the existence of σ_E in degree 1. The extendability says that the wedge product can be taken through a 'braid' crossing in the diagrammatic notation at the end of §3.4.2, in a similar manner to a braided algebra product in braided algebra (see §2.6). Finally, an easy induction tells us that in the extendable case, a morphism in the sense of (4.5) in fact obeys for all $m, n \geq 0$,

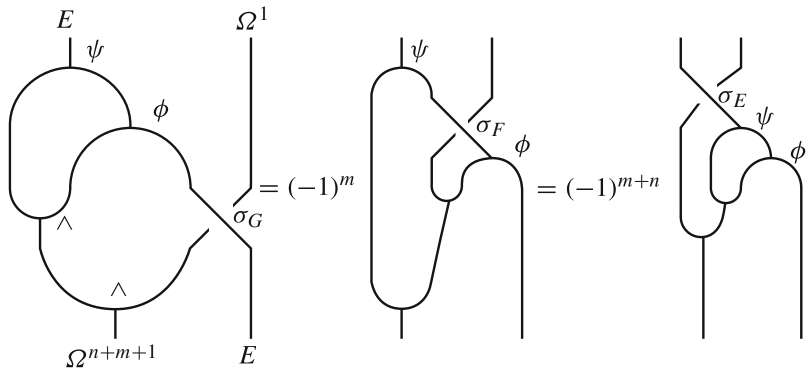
$$(\mathrm{id} \wedge \phi)\sigma_E = (-1)^{nm}(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}) : E \otimes_A \Omega^m \to \Omega^{n+m} \otimes_A F. \tag{4.6}$$

One final ingredient is that in ${}_A\mathcal{G}$ we had a picture of the curvature $R_E : E \to \Omega^2 \otimes_A E$ as a morphism of grade 2. We would still like this to be the case now, which requires among other things that R_E should now be a bimodule map. A necessary and sufficient condition for this was already given in (3.29) and is closely tied up with extendability, as we will see later in the case of the maximal prolongation, hence we build this in as a further condition on objects. We can summarise our proposed DG category by means of the following table.

Name	Objects	$\mathrm{Mor}_n((E, \nabla_E, \sigma_E), (F, \nabla_F, \sigma_F))$
${}_A\mathcal{G}_A$	(E, ∇_E, σ_E)	$\phi : E \to \Omega^n \otimes_A F$
	Extendable left bimodule connections	Bimodule map obeying (4.5)
	with R_E a bimodule map	$(\mathrm{id} \wedge \phi)\sigma_E = (-1)^n(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id})$

Theorem 4.11 *Let A be an algebra with a given exterior algebra. Then ${}_A\mathcal{G}_A$ as summarised is a DG category. Objects are bimodules with bimodule left connections that are extendable and have bimodule map curvature. Morphisms are $\mathbb{N} \cup \{0\}$-graded with $\phi \in \mathrm{Mor}_n(E, F)$ obeying (4.5) and composition the same as for ${}_A\mathcal{G}$ in Definition 4.2. The conclusions of Theorem 4.3 hold but now for ${}_A\mathcal{G}_A$.*

Proof We begin by checking that ${}_A\mathcal{G}_A$ is a category, i.e., that we can compose morphisms. Using the notation explained at the end of §3.4.2, (4.5) holds by



It is clear that $\phi \circ \psi$ is also a bimodule map if ϕ, ψ are. Lemma 4.12 shows that the curvature is a morphism and Lemma 4.13 shows that the differential of a morphism is a morphism. The rest follows just as for ${}_A\mathcal{G}$. □

Lemma 4.12 *Suppose that (E, ∇_E, σ_E) is an extendable left bimodule connection. Then R_E is a right module map if and only if*

$$\nabla_E{}^{[n]}\sigma_E = (\mathrm{id}\wedge\sigma_E)(\nabla_E\otimes\mathrm{id}) + \sigma_E(\mathrm{id}\otimes\mathrm{d}) : E\otimes_A \Omega^n \to \Omega^{n+1}\otimes_A E$$

holds for $n = 1$. In this case it holds for all $n \geq 0$ and

$$(\mathrm{id}\wedge R_E)\sigma_E = (\mathrm{id}\wedge\sigma_E)(R_E\otimes\mathrm{id}) : E\otimes_A \Omega^n \to \Omega^{n+2}\otimes_A E$$

also holds for all $n \geq 0$.

Proof Suppose that R_E is a right module map, and apply (3.29) to $eb\otimes \mathrm{d}a$ instead of $e\otimes\mathrm{d}a$, giving

$$\begin{aligned}(\mathrm{d}\otimes\mathrm{id} - \mathrm{id}\wedge\nabla_E)\sigma_E(eb\otimes\mathrm{d}a) &= (\mathrm{id}\wedge\sigma_E)(\nabla_E(eb)\otimes\mathrm{d}a)\\ &= (\mathrm{id}\wedge\sigma_E)(\nabla_E e\otimes b\mathrm{d}a) + (\mathrm{id}\wedge\sigma_E)(\sigma_E(e\otimes\mathrm{d}b)\otimes\mathrm{d}a).\end{aligned}$$

Given the definition of extendability, we can rewrite this as

$$(\mathrm{d}\otimes\mathrm{id} - \mathrm{id}\wedge\nabla_E)\sigma_E(e\otimes b\mathrm{d}a) = (\mathrm{id}\wedge\sigma_E)(\nabla_E e\otimes b\mathrm{d}a) + \sigma_E(e\otimes\mathrm{d}(b\mathrm{d}a)),$$

which replacing $b\mathrm{d}a$ by general $\xi\in\Omega^1$ is the $n = 1$ case of the stated condition. Now suppose that the stated condition holds for n, and consider it for $n+1$. By the exterior algebra condition it is enough to check it on products $\xi\wedge\eta$, for $\xi\in\Omega^1$ and $\eta\in\Omega^n$,

$$\begin{aligned}\nabla_E{}^{[n+1]}\sigma_E(e\otimes\xi\wedge\eta) &= \nabla_E{}^{[n+1]}(\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_E)(\sigma_E(e\otimes\xi)\otimes\eta)\\ &= (\mathrm{d}\wedge\sigma_E - \mathrm{id}\wedge\nabla_E{}^{[n]}\sigma_E)(\sigma_E(e\otimes\xi)\otimes\eta)\end{aligned} \tag{4.7}$$

by using extendability. Now by the $n = 1$ case and extendability,

$$(\mathrm{d}\otimes\mathrm{id})\sigma_E(e\otimes\xi) = (\mathrm{id}\wedge\nabla_E)\sigma_E(e\otimes\xi) + (\mathrm{id}\wedge\sigma_E)(\nabla_E e\otimes\xi) + \sigma_E(e\otimes\mathrm{d}\xi),$$

$$\begin{aligned}(\mathrm{d}\wedge\sigma_E)(\sigma_E(e\otimes\xi)\otimes\eta) &= (\mathrm{id}\wedge\sigma_E)((\mathrm{id}\wedge\nabla_E)\sigma_E(e\otimes\xi)\otimes\eta)\\ &\quad + (\mathrm{id}\wedge\sigma_E)((\mathrm{id}\wedge\sigma_E)(\nabla_E e\otimes\xi)\otimes\eta) + (\mathrm{id}\wedge\sigma_E)(\sigma_E(e\otimes\mathrm{d}\xi)\otimes\eta)\\ &= (\mathrm{id}\wedge\sigma_E)((\mathrm{id}\wedge\nabla_E)\sigma_E(e\otimes\xi)\otimes\eta)\\ &\quad + (\mathrm{id}\wedge\sigma_E)(\nabla_E e\otimes\xi\wedge\eta) + \sigma_E(e\otimes\mathrm{d}\xi\wedge\eta).\end{aligned}$$

Also, the inductive hypothesis for n and more extendability gives

$$\begin{aligned}&(\mathrm{id}\wedge\nabla_E{}^{[n]}\sigma_E)(\sigma_E(e\otimes\xi)\otimes\eta)\\ &\qquad = (\mathrm{id}\wedge((\mathrm{id}\wedge\sigma_E)(\nabla_E\otimes\mathrm{id}) + \sigma_E(\mathrm{id}\otimes\mathrm{d})))(\sigma_E(e\otimes\xi)\otimes\eta)\\ &\qquad = (\mathrm{id}\wedge\mathrm{id}\wedge\sigma_E)(\mathrm{id}\otimes\nabla_E\otimes\mathrm{id})(\sigma_E(e\otimes\xi)\otimes\eta)\\ &\qquad\quad + (\mathrm{id}\wedge\sigma_E)(\sigma_E\otimes\mathrm{id})(e\otimes\xi\otimes\mathrm{d}\eta)\end{aligned}$$

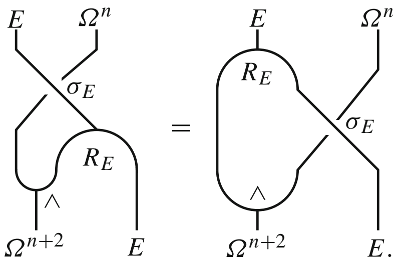
$$= (\mathrm{id} \wedge \sigma_E)((\mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \xi) \otimes \eta) + \sigma_E(e \otimes \xi \wedge \mathrm{d}\eta).$$

Combining these in (4.7) yields the stated condition on $\xi \wedge \eta$. So this holds for all n.

For the converse direction, suppose that the stated condition holds. The $n = 1$ case implies that R_E is a right module map in condition (3.29) since $\mathrm{d}^2 = 0$. Also in this case, using (4.3)

$$\begin{aligned}
(\mathrm{id} \wedge R_E)\sigma_E &= \nabla_E{}^{[2]}\nabla_E{}^{[1]}\sigma_E = \nabla_E{}^{[2]}(\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}) + \nabla_E{}^{[2]}\sigma_E(\mathrm{id} \otimes \mathrm{d}) \\
&= \nabla_E{}^{[2]}(\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}) + (\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{d}) + \sigma_E(\mathrm{id} \otimes \mathrm{d})(\mathrm{id} \otimes \mathrm{d}) \\
&= (\mathrm{d} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}) - (\mathrm{id} \wedge \nabla_E{}^{[1]}\sigma_E)(\nabla_E \otimes \mathrm{id}) + (\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{d}) \\
&= (\mathrm{d} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}) - (\mathrm{id} \wedge (\mathrm{id} \wedge \sigma_E)(\nabla_E \otimes \mathrm{id}))(\nabla_E \otimes \mathrm{id}) \\
&= (\mathrm{id} \wedge \sigma_E)((\mathrm{d} \otimes \mathrm{id})\nabla_E \otimes \mathrm{id}) - (\mathrm{id} \wedge \sigma_E)((\mathrm{id} \wedge \nabla_E)\nabla_E \otimes \mathrm{id}) \\
&= (\mathrm{id} \wedge \sigma_E)(\nabla^{[1]}\nabla_E \otimes \mathrm{id}) = (\mathrm{id} \wedge \sigma_E)(R_E \otimes \mathrm{id}),
\end{aligned}$$

which is the first nontrivial case of the second stated identity. The proof then extends to other n by induction, a tedious but straightforward exercise in associativity of wedge products and most easily done with the identity in the diagrammatic form



□

This says in particular that R_E obeys the morphism condition (4.5) if it is a right (and hence bi)-module map and the connection is extendable. Another application of Lemma 4.12 will be to obtain a cup product on sheaf cohomology, see Proposition 4.49. To conclude the proof of Theorem 4.11, we also need the following.

Lemma 4.13 *If $\phi \in \mathrm{Mor}_n(E, F)$ in ${}_A\mathcal{G}_A$, then $\mathbb{V}(\phi) \in \mathrm{Mor}_{n+1}(E, F)$.*

Proof First we check that $\mathbb{V}(\phi)$ is a right module map. For all $e \in E$ and $a \in A$ and setting $\phi e = \xi \otimes f$ (summation understood),

$$\begin{aligned}
\mathbb{V}(\phi)(e.a) &= \nabla_F^{[n]}(\phi(e.a)) - (\mathrm{id} \wedge \phi)\nabla_E(e.a) \\
&= \mathrm{d}\xi \otimes f.a + (-1)^n\, \xi \wedge \nabla_F(f.a) - (\mathrm{id} \wedge \phi)(\nabla_E e).a - (\mathrm{id} \wedge \phi)\sigma_E(e \otimes \mathrm{d}a)
\end{aligned}$$

$$= \mathbb{W}(\phi)(e).a + (-1)^n\ \xi \wedge \sigma_F(f \otimes \mathrm{d}a) - (\mathrm{id} \wedge \phi)\sigma_E(e \otimes \mathrm{d}a) = \mathbb{W}(\phi)(e).a,$$

where the last equality is the assumption (4.5). Now check that $\mathbb{W}(\phi)$ itself obeys the condition (4.5) in degree $n+1$. Here

$$(\mathrm{id} \wedge \mathbb{W}(\phi))\sigma_E = (\mathrm{id} \wedge \nabla_F^{[n]}\phi)\sigma_E - (\mathrm{id} \wedge \phi)(\mathrm{id} \wedge \nabla_E)\sigma_E : E \otimes_A \Omega^1 \to \Omega^{n+2} \otimes_A F.$$

From the definition of ${\nabla_F}^{[n+1]}$,

$$\nabla_F^{[n+1]}(\mathrm{id} \wedge \phi)\sigma_E = (\mathrm{d} \wedge \phi)\sigma_E - (\mathrm{id} \wedge \nabla_F^{[n]}\phi)\sigma_E,$$

so using Lemma 4.12,

$$\begin{aligned}(\mathrm{id} \wedge \mathbb{W}(\phi))\sigma_E &= (\mathrm{d} \wedge \phi)\sigma_E - \nabla_F^{[n+1]}(\mathrm{id} \wedge \phi)\sigma_E - (\mathrm{id} \wedge \phi)(\mathrm{id} \wedge \nabla_E)\sigma_E \\ &= (\mathrm{id} \wedge \phi){\nabla_E}^{[1]}\sigma_E - (-1)^n\ \nabla_F^{[n+1]}(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}) \\ &= (\mathrm{id} \wedge (\mathrm{id} \wedge \phi)\sigma_E)(\nabla_E \otimes \mathrm{id}) + (\mathrm{id} \wedge \phi)\sigma_E(\mathrm{id} \otimes \mathrm{d}) \\ &\quad - \ (-1)^n\ \nabla_F^{[n+1]}(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}).\end{aligned}$$

Taking some care over the signs and using (4.6),

$$\begin{aligned}(-1)^n\ (\mathrm{id} \wedge \mathbb{W}(\phi))\sigma_E &= (\mathrm{id} \wedge (\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}))(\nabla_E \otimes \mathrm{id}) \\ &\quad + (-1)^n\ (\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{d}) - \nabla_F^{[n+1]}(\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{id}) \\ &= (\mathrm{id} \wedge \sigma_F)(\mathrm{id} \wedge \phi \otimes \mathrm{id})(\nabla_E \otimes \mathrm{id}) + (-1)^n\ (\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{d}) \\ &\quad - (\mathrm{d} \wedge \sigma_F)(\phi \otimes \mathrm{id}) - (-1)^n\ (\mathrm{id} \wedge \nabla_F^{[1]}\sigma_F)(\phi \otimes \mathrm{id}) \\ &= (\mathrm{id} \wedge \sigma_F)(\mathrm{id} \wedge \phi \otimes \mathrm{id})(\nabla_E \otimes \mathrm{id}) + (-1)^n\ (\mathrm{id} \wedge \sigma_F)(\phi \otimes \mathrm{d}) \\ &\quad - (\mathrm{id} \wedge \sigma_F)((\mathrm{d} \otimes \mathrm{id})\phi \otimes \mathrm{id}) - (-1)^n\ (\mathrm{id} \wedge (\mathrm{id} \wedge \sigma_F)(\nabla_F \otimes \mathrm{id}))(\phi \otimes \mathrm{id}) \\ &\quad - (-1)^n\ (\mathrm{id} \wedge \sigma_F(\mathrm{id} \otimes \mathrm{d}))(\phi \otimes \mathrm{id}),\end{aligned}$$

which is $-(\mathrm{id} \wedge \sigma_F)(\mathbb{W}(\phi) \otimes \mathrm{id})$ as required. □

We now give a convenient class of examples where extendability of the bimodule connections is automatic once R_E is assumed to be a bimodule map.

Lemma 4.14 *Suppose that A is given the maximal prolongation differential calculus for some first-order differential calculus (see Lemma 1.32) and that (E, ∇_E, σ_E) is a left bimodule connection whose curvature R_E is a right module map. Then (E, ∇_E, σ_E) is an extendable bimodule connection.*

Proof We continue from Lemma 3.72 and suppose that $a\mathrm{d}b = (\mathrm{d}r)s \in \Omega^1$ (summation on both sides implicit, for some $a, b, r, s \in A$). We also set $\sigma_E(e \otimes$

$\mathrm{d}r) = \xi \otimes f$ (summation understood) as a shorthand. Then

$$\begin{aligned}(\mathrm{id} \wedge \sigma_E)(\sigma_E(e \otimes \mathrm{d}r) \otimes \mathrm{d}s) &= \xi \wedge (\nabla_E(fs) - (\nabla_E f)s)\\ &= \mathrm{d}\xi \otimes fs - (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes fs) - \xi \wedge (\nabla_E f)s\\ &= ((\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes f))s - (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)(\xi \otimes fs)\\ &= ((\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}r))s - (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes (\mathrm{d}r)s)\\ &= ((\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(e \otimes \mathrm{d}r))s - (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_E)\sigma_E(ea \otimes \mathrm{d}b).\end{aligned}$$

Hence Lemma 3.72 gives us

$$\begin{aligned}(\mathrm{id}\wedge\sigma_E)&(\sigma_E(e \otimes \mathrm{d}r) \otimes \mathrm{d}s)\\ &= (\mathrm{id} \wedge \sigma_E)(\nabla_E e \otimes (\mathrm{d}r)s - \nabla_E(ea) \otimes \mathrm{d}b)\\ &\quad + R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b\\ &= (\mathrm{id} \wedge \sigma_E)(\nabla_E e \otimes a\mathrm{d}b - \nabla_E(ea) \otimes \mathrm{d}b) = -(\mathrm{id} \wedge \sigma_E)(\sigma_E(e \otimes \mathrm{d}a) \otimes \mathrm{d}b)\end{aligned}$$

if R_E is a right module map. Hence in this case

$$(\mathrm{id} \wedge \sigma_E)(\sigma_E \otimes \mathrm{id})(e \otimes (\mathrm{d}a \otimes \mathrm{d}b + \mathrm{d}r \otimes \mathrm{d}s)) = 0$$

for all e and all sums of a, b, r, s obeying $a\mathrm{d}b = (\mathrm{d}r)s \in \Omega^1$ (summation implicit). But the relations for the wedge product of the maximal prolongation in Lemma 1.32 are given by quotienting out $\mathrm{d}a \otimes \mathrm{d}b + \mathrm{d}r \otimes \mathrm{d}s$ for such a, b, r, s. Hence under our assumption on R_E, we find that $(\mathrm{id} \otimes \sigma_E)(\sigma_E \otimes \mathrm{id})$ induces a well-defined map $\sigma_E : E \otimes_A \Omega^2 \to \Omega^2 \otimes_A E$ and have proven extendability in degree 2. The higher extendability in the case of the maximal prolongation is then automatic. Thus for Ω^3 consider the map

$$(\mathrm{id}\otimes\sigma_E)(\mathrm{id}\otimes\sigma_E\otimes\mathrm{id})(\sigma_E\otimes\mathrm{id}) : E\otimes_A\Omega^1\otimes_A\Omega^1\otimes_A\Omega^1 \to \Omega^1\otimes_A\Omega^1\otimes_A\Omega^1\otimes_A E.$$

By our degree 2 result, this sends $E \otimes_A \ker\wedge \otimes_A \Omega^1$ to $\ker\wedge \otimes_A \Omega^1 \otimes_A E$ and $E \otimes_A \Omega^1 \otimes_A \ker\wedge$ to $\Omega^1 \otimes_A \ker\wedge \otimes_A E$. Then by definition of the maximal prolongation, there is a well-defined map from $E \otimes_A \Omega^3$ to $\Omega^3 \otimes_A E$. Similarly for higher degree. □

We have succeeded in giving a DG version of the category ${}_A\mathcal{E}_A$ of bimodules with bimodule connections from §3.4.2 and a bimodule version of the DG category ${}_A\mathcal{G}$ from earlier in this section. Of course the whole point of using bimodules and bimodule connections is that one can tensor product them. We now show the same for ${}_A\mathcal{G}_A$ using the same tensor product of objects as for ${}_A\mathcal{E}_A$ (see Theorem 3.78) and an extended tensor product $\phi \boxtimes \psi$ of morphisms. We use a special symbol as this should not be confused with ordinary tensor products which feature in its construction. It should be clear that the unit object $(A, \mathrm{d}, \mathrm{id})$ of ${}_A\mathcal{E}_A$ will also be our unit object now.

Theorem 4.15 *The category ${}_A\mathcal{G}_A$ is a monoidal category, with tensor product of objects being the usual tensor product of bimodules with bimodule connection. The tensor product of morphisms $\phi \in \mathrm{Mor}_n(E, G)$ and $\psi \in \mathrm{Mor}_m(F, H)$ is*

$$\phi \boxtimes \psi = (\mathrm{id} \wedge \sigma_F \otimes \mathrm{id})(\phi \otimes \psi) : E \otimes_A F \to \Omega^{n+m} \otimes_A G \otimes_A H.$$

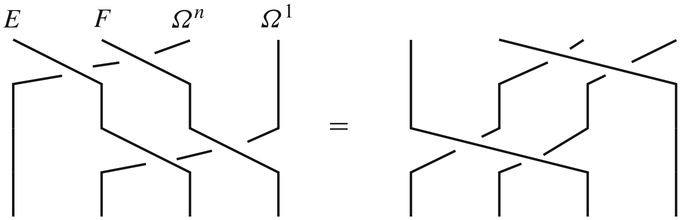
There is a signed rule for composition of tensor products,

$$(\phi \boxtimes \kappa) \circ (\psi \boxtimes \tau) = (-1)^{|\psi|\,|\kappa|}\,(\phi \circ \psi) \boxtimes (\kappa \circ \tau).$$

For objects E and F in ${}_A\mathcal{G}_A$, $R_{E\otimes F} = R_E \boxtimes \mathrm{id} + \mathrm{id} \boxtimes R_F$. For the differential in the DG category,

$$\mathbb{W}(\phi \boxtimes \psi) = \mathbb{W}(\phi) \boxtimes \psi + (-1)^n \phi \boxtimes \mathbb{W}(\psi).$$

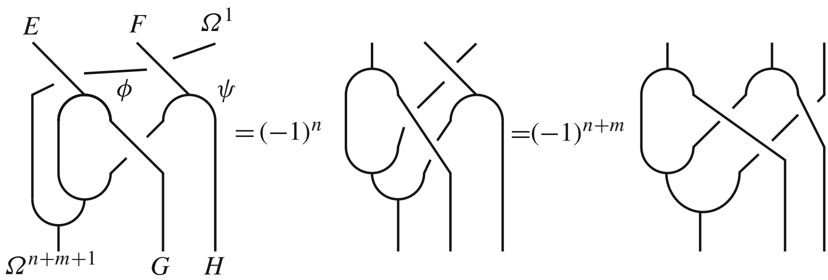
Proof We show that if (E, ∇_E, σ_E) and (F, ∇_F, σ_F) are objects in ${}_A\mathcal{G}_A$ then so is $(E \otimes_A F, \nabla_{E\otimes F}, \sigma_{E\otimes F})$. The extendable connection condition for $E \otimes_A F$ follows fairly immediately from rearranging the crossings for $E \otimes_A F$ (on the LHS of the following) in terms of the crossings for E and the crossings for F on the RHS,



Next we check the condition in Lemma 4.12 for $R_{E\otimes F}$ to be a bimodule map.

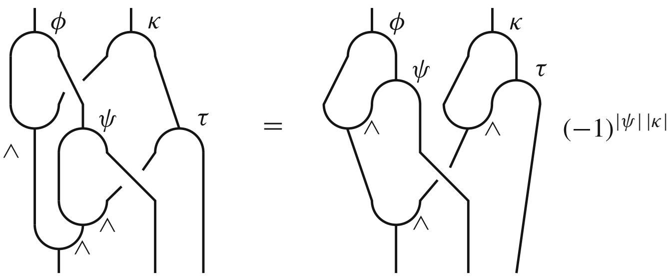
$$\begin{aligned}
\nabla^{[1]}_{E\otimes F}\sigma_{E\otimes F} &= (\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})\sigma_{E\otimes F} - (\mathrm{id}\wedge\nabla_{E\otimes F})\sigma_{E\otimes F} \\
&= (\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})\sigma_{E\otimes F} - (\mathrm{id}\wedge\nabla_E\otimes\mathrm{id})\sigma_{E\otimes F} \\
&\quad - (\mathrm{id}\wedge\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla_F)\sigma_{E\otimes F} \\
&= (\mathrm{id}\wedge\sigma_E\otimes\mathrm{id})(\nabla_E\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F) + (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{d}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F) \\
&\quad - ((\mathrm{id}\wedge\sigma_E)(\sigma_E\otimes\mathrm{id})\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla_F)(\mathrm{id}\otimes\sigma_F) \\
&= (\mathrm{id}\wedge\sigma_{E\otimes F})(\nabla_E\otimes\mathrm{id}\otimes\mathrm{id}) + (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{d}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F) \\
&\quad - (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\wedge\nabla_F)(\mathrm{id}\otimes\sigma_F) \\
&= (\mathrm{id}\wedge\sigma_{E\otimes F})(\nabla_E\otimes\mathrm{id}\otimes\mathrm{id}) + (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\nabla^{[1]}_F\sigma_F) \\
&= (\mathrm{id}\wedge\sigma_{E\otimes F})(\nabla_E\otimes\mathrm{id}\otimes\mathrm{id}) + (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes(\mathrm{id}\wedge\sigma_F)(\nabla_F\otimes\mathrm{id})) \\
&\quad + (\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_F(\mathrm{id}\otimes\mathrm{d})) \\
&= (\mathrm{id}\wedge\sigma_{E\otimes F})(\nabla_{E\otimes F}\otimes\mathrm{id}) + \sigma_{E\otimes F}(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{d}).
\end{aligned}$$

For morphisms, the formula for $\phi \boxtimes \psi$ shows that it is a bimodule map. We also check that it satisfies the σ condition to be a morphism, which diagrammatically is



At each stage we have used the morphism property for one of ϕ or ψ in the diagrammatic form that appears near the end of the proof of Lemma 4.12, and also associativity of the $\wedge$ product.

We also have to check associativity of $\boxtimes$ and the signed rule for compatibility with composition, both of which are routine exercises using the definitions and associativity of the wedge product (and best done diagrammatically). The more complicated case is the composition of tensor products, which is represented by the following diagram, where we use (4.6) and extendability,



Next we verify the formula for the derivative of a tensor product of morphisms. Suppose that $\phi \in \mathrm{Mor}_n(E, G)$ and $\psi \in \mathrm{Mor}_m(F, H)$ in ${}_A\mathcal{G}_A$. Then

$$
\begin{aligned}
&\nabla_{G\otimes H}{}^{[n+m]}(\phi \boxtimes \psi)\\
&\quad = (\mathrm{d} \wedge \sigma_G \otimes \mathrm{id})(\phi \otimes \psi) + (-1)^n (\mathrm{id} \wedge \nabla_G{}^{[m]}\sigma_G \otimes \mathrm{id})(\phi \otimes \psi)\\
&\qquad + (-1)^{n+m} (\mathrm{id} \wedge \sigma_G \otimes \mathrm{id})(\phi \otimes (\mathrm{id} \wedge \nabla_H)\psi)\\
&\quad = (\mathrm{d} \wedge \sigma_G \otimes \mathrm{id})(\phi \otimes \psi) + (-1)^n (\mathrm{id} \wedge (\mathrm{id} \wedge \sigma_G)(\nabla_G \otimes \mathrm{id}) \otimes \mathrm{id})(\phi \otimes \psi)\\
&\qquad + (-1)^n (\mathrm{id} \wedge \sigma_G(\mathrm{id} \otimes \mathrm{d}) \otimes \mathrm{id})(\phi \otimes \psi)\\
&\qquad + (-1)^{n+m} (\mathrm{id} \wedge \sigma_G \otimes \mathrm{id})(\phi \otimes (\mathrm{id} \wedge \nabla_H)\psi)\\
&\quad = (\mathrm{id} \wedge \sigma_G \otimes \mathrm{id})(\nabla_G{}^{[n]}\phi \otimes \psi + (-1)^n \phi \otimes \nabla_H{}^{[m]}\psi),
\end{aligned}
$$

$$
\begin{aligned}
&\big(\mathrm{id}\wedge(\phi\boxtimes\psi)\big)\nabla_{E\otimes F}\\
&\quad=\big(\mathrm{id}\wedge(\phi\boxtimes\psi)\big)(\nabla_E\otimes\mathrm{id})+\big(\mathrm{id}\wedge(\phi\boxtimes\psi)\big)(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)\\
&\quad=(\mathrm{id}\wedge\sigma_G\otimes\mathrm{id})((\mathrm{id}\wedge\phi)\nabla_E\otimes\psi)\\
&\qquad+\big(\mathrm{id}\wedge(\mathrm{id}\wedge\sigma_G\otimes\mathrm{id})(\phi\otimes\psi)\big)(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)\\
&\quad=(\mathrm{id}\wedge\sigma_G\otimes\mathrm{id})((\mathrm{id}\wedge\phi)\nabla_E\otimes\psi)+(-1)^n\,(\mathrm{id}\wedge\sigma_G\otimes\mathrm{id})(\phi\otimes(\mathrm{id}\wedge\psi)\nabla_F).
\end{aligned}
$$

Subtracting these gives $\mathbb{W}(\phi\boxtimes\psi)$.

Finally, for the formula for $R_{E\otimes F}$ we use Lemma 4.12:

$$
\begin{aligned}
&\nabla_{E\otimes F}{}^{[1]}\,\nabla_{E\otimes F}=\nabla_{E\otimes F}{}^{[1]}\,(\nabla_E\otimes\mathrm{id}+(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F))\\
&\quad=\nabla_E{}^{[1]}\nabla_E\otimes\mathrm{id}-(\mathrm{id}\wedge\sigma_E\otimes\mathrm{id})(\nabla_E\otimes\nabla_F)\\
&\qquad+(\nabla_E{}^{[1]}\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)-(\mathrm{id}\wedge\sigma_E\otimes\mathrm{id})(\sigma_E\otimes\nabla_F)(\mathrm{id}\otimes\nabla_F)\\
&\quad=\nabla_E{}^{[1]}\nabla_E\otimes\mathrm{id}+(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{d}\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)\\
&\qquad-(\mathrm{id}\wedge\sigma_E\otimes\mathrm{id})(\sigma_E\otimes\nabla_F)(\mathrm{id}\otimes\nabla_F)\\
&\quad=\nabla_E{}^{[1]}\nabla_E\otimes\mathrm{id}+(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{d}\otimes\mathrm{id})(\mathrm{id}\otimes\nabla_F)\\
&\qquad-(\sigma_E\otimes\mathrm{id})(\mathrm{id}\otimes(\mathrm{id}\wedge\nabla_F)\nabla_F),
\end{aligned}
$$

which we recognise as the sum of the curvatures R_E and R_F. □

We can use this machinery to find another of the symmetries of the Riemann curvature (we gave a version of the second Bianchi identity in Theorem 4.3 and the first Bianchi identity in Example 4.4). In this case we use a metric as defined in §1.3 and in the classical case the following is precisely the $R_{abcd}=-R_{bacd}$ symmetry of the Riemann tensor. We will return to this in Proposition 8.9.

Corollary 4.16 (Riemann Antisymmetry Identity) *Suppose that* $(\Omega^1,\nabla_{\Omega^1},\sigma_{\Omega^1})\in{}_A\mathcal{G}_A$ *and that* $g\in\Omega^1\otimes_A\Omega^1$ *is preserved in the sense that* $\nabla_{\Omega^1\otimes\Omega^1}g=0$. *Then*

$$(R_{\Omega^1}\otimes\mathrm{id}+(\sigma_{\Omega^1}\otimes\mathrm{id})(\mathrm{id}\otimes R_{\Omega^1}))g=0.$$

Proof Since g is preserved by the connection, we have $R_{\Omega^1\otimes\Omega^1}g=0$. Since $(\Omega^1,\nabla_{\Omega^1},\sigma_{\Omega^1})\in{}_A\mathcal{G}_A$, we can use Theorem 4.15 to write $R_{\Omega^1\otimes\Omega^1}=R_{\Omega^1}\boxtimes\mathrm{id}+\mathrm{id}\boxtimes R_{\Omega^1}$, which gives the answer. For the classical case mentioned in the lead up, use the expression for the curvature in Example 3.29. □

If we suppose that $g\in\Omega^1\otimes_A\Omega^1$ is central and that $(\Omega^1,\nabla_{\Omega^1},\sigma_{\Omega^1})\in{}_A\mathcal{G}_A$ but not necessarily metric compatible as in the corollary then we can ask when right multiplication by g is a 1-morphism $R_g\in\mathrm{Mor}_1(A,\Omega^1)$, where $(A,\mathrm{d},\mathrm{id})$ is the unit object. The condition (4.5) for this comes down to

$$\xi\wedge g=-(\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{\Omega^1})(g\otimes\xi)\in\Omega^2\otimes_A\Omega^1 \tag{4.8}$$

for all $\xi \in \Omega^1$. In this case applying the derivative gives

$$\mathbb{W}(R_g) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)g =: \mathrm{co}T_\nabla \in \mathrm{Mor}_2(A, \Omega^1), \tag{4.9}$$

i.e., we recover the 'cotorsion', the vanishing of which will arise naturally in the quantum frame bundle theory in Chap. 5 and in key examples there and in Chap. 8. A sufficient condition for (4.8) to hold is that $\nabla_{\Omega^1\otimes\Omega^1} g \in \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$ is central and that the torsion is a bimodule map. Here $\nabla_{\Omega^1\otimes\Omega^1}(ga) - (\nabla_{\Omega^1\otimes\Omega^1} g)a = \nabla_{\Omega^1\otimes\Omega^1}(ag) - a\nabla_{\Omega^1\otimes\Omega^1} g$ for all $a \in A$ tells us that $(\sigma_{\Omega^1}\otimes \mathrm{id})(\mathrm{id}\otimes\sigma_{\Omega^1})(g\otimes\xi) = \xi \otimes g$. We apply $\wedge \otimes \mathrm{id}$ to this and use Lemma 3.72.

Before discussing the use of metrics in performing traces, we should review quantum metrics in general. A bilinear (as opposed to Hermitian) metric for a bimodule E is defined in the same way as for Ω^1 in Definition 1.15, namely an element $g \in E \otimes_A E$ which we require to be invertible in the sense of a bilinear 'inner product' or inverse quantum metric. This is defined as a bimodule map $(\ ,\ ) : E \otimes_A E \to A$ such that $(\mathrm{id}\otimes(\ ,\ ))(g\otimes \mathrm{id}) : E \to E$ and $((\ ,\ )\otimes \mathrm{id})(\mathrm{id}\otimes g) : E \to E$ are both the identity. This forces g to be central while if we are just given g without necessarily having an inverse, we can require g to be central as a possibly degenerate 'quantum metric'. Such central metrics can be thought of equivalently as bimodule maps $g : A \to E \otimes_A E$ defined by $g(a) = ag(1)$ and $g(1)$ is the metric in our first point of view. We will use both notations for the meaning of g according to context.

In categorical terms, an (invertible) quantum metric g on E makes it a self-dual object in the category ${}_A\mathcal{M}_A$. When E has a left bimodule connection ∇_E, σ_E, we can ask for g to make E self-dual as an object in ${}_A\mathcal{G}_A$, which amounts to our maps being morphisms $g \in \mathrm{Mor}_0(A, E \otimes_A E)$ and $(\ ,\ ) \in \mathrm{Mor}_0(E \otimes_A E, A)$. This in turn amounts to their commuting with the generalised braiding in the sense that

$$\begin{aligned} (\ ,\ ) \otimes \mathrm{id} &= (\mathrm{id} \otimes (\ ,\ ))(\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E), \\ \mathrm{id} \otimes g &= (\sigma_E \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E)(g \otimes \mathrm{id}) \end{aligned} \tag{4.10}$$

as equivalently depicted in Fig. 4.1. One can also see these diagrams as saying that σ_E is invertible and

$$(\mathrm{id} \otimes (\ ,\ ))(\mathrm{id} \otimes \sigma_E \otimes \mathrm{id})(g \otimes \mathrm{id}) = \sigma_E^{-1} : \Omega^1 \otimes_A E \to E \otimes_A \Omega^1.$$

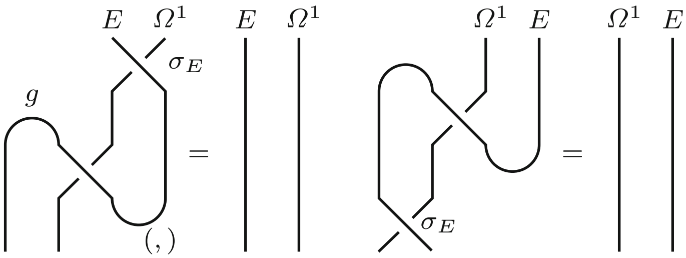


Fig. 4.1 Illustrating that $(\mathrm{id} \otimes (\ ,\ ))(\mathrm{id} \otimes \sigma_E \otimes \mathrm{id})(g \otimes \mathrm{id})$ being an inverse to σ_E is equivalent to g and $(\ ,\ )$ commuting with σ_E

As an application, we recall that we showed in Lemma 4.7 that given functions from a module to $\Bbbk$ satisfying a trace property we could take the trace of certain module maps, including curvatures. The problem, of course, is in finding enough suitable maps. Provided we are working in the category ${}_A\mathcal{G}_A$ and have a nondegenerate metric in that category, we can use the quantum metric to take traces and produce de Rham classes in the sense of the cohomology of the exterior algebra Ω of A.

Corollary 4.17 *Let $E \in {}_A\mathcal{G}_A$ and suppose that $g : A \to E \otimes_A E$ and its inverse $(\ ,\ ) : E \otimes_A E \to A$ are morphisms in ${}_A\mathcal{G}_A$ and that $\nabla_{E\otimes E}\, g(1) = 0$. Then we have an element of de Rham cohomology*

$$[(\mathrm{id} \otimes (\ ,\ ))(R_E{}^n \otimes \mathrm{id})g(1)] \in \mathrm{H}^{2n}_{\mathrm{dR}}(A).$$

Proof Firstly, $\nabla_{E\otimes E}\, g(1) = 0$ is equivalent to $\mathbb{W}(g) = 0$. From the equation $((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes g) = \mathrm{id} : E \to E$ we deduce

$$(\mathbb{W}((\ ,\ )) \boxtimes \mathrm{id}) \circ (\mathrm{id} \boxtimes g) + ((\ ,\ ) \boxtimes \mathrm{id}) \circ (\mathrm{id} \boxtimes \mathbb{W}(g)) = \mathbb{W}(\mathrm{id}) = 0$$

so $\mathbb{W}((\ ,\ )) = 0$ also holds. Now by Theorem 4.3 and Theorem 4.15,

$$\mathbb{W}((\mathrm{id} \otimes (\ ,\ ))(R_E{}^n \otimes \mathrm{id})g) = 0$$

for all $n \geq 0$, which implies the result. □

In the $n = 0$ case, we call this trace the 'quantum metric dimension' $\underline{\dim}_E = (\ ,\ )(g)$ of E since classically it will just recover the fibre dimension or rank of the bundle. We have $\mathrm{d}(\underline{\dim}_E) = 0$ which classically corresponds to the fibre dimension being locally constant. However, it is not clear what these noncommutative cohomology classes are measuring in general. We similarly used twisted cycles in §3.3.5, but these pair with an equivariant K-theory which incorporates the twisting automorphism, rather than the ordinary K-theory of the algebra. Both contexts take us away from the classical idea of a characteristic class being only a property of the isomorphism class of the module, due to dependence on additional data.

Example 4.18 We follow on from Example 3.76 for $A = \mathbb{C}(S_3)$ with its 3D calculus in Example 1.60 and the 5-parameter moduli of invariant bimodule connections on $E = \Omega^1$. This time we do not require that σ satisfies the braid relations, but rather

$$(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \Psi) = (\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}), \tag{4.11}$$

where $\Psi(e_u \otimes e_v) = e_w \otimes e_u$ etc. is the crossed-module braiding that defines the calculus. This implies that the connection is extendable for the canonical (Woronowicz) exterior algebra. To see this, recall from Corollary 2.72 and Theorem 2.74 that to obtain Λ^n we take the tensor product of n copies of Λ^1 and quotient by the kernel of $[n, -\Psi]!$. We take the case $n = 2$, and then from (4.11)

$$(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes [2, -\Psi]!) = ([2, -\Psi]! \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}),$$

where we have used $[2, -\Psi]! = \mathrm{id} - \Psi$ from Definition 2.67. Applying this to $u \otimes v \otimes w$ where $v \otimes w \in \ker[2, -\Psi]!$ (summation implicit) we see that

$$(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(u \otimes v \otimes w) \subseteq \ker[2, -\Psi]! \otimes \Lambda^1,$$

which we can also write as $(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(u \otimes v \otimes w) = 0$. This means that we have a well-defined σ on 2-forms given by

$$\sigma(u \otimes v \wedge w) = (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(u \otimes v \otimes w).$$

This extends directly to the more complicated $[n, -\Psi]!$ as these are also just sums of compositions of tensor products of Ψ. The converse also holds in nice cases such as our current S_3 example, i.e., it happens that we obtain all extendable σ by this method. On the other hand, solving (4.11) results in six 1-parameter cases:

(i) $b = e = c = d = 0$,
(ii) $e = q^2 b$, $c = d = a = qb$ where q is a cube root of unity (three cases),
(iii) $b = e = c = 0$, $a = d$,
(iv) $b = e = d = 0$, $a = c$.

which are, however, all instances of the respective cases (i)–(iv) of $R_\nabla = 0$ in the analysis of Example 3.76. We therefore have a six 1-parameter moduli of invariant connections making Ω^1 an object of ${}_{\mathbb{C}(S_3)}\mathcal{G}_{\mathbb{C}(S_3)}$, and they are all flat. If we impose torsion compatibility/freeness, as also covered in Example 3.76, then there are only two of these, namely case (ii) with $a = 1/(1-q)$ and two values of q. ◇

On the dual side, we have covered the translation-invariant differential structures on group algebras $\Bbbk G$ for a group G in §1.6.2.

Example 4.19 (Bimodule Connections on $\Bbbk G$) We recall that $\Omega^1 = G.\Lambda^1$ is generated by a right G-module Λ^1 with relations given by the action and d given by a group 1-cocycle $\zeta \in Z^1(G, \Lambda^1)$. The exterior algebra is $\Omega = G.\Lambda$, where Λ is the Grassmann algebra on Λ^1 obeying $\mathrm{d} = 0$ on Λ. Next, following Example 3.31, left-invariant left connections on $\Omega^1(\Bbbk G)$ of are in 1–1 correspondence with linear maps $\nabla^L : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$. To determine if this is a bimodule connection, we use Proposition 3.73 with $\varpi\pi_\epsilon g = g^{-1}\mathrm{d}g = \zeta(g)$ to find

$$\sigma^L(\xi \otimes \zeta(g)) = \zeta(g) \otimes \xi + \nabla^L \xi - (\nabla^L(\xi \triangleleft g^{-1})) \triangleleft g$$

for all $\xi \in \Lambda^1$ and $g \in G$. A little calculation using the cocycle condition shows that if this formula is well-defined then σ^L automatically preserves the action and is therefore a right (and hence bi-)module map. We focus on the case where ∇^L is a right module map, in which case σ^L is just flip map. The bimodule connection is then also extendable due to the Grassmann algebra form of the exterior algebra. Moreover, all the pieces making up the curvature preserve the action, so the curvature is necessarily a right (and hence bi-)module map and $\Omega^1(\Bbbk G) \in {}_{\Bbbk G}\mathcal{G}_{\Bbbk G}$.

For a concrete example, we set $G = PSL_2(\mathbb{Z})$ with generators s, t and relations $(st)^3 = s^2 = e$. This is represented on $\Lambda^1 = \mathrm{Row}^3(\mathbb{C})$ as right multiplication by

$$\rho(s) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho(t) = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

The relations for $PSL_2(\mathbb{Z})$ imply $\zeta(s) + \zeta(s){\triangleleft}s = 0$ and

$$0 = \zeta(s){\triangleleft}tstst + \zeta(t){\triangleleft}stst + \zeta(s){\triangleleft}tst + \zeta(t){\triangleleft}st + \zeta(s){\triangleleft}t + \zeta(t)$$

and as a result

$$\zeta(s) = (a, b, -a),\ \zeta(t) = (c, d, -\frac{b}{\sqrt{2}} - \mathrm{c} + \frac{d}{\sqrt{2}})$$

for any $a, b, c, d \in \mathbb{C}$. It is easy to check that this gives an inner calculus as in §1.6.2 if and only if $c = 0$, with $\theta = (\frac{d}{\sqrt{2}}, -\frac{b}{2}, a + \frac{d}{\sqrt{2}})$. Given this calculus, let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $\lambda \in \mathbb{C}$. Then

$$\nabla^L e_1 = \lambda e_1 \otimes e_2 - \lambda e_2 \otimes e_1,\ \nabla^L e_2 = \lambda e_1 \otimes e_3 - \lambda e_3 \otimes e_1,\ \nabla^L e_3 = \lambda e_2 \otimes e_3 - \lambda e_3 \otimes e_2$$

is a right module map and we obtain an object of ${}_{PSL_2(\mathbb{Z})}\mathcal{G}_{PSL_2(\mathbb{Z})}$ with curvature

$$R_\nabla e_1 = \lambda^2 (e_1 \wedge e_3 \otimes e_1 + e_2 \wedge e_1 \otimes e_2),$$

and similarly for the other basis vectors. ◇

In the remainder of this section we look at objects $L \in {}_A\mathcal{G}_A$ which are line A-modules in the sense of §3.5 and extend the Frölich map Φ_L in Corollary 3.94. We denote by $Z_A(\Omega)$ the elements of Ω that commute with all elements of A.

Corollary 4.20 *Let A be an algebra with exterior algebra Ω.*

(1) *Given a line module and extendable left bimodule connection (L, ∇_L, σ_L), there is an algebra map $\hat{\Phi}_L : Z_A(\Omega) \to Z_A(\Omega)$ defined by $\hat{\Phi}_L(\xi) \otimes e = \sigma_L(e \otimes \xi)$. This restricts on degree zero to Φ_L^{-1} and on tensor products $\hat{\Phi}_{L \otimes_A M} = \hat{\Phi}_L \circ \hat{\Phi}_M$.*
(2) *If in addition the curvature R_L is a bimodule map (so $L \in {}_A\mathcal{G}_A$) then there is a unique $\omega_L \in Z_A(\Omega^2)$ such that $R_L(e) = \omega_L \otimes e$. If (M, ∇_M, σ_M) is another such line module then so is their tensor product and $\omega_{L \otimes_A M}$ is given by $\omega_L + \hat{\Phi}_L(\omega_M)$.*

Proof Suppose that $\xi \in Z_A(\Omega^n)$. Then the map $e \mapsto \sigma_L(e \otimes \xi)$ is a right module map from L to $\Omega^n \otimes_A L$, so by Lemma 3.92 it is given by $e \mapsto \hat{\Phi}_L(\xi) \otimes e$ for some $\hat{\Phi}_L(\xi)$. Uniqueness gives the required properties for $\hat{\Phi}_L$. Now suppose that R_L is a bimodule map. The existence and uniqueness of $\omega_L \in \Omega^2$ follows from applying Lemma 3.92 in the right A-module map case to R_L. Taking this form of R_L and knowing that it is a left module map tells us that ω_L is central. For the curvature of the tensor product we use Theorem 4.15. □

The reader may have noted that the curvature in classical electromagnetism is often simply quoted as a 2-form rather than an operator. The result above says that this is a general feature for line modules $L \in {}_A\mathcal{G}_A$. Similarly in this case the de Rham class in Corollary 4.17 reduces to the class of ${\omega_L}^n$ times the metric dimension, as an element of $\mathrm{H}^{2n}_{\mathrm{dR}}(A)$.

Example 4.21 Let G be a finite group and take $\mathbb{C}(G)$ with calculus given by $\mathcal{C} \subseteq G \setminus \{e\}$ as in Proposition 1.52. We take a trivial line module $L = \mathbb{C}(G)$ with bimodule structure given by multiplication and a general connection of the form

$$\nabla f = \mathrm{d}f \otimes 1 + \sum_{a\in\mathcal{C}} f.e_a \otimes (1 - \gamma_a)$$

for all $f \in A$, where $\gamma_a \in \mathbb{C}(G)$ are some functional parameters. One can check that this is always a bimodule connection with

$$\sigma(1 \otimes e_a) = e_a \otimes \gamma_a.$$

To check if the connection is extendable for the canonical exterior algebra we use the same method as Example 4.18, which means that we have to check if applying σ twice commutes with the braiding $\Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a$ from §1.7.2. We have for the RHS of (4.11),

$$\begin{aligned}(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(1 \otimes e_a \otimes e_b) &= (\Psi \otimes \mathrm{id})(e_a \otimes e_b \otimes R_{b^{-1}}(\gamma_a)\gamma_b)\\ &= e_{aba^{-1}} \otimes e_a \otimes R_{b^{-1}}(\gamma_a)\gamma_b.\end{aligned}$$

Equating this with the composition on the other side gives the condition $R_{b^{-1}}(\gamma_a)\gamma_b = R_{a^{-1}}(\gamma_{aba^{-1}})\gamma_a$ for all $a, b \in \mathcal{C}$ for commuting with the braiding, which implies extendability. For the curvature, some calculation gives

$$R_\nabla(1) = \sum_{a\in\mathcal{C}} \mathrm{d}e_a \otimes (1 - \gamma_a) + \sum_{a\in\mathcal{C}} e_a \wedge \mathrm{d}\gamma_a \otimes 1 - \sum_{a,b\in\mathcal{C}} e_a \wedge (1 - \gamma_a)e_b \otimes (1 - \gamma_b),$$

showing that R_∇ a bimodule map is equivalent to the centrality of

$$\omega_\nabla = \sum_{a\in\mathcal{C}} \mathrm{d}e_a(1 - \gamma_a) + \sum_{a\in\mathcal{C}} e_a \wedge \mathrm{d}\gamma_a - \sum_{a,b\in\mathcal{C}} e_a \wedge (1 - \gamma_a)e_b(1 - \gamma_b) \in \Omega^2.$$

If this is central and the extendability condition above is satisfied then we will obtain a line bundle object of ${}_{\mathbb{C}(G)}\mathcal{G}_{\mathbb{C}(G)}$ with ω_∇ the curvature 2-form.

For example, we can take $G = S_3$ with its 3D calculus from Example 1.60 with $\gamma_a(e) = \alpha_0$, $\gamma_a(a) = \alpha_1$, $\gamma_a(b) = \alpha_2$ for all $a \neq b$ and $\gamma_a(uv) = \gamma_a(vu) = \alpha_3$ for all $a, b \in \mathcal{C}$ where $\alpha_i \in \mathbb{C}$ obey $\alpha_2\alpha_3 = \alpha_0\alpha_1$ to have extendability. In this case $R_\nabla = 0$, so we have a zero curvature line module in ${}_{\mathbb{C}(S_3)}\mathcal{G}_{\mathbb{C}(S_3)}$. ◇

In Example 3.86 we looked at a discrete approximation to the Möbius bundle. We now consider the continuous circle case.

Example 4.22 (Möbius Bundle) Recall that the algebraic circle $\mathbb{C}_q[S^1]$ in Example 1.11 refers to $\mathbb{C}[t, t^{-1}]$ as a $\mathbb{Z}$-graded algebra with $|t^n| = n$ and q-commutation relations for $\mathrm{d}t, t$. Clearly its even subalgebra is another circle $A = \mathbb{C}_{q^2}[S^1]$ as an algebra $\mathbb{C}[s, s^{-1}]$ where $s = t^2$ has grade 2 and $\mathrm{d}s, s$ have q^2-commutation relations. We let L be the odd grade elements of $\mathbb{C}_q[S^1]$ as a sub-bimodule over A, i.e., odd powers of t, t^{-1}. Following Proposition 3.98, we can write dual bases for L as column vectors $\underline{v}$ and $\underline{w}$, giving the projection matrix P for the module as

$$\underline{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} t \\ t^{-1} \end{pmatrix}, \quad \underline{w} = \frac{1}{\sqrt{2}}\begin{pmatrix} t^{-1} \\ t \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 1 & s \\ s^{-1} & 1 \end{pmatrix}$$

and demonstrating that L is indeed a line module. The product in the larger algebra $\mathbb{C}_q[S^1]$ gives an isomorphism $L \otimes_{\mathbb{C}_{q^2}[S^1]} L \cong \mathbb{C}_{q^2}[S^1]$. There is a left connection on $e \in L$ defined by d on $\mathbb{C}_q[S^1]$,

$$\nabla e = (\mathrm{d}e)t \otimes t^{-1} \in \Omega^1(\mathbb{C}_{q^2}[S^1]) \otimes_{\mathbb{C}_{q^2}[S^1]} L,$$

and this has zero curvature. For this to be a bimodule connection, we need

$$\begin{aligned}\sigma(e \otimes \mathrm{d}a) &= (\mathrm{d}(ea))t \otimes t^{-1} - (\mathrm{d}e)t \otimes t^{-1}a = (\mathrm{d}(ea))t \otimes t^{-1} - (\mathrm{d}e)at \otimes t^{-1} \\ &= e(\mathrm{d}a)t \otimes t^{-1},\end{aligned}$$

where we have shifted even degree factors across the tensor product. From this we see that $\sigma(e \otimes \xi) = e\xi t \otimes t^{-1}$ for all $e \in L$ and $\xi \in \Omega^1$ on A is a bimodule map and thus that ∇ is a bimodule connection. Next,

$$(\mathrm{id} \wedge \sigma)(\sigma \otimes \mathrm{id})(e \otimes \xi \otimes \eta) = e\xi t \wedge \sigma(t^{-1} \otimes \eta) = e\xi t \wedge t^{-1}\eta t \otimes t^{-1},$$

so that the connection is extendable, and as the curvature is zero we have a line bundle in $L \in {}_{\mathbb{C}_{q^2}[S^1]}\mathcal{G}_{\mathbb{C}_{q^2}[S^1]}$. Although this line bundle is trivial with free generator $t \in L$, it is not trivial as a real bundle which would require a hermitian generator as opposed to $t^* = t^{-1}$ as here. Nevertheless we have a $*$-preserving connection in the sense of Definition 3.85 and similar features for the nonexistence of flat sections apply as in the discrete approximation to the Möbius bundle in Example 3.86.

Classically, one can write $s = e^{i\theta}$ for $\theta \in [0, 2\pi]$ and t changes sign as we go around this circle. $\diamond$

This was a warm-up for the next example.

Example 4.23 The Hopf algebra $A = \mathbb{C}_q[SO_3]$ consists of the even degree elements of $\mathbb{C}_q[SU_2]$. The 3D calculus of Example 2.32 on the latter restricts to give $\Omega^1(\mathbb{C}_q[SO_3]) = \mathbb{C}_q[SO_3].\Lambda^1$ with the same space of invariant 1-forms Λ^1 as for $\mathbb{C}_q[SU_2]$. We define a sub-bimodule L over $\mathbb{C}_q[SO_3]$ as the odd degree elements of $\mathbb{C}_q[SU_2]$. It is a line module as $1 \in L.L$ and has a left flat connection given by d, so

$$\nabla e = (\mathrm{d}e)a \otimes d - q^{-1}(\mathrm{d}e)c \otimes b \in \Omega^1(\mathbb{C}_q[SO_3]) \otimes_{\mathbb{C}_q[SO_3]} L$$

for $e \in L$. By following just the same arguments as in the preceding example, we see that we have $L \in {}_{\mathbb{C}_q[SO_3]}\mathcal{G}_{\mathbb{C}_q[SO_3]}$. This time, however, L is not a singly generated trivial left module. This is because if there was an $e \in L$ such that $L = A.e$ and since $1 \in L.L = L.A.e = L.e$, this would imply that $e \in L$ is invertible. However, the only invertible elements in $\mathbb{C}_q[SU_2]$ are of the form of a nonzero number in $\mathbb{C}$ times the identity. Although we have not developed a theory of 'simply connectedness' in the noncommutative case, the existence of a nontrivial flat line bundle L in this sense is in line with the classical case where SO_3 is not simply connected. $\diamond$

We conclude with the tautological or 'q-monopole' line module on the q-sphere in Example 3.99 with Grassmann connection associated to a projector P.

Example 4.24 (Grassmann Connection on q-Sphere) We take $A = \mathbb{C}_q[S^2]$ with its standard real form. We recall that the line bundle L in Example 3.99 can be identified with a submodule of $\mathrm{Row}^2(A)$ by sending $f \in \mathbb{C}_q[SU_2]$ of degree -1 to $(fa, q^{-1}fc)$. Setting $e^1 = (1,0).P = (1 - q^2x, z)$, $e^2 = (0,1).P = (z^*, x)$ as in Example 3.27, the curvature of the Grassmann connection ∇ was $R_\nabla e^i = q^3\mathrm{Vol} \otimes e^i$ extended as a left module map. Since Vol on the q-sphere is central, this curvature of the Grassmann connection is a bimodule map. It remains to show that the Grassmann connection is a bimodule connection, then we will have $L \in {}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$ with curvature 2-form $\omega_\nabla = q^3\mathrm{Vol}$, in keeping with the classical picture if we remember that the geometric volume form is $\mathrm{i}\mathrm{Vol}$.

Indeed, with dual basis e_1, e_2 picking out the first and second entries of a row vector, the Grassmann connection in Proposition 3.24 is

$$\begin{aligned}\nabla(fa, q^{-1}fc) &= \mathrm{d}(e_1(fa, q^{-1}fc)) \otimes e^1 + \mathrm{d}(e_2(fa, q^{-1}fc)) \otimes e^2 \\ &= \mathrm{d}(fa) \otimes e^1 + q^{-1}\mathrm{d}(fc) \otimes e^2.\end{aligned}$$

Using the e_i in column vector form, where

$$e^1\begin{pmatrix} d \\ -b \end{pmatrix} = (1 - q^2x, z)\begin{pmatrix} d \\ -b \end{pmatrix} = d, \quad e^2\begin{pmatrix} d \\ -b \end{pmatrix} = (z^*, x)\begin{pmatrix} d \\ -b \end{pmatrix} = -b,$$

we transfer ∇ back to the degree -1 sub-bimodule L to obtain

$$\nabla f = \mathrm{d}(fa) \otimes d - q^{-1}\mathrm{d}(fc) \otimes b. \tag{4.12}$$

From this it is possible to show that ∇ is a bimodule connection and then by Lemma 4.14 that it is extendable. The formula for σ is still quite complicated as well as hard to check directly, but an example is

$$\begin{aligned}
\sigma(b \otimes \mathrm{d}z) &= \nabla(bz) - (\nabla b)z \\
&= \mathrm{d}(bza) \otimes d - q^{-1}\mathrm{d}(bzc) \otimes b - \mathrm{d}(ba) \otimes dz + q^{-1}\mathrm{d}(bc) \otimes bz \\
&= \mathrm{d}(cdab) \otimes d - \mathrm{d}(bcz) \otimes b - q(\mathrm{d}(ba))z \otimes d + q^{-2}(\mathrm{d}(bc))z \otimes b \\
&= -q^{-1}\mathrm{d}(zz^*) \otimes d + q\mathrm{d}(xz) \otimes b + q(\mathrm{d}z^*)z \otimes d - q^{-1}(\mathrm{d}x)z \otimes b \\
&= (q\mathrm{d}(xz) - q^{-1}(\mathrm{d}x)z) \otimes b + (q(\mathrm{d}z^*)z - q^{-1}\mathrm{d}(zz^*)) \otimes d \\
&= qx\mathrm{d}z \otimes b + q((q^2-1)x\mathrm{d}x - z^*\mathrm{d}z) \otimes d,
\end{aligned}$$

where we move z from the right using the commutation relations in $\mathbb{C}_q[SU_2]$ and through $\otimes_A$. We then identify resulting expressions in terms of the q-sphere generators, and use the module relations

$$(x-1)b = z^*d, \quad zb = -q^2xd$$

and the q-sphere relations for the final form. In the same way one can find

$$\begin{aligned}
\sigma(b \otimes \mathrm{d}z^*) &= q(x\mathrm{d}z^* - (q^2-1)z^*\mathrm{d}x) \otimes b - qz^*\mathrm{d}z^* \otimes d, \\
\sigma(b \otimes \mathrm{d}x) &= x\mathrm{d}x \otimes b - q^2z^*\mathrm{d}x \otimes d,
\end{aligned}$$

and similar formulae for $\sigma(d \otimes (\ ))$. Our σ are necessarily compatible with the sphere projector relation in Chap. 2 for the q-sphere. This example is the $n = 1$ case of a line module E_n in Example 5.26 consisting of the degree $-n$ elements of $\mathbb{C}_q[SU_2]$ (the q-monopole sections of charge n), and we will see that it has a bimodule connection in Example 5.51. The curvature is again a bimodule map so that $E_n \in {}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$ with curvature 2-form $\omega_\nabla = q^3[n]_{q^2}\mathrm{Vol}$. ◇

4.2 Coactions on the de Rham Complex

In this section we look at methods to reduce the noncommutative de Rham cohomology of a Hopf algebra H to its invariants, which is important in the classical compact Lie group case and achieved by averaging using the Haar integral. Thus we suppose that H has a normalised right-integral as in Definition 2.18 whereby

$(\int \otimes \mathrm{id})\Delta = \int \otimes 1$ and $\int 1 = 1$. Using this, on any left H-comodule E with coaction $\Delta_L : E \to H \otimes E$ we define the averaging map

$$\mathrm{av} = (\int \otimes \mathrm{id})\Delta_L : E \to {}^H E, \tag{4.13}$$

where ${}^H E$ is the subspace of invariants where $\Delta_L e = 1 \otimes e$. It is elementary using the definition of a right-integral to check that the image here is invariant. Now suppose that E is a cochain complex with the H-coaction preserving the grades E^n and its differential d is H-covariant. As d is a comodule map its image and kernel are both comodules, and hence the cohomology $\mathrm{H}(E)$ is also a comodule. It is also obvious from covariance that ${}^H E$ is a cochain complex and from covariance of d that $\mathrm{av} \circ \mathrm{d} = \mathrm{d} \circ \mathrm{av}$. The cochain complexes $({}^H E, \mathrm{d})$ and (E, d) are therefore related by the cochain maps $\mathrm{av} : (E, \mathrm{d}) \to ({}^H E, \mathrm{d})$ and the canonical inclusion $i : ({}^H E, \mathrm{d}) \to (E, \mathrm{d})$. Because the integral is normalised, $\mathrm{av} \circ i$ is the identity. Using [ ] to denote cohomology class, we obtain a cohomology map $\mathrm{H}(i) : \mathrm{H}({}^H E) \to \mathrm{H}(E)$ given by $\mathrm{H}(i)[e] = [i(e)]$.

If we now equip H with an exterior algebra Ω then we can consider $\Omega \otimes E$ as the tensor product cochain complex, meaning its degree n consists of $\oplus_{0 \le r \le n} \Omega^r \otimes E^{n-r}$ and has differential $\mathrm{d} \otimes \mathrm{id} + (-1)^r \mathrm{id} \otimes \mathrm{d}$. We now suppose there is a cochain map

$$\Delta_{L*} : E^n \to H \otimes E^n \bigoplus \Omega^1 \otimes E^{n-1} \bigoplus \cdots$$

where the first part is Δ_L and we write $\delta_L : E^n \to \Omega^1 \otimes E^{n-1}$ for the second term. That this commutes with d is

$$\Delta_L(\mathrm{d}e) + \delta_L(\mathrm{d}e) + \cdots = (\mathrm{d} \otimes \mathrm{id})\Delta_L e + (\mathrm{id} \otimes \mathrm{d})\Delta_L e - (\mathrm{id} \otimes \mathrm{d})\delta_L e + \cdots$$

where the omitted terms have degree ≥ 2 in Ω. Since d was a comodule map, this comes down to

$$(\mathrm{d} \otimes \mathrm{id})\Delta_L e = \delta_L(\mathrm{d}e) + (\mathrm{id} \otimes \mathrm{d})\delta_L e \tag{4.14}$$

plus constraints on $(\mathrm{d} \otimes \mathrm{id})\delta_L e$ that depend on the higher terms. We say that Δ_L *extends to degree 1* if we can find δ_L obeying (4.14).

Proposition 4.25 *Suppose that H has a normalised right-integral. If E is an H-covariant cochain complex as above then* $\mathrm{H}(i) : \mathrm{H}({}^H E) \to \mathrm{H}(E)$ *is injective with image* ${}^H\mathrm{H}(E)$. *If Δ_L extends to degree 1 and the calculus on H is connected then we can identify* $\mathrm{H}({}^H E) = \mathrm{H}(E)$ *via* $\mathrm{H}(i)$.

Proof If $f \in \ker \mathrm{d} : {}^H E^n \to {}^H E^{n+1}$, then $[f] \in \mathrm{H}^n(E)$ is a coinvariant. Injectivity follows from $\mathrm{H}(\mathrm{av}) \circ \mathrm{H}(i)$ being the identity. Now suppose that $[e] \in \mathrm{H}^n(E)$ is a coinvariant, so that $(\mathrm{id} \otimes [\,])\Delta_L(e) = 1 \otimes [e]$. Write $\Delta_L(e) = \sum_i h_i \otimes f_i$, where $\mathrm{d}f_i = 0$ (as $\ker \mathrm{d}$ is a comodule). Then $\sum_i h_i \otimes [f_i] = 1 \otimes [e]$, or $\sum_i h_i \otimes f_i =$

$1 \otimes e + \sum_j q_j \otimes \mathrm{d}g_j$ for some q_j, g_j. Applying $\int \otimes \mathrm{id}$ to this,

$$\mathrm{av}(e) = \sum_i (\int h_i) f_i = e + \sum_j (\int q_j) \mathrm{d}g_j = e + \mathrm{d}\big(\sum_j (\int q_j) g_j\big),$$

so $[\mathrm{av}(e)] = [e] \in \mathrm{H}^n(E)$, and $[\mathrm{av}(e)]$ is in the image of $\mathrm{H}(i)$.

For the second part we first show that if Δ_L extends to degree 1 then the image of $\Delta_L : \mathrm{H}^n(E) \to H \otimes \mathrm{H}^n(E)$ is contained in $(\ker \mathrm{d} : H \to \Omega^1_H) \otimes \mathrm{H}^n(E)$. Thus, consider $e \in E^n$ with $\mathrm{d}e = 0$ and write $\Delta_L e = \sum h_i \otimes f_i$ as before and $\delta_L e = \sum \xi_i \otimes g_i$. By Δ_{L*} a cochain map as elaborated above and $\mathrm{d}e = 0$, we have in component $H\otimes$ that $\sum_i h_i \otimes \mathrm{d}f_i = 0$ (which expresses that d on E is a comodule map) hence without loss of generality we can assume that $f_i \in \ker \mathrm{d}$. We have in the Ω^1 component that $\sum_i \mathrm{d}h_i \otimes f_i = \sum_j \xi_j \otimes \mathrm{d}g_j$ (which expresses (4.14)) and taking the cohomology class [ ] gives $\sum_i \mathrm{d}h_i \otimes [f_i] = 0 \in \Omega^1_H \otimes \mathrm{H}^n(E)$. It then follows that if the calculus on H is connected then for any class x we have $\Delta_L x = 1_H \otimes y \in H \otimes \mathrm{H}^n(E)$ for some y and applying $\epsilon \otimes \mathrm{id}$ shows that $y = x$, so x is invariant under Δ_L. Hence $\mathrm{H}(E) = {}^H\mathrm{H}(E)$, which completes the proof. □

Clearly, a main example is $E = \Omega$, the exterior algebra on H itself, if this happens to be left-covariant. The simplest case for the rest of the data is if the calculus is in fact bicovariant (so there are left and right translation coactions Δ_L, Δ_R) and Ω is a super-Hopf algebra with super-coproduct which we will now distinguish by Δ_* restricting to the coproduct Δ in degree 0 and $\Delta_L + \Delta_R$ in degree 1. We have seen that this holds for the canonical exterior algebra of a bicovariant first-order calculus in Theorem 2.74 and it also holds, for example, for the maximal prolongation exterior algebra of a bicovariant Ω^1.

Corollary 4.26 *Let H have a connected left-covariant exterior algebra Ω as in Chap. 2. If left translation extends to degree 1 and if H has a normalisable right-integral then $\mathrm{H}_{\mathrm{dR}}(H) = \mathrm{H}(\Lambda)$, where Λ is the subalgebra of left-invariant forms with the restricted* d. *If Ω is bicovariant and a super-Hopf algebra restricting to Δ, $\Delta_L + \Delta_R$ in degree 0,1 then $\Delta_{L*} = \Delta_*$ extends Δ_L to all degrees.*

Proof The first part is just the theory above applied to $E = \Omega$ with $\delta_L : \Omega^n \to \Omega^1 \otimes \Omega^{n-1}$ obeying (4.14). We check that the latter datum arises very naturally in the bicovariant super-Hopf algebra case. Since we supposed an exterior algebra as opposed to a general DGA, the super-coproduct $\Delta_* : \Omega \to \Omega \underline{\otimes} \Omega$ (where the underline refers to the graded tensor product), if it exists, will be uniquely determined by its graded-homomorphism property and necessarily commutes with d, i.e., is necessarily a map of cochain complexes. When the tensor product d is written out explicitly, the latter is the same as a 'supercoderivation property' in the explicit construction of Theorem 2.74 but in fact holds whenever we are in the super-Hopf algebra case. Thus in lowest degree, $\Delta_* \mathrm{d}h = \Delta_L \mathrm{d}h + \Delta_R \mathrm{d}h = h_{(1)} \otimes \mathrm{d}h_{(2)} + \mathrm{d}h_{(1)} \otimes h_{(2)} = \mathrm{d}\Delta h$ by left and right covariance of the first-order calculus. It then follows in higher degree by induction since $\Delta_*(\mathrm{d}(h\mathrm{d}\omega)) = (\Delta_* \mathrm{d}h)(\Delta_* \mathrm{d}\omega) = (\mathrm{d}\Delta h)(\mathrm{d}\Delta_* \omega) = \mathrm{d}((\Delta h)(\mathrm{d}\Delta_* \omega)) = \mathrm{d}\Delta_*(h\mathrm{d}\omega)$ on remembering to use the graded product. We are then free to expand

$$\Delta_*: \Omega^n \to H \otimes \Omega^n \bigoplus \Omega^1 \otimes \Omega^{n-1} \bigoplus \cdots \bigoplus \Omega^{n-1} \otimes \Omega^1 \bigoplus \Omega^n \otimes H,$$

where the first term is Δ_L, the next term is δ_L, through to the last term Δ_R. □

We do need that the calculus on H is connected in Proposition 4.25. For example, if we take $E = \Omega$ with $\Omega^n = 0$ for all $n > 0$, then clearly $\mathrm{H}^0_{\mathrm{dR}}(H) = H$ while ${}^H\Omega = \mathbb{C}.1$ and its cohomology is likewise $\mathbb{C}.1$.

Example 4.27 We compute the cohomology of $\mathbb{C}_q[SL_2]$ with its 4D bicovariant calculus in Example 2.77 and q generic. We know from Chap. 2 that this has a normalised integral and that the calculus is connected, so Corollary 4.26 applies. Clearly $\Lambda^0 = \mathbb{C}$ recovering $\mathrm{H}^0_{\mathrm{dR}} = \mathbb{C}.1$, which we already knew as it expresses that the calculus is connected. The kernel of d in Λ^1 is $\mathbb{C}.\theta$, giving $\mathrm{H}^1_{\mathrm{dR}} = \mathbb{C}.\theta$. A basis of Λ^2 is

$$e_c \wedge e_z, \quad e_z \wedge e_b, \quad e_b \wedge e_c, \quad \theta \wedge e_c, \quad \theta \wedge e_b,\ \theta \wedge e_z.$$

The first three are a basis of the kernel of d, and all three of these are in the image of d, giving $\mathrm{H}^2_{\mathrm{dR}} = 0$. A basis of Λ^3 is

$$\theta \wedge e_c \wedge e_z, \quad \theta \wedge e_z \wedge e_b, \quad \theta \wedge e_b \wedge e_c, \quad e_z \wedge e_b \wedge e_c.$$

The first three are a basis of the image of d, but all four are in the kernel, giving $\mathrm{H}^3_{\mathrm{dR}} = \mathbb{C}$ as spanned by the last. Finally, $\theta \wedge e_z \wedge e_b \wedge e_c$ is not exact and proportional to the chosen volume form, so $\mathrm{H}^4_{\mathrm{dR}} = \mathbb{C}.\mathrm{Vol}$. ◇

Motivated by the above theory we can now 'polarise' to let H coact on $E = \Omega_A$, the exterior algebra of an H-comodule algebra A.

Definition 4.28 Let H be a Hopf algebra and A an algebra, with exterior algebras Ω_H and Ω_A respectively. A left coaction $\Delta_L : A \to H \otimes A$ is called *differentiable* if it extends to a degree-preserving map $\Delta_{L*} : \Omega_A \to \Omega_H \underline{\otimes} \Omega_A$ of exterior algebras.

Here we assumed an exterior algebra on A so that the extension, if it exists, is unique. For example, on Ω^1_A we would need

$$\Delta_{L*}(a\mathrm{d}b) = a_{(\bar{1})} b_{(\bar{1})} \otimes a_{(\bar{\infty})} \mathrm{d}b_{(\bar{\infty})} + a_{(\bar{1})} \mathrm{d}b_{(\bar{1})} \otimes a_{(\bar{\infty})} b_{(\bar{\infty})} \tag{4.15}$$

to be well defined. This entails for the first term that Δ_L extends to Ω^1_A i.e., that the calculus is H-covariant and additionally it entails that the $\Omega^1_H \otimes$ term

$$\delta_L(a\mathrm{d}b) = a_{(\bar{1})} \mathrm{d}b_{(\bar{1})} \otimes a_{(\bar{\infty})} b_{(\bar{\infty})}, \quad \delta_L : \Omega^1_A \to \Omega^1_H \otimes A \tag{4.16}$$

is well defined. Classically, a Lie group action on a manifold generates vertical vector fields on A, and δ_L can be viewed as the generator of these in form terms.

Of course, we have the same notions for a right coaction $\Delta_R : A \to A \otimes H$ to be differentiable if it extends to a DGA map $\Delta_{R*} : \Omega_A \to \Omega_A \underline{\otimes} \Omega_H$.

Lemma 4.29 *Suppose that the map $\Delta_{L*} : \Omega^1_A \to (H \otimes \Omega^1_A) \bigoplus (\Omega^1_H \otimes A)$ given by (4.15) is well defined and that the differential calculus Ω_A is the maximal prolongation of Ω^1_A. Then Δ_{L*} extends to all degrees as a map of exterior algebras.*

Proof It is enough to prove it for extension to Ω^2_A since the maximal prolongation is quadratic, so we have to check that $\Delta_{L*}(\xi \wedge \eta) = (\Delta_{L*}\xi) \wedge (\Delta_{L*}\eta)$ is well defined for $\xi, \eta \in \Omega^1_A$. Now suppose that we have the relation $a.\mathrm{d}b - \mathrm{d}r.s = 0$ in Ω^1_A (summation implicit). Applying Δ_{L*} to this gives

$$0 = a_{(\bar{1})} b_{(\bar{1})} \otimes a_{(\bar{\infty})}.\mathrm{d}b_{(\bar{\infty})} - r_{(\bar{1})} s_{(\bar{1})} \otimes \mathrm{d}r_{(\bar{\infty})}.s_{(\bar{\infty})},$$
$$0 = a_{(\bar{1})}.\mathrm{d}b_{(\bar{1})} \otimes a_{(\bar{\infty})} b_{(\bar{\infty})} - \mathrm{d}r_{(\bar{1})}.s_{(\bar{1})} \otimes r_{(\bar{\infty})} s_{(\bar{\infty})}.$$

Applying $\mathrm{d} \otimes \mathrm{id}$ to the first equation, $\mathrm{id} \otimes \mathrm{d}$ to the second and subtracting gives

$$\begin{aligned} 0 = \mathrm{d}a_{(\bar{1})}.b_{(\bar{1})} \otimes a_{(\bar{\infty})}.\mathrm{d}b_{(\bar{\infty})} - r_{(\bar{1})}.\mathrm{d}s_{(\bar{1})} \otimes \mathrm{d}r_{(\bar{\infty})}.s_{(\bar{\infty})} \\ - a_{(\bar{1})}.\mathrm{d}b_{(\bar{1})} \otimes \mathrm{d}a_{(\bar{\infty})}.b_{(\bar{\infty})} + \mathrm{d}r_{(\bar{1})}.s_{(\bar{1})} \otimes r_{(\bar{\infty})}.\mathrm{d}s_{(\bar{\infty})}. \end{aligned}$$

This is exactly what the $\Omega^1_H \otimes \Omega^1_A$ part of $\Delta_{L*}(\mathrm{d}a \wedge \mathrm{d}b + \mathrm{d}r \wedge \mathrm{d}s)$ should be. Applying $\mathrm{id} \otimes \mathrm{d}$ to the first equation gives $a_{(\bar{1})} b_{(\bar{1})} \otimes \mathrm{d}a_{(\bar{\infty})} \wedge \mathrm{d}b_{(\bar{\infty})} + r_{(\bar{1})} s_{(\bar{1})} \otimes \mathrm{d}r_{(\bar{\infty})} \wedge \mathrm{d}s_{(\bar{\infty})}$, which is the $\Omega^0_H \otimes \Omega^2_A$ part, and applying $\mathrm{d} \otimes \mathrm{id}$ to the second equation gives the $\Omega^2_H \otimes \Omega^0_A$ part. Hence all the relations in the maximal prolongation map to zero. □

The classical differential calculus on a manifold is an example of a maximal prolongation, which is why a differentiable action in the usual sense extends automatically to higher forms.

Example 4.30 Let G be a finite group acting as automorphisms of a finite directed graph X, which we write as a left coaction of $H = \Bbbk(G)$ on $A = \Bbbk(X)$ with its graph calculus (see §1.4). The left action is given by $\Delta_L \delta_x = \sum_{g \in G} \delta_g \otimes \delta_{g^{-1}x}$ and we suppose that for every arrow $\omega_{x \to y}$ in X and every $g \in G$ there is an arrow $\omega_{gx \to gy}$, i.e., that G acts on the arrows in X and thus on $\Omega^1(X)$. We also fix a bicovariant calculus on $\Bbbk(G)$ by a choice of Ad-stable subset $\mathcal{C} \subseteq G \setminus \{e\}$. Differentiating the coaction on $\Omega^1(X)$, we have

$$\begin{aligned} \Delta_{L*}(\delta_x \mathrm{d}\delta_y) &= \sum_{g,h \in G} \big(\delta_g \mathrm{d}\delta_h \otimes \delta_{g^{-1}x} \delta_{h^{-1}y} + \delta_g \delta_h \otimes \delta_{g^{-1}x} \mathrm{d}\delta_{h^{-1}y}\big) \\ &= \sum_{g,h \in G} \Big(\delta_g \sum_{a \in \mathcal{C}} (\delta_{ha^{-1}} - \delta_h) e_a \otimes \delta_{g^{-1}x} \delta_{h^{-1}y} \\ &\quad + \delta_g \delta_h \otimes \delta_{g^{-1}x} \Big(\sum_{z: z \to h^{-1}y} \omega_{z \to h^{-1}y} - \sum_{z: h^{-1}y \to z} \omega_{h^{-1}y \to z} \Big)\Big), \end{aligned}$$

which we write as

$$\begin{aligned}
&\Delta_{L*}\Big(\delta_x\big(\sum_{z:z\to y}\omega_{z\to y}-\delta_{x,y}\sum_{z:y\to z}\omega_{y\to z}\big)\Big)\\
&\quad=\sum_{g\in G}\Big(\sum_{a\in\mathcal{C}}\delta_g e_a\otimes\delta_{g^{-1}x}\delta_{a^{-1}g^{-1}y}-\delta_{x,y}\sum_{a\in\mathcal{C}}\delta_g e_a\otimes\delta_{g^{-1}x}\\
&\qquad+\delta_g\otimes\delta_{g^{-1}x}\Big(\sum_{z:z\to g^{-1}y}\omega_{z\to g^{-1}y}-\delta_{x,y}\sum_{z:g^{-1}x\to z}\omega_{g^{-1}x\to z}\Big)\Big).
\end{aligned}$$

For this to be well defined (i.e., not to map zero to something nonzero) we need that if $x\neq y$ and $x\not\to y$ then $y\neq gag^{-1}x$, or to put this another way, *we need that there is an arrow $x\to ax$ for every $x\in X$ and $a\in\mathcal{C}$ such that $ax\neq x$*. Then

$$\Delta_{L*}\omega_{x\to y}=\sum_{g\in G}\Big(\sum_{a\in\mathcal{C}}\delta_g e_{g^{-1}ag}\otimes\delta_{g^{-1}x}\delta_{y,ax}+\delta_g\otimes\omega_{g^{-1}x\to g^{-1}y}\Big)\tag{4.17}$$

and a brief check shows that this is indeed a bimodule map and that Δ_{L*} is a differentiable action as in (4.15). Note that X can have arrows which are not of the form $\omega_{x\to ax}$ for $a\in\mathcal{C}$, but these are in the kernel of the $\Omega^1{}_{\Bbbk(G)}\otimes\Bbbk(X)$ component of Δ_{L*}. Whether the coaction is fully differentiable depends on the higher degrees of $\Omega(X)$ but, for example, the maximal prolongation will do by Lemma 4.29. $\diamond$

We pause to give a case where a coaction we have given previously is *not* differentiable.

Example 4.31 We consider the right $H=\mathbb{C}(\mathbb{Z}_N)$ coaction $\Delta_R z=z\otimes\chi$ on the deformed disk algebra $A=\mathbb{C}_q[D]$ from Example 3.100. If this coaction were differentiable with respect to some calculus on $\mathbb{C}(\mathbb{Z}_N)$ then we would have a well-defined map $\Omega^1_A\to\Omega^1_A\otimes H+A\otimes\Omega^1_H$ with $\mathrm{d}z\mapsto\mathrm{d}z\otimes\chi+z\otimes\mathrm{d}\chi$. Then

$$\begin{aligned}
z\mathrm{d}\bar z&\longmapsto z\mathrm{d}\bar z\otimes\chi\chi^*+z\bar z\otimes\chi\mathrm{d}\chi^*,\\
q^{-2}(\mathrm{d}\bar z)z&\longmapsto q^{-2}(\mathrm{d}\bar z)z\otimes\chi^*\chi+q^{-2}\bar z z\otimes(\mathrm{d}\chi^*)\chi,
\end{aligned}$$

and subtracting gives

$$0\longmapsto z\bar z\otimes(\chi\mathrm{d}\chi^*-(\mathrm{d}\chi^*)\chi)+(1-q^{-2}).1\otimes(\mathrm{d}\chi^*)\chi,$$

so we get $\mathrm{d}\chi^*=0$. As χ^* and 1 generate $\mathbb{C}(\mathbb{Z}_N)$ we would then have $\mathrm{d}=0$ on all of $\mathbb{C}(\mathbb{Z}_N)$. Thus for any nontrivial differential calculus on $\mathbb{C}(\mathbb{Z}_N)$ the coaction is *not* differentiable. $\diamond$

We also cannot apply Corollary 4.26 in the case $A=H$ with a calculus that is only left-covariant, because part of its assumptions on degree 1 require (4.15) to be well defined. The first term is well defined with coaction given by the coproduct as the calculus is left-covariant, so for the whole to be well defined we would need the second term to be, which is exactly right-covariance. Thus for any left-covariant first-order calculus on H, left translation is differentiable on degree 1 if and only if Ω^1 is bicovariant.

Example 4.32 The 3D calculus on $\mathbb{C}_q[SL_2]$ of Example 2.32 is a maximal prolongation of degree 1 but we cannot use Corollary 4.26 to find its de Rham cohomology as it is not differentiable for left translation. Given the integral, we can still calculate the invariant part of its de Rham cohomology simply from the invariant forms Λ by Proposition 4.25. The invariant 0-forms have basis $\{1\}$, so the 0th cohomology is $\mathrm{H}^0(\Lambda) = \mathbb{C}$. The left-invariant 1-forms and 2-forms are both 3-dimensional vector spaces, and d is a linear isomorphism between them, so $\mathrm{H}^1(\Lambda) = \mathrm{H}^2(\Lambda) = 0$. The top 3-forms have one basis element $\mathrm{Vol} = e^+ \wedge e^- \wedge e^0$, so $\mathrm{H}^3(\Lambda) = \mathbb{C}$. ⋄

This turns out in fact to be the full $\mathrm{H}_{\mathrm{dR}}(\mathbb{C}_q[SL_2])$ with its 3D calculus, but we cannot see this from the theory so far. Also, we have not stressed the $*$-structure but our constructions are compatible and we shall henceforth speak 'geometrically' about $\mathbb{C}_q[SU_2]$ to indicate that we have in mind the standard $*$-structure. Our approach will be to compute the cohomology of $\mathbb{C}_q[S^2]$, which we can do by hand, and then lift to the desired result. The idea is that when a Lie group acts on a manifold we can take the horizontal one-forms on the manifold, meaning forms that kill every vector along the orbits (i.e., annihilating the vertical vector fields induced by the Lie algebra action). In nice cases where the quotient space is a manifold, we will also be able to view the horizontal forms generated by those pulled back from the base, i.e., sections of the pull back bundle. These notions are more divergent in the noncommutative case so we distinguish between them: we define the *weakly horizontal 1-forms* to be the kernel of δ_L as the analogue of being killed by all vertical vector fields, reserving 'horizontal' for those that come up from the base, which in an algebraic language means generated by differentials of invariant functions on the total space.

We will explore this briefly, but to fit with conventions needed in Chap. 5 we will do it for differentiable right coactions Δ_R. On 1-forms this means that the calculus is right-covariant and that

$$\delta_R(a\mathrm{d}b) = a_{\bar{(0)}}b_{\bar{(0)}} \otimes a_{\bar{(1)}}\mathrm{d}b_{\bar{(1)}}, \quad \delta_R : \Omega^1_A \to A \otimes \Omega^1_H \tag{4.18}$$

is well defined. Then the (right) weakly horizontal forms are precisely the kernel of this. The expression vanishes for all $b \in B = A^H$ with any a, so clearly the weakly horizontal forms contain everything of the form $A\Omega^1_B$, where Ω^1_B is generated within Ω^1_A by $B, \mathrm{d}B$. In nice cases, this $A\mathrm{d}B$ is all of the weakly horizontal forms.

Example 4.33 We saw in the discussion after Proposition 2.33 that $\mathbb{C}_q[S^2] = \mathbb{C}_q[SU_2]^H$ in terms of a coaction $\Delta_R f = f \otimes t^{|f|}$ of $H = \mathbb{C}[t, t^{-1}]$ on homogeneous elements $f \in \mathbb{C}_q[SU_2]$ with degree $|f|$ and that the latter extends to a right covariance on $\Omega^1(\mathbb{C}_q[SU_2])$ with its 3D calculus in Example 2.32. This has basic left-invariant 1-forms $e^\pm, e^0$ and a short calculation from the formulae for these gives

$$\delta_R e^{\pm} = 0, \quad \delta_R e^0 = 1 \otimes t^{-1}\mathrm{d}t.$$

It is a general feature for this kind of coaction, which is of the form $(\mathrm{id} \otimes \pi)\Delta$ for a Hopf algebra map π, that $\delta_R(Sa_{(1)}\mathrm{d}a_{(2)}) = 1 \otimes S\pi(a)_{(1)}\mathrm{d}\pi(a)_{(2)}$. This sends left-invariant forms to left-invariant forms with the Maurer–Cartan forms connected by π. We still need to check that this map is well defined in terms of Δ_{R*} respecting the bimodule relations which, since Δ_R does, means checking that δ_R does. Indeed

$$\delta_R(e^0 f - q^{2|f|} f e^0) = \delta_R(e^0)\Delta_R f - q^{2|f|}(\Delta_R f)\delta_R(e^0) = f \otimes t^{|f|} e_t - q^{2|f|} f \otimes e_t t^{|f|}$$

vanishes precisely due to the relations $(\mathrm{d}t)t = q^2 t\mathrm{d}t$ (recall that we have denoted $\mathbb{C}[t, t^{-1}]$ with this calculus by $\mathbb{C}_{q^2}[S^1]$). Here $e_t = t^{-1}\mathrm{d}t$ is the left-invariant basic form on H. Clearly $\delta_R(\omega_i e^i) = \omega_0 \otimes t^{|\omega_0|-1}\mathrm{d}t$ so the kernel of δ_R and hence the space of weakly horizontal forms is precisely the submodule spanned by $e^{\pm}$. To fill out the picture, note that this implies that the invariant weakly horizontal forms comprise the submodule of the form $\mathbb{C}_q[SL_2]_{-2}e^+ \oplus \mathbb{C}_q[SL_2]_2 e^-$ where the subscripts refer to the degree components of $\mathbb{C}_q[SL_2]$. We saw in the proof of Proposition 2.35 that these are exactly $\Omega^1(\mathbb{C}_q[S^2])$ and this also tells us that the weakly horizontal forms themselves can be written as $\mathbb{C}_q[SU_2]\Omega^1(\mathbb{C}_q[S^2])$. ⋄

We will study weakly horizontal forms in generality in Chap. 5. We turn now to a direct calculation of the de Rham cohomology of $\mathbb{C}_q[S^2]$ for its 2D calculus but viewed as in Proposition 2.35 and now understood by Example 4.33 as the submodule of invariant weakly horizontal forms within the 3D calculus of $\mathbb{C}_q[SU_2]$.

Proposition 4.34 *The q-sphere $\mathbb{C}_q[S^2]$, with its 2D calculus in Proposition 2.35 and generic q, has* $\mathrm{H}^0_{\mathrm{dR}} = \mathbb{C}.1$, $\mathrm{H}^1_{\mathrm{dR}} = 0$ *and* $\mathrm{H}^2_{\mathrm{dR}} = \mathbb{C}.[\mathrm{Vol}] \neq 0$ *where* $\mathrm{Vol} = e_+ \wedge e_-$.

Proof We refer to the list of differentials of basis elements of $\mathbb{C}_q[SL_2]$ in Example 2.32. Take a finite linear combination of basis elements of $\mathbb{C}_q[S^2]$ in ker d. Suppose that there is a nonzero term with $d^n b^r c^s$ for some $n \geq 1$. Then choose the nonzero term with the largest power of d, and for that power of d the largest power of b. Now applying d to this $d^n b^r c^s$ term gives a nonzero $q^{2-s-r}[s]_{q^2} d^{n+1} b^r c^{s-1} e^+$ term, which cannot be cancelled by any other term (note $s = n + r \neq 0$). We conclude that our element of ker d cannot contain any basis elements with nonzero powers of d. A similar argument shows that it cannot contain any basis elements with nonzero powers of a. We are left with a sum of $b^r c^r$ terms, and taking the largest r among the nonzero terms similarly shows that we must have the largest $r = 0$. We conclude that $\mathrm{H}^0_{\mathrm{dR}} = \mathbb{C}$ with generator the class of 1.

For grade zero 1-forms we have

$$\mathrm{d}(a^n b^r c^s e^+) = -[r]_{q^2} q^{1-n} a^{n+1} b^{r-1} c^s e^+ \wedge e^- \tag{4.19}$$

and $r = 2 + n + s \neq 0$, hence $[a^n b^r c^s e^+ \wedge e^-] = 0$ for all $n \geq 1$. Similarly

$$\mathrm{d}(d^n b^r c^s e^-) = [s]_{q^2} q^{2-s-r} d^{n+1} b^r c^{s-1} e^+ \wedge e^- \tag{4.20}$$

and hence $[d^n b^r c^s e^+ \wedge e^-] = 0$ for all $n \geq 1$. Thus the only possible nonzero elements of $\mathrm{H}^2_{\mathrm{dR}}$ would be spanned by $[e^+ \wedge e^- b^r c^s]$. Next,

$$\mathrm{d}(ab^{s-1} c^s e^-) = \big([s+1]_{q^2} q^{2-2s} b^s c^s e^+ \wedge e^- + [s]_{q^2} q^{3-2s} b^{s-1} c^{s-1}\big) e^+ \wedge e^-$$

for all $s \geq 1$, hence at the level of cohomology classes,

$$[b^s c^s e^+ \wedge e^-] = -\frac{[s]_{q^2}}{q^{-1}[s+1]_{q^2}} [b^{s-1} c^{s-1} e^+ \wedge e^-]. \tag{4.21}$$

It follows that every cohomology class in $\mathrm{H}^2_{\mathrm{dR}}$ is a multiple of $[e^+ \wedge e^-]$. We also know that $[e^+ \wedge e^-] \neq 0$ by taking the cohomology of the left $\mathbb{C}_q[SU_2]$-invariant forms for $\mathbb{C}_q[S^2]$, using Proposition 4.25.

For the first cohomology we need to calculate more values, e.g., for $n \geq 1$,

$$\mathrm{d}(d^n b^r c^s e^+) = -q^2([r+n]_{q^2} q^{-n} d^{n-1} b^r c^{s+1} + [r]_{q^2} q^{n-1} d^{n-1} b^{r-1} c^s) e^+ \wedge e^-,$$

where $s - n - r = -2$. A tedious consideration of all possibilities on the linear basis of 1-forms gives the vanishing of the first cohomology. On the other hand, in Example 7.28 we will give a more efficient method using complex structures. □

We may now use differentiability of the right coaction in Example 4.33 to simplify the process of finding the de Rham cohomology of $\mathbb{C}_q[SU_2]$ with the left-covariant 3D calculus. Here we only find $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}_q[SU_2])$ and will later feed this into the Leray–Serre spectral sequence calculation in Example 4.68 to determine the rest of the de Rham cohomology.

Corollary 4.35 *The 3D left-covariant calculus on $\mathbb{C}_q[SU_2]$ has first de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}_q[SU_2]) = 0$.*

Proof By the differentiability of the right coaction in Example 4.33 and Proposition 4.25 (or rather the right-handed version of that result), the de Rham cohomology of $\mathbb{C}_q[SU_2]$ is the same as the cohomology of its right-invariant (i.e., grade zero) differential forms. Suppose that $\xi = he^0 + fe^+ + ge^-$ has grade zero and is in the kernel of d. Considering the $e^\pm \wedge e^0$ parts and using the formulae in Example 2.32, we obtain $\mathrm{d}h \wedge e^0 = 0$, implying that $\mathrm{d}h = 0$ as it has no e^0 component given that h is grade zero and therefore in $\mathbb{C}_q[S^2]$. We deduce that h is a multiple of the identity, and then

$$\mathrm{d}(fe^+ + ge^-) = -hq^3 e^+ \wedge e^-,$$

which contradicts $0 \neq [e^+ \wedge e^-] \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_q[S^2])$ unless $h = 0$. Thus any invariant 1-form in the kernel of d is also weakly horizontal and hence can be viewed as on the base $\mathbb{C}_q[S^2]$). Hence it is d of an invariant 0-form by $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}_q[S^2]) = 0$. □

In the remainder of this section, we shall look at the matter the other way around. Instead of assuming that there is an integral and applying it to find the cohomology, we shall find an integral from the cohomology. One way of viewing integration on a compact orientated manifold M of dimension m is by using the top-dimensional cohomology. Locally on the manifold, for coordinates $\{x^1, \ldots, x^m\}$, there is a top-dimensional form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. The definition of the manifold being orientated is just that coordinate patches can be chosen in such a way that any local top form is a positive multiple of any other local top form on the overlap of the charts. The result is that we can form a global top form Vol such that any other top form on M is a function times Vol, the volume form. As $\Omega^{m+1}(M) = 0$, all m-forms are closed hence for any $f \in C^\infty(M)$ we can take the cohomology class $[f\mathrm{Vol}] \in \mathrm{H}^m_{\mathrm{dR}}(M)$. Poincaré duality states, among other things, that $\mathrm{H}^m_{\mathrm{dR}}(M) \cong \mathbb{R}$ for a compact connected orientated m-dimensional manifold without boundary, allowing us to define $\int_M f = [f\mathrm{Vol}] \in \mathbb{C}$.

For a noncommutative algebra A with exterior algebra Ω, we similarly suppose that there is a top degree m (called the 'volume dimension' in Chap. 1) such that $\Omega^n = 0$ for $n > m$ and such that Ω^m has a single generator Vol with $\mathrm{H}^m_{\mathrm{dR}}(A) = \Bbbk[\mathrm{Vol}] \neq 0$. Now we may define a normalised integral $\int : A \to \Bbbk$ by

$$\int a = \frac{[\mathrm{Vol}\, a]}{[\mathrm{Vol}]} \in \Bbbk. \tag{4.22}$$

We could also have used $[a\mathrm{Vol}]$, which would give a different result in general. Now we suppose that the Hopf algebra H left coacts on Ω with d, a comodule map, and that Vol is invariant under the coaction of H. As $\mathrm{H}^m_{\mathrm{dR}}(A)$ is spanned by [Vol], it follows that H has a trivial coaction on $\mathrm{H}^m_{\mathrm{dR}}(A)$, and from this we have

$$\Delta_L[\mathrm{Vol}\, a] = 1 \otimes [\mathrm{Vol}\, a] = a_{(\bar{1})} \otimes [\mathrm{Vol}\, a_{(\bar{\infty})}],$$

which implies that

$$\int a = a_{(\bar{1})} \int a_{(\bar{\infty})} \tag{4.23}$$

for all $a \in A$. In the special case of $A = H$ and left translation, $\int$ is a normalised left-integral on the Hopf algebra H. (A left-integral is defined analogously to a right-integral in Definition 2.18 but now by $(\mathrm{id} \otimes \int)\Delta = 1_H \int : H \to H$.) Now in a case where we can calculate with the cohomology, we might be able to find a formula for the left-integral, and we shall do this for $\mathbb{C}_q[S^2]$.

Example 4.36 Assume that q is not a root of unity. On $\mathbb{C}_q[S^2]$ (see Lemma 2.34) with differential calculus given in Example 4.33, we take the $\mathbb{C}_q[SU_2]$-invariant volume form $\mathrm{Vol} = e^+ \wedge e^-$ and note that Ω^2 is generated by Vol as a right $\mathbb{C}_q[S^2]$-module. We also have $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_q[S^2]) \cong \mathbb{C}$ with cohomology class $[\mathrm{Vol}] \neq 0$, so we can apply (4.22) and (4.23) to obtain a normalised left-invariant integral on $\mathbb{C}_q[S^2]$. From (4.19) and (4.20), we see that

$$\int a^n b^m c^p = \int d^n b^m c^p = 0$$

for $n \geq 1$, while (4.21) gives

$$\int b^p c^p = -\frac{[p]_{q^2}}{q^{-1}[p+1]_{q^2}} \int b^{p-1} c^{p-1}.$$

As $\int b^0 c^0 = 1$, induction shows that

$$\int (bc)^p = \frac{(-q)^p}{[p+1]_{q^2}}.$$

This is the restriction of the Haar integral given in Example 2.21 and conversely could be extended to a definition of the latter constructed via the cohomology. $\diamond$

The next example is about the differential calculus on the deformed unit disk $\mathbb{C}_q[D]$ in Example 3.40 but it cannot be described so straightforwardly. We consider an integral on the deformed unit disk $\mathbb{C}_q[D]$ that is characterised by invariance under a left action of $U_q(su_{1,1})$. The problem is that this is not always defined—classically we would not have a bounded volume. Instead, imposing the $U_q(su_{1,1})$ action will force us in the direction of hyperbolic space and the noncompact open unit disk, rather than the compact closed unit disk.

Example 4.37 There is a left action of $U_q(su_{1,1})$ on the deformed unit disk $\mathbb{C}_q[D]$ in Example 3.40 which is unitary in the Hopf $*$-algebra sense (2.1), given by

$$X_{\pm} \triangleright 1 = 0,\ q^{\frac{H}{2}} \triangleright 1 = 1, \quad q^{\frac{H}{2}} \triangleright z = q^{-1} z, \quad q^{\frac{H}{2}} \triangleright \bar{z} = q\, \bar{z},$$
$$X_+ \triangleright z = q^{-1/2}, \quad X_+ \triangleright \bar{z} = -q^{-1/2} \bar{z}^2, \quad X_- \triangleright \bar{z} = q^{1/2}, \quad X_- \triangleright z = -q^{1/2} z^2.$$

Recalling that $w = 1 - \bar{z}z$ and using the coproduct on $U_q(su_{1,1})$, we have

$$q^{1/2} X_+ \triangleright w = -q^{-1} \bar{z} w, \quad q^{-1/2} X_- \triangleright w = -q^{-1} w z, \quad q^{\frac{H}{2}} \triangleright w = w,$$

which extends to the calculus by

$$q^{\frac{H}{2}} \triangleright \mathrm{d}z = q^{-1} \mathrm{d}z, \quad q^{\frac{H}{2}} \triangleright \mathrm{d}\bar{z} = q \mathrm{d}\bar{z}, \quad X_+ \triangleright \mathrm{d}z = 0, \quad X_- \triangleright \mathrm{d}\bar{z} = 0,$$
$$X_+ \triangleright \mathrm{d}\bar{z} = -q^{-1/2} (\bar{z} \mathrm{d}\bar{z} + (\mathrm{d}\bar{z})\bar{z}), \quad X_- \triangleright z = -q^{1/2} (z \mathrm{d}z + (\mathrm{d}z) z).$$

A partially defined integral $\int : \mathbb{C}_q[D] \to \mathbb{C}$, invariant under the $U_q(su_{1,1})$ action, is given by

$$\int w^{n+1} = \frac{1}{[n]_{q^{-2}}}, \quad n \geq 1,$$

as well as zero on any monomial of nonzero grade. To see how this comes about from invariance, we first use induction to show for $n \geq 0$ that

$$q^{1/2} X_+ \triangleright w^n = -q^{-1} \bar{z} [n]_{q^{-2}} w^n$$

and from this that

$$q^{1/2} X_+ \triangleright (z w^n) = w^n - z\bar{z}[n]_{q^{-2}} w^n = -q^{-2}[n-1]_{q^{-2}} w^n + q^{-2}[n]_{q^{-2}} w^{n+1}.$$

For invariance, the integral applied to this should give zero since $\epsilon(X_+) = 0$, which tells us that

$$\int [n-1]_{q^{-2}} w^n = \int [n]_{q^{-2}} w^{n+1},$$

in agreement with the form stated. There is also a twisting automorphism $\varsigma(b) = q^{2|b|} b$ on homogeneous elements of $\mathbb{C}_q[D]$ and more calculation shows that the integral is a twisted trace, i.e. $\int ab = \int \varsigma(b) a$ where defined. On the other hand, this integral is not defined on all of $\mathbb{C}_q[D]$. We will not try precisely to describe the domain of the integral here, but it does not even include 1. This is to be expected since classically on the interior of the disk the integral is the invariant measure on hyperbolic space, which has infinite area. We might also ask if our integral is compatible with a suitable cohomology according to the relationship

$$[f \mathrm{d}\bar{z} \wedge \mathrm{d}z] = \int f w^2,$$

where [ ] denotes taking a suitable equivalence class. For example,

$$\mathrm{d}(\bar{z} w^m \mathrm{d}z) = q^{2m}\big([m+1]_{q^{-2}} w^m - [m]_{q^{-2}} w^{m-1}\big) \mathrm{d}\bar{z} \wedge \mathrm{d}z.$$

and this element for $m \geq 1$ is zero in our desired cohomology since the integral of the coefficient of $\mathrm{d}\bar{z} \wedge \mathrm{d}z$ is zero. But for $m < 1$, we do not allow such elements, as the integral would not even be defined. By the same arguments, we similarly exclude $\xi \in \Omega^1$ that do not have enough powers of w. We also have $[\mathrm{d}\bar{z} \wedge \mathrm{d}z] \neq 0$ since $\int w^2 = 1$, so our putative cohomology is nontrivial. This is indeed true classically if we consider only forms on the open disk which have compact support. This *compactly supported cohomology* is indeed nonzero in degree 2. In practice we could use forms sufficiently rapidly decreasing towards the boundary, which in the quantum case we see as ξ having enough powers of w. To make the quantum case precise, we would need of course to say rather more about the integral.

Geometrically, $U_q(su_{1,1})$ acts as the quantum analogue of $SL_2(\mathbb{R})$ on the disk by Möbius transformations, which are isometries of hyperbolic space on the interior. Classically in the hyperbolic case, the continuous functions on the open disk include $w^{-1} = (1-r^2)^{-1}$ for r the distance from the origin, so we now adjoin w^{-1}. In this case there is a quantum symmetric metric for $\mathbb{C}_q[D]$, namely

$$g = w^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z), \tag{4.24}$$

and if $q \to 1$ we recover the classical hyperbolic metric. The w^{-2} factor is needed for the metric to be central, so we might consider that the calculus itself imposes a hyperbolic metric. The metric is invariant under the $U_q(sl_2)$ action, for example

$$\begin{aligned}
&q^{1/2}X_+\triangleright(w^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z))\\
&\quad= (q^{1/2}X_+\triangleright(w^{-2}))(q^{\frac{H}{2}}\triangleright(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z))\\
&\qquad+\ (q^{-\frac{H}{2}}\triangleright(w^{-2}))(X_+\triangleright(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z))\\
&\quad= q^3\bar{z}[2]_{q^{-2}}w^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z)\\
&\qquad+\ w^{-2}(q^{-\frac{H}{2}}\triangleright\mathrm{d}z \otimes X_+\triangleright\mathrm{d}\bar{z} + q^{-2}X_+\triangleright\mathrm{d}\bar{z} \otimes q^{\frac{H}{2}}\triangleright\mathrm{d}z))\\
&\quad= q^3\bar{z}[2]_{q^{-2}}w^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z)\\
&\qquad-\ w^{-2}(1+q^{-2})(q\mathrm{d}z \otimes \bar{z}.\mathrm{d}\bar{z} + q^{-3}\bar{z}.\mathrm{d}\bar{z} \otimes \mathrm{d}z))\\
&\quad= q^3\bar{z}w^{-2}([2]_{q^{-2}} - 1 - q^{-2})(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z) = 0\,. \qquad \diamond
\end{aligned}$$

We finish this section with a higher-dimensional example.

Example 4.38 (Connes–Landi Sphere) We let $q = \exp(2\pi \mathrm{i}\theta)$ with θ real and start with the $*$-algebra $\mathbb{C}_\theta[\mathbb{R}^5]$ generated by x^1, x^2, x^3, x^4, x^5 with x^3 central and

$$\begin{aligned}
x^1x^2 = qx^2x^1,\ \ x^1x^4 = q^{-1}x^4x^1,\ \ x^1x^5 = x^5x^1,\\
x^2x^5 = qx^5x^2,\ \ x^4x^5 = q^{-1}x^5x^4,\ \ x^2x^4 = x^4x^2,
\end{aligned}$$

and $(x^i)^* = x^{i'}$ where $i' = 6 - i$. For convenience, we define q_{ij} from the above relations by reading them as $x^ix^j = q_{ij}x^jx^i$ and define $\Omega(\mathbb{C}_\theta[\mathbb{R}^5])$ by the relations

$$(\mathrm{d}x^i)x^j = q_{ij}x^j\mathrm{d}x^i, \quad \mathrm{d}x^i \wedge \mathrm{d}x^j = -q_{ij}\mathrm{d}x^j \wedge \mathrm{d}x^i$$

as a $*$-calculus. The 4-forms

$$\omega_1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge \mathrm{d}x^5\,, \quad \omega_2 = -q\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge \mathrm{d}x^5\,,$$
$$\omega_3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \wedge \mathrm{d}x^5\,, \quad \omega_4 = -q^{-1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^5\,,$$
$$\omega_5 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$$

have the property that $\omega_k \wedge \mathrm{d}x^i = \delta_{i,k}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge \mathrm{d}x^5$ in the differential structure on $\mathbb{C}_\theta[\mathbb{R}^5]$. There is also a torus $S^1 \times S^1$ action on $\mathbb{C}_\theta[\mathbb{R}^5]$ by

$$(u, v){\triangleright}(x^1, x^2, x^3, x^4, x^5) = (ux^1, vx^2, x^3, v^*x^4, u^*x^5)$$

and a ‘radius squared’ central element

$$c = 2x^1x^5 + 2x^2x^4 + x^3x^3.$$

The Connes–Landi sphere $\mathbb{C}_\theta[S^4]$ is the quotient $\mathbb{C}_\theta[\mathbb{R}^5]$ by the relation $c = 1$, while $\mathrm{d}c = 0$ gives us $\Omega(\mathbb{C}_\theta[S^4])$. We define the volume form on the sphere as

$$\mathrm{Vol} = \sum_i x^i \omega_i.$$

Any 4-form can be written as an algebra element times Vol, which can most easily be seen by checking that $\omega_i = x^{i'}\mathrm{Vol}$. The above torus action restricts to the sphere, and if we rephrase it as a coaction of $\mathbb{C}[S^1 \times S^1]$ with its classical calculus, one can check that it is differentiable and that Proposition 4.25 applies. As a consequence, it is enough to calculate the cohomology on forms which are invariant under the torus action. Using this and the relation $c = 1$ to reduce products of the form x^1x^5 gives a dramatic reduction in the number of cases to follow. For example, the invariant elements of $\mathbb{C}_\theta[S^4]$ have a linear basis $(x^3)^t(x_2x_4)^p$ for integers $t, p \geq 0$.

We now determine a unique integration compatible with the cohomology assuming that $H_{\mathrm{dR}}(\mathbb{C}_\theta[S^4])$ is indeed 1-dimensional (as one can show). Thus, we calculate d on a linear basis of invariant 3-forms, e.g.,

$$\mathrm{d}\big((x^3)^t(x^2x^4)^p x^4x^5\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\big) = (x^3)^t\big((x^2x^4)^{p+1} - (p+1)(x^2x^4)^p x^5x^1\big)\mathrm{Vol},$$
$$\mathrm{d}\big((x^3)^t(x^2x^4)^p x^5\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4\big) = \big(t(x^3)^{t-1}(x^2x^4)^p x^5x^1 - (x^3)^{t+1}(x^2x^4)^p\big)\mathrm{Vol}.$$

Setting the integral of these to be zero and substituting for x^1x^5 gives recurrence relations for the integral, which we solve to find

$$\int (x^3)^t(x_2x_4)^p = \begin{cases} 0 & t \text{ odd}, \\ \frac{3p!\,(t-1)!!}{(t+2p+3)!!} & t \text{ even}, \end{cases}$$

where we have used the double factorial defined by $3!! = 3 \times 1$, $5!! = 5 \times 3 \times 1$ etc., with the convention $(-1)!! = 1$. The integral is zero on noninvariant monomials. Moving one x^i at a time from front to the back of the integrated monomial and using the fact that an invariant monomial must be central shows that the integral is a trace, i.e., $\int ab = \int ba$ for all $a, b \in \mathbb{C}_\theta[S^4]$. ◇

4.3 Sheaf Cohomology

In this section we generalise the quantum de Rham cohomology of an algebra A with differential calculus to a de Rham cohomology 'with coefficients'. We do this using the category ${}_A\mathcal{E}$ defined in §3.2, with objects consisting of a left A-module E and a left connection $\nabla_E : E \to \Omega^1 \otimes_A E$. Define a graded sequence of left A-modules $\Omega^n \otimes_A E$ with differential as in equation (4.2),

$$\nabla_E{}^{[n]} : \Omega^n \otimes_A E \to \Omega^{n+1} \otimes_A E, \qquad \xi \otimes e \mapsto \mathrm{d}\xi \otimes e + (-1)^n \xi \wedge \nabla_E e.$$

From the proof of Theorem 4.3 we note that if the curvature R_E were to vanish then every composition $\nabla_E{}^{[n+1]} \circ \nabla_E{}^{[n]}$ would be zero, i.e., we would have a cochain complex $(\Omega \otimes_A E, \nabla_E)$. As in the classical case, we call a connection ∇_E *flat* if its curvature is zero. It will be convenient to define a category ${}_A\mathcal{F}$ to express this:

Name	Objects	Morphisms
${}_A\mathcal{F}$	(E, ∇_E) left modules and left connections with zero curvature	Left module maps intertwining ∇_E

Definition 4.39 For $(E, \nabla_E) \in {}_A\mathcal{F}$, define $\mathrm{H}(A, E, \nabla_E)$ to be the cohomology of the complex

$$E \xrightarrow{\nabla_E} \Omega^1 \otimes_A E \xrightarrow{\nabla_E{}^{[1]}} \Omega^2 \otimes_A E \xrightarrow{\nabla_E{}^{[2]}} \ldots.$$

Note that $\mathrm{H}^0(A, E, \nabla_E) = \{e \in E : \nabla_E e = 0\}$, the flat sections of E.

We may abbreviate the notation to $\mathrm{H}(A, E)$ when ∇_E is understood as part of the data attached to E. We propose to view this construction as a noncommutative replacement for sheaf cohomology. As its purpose is to add coefficients to the de Rham cohomology, it is not surprising that the simplest example of an object in ${}_A\mathcal{F}$ is (A, d) and that $\mathrm{H}(A, A, \mathrm{d}) = \mathrm{H}_{\mathrm{dR}}(A)$, the usual de Rham cohomology. Another general example is the flat connection induced on any module by an inner calculus.

Example 4.40 Let A be an algebra with inner differential calculus Ω for an element $\theta \in \Omega^1$, and suppose that $\theta \wedge \theta = 0$. For any left A-module F we can extend the left connection ${}_\theta\nabla f = \theta \otimes f$ in Example 3.22 to ${}_\theta\nabla^{[n]} : \Omega^n \otimes_A F \to \Omega^{n+1} \otimes_A F$

by ${}_\theta\nabla^{[n]}(\xi \otimes f) = \theta \wedge \xi \otimes f$. Using $\theta \wedge \theta = 0$, we obtain a flat connection, with cohomology

$$\mathrm{H}^n(A, F, {}_\theta\nabla) = \frac{\ker(\theta\wedge : \Omega^n \otimes_A F \to \Omega^{n+1} \otimes_A F)}{\mathrm{image}(\theta\wedge : \Omega^{n-1} \otimes_A F \to \Omega^n \otimes_A F)}.$$

If F is flat as a left A-module, then

$$\mathrm{H}^n(A, F, {}_\theta\nabla) = \frac{\ker(\theta\wedge : \Omega^n \to \Omega^{n+1})}{\mathrm{image}(\theta\wedge : \Omega^{n-1} \to \Omega^n)} \otimes_A F = \mathrm{H}^n(A, {}_\theta\nabla) \otimes_A F,$$

where $\mathrm{H}^n(A, {}_\theta\nabla)$ is the complex Ω with $\theta\wedge$ and can be identified with $\mathrm{H}^n(A, A, {}_\theta\nabla)$.

For a concrete example we take $A = \mathbb{C}(S_3)$ and $\Omega(S_3)$ from Example 1.60 with left-invariant basic 1-forms e_u, e_v, e_w and $\theta = e_u + e_v + e_w$. Then $\theta\wedge : A \to \Omega^1$ is injective, so $\mathrm{H}^0(A, {}_\theta\nabla) = 0$. We also have

$$\begin{aligned}&\theta \wedge (e_u a + e_v b + e_w c)\\ &\qquad = e_u \wedge e_v(b - a) + e_v \wedge e_u(a - c) + e_v \wedge e_w(c - a) + e_w \wedge e_v(b - c)\end{aligned}$$

for any $a, b, c \in \mathbb{C}(S_3)$, which vanishes when $a = b = c$, so $\mathrm{H}^1(A, {}_\theta\nabla) = 0$. Next

$$\begin{aligned}&\theta \wedge (e_u \wedge e_v a + e_v \wedge e_u b + e_v \wedge e_w c + e_w \wedge e_v d)\\ &\quad = e_u \wedge e_v \wedge e_w(c + b) + e_v \wedge e_w \wedge e_u(d - b - a) + e_w \wedge e_u \wedge e_v(a - d - c),\end{aligned}$$

similarly giving $\mathrm{H}^2(A, {}_\theta\nabla) = 0$. Finally,

$$\begin{aligned}&\theta \wedge (e_u \wedge e_v \wedge e_w a + e_v \wedge e_w \wedge e_u b + e_w \wedge e_u \wedge e_v c)\\ &\qquad\qquad\qquad = e_u \wedge e_v \wedge e_u \wedge e_w(a + b + c),\end{aligned}$$

where we have used $e_u \wedge e_v \wedge e_w \wedge e_u = -e_u \wedge e_u \wedge e_v \wedge e_u - e_u \wedge e_w \wedge e_u \wedge e_u = 0$. This gives $\mathrm{H}^3(A, {}_\theta\nabla) = \mathrm{H}^4(A, {}_\theta\nabla) = 0$, so all the sheaf cohomology vanishes. This should be contrasted with the de Rham cohomology in Example 1.60. ◇

The Möbius line module L with zero curvature connection given in Example 4.22 has $\mathrm{H}^0(\mathbb{C}_{q^2}[S^1], L) = 0$ since the only elements of the bigger algebra $\mathbb{C}_q[S^1]$ in the kernel of d are in $\mathbb{C}.1$. Similarly $\mathrm{H}^0(\mathbb{C}_q[SO_3], L) = 0$ for the line module L in Example 4.23. Here is a nonvanishing example.

Example 4.41 Let $A = \mathbb{C}(S_3)$ with $\Omega(S_3)$ from Example 1.60 and $\theta = e_u + e_v + e_w$, the same as in Example 4.40. We take the left module Ω^1 with the 5-parameter family of connections ∇ in Example 3.76 and more specifically the flat connections identified there as cases (i)–(iv). The condition for the section $f = \sum_{g,a} f_{g,a}\delta_g e_a$

(where $f_{g,a} \in \mathbb{C}$) to obey $\nabla f = 0$ can be written as

$$\begin{pmatrix} f_{gx,x} \\ f_{gx,y} \\ f_{gx,z} \end{pmatrix} = \begin{pmatrix} a & b & b \\ e & d & c \\ e & c & d \end{pmatrix} \begin{pmatrix} f_{g,x} \\ f_{g,y} \\ f_{g,z} \end{pmatrix}$$

for all $g \in S_3$ and x, y, z any permutation of u, v, w. To illustrate a computation, we choose case (iv) where $a = c$ and $e = d = b$. In this case the above matrix has eigenvalues $a - b$ with eigenspace spanned by $(-2, 1, 1)^T$, $b - a$ with $(0, -1, 1)^T$, and $a + 2b$ with $(1, 1, 1)^T$. The space of flat sections can then be computed and depends on a, b. For example,

$$\begin{aligned} a = b = 0 : &\qquad \mathrm{H}^0(A, \Omega^1, \nabla) = 0, \\ a = b = \tfrac{1}{3} : &\qquad \mathrm{H}^0(A, \Omega^1, \nabla) = \mathbb{C}. \end{aligned} \qquad \diamond$$

In algebraic topology it is useful to have maps changing coefficients for cohomology on the same topological space. This can also be done with our generalised de Rham cohomology. Also these cohomologies are modules over the usual de Rham cohomology, which we noted is itself an algebra in Definition 1.30. We combine these ideas into a functor:

Proposition 4.42 *The cohomology* H *in Definition 4.39 is a functor from* ${}_A\mathcal{F}$ *to graded left* $\mathrm{H}_{\mathrm{dR}}(A)$*-modules. Here* $\mathrm{H}(A, E, \nabla_E)$ *is a left* $\mathrm{H}_{\mathrm{dR}}(A)$*-module by*

$$\wedge \otimes \mathrm{id} : \Omega^n \otimes \Omega^r \otimes_A E \to \Omega^{n+r} \otimes_A E.$$

Proof We write ${\nabla_E}^{[*]}$ for the appropriate ${\nabla_E}^{[n]}$. Then

$$\begin{aligned} {\nabla_E}^{[*]}(\xi \wedge \omega \otimes e) &= \mathrm{d}(\xi \wedge \omega) \otimes e + (-1)^{|\xi|+|\omega|} \xi \wedge \omega \wedge \nabla_E e \\ &= \mathrm{d}\xi \wedge (\omega \otimes e) + (-1)^{|\xi|} \xi \wedge {\nabla_E}^{[*]}(\omega \otimes e) \end{aligned}$$

tells us that we have a cochain map from the tensor product DGA $\Omega \otimes (\Omega \otimes_A E) \to \Omega \otimes_A E$, and thus a map of cohomologies. To expand on this point, if $\mathrm{d}\xi = 0$ and ${\nabla_E}^{[*]}(\omega \otimes e) = 0$ then ${\nabla_E}^{[*]}(\xi \wedge \omega \otimes e) = 0$. If ${\nabla_E}^{[*]}(\omega \otimes e) = 0$ then $\mathrm{d}\xi \wedge (\omega \otimes e)$ is in the image of ${\nabla_E}^{[*]}$. If $\mathrm{d}\xi = 0$ then $\xi \wedge {\nabla_E}^{[*]}(\omega \otimes e)$ is in the image of ${\nabla_E}^{[*]}$. Next suppose that $\phi : (E, \nabla_E) \to (F, \nabla_F)$ is a morphism in ${}_A\mathcal{F}$, i.e., it is a left module map such that $\nabla_F \circ \phi = (\mathrm{id} \otimes \phi) \circ \nabla_E : E \to \Omega^1 \otimes_A F$. Then $\mathrm{id} \otimes \phi : \Omega \otimes_A E \to \Omega \otimes_A F$ is a cochain map,

$$\begin{aligned} {\nabla_F}^{[*]}(\mathrm{id} \otimes \phi)(\eta \otimes e) &= {\nabla_F}^{[*]}(\eta \otimes \phi(e)) = \mathrm{d}\eta \otimes \phi(e) + (-1)^{|\eta|} \eta \wedge \nabla_F \phi(e) \\ &= \mathrm{d}\eta \otimes \phi(e) + (-1)^{|\eta|} \eta \wedge (\mathrm{id} \otimes \phi) \nabla_E e = (\mathrm{id} \otimes \phi) {\nabla_E}^{[*]}(\eta \otimes e). \end{aligned}$$

That $\mathrm{id} \otimes \phi$ commutes with the left action of Ω is immediate. $\square$

To further motivate our view of the above as some kind of noncommutative sheaf cohomology, we now show that indeed several results hold which are analogous with results in classical sheaf theory. One basic classical result is that a short exact sequence of coefficients gives rise to a long exact sequence of cohomology. To generalise this, we shall assume the technical condition that each Ω^n is *flat* as a right A-module. This simply means that applying $\Omega^n \otimes_A -$ to an exact sequence of left A-modules gives another exact sequence—see §3.6 for details.

Theorem 4.43 *If every Ω^n is flat as a right A-module then given a short exact sequence in ${}_A\mathcal{F}$*

$$0 \longrightarrow (E, \nabla_E) \xrightarrow{\phi} (F, \nabla_F) \xrightarrow{\psi} (G, \nabla_G) \longrightarrow 0,$$

there is a cohomology long exact sequence

$$\mathrm{H}^0(A, E) \to \mathrm{H}^0(A, F) \to \mathrm{H}^0(A, G) \to \mathrm{H}^1(A, E)$$
$$\to \mathrm{H}^1(A, F) \to \mathrm{H}^1(A, G) \to \mathrm{H}^2(A, E) \to \dots$$

Proof We form a diagram where the rows are exact (as every Ω^n is flat), and the columns form cochain complexes (i.e., the vertical maps compose to give zero):

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\phi"] \arrow[d, "\nabla_E"] & F \arrow[r, "\psi"] \arrow[d, "\nabla_F"] & G \arrow[r] \arrow[d, "\nabla_G"] & 0 \\
0 \arrow[r] & \Omega^1 \otimes_A E \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d, "\nabla_E{}^{[1]}"] & \Omega^1 \otimes_A F \arrow[r, "\mathrm{id} \otimes \psi"] \arrow[d, "\nabla_F{}^{[1]}"] & \Omega^1 \otimes_A G \arrow[r] \arrow[d, "\nabla_G{}^{[1]}"] & 0 \\
0 \arrow[r] & \Omega^2 \otimes_A E \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d, "\nabla_E{}^{[2]}"] & \Omega^2 \otimes_A F \arrow[r, "\mathrm{id} \otimes \psi"] \arrow[d, "\nabla_F{}^{[2]}"] & \Omega^2 \otimes_A G \arrow[r] \arrow[d, "\nabla_G{}^{[2]}"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

and we then use standard 'diagram chasing'. The maps between homology of the same degree are induced by either $\mathrm{id} \otimes \phi$ or $\mathrm{id} \otimes \psi$. The connecting maps $\mathrm{H}^n(A, G) \to \mathrm{H}^{n+1}(A, E)$ are given by $[(\mathrm{id} \otimes \phi^{-1})\nabla_F^{[n]}(\mathrm{id} \otimes \psi^{-1})(\xi \otimes g)]$, where $[\ ]$ denotes cohomology class, and $\xi \otimes g$ denotes an element (summation suppressed) of $\ker \nabla_G{}^{[n]}$. Since ψ is not injective, $\psi^{-1}(g)$ is not well defined but we can take any representative and the result in cohomology can be shown to be well defined. These are standard arguments from homological algebra and we omit the details. □

Now we give an example of a calculation by means of this long exact sequence for sheaf theory. We continue the story of Example 4.41, and on the way also illustrate connections on submodules and quotients.

Example 4.44 We continue with Example 4.41 with the moduli of flat connections ∇ on the 3D calculus $\Omega^1(S_3)$ on $A = \mathbb{C}(S_3)$. The connection ∇ there restricts to a submodule $E \subseteq \Omega^1$ if $\nabla(E) \subseteq \Omega^1 \otimes_A E$. We look at E generated by $e_u - e_v$ and $e_u - e_w$ (i.e., the coefficients restricted to $f_{g,u} + f_{g,v} + f_{g,w} = 0$ for all $g \in S_3$), to which the connection restricts precisely when $a - b - c - d + 2e = 0$. As before, we use case (ii) of the flat connections in Example 3.76 to illustrate the method, so the condition for restriction becomes $b + 2c = 0$ and $\{a, b, c, d, e\} = \{-2c, -2c, c, c, c\}$ is now a 1-parameter family of flat connections on E. This has

$$\nabla(e_u - e_v) = \theta \otimes (e_u - e_v), \quad \nabla(e_u - e_w) = \theta \otimes (e_u - e_w),$$

independently of c. By Theorem 3.114, there is a connection on the quotient module Ω^1/E with basis $[\theta]$ in the quotient, namely $\nabla[\theta] = \theta \otimes [\theta]$.

Now consider the short exact sequence of left A-modules with connection:

$$0 \longrightarrow E \xrightarrow{\text{inclusion}} \Omega^1 \longrightarrow \Omega^1/E \longrightarrow 0.$$

As left A-modules, this exact sequence splits and we have $\Omega^1 = E \oplus F$, where F is the submodule generated by θ. However,

$$\nabla\theta = (1 - 3c)\theta \otimes \theta + 9c(e_u \otimes e_u + e_v \otimes e_v + e_w \otimes e_w)$$

so that ∇ does not restrict to F. Hence, for $c \neq 0$, we cannot consider (Ω^1, ∇) as a direct sum of the two modules E and F with connection.

By Theorem 4.43, we have a long exact sequence in cohomology

$$\mathrm{H}^0(\mathbb{C}(S_3), E) \longrightarrow \mathrm{H}^0(\mathbb{C}(S_3), \Omega^1) \longrightarrow \mathrm{H}^0(\mathbb{C}(S_3), \Omega^1/E) \longrightarrow \mathrm{H}^1(\mathbb{C}(S_3), E)$$
$$\longrightarrow \mathrm{H}^1(\mathbb{C}(S_3), \Omega^1) \longrightarrow \mathrm{H}^1(\mathbb{C}(S_3), \Omega^1/E) \longrightarrow \dots$$

and the formula for ∇ on Ω^1/E shows that it is isomorphic to the sheaf in Example 4.40, i.e., all its cohomology vanishes. The formula for ∇ on E shows that it is isomorphic to the direct sum of two copies of the sheaf in Example 4.40, so all its cohomology vanishes also. We are left with the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathbb{C}(S_3), \Omega^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{H}^1(\mathbb{C}(S_3), \Omega^1) \longrightarrow 0 \longrightarrow \dots$$

so $\mathrm{H}^n(\mathbb{C}(S_3), \Omega^1) = 0$ for all n for our 1-parameter family of flat connections. ◇

Here is another example, this time of a long exact sequence with nonzero entries.

Example 4.45 Let A be an algebra with exterior algebra Ω and Ω^1 flat as a right A-module. Let E, F be left modules E, F with flat left connections and suppose that F is free with a single generator f^0 and connection given by $\nabla_F f^0 = 0$ (then its cohomology is just the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(A)$). Suppose that E is free with two generators e^1 and e^2 and has connection given by $\nabla_E e^1 = 0$ and $\nabla_E e^2 = \xi \otimes e^1$ for some closed $\xi \in \Omega^1$. The curvature of E is

$$R_E e^1 = 0, \quad R_E e^2 = \mathrm{d}\xi \otimes e^1 - \xi \wedge \nabla_E e^1 = \mathrm{d}\xi \otimes e^1 = 0$$

so this is also flat. Then we have a short exact sequence of sheaves

$$0 \longrightarrow F \xrightarrow{i} E \xrightarrow{\pi} F \longrightarrow 0$$

where $i(f^0) = e^1$, $\pi(e^1) = 0$ and $\pi(e^2) = f^0$. This gives a long exact cohomology sequence

$$\mathrm{H}^0_{\mathrm{dR}} \to \mathrm{H}^0(A, E) \to \mathrm{H}^0_{\mathrm{dR}} \to \mathrm{H}^1_{\mathrm{dR}} \to \mathrm{H}^1(A, E) \to \mathrm{H}^1_{\mathrm{dR}} \to \mathrm{H}^2_{\mathrm{dR}} \to \mathrm{H}^2(A, E) \to \dots$$

Now we specialise to the case where $\mathrm{H}^0_{\mathrm{dR}} = \mathbb{C}$ (i.e., A is connected), $\mathrm{H}^1_{\mathrm{dR}} \neq 0$ and $[\xi] \neq 0$. We calculate $\mathrm{H}^0(A, E)$ by starting with $ae^1 + be^2 \in \ker \mathrm{d}$, and then

$$(\mathrm{d}a + b\xi) \otimes e^1 + \mathrm{d}b \otimes e^2 = 0.$$

By connectedness, we have $b = \lambda \in \mathbb{C}$ and so $\mathrm{d}a = -\lambda\xi$, which is impossible as $[\xi] \neq 0$ unless $\lambda = 0$. The result is $a \in \mathbb{C}$ and $b = 0$, so $\mathrm{H}^0(A, E) = \mathbb{C}$.

For a concrete example, we may take A to be the commutative torus algebra, or equally the noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$ with irrational rotation in Example 1.36. Either way, the Betti numbers are $1, 2, 1, 0, \dots$ and the long exact sequence becomes

$$\mathbb{C} \xrightarrow{\cong} \mathbb{C} \xrightarrow{0} \mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathrm{H}^1(A, E) \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C} \longrightarrow \mathrm{H}^2(A, E) \longrightarrow \mathbb{C} \longrightarrow 0 \dots$$

We deduce that both $\mathrm{H}^1(A, E)$ and $\mathrm{H}^2(A, E)$ are nonzero. ◇

As well as maps changing coefficients over the same algebra, we also need to consider maps between different algebras, which for the setting above we take to be differentiable. Recall from Definition 1.30 that an algebra map $\varphi : A \to B$ is differentiable if it extends to a map $\varphi : \Omega_A \to \Omega_B$ of differential graded algebras, where in the more sophisticated point of view in this section, *we use the same symbol for the extension* (reserving φ_* for the direct image below). For example, the Hopf $*$-algebra map $\pi : \mathbb{C}_q[SU_2] \to \mathbb{C}_{q^2}[S^1]$ in Proposition 2.33 is differentiable while the comments after Proposition 1.52 give the conditions for a group homomorphism to induce a differentiable map between the algebras of functions on finite groups. Next recall that in topology, continuous functions between topological spaces induce maps on the cohomology by the 'pull back' construction, but using algebras reverses the arrows, so that it looks like a 'push forward'. We shall retain something of the usual sheaf notation by referring to the following construction as the *inverse image* sheaf. Having the arrows the other way gives the *direct image* sheaf, and this construction will be discussed after Lemma 4.65.

Proposition 4.46 (Inverse Image Functor) *A differentiable algebra map $\varphi : A \to B$ induces a functor $\varphi_* : {}_A\mathcal{E} \to {}_B\mathcal{E}$ defined on objects by $\varphi_*(E, \nabla_E) = (B \otimes_A E, \hat{\nabla}) \in {}_B\mathcal{E}$, where the right action of A on B is $b{\triangleleft}a = b\varphi(a)$ and*

$$\hat{\nabla} : B \otimes_A E \to \Omega^1_B \otimes_B B \otimes_A E = \Omega^1_B \otimes_A E, \quad \hat{\nabla}(b \otimes e) = b(\varphi \otimes \mathrm{id})(\nabla_E e) + \mathrm{d}b \otimes e.$$

The functor sends a morphism $\phi : E \to F$ to the morphism $\mathrm{id} \otimes \phi : B \otimes_A E \to B \otimes_A F$ *and restricts to a functor ${}_A\mathcal{F} \to {}_B\mathcal{F}$.*

Proof To show that $\hat{\nabla}$ is well defined on $B \otimes_A E$, we calculate

$$\begin{aligned}\hat{\nabla}(b\varphi(a) \otimes e) &= b\varphi(a)(\varphi \otimes \mathrm{id})(\nabla_E e) + \mathrm{d}(b\varphi(a)) \otimes e\\ &= b(\varphi \otimes \mathrm{id})(a\nabla_E e + \mathrm{d}a \otimes e) + \mathrm{d}b \otimes ae\\ &= b(\varphi \otimes \mathrm{id})\nabla_E(ae) + \mathrm{d}b \otimes ae = \hat{\nabla}(b \otimes ae)\end{aligned}$$

for all $a \in A$, $b \in B$ and $e \in E$. The Leibniz rule for $\hat{\nabla}$ follows from the definition. Now suppose that ∇_E is flat and set $\nabla_E e = \xi \otimes f$ (summation implicit). Then

$$\begin{aligned}\hat{\nabla}^{[1]}\hat{\nabla}(b \otimes e) &= \hat{\nabla}^{[1]}(b(\varphi \otimes \mathrm{id})(\nabla_E e) + \mathrm{d}b \otimes e)\\ &= \hat{\nabla}^{[1]}(b\xi \otimes f + \mathrm{d}b \otimes e) = \mathrm{d}(b.\xi) \otimes f + \mathrm{dd}b \otimes e - b\xi \wedge \nabla_E f - \mathrm{d}b \wedge \nabla_E e\\ &= \mathrm{d}b \wedge \xi \otimes f + b.\mathrm{d}\xi \otimes f - b\xi \wedge \nabla_E f - \mathrm{d}b \wedge \nabla_E e\\ &= b(\mathrm{d}\xi \otimes f - \xi \wedge \nabla_E f) = b{\nabla_E}^{[1]}\nabla_E e = 0.\end{aligned}$$

For a morphism $\phi : (E, \nabla_E) \to (F, \nabla_F)$ in ${}_A\mathcal{E}$, we have $\mathrm{id}\otimes\phi : B \otimes_A E \to B \otimes_A F$ is a morphism in ${}_B\mathcal{E}$ by

$$\begin{aligned}\hat{\nabla}(b \otimes \phi(e)) &= b(\varphi \otimes \mathrm{id})\nabla_F \phi(e) + \mathrm{d}b \otimes \phi(e)\\ &= b(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\nabla_E e + \mathrm{d}b \otimes \phi(e)\\ &= (\mathrm{id} \otimes \phi)(b(\varphi \otimes \mathrm{id})\nabla_E e + \mathrm{d}b \otimes e) = (\mathrm{id} \otimes \phi)\hat{\nabla}(b \otimes e).\end{aligned}$$

Composition is just $(\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \psi) = \mathrm{id} \otimes \phi \circ \psi$. □

For clarity, we may denote the B-A bimodule B in the theorem by B_φ to remind the reader of the right action $b{\triangleleft}a = b\varphi(a)$.

Example 4.47 The Möbius bundle Möb_N in Example 3.86 came with a flat connection $\nabla_{\text{Möb}_N}$ over the algebra $\mathbb{C}(\mathbb{Z}_N)$ and it is therefore an example of a sheaf in our sense. We also saw that $\nabla_{\text{Möb}_N} s = 0$ has no solutions for $s \in \text{Möb}_N$, i.e., $\mathrm{H}^0(\text{Möb}_N, \nabla_{\text{Möb}_N}) = 0$. Applying the same construction with N replaced by $2N$, we also have a module Möb_{2N} with connection $\nabla_{\text{Möb}_{2N}}$ over $\mathbb{C}(\mathbb{Z}_{2N})$. Both the algebras $\mathbb{C}(\mathbb{Z}_N)$ and $\mathbb{C}(\mathbb{Z}_{2N})$ are given the 2D differential calculus defined by the subset $\mathcal{C} = \{1, -1\}$. To illustrate Proposition 4.46, the double cover map

$\varphi : \mathbb{Z}_{2N} \to \mathbb{Z}_N$ given by $\varphi(n) = n \bmod N$ induces a differentiable algebra map also denoted $\varphi : \mathbb{C}(\mathbb{Z}_N) \to \mathbb{C}(\mathbb{Z}_{2N})$ (by the discussion after Proposition 1.24).

The inverse image functor here gives a $\mathbb{C}(\mathbb{Z}_{2N})$ module $\varphi_*\text{Möb}_N$ defined by

$$\varphi_*\text{Möb}_N = \mathbb{C}(\mathbb{Z}_{2N})_\varphi \otimes_{\mathbb{C}(\mathbb{Z}_N)} \text{Möb}_N = \mathbb{C}(\mathbb{Z}_{2N})_\varphi \otimes_{\mathbb{C}(\mathbb{Z}_N)} \mathbb{C}(\mathbb{Z}_N) = \mathbb{C}(\mathbb{Z}_{2N})$$

as a left $\mathbb{C}(\mathbb{Z}_{2N})$-module. The connection has $\nabla_{\text{Möb}_N}(1) = (\lambda_1 e_1 + \lambda_{-1}e_{-1}) \otimes 1$ where $\lambda_{\pm 1} = 1 - e^{\pm i\pi/N}$, so on the inverse image module we have

$$\varphi_*(\nabla_{\text{Möb}_N})(1 \otimes 1) = \mathrm{d}1 \otimes 1 + \varphi(\lambda_1 e_1 + \lambda_{-1}e_{-1}) \otimes 1 = (\lambda_1 e_1 + \lambda_{-1}e_{-1}) \otimes 1.$$

For a flat section $s = \sum_{x \in \mathbb{Z}_{2N}} s_x \delta_x$ we require $s_{x+1} = e^{i\pi/N} s_x$ with solution proportional to $s_x = e^{i\pi x/N}$ for $x \in \mathbb{Z}_{2N}$, hence $\mathrm{H}^0(\mathbb{C}(\mathbb{Z}_{2N}), \varphi_*\text{Möb}_N) \cong \mathbb{C}$. We return to the discrete Möbius bundle in Example 5.49. $\diamond$

As an application, we can use the inverse image functor to induce modules with connection over group algebras from the same data on a subgroup algebra.

Example 4.48 Let $G \supset \mathbb{Z}_m$ be a discrete group with a cyclic subgroup, F a 1-dimensional $\mathbb{Z}_m$-module with basis $f \in F$ and $\Bbbk G \otimes_{\Bbbk Z_m} F$ the induced G-module. We extend this construction to sheaves in our sense of modules equipped with flat connections. We take for $\Bbbk\mathbb{Z}_m = \Bbbk[z]/\langle z^m - 1\rangle$ its standard 1-dimensional calculus with basic 1-form $\xi = z^{-1}\mathrm{d}z$ and commutation relations $\xi z = qz\xi$, where we assume that there exists $q \in \Bbbk$ obeying $q^m = 1$ with $q \neq 1$ (as in Example 1.11 and Exercise E2.5). The canonical exterior algebra is $\Omega^2(\Bbbk\mathbb{Z}_m) = 0$. F as a $\Bbbk\mathbb{Z}_m$-module is specified by $z \triangleright f = \omega f$ with $\omega^m = 1$ and a connection on it by $\nabla f = \gamma\xi \otimes f$ for some $\gamma \in \Bbbk$. (We do not have general $\gamma \in \Bbbk\mathbb{Z}_m$ as group elements can be pushed to the right and just give a multiple in $\Bbbk$.) The left Leibniz rule requires

$$\omega\nabla f = \nabla(z \triangleright f) = \mathrm{d}z \otimes f + \gamma z\xi \otimes f = (\mathrm{d}z)z^{-1} \otimes z \triangleright f + \gamma z\xi z^{-1} \otimes z \triangleright f$$

so that $\gamma = 1/(q-1)$. Hence F has a unique (flat) connection $\nabla f = (q-1)^{-1}\xi \otimes f$.

Next we fix $\Omega^1(\Bbbk G)$ by a right G-module Λ^1_G and a cocycle $\zeta \in Z^1(G, \Lambda^1_G)$ as in Theorem 1.47. That the inclusion $i : \Bbbk\mathbb{Z}_m \hookrightarrow \Bbbk G$ is differentiable amounts to

$$(\zeta(i(z))) \triangleleft i(z) = q\zeta(i(z))$$

on requiring that $i_* : \Lambda_{\Bbbk\mathbb{Z}_m} \to \Lambda_G$ is a right module map. In this case Proposition 4.46 implies a flat left $\Bbbk G$-connection on $\Bbbk G \otimes_{\Bbbk\mathbb{Z}_m} F$,

$$\hat{\nabla}(g \otimes f) = \Big(\mathrm{d}g + \frac{g}{q-1}\zeta(i(z))\Big) \otimes f \ \in \ \Omega^1(\Bbbk G) \otimes_{\Bbbk\mathbb{Z}_m} F.$$

For a concrete example, we take $\mathbb{Z}_2 \subset S_3$ with its usual generators and relations $u^2 = v^2 = e$, $uvu = vuv$, and set $i(z) = u$. For $\Bbbk\mathbb{Z}_2$ we have only the universal calculus $q = -1$, and we take any calculus on $\Bbbk S_3$ (see Examples 1.48 and 1.50).

We have two $\mathbb{Z}_2$-modules $F_\pm$ for $\omega = \pm 1$ with generators $f_\pm$ respectively. We can write a basis of the inverse image module as $(uv)^j \otimes f_\pm$ for $j \in \{0, 1, 2\}$ with action

$$uv{\triangleright}((uv)^j \otimes f_\pm) = (uv)^{j+1} \otimes f_\pm, \quad u{\triangleright}((uv)^j \otimes f_\pm) = \pm(uv)^{2j} \otimes f_\pm$$

and we obtain a left connection on each 3-dimensional $\Bbbk S_3$-module,

$$\hat\nabla((uv)^j \otimes f_\pm) = \Big(\mathrm{d}((uv)^j) - \frac{1}{2}(uv)^j\zeta(u)\Big) \otimes f_\pm \in \Omega^1(\Bbbk S_3) \otimes_{\Bbbk\mathbb{Z}_2} F_\pm.$$

Here $u^2 = e$ implies that $\zeta(u){\triangleleft}u = -\zeta(u)$, so i here is always differentiable. $\diamond$

Finally, we mention another important idea in cohomology, the cup product, which we return to briefly when covering Serre duality in §7.4.1. The cup product is a generalisation of the H_{dR}-module structure of Proposition 4.42. To state the formula for the cup product, we use extendable left bimodule connections.

Proposition 4.49 *Let (E, ∇_E, σ_E) be an extendable left bimodule connection with zero curvature and (F, ∇_F) a left module connection with zero curvature. Then*

$$\mathrm{id} \wedge \sigma_E \otimes \mathrm{id} : \Omega^m \otimes_A E \otimes \Omega^n \otimes_A F \to \Omega^{n+m} \otimes_A E \otimes_A F$$

induces a map $\cup : \mathrm{H}^m(A, E, \nabla_E) \otimes \mathrm{H}^n(A, F, \nabla_F) \to \mathrm{H}^{n+m}(A, E \otimes_A F, \nabla_{E\otimes F})$.

Proof We use the first equation displayed in Lemma 4.12 to show that

$$\nabla_{E\otimes F}{}^{[n+m]}(\mathrm{id} \wedge \sigma_E \otimes \mathrm{id}) = (\mathrm{id} \wedge \sigma_E \otimes \mathrm{id})(\nabla_E{}^{[m]} \otimes \mathrm{id} + (-1)^m \mathrm{id} \otimes \nabla_F{}^{[n]}),$$

and the result then follows from routine algebra. We remark that extendability ensures that $\cup$ is associative in the obvious sense if we consider $G \otimes_A E \otimes_A F$. $\square$

Classically, the cup product is often used in conjunction with a product of sheaves, which in our formulation is a morphism $E \otimes_A F \to G$ in ${}_A\mathcal{F}$. This induces a map on cohomology which composes with the map from Proposition 4.49 to give a product $\cup : \mathrm{H}^m(A, E, \nabla_E) \otimes \mathrm{H}^n(A, F, \nabla_F) \to \mathrm{H}^{n+m}(A, G, \nabla_G)$.

4.4 Spectral Sequences and Fibrations

A spectral sequence is a machine for doing algebraic calculations in a pictorial manner. This is a large topic and we will give only the briefest of introductions, referring the reader to the excellent text of McCleary for more background. We also restrict our attention to vector spaces and zero entries outside the first quadrant. We use the shorter form im for the image of a map.

Definition 4.50 A double complex of bidegree (r, t) is a collection of vector spaces $E^{n,m}$ (assumed to be zero if either $n < 0$ or $m < 0$) and linear maps

$D : E^{n,m} \to E^{n+r,m+t}$ with $D \circ D = 0$. A spectral sequence of degree $\geq s$ is a collection $(E_r^{*,*}, D_r)$ of double complexes of bidegree $(r, 1-r)$ for $r \geq s$ such that

$$E_{r+1}^{n,m} \cong \mathrm{H}^{n,m}(E_r, D_r) = \frac{\ker D_r : E_r^{n,m} \to E_r^{n+r,m+1-r}}{\mathrm{im}\ D_r : E_r^{n-r,m+r-1} \to E_r^{n,m}}.$$

The double complex $(E_r^{n,m}, D_r)$ is called the r-th page of the spectral sequence. A picture of a first page of a spectral sequence would look like the following, with differential $D_1 : E_1^{n,m} \to E_1^{n+1,m}$, and $D_1 \circ D_1 = 0$:

$$\begin{tikzcd}
0 \arrow[r] & E_1^{0,2} \arrow[r] & E_1^{1,2} \arrow[r] & E_1^{2,2} \arrow[r] & {} \\
0 \arrow[r] & E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & {} \\
0 \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & {} \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

Page one

At the entry (n, m) we take the quotient of the kernel of D_1 divided by the image of D_1 (i.e., the cohomology), and the result is the second page entry $E_2^{n,m}$. The machinery of the spectral sequence makes the differential D_2. A picture of the second page of a spectral sequence, with differential $D_2 : E_2^{n,m} \to E_r^{n+2,m-1}$ and $D_2 \circ D_2 = 0$, would look like the following:

$$\begin{tikzcd}
{} \arrow[drr] & 0 \arrow[drr] & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & {} & {} \\
{} \arrow[drr] & 0 \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & {} & {} \\
{} & 0 \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & {} & {} \\
{} & 0 & 0 & 0 & 0 & {} & {}
\end{tikzcd}$$

Page two

At the entry (n, m) we take the quotient of the kernel of D_2 divided by the image of D_2, and the result is the third page entry $E_3^{n,m}$. The machinery of the spectral sequence makes the differential D_3. The process continues, with differential $D_r : E_r^{n,m} \to E_r^{n+r,m+1-r}$ on the r-th page. However, because we have taken all entries outside the first quadrant to be zero, for a given entry (n, m) the value will eventually be constant. Precisely, for $r > \max\{m + 1, n\}$ we have on the r-th page

$$0 = E_r^{n-r,m-1+r} \xrightarrow{D_r} E_r^{n,m} \xrightarrow{D_r} E_r^{n+r,m+1-r} = 0.$$

Taking the quotient of the kernel (which is all of $E_r^{n,m}$) and the image (which is 0) gives $E_{r+1}^{n,m} = E_r^{n,m}$, so for $r > \max\{m + 1, n\}$, the $E_r^{n,m}$ are independent of r, and we say that the entry has stabilised. In this way the spectral sequence tends to a limit as $r \to \infty$, and we call the (n, m) entry of the limit $E_\infty^{n,m}$. In an application of a spectral sequence, typically the initial data sets up the first or second page, and the answer is extracted from the limit. In many cases the limit can be calculated without much knowledge of the machinery of the spectral sequence. We recall two well-known constructions of spectral sequences, both of them covered with more background in the book by McCleary.

Example 4.51 (Spectral Sequences for a Double Complex) We start with a double complex $C^{n,m}$ (zero if either $n < 0$ or $m < 0$) with two differentials, d' of bidegree $(1, 0)$ and d'' of bidegree $(0, 1)$, which satisfy $\mathrm{d}'' \circ \mathrm{d}' + \mathrm{d}' \circ \mathrm{d}'' = 0$. The total complex T^s is defined as $T^s = \oplus_i C^{i,s-i}$ with differential $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$. We take the cohomologies in d' and d'' separately and label them

$$\begin{aligned}
\mathrm{H}_I^{n,m}(C) &= \mathrm{H}^{n,m}(C, \mathrm{d}') \\
&= (\ker \mathrm{d}' : C^{n,m} \to C^{n+1,m})/(\mathrm{im}\ \mathrm{d}' : C^{n-1,m} \to C^{n,m}), \\
\mathrm{H}_{II}^{n,m}(C) &= \mathrm{H}^{n,m}(C, \mathrm{d}'') \\
&= (\ker \mathrm{d}'' : C^{n,m} \to C^{n,m+1})/(\mathrm{im}\ \mathrm{d}'' : C^{n,m-1} \to C^{n,m}).
\end{aligned}$$

The map d'' induces a differential $\bar{\mathrm{d}}''$ on the complex $\mathrm{H}_I^{n,m}(C)$ by $\bar{\mathrm{d}}''[c] = [\mathrm{d}''c]$, and d' induces a differential $\bar{\mathrm{d}}'$ on the complex $\mathrm{H}_{II}^{n,m}(C)$ by $\bar{\mathrm{d}}'[c] = [\mathrm{d}'c]$, where $[c]$ denotes the equivalence class of $c \in C^{n,m}$ under the relevant quotient. Now there are two spectral sequences $({}_IE_r, {}_ID_r)$ and $({}_{II}E_r, {}_{II}D_r)$ for $r \geq 2$, beginning with ${}_IE_2^{n,m} \cong \mathrm{H}^{n,m}(\mathrm{H}_{II}(C), \bar{\mathrm{d}}')$ and ${}_{II}E_2^{n,m} \cong \mathrm{H}^{n,m}(\mathrm{H}_I(C), \bar{\mathrm{d}}'')$, and both converging to the cohomology of the total complex $\mathrm{H}^*(T, \mathrm{d})$, meaning $\oplus_i E_\infty^{i,n-i} = \mathrm{H}^n(T, \mathrm{d})$. ◇

Example 4.52 (Spectral Sequence for a Filtration) We start with a cochain complex $C = \oplus_n C^n$ (zero for $n < 0$) and $\mathrm{d} : C^n \to C^{n+1}$ with $\mathrm{d}^2 = 0$. Suppose C has a decreasing filtration $F^m C \subseteq C$ for $m \geq 0$ in the sense that

(1) $\mathrm{d}F^m C \subseteq F^m C$ for all $m \geq 0$,
(2) $F^{m+1} C \subseteq F^m C$ for all $m \geq 0$,
(3) $F^0 C = C$ and $F^m C^n := F^m C \cap C^n = \{0\}$ for all $m > n$.

Then there is a spectral sequence (E_r, d_r) with first page $E_1^{p,q}$ being

$$\mathrm{H}^{p+q}(F^p C/F^{p+1} C) = \frac{\ker \mathrm{d} : F^p C^{p+q}/F^{p+1} C^{p+q} \to F^p C^{p+q+1}/F^{p+1} C^{p+q+1}}{\mathrm{im}\, \mathrm{d} : F^p C^{p+q-1}/F^{p+1} C^{p+q-1} \to F^p C^{p+q}/F^{p+1} C^{p+q}}.$$

To describe the differential d_r on the r-th page, we first define

$$\begin{aligned}
Z_r^{p,q} &= F^p C^{p+q} \cap \mathrm{d}^{-1}(F^{p+r} C^{p+q+1}), \\
B_r^{p,q} &= F^p C^{p+q} \cap \mathrm{d}(F^{p-r} C^{p+q-1}), \\
E_r^{p,q} &= Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}),
\end{aligned}$$

where d^{-1} refers to the inverse image. The differential $\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is the map induced by quotienting $\mathrm{d} : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$. (Basically, the differential is always the original d, just applied to different subsets and with different quotients.) The spectral sequence converges to $\mathrm{H}^*(C, \mathrm{d})$ in the sense that

$$E_\infty^{p,q} \cong \frac{F^p\mathrm{H}^{p+q}(C,\mathrm{d})}{F^{p+1}\mathrm{H}^{p+q}(C,\mathrm{d})},$$

where $F^p\mathrm{H}(C, \mathrm{d})$ is the image of the map $\mathrm{H}(F^pC, \mathrm{d}) \to \mathrm{H}(C, \mathrm{d})$ induced by inclusion $F^pC \to C$. ◇

We will apply these ideas in noncommutative geometry, Example 4.51 to the van Est spectral sequence and Example 4.52 to the Leray–Serre and Frölicher spectral sequences. First we give a well-known application in algebra.

4.4.1 The Spectral Sequence of a Resolution

A resolution of an object F in an abelian category as in §3.6.2 is a collection of objects E^n and morphisms forming an exact sequence

$$0 \longrightarrow F \xrightarrow{\epsilon} E^0 \xrightarrow{i_0} E^1 \xrightarrow{i_1} E^2 \xrightarrow{i_2} \dots \tag{4.25}$$

This is sometimes called a right resolution, as opposed to a left resolution with the arrows reversed. Typically the objects E^n or maps i_n are chosen to have some property which makes calculating with them easier than with the original object F. (We have already seen an example, the bar resolution in Definition 3.48.) In our case, we take $F, E^n \in {}_A\mathcal{F}$ and morphisms $\epsilon, i_0, \cdots$ and tensor the exact sequence above with Ω^n, adding vertical maps, to obtain a double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2\otimes_A F \arrow[u, "\nabla^{[2]}"] \arrow[r, "\mathrm{id}\otimes\epsilon"] & \Omega^2\otimes_A E^0 \arrow[u, "-\nabla^{[2]}"] \arrow[r, "\mathrm{id}\otimes i_0"] & \Omega^2\otimes_A E^1 \arrow[u, "\nabla^{[2]}"] \arrow[r, "\mathrm{id}\otimes i_1"] & \Omega^2\otimes_A E^2 \arrow[u, "-\nabla^{[2]}"] \arrow[r, "\mathrm{id}\otimes i_2"] & \dots \\
0 \arrow[r] & \Omega^1\otimes_A F \arrow[u, "\nabla^{[1]}"] \arrow[r, "\mathrm{id}\otimes\epsilon"] & \Omega^1\otimes_A E^0 \arrow[u, "-\nabla^{[1]}"] \arrow[r, "\mathrm{id}\otimes i_0"] & \Omega^1\otimes_A E^1 \arrow[u, "\nabla^{[1]}"] \arrow[r, "\mathrm{id}\otimes i_1"] & \Omega^1\otimes_A E^2 \arrow[u, "-\nabla^{[1]}"] \arrow[r, "\mathrm{id}\otimes i_2"] & \dots \\
0 \arrow[r] & F \arrow[u, "\nabla^{[0]}"] \arrow[r, "\epsilon"] & E^0 \arrow[u, "-\nabla^{[0]}"] \arrow[r, "i_0"] & E^1 \arrow[u, "\nabla^{[0]}"] \arrow[r, "i_1"] & E^2 \arrow[u, "-\nabla^{[0]}"] \arrow[r, "i_2"] & \dots
\end{tikzcd}$$

We then take the double complex $C^{n,m} = \Omega^m \otimes_A E^n$ for $n, m \geq 0$ in Example 4.51, i.e., we throw away the column containing F. Note that we have added some minus signs to the connections to satisfy the equation $\mathrm{d}'' \circ \mathrm{d}' + \mathrm{d}' \circ \mathrm{d}'' = 0$. Next, keeping the notation of Example 4.51, if every Ω^n is flat as a right A-module then

$$\mathrm{H}_I^{n,m}(C) = \mathrm{H}^{n,m}(C, d') = \begin{cases} \Omega^m \otimes_A F & n = 0 \\ 0 & n \neq 0, \end{cases}$$

$$\mathrm{H}_{II}^{n,m}(C) = \mathrm{H}^{n,m}(C, d'') = \mathrm{H}^n(A, E^m, \nabla_{E^m}).$$

The zeros occur as the rows are exact because of the flatness of Ω^n. The map d'' induces a differential $\bar{\mathrm{d}}''$ on the complex $\mathrm{H}_I^{n,m}(C)$ by $\bar{\mathrm{d}}''[c] = [\nabla^{[*]}c]$, so we get a page two spectral sequence with ${}_I E^{0,m} = \mathrm{H}^m(A, F, \nabla_F)$ and all others zero, so the cohomology of the total complex $\mathrm{H}^n(T) = \mathrm{H}^n(A, F, \nabla_F)$.

The other spectral sequence has page two ${}_{II}E^{n,m}$ given by the cohomology of

$$\mathrm{H}^n(A, E^0, \nabla_{E^0}) \xrightarrow{[i_0]} \mathrm{H}^n(A, E^1, \nabla_{E^1}) \xrightarrow{[i_1]} \mathrm{H}^n(A, E^2, \nabla_{E^2}) \xrightarrow{[i_2]} \dots,$$

and this spectral sequence also converges to $\mathrm{H}^n(T) = \mathrm{H}^n(A, F, \nabla_F)$.

A standard application of the above, although arguably one we do not need quite this much machinery to derive, is for acyclic resolutions. In our context an object $(E, \nabla_E) \in {}_A\mathcal{F}$ is acyclic if $\mathrm{H}^i(A, E, \nabla_E) = 0$ for all $i > 0$.

Corollary 4.53 *If we are given* (F, ∇) *in* ${}_A\mathcal{F}$ *and a resolution* (4.25) *of* F *in that category by acyclic objects* (E^i, ∇_{E^i}) *then* $\mathrm{H}^n(A, F, \nabla_F)$ *is given by the cohomology of the sequence*

$$\mathrm{H}^0(A, E^0, \nabla_{E^0}) \xrightarrow{[i_0]} \mathrm{H}^0(A, E^1, \nabla_{E^1}) \xrightarrow{[i_1]} \mathrm{H}^0(A, E^2, \nabla_{E^2}) \xrightarrow{[i_2]} \dots.$$

Proof In this case the spectral sequence at the beginning of this section converges at the second page, as all differentials from that page on are zero. □

4.4.2 The van Est Spectral Sequence

The classical van Est spectral sequence is a spectral sequence for a Lie group G which links the topological cohomology of G to the Lie algebra cohomology and a version of the group cohomology. We start by adapting group cohomology to the Hopf algebra case. Let H be a Hopf algebra and $\Delta_L : F \to H \otimes F$ be a left H-comodule, which we write as $\Delta_L f = f_{(\bar{1})} \otimes f_{(\bar{\infty})}$.

Definition 4.54 Define $D^n = H^{\otimes n+1} \otimes F$ for $n \geq 0$, with the tensor product left H-coaction. The differential $\mathrm{d} : D^n \to D^{n+1}$ with $\mathrm{d}^2 = 0$ is defined by

$$\mathrm{d}(h_0 \otimes \dots \otimes h_n \otimes f) = \sum_{n+1 \geq i \geq 0} (-1)^i h_0 \otimes \dots \otimes h_{i-1} \otimes 1 \otimes h_i \otimes \dots \otimes h_n \otimes f.$$

As d is a left H-comodule map, we can restrict the complex to the invariants to give $({}^H D^n, \mathrm{d})$, and the cohomology of the invariants is called the *Hopf cochain cohomology* $\mathrm{H}_c(H; F)$ of H with coefficients in F.

Exercise E4.7 explains how this gives the usual group cohomology for the right representations F of a finite group G with $H = \Bbbk(G)$. It will also be useful to have an alternative definition of the Hopf cochain cohomology without taking invariants.

Proposition 4.55 *Define a cochain complex* $(G, \bar{\mathrm{d}})$ *by* $G^n = H^{\otimes n} \otimes F$ *for* $n \geq 0$ *with* $\bar{\mathrm{d}} f = 1_H \otimes f - \Delta_L f$ *for* $f \in F$ *and*

$$\bar{\mathrm{d}}(h_1 \otimes \cdots \otimes h_n \otimes f) = 1_H \otimes h_1 \otimes \cdots \otimes h_n \otimes f - \Delta h_1 \otimes \cdots \otimes h_n \otimes f + \ldots$$
$$+ (-1)^n h_1 \otimes \cdots \otimes \Delta h_n \otimes f - (-1)^n h_1 \otimes \cdots \otimes h_n \otimes \Delta_L f.$$

Then there is an invertible cochain map $\theta : (G, \bar{\mathrm{d}}) \to ({}^H D, \mathrm{d})$ *given by* $\theta f = S f_{(\bar{1})} \otimes f_{(\bar{\infty})}$ *and*

$$\theta(h_1 \otimes \cdots \otimes h_n \otimes f) = S h_{1(1)} \otimes h_{1(2)} S h_{2(1)} \otimes \cdots \otimes h_{n-1(2)} S h_{n(1)}$$
$$\otimes h_{n(2)} S f_{(\bar{1})} \otimes f_{(\bar{\infty})}.$$

Proof For $\mathrm{im}\,\theta$ the H-coinvariant part of D^n, it suffices to check

$$S h_{i(1)} \otimes h_{i(2)} \mapsto (S h_{i(2)}) h_{i(3)} \otimes S h_{i(1)} \otimes h_{i(4)} = 1 \otimes S h_{i(1)} \otimes h_{i(2)},$$
$$S f_{(\bar{1})} \otimes f_{(\bar{\infty})} \mapsto (S f_{(\bar{2})}) f_{(\bar{3})} \otimes S f_{(\bar{1})} \otimes f_{(\bar{\infty})} = 1 \otimes S f_{(\bar{1})} \otimes f_{(\bar{\infty})}.$$

To check that θ is a cochain map, we compute

$$\theta(1 \otimes h_1 \otimes \cdots \otimes h_n \otimes f) = 1 \otimes S h_{1(1)} \otimes h_{1(2)} S h_{2(1)} \otimes \cdots \otimes h_{n(2)} S f_{(\bar{1})} \otimes f_{(\bar{\infty})},$$
$$\theta(\Delta h_1 \otimes \cdots \otimes h_n \otimes f) = S h_{1(1)} \otimes 1_H \otimes h_{1(2)} S h_{2(1)} \otimes \cdots \otimes h_{n(2)} S f_{(\bar{1})} \otimes f_{(\bar{\infty})},$$
$$\theta(h_1 \otimes \cdots \otimes \Delta h_n \otimes f) = S h_{1(1)} \otimes h_{1(2)} S h_{2(1)} \otimes \cdots \otimes 1_H \otimes h_{n(2)} S f_{(\bar{1})} \otimes f_{(\bar{\infty})},$$
$$\theta(h_1 \otimes \cdots \otimes h_n \otimes \Delta_L f) = S h_{1(1)} \otimes h_{1(2)} S h_{2(1)} \otimes \cdots \otimes h_{n(2)} S f_{(\bar{1})} \otimes 1 \otimes f_{(\bar{\infty})}.$$

Adding these with alternating signs shows that $\theta \circ \bar{\mathrm{d}} = \mathrm{d} \circ \theta$. The inverse of θ is

$$\theta^{-1}(h_0 \otimes f) = (\epsilon h_0) f,$$
$$\theta^{-1}(h_0 \otimes h_1 \otimes f) = (\epsilon h_0) h_1 f_{(\bar{1})} \otimes f_{(\bar{\infty})},$$
$$\theta^{-1}(h_0 \otimes h_1 \otimes h_2 \otimes f) = (\epsilon h_0) h_1 h_{2(1)} f_{(\bar{1})} \otimes h_{2(2)} f_{(\bar{2})} \otimes f_{(\bar{\infty})},$$
$$\theta^{-1}(h_0 \otimes \cdots \otimes h_n \otimes f) = (\epsilon h_0) h_{1(1)} h_{2(1)} \ldots h_{n(1)} f_{(\bar{1})}$$
$$\otimes h_{2(2)} \ldots h_{n(2)} f_{(\bar{2})} \otimes h_{3(3)} \ldots h_{n(3)} f_{(\bar{3})} \otimes \cdots \otimes f_{(\bar{\infty})}$$

as we show for $n = 2$ (the general case is similar). Thus

$$\begin{aligned}
&\theta\theta^{-1}(h_0 \otimes h_1 \otimes h_2 \otimes f)\\
&\quad = (\epsilon h_0)\theta(h_1 h_{2(1)} f_{(\bar 1)} \otimes h_{2(2)} f_{(\bar 2)} \otimes f_{(\bar\infty)})\\
&\quad = (\epsilon h_0) S(h_{1(1)} h_{2(1)(1)} f_{(\bar 1)(1)}) \otimes h_{1(2)} h_{2(1)(2)} f_{(\bar 1)(2)} S(h_{2(2)(1)} f_{(\bar 2)(1)})\\
&\qquad \otimes h_{2(2)(2)} f_{(\bar 2)(2)} S f_{(\bar\infty)(\bar 1)} \otimes f_{(\bar\infty)(\bar\infty)}\\
&\quad = (\epsilon h_0) S(h_{1(1)} h_{2(1)} f_{(\bar 1)}) \otimes h_{1(2)} \otimes h_{2(2)} \otimes f_{(\bar\infty)}.
\end{aligned}$$

Since $h_0 \otimes h_1 \otimes h_2 \otimes f$ is invariant,

$$1 \otimes h_0 \otimes h_1 \otimes h_2 \otimes f = h_{0(1)} h_{1(1)} h_{2(1)} f_{(\bar 1)} \otimes h_{0(2)} \otimes h_{1(2)} \otimes h_{2(2)} \otimes f_{(\bar\infty)},$$

and applying S to the first part and taking the product of first and second, gives

$$h_0 \otimes h_1 \otimes h_2 \otimes f = (\epsilon h_0)(S f_{(\bar 1)})(S h_{2(1)}) S h_{1(1)} \otimes h_{1(2)} \otimes h_{2(2)} \otimes f_{(\bar\infty)}.$$

For the other way,

$$\begin{aligned}
\theta^{-1}\theta(h_1 \otimes h_2 \otimes f) &= \theta^{-1}(S h_{1(1)} \otimes h_{1(2)} S h_{2(1)} \otimes h_{2(2)} S f_{(\bar 1)} \otimes f_{(\bar\infty)})\\
&= \epsilon(S h_{1(1)}) h_{1(2)} (S h_{2(1)}) h_{2(2)(1)} (S f_{(\bar 1)})_{(1)} f_{(\bar\infty)(\bar 1)}\\
&\qquad \otimes h_{2(2)(2)} (S f_{(\bar 1)})_{(2)} f_{(\bar\infty)(\bar 2)} \otimes f_{(\bar\infty)(\bar\infty)} = h_1 \otimes h_2 \otimes f. \qquad \square
\end{aligned}$$

The idea behind the van Est spectral sequence is to take the complex for the Hopf cochain cohomology, and make the coefficients (which we have been calling F) into a cochain complex. We then get a double complex, as in Example 4.51. There we gave constructions for two spectral sequences, which have the same limit. For one of these spectral sequences it will easy to compute its limit, while the other has a very different character. Moreover, there is a case for the coefficients where the Hopf cochain cohomology in Definition 4.54 gives a simple answer. Recall from Definition 2.16 that F an H-Hopf module means that F has a left H-action and a left H-coaction which are related by $\Delta_L(h \triangleright f) = h_{(1)} f_{(\bar 1)} \otimes h_{(2)} \triangleright f_{(\bar\infty)}$.

Proposition 4.56 *Let F be an H-Hopf module. Then $\mathrm{H}^n_c(H, F) = 0$ for $n \geq 1$ and $\mathrm{H}^0_c(H, F) \cong {}^H F$, with the isomorphism mapping $f \in {}^H F$ to $[1 \otimes f] \in \mathrm{H}^0_c(H, F)$.*

Proof Define $\phi : H^{\otimes n+2} \otimes F = D^{n+1} \to H^{\otimes n+1} \otimes F = D^n$ for $n \geq -1$ by

$$\phi(h_0 \otimes \ldots \otimes h_{n+1} \otimes f) = (-1)^{n+1} h_0 \otimes \cdots \otimes h_n \otimes h_{n+1} \triangleright f.$$

Now compute

$$\phi\,\mathrm{d}(h_0 \otimes \ldots \otimes h_{n+1} \otimes f)$$
$$= \sum_{n+1 \geq i \geq 0} (-1)^{i+n} h_0 \otimes \cdots \otimes h_{i-1} \otimes 1 \otimes h_i \otimes \ldots \otimes h_{n+1} {\triangleright} f + h_0 \otimes \cdots \otimes h_{n+1} \otimes f,$$
$$\mathrm{d}\phi(h_0 \otimes \cdots \otimes h_{n+1} \otimes f) = (-1)^{n+1} 1 \otimes h_0 \otimes \ldots \otimes h_n \otimes h_{n+1} {\triangleright} f + \ldots$$
$$+ h_0 \otimes \cdots \otimes h_n \otimes 1 \otimes h_{n+1} {\triangleright} f$$
$$(\phi\mathrm{d} + \mathrm{d}\phi)(h_0 \otimes \ldots \otimes h_{n+1} \otimes f) = h_0 \otimes \ldots \otimes h_{n+1} \otimes f.$$

Since ϕ is left H-covariant, it restricts to the coinvariant complex ${}^H D$ and provides a cochain homotopy contracting that complex, so $\mathrm{H}^n_c(H; F) = 0$ for $n \geq 1$. To find $\mathrm{H}^0_c(H; F)$, note that $x = \mathrm{d}\phi(x) + \phi(\mathrm{d}x)$ for $x \in D^0$ (the definition of d extends to $D^{-1} = F$ in an obvious manner). If $x \in {}^H D^0$ and $\mathrm{d}x = 0$, then $x = 1 \otimes \phi(x)$, so $\ker \mathrm{d} : D^0 \to D^1$ is $1 \otimes {}^H F$. □

This allows us to simplify the double complex spectral sequence as follows.

Theorem 4.57 *Suppose that (F^n, d_F) is a cochain complex of left H-Hopf modules with the differential d_F being a comodule map. Then there is a spectral sequence beginning with $E_2^{n,m} = \mathrm{H}^n_c(H; \mathrm{H}^m(F, \mathrm{d}_F))$ (remembering that $\mathrm{H}^m(F, \mathrm{d}_F)$ is a left H-comodule) which converges to $\mathrm{H}^s({}^H F, \mathrm{d}_F)$.*

Proof We form a double complex $C^{n,m} = H^{\otimes n} \otimes F^m$ $(n, m \geq 0)$ with differentials $\mathrm{d}' : H^{\otimes n} \otimes F^m \to H^{\otimes n+1} \otimes F^m$ being the Hopf-cochain differential in Proposition 4.55 (written $\bar{\mathrm{d}}$ there) and $\mathrm{d}'' : H^{\otimes n} \otimes F^m \to H^{\otimes n} \otimes F^{m+1}$ being $(-1)^n \mathrm{id} \otimes \mathrm{d}_F$. This satisfies the conditions for the double complex in Example 4.51.

Following the notation of Example 4.51 and using Proposition 4.56, we have $\mathrm{H}_I^{n,m}(C) = \mathrm{H}^{n,m}(C, \mathrm{d}') = 0$ for $n > 0$ and $\mathrm{H}_I^{0,m}(C) \cong {}^H F^m$, with the isomorphism mapping $f \in {}^H F^m$ to $[1 \otimes f] \in \mathrm{H}_I^{0,m}(C)$ (switching to the D^n notation of Definition 4.54). By definition $\mathrm{d}''[1 \otimes f] = [1 \otimes \mathrm{d}_F f]$, so the isomorphism identifies the complexes $(\mathrm{H}_I^{0,m}(C), \mathrm{d}'')$ and $({}^H F^m, \mathrm{d}_F)$. Thus we have ${}_{II}E_2^{n,m} = 0$ for $n > 0$ and ${}_{II}E_2^{0,m} = \mathrm{H}^m({}^H F, \mathrm{d}_F)$. But now every map ${}_{II}D_r$ $(r \geq 2)$ must be zero, as it either maps into or out of zero. This means that the ${}_{II}E_r$ spectral sequence stabilises at $r = 2$, so the spectral sequence ${}_I E_r^{n,m}$ must converge to $\mathrm{H}^s({}^H F, \mathrm{d}_F)$. Now we identify ${}_I E_2^{n,m}$. The short exact sequence

$$0 \longrightarrow \mathrm{im}\,(\mathrm{d}_F : F^{m-1} \to F^m) \longrightarrow \ker(\mathrm{d}_F : F^m \to F^{m+1}) \longrightarrow \mathrm{H}^m(F, \mathrm{d}_F) \longrightarrow 0$$

remains exact if we tensor (over $\Bbbk$) on the left with $H^{\otimes n}$. Then

$$\mathrm{H}^m(H^{\otimes n} \otimes F, \mathrm{id} \otimes \mathrm{d}_F) = H^{\otimes n} \otimes \mathrm{H}^m(F, \mathrm{d}_F).$$

Applying the induced differential to this gives the result. □

The promised Van Est spectral sequence is now a corollary of Theorem 4.57.

Corollary 4.58 (Van Est Spectral Sequence for Hopf Algebras) *Suppose that H is a Hopf algebra with left-covariant differential calculus. Then there is a spectral sequence beginning with $E_2^{n,m} = \mathrm{H}_c^n(H, \mathrm{H}_{\mathrm{dR}}^m(H))$ which converges to the cohomology of the left-invariant forms* $\mathrm{H}^s(\Lambda_H, \mathrm{d})$.

Proof We set (F, d_F) in Theorem 4.57 equal to (Ω_H, d). The left H-action is the product : $H \otimes \Omega_H^n \to \Omega_H^n$. □

The invariant 1-forms for a compact Lie group are simply the dual of the Lie algebra. The cohomology $\mathrm{H}^s(\Lambda_H, \mathrm{d})$ of the left-invariant differential forms can similarly be viewed as some kind of 'Lie algebra' cohomology.

Example 4.59 We consider the Hopf algebra $H = \mathbb{C}_q[SU_2]$ (q not a root of unity) with its 3D calculus. In Example 4.32 we calculated the cohomology of the left-invariant forms as $\mathbb{C}, 0, 0, \mathbb{C}, 0, \ldots$. We will see later, in Example 4.68, that this is the same as $\mathrm{H}_{\mathrm{dR}}(\mathbb{C}_q[SU_2])$ with this calculus, hence the latter cohomology has trivial $\mathbb{C}_q[SU_2]$-coaction. In the next diagram we show all the possible nonzero places in the Van Est spectral sequence, together with the only possible nonzero differentials, on the fourth page.

$$\begin{tikzcd}
\mathrm{H}_c^0(H;\mathbb{C}) \arrow[rrrrddd] & \mathrm{H}_c^1(H;\mathbb{C}) \arrow[rrrrddd] & \mathrm{H}_c^2(H;\mathbb{C}) & \mathrm{H}_c^3(H;\mathbb{C}) & \mathrm{H}_c^4(H;\mathbb{C}) & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\mathrm{H}_c^0(H;\mathbb{C}) & \mathrm{H}_c^1(H;\mathbb{C}) & \mathrm{H}_c^2(H;\mathbb{C}) & \mathrm{H}_c^3(H;\mathbb{C}) & \mathrm{H}_c^4(H;\mathbb{C}) & \mathrm{H}_c^5(H;\mathbb{C})
\end{tikzcd}$$

Page 4

After we take the cohomology of this fourth page to get the fifth page, the sum of the diagonals of the fifth page must be $\mathbb{C}, 0, 0, \mathbb{C}, 0, \ldots$, the cohomology of the invariant forms. From this we immediately read off $\mathrm{H}_c^0(H;\mathbb{C}) = \mathbb{C}$, $\mathrm{H}_c^1(H;\mathbb{C}) = 0$ and $\mathrm{H}_c^2(H;\mathbb{C}) = 0$. After this we get two possible cases, and the information we have so far calculated sheds no light on which case happens. The first possibility is that $\mathrm{H}_c^n(H;\mathbb{C}) = 0$ for all $n > 0$, and the second possibility is that $\mathrm{H}_c^n(H;\mathbb{C}) = \mathbb{C}$ for $n \in \{0, 3\}$ mod 4 and $\mathrm{H}_c^n(H;\mathbb{C}) = 0$ for $n \in \{1, 2\}$ mod 4. ◇

4.4.3 *Fibrations and the Leray–Serre Spectral Sequence*

There are several ideas of fibration in topology, but the essential idea is that we have a map $\pi : E \to B$, where E is the *total space* and B is the *base* of the fibration. The *fibre* F is identified with each $\pi^{-1}\{b\}$ for $b \in B$ in a continuous fashion. As a concrete class of examples, a *locally trivial fibration* is defined in the same way as a locally trivial vector bundle, but substituting continuous functions for linear functions. Thus there is a cover of B by open sets such that for all $U \subseteq B$ in the cover , $\pi^{-1}U \cong F \times U$, where π is projection to the second coordinate. The homeomorphisms $\pi^{-1}U \cong F \times U$ and $\pi^{-1}W \cong F \times W$ for U, W in the cover are patched together on the intersection $U \cap W$ by transition functions

$$\pi^{-1}U \supset F \times (U \cap W) \xrightarrow{\phi_{UW}} F \times (U \cap W) \subseteq \pi^{-1}W$$

which are homeomorphisms and obey the property $\pi \circ \phi_{UW} = \pi$, where π is projection to the second coordinate. Every locally trivial vector bundle is an example of a locally trivial fibration, with fibre the vector bundle. A trivial fibration is of the form $\pi : F \times B \to B$. An example of a locally trivial fibration which is not trivial is the subset of the Möbius bundle on the circle given by points at distance one from the zero section. The fibre is the set of two points, so we have a *double cover*. The fibration is topologically the same as the square map from the unit circle $S^1 \subset \mathbb{C}$ to itself. The Hopf fibration $\pi : S^3 \to S^2$ with fibre S^1 is another example.

In noncommutative geometry we do not normally have a notion of open cover and hence have a problem in defining 'locally trivial'. In fact, without having a point to take an inverse image of, it is not obvious what the fibre actually is. As in sheaf theory, we shall use differential structures to get round this. The big disadvantage of such an approach is that we will have an idea of fibration only in the differentiable case. To introduce the definition, we start with a trivial commutative example.

Example 4.60 Take the fibration $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ given by projection to the first n coordinates by $\pi(x_1, \dots, x_n, y_1, \dots, y_m) = (x_1, \dots, x_n)$. Here the base space is $B = \mathbb{R}^n$, the fibre is $F = \mathbb{R}^m$, and the total space is $E = \mathbb{R}^{n+m}$. Write a basis for the differential forms on the total space, putting the B terms (the $\mathrm{d}x_i$) first. Thus the form $\mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_7 \wedge \mathrm{d}y_9$ is of degree 2 in the base, degree 3 in the fibre and total degree 5. We consider two elements of $\pi^*\Omega^2_B \wedge \Omega^3_E$,

$$\xi = \pi^*(\mathrm{d}x_2 \wedge \mathrm{d}x_4) \wedge (\mathrm{d}y_1 \wedge \mathrm{d}y_7 \wedge \mathrm{d}y_9),$$
$$\eta = \pi^*(\mathrm{d}x_2 \wedge \mathrm{d}x_4) \wedge (\mathrm{d}x_3 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_7)$$

which have classes $[\xi] \neq 0$, $[\eta] = 0$ in the vector space quotient

$$\frac{\pi^*\Omega^2_B \wedge \Omega^3_E}{\pi^*\Omega^3_B \wedge \Omega^2_E}$$

(because we could also write $\eta = \pi^*(\mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_3) \wedge (\mathrm{d}y_1 \wedge \mathrm{d}y_7)$). This illustrates how the quotient space can be identified with forms of degree 2 on the base and degree 3 on the fibre. ◇

Similarly for any differentiable locally trivial classical fibration, we can identify the quotient space

$$\frac{\pi^*\Omega_B^p \wedge \Omega_E^q}{\pi^*\Omega_B^{p+1} \wedge \Omega_E^{q-1}}$$

with the forms on the total space of degree p on the base and degree q on the fibre. In the noncommutative case we consider the map of DGAs $\iota : \Omega_B \to \Omega_A$, where now A is the 'total algebra' and B is the 'base algebra' and take the above as a definition.

Definition 4.61 For any DGA map $\iota : \Omega_B \to \Omega_A$, we define

$$N_{p,q} = \frac{\iota\,\Omega_B^p \wedge \Omega_A^q}{\iota\,\Omega_B^{p+1} \wedge \Omega_A^{q-1}}, \quad N_{p,0} = \iota\,\Omega_B^p \otimes_A A$$

and say that ι is a *differential fibration* if the induced maps

$$\Omega_B^p \otimes_B N_{0,q} \to N_{p,q}, \quad \xi \otimes [x] \mapsto [\iota(\xi) \wedge x]$$

are isomorphisms for all $p, q \geq 0$.

It is easy to check that the induced maps here are well defined. We will return to this definition in Theorem 5.61 for quantum principal bundles.

Example 4.62 The Heisenberg group Hg is a multiplicative matrix group

$$\mathrm{Hg} = \left\{ \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : n, m, k \in \mathbb{Z} \right\}$$

which we take with generators u, v, w given by

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here w is central and there is one more relation $uv = wvu$. For a calculus on the group algebra $\mathbb{C}\mathrm{Hg}$, we take Λ^1 with basis e^u, e^v, e^w and right action

$$e^u \triangleleft u = e^u, \quad e^v \triangleleft u = e^v - \frac{1}{2}e^w, \quad e^u \triangleleft v = e^u + \frac{1}{2}e^w, \quad e^v \triangleleft v = e^v,$$

all actions on e^w leaving it invariant, and $\triangleleft w = \mathrm{id}$ acting trivially. We define $\zeta(x) = e^x$ for all $x = u, v, w$, one can check that this extends as a cocycle in $Z^1(\mathrm{Hg}, \Lambda^1)$.

By Theorem 1.47, this gives us a calculus $\Omega^1(\mathbb{C}\mathrm{Hg})$ with

$$x^{-1}\mathrm{d}x = e^x, \quad xe^x = e^x x, \quad u^{-1}e^v u = e^v - \frac{1}{2}e^w, \quad v^{-1}e^u v = e^u + \frac{1}{2}e^w$$

for $x = u, v, w$, and with w, e^w central. $\Omega(\mathbb{C}\mathrm{Hg})$ has the $\{e^x\}$ anticommuting.

There is a differential fibration with total algebra $A = \mathbb{C}\mathrm{Hg}$ and base $\mathbb{C}[z, z^{-1}]$ with its classical differential structure, namely by the algebra map $\iota : \mathbb{C}[z, z^{-1}] \to \mathbb{C}\mathrm{Hg}$ given by $\iota(z) = w$. This is differentiable with $\iota(z^{-1}\mathrm{d}z) = e^w$ (in this DGA setting we use the same symbol for the extension to forms). To see that this gives a fibration we calculate the quotient $N_{p,q}$ in Definition 4.61. Using $A.\{\ \}$ for the left $\mathbb{C}\mathrm{Hg}$-module generated by a list of elements, the only nonzero $N_{p,q}$ are

$$N_{0,0} \cong A,\ N_{1,0} \cong A.\{e^w\}, \quad N_{0,1} = \frac{A.\{e^u, e^v, e^w\}}{A.\{e^w\}} \cong A.\{e^u, e^v\},$$

$$N_{1,1} = \frac{e^w \wedge A.\{e^w, e^u, e^v\}}{A.\{0\}} \cong A.\{e^w \wedge e^u, e^w \wedge e^v\},$$

$$N_{0,2} = \frac{A.\{e^u \wedge e^v, e^w \wedge e^u, e^w \wedge e^v\}}{A.\{e^w \wedge e^u, e^w \wedge e^v\}} \cong A.\{e^u \wedge e^v\},$$

$$N_{1,2} = \frac{e^w \wedge A.\{e^w \wedge e^u, e^w \wedge e^v, e^u \wedge e^v\}}{A.\{0\}} \cong A.\{e^w \wedge e^u \wedge e^v\}.$$

Then the map $\Omega^1(\mathbb{C}[z, z^{-1}]) \otimes_{\mathbb{C}[z,z^{-1}]} N_{0,n} \to N_{1,n}$ is one-to-one and onto, giving a differential fibration in the sense of Definition 4.61. In fact, we have a fibre in quite a classical sense as $\mathbb{C}[z, z^{-1}]$ is commutative with a generator z which can be regarded as a function on the unit circle, so we have a classical base space. If we substitute a particular value $z = \mathrm{e}^{-\mathrm{i}\theta} \in S^1$ then the 'fibre algebra' is given by substituting $w \mapsto \mathrm{e}^{-\mathrm{i}\theta}$ in the algebra relations, giving $uv = \mathrm{e}^{-\mathrm{i}\theta} vu$. As u, v are invertible this is exactly the noncommutative torus $\mathbb{C}_\theta[\mathbb{T}^2]$. We can take $N_{0,q}$ to define a differential structure on the fiber algebra, which is just the calculus for $\mathbb{C}_\theta[\mathbb{T}^2]$ in Example 1.36. We have obtained an explicit construction of a continuous family of noncommutative tori with their standard differential calculus.

This whole construction is equivariant under the group $SL_2(\mathbb{Z})$. For each

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

there is an isomorphism $\varphi : \mathrm{Hg} \longrightarrow \mathrm{Hg}$ given by $\varphi(u) = u^a v^b$, $\varphi(v) = u^c v^d$ and $\varphi(w) = w$ and this extends to the calculus (using the same symbol) by

$$\varphi(e^w) = e^w, \quad \varphi(e^u) = ae^u + be^v + \frac{1}{2}abe^w, \quad \varphi(e^v) = ce^u + de^v + \frac{1}{2}cde^w.$$

As φ acts trivially on the base, it acts on each $\mathbb{C}_\theta[\mathbb{T}^2]$ fibre. $\diamond$

We will see in Example 5.63 that the noncommutative Hopf fibration with the 3D calculus on $\mathbb{C}_q[SU_2]$ is another example of a differential fibration. For the moment we focus on setting up the machinery for a noncommutative version of the Leray–Serre spectral sequence. Classically, this is constructed from a sheaf on the total space of a fibration. Thus we take $(E, \nabla_E) \in {}_A\mathcal{F}$, where A is the total algebra of our differential fibration $\iota : \Omega_B \to \Omega_A$. Then we have a cochain complex $\nabla_E^{[n]}$: $\Omega_A^n \otimes_A E \to \Omega_A^{n+1} \otimes_A E$. Let (C, d) denote this cochain complex.

Lemma 4.63 *If $\iota : \Omega_B \to \Omega_A$ is a map of DGAs then*

$$F^m(\Omega_A^n \otimes_A E) = \begin{cases} \iota\,\Omega_B^m \wedge \Omega_A^{n-m} \otimes_A E & 0 \le m \le n; \\ 0 & \text{otherwise} \end{cases}$$

gives a filtration of $C^n = \Omega_A^n \otimes_A E$ satisfying the conditions of Example 4.52.

Proof We have $F^0(\Omega_A^n \otimes_A E) = \iota\,\Omega_B^0 \wedge \Omega_A^n \otimes_A E = \Omega_A^n \otimes_A E$ as $1 \in \iota\,\Omega_B^0 = \iota\,B$. The decreasing condition (2) holds as

$$\begin{aligned} F^{m+1}(\Omega_A^n \otimes_A E) &= \iota\,\Omega_B^{m+1} \wedge \Omega^{n-m-1}A \otimes_A E \\ &= \iota\,\Omega_B^m \wedge (\iota\,\Omega_B^1 \wedge \Omega_A^{n-m-1}) \otimes_A E \\ &\subseteq \iota\,\Omega_B^m \wedge \Omega_A^{n-m} \otimes_A E \subseteq F^m(\Omega_A^n \otimes_A E). \end{aligned}$$

For the d condition (1), if $\iota\,\xi \wedge \eta \otimes e \in F^m(\Omega_A^n \otimes_A E)$ with $\xi \in \Omega_B^m$ and $\eta \in \Omega_A^{n-m}$ then

$$\mathrm{d}(\iota\,\xi \wedge \eta \otimes e) = \iota\,\mathrm{d}\xi \wedge \eta \otimes e + (-1)^m \iota\,\xi \wedge \mathrm{d}\eta \otimes e + (-1)^n \iota\,\xi \wedge \eta \wedge \nabla_E e.$$

The first term is in $F^{m+1}C \subseteq F^m C$ and the other terms are already in $F^m C$. $\square$

Next, by Example 4.52, this filtration gives a spectral sequence converging to $\mathrm{H}(A, E)$. We have to find the first and second pages of this spectral sequence. We let

$$M_{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\iota\,\Omega_B^p \wedge \Omega_A^q \otimes_A E}{\iota\,\Omega_B^{p+1} \wedge \Omega_A^{q-1} \otimes_A E}. \tag{4.26}$$

The cohomology of

$$\cdots \xrightarrow{\mathrm{d}} M_{p,q-1} \xrightarrow{\mathrm{d}} M_{p,q} \xrightarrow{\mathrm{d}} M_{p,q+1} \xrightarrow{\mathrm{d}} \cdots \tag{4.27}$$

for p fixed provides the first page of the spectral sequence. Denote the quotient in $M_{p,q}$ by $[\ \ ]_{p,q}$, so if $x \in \iota\,\Omega_B^p \wedge \Omega_A^q \otimes_A E$ then $[x]_{p,q} \in M_{p,q}$. In preparation for the second page, we have the following lemma.

Lemma 4.64 *Suppose as above that E is a flat left A-module. Every Ω_B^p is flat as a right B-module and $\iota : \Omega_B \to \Omega_A$ is a differential fibration. Define the 'cohomology of the fibre' $\hat{\mathrm{H}}^q(M)$ as the cohomology at the $M_{0,q}$ position of the complex*

$$\cdots \xrightarrow{[\nabla_E{}^{[q-1]}]_{0,q}} M_{0,q} \xrightarrow{[\nabla_E{}^{[q]}]_{0,q+1}} M_{0,q+1} \xrightarrow{[\nabla_E{}^{[q+1]}]_{0,q+2}} \cdots$$

The differential is a left B-module map by $b.[\eta \otimes e]_{0,q} = [\iota(b)\eta \otimes e]_{0,q}$ for $b \in B$ and $\eta \otimes e \in \Omega_A^q \otimes_A E$. Moreover, $\Omega_B^p \otimes_B \hat{\mathrm{H}}^q(M)$ is isomorphic to the cohomology of (4.27) by

$$\xi \otimes \langle \eta \otimes e \rangle_{0,q} \mapsto \langle \iota\,\xi \wedge \eta \otimes e \rangle_{p,q}\,,$$

where $\xi \in \Omega_B^p$ and $\langle\ \rangle_{p,q}$ denotes the equivalence class in the cohomology of $M_{p,q}$.

Proof There is a short exact sequence

$$0 \longrightarrow \iota\,\Omega_B^{p+1} \wedge \Omega_A^{q-1} \xrightarrow{\mathrm{inc}} \iota\,\Omega_B^p \wedge \Omega_A^q \xrightarrow{[\ ]} N_{p,q} \longrightarrow 0$$

and tensoring this with E gives

$$0 \longrightarrow \iota\,\Omega_B^{p+1} \wedge \Omega_A^{q-1} \otimes_A E \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} \iota\,\Omega_B^p \wedge \Omega_A^q \otimes_A E \xrightarrow{[\ ]\otimes\mathrm{id}} N_{p,q} \otimes_A E \longrightarrow 0.$$

Thus $M_{p,q} \cong N_{p,q} \otimes_A E$ and using the definition of differential fibration shows that $\Omega_B^p \otimes_B N_{0,q} \otimes_A E \cong M_{p,q}$ via the isomorphism $\xi \otimes [x] \otimes e \mapsto [\iota\,\xi \wedge x \otimes e]_{p,q}$. The element $\xi \otimes [x] \otimes e \in \Omega_B^p \otimes_B N_{0,q} \otimes_A E$ (for $x \in \Omega_A^q$) corresponds to $[\iota\,\xi \wedge x \otimes e]_{0,q} \in M_{p,q}$. Applying $\nabla_E{}^{[p+q]}$ to $\iota\,\xi \wedge x \otimes e$ gives

$$\begin{aligned} &\mathrm{d}(\iota\,\xi \wedge x) \otimes e + (-1)^{p+q}\,\iota\,\xi \wedge x \wedge \nabla e \\ &\qquad = \iota\,\mathrm{d}\xi \wedge x \otimes e + (-1)^p\,\iota\,\xi \wedge \mathrm{d}x \otimes e + (-1)^{p+q}\,\iota\,\xi \wedge x \wedge \nabla_E e \end{aligned}$$

but the first term vanishes after applying $[\ ]_{p,q+1}$ as $\mathrm{d}\xi \in \Omega_B^{p+1}$, giving

$$\mathrm{d}[\iota\,\xi \wedge x \otimes e]_{p,q} = (-1)^p[\iota\,\xi \wedge (\mathrm{d}x \otimes e + (-1)^q x \wedge \nabla_E e)]_{p,q+1}.$$

It follows that the differential d on $M_{p,q}$ can be written as

$$\mathrm{d}(\xi \otimes [y]_{0,q}) = (-1)^p \xi \otimes [\nabla_E{}^{[q]} y]_{0,q+1},$$

where $\xi \in \Omega^p_B$ and $y \in \Omega^q_A \otimes_A E$, and in particular this differential is a left Ω_B-module map. For the next part we let $Z_{p,q} = \mathrm{im}\, \mathrm{d} : M_{p,q-1} \to M_{p,q}$ and $K_{p,q} = \ker \mathrm{d} : M_{p,q} \to M_{p,q+1}$. As $\mathrm{d} = [\nabla_E{}^{[q]}]_{0,q+1} : M_{0,q} \longrightarrow M_{0,q+1}$ is a left B-module map, there is an exact sequence of left B modules,

$$0 \longrightarrow K_{0,q} \xrightarrow{\mathrm{inc}} M_{0,q} \xrightarrow{\mathrm{d}} Z_{0,q+1} \longrightarrow 0,$$

and tensoring with Ω^p_B gives another exact sequence,

$$0 \longrightarrow \Omega^p_B \otimes_B K_{0,q} \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} \Omega^p_B \otimes_B M_{0,q} \xrightarrow{\mathrm{id}\otimes\mathrm{d}} \Omega^p_B \otimes_B Z_{0,q+1} \longrightarrow 0. \tag{4.28}$$

By the first part of the proof, $\mathrm{id} \otimes \mathrm{d} = (-1)^p \mathrm{d}$ on $M_{p,q}$ so that $Z_{p,q} = \Omega^p_B \otimes_B Z_{0,q}$ and $K_{p,q} = \Omega^p_B \otimes_B K_{0,q}$. By definition of $\hat{\mathrm{H}}^q(M)$, we have another short exact sequence

$$0 \longrightarrow Z_{0,q} \xrightarrow{\mathrm{inc}} K_{0,q} \longrightarrow \hat{\mathrm{H}}^q(M) \longrightarrow 0$$

and tensoring with Ω^p_B proves the last result as

$$0 \longrightarrow \Omega^p_B \otimes_B Z_{0,q} \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} \Omega^p_B \otimes_B K_{0,q} \longrightarrow \Omega^p_B \otimes_B \hat{\mathrm{H}}^q(M) \longrightarrow 0. \qquad \square$$

The second page of the spectral sequence consists of the cohomology of the previous cohomology, i.e., the cohomology of

$$\mathrm{d} : \mathrm{cohomology}(M_{p,q}) \longrightarrow \mathrm{cohomology}(M_{p+1,q}),$$

which is, using the isomorphism of Lemma 4.64,

$$\mathrm{d} : \Omega^p_B \otimes_B \hat{\mathrm{H}}^q(M) \longrightarrow \Omega^{p+1}_B \otimes_B \hat{\mathrm{H}}^q(M). \tag{4.29}$$

Motivated by the classical picture where $\hat{\mathrm{H}}^q(M)$ is a cohomology over the fibre, we treat it as a sheaf over the base space. Note that every differential in the spectral sequence of a filtration is basically the same differential, only in a different quotient.

Lemma 4.65 *The differential* d *and the isomorphism in Lemma 4.64 define a flat left connection*

$$\nabla_q : \hat{\mathrm{H}}^q(M) \longrightarrow \Omega^1_B \otimes_B \hat{\mathrm{H}}^q(M), \quad \langle \xi \otimes e \rangle_{0,q} \longmapsto \eta \otimes \langle \omega \otimes f \rangle_{0,q}\,,$$

where (with sum of terms implicit),

$$\mathrm{d}\xi \otimes e + (-1)^q \xi \wedge \nabla_E e = \iota\, \eta \wedge \omega \otimes f \in \iota\, \Omega^1_B \wedge \Omega^q_A \otimes_A E.$$

Proof We take $\langle x\rangle_{0,q} \in \hat{\mathrm{H}}^q(M)$, where $x \in K_{0,q} = \ker \mathrm{d} : M_{0,q} \to M_{0,q+1}$, and suppose $x = \xi \otimes e$, where $\xi \in \Omega^q_A$ and $e \in E$. Then

$$[\mathrm{d}x]_{0,q+1} = [\mathrm{d}\xi \otimes e + (-1)^q \xi \wedge \nabla_E e]_{0,q+1} = 0.$$

Hence for some $\eta \in \Omega^1 B$, $\omega \in \Omega^q_A$ and $f \in E$,

$$\mathrm{d}\xi \otimes e + (-1)^q \xi \wedge \nabla_E e = \iota\,\eta \wedge \omega \otimes f \in \iota\,\Omega^1_B \wedge \Omega^q_A \otimes_A E. \tag{4.30}$$

By Lemma 4.64, this corresponds to $\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1_B \otimes_B M_{0,q}$. As the curvature of E vanishes, applying ${\nabla_E}^{[q+1]}$ to (4.30) gives

$$\iota\,\mathrm{d}\eta \wedge \omega \otimes f - \iota\,\eta \wedge \mathrm{d}\omega \otimes f + (-1)^{q+1} \iota\,\eta \wedge \omega \wedge \nabla_E f = 0. \tag{4.31}$$

We take this as an element of $M_{1,q+1}$ and apply $[\quad]_{1,q+1}$ to (4.31). Then as the denominator of $M_{1,q+1}$ is $\iota\,\Omega^2_B \wedge \Omega^q_A \otimes_A E$, the first term of (4.31) gives zero, so

$$-[\iota\,\eta \wedge (\mathrm{d}\omega \otimes f + (-1)^q \omega \wedge \nabla_E f)]_{1,q+1} = 0.$$

By Lemma 4.64, this corresponds to

$$-\,\eta \otimes_B [\mathrm{d}\omega \otimes f + (-1)^q \omega \wedge \nabla_E f]_{0,q+1} = 0. \tag{4.32}$$

Thus $\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1_B \otimes_B M_{0,q}$ is in $\ker(\mathrm{id} \otimes \mathrm{d})$ in the exact sequence (4.28), so $\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1_B \otimes_B K_{0,q}$, and taking the cohomology class gives $\eta \otimes \langle \omega \otimes f\rangle_{0,q} \in \Omega^1_B \otimes_B \hat{\mathrm{H}}^q$ as required. Next we compute

$$\nabla_q(b.\xi \otimes e) = \mathrm{d}(b.\xi) \otimes e + (-1)^q b.\xi \wedge \nabla_E e = \mathrm{d}b \wedge \xi \otimes e + b.(\mathrm{d}\xi \otimes e + (-1)^q \xi \wedge \nabla_E e),$$

for $b \in B$. Hence

$$\nabla_q \langle b.\xi \otimes e\rangle_{0,q} = \mathrm{d}b \otimes \langle \xi \otimes e\rangle_{0,q} + b.\nabla_q \langle \xi \otimes e\rangle_{0,q}\,,$$

which means we have a left connection. Applying $\nabla_q^{[1]}$ to $\nabla_q \langle \xi \otimes e\rangle_{0,q} = \eta \otimes \langle \omega \otimes f\rangle_{0,q}$ gives curvature

$$R_{\nabla_q} \langle \xi \otimes e\rangle_{0,q} = \mathrm{d}\eta \otimes \langle \omega \otimes f\rangle_{0,q} - \eta \wedge \nabla_q \langle \omega \otimes f\rangle_{0,q}\,. \tag{4.33}$$

To find $\nabla_q \langle w \otimes f\rangle_{0,q}$, we note that (4.32) implies

$$\eta \otimes_B (\mathrm{d}\omega \otimes f + (-1)^q \omega \wedge \nabla_E f) \in \Omega^1_B \otimes_B (\iota\,\Omega^1_B \wedge \Omega^q_A \otimes_A E)$$

by tensoring the exact sequence

$$0 \longrightarrow \iota\,\Omega^1_B \wedge \Omega^q_A \otimes_A E \longrightarrow \Omega^{q+1}_A \otimes_A E \overset{[\ ]_{0,q+1}}{\longrightarrow} M_{0,q+1} \longrightarrow 0$$

on the left by Ω^1_B. Now

$$\eta \otimes (\mathrm{d}\omega \otimes f + (-1)^q \omega \wedge \nabla f) = \eta' \otimes (\iota\,\kappa \wedge \zeta \otimes g) \tag{4.34}$$

for some $\eta', \kappa \in \Omega^1_B$, $\zeta \in \Omega^q_A$ and $g \in E$ and hence, from the definition of ∇_q,

$$\eta \wedge \nabla_q \langle \omega \otimes f \rangle_{0,q} = \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}\,,$$

so that (4.33) becomes

$$R_{\nabla_q} \langle \xi \otimes e \rangle_{0,q} = \mathrm{d}\eta \otimes \langle \omega \otimes f \rangle_{0,q} - \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}\,. \tag{4.35}$$

Meanwhile, (4.34) implies that

$$\iota\,\eta \wedge (\mathrm{d}\omega \otimes f + (-1)^q \omega \wedge \nabla_E f) = \iota\,\eta' \wedge \iota\kappa \wedge \zeta \otimes g,$$

and substituting this into (4.31) gives

$$\iota\,\mathrm{d}\eta \wedge \omega \otimes f - \iota\,\eta' \wedge \iota\,\kappa \wedge \zeta \otimes g = 0.$$

Hence on taking equivalence classes in $M_{2,q}$, we find

$$\mathrm{d}\eta \otimes [\omega \otimes f]_{0,q} - \eta' \wedge \kappa \otimes [\zeta \otimes g]_{0,q} = 0,$$

which by (4.35) implies that $R_{\nabla_q} = 0$. □

Classically, the functor from sheaves on the total space of a fibration to sheaves on the base is an example of a *direct image* construction and can be formulated as 'integrating over the fibres' of the map—i.e., taking cohomology classes in the fibre direction. Lemma 4.65 can be viewed as a noncommutative version of this. We have now done all the work for the main theorem of this section.

Theorem 4.66 (The Leray–Serre Spectral Sequence) *Suppose we are given*

(1) *a map $\iota : B \longrightarrow A$ which is a differential fibration (see Definition 4.61);*
(2) *a flat left A-module E, with a flat left connection $\nabla_E : E \to \Omega^1_A \otimes_A E$;*
(3) *the exterior algebra Ω_B has each Ω^p_B flat as a right B module.*

There is a spectral sequence converging to $\mathrm{H}(A, E, \nabla_E)$ *with second page position* (p, q) *being* $\mathrm{H}^p(B, \hat{\mathrm{H}}^q(M), \nabla_q)$ *where* $\hat{\mathrm{H}}^q(M)$ *is the cohomology of*

$$\cdots \overset{\mathrm{d}}{\longrightarrow} M_{0,q} \overset{\mathrm{d}}{\longrightarrow} M_{0,q+1} \overset{\mathrm{d}}{\longrightarrow} \cdots$$

with

$$M_{0,q} = \frac{\Omega^q_A \otimes_A E}{\iota\, \Omega^1_B \wedge \Omega^{q-1} A \otimes_A E}, \quad \mathrm{d}[x \otimes e]_{0,q} = [\mathrm{d}x \otimes e + (-1)^q x \wedge \nabla_E e]_{0,q+1}$$

and $\nabla_q : \hat{\mathrm{H}}^q(M) \to \Omega^1_B \otimes_B \hat{\mathrm{H}}^q(M)$ *defined in Lemma 4.65.*

Proof The first part of the proof is given in Lemma 4.64. Now we need to calculate the cohomology of

$$\mathrm{d} : \Omega^p_B \otimes_B \hat{\mathrm{H}}^q(M) \longrightarrow \Omega^{p+1}_B \otimes_B \hat{\mathrm{H}}^q(M).$$

If $\xi \in \Omega^p_B$, $\eta \in \Omega^q_A$ and $e \in E$ then $\xi \otimes \langle \eta \otimes e \rangle_{0,q}$ corresponds to $\iota\,\xi \wedge \eta \otimes e$ and applying d to the latter gives

$$\iota\, \mathrm{d}\xi \wedge \eta \otimes e + (-1)^p \iota\, \xi \wedge \mathrm{d}\eta \otimes e + (-1)^{p+q} \iota\, \xi \wedge \eta \wedge \nabla_E e.$$

We calculated the effect of d on $\hat{\mathrm{H}}^q(M)$ in Lemma 4.65 and comparing expressions,

$$\mathrm{d}(\xi \otimes \langle \eta \otimes e \rangle_{0,q}) = \mathrm{d}\xi \otimes \langle \eta \otimes e \rangle_{0,q} + (-1)^p \xi \wedge \nabla_q \langle \eta \otimes e \rangle_{0,q},$$

and this is precisely the differential for calculating the cohomology of the sheaf $\hat{\mathrm{H}}^q(M)$ from Lemma 4.65. □

Note that the usual Serre spectral sequence for de Rham cohomology is a corollary of Theorem 4.66 given by setting (E, ∇_E) to be the trivial sheaf (A, d). The next Examples 4.67 and 4.68 are also quantum principal bundles and will be considered more closely in Examples 5.63 and 5.64.

Example 4.67 We look at the fibration involving the Heisenberg group in Example 4.62 to find the dimensions of the de Rham cohomology (i.e., the Betti numbers) for the Heisenberg group. Here the base B is generated by w, and to calculate the quotients we fix a normal order: e^w, w, u, v, e^u, e^v. In $M_{0,q}$ we have, as the quotient has removed the terms with e^w,

$$\begin{aligned}
\mathrm{d}[w^p u^n v^m] &= n[w^p u^n v^m e^u] + m[w^p u^n v^m e^v],\\
\mathrm{d}[w^p u^n v^m e^u] &= -m[w^p u^n v^m e^u \wedge e^v],\\
\mathrm{d}[w^p u^n v^m e^v] &= n[w^p u^n v^m e^u \wedge e^v],\\
\mathrm{d}[w^p u^n v^m e^u \wedge e^v] &= 0.
\end{aligned}$$

The general elements of the cohomology $\hat{\mathrm{H}}^q(M)$ of the $M_{0,q}$ are

$$f(w) \in \hat{\mathrm{H}}^0(M), \quad f(w)e^u + g(w)e^v \in \hat{\mathrm{H}}^1(M), \quad f(w)e^u \wedge e^v \in \hat{\mathrm{H}}^2(M),$$

where f, g are Laurent polynomials (i.e., allowing negative powers). The next stage is a completely classical cohomology computation over the circle with coordinate w and $\mathrm{d}w^n = ne^w w^n$. To summarise, the second page entries split into a tensor product of the cohomology of the base circle (with Betti numbers, the dimension of the cohomology, $1, 1, 0\ldots$) times the cohomology of the fibre (with Betti numbers $1, 2, 1, 0\ldots$). The Betti numbers are shown on the second page diagram in the picture

$$\begin{array}{|ccc}
0 & 0 & 0 \\
1 & 1 & 0 \\
2 & 2 & 0 \\
1 & 1 & 0 \\
\hline
\end{array}$$

We see that the spectral sequence stabilises at this page, as all maps are either from or to zero. We read off the Betti numbers of the Heisenberg group for this calculus by summing over the diagonals to get $1, 3, 3, 1, 0\ldots$ with trailing zeros. ◇

Example 4.68 We continue the story of the noncommutative Hopf fibration from Example 4.33, using the 3D calculus from Example 2.32. The proof that we actually have a differential fibration is best done using quantum principal bundles and hence is deferred to Example 5.63, while here we do the spectral sequence computation assuming this. The second page of the spectral sequence is $E_2^{p,q} = \mathrm{H}^p_{\mathrm{dR}}(\mathbb{C}_q[S^2]) \otimes \mathrm{H}^q_{\mathrm{dR}}(\mathbb{C}_{q^2}[z, z^{-1}])$. As the Betti numbers for $\mathbb{C}_q[S^2]$ from Proposition 4.34 are $1, 0, 1, 0, \ldots$ with trailing zeros, we get the second page

$$\begin{array}{|cccc}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}$$

This spectral sequence converges to $\mathrm{H}^p_{\mathrm{dR}}(\mathbb{C}_q[SU_2])$. There are two possible limits for the spectral sequence: If the page 2 differential from the top left 1 to the bottom right 1 is zero then the limit has Betti numbers $1, 1, 1, 1, 0, \ldots$ while if it is nonzero we get $1, 0, 0, 1, 0, \ldots$, in both cases with trailing zeros. In Corollary 4.35 we showed the latter is the case. ◇

In Example 5.45 we will meet two more calculations with the Leray–Serre spectral sequence, one for the de Rham cohomology of a noncommutative calculus for the Klein bottle and one for the cohomology of a line bundle over the Klein bottle.

4.5 Correspondences, Bimodules and Positive Maps

In classical geometry a vector field is both a section of the tangent bundle and a bundle map from the cotangent bundle to the trivial bundle. Geometry is full of things which can be considered both as objects and as maps, and sometimes this can be used to generalise the idea of a mapping. Such is the case for the idea of a correspondence between spaces in topology and algebraic geometry. Roughly speaking (omitting much detail and generalisation), a correspondence between X and Y is a subset $\mathcal{C} \subseteq X \times Y$. In terms of our algebraic picture, the projection to the first coordinate $\pi_1 : \mathcal{C} \to X$ gives a map of functions $\pi_1^* : C(X) \to C(\mathcal{C})$ and by using this we can regard functions on $\mathcal{C}$ as a module over $C(X)$. Projection to the second coordinate $\pi_2 : \mathcal{C} \to Y$ allows us to similarly regard functions on $\mathcal{C}$ as a module over $C(Y)$. These actions commute and $C(\mathcal{C})$ becomes a $C(X)$-$C(Y)$ bimodule. Tensoring with $C(\mathcal{C})$ gives a functor from $C(Y)$-modules to $C(X)$-modules. This point of view includes the idea of viewing a function $f : X \to Y$ as a graph $\{(x, f(x)) \in X \times Y : x \in X\}$, in which case the functor is the pull back. In the noncommutative case we can still consider a B-A bimodule M as a kind of generalised morphism between algebras A, B. If we have an actual algebra map $\varphi : A \to B$ then we construct a B-A bimodule B_φ by $B_\varphi = B$ as a left B-module, and right A-action given by $b.a = b\varphi(a)$. We have already used this for twisted homology in §3.3.5 (albeit the twist in ${}_\varsigma A$ was on the other side) and for the inverse image sheaf in Proposition 4.46. Thus bimodules can be constructed from algebra maps. But we are not limited to this case and can think of a general bimodule in the same spirit as a functor between the algebra representation categories. Bimodules can also be given differentiability properties, as we will see.

Another motivation comes from quantum mechanics, where a measurement on a system gives a projection to an eigenspace of the measurement operator and can be expressed as a completely positive map. We shall focus on the KSGNS construction, which deals with completely positive maps and links them to bimodules.

4.5.1 B-A Bimodules with Connections

Let A, B be algebras. We let ${}_B\mathcal{M}_A$ denote the category of B-A bimodules. So objects are just vector spaces on which B acts from the left, A from the right and the two actions commute, generalising our previous category ${}_A\mathcal{M}_A$ of A-bimodules. An example of an $M_n(\mathbb{C})$-$M_m(\mathbb{C})$ bimodule is the set of $n \times m$ complex matrices with the action being matrix multiplication on the appropriate side. A bimodule map $\phi : M \to N$ for $M, N \in {}_B\mathcal{M}_A$ is a linear map which is both a left B-module map and a right A-module map, i.e., $\phi(b.m) = b.\phi(m)$ and $\phi(m.a) = \phi(m).a$ for all $m \in M$, $a \in A$ and $b \in B$. We write ${}_B\mathrm{Hom}_A(M, N)$ for the collection of such maps. These are the morphisms of ${}_B\mathcal{M}_A$. Clearly, if you fix $M \in {}_B\mathcal{M}_A$ then $M\otimes_A : {}_A\mathcal{M} \to {}_B\mathcal{M}$ and $\otimes_B M : \mathcal{M}_B \to \mathcal{M}_A$ are functors with the latter generalising the notion of pull back along an algebra map $\varphi : A \to B$ (if we actually have φ then $M \otimes_B B_\varphi \cong M_\varphi$ and has right action $m \otimes_B 1.a. = m \otimes_B \varphi(a) = m.\varphi(a)$ the pull back right module). We next specify a category of B-A bimodules with connection, generalising our previous ${}_A\mathcal{E}_A$.

Definition 4.69 A left B-A bimodule connection on $M \in {}_B\mathcal{M}_A$ means

(1) $\nabla_M : M \to \Omega^1_B \otimes_B M$ a left connection on M over B as usual.
(2) $\sigma_M : M \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B M$ a B-A bimodule map such that

$$\nabla_M(m.a) = \nabla_M(m).a + \sigma_M(m \otimes \mathrm{d}a)$$

for all $m \in M, a \in A$.

The category ${}_B\mathcal{E}_A$ has such objects (M, ∇_M, σ_M) and morphisms $(M, \nabla_M, \sigma_M) \to (N, \nabla_N, \sigma_N)$ defined as bimodule maps $\phi : M \to N$ satisfying $\nabla_N \circ \phi = (\mathrm{id} \otimes \phi) \circ \nabla_M$. This implies that $\sigma_N \circ (\phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi) \circ \sigma_M$.

We have specified the simplest but not the only choice in requiring our bimodule maps to intertwine the connections. In summary:

Name	Objects	Morphisms
${}_B\mathcal{M}_A$	B-A bimodules	Bimodule maps ${}_B\mathrm{Hom}_A$
${}_B\mathcal{E}_A$	(M, ∇_M, σ_M) Left B-A bimodule connections	Bimodule maps intertwining the connections

If we are thinking of a B-A bimodule as a generalised map then it is natural to think of a B-A bimodule with connection as a 'differentiable map' and we will therefore use the term *differentiable B-A bimodule* for an object of ${}_B\mathcal{E}_A$. To further motivate our use of bimodules to replace maps, we show that they can be used to move between categories built with different algebras in a similar manner to the inverse image construction in Proposition 4.46.

Proposition 4.70 *Suppose that (M, ∇_M, σ_M) is a left differentiable B-A bimodule. Then we have a functor $(M, \nabla_M, \sigma_M)_* : {}_A\mathcal{E} \to {}_B\mathcal{E}$ sending an object $(E, \nabla_E) \in {}_A\mathcal{E}$ to $(M \otimes_A E, \nabla_{M\otimes E}) \in {}_B\mathcal{E}$ with*

$$\nabla_{M\otimes E}(m \otimes e) = \nabla_M m \otimes e + (\sigma_M \otimes \mathrm{id})(m \otimes \nabla_E e)$$

and sending a morphism $\phi : E \to F$ to $\mathrm{id} \otimes \phi : M \otimes_A E \to M \otimes_A F$.

Proof Checking the required properties of $(M \otimes_A E, \nabla_{M\otimes E})$ is almost identical to the first part of the proof of Theorem 3.78, while

$$\begin{aligned}\nabla_{M\otimes F}(m \otimes \phi(e)) &= \nabla_M m \otimes \phi(e) + (\sigma_M \otimes \mathrm{id})(m \otimes \nabla_F \phi(e)) \\ &= \nabla_M m \otimes \phi(e) + (\sigma_M \otimes \phi)(m \otimes \nabla_E e) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)\nabla_{M\otimes E}(m \otimes e)\end{aligned}$$

verifies the morphism condition □

Continuous or differentiable maps between spaces can be used to induce maps on cohomology. In Proposition 4.46 we showed that differentiable algebra maps give a functor on the category of modules with zero curvature connections, which we think of as sheaves on the algebra. If we want to present differentiable bimodules as a generalisation of differentiable algebra maps then we should see when the functor in Proposition 4.70 restricts to a functor on the category of modules with zero curvature connections. To this end, we provide a minimal adaptation of Definition 4.10.

Definition 4.71 A left differentiable B-A bimodule (M, ∇_M, σ_M) is called *extendable* if $\sigma_M : M \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B M$ extends to $\sigma_M : M \otimes_A \Omega^n_A \to \Omega^n_B \otimes_B M$ for all $n \geq 0$ such that for all $n, m \geq 0$,

$$(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_M)(\sigma_M \otimes \mathrm{id}) = \sigma_M(\mathrm{id} \otimes \wedge) : M \otimes_A \Omega^n_A \otimes_A \Omega^m_A \to \Omega^{n+m} B \otimes_B M.$$

Note that the extension, if it exists, must be unique if the higher forms are generated by the 1-forms. In addition, (M, ∇_M, σ_M) is said to be flat if its curvature $R_M = \nabla_M{}^{[1]}\nabla_M : M \to \Omega^2_B \otimes_B M$ vanishes.

The B-A version of Lemma 4.14 is as follows, with a parallel proof.

Corollary 4.72 *Suppose that A is given the maximal prolongation differential calculus for some given first-order differential calculus, and that (M, ∇_M, σ_M) is a left differentiable B-A bimodule whose curvature R_M is a right module map. Then (M, ∇_M, σ_M) is extendable.*

Now we can define the differentiable B-A bimodules which we shall use to give functors on the category of sheaves.

Name	Objects	Morphisms
${}_B\mathcal{F}_A$	(M, ∇_M, σ_M) zero curvature extendable left B-A bimodule connections	Bimodule maps intertwining the connections

Example 4.73 We check that an algebra map $\varphi : A \to B$ which is differentiable in the sense of extending to a map of DGAs $\varphi : \Omega_A \to \Omega_B$ gives rise to a object of ${}_B\mathcal{F}_A$. We already met the induced B-A bimodule $B_\varphi = B$ as a left B-module with $b.a = b\varphi(a)$ as a right A-module. We now add to this a bimodule connection

$$\nabla : B_\varphi \to \Omega^1_B \otimes_B B_\varphi, \quad \nabla b = \mathrm{d}b \otimes 1,$$
$$\sigma : B_\varphi \otimes_A \Omega_A \to \Omega_B \otimes_B B_\varphi, \quad \sigma(b \otimes \eta) = b\varphi(\eta) \otimes 1$$

which obeys $\nabla(b.a) = \nabla(b\varphi(a)) = \mathrm{d}(b\varphi(a)) \otimes 1 = (\mathrm{d}b)\varphi(a) \otimes 1 + b\varphi(\mathrm{d}a) \otimes 1 = (\nabla b)a + \sigma(b \otimes \mathrm{d}a)$ as required, and is clearly flat as $\mathrm{d}^2 = 0$. It is also easy to see that it is extendable given that φ extends to the DGAs. ◇

We also need the following lemma parallel to Lemma 4.12.

Lemma 4.74 *If a left differentiable B-A bimodule (M, ∇_M, σ_M) is extendable then*

$$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\sigma_M-(\mathrm{id}\wedge\sigma_M)(\nabla_M\otimes\mathrm{id})-\sigma_M(\mathrm{id}\otimes\mathrm{d}) : M\otimes_A\Omega^1_A\to\Omega^2_B\otimes_B M$$

vanishes if and only if the curvature R_M is a right A-module map.

Proof We first check that the displayed formula gives a right A-module map,

$$\begin{aligned}
&(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\sigma_M(m\otimes\xi a)=(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)(\sigma_M(m\otimes\xi)a)\\
&\qquad=((\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\sigma_M(m\otimes\xi))\,a-(\mathrm{id}\wedge\sigma_M)(\sigma_M(m\otimes\xi)\otimes\mathrm{d}a)\ ,\\
&((\mathrm{id}\wedge\sigma_M)(\nabla_M\otimes\mathrm{id})+\sigma_M(\mathrm{id}\otimes\mathrm{d}))\,(m\otimes\xi a)\\
&\qquad=\ (((\mathrm{id}\wedge\sigma_M)(\nabla_M\otimes\mathrm{id})+\sigma_M(\mathrm{id}\otimes\mathrm{d}))(m\otimes\xi))\,a-\sigma_M(m\otimes\xi\wedge\mathrm{d}a)
\end{aligned}$$

for $m\in M, a\in A$ and $\xi\in\Omega^1_A$. Subtracting these equations and using extendability gives right A-linearity. To show the vanishing of the displayed formula, we now only have to apply it to elements of the form $m\otimes\mathrm{d}a$. Thus,

$$\begin{aligned}
&(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\sigma_M(m\otimes\mathrm{d}a)=(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)(\nabla_M(m.a)-(\nabla_M m).a)\\
&=\ (\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\nabla_M(m.a)-(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla_M)\nabla_M(m).a\\
&\quad+(\mathrm{id}\wedge\sigma_M)(\nabla_M m\otimes\mathrm{d}a),
\end{aligned}$$

which is $R_M(m.a)-R_M(m).a$ on using $\mathrm{d}^2=0$. □

We are now ready to show that we do indeed obtain a functor between categories of sheaves.

Proposition 4.75 *If (M,∇_M,σ_M) is an object in ${}_B\mathcal{F}_A$, then $(M,\nabla_M,\sigma_M)_*$: ${}_A\mathcal{E}\to{}_B\mathcal{E}$ restricts to a functor ${}_A\mathcal{F}\to{}_B\mathcal{F}$.*

Proof Given (E,∇_E) in ${}_A\mathcal{F}$, we need to show that the tensor product connection $\nabla_{M\otimes E}$ on $M\otimes_A E$ has curvature $R_{M\otimes E}=0$. For $m\in M$ and $e\in E$,

$$\begin{aligned}
R_{M\otimes E}(m\otimes e)&={\nabla_{M\otimes E}}^{[1]}(\nabla_M m\otimes e+(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e))\\
&=(\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})(\nabla_M m\otimes e)+(\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e)\\
&\quad-(\mathrm{id}\wedge\nabla_M\otimes\mathrm{id})(\nabla_M m\otimes e)-(\mathrm{id}\wedge\nabla_M\otimes\mathrm{id})(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e)\\
&\quad-(\mathrm{id}\wedge\sigma_M\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla_E)(\nabla_M m\otimes e+(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e)).
\end{aligned}$$

As ∇_M has zero curvature, the first and third terms cancel, so $R_{M\otimes E}(m\otimes e)$ becomes

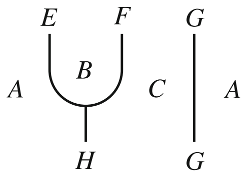
$$\begin{aligned}
&(\mathrm{d}\otimes\mathrm{id}\otimes\mathrm{id})(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e)-(\mathrm{id}\wedge\nabla_M\otimes\mathrm{id})(\sigma_M\otimes\mathrm{id})(m\otimes\nabla_E e)\\
&\quad-(\mathrm{id}\wedge\sigma_M\otimes\mathrm{id})(\nabla_M\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla_E e)\\
&\quad-(\mathrm{id}\wedge\sigma_M\otimes\mathrm{id})(\sigma_M\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla_E)(m\otimes\nabla_E e).
\end{aligned}$$

Using extendability, $R_{M\otimes E}(m \otimes e)$ becomes

$$
\begin{aligned}
&(\mathrm{d} \otimes \mathrm{id} \otimes \mathrm{id})(\sigma_M \otimes \mathrm{id})(m \otimes \nabla_E e) - (\mathrm{id} \wedge \nabla_M \otimes \mathrm{id})(\sigma_M \otimes \mathrm{id})(m \otimes \nabla_E e)\\
&\quad - (\mathrm{id} \wedge \sigma_M \otimes \mathrm{id})(\nabla_M \otimes \mathrm{id} \otimes \mathrm{id})(m \otimes \nabla_E e)\\
&\quad - (\sigma_M \otimes \mathrm{id})(m \otimes (\mathrm{id} \wedge \nabla_E)\nabla_E e)\\
&= (\mathrm{d} \otimes \mathrm{id} \otimes \mathrm{id})(\sigma_M \otimes \mathrm{id})(m \otimes \nabla_E e) - (\mathrm{id} \wedge \nabla_M \otimes \mathrm{id})(\sigma_M \otimes \mathrm{id})(m \otimes \nabla_E e)\\
&\quad - (\mathrm{id} \wedge \sigma_M \otimes \mathrm{id})(\nabla_M m \otimes \nabla_E e) - (\sigma_M \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{d} \otimes \mathrm{id})(m \otimes \nabla_E e),
\end{aligned}
$$

where we have used the zero curvature of ∇_E. The result can be written as $(T \otimes \mathrm{id})(m \otimes \nabla_E e)$, where T vanishes by Lemma 4.74, completing the proof. □

Previously, in §3.4, we only considered bimodule connections over the same algebra A and had a monoidal category ${}_A\mathcal{E}_A$. Now that we have mixed algebras, we no longer have a monoidal category as we cannot sensibly take the tensor product of an A-B bimodule and a C-D bimodule for different algebras A, B, C, D. However, we can consider the algebras as labels and only take the tensor product of bimodules where the labels are the same. In terms of our diagrammatic approach to tensor categories, we use labels or colours on the areas between the lines, for example in the next diagram we have marked the algebras between the bimodule lines. Taking bimodules $E \in {}_A\mathcal{M}_B$, $F \in {}_B\mathcal{M}_C$, $G \in {}_C\mathcal{M}_A$ and $H \in {}_A\mathcal{M}_C$, the next diagram illustrates a morphism $E \otimes_B F \otimes_C G \to H \otimes_C G$:

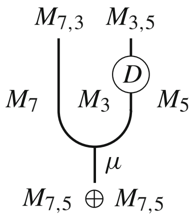


An ordinary category (or 1-category) has objects and morphisms between the objects. A 2-category has objects, 1-morphisms between the objects, and 2-morphisms between the 1-morphisms. Note that a 2-category with one object is the same as a monoidal category. The objects of the monoidal category become 1-morphisms of the 2-category, composition of 1-morphisms is then tensor product, and morphisms in the monoidal category become 2-morphisms of the 2-category, with the original composition. Following this path to making the above idea of a coloured monoidal category precise leads to the definition of a bicategory, a bit beyond the scope of this book. From our point of view, the coloured monoidal category picture is a convenient extension of the usual way of describing modules, for example:

Name	Colours	Objects	Morphisms
$\mathcal{M}$	Algebras	Bimodules	Bimodule maps
$\mathcal{E}$	Algebras with differential calculi	Bimodules with bimodule connection	Bimodule maps intertwining derivatives

As we are dealing with three different levels of structure, it may be helpful to give an example in $\mathcal{M}$ just to clarify the definition.

Example 4.76 We consider the entire collection of algebras of $n\times n$ matrices $M_n(\mathbb{C})$ for $n \geq 1$, and we use these as colours. Omitting the field, $M_{n,m}$, the $n\times m$ matrices, form an object in ${}_{M_n}\mathcal{M}_{M_m}$ and in the diagram this is represented by a line with the colour M_n on the left and M_m on the right. The doubling map $D : M_{3,5} \to M_{3,5} \oplus M_{3,5}$ simply sends a matrix to the direct sum of two copies of the matrix, and is a morphism in ${}_{M_3}\mathcal{M}_{M_5}$. There is also a matrix product $\mu : M_{7,3} \otimes_{M_3} M_{3,5} \to M_{7,5}$ which is a morphism in ${}_{M_7}\mathcal{M}_{M_5}$ and which can be extended to direct sums of matrices. The operation $X \otimes_{M_3} Y \mapsto (XY, XY)$ for $X \in M_{7,3}$, $Y \in M_{3,5}$ is



and is an M_7-M_5 bimodule map. ◇

4.5.2 *Hilbert C*-Bimodules and Positive Maps*

In the $*$-algebra setting it is natural to add more structure to a B-A bimodule so as to be able to complete it. This leads to the notion of a Hilbert C^*-bimodule and Theorem 4.81 will relate these to completely positive maps between C^*-algebras. We start with the one-sided case of Hilbert C^*-modules where a module E over a C^*-algebra A is equipped with a sesquilinear inner product generalising the notion of a Hilbert space with $\mathbb{C}$ replaced by A. We will work in standard conventions where A acts on E from the right and the left argument of the inner product is the conjugate one. Our exposition will use the notion of conjugate module $\overline{E}$ explained in Example 2.101, which we remind the reader has the same additive group as E but a conjugate action of $\mathbb{C}$ and a left action of A by $a.\overline{e} = \overline{e.a^*}$. We also recall that an element of a C^*-algebra A is said to be *positive* (one writes $a \geq 0$) if there is an element $b \in A$ such that $a = b^*b$. This notion also makes sense for a $*$-subalgebra of a C^*-algebra where we say that $a \geq 0$ if it is positive when viewed in the C^*-algebra. We restrict ourselves to unital algebras. More details about Hilbert C^*-modules and bimodules can be found in the excellent text by Lance.

Definition 4.77 Let A be unital dense $*$-subalgebra of a C^*-algebra with norm $\|\cdot\|_A$, E a right A-module and $\langle\ ,\ \rangle : \overline{E} \otimes E \to A$ an A-bimodule map.

(1) E is a right *semi-inner product* A-module if $\langle \overline{e}, e\rangle \geq 0$ for all $e \in E$;
(2) E is a right *inner product A-module* if in addition $\langle \overline{e}, e\rangle = 0$ implies $e = 0$;
(3) E is a right *Hilbert A-module* if in addition A is a C^*-algebra and E is complete with respect to the norm $\|e\|_E = \sqrt{\|\langle \overline{e}, e\rangle\|_A}$.

A Hilbert $\mathbb{C}$-module by this definition just means a complex Hilbert space. Any C^*-algebra A is a Hilbert A-module by defining $\langle \overline{a}, b\rangle = a^* b$, and this extends to n-column vectors $\mathrm{Col}^n(A)$ with entries in A by $\langle \overline{\underline{a}}, \underline{b}\rangle = \sum_i a_i^* b_i$ for $\underline{a}, \underline{b} \in \mathrm{Col}^n(A)$.

Lemma 4.78 *Let A be a unital dense $*$-subalgebra of a C^*-algebra and $\langle\ ,\ \rangle : \overline{E} \otimes E \to A$ a right semi-inner product A-module. Then for all $e, c \in E$ and $a \in A$,*

(1) $\langle \overline{e}, c\rangle^* = \langle \overline{c}, e\rangle$;
(2) $\langle \overline{e}, c\rangle\langle \overline{c}, e\rangle \leq \|\langle \overline{c}, c\rangle\|_A \langle \overline{e}, e\rangle$;
(3) $\|\langle \overline{e}, c\rangle\|_A \leq \|c\|_E \|e\|_E$;
(4) $\|ea\|_E \leq \|e\|_E \|a\|_A$;
(5) $\|e\|_E = \sup\left\{\|\langle \overline{e}, c\rangle\|_A : \|c\|_E \leq 1\right\}$.

Proof Since positive elements are also hermitian, the positivity of $\langle \overline{e + \lambda c}, e + \lambda c\rangle$ for all $\lambda \in \mathbb{C}$ implies that $\langle \overline{e}, c\rangle^* = \langle \overline{c}, e\rangle$ for all $e, c \in E$, so we obtain an A-valued hermitian inner product. For (2)

$$\begin{aligned} 0 \leq \langle \overline{ca - e}, ca - e\rangle &= a^*\langle \overline{c}, c\rangle a - \langle \overline{e}, c\rangle a - a^*\langle \overline{c}, e\rangle + \langle \overline{e}, e\rangle \\ &\leq \|c\|_E^2 a^* a - \langle \overline{e}, c\rangle a - a^*\langle \overline{c}, e\rangle + \langle \overline{e}, e\rangle. \end{aligned}$$

If $\|c\|_E = 0$, put $e = c$ and $a = \langle \overline{c}, e\rangle$ to get $\langle \overline{c}, e\rangle = 0$. If $\|c\|_E \neq 0$, put $a = \langle \overline{c}, e\rangle\|c\|_E^{-2}$. (3) then follows, while (4) follows from $\langle \overline{ea}, ea\rangle = a^*\langle \overline{e}, e\rangle a \leq a^*\|e\|_E{}^2 a$. (5) follows from (3), considering the case $\|e\|_E = 0$ separately. $\square$

To get from a semi-inner product module E to an inner product module, we can quotient by the submodule $N = \{e \in E : \|e\|_E = 0\}$, and then E/N with the inherited $\langle\ ,\ \rangle$ is an inner product A-module. The inner product on E/N is well defined by Lemma 4.78 part (3). To define Hilbert B-A bimodules, we first consider for a C^*-algebra A and a Hilbert A-module E the set $\mathcal{L}(E)$ of linear maps $T : E \to E$ which are *adjointable* in the sense that there is a map $T^* : E \to E$ such that

$$\langle \overline{T(e)}, c\rangle = \langle \overline{e}, T^*(c)\rangle \tag{4.36}$$

for all $e, c \in E$. Every such T is a right A-module map since

$$\langle \overline{T(e.a)}, c\rangle = \langle \overline{e.a}, T^*(c)\rangle = a^*\langle \overline{e}, T^*(c)\rangle = a^*\langle \overline{T(e)}, c\rangle = \langle \overline{T(e).a}, c\rangle$$

for all $a \in E$. If we define a norm

$$\|T\| = \sup\{\|T(e)\|_E : e \in E,\ \|e\|_E = 1\}$$

then $\mathcal{L}(E)$ is a C^*-algebra with product being composition. (To see that the norm of an adjointable operator exists is a bit beyond our scope but one can use the closed graph theorem.) Now given another C^*-algebra B, we define a Hilbert B-A bimodule as a Hilbert A-module E together with a unital $*$-algebra map $\lambda : B \to \mathcal{L}(E)$. This specifies a left B-module structure on E by $b.e = \lambda(b)(e)$ and the inner product then becomes an A-bimodule map $\langle\ ,\ \rangle : \overline{E} \otimes_B E \to A$.

For the case of unital dense $*$-subalgebras of C^*-algebras, we can still define semi-inner product and inner product bimodules by specifying a B-A bimodule E with a (semi-)inner product $\langle\ ,\ \rangle_E : \overline{E} \otimes_B E \to A$. With care, one can transfer many results about Hilbert A-modules to unital dense $*$-subalgebras. For example, tensor products as follows.

Lemma 4.79 *Let A and B be unital dense $*$-subalgebras of the C^*-algebras $\widehat{A}$ and $\widehat{B}$ respectively. Suppose that we have a right B-module F with inner product $\langle\ ,\ \rangle_F : \overline{F} \otimes F \to B$ which extends to a Hilbert $\widehat{B}$-module $\widehat{F}$ where $F \subseteq \widehat{F}$ as a $\widehat{B}$-module. Also suppose that we have a B-A bimodule E with inner product $\langle\ ,\ \rangle_E : \overline{E} \otimes_B E \to A$ which extends to a Hilbert $\widehat{B}$-$\widehat{A}$ bimodule $\widehat{E}$ where $E \subseteq \widehat{E}$ as a bimodule and the Hilbert $\widehat{B}$-$\widehat{A}$ bimodule inner product extends the original inner product on E. Then, recalling that $\Upsilon(\overline{f \otimes e}) = \overline{e} \otimes \overline{f}$,*

$$\langle\ ,\ \rangle_{F\otimes E} = \langle\ ,\ \rangle_E(\mathrm{id} \otimes \langle\ ,\ \rangle_F \otimes \mathrm{id})(\Upsilon \otimes \mathrm{id} \otimes \mathrm{id}) : \overline{F \otimes_B E} \otimes F \otimes_B E \to A$$

is a semi-inner product.

Proof We need to show that a finite sum $f_i \otimes e_i \in F \otimes E$ has (summing over i, j) $\langle \overline{f_i \otimes e_i}, f_j \otimes e_j \rangle_{F\otimes E} \geq 0$. Using the formula we need $\langle \overline{e_i}, \langle \overline{f_i}, f_j \rangle_F e_j \rangle_E \geq 0$. The matrix $\langle \overline{f_i}, f_j \rangle_F \in M_n(\widehat{B})$ is positive. To see this, given a row vector $(b_1^*, \dots, b_n^*)$ with entries in $\widehat{B}$, we have

$$\sum_{ij} b_i^* \langle \overline{f_i}, f_j \rangle_{\widehat{F}} b_j = \langle \overline{\sum_i f_i b_i}, \sum_j f_j b_j \rangle_{\widehat{F}} \geq 0$$

(this uses a standard result for the positivity of matrices on C^*-algebras). Using functional calculus, there is a positive square root $T \in M_n(\widehat{B})$ for this matrix, i.e., $T_{ik} T_{kj} = \langle \overline{f_i}, f_j \rangle_F$. Then

$$\sum_{i,j} \langle \overline{e_i}, \langle \overline{f_i}, f_j \rangle_F e_j \rangle_{\widehat{E}} = \sum_{i,j,k} \langle \overline{e_i}, T_{ik} T_{kj} e_j \rangle_{\widehat{E}} = \sum_k \langle \overline{\sum_i T_{ki} e_i}, \sum_j T_{kj} e_j \rangle_{\widehat{E}} \geq 0$$

verifies positivity. $\square$

Sesquilinear bimodule inner products underly the theory of Hermitian Riemannian structures in §8.4. For now we discuss their relation to positive maps.

Definition 4.80 Let A and B be unital C^*-algebras. A linear map $\phi : B \to A$ is said to be *positive* if it maps positive elements of B to positive elements of A. It is said to be completely positive if the induced map of matrix algebras $M_n(B) \to M_n(A)$ is positive for every $n \geq 1$. The completely bounded norm $\|\phi\|_{cb}$ is defined as the supremum of the norm of the induced maps $M_n(B) \to M_n(A)$ for all $n \geq 1$.

A $*$-algebra map $\psi : B \to A$ is automatically positive, as if $b = x^*x$ then $\psi(b) = \psi(x)^*\psi(x) \geq 0$. As its extension to $M_n(B) \to M_n(A)$ (by applying ψ to every entry) is a $*$-algebra map, ψ is also automatically completely positive. We now state the Kasparov–Stinespring–Gel'fand–Naĭmark–Segal theorem.

Theorem 4.81 (KSGNS Construction) *Let $\phi : B \to A$ be a completely positive map between unital C^*-algebras. Then there is a Hilbert B-A bimodule E and an $e \in E$ such that $\phi(b) = \langle \overline{e}, b.e\rangle$. Also BeA is dense in E and $\|\phi\|_{cb} = \|e\|_E^2$. Conversely, every map of the form $b \mapsto \langle \overline{e}, b.e\rangle$ for a Hilbert B-A bimodule is a completely positive map.*

Proof The algebraic tensor product $B \otimes A$ is a B-A bimodule by the usual left and right multiplication. It is made into a semi-inner product A-module by defining

$$\langle \overline{b_1 \otimes a_1}, b_2 \otimes a_2\rangle = a_1^*\phi(b_1^*b_2)a_2. \tag{4.37}$$

The completely positive property is needed to prove positivity of the inner product for sums of terms. This is done by taking the square root of a positive matrix in $M_n(A)$, where n is the number of terms summed, in a similar method to the proof of positivity in Lemma 4.79. Now set $e = 1 \otimes 1$, and $\langle \overline{1 \otimes 1}, b.1 \otimes 1\rangle = \phi(b)$. Then $B \otimes A$ is quotiented and completed to give a Hilbert B-A bimodule. We omit the technical analysis needed to complete the proof. For the converse direction, ϕ is clearly positive as $\phi(b^*b) = \langle \overline{b.e}, b.e\rangle \geq 0$. The argument for complete positivity is more involved and is based on $M_n(B)$ acting on an n-fold direct sum of the given Hilbert bimodule. □

The simplest case is $A = \mathbb{C}$, so that $\phi : B \to \mathbb{C}$ is a state (i.e., a $\mathbb{C}$ linear map sending positive elements of B to positive elements of $\mathbb{R}$), and the resulting E is a Hilbert space with a left B action. This gives us the following well-known construction as an example.

Example 4.82 (GNS Construction) Given a state $\phi : B \to \mathbb{C}$ on a unital C^*-algebra, we take a semi-inner product on B as $\langle \overline{c}, b\rangle = \phi(c^*b)$. Quotienting by the null (zero length) space and then completing gives a Hilbert space, which we call ${}_\phi\mathcal{H}$. If we denote by $[b]$ the equivalence class of $b \in B$ in ${}_\phi\mathcal{H}$, then the left action of B on ${}_\phi\mathcal{H}$ is $c.[b] = [cb]$. This means that ${}_\phi\mathcal{H}$ has a cyclic vector, i.e., a vector for which B acting on the vector gives a dense subset—the obvious choice is $[1]$. The reader may be wondering why we do not need to require ϕ to be *completely positive* as in the full KSGNS construction. In fact, completely positive is automatic in this case by $\mathbb{C}$-linearity of ϕ. For a matrix $(b_{ij}) \in M_n(B)$, we need to show that $\sum_j(\phi(b_{ij}b_{jk}{}^*)) \in M_n(\mathbb{C})$ is positive. However, this just means that for every vector $v_i \in \mathbb{C}$ we have $\sum_{i,k,j}\phi(v_i b_{ij} b_{jk}{}^* v_k{}^*) \geq 0$, which is implied by $\phi(c_j c_j{}^*) \geq 0$ for $c_j = \sum_i v_i b_{ij}$. ◇

We previously noted that $*$-algebra maps are completely positive. However, completely positive maps really are more general than algebra maps, as we see now.

Example 4.83 Consider two $*$-algebra maps $\phi, \psi : B \to A$ between unital C^*-algebras. Their sum is completely positive, but not in general an algebra map. (One way to show that the sum is completely positive is to take the direct sum of the corresponding Hilbert C^*-bimodules from the KSGNS construction.) Similarly, a positive multiple of a $*$-algebra map is completely positive. ◇

Now we consider the differential properties of bimodules with inner products and connections, for which purpose we assume that our algebras are unital dense $*$-subalgebras of C^*-algebras and take this version of the theory of bimodule inner products. We have also committed to the right-handed construction of Hilbert modules so at the differential level we now suppose that $(E, \tilde\nabla_E)$ is a right connection on a right A-module E. There is a corresponding left connection $\nabla_{\overline{E}} : \overline{E} \to \Omega^1_A \otimes_A \overline{E}$ given by $\nabla_{\overline{E}}(\overline{e}) = \eta^* \otimes \overline{c}$ if $\tilde\nabla_E(e) = c \otimes \eta$ (sum of terms understood) and one says that the right connection $\tilde\nabla_E$ preserves the bimodule map $\langle\, ,\rangle : \overline{E} \otimes E \to A$ if

$$\mathrm{d} \circ \langle\ ,\ \rangle = (\mathrm{id} \otimes \langle\ ,\ \rangle)(\nabla_{\overline{E}} \otimes \mathrm{id}) + (\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \tilde\nabla_E) : \overline{E} \otimes E \to \Omega^1_A. \tag{4.38}$$

Our first result is that we can still carry out the passage from a semi-inner product module to an inner product module by quotienting by the zero length elements as above but now in the presence of a connection.

Lemma 4.84 *Let A be a unital dense $*$-subalgebra of a C^*-algebra and $\langle\ ,\ \rangle : \overline{E} \otimes E \to A$ a semi-inner product A-module such that the right A-module connection $(E, \tilde\nabla_E)$ preserves $\langle\ ,\ \rangle$. Suppose that Ω^1_A is fgp as a left A-module. Then the submodule $N = \{e \in E : \|e\|_E = 0\}$ obeys $\tilde\nabla_E(N) \subseteq N \otimes_A \Omega^1_A$ and $\tilde\nabla_E$ descends to a right A-module connection on E/N.*

Proof Let $n \in N$, $e \in E$ and set $\tilde\nabla_E(n) = m \otimes \xi$, $\tilde\nabla_E(e) = c \otimes \eta$ (sums understood). Then $\mathrm{d}\langle \overline{e}, n\rangle = 0 = \langle \overline{e}, m\rangle\xi + \eta^*\langle \overline{c}, n\rangle = \langle \overline{e}, m\rangle\xi$. Hence given $\alpha \in (\Omega^1_A)^\flat$, we have $\langle \overline{e}, m\mathrm{ev}(\xi \otimes \alpha)\rangle = 0$ for all $e \in E$ and hence $m\mathrm{ev}(\xi \otimes \alpha) \in N$. As Ω^1_A is fgp as a left module, we have $\tilde\nabla_E(N) \subseteq N \otimes_A \Omega^1_A$. Now using flatness, the exact sequence

$$0 \longrightarrow N \otimes_A \Omega^1_A \longrightarrow E \otimes_A \Omega^1_A \longrightarrow E/N \otimes_A \Omega^1_A \longrightarrow 0$$

shows that we have a well-defined quotient connection $\tilde\nabla_E[e] = ([\ ] \otimes \mathrm{id})\tilde\nabla_E(e)$ where $[\ ]$ denotes the quotient in E/N. If we had a left rather than right connection we could simply have cited Theorem 3.114 to get a connection on the quotient. □

This allows us to take a semi-inner product A-module with a connection and quotient it to obtain an inner product A-module with a connection. Taking a completion to get a Hilbert C^*-module will normally destroy the differentiability in just the same way as taking the L^2 completion of smooth functions produces nondifferentiable functions, which is why we work with unital dense $*$-subalgebras of C^*-algebras. We define a completely positive map $\phi : B \to A$ for such $*$-algebras to be a map which continuously extends to a completely positive map of the ambient C^*-algebras. Next we address the problem of what it might mean for such a completely positive map $\phi : B \to A$ to be differentiable. If ϕ were an algebra map then we would suppose that it extends to a DGA map $\Omega_B \to \Omega_A$, but we do not want to assume this. Rather, using Theorem 4.81, we write a completely positive map as $\phi(b) = \langle \overline{e}, b.e \rangle$, and see what differentiable structure we can impose by using a connection. First, we need a right version of Definitions 4.69 and 4.10.

Definition 4.85 Let A, B be unital algebras and E a B-A bimodule. A right B-A bimodule connection $(E, \tilde{\nabla}_E, \tilde{\sigma}_E)$ means

(1) $\tilde{\nabla}_E : E \to E \otimes_A \Omega^1_A$ is a right connection on A.
(2) $\tilde{\sigma}_E : \Omega^1_B \otimes_B E \to E \otimes_A \Omega^1_A$ is a B-A bimodule satisfying

$$\tilde{\nabla}_E(b.e) = b.\tilde{\nabla}_E(e) + \tilde{\sigma}_E(\mathrm{d}b \otimes e).$$

This is extendable if there is a $\tilde{\sigma}_E : \Omega^n_B \otimes_B E \to E \otimes_A \Omega^n_A$ for all $n \geq 0$ such that

$$(\mathrm{id} \otimes \wedge)(\tilde{\sigma}_E \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{\sigma}_E) = \tilde{\sigma}_E(\wedge \otimes \mathrm{id}) : \Omega^n_B \otimes_B \Omega^m_B \otimes_B E \to E \otimes_A \Omega^{n+m}_A.$$

holds for all $n, m \geq 0$.

We are now ready for a differentiable extension of the converse part of the KSGNS construction in Theorem 4.81 in which the completely positive map is the degree 0 part of a cochain map. We make no attempt to fully characterise which cochain maps can be produced by this construction, which would be needed as a starting point for the other direction.

Proposition 4.86 *Suppose that A is a unital dense $*$-subalgebra of a C^*-algebra, $(E, \tilde{\nabla}_E, \tilde{\sigma}_E)$ a right B-A bimodule connection which is extendable with curvature $\tilde{R}_E$ a bimodule map and $\langle\ ,\ \rangle : \overline{E} \otimes_A E \to A$ a semi-inner product A-module structure preserved by $\tilde{\nabla}_E$. If $e \in E$ obeys $\tilde{\nabla}_E(e) = 0$ then*

$$\phi : \Omega_B \to \Omega_A, \quad \phi(\xi) = (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{e} \otimes \tilde{\sigma}_E(\xi \otimes e))$$

is a cochain map, i.e., $\mathrm{d} \circ \phi = \phi \circ \mathrm{d}$.

Proof As $\tilde{R}_E$ is assumed to be a left module map, extendability implies by a right analogue of Lemma 4.12 that

$$(\tilde{\nabla}_E \wedge \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\tilde{\sigma}_E = (-1)^{n+1}\tilde{\sigma}_E(\mathrm{d} \otimes \mathrm{id}) - (\tilde{\sigma}_E \wedge \mathrm{id})(\mathrm{id} \otimes \tilde{\nabla}_E)$$

as maps $\Omega^n_B \otimes_B E \to E \otimes_A \Omega^{n+1}_A$. Using this and the corresponding left connection $\nabla_{\overline{E}}$ on $\overline{E}$, we have

$$\begin{aligned} \mathrm{d}\phi(\xi) = &(\mathrm{id} \wedge \langle\ ,\ \rangle \wedge \mathrm{id})(\nabla_{\overline{E}}(\overline{e}) \otimes \tilde{\sigma}_E(\xi \otimes e)) \\ &+ (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{e} \otimes (\tilde{\nabla}_E \wedge \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\tilde{\sigma}_E(\xi \otimes e)) \\ = &(\mathrm{id} \wedge \langle\ ,\ \rangle \wedge \mathrm{id})(\nabla_{\overline{E}}(\overline{e}) \otimes \tilde{\sigma}_E(\xi \otimes e)) + (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{e} \otimes \tilde{\sigma}_E(\mathrm{d}\xi \otimes e)) \\ &- (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{e} \otimes (\tilde{\sigma}_E \wedge \mathrm{id})(\mathrm{id} \otimes \tilde{\nabla}_E)(\xi \otimes e)), \end{aligned}$$

and we now use $\tilde{\nabla}_E(e) = 0$. □

Remember that a cochain map induces maps on the de Rham cohomology, so this is still useful. However, if we start with a $*$-algebra map ϕ which is differentiable in the sense that it extends to a map of DGAs then Proposition 4.86 just recovers this. We illustrate this in degree 1.

Example 4.87 Suppose that $\phi : B \to A$ is a $*$-algebra map between unital dense $*$-subalgebras of C^*-algebras. Recall that the semi-inner product (4.37) on $B \otimes A$ in the proof of Theorem 4.81 factors through the B-A bimodule ${}_\phi A$ by the map $b \otimes a \mapsto \phi(b)a$, so the Hilbert C^*-bimodule will be a completion of a quotient of ${}_\phi A$. We use the connection $\mathrm{d} : {}_\phi A \to {}_\phi A \otimes_A \Omega^1_A$. The semi-inner product on ${}_\phi A$ is $\langle \overline{a}, a' \rangle = a^* a'$, and this is preserved by d. Now supposing that ϕ is a differentiable algebra map, we see that $\mathrm{d} : {}_\phi A \to {}_\phi A \otimes_A \Omega^1_A$ is a right B-A bimodule connection,

$$\mathrm{d}(b.a) = \mathrm{d}(\phi(b)a) = \mathrm{d}\phi(b)a + \phi(b)\mathrm{d}a = \tilde{\sigma}(\mathrm{d}b \otimes a) + b.\mathrm{d}a,$$

where $\tilde{\sigma} : \Omega^1_B \otimes_B {}_\phi A \to {}_\phi A \otimes_A \Omega^1_A$ is defined by $\tilde{\sigma}(\xi \otimes a) = \phi(\xi)a$. The element e mentioned in Theorem 4.81 is just $1 \in {}_\phi A$, and the connection applied to this element gives zero. The curvature of the connection is zero. The formula for $\tilde{\sigma}$ shows that the induced map in Proposition 4.86 on 1-forms is the DGA extension. ⋄

A less trivial example of a cochain map constructed by Proposition 4.86 makes use of a representation of $\mathbb{C}_q[SU_2]$ due to Woronowicz (which he used to study its C^* completion) except that in our case the representation space E is given an algebra-valued inner product.

Example 4.88 Let E be a complex vector space with basis $\psi_{n,k}$ for integers $k \in \mathbb{Z}$ and $n \geq 0$. Then E has a right action of the Hopf algebra $A = \mathbb{C}[z, z^{-1}]$ given by $\psi_{n,k} {\triangleleft} z^j = \psi_{n,k+j}$. Define a $\mathbb{C}[z, z^{-1}]$-valued inner product $\langle\ ,\ \rangle : \overline{E} \otimes E \to$

$\mathbb{C}[z,z^{-1}]$ making E into an inner product $\mathbb{C}[z,z^{-1}]$-module (see Definition 4.77) by

$$\langle\overline{\psi_{m,s}},\psi_{n,k}\rangle = z^{k-s}\,\delta_{n,m}.$$

Note that $\langle\overline{z^s},z^k\rangle = z^{k-s}$ gives a $\mathbb{C}[z,z^{-1}]$-valued inner product on $\mathbb{C}[z,z^{-1}]$, which is positive as $\langle\overline{f},f\rangle = f^*f$ for $f\in\mathbb{C}[z,z^{-1}]$. The inner product on E is the direct sum of copies of this, indexed by $n\geq 0$ where $\psi_{n,k}$ corresponds to z^k in the n-th copy, and so is positive. For $q\in\mathbb{R}$ with $|q|>1$, there is a left action of $B=\mathbb{C}_q[SU_2]$ on E by

$$a{\triangleright}\psi_{n,k} = \sqrt{1-q^{-2n}}\,\psi_{n-1,k},\quad c{\triangleright}\psi_{n,k} = q^{-n}\psi_{n,k+1}$$

and, using the $\mathbb{C}_q[SU_2]$ $*$-operation $a^*=d$ and $c^*=-qb$ of Proposition 2.13,

$$a^*{\triangleright}\psi_{n,k} = \sqrt{1-q^{-2n-2}}\,\psi_{n+1,k},\quad c^*{\triangleright}\psi_{n,k} = q^{-n}\psi_{n,k-1}.$$

A brief check shows that we have adjointable operators, with

$$\langle\overline{a^*{\triangleright}\psi_{m,s}},\psi_{n,k}\rangle = \langle\overline{\psi_{m,s}},a{\triangleright}\psi_{n,k}\rangle \text{ and } \langle\overline{c^*{\triangleright}\psi_{m,s}},\psi_{n,k}\rangle = \langle\overline{\psi_{m,s}},c{\triangleright}\psi_{n,k}\rangle.$$

Thus E becomes a Hilbert B-A bimodule. We next equip our algebras with differential structures and construct a right $\mathbb{C}_q[SU_2]-\mathbb{C}_{q^\alpha}[S^1]$ bimodule connection, using the left-covariant 3D calculus on $\mathbb{C}_q[SU_2]$ and a bicovariant calculus on $\mathbb{C}[z,z^{-1}]$ of the general form $(\mathrm{d}z)z = q^\alpha z\mathrm{d}z$ for some power α as indicated in the notation of Example 1.11. Now we examine the conditions to have a bimodule map

$$\tilde\sigma_E : \Omega^1_{\mathbb{C}_q[SU_2]}\otimes_{\mathbb{C}_q[SU_2]} E \to E\otimes_{\mathbb{C}[z,z^{-1}]}\Omega^1_{\mathbb{C}_{q^\alpha}[S^1]}.$$

This is determined by the values $\tilde\sigma_E(e^i\otimes\psi_{0,0})$ for $i\in\{\pm,0\}$. We have

$$a.\tilde\sigma_E(e^\pm\otimes\psi_{0,0}) = q^{-1}\tilde\sigma_E(e^\pm\otimes a{\triangleright}\psi_{0,0}) = 0$$

and similarly for e^0, so $\tilde\sigma_E(e^i\otimes\psi_{0,0}) = \psi_{0,0}\otimes\omega^i$ for some $\omega^i\in\Omega^1_{\mathbb{C}_{q^\alpha}(S^1)}$. Next,

$$c\tilde\sigma(e^\pm\otimes\psi_{0,0}) = q^{-1}\tilde\sigma(e^\pm\otimes c\psi_{0,0}) = q^{-1}\tilde\sigma(e^\pm\otimes\psi_{0,0})z,$$

and substituting the previous values gives

$$c{\triangleright}\psi_{0,0}\otimes\omega^\pm = \psi_{0,0}\otimes z.\omega^\pm = q^{-1}\psi_{0,0}\otimes\omega^\pm.z,$$

so $z\omega^\pm = q^{-1}\omega^\pm z$. An analogous calculation for the action on e^0 tells us that $z\omega^0 = q^{-2}\omega^0 z$. Since ω^i are functions of z, z^{-1} times $\mathrm{d}z$, this implies that either $\tilde\sigma_E$ vanishes identically or $\alpha = 1$ so that $\omega^\pm$ can be nonzero or $\alpha = 2$ so that ω^0 can be nonzero. We proceed with the latter possibility for the rest of our construction to work, i.e., we work with $\mathbb{C}_{q^2}[S^1]$. Next, from the e^0 term of $\mathrm{d}y$ in Example 2.32,

$$\tilde\sigma_E(\mathrm{d}y \otimes \psi_{0,0}) = [|y|]_{q^2}\tilde\sigma_E(ye^0 \otimes \psi_{0,0}) = \tilde\nabla_E(y\triangleright\psi_{0,0}) - y.\tilde\nabla_E(\psi_{0,0})$$

for $y \in \mathbb{C}_q[SU_2]$, which we rearrange as

$$\tilde\nabla_E(y\triangleright\psi_{0,0}) = y\triangleright\big([|y|]_{q^2}\tilde\sigma_E(e^0 \otimes \psi_{0,0}) + \tilde\nabla_E(\psi_{0,0})\big). \tag{4.39}$$

Putting $y = c^*c$ and using $c^*c\triangleright\psi_{0,0} = \psi_{0,0}$ shows $\tilde\nabla_E(\psi_{0,0}) = \psi_{0,0}\otimes\eta$ for some $\eta \in \Omega^1_{\mathbb{C}_{q^2}[S^1]}$. Now put $y = c^k$ and use $c^k\triangleright\psi_{0,0} = \psi_{0,0}.z^k$ in (4.39) to obtain

$$\psi_{0,0}\otimes \mathrm{d}(z^k) + \psi_{0,0}\otimes\eta.z^k = c^k\triangleright\big([k]_{q^2}\psi_{0,0}\otimes\omega^0 + \psi_{0,0}\otimes\eta\big),$$

which reduces, on using $\mathrm{d}(z^k) = [k]_{q^2}z^{k-1}\mathrm{d}z$, to

$$[k]_{q^2}z^{k-1}\mathrm{d}z + q^{2k}z^k\eta = [k]_{q^2}z^k\omega^0 + z^k\eta$$

for all $k \ge 0$. We take the case $\eta = 0$ and $\omega^0 = z^{-1}\mathrm{d}z$, and as a result

$$\tilde\nabla_E(y\triangleright\psi_{0,0}) = [|y|]_{q^2}\, y\triangleright\psi_{0,0}\otimes z^{-1}\mathrm{d}z,$$

so $\tilde\nabla_E(\psi_{n,k}) = [k-n]_{q^2}\psi_{n,k}\otimes z^{-1}\mathrm{d}z$. We find that the metric is preserved,

$$\begin{aligned}&\big((\langle\ ,\ \rangle\otimes\mathrm{id})(\mathrm{id}\otimes\tilde\nabla_E) + (\mathrm{id}\otimes\langle\ ,\ \rangle)(\nabla_{\overline{E}}\otimes\mathrm{id})\big)(\overline{\psi_{m,r}}\otimes\psi_{n,k})\\ &\qquad = [k-n]_{q^2}\langle\overline{\psi_{m,r}},\psi_{n,k}\rangle z^{-1}\mathrm{d}z - [r-m]_{q^2}z^{-1}\mathrm{d}z\langle\overline{\psi_{m,r}},\psi_{n,k}\rangle\\ &\qquad = \delta_{n,m}[k-r]_{q^2}z^{k-r-1}\mathrm{d}z = \mathrm{d}\,\langle\overline{\psi_{m,r}},\psi_{n,k}\rangle\end{aligned}$$

and also that the connection is automatically zero curvature and extendable.

We are now in position to apply Proposition 4.86 to construct a completely positive map extending to a cochain map on differential forms. Set $e_n = \psi_{n,n}$ for some $n \ge 0$. Then $\tilde\nabla_E(e_n) = 0$ and ϕ_n is defined by

$$\phi_n(y) = \langle\overline{e_n}, y\triangleright e_n\rangle = \langle\overline{\psi_{n,n}}, y\triangleright\psi_{n,n}\rangle.$$

For the standard basis for $\mathbb{C}_q[SU_2]$, we have $\phi_n(a^m b^r c^s) = 0$ and $\phi_n(d^m b^r c^s) = 0$ for $m > 0$ and $\phi_n(b^r c^s) = q^{-n(r+s)}(-q^{-1})^r z^{s-r}$. This is not an algebra map. The value of ϕ_n on 1-forms (it must vanish on higher forms) from Proposition 4.86 is given by

$$\phi_n(\xi) = (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{\psi_{n,n}} \otimes \tilde{\sigma}_E(\xi \otimes \psi_{n,n})),$$

so ϕ_n vanishes on any multiple of $e^{\pm}$. For e^0, we calculate

$$q^{2n}\tilde{\sigma}_E(e^0 \otimes d^n \triangleright \psi_{0,n}) = d^n \tilde{\sigma}_E(e^0 \otimes \psi_{0,n}) = d^n \triangleright \psi_{0,0} \otimes \omega^0 . z^n = q^{2n} d^n \triangleright \psi_{0,n} \otimes \omega^0,$$

so that $\tilde{\sigma}_E(e^0 \otimes \psi_{n,n}) = \psi_{n,n} \otimes \omega^0$. This gives

$$\phi_n(y.e^0) = (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{\psi_{n,n}} \otimes \tilde{\sigma}_E(y.e^0 \otimes \psi_{n,n})) = \phi_n(y).\omega^0 = \phi_n(y).z^{-1}\mathrm{d}z$$

as the degree 1 part of a cochain map $\Omega(\mathbb{C}_q[SU_2]) \to \Omega(\mathbb{C}_{q^2}[S^1])$. ◇

We conclude with an example of the construction for maps from $M_2(\mathbb{C})$ to itself.

Example 4.89 Consider $A = M_2(\mathbb{C})$. A completely positive map $\phi : A \to A$ can be written as $\phi(a) = \sum_i e_i{}^* a e_i$ for some $e_1 \oplus \cdots \oplus e_n \in E = M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C})$ as a Hilbert bimodule with inner product $\langle \overline{x_1 \oplus \cdots \oplus x_n}, y_1 \oplus \cdots \oplus y_n \rangle = \sum_i x_i^* y_i$. For simplicity we focus on $\phi(a) = e^* a e$ defined by just one copy $E = M_2(\mathbb{C})$ and some $e \in E$. We use the calculus on $A = M_2(\mathbb{C})$ in Corollary 1.9 with the maximal prolongation calculus quotiented by $s^2 = t^2 = 0$ as in Proposition 1.38. We set

$$\tilde{\nabla}_E x = 1 \otimes (\mathrm{d}x + (fs + gt).x),$$

for all $x \in M_2(\mathbb{C})$, where $f, g \in M_2(\mathbb{C})$ parameterise the right connection. To check if this is a bimodule connection, we use

$$\tilde{\sigma}_E(\mathrm{d}a \otimes x) = \tilde{\nabla}_E(a.x) - a.\tilde{\nabla}_E x = 1 \otimes (\mathrm{d}a + [fs + gt, a]).x.$$

A condition for $\tilde{\sigma}_E$ to exist is that $f_{11} = f_{22}$ and $g_{11} = g_{22}$, and then

$$\begin{aligned}\tilde{\sigma}_E(([E_{12}, a]s + [E_{21}, a]t) \otimes 1) = 1 \otimes \big(&(1 + f_{12})[E_{12}, a]s + (1 + g_{21})[E_{21}, a]t \\ &+ f_{21}[E_{21}, a]s + g_{12}[E_{12}, a]t\big),\end{aligned}$$

giving

$$\tilde{\sigma}_E((bs + ct) \otimes 1) = 1 \otimes \big(((1 + f_{12})b + f_{21}c)s + ((1 + g_{21})c + g_{12}b)t\big)$$

for all $b, c \in M_2(\mathbb{C})$. To see if this connection is extendable, we calculate

$$\begin{aligned}
&(\mathrm{id} \otimes \wedge)(\tilde{\sigma}_E \otimes \mathrm{id})((b's + c't) \otimes \tilde{\sigma}_E((bs + ct) \otimes 1)) \\
&\quad = 1 \otimes \big(((1 + f_{12})b' + f_{21}c')s \\
&\qquad + ((1 + g_{21})c' + g_{12}b')t\big) \wedge \big(((1 + f_{12})b + f_{21}c)s + ((1 + g_{21})c + g_{12}b)t\big) \\
&\quad = 1 \otimes \big(((1 + f_{12})b' + f_{21}c')((1 + g_{21})c + g_{12}b) \\
&\qquad + ((1 + g_{21})c' + g_{12}b')((1 + f_{12})b + f_{21}c)\big)st
\end{aligned}$$

and this must equal

$$\tilde{\sigma}_E\big((b's + c't) \wedge (bs + ct) \otimes 1\big) = (b'c + c'b)\tilde{\sigma}_E(st \otimes 1).$$

To solve this we have the conditions $(1 + f_{12})g_{12} = (1 + g_{21})f_{21} = 0$ and then

$$\tilde{\sigma}_E(st \otimes 1) = ((1 + f_{12})(1 + g_{21}) + g_{12}f_{21})1 \otimes st$$

to give extendability. The curvature of the connection is given by

$$\begin{aligned}
\tilde{R}(a) &= 1 \otimes \big(\mathrm{d}(fs + gt) + (fs + gt) \wedge (fs + gt)\big)a \\
&= 1 \otimes \big(E_{21}f + fE_{21} + E_{12}g + gE_{12} + fg + gf)sta,
\end{aligned}$$

and the condition for this to be a bimodule map, in addition to our earlier assumption $f_{11} = f_{22}$ and $g_{11} = g_{22}$ to force a bimodule connection, is that $g_{11}(1 + f_{12}) + f_{11}g_{12} = 0$ and $f_{21}g_{11} + f_{11}(1 + g_{21}) = 0$. Next we consider compatibility with the inner product, namely

$$\mathrm{d}(x^*y) = (\langle\ ,\ \rangle \otimes \mathrm{id})(\overline{x} \otimes \tilde{\nabla}_E y) + (\mathrm{id} \otimes \langle\ ,\ \rangle)\nabla\overline{x} \otimes y)$$

for all $x, y \in M_2(\mathbb{C})$, where $\nabla\overline{x} = \kappa^* \otimes \overline{z}$ if $\tilde{\nabla}_E x = z \otimes \kappa$. In our case this comes down to $(fs + gt)^* = -fs - gt$, which reduces to $g = f^*$. In this way, we obtain three cases for the bimodule connection to be extendable, preserve the inner product and have curvature a bimodule map, namely $g = f^*$ and

(a) $f_{21} = 0$ and $f_{11} = f_{22} = 0$;
(b) $f_{21} = 0$ and $f_{12} = -1$ and $f_{11} = f_{22}$;
(c) $f_{11} = f_{22} = 0$ and $f_{12} = -1$.

If we now look at $\tilde{\nabla}_E e = 0$, the solutions split up according to the previous cases:

(a) require $|1 + f_{12}| = 1$ for nonzero e;
(b) no nonzero e;
(c) require $|f_{21}| = 1$ for nonzero e,

and correspondingly e is required to be of the form

$$(\mathrm{a}):\quad e=\begin{pmatrix}\lambda(1+f_{12}) & 0\\ 0 & \lambda\end{pmatrix},\qquad (\mathrm{c}):\quad e=\begin{pmatrix}0 & \lambda\\ \lambda f_{21} & 0\end{pmatrix}$$

for any $\lambda\in\mathbb{C}$ (they are unitary when $|\lambda|=1$). Proposition 4.86 then extends $\phi(a)=e^*ae$ to a cochain map $\phi:\Omega_A\to\Omega_A$ as

$$\begin{aligned}\phi(bs+ct)&=(\langle\ ,\ \rangle\otimes\mathrm{id})(\overline{e}\otimes\tilde\sigma_E((bs+ct)\otimes e))\\&=\langle\overline{e},1\rangle\big(((1+f_{12})b+f_{21}c)s+((1+g_{21})c+g_{12}b)t\big)e\\&=e^*\big(((1+f_{12})b+f_{21}c)s+((1+\overline{f}_{12})c+\overline{f_{21}}b)t\big)e,\\ \phi(ast)&=(\langle\ ,\ \rangle\otimes\mathrm{id})(\overline{e}\otimes\tilde\sigma_E(ast\otimes e))\\&=\langle\overline{e},a\rangle((1+f_{12})(1+g_{21})+g_{12}f_{21})ste\\&=e^*a(|1+f_{12}|^2+|f_{21}|^2)est=e^*aest,\end{aligned}$$

given the restrictions. We see that ϕ on the coefficients in degree 2 is the same as in degree 0 while ϕ in degree 1 preserves the set of antihermitian elements of Ω^1_A. ◇

4.6 Relative Cohomology and Cofibrations

There are two short topics left to discuss in this chapter, one relatively straightforward and one for discussion. We begin with relative cohomology. Given cochain complexes (F,d) and (G,d) with a cochain map $\phi:F^n\to G^n$ for all n (i.e., $\mathrm{d}\phi=\phi\mathrm{d}$), we form a new complex $E^n=F^n\oplus G^{n-1}$ with $\mathrm{d}(f,g)=(\mathrm{d}f,\phi(f)-\mathrm{d}g)$. Then

$$\mathrm{d}^2(f,g)=\mathrm{d}(\mathrm{d}f,\phi(f)-\mathrm{d}g)=(\mathrm{d}^2f,\phi(\mathrm{d}f)-\mathrm{d}\phi(f)+\mathrm{d}^2g)=(0,0).$$

We call the cohomology of (E,d) the *relative cohomology* $\mathrm{H}(F,G,\phi)$.

Proposition 4.90 *Given a cochain map $\phi:F^n\to G^n$ for all n we have a long exact relative cohomology sequence*

$$\dots\mathrm{H}^{n-1}(G)\xrightarrow{i_2^*}\mathrm{H}^n(F,G,\phi)\xrightarrow{\pi_1^*}\mathrm{H}^n(F)\xrightarrow{\phi^*}\mathrm{H}^n(G)\xrightarrow{i_2^*}\mathrm{H}^{n+1}(F,G,\phi)\dots,$$

where $\pi_1:E^n\to F^n$ is $\pi(f,g)=f$ and $i_2:G^n\to E^{n+1}$ is $i_2(g)=(-1)^n(0,g)$.

Proof Standard algebraic manipulation. Looking at the kernel of $\phi^*:\mathrm{H}^n(F)\to\mathrm{H}^n(G)$ shows that $\mathrm{H}^n(F,G,\phi)$ is defined precisely to make this work. □

This is the standard definition of relative cohomology but it is not the only definition. Godbillon made a definition which in our notation would be the cohomology of the complex $(\ker\phi)^n = \{x \in F^n : \phi(x) = 0\}$. There is a cochain map $i_1 : (\ker\phi)^n \to E^n$ given by $i_1(x) = (x, 0)$, but in general this will not give an isomorphism in cohomology.

Proposition 4.91 *The map $i_1^* : \mathrm{H}^n(\ker\phi) \to \mathrm{H}^n(E)$ is an isomorphism for $n = 0$, and if ϕ is onto then i_1^* is an isomorphism for all $n \geq 0$.*

Proof The kernel of $\mathrm{d} : E^0 \to E^1$ is the set of $(x, 0) \in F^0 \oplus 0$ such that $\mathrm{d}(x, 0) = (\mathrm{d}x, \phi(x)) = (0, 0)$, so it is just the same as the kernel of $\mathrm{d} : (\ker\phi)^0 \to (\ker\phi)^1$. Now let Z^n_E and $Z^n_{\ker\phi}$ be the kernels of $\mathrm{d} : E^n \to E^{n+1}$ and $\mathrm{d} : (\ker\phi)^n \to (\ker\phi)^{n+1}$ respectively, and B^n_E and $B^n_{\ker\phi}$ be the images of $\mathrm{d} : E^{n-1} \to E^n$ and $\mathrm{d} : (\ker\phi)^{n-1} \to (\ker\phi)^n$. For $n \geq 1$ the cohomology map is

$$i_1^* : \mathrm{H}^n(\ker\phi) = \frac{Z^n_{\ker\phi}}{B^n_{\ker\phi}} \to \mathrm{H}^n(E) = \frac{Z^n_E}{B^n_E}.$$

This is injective when $i_1 Z^n_{\ker\phi} \cap B^n_E \subseteq i_1 B^n_{\ker\phi}$. If $(x, 0) \in i_1 Z^n_{\ker\phi} \cap B^n_E$, then $(x, 0) = (\mathrm{d}z, \phi(z) - \mathrm{d}w)$ for some $z \in F^{n-1}$ and $w \in G^{n-2}$, and also $\phi(x) = 0$. If ϕ is onto, then take $u \in F^{n-2}$ such that $\phi(u) = w$. Then $(x, 0) = (\mathrm{d}z, \phi(z - \mathrm{d}u))$, so $x = \mathrm{d}(z - \mathrm{d}u)$ and $z - \mathrm{d}u \in (\ker\phi)^{n-1}$.

The map i_1^* is surjective if, for all $(x, y) \in Z^n_E$, there is a $(z, w) \in E^{n-1}$ such that $(x, y) - \mathrm{d}(z, w) \in i_1 Z^n_{\ker\phi}$. For this it is sufficient to show that $y - \phi(z) + \mathrm{d}w = 0$. If ϕ is onto we can set $w = 0$ and choose z such that $y = \phi(z)$. □

The motivating context in which Godbillon made his definition was the inclusion of a submanifold in a manifold, and the de Rham complex. This is a case where i_1 does induce an isomorphism as usually shown by considering tubular neighbourhoods. In our case we see it from Proposition 4.91 as follows. Suppose that N (dimension n) is an embedded submanifold of M (dimension m). Then, essentially by definition, there is a collection of coordinate charts U for M with $N \cap U$ given by $\{(x_1, \ldots, x_m) \in U : x_{n+1} = \cdots = x_m = 0\}$. From this we can show that any form on N is the restriction of a form on M, so the restriction map for forms is surjective. We are really looking at the property of the embedding map $N \to M$, but in this case as N can be given the induced differential structure from M, confusion should not arise. We now give a noncommutative example, of which the classical limit is the embedding of the diagonal unitary matrices $U_1 \subset SU_2$.

Example 4.92 We consider the surjective differentiable Hopf $*$-algebra map $\pi : \mathbb{C}_q[SU_2] \to \mathbb{C}_{q^2}[S^1]$ given in Proposition 2.33. We use the De Rham cohomology of the 3D calculus of $\mathbb{C}_q[SU_2]$ as calculated in Corollary 4.35. Now we have the relative cohomology long exact sequence

$$\mathrm{H}^0(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \to \mathbb{C} \to \mathbb{C} \to \mathrm{H}^1(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \to 0 \to \mathbb{C} \to$$
$$\mathrm{H}^2(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \to 0 \to 0 \to \mathrm{H}^3(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \to \mathbb{C} \to 0 \to \dots$$

At the beginning of the sequence the map $\mathbb{C} \to \mathbb{C}$ sends the generator $1 \in \mathbb{C}_q[SU_2]$ to $1 \in \mathbb{C}_{q^2}[S^1]$, so we can read off, for $n \geq 4$,

$$\mathrm{H}^2(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \cong \mathrm{H}^3(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \cong \mathbb{C},$$
$$\mathrm{H}^1(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \cong \mathrm{H}^n(\mathbb{C}_q[SU_2], \mathbb{C}_{q^2}[S^1], \pi) \cong 0. \qquad \diamond$$

We close with a brief discussion of what would be needed for noncommutative homotopy theory. There is an abstract categorical approach to homotopy theory, but which is very dependent on just what category we use. The theory we want should reduce to classical homotopy theory to as great an extent as possible, so that is where we start.The reader may be familiar with the *homotopy extension property*. This is based on the following diagram of topological spaces and continuous functions,

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "i_0"'] & E \arrow[d, "\pi"] \\ Y \times [0,1] \arrow[r, "g"] \arrow[ur, dotted, "h"] & B, \end{tikzcd}$$

where $i_0(y) = (y, 0)$. If $Y = [0, 1]^n$ we can ask whether, given the solid arrow commuting diagram, is there necessarily an h (the dotted arrow) such that the diagram still commutes? If the answer is yes for all n, p, g then by definition $\pi : E \to B$ is a *Serre fibration*. Such fibrations are fundamental to homotopy theory. This, and other related ideas, was generalised by Quillen into the idea of a *model category*.

A model category $\mathcal{C}$ by definition has three special types of morphism, fibrations, cofibrations and weak equivalences. These are required to obey several conditions, with one of them being similar to the one we have already seen. Take the diagram of objects and morphisms in $\mathcal{C}$:

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "i"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "g"] \arrow[ur, dotted, "h"] & B. \end{tikzcd}$$

Here i is a cofibration, π is a fibration and p, g are any morphisms such that the solid arrows commute. Then in two cases we have to be able to find h such that the diagram still commutes. The first case is that i is also a weak equivalence, and the second is that π is also a weak equivalence. (The case of the Serre fibration corresponds to the case when i is a weak equivalence.)

Next we switch to continuous functions on our spaces, in which case maps are reversed to give algebra maps going the other way. However, when we talked about fibrations, such as the Hopf fibration, in noncommutative geometry we kept the connection to the fibration of spaces (i.e., we called the noncommutative analogue of the Hopf fibration a fibration). The net result is that our fibration arrows go the other way round, and as taking the opposite category (i.e., just redrawing the arrows the other way round) would only cause more confusion, we take the following diagram for a fibration in terms of algebras or DGAs:

$$\begin{tikzcd}
C & A \arrow[l, "p"'] \arrow[dl, dotted, "h"] \\
D \arrow[u, "\phi"] & B \arrow[l, "g"] \arrow[u, "\iota"']
\end{tikzcd}$$

This brings us to the natural question: is there a model category for doing homotopy theory in noncommutative geometry in which the fibrations are (generalisations of) the differential fibrations of §4.4? The weak equivalences should induce isomorphisms in the de Rham cohomology (and likely satisfy more conditions). We would also need a notion of cofibration, which classically generalises the notion of embedded submanifold where the manifold retracts to the submanifold, i.e., there is a continuous map from the manifold to the submanifold which is the identity on the submanifold. It is beyond our scope to address these matters here other than to note that usually in noncommutative geometry one does not have enough algebra maps compared to the classical case; one could expect to need more general notions of morphism such as completely positive maps and bimodules as discussed in this chapter.

Exercises for Chap. 4

E4.1 Use $a = c = d = 1$ and $b = e = 0$ in case (v) of the 5-parameter moduli of left and right-invariant connections ∇ on $\Omega^1(S_3)$ in Example 3.76 to illustrate that the curvature R_∇ obeys the first and second Bianchi identities (it suffices to evaluate them on e_u). Also do some checks to verify that $\int(a\mathrm{Vol}) = \sum_{x\in S_3} a(x)$ is a 4-cycle and that $\int \mathrm{Tr} R_\nabla^2 = 0$.

E4.2 Let $\mathbb{Z}_4 = \{1, z, z^2, z^3\}$ with $z^4 = 1$, and let $H = P = \Bbbk(\mathbb{Z}_4)$ with H left coacting on P by the coproduct, $\Delta_L(\delta_{z^j}) = \sum_i \delta_{z^i} \otimes \delta_{z^{j-i}}$. Give H and P Cayley graph calculi for the subsets $\mathcal{C}_H \subseteq \mathbb{Z}_4 \setminus \{1\}$ and $\mathcal{C}_P = \{z^1, z^2\}$ and associated left-invariant 1-forms $h_i \in \Omega^1_H$ for $z^i \in \mathcal{C}_H$ and $e_1, e_2 \in \Omega^1_P$ for

$z^1, z^2 \in \mathcal{C}_P$, and use the maximal prolongation calculi whereby $e_1 \wedge e_2 = e_2 \wedge e_1 = 0$ for P. Following Example 4.30, show that (i) if $\mathcal{C}_H = \{z^3\}$ then Δ_L is not differentiable; (ii) if $\mathcal{C}_H = \{z\}$ then Δ_L is differentiable and $\Delta_{L*}e_1 = 1 \otimes e_1 + h_1 \otimes 1$, $\Delta_{L*}e_2 = 1 \otimes e_2$.

E4.3 Show that the 2-parameter noncommutative torus $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$ (see Exercise E1.5) is differentiable for the right coaction of $\mathbb{C}_q[S^1] \otimes \mathbb{C}_q[S^1]$ corresponding to bigrading by the powers of u and v respectively in a monomial. Here q enters into the differential structure of each circle as in Example 1.11. [Differentiability in fact dictates this form of the calculus on $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$.]

E4.4 For $A = M_2(\Bbbk)$ with its standard calculus and a line module E with single generator e^1 as in Exercise E3.10, find the most general form of bimodule connection for which (E, ∇, σ) is an object in the category ${}_A\mathcal{G}_A$ (see Theorem 4.11) and show that some of these have nonzero curvature. [Hint: Use the same parameters γ_i and ${c_i}^j$ as in E3.10 and reuse some of the formulae from its solution.]

E4.5 The four-dimensional Sweedler–Taft algebra $u_{-1}(b_+)$ has a unique 2D bicovariant calculus (from E2.8 at $q = -1$ or E2.2 for the ideal $\{x - xt\}$). We take this with $\{e_1, e_2\}$ as basis of Λ^1 and relations

$$\mathrm{d}t = te_2, \quad \mathrm{d}x = xe_2 + e_1, \quad \mathrm{d}(xt) = te_1,$$

$$e_2 t = -te_2, \quad e_1 t = te_1, \quad e_1 x = xe_1, \quad e_2 x = -xe_2 - 2e_1,$$

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0, \quad e_1 \wedge e_2 = e_2 \wedge e_1, \quad \mathrm{d}e_2 = 0, \quad \mathrm{d}e_1 = -e_1 \wedge e_2.$$

Show that its cohomology $\mathrm{H}_{\mathrm{dR}}(u_{-1}(b_+))$ has dimensions $1 : 2 : 1$ in grades 0,1,2 (and zero for higher), and that the left $u_{-1}(b_+)$-coaction on the de Rham cohomology is trivial. Also calculate the cohomology of the left-invariant forms Λ and show that $\mathrm{H}^0(\Lambda) \cong \mathrm{H}^1(\Lambda) \cong \mathbb{C}$ with all others zero. [Corollary 4.26, which would have equated these cohomologies, does not apply as the integral cannot be normalised.]

E4.6 Write out the second page of the van Est spectral sequence for $u_{-1}(b_+)$ with the calculus in E4.5. Deduce that ${\mathrm{H}_c}^0(u_{-1}(b_+); \mathbb{C}) \cong \mathbb{C}$ (with trivial coaction on $\mathbb{C}$) and that if ${\mathrm{H}_c}^1(u_{-1}(b_+); \mathbb{C}) \cong 0$ (which is actually the case) then ${\mathrm{H}_c}^2(u_{-1}(b_+); \mathbb{C}) \cong \mathbb{C}$.

E4.7 For a finite group G, identify right G-representations F with $\Bbbk(G)$-comodules in the usual way (so $\Delta_L f \in \Bbbk(G) \otimes F$ evaluates at $g \in G$ to $f{\triangleleft}g$). Show that the cohomology of $H = \Bbbk(G)$ with coefficients in the left comodule F in Proposition 4.55 can be identified with the usual group cohomology $H_{\triangleleft}(G, F)$. This can be defined by cochain complex

(K, d) where $K^n = C(G^n, F)$ is functions on G^n with values in F and, for $\phi \in K^n$,

$$(\mathrm{d}\phi)(g_1, g_2, \ldots, g_{n+1}) = \phi(g_2, \ldots, g_{n+1}) - \phi(g_1 g_2, g_3, \ldots, g_{n+1}) \\ + \phi(g_1, g_2 g_3, \ldots, g_{n+1}) - \ \ldots \ - (-1)^n \phi(g_1, \ldots, g_n) \triangleleft g_{n+1}.$$

E4.8 On the Connes–Landi sphere in Example 4.38, consider an fgp module defined by the 4×4 projection matrix given in 2×2 block form as

$$P = \frac{1}{2}\begin{pmatrix} (1 + x^3)\mathrm{id} & Q \\ \mathbb{Q}^* & (1 - x^3)\mathrm{id} \end{pmatrix}; \quad Q = \sqrt{2}\begin{pmatrix} x^1 & -rx^2 \\ rx^4 & x^5 \end{pmatrix}$$

where $QQ^* = Q^*Q = (1-(x^3)^2)\mathrm{id}$ and $r \in \mathbb{C}$ with $r^2 = q$. For the 4-cycle $\phi(a.\mathrm{Vol}) = \int a$, show that

$$\begin{aligned} &32\phi \circ \mathrm{Tr}(\mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P.P) \\ &\quad = \phi \circ \mathrm{Tr}\big((1 + x^3)\mathrm{d}Q \wedge \mathrm{d}Q^* \wedge \mathrm{d}Q \wedge \mathrm{d}Q^* + 4\mathrm{d}x^3 \wedge \mathrm{d}Q \wedge \mathrm{d}Q^* \wedge \mathrm{d}Q.Q^* \\ &\qquad - 4\mathrm{d}x^3 \wedge \mathrm{d}Q^* \wedge \mathrm{d}Q \wedge \mathrm{d}Q^*.Q + (1 - x^3)\mathrm{d}Q^* \wedge \mathrm{d}Q \wedge \mathrm{d}Q^* \wedge \mathrm{d}Q\big). \end{aligned}$$

By using the properties of ϕ and the relation for Q, show that

$$16\phi \circ \mathrm{Tr}(\mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P.P) = 5\phi \circ \mathrm{Tr}(x^3\mathrm{d}Q \wedge \mathrm{d}Q^* \wedge \mathrm{d}Q \wedge \mathrm{d}Q^*)$$

[Hint: because of the matrix trace, you can move matrices from the front to the back using the rules for the 4-cycle on the matrix entries]. Now calculate $\mathrm{d}Q \wedge \mathrm{d}Q^*$ and hence show that

$$\phi \circ \mathrm{Tr}(\mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P \wedge \mathrm{d}P.P) = 15\phi((x^3)^2\mathrm{Vol}) = 3.$$

E4.9 For $u_{-1}(b_+)$ with its 2D calculus in E4.5, define a left connection on Ω^1 by

$$\nabla e_1 = xte_1 \otimes e_2, \quad \nabla e_2 = -e_2 \otimes e_2.$$

Show that this connection has zero curvature and preserves the map $\phi : \Omega^1 \to \Omega^1$ given by $\phi(e_1) = te_2$ and $\phi(e_2) = 0$. For the resulting short exact sequence of sheaves,

$$0 \longrightarrow K \longrightarrow \Omega^1 \longrightarrow \Omega^1/K \longrightarrow 0,$$

where $K = \ker\phi$, write down the quotient connection on Ω^1/K and calculate H^0 of all three sheaves. Deduce that the map $\mathrm{H}^0(\Omega^1/K) \to \mathrm{H}^1(K)$ in the corresponding long exact cohomology sequence is nonzero.

E4.10 Recall the definition of $B = \mathbb{C}_q[D]$ and the $*$-algebra map $\pi : \mathbb{C}_q[D] \to \mathbb{C}_{q^2}[S^1] = A$ given by $\pi(z) = t$, $\pi(\bar{z}) = t^* = t^{-1}$ in Example 3.40. Here A is $\mathbb{C}[t, t^{-1}]$ as an algebra viewed with the $*$-calculus in Example 1.11 (with q^2 instead of q there). Let $E = {}_\pi A$ be the B-A bimodule given by A as a vector space with right action the product in A and left action $b{\triangleright}a = \pi(b)a$. Show that $\langle \overline{a}, a' \rangle = a^* a'$ makes E into a B-A bimodule with inner product-valued in A and that for $a \in {}_\pi A$,

$$\tilde{\nabla} a = \mathrm{d}a \in {}_\pi \Omega^1_A \cong {}_\pi A \otimes_A \Omega^1_A$$

is a right B-A bimodule connection on E compatible with the inner product.

Notes for Chap. 4

Our use of the term DG category in §4.1, where d^2 is given by a commutator (a curved DGA), is in agreement with [42]. In particular, Theorem 4.3 means that each $(\mathrm{Mor}_*(E, E), \nabla, R_E)$ in ${}_A\mathcal{G}$ forms a curved DGA as in [42]. For more on DG categories see [158].

The theory of characteristic classes using differential forms and curvature was set out by Chern in [77]. The proof of the independence of the trace of the curvature in classical geometry under changing the connection is laid out in Kobayashi & Nomizu [168], which we mimic in the noncommutative case following [36]. The latter also contained the ${}_A\mathcal{G}_A$ construction and Bianchi identity results. One thing which is so far missing in the noncommutative case is a theory of classifying spaces. In Example 4.23 the fact that the only invertible elements of $\mathbb{C}_q[SU_2]$ are multiples of the identity is from [134].

When a Lie group acts on a manifold, the *basic* differential forms are defined to be the forms *horizontal and invariant* for the action [253]. Chevalley & Eilenberg in [78] considered Lie group actions on manifolds and de Rham cohomology. Some of those results were extended by the first author and Brzeziński in [22] to the coaction of Hopf algebras on algebras with differential structure, which forms the basis of §4.2. Heckenberger & Schüler [139] showed that the de Rham cohomology of the bicovariant calculi on the quantum groups $\mathbb{C}_q[SL_m]$, $\mathbb{C}_q[SO_m]$ and $\mathbb{C}_q[Sp_m]$ for q generic is the same as the cohomology of the left-invariant forms. Classically, these Lie groups are connected, while in the non-connected case for m odd they found that the de Rham cohomology of $\mathbb{C}_q[O_m]$ is the cohomology of the left coinvariant forms plus the cohomology of the left coinvariant forms times the quantum determinant.

Since we are working over a field, the existence of a normalised left-integral on a Hopf algebra here is equivalent to the latter being *cosemisimple* (i.e., a sum of simple coalgebras) [310, Sec. 14.0.3]. An integral on the undeformed unit disk which has classical limit the Lebesgue integral was given in [164] and used to examine noncommutative function theory on this example. The integral in

Example 4.37 is motivated by invariance under the $U_q(su_{1,1})$ action on the disk and is likened to a classical compactly supported de Rham cohomology class (see [50]), but we do not attempt to make this idea precise in noncommutative geometry. A more detailed account of the domain of the invariant integral can be found in [319]. The Connes–Landi sphere in Example 4.38 was first described in [87]. We use the notation and results from [9], which differs from the notation in [87] by $x^1 = \sqrt{2}\,\alpha$, $x^5 = \sqrt{2}\,\alpha^*$, $x^2 = \sqrt{2}\,\beta$, $x^4 = \sqrt{2}\,\beta^*$, and $x^3 = 2t - 1$. Note that Milnor in [255] showed that there were nonisomorphic differential structures on manifolds which were topologically the same compact Hausdorff topological space (exotic 7-spheres). However, their de Rham cohomology is the same, being isomorphic to topological cohomology. This is not true in the noncommutative case, e.g., the 3D and 4D calculus on $\mathbb{C}_q[SU_2]$ have different de Rham cohomologies, as we saw in Corollary 4.35 and Example 4.27.

§4.3 is based on [23], where the first author and Brzeziński show that a cohomology theory based on noncommutative differential calculi and zero curvature connections on modules behaves in many respects like classical sheaf cohomology. If we take the classical definition of a sheaf as a local homeomorphism of topological spaces $\pi : S \to M$ then given $s \in S$, there is a neighbourhood $s \in U$ such that $\pi : U \to M$ is a homeomorphism. We can use this homeomorphism to transfer the differential structure from the open set of a manifold M to U, and thus we get a flat connection on $\Omega(U)$, corresponding to d on $\Omega(\pi(U))$. The reader can find more about classical sheaves over topological spaces in [54].

To transfer information between different ways of studying algebras, it is important to consider other sheaf cohomology theories. In noncommutative algebraic geometry, a central role is taken by categories of quasicoherent sheaves (see Stafford & van den Bergh [307]). Another point of view is that quantales might potentially provide a calculable noncommutative sheaf cohomology for C^*-algebras without differential structure (for the general theory of quantales see [45] and [262]). For another approach to sheaves and C^*-algebras, see [7]. There is also an idea of sheaves and their homological algebra in graph theory [118] which should be of interest in connection with graph calculi.

For the classical cup product in cohomology, as well as a general view of topological cohomology theory, see [306]. Definition 4.50 and Examples 4.51 and 4.52 in §4.4 are taken, with minor modifications, from the excellent book on spectral sequences by McCleary [247], to which we refer the interested reader.

The cohomology of the left-invariant differential forms on a Lie group can be taken to define the Lie algebra cohomology [147]. The van Est spectral sequence in its original form relating the cohomologies of a Lie group, the Lie algebra and the group cohomology, can be found in [46]. Group cohomology is described in [58] and a key part of extending the van Est spectral sequence to noncommutative geometry in §4.4.2 is extending this cohomology to Hopf algebras, as we do in Corollary 4.58. The main results here are due to the first author and Brzeziński in [22]. In fact, the noncommutative van Est spectral sequence itself can be seen as a unital coalgebra version of the standard cohomology of augmented algebras. The idea of a differential fibration and the Serre spectral sequence in this context was introduced in [23], and the generalisation including sheaves in the spectral sequence of a fibration in [37]. Example 4.62 is adapted from [107, 108], where a definition of noncommutative fibration was given without needing a differential structure and where the base was a topological space.

A well-known illustration of the comment at the start of §4.5 about needing additional morphisms between algebras is the notion of *asymptotic morphisms* introduced by Connes & Higson [86] to allow enough morphisms to do E-theory. They were also used in a noncommutative analog of topological shape theory in [94] and [242]. More relevant to the chapter, Connes used bimodules as a more general notion of morphism, which he termed correspondences, to study representations of von Neumann algebras [84]. The idea of using differential bimodules to give maps on sheaf cohomology, as in §4.5.1, comes from [23]. The coloured monoidal category approach is used to define bicategories in [299].

In §4.5.2 we used the well-known book by Lance [174] and notes by Skoufranis [304] for Hilbert C^*-modules, and we refer to these for more background. Note that some authors use a different definition of Hilbert C^*-bimodule, see e.g., [156]. The criterion for an element of a matrix algebra of a C^*-algebra to be positive used in the proof of Lemma 4.79 is standard and can be found in [304] and many other places. In Example 4.89 we apply a very special case of Choi's result [79] that completely positive maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ are exactly those of the form $\phi(a) = \sum_i v_i{}^* a v_i$. Example 4.88 is taken from [4] and builds on an adaptation of the Hilbert space representation of $\mathbb{C}_q[SU_2]$ in [325] used by Woronowicz to study the properties of the C^* completion of $\mathbb{C}_q[SU_2]$. A Fock space construction can be used to give an inner product on the sum of tensor products of a Hilbert C^*-bimodule, and from here Toeplitz algebras and Cuntz–Krieger–Pimsner algebras can be defined [279]. For an argument that positive maps, other than completely positive maps, are important in quantum mechanics and quantum information theory, see [301].

§4.6 starts with the definition of relative cohomology and the exact sequence as in Bott & Tu [50], while Godbillon's definition is in [122] (we also benefitted from clarifying comments by J. Ebert relating the two). While we have presented a working definition of fibration, to have a model category for homotopy in the sense of Quillen [283], we would also need an idea of cofibration, which is lacking in our differential setting. A further reference for model categories for homotopy theory is [143] while a topological discussion of cofibrations is in [306]. An alternative approach to finding a model category might be to develop Brown's homotopy theory [57] in which ones requires only the existence of fibrations and weak equivalences, along with some other axioms. This is used to develop the homotopy theory of C^*-algebras in [316] and algebraic Kasparov K-theory [121]. A detailed discussion of homotopy theory for C^*-algebras, including an idea of cofibrations, is in the book by Østvær [275] and in [149]. There is also an approach to bivariant K-theory via triangulated categories [250, 251].

The exercises mostly extend the examples introduced in the exercises in earlier chapters, with E4.1 taken from [36]. Exercise E4.7 is a general result which is part of the classical theory of augmented algebras when Proposition 4.55 is seen as a coalgebra version of this. The result of E4.8 is in [87] but used now to illustrate the methods of the chapter.

Chapter 5
Quantum Principal Bundles and Framings

Vector bundles in classical geometry typically arise as associated to something deeper, a principal bundle. A connection on this then induces covariant derivatives on all associated bundles in a coherent way. This is the situation in Riemannian geometry where a 'spin connection' on the frame bundle induces the Levi-Civita connection on tensor fields but also a covariant derivative on the spinor bundle in the case of a spin manifold, leading to the Dirac operator. Similarly in gauge theory, a principal connection induces covariant derivatives on all associated matter fields.

Briefly, a principal G-bundle P over a manifold X is defined exactly like a vector bundle with a surjection $\pi : P \to X$ but each fibre $P_x = \pi^{-1}(x)$ now has the structure of a fixed group G. This is achieved by starting with a free right action of G on the manifold P such that $X = P/G$. Free here means any non-identity element of G acts without fixed points, which is equivalent to saying that the map

$$P \times G \to P \times P, \quad (p, g) \mapsto (p, p^g) \tag{5.1}$$

is an inclusion, where p^g denotes the right action of $g \in G$ on $p \in P$. A connection on P is defined concretely as an equivariant complement in $\Omega^1(P)$ to the 'horizontal forms' (those pulled back from $\Omega^1(X)$). This is, however, equivalent to $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ with certain properties, where $\mathfrak{g}$ is the Lie algebra of G. We will see details in the noncommutative case. Given this data, there is an associated vector bundle $E = P \times_G V$ and a connection ∇ on it, for any representation V of G. We will give the algebraic and potentially 'quantum' version of this notion where the structure group is now a Hopf algebra or 'quantum group' as in Chap. 2. We will then use this theory to understand the geometry of quantum homogeneous spaces and framed quantum manifolds more generally.

Before doing this, let us recall another concept where something like a principal bundles occurs, namely in Galois theory. Suppose you are working over the rationals $\mathbb{Q}$ and want to solve the equation $x^2 - 2 = 0$. You will not be able to, but one thing you can do is 'add in' the solution you want by extending the field to $\mathbb{Q}[\sqrt{2}]$ with

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the usual rules of a field extended to the symbol $\sqrt{2}$ and the rule that its square is 2. Now this bigger field, called the splitting field of the original polynomial equation, has automorphisms that fix the base field $\mathbb{Q}$ much like a gauge transformation in physics. The set of such transformations is the *Galois group* of the field extension. In this example it is $\mathbb{Z}_2$, where the nontrivial element sends $\sqrt{2} \mapsto -\sqrt{2}$. We will see that such Galois extensions are also examples of principal bundles when done algebraically, in this case a trivial quantum principal bundle given by a twisted group algebra $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\mathbb{Z}_2)^\chi$ with respect to a nontrivial 2-cocycle $\chi \in H^2(\mathbb{Z}_2, \mathbb{Q})$. Again, we can easily replace a group by a Hopf algebra or quantum group, i.e., a Hopf–Galois extension.

5.1 Universal Calculus Quantum Principal Bundles

The theory of quantum principal bundles properly requires algebras to be equipped with differential structures. However, there is a warm up which just takes the universal differential calculus in Proposition 1.5 for everything, more like the notion of a topological principal bundle. *Throughout §5.1–§5.3 Ω^1 will refer to this universal calculus, i.e., the label 'uni' should be understood.*

In this case, our only data for a bundle is a Hopf algebra H, an algebra P which is a right H-comodule algebra, i.e., equipped with a right coaction $\Delta_R : P \to P \otimes H$, subject to certain conditions. Here Δ_R is an algebra homomorphism and $A = P^H \subseteq P$ is the subalgebra of elements fixed under Δ_R. We recall that this means elements $a \in P$ such that $\Delta_R a = a \otimes 1$. The universal $\Omega^1_P \subset P \otimes P$ is given by the kernel of the product map with differential $\mathrm{d}p = 1 \otimes p - p \otimes 1$ for all $p \in P$. We also remark that when the Hopf algebra has invertible antipode, the right coaction Δ_R on P converts to a left coaction

$$\Delta_L p = S^{-1} p_{(\bar{1})} \otimes p_{(\bar{0})} \tag{5.2}$$

for all $p \in P$, making P a left H^{op}-comodule algebra. Then $\mathrm{id} \otimes \Delta_R$, $\Delta_L \otimes \mathrm{id}$ make $P \otimes P$ and Ω^1_P into H-bicomodules (and H^{op}-H-bicomodule algebras). Using the right side, we let $H^+ = \ker \epsilon$ and define

$$\mathrm{ver} : \Omega^1_P \to P \otimes H^+, \quad \mathrm{ver} = (\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_R),$$

where $\cdot$ denotes the product in P. Applying $\mathrm{id} \otimes \epsilon$ shows that the image is in $P \otimes H^+$ due to the restriction to Ω^1_P. Evaluating ver against elements of H^* would give maps $\Omega^1_P \to P$ for each such element, which play the role geometrically of the vertical vector fields on P generated (in the classical case) by the action of the Lie algebra of the structure group of the bundle. This is the geometric content of the map ver. We also define the 'horizontal forms' as the universal 1-forms Ω^1_A on the base extended to a sub-bimodule of the bigger algebra,

$$\Omega^1_{\mathrm{hor}}P := P\Omega^1_A P \subseteq \Omega^1_P,$$

and ver vanishes on this subspace since $\Delta_R(ap) = a\Delta_R(p)$ for all $a \in A$, $p \in P$.

Definition 5.1 A *universal quantum principal bundle* with structure Hopf algebra H is a right H-comodule algebra P such that

$$0 \to P\Omega^1_A P \to \Omega^1_P \xrightarrow{\mathrm{ver}} P \otimes H^+ \to 0$$

is a short exact sequence.

The surjectivity of ver in this definition corresponds classically to freeness of the action (no vertical vector field is trivial) where the map (5.1) was injective, while exactness in the middle side says that the 'horizontal' forms from the base are exactly the kernel of ver. This replaces the classical concept of the existence of a local trivialisation, which would normally be used to prove such transversality.

Next, we define a connection on a quantum bundle with universal calculus as an equivariant splitting of this sequence in the sense of Definition 3.110, i.e., a choice of equivariant complement to the horizontal 1-forms.

Definition 5.2 A connection on a universal quantum principal bundle (P, H, Δ_R) is $\Pi : \Omega^1_P \to \Omega^1_P$, where

(1) $\Pi^2 = \Pi$ and $\ker \Pi = P\Omega^1_A P$.
(2) Π is a left P-module map.
(3) Π commutes with the coaction of H, where H coacts on $\Omega^1_P \subset P \otimes P$ by the tensor product coaction.

A connection is called *strong* if $(\mathrm{id} - \Pi)\mathrm{d}P \subseteq \Omega^1_A P$.

As in classical geometry, there is a correspondence between such projections and connection 1-forms.

Lemma 5.3 *Let (P, H, Δ_R) be a (universal) quantum principal bundle. There is a 1–1 correspondence between connections Π and* connection forms $\omega : H^+ \to \Omega^1_P$ *such that*

(1) $\mathrm{ver} \circ \omega = 1 \otimes \mathrm{id}$ *as maps $H^+ \to P \otimes H^+$.*
(2) *ω commutes with the coaction of H, where H coacts on H^+ by the right adjoint coaction* $\mathrm{Ad}_R(h) = h_{(2)} \otimes (Sh_{(1)})h_{(3)}$.

Proof Given ω, we define

$$\Pi = (\cdot \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \omega) \circ \mathrm{ver}, \tag{5.3}$$

which is a left module map as ver is, and a projection in view of axiom (1) for ω. Also for this reason, by applying ver, it is clear that the kernel of Π coincides with the kernel of ver, which by assumption is $P\Omega^1_A P$. Conversely, given Π, we define

$$\omega(h) = \Pi \circ \mathrm{ver}^{-1}(1 \otimes h)$$

for all $h \in H^+$. The different liftings of ver^{-1} to Ω^1_P differ by an element of $P\Omega^1_A P$ and we assume this is the kernel of Π, so ω is well defined.

Moreover, the map ver is equivariant where $P \otimes H^+$ has the tensor product coaction $\Delta_{P\otimes H^+}$ of the given coaction on P and Ad_R,

$$\begin{aligned}\Delta_{P\otimes H^+}\mathrm{ver}(p \otimes q) &= \Delta_{P\otimes H^+}(pq_{(\bar 0)} \otimes q_{(\bar 1)})\\ &= p_{(\bar 0)}q_{(\bar 0)(\bar 0)} \otimes q_{(\bar 1)(2)} \otimes p_{(\bar 1)}q_{(\bar 0)(\bar 1)}(Sq_{(\bar 1)(1)})q_{(\bar 1)(3)}\\ &= p_{(\bar 0)}q_{(\bar 0)} \otimes q_{(\bar 1)(3)} \otimes p_{(\bar 1)}q_{(\bar 1)(1)}(Sq_{(\bar 1)(2)})q_{(\bar 1)(4)}\\ &= p_{(\bar 0)}q_{(\bar 0)(\bar 0)} \otimes q_{(\bar 0)(\bar 1)} \otimes p_{(\bar 1)}q_{(\bar 1)} = (\mathrm{ver} \otimes \mathrm{id})\Delta_R(p \otimes q)\end{aligned}$$

using the coaction properties. Note that a sum of such expressions should be understood (so as to lie in Ω^1_P) even though we checked it only on $p \otimes q$. Applying $(\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \omega)$ to (5.3) gives Π equivariant when ω is. Going the other way, the kernel of ver is stable under the coaction, so $\mathrm{ver}^{-1} : P \otimes H^+ \to \Omega^1_P/P\Omega^1_A P$ is equivariant and so is its restriction to $1 \otimes H^+$. □

In short, the exact sequence in Definition 5.1 is equivariant and a connection is an equivariant splitting of it, which can be seen as an equivariant map ω. Our first example is a Möbius band similar to Example 4.22 but now as a principal bundle. See also Example 5.49 and the Klein bottle in Example 5.11.

Example 5.4 (Double Cover of a Circle) Here $P = \mathbb{C}[t, t^{-1}]$ an algebraic circle and fibre $\mathbb{C}(\mathbb{Z}_2)$ which we treat equivalently as $H = \mathbb{C}\mathbb{Z}_2$ with generator g obeying $g^2 = 1$. We define $\Delta_R : P \to P \otimes H$ by $\Delta_R t^n = t^n \otimes g^n$ so that $A = P^H = \mathbb{C}[s, s^{-1}] \subset P$ by $s = t^2$. The universal calculus is

$$\Omega^1_P = \Big\{\sum_i a_i t^{n_i} \otimes t^{m_i} \,\Big|\, \sum_i a_i t^{n_i+m_i} = 0,\ a_i \in \mathbb{C},\ n_i, m_i \in \mathbb{Z}\Big\}$$

while $\mathrm{ver} : \Omega^1_P \to P \otimes H^+$ is $\mathrm{ver}(\sum_i a_i t^{n_i} \otimes t^{m_i}) = \sum_i a_i t^{n_i+m_i} \otimes g^{m_i}$. We next define $\omega : H^+ \to \Omega^1_P$ by $\omega(g-1) = t^{-1} \otimes t - 1 \otimes 1$, then $\mathrm{ver}(t^n\omega(g-1)) = t^n \otimes (g-1)$ proving surjectivity. Finally, we write an element of Ω^1_P in four parts,

$$\sum_i b_i t^{2n_i} \otimes t^{2m_i} + \sum_i d_i t^{2n_i+1} \otimes t^{2m_i} + \sum_i e_i t^{2n_i} \otimes t^{2m_i+1} + \sum_i f_i t^{2n_i+1} \otimes t^{2m_i+1}.$$

If this element is in the kernel of ver then

$$\sum_i b_i t^{2n_i+2m_i} + \sum_i d_i t^{2n_i+1+2m_i} = 0 = \sum_i e_i t^{2n_i+2m_i+1} + \sum_i f_i t^{2n_i+2m_i+2}$$

and by considering odd and even powers, all four sums must vanish, so elements of ker ver are of the form

$$\sum_i b_i t^{2n_i} \otimes t^{2m_i} + \sum_i t d_i t^{2n_i} \otimes t^{2m_i} + \sum_i e_i t^{2n_i} \otimes t^{2m_i} t + t \sum_i f_i t^{2n_i} \otimes t^{2m_i} t$$

explicitly exhibited as four terms each in $P\Omega^1_A P$. Hence we have a quantum principal bundle, and ω is a connection on it. ◇

We next define the *curvature form* of a connection form ω as

$$F_\omega(h) = \mathrm{d}\omega(h) + \omega(\pi_\epsilon h_{(1)}) \wedge \omega(\pi_\epsilon h_{(2)}) \in \Omega^2_P \tag{5.4}$$

for all $h \in H^+$, where we recall that $\pi_\epsilon(h) = h - 1\epsilon(h)$ is the *counit projection* to H^+. It is easy to see that the curvature obeys

$$\mathrm{d}F_\omega(h) + \omega(\pi_\epsilon h_{(1)}) F_\omega(\pi_\epsilon h_{(2)}) - F_\omega(\pi_\epsilon h_{(1)})\omega(\pi_\epsilon h_{(2)}) = 0.$$

If V is a right H-comodule, we define a left A-module E by

$$E = (P \otimes V)^H = P \otimes^H V,$$

which we will call an *associated vector bundle* to P. In the first expression we use invariants under the tensor product right coaction but it is also useful to think of this space of invariants as the cotensor product $P \otimes^H V$, where V has the left comodule structure $\Delta_L v := v_{(\bar{\infty})} \otimes v_{(\bar{1})} = S^{-1} v_{(\bar{1})} \otimes v_{(\bar{0})}$ in terms of the right coaction and $\otimes^H$ of a right H-comodule and a left one was defined in Definition 3.109 as the kernel of $\Delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_L$, so

$$E = \{s \in P \otimes V \mid (\Delta_R \otimes \mathrm{id})s = (\mathrm{id} \otimes \Delta_L)s\}. \tag{5.5}$$

In this form we could start with V a left comodule and avoid needing an invertible antipode. Either way, the left A action is $a.(p \otimes v) = ap \otimes v$. This association $\mathfrak{E} : \mathcal{M}^H \to {}_A\mathcal{M}$ is functorial and we will study it in more detail in §5.3.

Proposition 5.5 *If ω is a strong connection on a quantum principal bundle with corresponding idempotent Π_ω and $E = (P \otimes V)^H$ an associated bundle then*

$$\begin{aligned}\nabla s &= ((\mathrm{id} - \Pi_\omega)\mathrm{d} \otimes \mathrm{id})s \\ &= \mathrm{d}s^1 \otimes s^2 - s^1 \omega(\pi_\epsilon s^2{}_{(\bar{1})}) s^2{}_{(\bar{\infty})} \in (\Omega^1_A P \otimes V)^H = \Omega^1_A \otimes_A E\end{aligned}$$

for all $s = s^1 \otimes s^2 \in E$ (summation understood) defines a left connection or covariant derivative $\nabla : E \to \Omega^1_A \otimes_A E$. Moreover, E is a projective module and

$$R_\nabla s = -s^1{}_{(\bar{1})} F_\omega(\pi_\epsilon s^1{}_{(\bar{0})}) \otimes s^2 = -s^1 F_\omega(\pi_\epsilon s^2{}_{(\bar{1})}) \otimes s^2{}_{(\bar{\infty})}$$

is manifestly a left A-module map. Moreover, ∇ is a bimodule connection with the right action $(p \otimes v).a = pa \otimes v$ restricted to E.

Proof If $s \in P \otimes V$ then $\nabla s \in \Omega^1_A P \otimes V$ since our connection is assumed to be strong. Since Π_ω and d are H-covariant, so is the combination $(\mathrm{id} - \Pi_\omega)\mathrm{d}$, and hence also $\Delta_R \nabla = (\nabla \otimes \mathrm{id})\Delta_R$ for the tensor product coaction on $P \otimes V$. Hence if $s \in E$ then ∇s is Δ_R-invariant. On the other hand, in nice cases the product map

$$\Omega^1_A \otimes_A P \to \Omega^1_A P \tag{5.6}$$

is an isomorphism. It follows from later results, see Lemma 5.29, that this is always the case when the bundle has a strong connection. For now, we just note that if we have a map $H \to P \otimes_A P$ denoted by $h \mapsto h^{(1)} \otimes_A h^{(2)}$ with certain covariance properties and such that $h^{(1)}h^{(2)} = \epsilon(h)$ holds then we can define the inverse

$$\Omega^1_A P \to \Omega^1_A \otimes_A P, \quad \xi \mapsto \xi_{\bar{(0)}}(\xi_{\bar{(1)}})^{(1)} \otimes_A (\xi_{\bar{(1)}})^{(2)}. \tag{5.7}$$

Such maps arise from a strong connection, as we will see in Lemma 5.7. Then $\Omega^1_A P \otimes V = \Omega^1_A \otimes_A P \otimes V \subset A \otimes A \otimes_A P \otimes V = A \otimes 1 \otimes P \otimes V = A \otimes P \otimes V$ in a canonical way where the rightmost values in $A \otimes A$ are pushed through the $\otimes_A$ in favour of 1. In this way, the space of coinvariants of $\Omega^1_A P \otimes V$ is identified with a subspace of $A \otimes (P \otimes V)^H = A \otimes A \otimes_A (P \otimes V)^H$, namely $\Omega^1_A \otimes_A (P \otimes V)^H$. Hence ∇ is well defined as a map $E \to \Omega^1_A \otimes_A E$. That ∇ is a left-connection is given by $\nabla(as) = (\mathrm{id} - \Pi_\omega)(a\mathrm{d}s^1 \otimes s^2 + (\mathrm{d}a)s^1 \otimes s^2) = a\nabla s + (\mathrm{d}a)s$ since Π_ω is a left module map and $(\mathrm{d}a)s^1 \in \Omega^1_A P \subseteq \ker \Pi_\omega$. Again in the universal bundle case, we recall from the Cuntz–Quillen Theorem 3.26 that the existence of a covariant derivative ∇ implies that E is a projective module as expected for a vector bundle. Finally, to compute the curvature we use the shorthand notation $s_{\bar{(0)}} \otimes s_{\bar{(1)}} := s^1{}_{\bar{(0)}} \otimes s^2 \otimes s^1{}_{\bar{(1)}} = s^1 \otimes s^2{}_{\bar{(\infty)}} \otimes s^2{}_{\bar{(1)}}$ in terms of the right or left coactions (this is a coaction on $P \otimes V$ but will not generally restrict to a coaction on E, unless H is cocommutative) and $(\mathrm{id} - \Pi_\omega)\mathrm{d}s = \mathrm{d}s - s_{\bar{(0)}}\omega\pi_\epsilon s_{\bar{(1)}} = \xi \otimes_A e = \xi e$ by the above identifications, for $\xi \in \Omega^1_A$, $e \in E$ (a sum of such understood). Then

$$\begin{aligned} R_\nabla s &= (\mathrm{d}\xi)e - \xi \wedge (\mathrm{id} - \Pi_\omega)\mathrm{d}e = \mathrm{d}(\xi e) + \xi \wedge \Pi_\omega(\mathrm{d}e) \\ &= \mathrm{d}(\xi e) + \xi \wedge e_{\bar{(0)}}\omega\pi_\epsilon e_{(1)} = \mathrm{d}(\xi e) + (\xi e)_{\bar{(0)}} \wedge \omega\pi_\epsilon(\xi e)_{(1)} \\ &= -\mathrm{d}(s_{\bar{(0)}}\omega\pi_\epsilon s_{(1)}) + \mathrm{d}s_{\bar{(0)}} \wedge \omega\pi_\epsilon s_{(1)} - s_{\bar{(0)}}\omega(\pi_\epsilon s_{(1)}) \wedge \omega(\pi_\epsilon s_{(2)}) \end{aligned}$$

which using the Leibniz rule gives the stated result in terms of F_ω. The first version of the curvature makes it clear that this is just the curvature on P in the next lemma extended to $P \otimes V$ and restricted to invariants. The last part also follows from the next lemma and we defer to the end of the proof of that. □

Lemma 5.6 *A universal quantum principal bundle P can regarded as an associated bundle $P \cong (P \otimes H)^H$ where $p \mapsto p_{\bar{(0)}} \otimes Sp_{\bar{(1)}}$. In this case the connection ∇_P*

on P associated to any strong connection ω is a bimodule connection. Its curvature is an A-bimodule map if $[A, F_\omega(\pi_\epsilon(H^+))] = 0$.

Proof We have $\mathrm{ver}(\mathrm{d}p) = \mathrm{ver}(1 \otimes p - p \otimes 1) = p_{(\bar 0)} \otimes p_{(\bar 1)} - p \otimes 1$ so that if ω is a strong connection then P has an H-equivariant covariant derivative

$$\nabla_P p = (\mathrm{id} - \Pi_\omega)\mathrm{d}p = \mathrm{d}p - p_{(\bar 0)}\omega(\pi_\epsilon p_{(\bar 1)}) \in \Omega^1_A P = \Omega^1_A \otimes_A P.$$

The construction being right H-covariant means we get the same formula if we apply $(\mathrm{id} - \Pi_\omega)\mathrm{d}$ to the first factor of $(P \otimes H)^H$ and use the isomorphism to convert to P itself. Conversely, having a left-covariant derivative ∇_P on P is equivalent to having a strong connection which we can recover from ∇_P using the surjectivity of ver. The curvature from the above is $R_{\nabla_P} p = -p_{(\bar 0)} F_\omega(\pi_\epsilon p_{(\bar 1)})$ and $R_{\nabla_P}(pa) - (R_\nabla p)a = -p_{(\bar 0)}[a, F_\omega(\pi_\epsilon p_{(\bar 1)})]$, giving the condition stated for the curvature to be a bimodule map for the canonical bimodule structure on P.

For ∇ to be a bimodule connection in the sense of Definition 3.66, we require

$$\begin{aligned}\nabla_P(pa) &= \mathrm{d}(pa) - p_{(\bar 0)} a \omega \pi_\epsilon p_{(\bar 1)} \\ &= (\mathrm{d}p)a - p_{(\bar 0)}\omega(\pi_\epsilon p_{(\bar 1)})a + p\mathrm{d}a - p_{(\bar 0)}[a, \omega\pi_\epsilon p_{(\bar 1)}] \\ &= (\nabla_P p)a + \sigma_P(p \otimes \mathrm{d}a)\end{aligned}$$

for $\sigma_P : P \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A P$, and so we find

$$\sigma_P(p \otimes \mathrm{d}a) = p\mathrm{d}a - p_{(\bar 0)}[a, \omega\pi_\epsilon p_{(\bar 1)}] \in \Omega^1_A P = \Omega^1_A \otimes_A P,$$

where ∇_P being a strong connection shows that σ_P maps to $\Omega^1_A P$. It is right H-equivariant since ∇_P is and extends to a manifestly well-defined bimodule map

$$\sigma_P(p \otimes (b \otimes a)) = pb \otimes a + p_{(\bar 0)} b(\omega\pi_\epsilon p_{(\bar 1)})a \in \Omega^1_A P = \Omega^1_A \otimes_A P,$$

where $b \otimes a \in \Omega^1_A \subset A \otimes A$ (summation implicit) and $\mathrm{d}a = 1 \otimes a - a \otimes 1$ recovers our previous formula.

We now complete the proof of Proposition 5.5 where ω above implies a left connection ∇ on every associated bundle $E = (P \otimes V)^H$ (where V is a right H-comodule). We set

$$\sigma((p \otimes v) \otimes \eta) = \sigma_P(p \otimes \eta) \otimes v : (P \otimes V) \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A (P \otimes V)$$

and check that

$$\nabla((p \otimes v)a) - (\nabla(p \otimes v))a = (\mathrm{id} - \Pi_\omega)\mathrm{d}(pa) \otimes v - ((\mathrm{id} - \Pi_\omega)\mathrm{d}p)a \otimes v,$$

which is $\sigma_P(p \otimes \mathrm{d}a) \otimes v$, as required. These formulae restrict to E since ∇ and hence σ are right H-equivariant. Note also that if the condition for R_∇ on P to be a bimodule map holds then clearly the curvatures on all associated bundles are also bimodule maps for the given relatively trivial right action of A on E. □

Finally, in the $*$-algebra case over $\mathbb{C}$, there is a natural definition of a $*$-preserving connection form. Here H is assumed to be a Hopf $*$-algebra, P a $*$-algebra and Δ_R a $*$-algebra map. Recall from Proposition 1.5 that the universal differential calculus Ω^1_P also acquires a $*$-structure, namely $-\mathrm{flip}(*\otimes*)$. With this in mind, a connection form will be called *$*$-preserving* if $*\omega = \omega*$. If we think of Λ^1_H concretely as H^* via the Maurer–Cartan form ϖ then this becomes

$$\omega(*Sh) = \mathrm{flip}(*\otimes*)\omega(h) \tag{5.8}$$

for $h \in H^+$, which we will later tie up with our geometric definition from Chap. 3 of a $*$-preserving covariant derivative (see Proposition 5.54).

5.1.1 Hopf–Galois Extensions

The above concepts were expressed in terms of universal calculus in such a way that they can later be generalised to other calculi. However, the universal calculus itself is actually expressing what would be the classical topological idea of a principal bundle, which in the noncommutative case is called a Hopf–Galois extension. From this point of view, the previous geometric constructions must have an equivalent formulation more directly in terms of the comodule algebra, which we describe here. First of all, for any right H-comodule algebra P we associate the map $\mathrm{ver}^\sharp = (\cdot\otimes\mathrm{id})(\mathrm{id}\otimes\Delta_R) : P\otimes P \to P\otimes H$ extending the map ver given before and which we also descend to a map $\mathrm{ver}^\sharp : P\otimes_A P \to P\otimes H$.

Lemma 5.7 *Let P be a right H-comodule algebra.*

(1) *P is a universal quantum principal bundle by* ver *if and only if* $\mathrm{ver}^\sharp : P\otimes_A P \to P\otimes H$ *is a bijection. One says equivalently that P is a* Hopf–Galois extension *of A.*
(2) *In this case, connections correspond to unital covariant maps $\omega^\sharp : H \to P\otimes P$ where H coacts on itself by* Ad_R *and on $P\otimes P$ by the tensor product coaction, such that* $\mathrm{ver}^\sharp\omega^\sharp(h) = 1\otimes h$ *for all $h\in H$. We refer to $\omega^\sharp$ as the* connection map.

Proof The first thing to note is that $\mathrm{ver}^\sharp = \mathrm{ver}\,\pi_{\Omega^1} + \cdot\otimes 1$, where $\cdot$ is the product of P and $\pi_{\Omega^1}(p\otimes q) = p\otimes q - pq\otimes 1$ is the projection in $P\otimes P = \Omega^1_P \oplus (P\otimes 1)$, while ver is the restriction of $\mathrm{ver}^\sharp$. If $p\otimes q \in \ker\mathrm{ver}$ (summation understood) then its product is zero so it lies in Ω^1_P, hence the two maps have the same kernel. Next, because of the form of the universal calculus Ω^1_A, we can also write $P\Omega^1_A P = P(\mathrm{d}A)P$ and thereby interpret the exactness in the middle of the sequence before as the condition that $\ker\mathrm{ver}$ is the sum of elements of the form $p(1\otimes a - a\otimes 1)q$, i.e., that $\mathrm{ver}^\sharp$ descended to $P \otimes_A P$ is injective. Surjectivity of $\mathrm{ver}^\sharp$ is equivalent to surjectivity of ver because we can write $H = H^+ \oplus \Bbbk 1$ by the counit projection π_ϵ. So an element of $P \otimes H$ can be written (summation understood) as $p \otimes h = p \otimes \pi_\epsilon h + p \otimes 1\epsilon h = \mathrm{ver}(x) + p' \otimes 1 = \mathrm{ver}^\sharp(x + p' \otimes 1)$ for some $x \in \Omega^1_P$ and $p' = p\epsilon h \in P$, and similarly in the other direction. For the second part, working with a connection form ω is similarly equivalent to working with its extension $\omega^\sharp(h) = \epsilon(h)1 \otimes 1 + \omega(\pi_\epsilon h)$ to a unital map $\omega^\sharp : H \to P \otimes P$. Given $\omega^\sharp : H \to P \otimes P$ as stated, $\mathrm{ver}^\sharp \omega^\sharp(h) = 1 \otimes h$ means $h^{(1)} h^{(2)}{}_{(\bar 0)} \otimes h^{(2)}{}_{(1)} = 1 \otimes h$ if we use an explicit notation $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$, and $\mathrm{id} \otimes \epsilon$ on this implies that $\cdot\, \omega^\sharp(h) = 1\epsilon(h)$ and hence that its restriction is a map $\omega : H^+ \to \Omega^1_P$ as required. Going the other way, $\mathrm{ver}^\sharp \omega^\sharp(h) = \mathrm{ver}^\sharp(\epsilon(h)1 \otimes 1 + \omega(\pi_\epsilon h)) = 1 \otimes h$ for all $h \in H$. The covariance properties can easily be seen to match as well. □

The notion of strong connection similarly has an equivalent formulation in terms of the Hopf–Galois extension.

Lemma 5.8 *Let (P, H, Δ_R) be a universal quantum principal bundle (aka Hopf–Galois extension) with H having bijective antipode. Strong connections correspond to unital bicomodule maps $\omega^\sharp : H \to P \otimes P$ such that $\cdot\, \omega^\sharp(h) = 1\epsilon(h)$ for all $h \in H$. Here $P \otimes P$ is a left comodule by Δ_L on the left factor and a right comodule by Δ_R on the right one. Conversely, if P is a right H-comodule algebra and $\omega^\sharp$ has the properties stated then P is a quantum principal bundle with connection map $\omega^\sharp$.*

Proof Suppose we have a bundle and $\omega^\sharp$ a connection map. For the connection to be strong, we need $\mathrm{d}p - \Pi_\omega(\mathrm{d}p) = 1 \otimes p - p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)}) \in \Omega^1_A P$. For this we need invariance under Δ_R in the first factor so that we are in $A \otimes P$. In terms of the left coaction (5.2), we need $p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)})$ to be invariant under $\Delta_L \otimes \mathrm{id}$. If $(\Delta_L \otimes \mathrm{id})\omega^\sharp(h) = h_{(1)} \otimes \omega^\sharp(h_{(2)})$ then this holds as

$$\begin{aligned}(\Delta_L \otimes \mathrm{id})(p_{(\bar 0)}\omega^\sharp)(p_{(\bar 1)}) &= p_{(\bar 1)(1)} S^{-1} p_{(\bar 0)(1)} \otimes p_{(\bar 0)(\bar 0)} \omega^\sharp(p_{(\bar 1)(2)}) \\ &= p_{(\bar 1)(1)} S^{-1} p_{(\bar 1)(2)} \otimes p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)(3)}) = 1 \otimes p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)}).\end{aligned}$$

Conversely, requiring this left invariance tells us that $\omega^\sharp(h) := h^{(1)} \otimes h^{(2)}$ obeys $p_{(\bar 0)(\bar 1)} p_{(\bar 1)}{}^{(1)}{}_{(\bar 1)} \otimes p_{(\bar 0)(\bar 0)} p_{(\bar 1)}{}^{(1)}{}_{(\bar 0)} \otimes p_{(\bar 1)}{}^{(2)} = 1 \otimes p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)})$ for all $p \in P$, which by the coaction property, is equivalent to

$$p_{(\bar 1)(1)} p_{(\bar 1)(2)}{}^{(1)}{}_{(\bar 1)} \otimes p_{(\bar 0)} p_{(\bar 1)(2)}{}^{(1)}{}_{(\bar 0)} \otimes p_{(\bar 1)(2)}{}^{(2)} = 1 \otimes p_{(\bar 0)} \omega^\sharp(p_{(\bar 1)}).$$

By surjectivity of $\mathrm{ver}^\sharp$, for any $h \in H$ we can chose $\sum q_i \otimes p_i$ such that $q_i p_{i\,(\bar 0)} \otimes p_{i\,(\bar 1)} = 1 \otimes h$, then $1 \otimes q_i$ times the above applied to p_i tells us that $h_{(1)} h_{(2)}{}^{(1)}{}_{(\bar 1)} \otimes h_{(2)}{}^{(1)}{}_{(\bar 0)} \otimes h_{(2)}{}^{(2)} = 1 \otimes \omega^\sharp(h)$, which is equivalent to $h^{(1)}{}_{(\bar 1)} \otimes h^{(1)}{}_{(\bar 0)} \otimes h^{(2)} = Sh_{(1)} \otimes \omega^\sharp(h_{(2)})$, i.e., left covariance of $\omega^\sharp$. Hence a connection is strong if and only if $\omega^\sharp$ is Δ_L covariant. This combined with Ad_R covariance is equivalent to bicovariance and also implies $\cdot\, \omega^\sharp = 1\epsilon$. Conversely, given a bicovariant map, right covariance

$(\mathrm{id}\otimes\Delta_R)\omega^\sharp(h)=\omega^\sharp(h_{(1)})\otimes h_{(2)}$ implies that $\cdot\,\omega^\sharp(h_{(1)})\otimes h_{(2)}=\mathrm{ver}^\sharp\omega^\sharp(h)$. Hence the property $\mathrm{ver}^\sharp\omega^\sharp=1\otimes\mathrm{id}$ is equivalent in our context to $\cdot\,\omega^\sharp=1\epsilon$.

For the second part, given a right H-comodule algebra P and a bicomodule map $\omega^\sharp$ obeying $\cdot\,\omega^\sharp=1\epsilon$, we define $\mathrm{ver}^{\sharp-1}(p\otimes h):=p\omega^\sharp(h)$ in $P\otimes_A P$ and $\mathrm{ver}^\sharp\mathrm{ver}^{\sharp-1}(p\otimes h)=\mathrm{ver}^\sharp(p\omega^\sharp(h))=p((\cdot\otimes\mathrm{id})(\mathrm{id}\otimes\Delta_R)\omega^\sharp(h))=p(\cdot\,\omega^\sharp(h_{(1)}))\otimes h_{(2)}=p\otimes h$ from the right covariance and product splitting properties of $\omega^\sharp$. For the other way around, note that $q_{(\bar 0)}\omega^\sharp(q_{(\bar 1)})\in P\otimes P$ by the $\Delta_L\otimes\mathrm{id}$ invariance of $\omega^\sharp$ lies as we have seen in $A\otimes P$. Then $\mathrm{ver}^{\sharp-1}\mathrm{ver}^\sharp(p\otimes_A q)=\mathrm{ver}^{\sharp-1}(pq_{(\bar 0)}\otimes q_{(\bar 1)})=pq_{(\bar 0)}\omega^\sharp(q_{(\bar 1)})=p\otimes_A\cdot(q_{(\bar 0)}\omega^\sharp(q_{(\bar 1)}))=p\otimes_A q$. Note that the bicomodule structure on $P\otimes P$ is on the left and right copies of P respectively (with the former making P an H^{op}-comodule algebra via Δ_L). □

As in the last proof, we will often adopt the shorthand $\omega^\sharp(h):=h^{(1)}\otimes h^{(2)}$ for all $h\in H$. In these terms, the equivariance properties of a strong connection become

$$\begin{aligned}h^{(1)}\otimes h^{(2)}{}_{(\bar 0)}\otimes h^{(2)}{}_{(\bar 1)}&=h_{(1)}{}^{(1)}\otimes h_{(1)}{}^{(2)}\otimes h_{(2)}\,,\\ h^{(1)}{}_{(\bar 0)}\otimes h^{(1)}{}_{(\bar 1)}\otimes h^{(2)}&=h_{(2)}{}^{(1)}\otimes Sh_{(1)}\otimes h_{(2)}{}^{(2)}\end{aligned}\tag{5.9}$$

as elements of $P\otimes P\otimes H$ and $P\otimes H\otimes P$ respectively.

We next recall that a normalised left-integral on a Hopf algebra H is a linear map $\int:H\to\Bbbk$ such that $(\mathrm{id}\otimes\int)\Delta=\int\otimes 1$ and $\int 1=1$. In Definition 2.18 and in §4.2, we previously focussed on right-integrals. We are now ready to prove our main theorem, on the existence of quantum principal bundles and strong connections ω.

Theorem 5.9 *Suppose that H has normalised left-integral and bijective antipode, and that P is a right H-comodule algebra with* $\mathrm{ver}^\sharp$ *surjective. Then (P,H,Δ_R) is a universal quantum principal bundle and admits a strong connection.*

Proof We define a bilinear pairing $b(h,g)=\int hSg$, which obeys $h_{(1)}b(h_{(2)},g)=b(h,g_{(1)})g_{(2)}$. We use b to convert the left and right coactions on P to 'actions' $a_R(p\otimes h)=p_{(\bar 0)}b(p_{(\bar 1)},h)$ and $a_L(h\otimes p)=b(h,S^{-1}p_{(\bar 1)})p_{(\bar 0)}$. Then

$$\Delta_R a_R(p\otimes h)=a_R(p\otimes h_{(1)})\otimes h_{(2)},\quad \Delta_L a_L(h\otimes p)=h_{(1)}\otimes a_L(h_{(2)}\otimes p)\tag{5.10}$$

by elementary computations. Next, given the splitting $H=\Bbbk 1\oplus H^+$ by the counit projection π_ϵ, we define a linear map $H\to P\otimes P$, temporarily denoted $h\mapsto h^{(1)}\otimes h^{(2)}$, by $1^{(1)}\otimes 1^{(2)}=1\otimes 1$ and a choice of representatives in $P\otimes P$ such that $\mathrm{ver}^\sharp(h^{(1)}\otimes_A h^{(2)})=1\otimes h$ for all $h^+\in H$. This is possible as $P\otimes P\to P\otimes_A P\to P\otimes H$ (by $\mathrm{ver}^\sharp$) is surjective and amounts to $h^{(1)}h^{(2)}{}_{(\bar 0)}\otimes h^{(2)}{}_{(\bar 1)}=1\otimes h$ for all $h\in H$. We apply the coaction to the first term and the coproduct to the second, so

$$h^{(1)}{}_{(\bar 0)}h^{(2)}{}_{(\bar 0)}\otimes h^{(1)}{}_{(\bar 1)}h^{(2)}{}_{(1)}\otimes h^{(2)}{}_{(2)}\otimes h^{(2)}{}_{(3)}=1\otimes 1\otimes h_{(1)}\otimes h_{(2)}.$$

Applying S to the third term, then multiplying the second and third terms gives

$$h^{(1)}{}_{\bar{(0)}}h^{(2)}{}_{\bar{(0)}} \otimes h^{(1)}{}_{\bar{(1)}} \otimes h^{(2)}{}_{\bar{(1)}} = 1 \otimes Sh_{(1)} \otimes h_{(2)}. \tag{5.11}$$

We then define the unital map

$$\omega^\sharp(h) := a_L(h_{(1)} \otimes h_{(2)}{}^{(1)}) \otimes a_R(h_{(2)}{}^{(2)} \otimes h_{(3)})$$

which from (5.10) is clearly bicovariant for the actions in Lemma 5.8. We also have

$$\begin{aligned}\cdot(\omega^\sharp(h)) &= a_L(h_{(1)} \otimes h_{(2)}{}^{(1)})a_R(h_{(2)}{}^{(2)} \otimes h_{(3)})\\ &= h_{(2)}{}^{(1)}{}_{\bar{(0)}}h_{(2)}{}^{(2)}{}_{\bar{(0)}}b(h_{(1)}, S^{-1}h_{(2)}{}^{(1)}{}_{\bar{(1)}})b(h_{(2)}{}^{(2)}{}_{\bar{(1)}}, h_{(3)})\\ &= b(h_{(1)}, h_{(2)})b(h_{(3)}, h_{(4)}) = \int 1\epsilon(h_{(1)}) \int 1\epsilon(h_{(2)}) = \epsilon(h),\end{aligned}$$

where we have used (5.11). We then use Lemma 5.8 to see that we have a strong connection and that we have a bundle. □

Another important property of strong connections relates to faithful flatness. We have met this abstractly in Proposition 3.6.1 and in our context we are interested in P being faithfully flat as a left A-module. This means that a short sequence of right A-modules is exact if and only if $\otimes_A P$ of it is exact. Similarly, for coflatness of a comodule, see Definition 3.109.

Proposition 5.10 *If (P, H, Δ_R, ω) is a quantum principal bundle with universal calculus equipped with a strong connection then P is projective and faithfully flat as a left $A = P^H$ module. If in addition the antipode of H is bijective then we also have the statement for the right module structure. Moreover in this case, P is coflat as a right H-comodule.*

Proof The Cuntz–Quillen Theorem 3.26 gives projectivity, and then Corollary 3.104 gives flatness. Next, A is a direct summand of P as a left A-module, with inclusion $A \to P$ one way and the map $p \mapsto p_{\bar{(0)}}p_{\bar{(1)}}{}^{(1)}f(p_{\bar{(1)}}{}^{(2)})$ the other way, where the linear map $f : P \to \Bbbk$ is chosen with $f(1) = 1$ and $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ is the connection map. Then Proposition 3.106 gives the left result. For the right result, we note that given the right module version of the Cuntz–Quillen theorem, this would follow if we had a right connection for P for the universal calculus on A, for example

$$\tilde{\nabla} p = (S^{-1}p_{\bar{(1)}})^{(1)} \otimes (S^{-1}p_{\bar{(1)}})^{(2)}p_{\bar{(0)}} - p \otimes 1$$

(for more details on the opposite handed theory, see Exercise E5.9).

For coflatness, suppose that we have a short exact sequence of left H-comodules,

$$0 \longrightarrow D \xrightarrow{\theta} E \xrightarrow{\phi} F \longrightarrow 0,$$

and we are required to prove that the following sequence is exact,

$$0 \longrightarrow P \otimes^H D \xrightarrow{\mathrm{id}\otimes\theta} P \otimes^H E \xrightarrow{\mathrm{id}\otimes\phi} P \otimes^H F \longrightarrow 0. \tag{5.12}$$

Now $P \otimes^H D$ is a right A-module, with $(p \otimes d).a = pa \otimes d$ (similarly for E, F), and by faithful flatness of P as a left A-module, to show that (5.12) is exact it is enough to show exactness of

$$0 \longrightarrow (P \otimes^H D) \otimes_A P \xrightarrow{\mathrm{id}\otimes\theta\otimes\mathrm{id}} (P \otimes^H E) \otimes_A P \xrightarrow{\mathrm{id}\otimes\phi\otimes\mathrm{id}} (P \otimes^H F) \otimes_A P \longrightarrow 0.$$

As $\mathrm{ver}^\sharp : P \otimes_A P \to P \otimes H$ and the map $\psi : P \otimes H \to P \otimes H$ given by $\psi(p \otimes h) = p_{(\bar 0)} \otimes p_{(\bar 1)} Sh$ are both bijective, on composing these we find that $\mathrm{ver}^{\sharp\sharp} : P \otimes_A P \to P \otimes H$ given by $\mathrm{ver}^{\sharp\sharp}(p \otimes q) = p_{(\bar 0)} q \otimes p_{(\bar 1)}$ is also bijective. This shows that the right H-comodules $P^\bullet \otimes_A P$ and $P \otimes H^\bullet$ are isomorphic, where the bullet indicates the factor which the coaction is applied to. This gives

$$(P \otimes^H D) \otimes_A P \cong (P^\bullet \otimes_A P) \otimes^H D \cong (P \otimes H^\bullet) \otimes^H D \cong P \otimes D,$$

(similarly for E, F) which demonstrates that the required sequence is exact. □

Example 5.11 (Klein Bottle) Here we give a nontrivial classical example of a topological $\mathbb{Z}_2$-bundle where the total space is a torus and the base is the Klein bottle, but in our algebraic language as a quantum principal bundle with universal calculus and with a strong connection $\omega^\sharp$. Indeed, topologically, the Klein bottle can be considered as a quotient of the square, shown shaded in Fig. 5.1 with the edges glued as shown by the arrows. The gluing of the horizontal edges gives a tube, which is then glued again to form the Klein bottle. If the vertical edges had arrows in the same direction then we would get a torus. However, as the vertical arrows are in opposite directions, we have to reverse the orientation of the vertical circle before glueing the edges together. This is shown in the picture by the tube overshooting the meeting point and then doubling back, appearing to pass through itself in a manner illustrating the fact that the Klein bottle cannot be embedded in $\mathbb{R}^3$.

Now set $s = e^{i\theta/2}$ and $v = e^{i\phi}$. The gluing of the horizontal edges identifies $\phi = 0$ with $\phi = 2\pi$. We take the functions on the Klein bottle to be the invariant periodic

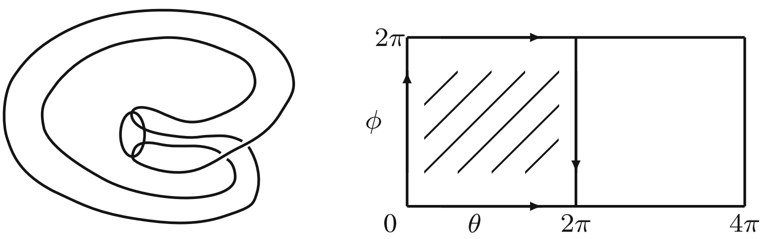

Fig. 5.1 Pictures of the classical Klein bottle, the second one as a quotient

functions on $[0, 4\pi] \times [0, 2\pi]$ (extended to periodic functions on $\mathbb{R}^2$) with $\mathbb{Z}_2$ acting by $\theta \mapsto \theta + 2\pi$ and $\phi \mapsto -\phi$. We express this as $P = \mathbb{C}[s, s^{-1}] \otimes \mathbb{C}[v, v^{-1}] = \mathbb{C}[s, s^{-1}, v, v^{-1}]$ (an algebraic torus) and the right coaction of $H = \mathbb{C}(\mathbb{Z}_2)$,

$$\Delta_R s = s \otimes \delta_0 - s \otimes \delta_1, \quad \Delta_R v = v \otimes \delta_0 + v^{-1} \otimes \delta_1. \tag{5.13}$$

We denote the fixed point subalgebra $A = \mathbb{C}[\text{Klein}] = P^H$, and this has linear basis

$$s^{2n}, \quad s^{2n} c_m, \quad s^{2n+1} s_m$$

for integers $n \in \mathbb{Z}$ and $m \geq 1$ where $c_m = \cos(m\phi) = \frac{1}{2}(v^m + v^{-m})$ and $s_m = \mathrm{i} \sin(m\phi) = \frac{1}{2}(v^m - v^{-m})$ in our algebraic formulation. These are invariant under $s \mapsto -s$ and $v \mapsto v^{-1}$. From the coaction, we get the canonical map

$$\mathrm{ver}^\sharp(s^{-1} \otimes s) = 1 \otimes \delta_0 - 1 \otimes \delta_1$$

so that $\mathrm{ver}^\sharp(1 \otimes 1 + s^{-1} \otimes s) = 2 \otimes \delta_0$ and $\mathrm{ver}^\sharp(1 \otimes 1 - s^{-1} \otimes s) = 2 \otimes \delta_1$. We use Lemma 5.8 to construct a quantum principal bundle with connection map

$$\omega^\sharp(\delta_0) = \frac{1}{2}(1 \otimes 1 + s^{-1} \otimes s), \quad \omega^\sharp(\delta_1) = \frac{1}{2}(1 \otimes 1 - s^{-1} \otimes s).$$

Here $\cdot\, \omega^\sharp(h) = \epsilon(h)$ clearly holds, and checking the equation $\omega^\sharp(h_{(1)}) \otimes h_{(2)} = (\mathrm{id} \otimes \Delta_R)\omega^\sharp(h)$ for the value $h = \delta_0$ gives $\frac{1}{2}(1 \otimes 1 \otimes 1 + s^{-1} \otimes s \otimes (\delta_0 - \delta_1))$ for both sides. From the right coaction in (5.13) and the antipode, we have a left coaction $\Delta_L s = \delta_0 \otimes s - \delta_1 \otimes s$ from which one can check that $\omega^\sharp$ is left and right-covariant. In addition, $\omega^\sharp$ is $*$-preserving according to (5.8). We compute its curvature form from (5.4) as $F_\omega(\delta_1) = 0$ on the single basis element $\delta_1 \in H^+$, so this is a flat connection.

It is known that the classical Klein bundle is nontrivial. We do have an algebra map $B = \mathbb{C}[t, t^{-1}] \to \mathbb{C}[\text{Klein}]$ given by $t \mapsto s^2 \otimes 1$, which corresponds to a surjection from the Klein bottle to the circle S^1 and which on our square picture is just the projection to the bottom horizontal line. Classically, this gives a circle bundle over the S^1 where the fibre circle corresponds to v, but not a principal bundle. This is because if we begin by rotating the fibre circle in the Klein bottle above a given point in the base circle in one direction, then by the flip in orientation of the fibre circle we must reverse the direction of rotation at some point as we travel round the base circle, and at that point we would have fixed points for the circle action, which is not allowed for a principal bundle. Nonetheless we still have a locally trivial fibration with base the circle, and we will see in Example 5.45 that with suitable differential structure it also satisfies our condition for a differential fibration. ◇

Example 5.12 We consider $P = \mathbb{C}\mathrm{Hg}$ from Example 4.62. This is the group algebra of the Heisenberg group with the unitary generators u, v, w and relations $uv = wuv$ with w central. Take $H = \mathbb{C}[\mathbb{T}^2] = \mathbb{C}[s, s^{-1}, t, t^{-1}] = \mathbb{C}\mathbb{Z}^2$, the algebraic torus with commuting unitary generators, coacting on P by

$$\Delta_R(u) = u \otimes s, \quad \Delta_R(v) = v \otimes t,\ \Delta_R(w) = w \otimes 1,$$

in which case $A = P^H = \mathbb{C}[w, w^{-1}]$ is the algebraic circle. The normalised integral on H is $\int s^i t^j = \delta_{i,0}\delta_{j,0}$. Theorem 5.9 tells us that we have a strong connection and universal quantum principal bundle or Hopf–Galois extension. Explicitly,

$$\mathrm{ver}^\sharp(u^r v^s \otimes_A u^n v^m w^p) = u^r v^s u^n v^m w^p \otimes s^n t^m = u^{n+r} v^{m+s} w^{p-sn} \otimes s^n t^m$$

with inverse and connection map

$$(\mathrm{ver}^\sharp)^{-1}(u^a v^b w^c \otimes s^n t^m) = u^{a-n} v^{b-m} \otimes u^n v^m w^{c+(b-m)n} \in P \otimes_A P,$$
$$\omega^\sharp(s^i t^j) = w^{-ij} u^{-i} v^{-j} \otimes u^i v^j \in P \otimes P.$$

This is a quantum homogeneous bundle, cf. Example 5.47 later. ◇

5.1.2 Trivial Quantum Principal Bundles

We first consider how our theory looks when the quantum principal bundle is trivial. The usual notion of the latter is that the total space is a direct product, which in our algebraic language means that $P = A \otimes H$ as algebras. However, there is a more useful weaker notion which we explain here.

We first recall that if Φ, Ψ are maps from a coalgebra (in our case H) to an algebra then so is the *convolution product* $\odot$ defined by

$$\Phi \odot \Psi = \cdot(\Phi \otimes \Psi)\Delta$$

and that Φ is *convolution-invertible* when there is an inverse Φ^{-1} such that $\Phi \odot \Phi^{-1} = \Phi^{-1} \odot \Phi = 1 \circ \epsilon$. The notation is extended to include the case where one of the maps has values in a module with $\cdot$ the module action. We also need the notion of a 2-cocycle on a Hopf algebra with values in an algebra A. In our case this means a pair $(\triangleright, \chi)$ where $\triangleright : H \otimes A \to A$, $\chi : H \otimes H \to A$ obey $h\triangleright(ab) = (h_{(1)}\triangleright a)(h_{(2)}\triangleright b)$, $h\triangleright 1 = \epsilon(h)1$ as for a module algebra and $1\triangleright a = a$, but

$$(h_{(1)}\triangleright(g_{(1)}\triangleright a))\chi(h_{(2)} \otimes g_{(2)}) = \chi(h_{(1)} \otimes g_{(1)})((h_{(2)}g_{(2)})\triangleright a), \tag{5.14}$$
$$(h_{(1)}\triangleright\chi(g_{(1)} \otimes f_{(1)}))\chi(h_{(2)} \otimes g_{(2)}f_{(2)}) = \chi(h_{(1)} \otimes g_{(1)})\chi(h_{(2)}g_{(2)} \otimes f) \tag{5.15}$$

and $\chi(1\otimes h) = \chi(h\otimes 1) = \epsilon(h)$. These displayed conditions can be written in terms of multiple convolution products. Such cocycles arise in the theory of extensions in Hopf algebra theory and provide a cocycle cross product $A_\chi{>\!\!\!\triangleleft} H$ generalising the left crossed product given in (2.2). This is built on $A \otimes H$ with product

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b)\chi(h_{(2)} \otimes g_{(1)}) \otimes h_{(3)} g_{(2)} \tag{5.16}$$

and unit $1 \otimes 1$, and forms a right H-comodule algebra under $\mathrm{id} \otimes \Delta$. The special case where $A = \Bbbk$ is the most well known and in this case the cocycle reduces to a 'Drinfeld cotwist' (see §9.6.3) and the cocycle cross product $\Bbbk_\chi \rtimes\!\!\!\lhd H$ to the twisted comodule algebra H^χ. There is a cohomology $\mathrm{H}^2(H, A)$ controlling the extensions, whereby two cocycles are cohomologous if they give isomorphic cross product algebras in a way compatible with the H-coaction and the subalgebra A.

Proposition 5.13 *Let P be a right H-comodule algebra equipped with a convolution-invertible right-comodule map $\Phi : H \to P$ with $\Phi(1) = 1$. Then P is a quantum principal bundle over $A = P^H$. We call it a* trivial bundle *with trivialisation Φ. In this case $P \cong A_\chi \rtimes\!\!\!\lhd H$ as a right module algebra, where the cocycle cross product has cocycle data $\chi : H \otimes H \to A$, $\triangleright : H \otimes A \to A$ given by*

$$h \triangleright a = \Phi(h_{(1)}) a \Phi^{-1}(h_{(2)}), \quad \chi(h \otimes g) = \Phi(h_{(1)})\Phi(g_{(1)})\Phi^{-1}(h_{(2)} g_{(2)}).$$

Moreover, if $\gamma : H \to A$ is unital and convolution-invertible then $\Phi^\gamma = \gamma \odot \Phi$ is another trivialisation of P and the resulting χ^γ is in the same class as $\chi \in$ $\mathrm{H}^2(H, A)$.

Proof This is a theorem from Hopf algebra theory, so we will be brief. It is easy to see that if $\triangleright$, χ are well defined by the stated formulae then they obey (5.14)–(5.15). The harder part is to check that their images lie in A not P, for which we need that if Φ is H-covariant then $\Delta_R \Phi^{-1}(h) = \Phi^{-1}(h_{(2)}) \otimes S h_{(1)}$ as Φ^{-1} is uniquely determined by Φ and this is required for invariance of 1ϵ. For example,

$$\begin{aligned}\Delta_R(\Phi(h_{(1)}) a \Phi^{-1}(h_{(2)})) &= \Phi(h_{(1)(1)}) a \Phi^{-1}(h_{(2)(2)}) \otimes h_{(1)(2)} S h_{(2)(1)} \\ &= \Phi(h_{(1)}) a \Phi^{-1}(h_{(4)}) \otimes h_{(2)} S h_{(3)} = \Phi(h_{(1)}) a \Phi^{-1}(h_{(2)}) \otimes 1\end{aligned}$$

so $h \triangleright a \in A$. It is clear that $\gamma \odot \Phi$ is another trivialisation and easy enough to see how the action and χ transform. The latter data up to such transformations amounts to the definition of $H^2(H, A)$ in explicit terms. Moreover, $\omega^\sharp(h) = \Phi^{-1}(h_{(1)}) \otimes \Phi(h_{(2)})$ is clearly a unital bicomodule map and $\cdot\, \omega^\sharp = 1\epsilon$. Hence, by Lemma 5.8, (P, H, Δ_R) is a universal quantum principal bundle with $\omega^\sharp$ a strong connection map on it. □

The classical case is where Φ is an algebra homomorphism, in which case $\chi = \epsilon \otimes \epsilon$ is trivial, and where P is commutative, in which case $\triangleright$ is the trivial action $h \triangleright a = \epsilon(h) a$. Then $P \cong A \otimes H$ as algebras. Note that when A is not central in P, even if Φ is an algebra map then Φ^γ need not be, so the tensor product form, even if it can be achieved, is not preserved under 'gauge transformations' γ. Moreover, it can be achieved precisely when $(\triangleright, \chi)$ is cohomologically trivial, which we will see is often not the case. We now look at connections etc. on such trivial bundles. As in §5.1.1 our conceptual picture is to work with curvature and connections as 'Lie algebra-valued', which for the universal calculus means maps from H^+. However, we now prefer to think of a map from H^+ equivalently as a map from H that vanishes on 1, by pulling back the original map along the counit projection $\pi_\epsilon : H \to H^+$.

Corollary 5.14 *On a trivial quantum principal bundle* (P, H, Δ_R, Φ) *strong connections are equivalent to maps* α *in the form*

$$\omega = \Phi^{-1}\odot \mathrm{d}\Phi + \Phi^{-1}\odot\alpha\odot\Phi, \quad \alpha : H \to \Omega^1_A,$$

where $\alpha(1) = 0$*, with curvature 2-form given by*

$$F_\omega = \Phi^{-1}\odot F(\alpha)\odot\Phi, \quad F(\alpha) = \mathrm{d}\alpha + \alpha\odot\alpha : H \to \Omega^2_A.$$

Sections s *of a bundle associated to a left comodule* V *are equivalent to elements* $\phi \in A \otimes V$*, along with the associated covariant derivative in these terms, by*

$$s = (\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \Phi \otimes \mathrm{id})\Delta_L\phi, \quad \nabla\phi = (\mathrm{d} \otimes \mathrm{id})\phi - (\cdot \otimes \mathrm{id})(\mathrm{id} \otimes \alpha \otimes \mathrm{id})\Delta_L\phi,$$

where Δ_L *is the coaction on* V *and the product is in* P *or* A*. Under a gauge transformation* γ*, the same connection, section and curvature are given by*

$$\alpha^\gamma = \gamma\odot\alpha\odot\gamma^{-1}+\gamma\odot\mathrm{d}\gamma^{-1}, \quad F(\alpha^\gamma) = \gamma\odot F(\alpha)\odot\gamma^{-1}, \quad \phi^\gamma = (\mathrm{id}\otimes\gamma^{-1})\Delta_L\phi.$$

Proof If $\omega^\sharp$ is a strong connection map then let $\alpha = \Phi\odot\omega^\sharp\odot\Phi^{-1} - 1\epsilon$. Since $\omega^\sharp$ is a bicomodule map, it follows by applying $\Delta_L \otimes \mathrm{id}$, $\mathrm{id} \otimes \Delta_R$ that $\alpha(h) \in A \otimes A$, while $\omega^\sharp$ unital tells us that $\alpha(1) = 0$. Since $\cdot\,\omega^\sharp = 1\epsilon$ it follows that $\cdot\,\alpha = 0$ so $\alpha(h) \in \Omega^1_A$ (the universal calculus). Rewriting tells us that $\omega^\sharp(h) = \Phi^{-1}(h_{(1)}) \otimes \Phi(h_{(2)}) + \Phi^{-1}\odot\alpha\odot\Phi$, or ω as stated. One can check in the other direction that $\omega^\sharp$ is a connection map if $\alpha : H \to \Omega^1_A$ obeys $\alpha(1) = 0$. The first term in the formula for $\omega^\sharp$ satisfies the conditions for a strong connection in (5.9), using the comodule map property of Φ and also $\Delta_R\Phi^{-1}(h) = \Phi^{-1}(h_{(2)}) \otimes Sh_{(1)}$, as discussed in the proof of Proposition 5.13. The associated connection form is given by restriction to H^+, where it has the form stated using $\mathrm{d}u = 1 \otimes u - u \otimes 1$ for the universal calculus. By (5.4), the curvature F_ω has the form shown. Next, if $s = s^1 \otimes s^2 \in (P \otimes V)^H$ (or rather, in the cotensor product) as in Proposition 5.5, then we let

$$\phi = s^1{}_{(\bar{0})}\Phi^{-1}(s^1{}_{(\bar{1})}) \otimes s^2 = s^1\Phi^{-1}(s^2{}_{(\bar{1})}) \otimes s^2{}_{(\bar{\infty})},$$

where we can use either the right coaction on the first factor of s or the left coaction in the second factor. From the first version and the covariance of Φ^{-1}, it is clear that $\phi \in A \otimes V$. We clearly recover $s = \phi^1\Phi(\phi^2{}_{(\bar{1})}) \otimes \phi^2{}_{(\bar{\infty})}$ as stated from the second version. We then compute from Proposition 5.5,

$$\begin{aligned}\nabla s &= \mathrm{d}(\phi^1\Phi(\phi^2{}_{(\bar{1})}))\otimes\phi^2{}_{(\bar{\infty})}-\phi^1\Phi(\phi^2{}_{(\bar{1})})\omega^\sharp(\phi^2{}_{(\bar{\infty})(\bar{1})})\phi^2{}_{(\bar{\infty})(\bar{\infty})}\\ &= (\mathrm{d}\phi^1)\Phi(\phi^1{}_{(\bar{1})})\otimes\phi^2{}_{(\bar{\infty})}-\phi^1\alpha(\phi^2{}_{(\bar{1})(1)})\Phi(\phi^2{}_{(\bar{1})(2)})\otimes\phi^2{}_{(\bar{\infty})}\\ &= (\mathrm{d}\phi^1)\Phi(\phi^1{}_{(\bar{1})})\otimes\phi^2{}_{(\bar{\infty})}-\phi^1\alpha(\phi^2{}_{(\bar{1})})\Phi(\phi^2{}_{(\bar{\infty})(\bar{1})})\otimes\phi^2{}_{(\bar{\infty})(\bar{\infty})}\\ &= (\nabla\phi)^1\Phi((\nabla\phi)^2{}_{(\bar{1})})\otimes(\nabla\phi)^2{}_{(\bar{\infty})},\end{aligned}$$

where we extend our previous correspondence and notation to $\nabla\phi\in\Omega^1_A\otimes V$ corresponding to $\nabla s\in\Omega^1_A\otimes_A(P\otimes V)^H$. The second equality uses the Leibniz rule on P and the definition of $\omega^\sharp$. Finally, if we change Φ to Φ^γ, the corresponding formula for α changes if we want the same $\omega^\sharp$. Similarly for F_ω and s. It follows, but it is also useful to see directly, that these gauge transformations behave as expected. Thus the direct proof that $F(\alpha^\gamma)=\gamma\odot F(\alpha)\odot\gamma^{-1}$ follows the same calculation as classically, just using the associativity and Leibniz rule for d with respect to the convolution product. For $\nabla\phi$, the calculation is again similar to the classical case if we extend the convolution product to include the coaction. Thus, $(\mathrm{id}\otimes\Delta_L)\phi^\gamma=\phi^1\gamma^{-1}(\phi^2{}_{(\bar{1})(1)})\otimes\phi^2{}_{(\bar{1})(2)}\otimes\phi^2{}_{(\bar{\infty})}$ using the coaction axioms, hence

$$\begin{aligned}\nabla_{\alpha^\gamma}\phi^\gamma &= \mathrm{d}(\phi^1\gamma^{-1}(\phi^2{}_{(\bar{1})}))\otimes\phi^2{}_{(\bar{\infty})}-\phi^1\gamma^{-1}(\phi^2{}_{(\bar{1})(1)})\alpha^\gamma(\phi^2{}_{(\bar{1})(2)})\otimes\phi^2{}_{(\bar{\infty})}\\ &= (\mathrm{d}\phi^1)\gamma^1(\phi^2{}_{(\bar{1})})\otimes\phi^2{}_{(\bar{\infty})}-\phi^1\alpha(\phi^2{}_{(\bar{1})(1)})\gamma^{-1}(\phi^2{}_{(\bar{1})(2)})\otimes\phi^2{}_{(\bar{\infty})}\\ &= (\mathrm{d}\phi-\phi^1\alpha(\phi^2{}_{(\bar{1})})\otimes\phi^2{}_{(\bar{\infty})})^\gamma=(\nabla\phi)^\gamma\end{aligned}$$

using the Leibniz rule and the formula for α^γ for the second equality. At the end, we used the same transformation for an element $\Omega^1_A\otimes V$ as we did for $\phi\in A\otimes V$ (just with the first component a 1-form). □

At present, the calculus is the universal one but the same 'local gauge theory' will apply when we come to nonuniversal calculi.

5.2 Constructions of Quantum Bundles with Universal Calculus

We have given the classical Klein bottle Example 5.11 and the general construction of trivial bundles in §5.1.2. Here we give some other specific constructions.

5.2.1 Galois Field Extensions as Quantum Bundles

Suppose that $\mathbb{E}$ is a field containing our ground field $\Bbbk$, so in particular $\mathbb{E}$ is a vector space over $\Bbbk$. Additionally suppose that the finite group G right acts on $\mathbb{E}$ by $\Bbbk$-linear field automorphisms. (These send 1 to 1, and therefore fix every element of $\Bbbk$.) Then the fixed points $\mathbb{F} = \mathbb{E}^G$ is also a field containing $\Bbbk$. We say that the field extension $\mathbb{F} \subseteq \mathbb{E}$ is Galois with Galois group G if (i) G acts faithfully on $\mathbb{E}$ (i.e., only the identity $e \in G$ acts as the identity) and (ii) the dimension of $\mathbb{E}$ as a vector space over $\mathbb{F}$ is the same as the size of G. We can of course rephrase the action of G as a coaction of $\Bbbk(G)$, the algebra of functions on G with pointwise product. Here

$$\Delta_R(e) = \sum_{g \in G} e{\triangleleft}g^{-1} \otimes \delta_g$$

so that the invariant elements of $\mathbb{E}$ for the G action are precisely the invariants for the $\Bbbk(G)$-coaction. Moreover, $\mathbb{E}$ and $\mathbb{F}$ are algebras over $\Bbbk$ and $\mathbb{E}$ becomes a right $\Bbbk(G)$-comodule algebra.

Proposition 5.15 *Let $\mathbb{F} \subseteq \mathbb{E}$ be a $\Bbbk$-linear field extension with $\mathbb{F} = \mathbb{E}^G$ and G finite. This is Galois if and only if $P = \mathbb{E}$ is a quantum principal bundle or Hopf–Galois extension of $A = \mathbb{F}$ with structure quantum group $H = \Bbbk(G)$.*

Proof First assume that $\mathbb{F} \subseteq \mathbb{E}$ is Galois with Galois group G. Write $G = \{g_1, \ldots, g_n\}$ and take a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{E}$ as a vector space over $\mathbb{F}$. Suppose that $\sum_i v_i \otimes e_i$ is in the kernel of ver, for $v_i \in \mathbb{E}$. Then for all k we have $\sum_i v_i(e_i{\triangleleft}g_k) = 0$. Now consider the matrix $N \in M_n(\mathbb{E})$ given by $N_{ik} = e_i{\triangleleft}g_k$. By Dedekind's Lemma (Lemma 5.16) the columns of N are linearly independent (as the action of G is faithful) so N is invertible. It follows that every $v_i = 0$, so the map ver is injective, and counting dimensions as vector spaces over $\mathbb{F}$ gives a 1–1 correspondence. Conversely suppose that ver is a 1–1 correspondence. Counting dimensions shows that the dimension of $\mathbb{E}$ as a vector space over $\mathbb{F}$ is the same as the size of G. Suppose that the action of G is not faithful—then there is a $g \neq e$ which has trivial action. Now, just taking those two elements, we get $a \otimes b \mapsto ab \otimes (\delta_e + \delta_g)$, so we would never get $1 \otimes \delta_e$ in the image, contradicting surjectivity. □

The required technical lemma for Proposition 5.15 is the following:

Lemma 5.16 (Dedekind's Lemma) *Any set of distinct field homomorphisms $\theta_i : \mathbb{E} \to \mathbb{E}$ is linearly independent over $\mathbb{E}$.*

Proof If there exists a nontrivial linear combination which is zero, take one involving the minimum number of homomorphisms, say $\{\theta_1, \ldots, \theta_m\}$, and write $\theta_1 = \sum_{2 \le i \le m} e_i.\theta_i$ for some $e_i \in \mathbb{E}$. Then for any $x, y \in \mathbb{E}$, we have

$$\theta_1(x)\theta_1(y) = \sum_{2 \le i \le m} e_i.\theta_i(x)\theta_i(y)$$

on applying the sum to xy, but we also have, applying the sum to y and multiplying the result by $\theta_1(x)$,

$$\theta_1(x)\theta_1(y) = \sum_{2\le i\le m} e_i.\theta_1(x)\theta_i(y).$$

Subtracting yields

$$\sum_{2\le i\le m} e_i.(\theta_1(x) - \theta_i(x))\theta_i(y)$$

for all x, y. By the minimality assumption, $\{\theta_2, \dots, \theta_m\}$ are linearly independent, so $e_i.(\theta_1(x) - \theta_i(x)) = 0$ for $2 \le i \le m$ and for all x. As the maps are distinct, we must have all $e_i = 0$, contradicting the existence of a nontrivial linear relation. □

Example 5.17 We set $\Bbbk = \mathbb{Q}$ and $\mathbb{E} = \mathbb{Q}[\sqrt{2}]$. The elements of $\mathbb{E}$ are of the form $r + s\sqrt{2}$ for rational r, s and form a field with $\frac{1}{r+s\sqrt{2}} = \frac{s\sqrt{2}-r}{2s^2-r^2}$, which is defined for nonzero $r + s\sqrt{2}$ since $\sqrt{2}$ is irrational. There is a field automorphism θ on $\mathbb{E}$ given by $\theta(r + s\sqrt{2}) = r - s\sqrt{2}$. Then θ^2 is the identity, so we get the cyclic group $G = \{\mathrm{id}, \theta\}$ acting, with fixed points $\mathbb{F} = \mathbb{Q}$.

Next, as a quantum bundle, $\mathbb{Q}[\sqrt{2}]$ is trivial with

$$\Phi : \mathbb{Q}(\mathbb{Z}_2) \to \mathbb{Q}[\sqrt{2}], \quad \Phi(\delta_0) = \frac{1}{2}(1 + \sqrt{2}), \quad \Phi(\delta_1) = \frac{1}{2}(1 - \sqrt{2}),$$

which is covariant under the $\mathbb{Z}_2$ action and convolution-invertible, namely by

$$\Phi^{-1}(\delta_0) = \frac{1}{2}(1 + (\sqrt{2})^{-1}), \quad \Phi^{-1}(\delta_1) = \frac{1}{2}(1 - (\sqrt{2})^{-1}).$$

It follows that $\mathbb{Q}[\sqrt{2}] \cong \mathbb{Q}(\mathbb{Z}_2)^\chi$, where we twist by a cocycle. To compute this, it is easier to write $\mathbb{Q}(\mathbb{Z}_2)$ as generated by $g = \delta_0 - \delta_1$. Then g is grouplike and $g^2 = 1 = \delta_0 + \delta_1$, i.e., we view our Hopf algebra as the group algebra $H = \mathbb{Q}\mathbb{Z}_2$. In this form we find more easily

$$\Phi(g) = \sqrt{2}, \quad \Phi^{-1}(g) = (\sqrt{2})^{-1}, \quad \chi(g^i \otimes g^j) = 2^{ij}, \quad i \in \{0, 1\}.$$

Here $\chi \in \mathrm{H}^2(\mathbb{Z}_2, \mathbb{Q})$ is nontrivial in group cohomology and $\mathbb{Q}[\sqrt{2}] = (\mathbb{Q}\mathbb{Z}_2)^\chi$ with product $h \bullet g = \chi(h_{(1)} \otimes g_{(1)})h_{(2)}g_{(2)}$, namely

$$1 \bullet 1 = 1, \quad 1 \bullet g = g \bullet 1 = g, \quad g \bullet g = 2,$$

gives a new product on the vector space of $\mathbb{Q}\mathbb{Z}_2$. ◇

Notice that χ had to be a nontrivial element of cohomology as $\mathbb{Q}[\sqrt{2}]$, being a field, cannot be isomorphic to a group algebra (or to any Hopf algebra since the augmentation ideal would have to be zero). Quantum bundles arising this way from classical Galois extensions are trivial quantum bundles in our sense but in a nontrivial cohomology class which prevents them from being a classical principal bundle over a point, where the total space would be the structure group and $\mathbb{E}$ would have the structure of a group function algebra.

There is a theory of Hopf–Galois extensions of fields defined in just the same way as for groups but with the action of a finite-dimensional Hopf algebra T in place of the group action, except that we should sum over a basis of T and dual basis, and use the antipode rather than group inverse. Replacing the action with the corresponding coaction, this notion is then equivalent to a quantum principal bundle or Hopf–Galois extension with structure quantum group $H = T^*$.

Example 5.18 Let $\Bbbk = \mathbb{Q}$ and T be the unital commutative 4-dimensional Hopf algebra over $\Bbbk$ with generators $\{s, c\}$ and relations and coalgebra

$$s^2 + c^2 = 1, \quad sc = 0, \quad \Delta c = c \otimes c - s \otimes s, \quad \Delta s = s \otimes c + c \otimes s, \quad \epsilon c = 1, \quad \epsilon s = 0,$$

and antipode $Sc = c$, $Ss = -s$ (this is a quotient of the 'trigonometric Hopf algebra'). Then T acts on the field $\mathbb{E} = \mathbb{Q}[\omega]$ where $\omega = 2^{1/4}$ by the following table

x	1	ω	ω^2	ω^3
$x{\triangleleft}c$	1	0	$-\omega^2$	0
$x{\triangleleft}s$	0	$-\omega$	0	ω^3

The invariants for this action are $\mathbb{F} = \mathbb{Q}$, and the field extension $\mathbb{Q} \subset \mathbb{E}$ is Galois with Hopf algebra T. Note that the elements of $\mathbb{E}$ can be written $r_0 + r_1\omega + r_2\omega^2 + r_3\omega^3$ for $r_i \in \mathbb{Q}$. The only nontrivial $\Bbbk$ linear field automorphism is given by $\theta(\omega) = -\omega$. There is no way to recover $\mathbb{F} = \mathbb{Q}$ as the fixed points of a group of automorphisms, so this example strictly needs the Hopf–Galois theory. In fact, $T = \mathbb{Q}(\mathbb{Z}_4)$ by writing

$$c = \delta_e - \delta_{g^2}, \quad s = \delta_g - \delta_{g^3}$$

in terms of a basis of δ-functions, where $\mathbb{Z}_4$ is generated by g with $g^4 = 1$. So the dual of T is the group algebra $H = \mathbb{Q}\mathbb{Z}_4$ and it is this which is the structure quantum group of the quantum principal bundle. Its coaction just means a $\mathbb{Z}_4$-grading, here given by $|\omega| = g$. From this is it clear that the map $\mathrm{ver}(u \otimes v) = uv \otimes g^{|v|}$ on homogeneous elements is an isomorphism so that we have a quantum bundle. Next, we look for a map $\Phi : \mathbb{Q}\mathbb{Z}_4 \to \mathbb{Q}[\omega]$ which is degree-preserving and convolution-invertible, which just means invertible on each power of g. An obvious choice is $\Phi(g^i) = \omega^i$, $i = 0, 1, 2, 3$, which is not an algebra map as $\omega^4 = 2$. Then

$$\chi(g^i, g^j) = \omega^i \omega^j (\omega^{i+j \bmod 4})^{-1} = \begin{cases} 1 & i + j < 4, \\ 2 & i + j \geq 4, \end{cases}$$

which is nontrivial in $\mathrm{H}^2(\mathbb{Z}_4, \mathbb{Q})$ and gives $\mathbb{Q}[\omega] \cong (\mathbb{Q}\mathbb{Z}_4)^\chi$ as a twisted group algebra by the map Φ. The right-hand side has the new product $g^i \bullet g^j = \chi(g^i, g^j)g^{i+j}$ which gives the same products as we expect of ω, for example $g \bullet g^3 = 2$ corresponding to $\omega\omega^3 = 2$ in $\mathbb{E}$. ◇

For a noncommutative quantum bundle, we take $\mathbb{E}$ a division algebra.

Example 5.19 Let $\mathbb{F} = \mathbb{R} \subset \mathbb{E} = \mathbb{H}$, the quaternions, as an extension of $\mathbb{R}$. This is a quantum bundle or Hopf–Galois extension over $\Bbbk = \mathbb{R}$ with structure quantum group $H = \mathbb{R}\mathbb{Z}_2^2$ (the group algebra of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$). Here $\mathbb{H} = \mathbb{R}\langle i, j\rangle$ with the relations $i^2 = j^2 = ijk = -1$. This implies $ij = k = -ji$, etc. The coaction is given by a $\mathbb{Z}_2^2$-grading here $|i| = 01, |j| = 10, |k| = 11$ in a compact notation for the elements of the group. The degree 0 component is $\mathbb{F} = \mathbb{R}$ as the fixed subalgebra and the map $\mathrm{ver} : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}\mathbb{Z}_2^2$ sending $u \otimes v \to uv \otimes |v|$ for a homogeneous element is clearly surjective hence bijective by dimensions. The trivialisation $\Phi(00) = 1, \Phi(01) = i, \Phi(10) = j, \Phi(11) = k$ is degree-preserving and point-wise invertible, so does the job. The associated cocycle is

$$\chi(01, 10) = ijk^{-1} = 1, \quad \chi(10, 01) = jik^{-1} = -1,$$
$$\chi(01, 11) = ikj^{-1} = -1, \quad \chi(11, 01) = kij^{-1} = 1,$$
$$\chi(10, 11) = jki^{-1} = 1, \quad \chi(11, 10) = kji^{-1} = -1,$$

and $\chi(00, 00) = 1$, $\chi(a, a) = -1$ for $a \neq 01, 10, 11$, or $\chi(a, b) = (-1)^{\sum_{i\le j} a_i b_j}$ for $a = a_0a_1$ and $b = b_0b_1$. Here $\chi \in \mathrm{H}(\mathbb{Z}_2^2, \mathbb{R})$ and makes the quaternions a twisted version of $\mathbb{R}\mathbb{Z}_2^2$. E.g., $01 \bullet 01 = \chi(01, 01)00$ corresponds to $i^2 = -1$.

This construction generalises to the Albuquerque–Majid description of the octonions as a twisted version of $H = \mathbb{R}\mathbb{Z}_2^3$ and means that the octonions can be similarly be seen as a quantum principal bundle albeit in a quasiassociative algebra setting (as touched upon in §9.6.3 and Exercise E9.9). ◇

The way to think about all these examples so far is that the structure quantum group algebra is $H = \Bbbk\Sigma$ for some group Σ. Then we found that $\mathbb{E}$ is a twisted group algebra $\mathbb{E} = (\mathbb{F}\Sigma)^\chi$ controlled by $\chi \in \mathrm{H}^2(\Sigma, \mathbb{F})$. Here Σ is the dual of the classical Galois group in the finite abelian case but otherwise this is a different theory. Here is an example on the other side where $H = \Bbbk(G)$ and the classical Galois group G is nonabelian, in which case the cohomology theory still applies but χ is not any kind of classical group cocycle as there is no dual group Σ and *has* to be handled by Hopf algebra methods.

Example 5.20 $\mathbb{E} = \mathbb{Q}(\omega, \eta)$ as an extension of $\mathbb{Q}$ by $\omega = \sqrt[3]{2}$ and $\eta = e^{\frac{2\pi i}{3}}$ has Galois group $G = S_3$, which in our usual generators $u = (12), v = (23)$ acts as

$\omega \triangleleft u = \eta\omega$, $\eta \triangleleft u = \eta^2$, $\omega \triangleleft v = \omega$, $\eta \triangleleft v = \eta^2$. We let

$$\Phi(\delta_e) = \sum_{ij} \phi_{ij}\omega^i\eta^j, \quad \Phi(\delta_u) = \sum_{ij} \phi_{ij}\omega^i\eta^{i+2j}, \quad \Phi(\delta_v) = \sum_{ij} \phi_{ij}\omega^i\eta^{2j},$$

$$\Phi(\delta_{uv}) = \sum_{ij} \phi_{ij}\omega^i\eta^{2i+j}, \quad \Phi(\delta_{vu}) = \sum_{ij} \phi_{ij}\omega^i\eta^{i+j}, \quad \Phi(\delta_{uvu}) = \sum_{ij} \phi_{ij}\omega^i\eta^{2(i+j)},$$

where $i = 0, 1, 2$, $j = 0, 1$ and the values are determined from the first one by S_3-equivariance from the right, where $\delta_g \triangleleft h = \delta_{gh}$. This ensures covariance

$$\Delta_R \Phi(\delta_g) = \sum_h \Phi(\delta_g) \triangleleft h^{-1} \otimes \delta_h = \sum \Phi(\delta_{gh^{-1}}) \otimes \delta_h = (\Phi \otimes \mathrm{id})\Delta\delta_g$$

under the corresponding right coaction. Then $\Phi(1) = 1$ gives the condition $\phi_{00} = \frac{1}{2}(\frac{1}{3} + \phi_{01})$, which we suppose, giving us a 5-dimensional $\mathbb{Q}$ moduli space of equivariant maps fixing 1. Now suppose another function Φ^{-1} where $\Phi^{-1}(1) = 1$ is required as $(\Phi^{-1} \odot \Phi)(1) = 1$, and $\Phi^{-1}(\delta_g) \in \mathbb{E}$ for $g \neq e$ are required to obey the five equations $(\Phi^{-1} \odot \Phi)(\delta_g) = 0$ at each $g \neq e$. Since $\mathbb{E}$ is a field, the solution exists generically, according to the nonvanishing of a determinant in $\mathbb{E}$, and the convolution-inverse in this case will necessarily obey $\Phi^{-1}(\delta_g) \triangleleft h = \Phi^{-1}(\delta_{h^{-1}g})$ as a useful check. For example, $\phi_{00} = \phi_{01} = \phi_{11} = \phi_{21} = 1/3$ and the rest zero has

$$\Phi^{-1}(\delta_u) = \frac{(\omega + \omega^2)(\eta + 2) - (2\eta + 1)}{9\omega^2}, \quad \Phi^{-1}(\delta_v) = \frac{\omega^2(\eta + 2) + (1 + \omega)(\eta - 1)}{9\omega^2},$$

$$\Phi^{-1}(\delta_{uv}) = \frac{(1 + \omega^2)(1 - \eta) + \omega(2\eta + 1)}{9\omega^2}, \quad \Phi^{-1}(\delta_{vu}) = \frac{(\omega + \omega^2)(1 - \eta) + 2\eta + 1}{9\omega^2},$$

$$\Phi^{-1}(\delta_{uvu}) = \frac{(1 + \omega^2)(\eta + 2) - \omega(2\eta + 1)}{9\omega^2}.$$

We then compute the 2-cocycle on $H = \mathbb{Q}(S_3)$ for this and find that it has values in $\mathbb{Q}$ as it should, namely in basis order $\delta_e, \delta_u, \delta_v, \delta_{uv}, \delta_{vu}, \delta_{uvu}$,

$$\chi = \frac{1}{9}\begin{pmatrix} 0 & 1 & 0 & 2 & 4 & 2 \\ 1 & -4 & -2 & -3 & 8 & 0 \\ 0 & -2 & 6 & -1 & -2 & -1 \\ 2 & -3 & -1 & -5 & 0 & 7 \\ 4 & 8 & -2 & 0 & -7 & -3 \\ 2 & 0 & -1 & 7 & -3 & -5 \end{pmatrix}$$

where the first row and column sum to 1 and the rest to 0 according to $\chi(1 \otimes \delta_g) = \chi(\delta_g \otimes 1) = \delta_{g,e}$ for all $g \in S_3$. According to our theory, $\mathbb{Q}(\omega, \eta) \cong \mathbb{Q}(S_3)^\chi$, where we twist the product by this Hopf algebra cocycle $\chi \in \mathrm{H}^2(H, \mathbb{Q})$. Note that we can also view $\chi \in \mathbb{Q}S_3 \otimes \mathbb{Q}S_3$, where it is a 2-cocycle *in* the group Hopf algebra $\mathbb{Q}S_3$. ⋄

As the base algebra in all these examples is the same as the ground field, the calculus on the base is the zero calculus. Hence there is only one connection $\omega(h) = \mathrm{ver}^{-1}(1 \otimes h)$ for all $h \in H^+$. Thus in the quaternion example, the universal calculus is 3-dimensional with basis $\mathrm{d}i, \mathrm{d}j, \mathrm{d}k$ and relations

$$\mathrm{d}i.i = -i\mathrm{d}i, \quad \mathrm{d}i.j = \mathrm{d}k - i\mathrm{d}j, \quad \mathrm{d}i.k = -\mathrm{d}j - i\mathrm{d}k,$$

and cyclic rotations $i \to j \to k \to i$ of these. These are all obtained from differentiating the quaternion relations. We have $\omega_i = i\mathrm{d}i, \omega_j = j\mathrm{d}j, \omega_k = k\mathrm{d}k$ for the three values of ω on H^+. This is just the 'zero' or flat connection defined by the trivialisation and there are no others. On the other hand, we can take other calculi, for example using twisting as in §9.6.3 to transfer any bicovariant calculus on H to a covariant calculus on $\mathbb{H}$ and similarly for our other extensions.

5.2.2 Quantum Homogeneous Bundles with Universal Calculus

Now we turn to another general construction, this time with P also a Hopf algebra.

Lemma 5.21 *Let $\pi : P \to H$ be a Hopf algebra surjection, $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ and $A = P^H$. Then*

(1) *(P, H, Δ_R) is a universal quantum principal bundle if the product map $\ker \pi|_A \otimes P \to \ker \pi$ is a surjection.*
(2) *If furthermore $i : H^+ \to P^+$ obeys $\pi \circ i = \mathrm{id}$ and $(\mathrm{id} \otimes \pi)\mathrm{Ad} \circ i = (i \otimes \mathrm{id}) \circ \mathrm{Ad}$, then*

$$\omega(h) = (Si(h)_{(1)})\mathrm{d}i(h)_{(2)}$$

for all $h \in H^+$ provides a connection form. Here $i(h)_{(1)} \otimes i(h)_{(2)} = \Delta \circ i(h)$.
(3) *If $i : H \to P$ is a unital left and right comodule map under $\Delta_L = (\pi \otimes \mathrm{id})\Delta$ and Δ_R such that $\pi \circ i = \mathrm{id}$ then the above two conditions hold and we have a quantum bundle with strong connection.*

Proof Let us note first that $\mathrm{ver}^\sharp = (\mathrm{id} \otimes \pi)\Theta$, where $\Theta : P \otimes P \to P \otimes P$ is $\Theta(p \otimes q) = pq_{(1)} \otimes q_{(2)}$ with inverse $\Theta^{-1}(p \otimes q) = pSq_{(1)} \otimes q_{(2)}$ (this is the map used in the theory of differential calculi in Theorem 2.26). Hence $\mathrm{ver}^\sharp$ is clearly surjective since π is, and writing $\rho \in \ker \mathrm{ver}^\sharp$ as $\rho = \Theta^{-1}(\sum p_i \otimes q_i)$ with p_i linearly independent, we conclude that $\pi(q_i) = 0$. Then if our assumption on $\ker \pi$ holds, we have $q_i = \sum_j a_{ij} w_{ij}$ where $a_{ij} \in \ker \pi|_A$ and hence our original element can be written as $\rho = \sum p_i(Sw_{ij\,(1)})(Sa_{ij\,(1)}) \otimes a_{ij\,(2)} w_{ij\,(2)}$. Hence the image of

$\rho \in P \otimes_A P$ is $\sum p_i S w_{ij\,(1)} \otimes w_{ij\,(2)} \epsilon(a_{ij})$. This vanishes because if $a \in A$, it means $(\mathrm{id} \otimes \pi)\Delta a = a \otimes 1$, which applying π implies $\pi(a) \otimes 1 = \Delta\pi(a)$ and hence $\epsilon(a) = \epsilon\pi(a) = \pi(a)$ so that $\ker \pi|_A = A^+$. Hence we have proven the Hopf–Galois property under our assumption. For the second part, we verify covariance of ω as

$$\begin{aligned}\Delta_R \omega(h) &= Si(h)_{(2)} \mathrm{d}i(h)_{(3)} \otimes \pi(Si(h)_{(1)})\pi \circ i(h)_{(4)}\\ &= Si(h_{(2)})_{(1)} \mathrm{d}i(h_{(2)})_{(2)} \otimes (Sh_{(1)})h_{(3)} = \omega(h_{(2)}) \otimes (Sh_{(1)})h_{(3)}\end{aligned}$$

for all $h \in H^+$. In the second equality we used the covariance property of i. Moreover,

$$\begin{aligned}\mathrm{ver}\omega(h) &= (Si(h)_{(1)})i(h)_{(2)} \otimes \pi(i(h)_{(3)}) - (Si(h)_{(1)})i(h)_{(2)} \otimes 1\\ &= 1 \otimes \pi(i(h)) = 1 \otimes h\end{aligned}$$

for all $h \in H^+$. We used the explicit form of d. For the third part we prefer to go via Lemma 5.8 where, clearly, i a unital bicomodule map gives $\omega^\sharp(h) = Si(h)_{(1)} \otimes i(h)_{(2)}$ a unital bicomodule map with $\cdot\,\omega^\sharp = \epsilon(h)$ since $\epsilon i(h) = \epsilon(h)$ easily follows from the left (say) covariance of i and $\pi \circ i = \mathrm{id}$. Hence, given a bicovariant i, we have a quantum principal bundle and a strong connection on it. Clearly, such a map is also Ad-covariant in the sense of (2) and moreover one can prove that if we have a quantum homogeneous bundle and a unital map right comodule map i then condition (1) holds so case (3) is in fact a special case. □

Our first example is still a trivial bundle but serves as a useful warm-up. It also illustrates that for quantum homogenous bundles the associated bundles carry a left coaction from $\Delta_L = (\pi \otimes \mathrm{id})\Delta$ on P, as studied at the end of the next section.

Example 5.22 (Finite Group Homogeneous Bundle) Let X be a finite group and G a subgroup. Let $\pi : P = \mathbb{C}(X) \to H = \mathbb{C}(G)$ be given by $\pi(\delta_x) = \delta_x$ if $x \in G$ and zero otherwise. The induced right coaction is

$$\Delta_R(\delta_x) = \sum_{g \in G} \delta_{xg^{-1}} \otimes \delta_g$$

and $A = P^H$ can be identified with functions on the coset space $A = \mathbb{C}(X/G)$. We choose coset representatives $x_j \in X$, labelling the cosets by j and including $x_0 = e$, the group identity. One can check that condition (1) in the lemma holds. The bundle is in fact trivial with $\Phi(\delta_g) = \sum_j \delta_{x_j g}$ viewed in $\mathbb{C}(G)$ with $\Phi^{-1}(\delta_g) = \sum_j \delta_{x_j g^{-1}} = \Phi(\delta_{g^{-1}})$. Here Φ is a right comodule map and also an algebra map (so there is no cocycle or action in the trivial bundle). We can also set $i = \Phi$ and check that conditions (2) or (3) in the lemma both then amount to requiring that $\{x_j\}$ is stable under conjugation by elements of G, i.e., $g x_j g^{-1} = x_{j'}$ for some index j'. In this case, we have a strong connection

$$\omega(\delta_g) = \sum_{x\in X, j} \delta_x \mathrm{d}\delta_{x x_j g}$$

for $g \in G \setminus \{e\}$. This is not necessarily the same as the trivial connection $\Phi^{-1}\odot\mathrm{d}\Phi$, unless the set of coset representatives is closed under products.

Next, a right $\mathbb{C}(G)$-comodule has the form $\Delta_R v = \sum_{g\in G} g\triangleright v \otimes \delta_g$ for V a left G-module. From the tensor product $\mathbb{C}(G)$-coaction $\Delta_R(\sum_{x\in X} \delta_x \otimes v_x) = \sum_{x\in X,\ g\in G} \delta_{xg^{-1}} \otimes g\triangleright v_x \otimes \delta_g$, the associated bundle $E = (\mathbb{C}(X) \otimes V)^{\mathbb{C}(G)}$ by Proposition 5.5 consists of elements $\sum \delta_x \otimes v_x$ such that $g\triangleright v_x = v_{xg^{-1}}$ for all $g \in G, x \in X$, i.e., can be identified with the space $\mathbb{C}_G(X, V)$ of equivariant functions on X with values in V. We then have a bimodule covariant derivative

$$\begin{aligned}\nabla(\sum_x \delta_x \otimes v_x) &= \sum_x \mathrm{d}\delta_x \otimes v_x - \sum_{x,g} \delta_x \omega(\pi_\epsilon \delta_g) \otimes v_{xg}\\ &= \sum_x \mathrm{d}\delta_x \otimes v_x - \sum_{g\neq e, x, j} \delta_x \mathrm{d}\delta_{x x_j g} \otimes (v_{xg} - v_x).\end{aligned}$$

Moreover, E carries a left coaction of $\mathbb{C}(X)$ or a right action of X given by $\delta_x \triangleleft y = \delta_{y^{-1}x}$ and our above constructions are equivariant. This is the geometric version of the idea that a representation of G on a vector space W leads to an induced representation $\mathbb{C}X \otimes_{\mathbb{C}G} W$ and indeed the $E = \mathbb{C}(X) \otimes^{\mathbb{C}(G)} V$ as cotensor product is just the dual construction to this if W is dual to V.

For a concrete example, we let $X = S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ with its usual generators u, v and $G = \mathbb{Z}_2 = \{e, u\}$, and coset representatives $x_j = t^j$ where $t = uv$ and $j = 0, 1, 2 \bmod 3$. This identifies $X/G \cong \mathbb{Z}_3$ as sets but not as a group since there are cross relations $ut = t^{-1}u$. We identify the base as $A \cong \mathbb{C}(\mathbb{Z}_3)$ as spanned by $\delta_j := \delta_{t^j} + \delta_{t^j u}$ constant on each coset. The above connection comes out as $\omega(\delta_u) = \delta \mathrm{d}\tilde{\delta} + \tilde{\delta}\mathrm{d}\delta$, where $\delta = i(\delta_e) = \sum_j \delta_{t^j}$ and $\tilde{\delta} = i(\delta_u) = \sum_j \delta_{t^j u}$ and agrees with $\Phi^{-1}\odot\mathrm{d}\Phi$, hence is flat. There is one nontrivial representation $V = \mathbb{k}.c$ where u acts on c as -1. However, $\delta - \tilde{\delta}$ also changes sign under the action of u so that $\xi := (\delta - \tilde{\delta}) \otimes c$ is invariant and a basis of E over A (a linear basis of E is then $\{\delta_j \xi\}$). The covariant derivative comes out using $\delta\tilde{\delta} = 0$, $\delta^2 = \delta$, $\tilde{\delta}^2 = \tilde{\delta}$ and $\delta + \tilde{\delta} = \sum_j \delta_j = 1$ as

$$\nabla\xi = 0, \quad \sigma(\xi \otimes \mathrm{d}\delta_j) = \mathrm{d}\delta_j \otimes \xi$$

on the basis element. Further strong connections can be obtained by adding $\Phi^{-1}\odot\alpha\odot\Phi$ for a free choice of $\alpha(\delta_u) = -\alpha(\delta_e) \in \Omega^1_A$, in which case $\nabla\xi = 2\alpha(\delta_u) \otimes \xi$. Finally, the right S_3-action on E comes out as permutations $u = -(23)$, $v = -(13)$ of the basis $\delta_0\xi, \delta_1\xi, \delta_2\xi$, i.e., a standard representation tensored with the sign representation. ◇

For classical homogeneous spaces, the existence of a canonical connection is based on local triviality; one builds it near the identity and then translates it around to other coordinate patches. In our construction we use a linear covariant map i, which is a new approach even for the $q = 1$ case of the next example.

Example 5.23 Over $\mathbb{C}$, let $P = \mathbb{C}_q[SL_2]$ and $H = \mathbb{C}\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ with coproduct $\Delta t = t \otimes t$. We have already seen the Hopf algebra surjection and push-forward coaction on the generators

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \Delta_R\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes t & b \otimes t^{-1} \\ c \otimes t & d \otimes t^{-1} \end{pmatrix}$$

in Proposition 2.33, which is equivalent to a $\mathbb{Z}$-grading $|a| = |c| = 1$ and $|b| = |d| = -1$. In this case, we know from the discussion after Proposition 2.33 that $A = P^H = \mathbb{C}_q[SL_2]_0 = \mathbb{C}_q[S^2]$, the q-sphere as degree zero subspace. We have the quantum bundle and canonical strong connection (the *q-monopole*) defined by the unital bicomodule map

$$i(t^n) = a^n, \quad i(t^{-n}) = d^n,$$

where $\Delta_R i(t^n) = \Delta_R a^n = (\mathrm{id} \otimes \pi)(a \otimes a + b \otimes c)^n = a^n \otimes t^n = i(t^n) \otimes t^n$ (and similarly for the other side and for d^n). Here, for example,

$$\omega(t-1) = d\mathrm{d}a - qb\mathrm{d}c = D - 1 \otimes 1, \quad \omega(t^{-1}-1) = a\mathrm{d}d - q^{-1}c\mathrm{d}b = \tilde{D} - 1 \otimes 1,$$

where $D = d \otimes a - qb \otimes c$ is the split q-determinant already encountered in the proof of Proposition 2.36 and $\tilde{D} = a \otimes d - q^{-1}c \otimes b$ is another similar split q-determinant. The connection map as in Lemma 5.8 is just $\omega^\sharp(t) = D$ and $\omega^\sharp(t^{-1}) = \tilde{D}$. Similarly on other t as a nested product of these, as easily proven by induction, for example using $Sa^n{}_{(1)} \otimes a^n{}_{(2)} = D_1 Sa^{n-1}{}_{(1)} \otimes a^{n-1}{}_{(2)} D_2$. Thus $\omega^\sharp(t^2) = D_1 D_1' \otimes D_2' D_2$, where $D = D_1 \otimes D_2$ and D' is another copy just as we used in Proposition 2.36.

Associated bundles are given by 1-dimensional vector spaces $V_n = \mathbb{C}v$ with coaction $\Delta_R v = v \otimes t^n$ for each $n \in \mathbb{Z}$ and hence the associated bundle is

$$E_n = (\mathbb{C}_q[SL_2] \otimes V_n)_0 = \mathbb{C}_q[SL_2]_{-n},$$

the degree $-n$ subspace. Here the left coaction on V_n is $\Delta_L v = t^{-n} \otimes v$ and the covariant derivative $\nabla : E_n \to \Omega^1_A \otimes_A E_n$ from Proposition 5.5 is therefore

$$\nabla s = \mathrm{d}s - s\omega(t^{-n} - 1)$$

for all $s \in \mathbb{C}_q[SL_2]_{-n}$, after we identify s of degree $-n$ with $s \otimes v \in E_n$ and view everything in Ω^1_P. For example, on E_1 we have

$$\begin{aligned}\nabla d &= \mathrm{d}d - d(a\mathrm{d}d - q^{-1}c\mathrm{d}d) = (\mathrm{d}d)(ad - q^{-1}cb) + d(\mathrm{d}a)d - q^{-1}d(\mathrm{d}c)b \\ &= (\mathrm{d}(da))d - q^{-1}(\mathrm{d}(dc))b = (\mathrm{d}(da)) \otimes_A d - q^{-1}(\mathrm{d}(dc)) \otimes_A b,\end{aligned}$$

where we used $ad - q^{-1}cb = 1$ and $\mathrm{d}(ad - q^{-1}cb) = 0$ for the second equality, then expanded out and recombined with the Leibniz rule. Once we have everything manifestly in the form $\Omega^1_A P$ by a product, we can undo the product and we know that the result is unique if we use $\otimes_A$. For general $s \in E_1$ we similarly have

$$\begin{aligned}\nabla s &= \mathrm{d}s - s\tilde{D}_1\mathrm{d}\tilde{D}_2 = \mathrm{d}s - \mathrm{d}(s\tilde{D}_1\tilde{D}_2) + (\mathrm{d}(s\tilde{D}_1))\tilde{D}_2 = (\mathrm{d}(s\tilde{D}_1))\tilde{D}_2\\ &= \mathrm{d}(s\tilde{D}_1)\otimes_A \tilde{D}_2 = \mathrm{d}(sa)\otimes_A d - q^1\mathrm{d}(sc)\otimes_A b,\end{aligned}$$

which is exactly the formula we obtained for the Grassmann connection in Example 4.24. The penultimate expression is manifestly a product of a differential of an element of degree 0 and an element of degree -1 allowing us to recognise the answer. Alternatively, even if an expression is not manifestly in the right form, we can undo the product as in (5.7) by using, say, the above $\omega^\sharp$. Here we realise that our original formula is actually $\cdot\nabla s = (\mathrm{id} - \Pi_\omega)\mathrm{d} \in \Omega^1_P$ if we write the product explicitly, and $\Delta_R(\cdot\nabla s) = (\cdot\nabla s)\otimes t^{-1}$ since Π_ω, d are right-covariant. Then

$$\begin{aligned}\nabla s &= (\cdot\nabla s)\omega^\sharp(t^{-1}) = (\mathrm{d}s - s\tilde{D}_1\mathrm{d}\tilde{D}_2)\tilde{D}'_1\otimes_A \tilde{D}'_2\\ &= (\mathrm{d}s)\tilde{D}'_1\otimes_A \tilde{D}'_2 - s\tilde{D}_1(\mathrm{d}\tilde{D}_2)\tilde{D}'_1\otimes_A \tilde{D}'_2\\ &= ((\mathrm{d}s)\tilde{D}'_1 - s(\mathrm{d}(\tilde{D}_1\tilde{D}_2))\tilde{D}'_1 + s(\mathrm{d}\tilde{D}_1)\tilde{D}_2\tilde{D}'_1)\otimes_A \tilde{D}'_2\\ &= (\mathrm{d}s)\tilde{D}'_1\otimes_A \tilde{D}_2 + s\mathrm{d}\tilde{D}_1\otimes_A D_2\tilde{D}'_1\tilde{D}'_2\\ &= (\mathrm{d}s)\tilde{D}_1\otimes_A \tilde{D}_2 + s\mathrm{d}\tilde{D}_1\otimes_A \tilde{D}_2 = \mathrm{d}(s\tilde{D}_1)\otimes_A \tilde{D}_2,\end{aligned}$$

which is a longer route to the same answer. Here $\tilde{D}_2\tilde{D}'_1$ has degree 0 so we could move it through the $\otimes_A$. We have the same formula for E_{-1} with D in place of $\tilde{D}$.

In particular, using $z = cd$, $z^* = -qab$, $x = -q^{-1}bc$ in the q-sphere, we have

$$\nabla\begin{pmatrix} d\\ -b\end{pmatrix} = \mathrm{d}\begin{pmatrix}1-q^2x & z\\ z^* & x\end{pmatrix}\otimes\begin{pmatrix} d\\ -b\end{pmatrix} = \mathrm{d}P\otimes\begin{pmatrix} d\\ -b\end{pmatrix}$$

for the same projector matrix P as in Example 3.15. This gives an isomorphism $E_1 \cong A^2e$ by $fd - gb \mapsto (f, g)e$ making ∇ the Grassmann connection. We have mentioned this for the classical and 3D calculus in Examples 3.27 and 4.24 but here we have the result in a way that works for any bundle calculus including the universal calculus. By the Cuntz–Quillen theorem, we similarly have and could construct projectors for all the E_n. ◇

Note that just as SU_2/S^1 still has a left action of SU_2, similarly the E_n above all carry left coactions of $\mathbb{C}_q[SL_2]$, which are given by the $\mathbb{C}_q[SL_2]$ coproduct and which in the $*$-algebra are unitary in the sense explained in Chap. 2 (that the coaction is a $*$-algebra map). In Chap. 8 we will see that the holomorphic such sections will give a geometric construction of the irreducible corepresentations of $\mathbb{C}_q[SL_2]$ (the

q-Borel–Weil–Bott theorem). It is beyond out scope but it is possible to further generalise the quantum principal theory in the case of the universal calculus to 'coalgebra bundles' where H need only be a coalgebra, and this is needed to obtain the nonstandard $\mathbb{C}_{q,\lambda}[S^2]$ spheres at the end of Chap. 2 in a similar way to the above. Meanwhile, Example 5.12 is another and more immediate example of a quantum homogeneous bundle with $\pi(u) = s$, $\pi(v) = t$, $\pi(w) = 1$ and $i(s^m t^n) = u^m v^n$ giving $\omega^\sharp$ there.

We now turn to a complementary point of view. Classically, if a group G acts from the right on a space X then choosing a point $x_0 \in X$ allows us to give a map $f_0 : G \to X$ by $f_0(g) = x_0 \triangleleft g$. The action is called *transitive* if f_0 is surjective and in this case it provides an isomorphism $G/K_0 \cong X$, where K_0 is the *stabiliser* subgroup (leaving x_0 fixed). Thus, SU_2 acts on $\mathbb{CP}^1$ and if we take (the class of) $x_0 = (1, 0)$ then K_0 is a circle and $SU_2/S^1 \cong \mathbb{CP}^1$. We can equally well think of SU_2 acting on S^2 and x_0 the north pole (say) which gives $SU_2/S^1 \cong S^2$. This extends to the two points of view on $\mathbb{C}_q[S^2]$ already covered after Proposition 2.33 and in Example 3.15. In doing the quantum homogeneous bundle construction, the counit provides the 'classical point' of $\mathbb{C}_q[S^2]$ after the fact.

Similarly, we can start classically with the group $SL_2(\mathbb{C})$ acting by Möbius transformations on the Riemann sphere $\mathbb{C}_\infty$ (the complex numbers with infinity) by

$$z \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{az + b}{cz + d}.$$

The condition for this map to send the unit circle to itself is $a^* = d$ and $b^* = c$, which is the subgroup $SU_{1,1}$ now seen as acting on the open unit disk D. Given a choice of base point, say $x_0 = 0$ in the disk, we have a map from $SU_{1,1}$ to the open disk given by $z = bd^{-1}$. (This is well defined as $|d|^2 = 1 + |b|^2$ on $SU_{1,1}$.) The stabiliser subgroup of $SU_{1,1}$ is the subgroup of diagonal matrix elements of $SU_{1,1}$, which is isomorphic to the circle group, so $SU_{1,1}/S^1 \cong D$. This construction is rather more complicated in the q-deformed case as we see next.

Example 5.24 One reason why the quantum disk $\mathbb{C}_q[D]$ in Example 3.40 is more complicated than the classical disk is that the only $*$-algebra maps $\phi : \mathbb{C}_q[D] \to \mathbb{C}$ for generic q are given by $\phi(z) = \alpha$ for $|\alpha| = 1$. These factor through the map $\mathbb{C}_q[D] \to \mathbb{C}_{q^2}[S^1]$ sending the generator $z \in \mathbb{C}_q[D]$ to the usual generator of $\mathbb{C}_{q^2}[S^1]$ as previously discussed, so may be thought of as points on the boundary of the closed quantum disk rather than points in the open one (which are not suitable since classically their orbits would be the boundary circle and not the whole disk).

In spite of this, we could still consider constructing a related quantum principal bundle even if we no longer expect it to be a quantum homogeneous bundle. For this, we might try the quantum group $\mathbb{C}_q[SU_{1,1}]$ as described in Proposition 2.13 but $d \in \mathbb{C}_q[SU_{1,1}]$ is not invertible, so we cannot simply set $z = bd^{-1}$ as in the classical case. We could modify the algebra by adjoining d^{-1} as a generator, which is possible but problematic in practice given the algebraic structure of $\mathbb{C}_q[SU_{1,1}]$. Rather, we prefer to make c (and therefore also $b = q^{-1}c^*$) invertible as we then have the following more simple reordering rules obtained by multiplying the existing relations by the inverse (and similarly for b^{-1}),

$$ac^{-1} = qc^{-1}a, \quad dc^{-1} = q^{-1}c^{-1}d, \quad bc^{-1} = c^{-1}b, \quad b^{-1}c^{-1} = c^{-1}b^{-1}.$$

These allow us to take any product involving inverses of b and c and reorder it into a standard form by putting all powers of b and c (positive or negative) together. Classically, inverting b, c instead of d means that now the Möbius transform maps $x_0 = \infty$ to $\bar{z} = ac^{-1}$ (say) and this is indeed in the interior of the disc (so $|a| < |c|$) provided we work with matrices of determinant -1 rather than working with $SU_{1,1}$. So this will also be our new point of view in the quantum case. Inverting $\{b, c\}$, we realise $\mathbb{C}_q[D]$ as a subalgebra of $P = \mathbb{C}_q[U_{1,1}][b^{-1}, c^{-1}]/\langle \det_q +1\rangle$ whereby $\det_q = ad - q^{-1}bc = -1$, by setting $\bar{z} = ac^{-1}$ and $z = db^{-1}$ and checking that

$$\begin{aligned} z\bar{z} - q^{-2}\bar{z}z &= qdb^{-1}c^{-1}a - q^{-1}c^{-1}adb^{-1} \\ &= q^{-1}(c^{-1}dab^{-1} - c^{-1}adb^{-1}) = 1 - q^{-2}. \end{aligned}$$

Given the $*$-operation, we have $c^*c = 1 + a^*a$ as the analogue of the classical $|a| < |c|$. Now we have a linear basis of the image of $\mathbb{C}_q[D]$ in P as

$$a^{m-n}b^{-n}c^{-m}, \quad m \geq n \geq 0, \quad d^{n-m}b^{-n}c^{-m}, \quad n > m \geq 0.$$

This is a subalgebra of the degree zero subalgebra of P. To identify the whole degree zero subalgebra, we note that that $w = 1-\bar{z}z = qc^{-1}b^{-1}$. So inverting w the q-disk algebra is the same as including the missing positive powers of bc, i.e., the whole degree zero subalgebra in P is the open unit disk algebra.

Finally, there is a $\mathbb{C}[S^1]$-coaction on P is given by saying that a, c have degree 1 and b, d have degree -1 (so $\Delta_R a = a \otimes t$, etc.) just as for the q-Hopf fibration. The invariant subalgebra is the degree zero subalgebra of P as above and this gives us a Hopf–Galois extension. We use Lemma 5.8 and define

$$\omega^\sharp(t) = -D, \quad \omega^\sharp(t^{-1}) = -\tilde{D},$$

using our split q-determinants $D, \tilde{D}$, extending to higher powers by nesting as in the standard q-Hopf fibration case. This has the same form as before but with map $i(t^n) = (-a)^n$ and $i(t^{-n}) = (-d)^n$. One can check bicovariance as in (5.9). $\diamond$

5.2.3 *Line Bundles and Principal Bundles*

We next consider a result about group algebra principal bundles. As in Example 2.9, a $\Bbbk G$-comodule algebra is just a G-graded algebra P where $P_h.P_g \subseteq P_{hg}$ for degrees $h, g \in G$. This is said to be *strongly graded* if in fact $P_h.P_g = P_{hg}$ for all $h, g \in G$.

Lemma 5.25 *Let G be a group G and P be a G-graded algebra. Then the $\Bbbk G$-coaction $\Delta_R p = p \otimes g$ for $p \in P_g$ makes P into a $\Bbbk G$ quantum principal bundle over $A = P_e$ ($e \in G$ the identity) if and only if the grading is strong.*

Proof The condition $P_h.P_g = P_{hg}$ is equivalent to the surjectivity of the map ver, so the only issue is the injectivity of ver, which reduces to showing the injectivity of the product $\cdot : P_h \otimes_A P_g \to P_{hg}$ for all $g, h \in G$. By the surjectivity of each $\cdot : P_{g^{-1}} \otimes_A P_g \to P_e$, we can choose $a_{i,g} \in P_{g^{-1}}$ and $b_{i,g} \in P_g$ such that $\sum_i a_{i,g} b_{i,g} = 1$. Then $p \mapsto \sum_i p.a_{i,g} \otimes b_{i,g}$ is inverse to $\cdot : P_h \otimes_A P_g \to P_{hg}$. □

Moreover, it is also clear from Proposition 3.98 that in the strongly graded case each $P_{g^{-1}}$ becomes a line A-module with dual P_g and we can think of these as associated line bundles to the above principal bundle for $V = \Bbbk$ of degree g and g^{-1} respectively.

Example 5.26 The reader may recall that the degree makes $\mathbb{C}_q[SL_2]$ into an integer graded module over $\mathbb{C}_q[S^2]$, and that the degree $-n$ submodule was called E_n. In Example 3.99 we showed that E_1 was a line module by a construction which effectively involved writing dual bases in terms of column vectors. Here we give a generalisation of this construction, omitting the column vectors. To show that each E_n is a line module using Lemma 5.25 we need to show that $E_n.E_m = E_{n+m}$. In fact all we need is that $1 \in E_n.E_{-n}$, as then we have

$$E_{n+m} \subseteq 1.E_{n+m} \subseteq E_n.E_{-n}.E_{n+m} \subseteq E_n.E_m.$$

More precisely, we need 1 to be in the image of the product map $E_n \otimes E_{-n} \to A$ and we have already seen this. For example, the split q-determinant $D_1 \otimes D_2 \in E_1 \otimes E_{-1}$ maps to $1 \in E_1.E_{-1}$ and similarly $\tilde{D}_1 \otimes \tilde{D}_2 \in E_{-1} \otimes E_1$ maps to $1 \in E_{-1}.E_1$. The same for products if we nest the copies, for example $D_1 D'_1 \otimes D'_2 D_2 \in E_2 \otimes E_{-2}$ maps to $1 \in E_2.E_{-2}$ and $\tilde{D}_1 \tilde{D}'_1 \tilde{D}''_1 \otimes \tilde{D}''_2 \tilde{D}'_2 \tilde{D}_2 \in E_{-3} \otimes E_3$ maps to $1 \in E_{-3}.E_3$. It is also worth noting that $D, \tilde{D}$ have the form of a dot product of the column vectors $\underline{v}$ and $\underline{w}$ in Example 3.99 but not multiplying the entries, so $D = \underline{v}^T \otimes \underline{w}$ and $\tilde{D} = \underline{w}^T \otimes \underline{v}$ map to $\underline{v}^T \underline{w} = \underline{w}^T \underline{v} = 1$ under the product. ◇

We now consider the converse direction. Given a line bundle L over an algebra A as discussed in §3.5, we use the following lemma to construct a quantum principal bundle P for the group algebra $H = \mathbb{C}\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ with coaction $\Delta_R p = p \otimes t^n$ on $p \in P_n$ and invariant subalgebra $A = P_0$. As in the above example, we think of $\mathbb{C}[t, t^{-1}]$ as functions on a circle in an algebraic form. To this end, we define $P = T_{\mathbb{Z}}(L)$ to be the $\mathbb{Z}$-graded vector space with components

$$T_{\mathbb{Z}}(L)_n = \begin{cases} A & n = 0 \\ L^{\otimes_A^n} & n > 0 \\ (L^\flat)^{\otimes_A^{-n}} & n < 0. \end{cases} \tag{5.17}$$

We wish to make $T_{\mathbb{Z}}(L)$ into an algebra. The product of two positive graded elements is just the usual tensor product $\otimes_A$ of tensor products of Ls, and likewise the product of two negative graded elements is just the usual tensor product of $L^\flat$s. The problem is the product of opposite signed elements. One way round we take repeated applications of $\mathrm{ev}: L \otimes_A L^\flat \to A$, and the other way we take $\mathrm{coev}^{-1}: L^\flat \otimes_A L \to A$. The problem in showing associativity of the product is that we have cases such as $L^\flat \otimes_A L \otimes_A L^\flat$ where the product to take is not unique.

Lemma 5.27 *For a line bundle L over the unital algebra A, $P = T_{\mathbb{Z}}(L)$ is a unital algebra and a $\mathbb{C}\mathbb{Z}$ Hopf–Galois extension of A. Moreover, $L = (P \otimes \Bbbk)_0$ is an associated bundle, where the elements of $\Bbbk$ have degree -1.*

Proof The previously mentioned product (essentially just $\otimes_A$) is associative, by considering cases of lengths of tensor products and Proposition 3.97, where the choice of the product in the awkward cases is shown not to matter. The Hopf–Galois extension result is by Lemma 5.25, using the fact that $\mathrm{ev}: L \otimes_A L^\flat \to A$ and $\mathrm{coev}^{-1}: L^\flat \otimes_A L \to A$ are surjective. □

For example, we can recover $\mathbb{C}_q[SL_2]$ with its usual grading as $T_{\mathbb{Z}}(E_{-1})$ over $A = \mathbb{C}_q[S^2]$. We now put these pieces together.

Theorem 5.28 *There is a 1–1 correspondence between:*

(a) *Autoequivalences of the category ${}_A\mathcal{M}$ of left A-modules.*
(b) *Line bundles over A.*
(c) *$\mathbb{C}\mathbb{Z}$ Hopf–Galois extensions of A.*

Proof (a)⇒(b) From the material on Morita contexts in §3.5, autoequivalences of the category ${}_A\mathcal{M}$ correspond to strict Morita contexts $(A, A, E, F, \mu_1, \mu_2)$, where E is an fgp left A-module, and there are A-bimodule isomorphisms $\mu_1: E \otimes_A F \to A$ and $\mu_2: F \otimes_A E \to A$ obeying the conditions of Definition 3.96. Labelling $\mathrm{ev} = \mu_1: E \otimes_A F \to A$ and $\mathrm{coev} = \mu_2^{-1}: A \to F \otimes_A E$, we see $F = E^\flat = {}_A\mathrm{End}(E, A)$. The 'associativity' conditions in Definition 3.96 ensure the usual rules for the evaluation and coevaluation maps. Then E satisfies the conditions to be a line bundle in Definition 3.90.

(b)⇒(c) Given a left line module L, $T_{\mathbb{Z}}(L)$ is a $\mathbb{C}\mathbb{Z}$ Hopf–Galois extension of A by Lemma 5.27.

(c)⇒(a) Given P, a $\mathbb{C}\mathbb{Z}$ Hopf–Galois extension of A, we have $P_0 = A$, and set $L = P_1$ and $L^\flat = P_{-1}$. Then L and $L^\flat$ are A-bimodules, and the multiplication maps $\mu_1: L \otimes_A L^\flat \to A$ and $\mu_2: L^\flat \otimes_A L \to A$ are onto by the Hopf–Galois condition in Lemma 5.25. The 'associativity' conditions in Definition 3.96 are implied by the associativity of P. □

Geometrically, a principal circle bundle has associated line bundles by taking representations of the circle group on $\mathbb{C}$. Also any complex line bundle with hermitian metric gives a principal circle bundle, by taking the norm 1 points on every fibre. In the noncommutative construction above, we should view $T_{\mathbb{Z}}(L)$ as the algebra of functions on the circle bundle given by the line bundle, where the t^n

grading gives the Fourier modes. For the full picture, we will need a $*$-operation on $T_{\mathbb{Z}}(L)$ as well as a hermitian metric on the line bundle. We return to this in Proposition 8.36.

5.3 Associated Bundle Functors

In this section we will look more abstractly at the functor $\mathfrak{E}$ that constructs associated bundles from corepresentations as in Proposition 5.5. *Throughout this section we shall assume that H is a Hopf algebra with invertible antipode.* We start with a generalisation of the Hopf module Lemma 2.17 to associated bundles in the form of a category $\mathcal{M}_P^H$ consisting of objects E which are right P-modules that are also right H-comodules, with the action a comodule map, i.e., $\Delta_R(e\triangleleft p) = e_{(\bar 0)}\triangleleft p_{(\bar 0)} \otimes e_{(\bar 1)}p_{(\bar 1)}$. Morphisms are maps that are equivariant for the action and coaction. As usual, E^H denotes the coinvariants. The following lemma fills in the proof of (5.6) being an isomorphism as promised there, by taking $E = \Omega^1_A P$ (with $E^H = \Omega^1_A$).

Lemma 5.29 *Let P be a universal principal bundle with strong connection and $E \in \mathcal{M}_P^H$. Then $E \cong E^H \otimes_A P$, where the map in one direction is the right action.*

Proof We use the shorthand $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ from (5.9) and consider the map $E \to E\otimes P$ defined by $e \mapsto e_{(\bar 0)}\triangleleft e_{(\bar 1)}{}^{(1)} \otimes e_{(\bar 1)}{}^{(2)}$. We show that this maps into $E^H \otimes P$ by applying the right coaction to the first factor,

$$\begin{aligned} e_{(\bar 0)}\triangleleft e_{(\bar 2)}{}^{(1)}{}_{(\bar 0)} \otimes e_{(\bar 1)}e_{(\bar 2)}{}^{(1)}{}_{(\bar 1)} \otimes e_{(\bar 2)}{}^{(2)} &= e_{(\bar 0)}\triangleleft e_{(\bar 2)(2)}{}^{(1)} \otimes e_{(\bar 1)}S(e_{(\bar 2)(1)}) \otimes e_{(\bar 2)(2)}{}^{(2)} \\ &= e_{(\bar 0)}\triangleleft e_{(\bar 1)}{}^{(1)} \otimes 1 \otimes e_{(\bar 1)}{}^{(2)}. \end{aligned}$$

We now quotient this further to a map $E \to E^H \otimes_A P$ and check that this is inverse to the action $\triangleleft : E^H \otimes_A P \to E$. Applying the action second and using $h^{(1)}h^{(2)} = \epsilon(h).1$ for all $h \in H$ leads easily to the identity on E. Going the other way, we let $f \in E^H$, $p \in P$ and have $(f\triangleleft p)_{(\bar 0)}\triangleleft(f\triangleleft p)_{(\bar 1)}{}^{(1)} \otimes (f\triangleleft p)_{(\bar 1)}{}^{(2)} = f\triangleleft p_{(\bar 0)}p_{(\bar 1)}{}^{(1)} \otimes p_{(\bar 1)}{}^{(2)}$ by $f \in E^H$. Now applying the right coaction to the first factor of $p_{(\bar 0)}p_{(\bar 1)}{}^{(1)} \otimes p_{(\bar 1)}{}^{(2)} \in P \otimes P$ (in the same manner as in the previous part) shows that this actually lies in $A \otimes P$, so

$$f\triangleleft p_{(\bar 0)}p_{(\bar 1)}{}^{(1)} \otimes p_{(\bar 1)}{}^{(2)} = f \otimes p_{(\bar 0)}p_{(\bar 1)}{}^{(1)}p_{(\bar 1)}{}^{(2)} = f \otimes p. \qquad \square$$

The case $P = H$ is the usual Hopf module lemma and we similarly get an equivalence of categories, this time to the category of right A-modules. We are going prove a version of this, and its generalisations. Note that in doing geometry equivariantly (i.e., keeping track of the Hopf algebra coaction in our case) we can work at three levels. We can look at the 'fibres' of the bundles, we can work 'upstairs' with the equivariant geometry of P-modules, or we can work 'downstairs' with A-modules. (The terms 'upstairs' and 'downstairs' come from the topologist's habit of drawing the total space of a fibration above the base space.) The associated module construction is a method of going from 'fibre' to 'downstairs'. The philosophy is summarised in (5.18), where we give categories associated to the three levels and functors between them. The first line is for left modules, and the second for bimodules.

$$\begin{tikzcd} \text{fibre} & \text{upstairs} & \text{downstairs} \\ \mathcal{M}^H \arrow[r, bend left, "P\otimes"] & {}_P\mathcal{M}^H \arrow[l, bend left, "\text{Forget}"] \arrow[r, bend left, "()^H"] & {}_A\mathcal{M} \arrow[l, bend left, "P\otimes_A"] \\ \mathcal{M}^H_H \arrow[r, "P\otimes"] & {}_P\mathcal{M}^H_P \arrow[r, "()^H"] & {}_A\mathcal{M}_A \end{tikzcd} \tag{5.18}$$

The categories ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_A$ are the usual A-module categories and $\mathcal{M}^H$ the usual H comodule category. The category $\mathcal{M}^H_H$ is in Example 2.46, ${}_P\mathcal{M}^H$ is a left module version of $\mathcal{M}^H_P$ already discussed and ${}_P\mathcal{M}^H_P$ is a bimodule version of it:

Name	Objects	Morphisms
${}_P\mathcal{M}^H$	Left P-modules and right H-comodules with P-action an H-comodule map	Left module maps which are also right comodule maps
${}_P\mathcal{M}^H_P$	P-bimodules and right H-comodules with P-actions both H-comodule maps	Bimodule maps which are also right comodule maps

The functor $()^H$ in the upper part of (5.18) assigns to a right H-comodule its invariant part V^H. If V were a P-module (e.g. on the left), then V^H is naturally an A-module as $A = P^H$. For $v \in V^H$ and $a \in A$ we see that $a{\triangleright}v \in V^H$ as the action is a comodule map. Going the other way, the functor $P\otimes_A$ assigns to a left A-module E the left P-module $P \otimes_A E$, with left P action $q{\triangleright}(p \otimes e) = qp \otimes e$ and right H coaction $p \otimes e \mapsto p_{(\bar 0)} \otimes e \otimes p_{(\bar 1)}$ for $p, q \in P$ and $e \in E$. A more general version of the following theorem, involving faithful flatness in place of the existence of a strong connection, is known as *Schneider's theorem*. We continue to assume that the antipode of H is invertible.

Theorem 5.30 *Let (P, H, Δ_R) be a Hopf–Galois extension with a strong connection. Then the functors $P\otimes_A : {}_A\mathcal{M} \to {}_P\mathcal{M}^H$ and $()^H : {}_P\mathcal{M}^H \to {}_A\mathcal{M}$ form an equivalence of categories.*

Proof Applying $P\otimes_A$ and then $()^H$ to $E \in {}_A\mathcal{M}$ gives $(P \otimes_A E)^H$. There is an obvious morphism $1\otimes : E \to (P \otimes_A E)^H$ in ${}_A\mathcal{M}$. To construct an inverse for this, we use the shorthand $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ from (5.9). For $p \in P$, $(S^{-1}p_{(\bar 1)})^{(1)} \otimes (S^{-1}p_{(\bar 1)})^{(2)}p_{(\bar 0)} \in P \otimes A$, so given a linear map $f : P \to \Bbbk$ with $f(1) = 1$, we have

$$f((S^{-1}p_{(\bar 1)})^{(1)})(S^{-1}p_{(\bar 1)})^{(2)}p_{(\bar 0)} \in A.$$

Now define $\theta_f : P \otimes_A E \to E$ by

$$\theta_f(p \otimes e) = f((S^{-1}p_{(\bar{1})})^{(1)})(S^{-1}p_{(\bar{1})})^{(2)}p_{(\bar{0})}.e.$$

We claim that the restriction $\tilde{\theta}_f : (P \otimes_A E)^H \to E$ of θ_f is inverse to $1\otimes : E \to (P \otimes_A E)^H$. The inverse one way amounts to $f(1) = 1$. For the other way, starting with $p \otimes e \in (P \otimes_A E)^H$, we have to show that

$$p \otimes e = 1 \otimes f((S^{-1}p_{(\bar{1})})^{(1)})(S^{-1}p_{(\bar{1})})^{(2)}p_{(\bar{0})}.e \in (P \otimes_A E)^H.$$

This holds because $p_{(\bar{0})} \otimes e \otimes p_{(\bar{1})} = p \otimes e \otimes 1 \in P \otimes_A E \otimes H$.

Next, taking the functors the other way around and applying $(\ )^H$ followed by $P\otimes_A$ to $V \in {}_P\mathcal{M}^H$ gives $P \otimes_A V^H$. There is a morphism $\psi : V \to P \otimes_A V^H$ with inverse $\phi : P \otimes_A V^H \to V$ given by

$$\psi(v) = (S^{-1}v_{(\bar{1})})^{(1)} \otimes (S^{-1}v_{(\bar{1})})^{(2)} \triangleright v_{(\bar{0})}, \quad \phi(p \otimes v) = p \triangleright v.$$

To check that the image of ψ is in the correct space, we use the flatness of P (from Proposition 5.10) and Example 3.105. The nonobvious part of the inverse is to check

$$\begin{aligned}\psi\phi(p \otimes v) &= (S^{-1}p_{(\bar{1})})^{(1)} \otimes (S^{-1}p_{(\bar{1})})^{(2)}p_{(\bar{0})} \triangleright v\\ &= (S^{-1}p_{(\bar{1})})^{(1)}(S^{-1}p_{(\bar{1})})^{(2)}p_{(\bar{0})} \otimes v = p \otimes v \in P \otimes_A V^H. \qquad \square\end{aligned}$$

Returning to the upper part of (5.18), $P\otimes$ assigns to a right H-comodule V an object $P \otimes V \in {}_P\mathcal{M}^H$ with left P-action $q \triangleright (p \otimes v) = qp \otimes v$ and right H-coaction $p \otimes v \mapsto p_{(\bar{0})} \otimes v_{(\bar{0})} \otimes p_{(\bar{1})}v_{(\bar{1})}$ for $p, q \in P$ and $v \in V$. Forget simply forgets about the left P-action. The composition of the right arrows in the top line of (5.18) gives the associated module functor. Recall that we assume an invertible antipode.

Theorem 5.31 *Let (P, H, Δ_R) be a Hopf–Galois extension with a strong connection.* (1) *The functors $\mathfrak{E} : \mathcal{M}^H \to {}_A\mathcal{M}$ and $P\otimes_A : {}_A\mathcal{M} \to \mathcal{M}^H$ form a left/right adjoint pair, i.e., there is a natural 1–1 correspondence between the morphism sets ${}_A\mathrm{Hom}(\mathfrak{E}(V), E)$ and $\mathrm{Hom}^H(V, P \otimes_A E)$.* (2) *There is a 1–1 correspondence between the morphism sets ${}_A\mathrm{Hom}(\mathfrak{E}(V), \mathfrak{E}(W))$ and $\mathrm{Hom}^H(V, P \otimes W)$, where $P \otimes W$ is the tensor product comodule.*

Proof (1) First, the functors $P\otimes : \mathcal{M}^H \to {}_P\mathcal{M}^H$ and Forget $: {}_P\mathcal{M}^H \to \mathcal{M}^H$ are adjoint, i.e., there is a 1–1 correspondence of morphism sets between ${}_P\mathrm{Hom}^H(P \otimes V, F)$ and $\mathrm{Hom}^H(V, F)$ for any $V \in \mathcal{M}^H$ and $F \in {}_P\mathcal{M}^H$. This sends $g \in {}_P\mathrm{Hom}^H(P \otimes V, F)$ to $\tilde{g} \in \mathrm{Hom}^H(V, F)$ by $\tilde{g}(v) = g(1 \otimes v)$, and $f \in \mathrm{Hom}^H(V, F)$ to $\hat{f} \in {}_P\mathrm{Hom}^H(P \otimes V, F)$ by $\hat{f}(p \otimes v) = p \triangleright f(v)$. These maps are inverse: $\hat{f}(1 \otimes v) = 1 \triangleright f(v) = f(v)$ and $p \triangleright \tilde{g}(v) = p \triangleright g(1 \otimes v) = g(p \otimes v)$. Then

$$\mathrm{Hom}^H(V, P \otimes_A E) \xrightarrow{\cong} {}_P\mathrm{Hom}^H(P \otimes V, P \otimes_A E) \xrightarrow{\cong} {}_A\mathrm{Hom}((P \otimes V)^H, E)$$

for all $E \in {}_A\mathcal{M}$, where the second isomorphism is from Theorem 5.30.

(2) Given part (1), it suffices to have an isomorphism $P \otimes_A (P \otimes W)^H \cong P \otimes W$ as comodules for any $W \in \mathcal{M}^H$. Here $p \otimes (q \otimes w) \mapsto pq \otimes w$ and inversely

$$q \otimes w \mapsto q(S^{-1}w_{(\bar{1})})^{(1)} \otimes ((S^{-1}w_{(\bar{1})})^{(2)} \otimes w_{(\bar{0})})$$

using the shorthand $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ in (5.9). That these are inverse is easy starting in $P \otimes W$, while to obtain the identity starting with $p \otimes (q \otimes w) \in P \otimes_A (P \otimes W)^H$, we use $q_{(\bar{0})} \otimes w \otimes S(q_{(\bar{1})}) = q \otimes w_{(\bar{0})} \otimes w_{(\bar{1})}$. $\square$

We now turn to the lower part of (5.18). All the categories here are monoidal, in the middle case by $\otimes_P$ and the tensor product of H-coactions. We first check that

$$P \otimes : \mathcal{M}_H^H \to {}_P\mathcal{M}_P^H, \quad V \mapsto P \otimes V$$

is monoidal. Here if V is a right H-crossed module then in addition to $P \otimes V \in {}_P\mathcal{M}^H$ as before, we also have a right action of P by $(p \otimes v) \triangleleft q = pq_{(\bar{0})} \otimes v \triangleleft q_{(\bar{1})}$ for all $p, q \in P$ and $v \in V$. The required natural isomorphism between $(P \otimes V) \otimes_P (P \otimes W)$ and $P \otimes V \otimes W$ is then given by

$$(p \otimes v) \otimes (q \otimes w) \mapsto p.q_{(\bar{0})} \otimes v \triangleleft q_{(\bar{1})} \otimes w, \quad p \otimes v \otimes w \mapsto (p \otimes v) \otimes (1 \otimes w).$$

By monoidal we mean the strict version as discussed just before Example 2.42. It is also fairly obvious that $(\ )^H : {}_P^H\mathcal{M}_P \to {}_A\mathcal{M}_A$ need not be monoidal. However, we now show that the composition $\mathfrak{E} := (\ )^H \circ P\otimes : \mathcal{M}_H^H \to {}_A\mathcal{M}_A$ *is* monoidal. Here $\mathfrak{E}$ agrees with our previous associated bundle construction $\mathfrak{E}(V) = (P \otimes V)^H$ as in Proposition 5.5 where the right action of the bimodule is just the product in P from the right and does not by itself need V to be more than an H-comodule, but its monoidal property does. We continue to assume an invertible antipode.

Proposition 5.32 *Let (P, H, Δ_R) be a Hopf–Galois extension with a strong connection. The associated module functor $\mathfrak{E} : \mathcal{M}_H^H \to {}_A\mathcal{M}_A$ is monoidal with*

$$c_{V,W} : \mathfrak{E}(V) \otimes_A \mathfrak{E}(W) \to \mathfrak{E}(V \otimes W), \quad c_{V,W}((p \otimes v) \otimes (q \otimes w)) = pq_{(\bar{0})} \otimes v \triangleleft q_{(\bar{1})} \otimes w.$$

Proof Writing $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$, we show that the inverse for $c_{V,W}$ is

$$r \otimes v \otimes w \longmapsto r(S^{-1}w_{(1)})^{(1)} \otimes v \triangleleft w_{(2)} \otimes (S^{-1}w_{(1)})^{(2)} \otimes w_{(\bar{0})},$$

where (5.9) show that this lands in the appropriate space. Starting in $\mathfrak{E}(V)\otimes_A\mathfrak{E}(W)$,

$$(p\otimes v)\otimes(q\otimes w)\longmapsto pq_{\bar{(0)}}(S^{-1}w_{(1)})^{(1)}\otimes v{\triangleleft}q_{\bar{(1)}}w_{(2)}\otimes(S^{-1}w_{\bar{(1)}})^{(2)}\otimes w_{\bar{(0)}}$$
$$\longmapsto(p\otimes v){\triangleleft}q(S^{-1}w_{\bar{(1)}})^{(1)}\otimes((S^{-1}w_{\bar{(1)}})^{(2)}\otimes w_{\bar{(0)}}).$$

But as in the proof of preceding theorem,

$$q(S^{-1}w_{\bar{(1)}})^{(1)}\otimes((S^{-1}w_{\bar{(1)}})^{(2)}\otimes w_{\bar{(0)}})\in A\otimes(P\otimes W)^H,$$

so we recover $(p\otimes v)\otimes(q\otimes w)$. The other direction is straightforward. □

Example 5.33 If $g\in H$ is grouplike and central, we can make $\Bbbk$ into right H-crossed $\Bbbk_g$ with coaction $\Delta_R\lambda=\lambda\otimes g$ and trivial action $\lambda{\triangleleft}h=\lambda\epsilon h$. Then

$$\mathfrak{E}(\Bbbk_g)=(P\otimes\Bbbk_g)^H=\{p\in P:\Delta_R p=p\otimes g^{-1}\},$$

and in particular $\mathfrak{E}(\Bbbk_1)=A$. Then $\Bbbk_g\otimes\Bbbk_{g^{-1}}\cong\Bbbk_1\cong\Bbbk_{g^{-1}}\otimes\Bbbk_g$ and $\mathfrak{E}(\Bbbk_g)$ is a line module in ${}_A\mathcal{M}_A$ as in Lemma 5.25 with dual $\mathfrak{E}(\Bbbk_{g^{-1}})$. ◇

For P a $*$-algebra and H a $*$-Hopf algebra, the category ${}_P\mathcal{M}_P^H$ is a bar category with the usual operations for ${}_P\mathcal{M}_P$ and coaction $\Delta_R(\overline{e})=\overline{e_{\bar{(0)}}}\otimes e_{\bar{(1)}}{}^*$. We saw in Example 2.105 that $\mathcal{M}_H^H$ is a bar category.

Proposition 5.34 *In the $*$-algebra setting, there is natural isomorphism θ :* bar $\circ$ $(P\otimes)\Rightarrow(P\otimes)\circ$ bar *of functors $\mathcal{M}_H^H\to{}_P\mathcal{M}_P^H$, i.e., functorial isomorphisms*

$$\theta_V:\overline{P\otimes V}\to P\otimes\overline{V},\quad\theta_V(\overline{p\otimes v})=p_{\bar{(0)}}{}^*\otimes\overline{v}{\triangleleft}(p_{\bar{(1)}}{}^*).$$

Restricting θ_V to the invariants gives a natural isomorphism bar $\circ\,\mathfrak{E}\Rightarrow\mathfrak{E}\circ$ bar *of functors $\mathcal{M}_H^H\to{}_A\mathcal{M}_A$.*

Proof It is straightforward to check that θ_V is a P-bimodule map and a right H-comodule map. The inverse is given by

$$\theta_V{}^{-1}(q\otimes\overline{w})=\overline{q_{\bar{(0)}}{}^*\otimes w{\triangleleft}q_{\bar{(1)}}{}^*}.$$

This fits with $()^H:{}_P\mathcal{M}_P^H\to{}_A\mathcal{M}_A$ using $(\overline{E})^H=\overline{E^H}$ for $E\in{}_P\mathcal{M}_P^H$. □

We now specialise the above constructions to quantum homogeneous bundles, resulting in a different associated module construction from the one in §5.2.2. Thus, let $\pi:P\to H$ be a surjective map of Hopf algebras with induced coaction $\Delta_R=(\mathrm{id}\otimes\pi)$ forming a Hopf–Galois extension over $A=P^H$. Note from coassociativity of the coproduct of P that $\Delta A\subseteq P\otimes A$ and makes A canonically a left P-comodule. Hence $E\in{}^P\mathcal{M}$ then both $A\otimes E$ and $E\otimes A$ have canonical tensor product P-coactions. In this context, we have the following categories.

Name	Objects	Morphisms
${}^P_A\mathcal{M}_A$	A-bimodules and left P-comodules with both A-actions being P-comodule maps	Bimodule and comodule maps
$\mathcal{M}_A^H$	Right A-modules and right H-comodules with $\Delta_R(e{\triangleleft}a) = e_{(\bar{0})}{\triangleleft}a_{(2)} \otimes \pi(Sa_{(1)})e_{(\bar{1})}$	Ditto
$\mathcal{M}_P^H$	Right P-modules and right H-comodules with $\Delta_R(e{\triangleleft}p) = e_{(\bar{0})}{\triangleleft}p_{(2)} \otimes \pi(Sp_{(1)})e_{(\bar{1})}\pi(p_{(3)})$	Ditto

The following is a version of a more general result known as *Takeuchi's equivalence of categories*. We continue to assume an invertible antipode.

Theorem 5.35 *Let $\pi : P \to H$ define a quantum homogeneous bundle with universal calculus for which there exists a strong connection. Then the functor $\mathfrak{E} : \mathcal{M}_A^H \to {}^P_A\mathcal{M}_A$ sending $V \in \mathcal{M}_A^H$ to $(P \otimes V)^H$ with actions and coactions*

$$a.(p \otimes v) = ap \otimes v, \quad (p \otimes v).a = pa_{(1)} \otimes v{\triangleleft}a_{(2)}, \quad \Delta_L(p \otimes v) = p_{(1)} \otimes p_{(2)} \otimes v$$

and the functor $\mathfrak{F} : {}^P_A\mathcal{M}_A \to \mathcal{M}_A^H$ sending $E \in {}^P_A\mathcal{M}_A$ to $E/(A^+E) \in \mathcal{M}_A^H$ with action and coaction

$$[e].a = [e.a], \quad \Delta_R([e]) = [e_{(\bar{\infty})}] \otimes \pi(Se_{(\bar{1})})$$

for all $e \in E$ and $\Delta_L e = e_{(\bar{1})} \otimes e_{(\bar{\infty})}$ are an equivalence of categories.

Proof Using [ ] to denote an equivalence class, there is a natural transformation with $\phi : \mathfrak{F}\mathfrak{E}V \to V$ for each V given by $\phi([p \otimes v]) = \epsilon(p)v$. We define its inverse $\psi(v) = [\epsilon((S^{-1}v_{(\bar{1})})^{(1)})(S^{-1}v_{(\bar{1})})^{(2)} \otimes v_{(\bar{0})}]$ in terms of the universal strong connection $\omega^\sharp(h) := h^{(1)} \otimes h^{(2)}$ in (5.9), checking first that the quantity in the square brackets is in the space of invariants for the right H-coaction,

$$\begin{aligned}
&\epsilon((S^{-1}v_{(\bar{1})})^{(1)})\Delta_R((S^{-1}v_{(\bar{1})})^{(2)} \otimes v_{(\bar{0})}) \\
&\quad = \epsilon((S^{-1}v_{(\bar{2})})^{(1)})(S^{-1}v_{(\bar{2})})^{(2)}{}_{(\bar{0})} \otimes v_{(\bar{0})} \otimes (S^{-1}v_{(\bar{2})})^{(2)}{}_{(\bar{1})}v_{(\bar{1})} \\
&\quad = \epsilon((S^{-1}v_{(\bar{2})})_{(1)}{}^{(1)})(S^{-1}v_{(\bar{2})})_{(1)}{}^{(2)} \otimes v_{(\bar{0})} \otimes (S^{-1}v_{(\bar{2})})_{(2)}v_{(\bar{1})} \\
&\quad = \epsilon((S^{-1}v_{(\bar{1})})^{(1)})(S^{-1}v_{(\bar{1})})^{(2)} \otimes v_{(\bar{0})} \otimes 1.
\end{aligned}$$

We then have $\phi \circ \psi = \mathrm{id}$ using $\epsilon(h^{(1)})\epsilon(h^{(2)}) = \epsilon(h)$, while for $p \otimes v \in (P \otimes V)^H$,

$$\begin{aligned}
\psi \circ \phi([p \otimes v]) &= \epsilon(p)[\epsilon((S^{-1}v_{(\bar{1})})^{(1)})(S^{-1}v_{(\bar{1})})^{(2)} \otimes v_{(\bar{0})}] \\
&= [\epsilon(p_{(\bar{0})}(p_{(\bar{1})})^{(1)})(p_{(\bar{1})})^{(2)} \otimes v].
\end{aligned}$$

Since $p_{(\bar{0})}(p_{(\bar{1})})^{(1)} \otimes (p_{(\bar{1})})^{(2)} \in A \otimes P$, we also have as required that

$$p_{(\bar{0})}(p_{(\bar{1})})^{(1)}(p_{(\bar{1})})^{(2)} - \epsilon(p_{(\bar{0})}(p_{(\bar{1})})^{(1)})(p_{(\bar{1})})^{(2)} = p - \epsilon(p_{(\bar{0})}(p_{(\bar{1})})^{(1)})(p_{(\bar{1})})^{(2)} \in A^+P.$$

Next, we have a natural transformation with $\theta : E \to \mathfrak{E}\mathfrak{F}E$ for each E given by $\theta(e) = e_{(\bar{1})} \otimes [e_{(\bar{\infty})}]$ and for its inverse we first define $f : P \to A$ by

$$f(p) = \epsilon((S^{-1}p_{(\bar{1})})^{(1)})(S^{-1}p_{(\bar{1})})^{(2)}p_{(\bar{0})},$$

which lies in A for reasons given in the proof of Theorem 5.30. Note that $f|_A = \mathrm{id}$ and that f is a right A-module map. Now define $\vartheta : (P \otimes (E/(A^+E)))^H \to E$ by $\vartheta(p \otimes [e]) = f(p(Se_{(\bar{1})}))\triangleright e_{(\bar{\infty})}$, which is well defined since

$$\vartheta(e \otimes [ae]) = f(p(Se_{(\bar{1})})(Sa_{(1)}))\triangleright a_{(2)}e_{(\bar{\infty})} = f(p(Se_{(\bar{1})})(Sa_{(1)})a_{(2)})\triangleright e_{(\bar{\infty})} = 0$$

for $a \in A^+$, using $\Delta A \subset P \otimes A$. Then

$$\vartheta \circ \theta(e) = \vartheta(e_{(\bar{1})} \otimes [e_{(\bar{\infty})}]) = f(e_{(\bar{1})}(Se_{(\bar{2})}))\triangleright e_{(\bar{\infty})} = f(1)\triangleright e = e.$$

$$\begin{aligned}\theta \circ \vartheta(p \otimes [e]) &= \theta(f(p(Se_{(\bar{1})}))\triangleright e_{(\bar{\infty})}) = f(p(Se_{(\bar{1})}))_{(1)}e_{(\bar{2})} \otimes [f(p(Se_{(\bar{1})}))_{(2)}\triangleright e_{(\bar{\infty})}]\\ &= f(p(Se_{(\bar{1})}))e_{(\bar{2})} \otimes [e_{(\bar{\infty})}] = \epsilon((S^{-1}h)^{(1)})(S^{-1}h)^{(2)}p_{(1)}(Se_{(\bar{2})})e_{(\bar{3})} \otimes [e_{(\bar{\infty})}]\\ &= \epsilon((S^{-1}h)^{(1)})(S^{-1}h)^{(2)}p_{(1)} \otimes [e_{(\bar{\infty})}]\end{aligned}$$

using $\Delta A \subset P\otimes A$ again and $[a\triangleright e] = \epsilon(a)[e]$ for $a \in A$. We also used the shorthand $h = \pi(p_{(2)}(Se_{(\bar{1})}))$. We then recover $p \otimes [e]$ since $p_{(1)} \otimes e_{(\bar{\infty})} \otimes \pi(p_{(2)}(Se_{(\bar{1})})) = p \otimes e \otimes 1$ holds by right H-invariance of $p \otimes [e]$. $\square$

Note that any $V \in \mathcal{M}^H$ can be viewed as an object of ${}_A\mathcal{M}^H_A$ by the trivial action via ϵ since $\pi|_A = 1.\epsilon$ (as one may in turn deduce from $A = P^H$), in which case $\mathfrak{E}(V)$ has the bimodule structure given by the right product in P as in Proposition 5.5 and Proposition 5.32. Indeed, the functor ${}_H\mathcal{M}^H_H \to {}_P\mathcal{M}^H_P$ given by pull back of an H-module to a P-module via π is clearly monoidal (where ${}_P\mathcal{M}^H_P$ is monoidal with the usual tensor product action and coaction). We also have a functor ${}_P\mathcal{M}^H_P \to {}_A\mathcal{M}^H_A$ by restricting the action of P to an action of $A \subseteq P$. Here ${}_A\mathcal{M}^H_A$ is not typically monoidal as $\Delta A \not\subseteq A \otimes A$ in general. Composition of these two functors forgets the crossed module H action and gives an object of ${}_A\mathcal{M}^H_A$ with trivial action of A as discussed. Hence $\mathfrak{E} : {}_A\mathcal{M}^H_A \to {}^P_A\mathcal{M}_A$ in Theorem 5.35 pulls back to two other functors also denoted $\mathfrak{E}$ according to

$$\begin{tikzcd} {}_H\mathcal{M}^H_H \arrow[r] \arrow[dr, "\mathfrak{E}"'] & {}_P\mathcal{M}^H_P \arrow[r] \arrow[d, "\mathfrak{E}"] & {}_A\mathcal{M}^H_A \arrow[dl, "\mathfrak{E}"] \\ & {}^P_A\mathcal{M}_A & \end{tikzcd}$$

where the one on the left recovers $\mathfrak{E}$ as in Proposition 5.32 but now in our context and with an additional left P-coaction $\Delta_L(p\otimes v)=p_{(1)}\otimes p_{(2)}\otimes v$. The category ${}_A^P\mathcal{M}_A$ is monoidal by $\otimes_A$ as usual and the usual tensor product P-coaction.

Proposition 5.36 *Given the same data as the preceding theorem, the functor* $\mathfrak{E}:{}_{\mathfrak{M}}{}_P^H\to{}_A^P\mathcal{M}_A$ *is monoidal, with the natural isomorphisms*

$$c_{V,W}:\mathfrak{E}(V)\otimes_A\mathfrak{E}(W)\to\mathfrak{E}(V\otimes W),\quad c_{V,W}\big(p\otimes v\otimes q\otimes w\big)=pq_{(1)}\otimes v\triangleleft q_{(2)}\otimes w.$$

Proof Here $c_{V,W}$ maps into $(P\otimes V\otimes W)^H$ since

$$\begin{aligned}\Delta_R(pq_{(1)}\otimes v\triangleleft q_{(2)}\otimes w)&=p_{(1)}q_{(1)}\otimes v_{(\bar 0)}\triangleleft q_{(2)}\otimes w_{(\bar 0)}\otimes\pi(p_{(2)})v_{(\bar 1)}\pi(q_{(3)})w_{(\bar 1)}\\&=pq_{(1)}\otimes v\triangleleft q_{(2)}\otimes w\otimes 1.\end{aligned}$$

That $c_{V,W}$ is a left A-module map is immediate, while the left P-comodule property

$$p_{(1)}q_{(1)}\otimes c_{V,W}\big(p_{(2)}\otimes v\otimes q_{(2)}\otimes w\big)=p_{(1)}q_{(1)}\otimes p_{(2)}q_{(2)}\otimes v\triangleleft q_{(3)}\otimes w$$

is also not difficult. It is a right A-module map as

$$\begin{aligned}c_{V,W}\big((p\otimes v)\otimes(q\otimes w).a\big)&=c_{V,W}\big(p\otimes v\otimes qa_{(1)}\otimes w\triangleleft a_{(2)}\big)\\&=pq_{(1)}a_{(1)}\otimes v\triangleleft q_{(2)}a_{(2)}\otimes w\triangleleft a_{(3)}=c_{V,W}\big((p\otimes v)\otimes(q\otimes w)\big).a.\end{aligned}$$

The inverse for $c_{V,W}$ is $d_{V,W}:\mathfrak{E}(V\otimes W)\to\mathfrak{E}(V)\otimes_A\mathfrak{E}(W)$ given by

$$d_{V,W}(p\otimes v\otimes w)=(p\otimes v).(S^{-1}w_{(\bar 1)})^{(1)}\otimes((S^{-1}w_{(\bar 1)})^{(2)}\otimes w_{(\bar 0)}),$$

where we have extended the right action of A on $P\otimes V$ to a P-action by $(p\otimes v).q=pq_{(1)}\otimes v\triangleleft q_{(2)}$. Now using $\Delta_R((p\otimes v).q)=(\Delta_R(p\otimes v)).\Delta_R q$ and (5.9) shows that $d_{V,W}$ maps into $(P\otimes V)^H\otimes_A(P\otimes W)^H$, as required. To check the inverse,

$$\begin{aligned}d_{V,W}c_{V,W}\big(p\otimes v\otimes q\otimes w\big)&=d_{V,W}(pq_{(1)}\otimes v\triangleleft q_{(2)}\otimes w)\\&=(p\otimes v).q(S^{-1}w_{(\bar 1)})^{(1)}\otimes((S^{-1}w_{(\bar 1)})^{(2)}\otimes w_{(\bar 0)})\\&=p\otimes v\otimes q(S^{-1}w_{(\bar 1)})^{(1)}(S^{-1}w_{(\bar 1)})^{(2)}\otimes w_{(\bar 0)}=p\otimes v\otimes q\otimes w.\end{aligned}$$

Here we have used $q\otimes w\in(P\otimes W)^H$ and moved an element of A from one side of $\otimes_A$ to the other side. Finally

$$\begin{aligned}c_{V,W}d_{V,W}(p\otimes v\otimes w)&=c_{V,W}\big((p\otimes v).(S^{-1}w_{(\bar 1)})^{(1)}\otimes((S^{-1}w_{(\bar 1)})^{(2)}\otimes w_{(\bar 0)})\big)\\&=(p\otimes v).(S^{-1}w_{(\bar 1)})^{(1)}(S^{-1}w_{(\bar 1)})^{(2)}\otimes w_{(\bar 0)}=p\otimes v\otimes w\end{aligned}$$

completes the proof. $\square$

This is compatible with Proposition 5.32 and means that the left commutative triangle in the diagram above is one of monoidal functors.

Example 5.37 Given our data $\pi : P \to H$ as above, we can always take $P \in \mathcal{M}^P_P$ as a crossed module by Ad_R and right multiplication as explained below Definition 2.22 and push out the coaction by π to give $P \in \mathcal{M}^H_P$. Then $\mathfrak{E}(P) = (P \otimes P)^H$ becomes a left P-covariant A-bimodule as an object in ${}^P_A\mathcal{M}_A$ by

$$\Delta_L(p \otimes q) = p_{(1)} \otimes p_{(2)} \otimes q, \quad a.(p \otimes q) = ap \otimes q, \quad (p \otimes q).a = pa_{(1)} \otimes qa_{(2)}$$

for invariant $p \otimes q$ (sum of such terms understood). Later on, a related example of Theorem 5.35 will be the quotient V of $P^+ \cap A$ in Theorem 5.77 and Corollary 5.78, where $V \in \mathcal{M}^H_A$ and $\mathfrak{E}(V) = \Omega^1_A$ is a left P-covariant calculus on A. ◇

Example 5.38 We return to the finite group homogeneous bundle in Example 5.22 based on $G \subseteq X$ with $\pi : P = \mathbb{C}(X) \to H = \mathbb{C}(G)$ given by $\pi(\delta_g) = \delta_g$ if $g \in G$ and zero otherwise. An object $V \in \mathcal{M}^H_P$ in this case just means a left X-graded crossed G-module, i.e., where the grading and action obey $|g{\triangleright}v| = g|v|g^{-1}$. We have $\mathfrak{E}(V) = (\mathbb{C}(X) \otimes V)^{\mathbb{C}(G)}$ consisting as before of $\sum_x \delta_x \otimes v_x$ where $g{\triangleright}v_x = v_{xg^{-1}}$ for all g, x. The theory above tells us that now $\mathfrak{E}(V)$ is an object of ${}^P_A\mathcal{M}_A$ and this comes out as $\mathbb{C}(X)$ coacting on itself via its coproduct and $\mathbb{C}(X/G)$-bimodule structure

$$f.(\sum \delta_x \otimes v_x) = \sum f(x)\delta_x \otimes v_x, \quad (\sum \delta_x \otimes v_x).f = \sum f(x|v_x|)\delta_x \otimes v_x$$

on v_x of homogenous degree. Proposition 5.36 tells us that we have a monoidal functor from X-graded crossed G-modules with tensor product action and product grading to G-covariant $\mathbb{C}(X/G)$-bimodules with tensor product over $\mathbb{C}(X/G)$.

The simplest special case is to start with $V \in \mathcal{M}^H_H$, i.e., a G-crossed module. Then pull back along π just means the G-crossed module grading is viewed as an X-grading. In this case the bimodule structure of $\mathfrak{E}(V)$ becomes commutative and agrees with the one in Example 5.22. More generally in Theorem 5.35, an object $V \in \mathcal{M}^H_A$ is an X/G-graded G-module with $|g{\triangleright}v| = g|v|$ and $\mathbb{C}(X/G)$-action $v{\triangleleft}f = vf(|v|)$. Then $\mathfrak{E}(V)$ has the bimodule structure and left coaction as above. ◇

5.4 Quantum Bundles with Nonuniversal Calculus

From now on in this chapter, *we no longer assume that our differential calculi are universal and if we refer to the latter, we will label them as such*. Thus, let P be an algebra over $\Bbbk$ with calculus Ω^1_P, and H be a Hopf algebra with bicovariant calculus Ω^1_H. Recall from Chap. 2 that the latter have the form $H.\Lambda^1_H$ and come with a Maurer–Cartan form $\varpi : H^+ \to \Lambda^1_H$ to the left-invariant 1-forms.

5.4.1 *Quantum Principal Bundles*

Definition 5.39 A 'quantum principal bundle' over a unital algebra A means:

(1) P a right H-comodule algebra under $\Delta_R : P \to P \otimes H$ such that $A = P^H$.
(2) Differential structures where $\Omega^1_H = H.\Lambda^1_H$ is bicovariant and Ω^1_P is right H-covariant and the two are compatible in the sense that

$$\mathrm{ver} : \Omega^1_P \to P \otimes \Lambda^1_H, \quad \mathrm{ver}(p\mathrm{d}q) = pq_{\bar{(0)}} \otimes (Sq_{\bar{(1)}(1)})\mathrm{d}q_{\bar{(1)}(2)} \tag{5.19}$$

for all $p, q \in P$ is well defined.
(3) There is a short exact sequence of left P-modules

$$0 \longrightarrow P\Omega^1_A P \longrightarrow \Omega^1_P \xrightarrow{\mathrm{ver}} P \otimes \Lambda^1_H \longrightarrow 0$$

where $\Omega^1_A = A(\mathrm{d}A)A \subseteq \Omega^1_P$.

Note that we could fix Ω^1_A in advance, in which case the last part of (3) becomes a condition. However, in practice it is useful to use the quantum principal bundle to generate the calculus as shown. The definition in the case of the universal calculus amounts to the same as we had in §5.1. In fact, the two conditions on Ω^1_P in (2) are together equivalent to the idea in Definition 4.28 (with a left-right swap) that Δ_R is differentiable up to degree 1 in Ω_P, i.e., that $\Delta_R : P \to P \otimes H$ on degree zero extends to a coaction $\Delta_{R*} : \Omega_P \to \Omega_P \otimes \Omega_H$ up to degree 1 in a manner that commutes with d. Explicitly, it means that $\Delta_R : \Omega^1_P \to \Omega^1_P \otimes H$ is well defined by $\Delta_R(p\mathrm{d}q) = p_{\bar{(0)}}\mathrm{d}q_{\bar{(0)}} \otimes p_{\bar{(1)}}q_{\bar{(1)}}$, in which case it is a coaction (this is the notion of a right-covariant calculus encountered in Chaps. 2 and 4) *and* that $\delta_R : \Omega^1_P \to P \otimes \Omega^1_H$ is well defined by $\delta_R(p\mathrm{d}q) = p_{\bar{(0)}}q_{\bar{(0)}} \otimes p_{\bar{(1)}}\mathrm{d}q_{\bar{(1)}}$. In fact the image of this second map is in the cotensor product

$$P \otimes^H \Omega^1_H \cong P \otimes^H H \otimes \Lambda^1_H \cong P \otimes \Lambda^1_H \tag{5.20}$$

in view of the Hopf-module Lemma 2.17 and $P \otimes^H H \cong P$ provided by $\mathrm{id} \otimes \epsilon$ with inverse by Δ_R. Then ver in (5.19) is just δ_R after allowing for this isomorphism. Cotensor products were discussed before Proposition 5.5.

To consider what Definition 5.39 entails beyond the universal calculus, recall from Chap. 1 that a general calculus on P is a quotient of the universal one by a sub-bimodule $N_P \subseteq \Omega_{\mathrm{uni},P}$, say, and from Chap. 2 that a bicovariant calculus on H is defined by an Ad-stable right ideal $I_H \subseteq H^+$. Then clearly the conditions in (2) for the differential calculi amount to

$$\Delta_R N_P \subseteq N_P \otimes H, \quad \mathrm{ver}_{\mathrm{uni}}(N_P) \subseteq P \otimes I_H$$

for Δ_R and ver to be well defined. By definition, we also have

$$N_A = N_P \cap \Omega_{\mathrm{uni},A}$$

for the specified calculus on A. Also note that $\mathrm{ver}(P(\mathrm{d}A)P) = 0$ and hence $P\Omega^1_A P \subseteq \ker \mathrm{ver}$ as an immediate consequence of (5.19) and the Leibniz rule in P. Hence the further exactness condition (3) amounts to a 'differential Hopf–Galois' condition that ver descends to an isomorphism

$$\frac{\Omega^1_P}{P\Omega^1_A P} \cong P \otimes \Lambda^1_H. \tag{5.21}$$

Lemma 5.40 *Let (P, H, Δ_R) be a universal quantum principal bundle and suppose that the coaction is differentiable. Then the exact sequence condition holds if and only if* $\mathrm{ver}_{\mathrm{uni}}(N_P) = P \otimes I_H$ *(i.e., an equality, not merely an inclusion).*

Proof We begin with the vector space isomorphism

$$\frac{\Omega_{\mathrm{uni},P}}{P\Omega_{\mathrm{uni},A}P + N_P} \cong \frac{\left(\frac{\Omega_{\mathrm{uni},P}}{N_P}\right)}{\left(\frac{P\Omega_{\mathrm{uni},A}P}{P\Omega_{\mathrm{uni},A}P\cap N_P}\right)} = \frac{\Omega^1_P}{P\Omega^1_A P}.$$

The isomorphism (5.21) holds if and only if the quotient of $\mathrm{ver}_{\mathrm{uni}}$,

$$\frac{\Omega_{\mathrm{uni},P}}{P\Omega_{\mathrm{uni},A}P + N_P} \to \frac{P \otimes H^+}{P \otimes I_H} \cong P \otimes \Lambda^1_H$$

is an isomorphism. This is always surjective as $\mathrm{ver}_{\mathrm{uni}} : \Omega_{\mathrm{uni},P} \to P \otimes H^+$ is surjective, so the condition we need is just for injectivity, i.e., for when all $\xi \in \Omega_{\mathrm{uni},P}$ such that $\mathrm{ver}_{\mathrm{uni}}(\xi) \in P \otimes I_H$ lie in $\xi \in P\Omega_{\mathrm{uni},A}P + N_P$.

Now suppose that $\mathrm{ver}_{\mathrm{uni}}(N_P) = P \otimes I_H$ and $\mathrm{ver}_{\mathrm{uni}}(\xi) \in P \otimes I_H$. Then there is an $\eta \in N_P$ such that $\mathrm{ver}_{\mathrm{uni}}(\xi - \eta) = 0$, from which $\xi - \eta \in P\Omega_{\mathrm{uni},A}P$. Hence injectivity of the quotient map holds. If, however, $\mathrm{ver}_{\mathrm{uni}}(N_P) \neq P \otimes I_H$ then choose $x \in P \otimes I_H$ but not in $\mathrm{ver}_{\mathrm{uni}}(N_P)$. By surjectivity of $\mathrm{ver}_{\mathrm{uni}}$, there is a $\kappa \in \Omega_{\mathrm{uni},P}$ such that $\mathrm{ver}_{\mathrm{uni}}(\kappa) = x$, and then note that the class of κ is not zero in the quotient but maps to zero under the quotient map. □

As for the universal calculus in §5.1, we define a connection as a left P-module and right H-comodule map

$$\Pi : \Omega^1_P \to \Omega^1_P, \quad \Pi^2 = \Pi, \quad \ker \Pi = P\Omega^1_A P \tag{5.22}$$

i.e., an equivariant complement to the horizontal forms or an equivariant splitting of the exact sequence in Definition 5.39. In general, the abstract definitions of all our concepts are the same, just with general calculi in place of the universal one and Λ^1_H in place of H^+. For example, a connection is strong if $(\mathrm{id} - \Pi)\mathrm{d}P \subseteq \Omega^1_A P$, and in this case $\nabla_P = (\mathrm{id} - \Pi)\mathrm{d}$ is a connection on P as an associated bundle as in Lemma 5.6, provided we still have (5.6) as before (e.g. if we have a strong connection on the bundle with the universal calculus so that we can use

Lemma 5.29). Also recall that there is an adjoint coaction as part of Λ^1_H a right H-crossed module.

Proposition 5.41 *Connections on a quantum principal bundle P are in 1–1 correspondence with right comodule maps $\omega : \Lambda^1_H \to \Omega^1_P$ such that* $\mathrm{ver}\circ\omega = 1\otimes\mathrm{id}$. *Here $\Pi(\mathrm{d}p) = p_{(\bar 0)}\omega(\varpi\pi_\epsilon p_{(\bar 1)})$. Viewing the calculus on P as a quotient of the universal calculus, a universal connection ω_{uni} descends to Λ^1_H if and only if $\omega_{\mathrm{uni}}(I_H) \subseteq N_P$. A strong universal connection that descends does so to a strong connection.*

Proof We first check that ver is an H-comodule map, where $P\otimes\Lambda^1_H$ has the tensor product of Δ_R and the right adjoint coaction on Λ^1_H. It is enough to check this on exact differentials where $\mathrm{ver}(\mathrm{d}p) = p_{(\bar 0)}\otimes\varpi\pi_\epsilon p_{(\bar 1)}$,

$$\begin{aligned}\Delta_{P\otimes\Lambda^1_H}\mathrm{ver}(\mathrm{d}p) &= p_{(\bar 0)(\bar 0)}\otimes\varpi\pi_\epsilon p_{(\bar 1)(2)}\otimes p_{(\bar 0)(\bar 1)}(Sp_{(\bar 1)(1)})p_{(\bar 1)(3)}\\ &= p_{(\bar 0)(\bar 0)}\otimes\varpi\pi_\epsilon p_{(\bar 0)(\bar 1)}\otimes p_{(\bar 1)} = \mathrm{ver}(\mathrm{d}p_{(\bar 0)})\otimes p_{(\bar 1)}.\end{aligned}$$

Any ω a defines a map $\cdot(\mathrm{id}\otimes\omega) : P\otimes\Lambda^1_H \to \Omega^1_P$ which by equivariance of ω is equivariant and by its further property obeys $\mathrm{ver}\circ\cdot(\mathrm{id}\otimes\omega) = \mathrm{id}$. The map Π is the composition the other way, hence is an equivariant projection. Conversely, given our assumption on Π and exactness of our sequence in Definition 5.39, we define

$$\omega(v) = \Pi\circ\mathrm{ver}^{-1}(1\otimes v),\quad v\in\Lambda^1,$$

where $\mathrm{ver}^{-1} : P\otimes\Lambda^1_H \to \Omega^1_P/\ker\Pi$. This is equivariant since ver and Π are.

Clearly a universal connection ω_{uni} descends if and only if $\omega_{\mathrm{uni}}(I_H) \subseteq N_P$. Further, if $(\mathrm{id}-\Pi_{\omega_{\mathrm{uni}}})(\mathrm{d}_{\mathrm{uni}}p) \subseteq \Omega^1_{\mathrm{uni},A}P$ then its quotient maps into $\Omega^1_A P$. □

Note that we may very well be interested in both a strong universal connection ω_{uni} in a 'topological' role needed for nice properties of the bundle and a connection $\omega : \Lambda^1_H \to \Omega^1_P$ in a more geometric role, without assuming that the former descends to the latter. We are, however, interested in the case where the bundle itself descends from a Hopf–Galois extension or quantum bundle with universal calculus as in §5.1.1, just as classically in differential geometry we require that a principal bundle manifold is based on a topological principal bundle. Thus we say that a quantum principal bundle is *regular* if it descends from a universal principal bundle and the latter has a strong connection.

Connections typically exist, certainly when for example H has a normalised integral and invertible antipode. Here we choose any linear map $\hat\omega : \Lambda^1_H \to \Omega^1_P$ such that $\mathrm{ver}\circ\hat\omega = 1\otimes\mathrm{id}$ and then average $\hat\omega$ to give a comodule map ω, and this will satisfy the required equation $\mathrm{ver}\circ\omega = 1\otimes\mathrm{id}$. We use the following lemma.

Lemma 5.42 *Suppose that H has normalised left-integral and bijective antipode. Given a linear map $\psi : V \to W$ between right H-comodules, the map b in the* *proof of Theorem 5.9 can be used to 'average' ψ to give a comodule map*

$$\mathrm{av}(\psi)(v) = (\psi(v_{(\bar 0)}))_{(\bar 0)}\, b((\psi(v_{(\bar 0)}))_{(\bar 1)}, v_{(\bar 1)}).$$

Further if $\phi : W \to U$ is a comodule map, then $\mathrm{av}(\phi) = \phi$ *and* $\mathrm{av}(\phi \circ \psi) = \phi \circ \mathrm{av}(\psi)$.

Proof The proof that this is a comodule map is the following equation:

$$\begin{aligned}\mathrm{av}(\psi)(v_{(\bar 0)}) \otimes v_{(\bar 1)} &= \psi(v_{(\bar 0)})_{(\bar 0)} \otimes b(\psi(v_{(\bar 0)})_{(\bar 1)}, v_{(\bar 1)}) v_{(\bar 2)}\\ &= \psi(v_{(\bar 0)})_{(\bar 0)} \otimes \psi(v_{(\bar 0)})_{(\bar 1)} b(\psi(v_{(\bar 0)})_{(\bar 2)}, v_{(\bar 1)}).\end{aligned}$$ □

We now give two examples of quantum principal bundles with nonuniversal calculus in the sense of Definition 5.39. Both of them are regular.

Example 5.43 We write $S_3 = \mathbb{Z}_3 {>\!\!\!\triangleleft} \mathbb{Z}_2$ where u generates $\mathbb{Z}_2$ and $t = uv$ generates $\mathbb{Z}_3$ (with cross relations $ut = t^2 u$). $P = \mathbb{C}S_3 = \mathbb{C}\mathbb{Z}_3 {>\!\!\!\triangleleft} \mathbb{C}\mathbb{Z}_2$ is a trivial bundle with universal calculus as in Proposition 5.13 with $H = \mathbb{C}\mathbb{Z}_2$ having generator z and base $A = \mathbb{C}\mathbb{Z}_3$. We take coaction $\Delta_R t^i = t^i \otimes 1$ and $\Delta_R u = u \otimes z$ (this is of the form $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ for $\pi(u) = z$ and $\pi(t) = 1$ if one thinks of this as a quantum homogeneous bundle). We have a strong connection map $\omega^\sharp_{\mathrm{uni}}(z) = u \otimes u$ coming from Corollary 5.14 for the trivialisation given by inclusion of the second factor.

For the differential structures, there is only the universal calculus for Ω^1_H, which is 1-dimensional and corresponds to the sign representation in the construction of Theorem 1.47 (so there is one basis element $\zeta(z)$ with $\mathrm{d}z = z\zeta(z)$ and $\zeta(z)z = -z\zeta(z)$). For Ω^1_P we use Theorem 1.47 with the direct sum of the sign and 2D irreducible representations, namely with basis $\{f_0, f_1, f_2\}$, action and cocycle

$$f_i {\triangleleft} t = q^i f_i, \quad f_0 {\triangleleft} u = -f_0, \quad f_1 {\triangleleft} u = f_2, \quad f_2 {\triangleleft} u = f_1; \qquad q = e^{\frac{2\pi i}{3}},$$

$$\zeta(u) = f_0 + a(f_1 - f_2), \quad \zeta(t) = (a(1-q) - qb) f_1 + b f_2$$

for parameters a, b such that $a(1-q) - qb \neq 0$ and $b \neq 0$ (this ensures the surjectivity condition). In this case $\zeta(u), \zeta(t), \zeta(t^2)$ are linearly independent and can also be taken as a basis of Λ^1_P. The commutation relations are $f_i t = q^i t f_i$, $f_0 u = -u f_0$ and $f_1 u = u f_2$, $f_2 u = u f_1$ from the representation and $\mathrm{d}u = u\zeta(u)$, $\mathrm{d}t = t\zeta(t)$ in the construction. The calculus is not inner but is H-covariant under Δ_R. To have a bundle, we also need ver to be well defined by $\mathrm{ver}(p\mathrm{d}q) = pq \otimes \pi(q)^{-1}\mathrm{d}\pi(q) = pq \otimes \zeta(\pi(q))$ according to (5.19). If we identify $\Omega^1_P \cong P \otimes \Lambda^1_P$ then $\mathrm{ver} : P \otimes \Lambda^1_P \to P \otimes \Lambda^1_H$ is just the identity on the first factor and we need a well-defined map, which we denote by π_*, such that $\pi_*(\zeta(q)) = \zeta(\pi(q))$ on the second factor. We define this as $\pi_*(f_0) = \zeta(z)$ and $\pi_*(f_i) = 0$ for $i = 1, 2$ and note that $\zeta(t^i u^j) = \zeta(t^i){\triangleleft} u^j + \zeta(u^j)$ which π_* sends to $\zeta(z^j)$ as required. Finally, the exact sequence in Definition 5.39 holds as $\ker \mathrm{ver} = P.\{f_1, f_2\} = P.\{\zeta(t), \zeta(t^2)\} =$

$P\Omega^1_A$. By the commutation relations, this is the same as $P\Omega^1_A P$ (and also the same as $\Omega^1_A P$).

By Proposition 5.41, the connections are in 1–1 correspondence with right comodule maps $\omega : \Lambda^1_H \to \Omega^1_P$ such that $\mathrm{ver} \circ \omega = 1 \otimes \mathrm{id}$. The latter condition holds for anything of the form

$$\omega(\zeta(z)) = \zeta(u) + \alpha; \quad \alpha \in P\Omega^1_A.$$

As Λ^1_H is invariant under the right coaction, the H-covariance of ω requires that $\alpha \in \Omega^1_A$. Thus the moduli of connections in this example is labelled by 1-forms on the base. These connections are automatically strong by our remark on $\Omega^1_A P$. ⋄

Example 5.44 Let X be a graph and $P = \Bbbk(X)$ with its graph calculus as in Proposition 1.24. Let $H = \Bbbk(G)$ be functions on a finite group with bicovariant calculus given by an Ad-stable set of generators $\mathcal{C} \subseteq G \setminus \{e\}$ as in §1.7, and suppose that G acts from the right on X (taking arrows to arrows i.e., if $x \to y$ is an arrow then so is every $x^g \to y^g$). This is the data for Ω^1_P to be covariant under G (viewing the canonical left action of G on P as a right coaction of H). It follows that $A = P^H = \Bbbk(X/G)$ with inherited calculus given by the quotient graph (by definition this has an arrow between two orbits if and only if there is an arrow between any two representatives). This is because the forms on A are generated by functions δ_i on X defined as a constant 1 on orbit $\mathcal{O}_i$ and zero elsewhere, and differentials

$$\mathrm{d}\delta_i = \Big(\sum_{x\to y, x\notin\mathcal{O}_i, y\in\mathcal{O}_i} \omega_{x\to y}\Big) - \Big(\sum_{x\to y, x\in\mathcal{O}_i, y\notin\mathcal{O}_i} \omega_{x\to y}\Big) = \sum_{j\neq i} \big(e_{j\to i} - e_{i\to j}\big)$$

with $e_{i\to j} = \sum_{x\to y, x\in\mathcal{O}_i, y\in\mathcal{O}_j} \omega_{x\to y}$ if an arrow from $\mathcal{O}_i$ to $\mathcal{O}_j$ exists.

Next, for a Hopf–Galois extension or quantum bundle at the universal level, we want the action of G to be free, which happens if and only if each orbit i has cardinality $|G|$, which in turn means each orbit is bijective with G on fixing a base point x_i. In this case we have a bijection $(X/G)\times G \to X$ by $i \times g \mapsto x_i^g$ and hence, at least when X finite, that $P \cong A \otimes H$ as an algebra. Here P has a trivialisation

$$\Phi(\delta_g) = \sum_{i\in X/G} \delta_{x_i^g}, \quad \Phi^{-1}(\delta_g) = \sum_{i\in X/G} \delta_{x_i^{g^{-1}}} \tag{5.23}$$

which one can check obeys the required conditions in Proposition 5.13. This gives us a flat connection $\omega_{\mathrm{uni}} = \Phi^{-1}\odot \mathrm{d}\Phi$ and 'local gauge theory' as in Corollary 5.14 at the level of the universal calculus (i.e., for the complete graph on X and hence on X/G). We have already seen this concretely in Example 5.22.

We will now see that we have a quotient of the universal bundle if and only if the graph on X within each orbit has valency $|\mathcal{C}|$ with arrows $x \to x^a$ for $a \in \mathcal{C}$, i.e., if and only if each orbit is not only isomorphic to G as a set but as a graph obtained by restriction from X (dropping any arrows that do not lie within the orbit). Indeed,

$$\mathrm{ver}(\omega_{x\to y}) = \delta_x \sum_{a\in\mathcal{C}} \delta_{y^{a^{-1}}} \otimes e_a = \sum_{a\in\mathcal{C},\, y=x^a} \delta_x \otimes e_a, \tag{5.24}$$

where we use the definition in terms the coaction Δ_R projected down to Λ^1_H. By freeness, there is only one possible a that can contribute so $\mathrm{ver}(\omega_{x\to x^a}) = \delta_x \otimes e_a$ and is zero on arrows not of this form. Meanwhile, the horizontal forms $P\Omega^1_A P$ are spanned by all arrows from one orbit to another (since these are picked out by $\delta_x e_{i\to j}\delta_y$ as we vary $x, y \in X$). If we want this to be exactly the kernel of ver then we need all arrows $x \to y$ where x, y are in the same orbit to be exactly those of the form $x \to x^a$ for some $a \in \mathcal{C}$. This proves our assertion. There is also a canonical flat connection form

$$\omega(e_a) = \sum_{x\in X} \omega_{x\to x^a} \tag{5.25}$$

where $\mathrm{ver}\,\omega(e_a) = \sum_x \delta_x \otimes e_a = 1 \otimes e_a$ and

$$\Delta_R\omega(e_a) = \sum_{g,x} \omega_{x^g\to x^{ag}} \otimes \delta_g = \sum_{x',g} \omega_{x'\to x'^{g^{-1}ag}} \otimes \delta_g = \sum_g \omega(e_{g^{-1}ag}) \otimes \delta_g$$

as required. But *note that this connection may not be strong* (see Example 5.49). ◇

The next example of a regular quantum principal bundle does not satisfy Poincaré duality as it corresponds classically to a nonorientable manifold.

Example 5.45 This is a nonuniversal calculus version of the Klein bottle in Example 5.11. For $H = \mathbb{C}(\mathbb{Z}_2)$ have the universal calculus as the only nonzero choice. For $P = \mathbb{C}[s, s^{-1}, v, v^{-1}] = \mathbb{C}_q[S^1] \otimes \mathbb{C}_{q'}[S^1]$ we have the standard circle calculi as in Example 1.11 with parameters $q, q' \in \mathbb{C}^\times$ respectively, so $(\mathrm{d}s)s = qs\mathrm{d}s$ and $(\mathrm{d}v)v = q'v\mathrm{d}v$. We ask for these to be covariant. Recalling that $\Delta_R s = s \otimes \delta_0 - s \otimes \delta_1$ we find no restriction on q while $\Delta_R v = v \otimes \delta_0 + v^{-1} \otimes \delta_1$ tells us

$$\Delta_R((\mathrm{d}v)v) = (\mathrm{d}v)v \otimes \delta_0 + (\mathrm{d}v^{-1})v^{-1} \otimes \delta_1 = (\mathrm{d}v)v \otimes \delta_0 - v^{-1}(\mathrm{d}v)v^{-2} \otimes \delta_1,$$
$$\Delta_R(v\mathrm{d}v) = v\mathrm{d}v \otimes \delta_0 + v^{-1}\mathrm{d}v^{-1} \otimes \delta_1 = v\mathrm{d}v \otimes \delta_0 - v^{-2}(\mathrm{d}v)v^{-1} \otimes \delta_1,$$

which using the relations forces $q' = 1$. We can also check that we need $\mathrm{d}v$ to commute with s and $\mathrm{d}s$ with v. Thus $P = \mathbb{C}_q[S^1] \otimes \mathbb{C}[S^1]$, where the first factor with generators s, s^{-1} has a q-differential structure, the other factor with v, v^{-1} generators is forced to be classical, and the two along with their differentials mutually commute. We denote the invariant algebra for the coaction now with this calculus $\mathbb{C}_q[\mathrm{Klein}]$. Some calculation with its elements shows that a linear basis of the 1-forms is

$$s^{2n}e_s,\quad s^{2n}c_m e_s,\quad s^{2n+1}s_m e_s,\quad s^{2n}s_m e_v,\quad s^{2n+1}c_m e_v,\quad s^{2n+1}e_v$$

for integers $m \geq 1$ and $n \in \mathbb{Z}$, where $e_s = s^{-1}\mathrm{d}s$, $e_v = v^{-1}\mathrm{d}v$ and $c_m(v)$, $s_m(v)$ are our sine and cosine in algebraic terms as before. The map ver is necessarily zero as we took the zero calculus on $\mathbb{Z}_2$, hence well defined, while $P\Omega^1(\mathbb{C}_q[\mathrm{Klein}])P = \Omega^1_P$ since using powers of s from P and the above basis we can obtain all powers of s, v times both e_s, e_v. Hence we have a quantum principal bundle. For the 2-forms on $\mathbb{C}_q[\mathrm{Klein}]$, the linear basis is

$$s^{2n+1}e_s \wedge e_v,\quad s^{2n+1}c_m e_s \wedge e_v,\quad s^{2n}s_m e_s \wedge e_v.$$

From the bases, it is quite easy to see that $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}_q[\mathrm{Klein}]) = 0$ as classically.

We now show that the algebra map $\iota : B = \mathbb{C}_{q^2}[S^1] \to A = \mathbb{C}_q[\mathrm{Klein}]$, where the former is $\mathbb{C}[t, t^{-1}]$ with $(\mathrm{d}t)t = q^2\mathrm{d}t$ and the map is given by $t \mapsto s^2$, is a differential algebra fibration according to Definition 4.61. Set

$$N_{p,k} = \frac{\iota\,\Omega^p_B \wedge \Omega^k_A}{\iota\,\Omega^{p+1}_B \wedge \Omega^{k-1}_A},\qquad N_{p,0} = \iota\,\Omega^p_B \otimes_A A$$

and note that $N_{0,0} \cong A$ and a linear basis of $N_{1,0}$ is

$$s^{2n}e_s,\quad s^{2n}c_m e_s,\quad s^{2n+1}s_m e_s.$$

The only other nonzero $N_{p,k}$ are $N_{1,1} \cong \Omega^2_A$ and $N_{0,1}$ with linear basis

$$[s^{2n}s_m e_v],\quad [s^{2n+1}c_m e_v],\quad [s^{2n+1}e_v].$$

We have a differential fibration as the map $\xi \otimes [x] \mapsto [\iota(\xi) \wedge x]$ is an isomorphism from $\Omega^1_B \otimes_B N_{0,k}$ to $N_{1,k}$ for all $k \geq 0$. The $E_1^{p,k}$ term of the Leray–Serre spectral sequence for the de Rham cohomology is the cohomology of

$$\cdots \xrightarrow{\mathrm{d}} N_{p,k-1} \xrightarrow{\mathrm{d}} N_{p,k} \xrightarrow{\mathrm{d}} N_{p,k+1} \xrightarrow{\mathrm{d}} \cdots$$

and using the linear bases above, we obtain

$$E_1^{1,1} = \mathbb{C}.[s^{2n+1}e_s \wedge e_v],\ E_1^{1,0} = \mathbb{C}.[s^{2n}e_s],\ E_1^{0,1} = \mathbb{C}.[s^{2n+1}e_v],\ E_1^{0,0} = \mathbb{C}.[s^{2n}]$$

for $n \in \mathbb{Z}$. Now $\mathrm{d} : E_1^{0,1} \to E_1^{1,1}$ is an isomorphism so $E_2^{0,1} = E_2^{1,1} = 0$, and considering $\mathrm{d} : E_1^{0,0} \to E_1^{1,0}$ we find $E_2^{0,0} = \mathbb{C}.1 \cong \mathbb{C}$ and $E_2^{1,0} = \mathbb{C}.e_s \cong \mathbb{C}$. Then the spectral sequence converges at the second page and we read off $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{C}_q[\mathrm{Klein}]) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}_q[\mathrm{Klein}]) \cong \mathbb{C}$ with all others zero. The reader may note that the noncommutativity of the calculus has played virtually no part in this calculation, all we have used is that q is generic so that various terms do not have coefficient zero.

We now continue down a classically well-trodden path and consider repairing some of the damage caused by nonorientability by means of a line bundle. Recall that in Example 5.11 there was an algebra of periodic functions on a rectangle which split into two parts under a $\mathbb{C}(\mathbb{Z}_2)$ coaction. One part is $A = \mathbb{C}_q[\mathrm{Klein}]$ and the other part is an A-module which we call L, with linear basis

$$s^{2n+1}, \quad s^{2n+1}c_m, \quad s^{2n}s_m$$

for integer $m \geq 1$ and $n \in \mathbb{N}$. Then L can be given a left connection by the usual d on $\mathbb{C}_q[S^1]$ and the classical d on ϕ (i.e., using the differential structure of the functions on the rectangle). We can use the Leray–Serre spectral sequence to find the cohomology of L by using

$$M_{p,k} = \frac{\iota\,\Omega^p_B \wedge \Omega^k_A \otimes_A L}{\iota\,\Omega^{p+1}_B \wedge \Omega^{k-1}_A \otimes_A L}, \quad M_{p,0} = \iota\,\Omega^p_B \otimes_A A \otimes_A L$$

in place of the $N_{p,k}$ used earlier. Note that $M_{0,0} \cong L$ and a linear basis of $M_{1,0}$ is

$$s^{2n+1}e_s, \quad s^{2n+1}c_m e_s, \quad s^{2n}s_m e_s.$$

The only other nonzero $M_{p,k}$ are $M_{1,1}$ with linear basis

$$[s^{2n}e_s \wedge e_v], \quad [s^{2n}c_m e_s \wedge e_v], \quad [s^{2n+1}s_m e_s \wedge e_v]$$

and $M_{0,1}$ with linear basis

$$[s^{2n+1}s_m e_v], \quad [s^{2n}c_m e_v], \quad [s^{2n}e_v].$$

The $E_1^{p,k}$ term of the Leray–Serre spectral sequence for the cohomology of L is the cohomology of

$$\cdots \xrightarrow{\mathrm{d}} M_{p,k-1} \xrightarrow{\mathrm{d}} M_{p,k} \xrightarrow{\mathrm{d}} M_{p,k+1} \xrightarrow{\mathrm{d}} \cdots$$

and using the linear bases, we have the generators

$$E_1^{1,1} = \mathbb{C}.[s^{2n}e_s \wedge e_v],\ E_1^{1,0} = \mathbb{C}.[s^{2n+1}e_s],\ E_1^{0,1} = \mathbb{C}.[s^{2n}e_v],\ E_1^{0,0} = \mathbb{C}.[s^{2n+1}]$$

for $n \in \mathbb{Z}$. Now $\mathrm{d} : E_1^{0,0} \to E_1^{1,0}$ is an isomorphism so $E_2^{0,0} = E_2^{1,0} = 0$, and considering $\mathrm{d} : E_1^{0,1} \to E_1^{1,1}$, we find $E_2^{0,1} = \mathbb{C}.e_v \cong \mathbb{C}$ and $E_2^{1,1} = \mathbb{C}.e_s \wedge e_v \cong \mathbb{C}$. Then the spectral sequence converges at the second page and we read off

$\mathrm{H}^2(L) \cong \mathrm{H}^1(L) \cong \mathbb{C}$ and others zero. The de Rham cohomology of $\mathbb{C}_q[\mathrm{Klein}]$ and the cohomology of L add up to that of a torus with Betti numbers $1, 2, 1, 0, \ldots$, as expected since $\mathbb{C}_q[\mathrm{Klein}]$ and L were defined by splitting the torus algebra. ⋄

We now turn to quantum homogeneous spaces, where we have already covered the universal calculus case in §5.2.2. Recall that now the total space algebra P is itself a quantum group and there is a Hopf algebra surjection $\pi : P \to H$ with induced push-forward coaction $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ and base algebra $A = P^H$.

Lemma 5.46 *A universal quantum homogeneous bundle $\pi : P \to H$ as in Lemma 5.21 descends to one with general left and bicovariant calculi respectively defined by I_P, I_H if and only if*

$$(\mathrm{id} \otimes \pi)\mathrm{Ad}_R(I_P) \subseteq I_P \otimes H, \quad \pi(I_P) = I_H.$$

The strong connection on the universal bundle provided by a unital bicomodule splitting map $i : H \to P$ descends if

$$i(I_H) \subseteq I_P,$$

giving a strong connection $\omega(h) = Si(h)_{(1)}\mathrm{d}i(h)_{(2)}$ on representatives $h \in H^+$.

Proof The first is the condition for $\Delta_R N_P \subseteq N_P \otimes H$ in terms of I_P given that the calculus is already left-covariant. The second is Lemma 5.40 in terms of I_P, I_H. □

We can now revisit the noncommutative Example 5.12, this time with nonuniversal calculi. Here H is the algebraic torus with its classical calculus and A is the algebraic circle, each with their classical calculi, but P is mildly noncommutative.

Example 5.47 As in Examples 4.62, 5.12, we consider the Heisenberg group algebra $P = \mathbb{C}\mathrm{Hg}$ with unitary generators u, v, w and relations $uv = vuw$, and $H = \mathbb{C}[s, s^{-1}, t, t^{-1}] = \mathbb{C}\mathbb{Z}^2$ the algebraic torus with commuting unitary generators. We take $\pi(u) = s$, $\pi(v) = t$, $\pi(w) = 1$ making P into a right H-comodule algebra as before and have a quantum bundle with strong connection at the universal level by $i(s^m t^n) = u^m v^n$. We now quotient to the calculus on P in Example 4.62 and the classical calculus on H (with basic 1-forms $e_1 = s^{-1}\mathrm{d}s$, $e_2 = t^{-1}\mathrm{d}t$). It can be checked that these maps descend according to Lemma 5.46, or we can just check directly that the resulting maps are consistent with the calculi. In the latter case we have the left P-module map $\mathrm{ver} : \Omega^1_P \to P \otimes \Lambda^1_H$ in (5.19) as

$$\mathrm{ver}(u^{-1}\mathrm{d}u) = 1 \otimes e_1, \quad \mathrm{ver}(v^{-1}\mathrm{d}v) = 1 \otimes e_2, \quad \mathrm{ver}(w^{-1}\mathrm{d}w) = 0.$$

It follows that the kernel of ver is $P(\mathrm{d}w)P = (\mathrm{d}w)P$. We can split this by the connection form $\omega(e_1) = u^{-1}\mathrm{d}u$ and $\omega(e_2) = v^{-1}\mathrm{d}v$. The inherited calculus on A is just the classical one since $\mathrm{d}w$ is central in the calculus on P. Note that to proceed more formally from the lemma, the classical calculus has

$$I_H = \langle (s-1)^2, (s-1)(t-1), (t-1)^2 \rangle$$

with e_1 represented by $s-1$, e_2 by $t-1$ while I_P can be obtained from the calculus in Example 4.62. $\diamond$

As in the case of universal calculi in Proposition 5.5, we now consider the connection and curvature on a module $E = (P \otimes V)^H$ associated to the principal bundle P with general calculi, where V is a right H-comodule. The H-invariant space is taken as before with respect to the tensor product right H-comodule structure. Alternatively, we can use the right H-comodule structure on P and a left H-comodule structure on V according to (5.5).

Proposition 5.48 *Let (P, H, Δ_R) be a regular quantum principal bundle with differential structures and ω a strong connection with associated Π_ω. If V is a right H-comodule then $E = (P \otimes V)^H$ acquires an associated covariant derivative $\nabla : E \to \Omega^1_A \otimes_A E$ given by*

$$\nabla(p \otimes v) = (\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v \in (\Omega^1_A P \otimes V)^H = \Omega^1_A \otimes_A (P \otimes V)^H.$$

Its curvature has the form

$$R_\nabla(p \otimes v) = -p_{(\bar{0})} F_\omega(\pi_\epsilon p_{(\bar{1})}) \otimes v,$$

where $F_\omega : H^+ \to \Omega^2_P$ is given by

$$F_\omega(h) = \mathrm{d}\omega(\varpi h) + \omega(\varpi \pi_\epsilon h_{(1)}) \wedge \omega(\varpi \pi_\epsilon h_{(2)}).$$

Proof We use the shorthand $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ as in (5.9). First we show that the space of H-invariants in $\Omega^1_A P$ is Ω^1_A. To do this, choose a linear map $f : P \to \mathbb{k}$ so that $f(1) = 1$, and from this define a map $\hat{f} : \Omega^1_P \to \Omega^1_P$ by $\hat{f}(\xi) = \xi_{(\bar{0})} \xi_{(\bar{1})}{}^{(1)} f(\xi_{(\bar{1})}{}^{(2)})$ where we use the shorthand $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ from (5.9). As $\omega^\sharp(1) = 1 \otimes 1$, this map fixes all invariant 1-forms. Now suppose that $\xi = \sum_i \eta_i . p_i$ is invariant for $\eta_i \in \Omega^1_A$ and $p_i \in P$. Then

$$\xi = \hat{f}\Big(\sum_i \eta_i . p_i\Big) = \sum_i \eta_i . p_{i\,(\bar{0})} p_{i\,(\bar{1})}{}^{(1)} f(p_{i\,(\bar{1})}{}^{(2)})$$

and using (5.9) we see $p_{i\,(\bar{0})} p_{i\,(\bar{1})}{}^{(1)} f(p_{i\,(\bar{1})}{}^{(2)}) \in A$, so we have the result.

Next we establish for our nonuniversal calculi that the product map

$$\Omega^1_A \otimes_A (P \otimes V)^H \to (\Omega^1_A P \otimes V)^H \tag{5.26}$$

is an isomorphism, as this is needed in the proof. We use the map

$$\xi \otimes v \longmapsto \xi_{(\bar{0})}\xi_{(\bar{1})}{}^{(1)} \otimes \xi_{(\bar{1})}{}^{(2)} \otimes v$$

which maps $\Omega^1_A P \otimes V$ into $\Omega^1_A \otimes P \otimes V$. Applying the right coaction to the last two factors, we get

$$\xi_{(\bar{0})}\xi_{(\bar{1})}{}^{(1)} \otimes \xi_{(\bar{1})}{}^{(2)}{}_{(\bar{0})} \otimes v_{(\bar{0})} \otimes \xi_{(\bar{1})}{}^{(2)}{}_{(\bar{1})}v_{(\bar{1})} = \xi_{(\bar{0})}\xi_{(\bar{1})}{}^{(1)} \otimes \xi_{(\bar{1})}{}^{(2)} \otimes v_{(\bar{0})} \otimes \xi_{(\bar{2})}v_{(\bar{1})}.$$

This shows that if $\xi \otimes v \in (\Omega^1_A P \otimes V)^H$ then $\xi_{(\bar{0})}\xi_{(\bar{1})}{}^{(1)} \otimes (\xi_{(\bar{1})}{}^{(2)} \otimes v) \in \Omega^1_A \otimes_A (P \otimes V)^H$, giving what we require for the inverse map.

After this, the computations are analogous to those in Proposition 5.5. Thus,

$$\nabla(a(p \otimes v)) = (\mathrm{id} - \Pi_\omega)(a\mathrm{d}p + (\mathrm{d}a)p) \otimes v = (\mathrm{d}a)p \otimes v + a(\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v$$

checks that ∇ as stated is indeed a covariant derivative. When written in this way, its form is that of the connection on P tensor the identity in V, restricted to $E = (P \otimes V)^H$. The curvature is similarly the curvature on P applied in the same way. To calculate the latter, for the remainder of the proof we set $\nabla_P p = (\mathrm{id} - \Pi_\omega)\mathrm{d}p = \eta q$ say, where $\eta \in \Omega^1_A$ and $q \in P$ (sum of such terms understood). Then

$$\begin{aligned}
R_{\nabla_P}(p) &= (\mathrm{d}\eta)q - \eta \wedge \nabla_P q = (\mathrm{d}\eta)q - \eta \wedge (\mathrm{id} - \Pi_\omega)\mathrm{d}q = \mathrm{d}(\eta q) + \eta \wedge \Pi_\omega(\mathrm{d}q) \\
&= \mathrm{d}(\eta q) + \eta \wedge q_{(\bar{0})}\omega\varpi\pi_\epsilon(q_{(\bar{1})}) = \mathrm{d}(\eta q) + (\eta q)_{(\bar{0})} \wedge \omega\varpi\pi_\epsilon((\eta q)_{(\bar{1})}) \\
&= -\mathrm{d}(p_{(\bar{0})}\omega\varpi\pi_\epsilon(p_{(\bar{1})})) + \mathrm{d}(p_{(\bar{0})}) \wedge \omega\varpi\pi_\epsilon(p_{(\bar{1})}) \\
&\quad - p_{(\bar{0})}\omega(\varpi\pi_\epsilon(p_{(\bar{2})})_{(\bar{0})}) \wedge \omega\varpi\pi_\epsilon(p_{(\bar{1})}\varpi\pi_\epsilon(p_{(\bar{2})})_{(\bar{1})}) \\
&= -p_{(\bar{0})}\mathrm{d}\omega\varpi\pi_\epsilon(p_{(\bar{1})}) - p_{(\bar{0})}\omega\varpi\pi_\epsilon(p_{(\bar{1})}) \wedge \omega\varpi\pi_\epsilon(p_{(\bar{2})}) = -p_{(\bar{0})}F_\omega(\pi_\epsilon p_{(\bar{1})}).
\end{aligned}$$

Note that the image of F_ω lies in $\Omega^2_A P$ by construction. □

As with the universal case, the above ∇ on associated bundles are all bimodule connections if ∇_P on P itself is, and their curvatures are all bimodule maps if the curvature on P is. These are conditions on ω which we will study further in Lemma 5.50. Also note that Proposition 5.48 computes the curvature R_∇ in terms of a particular map $F_\omega : H^+ \to \Omega^2_P$ but by comparison with classical geometry, we would ideally like F_ω to be a 'Lie algebra-valued' 2-form. Thus, in our language, we would ideally like F_ω to descend to a map from Λ^1_H, which requires

$$F_\omega(I_H) = 0 \tag{5.27}$$

when we identify $H^+/I_H \cong \Lambda^1_H$ via ϖ. We do not limit ourselves to this case but when it happens, we say that ω is *regular*. Also note that if $\omega : \Lambda^1_H \to \Omega^1_P$ were to extend to a map of DGAs, then the curvature in the form $F_\omega : \Lambda^1_H \to \Omega^2_P$ would vanish due to $\mathrm{d}\varpi(h) + \varpi(\pi_\epsilon h_{(1)}) \wedge \varpi(\pi_\epsilon h_{(2)}) = 0$ for all $h \in H^+$ (the Maurer–Cartan equations in Proposition 2.31). Thus the curvature can be considered as an obstruction to this extension. After the above brief account of the general theory, we now turn to examples of connections with nonuniversal differentials.

Example 5.49 Let $G \subseteq X$ be a nontrivial subgroup of a finite group and $\mathcal{C}$ define a bicovariant calculus on G and $\mathcal{C}_X$ a left-covariant calculus on X. Then $X \to X/G$ gives a homogeneous quantum principal bundle if and only if (i) $\mathcal{C}_X$ is stable under conjugation by G and (ii) $\mathcal{C} = \mathcal{C}_X \cap G$. Here $P = \mathbb{C}(X)$, $H = \mathbb{C}(G)$ and $A = \mathbb{C}(X/G)$. Condition (i) is for right G-covariance of the calculus on X while (ii) is the orbit graph condition in Example 5.44 since the arrows in X are of the form $x \to xb$ where $b \in \mathcal{C}_X$ and those that stay in the same orbit are those where $b \in G$. This is also clearly the content of the two conditions in Lemma 5.46 in our case. The quotient graph has arrows labelled by equivalence classes of pairs (x, a) where $x \in X$ and $a \in \mathcal{D} = \mathcal{C}_X \setminus \mathcal{C}$ for the equivalence relation $(x, a) \sim (y, b)$ if $y = xg$ and $gb = ah$ for some $g, h \in G$. This is the condition for $\omega_{x\to xa}$ and $\omega_{y\to yb}$ to both be arrows from the same orbits $[x] = [y]$ to $[xa] = [yb]$, i.e., to be the same arrow in the quotient graph, where $[x] = xG$ denotes the orbit containing x, i.e., the coset of G in X. The corresponding 1-form in Ω^1_A is

$$\omega_{[x]\to[xa]} = \delta_{[x]}\mathrm{d}\delta_{[xa]} = \sum_{(y,b)\sim(x,a)} \delta_y e_b = \sum_{(y,b)\sim(x,a)} \omega_{y\to yb}$$

in terms of the basic left-invariant 1-forms in Ω^1_P given as usual by $e_a = \sum_{x\in X} \omega_{x\to xa}$ for all $a \in \mathcal{C}_X$, here restricted to $a \in \mathcal{D}$ (then also only $b \in \mathcal{D}$ can appear in the sum).

One special case of interest is where G is normal in X (at least in the sense that $aGa^{-1} \subseteq G$ for all $a \in \mathcal{C}_X$, but for a connected calculus this would be the same thing). In this case, the arrows of the quotient graph are labelled by cosets $[x] = xG$ as the points of X/G and $[a] = aG$ for $a \in \mathcal{D}$ labelling the arrows out of each point (so the latter cosets are the elements of $\mathcal{C}_{X/G}$). Moreover, setting

$$e_{[a]} = \sum_{b\in aG\cap\mathcal{C}_X} e_b \in \Omega^1_A,$$

there is a well-defined right translation $R_{[a]}$ on X/G given by right-multiplication by any representative of $[a]$ and one can check that

$$\mathrm{d}f = \sum_{[a]} \partial_{[a]} f e_{[a]}$$

confirming that $\{e_{[a]} \mid a \in \mathcal{D}\}$ are in this case the basic forms on X/G viewed upstairs on X, with $\delta_{[x]}e_{[a]} = \omega_{[x]\to[x][a]} = \omega_{[x]\to[xa]}$. Also, if $\{E_a \mid a \in \mathcal{C}\}$ denotes the basis of Λ^1_H where $E_a = \varpi\pi_\epsilon\delta_a$, then the canonical connection from Example 5.44 becomes $\omega(E_a) = e_a$ for all $a \in \mathcal{C}$. We also have $\mathrm{ver}(e_a) = 1 \otimes E_a$

for $a \in \mathcal{C}$ and zero otherwise. Then the associated canonical projection is

$$\Pi_\omega(\mathrm{d}p) = p_{(\bar{0})}\omega(\varpi\pi_\epsilon p_{(\bar{1})}) = \sum_g R_g(p)\omega(\varpi\pi_\epsilon\delta_g) = \sum_{a\in\mathcal{C}}(\partial_a p)e_a$$

for all $p \in P$, where $\delta_e = 1 - \sum_{g\neq e}\delta_g$ on the group G and where ϖ projects out directions not in $\mathcal{C}$. Hence

$$(\mathrm{id} - \Pi_\omega)(\mathrm{d}p) = \sum_{a\in\mathcal{C}_X\setminus\mathcal{C}}(\partial_a p)e_a = [D, p]; \quad D = \sum_{a\in\mathcal{C}_X\setminus\mathcal{C}} e_a = \sum_{[a]} e_{[a]}.$$

Here $D \in \Omega^1_A$ but $(\mathrm{id} - \Pi_\omega)\mathrm{d}p$ is not necessarily in $\Omega^1_A P$.

To be concrete, we now consider the classical bundle $X = \mathbb{Z}_{2N} \to \mathbb{Z}_N$ by taking mod N as in the Möbius bundle Examples 3.86, 4.47 and $N > 2$, and we take the structure subgroup $G = \mathbb{Z}_2 \subset \mathbb{Z}_{2N}$ embedded as $\{0, N\}$. We have a quantum bundle $\mathbb{C}(\mathbb{Z}_N) \subset \mathbb{C}(\mathbb{Z}_{2N}) = P$ with $H = \mathbb{C}(\mathbb{Z}_2)$ with differential calculi $\mathcal{C}_X = \{1, N-1, N, N+1, 2N-1\}$ for $\mathbb{Z}_{2N}$ and the universal calculus for $\mathbb{Z}_2$ given by $\mathcal{C} = \{N\}$ viewed in X. This meets our condition as $\mathcal{C}_X \cap G = \{N\}$. The quotient space is identified with $\mathbb{Z}_N$ labelling cosets $\mathcal{O}_i = [i] = \{i, i+N\} \subset \mathbb{Z}_{2N}$ and has quotient graph the Cayley graph on $\mathbb{Z}_N$ generated by $\{-1, 1\}$. These elements appear as the cosets $\{1, N+1\}$ and $\{N-1, 2N-1\}$ and the associated forms for $i = 0, \cdots, N-1$,

$$\omega_{[i]\to[i\pm1]} = \omega_{i\to(i\pm1)} + \omega_{(i+N)\to(i\pm1)} + \omega_{i\to(i\pm1+N)} + \omega_{(i+N)\to(i\pm1+N)},$$

where orbits are labelled mod N. From (5.25) the canonical connection form is

$$\omega = \sum_{i=0}^{N-1}\omega_{i\to(i+N)} + \sum_{i=0}^{N-1}\omega_{(i+N)\to i},$$

where we just give the value of ω as a map on the unique basic 1-form E_1 for the calculus on G. Moreover, $e_1, e_{N-1}, e_N, e_{N+1}, e_{2N-1}$ are the basic 1-forms on $\mathbb{Z}_{2N}$ and $\omega = e_N$. The basic forms on the base are $e_{[1]} = e_1 + e_{N+1}, e_{[N-1]} = e_{N-1} + e_{2N-1}$ and one finds by explicit calculation that this ω by itself is not strong.

On the other hand, the principal bundle at the universal level is trivial with $\Phi(\delta_0) = \sum_{i=0}^{N-1}\delta_{x_i} = \Phi^{-1}(\delta_0)$ where each $x_i \in \{i, i+N\}$. The values on δ_1 are complementary and the induced strong flat connection evaluates on $E_1 = \delta_1 \in \mathbb{C}(G)^+$ as the 1-form

$$\omega_\Phi = \Phi^{-1}(\delta_0)\mathrm{d}\Phi(\delta_1) + \Phi^{-1}(\delta_1)\mathrm{d}\Phi(\delta_0) = \sum_{i,j=0}^{N-1}\left(\omega_{x_i\to(x_j+N)} + \omega_{(x_j+N)\to x_i}\right)$$

wherever the arrows are valid. Here the calculus on H is the universal one and ω_Φ at the universal level then necessarily descends to a strong connection on P with its given calculus. We similarly have a strong connection $\omega_\alpha = \omega_\alpha(\delta_1)$ for any $\alpha = \alpha(\delta_1) \in \Omega^1_A$ (firstly at the universal calculus level by the formula in Corollary 5.14 then projecting down to Ω^1_P) for which the associated projection is

$$(\mathrm{id} - \Pi_\alpha)\mathrm{d}p = \mathrm{d}p - p\omega_\alpha(\pi_\epsilon\delta_0) + p\omega_\alpha(\varpi\delta_1) = \mathrm{d}p + 2p\omega_\alpha.$$

Now let $V = \mathbb{C}v$ with the unique nontrivial representation of $\mathbb{Z}_2$. Then $E = (P \otimes V)^{\mathbb{Z}_2}$ is the space of functions p on $\mathbb{Z}_{2N}$ that change sign under shift by N. Using $\Delta_L v = (1 - 2\delta_1) \otimes v$ in Corollary 5.14 a section $p \otimes v$ 'upstairs' then corresponds to a function $s \in \mathbb{C}(\mathbb{Z}_N)$ by $p \otimes v = s\Phi \otimes v$, where $\Phi = \Phi(1 - 2\delta_1) \in P$ (here $p \otimes v, s$ were denoted s, ϕ in Corollary 5.14). To be concrete we could choose $x_i = i$ then Φ is 1 on $\{0, \cdots, N-1\}$ and -1 on the rest of $\mathbb{Z}_{2N}$ and the correspondence becomes $p(i) = s(i)$, $p(i + N) = -s(i)$ for $i = 0, \cdots, N-1$. We obtain a similar correspondence and the same effective ∇ for any choice of trivialisation. Proceeding as in the proof of Corollary 5.14, but now with a general calculus, the $\Phi \odot \Phi^{-1}$ cancel as before and $\nabla(p \otimes v) = (\mathrm{d}s)\Phi \otimes v - s\alpha(1 - 2\delta_1)\Phi \otimes v = (\mathrm{d}s)\Phi \otimes v + 2s\alpha\Phi \otimes v$ for any $\alpha = \alpha_1 e_{[1]} + \alpha_{-1}e_{[N-1]} \in \Omega^1_A$ and d now the one on $\mathbb{Z}_N$. In terms of s, this effectively means a covariant derivative

$$\nabla s = \mathrm{d}s + 2s\alpha$$

for all $s \in \mathbb{C}(\mathbb{Z}_N)$. In particular,

$$\alpha_{\pm 1} = \frac{1}{2}\lambda^{\pm 1} = \frac{1}{2}(1 - \mathrm{e}^{\pm\frac{\mathrm{i}\pi}{N}})$$

reproduces the covariant derivative in Examples 3.86, 4.47 now as associated to our principal bundle. $\diamond$

As in the universal case, P itself is an example of an associated bundle and it is natural to ask when a strong connection ω gives a bimodule connection on P.

Lemma 5.50 *Let P be a regular quantum principal bundle and ω a strong connection. We consider the connection ∇_P defined on $P = (P \otimes H)^H$ by the formula in Lemma 5.6, now in the nonuniversal case. Then ∇_P is a bimodule connection if $\sum a_i\omega(\Lambda^1_H)b_i = 0$ for all $\sum a_i \otimes b_i \in N_A$ and in this case the connection induced on any associated bundle in Proposition 5.48 is also a bimodule connection. The curvature R_{∇_P} of the connection on P is a bimodule map if $[A, F_\omega(H^+)] = 0$ in Ω^2_P and in this case the curvature on any associated bundle is a bimodule map.*

Proof The connection is $\nabla_P\mathrm{d}p = \mathrm{d}p - p_{(\bar{0})}\omega\varpi\pi_\epsilon p_{(\bar{1})}$ and the curvature has the form stated in Proposition 5.48. We look more carefully at the bimodule features.

Begin with, for $p \in P$ and $a \in A$,

$$\nabla_P(pa) - (\nabla_P p)a = (\mathrm{id} - \Pi_\omega)(p\mathrm{d}a + (\mathrm{d}p)a) - ((\mathrm{id} - \Pi_\omega)\mathrm{d}p)a$$
$$= p\mathrm{d}a - \Pi_\omega((\mathrm{d}p)a) + (\Pi_\omega \mathrm{d}p)a = p\mathrm{d}a + p_{(\bar{0})}(\omega(\varpi\pi_\epsilon p_{(1)}))a - p_{(\bar{0})}a\omega(\varpi\pi_\epsilon p_{(1)}).$$

Now Lemma 3.67 gives that this is a bimodule connection with

$$\sigma_P(p \otimes (\mathrm{d}a_i)b_i) = p(\mathrm{d}a_i)b_i - p_{(\bar{0})}[a_i, \omega\varpi\pi_\epsilon p_{(\bar{1})}]b_i \tag{5.28}$$

provided this gives a well-defined function of $(\mathrm{d}a_i)b_i \in \Omega^1_A$ (summing over i understood). Without loss of generality, we assume here that $a_i b_i = 0$. Then in terms of the quotient of the universal calculus $(\mathrm{d}a_i)b_i = 0$ precisely when $(\mathrm{d}_{\mathrm{uni}}a_i)b_i \in N_A$ or $a_i \otimes b_i \in N_A$ from the form of $\mathrm{d}_{\mathrm{uni}}$, giving the result stated. It is clear that if the above condition holds for ∇_P on P to be a bimodule connection then the same formula and proof as for the universal case in Lemma 5.6 gives us the generalised braiding for ∇ in Proposition 5.48 on any associated bundle. Similarly, the bimodule condition on R_{∇_P} is clear from its form as in Lemma 5.6 and that it then holds generally is clear from the form of R_∇ on an associated bundle. Such ω obeying these conditions are particularly nice and could be called 'superstrong'. □

This ∇_P will be sufficient to cover the q-Hopf fibration bundle since the different charge associated bundles E_n are graded components of $P = \mathbb{C}_q[SL_2]$ in this case.

Example 5.51 (q-Hopf Fibration with the 3D Calculus) Over $\mathbb{C}$, we let $P = \mathbb{C}_q[SL_2]$ with its $3D$ left-covariant calculus and $H = \mathbb{C}_{q^2}[S^1]$ with generators t, t^{-1} and calculus $\mathrm{d}t.t = q^2 t\mathrm{d}t$ and π the diagonal projection as in Examples 1.11, 2.32, 2.33. The ideal I_P was stated as $\mathcal{I}$ in Example 2.32, from which $\pi(I_P) = \langle t + q^2 t^{-1} - (1+q^2)\rangle = \langle (t-1)(t-q^2)\rangle = I_H$. The first condition in Lemma 5.46 also holds as a vestigial bicovariance of the 3D calculus in Remark 2.37. Hence the universal quantum monopole bundle in Example 5.23 descends with these calculi. The unital bicovariant splitting map i there has

$$i(t^m(t + q^2t^{-1} - (1+q^2))) = \begin{cases} a^{m-1}(a^2 + q^2 - (1+q^2)a) & m > 0 \\ a + q^2 d - (1+q^2) & m = 0 \\ d^{1-m}(d^2 + q^2 - (1+q^2)d) & m < 0. \end{cases}$$

The middle case is manifestly in I_P. Multiplying it by a gives $a^2 + q^2 - (1 + q^2)a \in I_P$ given that $bc \in I_P$, similarly for the other case. Hence $i(I_H) \subseteq I_P$, so i descends. It follows from Lemma 5.46 that our previous universal bundle connection in Example 5.23 now descends to our nonuniversal calculus. Note that $\varpi(t^n - 1) = t^{-n}\mathrm{d}t^n = [n]_{q^2}t^{-1}\mathrm{d}t$ for the map $H^+ \to \Lambda^1_H$, so we must have $\omega(t^n - 1) = [n]_{q^2}\omega(t^{-1}\mathrm{d}t) = [n]_{q^2}e^0$ as a map $H^+ \to \Omega^1_P$. Here we denoted the descended map also as $\omega : \Lambda^1_H \to \Omega^1_P$ and we identified e^0 as the value of $\omega(t-1)$ given in

Example 5.23. One can also compute this formula for $\omega(t^n - 1)$ inductively from the formulae in Example 5.23 and the relations of the 3D calculus to see that it factors through $\omega : \Lambda^1_H \to \Omega^1_P$, as it must. The curvature form comes out as

$$F_\omega : H^+ \to \Omega^2_P, \quad F_\omega(t^n - 1) = q^3[n]_{q^2} e^+ \wedge e^-$$

for all $n \in \mathbb{Z}$ and hence also factors through Λ^1_H, i.e., the connection is regular, with $F_\omega(t^{-1}\mathrm{d}t) = q^3 e_+ \wedge e_- = q^3\mathrm{Vol}$. This is in fact valued in Ω^2_A and is precisely the curvature 2-form ω_∇ from the line module approach in Example 4.24.

It is also possible to not refer to the ideals but construct the bundle exact sequence and the connection as its splitting, working directly with the 3D calculus. In Example 4.33, we have already calculated the differential of the coaction of $H = \mathbb{C}_{q^2}[S^1]$ on $P = \mathbb{C}_q[SL_2]$ with the 3D calculus and in our current language this is

$$\mathrm{ver}(e^\pm) = 0, \quad \mathrm{ver}(e^0) = 1 \otimes t^{-1}\mathrm{d}t.$$

We also identified its kernel as $P\Omega^1_A = P\Omega^1_A P$ given the relations of the calculus. Hence we have the short exact sequence needed for the bundle, and taking $\omega(t^{-1}\mathrm{d}t) = e^0$ now as a definition of ω, we see that $\Pi_\omega(e^\pm) = 0$ and $\Pi_\omega(e^0) = e^0$, from which one can see that we have an equivariant splitting and hence a connection.

It follows that the covariant derivative ∇_P on P in Lemma 5.50 is a bimodule connection because e_0 commutes with elements of $A = \mathbb{C}_q[S^2]$ so that the condition there holds (it likewise follows that its curvature is a bimodule map). Here $\cdot\nabla_P p$ is $\mathrm{d}p$ followed by using $\mathrm{id} - \Pi_\omega$ to project out the e^0 term, and when restricted to grade $-n$ it gives the connection ∇ on the associated bundle E_n at our nonuniversal level. The formula for ∇ is the same as for the universal calculus in Example 5.23 but now with $\omega(t^{-n} - 1) = [-n]_{q^2} e^0 = -q^{-2n}[n]_{q^2} e_0$, which projects out the e^0 term. It remains to give this explicitly. We write for any $s \in E_n$,

$$\mathrm{d}s = s_+ e^+ + s_- e^- + s_0 e^0, \tag{5.29}$$

then as discussed we apply $\mathrm{id} - \Pi_\omega$, which just means discarding the e^0 term, to obtain working in Ω^1_P,

$$\begin{aligned}\cdot\nabla s = s_+ e^+ + s_- e^- &= q^{n+2} e^+ s_+ + q^{n-2} e^- s_- \\ &= q^{n+2}(e^+ D_1 D_1') D_2' D_2 s_+ + q^{n-2}(e^- \tilde{D}_1 \tilde{D}_1') \tilde{D}_2' \tilde{D}_2 s_-,\end{aligned}$$

where we used the commutation relations of the 3D calculus and that $s_\pm$ have degree $-n \mp 2$, and then inserted $1 = D_1 D_1' D_2' D_2 = \tilde{D}_1 \tilde{D}_1' \tilde{D}_2' \tilde{D}_2$. The bracketed expressions have degree 0 and hence lie in Ω^1_A, as explained in Proposition 2.35. This is now manifestly of the right form, namely in $\Omega^1_A P$, and we can insert a $\otimes_A$

to undo the product, to give

$$\nabla s = q^{n+2}e^+ D_1 D_1' \otimes_A D_2' D_2 s_+ + q^{n-2}e^- \tilde{D}_1 \tilde{D}_1' \otimes_A \tilde{D}_2' \tilde{D}_2 s_-.$$

By construction, this obeys the left Leibniz rule for multiplication by $f \in \mathbb{C}_q[S^2]$, although this is not so obvious from the formula given. If one wanted to check it explicitly, using the notation in (5.29) whereby $(fs)_\pm = q^{-n} f_\pm s + f s_\pm$, we have

$$\begin{aligned}\nabla(fs) - f\nabla s &= q^2 e^+ D_1 D_1' \otimes D_2' D_2 f_+ s + q^{-2} e^- \tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2 f_- s \\ &= q^2 e^+ D_1 D_1' D_2' D_2 f_+ \otimes s + q^{-2} e^- \tilde{D}_1 \tilde{D}_1' \tilde{D}_2' \tilde{D}_2 f_- \otimes s = \mathrm{d} f \otimes s.\end{aligned}$$

For $n = 1$, we obtain the Grassmann connection (4.12) in Example 4.24 as already explained in Example 5.23 for the universal calculus but in a way that works for other calculi. That the two connections coincide in our current description comes down to

$$qbe^+ \otimes d - de^+ \otimes b = qbe^+ \otimes d\tilde{D}_1 \tilde{D}_2 - de^+ \otimes b\tilde{D}_1 \tilde{D}_2 = (qbe^+ d\tilde{D}_1 - de^+ b\tilde{D}_1) \otimes \tilde{D}_2$$

vanishing by the relations of the calculus. Also, we have from (5.28) that $\cdot \sigma_P(p \otimes \xi) = p\xi$ for a 1-form ξ on $\mathbb{C}_q[S^2]$, so to find σ explicitly we have to undo the product. Following a similar procedure to the covariant derivative above gives

$$\sigma(s \otimes e^+ r) = q^n e^+ D_1 D_1' \otimes D_2' D_2 sr, \quad \sigma(s \otimes e^- p) = q^n e^- \tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2 sp$$

for r of grade -2 and p of grade 2. Finally, from Proposition 5.48 the curvature is

$$R_\nabla s = -s_{\bar{(0)}} F_\omega(\pi_\epsilon s_{\bar{(1)}}) \otimes v = -s F_\omega \pi_\epsilon(t^{-n}) \otimes v = q^3 [n]_{q^2} \mathrm{Vol} \otimes_A s$$

for $s \in E_n$ and v basis element of V_n involved in the identification of $E_n = (P \otimes V_n)^H$ as in Example 5.23. Here $\mathrm{Vol} = e^+ \wedge e^-$ is the volume form in our (non-geometric) normalisation and is central, which fits with our claim that this is a bimodule map. As the exterior algebra is a maximal prolongation of a first-order calculus, Lemma 4.14 tells us that σ on E_n is extendable. This comes out as

$$\sigma(s \otimes \mathrm{Vol}) = q^{2n} \mathrm{Vol} \otimes s$$

and makes $E_n \in {}_A\mathcal{G}_A$ as discussed in Example 4.24. ◇

We close with some categorical remarks. Following on from our discussion of associated module functors in §5.3, we should consider how that categorical picture extends to include connections. We recall the categories ${}_A\mathcal{E}$ and ${}_A\mathcal{E}_A$ of modules with connections and bimodule connections respectively from Chap. 3. Now we define related categories where the objects have H-coactions:

Name	Objects	Morphisms
${}_A\mathcal{E}^H$	Left A-modules and right H-comodules with A-action an H-comodule map and H-covariant left A-connection	Left module maps which are also right comodule maps intertwining the connections
${}_A\mathcal{E}^H_A$	A-bimodules and right H-comodules with A-actions both H-comodule maps & H-covariant A-bimodule connection	Bimodule maps which are also right comodule maps intertwining the connections

Following Proposition 5.48, we can now present the associated bundle functor for nonuniversal connections in a manner extending (5.18) in the universal case.

Proposition 5.52 *Let P be a regular quantum principal bundle with a strong connection ω and associated Π_ω, and suppose that Ω^1_A is flat as a right A-module. Then we have functors*

$$\mathcal{M}^H \xrightarrow{P_\omega\otimes} {}_A\mathcal{E}^H \xrightarrow{()^H} {}_A\mathcal{E}$$

where, in addition to (5.18), we have a connection $\nabla(p\otimes v) = (\mathrm{id} - \Pi_\omega)\mathrm{d}p\otimes v$ on $P\otimes V$ for all $V \in \mathcal{M}^H$ in the first step and restriction to H-invariants of the connection associated to an object in the second step. If Π_ω gives a bimodule connection on P then we similarly have

$$\mathcal{M}^H_H \xrightarrow{P_\omega\otimes} {}_A\mathcal{E}^H_A \xrightarrow{()^H} {}_A\mathcal{E}_A$$

where $V \in \mathcal{M}^H_H$. In both cases, the composition of the two functors is the associated bundle functor $\mathfrak{E}$ in agreement with Proposition 5.48.

Proof Mainly this summarises results in the section or is automatic using H-covariance of our connections. Flatness of Ω^1_A is required to show that the operation $\Omega^1_A\otimes_A$ commutes with taking the H-invariants, as in Example 3.105. □

The categories $\mathcal{M}^H_H$ and ${}_A\mathcal{E}_A$ are both monoidal, and in Proposition 5.56 we will give conditions for the associated bundle functor $\mathfrak{E}$ in this case to be monoidal.

5.4.2 Strong Quantum Principal Bundles

In this section we look at a class of principal bundles where the commutation relations between the base algebra A and forms on P are sufficiently strong or close to classical that

$$P\Omega^1_A = \Omega^1_A P \tag{5.30}$$

as subspaces of Ω^1_P. We say in this case that the quantum principal bundle is *strong*. We have already noted that this holds for the q-Hopf fibration in the preceding section, and we will see in Example 5.55 that this also holds for all tensor product bundles. On the other hand, the universal calculus on P will never give a strong bundle unless the base or fibre is trivial. Clearly, every connection on a strong bundle is automatically strong, which is another advantage of this setting. In fact there is a quite natural interpretation of strong quantum principal bundles in terms of the weakly horizontal 1-forms Ω^1_{hor} defined as the kernel of ver : $\Omega^1_P \to P \otimes \Lambda^1_H$.

Corollary 5.53 *If P is a regular quantum principal bundle and H has invertible antipode then $(\Omega^1_{\mathrm{hor}})^H = \Omega^1_A$ (i.e., the weakly horizontal invariant 1-forms are the 1-forms on the base) if and only if the bundle is strong.*

Proof If $(\Omega^1_{\mathrm{hor}})^H = \Omega^1_A$ and using Lemma 5.29, we have $\Omega^1_{\mathrm{hor}} \cong (\Omega^1_{\mathrm{hor}})^H \otimes_A P = \Omega^1_A \otimes_A P = \Omega^1_A P$ with isomorphisms given by the product. A similar observation with left-right reversed then implies that $P\Omega^1_A = \Omega^1_A P$. This uses a left module version of Lemma 5.29, which we shall not prove separately but which follows from the symmetries between the first and last tensor factors of $\omega^\sharp$ inherent from its definition, as displayed explicitly in (5.9), and that S is invertible.

Now suppose that the bundle is strong, in which case $\Omega^1_{\mathrm{hor}} = P\Omega^1_A P$ in the definition of a quantum bundle tells us that $\Omega^1_{\mathrm{hor}} = \Omega^1_A P$. To complete the proof we note that $(\Omega^1_A P)^H = \Omega^1_A$ as shown at the start of the proof of Proposition 5.48. □

Another motivation for the strong bundle property comes from Lemma 5.50 where we have conditions for ∇_P on P to be a bimodule connection. In a strong bundle $\Omega^1_A \otimes_A P$ and $P \otimes_A \Omega^1_A$ are canonically isomorphic as A-bimodules and right H-comodules on being identified via the product map with the two sides of $\Omega^1_A P = P\Omega^1_A$. This leads to a stronger version of Lemma 5.50 as follows.

Proposition 5.54 *Let ω be a connection on a regular quantum principal bundle and H have invertible antipode.*

(1) *Π_ω is a P-bimodule map if and only if*

$$p_{(\bar 0)}\omega(v \triangleleft p_{(\bar 1)}) = \omega(v)p$$

for $v \in \Lambda^1_H$ and $p \in P$.

(2) *If Π_ω is a bimodule map and ω is strong then P is strong and the associated bimodule connection on P has invertible σ_P.*

(3) *In the $*$-algebra case of a strong bundle in part (2), if ω commutes with $*$ then the connection on P is $*$-preserving and Π_ω commutes with $*$.*

Proof (1) Comparing $\Pi_\omega((\mathrm{d}p)q) = p_{(\bar 0)}q_{(\bar 0)}\omega(\varpi\pi_\epsilon(p_{(\bar 1)}) \triangleleft q_{(\bar 1)})$ and $(\Pi_\omega \mathrm{d}p)q = p_{(\bar 0)}\omega(\varpi\pi_\epsilon(p_{(\bar 1)}))q$, we clearly have a bimodule map if the stated condition holds. Now recall that a regular quantum principal bundle comes with strong universal connection, which we write as $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ and which obeys (5.9). Hence in the converse direction, if Π_ω is a bimodule map then $h^{(1)}\Pi_\omega((\mathrm{d}h^{(2)})q) = h^{(1)}(\Pi_\omega \mathrm{d}h^{(2)})q$, which gives the stated condition on using the properties of $\omega^\sharp$.

(2) If ω is strong and obeys the stated condition then clearly $[A, \omega(\Lambda^1_H)] = 0$, so Lemma 5.50 applies. The generalised braiding σ_P in (5.28) simplifies to the identity map $\sigma_P(p \otimes \xi) = p\xi$ for all $p \in P$ and $\xi \in \Omega^1_A$, and as the connection is strong, this must be contained in $\Omega^1_A P$. Thus $P\Omega^1_A \subseteq \Omega^1_A P$ and hence $\Omega^1_{\mathrm{hor}} = P\Omega^1_A P \subseteq \Omega^1_A P$. Then by the proof of Proposition 5.48 we have $(\Omega^1_{\mathrm{hor}})^H \subseteq \Omega^1_A$ but also $\Omega^1_A \subseteq (\Omega^1_{\mathrm{hor}})^H$ so we have equality. The statement of Corollary 5.53 now gives that the bundle P is strong. In this case $P\Omega^1_A = \Omega^1_A P$ so that the map σ_P is invertible.

(3) In the $*$-algebra setting, we assume that H is a Hopf $*$-algebra, the calculi are $*$-calculi and the coactions are $*$-algebra maps. A bimodule connection being $*$-preserving in Definition 3.85 depends on the braiding which in our case is given in part (2). So if $\nabla_P p = \mathrm{d}a \otimes q$ (sum of such terms) then $\nabla_P(p^*) = \sigma_P(q^* \otimes \mathrm{d}a^*)$ comes down to $\cdot\nabla_P(p^*) = q^*\mathrm{d}a^* = (\cdot\nabla_P p)^*$ or $[\cdot\nabla_P, *] = 0$. Thus we require

$$\mathrm{d}p^* - p^*{}_{(\bar{0})}\omega(\varpi(\pi_\epsilon p^*{}_{(\bar{1})})) = \big(\mathrm{d}p - p_{(\bar{0})}\omega(\varpi(\pi_\epsilon p_{(\bar{1})}))\big)^*$$

for all $p \in P$, or equivalently, equality in the middle of

$$(\Pi_\omega(\mathrm{d}p^*))^* = (\omega(\varpi\pi_\epsilon(p_{(\bar{1})}{}^*)))^* p_{(\bar{0})} = p_{(\bar{0})}\omega(\varpi(\pi_\epsilon p_{(\bar{1})})) = \Pi_\omega(\mathrm{d}p).$$

We use $(\varpi\pi_\epsilon(h^*))^* = -\varpi\pi_\epsilon(S^{-1}h)$ for all $h \in H$ and our assumption that ω commutes with $*$ to rewrite the left-hand side as

$$-\omega(\varpi\pi_\epsilon S^{-1}p_{(\bar{1})})p_{(\bar{0})} = -p_{(\bar{0})}\omega((\varpi\pi_\epsilon S^{-1}p_{(\bar{1})(2)}) \triangleleft p_{(\bar{1})(1)})$$

since Π_ω is a bimodule map so that the condition in (1) holds. This is the right-hand side of our condition since $(\varpi\pi_\epsilon(S^{-1}h_{(2)}))\triangleleft h_{(1)} = -\varpi\pi_\epsilon(h)$. □

The Π_ω bimodule condition is a rather strong one, but it does hold for the q-Hopf fibration. We saw for the q-monopole connection in Example 5.51 that Π_ω simply projects out multiples of $e^\pm$ and is the identity on multiples of e^0, hence is clearly a bimodule map. When q is real, we also have $\omega(t^{-1}\mathrm{d}t) = e^0$, which commutes with $*$ using $t^* = t^{-1}$ and $e^{0*} = -e^0$ according to the respective standard compact real forms and $*$-calculi for $H = \mathbb{C}_{q^2}[S^1]$ and $\mathbb{C}_q[SU_2]$ in Chap. 2.

The simplest example of a strong principal bundle is the following tensor product bundle. Connections here correspond to 'Lie algebra-valued' 1-forms α or 'gauge fields' on the base. Such bundles will be studied further in §5.6.2.

Example 5.55 If (A, Ω^1_A) is a differential algebra and (H, Ω^1_H) is a bicovariant calculus then $P = A \otimes H$ with its canonical right coaction and tensor product differential calculus is easily seen to be a strong trivial quantum principal bundle.

There is also a strong universal connection given by

$$\omega^\sharp(h) = 1 \otimes Sh_{(1)} \otimes 1 \otimes h_{(2)},$$

and using $\Omega^1_P = \Omega^1_A \otimes H \bigoplus A \otimes \Omega^1_H$, the horizontal part is $\Omega^1_A \otimes H = \Omega^1_A(A \otimes H)$. The strong connection descends to $\omega : \Lambda^1_H \to \Omega^1_P$ given by $\omega(\xi) = 1 \otimes \xi \in A \otimes \Omega^1_H$. More generally, we can add a gauge potential $\alpha : \Lambda^1_H \to \Omega^1_A$ to give

$$\omega(\xi) = 1 \otimes \xi + (\alpha \otimes \mathrm{id})\mathrm{Ad}_R(\xi)$$

for all $\xi \in \Lambda^1_H$. Here $\mathrm{Ad}_R : \Lambda^1_H \to \Lambda^1_H \otimes H$ is defined by $\mathrm{Ad}_R(\varpi\pi_\epsilon(h)) = \varpi\pi_\epsilon(h_{(2)}) \otimes (Sh_{(1)})h_{(3)}$.

In the notation of §5.1.2 we have a trivialisation $\Phi : H \to P = A \otimes H$ given by $\Phi(h) = 1 \otimes h$, with convolution inverse $\Phi(h) = 1 \otimes Sh$. From Proposition 5.13, we have trivial cocycle data $h \triangleright a = \epsilon(h)a$, $\chi(h \otimes g) = \epsilon(h)\epsilon(g)$ for all $h, g \in H$. From Proposition 5.48, the curvature of the connection on P is given by $R_\nabla(p) = -p_{(\bar 0)} F_\omega(\pi_\epsilon p_{(\bar 1)})$, where

$$F_\omega(h) = \mathrm{d}\omega\varpi\pi_\epsilon(h) + \omega\varpi\pi_\epsilon(h_{(1)}) \wedge \omega\varpi\pi_\epsilon(h_{(2)})$$

for all $h \in H^+$. For the above ω we have

$$\begin{aligned}
\omega\varpi\pi_\epsilon(h) &= 1 \otimes \varpi\pi_\epsilon(h) + \alpha(\varpi\pi_\epsilon(h_{(2)})) \otimes (Sh_{(1)})h_{(3)},\\
\mathrm{d}\omega\varpi\pi_\epsilon(h) &= 1 \otimes \mathrm{d}\varpi\pi_\epsilon(h) + \mathrm{d}\alpha(\varpi\pi_\epsilon(h_{(2)})) \otimes (Sh_{(1)})h_{(3)}\\
&\quad - \alpha(\varpi\pi_\epsilon(h_{(2)})) \otimes \big((\mathrm{d}Sh_{(1)})h_{(3)} + (Sh_{(1)})\mathrm{d}h_{(3)}\big)
\end{aligned}$$

for all $h \in H$. As $\mathrm{d}\varpi\pi_\epsilon(h) = -\varpi\pi_\epsilon(h_{(1)}) \wedge \varpi\pi_\epsilon(h_{(2)})$, the terms in $F_\omega(h)$ which do not contain α cancel and remembering the sign for the wedge product of tensor products (explicitly noted in Definition 4.28), we find

$$F_\omega(h) = \mathrm{d}\alpha(\varpi\pi_\epsilon(h_{(2)})) \otimes (Sh_{(1)})h_{(3)} + \alpha(\varpi\pi_\epsilon(h_{(2)})) \wedge \alpha(\varpi\pi_\epsilon(h_{(3)})) \otimes (Sh_{(1)})h_{(4)}$$

for all $h \in H^+$. Now suppose that V is a right H-comodule. For the connection on the associated module $(A \otimes H \otimes V)^H$ we have

$$\begin{aligned}
\nabla(a \otimes h \otimes v) &= (\mathrm{id} - \Pi_\omega)\mathrm{d}(a \otimes h) \otimes v\\
&= (\mathrm{id} - \Pi_\omega)(\mathrm{d}a \otimes h + a \otimes h_{(1)}\varpi\pi_\epsilon(h_{(2)})) \otimes v\\
&= (\mathrm{d}a \otimes h + a \otimes h_{(1)}\varpi\pi_\epsilon(h_{(2)}) - (a \otimes h_{(1)})\omega(\varpi\pi_\epsilon(h_{(2)}))) \otimes v\\
&= (\mathrm{d}a \otimes h - a\alpha(\varpi\pi_\epsilon(h_{(1)})) \otimes h_{(2)}) \otimes v.
\end{aligned}$$

One can also look at the associated module in another way, using an isomorphism $(A \otimes H \otimes V)^H \cong A \otimes V$ afforded by $a \otimes h \otimes v \mapsto a\epsilon(h) \otimes v$ and $a \otimes v \mapsto$

$a \otimes S^{-1}v_{(\bar{1})} \otimes v_{(\bar{0})}$ when H has invertible antipode. The connection and its curvature then transfer to $A \otimes V$ as

$$\begin{aligned}\nabla(a \otimes v) &= \mathrm{d}a \otimes v - a\alpha(\varpi\pi_\epsilon(S^{-1}v_{(\bar{1})})) \otimes v_{(\bar{0})},\\ R_\nabla(a \otimes v) &= (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})R_\nabla(a \otimes S^{-1}v_{(\bar{1})} \otimes v_{(\bar{0})})\\ &= -(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})\big((a \otimes (S^{-1}v_{(\bar{1})})_{(1)})F_\omega(\pi_\epsilon(S^{-1}v_{(\bar{1})})_{(2)}) \otimes v_{(\bar{0})}\big)\\ &= -aF_\alpha(\pi_\epsilon S^{-1}v_{(\bar{1})}) \otimes v_{(\bar{0})},\end{aligned}$$

where we have written

$$F_\alpha(\pi_\epsilon h) = \mathrm{d}\alpha(\varpi\pi_\epsilon h) + \alpha(\varpi\pi_\epsilon h_{(1)}) \wedge \alpha(\varpi\pi_\epsilon h_{(2)}).$$

In terms of the notation in §5.1.2, we can now replace $a \otimes v$ by a formal sum $\phi = \phi^1 \otimes \phi^2$ and using our standard equivalent left coaction on V via the antipode, we have analogously to Corollary 5.14 (but now with nontrivial calculi)

$$\nabla\phi = \mathrm{d}\phi^1 \otimes \phi^2 - \phi^1\alpha(\varpi\pi_\epsilon\phi^2{}_{(\bar{1})}) \otimes \phi^2{}_{(\bar{\infty})}, \quad R_\nabla\phi = -\phi^1F_\alpha(\pi_\epsilon\phi^2{}_{(\bar{1})}) \otimes \phi^2{}_{(\bar{\infty})}.$$

A brief calculation shows that, for $a \in A$ and right action $(\phi^1 \otimes \phi^2)a = \phi^1 a \otimes \phi^2$,

$$\nabla(\phi a) - (\nabla\phi)a = \phi^1\mathrm{d}a \otimes \phi^2 + \phi^1\big[\alpha(\varpi\pi_\epsilon\phi^2{}_{(\bar{1})}), a\big] \otimes \phi^2{}_{(\bar{\infty})},$$

and if this is $\sigma(\phi \otimes \mathrm{d}a)$ for a bimodule map $\sigma : (A \otimes V) \otimes_A \Omega^1 \to \Omega^1 \otimes_A (A \otimes V)$ then we have a bimodule connection.

The bimodule map condition for Π_ω holds when $[A, \alpha(\Lambda^1)] = 0$ in which case the braiding σ is trivial. In this case and in a $*$-algebra setting with $(a \otimes h)^* = a^* \otimes h^*$, ω commutes with $*$ whenever α on the base commutes with $*$. ◇

We close with some categorical remarks. As previously mentioned, the functor $()^H$ in the second case of Proposition 5.52 will not in general be monoidal, but the composition along the bottom line will be in the strong case, as we now show. We also functor to the subcategory ${}_A\mathcal{E}\mathcal{I}_A$, not just ${}_A\mathcal{E}_A$. Recall that the right action of A in this context is by right multiplication on P in $\mathfrak{E}(V) = (P \otimes V)^H$.

Proposition 5.56 *Let P be a regular strong quantum bundle with H having invertible antipode and let ω be a connection such that Π_ω is a P bimodule map, then the associated module functor $\mathfrak{E} : \mathcal{M}_H^H \to {}_A\mathcal{E}\mathcal{I}_A$ is a monoidal functor.*

Proof We know from Proposition 5.54 that the connection on P is a bimodule connection with invertible σ_P. Then, as in Proposition 5.52, there is a bimodule connection induced on all associated bundles with

$$\sigma(p \otimes v \otimes \xi) = p\xi \otimes v \in (\Omega^1_A P \otimes V)^H = \Omega^1_A \otimes_A (P \otimes V)^H$$

for $\xi \in \Omega^1_A$, using the isomorphism in the proof of Proposition 5.48 and that the bundle is strong to reorder, which is σ_P in our current context. It remains to show that the maps $c_{V,W} : \mathfrak{E}(V) \otimes_A \mathfrak{E}(W) \to \mathfrak{E}(V \otimes W)$ in Proposition 5.32 intertwine the relevant connections. Writing $(\mathrm{id} - \Pi_\omega)\mathrm{d}q = \mathrm{d}a \otimes r$ (a sum of such terms),

$$\begin{aligned}
&(\mathrm{id} \otimes c_{V,W})\nabla_{\mathfrak{E}(V)\otimes\mathfrak{E}(W)}((p \otimes v) \otimes (q \otimes w))\\
&= (\mathrm{id} \otimes c_{V,W})\big(((\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v) \otimes (q \otimes w) + \sigma_{\mathfrak{E}(V)}(p \otimes v \otimes \mathrm{d}a) \otimes r \otimes w\big)\\
&= (\mathrm{id} \otimes c_{V,W})\big(((\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v) \otimes (q \otimes w) + p\mathrm{d}a \otimes v \otimes r \otimes w\big)\\
&= (\mathrm{id} - \Pi_\omega)((\mathrm{d}p)q_{(\bar 0)}) \otimes v{\triangleleft}q_{(1)} \otimes w + p(\mathrm{d}a)r_{(\bar 0)} \otimes v{\triangleleft}r_{(1)} \otimes w\\
&= (\mathrm{id} - \Pi_\omega)((\mathrm{d}p)q_{(\bar 0)}) \otimes v{\triangleleft}q_{(1)} \otimes w + (\mathrm{id} - \Pi_\omega)(p\mathrm{d}q_{(\bar 0)}) \otimes v{\triangleleft}q_{(1)} \otimes w\\
&= (\mathrm{id} - \Pi_\omega)\mathrm{d}(pq_{(\bar 0)}) \otimes v{\triangleleft}q_{(1)} \otimes w = \nabla_{\mathfrak{E}(V\otimes W)}c_{V,W}((p \otimes v) \otimes (q \otimes w)). \qquad \square
\end{aligned}$$

We also consider associated bundles in our $*$-algebra setting, where it is natural to ask whether the associated bundle functor in Proposition 5.56 preserves the bar category structures on ${}\mathcal{M}^H_H$ given in Example 2.105 and on ${}_A\mathcal{EI}_A$ in Theorem 3.84.

Proposition 5.57 *Let P be a regular strong quantum principal bundle in the $*$-algebra setting and ω a connection commuting with $*$ such that Π_ω is a bimodule map. Then there is an invertible natural transformation $\theta_V : \overline{(P \otimes V)^H} \to (P \otimes \overline{V})^H$ between the functors* bar $\circ\, \mathfrak{E}$ *and* $\mathfrak{E} \circ$ bar $: \mathcal{M}^H_H \to {}_A\mathcal{EI}_A$, *given by*

$$\theta_V(\overline{p \otimes v}) = p_{(\bar 0)}{}^* \otimes \overline{v}{\triangleleft}(p_{(\bar 1)}{}^*).$$

Proof Following Proposition 5.34, it only remains to show that

$$\nabla_{\mathfrak{E}(\overline{V})}\,\theta_V = (\mathrm{id} \otimes \theta_V)\nabla_{\overline{\mathfrak{E}(V)}} : \overline{(P \otimes V)^H} \to \Omega^1_A \otimes_A (P \otimes \overline{V})^H.$$

We begin with

$$\nabla_{\mathfrak{E}(\overline{V})}\,\theta_V(\overline{p \otimes v}) = \nabla_{\mathfrak{E}(\overline{V})}\big(p_{(\bar 0)}{}^* \otimes \overline{v}{\triangleleft}p_{(\bar 1)}{}^*\big) = (\mathrm{id} - \Pi_\omega)(\mathrm{d}p_{(\bar 0)}{}^*) \otimes \overline{v}{\triangleleft}p_{(\bar 1)}{}^*$$

for $p \otimes v \in (P \otimes V)^H$ (sum understood). Writing $(\mathrm{id} - \Pi_\omega)(\mathrm{d}p_{(\bar 0)}{}^*) \otimes \overline{v}{\triangleleft}p_{(\bar 1)}{}^* = \xi q \otimes \overline{w}$ (a sum of such terms) with $\xi \in \Omega^1_A$, $q \in P$ and $w \in V$,

$$(\mathrm{id} \otimes \theta_V{}^{-1})\nabla_{\mathfrak{E}(\overline{V})}\,\theta_V(\overline{p \otimes v}) = \xi \otimes \overline{q_{(\bar 0)}{}^* \otimes w{\triangleleft}q_{(\bar 1)}{}^*}$$

using the formula for $\theta_V{}^{-1}$ in Proposition 5.34. Now we calculate

$$\begin{aligned}
\overline{\sigma_{\mathfrak{E}(V)}}\,\Upsilon^{-1}(\star \otimes \mathrm{id})(\mathrm{id} \otimes \theta_V{}^{-1})\nabla_{\mathfrak{E}(\overline{V})}\,\theta_V(\overline{p \otimes v}) &= \overline{\sigma_{\mathfrak{E}(V)}(q_{(\bar 0)}{}^* \otimes w{\triangleleft}q_{(\bar 1)}{}^* \otimes \xi^*)}\\
&= \overline{q_{(\bar 0)}{}^*\xi^* \otimes w{\triangleleft}q_{(\bar 1)}{}^*} = \overline{(\xi q)_{(\bar 0)}{}^* \otimes w{\triangleleft}(\xi q)_{(\bar 1)}{}^*}
\end{aligned}$$

$$= \overline{((\mathrm{id} - \Pi_\omega)\mathrm{d}p_{(\bar 0)}{}^*)^* \otimes v{\triangleleft}S^{-1}(p_{(\bar 2)})p_{(\bar 1)}}$$
$$= \overline{((\mathrm{id} - \Pi_\omega)\mathrm{d}p^*)^* \otimes v} = \overline{(\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v} = \overline{\nabla_{\mathfrak{E}(V)}(p \otimes v)}. \qquad \square$$

As a corollary, suppose that V is a star object in $\mathcal{M}_H^H$, which means there is a morphism $\star : V \to \overline{V}$ obeying the conditions of Definition 2.102. In ordinary terms, this means an antilinear map $v \mapsto v^*$ obeying $v^*{}_{(\bar 0)} \otimes v^*{}_{(\bar 1)} = v_{(\bar 0)}{}^* \otimes v_{(\bar 1)}{}^*$ and $(v{\triangleleft}h)^* = v^*{\triangleleft}S^{-1}(h^*)$ for all $v \in V$ and $h \in H$. If the conditions of Proposition 5.57 hold then

$$\mathfrak{E}(V) \xrightarrow{\mathfrak{E}(\star)} \mathfrak{E}(\overline{V}) \xrightarrow{\theta_V{}^{-1}} \overline{\mathfrak{E}(V)}$$

as isomorphisms in the category ${}_A\mathcal{E}\mathcal{I}_A$. The composition here amounts to an antilinear $*$-operation $(p \otimes v)^* = p_{(\bar 0)}{}^* \otimes v^*{\triangleleft}p_{(\bar 1)}{}^*$ on the associated bundle $\mathfrak{E}(V)$.

5.5 Principal Bundles and Spectral Sequences

A classical principal bundle is a fibre bundle over the base with fibre the structure group, while for a Hopf algebra differentiable fibration in the sense of Definition 4.61, we have the Leray–Serre spectral sequence in Theorem 4.66. It is natural to ask if a quantum principal bundle with general differential structures is a differentiable fibration, and if it is then to identify the pieces in the Leray–Serre spectral sequence.

We assume that we have a strongly bicovariant exterior algebra Ω_H on a Hopf algebra H whereby $\Delta_*^{r,s} : \Omega_H^{r+s} \to \Omega_H^r \otimes \Omega_H^s$ extends the coproduct to all forms as a super one (for example the canonical one in Theorem 2.74). We also assume that P is a comodule algebra with coaction $\Delta_R : P \to P \otimes H$ differentiable on all forms, i.e., it extends to a map of DGAs $\Delta_{R*} : \Omega_P \to \Omega_P \otimes \Omega_H$ (see Definition 4.28). Then Ω_P becomes a right super-Ω_H-comodule algebra, with super-comodule property

$$(\Delta_{R*} \otimes \mathrm{id})\Delta_{R*} = (\mathrm{id} \otimes \Delta_*)\Delta_{R*} : \Omega_P \to \Omega_P \otimes \Omega_H \otimes \Omega_H.$$

Lemma 5.58 (Spectral Sequence of a Differentiable Coaction) *Suppose that an H-comodule algebra P has calculi with coaction $\Delta_R : P \to P \otimes H$ differentiable as above. Then*

$$F^m(\Omega_P^n) = \begin{cases} \bigcap_{k>n-m} \ker \Delta_{R*}^{n-k,k} : \Omega_P^n \to \Omega_P^{n-k} \otimes \Omega_H^k & 0 \le m \le n; \\ 0 & \text{otherwise} \end{cases}$$

defines a filtration of Ω_P satisfying the conditions of Example 4.52. Hence there is a spectral sequence for the filtration converging to $\mathrm{H}_{\mathrm{dR}}(P)$.

Proof First, if $\xi \in F^m(\Omega^n_P)$ for $0 \le m \le n$ then by definition

$$\Delta_{R*}\xi \in \Omega^n_P \otimes \Omega^0_H \bigoplus \Omega^{n-1}_P \otimes \Omega^1_H \bigoplus \cdots \bigoplus \Omega^m_P \otimes \Omega^{n-m}_H,$$

and applying d gives

$$\Delta_{R*}(\mathrm{d}\xi) = \mathrm{d}\Delta_{R*}(\xi) \in \Omega^{n+1}_P \otimes \Omega^0_H \bigoplus \Omega^n_P \otimes \Omega^1_H \bigoplus \cdots \bigoplus \Omega^m_P \otimes \Omega^{n+1-m}_H,$$

so $\mathrm{d}f \in F^m(\Omega^{n+1}_P)$, verifying condition (1) of Example 4.52. Conditions (2) and (3) are obvious. □

We show that under some assumptions, this spectral sequence is the same as the Leray–Serre spectral sequence for an associated differential fibration with total algebra P. We first define

$$\mathrm{ver}^{m,n-m} : \Omega^n_P \to \Omega^m_P \otimes \Lambda^{n-m}_H$$

in terms of the left H coaction on Ω^{n-m}_H as $\Delta^{m,n-m}_{R*} : \Omega^n_P \to \Omega^m_P \otimes \Omega^{n-m}_H$ followed by $\xi \otimes \eta \mapsto \xi \otimes (S\eta_{(\bar{1})})\eta_{(\bar{\infty})}$. Our usual map ver is $\mathrm{ver}^{0,1}$.

Lemma 5.59 *If P is a regular quantum principal bundle with differentiable coaction as above then the filtration in the preceding lemma is such that*

$$\begin{tikzcd}
0 \arrow[r] & F^{m+1}(\Omega^n_P) \arrow[r, "i"] & F^m(\Omega^n_P) \arrow[r, "\mathrm{ver}^{m,n-m}"] & F^m(\Omega^m_P) \otimes \Lambda^{n-m}_H \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{m+1}_A \wedge \Omega^{n-m-1}_P \arrow[r, "i"] \arrow[u, "i"] & \Omega^m_A \wedge \Omega^{n-m}_P \arrow[u, "i"] & &
\end{tikzcd}$$

commutes with rows exact. Here i denotes inclusion.

Proof By surjectivity of the exterior algebra, $\Omega^{m+1}_A = \Omega^m_A \wedge \Omega^1_A \subseteq \Omega^m_A \wedge \Omega^1_P$ gives the inclusion on the bottom line. Under the conditions of Lemma 5.58, $\Omega^m_A \wedge \Omega^{n-m}_P$ is contained in $F^m(\Omega^n_P)$ due to $\Delta_{R*}(\eta \wedge \kappa) = \eta \wedge \Delta_{R*}\kappa$ for $\eta \in \Omega_A$. Then by definition of $F^m(\Omega^n_P)$ we have the following with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F^{m+1}(\Omega^n_P) \arrow[r, "i"] & F^m(\Omega^n_P) \arrow[r, "\Delta^{m,n-m}_{R*}"] & \Omega^m_P \otimes \Omega^{n-m}_H \\
0 \arrow[r] & \Omega^{m+1}_A \wedge \Omega^{n-m-1}_P \arrow[r, "i"] \arrow[u, "i"] & \Omega^m_A \wedge \Omega^{n-m}_P . \arrow[u, "i"] &
\end{tikzcd}$$

Since $\Delta_* : \Omega_H \to \Omega_H \otimes \Omega_H$ preserves the total degree of forms, we see from the super-comodule property of Δ_{R*} that

$$(\Delta_{R*} \otimes \mathrm{id})\Delta_{R*}\xi \in \bigoplus_{r+s \le n-m} \Omega^{n-r-s}_P \otimes \Omega^r_H \otimes \Omega^s_H$$

for all $\xi \in F^m(\Omega_P^n)$. It follows that the image of $\Delta_{R*}^{m,n-m} : F^m(\Omega_P^n) \to \Omega_P^m \otimes \Omega_H^{n-m}$ is contained in $F^m(\Omega_P^m) \otimes \Omega_H^{n-m}$.

Now note that the image of $\Delta_{R*}^{m,n-m} : \Omega_P^n \to \Omega_P^m \otimes \Omega_H^{n-m}$ is actually contained in the cotensor product $\Omega_P^m \otimes^H \Omega_H^{n-m}$, which in terms of the ordinary H-coactions is the kernel of

$$\Delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_L : \Omega_P^m \otimes \Omega_H^{n-m} \to \Omega_P^m \otimes H \otimes \Omega_H^{n-m}.$$

Furthermore, this cotensor product is isomorphic to $\Omega_P^m \otimes \Lambda_H^{n-m}$ via the map $\xi \otimes \eta \mapsto \xi \otimes (S\eta_{(\bar{1})})\eta_{(\bar{\infty})}$, and this is the map appearing in the definition of ver.

To see that the image of $\mathrm{ver}^{m,n-m}$ is all of $F^m(\Omega_P^m) \otimes \Lambda_H^{n-m}$, take $\xi \in F^m(\Omega_P^m)$, $h_i \in H$ for $1 \le i \le n-m$ and $\omega^\sharp(h) = h^{(1)} \otimes h^{(2)}$ a strong universal connection. Using two applications of (5.9) we find

$$\Delta_{R*}^{0,1}(h^{(1)}\mathrm{d}h^{(2)}) = h^{(1)}{}_{(\bar{0})}h^{(2)}{}_{(\bar{0})} \otimes h^{(1)}{}_{(\bar{1})}\mathrm{d}h^{(2)}{}_{(\bar{1})} = 1 \otimes \varpi\pi_\epsilon(h),$$

for all $h \in H$, and using this we obtain as required that

$$\begin{aligned}&\mathrm{ver}^{m,n-m}(\xi \wedge h_1{}^{(1)}\mathrm{d}h_1{}^{(2)} \wedge \cdots \wedge h_{n-m}{}^{(1)}\mathrm{d}h_{n-m}{}^{(2)})\\ &\qquad = \xi \otimes \varpi\pi_\epsilon(h_1) \wedge \cdots \wedge \varpi\pi_\epsilon(h_{n-m}) \qquad \square\end{aligned}$$

To get further, we need to impose some condition on the higher forms on P, which is really just the extension to higher forms of the idea of invariant weakly horizontal 1-forms in Ω_P^1 being Ω_A^1 in the strong bundle case.

Lemma 5.60 *Let P be a regular quantum principal bundle with differentiable coaction and suppose that every Ω_A^n is flat as a right A-module, that the conditions of Lemma 5.59 apply, and that*

$$0 \longrightarrow \Omega_A^1 \wedge \Omega_P^{m-1} \xrightarrow{i} \Omega_P^m \xrightarrow{\mathrm{ver}^{0,m}} P \otimes \Lambda_H^m \longrightarrow 0$$

is exact for all $m \ge 1$. Then

$$0 \longrightarrow \Omega_A^{n+1} \wedge \Omega_P^{m-1} \xrightarrow{i} \Omega_A^n \wedge \Omega_P^m \xrightarrow{\mathrm{ver}^{n,m}} \Omega_A^n P \otimes \Lambda_H^m \longrightarrow 0$$

is also exact for all n and $m \ge 1$, and $F^r(\Omega_P^n) = \Omega_A^r \wedge \Omega_P^{n-r}$ for all $n \ge r \ge 0$.

Proof Applying $\Omega_A^n \otimes_A$ to the given exact sequence gives an exact row

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^n \otimes_A \Omega_A^1 \wedge \Omega_P^{m-1} \arrow[r, "\mathrm{id}\otimes i"] & \Omega_A^n \otimes_A \Omega_P^m \arrow[r, "\mathrm{id}\otimes\mathrm{ver}^{0,m}"] \arrow[d, "\wedge"] & \Omega_A^n \otimes_A P \otimes \Lambda_H^m \arrow[r] \arrow[d, "\cong"', "\cdot\otimes\mathrm{id}"] & 0\\
 & & \Omega_A^n \wedge \Omega_P^m \arrow[r, "\mathrm{ver}^{n,m}"] & \Omega_A^n P \otimes \Lambda_H^m &
\end{tikzcd}$$

and we have a commuting square because $\Delta_{R*}(\eta\wedge\kappa)=\eta\wedge\Delta_{R*}\kappa$ for $\eta\in\Omega^n_A$. The map $\cdot\otimes\mathrm{id}$ is an isomorphism by Lemma 5.29 together with the H-invariant part of $\Omega^n_A P$ being Ω^n_A. Then $\mathrm{ver}^{n,m}$ in the bottom line is onto. Suppose that $\sum_i \xi_i\wedge\eta_i$ is in the kernel of $\mathrm{ver}^{n,m}$ for $\xi_i\in\Omega^n_A$ and $\eta_i\in\Omega^m_P$. Then $\sum_i \xi_i\otimes\eta_i$ is in the kernel of $\mathrm{id}\otimes\mathrm{ver}^{0,m}$, so by exactness $\sum_i \xi_i\wedge\eta_i\in\Omega^{n+1}_A\wedge\Omega^{m-1}_P$, giving the second exact sequence in the statement.

For the last part of the statement, $F^1(\Omega^n_P)$ is the kernel of $\mathrm{ver}^{0,n}:\Omega^n_P\to P\otimes\Lambda^n_H$, so $F^1(\Omega^n_P)=\Omega^1_A\wedge\Omega^{n-1}_P$. Now $F^2(\Omega^n_P)\subseteq F^1(\Omega^n_P)$, so by the hypothesis

$$0\longrightarrow\Omega^2_A\wedge\Omega^{n-2}_P=F^2(\Omega^n_P)\xrightarrow{i}\Omega^1_A\wedge\Omega^{n-1}_P\xrightarrow{\mathrm{ver}^{1,n-1}}\Omega^1_A P\otimes\Lambda^{n-1}_H\longrightarrow 0.$$

The proof then follows by induction. □

The exactness condition in Lemma 5.60 for $m=1$ implies that the weakly horizontal forms in Ω^1_P are just $\Omega^1_A P$, and thus implies the condition for a strong bundle. For $m=2$ it implies a reordering principle $\Omega^1_P\wedge\Omega^1_A\subseteq\Omega^1_A\wedge\Omega^1_P$. We now prove our main theorem.

Theorem 5.61 *Given the conditions of Lemma 5.60, the inclusion $i:\Omega_A\to\Omega_P$ induces an isomorphism*

$$\Omega^n_A\otimes_A\frac{\Omega^m_P}{\Omega^1_A\wedge\Omega^{m-1}_P}\cong\frac{\Omega^n_A\wedge\Omega^m_P}{\Omega^{n+1}_A\wedge\Omega^{m-1}_P},\qquad \xi\otimes[\eta]\mapsto[i(\xi)\wedge\eta]$$

for $\xi\in\Omega_A$ and $\eta\in\Omega_P$. Here i satisfies the conditions for a differential fibration in Definition 4.61 and the m-forms in the fibre direction are of the form

$$\frac{\Omega^m_P}{\Omega^1_A\wedge\Omega^{m-1}_P}\cong P\otimes\Lambda^m_H.$$

Proof Applying $\Omega^n_A\otimes_A$ to the exact sequence

$$0\longrightarrow\Omega^1_A\wedge\Omega^{m-1}_P\xrightarrow{i}\Omega^m_P\xrightarrow{\mathrm{ver}^{0,m}}P\otimes\Lambda^m_H\longrightarrow 0$$

and using Lemma 5.60 gives the row exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{n+1}_A\wedge\Omega^{m-1}_P & \xrightarrow{i} & \Omega^n_A\wedge\Omega^m_P & \xrightarrow{\mathrm{ver}^{n,m}} & \Omega^n_A P\otimes\Lambda^m_H & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\wedge} & & \uparrow{\scriptstyle\wedge} & & \cong\uparrow{\scriptstyle\cdot\otimes\mathrm{id}} & & \\
0 & \longrightarrow & \Omega^n_A\otimes_A\Omega^1_A\wedge\Omega^{m-1}_P & \xrightarrow{i} & \Omega^n_A\otimes_A\Omega^m_P & \xrightarrow{\mathrm{id}\otimes\mathrm{ver}^{0,m}} & \Omega^n_A\otimes_A P\otimes\Lambda^m_H & \longrightarrow & 0.
\end{array}$$

The last vertical map provides the required isomorphism. □

Now using Theorem 4.66, we can write a spectral sequence starting with the cohomology of the fibre H and converging to the cohomology of P. Under certain assumptions, we can give its second page in a simple manner as follows.

Corollary 5.62 *Let P be a regular quantum principal bundle with differentiable coaction such that the conditions of Lemma 5.60 apply. If $\mathrm{H}_{\mathrm{dR}}(H)$ is left-invariant then we get a spectral sequence with second page $E_2^{pq} = \mathrm{H}^p_{\mathrm{dR}}(A) \otimes \mathrm{H}^q_{\mathrm{dR}}(H)$ which converges to $\mathrm{H}_{\mathrm{dR}}(P)$.*

Proof In terms of Theorem 4.66, the cohomology bundle $\hat{\mathrm{H}}^q$ is the cohomology of the cochain complex

$$\cdots \xrightarrow{\mathrm{d}} P \otimes^H \Omega_H^{q-1} \xrightarrow{\mathrm{id}\otimes\mathrm{d}} P \otimes^H \Omega_H^{q} \xrightarrow{\mathrm{id}\otimes\mathrm{d}} P \otimes^H \Omega_H^{q+1} \xrightarrow{\mathrm{id}\otimes\mathrm{d}} \cdots$$

where we use the cotensor product $P \otimes^H \Omega_H$ version rather than the isomorphic $P \otimes \Lambda_H$ version, as the differential is easier. By coflatness of P as a right H-comodule (see Proposition 5.10) applied to the short exact sequences of left H-comodules

$$0 \longrightarrow \ker \mathrm{d} \longrightarrow \Omega_H^q \xrightarrow{\mathrm{d}} \mathrm{im}\, \mathrm{d} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{im}\, \mathrm{d} \longrightarrow \ker \mathrm{d} \longrightarrow \mathrm{H}^q_{\mathrm{dR}}(H) \longrightarrow 0$$

we find $\hat{\mathrm{H}}^q = P \otimes^H \mathrm{H}^q_{\mathrm{dR}}(H) = A \otimes \mathrm{H}^q_{\mathrm{dR}}(H)$ since $\mathrm{H}_{\mathrm{dR}}(H)$ is left-invariant. Using the formula at the end of the proof of Theorem 4.66, the derivative on $\mathrm{H}^q_{\mathrm{dR}}(H)$ vanishes, hence taking the cohomology gives $\mathrm{H}_{\mathrm{dR}}(A) \otimes \mathrm{H}_{\mathrm{dR}}(H)$. The result follows by Theorem 4.66. □

Next we shall use the results of this section to show that the Hopf fibration is a differential fibration, a result which we have already used in Example 4.68.

Example 5.63 In Example 5.23, we showed that the q-Hopf fibration was a Hopf–Galois extension with $A = \mathbb{C}_q[S^2]$, $H = \mathbb{C}_{q^2}[S^1]$ and $P = \mathbb{C}_q[SU_2]$, and in Example 5.51 we showed that it was a quantum principal bundle for the 3D calculus on $\mathbb{C}_q[SU_2]$. The key part now is to check the first exact sequence condition in Lemma 5.60 for $m \geq 2$ (it has already been noted for $m = 1$, giving Ω^1_A as the invariant weakly horizontal part of Ω^1_P). From Example 4.33, we have $\mathrm{ver}^{0,1}(e^{\pm}) = 0$ and $\mathrm{ver}^{0,1}(e^0) = 1 \otimes t^{-1}\mathrm{d}t$ and the kernel of $\mathrm{ver}^{0,2}$ is generated by $e^+ \wedge e^-$, $e^+ \wedge e^0$ and $e^- \wedge e^0$, so the exact sequence holds here. The kernel of $\mathrm{ver}^{0,3}$ is generated by $e^+ \wedge e^- \wedge e^0$, so this top case also holds. ◇

It might be thought that the exact sequence in the statement of Lemma 5.60 was only true in a near-classical case such as with q-commutation relations between the basis 1-forms. This is not the case, as the following example shows for a quantum bundle based on symmetric groups with their standard calculi from Proposition 1.63.

Example 5.64 In the general construction of Example 5.49, we take $G = S_n \subset X = S_{n+1}$ where the latter is the group of permutations of $\{1, 2, \ldots, n+1\}$ and G leaves

$n+1$ fixed. We take the standard 2-cycles calculi on S_n, S_{n+1} so that $\mathcal{C}_X$ is the set of all transpositions in X and $\mathcal{C} = \mathcal{C}_X \cap G$ is the set of transpositions not containing $n+1$, thereby satisfying the conditions in Example 5.49 for a homogeneous quantum principal bundle. Here G is not normal in X so this is not the special case treated before (and in fact turns out to give a strong bundle, which was not the case for the concrete example given before). Right multiplication by S_n gives us the coaction $\Delta_R : \Bbbk(S_{n+1}) \to \Bbbk(S_{n+1}) \otimes \Bbbk(S_n)$ with differential on $\Omega^1(S_{n+1})$ given by

$$\Delta_{R*}e_{(ij)} = \sum_{g \in G} e_{g(ij)g^{-1}} \otimes \delta_g + 1 \otimes e_{(ij)}\pi_*((ij)), \tag{5.31}$$

where $\pi_*((ij)) = 1$ if $(ij) \in \mathcal{C}$ and zero otherwise. We want to show that this extends to the full exterior algebra in Proposition 1.63 and to prove this we use that they are quotients by the relations $e_a^2 = 0$ (where a is a transposition) of the maximal prolongations of the respective first-order calculi. We already know from Lemma 4.29 (or rather its right-handed version) that Δ_{R*} extends to the exterior algebra using the maximal prolongations on $\Bbbk(S_{n+1})$ and $\Bbbk(S_n)$. We need to check that the additional relations still give a differentiable action, i.e., that $\Delta_{R*}(e_a^2) = 0$ on applying the relations. This can be verified from (5.31) on accounting for the sign in the tensor product of DGAs. Thus we have a quantum principal bundle with our standard exterior algebras and differentiable extended coaction. The base can be identified as $A = \Bbbk(\mathbb{Z}_{n+1})$ since every element of S_{n+1} can be uniquely expressed as the product of a power of the cycle $(12\cdots n+1)$ and an element of S_n. Its calculus at first order is the universal one, being n-dimensional.

Next we check the exactness condition needed to apply Theorem 5.61 so that we have a differential fibration. Using $\mathcal{D}$ as notation for the transpositions containing $n+1$, we have a disjoint union $\mathcal{C}_X = \mathcal{C} \sqcup \mathcal{D}$. For the set ${\mathcal{C}_X}^k$ of words of length k in the alphabet $\mathcal{C}_X$, we define $c_s : {\mathcal{C}_X}^k \to \mathbb{N} \cup \{0\}$ for $1 \le s \le k$ by

$$c_s(\underline{w}) = \begin{cases} r \text{ if there are } r \text{ elements of } \mathcal{C} \text{ before the } s^{\text{th}} \text{ element of } \mathcal{D} \text{ in } \underline{w} \\ 0 \text{ if there is no } s^{\text{th}} \text{ element of } \mathcal{D} \text{ in } \underline{w}. \end{cases}$$

We write any element of $\Lambda^k(S_{n+1})$ explicitly as a sum of monomials in e_a of length k (i.e., words as previously mentioned), and then we perform rewrite rules, which are the relations in $\Lambda^2(S_{n+1})$ from Proposition 1.63,

$$e_{(ij)} \wedge e_{(n+1\,k)} \mapsto -e_{(n+1\,k)} \wedge e_{(ij)},$$
$$e_{(ij)} \wedge e_{(n+1\,i)} \mapsto -e_{(n+1\,i)} \wedge e_{(n+1\,j)} - e_{(n+1\,j)} \wedge e_{(ij)},$$

for $\{n+1, i, j, k\}$ all different. We choose the rewrites successively to decrease c_1 of the monomials to 0. Each use of a rule may increase the number of monomials, but each new monomial has a smaller c_1 than the old monomial. Then we reduce c_2 to zero and so on, until all the transpositions containing $n+1$ have been ordered before the transpositions which do not contain $n+1$. Denoting by $\Lambda_{\mathcal{D}}$ the subalgebra of wedge products of the e_a with $a \in \mathcal{D}$, we have

$$\Lambda^m(S_{n+1}) = \Lambda^m_{\mathcal{D}} + \Lambda^{m-1}_{\mathcal{D}} \wedge \Lambda^1(S_n) + \Lambda^{m-2}_{\mathcal{D}} \wedge \Lambda^2(S_n) + \cdots + \Lambda^m(S_n).$$

From this it follows that the exact sequence condition in Lemma 5.60 holds for all $m \geq 1$. Hence we have a differential fibration by Theorem 5.61. It follows that Corollary 5.62 applies since H_{dR} is left-invariant by Proposition 4.25 (since there is an integral and the calculus is connected).

Exercise E5.6 shows that this construction gives a strong bundle, and in addition the invariant integral for $\Bbbk(S_n)$ implies that the free finitely-generated right module of weakly horizontal forms contains Ω^1_A as a complemented right submodule, and hence that Ω^1_A is right fgp and flat. The particular case of $S_2 \subseteq S_3$ and its Leray–Serre spectral sequence is considered in Exercise E5.7. ◇

5.6 Quantum Homogeneous Spaces as Framed Quantum Spaces

In this section we are going to apply our machinery of quantum principal bundles to quantise the notion of 'frame bundle' as our first approach to quantum Riemannian geometry. When it works, it gives more geometrical insight than simply writing down a linear connection on Ω^1 as we do in Chap. 8. We first explain the classical set-up but in a slightly generalised form where the frame group can be a more general Lie group G and where the metric need not be symmetric. Thus, we let M be a manifold and $\pi : P \to M$ a principal bundle with structure group G and suppose that there is a left G-module V and a bundle isomorphism $T^*M \cong E = P \times_G V$ of the cotangent bundle with the associated bundle. Normally one would speak in terms of the tangent bundle, but we prefer to focus on the cotangent bundle. As above, we will work at the level of sections where $\Gamma(E) = C^\infty_G(P, V)$, the G-equivariant functions on P with values in V. We similarly say that an element of $\Omega^1(P, V^*)$ (i.e., a 1-form on P with values in V^*) is *tensorial* if it is both horizontal and equivariant for the G action, where G acts on 1-forms from the left as induced by the right action on P.

Lemma 5.65 *$\theta \in \Omega^1_{\text{tensorial}}(P, V^*)$ are in correspondence with bundle maps $E \to T^*M$ and, at the level of sections, with $C^\infty(M)$-module maps*

$$C^\infty_G(P, V) \to \Omega^1(M).$$

Proof Explicitly, a V-valued equivariant function ϕ maps to the one form which at x has values $(\pi^*)^{-1}\langle \phi_p, \theta_p \rangle$ where we chose any $p \in \pi^{-1}(x)$ and identify $\langle \phi_p, \theta_p \rangle$ as in the image of $\pi^* : T^*_x M \to T^*_p P$. Here $\langle\ ,\ \rangle$ denotes the evaluation of V with V^*. In functional terms it means ϕ maps to $\langle \phi, \theta \rangle$ with pointwise product. □

Example 5.66 On any manifold M, a frame at $x \in M$ is a linear isomorphism $\mathbb{R}^n \to T_xM$ (i.e., a choice of basis of T_xM). The frame bundle $P = FM$ is a principal bundle over M with structure group $G = GL_n$ and fibre over x given by the set of all frames at x. In a local patch where the bundle is trivial, we denote by $\pi_2(p) : \mathbb{R}^n \to T_{\pi(p)}M$ the corresponding frame. The canonical 1-form $\theta \in \Omega^1_{\rm tensorial}(P, \mathbb{R}^n)$ is locally defined by

$$\theta_p(X) = \pi_2(p)^{-1}\pi_*(X)(\pi(p))$$

for all $X \in TP$. One may check that it is tensorial and globally defined.

One also has different examples with different structure groups. For example, one may take the bundle of affine frames with structure group $\mathbb{R}^n{>\!\!\!\triangleleft} GL_n$ or, for a manifold admitting a metric, the bundle of orthogonal frames with structure group O_n. In noncommutative geometry, we do not fix the structure quantum group as we might have several different candidates, but this freedom is also useful classically.

Definition 5.67 Let M be a manifold. We call any (P, G, V, θ) inducing an isomorphism via Lemma 5.65 a *frame resolution* of the tangent bundle of M, or G*-framing*.

Given a frame resolution (P, G, V, θ), other tensor products of 1-forms and vector fields on M may then be similarly expressed as V or V^*-valued equivariant functions on P. For example,

$$\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M) \cong \Omega^1_{\rm tensorial}(P, V), \tag{5.32}$$

$$\Omega^1(M) \otimes_{C^\infty(M)} \Omega^{-1}(M) \cong \Omega^1_{\rm tensorial}(P, V^*), \tag{5.33}$$

$$\Omega^2(M) \otimes_{C^\infty(M)} \Omega^1(M) \cong \Omega^2_{\rm tensorial}(P, V), \tag{5.34}$$

$$\Omega^2(M) \otimes_{C^\infty(M)} \Omega^{-1}(M) \cong \Omega^2_{\rm tensorial}(P, V^*). \tag{5.35}$$

The canonical form $\theta \in \Omega^1_{\rm tensorial}(P, V^*)$ corresponds under (5.33) to the constant section of $\Omega(M)\otimes_{C^\infty(M)}\Omega^{-1}(M)$ given over each point x by the canonical element of $T^*_xM \otimes T_xM$ (here $\Omega^{-1}(M)$ denotes vector fields). For another example, when P is equipped with a connection ω, it induces as we have seen a covariant derivative on sections of any associated bundle, which throughout this section we will denote by D, and in particular we can transfer this over using the framing isomorphism to a connection on $\Omega^1(M)$ and associated tensors, for which we will reserve the symbol ∇. Here $\nabla\eta = \langle D\phi, \theta\rangle$ for $\eta = \langle\phi, \theta\rangle \in \Omega^1(M)$ corresponding to $\phi \in C^\infty_G(P, V)$. Its curvature $R_\nabla\eta \in \Omega^2(M) \otimes_{C^\infty(M)} \Omega^1(M)$ similarly corresponds under (5.34) to $F_\omega.\phi \in \Omega^2_{\rm tensorial}(P, V)$, where F_ω is a 2-form with Lie algebra value acting on V. Similarly, we have an induced $D\wedge$ on form-valued sections of which $\theta \in \Omega^1_{\rm tensorial}(P, V^*)$ is itself an example.

Proposition 5.68 *Under the framing isomorphisms* $-D\wedge\theta\in\Omega^2(P,V^*)$ *corresponds under (5.35) to the torsion tensor* $T_\nabla\in\Omega^2(M)\otimes_{C^\infty(M)}\Omega^{-1}(M)$.

Proof If $\eta\in\Omega^1(M)$ corresponds to $\phi\in C^\infty_G(P,V)$ under the correspondence in Lemma 5.65 and we use a similar correspondence for $\nabla\wedge\eta$, we have $\pi^*\nabla\wedge\eta=(D\phi)\wedge\theta=(\mathrm{d}\phi)\wedge\theta+\omega.\phi\wedge\theta=\mathrm{d}(\phi\theta)-\phi\mathrm{d}\theta-\phi\omega\wedge\theta=\pi^*\mathrm{d}\eta-\phi D\wedge\theta$, as required, where the pairing between V,V^* should be understood and where the action of the Lie algebra value of ω on V is minus the action on V^*. □

To consider metrics, a standard approach is to identify $\Omega^1(M)\otimes_{C^\infty(M)}\Omega^1(M)$ with $C^\infty_G(P,V\otimes V)$ by a framing isomorphism similar to the one for $\Omega^1(M)$ and implied by it. A connection ω on P in this case induces a covariant derivative in the vector bundle associated to $V\otimes V$ and a metric appears as a flat section of this. We do not need the metric to be symmetric but the section should be pointwise invertible. The existence of a flat section imposes conditions on the holonomy, for example we can look at functions which are given by a constant G-invariant element $\eta\in V\otimes V$, which implies a constraint on the structure group G. Normally, one would assume that the section values are symmetric so that g is symmetric but we do not necessarily assume this and refer to g as a *generalised metric* for this reason.

We now describe an alternative approach which is more symmetric between tangent and cotangent spaces and hence more compatible with self-duality ideas in noncommutative geometry. Namely, using (5.32), we can think of g as equivalent to an element $\theta^*\in\Omega^1_{\rm tensorial}(P,V)$ which looks much like the data we assumed for a framing but with V swapped with V^*. The correspondence is via the formula

$$g=\langle\theta^*,\theta\rangle\in\Omega^1(M)\otimes_{C^\infty(M)}\Omega^1(M),$$

where we evaluate V^* with V. By a version of Lemma 5.65, such θ^* correspond to bundle maps $E^*=P\times_G V^*\to T^*M$ or dually $TM\to P\times_G V=E\cong T^*M$ (as we already assumed a framing θ). Here g being nondegenerate as a map $TM\to T^*M$ corresponds to (θ^*,V^*) being another framing, which we will call a *coframing*. Thus, G-framed manifolds equipped with a nondegenerate not necessarily symmetric 1–1 tensor g (i.e., a generalised metric) are equivalent to manifolds which are both framed and coframed by data (P,G,V,θ,θ^*). This is equivalent to our first point of view, where g corresponds to a section η with values in $V\otimes V$ by $\theta^*=(\theta\otimes\mathrm{id})\eta$. Also in this situation, a connection ω on P induces *two* connections, one the original connection ∇ via the framing and another, ∇^*, via θ^*. This leads to the concept of the cotorsion of ∇ as, essentially, the torsion of ∇^*.

Lemma 5.69 *When M is both framed and coframed and ω a connection on the framing bundle, the two induced connections on $\Omega^1(M)$ are related by* $(\nabla^*_X\otimes\mathrm{id})g+(\mathrm{id}\otimes\nabla_X)g=0$ *for all vector fields X. The* cotorsion *of ∇ is*

$$\mathrm{co}T_\nabla=-(T_{\nabla^*}\otimes\mathrm{id})g=(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla)g\in\Omega^2(M)\otimes_{C^\infty(M)}\Omega^1(M),$$

where T_{∇^*} is the torsion of ∇^*, and corresponds under (5.35) to $D\wedge\theta^*$.

Proof That the torsion of ∇^* corresponds to $D \wedge \theta^*$ is clear from the swapped roles of θ, θ^*. Assuming ∇, ∇^* are related as stated and using the shorthand $g = g^1 \otimes g^2$ and $\nabla\xi = \xi_{(\bar{1})} \otimes \xi_{(\bar{\infty})}$ for $\xi \in \Omega^1(M)$ (with sum of terms, and tensor products over $C^\infty(M)$ understood), we have

$$T_{\nabla^*} g^1 \otimes g^2 = (\nabla^* \wedge g^1 - \mathrm{d}g^1) \otimes g^2 = -g^2{}_{(\bar{1})} \wedge g^1 \otimes g^2{}_{(\bar{\infty})} - \mathrm{d}g^1 \otimes g^2 = -\mathrm{co}T_\nabla.$$

It remains to check ∇^*. Here $\psi \in C^\infty_G(P, V^*)$ corresponds to a 1-form $\langle\theta^*, \psi\rangle$ and $\nabla^*_X\langle\theta^*, \psi\rangle = \langle\theta^*, D_{\tilde{X}}\psi\rangle = \langle\theta^*, i_{\tilde{X}}(\mathrm{d}\psi + \omega\psi)\rangle$ for any lift $\tilde{X}$ of a vector field on M to one on P. We write $\theta = \psi_i\zeta^i$ and $\theta^* = \eta^i\phi_i$ with $\eta^i, \zeta^i \in \Omega^1(M)$ so that $g = \langle\theta^*, \theta\rangle = \eta^i\langle\phi_i, \psi_j\rangle \otimes \zeta_j$, where $\langle\phi_i, \psi_j\rangle \in C^\infty(M)$. Then

$$\begin{aligned}
&\nabla^*_X g^1 \otimes g^2 + g^1 \otimes \nabla_X g^2 = \nabla^*_X(\eta^i\langle\phi_i, \psi_j\rangle) \otimes \zeta^j + \eta^i\langle\phi_i, \psi_j\rangle \otimes \nabla_X\zeta^j \\
&= \nabla^*_X(\eta^i\langle\phi_i, \psi_j\rangle) \otimes \zeta^j + \eta^i \otimes \nabla_X(\langle\phi_i, \psi_j\rangle\zeta^j) - \eta^i \otimes X(\langle\phi_i, \psi_j\rangle)\zeta^j \\
&= \langle\theta^*, i_{\tilde{X}}(\mathrm{d}\psi_j + \omega.\psi_j\rangle \otimes \zeta^i + \eta^i \otimes \langle i_{\tilde{X}}(\mathrm{d}\phi_i + \omega.\phi_i), \theta\rangle - \eta^i \otimes X(\langle\phi_i, \psi_j\rangle)\zeta^j \\
&= \langle\theta^*, i_{\tilde{X}}(\omega).\theta\rangle + \langle i_{\tilde{X}}(\omega).\theta^*, \theta\rangle = 0,
\end{aligned}$$

where we used the derivation rule for $X(\langle\phi_i, \psi_j\rangle)$ and at the end the Lie algebra value of ω acts with a minus sign on V^* compared to its action on V. □

One can take the formulae stated in the lemma as definitions of the adjoint ∇^* and cotorsion of any connection ∇ on Ω^1 in the presence of a generalised (i.e., not necessarily symmetric) metric on a manifold, i.e., without the above framing context in which it arises. This amounts to a generalisation of conventional Riemannian geometry, which can be translated back to vector fields rather than forms.

Corollary 5.70 *Let M be a manifold with linear connection ∇ and a generalised (i.e., not necessarily symmetric) metric g and let ∇^* be defined by*

$$X(g(Y, Z)) = g(\nabla^*_X Y, Z) + g(Y, \nabla_X Z).$$

Then the cotorsion tensor co$T$ *of* ∇ *defined as the torsion tensor of* ∇^* *obeys*

$$g(\mathrm{co}T(X, Y), Z) = (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, \ ) + g(T(X, Y), Z)$$

for vector fields X, Y, Z. We say that ∇ is a weak Levi-Civita connection *for g if it is torsion and cotorsion free, i.e., it is torsion free and*

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0$$

for all X, Y, Z. In local coordinates, this is $g_{\mu\nu;\rho} = g_{\rho\nu;\mu}$.

Proof This follows from the framing and coframing theory but we now verify it directly. Thus

$$
\begin{aligned}
g(\mathrm{co}T(X,Y),Z) &= g(\nabla^*_X Y - \nabla^*_Y X - [X,Y], Z)\\
&= X(g(Y,Z)) - g(Y,\nabla_X Z) - Y(g(X,Z)) + g(X,\nabla_Y Z) - g([X,Y],Z)\\
&= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(\nabla_X Y - \nabla_Y X - [X,Y], Z).
\end{aligned}
$$

We also check that this agrees with our previous formula on forms. Using similar computations as in the proof of Example 3.29, we have

$$
\begin{aligned}
& i_Y i_X \mathrm{co}T_\nabla = i_Y i_X(\mathrm{d}\otimes\mathrm{id} - \mathrm{id}\wedge\nabla)g = (i_Y i_X \mathrm{d}g^1)g^2 - i_Y i_X(g^1\wedge g^2{}_{(\bar{1})})g^2{}_{(\bar{\infty})}\\
&= (X(\langle Y,g^1\rangle) - Y(\langle X,g^1\rangle) - \langle [X,Y],g^1\rangle)g^2\\
&\quad - (\langle X,g^1\rangle\langle Y,g^2{}_{(\bar{1})}\rangle - \langle Y,g^1\rangle\langle X,g^2{}_{(\bar{1})}\rangle)g^2{}_{(\bar{\infty})}\\
&= (X(\langle Y,g^1\rangle) - Y(\langle X,g^1\rangle) - \langle [X,Y],g^1\rangle)g^2 + \langle Y,g^1\rangle\nabla_X g^2 - \langle X,g^1\rangle\nabla_Y g^2\\
&= (\nabla_X g)(Y,\ ) - (\nabla_Y g)(X,\ ) + \langle g^1,\nabla_X Y\rangle g^2 - \langle g^1,\nabla_Y X\rangle g^2 - \langle [X,Y],g^1\rangle g^2\\
&= (\nabla_X g)(Y,\ ) - (\nabla_Y g)(X,\ ) + \langle g^1, T(X,Y)\rangle g^2
\end{aligned}
$$

where we recognised

$$
\begin{aligned}
(\nabla_X g)(Y,) &= \langle \nabla_X g^1, Y\rangle g^2 + \langle Y,g^1\rangle\nabla_X g^2\\
&= X(Y(g^1))g^2 - \langle g^1,\nabla_X Y\rangle g^2 + \langle Y,g^1\rangle\nabla_X g^2.
\end{aligned}
$$

Note that $\langle T(X,Y),\ \rangle = i_X i_Y T_\nabla$ in Example 3.29 and similarly $g(\mathrm{co}T(X,Y),\ ) = i_X i_Y T_{\nabla^*}(g^1)g^2 = i_Y i_X \mathrm{co}T_\nabla$ in our sign conventions. □

We should not expect that a 'weak Levi-Civita connection' necessarily exists if the generalised metric is not symmetric, nor that it is unique; rather we regard the metric and the connection as independent data with a compatibility condition. The theory essentially includes, however, symplectic geometry as an extreme case.

Example 5.71 If g is antisymmetric and there exists a torsion free and cotorsion free (or weak Levi-Civita) linear connection then $\mathrm{d}g = 0$, i.e., g is symplectic. This is immediate from $\mathrm{co}T_\nabla = (\mathrm{d}\otimes\mathrm{id} - \mathrm{id}\wedge\nabla))g$ projected to $\Omega^3(M)$ and using torsion freeness again in the form $\wedge\nabla = \mathrm{d}$. ◇

Although a framing is not necessary at this level, it is the setting where the full symmetry between torsion and cotorsion in the above generalised Riemannian geometry is realised. Indeed, we are free to reverse the roles of θ and θ^* (and V^* and V) in all of the above, regarding θ^* as the frame resolution and θ as corresponding to a generalised metric. From this point of view, it is natural to replace $D\theta = 0$ and $\nabla g = 0$ by more 'self-dual' conditions $D\theta = 0$, $D\theta^* = 0$ which is our notion of weak Levi-Civita connection as above when seen from the point of view of ∇.

Finally, when the frame resolution bundle is trivial, M must be parallelisable. Indeed, equivariant functions in $C^\infty_G(P, V)$ correspond for trivial bundles to functions $C^\infty(M, V)$ on the base with values in V as we have seen even in the quantum case. Then $\theta \in \Omega^1_{\mathrm{tensorial}}(P, V^*)$ corresponds similarly to a *bein* (or V-bein) $e \in \Omega^1(M, V^*)$. If we chose a basis of V then this just means an n-bein where n is the dimension of M and of V. Choosing an n-bein e is equivalent to choosing an isomorphism $\Omega^1(M) \cong C^\infty(M, V)$ where we send a 1-form to its coefficients in the n-bein basis. Similarly, θ^* corresponds to a *cobein* $f \in \Omega^1(M, V)$ and $g = f \otimes_{C^\infty(M)} e$ is the metric or generalised metric. The vanishing of torsion and cotorsion for a connection on P given by a gauge field α on the base is then the symmetric condition

$$D_\alpha \wedge e = 0, \quad D_\alpha \wedge f = 0, \tag{5.36}$$

where D_α denotes the associated covariant derivative.

Example 5.72 If G is a semisimple Lie group, consider $P = G \times G$ as a principal G-bundle (where G acts on the right factor from the right). We take $V^* = \mathfrak{g}$ with the adjoint action and let $e = \omega$, the canonical Maurer–Cartan form on G with values in $\mathfrak{g}$. We let $f = \eta \circ e$, where η is the Killing form. This defines the usual Killing metric on G (Riemannian in the compact case). One also has a natural gauge field $\alpha = \omega$ which has zero curvature $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$ but both torsion and cotorsion. Another choice is $\alpha = \frac{1}{2}\omega$, which has curvature but is then torsion and cotorsion free and in fact metric compatible, i.e. the Levi-Civita connection for the Killing metric. ◇

5.6.1 *Framed Quantum Manifolds*

With the above theory of G-framed manifolds as motivation, we are clearly in position to write down the quantum version of these ideas. Here we consider an algebra A equipped with a differential structure (Ω^1, d) as in Chap. 1 in place of the 'manifold'. In contrast to the classical geometry in the preceding section, we will work in the quantum case entirely with differential forms, albeit we recall from §2.7 that there is still a notion of vector fields $\Omega^{-1} = \mathfrak{X}^L = {}_A\mathrm{Hom}(\Omega^1, A)$, and we will have to be more careful about left and right versions. *Unadorned Ω will refer throughout this section to differential forms on A.* We will also need the right dual $V^\flat$ in the category of finite-dimensional right comodules V over a Hopf algebra H, see §2.4, with evaluation $V \otimes V^\flat \to \Bbbk$ and coevaluation $\Bbbk \to V^\flat \otimes V$.

Lemma 5.73 *Let (P, H, Δ_R) be a regular quantum principal bundle with $A = P^H$, V a finite-dimensional right H-comodule and $\theta : V \to P\Omega^1$ equivariant. There is an induced left A-module map $s_\theta = \cdot(\mathrm{id} \otimes \theta) : (P \otimes V)^H \to \Omega^1$. If this is an isomorphism then Ω^1 is fgp and*

$$(\Omega^n P \otimes V)^H \cong \Omega^n \otimes_A \Omega^1, \quad (V^\flat \otimes P\Omega^n)^H \cong {}_A\mathrm{Hom}(\Omega^1, \Omega^n).$$

Proof By construction, $s_\theta : (P\otimes V)^H \to P\Omega^1$ and its image is invariant under Δ_R by equivariance, hence in Ω^1 as in the proof of Corollary 5.53. The first stated isomorphism follows from $(\Omega^n P\otimes V)^H \cong \Omega^n \otimes_A (P\otimes V)^H$, which similarly holds using the strong universal connection $\omega^\sharp(h) = h^{(1)}\otimes h^{(2)}$ assumed as part of the regularity of P. We apply $\mathrm{id}\otimes s_\theta$ to the right-hand side.

We next show that $(P\otimes V)^H$ as a left A-module is fgp with right dual $(V^\flat\otimes P)^H$ and hence that the latter is isomorphic to ${}_A\mathrm{Hom}(\Omega^1, A)$ as a right A-module. We can certainly map $(P\otimes V)^H\otimes (V^\flat\otimes P)^H \to P$ by evaluation of V with $V^\flat$ and the product in P, and the result actually lies in A due to the equivariance properties. For the 'coevaluation', write e_i a basis of $V^\flat$ and f^i the dual basis in V. Then define

$$\mathrm{coev} = e_{i\,\bar{(0)}}\otimes e_{i\,\bar{(1)}}{}^{(1)}\otimes e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i$$

using the strong connection. We check from the properties (5.9) of $\omega^\sharp$ that this actually lies in $(V^\flat\otimes P)^H\otimes (P\otimes V)^H$. For example on the left side,

$$\begin{aligned}(\Delta_R\otimes\mathrm{id})\mathrm{coev} &= e_{i\,\bar{(0)}\bar{(0)}}\otimes e_{i\,\bar{(1)}}{}^{(1)}{}_{\bar{(0)}}\otimes e_{i\,\bar{(0)}\bar{(1)}}e_{i\,\bar{(1)}}{}^{(1)}{}_{\bar{(1)}}\otimes e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i\\
&= e_{i\,\bar{(0)}\bar{(0)}}\otimes e_{i\,\bar{(1)}(2)}{}^{(1)}\otimes e_{i\,\bar{(0)}\bar{(1)}}(Se_{i\,\bar{(1)}(1)})\otimes e_{i\,\bar{(1)}(2)}{}^{(2)}\otimes f^i\\
&= e_{i\,\bar{(0)}}\otimes e_{i\,\bar{(1)}}{}^{(1)}\otimes 1\otimes e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i.\end{aligned}$$

On the right side, we need to use $e_i\otimes f^i\in (V^\flat\otimes V)^H$ as the coevaluation map $\Bbbk\to V^\flat\otimes V$ is an H-comodule map. Now we check that

$$\begin{aligned}&(\mathrm{ev}\otimes\mathrm{id})(p\otimes v\otimes e_{i\,\bar{(0)}}\otimes e_{i\,\bar{(1)}}{}^{(1)}\otimes e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i)\\
&\qquad = pe_{i\,\bar{(1)}}{}^{(1)}e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i e_{i\,\bar{(1)}}(v) = p\otimes f^i e_i(v) = p\otimes v\end{aligned}$$

for all $p\otimes v\in (P\otimes V)^H$. This shows that $E=(P\otimes V)^H$ is fgp with dual bases $e_{i\,\bar{(1)}}{}^{(2)}\otimes f^i\in E$ and $e_{i\,\bar{(0)}}\otimes e_{i\,\bar{(1)}}{}^{(1)}$ viewed in $E^\flat$ via the evaluation map. By the left version of the remark after Definition 3.1, these span all of $E^\flat$, which we therefore identify with $(V^\flat\otimes P)^H$. Finally, $(V^\flat\otimes P\Omega^n)^H\cong {}_A\mathrm{Hom}(\Omega^1, A)\otimes_A\Omega^n$, and by the usual theory of fgp modules this is ${}_A\mathrm{Hom}(\Omega^1,\Omega^n)$. □

Similarly, with the roles of $V, V^\flat$ and left and right swapped, we have a notion of a 'coframing'. Motivated by the preceding classical theory, we think of a 'quantum manifold' as an algebra equipped a differential calculus and a framing, and 'generalised quantum Riemannian manifold' as one with in addition a coframing.

Definition 5.74 A *framing* of A is a regular quantum principal bundle (P, H, Δ_R) with $A = P^H$ and the antipode of H invertible, a finite-dimensional right H-comodule V and an equivariant map $\theta : V\to P\Omega^1$ such that s_θ in Lemma 5.73 is an isomorphism of left A-modules. A *coframing* is a regular quantum principal bundle, a finite-dimensional right comodule $V^\flat$ and an equivariant map $\theta^* : V^\flat \to \Omega^1 P$ with $s_{\theta^*} = (\theta^* \otimes \mathrm{id}) : (V^\flat \otimes P)^H \to \Omega^1$ an isomorphism of right A-modules.

Next, in view of Lemma 5.73, a framed and coframed algebra is the same as a framed algebra with generalised quantum metric

$$g \in \Omega^1 \otimes_A \Omega^1, \quad g = \langle \theta^*, \theta \rangle,$$

where we apply $\theta^\flat \otimes \theta$ to the coevaluation element in $V^\flat \otimes V$ and land in $\Omega^1 P \otimes_P P\Omega^1 = \Omega^1 \otimes_A P \otimes_A \Omega^1$. By equivariance properties and similar arguments using the strong universal connection as in the proof of the lemma, the result actually lies in $\Omega^1 \otimes_A A \otimes_A \Omega^1 = \Omega^1 \otimes_A \Omega^1$. Here g being nondegenerate is expressed as s_{θ^*} being an isomorphism; the latter implies that $(V^\flat \otimes P)^H \cong \Omega^1$, which we know from the lemma is isomorphic to $\Omega^{-1} := {}_A\mathrm{Hom}(\Omega^1, A)$. There is, however, no particular reason to demand quantum symmetry $\wedge(g) = 0$ from this point of view.

In the framed setting, a connection form $\omega : \Lambda^1_H \to \Omega^1_P$ induces a left connection D on $(P \otimes V)^H$ and hence a connection ∇ on Ω^1 via s_θ. It also induces a right connection D_R on the dual bundle $(V^\flat \otimes P)^H$ and, moreover, we can view $\theta \in (V^\flat \otimes P\Omega^1)^H = (V^\flat \otimes P)^H \otimes_A \Omega^1$ as a form-valued section of this dual bundle. This entails a theory of associated bundles with left and right reversed in the tensor factors, with principal bundle $P_R = P$ as an algebra but with an $H_R = H^{op}$ coaction $\Delta_L p = S^{-1} p_{(\bar 1)} \otimes p_{(\bar 0)} = p_{(\bar 1)} \otimes p_{(\bar\infty)}$ (the same formula by which we view a right coaction on V as a left coaction $\Delta_L v = v_{(\bar 1)} \otimes v_{(\bar\infty)}$). See Exercise E5.9 for a closer look at this reversed bundle. If we also have a coframing then D_R can be viewed via s_{θ^*} as a right connection ∇^* on Ω^1 and θ^* can be viewed as an element of $(\Omega^1 P \otimes V)^H = \Omega^1 \otimes_A (P \otimes V)^H$, i.e., a form-valued section of $(P \otimes V)^H$.

Theorem 5.75 *Let A be framed by (P, V, θ), ω a strong connection on P, D the induced covariant derivative on the associated bundle $(P \otimes V)^H$, and ∇ defined on Ω^1 by the isomorphism $s_\theta : (P \otimes V)^H \to \Omega^1$.*

(1) $T_\nabla = \wedge\nabla - \mathrm{d}$ *corresponds via framing isomorphisms to* $-(\mathrm{id} \otimes \mathrm{d} + D_R \wedge \mathrm{id})\theta$.
(2) $R_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$ *corresponds to the curvature of D (via F_ω).*

If there is also a coframing θ^* and we let $g = \langle \theta^*, \theta \rangle$ then

(3) $(\nabla^* \otimes \mathrm{id} + \mathrm{id} \otimes \nabla)g = 0$.
(4) $\mathrm{co}T_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)g$ *corresponds to* $(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge D)\theta^*$.

Proof (1) Using s_θ and induced framing isomorphisms as in the lemma, we write $\xi \in \Omega^1$ as corresponding to $p \otimes v \in (P \otimes V)^H$ and $X \in \Omega^{-1}$ to $x \otimes q \in (V^\flat \otimes P)^H$ (sum of such terms understood). Then $\nabla\xi$ corresponds to $D(p \otimes v) = (\mathrm{id} - \Pi_\omega)\mathrm{d}p \otimes v$ as usual for an associated bundle, while on the other side the right connection $\tilde\nabla X$ corresponds to $D_R(x \otimes q) = x \otimes (\mathrm{id} - \Pi_{R\omega})\mathrm{d}q$ using the reversed bundle theory on P_R. Given a strong connection ω, we use the left P-module splitting $\Pi_\omega(\mathrm{d}p) = p_{(\bar 0)}\omega\pi_\epsilon p_{(\bar 1)}$ as usual and a right P-module spitting $\Pi_{R\omega}(\mathrm{d}p) = -\omega(\varpi\pi_\epsilon p_{(\bar 1)})p_{(\bar\infty)}$ in the right-handed theory.

We now check that $\tilde{\nabla}$, ∇ are dual in the sense of Proposition 3.32 (so, although $\tilde{\nabla}$ has a tilde to emphasize that it is a right connection, it is just ∇ on Ω^1 in its dual form on vector fields). Writing evaluation as a bimodule map $\mathrm{ev} : \Omega^1 \otimes \Omega^{-1} \to A$, this is

$$(\mathrm{id} \otimes \mathrm{ev})(\nabla\xi \otimes X) + (\mathrm{ev} \otimes \mathrm{id})(\xi \otimes \tilde{\nabla} X) = \mathrm{d} \circ \mathrm{ev}(\xi \otimes X)$$

which translates to

$$((\mathrm{id} - \Pi_\omega)(\mathrm{d}p)q + p(\mathrm{id} - \Pi_{R\omega})(\mathrm{d}q))x(v) = \mathrm{d}(pq)x(v)$$

or $\Pi_\omega(\mathrm{d}(x(v)p))q + x(v)p\Pi_{R\omega}(\mathrm{d}q) = 0$ where (the sum of terms) $px(v) \otimes q$ is an invariant element of $P \otimes P$ (since it is obtained from an invariant element of $P \otimes V \otimes V^\flat \otimes P$ by a covariant operation.) Invariance also means $x(v)p_{(\bar{0})} \otimes p_{(\bar{1})} \otimes q = x(v)p \otimes q_{(\bar{1})} \otimes q_{(\bar{\infty})}$ which implies the result on using the formulae for Π_ω, $\Pi_{R\omega}$.

Now note that θ under the framing isomorphism in the lemma is just a 'coevaluation' dual bases element $e_i \otimes e^i \in \Omega^{-1} \otimes_A \Omega$ for Ω^1 as left fgp (summation understood), where $\mathrm{ev}(\xi \otimes e_i)e^i = \xi$ for all $\xi \in \Omega^1$. In these terms, $(D_R \wedge \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\theta$ corresponds to $(\tilde{\nabla} e_i) \wedge e^i + e_i \otimes \mathrm{d}e^i$. If we evaluate this against $\xi \in \Omega^1$ on the left, we have by our duality result that

$$\begin{aligned}(\mathrm{ev} \otimes \mathrm{id})&(\xi \otimes \tilde{\nabla} e_i) \wedge e^i + \mathrm{ev}(\xi \otimes e_i)\mathrm{d}e^i \\ &= \mathrm{d}(\mathrm{ev}(\xi \otimes e_i)e^i) - (\mathrm{id} \otimes \mathrm{ev})(\nabla\xi \otimes e_i) \wedge e^i = \mathrm{d}\xi - \wedge\nabla\xi = -T_\nabla \xi.\end{aligned}$$

(2) This is a special case of Proposition 5.48 transferred to Ω^1 via the framing.

(3) If we use s_{θ^*} then $(V^\flat \otimes P)^H$ corresponds to Ω^1 and D_R corresponds to a right-covariant derivative ∇^* on it. Moreover the metric corresponds to the canonical dual bases element of $(V^\flat \otimes P)^H \otimes_A (P \otimes V)^H$ when we use s_{θ^*} on the first factor and s_θ as usual in the second. The statement then reduces to D, D_R dual as proven in part (1) in the equivalent form of preserving the canonical element as in (3.15).

(4) Here $g = (\mathrm{id} \otimes s_\theta)\theta^*$, so this amounts to the definition of ∇ as equivalent by s_θ to the connection D on $(P \otimes V)^H$. □

Underlying these constructions is the idea that a principal bundle equipped with a strong connection induces covariant derivatives on all associated bundles in a functorial manner. Moreover, since Ω^1 is in fact a bimodule, we are interested in when the induced ∇ is a bimodule connection.

Corollary 5.76 *Let A be framed by $(P, H, \Delta_R, V, \theta)$ and suppose that the image of θ commutes with A.*

(1) *If ω is a strong connection such that the connection on P is a bimodule connection with generalised braiding σ_P (as in Lemma 5.50) then the induced ∇ on Ω^1 is a bimodule connection. Here*

$$\sigma(s_\theta(p\otimes v)\otimes\xi)=\zeta\otimes s_\theta(q\otimes v)$$

if $\sigma_P(p\otimes\xi)=\zeta\otimes q\in\Omega^1\otimes_A P$ for $\zeta\in\Omega^1$ and $q\in P$ (sum of terms implicit).

(2) *If the bundle is strong and $\eta: V^\flat\to V$ is an isomorphism of H-comodules (i.e., there is a nondegenerate invariant element $\eta\in V\otimes V$) then $\theta^*=\theta\circ\eta$ is a coframing and there is an induced quantum metric $g=(\theta\otimes\theta)(\eta)$.*

Proof Centrality of the image of θ gives $(s_\theta(p\otimes v))a=p\theta(v)a=pa\theta(v)=s_\theta(pa\otimes v)$. Then Ω^1 is isomorphic as a bimodule to the relatively trivial bimodule structure on $(P\otimes V)^H$ where we multiply P by A from either side. In this case, if the condition in Lemma 5.50 holds, we have a bimodule connection on the associated bundle which we transfer via s_θ. For the metric, clearly an isomorphism η as stated turns θ into a map $V^\flat\to P\Omega^1$ and we assume the bundle is strong to identify this with $\Omega^1 P$. Such a metric is central by the assumption on the image of θ. □

We will cover two general classes of examples. One will be tensor product bundles in the next section and the other will be quantum homogenous space bundles in the remainder of the present section.

Theorem 5.77 (Majid) *Let $\pi: P\to H$ be a regular quantum homogeneous bundle as in Lemma 5.46. Then $A=P^H$ is framed by the bundle and*

$$V=(P^+\cap A)/(I_P\cap A),\quad \Delta_R v=\tilde v_{(2)}\otimes S\pi(\tilde v_{(1)}),\quad \theta(v)=S\tilde v_{(1)}\mathrm{d}\tilde v_{(2)},$$

where $\tilde v$ is a representative of v in $P^+\cap A$. The bimodule structure on Ω^1 is isomorphic via s_θ to left multiplication on $P\otimes V$ and the tensor product right multiplication on $P\otimes V$.

Proof First observe that $v\in A$ means by definition $v_{(1)}\otimes\pi(v_{(2)})=v\otimes 1$. Moreover, if $v\in A$ then we have $v_{(1)}\otimes v_{(2)}\in P\otimes A$ because $v_{(1)}\otimes v_{(2)(1)}\otimes\pi(v_{(2)(2)})=v_{(1)(1)}\otimes v_{(1)(2)}\otimes\pi(v_{(2)})=v_{(1)}\otimes v_{(2)}\otimes 1$, and if $v\in P^+\cap A$ then $\epsilon(v_{(2)})\pi(Sv_{(1)})=\pi(Sv)=S\pi(v_{(2)})\epsilon(v_{(1)})=1\epsilon(v)=0$ so that Δ_R restricts to $P^+\cap A$ in the first place. Similarly,

$$\Delta_R v=v_{(1)}\otimes\pi(Sv_{(1)})=v_{(1)(2)}\otimes\pi(Sv_{(1)(1)})\pi(v_{(2)})=v_{(2)}\otimes\pi(Sv_{(1)}v_{(3)}),$$

which is the projected adjoint action. I_P is stable under this, hence if $v\in I_P\cap A$ then $\Delta_R v\in I_P\cap A\otimes H$ and Δ_R descends to V. Meanwhile, if $v\in I_P$ then $S\tilde v_{(1)}\otimes\tilde v_{(2)}\in N_P$ and hence $\theta(v)=0$ in Ω^1_P, so this is well defined. Moreover, if $\tilde v\in A$ is a representative of $v\in V$ then as remarked $\Delta\tilde v\in P\otimes A$ so that $\theta(v)=S\tilde v_{(1)}\mathrm{d}\tilde v_{(2)}\in P\Omega^1$, as required. Hence all maps are defined as required and we have $s_\theta:(P\otimes V)^H\to\Omega^1$. It remains to give its inverse, which we do by quotienting the inverse in the universal calculus case, namely

$$s_\theta^{-1}(a\mathrm{d}b)=ab_{(1)}\otimes[\pi_\epsilon b_{(2)}]$$

for all $a,b\in A$, where the expression in square brackets lies in $P\otimes P^+\cap A$ (again using the observation above) and [ ] denotes the equivalence class modulo $I_P\cap A$ in the second tensor factor. That the result actually lies in $(P\otimes V)^H$ and gives the inverse of s_θ is then a direct verification. Finally, it is easy to see that $(P\otimes V)^H$ has a right action of A as well as the usual left one, making it a bimodule, with

$$s_\theta(p\otimes v).a=pS\tilde v_{(1)}(\mathrm{d}\tilde v_{(2)})a=pa_{(1)}Sa_{(2)}S\tilde v_{(1)}\mathrm{d}(\tilde v_{(2)}a_{(2)})=s_\theta(pa_{(1)}\otimes va_{(2)}),$$

so s_θ maps the bimodule structure to that of Ω^1. That $pa_{(1)}\otimes va_{(2)}$ is H-invariant and that Ω^1 acquires in this setting a left P-coaction can also be checked. □

We also assume for a framing that the antipode of H is invertible, although this is not used in the main part of the theorem itself. It is an underlying assumption, however, for functorial results on associated bundles developed earlier.

Corollary 5.78 *$V=(P^+\cap A)/(I_P\cap A)$ in Theorem 5.77 is a Takeuchi crossed module $V\in{}_A\mathcal{M}_A^H$ such that $\mathfrak{E}(V)=\Omega^1$ as a bimodule by Theorem 5.35. If P is a strong bundle then $V\cong\Lambda^1_{\mathrm{hor}}=\Lambda^1_P\cap P\Omega^1$ (the weakly horizontal forms on P) and an object in ${}_P\mathcal{M}_P^H$ restricting to $V\in{}_A\mathcal{M}_A^H$.*

Proof It is easy to check directly that V is an object in the category ${}_A\mathcal{M}_A^H$ of H-A crossed modules in the sense of Theorem 5.35 by $\Delta_R v$ the push-forward $(\mathrm{id}\otimes\pi)\mathrm{Ad}_R$ on P^+ and $\triangleleft$ the restriction to A of the right action of P on P^+; it is the reason why $(P\otimes V)^H$ is naturally a bimodule, which we saw in the proof of the preceding theorem matches via s_θ to the bimodule structure of Ω^1. It is also clear that this crossed-module structure is inherited ultimately from P^+ a P-crossed module which we saw in Example 5.37 pushed out to an object of ${}_P\mathcal{M}_P^H$, which we then intersect with A and quotient. We can also restrict the Maurer–Cartan form ϖ on P to an injective map $\varpi:(P^+\cap A)/(I_P\cap A)\to\Lambda^1_P$ where the image now lies in the kernel of $\mathrm{ver}:\Omega^1_P\to P\otimes\Lambda^1_H$, i.e., is contained in the horizontal forms. If P is strong then the horizontal forms are $P\Omega^1$ and if such $\xi=p\mathrm{d}a$ (sum of such terms) is also left-invariant then it means $1\otimes p\mathrm{d}a=p_{(1)}a_{(1)}\otimes p_{(2)}\mathrm{d}a_{(2)}$, which implies that $\xi=S(1)p\mathrm{d}a=(S(p_{(1)}a_{(1)}))p_{(2)}\mathrm{d}a_{(2)}=\epsilon(p)(Sa_{(1)})\mathrm{d}a_{(2)}=\varpi\pi_\epsilon(\epsilon(p)a)$. Hence the image of ϖ is all of the space Λ^1_{hor} of horizontal elements of Λ^1_P. Now Λ^1_P carries a right P-action since the calculus on P is assumed to be left-covariant, given by $\xi\triangleleft p=(Sp_{(1)})\xi p_{(2)}$, and such conjugation clearly preserves horizontality, so the action restricts to Λ^1_{hor}. We also have a right H-coaction on P as part of the bundle and hence on the horizontal forms in the strong bundle case. This right coaction commutes with the left P-coaction hence also restricts to Λ^1_{hor}. Equivalently, the right H-coaction on Ω^1_P as part of the bundle commutes with the left P-coaction and hence always restricts to Λ^1_P; in the strong case it restricts further to the coaction on Λ^1_{hor}. Since ϖ intertwines the standard P-action and H-coaction on Λ^1_P with that on P^+ and P^+/I_P, its restriction is equivariant and therefore makes Λ^1_{hor} an object of ${}_A\mathcal{M}_P^H$ isomorphic as such to $(P^+ \cap A)/(I_P \cap A)$. One can also check explicitly that $\Lambda^1_{\rm hor} \in {}_A\mathcal{M}_P^H$ since the crossed module condition holds,

$$\begin{aligned}\Delta_R(\xi \triangleleft p) &= (Sp_{(1)})_{(1)}\xi_{\bar{(0)}}p_{(2)(1)} \otimes \pi((Sp_{(1)})_{(2)})\xi_{\bar{(1)}}\pi(p_{(2)(2)})\\ &= \xi_{\bar{(0)}} \triangleleft p_{(2)} \otimes \pi(Sp_{(1)})\xi_{\bar{(1)}}\pi(p_{(3)}). \qquad \square\end{aligned}$$

To illustrate the theory, we use the q-sphere in Example 5.51 as a quantum homogeneous bundle. We have not developed the $*$-algebra aspects of the theory, but this is visible in examples.

Example 5.79 (Framing of the Standard q-Sphere) We return to the q-Hopf fibration on $A = \mathbb{C}_q[S^2]$ with $P = \mathbb{C}_q[SU_2]$, $H = \mathbb{C}_{q^2}[S^1]$ and calculi as in Example 5.51.

(i) We use Theorem 5.77 for the framing. First of all, $A^+ = \langle z, z^*\rangle$ as an ideal. Looking at the ideal I_P for this calculus in Example 2.32, we focus on b^2, c^2, bc and note that elements of degree 0 in $\langle b^2\rangle$ include $b^2\{a^2, ac, c^2\}$ so that z^{*2}, z^*x, x^2 lie in $I_P \cap A$. Similarly looking at $\langle c^2\rangle$ tells us that z^2, xz also lie in $I_P \cap A$. Here x is already in the ideal and we find that $V = \langle z, z^*\rangle/\langle z^2, z^{*2}, x\rangle$ is 2-dimensional. We take $v^+ = [z]$, $v^- = [-q^{-1}z^*]$ as a basis. Next, the coproduct of $\mathbb{C}_q[SL_2]$ already gave us a left coaction on $\mathbb{C}_q[S^2_{\mathbb{C}}]$ as

$$\Delta\begin{pmatrix}-q^{-1}z^*\\ -qx\\ z\end{pmatrix} = \begin{pmatrix}a^2 & (2)_q ab & b^2\\ ca & 1+(2)_q bc & db\\ c^2 & (2)_q cd & d^2\end{pmatrix} \otimes \begin{pmatrix}-q^{-1}z^*\\ -qx\\ z\end{pmatrix}$$

in Proposition 2.36; we apply S to these coproducts and compute the framing as

$$\theta(v^+) = Sz_{(1)}\mathrm{d}z_{(2)} = a^2\mathrm{d}z - q^{-3}c^2\mathrm{d}z^* + (2)_q ac\mathrm{d}x = e^+,$$
$$\theta(v^-) = -q^{-1}Sz^*{}_{(1)}\mathrm{d}z^*{}_{(2)} = -q^{-1}d^2\mathrm{d}z^* + q^2b^2\mathrm{d}z + q^2(2)_q bd\mathrm{d}x = e^-,$$

using $\mathrm{d}x$ etc. in the proof of Proposition 2.35. We can also see that θ has its values in $\mathbb{C}_q[SL_2]_{\pm 2}\Omega^1(\mathbb{C}_q[S^2_{\mathbb{C}}])$. Now s_θ extends this to a map $(P \otimes V)^H \to \Omega^1$, where invariant elements of the form $s \otimes v^+ + t \otimes v^-$ for $s, t \in \mathbb{C}_q[SL_2]$ of degrees $-2, 2$ respectively map to $se^+ + te^-$. In practice, we identify sections of $E_{\pm 2}$ as elements of degree $-2, 2$ and in this case the framing isomorphism is to simply identify

$$E_2 \oplus E_{-2} = \Omega^1, \quad s \oplus t \mapsto se^+ + te^-.$$

We further identify the summands with the holomorphic and antiholomorphic differentials $\Omega^{1,0}, \Omega^{0,1}$ respectively in the double complex in Proposition 2.35, so we see now how that decomposition naturally arose.

(ii) Next, Lemma 5.73 tells us that the quantum metric on $\mathbb{C}_q[S^2_{\mathbb{C}}]$

$$g = q\mathrm{d}z^* \otimes \mathrm{d}z + q^{-1}\mathrm{d}z \otimes \mathrm{d}z^* + q^2(2)_q \mathrm{d}x \otimes \mathrm{d}x,$$

previously found in Proposition 2.36, necessarily corresponds to a coframing

$$\begin{aligned}
\theta^* &= g^1 s_\theta^{-1}(g^2) \\
&= q(\mathrm{d}z^*)z_{(1)} \otimes [\pi_\epsilon z_{(2)}] + q^{-1}(\mathrm{d}z)z^*{}_{(1)} \otimes [\pi_\epsilon z^*{}_{(2)}] + q^2(2)_q(\mathrm{d}x)x_{(1)} \otimes [\pi_\epsilon x_{(2)}] \\
&= q\mathrm{d}z^*(c^2 \otimes v^- + d^2 \otimes v^+) - \mathrm{d}z(a^2 \otimes v^- + b^2 \otimes v^+) \\
&\quad - q(2)_q \mathrm{d}x(ca \otimes v^- + db \otimes v^+) \\
&= (q(\mathrm{d}z^*)c^2 - (\mathrm{d}z)a^2 - q(2)_q(\mathrm{d}x)ca) \otimes v^- \\
&\quad + (q(\mathrm{d}z^*)d^2 - (\mathrm{d}z)b^2 - q(2)_q(\mathrm{d}x)db) \otimes v^+
\end{aligned}$$

which exhibits $\theta^* \in (\Omega^1 P \otimes V)^H$. Conversely, we can recover $g = \langle \theta^*, \theta\rangle = (\mathrm{id}\otimes_A \theta)\theta^*$ provided we note that while the expression looks like it lives in $\Omega^1_A P \otimes_P P\Omega^1_A = \Omega^1_A \otimes_A P \otimes_A \Omega^1_A$, it is actually invariant and hence lives in $\Omega^1_A \otimes_A \Omega^1_A$.

In fact there is a shorter way to do these calculations if we fully embrace the framing isomorphisms and work 'upstairs' on P. Thus

$$\begin{aligned}
e^+ &= \tilde{D}_1\tilde{D}'_1.(\tilde{D}'_2\tilde{D}_2 e^+) = (e^+ D_1 D'_1).D'_2 D_2, \\
e^- &= D_1 D'_1.(D'_2 D_2 e^-) = (e^- \tilde{D}_1\tilde{D}'_1).\tilde{D}'_2\tilde{D}_2
\end{aligned}$$

exhibits these elements in both $P\Omega^1_A$ and $\Omega^1_A P$, where $D = d \otimes a - qb \otimes c$ and $\tilde{D} = a\otimes d - q^{-1}c\otimes b$ are the split q-determinants as in the proof of Proposition 2.36 and primes denote second copies. We can then write the coframing and metric as

$$\begin{aligned}
\theta^* &= -q^2 e^+ \otimes v^- - e^- \otimes v^+ \\
&= -q^2(e^+ D_1 D'_1).D'_2 D_2 \otimes v^- - (e^- \tilde{D}_1\tilde{D}'_1).\tilde{D}'_2\tilde{D}_2 \otimes v^+, \\
g &= \langle \theta^*, \theta\rangle = -q^2 e^+ \otimes_P e^- - e^- \otimes_P e^+ \\
&= -q^2 e^+ D_1 D'_1 \otimes_A D'_2 D_2 e^- - e^- \tilde{D}_1\tilde{D}'_1 \otimes_A \tilde{D}'_2\tilde{D}_2 e^+
\end{aligned}$$

on the one hand working over P and on the other placing expressions manifestly in $\Omega^1_A \otimes_A \Omega^1_A$. We used this method already in the proof of Proposition 2.35 for the metric and in Example 5.51 for the connection on an associated bundle.

(iii) The q-monopole connection in Example 5.51 on $E_2 \oplus E_{-2}$ is

$$\nabla(se^+ + te^-) = ((\mathrm{id} - \Pi)\mathrm{d}s)\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2 e^+ + ((\mathrm{id} - \Pi)\mathrm{d}t)D_1D_1' \otimes D_2'D_2 e^- \tag{5.37}$$

for s, t of grades $-2, 2$ respectively, where $\mathrm{id} - \Pi$ is the horizontal projection that kills e^0. This transfers via s_θ to the linear connection

$$\nabla \mathrm{d}z = -(2)_q zg, \quad \nabla \mathrm{d}z^* = -(2)_q z^* g, \quad \nabla \mathrm{d}x = -((2)_q x - q^{-1})g \tag{5.38}$$

after a short computation. For example, from the relations of the 3D calculus we have $\mathrm{d}z = d^2e^+ + c^2c^-$ and $\mathrm{d}d^2 = ce^-d + dce^- + \cdots = (2)_q ze^- + \cdots$ where we suppressed terms in e^0. Similarly $\mathrm{d}c^2 = q^2(2)_q ze^+ + \cdots$. Hence

$$\begin{aligned}\nabla \mathrm{d}z &= \nabla(d^2e^+ + c^2e^-)\\ &= ((\mathrm{id} - \Pi)\mathrm{d}d^2)\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2 e^+ + ((\mathrm{id} - \Pi)\mathrm{d}c^2)D_1D_1' \otimes D_2'D_2 e^-\\ &= -(2)_q zg.\end{aligned}$$

This q-deforms the classical Levi-Civita connection on the sphere and is clearly quantum torsion free as $\wedge(g) = 0$, so that $T_\nabla(\mathrm{d}z) = \mathrm{d}^2 z = 0$ etc. Its curvature using the formula for F_ω in Example 5.51 becomes

$$R_\nabla(\partial f) = q^4(2)_q \mathrm{Vol} \otimes \partial f, \quad R_\nabla(\bar{\partial} f) = -(2)_q \mathrm{Vol} \otimes \bar{\partial} f \tag{5.39}$$

for the restriction to $\Omega^{1,0}$ and $\Omega^{0,1}$ in the description of Proposition 2.35.

We check that ∇ is quantum cotorsion free. The first factor of g is killed by d and applying ∇ to the second factor, we have

$$\mathrm{co}T_\nabla = -\wedge(\mathrm{id} \otimes \nabla)g = \Big(q(\mathrm{d}z^*)z + q^{-1}(\mathrm{d}z)z^* + q^2(2)_q(\mathrm{d}x)((2)_q x - q^{-1})\Big) \wedge (2)_q g,$$

which vanishes in view of (2.11). In fact, the connection is a bimodule connection and fully quantum metric compatible, as we will see shortly.

Another remark is that from ∇ and the quantum inverse metric $(,)$ we have a *geometric quantum Laplacian* $\Delta = (,)\nabla \mathrm{d}$ and we will see in Chap. 8 that in the case of a bimodule connection, it meets the definition of a second-order operator in Chap. 1 with respect to the bimodule map $\frac{1}{2}(,)(\mathrm{id} + \sigma)$. In our case, we find

$$\Delta z = \lambda z, \quad \Delta z^* = \lambda z^*, \quad \Delta((2)_q x - q^{-1}) = \lambda((2)_q x - q^{-1}), \tag{5.40}$$

where $\lambda = -(2)_q(,)(g) = -(2)_q(2)_{q^2}$.

(iv) On the technical side, it is clear from the q-commutation relations of the 3D calculus that the bundle in this example is strong and the centrality condition in Corollary 5.76 holds. In this case, we can write

$$\theta^* = \theta(\eta^1) \otimes \eta^2, \quad \eta = -q^2 v^+ \otimes v^- - v^- \otimes v^+ \in V \otimes V$$

for an invariant 'local metric' η. We also saw in Remark 2.37 that the q-Hopf fibration has an associated H-crossed module on Λ^1_P. This restricts to $e^\pm$ as a crossed module structure or to $v^\pm$ via θ, such that $v^\pm{\triangleleft}t = qv^\pm$ and $v^\pm$ have degree ± 2. Then we already noted in Remark 2.37 that η spans the kernel of the restricted $\mathrm{id} - \Psi$ which underlies the fact that the metric is quantum symmetric. Finally, we know on general grounds that the H-A crossed module structure in this example, being in the image of $\mathcal{M}^H_H \to \mathcal{M}^H_A$, has trivial action $v{\triangleleft}a = v\epsilon(a)$, as one can easily check. For example $v^+{\triangleleft}z = [z^2] = 0$, $v^+{\triangleleft}z^* = [q^2x(1-q^2x)] = 0$ and $v^+{\triangleleft}x = [zx] = 0$ in the quotient. This fits with the fact that $e^\pm$ as the image of θ is central in the 3D calculus for multiplication by elements of A. We also know from Corollary 5.76 and the fact that the underlying q-monopole connections are bimodule connections that ∇ is a bimodule connection. The generalised braiding from Example 5.51 in this form is

$$\sigma(se^+ \otimes re^+) = q^2 D_1 D_1' e^+ \otimes D_2' D_2 s r e^+, \tag{5.41}$$

$$\sigma(se^+ \otimes pe^-) = q^2 \tilde{D}_1 \tilde{D}_1' e^- \otimes \tilde{D}_2' \tilde{D}_2 s p e^+, \tag{5.42}$$

$$\sigma(te^- \otimes re^+) = q^{-2} D_1 D_1' e^+ \otimes D_2' D_2 t r e^-, \tag{5.43}$$

$$\sigma(te^- \otimes pe^-) = q^{-2} \tilde{D}_1 \tilde{D}_1' e^- \otimes \tilde{D}_2' \tilde{D}_2 t p e^- \tag{5.44}$$

for s, r of grade -2 and p, t of grade 2.

(v) We have already seen in Example 5.51 that the q-monopole connections on the E_n are extendable and this therefore also applies to our bimodule connection on Ω^1 in part (iii), making it an object of ${}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$. Corollary 4.16 then tells us that the Riemann curvature is antisymmetric in the sense that

$$(R_\nabla \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes R_\nabla))g = 0,$$

as one can also see directly given that g has only mixed terms in the decomposition into holomorphic and holomorphic parts. For example, a holomorphic part in the first tensor factor of g will contribute $q^4(2)_q\mathrm{Vol}\otimes$ for the first term of the above, while R_∇ for the second term will contribute $-(2)_q\mathrm{Vol}\otimes$, which followed by the + case of $\sigma((\ ) \otimes \mathrm{Vol}) = q^{\pm 4}\mathrm{Vol} \otimes (\ )$ on $E_{\pm 2}$ cancels to give zero for this part. ◇

The frame bundle approach expresses the torsion and cotorsion in a symmetric way and is a useful method to find 'weak quantum Levi-Civita' connections (where these vanish). If we are lucky then this may in fact be quantum metric compatible or 'quantum Levi-Civita'. This happens for the q-sphere.

Proposition 5.80 *For the q-sphere $\mathbb{C}_q[S^2]$ with its calculus and quantum metric as in Proposition 2.36, the above torsion-free cotorsion-free bimodule connection ∇ on $\Omega^1 = E_2 \oplus E_{-2}$ induced by the q-monopole on the principal bundle is quantum metric compatible or 'quantum Levi-Civita' and $*$-preserving.*

Proof We use (5.37) and a shorthand as in Example 5.51 for the partial derivatives in the $e^\pm$ directions whereby $(\mathrm{id} - \Pi)\mathrm{d}f = f_\pm e^\pm$ (summing over $\pm$) for any f. Note that these partial derivatives $(\ )_\pm$ change grade by ∓ 2 and obey a braided-Leibniz rule $(fh)_\pm = q^{|h|} f_\pm h + f h_\pm$ for all $f, h \in \mathbb{C}_q[SL_2]$. We focus on $g_{+-} = -D_1 D_1' e^+ \otimes D_2' D_2 e^-$ as a piece of g and show that $\nabla g_{+-} = 0$. The calculation for the remaining g_{-+} part is strictly analogous so we do not repeat this, a strategy which we already employed in the proof of Proposition 2.36. Thus, we look at

$$
\begin{aligned}
-\nabla g_{+-} &= (\mathrm{id} - \Pi)\mathrm{d}(D_1 D_1')\tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2 e^+ \otimes D_2' D_2 e^- \\
&\quad + \sigma(D_1 D_1' e^+ \otimes (\mathrm{id} - \Pi)\mathrm{d}(D_2' D_2) D_1'' D_1''') \otimes D_2''' D_2'' e^- \\
&= q^2 (D_1 D_1')_\pm \tilde{D}_1 \tilde{D}_1' e^\pm \otimes \tilde{D}_2' \tilde{D}_2 e^+ \otimes D_2' D_2 e^- \\
&\quad + q^{-2} \sigma(D_1 D_1' e^+ \otimes (D_2' D_2)_\pm D_1'' D_1''' e^\pm) \otimes D_2''' D_2'' e^- \\
&= q^2 (D_1 D_1')_\pm \tilde{D}_1 \tilde{D}_1' e^\pm \otimes \tilde{D}_2' \tilde{D}_2 e^+ \otimes D_2' D_2 e^- \\
&\quad + D_1^{iv} D_1^{v} e^+ \otimes D_2^{v} D_2^{iv} D_1 D_1' (D_2' D_2)_+ D_1'' D_1''' e^+ \otimes D_2''' D_2'' e^- \\
&\quad + \tilde{D}_1 \tilde{D}_1' e^- \otimes \tilde{D}_2' \tilde{D}_2 D_1 D_1' (D_2' D_2)_- D_1'' D_1''' e^+ \otimes D_2''' D_2'' e^- \\
&= q^2 (D_1 D_1')_\pm \tilde{D}_1 \tilde{D}_1' e^\pm \otimes \tilde{D}_2' \tilde{D}_2 e^+ \otimes D_2' D_2 e^- \\
&\quad - q^2 D_1^{iv} D_1^{v} e^+ \otimes D_2^{v} D_2^{iv} (D_1 D_1')_+ D_2' D_2 D_1'' D_1''' e^+ \otimes D_2''' D_2'' e^- \\
&\quad - q^2 \tilde{D}_1 \tilde{D}_1' e^- \otimes \tilde{D}_2' \tilde{D}_2 (D_1 D_1')_- D_2' D_2 D_1'' D_1''' e^+ \otimes D_2''' D_2'' e^- \\
&= q^2 (D_1 D_1')_\pm \tilde{D}_1 \tilde{D}_1' e^\pm \otimes \tilde{D}_2' \tilde{D}_2 e^+ \otimes D_2' D_2 e^- \\
&\quad - q^2 D_1^{iv} D_1^{v} e^+ \otimes D_2^{v} D_2^{iv} (D_1 D_1')_+ e^+ \otimes D_2' D_2 e^- \\
&\quad - q^2 \tilde{D}_1 \tilde{D}_1' e^- \otimes \tilde{D}_2' \tilde{D}_2 (D_1 D_1')_- e^+ \otimes D_2' D_2 e^- = 0
\end{aligned}
$$

using our notations and the first two lines of σ for the 3rd equality, followed by the braided Leibniz rule for the 4th. For the 5th equality, we moved some degree 0 pieces through to $\otimes_A$ to the right and cancelled the q-determinants. Finally we replace the rightmost e^+ in the final two terms with $\tilde{D}_1'' \tilde{D}_1''' \tilde{D}_2''' \tilde{D}_2'' e^+$, and then moving the grade zero factors $D_2^{v} D_2^{iv} (D_1 D_1')_+ \tilde{D}_1'' \tilde{D}_1'''$ and $\tilde{D}_2' \tilde{D}_2 (D_1 D_1')_- \tilde{D}_1'' \tilde{D}_1'''$ over the leftmost tensor product of the last two terms gives the result.

Over $\mathbb{C}$, if we take the compact $*$-structure for $\mathbb{C}_q[SU_2]$ and $\mathbb{C}[S^1]$ for the fibre, we already know that ω on the principal bundle commutes with $*$. This transfers via the general theory to ∇ $*$-preserving by the note after Proposition 5.56.

Here we simply verify it directly for our connection. We need to show that $(*\otimes *)\mathrm{flip}\,\sigma^{-1}\nabla(\xi^*)=\nabla\xi$ where we set $\xi=se^++te^-$, so $|s|=-2$ and $|t|=2$. Then

$$\begin{aligned}\nabla(\xi^*)&=-\nabla(q^{-1}t^*e^++qs^*e^-)\\&=-(\mathrm{id}-\Pi)q^{-1}\mathrm{d}t^*\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2e^+-(\mathrm{id}-\Pi)q\mathrm{d}s^*D_1D_1'\otimes D_2'D_2e^-.\end{aligned}$$

Now we set $\mathrm{d}t=t_\pm e^\pm+t_\pm e^\pm$ and $\mathrm{d}s=s_\pm e^\pm$ and then $\mathrm{d}t^*=-q^{-1}t_+{}^*e^--q^{-3}t_-{}^*e^+$ and $\mathrm{d}s^*=-q^3s_+{}^*e^--q^1s_-{}^*e^+$. We show the calculation for the term in $\nabla(\xi^*)$ containing t_-, suppressing a further three similar terms. Thus,

$$\begin{aligned}\nabla(\xi^*)&=q^{-4}t_-{}^*e^+\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2e^++\dots,\\ \sigma^{-1}\nabla(\xi^*)&=q^{-6}t_-{}^*e^+\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2e^++\dots,\\ (*\otimes *)\mathrm{flip}\,\sigma^{-1}\nabla(\xi^*)&=q^{-8}e^-\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2e^-t_-+\dots\\&=t_-e^-\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2e^-+\dots=\nabla(\xi),\end{aligned}$$

where we use the σ on the appropriate E_n from Example 5.51. □

5.6.2 *Trivially Framed Quantum Manifolds*

Here we specialise to the case of a differentiable algebra (A,Ω^1,d) which is parallelisable as in Definition 1.2. We consider this as framed by any Hopf algebra H with tensor product bundle $P=A\otimes H$ as in Example 5.55. We will later specialise to $A=H$ for the intrinsic framed quantum geometry of a Hopf algebra.

We first state the notion of left-parallelisability more formally as $A\otimes V\cong\Omega^1$ by $s_e(a\otimes v)=ae(v)$ for some map $e:V\to\Omega^1$ (called a 'V-bein'.). We also consider the right-parallelisable case, so $W\otimes A\cong\Omega^1$ by $s_f(w\otimes a)=f(w)a$ for $f:W\to\Omega^1$. If we choose $W=V^\flat$ then we say that f is a 'V-cobein'. In the case of a classical manifold, such a bein (e.g. in dimension 4 a 'vierbein') exists locally and we would want V to carry a representation of the local frame group.

Corollary 5.81 *If (A,Ω^1) is left-parallelisable by (V,e) and H is any Hopf algebra with invertible antipode coacting on V from the right then $P=A\otimes H$ with coaction $\Delta_R=\mathrm{id}\otimes\Delta$ and $\theta(v)=v_{(\bar 1)}e(v_{(\bar 0)})$ gives a framing of A with P strong. If there is also a V-cobein f then A is also coframed and there is a generalised quantum metric $g=\langle f,e\rangle$. A linear map $\alpha:\Lambda^1_H\to\Omega^1$ induces a weak quantum Levi-Civita (i.e., torsion free cotorsion free) connection if and only if $D_\alpha\wedge e=D_\alpha\wedge f=0$.*

Proof (i) Let V be a right H-comodule which we can also view as usual as a left comodule with $\Delta_L v = v_{(\bar{1})} \otimes v_{(\bar{\infty})} = S^{-1} v_{(\bar{1})} \otimes v_{(\bar{0})}$. Then $\theta(v) = v_{(\bar{1})}.e(v_{(\bar{0})})$, where we view $H \subseteq P$, is a framing. To prove this, we first check equivariance as

$$\Delta_R \theta(v) = v_{(\bar{1})(1)} e(v_{(\bar{0})}) \otimes v_{(\bar{1})(2)} = v_{(\bar{0})(\bar{1})} e(v_{(\bar{0})(\bar{0})}) \otimes v_{(\bar{1})} = \theta(v_{(\bar{0})}) \otimes v_{(\bar{1})}.$$

We also identify $(H \otimes V)^H = V$ by $\epsilon \otimes \mathrm{id}$ in one direction and Δ_L in the other so that $(P \otimes V)^H = A \otimes V$ as in Example 5.55. Then

$$s_\theta(a \otimes v_{(\bar{1})} \otimes v_{(\bar{\infty})}) = a v_{(\bar{1})} (S v_{(\bar{\infty})(\bar{1})}) e(v_{(\bar{\infty})(\bar{\infty})}) = a v_{(\bar{1})(1)} (S v_{(\bar{1})(2)}) e(v_{(\bar{\infty})}) = a e(v)$$

reduces to the s_e isomorphism. We also know from Example 5.55 that the bundle is strong and regular.

(ii) We know from Example 5.55 that a connection on P is given by a linear map $\alpha : \Lambda^1_H \to \Omega^1$ and the framing isomorphism now gives us

$$\nabla(\phi^1 e(\phi^2)) = \mathrm{d}\phi^1 \otimes e(\phi^2) - \phi^1 \alpha(\varpi \pi_\epsilon \phi^2{}_{(\bar{1})}) \otimes e(\phi^2{}_{(\bar{\infty})})$$

for all $\phi \in A \otimes V$. The torsion computed from ∇ is immediately seen to be

$$T_\nabla(\phi^1 e(\phi^2)) = -\phi^1 (D_\alpha \wedge e)(\phi^2), \quad (D_\alpha \wedge e)(v) = \mathrm{d} \wedge e(v) + \alpha(\varpi \pi_\epsilon v_{(\bar{1})}) \wedge e(v_{(\bar{\infty})})$$

so that torsion-free corresponds to $D_\alpha \wedge e = 0$. Part (1) of Theorem 5.75 reduces to this for the stated form of θ and ω built from α. Its curvature from Example 5.55 is

$$R_\nabla(\phi) = -\phi^1 F_\alpha(\varpi \pi_\epsilon \phi^2{}_{(\bar{1})}) \otimes e(\phi^2{}_{(\bar{\infty})}).$$

(iii) A V-cobein can be viewed as $f = f^1 \otimes f^2 \in \Omega^1 \otimes V$ giving an isomorphism $\Omega^1 \cong V^\flat \otimes A$ according to coframing $\theta^* = \cdot(\mathrm{id} \otimes \Delta_L) f \in \Omega^1 P \otimes V$. The generalised quantum metric is then $g = \langle \theta^*, \theta \rangle = f^1 \otimes_A e(f^2) = \langle f, e \rangle$ after a cancellation. Then

$$\mathrm{co}T_\nabla = (\mathrm{id} \otimes e)(D_\alpha \wedge f); \quad D_\alpha \wedge f = \mathrm{d}f^1 \otimes f^2 + f^1 \wedge \alpha(\varpi \pi_\epsilon f^2{}_{(\bar{1})}) \otimes f^2{}_{(\bar{\infty})}$$

so that vanishing of cotorsion is equivalent to $D_\alpha \wedge f = 0$.

(iv) For reference, we let $\{v^i\}$ be a basis of V so that $e^i = e(v^i)$ is a basis of Ω^1 over A and $\phi = \phi_i e^i$ for any $\phi \in \Omega^1$ (summation understood). If $\Delta_L v^i = t^i{}_j \otimes v^j$, say for $\{t^i{}_j \in H\}$, then the above become

$$\nabla(\phi) = \nabla\phi_i \otimes e^i, \quad \nabla\phi_i = \mathrm{d}\phi_i - \phi_j \alpha(\varpi\pi_\epsilon t^j{}_i)),$$
$$T_\nabla(\phi) = -\phi_i D_\alpha \wedge e^i, \quad D_\alpha \wedge e^i = \mathrm{d}e^i + \alpha(\varpi\pi_\epsilon t^i{}_j) \wedge e^j,$$
$$R_\nabla(\phi) = -\phi_i F_\alpha(\varpi\pi_\epsilon t^i{}_j) \wedge e_j,$$
$$\mathrm{co}T_\nabla = D_\alpha \wedge f_i \otimes e^i, \quad D_\alpha \wedge f_i = \mathrm{d}f_i + f_j \wedge \alpha(\varpi\pi_\epsilon t^j{}_i). \qquad \square$$

We now consider the special case where $A = H$ with a bicovariant calculus Ω^1. Indeed, we have already proven in Theorem 2.26 that Ω^1 is parallelisable, $\Omega^1 \cong H \otimes \Lambda^1$ where $V = H^+/I \cong \Lambda^1$ for some Ad_R-stable right ideal I and the isomorphism is afforded by the Maurer–Cartan form, which now becomes the V-bein $e(v) = \varpi(\tilde{v}) = S\tilde{v}_{(1)}\mathrm{d}\tilde{v}_{(2)}$ for any representative $\tilde{v} \in H^+$. Here $\mathrm{Ad}_R(h) = h_{(2)} \otimes (Sh_{(1)})h_{(3)}$ is the right adjoint coaction. We also saw that $e(v)h = h_{(1)}e(v \triangleleft h_{(2)})$ in current notations, where the right action of H is by right multiplication on H^+ and makes $V \in \mathcal{M}^H_H$. Here the bimodule structure on Ω^1 in terms of $\phi \in H \otimes V$ is $\phi.h = \phi^1 h_{(1)} \otimes \phi^2 \triangleleft h_{(2)}$, which is more general than the trivial action case considered at the end of Example 5.55.

Recall also that we found it useful in Chap. 2 to absorb the isomorphism provided by e and work directly with $\Lambda^1 \subset \Omega^1$ (the left-invariant 1-forms) as the relevant crossed module $\Lambda^1 \in \mathcal{M}^H_H$, in which case e becomes the identity map. These points of view are useful and we will use both. In particular, it will be useful to have 'quantum Lie cobrackets' $\mathrm{ad}_R, \mathrm{ad}_L : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ adjoint to the quantum Lie bracket on Λ^{1*} in Theorem 2.85 or its right-handed version. These are defined by

$$\mathrm{ad}_R(v) = v_{(\bar{0})} \otimes \varpi\pi_\epsilon v_{(\bar{1})}, \quad \mathrm{ad}_L(v) = \varpi\pi_\epsilon v_{(\bar{1})} \otimes v_{(\bar{\infty})} = \varpi\pi_\epsilon S^{-1} v_{(\bar{1})} \otimes v_{(\bar{0})}$$

in terms of the right crossed module coaction transferred to Λ^1 and its left-handed version. Also recall that the Maurer–Cartan equation $\mathrm{d}v = -(\varpi\pi_\epsilon\tilde{v}_{(1)}) \wedge (\varpi\pi_\epsilon\tilde{v}_{(2)}) \in \Lambda^2$ for any representative $\tilde{v} \in H^+$ such that $v = \varpi\tilde{v}$. It is useful to define

$$\delta : H^+ \to \Lambda^1 \otimes \Lambda^1, \quad \delta = (\varpi\pi_\epsilon \otimes \varpi\pi_\epsilon)\Delta$$

so that the Maurer–Cartan equation becomes $\mathrm{d}v = - \wedge \delta\tilde{v}$. We take the canonical exterior algebra defined by the crossed module braiding Ψ in Theorem 2.74.

For the metric, we are particularly interested in the intrinsic geometry of the quantum group and hence in metrics which are at least left-invariant, i.e., $g \in \Lambda^1 \otimes \Lambda^1$, which we can also view as a V-cobein $f = (e \otimes \mathrm{id})\eta \in \Omega^1 \otimes V$ and hence as $g = (e \otimes e)(\eta)$, where $\eta \in V \otimes V$. We require the latter to be nondegenerate and invariant under the right action of H (since an invertible quantum metric has to be central). We also know from Corollary 5.81 with $A = H$ that connections are given by linear maps $\alpha : \Lambda^1 \to \Omega^1$ and induce covariant derivatives ∇ on Ω^1.

Corollary 5.82 *Let H be a Hopf algebra with invertible antipode and bicovariant Ω^1, framed by $P = H\otimes H$ according to Corollary 5.81 with $V = H^+/I$, $e = \varpi$.*

(1) *g defined by η is biinvariant if and only if $\eta \in \mathcal{M}_H^H$ is invariant. In this case g is quantum symmetric if and only if* $(\mathrm{id}\otimes S^2)\mathrm{flip}(\eta) = \eta$.
(2) *∇ defined by α is weak quantum Levi-Civita if and only if*

$$(\alpha \wedge \mathrm{id})\mathrm{ad}_L(v) = -\mathrm{d}v = (\mathrm{id} \wedge \alpha)\mathrm{ad}_R(v)$$

for all $v \in \Lambda^1$. Here α is regular if and only if $(\alpha\wedge\alpha)\delta(I) = 0$.
(3) *∇ defined by α is left-covariant if and only if* $(\alpha \otimes \mathrm{id})\mathrm{ad}_L : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$, *e.g. $\alpha : \Lambda^1 \to \Lambda^1$. It is a bimodule connection if and only if*

$$\big((\alpha \otimes \mathrm{id})\mathrm{ad}_L(v{\triangleleft}Sh_{(1)})\big)\triangleleft h_{(2)} = 0$$

for all $h \in I$ and $v \in \Lambda^1$, where $h_{(2)}$ has the tensor product action.

Proof (1) $g = (\varpi \otimes \varpi)(\eta)$ is already by construction left-invariant. To also be right-invariant under the coaction induced by the coproduct is equivalent to Ad_R-invariance, which means η is invariant under the tensor product crossed module coaction. Next, the braiding Ψ on g corresponds to the crossed-module braiding on V. Hence in the canonical exterior algebra, $\wedge(g) = 0$ if and only if $\Psi(\eta) = \eta$, where

$$\Psi(\eta) = \eta^2{}_{(\bar 0)} \otimes \eta^1{\triangleleft}\eta^2{}_{(\bar 1)} = \eta^2 \otimes \eta^1{}_{(\bar 0)}{\triangleleft}S\eta^1{}_{(\bar 1)} = \eta^2 \otimes \eta^1{}_{(2)}(S\eta^1{}_{(3)})S^2\eta^1{}_{(1)} = \eta^2 \otimes S^2\eta^1$$

using invariance of η under the crossed module coaction and the specific form of the coaction and action on V. Note that $S^2(I) \subseteq I$ as a consequence of I being Ad_R stable, so this condition is well defined on the quotient V.

(2) We specialise Corollary 5.81 to $P = H \otimes H$. Then $D \wedge e = 0$ is

$$\alpha(\varpi\pi_\epsilon(S^{-1}h_{(3)})h_{(1)}) \wedge \varpi(\pi_\epsilon h_{(2)}) = \varpi(\pi_\epsilon h_{(1)}) \wedge \varpi(\pi_\epsilon h_{(2)})$$

for all $h \in H^+$, given that ϖ obeys the Maurer–Cartan equations in Proposition 2.31 for $\mathrm{d}\varpi$. We can write this more compactly as the first of the two displayed conditions given the definitions of ad_L. Next we assume that $f = (e\otimes\mathrm{id})\eta$ where η is invariant in the crossed module category. This means both under the action which clearly ensures that the metric is central and under the coaction, which we use to eliminate η from the $D \wedge f = 0$ equation to give $\mathrm{d}e(v) + e(v_{(\bar 0)}) \wedge \alpha(\varpi\pi_\epsilon v_{(\bar 1)}) = 0$ or

$$\varpi(\pi_\epsilon h_{(1)}) \wedge \varpi(\pi_\epsilon h_{(2)}) = \varpi(\pi_\epsilon h_{(2)}) \wedge \alpha(\varpi\pi_\epsilon(Sh_{(1)})h_{(3)})$$

for all $h \in H^+$. This derives the second of the displayed equalities. The regularity condition from Example 5.55 is $\alpha(\varpi\pi_\epsilon h_{(1)}) \wedge \alpha(\varpi\pi_\epsilon h_{(2)}) = 0$ for all $h \in I$ and can again be written compactly as stated.

(3) Writing an element of Ω^1 as $ae(v)$ (sum of such terms understood), Corollary 5.81 gives us a connection as

$$\nabla(ae(v)) = \mathrm{d}a \otimes e(v) - a\alpha\varpi\pi_\epsilon S^{-1}v_{(\bar{1})} \otimes e(v_{(\bar{0})})$$

for any linear map $\alpha : \Lambda^1 \to \Omega^1$, with restriction $\nabla^L e(v) = -\alpha\varpi\pi_\epsilon S^{-1}v_{(\bar{1})} \otimes e(v_{(\bar{0})})$ for $v \in V$. This is equivalent to $\nabla^L v = -(\alpha \otimes \mathrm{id})\mathrm{ad}_L(v)$ for $v \in \Lambda^1$. We already observed in Example 3.31 that translation-invariant ∇ correspond to $\nabla^L : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$, which is equivalent to the stated condition on α. Proposition 3.73 restricted to left-invariant forms tells us the form of σ as

$$\sigma(e(v) \otimes e(w)) = \Psi(e(v) \otimes e(w)) - (\nabla^L(e(v){\triangleleft}S\tilde{w}_{(1)})){\triangleleft}\tilde{w}_{(2)},$$

which we need to be well defined for all $v, w \in V$. Here the first term in Proposition 3.73 is the braiding Ψ while the second term involves a lift $\tilde{w} \in H^+$ of $w \in V$. Requiring independence of the choice of $\tilde{w}$ is the stated ideal condition on α. Note that $\theta(v) = v_{(\bar{1})}e(v_{(\bar{0})})$ viewed in $H\Omega^1 \subset P\Omega^1$ is not central so long as the crossed module action of H on V is nontrivial, so we could not use Corollary 5.76. □

Here $\alpha = 0$ gives $\sigma = \Psi$ and $\alpha = \mathrm{id}$ gives $\sigma = \Psi^{-1}$, recovering the first two cases of Example 3.74. We still have to solve these equations to construct quantum Riemannian geometries in the frame bundle approach. We content ourselves with two main cases, namely finite groups and coquasitriangular Hopf algebras. In both cases there are natural choices of invariant metrics.

Example 5.83 We take $A = H = \Bbbk(G)$, the functions on a finite group with calculus $\Omega(G)$ defined as in §1.7.2 and Example 2.29 by an Ad-stable subset $\mathcal{C} \subseteq G \setminus \{e\}$. Here $\{e_a \mid a \in \mathcal{C}\}$ is a basis of left-invariant 1-forms and constitutes the V-bein, where $V = \mathrm{span}\{\delta_a\}$ modulo relations setting to zero those elements of $G \setminus \{e\}$ not in $\mathcal{C}$. The crossed module action and coaction are $\Delta_R e_a = \sum e_{gag^{-1}} \otimes \delta_g$ and $e_a{\triangleleft}f = f(a)e_a$ for all $f \in \Bbbk(G)$. The quantum Lie cobrackets and δ are

$$\mathrm{ad}_L e_a = \sum_b e_b \otimes (e_{b^{-1}ab} - e_a),\quad \mathrm{ad}_R e_a = \sum_{b\in\mathcal{C}} (e_{bab^{-1}} - e_a) \otimes e_b,\quad \delta(\delta_g) = \sum_{a,b\in\mathcal{C}, ab=g} e_a \otimes e_b$$

if $g \notin \mathcal{C} \cup \{e\}$. We used that $\varpi\pi_\epsilon\delta_e = -\theta$, $\theta = \sum_{a\in\mathcal{C}} e_a$. If a connection α has values $\alpha_a = \alpha(e_a) \in \Omega^1$ then the associated linear connection on our basis is

$$\nabla e_a = -\sum_b \alpha_b \otimes (e_{b^{-1}ab} - e_a)$$

with curvature $R_\nabla e_a = \sum_b F_b \otimes (e_{b^{-1}ab} - e_a)$, where

$$F_a = F(\delta_a) = \mathrm{d}\alpha_a + \sum_{c,d\in\mathcal{C},cd=a} \alpha_c \wedge \alpha_d - \sum_b(\alpha_a \wedge \alpha_b + \alpha_b \wedge \alpha_a)$$

and the condition for α to be regular becomes

$$\sum_{a,b\in\mathcal{C},ab=g} \alpha_a \wedge \alpha_b = 0 \tag{5.45}$$

for all $g \notin \mathcal{C}$ and $g \neq e$. Vanishing torsion in Corollary 5.82 becomes

$$\sum_{b\neq a} \alpha_b \wedge (e_{b^{-1}ab} - e_a) + e_b \wedge e_a + e_a \wedge e_b = 0. \tag{5.46}$$

We take the central bi-invariant metric $\eta_{a,b} = \delta_{a,b^{-1}}$ so $g = \sum_a e_a \otimes_A e_{a^{-1}}$, which is symmetric when $\mathcal{C}$ has only elements of order 2 by the above. The condition for vanishing cotorsion is then

$$\sum_{b\neq a}(e_{bab^{-1}} - e_a) \wedge \alpha_b + e_b \wedge e_a + e_a \wedge e_b = 0. \tag{5.47}$$

We now solve these conditions (5.45)–(5.47) for α for $G = S_3$ with its 3D bicovariant calculus $\Omega^1 = \mathbb{C}(S_3)\{e_u, e_v, e_w\}$ labelled by the 2-cycles $u = (12)$, $v = (23)$, $w = (13)$ as in Example 1.60. Here $V = \Lambda^1$ with basis $\{e_a\}$, $a = u, v, w$ and we take the co-bein to be the same as $\eta_{a,b} = \delta_{a,b}$ since $a^2 = e$ (the group identity element) for all $a \in \mathcal{C}$ in this example. Taking a basis of Ω^2, which is 4-dimensional, and coefficients $\alpha_a = \alpha_a{}^b e_b$, the system of equations for zero torsion and cotorsion are linear in α easily solved. Solving for zero torsion gives a 12-parameter moduli

$$\alpha_u = (\mu + 1)e_u + \nu e_v + \rho e_w, \quad \alpha_v = \nu e_u + (\rho + 1)e_v + \mu e_w,$$

$$\alpha_w = \rho e_u + \mu e_w + (\nu + 1)e_w,$$

where $\mu + \nu + \rho = -1$ for functional parameters μ, ν, ρ. The moduli of cotorsion free connections is similar and also 12-dimensional with functional parameters μ', ν', ρ' on the right. From the bimodule relations, one finds that their intersection requires $\mu = \mu', \nu = \nu', \rho = \rho'$ and that these be constants, so the intersection is a 2-dimensional moduli space of weak quantum Levi-Civita connections in our frame bundle approach. If we let $\beta := \mu e_u + \rho e_v + \nu e_w = \beta_a e_a$ (so we regard our parameters as a 3-vector with the constraint $\sum_a \beta_a = -1$) then we have

$$\nabla e_a = -\Psi^{-1}(e_a \otimes \theta) - \sum_b \beta_b \sum_{cd=ab} e_c \otimes e_d$$

for our 2-parameter moduli, where $\theta = e_u + e_v + e_w$ (not to confused with the framing) and $\Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a$ is the crossed module braiding.

Finally, when we impose regularity (5.45) we obtain a *unique* regular weak quantum Levi-Civita connection given by $\mu = \nu = \rho = -1/3$. Clearly now

$$\nabla e_a = -\Psi^{-1}(e_a \otimes \theta) + \frac{1}{3}\theta \otimes \theta,$$

which is a bimodule connection with

$$\sigma(e_a \otimes e_b) = \Psi(e_a \otimes e_b) + \Psi^{-1}(e_a \otimes e_b) - \frac{1}{3}\sum_{cd=ab} e_c \otimes e_d$$

(this agrees with the general formula in Corollary 5.82) and curvature

$$R_\nabla e_a = -(\mathrm{d} \otimes \mathrm{id})\Psi^{-1}(e_a \otimes \theta).$$

This is the canonical quantum Riemannian geometry of S_3 in the frame bundle approach and is left-covariant as the α_a have constant coefficients in our left-invariant basis. This unique regular point lies in the moduli of Ad-invariant connections for this calculus in Example 3.76 as explained further in Example 8.21.

One can check that this natural bimodule connection is not metric compatible as such for the Euclidean metric, so we do need a weaker notion of quantum Levi-Civita connection. Its associated geometric Laplacian $\Delta = (\ ,\ )\nabla$ is

$$\Delta f = (\ ,\ )\nabla(\theta f - f\theta) = (\ ,\ )[\nabla\theta, f] + (\ ,\ )\sigma(\theta\otimes \mathrm{d}f) - (\mathrm{d}f, \theta) = -2(\mathrm{d}f, \theta) = {}_\theta\Delta f$$

which agrees with ${}_\theta\Delta = \Delta_\theta$ in Example 1.60. We used $\sigma(\theta\otimes e_b) = \Psi(\theta\otimes e_b) + e_b\otimes\theta - \frac{1}{3}\theta\otimes\theta$ to find $(\ ,\ )\sigma(\theta\otimes e_b) = 1$ so that $(\ ,\ )\sigma(\theta\otimes \mathrm{d}f) = -\sum_b \partial^b f = -(\mathrm{d}f, \theta)$. Note, however, that if we work over $\mathbb{C}$ with the usual $*$-differential calculus then our connection is *not* $*$-preserving, as can be seen from the classification of the latter in Example 3.88 with $(a, b, c, d, e) = (\frac{5}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. $\diamond$

We conclude with a much more complicated example, $H = \mathbb{C}_q[SU_2]$. Much of the structure we need was already computed in Example 2.59, with the main new ingredient being the map $\varpi\pi_\epsilon$. For any Hopf algebra H, if Λ^1 has basis $\{e_i\}$ and ∂^i are the associated partial derivatives then

$$\varpi\pi_\epsilon h = \sum_i e_i \epsilon(\partial^i h), \quad \delta h = \sum_{i,j} e_i \otimes e_j \epsilon(\partial^i \partial^j h)$$

for all $h \in H^+$. This follows from $\mathrm{d}h = \sum_i \partial^i h \otimes e_i = h_{(1)} \otimes \varpi\pi_\epsilon h_{(2)}$ on applying ϵ to obtain the first formula and $\varpi\pi_\epsilon$ to obtain the second.

Lemma 5.84 *For the 4D calculus on $\mathbb{C}_q[SL_2]$ in Example 2.59 but with* d *normalised to be λ^{-1} times the differentials listed there (this is the geometric normalisation with classical limit), we have*

$$\mathrm{ad}_R(e_a) = e_c \otimes e_b - e_b \otimes e_c + \lambda q^2 e_z \otimes e_a,$$
$$\mathrm{ad}_R(e_d) = e_b \otimes e_c - e_c \otimes e_b - \lambda q^2 e_z \otimes e_a,$$
$$\mathrm{ad}_R(e_b) = e_b \otimes (q^2 e_a - e_d) - e_z \otimes e_b,$$
$$\mathrm{ad}_R(e_c) = e_c \otimes ((q^2 - 1 - q^{-2})e_a + q^2 e_d) + q^2 e_z \otimes e_c,$$
$$\mathrm{ad}_L(e_a) = e_c \otimes e_b - e_b \otimes e_c + \lambda q^2 e_a \otimes e_z,$$
$$\mathrm{ad}_L(e_d) = e_b \otimes e_c - e_c \otimes e_b - \lambda q^2 e_a \otimes e_z,$$
$$\mathrm{ad}_L(e_c) = (q^2 e_a - e_d) \otimes e_c - e_c \otimes e_z,$$
$$\mathrm{ad}_L(e_b) = ((q^2 - 1 - q^{-2})e_a + q^2 e_d) \otimes e_b + q^2 e_b \otimes e_z,$$

where $e_z = q^{-2} e_a - e_d$. We also have the crossed module actions

$$e_a \triangleleft a = q e_a, \quad e_a \triangleleft b = 0, \quad e_a \triangleleft c = 0, \quad e_a \triangleleft d = q^{-1} e_a,$$
$$e_b \triangleleft a = e_b, \quad e_b \triangleleft b = q\lambda e_a, \quad e_b \triangleleft c = 0, \quad e_b \triangleleft d = e_b,$$
$$e_c \triangleleft a = e_c, \quad e_c \triangleleft b = 0, \quad e_c \triangleleft c = q\lambda e_a, \quad e_c \triangleleft d = e_c,$$
$$e_d \triangleleft a = q^{-1} e_d, \quad e_d \triangleleft b = \lambda e_c, \quad e_d \triangleleft c = \lambda e_b, \quad e_d \triangleleft d = q e_d + q\lambda^2 e_a.$$

Proof Recall that $\lambda = 1 - q^{-2}$ and we had used the normalisation coming from the general construction which needs to be adjusted to have the standard classical limit. The partial derivatives are then read off from Lemma 2.61 and composing these with ϵ we can easily read off

$$\varpi\pi_\epsilon(a) = \frac{q}{[2]_q}(q e_a - e_d), \quad \varpi\pi_\epsilon(c) = e_b,$$
$$\varpi\pi_\epsilon(b) = e_c, \quad \varpi\pi_\epsilon(d) = \frac{1}{[2]_q}(q^2 e_d + (q^2 - q^{-1} - 1)e_a)$$

and on products such as

$$\varpi\pi_\epsilon(b^2) = 0, \quad \varpi\pi_\epsilon(bd) = e_c, \quad \varpi\pi_\epsilon(d^2) = (q^2 - q^{-2} - 1)e_a + q^2 e_d,$$

which, with $\Delta_R e_c = -qbe_b \otimes b^2 + q^2 e_z \otimes bd + e_c \otimes d^2$, gives us $\mathrm{ad}_R(e_c)$. Similarly for the rest of the $\mathrm{ad}_R, \mathrm{ad}_L$. The crossed module actions are implicit in the commutation relations in Example 2.59 and can be reconstructed from these and the formula $e_i f = f_{(1)}(e_i \triangleleft f_{(2)})$ where $i = a, b, c, d$. They can also be computed from the general formula in the proof of Corollary 2.57. □

We also have a bicovariant central quantum-symmetric metric

$$g = e_c \otimes e_b + q^2 e_b \otimes e_c + \frac{q^3}{(2)_q}(e_z \otimes e_z - \theta \otimes \theta)$$

for this calculus found in Proposition 2.60. What is important is that it is invariant under Δ_R and the crossed module action. Then we can apply Corollary 5.82.

Proposition 5.85 *For $\mathbb{C}_q[SL_2]$ with generic q and 4-dimensional calculus with its invariant central metric as above, there is a unique weak quantum Levi-Civita connection given by*

$$\alpha_a = -\alpha_d = \frac{q}{(4)_q} e_z, \quad \alpha_b = \frac{1}{(2)_{q^2}} e_b, \quad \alpha_c = \frac{1}{(2)_{q^2}} e_c,$$

$$\nabla e_a = -\nabla e_d = \frac{1}{(2)_{q^2}}\Big(e_b \otimes e_c - e_c \otimes e_b - \lambda \frac{q^3}{(2)_q} e_z \otimes e_z\Big),$$

$$\nabla e_b = \frac{1}{(2)_{q^2}}\Big(e_z \otimes e_b - q^2 e_b \otimes e_z\Big), \quad \nabla e_c = \frac{1}{(2)_{q^2}}\Big(-q^2 e_z \otimes e_c + e_c \otimes e_z\Big)$$

with curvature

$$R_\nabla e_a = -R_\nabla(e_d) = -\frac{q^2}{(2)_{q^2}^2}(e_c \wedge e_z \otimes e_b + q^{-2} e_b \wedge e_z \otimes e_c + \lambda e_b \wedge e_c \otimes e_z),$$

$$R_\nabla e_b = \frac{q^2}{(2)_{q^2}^2}(e_b \wedge e_z \otimes e_z + q^{-3}(2)_q e_b \wedge e_c \otimes e_b),$$

$$R_\nabla e_c = \frac{q^2}{(2)_{q^2}^2}(e_c \wedge e_z \otimes e_z - q^{-1}(2)_q e_b \wedge e_c \otimes e_c).$$

The connection is not regular nor a bimodule connection.

Proof We focus on finding the solutions; to show that they are the only ones is rather involved and we will only say a few words. We define the components of the connection by $\alpha_i = \alpha^j{}_i e_j$ and have to solve for this 4×4 matrix of elements of $\mathbb{C}_q[SL_2]$. Looking first at the torsion equation, we see that the coefficients of α are all to the left and hence its functional dependence is immaterial. We write out the equations using the form of ad_L and $\mathrm{d}e_a$ from Lemma 5.84 and match coefficients of a basis of Λ^2. This is a linear system allowing us to solve the torsion equation as

$$\alpha = \begin{pmatrix} x & q^2\lambda y & -q^{-2}\lambda z & -q^{-2}w \\ z & \frac{q^4+x(1-q^8)}{1+q^4} & 0 & q^{-6}z \\ y & 0 & \frac{1+x(q^6-q^{-2})}{1+q^4} & (q^4-1+q^{-2})y \\ -q^2x & -q^4\lambda y & \lambda z & w \end{pmatrix}$$

where x, y, z are arbitrary elements of $\mathbb{C}_q[SL_2]$ and $w = (1+q^6)/[4]_{q^2} - x(q^4 - q^2+q^{-2})$. If we assume for the moment that $\alpha^j{}_i$ are numbers (i.e., commute with the 1-forms) then the cotorsion equation is entirely similar, now with ad_R. Inserting the above gives a joint solution of this form if and only if $y = z = 0$ and $x = q^2/[4]_{q^2}$, which then simplifies to the solution stated in the theorem as the unique solution with constant coefficients.

To show that these are the only solutions, the method is as follows. Denoting our stated solution by α_0, we write $\alpha = \alpha_0 + \tau$ and write the above general form of α from the torsion equation in terms of τ with coefficients linear in $x' = x - q^2/[4]_{q^2}, y, z$. We then write out the cotorsion equation for τ but now, instead of assuming constant coefficients, we use commutation relations in the form $e_z f = \sum_i C_z{}^i(f)e_i$ for any element $f \in \mathbb{C}_q[SL_2]$, etc. to pass all coefficients to the left in a 2-form basis $\{e_{ab}, e_{ac}, e_{bd}, e_{cd}, e_{ad}, e_{bc}\}$, where $e_{ab} := e_a\wedge e_b$ is a shorthand. This gives us up to 18 linear equations for the three coefficients. They include $C_z{}^d(y) \propto y$ and $C_z{}^d(z) \propto z$ which, for generic q and the specific form of these operators from the commutation relations in the proof of Lemma 2.61, leads to $y, z = 0$. A similar argument gives $x' = 0$.

The covariant derivative ∇ is then computed from α according to the formula in the proof of Corollary 5.82, using ad_L listed above. That the connection is not regular is a direct computation using the form of α and the degree 2 exterior algebra relations and we omit the details. Its curvature is from F_α and the formulae in the proof of Corollary 5.81. One can also compute the curvature directly, remembering that d is in the geometric normalisation without the λ factor. That the connection is not a bimodule is a computation from the condition in Corollary 5.82. □

We see this more general theory of 'weak Levi-Civita connections' based on torsion and cotorsion freeness of a left connection is strictly needed and that there is a unique answer, so we cannot do better. It moreover obeys $\nabla\eta = 0$ in the classical limit, so it deforms the classical Levi-Civita connection. Indeed, $\alpha_z = q^{-2}\alpha_a - \alpha_d, \alpha_b, \alpha_c$ tend to the classical spin connection on SU_2 when suitable linear combinations of α_b, α_c are taken, thus deforming Example 5.72. As for our S_3 example, we also have $\nabla\theta = 0$ so that our extra cotangent direction is covariantly constant. We will see in Chap. 8 that in many cases one has a reasonable Ricci tensor built from R_∇ and in our $\mathbb{C}(S_3)$ and $\mathbb{C}_q[SL_2]$ examples this will be 'essentially' proportional to the metric, i.e., an 'Einstein space'.

Exercises for Chap. 5

E5.1 Show in the $*$-algebra case that the condition for a strong connection ω to be $*$-preserving in the sense of (5.8) has the same form in terms of the connection map $\omega^\sharp$ in Lemma 5.8. Show that if $\omega^\sharp$ is a strong connection map then

$$\omega^\sharp_{\mathrm{new}}(h) = \frac{1}{2}\left(\omega^\sharp(h) + \mathrm{flip}(* \otimes *)\omega^\sharp(*Sh)\right)$$

is such a $*$-preserving strong connection map.

E5.2 Show that in addition to our usual integer grading $|\ |$ on $\mathbb{C}_q[SL_2]$, there is a second one defined by $\|a\| = 1$, $\|d\| = -1$ and $\|b\| = \|c\| = 0$. Show that the combined $\mathbb{Z}^2$-grading $(|\ |, \|\ \|)$ does *not* give a strongly graded algebra in the sense of Lemma 5.25. [Hint: use the usual linear basis to write the general form of an element of grade $(1, 1)$ and an element of grade $(-1, -1)$.]

E5.3 Let $\mathbb{Z}_4 = \{1, z, z^2, z^3\}$ with $z^4 = 1$ and $H = P = \Bbbk(\mathbb{Z}_4)$ with a right coaction of H on P by the coproduct, $\Delta_R \delta_{z^j} = \sum_i \delta_{z^i} \otimes \delta_{z^{j-i}}$. Show that this gives a quantum principal bundle for the universal calculus with base $A = \Bbbk.1$ and find a universal strong connection. Now give H and P nonuniversal calculi by the subsets $\mathcal{C}_H = \{z^1\}$ and $\mathcal{C}_P = \{z^1, z^2\}$, where the coaction is differentiable (by left/right reversal of Exercise E4.2) with $\Delta_{R*} e_1 = 1 \otimes h_1 + e_1 \otimes 1$ and $\Delta_{R*} e_2 = e_2 \otimes 1$. Explain why this does *not* give a quantum principal bundle for these calculi.

E5.4 Let $P = \Bbbk(\mathbb{Z}_4)$ as in E5.3, $\mathbb{Z}_2 = \{1, t\}$ with $t^2 = 1$ and $H = \Bbbk(\mathbb{Z}_2)$ coact on P by $\Delta_R \delta_{z^i} = \delta_{z^i} \otimes \delta_{t^0} + \delta_{z^{i+2}} \otimes \delta_t$. Let $A \subseteq P$ be the invariant subalgebra. Give P and H their universal calculi with bases e^i and E^1 respectively corresponding to $\mathcal{C}_P = \{z, z^2, z^3\}$ and $\mathcal{C}_H = \{t\}$. [This is the $N = 2$ case of the discrete Möbius bundle as described in Example 5.49.] (i) Show that $\mathrm{ver}(e_a) = \delta_{a,2} E_1$. (ii) Show that there is a unital convolution invertible right comodule map $\Phi : H \to P$ with $\Phi(\delta_t) = \delta_{z^0} + \delta_{z^1}$, and find Φ^{-1}. (iii) Use Corollary 5.14 to find a strong connection (set $\alpha = 0$ and note that the formula there gives the connection map $\omega^\sharp(\delta_t)$ rather than the connection form $\omega(E_1)$ itself). (iv) Find $\nabla \delta_{z^1}$ for the associated connection on P.

E5.5 In Examples 5.12 and 5.47, we considered $P = \mathbb{C}\mathrm{Hg}$ as a principal bundle for $H = \mathbb{C}[\mathbb{T}^2]$ with unitary generators s, t and invariants $A = \mathbb{C}[S^1]$. Calculate the map $\mathrm{ver}^{0,m} : \Omega^m_{\mathbb{C}\mathrm{Hg}} \to \mathbb{C}\mathrm{Hg} \otimes \Lambda^m_H$ for $m = 1, 2$ applied to wedge products of e^u, e^v, e^w. Find its kernel and hence show that the following sequence in the statement of Lemma 5.60 is exact, namely in this case

$$0 \longrightarrow \Omega^1_{\mathbb{C}(S^1)} \wedge \Omega^{m-1}_{\mathbb{C}\mathrm{Hg}} \xrightarrow{i} \Omega^m_{\mathbb{C}\mathrm{Hg}} \xrightarrow{\mathrm{ver}^{0,m}} \mathbb{C}\mathrm{Hg} \otimes \Lambda^m_H \longrightarrow 0.$$

[Then Theorem 5.61 gives an alternative (and less computationally intensive) proof that we have a differential fibration to the one in Example 4.67.]

E5.6 Show in the general construction of Example 5.49 that the bundle $X \to X/G$ is strong if and only if $\mathcal{D}\mathcal{D}^{-1} \cap G = \{e\}$ (this says that if two arrows from within one orbit to within another have the same endpoint then they have the same starting point). [Hint: consider $\delta_{[x]}.\mathrm{d}\delta_{[y]}$ for equivalence classes $[x]$ and $[y]$.] Check that the condition does *not* hold for the Möbius example given in Example 5.49 (this is compatible with its canonical connection not being strong, as claimed in the text). Show, by contrast, that the condition *does* hold for $S_{n+1} \to S_{n+1}/S_n$ in Example 5.64, which is therefore a strong bundle.

E5.7 For the $n = 2$ case of Example 5.64 and $u = (12)$, $v = (23)$, $w = (13)$ in S_3, show that $\Lambda_{\mathcal{D}}$ has the relation $e_w \wedge e_v \wedge e_w = e_v \wedge e_w \wedge e_v$ in addition to $e_w^2 = e_v^2 = 0$. Show, by choosing as coset representatives powers of $uv = (123)$, that the subalgebra A has basis elements $\delta_{(uv)^m} + \delta_{(uv)^m u}$ which can be identified with $\delta_m \in \Bbbk(\mathbb{Z}_3)$, and an inherited calculus generated by

$$e_1 = xe_w + (1-x)e_v, \quad e_2 = xe_v + (1-x)e_w; \quad x = \delta_e + \delta_{uv} + \delta_{vu}.$$

Describe the corresponding calculus on $\Bbbk(\mathbb{Z}_3)$ and show that $\Lambda(\mathbb{Z}_3)$ has dimensions $1 : 2 : 2 : 1$ in the different degrees (and top degree 3). Show that $\mathrm{d}e_1 = 2e_1 \wedge e_1 - e_2 \wedge e_2$ and $\mathrm{d}e_2 = 2e_2 \wedge e_2 - e_1 \wedge e_1$. If $\Bbbk$ is has characteristic $\neq 2$, show that $\mathrm{H}_{\mathrm{dR}}(\mathbb{Z}_3)$ has dimensions $1 : 0 : 0 : 1$ in the different degrees. Use the Leray–Serre spectral sequence to show that $\mathrm{H}_{\mathrm{dR}}(S_3)$ with its usual 2-cycles calculus has dimensions $1 : 1 : 0 : 1 : 1$, as claimed in Example 1.60.

E5.8 Let M be a manifold and $P = C^\infty(M) \otimes \mathbb{C}(S_3)$ with their usual calculi a tensor product bundle by Example 5.55. Write a connection $\alpha : \Lambda^1_{S_3} \to \Omega^1(M)$ as three classical 1-forms $\alpha_a = \alpha(e_a)$ for $a = u, v, w$ and find the curvature 2-forms $F(\alpha)_a = F(\alpha)(\delta_a)$ for $a \neq e$ in terms of them. If ρ is a 2D representation of S_3 (such as in Example 1.48), find the corresponding sections and covariant derivative in terms of component classical fields on M.

E5.9 Prove that if (P, H, Δ_R) is a regular quantum principal bundle with the antipode of H invertible then there is a right-handed regular quantum principal bundle $P_R = P$ as an algebra but with structure quantum group $H_R = H^{\mathrm{op}}$ (i.e., with reversed product and usual coproduct) and $\Delta_L p = S^{-1} p_{(\bar{1})} \otimes p_{(\bar{0})}$. [This was used in Theorem 5.75.]

E5.10 Let $P = \mathbb{C}_{q,\theta}[\mathbb{T}^2]$ be the q-deformed noncommutative torus differential algebra as in Exercise E1.5 and $H = \mathbb{C}_q[S^1]$ with generators t, t^{-1} and coaction $\Delta_R(u^m v^n) = u^m v^n \otimes t^n$ (i.e., according to grade $|u| = 0, |v| = 1$). (i) Show that this is a strong quantum principal bundle with base $A = \mathbb{C}_q[S^1] = \mathbb{C}_{q,\theta}[\mathbb{T}^2]_0$ (the grade 0 subalgebra) and that it is trivial in the sense of §5.1.2 but now with the given differentials and $\Phi(t^m) = v^m$ differentiable.

(ii) Show that A is framed with $V = \mathbb{C}$ of grade 1 so that $\Omega^1 \cong \mathbb{C}_{q,\theta}[\mathbb{T}^2]_{-1}$ (the subspace of grade -1) via framing $\theta = vu^{-1}\mathrm{d}u$ and coframed by $\theta^* = f.(\mathrm{d}u)v^{-1}$ for any invertible $f \in A$ and find the associated quantum metric.
(iii) Show that $\omega(t^{-1}\mathrm{d}t) = v^{-1}\mathrm{d}v + \alpha\mathrm{d}u$ is a connection for any $\alpha \in A$ and find the associated covariant derivative on Ω^1. [See also Example 8.5 for which of these noncentral metrics on $\mathbb{C}_q[S^1]$ admit metric compatible connections.]

Notes for Chap. 5

The topological theory of classical principal and associated bundles is well established, and the reader may refer to [308]. The quantum group version as in Definition 5.39 and connections on them or 'quantum group gauge theory' was introduced by Brzeziński and the second author in the early 1990s [62, 63]. Hopf–Galois extensions had an earlier independent origin in the Hopf algebra literature as a generalisation of Galois theory and their role as generalising homogeneous spaces was particularly explored by Schneider [294] albeit without the geometric picture of a differential calculus and connections. It was later realised that they are a special case of quantum principal bundles, namely with the universal calculus. Apart from the change in context, the first part of Theorem 5.9 is due to [294]. Associated vector bundles $E = (P \otimes V)^H$ and connections in them were also in [62, 63] subject to a constraint on the connection. The constraint in the case of $E = P$ was termed a 'strong connection' in [133] and we have adopted this terminology.

The key property of strong connections on universal bundles in Lemma 5.8 as equivalent for the universal calculus to either of the bicomodule identities (5.9) appeared in work of the second author [207, Prop. 3.4]. This allows one to replace some of the more technical conditions in Hopf–Galois theory by the geometric assumption of the existence of a strong connection. The statement and proof of Proposition 5.10 ensuring that P is projective and faithfully flat in this case are taken from [92], with the exception of the coflatness of P, which is from [294].

Gauge theory on tensor product bundles was already in [62, 63], while the more general interpretation of cocycle extensions as trivial bundles and hence a nonabelian cohomology of trivial bundles was in [199], from which we took Proposition 5.13. The link between classical Galois theory and Hopf–Galois theory as in Proposition 5.15 has a long history, see [76, 169] and a nice account in the review by Montgomery [258]. The proof of Dedekind's lemma was taken from [82] and Example 5.18 from [127]. The interpretation of classical Galois extensions as cocycle cross products and hence trivial quantum bundles, as well as the remaining examples of the section, are our own account, but see [295].

The q-Hopf fibration example goes back in some version to [62] both with the universal calculus in Example 5.23 and later with the 3D calculus in Example 5.51, although cleaned up in later works such as [64, 134, 216]. For more on the *localisation* procedure of adding inverses to an algebra or ring, which we used in Example 5.24, see [125]. Strong grading as an approach to line bundles in Lemma 5.25 is from [315]. The argument that P_g invertible over A implies a strong grading as in Example 5.26 appeared in [264]. Theorem 5.28 is from [25].

The equivalence of categories between ${}_P\mathcal{M}^H$ and ${}_A\mathcal{M}$ in Theorem 5.30 is due to Schneider [294] but we have given a more accessible version assuming the existence of a strong connection for the universal calculus (our notion of a *regular* quantum principal bundle). A reference for the functorial results in §5.3, including the monoidal functor results and the results on crossed modules (often in more generality than we have covered) is Schauenburg's article [292]. For Takeuchi's equivalence of categories, see [313]. The particular variant of this equivalence given in Theorem 5.35 is taken from [270]. Giving categorical equivalences for the bimodule (bottom) line of (5.18) is more awkward than for the left module (top) line. Schauenburg [293] provides subcategories of the categories on the bottom line by adding more structure so that they are then categorically equivalent.

The interaction of quantum principal bundle theory and bimodule connections occupying much of the chapter is new. We have coined the term 'superstrong' for when the connection on P is a bimodule connection as characterised in Lemma 5.50, resulting in associated bundles as in Proposition 5.48 acquiring induced bimodule connections and the functor $\mathcal{M}^H \to {}_A\mathcal{E}_A^H$ in Proposition 5.52. The extra nice case of *strong bundles* in §5.4.2 grew out of experience with tensor bundles in [210] and the q-Hopf fibration but the general theory such as the monoidal functor $\mathcal{M}_H^H \to {}_A\mathcal{E}\mathcal{I}_A$ in Proposition 5.56 is new, as are the formulation of $*$-structures at this level and the spectral theory in §5.5. Examples 5.44 and 5.49 on the general construction of finite set principal bundles are from [239] but the discrete Möbius bundle and the finite quantum homogeneous bundles $S_n \subset S_{n+1} \to \mathbb{Z}_{n+1}$ in Example 5.64 are new.

The theory of quantum frame bundles is due to the second author in [207, 210, 216]. Its classical roots as well as ordinary frame bundle theory are in parabolic geometry, replacing SO_n by more general frame groups. The classical theory developing this into a generalised Riemannian geometry in which the metric need not be symmetric and the metric compatibility is a weaker 'cotorsion free' condition is due to the second author in [207]. This is of interest as a classical generalisation of Riemannian geometry that is self-dual under swap of vectors and covectors. This work introduced the concept of cotorsion and also covered the quantum case with universal calculus. Reference [210] covered tensor frame bundles and the Riemannian geometry of quantum groups, including $\mathbb{C}(S^3)$ in Example 5.83, while [216] covered the q-sphere. Theorem 5.77 that every quantum homogeneous space has a framing so that Ω^1 is an associated bundle is in [207, 216]. Proposition 5.85 for the Riemannian geometry of $\mathbb{C}_q[SL_2]$ is from [211] and more details of the proof can be found there. These connections arising in the frame bundle approach are studied further in Chap. 8. Another finite group example, which was Ricci flat, appeared in [268].

Quantum principal bundles and Hopf–Galois extensions in particular are a vast topic and we have only covered the basic theory with a slant towards the frame bundle theory and bimodule connections, both more relevant to the rest of the book.

In particular, it was quickly understood that to realise the monopole bundle over the 2-parameter Podleś spheres (see Example 3.16) one needed a more general theory where the fibre is merely a coalgebra. This leads to a theory of 'coideal subalgebras' but for which the differential geometry remains unclear. One can go a certain way with the bundle theory, however, see [65]. Another direction is braided Hopf algebras as fibres, of which the first step is to formulate the gauge theory in a braided category[206]. We have also not had room to cover important examples such as the q-deformation of the next Hopf fibration $S^7 \to S^4$ [176] among many others.

The exercises are mainly fresh examples but E5.6 is a general new result adding to the theory in [239] and E5.8 is a general result needed in the quantum frame bundle theory [207] but not previously worked out in detail.

Chapter 6
Vector Fields and Differential Operators

So far we have presented a largely one-sided view of differential geometry based on differential forms with little emphasis on vector fields. However, we did define a space of vector fields $\mathfrak{X}$ in terms of a choice of differential calculus Ω^1 in §2.7, which were also used in §5.6.1 under the notation Ω^{-1}. There are both left and right-handed versions and in the present chapter we work exclusively with the right vector fields, so $\mathfrak{X} = \mathfrak{X}^R$ in what follows. Classically, vector fields are dual to 1-forms, or rather in the classical case there is a duality between the tangent space and cotangent space at every point of a manifold, but this amounts to a duality over the coordinate algebra. To have this duality in the noncommutative world, we assume that Ω^1 is finitely generated projective as a right module throughout this chapter.

Vector fields on a classical manifold give derivations on the algebra of functions, but from our point of view this is an accident due to the fact that classically the left and right actions of functions on 1-forms coincide. Thus, classically, the elements of $\mathfrak{X}$ are bimodule maps from Ω^1 to A and we will see in (6.1) that such bimodule maps act as derivations on the algebra. Also recall from Chap. 4 that there are often not enough algebra maps in the noncommutative case and one has alternative notions of morphism such as asymptotic algebra maps, bimodules or positive maps. A derivation is an infinitesimal algebra map and similarly there are often not enough of them for a useful theory.

As a first application, recall that in §2.7 we studied invariant vector fields as a kind of Lie algebra of a Hopf algebra. In the present chapter we shall see how invariant vector fields on a Hopf algebra induce vector fields on any algebra equipped with a differentiable coaction of the Hopf algebra. Also as in the classical case, we can define the divergence of a vector field if we are given a right connection on $\mathfrak{X}$. Similarly, given a metric in the form of an element of $\mathfrak{X} \otimes_A \mathfrak{X}$, we can define the gradient of an element of the algebra as a vector field.

Next we consider differential operators defined using vector fields. Studying classical differential operators of arbitrary order already needs a lot of bookkeeping and the noncommutative case is not any easier. We start with a canonical action of

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vector fields on A-modules with left connection by covariant differentiation along the vector field. We then introduce a noncommutative filtered algebra $T\mathfrak{X}_\bullet$ built on the tensor algebra $T\mathfrak{X}$ of multivector fields but with a new product $\bullet$ and acting on every A-module with connection (E, ∇_E) in such a way as to give an isomorphism of categories ${}_A\mathcal{E} \cong {}_{T\mathfrak{X}_\bullet}\mathcal{M}$ (this is Proposition 6.15). In the classical case, ${}_A\mathcal{E}$ is a monoidal category (one can tensor product vector bundles with connection) which means that morally speaking $T\mathfrak{X}_\bullet$ is a cocommutative bialgebra over A. A similar situation holds for our algebra of differential operators $\mathcal{D}_A$, where $\mathcal{D}_A$-modules can be identified with A-modules with flat connection, i.e., ${}_A\mathcal{F} \cong {}_{\mathcal{D}_A}\mathcal{M}$ (this is Corollary 6.24). The latter classically have a tensor product and indeed $\mathcal{D}_A$ classically is again a cocommutative bialgebra over A or bialgebroid. Moreover, the associated graded algebra of $T\mathfrak{X}_\bullet$ is the tensor algebra $T\mathfrak{X}$ over A while classically the associated graded algebra of $\mathcal{D}_A$ is the symmetric algebra over A, so $T\mathfrak{X}_\bullet$ should be thought of as a 'free' version of the algebra of differential operators needed to encode all bundles with connection, not just flat ones.

In the general case, however, ${}_A\mathcal{E}$ is not monoidal and $T\mathfrak{X}_\bullet$ is not a bialgebra over A. We instead work with the monoidal category ${}_A\mathcal{E}_A$ of bimodules with bimodule connections, but this category is no longer isomorphic to the representations of $T\mathfrak{X}_\bullet$. Rather, we show that the latter is naturally a braided-commutative algebra in the prebraided category $\mathcal{Z}({}_A\mathcal{E}_A)$ given by the monoidal dual or centre construction in Theorem 2.47. The motivation behind this is technical and deferred to the start of §6.3, but as an object it means that we have a bimodule connection on $T\mathfrak{X}_\bullet$ which is a 'free' version of a canonical flat bimodule connection on $\mathcal{D}_A$. The latter is our version of the canonical flat connection on the algebra of differential operators in the classical theory of $\mathcal{D}$-modules.

As in §4.3, we think of an A-module with flat connection as a 'differential sheaf' over A and in view of its canonical flat bimodule connection, we refer to $\mathcal{D}_A$ more precisely as the 'sheaf of differential operators' on A in our setting. We construct it as a quotient of $T\mathfrak{X}_\bullet$ by a 2-sided ideal (which classically would just be generated by the commutator of partial derivatives $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$). In a representation of $T\mathfrak{X}_\bullet$, any vector fields become covariant derivatives and we accordingly define $\mathcal{D}_A$ in such a way that its modules are the flat left A-modules with connection among all left A-modules with connection. This time $\mathcal{D}_A$ with its canonical flat bimodule connection becomes a unital associative braided commutative algebra in ${}_A\mathcal{F}_A$, the centre of the category of flat extendable bimodule connections.

We will use $\mathbb{C}_q[S^2]$ as a nontrivial but not too complicated example to illustrate the theory at the end of each of the first three main sections of the chapter, see Examples 6.14, 6.20 and 6.29. The final section concludes with a variety of other examples. §6.5.1 covers invariant differential operators for the left-covariant calculus on a Hopf algebra, which classically would give the universal enveloping algebra of the Lie algebra of a Lie group. We see for example how the quantum and braided Lie algebras for $\Omega^1(S_3)$ from Chap. 2 enter. §6.5.2 shows that the sheaf of differential operators on a circle $\mathbb{C}_q[S^1]$ (with its q-calculus) comes out, not surprisingly, as a q-Weyl algebra in one variable, and identify a natural q-Witt algebra that maps to it. Finally, §6.5.3 covers differential operators on the algebra

$M_2(\mathbb{C})$. We show that $\mathcal{D}_{M_2} \cong M_2 \otimes B$, where B is the 'fermionic commutation relations algebra' in the sense of generators X, Y and anticommutator relations $\{X, Y\} = 1$.

6.1 Vector Fields and Their Action

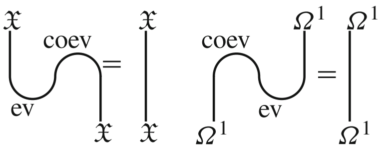
Classical vector fields are sections of the tangent bundle of a manifold, which is the dual bundle to the cotangent bundle. Locally, given coordinates $x^1, \ldots, x^m$ for a smooth manifold M, the cotangent bundle has basis $\mathrm{d}x^i$ and the dual basis for the tangent bundle is $\frac{\partial}{\partial x^j}$ (the pairing between these being 1 if $i = j$ and 0 if $i \neq j$). The vector fields act on functions by partial differentiation, and in particular they act as a derivation, i.e., $\frac{\partial}{\partial x^j}(fg) = \frac{\partial f}{\partial x^j} g + f \frac{\partial g}{\partial x^j}$ for all functions f, g. In fact, being a derivation on the algebra $C^\infty(M)$ and being dual to the 1-forms are equivalent.

In noncommutative geometry, we need to choose a left or right-handedness to express the duality between the vector fields and the 1-forms. Following the discussion in §2.7, we define the vector fields $\mathfrak{X}^R = \mathrm{Hom}_A(\Omega^1, A)$, the right module maps from Ω^1 to A. As mentioned earlier, having chosen this side we now just refer to $\mathfrak{X}$. This means that there is an evaluation map $\mathrm{ev} : \mathfrak{X} \otimes_A \Omega^1 \to A$, and using this we define the derivative of an algebra element $a \in A$ in the direction of a vector field $v \in \mathfrak{X}$ by $v \triangleright a = \mathrm{ev}(v \otimes \mathrm{d}a)$. This action of the vector fields on the algebra A is not by derivations in general since

$$v \triangleright (ab) = \mathrm{ev}(v \otimes (\mathrm{d}a.b + a.\mathrm{d}b)) = (v \triangleright a)b + \mathrm{ev}(v \otimes a.\mathrm{d}b); \tag{6.1}$$

in general, we cannot swap the order of v and a in the second term. Of course, if evaluation with v was also a left module map (hence a bimodule map) then v would act as a derivation, but this is too strong a condition to require in general.

In §3.1, we studied fgp modules and showed that their duals were well behaved (i.e., also fgp modules with evaluation and coevaluation maps). We assume throughout this chapter that the bimodule Ω^1 is fgp as a right A-module, which then means that we have both an evaluation map $\mathrm{ev} : \mathfrak{X} \otimes_A \Omega^1 \to A$ and a coevaluation map $\mathrm{coev} : A \to \Omega^1 \otimes_A \mathfrak{X}$ as previously discussed. In the diagrammatic notation of §2.4, the evaluation and coevaluation maps are related by



There is of course a certain symmetry here in that we could have defined the 1-forms Ω^1 as the dual of the vector fields. The idea of a differential calculus as a differential graded algebra extending A has proven to be the more popular so we regard Ω^1 as the fundamental object. However, a recurring theme in this chapter is that some constructions can be done both in terms of forms and in terms of fields, and classically having both approaches frequently makes things easier to understand.

In §2.7, we considered the tangent Lie algebra of a quantum group. Recall that a vector field $v : \Omega^1_H \to H$ is called left-invariant (we say $v \in {}^H\mathfrak{X}$) if it intertwines the left actions as follows,

$$\begin{tikzcd} \Omega^1_H \arrow[r, "\Delta_L"'] \arrow[d, "v"] & H \otimes \Omega^1_H \arrow[d, "\mathrm{id}\otimes v"] \\ H \arrow[r, "\Delta"] & H \otimes H \end{tikzcd}$$

It was explained in Lemma 2.82 that the left-invariant vector fields are the dual of the left-invariant 1-forms Λ^1 and form a quantum Lie algebra $\mathfrak{h}$. Classically on a Lie group, there is a 1-1 correspondence between the left-invariant vector fields and the tangent space at the group identity, which latter point of view can be useful. On a Hopf algebra, the counit ϵ plays the role of evaluation at the identity and we can make a similar definition of vector fields or 'tangent space' $T_\epsilon H$ at the identity.

Lemma 6.1 *Let Ω^1_H be a left-covariant calculus on a Hopf algebra H and $\Lambda^1 \subset \Omega^1_H$ the subspace of left-invariant 1-forms.*

(1) *There is a 1-1 correspondence between left-invariant vector fields $v : \Omega^1_H \to H$ and linear maps*

$$T_\epsilon H := \{\alpha : \Omega^1_H \to \mathbb{k} : \alpha(\xi.h) = \alpha(\xi).\epsilon(h) \text{ for all } h \in H,\ \xi \in \Omega^1_H\}$$

given by $\alpha = \epsilon \circ v$ and $v = (\mathrm{id} \otimes \alpha)\Delta_L$.

(2) *If the antipode of H is invertible then the right factorisation*

$$\Omega^1_H = \Lambda^1.H, \quad (\mathrm{d}h).g = ((\mathrm{d}h_{(3)})S^{-1}h_{(2)}).h_{(1)}g$$

induces a further identification $T_\epsilon H = (\Lambda^1)^*$.

Proof (1) We check that the stated formulae land in the correct spaces. First for $\xi \in \Omega^1_H$ and $h \in H$, $\epsilon \circ v(\xi.h) = \epsilon(v(\xi)h) = \epsilon(v(\xi))\epsilon(h)$. Going the other way,

$$(\mathrm{id} \otimes \alpha)\Delta_L(\xi.h) = \xi_{(1)}h_{(1)}.\alpha(\xi_{(\bar{\infty})}h_{(2)}) = (\mathrm{id} \otimes \alpha)\Delta_L(\xi).h.$$

To check the 1-1 correspondence,

$$\epsilon(\mathrm{id}\otimes\alpha)\Delta_L(\xi)=\epsilon(\xi_{(1)}).\alpha(\xi_{(\bar\infty)})=\alpha(\xi),$$
$$(\mathrm{id}\otimes\epsilon\circ v)\Delta_L(\xi)=\xi_{(1)}.\epsilon\circ v(\xi_{(\bar\infty)})=v(\xi)_{(1)}.\epsilon(v(\xi)_{(2)})=v(\xi)$$

as v intertwines the left action. Finally, for $(\mathrm{id}\otimes\alpha)\Delta_L$ to intertwine the coaction,

$$\Delta(\mathrm{id}\otimes\alpha)\Delta_L(\xi)=\Delta(\xi_{(1)}).\alpha(\xi_{(\bar\infty)})=\xi_{(1)}\otimes\xi_{(2)}.\alpha(\xi_{(\bar\infty)}),$$
$$(\mathrm{id}\otimes(\mathrm{id}\otimes\alpha)\Delta_L)\Delta_L(\xi)=\xi_{(1)}\otimes(\mathrm{id}\otimes\alpha)\Delta_L(\xi_{(\bar\infty)})=\xi_{(1)}\otimes\xi_{(2)}.\alpha(\xi_{(\bar\infty)}).$$

(2) Given $\alpha\in T_\epsilon H$, we can obviously just restrict it to Λ^1. Conversely, given the right factorisation, any linear map $\alpha:\Lambda^1\to\Bbbk$ extends to an element of $T_\epsilon H$ by $\alpha(\omega.h)=\alpha(\omega)\epsilon(h)$ for all $\omega\in\Lambda^1$ and $h\in H$.

It is clear that $(\mathrm{d}h_{(2)})S^{-1}h_{(1)}=-h_{(2)}\mathrm{d}S^{-1}h_{(1)}=-\varpi(S^{-1}h)\in\Lambda^1$ for all $h\in H$ giving the factorisation stated, where $\varpi(h)=Sh_{(1)}\mathrm{d}h_{(2)}$ is the usual construction of left-invariant 1-forms in Chap. 2. This right factorisation is equivalent the left factorisation $\Omega^1_H=H.\Lambda^1$ used in most of the book since, if the antipode is invertible, the right crossed module braiding

$$\Psi_{\Lambda^1,H}:\Lambda^1\otimes H\to H\otimes\Lambda^1,\quad \Psi_{\Lambda^1,H}(\omega\otimes h)=h_{(1)}\otimes\omega\triangleleft h_{(2)}$$

inherent in the commutation relations $\omega h=h_{(1)}(\omega\triangleleft h_{(2)})$ is invertible. □

Given a differentiable action of a Lie group on a manifold, an element of the Lie algebra gives a vector field on the manifold. The next result is a left-handed noncommutative version of this with vector fields $\mathfrak{X}_A$ on an algebra A.

Corollary 6.2 *Given a left H-comodule algebra A with differentiable left H-coaction $\Delta_L:A\to H\otimes A$ (see Definition 4.28), there is a map* ver $:T_\epsilon H\to\mathfrak{X}_A$ *with* ver(α) *given by*

$$\Omega^1_A\xrightarrow{\delta_L}\Omega^1_H\otimes A\xrightarrow{\alpha\otimes\mathrm{id}}A.$$

Proof For $\alpha\in T_e H$, we have

$$\mathrm{ver}(\alpha):(\mathrm{d}a)b\longmapsto(\mathrm{d}a_{(1)})b_{(1)}\otimes a_{(\bar\infty)}b_{(\bar\infty)}\longmapsto\alpha(\mathrm{d}a_{(1)})a_{(\bar\infty)}b$$

for all $a,b\in H$, so that $\mathrm{ver}(\alpha)((\mathrm{d}a)b)=(\mathrm{ver}(\alpha)(\mathrm{d}a))b$ and $\mathrm{ver}(\alpha)\in\mathfrak{X}_A$. □

An easy example is provided by a finite group acting differentially in a graph.

Example 6.3 Let G be a finite group with calculus given by $\mathcal{C} \subseteq G \setminus \{e\}$ acting as automorphisms of a finite directed graph X, as in Example 4.30. For $b \in \mathcal{C}$ we have a dual basis f_b for the $e_a \in \Lambda^1_G$ with $\mathrm{ev}(f_b \otimes e_a) = \delta_{b,a}$ The vector field f_b corresponds to $\epsilon \circ f_b : \Omega^1_G \to \Bbbk$, and so has value on $\omega_{w\to z}$ given by

$$\begin{aligned}\mathrm{ver}(\epsilon \circ f_b)(\omega_{w\to z}) &= (\epsilon \circ \mathrm{ev} \otimes \mathrm{id})(f_b \otimes \Delta_L(\omega_{w\to z}))\\ &= (\epsilon \circ \mathrm{ev} \otimes \mathrm{id})(f_b \otimes \sum_{c\in\mathcal{C}} \delta_{z,cw} \sum_{g\in G} \delta_{c^{-1}g}.e_{g^{-1}cg} \otimes \delta_{g^{-1}z})\\ &= \sum_{c\in\mathcal{C}} \delta_{z,cw} \sum_{g\in G} \epsilon \circ \mathrm{ev}(f_b \otimes e_{g^{-1}cg}.\delta_g) \otimes \delta_{g^{-1}z}\\ &= \sum_{c\in\mathcal{C}} \delta_{z,cw} \sum_{g\in G} \delta_{b,g^{-1}cg}\delta_{g,e}\delta_{g^{-1}z} = \delta_{z,bw}\delta_z\end{aligned}$$

which, as expected, is a right module map from 1-forms to functions on the graph. ◇

Corollary 6.2 works just as well for right coactions, in which case the above notation ver aligns with Chap. 5 on quantum principal bundles and framings. Thus, given a differentiable *right* coaction Δ_R of H on P in Definition 4.28 (with left and right swapped), we have

$$\mathrm{ver}(\alpha) : \Omega^1_P \xrightarrow{\delta_R} P \otimes \Omega^1_H \xrightarrow{\mathrm{id}\otimes\alpha} P \tag{6.2}$$

whereby an element $\alpha \in T_\epsilon H$ gives a right vector field on P. In Chap. 5 we called these vector fields 'vertical' as they give zero when evaluated against the weakly horizontal forms (i.e., classically they are in the fibre direction). This is equivalent to the map $\mathrm{ver} : \Omega^1_P \to P \otimes \Lambda^1$ in Chap. 5 where we followed δ_R by the isomorphism (5.20) based on the usual identification $\Omega^1_H = H.\Lambda^1$, whereas now we use $T_\epsilon H$.

Example 6.4 We explicitly calculate the vertical vector fields, their action, and then the weakly horizontal forms for the quantum Hopf fibration in Example 5.23. Here $H = \mathbb{C}_{q^2}[S^1]$, with generator t, and we have a single basis element $\alpha \in T_\epsilon H$ given by $\alpha(t^{-1}\mathrm{d}t) = 1$. The 3D calculus on $P = \mathbb{C}_q[SU_2]$ in Example 2.32 has left-invariant generators $e^0, e^\pm$. The dual basis is $f_0, f_\pm$ as vector fields

$$f_i(e^j.a) = \delta_{ij}a, \quad f_0.a = q^{-2|a|}a.f_0, \quad f_\pm.a = q^{-|a|}a.f_\pm$$

for $a \in \mathbb{C}_q[SU_2]$ of grade $|a|$. The extension of Δ_R by δ_R on Ω^1_P from Example 4.33 can be used to calculate the vector field corresponding to α using (6.2). It sends

$$e^0 \longmapsto 1 \otimes t^{-1}\mathrm{d}t \longmapsto 1, \quad e^\pm \longmapsto 0 \longmapsto 0,$$

and so is just f_0. Hence the vertical vector fields are spanned over P by f_0 and from this point of view the weakly horizontal forms are those which give zero when evaluated against f_0. Hence the weakly horizontal forms are generated by $e^{\pm}$. $\diamond$

Next, the reader will likely be familiar with divergence and gradient from vector calculus. The divergence assigns a scalar field to a vector field, and the gradient assigns a vector field v to a scalar field or function f. In the simple case of $\mathbb{R}^3$ with coordinates (x, y, z) and the Euclidean metric we have the formulae

$$\mathrm{div}(v_1, v_2, v_3) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}, \quad \mathrm{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

If we have a right connection $\tilde\nabla : \mathfrak{X} \to \mathfrak{X} \otimes_A \Omega^1$ then the noncommutative divergence of a vector field is given by $\mathrm{div} = \mathrm{ev} \circ \tilde\nabla : \mathfrak{X} \to A$. For the gradient, we need a metric or similar construction to give a map from Ω^1 to $\mathfrak{X}$ which can be composed with $\mathrm{d} : A \to \Omega^1$ so that the result is a vector field. As for framings in §5.6, it is convenient to express a metric as a bimodule isomorphism $\tilde g : \mathfrak{X} \to \Omega^1$. Then the divergence of the gradient is yet another route to arriving at a Laplacian.

Example 6.5 On the q-sphere $A = \mathbb{C}_q[S^2]$ we take the standard calculus in Proposition 2.35 given by zero grade elements of the module generated by e^+ and e^- in Example 6.4. We adopt the notation $\mathrm{d}s = \partial_+ s e^+ + \partial_- s e^- + \partial_0 s e^0$ for the calculus on $\mathbb{C}_q[SL_2]$ and write $\xi = \xi_+ e^+ + \xi_- e^- \in E_2 \oplus E_{-2} = \Omega^1$, where $|\xi_\pm| = \mp 2$, then the natural quantum Levi-Civita bimodule connection ∇ on Ω^1 in Example 5.79 defined by horizontal projection $(\mathrm{id} - \Pi)$ of $\mathrm{d}s$ (killing the e^0 component) is

$$\nabla(\xi_+ e^+) = (\partial_+\xi_+ e^+ + \partial_-\xi_+ e^-)\tilde D_1 \tilde D_1' \otimes \tilde D_2' \tilde D_2 e^+, \tag{6.3}$$

$$\nabla(\xi_- e^-) = (\partial_+\xi_- e^+ + \partial_-\xi_- e^-) D_1 D_1' \otimes D_2' D_2 e^-, \tag{6.4}$$

where as usual $\tilde D_1 \otimes \tilde D_2 = a \otimes d - q^{-1} c \otimes b$ and $D_1 \otimes D_2 = d \otimes a - q b \otimes c$.

We now dualise this to obtain a right quantum Levi-Civita connection $\tilde\nabla$ on vector fields. The most natural way to do this gives $\tilde\nabla$ on the left vector fields $\Omega^{-1} = \mathfrak{X}^L$ as we did effectively in Theorem 5.75. For our present purposes, we first convert the above quantum Levi-Civita connection on Ω^1 to a right version $\sigma^{-1}\nabla$ using its generalised braiding, then the natural dual of *that* is a left bimodule connection on $\mathfrak{X}$ (both of these are stated later when we need them, see Example 6.14) which we convert by its σ^{-1} to a right bimodule connection on $\mathfrak{X}$. The result is

$$\tilde\nabla(f_+ v_+) = f_+ \tilde D_1 \tilde D_1' \otimes \tilde D_2' \tilde D_2 (\partial_+ v_+ e^+ + \partial_- v_+ e^-),$$

$$\tilde\nabla(f_- v_-) = f_- D_1 D_1' \otimes D_2' D_2 (\partial_+ v_- e^+ + \partial_- v_- e^-),$$

where $\{f_+, f_-\}$ is a left-invariant dual basis to $\{e^+, e^-\}$ and we write $v = f_+v_+ + f_-v_- \in \mathfrak{X}$ with $|v_\pm| = \pm 2$. This amounts to $\mathfrak{X} = E_{-2} \oplus E_2$ and

$$\tilde{\nabla}(f_+v_+ + f_-v_-) = f_+\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2(\mathrm{id} - \Pi)\mathrm{d}v_+ + f_-D_1D_1' \otimes D_2'D_2(\mathrm{id} - \Pi)\mathrm{d}v_-$$

as a right-handed version of the associated bundle connections in Example 5.51.

To have a gradient, we take the metric g from Proposition 2.36, which can be written in terms of dual bases as $(\mathrm{id} \otimes \tilde{g})\mathrm{coev}$, where

$$\mathrm{coev} = e^+D_1D_1' \otimes D_2'D_2f_+ + e^-\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2f_- \in \Omega^1 \otimes_A \mathfrak{X}$$

and the A-bimodule map $\tilde{g} : \mathfrak{X} \to \Omega^1$ is given by $\tilde{g}(f_+v_+) = -qv_+e^-$ and $\tilde{g}(f_-v_-) = -qv_-e^+$ for $v_\pm \in \mathbb{C}_q[SU_2]$ of grade ± 2 respectively. Then the gradient of $y \in A$ is given by $\mathrm{grad}\, y = \tilde{g}^{-1}(\mathrm{d}y)$. On $z = cd \in \mathbb{C}_q[S^2]$ from Proposition 2.35, we have $\mathrm{d}z = d^2e^+ + c^2e^-$, hence

$$\mathrm{grad}\, z = -d^2f_- - q^{-2}c^2f_+ = -q^{-2}f_-d^2 - f_+c^2.$$

Composing the gradient and divergence gives a Laplacian

$$\mathrm{div} \circ \mathrm{grad}\, z = \mathrm{ev}\big(-q^{-2}f_- \otimes \mathrm{d}(d^2) - f_+ \otimes \mathrm{d}(c^2)\big) = -(q + q^{-1})(q^2 + q^{-2})z$$

in agreement with the Laplacian calculation in (5.40) as one might expect due to the use of the quantum Levi-Civita connection. In principle, the divergence approach to the Laplacian can still differ from the geometric Laplacian in Lemma 8.6 due to the placement of the metric. ◇

Classical covariant derivatives are often expressed in the direction of a vector field, with $\nabla_v s$ being the covariant derivative of the section s in the direction of v. We can do the same thing in noncommutative geometry; given (E, ∇_E) a left connection on a left A-module E and $v \in \mathfrak{X}$, $e \in E$, we define

$$\nabla_v e := (\mathrm{ev} \otimes \mathrm{id})(v \otimes \nabla_E e).$$

One matter to consider is the classical definition of curvature in terms of vector fields, as given in (3.13), and how it is related to the form definition in Definition 3.18.

Proposition 6.6 *Consider an antisymmetric tensor product of vectors $v \otimes w \in \Lambda^2\mathfrak{X} \subseteq \mathfrak{X} \otimes \mathfrak{X}$ (summation implicit), and (E, ∇_E) a left connection on a left A-module E. Then, using the pairing of antisymmetric tensor products of vectors with Ω^2 and the generalisation of Lie bracket of vector fields given in §2.7,*

$$\nabla_v\nabla_w - \nabla_{[\![v,w]\!]_R} = -(\mathrm{ev} \otimes \mathrm{id})(v \otimes w \otimes R_E).$$

Proof Writing $\nabla_E e = \xi \otimes f$ (summation implicit),

$$\begin{aligned}\nabla_v \nabla_w e &= \nabla_v(\mathrm{ev}(w \otimes \xi) f) \\ &= \mathrm{ev}(v \otimes \mathrm{d}\,\mathrm{ev}(w \otimes \xi)) f + (\mathrm{ev} \otimes \mathrm{id})(v \otimes \mathrm{ev}(w \otimes \xi)\nabla f).\end{aligned}$$

We also have, for $\mathrm{d}\xi = \eta \wedge \kappa$ (summation implicit),

$$\begin{aligned}\nabla_{[\![v,w]\!]_R} e &= (\mathrm{ev} \otimes \mathrm{id})([\![v, w]\!]_R \otimes \xi \otimes f) \\ &= \mathrm{ev}(\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id})(v \otimes w \otimes \eta \otimes \kappa) f + \mathrm{ev}(v \otimes \mathrm{d}\,\mathrm{ev}(w \otimes \xi)) f.\end{aligned}$$

Now we have

$$(\nabla_v \nabla_w - \nabla_{[\![v,w]\!]_R})e = (\mathrm{ev} \otimes \mathrm{id})(v \otimes \mathrm{ev}(w \otimes \xi)\nabla f) - \mathrm{ev}(v \otimes \mathrm{ev}(w \otimes \eta)\kappa) f.$$

We then use $R_E(e) = \eta \wedge \kappa \otimes f - \xi \wedge \nabla f$ and that $v \otimes w$ has a well-defined evaluation with Ω^2. □

The minus sign in Proposition 6.6 comes from the order of pairing used for the forms and vector fields, and that order is chosen to avoid the need for any generalised braidings. Later on it will be convenient to write $\nabla_v e = v \triangleright e$ using the notation for an action.

We have noted that one reason why it has been more popular to work with differential forms rather than vector fields as the primary object in noncommutative differential geometry is that the forms form an algebra with d a derivation, whereas the derivation-type properties for vector fields are more complicated. Also, for vector fields a $*$ operation naturally swaps left and right vector fields since if $v \in \mathfrak{X}^R$ then the map $\xi \mapsto v(\xi^*)^*$ for $\xi \in \Omega^1$ gives an element of $\mathfrak{X}^L$.

6.2 Higher Order Differential Operators

In the last section, we considered the action of a single vector field on an algebra, and later we will also have actions on modules with connection. However, to have an action we should have an algebra acting, and this requires actions of multiple vector fields giving higher order differential operators. It is reasonable to wish to view these operators as actions of tensor products of vector fields, but such an identification is not entirely trivial. In particular, we will require a connection (called $\heartsuit$ here) on vector fields and 1-forms to achieve this. As we will require several copies of 1-forms and vector fields on A, we define

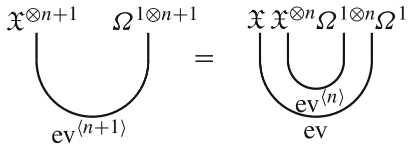
$$\begin{aligned}&\mathfrak{X}^{\otimes 0} = A, \quad \mathfrak{X}^{\otimes n} = \mathfrak{X} \otimes_A \mathfrak{X} \otimes_A \cdots \otimes_A \mathfrak{X}, \\ &\Omega^{1\otimes 0} = A, \quad \Omega^{1\otimes n} = \Omega^1 \otimes_A \Omega^1 \otimes_A \cdots \otimes_A \Omega^1,\end{aligned}$$

where we have n copies of $\mathfrak{X}$ and Ω^1. Note that the definition of $\mathfrak{X}^{\otimes n}$ and $\Omega^{1\otimes n}$ uses $\otimes_A$, the tensor product over the algebra. We will sometimes use $\mathrm{id}^{\otimes n}$ as the identity on $\mathfrak{X}^{\otimes n}$ or $\Omega^{1\otimes n}$. As for the duality of braided tensor algebras in §2.6, we define n-fold evaluation $\mathrm{ev}^{\langle n\rangle} : \mathfrak{X}^{\otimes n} \otimes_A \Omega^{1\otimes n} \to A$ and n-fold coevaluation $\mathrm{coev}^{\langle n\rangle} : A \to \Omega^{1\otimes n} \otimes_A \mathfrak{X}^{\otimes n}$ in a nested way, which we specify recursively by

$$\mathrm{ev}^{\langle 1\rangle} = \mathrm{ev}, \quad \mathrm{ev}^{\langle n+1\rangle} = \mathrm{ev}(\mathrm{id} \otimes \mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id}),$$

$$\mathrm{coev}^{\langle 1\rangle} = \mathrm{coev}, \quad \mathrm{coev}^{\langle n+1\rangle} = (\mathrm{id} \otimes \mathrm{coev}^{\langle n\rangle} \otimes \mathrm{id})\mathrm{coev}.$$

In diagrammatic terms, the first of these is



and there is a similar upside down version of this for coevaluation.

In classical geometry, we form differential operators locally by using partial derivatives with local coordinates. We do not have this option in noncommutative geometry and instead we use connections. Here a single derivative for a bundle E is given by a connection $\nabla_E : E \to \Omega^1 \otimes_A E$ but for a second derivative $\nabla_E^{(2)} : E \to \Omega^{1\otimes 2} \otimes_A E$ we will need a bimodule connection on Ω^1 as well as ∇_E. We use the symbol $\heartsuit$ for this required right bimodule connection on Ω^1. There will be an induced dual left connection on the vector fields $\mathfrak{X}$ which we also denote by $\heartsuit$. There are also induced bimodule connections on tensor products of $\mathfrak{X}$ and tensor products of Ω^1 in the usual way and we will denote these by $\heartsuit^{\langle n\rangle}$.

To this end, we let $(\Omega^1, \heartsuit, \sigma^{-1})$ be a right bimodule covariant derivative, which we recall means that $\heartsuit : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ obeys

$$\heartsuit(\xi.a) = \heartsuit(\xi).a + \xi \otimes \mathrm{d}a, \quad \heartsuit(a.\xi) = a.\heartsuit(\xi) + \sigma^{-1}(\mathrm{d}a \otimes \xi)$$

for $\xi \in \Omega^1$ and $a \in A$. Here $\sigma^{-1} : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ is a bimodule map, and we shall assume that it is invertible with inverse σ. The use of σ^{-1} is a convention so as to be compatible with the use of σ for left bimodule covariant derivatives. Then by (3.17) there is a dual left-covariant derivative $\heartsuit : \mathfrak{X} \to \Omega^1 \otimes_A \mathfrak{X}$ such that (3.16) holds, which in our case is shown in Fig. 6.1. Here $\heartsuit$ on $\mathfrak{X}$ is a left bimodule

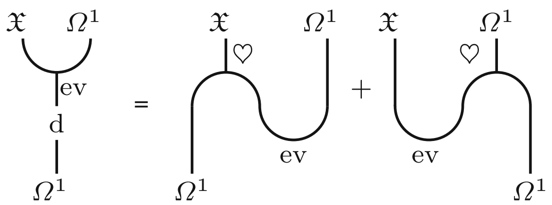
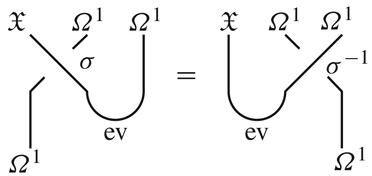


Fig. 6.1 Duality for the $\heartsuit$ bimodule connections on one forms and vector fields

covariant derivative with another $\sigma : \mathfrak{X} \otimes_A \Omega^1 \to \Omega^1 \otimes_A \mathfrak{X}$ defined by

$$\sigma = (\mathrm{ev} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{coev}). \tag{6.5}$$

Note that this σ and the map σ^{-1} that was part of the connection on Ω^1 are not inverses of each other, as they have different domains. Equivalently, using the usual properties of evaluation and coevaluation, one has



For multiple derivatives, we need to set up machinery to allow us to differentiate tensor products $\Omega^{1\otimes n}$ and $\mathfrak{X}^{\otimes n}$ as easily as a single copy of Ω^1. A railway analogy (in line with our diagrammatic notation) would be that we are converting a single track to multiple tracks. We extend $\heartsuit$ in the usual way for a right bimodule covariant derivative to $\heartsuit^{\langle n\rangle} : \Omega^{1\otimes n} \to \Omega^{1\otimes n+1}$, which means recursively by

$$\heartsuit^{\langle 0\rangle} = \mathrm{d}, \quad \heartsuit^{\langle 1\rangle} = \heartsuit, \quad \heartsuit^{\langle n+1\rangle} = \mathrm{id}^{\otimes n} \otimes \heartsuit + (\mathrm{id}^{\otimes n} \otimes \sigma^{-1})(\heartsuit^{\langle n\rangle} \otimes \mathrm{id}^{\otimes 1}).$$

Similarly, $\heartsuit$ on vector fields extends as usual for a left bimodule connection to $\heartsuit^{\langle n\rangle} : \mathfrak{X}^{\otimes n} \to \Omega^1 \otimes_A \mathfrak{X}^{\otimes n}$, which means recursively by

$$\heartsuit^{\langle 0\rangle} = \mathrm{d}, \quad \heartsuit^{\langle 1\rangle} = \heartsuit, \quad \heartsuit^{\langle n+1\rangle} = \heartsuit \otimes \mathrm{id}^{\otimes n} + (\sigma \otimes \mathrm{id}^{\otimes n})(\mathrm{id}^{\otimes 1} \otimes \heartsuit^{\langle n\rangle}).$$

Our notation here is justified since $\heartsuit^{\langle n\rangle}$ on $\mathfrak{X}^{\otimes n}$ is the dual left bimodule connection to $\heartsuit^{\langle n\rangle}$ on $\Omega^{1\otimes n}$ (in the sense of Fig. 6.1 but on tensor products), i.e.,

$$\mathrm{d}\,\mathrm{ev}^{\langle n\rangle} = (\mathrm{id} \otimes \mathrm{ev}^{\langle n\rangle})(\heartsuit^{\langle n\rangle} \otimes \mathrm{id}) + (\mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id})(\mathrm{id} \otimes \heartsuit^{\langle n\rangle}) \tag{6.6}$$

as maps $\mathfrak{X}^{\otimes n} \otimes_A \Omega^{1\otimes n} \to \Omega^1$. This follows by induction on n, where the case $n = 1$ is true by definition of $\heartsuit$ on $\mathfrak{X}$. Assuming the result for n, the $n + 1$ case follows by Corollary 3.81 with $E = \mathfrak{X}$ and $F = \mathfrak{X}^{\otimes n}$, recalling that $\mathfrak{X}^\flat = \Omega^1$.

We now define the multiple derivative $\nabla_E{}^{(n)} : E \to \Omega^{1\otimes n} \otimes_A E$ of a left module with connection (E, ∇_E) recursively by

$$\nabla_E{}^{(1)} = \nabla_E, \quad \nabla_E{}^{(n+1)} = (\heartsuit^{\langle n\rangle} \otimes \mathrm{id} + \mathrm{id}^{\otimes n} \otimes \nabla_E)\, \nabla_E{}^{(n)}. \tag{6.7}$$

This should not be confused with ${\nabla_E}^{[n]}$ defined in (4.2). We check that $\heartsuit^{\langle n\rangle}\otimes \mathrm{id}+\mathrm{id}^{\otimes n}\otimes\nabla_E$ is a well-defined operation on $\Omega^{1\otimes n}\otimes_A E$,

$$(\heartsuit^{\langle n\rangle}\otimes \mathrm{id}+\mathrm{id}^{\otimes n}\otimes\nabla_E)(\underline{\xi}.a\otimes e)=\heartsuit^{\langle n\rangle}(\underline{\xi}).a\otimes e+\underline{\xi}\otimes \mathrm{d}a\otimes e+\underline{\xi}.a\otimes\nabla_E e,$$
$$(\heartsuit^{\langle n\rangle}\otimes \mathrm{id}+\mathrm{id}^{\otimes n}\otimes\nabla_E)(\underline{\xi}\otimes a.e)=\heartsuit^{\langle n\rangle}(\underline{\xi})\otimes a.e+\underline{\xi}\otimes \mathrm{d}a\otimes e+\underline{\xi}\otimes a.\nabla_E e$$

for $a\in A$, $e\in E$ and $\underline{\xi}\in\Omega^{1\otimes n}$, where we underline to indicate the multiple tensor product. We have an 'action' of $\underline{v}\in\mathfrak{X}^{\otimes n}$ on $e\in E$ by this multiple derivative,

$$\underline{v}\triangleright e=(\mathrm{ev}^{\langle n\rangle}\otimes\mathrm{id}_E)(\underline{v}\otimes\nabla_E^{(n)}e). \tag{6.8}$$

Example 6.7 We consider partial derivative as a classical connection ∇ on $C^\infty(\mathbb{R}^n)$ with coordinates $(x^1,\dots,x^n)$, and take $\heartsuit$ characterised by $\heartsuit(\mathrm{d}x^i)=0$ and $\heartsuit(\frac{\partial}{\partial x^i})=0$. Then the formulae for $\nabla^{(2)}$ and the action (6.8) give

$$\nabla^{(2)}(f)=\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\frac{\partial^2 f}{\partial x^i\partial x^j},\quad (\frac{\partial}{\partial x^j}\otimes\frac{\partial}{\partial x^i})\triangleright f=\frac{\partial^2 f}{\partial x^i\partial x^j}. \qquad\diamond$$

Example 6.8 Let H be a Hopf algebra with invertible antipode and left-covariant differential calculus $(\Omega,\mathrm{d},\wedge)$. We first construct a functor ${}_H\mathcal{M}^H\to{}_H\mathcal{F}$ from the category of Hopf modules to the category of left H-modules with flat connection. Here ${}_H\mathcal{M}^H$ denotes the category of left-right Hopf modules, i.e., right H-comodules E which are also left H-modules and such that the action $\triangleright: H\otimes E\to E$ is a comodule map (similarly to the left-left case in Definition 2.16). Morphisms intertwine both the action and the coaction. We map such an object $(E,\triangleright,\Delta_R)$ to (E,∇_E), where

$$\nabla_E e=(\mathrm{d}e_{(\bar 1)(2)})S^{-1}e_{(\bar 1)(1)}\otimes e_{(\bar 0)}\in\Omega^1\otimes_H E$$

for all $e\in E$, which actually lives in $\Lambda^1\otimes E$. We check that this is a left connection,

$$\begin{aligned}\nabla_E(h.e)&=(\mathrm{d}(h_{(3)}e_{(\bar 1)(2)}))S^{-1}(h_{(2)}e_{(\bar 1)(1)})\otimes h_{(1)}.e_{(\bar 0)}\\&=(\mathrm{d}(he_{(\bar 1)(2)}))S^{-1}e_{(\bar 1)(1)}\otimes e_{(\bar 0)}\\&=(\mathrm{d}h)e_{(\bar 1)(2)}S^{-1}e_{(\bar 1)(1)}\otimes e_{(\bar 0)}+h(\mathrm{d}e_{(\bar 1)(2)})S^{-1}e_{(\bar 1)(1)}\otimes e_{(\bar 0)}\\&=\mathrm{d}h\otimes e+h.\nabla_E e\end{aligned}$$

for $h\in H$ and $e\in E$, and that

$$\begin{aligned}R_E e=&\,\mathrm{d}(\mathrm{d}e_{(\bar 1)(2)}(S^{-1}e_{(\bar 1)(1)}))\otimes e_{(\bar 0)}\\&-(\mathrm{d}e_{(\bar 4)})(S^{-1}e_{(\bar 3)})\wedge(\mathrm{d}e_{(\bar 1)(2)})(S^{-1}e_{(\bar 1)(1)})\otimes e_{(\bar 0)}\end{aligned}$$

$$\begin{aligned}&= -\mathrm{d}e_{(\bar{1})(2)} \wedge \mathrm{d}(S^{-1}e_{(\bar{1})(1)}) \otimes e_{(\bar{0})}\\ &\quad - \mathrm{d}e_{(\bar{4})} \wedge (S^{-1}e_{(\bar{3})})(\mathrm{d}e_{(\bar{1})(2)})(S^{-1}e_{(\bar{1})(1)}) \otimes e_{(\bar{0})}\\ &= 0.\end{aligned}$$

It is easy to check that morphisms are sent to morphisms. Moreover, (6.8) provides a left 'action' of $v \in \mathfrak{X}$ on any left module with connection (E, ∇_E), in our case

$$v{\triangleright}e = (\mathrm{ev} \otimes \mathrm{id})(v \otimes \nabla_E e) = v((\mathrm{d}e_{(\bar{1})(2)})S^{-1}e_{(\bar{1})(1)}).e_{(\bar{0})}.$$

Note that if v is left-invariant then we can define $x \in H^*$ by $x(h) = v((\mathrm{d}h_{(2)})S^{-1}h_{(1)})$ for all $h \in H$. In this case our 'action' of vector fields (6.8) restricts to the action of such elements $x \in H^*$ acting naturally on objects in ${}_H\mathcal{M}^H$ by evaluation against the coaction, $x{\triangleright}e = \langle x, e_{(\bar{1})}\rangle e_{(\bar{0})}$. ◇

We now turn to defining a suitable algebra for which the above $\underline{v}{\triangleright}e$ is an algebra action. As a vector space we take the direct sum of tensor products

$$T\mathfrak{X} = \bigoplus_{n\geq 0} \mathfrak{X}^{\otimes n},$$

but the product $\bullet$ will be rather more complicated than the tensor algebra over A. The key to understanding its structure is the following lemma.

Lemma 6.9 *For the operation* $\triangleright$ *in (6.8), with* $\underline{v} \in \mathfrak{X}^{\otimes n}$, $w \in \mathfrak{X}$ *and* $e \in E$,

$$w{\triangleright}(\underline{v}{\triangleright}e) = (w \otimes \underline{v}){\triangleright}e + \big((\mathrm{ev} \otimes \mathrm{id}^{\otimes n})(w \otimes \heartsuit^{\langle n\rangle}\underline{v})\big){\triangleright}e.$$

Proof By Proposition 6.6 and (6.8),

$$\begin{aligned}\nabla_E(\underline{v}{\triangleright}e) &= (\mathrm{d}\,\mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id} + \mathrm{ev}^{\langle n\rangle} \otimes \nabla_E)\,(\underline{v} \otimes \nabla_E{}^{(n)}e)\\ &= (\mathrm{id} \otimes \mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id})(\heartsuit^{\langle n\rangle}\underline{v} \otimes \nabla_E{}^{(n)}e)\\ &\qquad + (\mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id} \otimes \mathrm{id})(\underline{v} \otimes (\heartsuit^{\langle n\rangle} \otimes \mathrm{id} + \mathrm{id}^{\otimes n} \otimes \nabla_E)\nabla_E{}^{(n)}e)\\ &= (\mathrm{id} \otimes \mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id})(\heartsuit^{\langle n\rangle}\underline{v} \otimes \nabla_E{}^{(n)}e) + (\mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id} \otimes \mathrm{id})(\underline{v} \otimes \nabla_E{}^{(n+1)}e).\end{aligned}$$

We use this to calculate

$$\begin{aligned}w{\triangleright}(\underline{v}{\triangleright}e) = (\mathrm{ev} \otimes \mathrm{id})(w \otimes \nabla_E(\underline{v}{\triangleright}e)) &= (\mathrm{ev} \otimes \mathrm{ev}^{\langle n\rangle} \otimes \mathrm{id})(w \otimes \heartsuit^{\langle n\rangle}\underline{v} \otimes \nabla_E{}^{(n)}e)\\ &\quad + (\mathrm{ev}^{\langle n+1\rangle} \otimes \mathrm{id})(w \otimes \underline{v} \otimes \nabla_E{}^{(n+1)}e),\end{aligned}$$

and the definition of $\triangleright$ gives the result. □

We have given this in full detail and we now proceed similarly with a little less detail and again recursively as follows.

Proposition 6.10 *Let (A, Ω^1) be an algebra with fgp calculus. $T\mathfrak{X}$ has an associative product $\bullet : T\mathfrak{X} \otimes T\mathfrak{X} \to T\mathfrak{X}$ such that $\triangleright$ in (6.8) is a left action on any left A-module with connection (E, ∇_E). Here $\underline{v} \bullet \underline{w} = \sum_{k=0}^{n+m} \underline{v} \bullet_k \underline{w}$ has components $\bullet_k : \mathfrak{X}^{\otimes n} \otimes \mathfrak{X}^{\otimes m} \to \mathfrak{X}^{\otimes k}$ defined by*

$$n = 0, \quad a \bullet_k \underline{w} = \begin{cases} a.\underline{w} & k = m \\ 0 & \text{otherwise} \end{cases}$$

$$n = 1, \quad u \bullet_k \underline{w} = \begin{cases} u \otimes \underline{w} & k = m+1 \\ (\mathrm{ev} \otimes \mathrm{id}^{\otimes m})(u \otimes \heartsuit^{\langle m \rangle} \underline{w}) & k = m \\ 0 & \text{otherwise} \end{cases}$$

for $a \in A$, $u \in \mathfrak{X}$ and recursively if defined for $\underline{v} \in \mathfrak{X}^{\otimes n}$ (setting $\bullet_{-1}$ to be zero) by

$$(u \otimes \underline{v}) \bullet_k \underline{w} = u \otimes (\underline{v} \bullet_{k-1} \underline{w}) + u \bullet_k (\underline{v} \bullet_k \underline{w}) - (u \bullet_n \underline{v}) \bullet_k \underline{w}.$$

We let $T\mathfrak{X}_\bullet$ denote $T\mathfrak{X}$ with the $\bullet$ product. It is filtered by $T\mathfrak{X}$ grade $\le n$ for each n.

Proof We prove the statements $P(n)$ that $\bullet_k : \mathfrak{X}^{\otimes n} \otimes \mathfrak{X}^{\otimes m} \to \mathfrak{X}^{\otimes k}$ is well defined and $Q(n)$ that $(a.\underline{v}) \bullet_k \underline{w} = a.(\underline{v} \bullet_k \underline{w})$ for all $a \in A$ and $\underline{v} \in \mathfrak{X}^{\otimes n}$. For $P(n+1)$ we have to show equality of the two expressions

$$(u.a \otimes \underline{v}) \bullet_k \underline{w} = u.a \otimes (\underline{v} \bullet_{k-1} \underline{w}) + (u.a) \bullet_k (\underline{v} \bullet_k \underline{w}) - ((u.a) \bullet_n \underline{v}) \bullet_k \underline{w},$$

$$(u \otimes a.\underline{v}) \bullet_k \underline{w} = u \otimes ((a.\underline{v}) \bullet_{k-1} \underline{w}) + u \bullet_k ((a.\underline{v}) \bullet_k \underline{w}) - (u \bullet_n (a.\underline{v})) \bullet_k \underline{w}.$$

Under the assumption $Q(n)$ the last line becomes

$$(u \otimes a.\underline{v}) \bullet_k \underline{w} = u.a \otimes (\underline{v} \bullet_{k-1} \underline{w}) + u \bullet_k (a.(\underline{v} \bullet_k \underline{w})) - (u \bullet_n (a.\underline{v})) \bullet_k \underline{w}$$

and then $P(n+1)$ holds by the $n = 1$ definitions. To verify $Q(n+1)$, we use $P(n+1)$ and $Q(n)$ to get

$$\begin{aligned}(a.u \otimes \underline{v}) \bullet_k \underline{w} &= a.u \otimes (\underline{v} \bullet_{k-1} \underline{w}) + (a.u) \bullet_k (\underline{v} \bullet_k \underline{w}) - ((a.u) \bullet_n \underline{v}) \bullet_k \underline{w} \\ &= a.\big((u \otimes \underline{v}) \bullet_k \underline{w}\big).\end{aligned}$$

Thus $P(n)$ and $Q(n)$ follow from the starting cases by induction. The proof of associativity $\underline{v} \bullet (\underline{w} \bullet \underline{x}) = (\underline{v} \bullet \underline{w}) \bullet \underline{x}$ follows by induction on the degree n for

$\underline{v} \in \mathfrak{X}^{\otimes n}$ with base case already proven as the assertions Q and uses

$$\begin{aligned}(u \otimes \underline{v}) \bullet_k (\underline{w} \bullet_s \underline{x}) &= u \otimes (\underline{v} \bullet_{k-1} (\underline{w} \bullet_s \underline{x})) + u \bullet_k (\underline{v} \bullet_k (\underline{w} \bullet_s \underline{x})) \\ &\quad - (u \bullet_n \underline{v}) \bullet_k (\underline{w} \bullet_s \underline{x}), \\ ((u \otimes \underline{v}) \bullet_k \underline{w}) \bullet_s \underline{x} &= \big(u \otimes (\underline{v} \bullet_{k-1} \underline{w}) + u \bullet_k (\underline{v} \bullet_k \underline{w}) - (u \bullet_n \underline{v}) \bullet_k \underline{w}\big) \bullet_s \underline{x} \\ &= u \otimes ((\underline{v} \bullet_{k-1} \underline{w}) \bullet_{s-1} \underline{x}) + u \bullet_s ((\underline{v} \bullet_{k-1} \underline{w}) \bullet_s \underline{x}) \\ &\quad - (u \bullet_{k-1} (\underline{v} \bullet_{k-1} \underline{w})) \bullet_s \underline{x} + \big(u \bullet_k (\underline{v} \bullet_k \underline{w}) - (u \bullet_n \underline{v}) \bullet_k \underline{w}\big) \bullet_s \underline{x}.\end{aligned}$$

The proof that $\underline{v} \triangleright (\underline{w} \triangleright e) = (\underline{v} \bullet \underline{w}) \triangleright e$ is also by induction on n, using

$$(u \otimes \underline{v}) \triangleright (\underline{w} \triangleright e) = u \triangleright (\underline{v} \triangleright (\underline{w} \triangleright e)) - \big((\mathrm{ev} \otimes \mathrm{id}^{\otimes n})(u \otimes \heartsuit^{\langle n \rangle} \underline{v})\big) \triangleright (\underline{w} \triangleright e)$$

from Lemma 6.9. □

The associated graded algebra of $T\mathfrak{X}_\bullet$ is just $T\mathfrak{X}$ since the modifications in $\bullet$ are in lower degree. We illustrate the recursive definition in low degree.

Example 6.11 Let $u, v, w \in \mathfrak{X}$. The product $\bullet : \mathfrak{X} \otimes \mathfrak{X}^{\otimes 2} \to T\mathfrak{X}$ is defined by

$$u \bullet_k (v \otimes w) = \begin{cases} u \otimes v \otimes w & k = 3 \\ (\mathrm{ev} \otimes \mathrm{id}^{\otimes 2})(u \otimes \heartsuit^{\langle 2 \rangle}(v \otimes w)) & k = 2 \\ 0 & \text{otherwise,} \end{cases}$$

as an instance of Proposition 6.10, where as for any left bimodule connection,

$$\heartsuit^{\langle 2 \rangle}(v \otimes w) = \heartsuit(v) \otimes w + (\sigma \otimes \mathrm{id})(v \otimes \heartsuit(w)).$$

The product $\bullet : \mathfrak{X}^{\otimes 2} \otimes \mathfrak{X} \to T\mathfrak{X}$ is then obtained recursively as

$$\begin{aligned}(u \otimes v) \bullet_3 w &= u \otimes (v \bullet_2 w) + u \bullet_3 (v \bullet_3 w) - (u \bullet_1 v) \bullet_3 w = u \otimes v \otimes w, \\ (u \otimes v) \bullet_2 w &= u \otimes (v \bullet_1 w) + u \bullet_2 (v \bullet_2 w) - (u \bullet_1 v) \bullet_2 w \\ &= u \otimes (v \bullet_1 w) + u \bullet_2 (v \otimes w) - (u \bullet_1 v) \otimes w, \\ (u \otimes v) \bullet_1 w &= u \otimes (v \bullet_0 w) + u \bullet_1 (v \bullet_1 w) - (u \bullet_1 v) \bullet_1 w \\ &= u \bullet_1 (v \bullet_1 w) - (u \bullet_1 v) \bullet_1 w, \\ (u \otimes v) \bullet_0 w &= u \otimes (v \bullet_{-1} w) + u \bullet_0 (v \bullet_0 w) - (u \bullet_1 v) \bullet_0 w = 0,\end{aligned}$$

where $u \bullet_1 v = (\mathrm{ev} \otimes \mathrm{id})(u \otimes \heartsuit v)$, so $(u \otimes v) \bullet_1 w$ has double derivatives of w. ⋄

This helps to compute low degrees for some actual examples, which we now do starting with the classical case.

Example 6.12 Let $A = C^\infty(M)$ for a classical manifold M with its classical 1-forms Ω^1 and vector fields $\mathfrak{X}$, and let $\heartsuit$ be a fixed connection on $\mathfrak{X}$ and on its tensor powers. Then $T\mathfrak{X}_\bullet$ consists of all multivector fields made into a noncommutative algebra over A which acts on any vector bundle with connection (E, ∇) on M. Here

$$a \bullet (v_1 \otimes \cdots \otimes v_n) = a(v_1 \otimes \cdots \otimes v_n),$$

$$u \bullet (v_1 \otimes \cdots \otimes v_n) = u \otimes v_1 \otimes \cdots \otimes v_n + \heartsuit_u(v_1 \otimes \cdots \otimes v_n),$$

$$u \bullet a = au + u(a), \quad (u \otimes v) \bullet a = a(u \otimes v) + u(a)v + uv(a) + u(v(a)) - (\heartsuit_u v)(a),$$

$$(u \otimes v) \bullet w = u \otimes v \otimes w + v \otimes \heartsuit_u w + u \otimes \heartsuit_v w + \heartsuit_u \heartsuit_v w - \heartsuit_{\heartsuit_u v} w,$$

$$a \triangleright e = ae, \quad u \triangleright e = \nabla_u e, \quad (u \otimes v) \triangleright e = \nabla_u \nabla_v e - \nabla_{\heartsuit_u v} e,$$

etc. in some low degree cases where $u, v, w, v_i \in \mathfrak{X}$ and $a \in C^\infty(M)$.

In the parallelisable case, say $M = \mathbb{R}^n$, one has $T\mathfrak{X}_\bullet \cong C^\infty(\mathbb{R}^n) \rtimes \mathbb{C}\langle \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \rangle$, where the $\frac{\partial}{\partial x^i}$ have an additive coproduct and act as usual on functions (which gives the usual cross relations) but do not mutually commute. Their product in $\mathbb{C}\langle \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \rangle$ is the $\bullet$ product and equates to $\otimes$ of $T\mathfrak{X}$ if $\heartsuit = 0$ on the generators as in Example 6.7 but we have the same algebra in terms of generators and relations independently of $\heartsuit$. The cross product acts on sections of a bundle by multiplication by functions and covariant derivative. This is also the local picture in general, with $T\mathfrak{X}_\bullet$ independent of $\heartsuit$ up to isomorphism but not so easily described. ◇

Example 6.13 Let $A = \mathbb{C}_\theta[\mathbb{T}^2]$ with calculus and central basis $e_1 = u^{-1}\mathrm{d}u, e_2 = v^{-1}\mathrm{d}v \in \Omega^1$ as in Example 1.36. The dual basis f_1, f_2 of $\mathfrak{X}$ obeys

$$\mathrm{ev}(f_i.a \otimes e_j) = \mathrm{ev}(f_i \otimes a.e_j) = \mathrm{ev}(f_i \otimes e_j.a) = \delta_{ij} a = \mathrm{ev}(a.f_i \otimes e_j),$$

for all $a \in \mathbb{C}_\theta[T^2]$, so the f_i are also central. Define a right connection $\heartsuit : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ by $\heartsuit(e_1) = \heartsuit(e_2) = 0$. Then applying Fig. 6.1 to $f_i \otimes e_j$ gives $\heartsuit(f_i) = 0$ for the left connection on $\mathfrak{X}$, and

$$\sigma(f_i \otimes \mathrm{d}a) = \heartsuit(f_i.a) - \heartsuit(f_i).a = \heartsuit(a.f_i) = \mathrm{d}a \otimes f_i,$$

and as f_i is central, we find $\sigma(f_i \otimes e_j) = e_j \otimes f_i$. Next,

$$a \bullet f_i = af_i, \quad f_i \bullet a = f_i a + f_i \triangleright a = f_i a + \mathrm{ev}(f_i \otimes \mathrm{d}a)$$

$$f_1 \bullet (uf_2) = f_1 \otimes uf_2 + f_1 \triangleright (uf_2) = f_1 \otimes uf_2 + (\mathrm{ev} \otimes \mathrm{id})(f_1 \otimes \heartsuit(uf_2))$$

$$= f_1 \otimes uf_2 + (\mathrm{ev} \otimes \mathrm{id})(f_1 \otimes ue_1 \otimes f_2)) = uf_1 \otimes f_2 + uf_2.$$

Using Example 6.11 and $\mathrm{d}(uv^2) = (e_1 + 2e_2)uv^2$, we can write

$$
\begin{aligned}
(f_1 \otimes f_2) \bullet_3 (uv^2 f_2) &= f_1 \otimes f_2 \otimes uv^2 f_2 = uv^2 f_1 \otimes f_2 \otimes f_2,\\
(f_1 \otimes f_2) \bullet_2 (uv^2 f_2) &= f_1 \otimes (f_2 \bullet_1 (uv^2 f_2)) + f_1 \bullet_2 (f_2 \otimes (uv^2 f_2))\\
&\quad - (f_1 \bullet_1 f_2) \otimes (uv^2 f_2)\\
&= uv^2(2f_1 \otimes f_2 + f_2 \otimes f_2),\\
(f_1 \otimes f_2) \bullet_1 (uv^2 f_2) &= f_1 \bullet_1 (f_2 \bullet_1 (uv^2 f_2)) - (f_1 \bullet_1 f_2) \bullet_1 (uv^2 f_2)\\
&= f_1 \bullet_1 (2uv^2 f_2) - 0 = 2uv^2 f_2.
\end{aligned}
$$

The sum of these three expressions is $(f_1 \otimes f_2) \bullet (uv^2 f_2)$. ◇

Next we continue Example 6.5, where we described $\Omega^1, \mathfrak{X}$ for $\mathbb{C}_q[S^2]$ and the quantum Levi-Civita connection in the former, a version of which we take for $\heartsuit$.

Example 6.14 On $\Omega^1(\mathbb{C}_q[S^2])$ let ∇ be the left quantum Levi-Civita bimodule as in Examples 5.79 and 6.5. The corresponding right connection $\heartsuit = \sigma^{-1}\nabla$ is

$$
\begin{aligned}
\heartsuit(e^+\xi_+) &= e^+ D_1 D_1' \otimes D_2' D_2(\partial_+\xi_+ e^+ + \partial_-\xi_+ e^-),\\
\heartsuit(e^-\xi_-) &= e^- \tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2(\partial_+\xi_- e^+ + \partial_-\xi_- e^-),
\end{aligned}
$$

where now we define coefficients by $\xi = e^+\xi_+ + e^-\xi_- \in \Omega^1$ with $|\xi_\pm| = \mp 2$. If we also write $v = v_+ f_+ + v_- f_- \in \mathfrak{X}$ with $|v_\pm| = \pm 2$ then the evaluation between vector fields and 1-forms is

$$
\mathrm{ev}\big((v_+ f_+ + v_- f_-) \otimes (e^+\xi_+ + e^-\xi_-)\big) = v_+\xi_+ + v_-\xi_-
$$

and the resulting dual left connection on $\mathfrak{X}$ has to obey

$$
(\mathrm{id} \otimes \mathrm{ev})(\heartsuit v \otimes \xi) + (\mathrm{ev} \otimes \mathrm{id})(v \otimes \heartsuit\xi) = \mathrm{d}(v_+\xi_+ + v_-\xi_-)
$$

from which we find

$$
\begin{aligned}
\heartsuit(v_+ f_+) &= (\partial_+ v_+ e^+ + \partial_- v_+ e^-) D_1 D_1' \otimes D_2' D_2 f_+,\\
\heartsuit(v_- f_-) &= (\partial_+ v_- e^+ + \partial_- v_- e^-) \tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2 f_-.
\end{aligned}
$$

Using a covariant derivative notation $\heartsuit_w v := (\mathrm{ev} \otimes \mathrm{id})(w \otimes \heartsuit v)$ and taking care with the grades, we have

$$
\heartsuit_{w_+ f_+}(v_+ f_+ + v_- f_-) = w_+(\partial_+ v_+) f_+ + q^4 w_+(\partial_+ v_-) f_-, \tag{6.9}
$$

$$
\heartsuit_{w_- f_-}(v_+ f_+ + v_- f_-) = q^{-4} w_-(\partial_- v_+) f_+ + w_-(\partial_- v_-) f_- \tag{6.10}
$$

and the resulting $\bullet$ products

$$w_+f_+ \bullet (v_+f_+ + v_-f_-) = w_+(\partial_+v_+)f_+ + q^4 w_+(\partial_+v_-)f_- \\ + w_+f_+ \otimes (v_+f_+ + v_-f_-),$$

$$w_-f_- \bullet (v_+f_+ + v_-f_-) = q^{-4} w_-(\partial_-v_+)f_+ + w_-(\partial_-v_-)f_- \\ + w_-f_- \otimes (v_+f_+ + v_-f_-).$$

We will continue with vector fields on $\mathbb{C}_q[S^2]$ in Example 6.20. $\diamond$

Because A is a subalgebra of an associative algebra, in our case $T\mathfrak{X}_\bullet$, the latter is automatically an algebra over A (an algebra in the A-bimodule category). This keeps $T\mathfrak{X}_\bullet$ geometric or 'tensorial' just like the exterior algebra, but now filtered rather than graded. It is also unital with $1 \in A = \mathfrak{X}^{\otimes 0}$. We also see in the examples that the bimodule structure—even in the classical case—is noncommutative. Here $a \bullet \underline{v} = a\underline{v}$ for $a \in A$ coincides with the original left module structure of $T\mathfrak{X}$ while $\underline{v} \bullet a \neq \underline{v}a$ when $\underline{v}$ had degree ≥ 1 and hence does not coincide with the original right module structure on $T\mathfrak{X}$ (it contains terms where a is differentiated).

Proposition 6.15 *Let (A, Ω^1) be an algebra with fgp calculus. There is an isomorphism of categories ${}_A\mathcal{E} \cong {}_{T\mathfrak{X}_\bullet}\mathcal{M}$ compatible with the forgetful functor to* vector spaces.

Proof Starting with a left A-module with connection $(E, \nabla_E) \in {}_A\mathcal{E}$, we already have $\triangleright : T\mathfrak{X}_\bullet \otimes_A E \to E$ in (6.8) is an action on E. A morphism $\phi : E \to F$ is a left-module map intertwining the covariant derivatives by $(\mathrm{id} \otimes \phi)\nabla_E = \nabla_F \phi$, in which case $(\mathrm{id}\otimes\phi){\nabla_E}^{(n)} = {\nabla_F}^{(n)}\phi$ and hence $\underline{v}\triangleright\phi(e) = \phi(\underline{v}\triangleright e)$ hold for all $e \in E$ and $\underline{v} \in T\mathfrak{X}$, i.e., the construction is functorial. Note in passing that the operation $v\triangleright : E \to E$ for fixed $v \in \mathfrak{X}$ is not itself a morphism in ${}_A\mathcal{E}$ as applying ∇_E to $v\triangleright e$ would also differentiate v.

Going the other way, a representation E of $T\mathfrak{X}_\bullet$ is also an A-module by restriction while the action of $\mathfrak{X}$ gives our covariant derivatives $\nabla_E e = e_i \otimes f_i\triangleright e$ where $\mathrm{coev} = e_i \otimes f_i \in \Omega^1 \otimes_A \mathfrak{X}$ (summing over i) is our coevaluation element. This is independent of the expression of coev since $(a.v)\triangleright e = (a \bullet v)\triangleright e = a\triangleright(v\triangleright e) = a.(v\triangleright e)$ for any $v \in \mathfrak{X}$. Using the relations in $T\mathfrak{X}_\bullet$ and the centrality of coev, we check

$$\nabla_E(a.e) = e_i \otimes f_i\triangleright(a\triangleright e) = e_i \otimes (f_i.a)\triangleright e + e_i \otimes \mathrm{ev}(f_i \otimes \mathrm{d}a).e \\ = a.e_i \otimes f_i\triangleright e + e_i.\mathrm{ev}(f_i \otimes \mathrm{d}a) \otimes e = a.\nabla_E(e) + \mathrm{d}a \otimes e.$$

It is obvious that the two constructions are inverse. Starting with a connection ∇_E, we have an induced action of $T\mathfrak{X}_\bullet$ and construct a connection, and vice versa,

$$e \mapsto e_i \otimes f_i\triangleright e = e_i \otimes (\mathrm{ev} \otimes \mathrm{id})(f_i \otimes \nabla_E(e)) = \nabla_E(e),$$

$$v \otimes e \mapsto (\mathrm{ev} \otimes \mathrm{id})(v \otimes \nabla_E(e)) = (\mathrm{ev} \otimes \mathrm{id})(v \otimes e_i \otimes f_i\triangleright e) = v\triangleright e. \qquad \square$$

Finally, no discussion of noncommutative differential operators would be complete without a second-order Laplacian and we will now see how to generate examples of these in the sense of Definition 1.17. Given the actions and $\heartsuit$ connections in this section, for any fixed element $v\otimes w \in \mathfrak{X}^{\otimes 2}$ (summation implicit) which commutes with elements of the algebra (i.e., $a.v\otimes w = v\otimes w.a$ for all $a \in A$), we define

$$\Delta a = (v \otimes w)\triangleright a = \mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes \heartsuit \mathrm{d}a\big) = \mathrm{ev}(v \otimes \heartsuit_w \mathrm{d}a)$$

from (6.8) applied to $E = A$ and $\nabla_E a = \mathrm{d}a \otimes 1$ as a bundle with connection. Then

$$\begin{aligned}\Delta(ab) = \mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes \heartsuit \mathrm{d}(ab)\big) &= \mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes \heartsuit(a(\mathrm{d}b) + (\mathrm{d}a)b)\big)\\ &= a.\mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes \heartsuit \mathrm{d}b\big) + \mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes \heartsuit \mathrm{d}a\big).b\\ &\quad + \mathrm{ev}^{\langle 2\rangle}\big(v \otimes w \otimes (\mathrm{id} \otimes \mathrm{id} + \sigma^{-1})(\mathrm{d}a \otimes \mathrm{d}b)\big)\end{aligned}$$

for all $a, b \in A$, so Δ satisfies the condition to be a second-order Laplacian. This is a special case of a general theory of geometric Laplacians in Chap. 8 defined by a bimodule inner product (not assumed to be positive) and a connection. In our case this is $(\ ,\ ) = \mathrm{ev}^{\langle 2\rangle}(v \otimes w \otimes - \otimes -) : \Omega^{1\otimes 2} \to A$.

6.3 $T\mathfrak{X}_\bullet$ as an Algebra in $\mathcal{Z}({}_A\mathcal{E}_A)$

In general, the category ${}_A\mathcal{E}$ in Proposition 6.15 is not monoidal and so we should not expect that $T\mathfrak{X}_\bullet$ is a bialgebra over A in any obvious sense. But it is in the classical case, so we might expect some remnant of this to remain in general. The aspect that we focus on is that any actual Hopf algebra H is a braided-commutative algebra in the prebraided category of H-crossed modules or, better, of the centre $\mathcal{Z}({}_H\mathcal{M})$, see Example 2.48. Recall that an object of the centre means one of the original monoidal category, here $H \in {}_H\mathcal{M}$ by the adjoint action, and a natural transformation, here λ_H as a collection $\lambda_{H,V} : H \otimes V \to V \otimes H$ for all other objects V, which 'represents' the monoidal product by composition. Explicitly, $\lambda_{H,V} = h_{(1)}\triangleright v \otimes h_{(2)}$ from which one can see that $\cdot\lambda_{H,H} = \cdot$ (the braided-commutativity of H) and the useful identity

$$(\mathrm{id} \otimes \triangleright)(\lambda_{H,V} \otimes \mathrm{id})(h \otimes v \otimes w) = h\triangleright(v \otimes w) \tag{6.11}$$

for all $V, W \in {}_H\mathcal{M}$. In our case now, the first step is to work instead with ${}_A\mathcal{E}_A$ which is monoidal. If this were to be identified with the modules of some Hopf-like algebra H_A over A then we might ask if $H_A \in \mathcal{Z}({}_A\mathcal{E}_A)$ as a braided commutative algebra in analogy with the above. In fact we will not study this here but rather we find similarly that $T\mathfrak{X}_\bullet \in \mathcal{Z}({}_A\mathcal{E}_A)$ as a braided-commutative algebra.

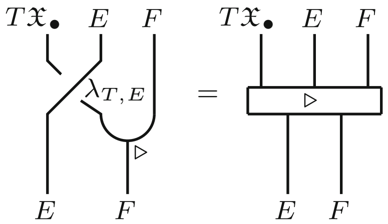


Fig. 6.2 The characteristic property for the desired $\lambda_{T,E}$ in relation to the action of $T\mathfrak{X}_\bullet$

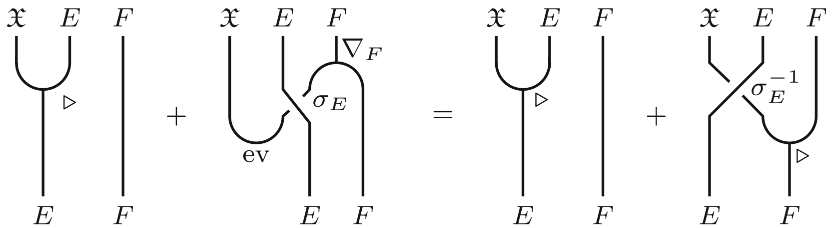


Fig. 6.3 The right-hand side of Fig. 6.2 in the case of a single vector field and the action on E and F separately

Lemma 6.16 *Let (A, Ω^1) be an algebra with fgp calculus. For the coevaluation* $\mathrm{coev} : A \to \Omega^1 \otimes_A \mathfrak{X}$*, the induced left connection* $\nabla(\underline{v}) = \mathrm{coev} \bullet \underline{v}$ *on* $T\mathfrak{X}_\bullet$ *corresponding by Proposition 6.15 to the left action of* $T\mathfrak{X}_\bullet$ *on itself is a right module map, i.e.,* $(T\mathfrak{X}_\bullet, \nabla, 0)$ *is an object in* ${}_A\mathcal{E}_A$.

Proof Proposition 6.15 for $\bullet$ as an action of $T\mathfrak{X}_\bullet$ on itself gives us a left connection $\nabla(\underline{v}) = \xi \otimes (u \bullet \underline{v})$ where $\mathrm{coev} = \xi \otimes u \in \Omega^1 \otimes_A \mathfrak{X}$ (summation implicit). The new part is that

$$\nabla(\underline{v} \bullet a) - \nabla(\underline{v}) \bullet a = \xi \otimes (u \bullet (\underline{v} \bullet a)) - \xi \otimes ((u \bullet \underline{v}) \bullet a) = 0$$

so that we have a bimodule connection with $\sigma = 0$. $\square$

Next we look for a natural transformation λ_T making $(T\mathfrak{X}_\bullet, \lambda_T)$ an object of the centre of $\mathcal{Z}({}_A\mathcal{E}_A)$. The idea is that although $T\mathfrak{X}_\bullet$ does not have an obvious coproduct, it does act on the tensor product $(E \otimes_A F, \nabla_{E\otimes F})$ of any two objects as this is another object of ${}_A\mathcal{E}_A$. Thus we seek a natural transformation λ_T obeying Fig. 6.2 in analogy with (6.11). We have depicted the 'braidings' $\lambda_{T,E}$ for any $E \in {}_A\mathcal{E}_A$ by an inverse braid crossing in the figure because when the box is expanded out as in Fig. 6.3, we see that its relevant component has a map $\sigma_E^{-1} : \mathfrak{X}\otimes_A E \to E\otimes_A \mathfrak{X}$ defined by

$$(\mathrm{id} \otimes \mathrm{ev})(\sigma_E^{-1} \otimes \mathrm{id}) = (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E) : \mathfrak{X} \otimes_A E \otimes_A \Omega^1 \to E. \tag{6.12}$$

This is depicted as an inverse braid in the diagrammatic notation as it is built from $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ depicted as a braid. Similarly to our comment after

(6.5), we are not assuming that either map is invertible as they are associated to different domains.

Lemma 6.17 *For $(E, \nabla_E, \sigma_E) \in {}_A\mathcal{E}_A$, define $\lambda_{T,E}$ on $A \oplus \mathfrak{X} \subset T\mathfrak{X}_\bullet$ by*

$$\lambda_{T,E}(a \otimes e) = a.e \otimes 1, \quad \lambda_{T,E}(v \otimes e) = v \triangleright e \otimes 1 + \sigma_E^{-1}(v \otimes e)$$

for $a \in A$, $v \in \mathfrak{X}$ and $e \in E$. This is a well-defined bimodule map and extends recursively to $\lambda_{T,E} : T\mathfrak{X}_\bullet \otimes_A E \to E \otimes_A T\mathfrak{X}_\bullet$ by

$$\lambda_{T,E}(u \bullet \underline{v} \otimes e) = (\lambda_{T,E} \bullet \mathrm{id})(u \otimes \lambda_{T,E}(\underline{v} \otimes e)).$$

Moreover, $\lambda_{T,E}$ obeys the condition in Fig. 6.2.

Proof To see that $\lambda_{T,E}$ is well defined, we calculate

$$\begin{aligned}\lambda_{T,E}(v \bullet a \otimes e) &= \lambda_{T,E}((v.a + (v \triangleright a)) \otimes e) = \lambda_{T,E}(v.a \otimes e) + (v \triangleright a)e \otimes 1 \\ &= (v.a) \triangleright e \otimes 1 + (v \triangleright a)e \otimes 1 + \sigma_E^{-1}(v.a \otimes e) \\ &= v \triangleright (a.e) \otimes 1 + \sigma_E^{-1}(v \otimes a.e) = \lambda_{T,E}(v \otimes a.e).\end{aligned}$$

The left module map property is straightforward while the right module property is

$$\begin{aligned}\lambda_{T,E}(v \otimes e.a) &= (\mathrm{ev} \otimes \mathrm{id}_E)(v \otimes \nabla_E(e.a)) + \sigma_E^{-1}(v \otimes e.a) \\ &= \lambda_{T,E}(v \otimes e).a + (\mathrm{ev} \otimes \mathrm{id}_E)(\mathrm{id} \otimes \sigma_E)(v \otimes e \otimes \mathrm{d}a) \\ &= \lambda_{T,E}(v \otimes e).a + (\mathrm{id}_E \otimes \mathrm{ev})(\sigma_E^{-1}(v \otimes e) \otimes \mathrm{d}a) = \lambda_{T,E}(v \otimes e) \bullet a.\end{aligned}$$

For a single vector field, Fig. 6.3 shows that the above definition of $\lambda_{T,E}$ satisfies Fig. 6.2. For the general case we use induction and the recursive definition of $\lambda_{T,E}$. Thus, assume that Fig. 6.2 applies for products of $\leq n$ vector fields. We show that it applies for products of $\leq n+1$ by applying Fig. 6.4 to inputs of the form $u \otimes \underline{v} \otimes e \otimes f$, where $\underline{v}$ is a product of $\leq n$ vector fields. The first equality in Fig. 6.4 is the recursive definition of $\lambda_{T,E}$, the second is the relation between the $\bullet$ product and the action, the third is by the inductive hypothesis, and the last is the relation between the $\bullet$ product and the action again. □

Note that Figs. 6.2, 6.3 and 6.4 are merely a pictorial representation of various formulae with tensor products and bimodule maps and λ there is not a braiding, albeit it will become one in the centre.

Theorem 6.18 *Let (A, Ω^1) be an algebra with fgp calculus. $\lambda_{T,E}$ in Lemma 6.17 are morphisms in the category ${}_A\mathcal{E}_A$ and define a natural transformation $\lambda_T : T\mathfrak{X}_\bullet \otimes_A \to \otimes_A T\mathfrak{X}_\bullet$ making $(T\mathfrak{X}_\bullet, \nabla, 0)$ in Lemma 6.16 a unital associative braided-commutative algebra in the centre $\mathcal{Z}({}_A\mathcal{E}_A)$ of ${}_A\mathcal{E}_A$.*

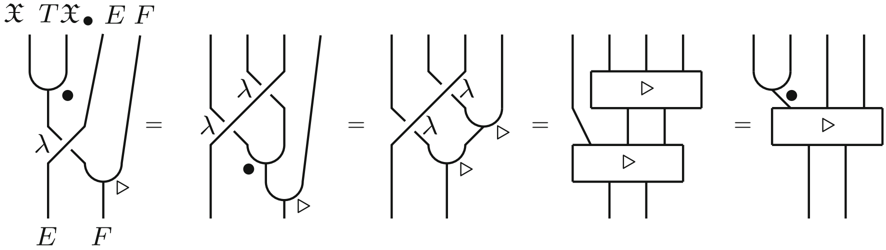


Fig. 6.4 The inductive step in the proof of Lemma 6.17, with $\lambda_{T,E}$ shortened to just λ

Proof To show that $\lambda_{T,E}$ is a morphism, we write $\xi \otimes u \in \Omega^1 \otimes_A \mathfrak{X}$ (sum of such elements understood) for the coevaluation element that defines ∇ and calculate

$$\begin{aligned}(\mathrm{id} \otimes \lambda_{T,E})\nabla_{T\mathfrak{X}_\bullet \otimes E}(\underline{v} \otimes e) &= \xi \otimes \lambda_{T,E}(u \bullet \underline{v} \otimes e)\\ &= \xi \otimes (\lambda_{T,E} \bullet \mathrm{id})(u \otimes \lambda_{T,E}(\underline{v} \otimes e))\end{aligned}$$

for $\underline{v} \otimes e \in T\mathfrak{X}_\bullet \otimes E$. Writing $\lambda_{T,E}(\underline{v} \otimes e) = g \otimes \underline{w}$ (sum understood), we have

$$\begin{aligned}\xi \otimes (u{\triangleright}g) \otimes \underline{w} + \xi \otimes \sigma_E^{-1}(u \otimes g) \bullet \underline{w} &= \nabla_E g \otimes \underline{w} + \sigma_E(g \otimes \xi) \otimes u \bullet \underline{w}\\ &= \nabla_{E \otimes T\mathfrak{X}_\bullet}\lambda_{T,E}(\underline{v} \otimes e)\end{aligned}$$

as required. To show functoriality, by the recursive definition of $\lambda_{T,E}$ it is enough to show functoriality for the single vector field case, and that is straightforward as a morphism in the category commutes with both the action $\triangleright$ and σ^{-1}. Also by the recursive definition of λ_T, it is enough for the tensor product condition for λ to check it on vector fields, i.e., to prove

$$(\mathrm{id} \otimes \lambda_{T,F})(\lambda_{T,E}(u \otimes e) \otimes f) = \lambda_{T,E\otimes F}(u \otimes e \otimes f)$$

for $e \in E$, $f \in F$ and $u \in \mathfrak{X}$. The left-hand side is

$$\begin{aligned}(\mathrm{id} \otimes \lambda_{T,F})(\lambda_{T,E}(u \otimes e) \otimes f) &= (\mathrm{id} \otimes \lambda_{T,F})(u{\triangleright}e \otimes f + \sigma_E^{-1}(u \otimes e) \otimes f)\\ &= u{\triangleright}e \otimes f + \sigma_E^{-1}(u \otimes e){\triangleright}f + \sigma_{E\otimes F}^{-1}(u \otimes e \otimes f),\end{aligned}$$

which agrees with the right-hand side on noting that

$$\begin{aligned}u{\triangleright}(e \otimes f) &= (\mathrm{ev} \otimes \mathrm{id})(u \otimes \nabla_E e \otimes f + u \otimes (\sigma_E \otimes \mathrm{id})(e \otimes \nabla_F f))\\ &= u{\triangleright}e \otimes f + (\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id})(\sigma_E^{-1}(u \otimes e) \otimes \nabla_F f).\end{aligned}$$

To show that the product of $T\mathfrak{X}_\bullet$ is a morphism in ${}_A\mathcal{E}_A$ means verifying $\nabla(\underline{v} \bullet \underline{w}) = (\mathrm{id} \otimes \bullet)\nabla(\underline{v} \otimes \underline{w})$ and this holds by definition of the bimodule connection in

Lemma 6.16. The monoidal structure of the centre in Theorem 2.47 tells us that $\lambda_{T\otimes T,E} = (\lambda_{T,E}\otimes \mathrm{id})(\mathrm{id}\otimes\lambda_{T,E})$, after which one can verify that $\bullet$ is moreover a morphism in the centre using the inductive definition of λ_T, E. The braided-commutativity holds as $\lambda_{T,T\mathfrak{X}_\bullet}(\underline{v}\otimes\underline{w}) = \underline{v}\bullet\underline{w}\otimes 1$. $\square$

In Theorem 2.47, we make a distinction between braiding and prebraiding for the centre of a monoidal category, with the former if the λ are isomorphisms. Hence it is natural to ask whether λ_T is invertible.

Lemma 6.19 *If $\sigma_E : E\otimes_A \Omega^1 \to \Omega^1\otimes_A E$ is invertible, then the map $\lambda_{T,E} : T\mathfrak{X}_\bullet \otimes_A E \to E\otimes_A T\mathfrak{X}_\bullet$ defined in Lemma 6.17 is invertible.*

Proof Set $V_n = T\mathfrak{X}_\bullet^{\otimes n}\otimes_A E$ and $W_n = E\otimes_A T\mathfrak{X}_\bullet^{\otimes n}$, and consider ${\lambda_{T,E}}^{(n)} : \oplus_{0\le m\le n} V_m \to \oplus_{0\le m\le n} W_m$. If we restrict and project ${\lambda_{T,E}}^{(n)}$ to a map from V_n to W_n we simply repeat the application of the invertible map $\sigma_E^{-1} : \mathfrak{X}\otimes_A E\to E\otimes_A\mathfrak{X}$. If we call this invertible composition K_n then writing an element of $\oplus_{0\le m\le n+1}V_m$ as a column vector with V_{n+1} at the top, and similarly for $\oplus_{0\le m\le n+1}W_m$, we have

$$ {\lambda_{T,E}}^{(n+1)} = \begin{pmatrix} K_{n+1} & 0 \\ B_n & {\lambda_{T,E}}^{(n)} \end{pmatrix} $$

in block triangular form for some B_n. Hence ${\lambda_{T,E}}^{(n)}$ is invertible by induction. $\square$

However, we cannot restrict to a category where the generalised braidings $\sigma_E : E\otimes_A\Omega^1 \to \Omega^1\otimes_A E$ are always invertible, as $\sigma = 0$ for the connection on $T\mathfrak{X}_\bullet$ in Lemma 6.16. Thus, although we have loosely spoken about braided-commutativity, we actually use the pre-braided version of the centre construction.

Example 6.20 For $\mathbb{C}_q[S^2]$ in Example 6.14 we set $(E,\nabla_E) = (\mathfrak{X},\heartsuit)$ in the above. Remembering that elements $a\in\mathbb{C}_q[S^2]$ commute with $e^\pm$ and $f_\pm$,

$$ \heartsuit(v_+f_+a) = \heartsuit(v_+af_+) = (\partial_+(v_+a)e^+ + \partial_-(v_+a)e^-)D_1D_1'\otimes D_2'D_2f_+, $$
$$ \heartsuit(v_-f_-a) = \heartsuit(v_-af_-) = (\partial_+(v_-a)e^+ + \partial_-(v_-a)e^-)\tilde D_1\tilde D_1'\otimes \tilde D_2'\tilde D_2f_-, $$

so we need

$$ \sigma_{\mathfrak{X}}(v_+f_+\otimes \mathrm{d}a) = (v_+(\partial_+a)e^+ + v_+(\partial_-a)e^-)D_1D_1'\otimes D_2'D_2f_+, $$
$$ \sigma_{\mathfrak{X}}(v_-f_-\otimes \mathrm{d}a) = (v_-(\partial_+a)e^+ + v_-(\partial_-a)e^-)\tilde D_1\tilde D_1'\otimes \tilde D_2'\tilde D_2f_-. $$

It follows that $(\mathfrak{X},\heartsuit,\sigma_{\mathfrak{X}})$ is in ${}_A\mathcal{E}_A$ with, for all $\xi\in\Omega^1$,

$$ \sigma_{\mathfrak{X}}((v_+f_+ + v_-f_-)\otimes\xi) = v_+\xi D_1D_1'\otimes D_2'D_2f_+ + v_-\xi\tilde D_1\tilde D_1'\otimes\tilde D_2'\tilde D_2f_-. $$

Now inserting a coevaluation in (6.12) we define $\sigma_{\mathfrak{X}}^{-1}:\mathfrak{X}\otimes_A\mathfrak{X}\to\mathfrak{X}\otimes_A\mathfrak{X}$ by

$$\sigma_{\mathfrak{X}}^{-1}(w\otimes v)=(\mathrm{ev}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_E\otimes\mathrm{id})(w\otimes v\otimes\mathrm{coev})$$

and if we use $v=v_+f_++v_-f_-$, $w=w_+f_++w_-f_-$ and

$$\mathrm{coev}=e^+D_1D_1'\otimes D_2'D_2f_++e^-\tilde{D}_1\tilde{D}_1'\otimes\tilde{D}_2'\tilde{D}_2f_-$$

we find

$$\begin{aligned}\sigma_{\mathfrak{X}}^{-1}(w\otimes v)&=(\mathrm{ev}\otimes\mathrm{id})(w\otimes\sigma_{\mathfrak{X}}(v\otimes e^+D_1D_1'))\otimes D_2'D_2f_+\\&\quad+(\mathrm{ev}\otimes\mathrm{id})(w\otimes\sigma_E(v\otimes e^-\tilde{D}_1\tilde{D}_1'))\otimes\tilde{D}_2'\tilde{D}_2f_-\\&=q^{-2}w_+v_+D_1D_1'f_+\otimes D_2'D_2f_++q^2w_+v_-D_1D_1'f_-\otimes D_2'D_2f_+\\&\quad+q^{-2}w_-v_+\tilde{D}_1\tilde{D}_1'f_+\otimes\tilde{D}_2'\tilde{D}_2f_-+q^2w_-v_-\tilde{D}_1\tilde{D}_1'f_-\otimes\tilde{D}_2'\tilde{D}_2f_-.\end{aligned}$$

Now the formula for $\lambda_{T,\mathfrak{X}}(w\otimes v)$ in Lemma 6.17 gives

$$\begin{aligned}\lambda_{T,\mathfrak{X}}(w\otimes v)&=\big(w_+(\partial_+v_+)f_++q^4w_+(\partial_+v_-)f_-+q^{-4}w_-(\partial_-v_+)f_+\\&\quad+w_-(\partial_-v_-)f_-\big)\otimes 1\\&\quad+q^{-2}w_+v_+D_1D_1'f_+\otimes D_2'D_2f_++q^2w_+v_-D_1D_1'f_-\otimes D_2'D_2f_+\\&\quad+q^{-2}w_-v_+\tilde{D}_1\tilde{D}_1'f_+\otimes\tilde{D}_2'\tilde{D}_2f_-+q^2w_-v_-\tilde{D}_1\tilde{D}_1'f_-\otimes\tilde{D}_2'\tilde{D}_2f_-\end{aligned}$$

on using (6.9) in Example 6.14. ◇

6.4 The Sheaf of Differential Operators $\mathcal{D}_A$

So far we have no relations for differential operators corresponding classically to commutativity of partial derivatives. When $T\mathfrak{X}_\bullet$ is represented on itself, the commutator of covariant derivatives gives curvature, so our first task is to write this as a differential operator, which we do with the help of the torsion $T_\heartsuit=\mathrm{d}+\wedge\heartsuit:\Omega^1\to\Omega^2$, a right module map.

Proposition 6.21 *Let (A,Ω^1) be an algebra with fgp calculus. There is a central element $\mathcal{R}\in\Omega^2\otimes_A T\mathfrak{X}_\bullet$ given by*

$$\mathcal{R}=\mathrm{d}e_i\otimes f_i-e_i\wedge e_j\otimes f_j\bullet f_i=T_\heartsuit(e_i)\otimes f_i-e_i\wedge e_j\otimes(f_j\otimes f_i)$$

where coev $=e_i\otimes f_i\in\Omega^1\otimes_A\mathfrak{X}$ *are dual bases and $e_j\otimes f_j$ is another, such that the curvature R_∇ on $T\mathfrak{X}_\bullet$ in Lemma 6.16 is given by $R_\nabla(\underline{v})=\mathcal{R}\bullet\underline{v}$.*

Proof The connection ∇ on $T\mathfrak{X}_\bullet$ in Lemma 6.16 is $\nabla(\underline{v}) = e_i \otimes (f_i \bullet \underline{v})$ with curvature $R_\nabla : T\mathfrak{X}_\bullet \to \Omega^2 \otimes_A T\mathfrak{X}_\bullet$ given by

$$R_\nabla(\underline{v}) = \mathrm{d}e_i \otimes (f_i \bullet \underline{v}) - e_i \wedge \nabla(f_i \bullet \underline{v}) = \mathrm{d}e_i \otimes (f_i \bullet \underline{v}) - e_i \wedge e_j \otimes (f_j \bullet f_i \bullet \underline{v}).$$

Now, from the formula for $\bullet$,

$$\begin{aligned} e_i \otimes e_j \otimes f_j \bullet f_i &= e_i \otimes e_j \otimes f_j \otimes f_i + e_i \otimes e_j \otimes (\mathrm{ev} \otimes \mathrm{id})(f_j \otimes \heartsuit f_i) \\ &= e_i \otimes e_j \otimes f_j \otimes f_i + (\mathrm{id}^{\otimes 2} \otimes \mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev} \otimes \mathrm{id}^{\otimes 2})(e_i \otimes \heartsuit f_i) \\ &= e_i \otimes e_j \otimes f_j \otimes f_i + e_i \otimes \heartsuit f_i = e_i \otimes e_j \otimes f_j \otimes f_i - \heartsuit e_i \otimes f_i, \end{aligned}$$

where we have used the usual equations for the evaluation and coevaluation. Using the torsion $T_\heartsuit$, we can rewrite $\mathcal{R}$ as

$$\mathcal{R} = \mathrm{d}e_i \otimes f_i - e_i \wedge e_j \otimes (f_j \otimes f_i) + \wedge \heartsuit e_i \otimes f_i = T_\heartsuit(e_i) \otimes f_i - e_i \wedge e_j \otimes (f_j \otimes f_i).$$

Since R_∇ is a left module map, applying R_∇ to $a \in A$ gives

$$\mathcal{R} \bullet a = R_\nabla(a) = R_\nabla(a.1) = a.R_\nabla(1) = a.\mathcal{R} \bullet 1 = a.\mathcal{R}. \qquad \square$$

The action of this differential operator form of the curvature gives the curvature on any left module with connection.

Proposition 6.22 *Suppose that (E, ∇_E) is a left A-module with left-covariant derivative, with curvature R_E. Then $\mathcal{R} \triangleright e = R_E(e)$ where the $T\mathfrak{X}_\bullet$ factor of $\mathcal{R}$ acts on E as usual by (6.8).*

Proof Acting on $e \in E$ and writing $\nabla_E e = \eta \otimes f$ (summation understood),

$$\begin{aligned} \mathcal{R} \triangleright e &= \mathrm{d}e_i \otimes f_i \triangleright e - e_i \wedge e_j \otimes f_j \bullet f_i \triangleright e \\ &= \mathrm{d}e_i \otimes \mathrm{ev}(f_i \otimes \eta).f - e_i \wedge e_j \otimes f_j \triangleright (\mathrm{ev}(f_i \otimes \eta).f) \\ &= \mathrm{d}e_i \otimes \mathrm{ev}(f_i \otimes \eta).f - e_i \wedge e_j \otimes \mathrm{ev}(f_j \otimes \mathrm{d}(\mathrm{ev}(f_i \otimes \eta))).f \\ &\quad - e_i \wedge e_j \otimes (\mathrm{ev} \otimes \mathrm{id})(f_j \otimes (\mathrm{ev}(f_i \otimes \eta).\nabla_E f)) \\ &= \mathrm{d}e_i \otimes \mathrm{ev}(f_i \otimes \eta).f - e_i \wedge \mathrm{d}(\mathrm{ev}(f_i \otimes \eta)) \otimes f - \eta \wedge e_j \otimes (\mathrm{ev} \otimes \mathrm{id})(f_j \otimes \nabla_E f) \\ &= \mathrm{d}(e_i.\mathrm{ev}(f_i \otimes \eta)) \otimes f - \eta \wedge \nabla_E f = \mathrm{d}\eta \otimes f - \eta \wedge \nabla_E f = R_E(e). \qquad \square \end{aligned}$$

Now we turn the curvature into relations on $T\mathfrak{X}_\bullet$. Let $\Omega^{2\sharp} := \mathrm{Hom}_A(\Omega^2, A)$ (the right A-module maps) and define $\widehat{\mathcal{R}} : \Omega^{2\sharp} \to T\mathfrak{X}_\bullet$ by $\widehat{\mathcal{R}}(\alpha) = (\alpha \otimes \mathrm{id})\mathcal{R}$. Then $\widehat{\mathcal{R}}$ is obviously a left module map, but in fact it is also a right module map. Writing $\mathcal{R} = \omega \otimes \underline{v}$ (summation understood), centrality of $\mathcal{R}$ in Proposition 6.21

appears as $a.\omega \otimes \underline{v} = \omega \otimes \underline{v} \bullet a$ for all in A and hence

$$\widehat{\mathcal{R}}(\alpha.a) = (\alpha.a \otimes \mathrm{id})\mathcal{R} = \alpha(a.\omega).\underline{v} = \alpha(\omega).\underline{v} \bullet a = \widehat{\mathcal{R}}(\alpha) \bullet a.$$

We now define $\mathcal{W} \subset T\mathfrak{X}_\bullet$ to be the 2-sided ideal (for the $\bullet$ product) generated by the image of $\widehat{\mathcal{R}}$, and the *algebra of differential operators* on A as

$$\mathcal{D}_A = T\mathfrak{X}_\bullet/\mathcal{W}. \tag{6.13}$$

It inherits a filtration from $T\mathfrak{X}_\bullet$. We will use $[\underline{v}]$ to denote the equivalence class in $\mathcal{D}_A$ containing $\underline{v} \in T\mathfrak{X}_\bullet$.

Example 6.23 (Classical Differential Operators) Setting $A = C^\infty(\mathbb{R}^n)$ as in Example 6.7 with $\heartsuit = 0$ on the coordinate bases and coevaluation $\mathrm{d}x^i \otimes \frac{\partial}{\partial x^i}$, Proposition 6.21 gives

$$\mathcal{R} = -\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \frac{\partial}{\partial x^j} \bullet \frac{\partial}{\partial x^i} \in \Omega^2 \otimes T_{C^\infty(\mathbb{R}^n)}\mathfrak{X},$$

where $\mathfrak{X} = \mathrm{Vect}(\mathbb{R}^n)$. Thus the ideal $\mathcal{W}$ is generated by commutators and hence $\mathcal{D}_{C^\infty(\mathbb{R}^n)}$ has the expected relations

$$\frac{\partial}{\partial x^j} \bullet \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} \bullet \frac{\partial}{\partial x^j} = 0$$

among the generators. $\mathcal{D}_{C^\infty(\mathbb{R}^n)}$ also contains the $x^i \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x^i} \bullet x^j = \frac{\partial}{\partial x^i} x^j + \mathrm{ev}(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j) = x^j \bullet \frac{\partial}{\partial x^i} + \mathrm{ev}(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j) = x^j \bullet \frac{\partial}{\partial x^i} + \delta_{ij}$$

from Proposition 6.10. Thus $\mathcal{D}_{C^\infty(\mathbb{R}^n)}$ recovers the differential operators on $\mathbb{R}^n$. ◇

Classically, having vector fields such as $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ commuting comes at a cost. It means that the algebra of differential operators no longer acts on all bundles with connection even though the vector fields separately act by covariant differentiation; it can only act where the latter commute. The same happens in the noncommutative setting and gives the following characterisation of $\mathcal{D}_A$-modules as differential sheaves in the sense of §4.3. Recall that $T\mathfrak{X}_\bullet$ has a left action $\triangleright$ on all objects of ${}_A\mathcal{E}$.

Corollary 6.24 *Suppose that Ω^1 and Ω^2 are fgp as right A-modules. There is an isomorphism of categories ${}_A\mathcal{F} \cong {}_{\mathcal{D}_A}\mathcal{M}$ given by restriction of Proposition 6.15.*

Proof We first show that if (E, ∇_E) has zero curvature then $[\underline{v}]\triangleright e = \underline{v}\triangleright e$ by the corresponding action of $T\mathfrak{X}_\bullet$ in Proposition 6.15 descends to an action of $\mathcal{D}_A$. From Proposition 6.22, we know that $(\alpha \otimes \mathrm{id})\mathcal{R}\triangleright e = (\alpha \otimes \mathrm{id}_E)R_E(e)$ for all $\alpha \in \Omega^{2\sharp}$. Thus if $R_E = 0$ then the image of $\widehat{\mathcal{R}}$ annihilates every $e \in E$, so the action of $\mathcal{D}_A$ is well defined.

Conversely, if Ω^2 is fgp as a right A-module and $[\underline{v}]\triangleright e = \underline{v}\triangleright e$ defines an action of $\mathcal{D}_A$ on (E, ∇_E) then $(\alpha \otimes \mathrm{id})R_E(e) = 0$ for all $\alpha \in \Omega^{2\sharp}$ tells us that $R_E = 0$. Hence if given a $\mathcal{D}_A$-module, we pull it back to a $T\mathfrak{X}_\bullet$-module which must come from some (E, ∇_E) and by our observation this must be flat. $\square$

Example 6.25 Since the connection ∇ in Lemma 6.16 is defined by a product in $T\mathfrak{X}_\bullet$, it automatically descends to any quotient by an ideal, so in our case we inherit a connection $\nabla_{\mathcal{D}}$ on $\mathcal{D}_A$. Also since the action $\triangleright$ of $T\mathfrak{X}_\bullet$ is by $\bullet$, it descends to an action of $\mathcal{D}_A$ and $(\mathcal{D}_A, \nabla_D)$ is in the image of the inverse of the functor in Corollary 6.24 as it comes from the left action of $\mathcal{D}_A$ on itself. Hence if Ω^2 is finitely generated projective as a right A-module then $(\mathcal{D}_A, \nabla_{\mathcal{D}})$ has zero curvature. One can also see this directly, beginning with $\nabla_{\mathcal{D}}([\underline{v}]) = e_i \otimes [f_i \bullet \underline{v}]$, where $\mathrm{coev} = e_i \otimes f^i$ as before. Then the curvature is given by

$$\begin{aligned} R_{\mathcal{D}}([\underline{v}]) &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_{\mathcal{D}})\nabla_{\mathcal{D}}([\underline{v}]) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_{\mathcal{D}})\big(e_i \otimes [f_i \bullet \underline{v}]\big) \\ &= \mathrm{d}e_i \otimes [f_i \bullet \underline{v}] - e_i \wedge e_j \otimes [f_j \bullet f_i \bullet \underline{v}] = \big(\mathrm{d}e_i \otimes [f_i] - e_i \wedge e_j \otimes [f_j \bullet f_i]\big) \bullet [\underline{v}] \\ &= ((\mathrm{id} \otimes [\ ])\mathcal{R}) \bullet [\underline{v}], \end{aligned}$$

which vanishes since $\mathcal{R} \in \Omega^2 \otimes_A \mathcal{W}$ on writing $\mathcal{R} = \beta_k \otimes (\alpha_k \otimes \mathrm{id})\mathcal{R} = \beta_k \otimes \widehat{\mathcal{R}}(\alpha_k)$ for dual bases $\beta_k \otimes \alpha_k \in \Omega^2 \otimes_A \Omega^{2\sharp}$. The flat connection $(\mathcal{D}_A, \nabla_{\mathcal{D}})$ justifies calling $\mathcal{D}_A$ a 'sheaf' of differential operators. $\diamond$

We now provide an alternative form of the relations for $\mathcal{D}_A$ in terms of the generalised Lie bracket $[\![\ ,\ ]\!]_R$ of right vector fields given in §2.7. This was defined on the subspace $\Lambda^2\mathfrak{X} \subseteq \mathfrak{X} \otimes_{\Bbbk} \mathfrak{X}$ given by $u \otimes v \in \mathfrak{X} \otimes_{\Bbbk} \mathfrak{X}$ (summation implicit) such that $\mathrm{ev}^{\langle 2\rangle}(u \otimes v \otimes g) = 0$ for all g in the kernel of $\wedge : \Omega^{1\otimes 2} \to \Omega^2$. In this notation, we can rephrase (2.27) as

$$[\![u, v]\!]_R(\xi) = \mathrm{ev}^{\langle 2\rangle}(u \otimes v \otimes i(\mathrm{d}\xi)) + u(\mathrm{d}v(\xi)), \tag{6.14}$$

where $i : \Omega^2 \to \Omega^{1\otimes 2}$ is any lift. We similarly define a subset of $\Lambda^2_A\mathfrak{X} \subset \mathfrak{X} \otimes_A \mathfrak{X}$ consisting of x for which $\mathrm{ev}^{\langle 2\rangle}(x \otimes g) = 0$ for all g as before.

Proposition 6.26 *Suppose that Ω^1 and Ω^2 are fgp as right A-modules. Then*

$$\Lambda^2_A\mathfrak{X} = \big\{\alpha(e_i \wedge e_j) f_j \otimes f_i \in \mathfrak{X} \otimes_A \mathfrak{X} : \alpha \in \Omega^{2\sharp}\big\}$$

for dual bases $\mathrm{coev} = e_i \otimes f_i \in \Omega^1 \otimes_A \mathfrak{X}$. *For all $v \otimes w \in \Lambda^2\mathfrak{X}$ (summation implicit), the combination $[\![v \otimes w]\!]_R - v \bullet w$ depends only on $v \otimes_A w \in \Lambda^2_A\mathfrak{X}$. The relations in $\mathcal{D}_A$ take the form*

$$[\![\alpha^k{}_{ij} f_j \otimes f_i]\!]_R - \alpha^k{}_{ij} f_j \bullet f_i = 0; \quad \alpha^k{}_{ij} = \alpha_k(e_i \wedge e_j)$$

for a set $\{\alpha_k\}$ of left generators for $\Omega^{2\sharp} = \mathrm{Hom}_A(\Omega^2, A)$.

Proof (i) The short exact sequence of right modules

$$0 \longrightarrow \ker\wedge \longrightarrow \Omega^1 \otimes_A \Omega^1 \xrightarrow{\wedge} \Omega^2 \longrightarrow 0$$

by Proposition 3.107 and the discussion after Definition 3.110 has a splitting right module map, i.e., a lift $i : \Omega^2 \to \Omega^{1\otimes 2}$. The map $\phi : \Lambda^2_A \mathfrak{X} \to \Omega^{2\sharp}$ and $\psi : \Omega^{2\sharp} \to \Lambda^2_A \mathfrak{X}$ defined by

$$\phi(x)(\omega) = \mathrm{ev}^{\langle 2\rangle}(x \otimes i(\omega)), \quad \psi(\alpha) = \alpha(e_i \wedge e_j) f_j \otimes f_i$$

are inverse since $\phi(\psi(\alpha)) = \alpha$ on one side, while using the dual bases of Ω^1,

$$\psi(\phi(x)) = \phi(x)(e_i \wedge e_j) f_j \otimes f_i = \mathrm{ev}^{\langle 2\rangle}(x \otimes e_i \otimes e_j) f_j \otimes f_i = x.$$

(ii) Suppose $x.a \otimes y \in \Lambda^2\mathfrak{X}$ for $a \in A$ (summation implicit), and therefore also $x \otimes a.y \in \Lambda^2\mathfrak{X}$. Then

$$\big(\llbracket x \otimes a.y \rrbracket_R - \llbracket x.a \otimes y \rrbracket_R\big)(\eta) = x(\mathrm{d}a.y(\eta)) = x(\mathrm{d}a)y(\eta) = ((x\triangleright a)y)(\eta)$$

by definition of the bracket, and $x \bullet (a.y) - (x.a) \bullet y = (x\triangleright a)y$.

(iii) From the definition of $\llbracket\ ,\ \rrbracket_R$ we have for all $\eta \in \Omega^1$,

$$\begin{aligned}
&\llbracket \alpha(e_i \wedge e_j) f_j \otimes f_i \rrbracket_R(\eta) \\
&\quad = \mathrm{ev}^{\langle 2\rangle}(\alpha(e_i \wedge e_j) f_j \otimes f_i \otimes i(\mathrm{d}\eta)) + \alpha(e_i \wedge e_j) f_j(\mathrm{d}(f_i(\eta))) \\
&\quad = \alpha(\wedge i(\mathrm{d}\eta)) + \alpha(e_i \wedge \mathrm{d}(f_i(\eta))) = \alpha(\mathrm{d}\eta) + \alpha(e_i \wedge \mathrm{d}(f_i(\eta))) \\
&\quad = \alpha(\mathrm{d}(e_i.f_i(\eta))) + \alpha(e_i \wedge \mathrm{d}(f_i(\eta))) = \alpha(\mathrm{d}e_i)\ f_i(\eta),
\end{aligned}$$

which we insert into

$$\widehat{\mathcal{R}}(\alpha) = (\alpha \otimes \mathrm{id})\mathcal{R} = \alpha(\mathrm{d}e_i)\ f_i - \alpha(e_i \wedge e_j)\ f_j \bullet f_i$$

from Proposition 6.21. As $\widehat{\mathcal{R}} : \Omega^{2\sharp} \to T\mathfrak{X}_\bullet$ is a right A-module map, to specify the relations of $\mathcal{D}_A$ it is enough to set $\widehat{\mathcal{R}}(\alpha) = 0$ on a basis $\{\alpha_k\}$. □

In a similar manner to $T\mathfrak{X}_\bullet$ acting on objects in the category ${}_A\mathcal{E}$ of left A-modules, $\mathcal{D}_A$ acts on objects in the subcategory ${}_A\mathcal{F}$ of left A-modules with flat connection or 'differentiable sheaves' as studied in §4.3. We will also be interested in a monoidal version using flat objects in ${}_A\mathcal{E}_A$, but for this we need an extra condition so that the tensor product of flat connections remains flat. This is the notion of extendability in Definition 4.10, where the end of the proof of Theorem 4.15 about the additivity of the curvature under tensor products tells us that flat extendable bimodule connection are closed under tensor products. This is the category ${}_A\mathcal{F}_A$, a special case of the category ${}_B\mathcal{F}_A$ of B-A bimodules with flat extendable bimodule connections in §4.5.1. Note that for our purposes here, we only really use extendability up to Ω^2.

We showed in Theorem 6.18 that the object $(T\mathfrak{X}_\bullet, \nabla, 0)$ is in the centre $\mathcal{Z}({}_A\mathcal{E}_A)$ of ${}_A\mathcal{E}_A$. It is natural to ask if we have a similar result that $\mathcal{D}_A \in \mathcal{Z}({}_A\mathcal{F}_A)$.

Lemma 6.27 *Suppose that Ω^1 and Ω^2 are fgp as right A-modules and that the extendability condition on $\sigma_E : E \otimes_A \Omega^2 \to \Omega^2 \otimes_A E$ in Definition 4.10 holds for $(E, \nabla_E, \sigma_E) \in {}_A\mathcal{E}_A$, and that the curvature $R_E : E \to \Omega^2 \otimes_A E$ is a right module map. Then*

$$\lambda_{T,E}(\widehat{\mathcal{R}} \otimes \mathrm{id}) - \widehat{\mathcal{R}}{\triangleright}\mathrm{id} \otimes 1 = (\mathrm{id} \otimes \widehat{\mathcal{R}})\sigma_E{}^{-1} : \Omega^{2\sharp} \otimes_A E \to E \otimes_A T\mathfrak{X}_\bullet.$$

Proof If Ω^2 is finitely generated projective as a right A-module then from $\sigma_E : E \otimes_A \Omega^2 \to \Omega^2 \otimes_A E$ we can construct $\sigma_E^{-1} : \Omega^{2\sharp} \otimes_A E \to E \otimes_A \Omega^{2\sharp}$ such that

$$(\mathrm{id}_E \otimes \mathrm{ev})(\sigma_E^{-1} \otimes \mathrm{id}) = (\mathrm{ev} \otimes \mathrm{id}_E)(\mathrm{id} \otimes \sigma_E) : \Omega^{2\sharp} \otimes_A E \otimes_A \Omega^2 \to E.$$

Now for $\alpha \in \Omega^{2\sharp}$ and $e \in E$, and using the formula for $\mathcal{R}$ in Proposition 6.21,

$$\begin{aligned}
\lambda_{T,E}((\alpha \otimes \mathrm{id})\mathcal{R} \otimes e) &= \alpha(\mathrm{d}e_i)\lambda_{T,E}(f_i \otimes e) - \alpha(e_i \wedge e_j)\lambda_{T,E}(f_j \bullet f_i \otimes e) \\
&= \alpha(\mathrm{d}e_i)\lambda_{T,E}(f_i \otimes e) - \alpha(e_i \wedge e_j)(\triangleright \otimes \mathrm{id} + \sigma_E^{-1} \bullet \mathrm{id})(f_j \otimes \lambda_{T,E}(f_i \otimes e)) \\
&= \alpha(\mathrm{d}e_i)(f_i{\triangleright}e + \sigma_E^{-1}(f_i \otimes e)) \\
&\quad - \alpha(e_i \wedge e_j)(\triangleright \otimes \mathrm{id} + \sigma_E^{-1} \bullet \mathrm{id})(f_j \otimes (f_i{\triangleright}e + \sigma_E^{-1}(f_i \otimes e))),
\end{aligned}$$

and from this we have

$$\begin{aligned}
&\lambda_{T,E}((\alpha \otimes \mathrm{id})\mathcal{R} \otimes e) - (\alpha \otimes \mathrm{id})\mathcal{R}{\triangleright}e \otimes 1 \\
&\qquad = \alpha(\mathrm{d}e_i)\sigma_E^{-1}(f_i \otimes e) - \alpha(e_i \wedge e_j)\big(\sigma_E^{-1}(f_j \otimes (f_i{\triangleright}e)) \\
&\qquad\quad + (\triangleright \otimes \mathrm{id} + \sigma_E^{-1} \bullet \mathrm{id})(f_j \otimes \sigma_E^{-1}(f_i \otimes e))\big). \qquad (6.15)
\end{aligned}$$

If we set $\sigma_E^{-1}(f_i \otimes e) = e_i \otimes v_i$, then the first and third terms out of four on the right-hand side of (6.15) sum to

$$\begin{aligned}
\alpha(\mathrm{d}e_i)e_i \otimes v_i - \alpha(e_i \wedge e_j)f_j{\triangleright}e_i \otimes v_i &= (\alpha \otimes \mathrm{id})(\mathrm{d}e_i \otimes e_i - e_i \wedge \nabla_E e_i) \otimes v_i \\
&= (\alpha \otimes \mathrm{id})(\nabla_E{}^{[1]} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E^{-1})(e_i \otimes f_i \otimes e) \\
&= (\alpha \otimes \mathrm{id})(\nabla_E{}^{[1]} \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id})(e \otimes e_i \otimes f_i) \\
&= (\alpha \otimes \mathrm{id})(\mathrm{id} \wedge \sigma_E \otimes \mathrm{id})(\nabla_E e \otimes e_i \otimes f_i) + (\alpha \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id})(e \otimes \mathrm{d}e_i \otimes f_i),
\end{aligned}$$

where we used the definition of ${\sigma_E}^{-1}$ for the next to last line, and Lemma 4.12 for the last step. The second term of (6.15) is, using the definition of ${\sigma_E}^{-1}$ again,

$$-\alpha(e_i \wedge e_j)\sigma_E^{-1}(f_j \otimes (f_i{\triangleright}e)) = -(\alpha \otimes \sigma_E^{-1})(\mathrm{id} \wedge \mathrm{coev} \otimes \mathrm{id})\nabla_E(e)$$
$$= -(\alpha \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \wedge \sigma_E \otimes \mathrm{id})(\nabla_E(e) \otimes e_i \otimes f_i).$$

The fourth term is

$$-((\alpha \circ \wedge) \otimes \mathrm{id} \otimes \bullet)(\mathrm{id} \otimes \mathrm{id} \otimes (\sigma_E^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E^{-1}))(e_i \otimes e_j \otimes f_j \otimes f_i \otimes e)$$
$$= -((\alpha \circ \wedge) \otimes \mathrm{id} \otimes \bullet)((\mathrm{id} \otimes \sigma_E)(\sigma_E \otimes \mathrm{id}) \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes e_i \otimes e_j \otimes f_j \otimes f_i)$$
$$= -(\alpha \otimes \mathrm{id} \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes e_i \wedge e_j \otimes f_j \bullet f_i).$$

Thus the right-hand side of (6.15) becomes

$$(\alpha \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id})(e \otimes \mathrm{d}e_i \otimes f_i) - (\alpha \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id})(e \otimes e_i \wedge e_j \otimes f_j \bullet f_i),$$

so that we obtain

$$\lambda_{T,E}((\alpha \otimes \mathrm{id})\mathcal{R} \otimes e) - (\alpha \otimes \mathrm{id})\mathcal{R}{\triangleright}e \otimes 1 = (\alpha \otimes \mathrm{id})(\sigma_E \otimes \mathrm{id})(e \otimes \mathcal{R}).$$

Finally, we convert the right-hand side to the required form using the identity

$$(\mathrm{ev} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_E \otimes \mathrm{id})(\alpha \otimes e \otimes \mathcal{R}) = (\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id})(\sigma_E^{-1} \otimes \mathcal{R})(\alpha \otimes e). \quad \square$$

Proposition 6.28 *Suppose Ω^1 and Ω^2 are fgp as right A-modules. Then the natural transformation λ_T descends to make $(\mathcal{D}_A, \nabla_{\mathcal{D}}, 0)$ an object of the centre of ${}_A\mathcal{F}_A$.*

Proof The curvature of the connection $\nabla_{\mathcal{D}}$ is zero, as shown in Example 6.25, and as the corresponding $\sigma_{\mathcal{D}}$ is zero, the extendability condition in Definition 4.10 is satisfied. Now we show for $(E, \nabla_E, \sigma_E) \in {}_A\mathcal{F}_A$ that

$$\lambda_{T,E}(\mathcal{W} \otimes_A E) \subseteq E \otimes_A \mathcal{W}.$$

Since $\lambda_{T,E}$ in Theorem 6.18 are morphisms, in particular bimodule maps, we only need show that $\lambda_{T,E}(\widehat{\mathcal{R}}(\alpha) \otimes e) \in E \otimes_A \mathcal{W}$ for all $\alpha \in \Omega^{2\sharp}$ and $e \in E$. But from Lemma 6.27, since $\widehat{\mathcal{R}}{\triangleright}\mathrm{id} : \Omega^{2\sharp} \otimes_A E \to E$ is zero, we have

$$\lambda_{T,E}(\widehat{\mathcal{R}}(\alpha) \otimes e) = (\mathrm{id} \otimes \widehat{\mathcal{R}}){\sigma_E}^{-1}(\alpha \otimes e),$$

which lands as required. $\square$

Example 6.29 (Differential Operators on the q-Sphere) Continuing the story of Example 6.20, for $\mathbb{C}_q[S^2]$ we calculate $\mathcal{R}$ and the relations defining $\mathcal{D}_A$. Using the formula $T_\heartsuit = \mathrm{d} + \wedge\heartsuit$ the torsion is

$$T_\heartsuit(e^+\xi_+) = -e^+ \wedge \mathrm{d}\xi_+ + e^+ \wedge (\partial_+\xi_+).e^+ + e^+ \wedge (\partial_-\xi_+).e^- = 0,$$
$$T_\heartsuit(e^-\xi_-) = -e^- \wedge \mathrm{d}\xi_- + e^- \wedge (\partial_+\xi_-).e^+ + e^- \wedge (\partial_-\xi_-).e^- = 0.$$

From Proposition 6.21, moving elements of $\mathbb{C}_q[S^2]$ across the $\otimes_A$

$$\begin{aligned}\mathcal{R} = &-e^+ D_1 D_1' \wedge e^+ D_1'' D_1''' \otimes (D_2''' D_2'' f_+ \otimes D_2' D_2 f_+)\\ &- e^+ D_1 D_1' \wedge e^- \tilde{D}_1 \tilde{D}_1' \otimes (\tilde{D}_2' \tilde{D}_2 f_- \otimes D_2' D_2 f_+)\\ &- e^- \tilde{D}_1 \tilde{D}_1' \wedge e^+ D_1 D_1' \otimes (D_2' D_2 f_+ \otimes \tilde{D}_2' \tilde{D}_2 f_-)\\ &- e^- \tilde{D}_1 \tilde{D}_1' \wedge e^- \tilde{D}_1'' \tilde{D}_1''' \otimes (\tilde{D}_2''' \tilde{D}_2'' f_- \otimes \tilde{D}_2' \tilde{D}_2 f_-)\\ = &\; e^+ \wedge e^- \otimes (\tilde{D}_1 \tilde{D}_1' f_+ \otimes \tilde{D}_2' \tilde{D}_2 f_- - q^2 D_1 D_1' f_- \otimes D_2' D_2 f_+).\end{aligned}$$

Now Ω^2 is a free module generated by $e^+ \wedge e^-$, so $\Omega^{2\sharp}$ is generated by α, where $\alpha(e^+ \wedge e^-) = 1$. Then

$$\mathcal{R}(\alpha) = \tilde{D}_1 \tilde{D}_1' f_+ \otimes \tilde{D}_2' \tilde{D}_2 f_- - q^2 D_1 D_1' f_- \otimes D_2' D_2 f_+$$

is the only relation defining $\mathcal{D}_A$. Then $\mathcal{D}_A$ is the quotient of $T\mathfrak{X}_\bullet$ by the 2-sided ideal generated by $\mathcal{R}(\alpha)$ under the $\bullet$ product. ◇

6.5 More Examples of Algebras of Differential Operators

We have focussed on the q-sphere as our main example for the constructions in this chapter. Here we give some parallelisable examples where $T\mathfrak{X}_\bullet$ and $\mathcal{D}_A$ can be computed more explicitly.

6.5.1 Left-Invariant Differential Operators on Hopf Algebras

Let H be a Hopf algebra with left-covariant calculus and with invertible antipode. We already know from Lemma 6.1 that left-invariant right vector fields ${}^H\mathfrak{X} \cong \Lambda^{1*}$ coming from a nonstandard factorisation $\Omega^1 = \Lambda^1.H \cong \Lambda^1 \otimes H$ as a vector space whereby we identify $v \otimes h$ with vh (in the 'wrong' order). We accordingly identify $\mathfrak{X} = H.\Lambda^{1*} \cong H \otimes \Lambda^{1*}$ as a vector space by sending $g \otimes x$ to the right vector field $vh \mapsto \langle g \otimes x, v \otimes h\rangle = g\langle x, v\rangle h \in H$ for $h, g \in H$, $v \in \Lambda^1$, $x \in \Lambda^{1*}$. Moreover, the commutation rules for Ω^1 between Λ^1 and H given by an action $\triangleleft$ dualise and provide the bimodule relations of $\mathfrak{X}$ for the dual action,

$$v.h = h_{(1)}.(v\triangleleft h_{(2)}); \quad (v\triangleleft h)(\omega) = v(\omega\triangleleft S^{-1}h)$$

for all $h \in H$, $v \in {}^H\mathfrak{X}$ and $\omega \in \Lambda^{1*}$.

Since $\mathfrak{X}$ is parallelisable, we can use the commutation relations

$$v \bullet h = vh + v(\mathrm{d}h) = h_{(1)}v\triangleleft h_{(2)} + v(\mathrm{d}h) = h_{(1)} \bullet (v\triangleleft h_{(2)}) + v(\mathrm{d}h)$$

for $h \in H$, $v \in \Lambda^{1*}$ to normal order elements of $T\mathfrak{X}_\bullet$ and give the latter an algebra factorisation into H and the left-invariant subalgebra $T\Lambda^{1*}$,

$$T\mathfrak{X}_\bullet = H.T\Lambda^{1*} \tag{6.16}$$

(as a vector space this is a tensor product of the two factors). If we choose a basis $\{e^i\}$ of Λ^1 with dual basis $\{f_i\}$ then $T\Lambda^{1*} = \Bbbk\langle f_i\rangle$ is the free associative algebra in the f_i and the 'quantum Weyl algebra' cross relations are

$$f_i \bullet h = h_{(1)} \bullet (f_i\triangleleft h_{(2)}) + \tilde{\partial}_i(h), \tag{6.17}$$

where $\mathrm{d}h = e^i\tilde{\partial}_i(h)$ (sum over repeated indices).

Next, since the calculus is left-invariant, we can also factorise $\Omega^2 = \Lambda^2.H = H.\Lambda^2$ for a left-invariant part Λ^2, so this is free as an H-module. We let $\{\omega^k\}$ be a basis of Λ^2 and define structure constants $d^i{}_k, c^{ij}{}_k \in \Bbbk$ by

$$\mathrm{d}e^i = d^i{}_k\omega^k, \quad e^i \wedge e^j = c^{ij}{}_k\omega^k \tag{6.18}$$

(sum over repeated indices). As usual, let $\mathrm{coev} = e^i \otimes f_i \in \Lambda^1 \otimes \Lambda^{1*}$ (summing over i). Then the first expression for $\mathcal{R}$ in Proposition 6.21 immediately tells us the relations for the left-invariant part ${}^H\mathcal{D}_H$ are

$$d^i{}_k f_i = c^{ij}{}_k f_j \bullet f_i. \tag{6.19}$$

Similarly to $T\mathfrak{X}_\bullet$, the full $\mathcal{D}_H$ factorises into H and ${}^H\mathcal{D}_H$ with the same cross relations as before. On a classical Lie group G, (6.19) gives the relations of its Lie algebra $\mathfrak{g}$, the left-invariant subalgebra of differential operators can be identified with the enveloping algebra $U(\mathfrak{g})$ and the full algebra is a semidirect product of this acting on functions.

Example 6.30 Consider $\mathfrak{h} = \Lambda^{1*}$ for $H = \Bbbk(S_3)$ with differential calculus in Example 1.60. Take a basis $\{f_u, f_v, f_w\}$ for $\mathfrak{h}$ dual to the given basis $\{e_u, e_v, e_w\}$ of the left-invariant 1-forms. The space of antisymmetric elements of $\mathfrak{h} \otimes \mathfrak{h}$ is

4-dimensional, with the brackets,

$$[\![f_v \otimes f_u - f_w \otimes f_v]\!]_R = f_u - f_w, \quad [\![f_v \otimes f_u - f_u \otimes f_w]\!]_R = f_v - f_w,$$
$$[\![f_u \otimes f_v - f_w \otimes f_u]\!]_R = f_v - f_w, \quad [\![f_u \otimes f_v - f_v \otimes f_w]\!]_R = f_u - f_w.$$

We take the basis of 2-forms

$$\omega_{uv} = e_u \wedge e_v, \quad \omega_{vw} = e_v \wedge e_w, \quad \omega_{vu} = e_v \wedge e_u, \quad \omega_{uw} = e_u \wedge e_w.$$

The only nonzero $c^{ij}{}_k$ in (6.18) are

$$c^{u,v}{}_{uv} = c^{v,w}{}_{vw} = c^{v,u}{}_{vu} = c^{u,w}{}_{uw} = 1,$$
$$c^{w,u}{}_{uv} = c^{w,u}{}_{vw} = c^{w,v}{}_{vu} = c^{w,v}{}_{uw} = -1$$

and the only nonzero $d^i{}_k$ are

$$d^u{}_{vw} = d^v{}_{uw} = d^w{}_{uv} = d^w{}_{vu} = -1, \quad d^u{}_{vu} = d^u{}_{uw} = d^v{}_{uv} = d^v{}_{vw} = 1.$$

The relations for the left-invariant part of $\mathcal{D}_H$ are

$$f_v - f_w = f_v \bullet f_u - f_u \bullet f_w, \quad f_u - f_w = f_u \bullet f_v - f_v \bullet f_w,$$
$$f_u - f_v = f_u \bullet f_w - f_w \bullet f_v, \quad f_v - f_u = f_v \bullet f_w - f_w \bullet f_u.$$

On the other hand, as this calculus is bicovariant, by Theorem 2.85 there is a left-covariant right-quantum Lie algebra on $\mathfrak{h}$ which is computed for the finite group case in Example 2.86. From there we find in our case that

$$[f_a, f_b]_R = f_c - f_a, \ [f_a, f_a]_R = 0, \ \sigma_R(f_a \otimes f_b) = f_b \otimes f_c, \ \sigma(f_a \otimes f_a) = f_a \otimes f_a,$$

for a, b, c distinct elements of $\{u, v, w\}$. The corresponding enveloping algebra $U(\mathfrak{h})$ has relations

$$f_a f_b - f_b f_c = f_c - f_a$$

which has the same form as the $\mathcal{D}_H$ relations aside from an overall sign on the right-hand side. Thus the left-invariant part ${}^H\mathcal{D}_H$ is naturally identified in our conventions with the enveloping algebra for the opposite of the right quantum Lie algebra (i.e., with the opposite sign to $[\ ,\ ]_R$). This is isomorphic to $U(\mathfrak{h})$ by a sign-reversing automorphism on the generators, so we still have ${}^H\mathcal{D}_H \cong U(\mathfrak{h})$.

The full $\mathcal{D}_H$ for $H = k(G)$ and a finite group G is generated by functions and ${}^H\mathcal{D}_H$ with cross relations

$$f_a \bullet f = R_{a^{-1}}(f) \bullet f_a + (\mathrm{id} - R_{a^{-1}}) f$$

albeit for S_3 above we have $a = a^{-1}$. By the right version of Example 2.93, we also have a right 'finite Lie algebra' or quandle $\mathcal{L}$ with grouplike generators $\tilde{f}_a = 1 - f_a$ for the enveloping bialgebra $U(\mathcal{L})$ with relations $\tilde{f}^a \tilde{f}^b = \tilde{f}^b \tilde{f}^{b^{-1}ab}$ in general, or $\tilde{f}^a \tilde{f}^b = \tilde{f}^a \tilde{f}^c$ in the case of S_3. These act by $\tilde{f}^a \triangleright f = R_{a^{-1}}(f)$ so that $\mathcal{D}_H \cong \mathbb{k}(G) \rtimes U(\mathcal{L})$ with the cross relation $\tilde{f}_a \bullet f = R_{a^{-1}}(f) \bullet \tilde{f}_a$. $\diamond$

The next example has a nonbicovariant calculus, so a right-handed version of Theorem 2.85 does not apply in any case.

Example 6.31 We consider $H = \mathbb{C}_q[SU_2]$ with its 3D calculus in Example 2.32 and set $\omega_0 = e^+ \wedge e^-$ and $\omega_\pm = e^\pm \wedge e^0$, and take f_0, f_+, f_- to be the dual basis to e^0, e^+, e^- respectively. The structure constants in (6.18) are given by $c^{ii}{}_k = 0$ and

$$d^0{}_k = \begin{cases} q^3 & k = 0 \\ 0 & \text{otherwise}, \end{cases} \qquad d^\pm{}_k = \begin{cases} \mp q^{\pm 2}(1 + q^{-2}) & k = \pm \\ 0 & \text{otherwise}, \end{cases}$$

$$c^{\pm 0}{}_k = \begin{cases} 1 & k = \pm \\ 0 & \text{otherwise}, \end{cases} \qquad c^{0\pm}{}_k = \begin{cases} -q^{\pm 4} & k = \pm \\ 0 & \text{otherwise}, \end{cases}$$

$$c^{+-}{}_k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise}, \end{cases} \qquad c^{-+}{}_k = \begin{cases} -q^2 & k = 0 \\ 0 & \text{otherwise}. \end{cases}$$

From (6.19) we find the relations

$$f_- \bullet f_+ - q^2 f_+ \bullet f_- = q^3 f_0, \quad f_0 \bullet f_\pm - q^{\pm 4} f_\pm \bullet f_0 = \mp q^{\pm 2}(1 + q^{-2}) f_\pm$$

for the left-invariant part ${}^H\mathcal{D}_H$. $\diamond$

We next illustrate constructing representations of $\mathcal{D}_H$. By Corollary 6.24, this amounts to finding flat connections on left modules.

Example 6.32 We consider the restricted class of left bimodule connections on Ω^1 for the 3D calculus on $\mathbb{C}_q[SU_2]$ that are left-invariant and invariant under the right $\mathbb{C}\mathbb{Z}$ coaction. As can be seen in Example 3.77, if q is not a root of unity and using ∇^L for the restriction to the left-invariant forms, these are of the form

$$\begin{aligned} \nabla^L(e^0) &= \gamma e^0 \otimes e^0 + \mu_+ e^+ \otimes e^- + \mu_- e^- \otimes e^+, \\ \nabla^L(e^\pm) &= \alpha_\pm e^0 \otimes e^\pm + \beta_\pm e^\pm \otimes e^0 \end{aligned} \tag{6.20}$$

for various complex parameters. The curvature of these connections is

$$\begin{aligned} R(e^\pm) = &\big(\alpha_\pm q^3 e^+ \wedge e^- - \mu_\mp \beta_\pm e^\pm \wedge e^\mp\big) \otimes e^\pm \\ &+ \beta_\pm\big(\mp q^{\pm 2}(1 + q^{-2}) + \alpha_\pm q^{\pm 4} - \gamma\big) e^\pm \wedge e^0 \otimes e^0, \end{aligned}$$

$$R(e^0) = \big(\gamma q^3 - \mu_+\beta_- + \mu_-\beta_+ q^2\big)e^+ \wedge e^- \otimes e^0$$
$$+ \mu_\pm\big(\mp q^{\pm 2}(1+q^{-2}) + \gamma q^{\pm 4} - \alpha_\mp\big)e^\pm \wedge e^0 \otimes e^\mp.$$

The curvature vanishes in four cases:

(a) $\nabla^L = 0$;
(b) $\alpha_- = -q^2(1+q^{-2}) = -\mu_+\beta_- q^{-1}, \alpha_+ = q^{-2}(1+q^{-2}) = q^{-3}\mu_-\beta_+, \gamma = 0$;
(c) $\alpha_- = \beta_- = \mu_+ = 0, \alpha_+ = q^{-2} = q^{-3}\mu_-\beta_+, \gamma = -1$;
(d) $\alpha_+ = \beta_+ = \mu_- = 0, \alpha_- = -1 = -\mu_+\beta_- q^{-1}, \gamma = q^{-2}$.

Given a flat connection (6.20) we have an action on Ω^1 which restricts to Λ^1. This is

$$f_+ \mapsto \begin{pmatrix} 0 & 0 & 0 \\ \beta_+ & 0 & 0 \\ 0 & \mu_+ & 0 \end{pmatrix}, \quad f_0 \mapsto \begin{pmatrix} \alpha_+ & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \alpha_- \end{pmatrix}, \quad f_- \mapsto \begin{pmatrix} 0 & \mu_- & 0 \\ 0 & 0 & \beta_- \\ 0 & 0 & 0 \end{pmatrix}$$

for the action of each f_i as a matrix acting by left multiplication on a column vector of coefficients of an element in the ordered basis e^+, e^0, e^-. Equivalently one can read these as acting from the right on (e^+, e^0, e^-) as a row vector, so for example $f_+ \triangleright e^+ = \beta_+ e_0$ from the first row of the matrix for f_+. We can now check our calculations by substituting these matrices into the relations in Example 6.31. This gives the following series of equations;

$$\beta_+\mu_- = \alpha_+ q^3, \quad \beta_-\mu_+ - \beta_+\mu_- q^2 - q^3\gamma = 0, \quad \beta_-\mu_+ = -\alpha_- qm,$$
$$\beta_+(-1 - q^2 + \alpha_+ q^4 - \gamma) = 0, \quad \mu_+(1 + \alpha_- + q^2 - q^4\gamma) = 0,$$
$$\mu_-(-1 - q^2 + \alpha_+ q^4 - \gamma) = 0, \quad \beta_-(1 + \alpha_- + q^2 - q^4\gamma) = 0,$$

which is precisely the same as the conditions for zero curvature. ◇

6.5.2 A Noncommutative Witt Enveloping Algebra

We start with the algebra of differential operators of the differential algebra $A = \mathbb{C}_q[S^1]$ in Example 1.11. This is $\mathbb{C}[z, z^{-1}]$ as an algebra (we denote the generator by z in the present context) and $q \neq \pm 1$ refers to its noncommutative differential calculus $(\mathrm{d}z)z = qz\mathrm{d}z$. This is a Hopf algebra with z grouplike and the preceding section applies. It also has to have $\Omega^2 = 0$ as $q \neq -1$. As left-invariant basic 1-form we take $e_1 = \mathrm{d}z.z^{-1} \in \Lambda^1$ so that $\mathrm{d}f = e_1\tilde{\partial}_z f$ where $\tilde{\partial}_z = z\partial_{q^{-1}}$ in terms of a usual q-derivative. The action of the dual basis vector f_1 of $\mathfrak{X}$ is by $\tilde{\partial}_z$ and has relations $f_1.z = q^{-1}z.f_1$. Then

$$\mathcal{D}_A = T\mathfrak{X}_\bullet = \mathbb{C}[z, z^{-1}] \rtimes \mathbb{C}[f_1], \quad f_1 \bullet z = q^{-1}z \bullet f_1 + z$$

for its description in the form of a q-deformed Weyl algebra on a circle. To describe the bialgebra structure on $\mathbb{C}[f_1]$ it is convenient to change variables,

$$L = \frac{1}{1-q^{-1}} - f_1; \quad L \bullet z = q^{-1} z \bullet L, \quad \mathcal{D}_A = \mathbb{C}[z, z^{-1}] {>\!\!\!\triangleleft} \mathbb{C}[L],$$

where $\mathbb{C}[L]$ is a bialgebra with L grouplike and acts on A by $L \triangleright z^n = q^{-n} z^n$. We can now see that representations of $\mathcal{D}_A$ are (E, Z, γ), where $Z, \gamma : E \to E$ with Z invertible and $Z\gamma = q^{-1}\gamma Z$ represent z, L respectively. The corresponding flat connection by Proposition 6.15 or Corollary 6.24 is

$$\nabla_E(e) = e_1 \otimes f_1 \triangleright e = (\mathrm{d}z)z^{-1} \otimes \left(\frac{e}{1-q^{-1}} - \gamma(e)\right).$$

Proposition 6.33

(1) *(E, ∇_E) defined by the triple (Z, γ) with $Z\gamma = q^{-1}\gamma Z$ and made into a bimodule by $e.z = Y(e)$, where Y is invertible, is a bimodule connection if and only if*

$$[Y\gamma - q\gamma Y, Y] = 0; \quad [Y, Z] = 0$$

and in this case $\sigma_E(e \otimes \mathrm{d}z) = (\mathrm{d}z)z^{-1} \otimes (Y\gamma - \gamma Y)(e)$.

(2) *The canonical representation of $\mathcal{D}_A$ on itself and the resulting bimodule connection are*

$$Z(z^m L^n) = z^{m+1} L^n, \ \ \gamma(z^m L^n) = q^{-m} z^m L^{n+1}, \ \ Y(z^m L^n) = q^{-n} z^{m+1} L^n,$$

$$\nabla_{\mathcal{D}}(z^m L^n) = (\mathrm{d}z)z^{-1} \otimes z^m \left(\frac{1}{1-q^{-1}} - q^{-m} L\right) L^n,$$

giving an object of ${}_A\mathcal{F}_A$. This becomes an object of $\mathcal{Z}({}_A\mathcal{F}_A)$ when supplemented with the natural transformation

$$\lambda_{\mathcal{D},E}(f_1 \otimes e) = \left(\frac{e}{1-q^{-1}} - \gamma(e)\right) \otimes 1 + (\gamma - Y^{-1}\gamma Y(e)) \otimes f_1.$$

Proof (1) The braiding if it exists is given by the connection and becomes $\sigma(e \otimes \mathrm{d}z) = (\mathrm{d}z)z^{-1} \otimes (\gamma(e)z - \gamma(ez))$. To be a bimodule map, it is sufficient to check that it is well defined for $\otimes_A$ as a right module map, $\sigma_E(ez \otimes \mathrm{d}z) = \sigma_E(e \otimes z\mathrm{d}z) = q^{-1}\sigma_E(e \otimes \mathrm{d}z)z$, which is equivalent to $\gamma(e)z^2 - (1+q)\gamma(ez)z + q\gamma(ez^2)$. When translated to the Y, γ notation, this gives the stated result.

(2) From (1) we have $\sigma_E(e \otimes \mathrm{d}e_1) = e_1 \otimes (\gamma(e) - \gamma(ez)z^{-1})$ which dualises to

$$\sigma_E^{-1}(f_1 \otimes e) = (\gamma(e) - \gamma(ez)z^{-1} \otimes f_1.$$

Then Lemma 6.17 tells us that

$$\lambda_{\mathcal{D},E}(f \otimes e) = \big(\frac{e}{1-q^{-1}} - \gamma(e)\big) \otimes 1 + (\gamma(e) - \gamma(ez)z^{-1}) \otimes f_1,$$

which we state in terms of matrices. □

Finally, we recall that classically on any manifold $U(\text{diff})$, where $\text{diff} = \mathfrak{X}$ as a Lie algebra in the usual way, can be realised as operators, so there is an algebra map $U(\text{diff}) \to \mathcal{D}_A$. In the case of the circle $\mathfrak{X}$ has bracket given by the Witt algebra with basis $L_n = -z^{n+1}\frac{\partial}{\partial z}$ (here $n \in \mathbb{Z}$), where z is the complex coordinate for the circle, viewed as the unit circle in $\mathbb{C}$, and

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{6.21}$$

The Witt enveloping algebra is the associative algebra with these relations. We similarly define a q-Witt enveloping algebra abstractly by

$$q^{m-n}L_nL_m - L_mL_n = [n-m]_{q^{-1}}L_{n+m}. \tag{6.22}$$

Proposition 6.34 *The q-Witt algebra is realised naturally in $\mathcal{D}_A$ by $L_n = -z^n \bullet f_1$.*

Proof We use the commutation relations in $\mathcal{D}_A$, denoting its product $\bullet$ by omission. As well as the relations previously stated, we also have $f_1z^m = q^{-m}f_1 + \tilde{\partial}_z z^m = q^{-m}z^mf_1 + [m]_{q^{-1}}z^m$. Using this,

$$\begin{aligned} q^{m-n}L_nL_m - L_mL_n &= q^{m-n}z^nf_1z^mf_1 - z^mf_1z^nf_1 \\ &= q^{m-n}[m]_{q^{-1}}z^{n+m}f_1 - [n]_{q^{-1}}z^{n+m}f_1 \\ &= \big(\frac{q^{m-n}(1-q^{-m}) - 1 + q^{-n}}{1-q^{-1}}\big)z^{n+m}f_1 \\ &= [n-m]_{q^{-1}}L_{n+m}, \end{aligned}$$

where the f_1^2 terms cancel. □

Clearly, if $\mathcal{D}_A$ is represented on (E, ∇_E) then this pulls back to a representation of q-Witt where f_1 acts by covariant differentiation. For example, $f_1 = z\partial_{q^{-1}}$ on functions as above.

6.5.3 Differential Operators on $M_2(\mathbb{C})$

For our final example, we look at our standard differential calculus on $A = M_2(\mathbb{C})$ in Example 1.38, namely the maximal prolongation in Example 1.37 along with $s^2 = t^2 = 0$. Take a basis of vector fields f_s, f_t dual to s, t respectively. These are central since s, t are central in Ω^1. Remembering that $s \wedge t = t \wedge s$ (a basis of Ω^2 over the algebra), we see that $f_s \otimes f_t + f_t \otimes f_s$ is a basis of the antisymmetric tensor products of vector fields $\Lambda^2\mathfrak{X}$. Finally, $\mathrm{d}s = 2s \wedge t E_{21}$ and $\mathrm{d}t = 2s \wedge t E_{12}$.

(a) *The Lie bracket.* From §2.7, we can calculate the bracket $[\![\ ,\ ]\!]_R : \Lambda^2\mathfrak{X} \to \mathfrak{X}$ defined in (2.27) as

$$\begin{aligned}[\![f_s \otimes f_t + f_t \otimes f_s]\!]_R(s) &= \mathrm{ev}^{\langle 2\rangle}((f_s \otimes f_t + f_t \otimes f_s) \otimes 2s \otimes t E_{21}) \\ &\quad + \mathrm{ev}(f_s \otimes \mathrm{d}\,\mathrm{ev}(f_t, s)) + \mathrm{ev}(f_t \otimes \mathrm{d}\,\mathrm{ev}(f_s, s)) = 2E_{21}, \\ [\![f_s \otimes f_t + f_t \otimes f_s]\!]_R(t) &= 2E_{12},\end{aligned}$$

so that

$$[\![f_s \otimes f_t + f_t \otimes f_s]\!]_R = 2E_{21} f_s + 2E_{12} f_t.$$

(b) *The algebra $T\mathfrak{X}_\bullet$.* Because the calculus is parallelisable and f_s, f_t central in $\mathfrak{X}$, we know that $T\mathfrak{X}_\bullet = M_2.\mathbb{C}\langle f_s, f_t\rangle$ as an algebra factorisation with cross relations

$$f_s \bullet x = x \bullet f_s + [E_{12}, x], \quad f_t \bullet x = x \bullet f_t + [E_{21}, x] \tag{6.23}$$

for all $x \in M_2$. Here $f_s(\mathrm{d}x) = [E_{12}, x]$ etc. as the partial derivatives. For an example, the quantum Levi-Civita connection on Ω^1 which we will encounter Example 8.13 corresponds to a $T\mathfrak{X}_\bullet$-module with action on the generators given by

$$f_s {\triangleright} s = 2E_{12}s, \quad f_s {\triangleright} t = 2E_{12}t, \quad f_t {\triangleright} s = 2E_{21}s, \quad f_t {\triangleright} t = 2E_{21}t$$

and extended so as to represent (6.23), for example $f_s {\triangleright} (xs) = \{E_{12}, x\}s$.

(c) *The algebra of differential operators.* Next, using Proposition 6.21 and the alternative notation $e^1 = s$, $e^2 = t$, $f_1 = f_s$, $f_2 = f_t$,

$$\begin{aligned}\mathcal{R} &= \sum_i \mathrm{d}e^i \otimes f_i - \sum_{i,j} e^i \wedge e^j \otimes f_j \bullet f_i \\ &= 2s \wedge t \otimes (E_{21} f_s + E_{12} f_t) - s \wedge t \otimes (f_t \bullet f_s + f_s \bullet f_t).\end{aligned}$$

This gives us $\mathcal{D}_{M_2}$ as $T\mathfrak{X}_\bullet$ with the additional relation

$$f_t \bullet f_s + f_s \bullet f_t = 2E_{12} \bullet f_t + 2E_{21} \bullet f_s. \tag{6.24}$$

To understand its structure, we change variables and find relations

$$\tilde{f}_s = f_s - E_{12}, \quad \tilde{f}_t = f_t - E_{21},$$
$$\tilde{f}_s \bullet x = x \bullet \tilde{f}_s, \quad \tilde{f}_t \bullet x = x \bullet \tilde{f}_t, \quad \tilde{f}_s \bullet \tilde{f}_t + \tilde{f}_t \bullet \tilde{f}_s = 1$$

for all $x \in M_2$. Hence $\mathcal{D}_{M_2} = M_2 \otimes B = M_2(B)$, where B is the *fermionic canonical commutation relations algebra* generated by $\tilde{f}_s, \tilde{f}_t$ and the anticommutator relation $\{\tilde{f}_s, \tilde{f}_t\} = 1$. We have not considered $*$-structures in this chapter but in a physical context it would be natural to have $\tilde{f}_t = (\tilde{f}_s)^*$.

(d) *$\mathcal{D}_{M_2}$-modules.* Clearly, a left module of $\mathcal{D}_{M_2}$ means a vector space E and two operators $X, Y : E \to E$ obeying $XY + YX = \mathrm{id}$, representing $\tilde{f}_s, \tilde{f}_t$ respectively. In addition, M_2 left acts on E and commutes with X, Y. The corresponding flat connection is

$$\nabla_E(e) = s \otimes f_s \triangleright e + t \otimes f_t \triangleright e = s \otimes (X(e) + E_{12}.e) + t \otimes (Y(e) + E_{21}.e). \qquad (6.25)$$

For a simple class of concrete examples, we can take $E = \mathbb{C}^2 \otimes V = V \oplus V$ where B is represented on V by X, Y and M_2 acts on $\mathbb{C}^2$ as usual. Then in matrix form

$$\nabla_E \begin{pmatrix} v \\ w \end{pmatrix} = s \otimes \begin{pmatrix} X(v) + w \\ X(w) \end{pmatrix} + t \otimes \begin{pmatrix} Y(v) \\ v + Y(w) \end{pmatrix}$$

is a flat connection in ${}_{M_2}\mathcal{F}$. For example, one can take the 2-state fermion system where $V = \mathbb{C}^2$ and $X(v) = E_{12}v$, $Y(v) = E_{21}v$, or another example would be the fermionic Fock space V.

(e) *Bimodule connections from $\mathcal{D}_{M_2}$-modules.* For bimodule connections we similarly consider the simplest case of $E = M_2 \otimes V$, where B is represented on V by operators $X, Y : V \to V$ and M_2 has its usual left and right multiplication.

Proposition 6.35

(1) *Let $E = M_2 \otimes V$, where V is a B-module by operators X, Y. Then*

$$\nabla_E(x \otimes v) = s \otimes (x \otimes X(v) + E_{12}x \otimes v) + t \otimes (x \otimes Y(v) + E_{21}x \otimes v)$$

is a flat bimodule connection in ${}_{M_2}\mathcal{F}_{M_2}$ with $\sigma_E = 0$.

(2) *The canonical connection coming from $\mathcal{D}_{M_2} = M_2 \otimes B$ acting on itself is an example of (1) with*

$$\nabla_{\mathcal{D}}(x \otimes b) = s \otimes (x \otimes \tilde{f}_s b + E_{12}x \otimes b) + t \otimes (x \otimes \tilde{f}_t b + E_{21}x \otimes b)$$

for all $x \in M_2$, $b \in B$. When restricted to E in the subcategory in (1), we have

$$\lambda_{\mathcal{D},E}(\tilde{f}_s \otimes (x \otimes b)) = (x \otimes \tilde{f}_s b) \otimes 1, \quad \lambda_{\mathcal{D},E}(\tilde{f}_t \otimes (x \otimes b)) = (x \otimes \tilde{f}_t b) \otimes 1.$$

Proof The flat connection in (1) is immediate from specialising the formula in (d) above to the case of $E = M_2 \otimes V$. Further, as the right M_2 action commutes with the left action of $\mathcal{D}_{M_2}$, the connection is a right module map and therefore $\sigma_E = 0$. This is clearly extendable with $\sigma_E = 0$ on all the higher forms. For (2), Lemma 6.17 immediately tells us that $\lambda_{\mathcal{D},E}(f_s \otimes (x \otimes b)) = (x \otimes \tilde{f}_s b + E_{12}x \otimes b) \otimes 1$ and we then shift to $\tilde{f}_s$ using that $\lambda_{\mathcal{D},E}$ is a (bi)module map. Similarly for $\tilde{f}_t$. $\square$

We can obviously take any representation of B, such as the spinor representation. The formula for $\lambda_{\mathcal{D},E}$ on a general $E \in {}_{M_2}\mathcal{F}_{M_2}$ is in Lemma 6.17 but does not particularly simplify. We close with just one example of a more general object $E \in {}_{M_2}\mathcal{F}_{M_2}$ beyond the tensor product form. We set $E = M_2$ with its left and right module structure and $X(e) = eE_{12}$, $Y(e) = eE_{21}$ in the general form (6.25) for an object of ${}_{M_2}\mathcal{F}$. In this case we have a bimodule connection,

$$\nabla_{M_2} e = s \otimes \{E_{12}, e\} + t \otimes \{E_{21}, e\}, \quad \sigma_{M_2}(e \otimes s) = -s \otimes e, \quad \sigma_{M_2}(e \otimes t) = -t \otimes e$$

using anticommutators. It is easy to see that σ_{M_2} here is extendable and therefore we obtain an object of the category ${}_{M_2}\mathcal{F}_{M_2}$. On this object, one may further compute that

$$\lambda_{\mathcal{D},M_2}(\tilde{f}_s \otimes e) = -e \otimes \tilde{f}_s, \quad \lambda_{\mathcal{D},M_2}(\tilde{f}_t \otimes e) = -e \otimes \tilde{f}_t.$$

Exercises for Chap. 6

E6.1 Let $A = \mathbb{C}(X)$, where $X = \{x, y, z\}$ with the directed graph calculus in Exercises E1.8 and E3.7. The vector fields $\mathfrak{X}$ have basis $\{f_{z\leftarrow y}, f_{z\leftarrow x}, f_{y\leftarrow x}\}$ where $\mathrm{ev}(f_{p\leftarrow q} \otimes \omega_{r\to s}) = \delta_{q,r}\delta_{p,s}\delta_p$. Give the bimodule structure on $\mathfrak{X}$ and the dual bases element $\mathrm{coev} \in \Omega^1 \otimes_A \mathfrak{X}$. Let $\tilde{\nabla}$ be a right connection on $\mathfrak{X}$ defined by $\tilde{\nabla}(v) = -v \otimes \theta$, where θ is the inner element of Ω^1, and find the divergence $\mathrm{div} : \mathfrak{X} \to A$ for this connection. Explain why no right connection on $\mathfrak{X}$ can have vanishing divergence of $f_{z\leftarrow x}$.

E6.2 For the 2D calculus on $H = \mathbb{C}S_3$ in Example 1.48 (with anticommuting basic 1-forms), give the relations for the left-invariant part of $\mathcal{D}_H$ as described in §6.5.1. Use the standard basis $e^1 = (1, 0)$ and $e^2 = (0, 1)$ of $\mathbb{C}^2$ for the forms and a dual basis f_1, f_2 for the vector fields.

E6.3 For the disk algebra $\mathbb{C}_q[D]$ of Example 3.40, take generators $\{v_z, v_{\bar{z}}\}$ of the vector fields $\mathfrak{X}^R$ defined by $v_z(\mathrm{d}z) = 1$ and $v_z(\mathrm{d}\bar{z}) = 0$ and similarly swapping z and $\bar{z}$ for $v_{\bar{z}}$. Show that $v_z \otimes v_{\bar{z}} - q^{-2} v_{\bar{z}} \otimes v_z$ is in $\Lambda^2\mathfrak{X}$ and find the value of $[\![-, -]\!]_R$ applied to it.

E6.4 Suppose that A is equipped with a parallelisable differential calculus such that $\mathfrak{X}$ is free as a left module with a finite basis f_i and has a left bimodule connection $\heartsuit$. Outline why $T\mathfrak{X}_\bullet$ defined on the vector space $T\mathfrak{X}$ factorises as $T\mathfrak{X}_\bullet = A.\Bbbk\langle f_i\rangle$ into subalgebras A and $\Bbbk\langle f_i\rangle$, i.e., every element can be expressed uniquely in the form $\sum a_{i_1\cdots i_n} \bullet f_{i_1} \bullet \cdots \bullet f_{i_n}$; exhibit this identification explicitly up to degree 2 in both directions, including what $v \otimes w = v^i f_i \otimes w_j f_j \in T\mathfrak{X}_\bullet$ maps to in $A.\Bbbk\langle f_i\rangle$ using $\heartsuit$. You may suppose that the bimodule relations of $\mathfrak{X}$ have the form $f_i a = C_i{}^j(a) f_j$ for some $C_i{}^j : A \to A$ and that $\heartsuit(f_i) = \Gamma^j{}_i \otimes f_j$ for some $\Gamma^j{}_i \in \Omega^1$. Establish the commutation relation

$$f_i \bullet a = C_i{}^k(a) \bullet f_k + \tilde{\partial}_i(a)$$

in $T\mathfrak{X}_\bullet$, where $\tilde{\partial}_i(a) = f_i(\mathrm{d}a)$ for $a \in A$.

E6.5 In the classification of Example 3.76 for connections on $\Omega^1(S_3)$ with its standard 3D calculus, let ∇ be the flat invariant left connection given by parameter values $a = c = 0$ and $e = d = b = 1$. Find the corresponding actions of $f_u, f_v, f_w \in T\mathfrak{X}_\bullet$ restricted to Λ^1 and check that they obey the relations found in Example 6.30.

E6.6 Following on from Example 6.25, show that the sheaf cohomology $\mathrm{H}(A, \mathcal{D}_A)$ is a right $\mathcal{D}_A$-module.

Notes for Chap. 6

The definition of vector fields on algebras with differential calculi used here was first given by Borowiec [47], in terms of a generalised idea of derivation (a Cartan pair). This was used to define certain cases of Lie brackets by Jara & Llena [148]. The particular case of vector fields on Hopf algebras had been considered in [11, 184, 204]. Some related ideas here are skew derivations (see e.g. [125]) and vector fields induced by coactions [20]. At the end of §6.1, we noted that $*$ naturally swaps left and right vector fields. Similarly when working with DGAs, there can be some settings where $*$ more reasonably maps a left-handed version of Ω^1 to a different right-handed one. Making a $*$-differential calculus in this case effectively forces a doubling of Ω^1 and we have already seen this for the $\mathbb{Z}^n$ lattice in Example 1.62.

§6.2 and §6.3 on $T\mathfrak{X}_\bullet$ are based on [24] by Brzeziński and the first author but the fuller picture including Proposition 6.15 that ${}_A\mathcal{E} \cong {}_{T\mathfrak{X}_\bullet}\mathcal{M}$ is new. §6.4 on $\mathcal{D}_A$ is mainly based on the follow-up work [21], while Corollary 6.24 that ${}_A\mathcal{F} \cong {}_{\mathcal{D}_A}\mathcal{M}$ is new. This last result extends to the noncommutative case the key feature of the classical algebra of differential operators underlying the theory of classical $\mathcal{D}$-modules (e.g. see [186]). Frequently used in the classical theory is that the algebra of differential operators itself has a flat connection and Example 6.25 provides a noncommutative version as a flat connection on $\mathcal{D}_A$. In many places in the literature one in fact refers to a 'sheaf of differential operators', see for example [142]. So it is no surprise that we get a differential sheaf in our noncommutative sense by virtue of this connection on $\mathcal{D}_A$.

We have avoided going in the direction of $T\mathfrak{X}_\bullet$ or $\mathcal{D}_A$ as some form of quantum bialgebroid, as it is a little outside our scope. In fact, suitable versions exist in principle and in a similar way but with module categories isomorphic to ${}_A\mathcal{E}_A$ and ${}_A\mathcal{F}_A$ respectively rather than ${}_A\mathcal{E}$ and ${}_A\mathcal{F}$ in our treatment. There are many works spanning the relevant and related algebraic notions and we mention $\times_A$-bialgebras [312], quantum groupoids [182], weak Hopf algebras [44] and more recently biHom-bialgebras [69, 126], see also [43] for a review and [327], where the classical algebra of differential operators was one of the motivations. There is also a more categorical literature on bimonads.

Vector fields for the 3D calculus on $\mathbb{C}_q[SU_2]$ in Example 6.31 go back to Woronowicz [325, 326]. Example 6.32 is largely taken from [31]. The appearance of the q-Witt algebra in §6.5.2 is new but not surprising once we showed that the algebra of differential operators on $\mathbb{C}_q[S^1]$ with its q-differential structure is the usual q-Weyl or q-Heisenberg algebra for the q-derivative. Our approach to the q-Witt algebra does raise the interesting question of whether there is similarly a quantum geometric picture of central extensions that might underly some kind of q-Virasoro algebra. An existing definition of a deformed Virasoro algebra can be found in [302]. Meanwhile, some ideas for noncommutative Witt and Virasoro algebras using nonassociative structures are in [126]. The calculation of $\mathcal{D}_{M_2}$ in §6.5.3 and the appearance there of fermionic annihilation and creation operators is new.

There is an existing terminology of quantum D-modules arising by considering the set of rational curves in a Kähler manifold and relating to quantum cohomology [130]. This does not seem to have an immediate relation to the noncommutative algebra $\mathcal{D}_A$ discussed here.

The exercises develop some small examples, aside from a general result in E6.4 which gives the structure of $T\mathfrak{X}_\bullet$ in the case of a parallelisable differential calculus, and E6.6 which illustrates an aspect of the construction in Example 6.25.

Chapter 7
Quantum Complex Structures

A classical complex manifold has charts which are open subsets of $\mathbb{C}^n$ with holomorphic change of coordinate maps. Such a manifold is also a smooth real $2n$-dimensional manifold M. The data that we need to recover the complex structure from the real manifold is the 'multiplication by i' map on the tangent space of the manifold, which we call the *almost complex structure* map $J : TM \to TM$ with $J^2 = -\mathrm{id}$. However there is one other condition which we need to ensure that complex coordinates can be constructed on the real manifold—the Newlander–Nirenberg integrability condition. We can then use such a complex structure to give a bigrading $\Omega^{p,q}$ for the forms, and to split the derivative d into $\mathrm{d} = \partial + \overline{\partial}$, where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. We have seen an example of such a splitting in the quantum case, for $\mathbb{C}_q[S^2]$ in Proposition 2.35. We start with a noncommutative formulation in line with this, and some examples of integrable almost complex structures.

Next we consider holomorphic modules, which classically arise as sections of bundles with holomorphic transition functions. In noncommutative geometry we shall characterise these as modules equipped with a $\overline{\partial}$-connection with zero curvature, which is classically equivalent by the Koszul–Malgrange theorem. Using the bigrading of the forms, we can define the bigraded Dolbeault cohomology, and this can have coefficients in the form of a holomorphic module. Associated to the Dolbeault cohomology we have the Frölicher spectral sequence.

Then we discuss sections of line modules, and their possible relevance to a noncommutative version of Serre's famous *Géométrie algébrique et géométrie analytique* or GAGA. Classically, this is a connection between algebraic geometry and complex manifolds showing that the algebraic and analytic properties of a smooth complex subvariety of $\mathbb{CP}^n$ are essentially the same. This bridge between complex differential geometry and algebraic geometry is evidence that the 'correct' definitions have been used on both sides. This is less clear in noncommutative geometry.

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Complex structures classically are usually described in terms of vector fields rather than differential forms as we do. We show in §7.3 that the integrability condition in the quantum case can also be formulated in terms of noncommutative vector fields. We examine when a such a vector field should be said to be holomorphic, in terms of the basic idea that the derivative of a holomorphic function in the direction of a holomorphic vector field should be holomorphic.

§7.4.2 considers the ∂ and $\overline{\partial}$ operators for a particular model of the noncommutative complex plane as unbounded operators on a hermitian inner product space (or Hilbert C^*-module on a suitable functional-analytic completion). §7.4.1 is about the quantum version of the Borel–Weil–Bott theorem for the case $\mathbb{C}_q[SU_2]$. Classically this theorem realises irreducible representations of compact Lie groups as holomorphic sections of line bundles (more generally, cohomologies of the line bundles) over complex manifolds which are quotients of the group. This raises the matter of Serre duality, which we verify in the specific case of line modules on $\mathbb{C}_q[S^2]$. Although we do not cover it in the book, the same method can be used to give an integrable almost complex structure on higher-dimensional complex projective spaces $\mathbb{C}_q[\mathbb{CP}^{n-1}]$ as homogeneous spaces for the quantum groups $\mathbb{C}_q[SU_n]$ with calculus given by the restriction of a calculus on the latter, in just the same way as the 2D calculus on $\mathbb{C}_q[S^2]$ is a restriction of the 3D calculus on $\mathbb{C}_q[SU_2]$.

7.1 Complex Structures and the Dolbeault Double Complex

We start by reviewing the case of a classical complex manifold, where we can work locally in open subsets of $\mathbb{C}^m$ and use holomorphic maps to change between local coordinates. The most well-known nontrivial example is the Riemann sphere $\mathbb{CP}^1$ or $\mathbb{C}_\infty$. This has two open charts, $U = \mathbb{C}$ and $V = \mathbb{C}$, and the change of coordinate map is such that $z \in U \setminus \{0\}$ corresponds to $z^{-1} \in V \setminus \{0\}$. We shall write the local complex coordinates as $z^1, \dots, z^m$ for a complex manifold M of dimension m. Then M also has the structure of a real $2m$-dimensional manifold, as we have $z^j = x^j + \mathrm{i}y^j$ for real coordinates $x^1, \dots, x^m, y^1, \dots, y^m$.

Locally the complex-valued 1-forms on M have basis $\mathrm{d}z^j = \mathrm{d}x^j + \mathrm{i}\mathrm{d}y^j$ and its conjugate $\mathrm{d}\bar{z}^j = \mathrm{d}x^j - \mathrm{i}\mathrm{d}y^j$. Dually,

$$\frac{\partial}{\partial z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j}), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial y^j})$$

for vector fields. It is common to abbreviate $\partial_j = \frac{\partial}{\partial z^j}$ and $\overline{\partial}_j = \frac{\partial}{\partial \bar{z}^j}$. The Cauchy–Riemann equations are the condition for a complex-valued function of a complex variable to be analytic. They are usually written in terms of splitting the function into real parts $u + \mathrm{i}v$ and the single complex variable $z = x + \mathrm{i}y$ as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These can be written using the $\overline{\partial}$ operators in several complex variables for a complex-valued function f on M as $\overline{\partial}_j f = 0$ for all j. By writing the n-forms in terms of $\mathrm{d}z^j$ and $\mathrm{d}\bar{z}^j$, we split $\Omega^n(M)$ into the direct sum of $\Omega^{p,q}(M)$ for $p+q=n$. Here p is the number of $\mathrm{d}z^j$ and q is the number of $\mathrm{d}\bar{z}^j$, so $\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$ is in $\Omega^{3,0}(M)$ and $\mathrm{d}\bar{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^3$ is in $\Omega^{1,2}(M)$. Now define derivatives $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ by

$$\partial \xi = \mathrm{d}z^j \wedge \partial_j \xi, \quad \overline{\partial}\xi = \mathrm{d}\bar{z}^j \wedge \overline{\partial}_j \xi$$

summed over j. It is fairly easy to check that $\partial + \overline{\partial} = \mathrm{d}$ and that $\partial^2 = \overline{\partial}^2 = 0$.

If we were given a real $2m$-dimensional manifold N, how could we recognise whether it is a complex manifold? One way is to look for an *almost complex structure* and then check whether it is *integrable*. If so, the Newlander–Nirenberg integrability condition says that N is a complex manifold, i.e., there exist complex coordinates and holomorphic transition functions. An almost complex structure is a map of vector bundles $J : T^*_{\mathbb{C}}N \to T^*_{\mathbb{C}}N$ ($T^*_{\mathbb{C}}N$ is the complexified cotangent bundle) with $J^2 = -\mathrm{id}$. It is more common to define J on the complexified tangent bundle but we prefer to work equivalently with 1-forms. On M, which is already a complex manifold, it is easy to describe J by $J(\mathrm{d}z^j) = \mathrm{i}\mathrm{d}z^j$ and $J(\mathrm{d}\bar{z}^j) = -\mathrm{i}\mathrm{d}\bar{z}^j$.

We now consider how this theory for classical complex manifolds can be extended to noncommutative differential geometry. We replace $C^\infty(N)$ by a possibly noncommutative $*$-algebra A over $\mathbb{C}$ with a $*$-differential calculus $(\Omega, \mathrm{d}, \wedge, *)$. As A is already an algebra over $\mathbb{C}$, we have no need to complexify the forms to define an almost complex structure. Note that here we insist on the property $\Omega^1 \wedge \Omega^n = \Omega^{n+1}$ for all $n \geq 1$, which was the surjectivity condition (3) in Definition 1.30. Thus we suppose that we begin with an exterior algebra.

Definition 7.1 An *almost complex structure* on $(\Omega, \mathrm{d}, \wedge, *)$ is a map $J : \Omega^n \to \Omega^n$ for all n such that

(1) $J(\xi \wedge \eta) = J(\xi) \wedge \eta + \xi \wedge J(\eta)$ for all $\xi, \eta \in \Omega$ (i.e., J is a derivation);
(2) J is identically 0 on A (and hence an A-bimodule map given (1));
(3) $J^2 = -\mathrm{id}$ on Ω^1;
(4) $J(\xi^*) = (J\xi)^*$ for $\xi \in \Omega^1$ (or, equivalently, $\overline{J}\star = \star J : \Omega^1 \to \overline{\Omega^1}$ using Definition 2.102).

In this case the map $e = \frac{1}{2}(\mathrm{id} - \mathrm{i}J) : \Omega^1 \to \Omega^1$ is an idempotent, $e^2 = e$, so Ω^1 splits as a direct sum of the $\pm\mathrm{i}$ eigenspaces of J. Write $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ where

$$\Omega^{1,0} = \{\xi \in \Omega^1 \mid J\xi = \mathrm{i}\xi\}, \quad \Omega^{0,1} = \{\xi \in \Omega^1 \mid J\xi = -\mathrm{i}\xi\}.$$

This is extended to more general forms by

$$\Omega^{p,q} = \{\xi \in \Omega^{p+q} \mid J(\xi) = (p-q)\mathrm{i}\xi\} \tag{7.1}$$

for integers $p, q \geq 0$. As J is a derivation, $\Omega^{p,q} \wedge \Omega^{p',q'} \subseteq \Omega^{p+p',q+q'}$. To see that we get a direct sum decomposition $\Omega^n = \oplus_{p+q=n}\Omega^{p,q}$, use $\Omega^m \wedge \Omega^1 = \Omega^{m+1}$ for all $m \geq 1$ to write any element of Ω^n as a sum of wedge products of elements of Ω^1. Then the decomposition of Ω^1 into $\Omega^{1,0}$ and $\Omega^{0,1}$ gives the direct sum. Associated to this direct sum we have projections

$$\pi^{p,q} : \Omega^{p+q} \to \Omega^{p,q}. \tag{7.2}$$

We also have $J(\xi^*) = (J(\xi))^* = (\mathrm{i}\xi)^* = -\mathrm{i}\xi^*$ for any $\xi \in \Omega^{1,0}$, so $\xi^* \in \Omega^{0,1}$. Similarly, from (7.1) we have $(\Omega^{p,q})^* \subseteq \Omega^{q,p}$ for all $p, q \geq 0$. Since $\xi^{**} = \xi$, we conclude that $(\Omega^{p,q})^* = \Omega^{q,p}$. Using the projections in (7.2), we define analogous operators to the classical ∂ and $\overline{\partial}$ by

$$\partial = \pi^{p+1,q}\mathrm{d} : \Omega^{p,q} \to \Omega^{p+1,q}, \quad \overline{\partial} = \pi^{p,q+1}\mathrm{d} : \Omega^{p,q} \to \Omega^{p,q+1}. \tag{7.3}$$

Next, the Newlander–Nirenberg integrability condition for a classical real manifold with an almost complex structure J is about existence of local complex coordinates, which does not immediately translate into noncommutative geometry. However, there is there several well-known equivalent conditions which do translate and which we take as a starting point. There is also a definition in terms of vector fields, and we shall return to this in Proposition 7.32.

Lemma 7.2 *For J an almost complex structure on $(A, \Omega, \mathrm{d}, *)$, the following conditions are equivalent, and if any of these hold then we say that J is integrable:*

(1) $0 = \overline{\partial}^2 : A \to \Omega^2$;
(2) $0 = \partial^2 : A \to \Omega^2$;
(3) $\mathrm{d} = \partial + \overline{\partial} : \Omega^1 \to \Omega^2$;
(4) $\mathrm{d}\Omega^{1,0} \subseteq \Omega^{2,0} \oplus \Omega^{1,1}$;
(5) $\mathrm{d}\Omega^{0,1} \subseteq \Omega^{1,1} \oplus \Omega^{0,2}$.

Proof We prove this in stages, starting with $\pi^{2,0}\mathrm{d}(\partial a + \overline{\partial} a) = \pi^{1,1}\mathrm{d}(\partial a + \overline{\partial} a) = \pi^{0,2}\mathrm{d}(\partial a + \overline{\partial} a) = 0$ for all $a \in A$ since $\mathrm{d}^2 a = 0$, which implies that

$$\partial^2 a + \pi^{2,0}\mathrm{d}\overline{\partial} a = \overline{\partial}\partial a + \partial\overline{\partial} a = \pi^{0,2}\mathrm{d}\partial a + \overline{\partial}^2 a = 0.$$

(1)⇔(4) (1) implies $\pi^{0,2}\mathrm{d}\partial a = 0$, so $\mathrm{d}(\partial a) \in \Omega^{2,0} \oplus \Omega^{1,1}$. If $b \in A$,

$$\mathrm{d}(b.\partial a) = \mathrm{d}b \wedge \partial a + b.\mathrm{d}(\partial a) \in \Omega^1 \wedge \Omega^{1,0} \subseteq \Omega^{2,0} \oplus \Omega^{1,1}.$$

As $\Omega^{1,0}$ is spanned by $b.\partial a$ for $a, b \in A$, (4) follows. Similarly, if $\mathrm{d}\Omega^{1,0} \subseteq \Omega^{2,0} \oplus \Omega^{1,1}$ then $\pi^{0,2}\mathrm{d}(\partial a) = 0$ for all $a \in A$ so $\overline{\partial}^2 a = 0$ by our observation. (2) ⇔ (5) is similar.
(4)⇒(5) If (4) holds then $(\mathrm{d}\Omega^{1,0})^* \subseteq (\Omega^{2,0})^* \oplus (\Omega^{1,1})^*$. As $(\Omega^{p,q})^* = \Omega^{q,p}$, it follows that (5) holds. (5) ⇒ (4) is similar.
(3)⇒(4) If $\mathrm{d} = \partial + \overline{\partial} : \Omega^1 \to \Omega^2$, then $\mathrm{d}\Omega^{1,0} \subseteq \partial\Omega^{1,0} + \overline{\partial}\Omega^{1,0} \subseteq \Omega^{2,0} + \Omega^{1,1}$.

(4)⇒(3) If $\omega \in \Omega^{1,0}$ and $\eta \in \Omega^{0,1}$ then $\mathrm{d}\omega \in \mathrm{d}\Omega^{1,0} \subseteq \Omega^{2,0} + \Omega^{1,1}$ so that $\mathrm{d}\omega = \partial\omega + \overline{\partial}\omega$. As (4) holds, so does (5), hence $\mathrm{d}\eta \in \Omega^{1,1} + \Omega^{0,2}$ and $\mathrm{d}\eta = \partial\eta + \overline{\partial}\eta$. Then $\mathrm{d}(\omega + \eta) = (\partial + \overline{\partial})(\omega + \eta)$. □

To check integrability, it is useful to have a module map the vanishing of which is equivalent to integrability, as then we need only check vanishing on a set of generators.

Lemma 7.3 *For J an almost complex structure on $(A, \Omega, \mathrm{d}, *)$, the following formulae give bimodule maps from Ω^1 to Ω^2,*

$$(1 - J \wedge J)\mathrm{d}J - J\mathrm{d}, \quad J^2\mathrm{d}J + 2J\mathrm{d}, \quad J^2\mathrm{d} - 2J\mathrm{d}J$$

and J is integrable if and only if any one of these vanishes.

Proof As J is a derivation,

$$J^2(\xi \wedge \eta) = J(J\xi \wedge \eta + \xi \wedge J\eta) = J^2\xi \wedge \eta + J\xi \wedge J\eta + J\xi \wedge J\eta + \xi \wedge J^2\eta$$

for all $\xi, \eta \in \Omega^1$, so the second stated operator is -2 times the first. The third is obtained from the second by composing on the right by J, so we only have to prove the bimodule result for the first operator. For the left module property,

$$\begin{aligned}\big((1 - J \wedge J)\mathrm{d}J - J\mathrm{d}\big)(a\xi) &= (1 - J \wedge J)\mathrm{d}(aJ\xi) - J(\mathrm{d}a \wedge \xi + a\mathrm{d}\xi)\\ &= (1 - J \wedge J)(\mathrm{d}a \wedge J\xi + a\mathrm{d}J\xi) - J(\mathrm{d}a \wedge \xi + a\mathrm{d}\xi)\\ &= \mathrm{d}a \wedge J\xi + J\mathrm{d}a \wedge \xi + a(1 - J \wedge J)\mathrm{d}J\xi - J(\mathrm{d}a \wedge \xi + a\mathrm{d}\xi)\\ &= a(1 - J \wedge J)\mathrm{d}J\xi - aJ\mathrm{d}\xi\end{aligned}$$

for $a \in A$. Similarly $(1 - J \wedge J)\mathrm{d}J - J\mathrm{d}$ is a right module map.

We have seen that the vanishing of the three operators is equivalent. Now suppose that J is integrable. As $\Omega^{1,0} = (J + \mathrm{i})\Omega^1$ and $\Omega^{2,0}$, $\Omega^{1,1}$ and $\Omega^{0,2}$ are the $2\mathrm{i}$, 0 and $-2\mathrm{i}$ eigenspaces respectively for J acting on Ω^2, Lemma 7.2(4) and (5) imply that $(J - 2\mathrm{i})J\mathrm{d}(J + \mathrm{i}) = 0$ and $(J + 2\mathrm{i})J\mathrm{d}(J - \mathrm{i}) = 0$ on Ω^1. Expanding this gives

$$\begin{aligned}(J - 2\mathrm{i})J\mathrm{d}(J + \mathrm{i}) &= J^2\mathrm{d}J + \mathrm{i}J^2\mathrm{d} - 2\mathrm{i}J\mathrm{d}J + 2J\mathrm{d} = 0,\\ (J + 2\mathrm{i})J\mathrm{d}(J - \mathrm{i}) &= J^2\mathrm{d}J - \mathrm{i}J^2\mathrm{d} + 2\mathrm{i}J\mathrm{d}J + 2J\mathrm{d} = 0,\end{aligned}$$

which implies that both the last two operators vanish.

Now assume that the last two operators vanish. An element $\omega \in \Omega^2$ belongs to $\Omega^{2,0} \oplus \Omega^{1,1}$ if and only if $(J - 2\mathrm{i})J\omega = 0$. As $\Omega^{1,0} = (J + \mathrm{i})\Omega^1$, Lemma 7.2(4) holds if and only if $(J - 2\mathrm{i})J\mathrm{d}(J + \mathrm{i})$ vanishes identically on Ω^1. But

$$(J - 2\mathrm{i})J\mathrm{d}(J + \mathrm{i}) = \big(J^2\mathrm{d}J + 2J\mathrm{d}\big) + \mathrm{i}\big(J^2\mathrm{d} - 2J\mathrm{d}J\big) = 0. \qquad \square$$

We also have a useful consequence of integrability.

Corollary 7.4 *Suppose* $(A, \Omega, \mathrm{d}, *)$ *is a differential* $*$*-calculus with an integrable almost complex structure* J. *Then* $\mathrm{d} = \partial + \overline{\partial}$ *on all* $\Omega^{p,q}$, *or equivalently,*

$$\mathrm{d}(\Omega^{p,q}) \ \subseteq \ \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

Proof This is by induction on $p + q = n$. The case $n = 1$ is true by the definition of integrability. Suppose $n \geq 2$ and that the result is true for $n - 1$. Now

$$\Omega^n = \Omega^1 \wedge \Omega^{n-1} = \big(\Omega^{1,0} \oplus \Omega^{0,1}\big) \wedge \Omega^{n-1},$$

and it follows from addition of indices under the wedge product that, for $p+q = n$,

$$\Omega^{p,q} = \Omega^{1,0} \wedge \Omega^{p-1,q} + \Omega^{0,1} \wedge \Omega^{p,q-1}.$$

Using $\mathrm{d}\Omega^{1,0} \subseteq \Omega^{2,0}\oplus\Omega^{1,1}$ and $\mathrm{d}\Omega^{0,1} \subseteq \Omega^{1,1}\oplus\Omega^{0,2}$ and the inductive hypothesis,

$$\mathrm{d}\Omega^{p,q} \subseteq \mathrm{d}\Omega^{1,0} \wedge \Omega^{p-1,q} + \Omega^{1,0} \wedge \mathrm{d}\Omega^{p-1,q} + \mathrm{d}\Omega^{0,1} \wedge \Omega^{p,q-1} + \Omega^{0,1} \wedge \mathrm{d}\Omega^{p,q-1},$$

which is a subset of $\Omega^{p+1,q} + \Omega^{p,q+1}$. $\square$

The following result is important as it allows us to form separate DGAs with the ∂ and $\overline{\partial}$ operators, and shows that the ∂ and $\overline{\partial}$ operators are linked by the $*$-operation:

Proposition 7.5 *Suppose* $(A, \Omega, \mathrm{d}, *)$ *is a differential* $*$*-calculus with an integrable almost complex structure* J. *Then* $\overline{\partial}$ *and* ∂ *are graded derivations with* $\partial^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0$ *and* $\overline{\partial}^2 = 0$. *Also* $(\overline{\partial}\xi)^* = \partial(\xi^*)$ *and* $(\partial\xi)^* = \overline{\partial}(\xi^*)$ *for all* $\xi \in \Omega$.

Proof By Corollary 7.4, $\mathrm{d} = \partial + \overline{\partial}$, so

$$0 = \mathrm{d}^2\Omega^{p,q} \subseteq \partial^2\Omega^{p,q} + (\partial\overline{\partial} + \overline{\partial}\partial)\Omega^{p,q} + \overline{\partial}^2\Omega^{p,q},$$

and as these terms are in $\Omega^{p+2,q}$, $\Omega^{p+1,q+1}$ and $\Omega^{p,q+2}$ respectively, they must all vanish. Next, for $\xi \in \Omega^{p,q}$ and $\eta \in \Omega^{p',q'}$,

$$\begin{aligned}\partial(\xi\wedge\eta) &= \pi^{p+p'+1,q+q'}\mathrm{d}(\xi\wedge\eta) = \pi^{p+p'+1,q+q'}\big(\mathrm{d}\xi\wedge\eta + (-1)^{|\xi|}\xi\wedge\mathrm{d}\eta\big)\\ &= \big(\pi^{p+1,q}\mathrm{d}\xi\big)\wedge\eta + (-1)^{|\xi|}\xi\wedge\big(\pi^{p'+1,q'}\mathrm{d}\eta\big) = \partial\xi\wedge\eta + (-1)^{|\xi|}\xi\wedge\partial\eta.\end{aligned}$$

There are analogous results for $\bar\partial$. Next for $\xi \in \Omega^{p,q}$, as $\mathrm{d}(\xi^*) = (\mathrm{d}\xi)^*$, we have

$$(\partial\xi)^* + (\overline{\partial}\xi)^* = \partial(\xi^*) + \overline{\partial}(\xi^*).$$

But $\partial(\xi^*), (\overline{\partial}\xi)^* \in \Omega^{q+1,p} = (\Omega^{p,q+1})^*$ while $\overline{\partial}(\xi^*), (\partial\xi)^* \in \Omega^{q,p+1} = (\Omega^{p+1,q})^*$. □

Note that while Proposition 7.5 starts from the J operator, we could also have defined an integrable almost complex structure map J starting with a double complex by (7.1). In fact that way round would be more general, as it does not require the optional surjectivity condition of Definition 1.30, i.e., we do not require an exterior algebra. Considering this, we do not assume the optional surjectivity condition, and simply define a Dolbeault double complex to be a bigraded $*$-differential algebra $\Omega^{p,q}$ for $p, q \geq 0$ with $(\Omega^{p,q})^* = \Omega^{q,p}$ and graded derivations $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ satisfying the conditions of Proposition 7.5.

Example 7.6 Recall the differential $*$-calculus on $A = M_2(\mathbb{C})$ in Proposition 1.38 which has $\Omega^1 = M_2 \oplus M_2$ with central generators $s = 1 \oplus 0$ and $t = 0 \oplus 1$ with $\mathrm{d}s = \mathrm{d}t = 0$. There is an almost complex structure $J(a \oplus b) = (\mathrm{i}a) \oplus (-\mathrm{i}b)$, i.e., $J(s) = \mathrm{i}s$ and $J(t) = -\mathrm{i}t$. As $s^* = -t$, we have condition (4) in Definition 7.1. Now $\Omega^{p,q}$ has a single generator $s^p t^q$, and as $\mathrm{d}(as^p t^q) = \mathrm{d}a \wedge s^p t^q$, which is explicitly contained in $\Omega^{p+1,q} \oplus \Omega^{p,q+1}$, we have integrability by Lemma 7.2. Note that even if we impose the additional relations $s^2 = t^2 = 0$ as in Proposition 1.38, we still have an integrable almost complex structure. ◇

Next we have an example which is not too different from a classical manifold.

Example 7.7 For an $n \times n$ real antisymmetric matrix Θ_{ij}, the algebraic noncommutative n-torus $\mathbb{C}_\Theta[\mathbb{T}^n]$ (a higher-dimensional generalisation of $\mathbb{C}_\theta[\mathbb{T}^2]$ in Example 1.36) has unitary generators t_i for $1 \leq i \leq n$ and relations

$$t_j t_i = e^{\mathrm{i}\Theta_{ij}} t_i t_j.$$

Ω^1 has central generators $e_i = t_i^{-1}\mathrm{d}t_i$ for $1 \leq i \leq n$. This extends to a higher order calculus with the relations $e_i \wedge e_j = -e_j \wedge e_i$. There is a $*$-operation $e_i^* = -e_i$ which makes Ω into a $*$-differential calculus.

Let $J = (J_i{}^k)$ be an $n \times n$ complex matrix such that $J^2 = -\mathrm{id}$ and let $J : \Omega^1 \to \Omega^1$ be the $\mathbb{C}_\Theta[\mathbb{T}^n]$-bimodule map defined by

$$J(e_i) = J_i{}^k e_k.$$

Then J extends uniquely to a derivation on all of Ω, as a brief check shows that it preserves the relations. Next, $*$-compatibility,

$$(J(e_i))^* = (J_i{}^k e_k)^* = (J_i{}^k)^* e_k{}^*,$$

requires that the matrix coefficients $J_i{}^k$ are real, which can only be the case when n is even, as in the odd case we would have $(\det J)^2 = -1$. Integrability by using Lemma 7.3 is automatic as each $\mathrm{d}J_i{}^k = 0$ and $\mathrm{d}e_i = -e_i \wedge e_i = 0$. Thus, any real-valued matrix J squaring to $-\mathrm{id}$ provides a complex structure. $\diamond$

We can also Θ-deform an already complex space, in the next example giving a Θ-deformed complex projective space.

Example 7.8 Suppose that Θ is a real antisymmetric $n \times n$ matrix, and define the $*$-algebra $\mathbb{C}_\Theta[\mathbb{C}^n]$ by generators z^μ and $\bar{z}^\mu$ for $1 \le \mu \le n$, with relations

$$z^\mu z^\nu = e^{i\Theta_{\mu\nu}} z^\nu z^\mu, \quad \bar{z}^\mu \bar{z}^\nu = e^{i\Theta_{\mu\nu}} \bar{z}^\nu \bar{z}^\mu, \quad \bar{z}^\mu z^\nu = e^{i\Theta_{\nu\mu}} z^\nu \bar{z}^\mu, \quad (z^\mu)^* = \bar{z}^\mu$$

and the $*$-differential calculus $\Omega^1(\mathbb{C}_\Theta[\mathbb{C}^n])$ with generators $\mathrm{d}z^\mu$, $\mathrm{d}\bar{z}^\mu$ and relations

$$z^\mu \mathrm{d}z^\nu = e^{i\Theta_{\mu\nu}} \mathrm{d}z^\nu z^\mu, \quad \bar{z}^\mu \mathrm{d}\bar{z}^\nu = e^{i\Theta_{\mu\nu}} \mathrm{d}\bar{z}^\nu \bar{z}^\mu,$$
$$\bar{z}^\mu \mathrm{d}z^\nu = e^{i\Theta_{\nu\mu}} \mathrm{d}z^\nu \bar{z}^\mu, \quad z^\mu \mathrm{d}\bar{z}^\nu = e^{i\Theta_{\nu\mu}} \mathrm{d}\bar{z}^\nu z^\mu,$$
$$\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu + e^{i\Theta_{\mu\nu}} \mathrm{d}z^\nu \wedge \mathrm{d}z^\mu = 0, \quad \mathrm{d}\bar{z}^\mu \wedge \mathrm{d}\bar{z}^\nu + e^{i\Theta_{\mu\nu}} \mathrm{d}\bar{z}^\nu \wedge \mathrm{d}\bar{z}^\mu = 0,$$
$$\mathrm{d}\bar{z}^\mu \wedge \mathrm{d}z^\nu + e^{i\Theta_{\nu\mu}} \mathrm{d}z^\nu \wedge \mathrm{d}\bar{z}^\mu = 0, \quad (\mathrm{d}z^\mu)^* = \mathrm{d}\bar{z}^\mu.$$

Define an almost complex structure on this calculus by $J(\mathrm{d}z^\mu) = i\mathrm{d}z^\mu$ and $J(\mathrm{d}\bar{z}^\mu) = -i\mathrm{d}\bar{z}^\mu$. To check this is integrable using (4) in Lemma 7.2, note that every element in $\Omega^{1,0}(\mathbb{C}_\Theta[\mathbb{C}^n])$ can be written as $f_\mu \mathrm{d}z^\mu$ for some $f_\mu \in \mathbb{C}_\Theta[\mathbb{C}^n]$, and applying d gives $\mathrm{d}f_\mu \wedge \mathrm{d}z^\mu$, which is in $\Omega^{2,0}(\mathbb{C}_\Theta[\mathbb{C}^n]) \oplus \Omega^{1,1}(\mathbb{C}_\Theta[\mathbb{C}^n])$.

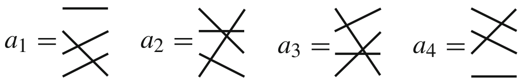
Now quotient $\mathbb{C}_\Theta[\mathbb{C}^n]$ to give the Θ-deformed sphere $\mathbb{C}_\Theta[S^{2n-1}]$ by adding the relation $\sum_\mu z^\mu \bar{z}^\mu = 1$ (recall each $\bar{z}^\mu z^\mu = z^\mu \bar{z}^\mu$ is central in $\mathbb{C}_\Theta[\mathbb{C}^n]$). Differentiating this gives an additional relation $\sum_\mu \mathrm{d}z^\mu \bar{z}^\mu + \sum_\mu z^\mu \mathrm{d}\bar{z}^\mu = 0$ on $\Omega^1(\mathbb{C}_\Theta[S^{2n-1}])$.

The group of unit norm complex numbers U_1 acts as $*$-algebra automorphisms of $\mathbb{C}_\Theta[\mathbb{C}^n]$ by $\lambda \triangleright z^\mu = \lambda z^\mu$ and $\lambda \triangleright \bar{z}^\mu = \lambda^* \bar{z}^\mu$. This action extends to the differential calculus with $\lambda \triangleright \mathrm{d}z^\mu = \lambda \mathrm{d}z^\mu$ and $\lambda \triangleright \mathrm{d}\bar{z}^\mu = \lambda^* \mathrm{d}\bar{z}^\mu$. As $\sum_\mu z^\mu \bar{z}^\mu - 1$ and $\sum_\mu \mathrm{d}z^\mu \bar{z}^\mu + \sum_\mu z^\mu \mathrm{d}\bar{z}^\mu$ are invariant under the action, we get a $*$ automorphism of $\mathbb{C}_\Theta[S^{2n-1}]$ and its differential calculus.

Define $\mathbb{C}_\Theta[\mathbb{CP}^{n-1}]$ to be the U_1-invariant $*$-subalgebra of $\mathbb{C}_\Theta[S^{2n-1}]$. Then $\Omega(\mathbb{C}_\Theta[\mathbb{CP}^{n-1}])$ is the U_1-invariant subalgebra of the calculus on $\mathbb{C}_\Theta[S^{2n-1}]$ with an additional relation, which gives $\mathbb{C}_\Theta[\mathbb{CP}^{n-1}]$ a complex structure inherited from that on $\mathbb{C}_\Theta[\mathbb{C}^n]$. Applying J to the relation $\sum_\mu \mathrm{d}z^\mu \bar{z}^\mu + \sum_\mu z^\mu \mathrm{d}\bar{z}^\mu = 0$ on the $\mathbb{C}_\Theta[S^{2n-1}]$ calculus shows that on $\mathbb{C}_\Theta[\mathbb{CP}^{n-1}]$ we require both $\sum_\mu \mathrm{d}z^\mu \bar{z}^\mu = 0$ and $\sum_\mu z^\mu \mathrm{d}\bar{z}^\mu = 0$. From §3.1 we see that $\Omega^{1,0}(\mathbb{C}_\Theta[\mathbb{CP}^{n-1}])$ and $\Omega^{0,1}(\mathbb{C}_\Theta[\mathbb{CP}^{n-1}])$ are right fgp modules, with respective bases $\mathrm{d}z^\mu$ and $\mathrm{d}\bar{z}^\mu$ and projection matrices

$$Q_{\mu\nu} = \delta_{\mu\nu} - \bar{z}^\mu z^\nu, \quad P_{\mu\nu} = \delta_{\mu\nu} - z^\mu \bar{z}^\nu. \qquad \diamond$$

Example 7.9 Let A_4 be the group of even permutations of 4 objects and set the conjugacy class $\mathcal{C}^{1,0} = \{a_1, a_2, a_3, a_4\}$ given by



Then $\mathcal{C}^{1,0}$ gives a bicovariant first-order calculus on the algebra $A = \mathbb{C}(A_4)$ which we denote by $(\Omega^{1,0}, \partial)$. Since $\mathcal{C}^{1,0}$ is not closed under inverses it is not a $*$-calculus, see Proposition 1.52. Also set $\mathcal{C}^{0,1} = \{a_1{}^{-1}, a_2{}^{-1}, a_3{}^{-1}, a_4{}^{-1}\}$ and denote its bicovariant first-order calculus as $(\Omega^{0,1}, \bar{\partial})$. Now $\mathcal{C} = \mathcal{C}^{1,0} \cup \mathcal{C}^{0,1}$ gives a bicovariant $*$-calculus $(\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}, \mathrm{d})$ and this can be extended to the maximal prolongation calculus, as described in Proposition 1.53.

The sum $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ gives an almost complex structure. To see this we note that taking products of pairs of elements $\mathcal{C}^{1,0}\mathcal{C}^{1,0} = \mathcal{C}^{0,1}$, $\mathcal{C}^{0,1}\mathcal{C}^{0,1} = \mathcal{C}^{1,0}$ and $\mathcal{C}^{1,0}\mathcal{C}^{0,1} = \mathcal{C}^{0,1}\mathcal{C}^{1,0}$ is the identity union the conjugacy class of two disjoint transpositions. We use these to check that applying J to the quadratic relations for the maximal prolongation Ω^2 in Proposition 1.53 and setting the result to be zero does not force any further relations. However, this almost complex structure is not integrable. To see this, note that the formula for $\mathrm{d}e_{a_1}$ in Proposition 1.53 is

$$\mathrm{d}e_{a_1} = \theta \wedge e_{a_1} + e_{a_1} \wedge \theta - \sum_{a,b\in\mathcal{C}:\, ab=a_1} e_a \wedge e_b$$

and that while the first two terms are in $\Omega^{1,1} \oplus \Omega^{2,0}$ as required, the third term is a nonzero element of $\Omega^{0,2}$ as we must have $a, b \in \mathcal{C}^{0,1}$.

Now consider the quotient of the maximal prolongation calculus in Corollary 1.54, which does give an integrable almost complex structure. For example, one of the relations in Corollary 1.54 shows that the third term in the displayed equation for $\mathrm{d}e_{a_1}$ is zero. Note that as $\mathcal{C}^{0,1}$ generates the whole group, the kernel of $\bar{\partial}$ on the functions is the constants, so the only holomorphic functions are constants. $\diamond$

We modify Definition 1.30 to give a definition of a holomorphic algebra map.

Definition 7.10 An algebra map $\varphi : A \to B$ between algebras with integrable complex structure is *holomorphic* if it extends to a map $\varphi : \Omega_A \to \Omega_B$ of differential bigraded algebras, i.e., $\varphi\Omega_A^{p,q} \subseteq \Omega_B^{p,q}$, $\mathrm{d}\varphi(\xi) = \varphi(\mathrm{d}\xi)$ and $\varphi(\xi) \wedge \varphi(\eta) = \varphi(\xi \wedge \eta)$.

Classically, there can be many complex structures on a given smooth manifold. For example, the moduli space of complex structures on the ordinary 2-torus $\mathbb{T}^2$ is the upper half complex plane H^+ modulo the action of $SL_2(\mathbb{Z})$ by Möbius transformations. The complex structure on $\mathbb{T}^2$ corresponding to $\tau \in \mathrm{H}^+$ is obtained by taking $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with the complex structure inherited from that on $\mathbb{C}$. The $n = 2$ case of Example 7.7 already tells us the form of complex structures on $\mathbb{C}_\theta[\mathbb{T}^2]$ assuming J has constant entries. Here we look at this in more detail.

Example 7.11 We recall $\mathbb{C}_\theta[\mathbb{T}^2]$ defined in Example 1.36 with unitary generators u, v obeying $vu = e^{i\theta}uv$ and antihermitian basis $e_1 = u^{-1}\mathrm{d}u$, $e_2 = v^{-1}\mathrm{d}v$ of Ω^1 commuting with the algebra. We represent J (as a right module map) by a matrix with entries in the algebra left-multiplying column vectors in this basis. As e_i is central and J is a bimodule map, we must have $a.J(e_i) = J(e_i).a$, so the entries of the matrix for J must be central in the algebra. For $\mathbb{C}_\theta[\mathbb{T}^2]$ with θ/π irrational, the central elements are just complex numbers. Hence in this case we are in the setting of Example 7.7 where J then has to have real entries and $J^2 = -\mathrm{id}$, so of the form

$$J = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \quad x^2 + yz = -1, \quad x, y, z \in \mathbb{R}. \tag{7.4}$$

A choice of such a J corresponds uniquely to a complex number

$$\tau = \frac{x + i}{z},$$

noting that z cannot be zero. By choosing a sign for $\pm J$, we assume that the imaginary part of τ is positive. (Remember that multiplication by $+i$ and $-i$ both give complex structures on $\mathbb{C}$—this is really choosing an orientation.)

Next, corresponding to

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

we have an automorphism f_M of $\mathbb{C}_\theta[\mathbb{T}^2]$ given by

$$f_M(u) = e^{iac\theta/2}u^a v^c, \quad f_M(v) = e^{ibd\theta/2}u^b v^d.$$

A little calculation shows that $f_M(u^{-1})\mathrm{d}f_M(u) = ae_1 + ce_2$, and the derivative of f_M acts by the matrix M in our basis for Ω^1. Thus $f_M : \mathbb{C}_\theta[\mathbb{T}^2] \to \mathbb{C}_\theta[\mathbb{T}^2]$ will be a holomorphic algebra map according to Definition 7.10 if we change the matrix giving the complex structure on the second $\mathbb{C}_\theta[\mathbb{T}^2]$ to MJM^{-1}. This corresponds to the Möbius transformation

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Hence the integrable complex structures J and MJM^{-1} are identified by a holomorphic automorphism. It follows that integrable complex structures are indexed by $SL_2(\mathbb{Z})\backslash \mathrm{H}^+$, although we do not attempt to show that these are all different. $\diamond$

Taking the one point compactification of the complex plane $\mathbb{C}$ to give the Riemann sphere $\mathbb{C}_\infty \cong \mathbb{CP}^1$ (which is topologically S^2) is another familiar construction from classical complex geometry. A simple way of visualising an identification of the complex plane with the sphere with one point removed is to use stereographic projection. We begin by describing this classical procedure, and then we apply it to the noncommutative case. As far as the q-sphere is concerned, we already gave the differential calculus on $\mathbb{C}_q[S^2]$ as a Dolbeault complex in Proposition 2.35, and this corresponds to an integrable almost complex structure given by J acting as multiplication by i on $\Omega^{1,0}$ and $-$i on $\Omega^{0,1}$. We also saw that this splitting arises naturally from the q-Hopf fibration in Example 5.51 as $\Omega^1 \cong E_{-2} \oplus E_2$, where $\Omega^{1,0} \cong E_2$ and $\Omega^{0,1} \cong E_{-2}$. Next we relate this almost complex structure on the standard q-sphere to an almost complex structure on a version of the complex plane.

Example 7.12 (Noncommutative Stereographic Projection) Figure 7.1 illustrates classical stereographic projection. Place a light at the point $(0, 0, -\frac{1}{2})$ of a glass sphere $x_1^2 + x_2^2 + x_3^2 = \frac{1}{4}$, and then every other point (x_1, x_2, x_3) on the sphere projects to a point $(w_1, w_2, 0)$ on the x_1x_2 plane. It is convenient to write this 1-1 correspondence in terms of complex variables as follows: In Example 3.13 we explained that classical $\mathbb{CP}^1$ corresponds to the sphere we have described with $z = x_1 + \mathrm{i}x_2$ and $x = x_3 + \frac{1}{2}$. Then a little geometry gives $w = w_1 + \mathrm{i}w_2 = \frac{1}{2}zx^{-1}$.

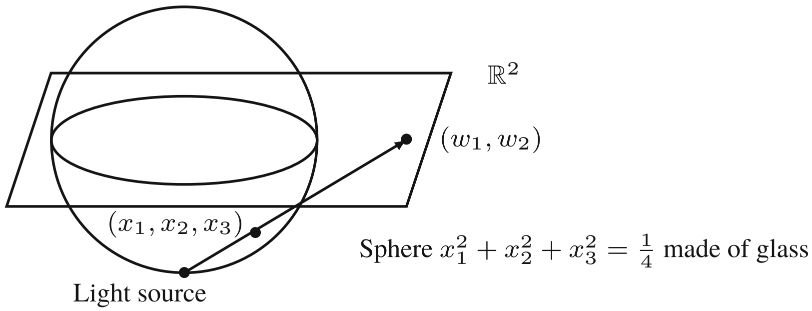


Fig. 7.1 Stereographic projection from the sphere $x_1^2+x_2^2+x_3^2 = \frac{1}{4}$ with light source at $(0, 0, -\frac{1}{2})$ to the $(x_1, x_2, 0)$ plane

We use the q-deformation of the classical generators from Lemma 2.34 to give generators $x = x^*$ and z of $\mathbb{C}_q[S^2]$. As in commutative stereographic projection, in the noncommutative case we would like to define $w^* = zx^{-1}$ (taking the liberty of rescaling to omit the $\frac{1}{2}$ and adding a $*$ to fit historical conventions). This inversion of x constitutes taking a *localisation* of the algebra, which classically corresponds to taking the subset where $x \neq 0$. In this case x^{-1} obeys the reordering rules $x^{-1}z = q^2 zx^{-1}$ and $x^{-1}z^* = q^{-2}z^*x^{-1}$, so we can reorder expressions with x^{-1} and cancel with any xs, so the localisation is quite well behaved. Now we calculate

$$ww^* = x^{-1}z^*zx^{-1} = x^{-1} - 1, \quad w^*w = zx^{-2}z^* = q^{-4}x^{-2}zz^* = q^{-2}x^{-1} - 1,$$

and from this $x^{-1} = ww^* + 1 = q^2(w^*w + 1)$, which can be rearranged as

$$ww^* - q^2 w^*w = q^2 - 1.$$

Up to normalisation, this is the q^2-Heisenberg algebra relation. Now our algebra of functions on the complex plane is the $*$-algebra generated by w and $(w^*w + 1)^{-1}$ subject to this q^2-Heisenberg algebra relation. We denote this algebra by $\mathbb{C}_q[S^2 \backslash \infty]$, a notation indicating its origin from the sphere rather than another version of a noncommutative complex plane. Corresponding to the classical inclusion $\mathbb{C} \to \mathbb{C}_\infty$ there is a map from $\mathbb{C}_q[S^2]$ to $\mathbb{C}_q[S^2 \backslash \infty]$ given by

$$x \longmapsto (ww^* + 1)^{-1}, \quad z \longmapsto w^*(ww^* + 1)^{-1}.$$

The easiest way to calculate the differential calculus is to use the description of the forms on $\mathbb{C}_q[S^2]$ as the horizontal invariant forms on $\mathbb{C}_q[SU_2]$, and then

$$\mathrm{d}w^* = q^{-2}c^2x^{-2}e^-, \quad \mathrm{d}w = -q^{-1}b^2x^{-2}e^+, \quad \mathrm{d}x = -bde^+ - ace^-$$

using Proposition 2.35, which leads to the commutation relations

$$(\mathrm{d}w^*)w^* = q^2 w^*\mathrm{d}w^*, \quad (\mathrm{d}w^*)w = q^{-2}w\mathrm{d}w^*, \quad \mathrm{d}x = -x(w\mathrm{d}w^* + (\mathrm{d}w)w^*)x$$

as well as $*$ applied to these. The formula $\mathrm{d}w = -q^{-1}b^2x^{-2}e^+$ is the reason why we had the extra $*$ in the definition of w earlier—traditionally e^+ is associated with the $+\mathrm{i}$ eigenspace of J. If we take $\mathrm{d}w$ in $\Omega^{1,0}$ and $\mathrm{d}w^*$ in $\Omega^{0,1}$ then the map from $\mathbb{C}_q[S^2]$ to $\mathbb{C}_q[S^2 \backslash \infty]$ is holomorphic. ◇

7.2 Holomorphic Modules and Dolbeault Cohomology

Throughout this section, $(\Omega, \mathrm{d}, *, J)$ is an integrable almost complex structure on A. The noncommutative equivalent of the classical Cauchy–Riemann condition for $a \in A$ to be holomorphic is that $\bar{\partial} a = 0$, and we call the collection of such holomorphic elements $A_{\rm hol}$. As $\overline{\partial}$ is a derivation, $A_{\rm hol}$ is a subalgebra of A. Similarly define $\Omega^p_{\rm hol}$, the holomorphic p-forms, as the elements of $\xi \in \Omega^{p,0}$ for which $\overline{\partial}\xi = 0$. The holomorphic forms form a sub-DGA of the de Rham complex, the *holomorphic de Rham complex*,

$$0 \longrightarrow A_{\rm hol} \xrightarrow{\partial} \Omega^1_{\rm hol} \xrightarrow{\partial} \Omega^2_{\rm hol} \xrightarrow{\partial} \cdots .$$

For a complex analytic manifold M it is natural to consider vector bundles with fibre $\mathbb{C}^n$ where the transition functions are holomorphic. In real differential geometry, the 'obvious' way to differentiate sections of a vector bundle is to take a trivialising open set, and then simply apply partial derivative $\frac{\partial}{\partial x^i}$ to the components of the section. Of course, this does not work globally as the derivatives of the transition functions enter, forcing us to use covariant derivatives and Christoffel symbols. However, in complex differential geometry, the 'obvious' thing to do works, with one condition. If we take the $\overline{\partial}$ derivatives of the components of the section, we get $\overline{\partial}_E : E \to \Omega^{0,1} \otimes_A E$ defined locally by

$$\overline{\partial}_E(v) = \mathrm{d}\overline{z}^i \otimes \frac{\partial v^j}{\partial \overline{z}^i} e_j, \tag{7.5}$$

where e_j is the local basis of the vector bundle, E is the sections of the bundle and $v = v^j e_j \in E$. Furthermore, this formula is perfectly well behaved under *holomorphic* change of basis as the $\overline{\partial}$ derivatives of the transition functions are zero, so we get a globally defined derivative. Thus, every complex vector bundle with holomorphic transition functions (we will just say *holomorphic vector bundle*) has a well-defined operator $\overline{\partial}_E$ satisfying the left $\overline{\partial}$-Leibniz rule, for $v \in E$ and $a \in A$,

$$\partial_E(a.v) = \overline{\partial} a \otimes v + a.\partial_E(v). \tag{7.6}$$

We could not have done this with the ∂ derivative instead of $\overline{\partial}$, as we then get the derivatives of the transition functions entering as in the real case. There is another important property of $\overline{\partial}_E$. Remember that $\overline{\partial}_E$ is defined in (7.5) by applying $\overline{\partial}$ to every component v^j, and applying $\overline{\partial}$ twice to a function gives zero. Thus we get the vanishing of the following classical quantity, called the *holomorphic curvature*,

$$(\overline{\partial} \otimes \mathrm{id} - \mathrm{id} \wedge \overline{\partial}_E)\, \overline{\partial}_E : E \to \Omega^{0,2} \otimes_A E. \tag{7.7}$$

Example 7.13 The tautological bundle γ on classical complex projective space, at a point $v \in \mathbb{CP}^n$, consists of the 1-dimensional vector space corresponding to the point v. Given homogeneous coordinates $(z_0, z_1, \ldots, z_n)$, on the subset where $z_i \neq 0$ we choose the section $e^i = (z_0, z_1, \ldots, 1, \ldots, z_n)$, where the 1 occurs at position i. The transition functions between these sections are holomorphic, as we check explicitly between e^1 and e^0. At the point with homogeneous coordinates

$(1, z_1, \ldots, z_n)$ we have $e^0 = (1, z_1, \ldots, z_n) = z_1(z_1^{-1}, 1, z_1^{-1}z_2, \ldots) = z_1 e^1$. We see that it is perfectly consistent to define $\overline{\partial}_\gamma(e^i) = 0$ for all i, as each $\overline{\partial} z^j = 0$. $\diamond$

Still classically, going from holomorphic transition functions to a $\overline{\partial}$-connection is the easy part. The difficult part is going the other way, that is, if we have a complex bundle with an operator $\overline{\partial}_E : E \to \Omega^{0,1} \otimes_A E$ obeying (7.6) and the vanishing of (7.7), then local bases can be chosen so that the transition functions are holomorphic, i.e., we have a holomorphic vector bundle, and further that $\overline{\partial}_E$ is just given by partial differentiation, as above, with respect to this basis. This is the content of the *Koszul–Malgrange theorem*. It justifies the following *definition* in noncommutative geometry, as we do not have local complex coordinate patches.

Definition 7.14 A $\overline{\partial}$-*operator* on a left A-module E is a linear map $\overline{\partial}_E : E \to \Omega^{0,1} \otimes_A E$ obeying the left $\overline{\partial}$-Leibniz rule (7.6). Its holomorphic curvature is defined as the left A-module map (7.7). Furthermore

(1) If the holomorphic curvature vanishes, then $(E, \overline{\partial}_E)$ is said to be a holomorphic left A-module and $e \in E$ with $\overline{\partial}_E e = 0$ is said to be holomorphic.
(2) If E is also a bimodule and there is a bimodule map $\sigma_E : E \otimes_A \Omega^{0,1} \to \Omega^{0,1} \otimes_A E$ such that $\overline{\partial}_E(e.a) = (\overline{\partial}_E e).a + \sigma_E(e \otimes \overline{\partial} a)$ for all $a \in A$ and $e \in E$, then $(E, \overline{\partial}_E, \sigma_E)$ is said to be a left holomorphic bimodule.

We shall also use the term *holomorphic structure* on a left/bi-module to mean equipping the module with a $\overline{\partial}$-operator satisfying these conditions. We can similarly give the right version of all these definitions for a right $\overline{\partial}$-operator ${}_F\overline{\partial} : F \to F \otimes_A \Omega^{0,1}$ on a right A-module F. The right $\overline{\partial}$-Leibniz rule and the corresponding curvature condition for a holomorphic right A-module are

$$ {}_F\overline{\partial}(f.a) = f \otimes \overline{\partial} a + ({}_F\overline{\partial} f).a, \quad (\mathrm{id} \otimes \overline{\partial} + {}_F\overline{\partial} \wedge \mathrm{id}){}_F\overline{\partial} = 0 $$

for $f \in F$ and $a \in A$. If F is also a bimodule and there is a bimodule map ${}_F\sigma : \Omega^{0,1} \otimes_A F \to F \otimes_A \Omega^{0,1}$ such that ${}_F\overline{\partial}(a.f) = a.{}_F\overline{\partial} f + {}_F\sigma(\overline{\partial} a \otimes f)$, then $(F, {}_F\overline{\partial}, {}_F\sigma)$ is a right holomorphic bimodule. Now we define a category of holomorphic modules:

Name	Objects	Morphisms
${}_A\mathrm{Hol}$	$(E, \overline{\partial}_E)$ holomorphic left A-modules	Left A-module maps intertwining $\overline{\partial}$ operators

A left module map $\phi : E \to F$ intertwines the $\overline{\partial}$ operators $\overline{\partial}_E$ and $\overline{\partial}_F$ if $(\mathrm{id} \otimes \phi)\overline{\partial}_E = \overline{\partial}_F \phi$.

Proposition 7.15 *If $\Omega^{0,1}$ and $\Omega^{0,2}$ are flat right A-modules then ${}_A\mathrm{Hol}$ is an abelian category.*

Proof Let $\phi : (E, \overline{\partial}_E) \to (F, \overline{\partial}_F)$ be a morphism in ${}_A\mathrm{Hol}$. Let K be the kernel and C the cokernel of ϕ in the category of left A-modules. The construction of $\overline{\partial}_K$ and $\overline{\partial}_C$ are given similarly to the proof of Theorem 3.114, using $\Omega^{0,1}$ in place of Ω^1 and $\overline{\partial}$ in place of d. It remains to show that the corresponding holomorphic curvatures $R_{(K,\overline{\partial}_K)}$ and $R_{(C,\overline{\partial}_C)}$ vanish. Since $\Omega^{0,2}$ is flat, the bottom row in the following commuting diagram is exact,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "R_{(K,\overline{\partial}_K)}"'] & E \arrow[r, "\phi"] \arrow[d, "0"'] & F \arrow[r, "\pi"] \arrow[d, "0"] & C \arrow[r] \arrow[d, "R_{(C,\overline{\partial}_C)}"] & 0 \\
0 \arrow[r] & \Omega^{0,2} \otimes_A K \arrow[r, "\mathrm{id}\otimes i"] & \Omega^{0,2} \otimes_A E \arrow[r, "\mathrm{id}\otimes\phi"] & \Omega^{0,2} \otimes_A F \arrow[r, "\mathrm{id}\otimes\pi"] & \Omega^{0,2} \otimes_A C \arrow[r] & 0
\end{tikzcd}$$

From this it follows that $R_{(K,\overline{\partial}_K)}$ and $R_{(C,\overline{\partial}_C)}$ vanish. □

Next we observe that usual connections $\nabla_E : E \to \Omega^1 \otimes_A E$ and $\overline{\partial}$-operators are both defined in terms of a Leibniz rule. We can use the projections $\pi^{p,q} : \Omega^{p+q} \to \Omega^{p,q}$ to relate the two definitions.

Proposition 7.16 *Let (E, ∇_E) be a left A-module with connection. Then the $\overline{\partial}$-operator on E defined by*

$$\overline{\partial}_E = (\pi^{0,1} \otimes \mathrm{id})\nabla_E : E \to \Omega^{0,1} \otimes_A E$$

has holomorphic curvature given in terms of the curvature R_E of (E, ∇_E) by

$$(\pi^{0,2} \otimes \mathrm{id})R_E : E \to \Omega^{0,2} \otimes_A E.$$

Hence if the $\Omega^{0,2}$ part of R_E vanishes then $(E, \overline{\partial}_E)$ is a holomorphic left A-module.

Proof Let $e \in E$ and write $\nabla e = \xi \otimes f$ (sum of terms understood) so that $R_E(e) = \mathrm{d}\xi \otimes f - \xi \wedge \nabla_E f$. Then

$$\begin{aligned}
(\pi^{0,2} \otimes \mathrm{id})R_E(e) &= \pi^{0,2}(\mathrm{d}\xi) \otimes f - (\pi^{0,2} \otimes \mathrm{id})(\xi \wedge \nabla_E f) \\
&= \pi^{0,2}(\mathrm{d}(\pi^{0,1}\xi) + \mathrm{d}(\pi^{1,0}\xi)) \otimes f - \pi^{0,1}(\xi) \wedge (\pi^{0,1} \otimes \mathrm{id})\nabla_E f \\
&= \pi^{0,2}(\mathrm{d}(\pi^{0,1}\xi)) \otimes f - \pi^{0,1}(\xi) \wedge \overline{\partial}_E f = \overline{\partial}(\pi^{0,1}\xi) \otimes f - \pi^{0,1}(\xi) \wedge \overline{\partial}_E f \\
&= (\overline{\partial} \otimes \mathrm{id} - \mathrm{id} \wedge \overline{\partial}_E)\overline{\partial}_E e,
\end{aligned}$$

which is the holomorphic curvature given in (7.7). □

The construction of Chern connections in §8.6 will be a partial converse to this result, where given a hermitian metric we construct an ordinary connection from a holomorphic left A-module.

Example 7.17 In Example 5.51, we saw that $\mathbb{C}_q[SU_2]$ is a module over $A = \mathbb{C}_q[S^2]$ with a connection whose curvature is in $\Omega^{1,1}$. Thus by Proposition 7.16, we can construct a holomorphic left $\mathbb{C}_q[S^2]$-module. By the remarks in Example 5.51, $\mathbb{C}_q[SU_2]$ splits into a direct sum of line modules E_n (the integer graded submodules of $\mathbb{C}_q[SU_2]$), and each of these has a connection giving a holomorphic left module. The formula for the $\overline{\partial}$-operator is given by applying the projection $\pi^{0,1}$ to the connection in Example 5.51, giving

$$\overline{\partial}_{E_n} s = q^{n-2} e^- \tilde{D}_1 \tilde{D}'_1 \otimes \tilde{D}'_2 \tilde{D}_2 s_- \ \in\ \Omega^{0,1} \otimes_A E_n,$$

where $s \in E_n$ and $\mathrm{d}s = s_+ e^+ + s_- e^- + s_0 e^0$ in $\Omega^1(\mathbb{C}_q[SU_2])$ and $\tilde{D}_1 \otimes \tilde{D}_2 = \tilde{D}'_1 \otimes \tilde{D}'_2 = a \otimes d - q^{-1} c \otimes b$. This is a bimodule connection with σ_E as defined in Example 5.51, so we conclude that we have a left holomorphic bimodule. ◇

As $\overline{\partial}$-operators behave in many regards like ordinary connections, some definitions and results are almost automatic from the usual connection case. (Not surprisingly, as we can regard $(\Omega^{0,*}, \overline{\partial})$ as an alternative calculus to (Ω, d).) In particular, we have duals of fgp holomorphic modules provided we allow right modules as well as left, as the dual $E^\flat$ of a left fgp module E is naturally a right fgp module. For example, adapting the equation for the left and right curvatures in Proposition 3.32, we have the following for a $\overline{\partial}$-operator preserving the evaluation map $\mathrm{ev} : E \otimes E^\flat \to A$.

Proposition 7.18 *Let E be a left fgp A-module with a left $\overline{\partial}$-operator $\overline{\partial}_E$. Then there is a unique right $\overline{\partial}$-operator ${}_{E^\flat}\overline{\partial} : E^\flat \to E^\flat \otimes_A \Omega^{0,1}$ such that*

$$\overline{\partial} \circ \mathrm{ev} = (\mathrm{id} \otimes \mathrm{ev})(\overline{\partial}_E \otimes \mathrm{id}) + (\mathrm{ev} \otimes \mathrm{id})(\mathrm{id} \otimes {}_{E^\flat}\overline{\partial}) : E \otimes E^\flat \to \Omega^{0,1}.$$

Moreover, if $(E, \overline{\partial}_E)$ is a holomorphic left module then $(E^\flat, {}_{E^\flat}\overline{\partial})$ is a holomorphic right module.

The reader may recall from §4.3 that from a connection (E, ∇_E) we can construct a cohomology theory with coefficients in E in the case where the curvature R_E vanished. We can similarly construct a cohomology theory from a $\overline{\partial}$-operator $(E, \overline{\partial}_E)$ when the holomorphic curvature vanishes. The first step is to extend the $\overline{\partial}$-operator on E to $\overline{\partial}_E{}^{[p,q]} : \Omega^{p,q} \otimes_A E \to \Omega^{p,q+1} \otimes_A E$ by

$$\overline{\partial}_E{}^{[p,q]}(\xi \otimes e) = \overline{\partial}\xi \otimes e + (-1)^{p+q} \xi \wedge \overline{\partial}_E(e) \tag{7.8}$$

with $\overline{\partial}_E{}^{[0,0]} = \overline{\partial}_E$. In just the same way as in the proof of Theorem 4.3, we can show that if the holomorphic curvature vanishes then the composition $\overline{\partial}_E{}^{[p,q+1]} \circ \overline{\partial}_E{}^{[p,q]} = 0$, so we get a cochain complex

$$0 \longrightarrow \Omega^{p,0} \otimes_A E \xrightarrow{\overline{\partial}_E{}^{[p,0]}} \Omega^{p,1} \otimes_A E \xrightarrow{\overline{\partial}_E{}^{[p,1]}} \Omega^{p,2} \otimes_A E \longrightarrow \cdots \tag{7.9}$$

Now we can define the cohomologies related to the $\overline{\partial}$-operator.

Definition 7.19 For all $p, q \in \mathbb{N}$, the (p, q)-*Dolbeault cohomology* $\mathrm{H}^{p,q}(A, \overline{\partial})$ is the q-th cohomology of the p-th Dolbeault complex

$$0 \longrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \longrightarrow \cdots$$

If E is a holomorphic left A-module then the Dolbeault cohomology $\mathrm{H}^{p,q}(A, E, \overline{\partial}_E)$ with coefficients in E is the q-th cohomology of the cochain complex (7.9). In the special case $p = 0$, the cohomology $\mathrm{H}^n(E, \overline{\partial}_E)$ of the complex

$$0 \longrightarrow E \xrightarrow{\overline{\partial}_E} \Omega^{0,1} \otimes_A E \xrightarrow{\overline{\partial}_E{}^{[0,1]}} \Omega^{0,2} \otimes_A E \xrightarrow{\overline{\partial}_E{}^{[0,2]}} \cdots$$

is the *holomorphic sheaf cohomology*.

Note that $\Omega^p_{\mathrm{hol}} = \mathrm{H}^{p,0}(A, \overline{\partial})$, while the dimensions $h^{p,q}$ of $\mathrm{H}^{p,q}(A, \overline{\partial})$ are the Hodge numbers. If $\phi : (E, \overline{\partial}_E) \to (F, \overline{\partial}_F)$ is a morphism in ${}_A\mathrm{Hol}$ (i.e., a left module map intertwining the $\overline{\partial}$ operators), then $\mathrm{id}\otimes\phi$ is a map of cochain complexes from (7.9) to the corresponding complex for F, so it induces a map in cohomology

$$\phi : \mathrm{H}^{p,q}(A, E, \overline{\partial}_E) \to \mathrm{H}^{p,q}(A, F, \overline{\partial}_F).$$

For holomorphic modules, we have the usual long exact sequence property for cohomology following the same pattern as the nonholomorphic case in Theorem 4.43.

Proposition 7.20 *If every $\Omega^{0,n}$ is flat as a right A-module, then every short exact sequence*

$$0 \longrightarrow (E, \overline{\partial}_E) \xrightarrow{\ \phi\ } (F, \overline{\partial}_F) \xrightarrow{\ \psi\ } (G, \overline{\partial}_G) \longrightarrow 0$$

in ${}_A\mathrm{Hol}$ *gives a long exact sequence of left* A_{hol}*-modules*

$$0 \longrightarrow \mathrm{H}^0(E, \overline{\partial}_E) \xrightarrow{\phi} \mathrm{H}^0(F, \overline{\partial}_F) \xrightarrow{\psi} \mathrm{H}^0(G, \overline{\partial}_G) \xrightarrow{\delta} \mathrm{H}^1(E, \overline{\partial}_E) \xrightarrow{\phi} \ldots$$

Proof By applying $\Omega^{0,n}\otimes_A$, we get an exact sequence

$$0 \longrightarrow \Omega^{0,n} \otimes_A E \longrightarrow \Omega^{0,n} \otimes_A F \longrightarrow \Omega^{0,n} \otimes_A G \longrightarrow 0$$

of cochain complexes. Then use homological algebra as in Theorem 4.43. □

We will use these long exact sequences in Example 7.41. To further study Dolbeault cohomology and its relation to holomorphic modules, we also need a further factorisability condition which is true for classical complex manifolds.

Definition 7.21 A calculus with an integrable almost complex structure is said to satisfy the *factorisability condition* if the wedge products

$$\wedge : \Omega^{0,q} \otimes_A \Omega^{p,0} \to \Omega^{p,q}, \quad \wedge : \Omega^{p,0} \otimes_A \Omega^{0,q} \to \Omega^{p,q}$$

are bimodule isomorphisms for all $p, q \geq 0$. We denote the inverses by

$$\Theta^{0qp0} : \Omega^{p,q} \to \Omega^{0,q} \otimes_A \Omega^{p,0}, \quad \Theta^{p00q} : \Omega^{p,q} \to \Omega^{p,0} \otimes_A \Omega^{0,q}.$$

Example 7.22 To get the map Θ^{0qp0} on a classical complex manifold with local complex coordinates z_i, just permute all the $\mathrm{d}\bar{z}_i$ to the left, with the appropriate power of -1, and replace the $\wedge$ separating the $\mathrm{d}\bar{z}_i$ from the $\mathrm{d}z_i$ by $\otimes$. For example,

$$\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 \longmapsto -\mathrm{d}\bar{z}_2 \otimes \mathrm{d}z_1,$$
$$\mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 \longmapsto -\mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2 \otimes \mathrm{d}z_1,$$
$$\mathrm{d}z_3 \wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}\bar{z}_4 \longmapsto \mathrm{d}\bar{z}_2 \wedge \mathrm{d}\bar{z}_4 \otimes \mathrm{d}z_3 \wedge \mathrm{d}z_1.$$

For Θ^{p00q}, similarly permute all the $\mathrm{d}\bar{z}_i$ to the right. ◇

Example 7.23 The integrable almost complex structure on $\mathbb{C}_q[S^2]$ given in Proposition 2.35 satisfies the factorisability condition, where

$$\Theta^{1001}(e^+ \wedge e^-) = e^+ D_1 D_1' \otimes D_2' D_2 e^-, \ \ \Theta^{0110}(e^+ \wedge e^-) = -q^{-2} e^- \tilde{D}_1 \tilde{D}_1' \otimes \tilde{D}_2' \tilde{D}_2 e^+$$

using the D and $\tilde{D}$ notation of Example 5.26. ◇

We use the factorisability condition to construct holomorphic modules.

Lemma 7.24 *Suppose that a calculus on A with an integrable almost complex structure satisfies the factorisability condition. Then for all holomorphic modules $(E, \overline{\partial}_E)$, each $\Omega^{p,0} \otimes_A E$ is a holomorphic left A-module for the $\overline{\partial}$-operator*

$$\overline{\partial}_{p,E} = (\Theta^{01p0} \otimes \mathrm{id})\overline{\partial}_E{}^{[p,0]} : \Omega^{p,0} \otimes_A E \to \Omega^{0,1} \otimes_A \Omega^{p,0} \otimes_A E.$$

Proof First we check the Leibniz condition,

$$\begin{aligned}\overline{\partial}_{p,E}(a.\xi \otimes e) &= (\Theta^{01p0} \otimes \mathrm{id})(a.\overline{\partial}_E{}^{[p,0]} + \overline{\partial} a \wedge \xi \otimes e)\\ &= a.\overline{\partial}_{p,E}(\xi \otimes e) + \Theta^{01p0}(\overline{\partial} a \wedge \xi) \otimes e = a.\overline{\partial}_{p,E}(\xi \otimes e) + \overline{\partial} a \otimes \xi \otimes e\end{aligned}$$

for all $a \in A$ and $\xi \otimes e \in \Omega^{p,0} \otimes_A E$. Next we check that the holomorphic curvature vanishes. This needs $\overline{\partial}_{p,E}{}^{[0,1]} \circ \overline{\partial}_{p,E} : \Omega^{p,0} \otimes_A E \to \Omega^{0,2} \otimes_A \Omega^{p,0} \otimes_A E$ to vanish. As $\wedge : \Omega^{0,2} \otimes_A \Omega^{p,0} \to \Omega^{p,2}$ is an isomorphism, it suffices to show that

$(\wedge\otimes\mathrm{id})\overline{\partial}_{p,E}{}^{[0,1]}\circ\overline{\partial}_{p,E}:\Omega^{p,0}\otimes_A E\to\Omega^{p,2}\otimes_A E$ vanishes. Now

$$\begin{aligned}(\wedge\otimes\mathrm{id})\overline{\partial}_{p,E}{}^{[0,1]}(\eta\otimes\xi\otimes e)&=\overline{\partial}\eta\wedge\xi\otimes e-\eta\wedge\overline{\partial}\xi\otimes e+(-1)^{p+1}\eta\wedge\xi\wedge\overline{\partial}_E e\\&=\overline{\partial}_E{}^{[p,1]}(\eta\wedge\xi\otimes e)=\overline{\partial}_E{}^{[p,1]}(\wedge\otimes\mathrm{id})(\eta\otimes\xi\otimes e)\end{aligned}$$

for all $\eta\otimes\xi\otimes e\in\Omega^{0,1}\otimes_A\Omega^{p,0}\otimes_A E$, and

$$(\wedge\otimes\mathrm{id})\overline{\partial}_{p,E}(\xi\otimes e)=\overline{\partial}\xi\otimes e+(-1)^p\xi\wedge\overline{\partial}_E e=\overline{\partial}_E{}^{[p,0]}e,$$

so we need $\overline{\partial}_E{}^{[p,1]}\circ\overline{\partial}_E{}^{[p,0]}$ to vanish, which is just (7.9). □

Example 7.25 The factorisability condition is satisfied for $A=M_2(\mathbb{C})$ with its standard calculus and complex structure given in Example 7.6, with $\Theta^{0110}:xst\mapsto xt\otimes s$ for $x\in M_2(\mathbb{C})$. By Lemma 7.24, we can put a holomorphic structure on the bimodule $E=\Omega^{1,0}$ by defining the holomorphic connection $\overline{\partial}_E$ as

$$\Omega^{1,0}\xrightarrow{\overline{\partial}}\Omega^{1,1}\xrightarrow{\Theta^{0110}}\Omega^{0,1}\otimes_{M_2(\mathbb{C})}\Omega^{1,0}.$$

This gives $\overline{\partial}_E(sx)=\{E_{21},x\}t\otimes s$. The holomorphic curvature vanishes,

$$\begin{aligned}(\overline{\partial}\otimes\mathrm{id}-\mathrm{id}\wedge\overline{\partial}_E)\overline{\partial}_E(sx)&=(\overline{\partial}\otimes\mathrm{id}-\mathrm{id}\wedge\overline{\partial}_E)(\{E_{21},x\}t)\otimes s)\\&=\{E_{21},\{E_{21},x\}\}t^2\otimes s-2\{E_{21},x\}t\wedge E_{21}t\otimes s=0\end{aligned}$$

as ${E_{21}}^2=0$, so we get a holomorphic module. ◇

We can also use Lemma 7.24 to give an alternative description of the Dolbeault cohomology as holomorphic sheaf cohomology.

Corollary 7.26 *Let A have an integrable almost complex structure obeying the factorisability condition in Definition 7.21. Then for all $p\in\mathbb{N}$, the $\overline{\partial}$-operator*

$$\overline{\partial}_{p,A}=\Theta^{01p0}\circ\overline{\partial}:\Omega^{p,0}\to\Omega^{0,1}\otimes_A\Omega^{p,0}$$

makes $\Omega^{p,0}$ into a holomorphic module and $\mathrm{H}^q(\Omega^{p,0},\overline{\partial}_{p,A})=\mathrm{H}^{p,q}(A,\overline{\partial})$.

Proof Lemma 7.24 shows that we get a holomorphic module. That the p-th Dolbeault complex in Definition 7.19 is isomorphic to

$$\Omega^{p,0}\xrightarrow{\overline{\partial}_{p,A}}\Omega^{0,1}\otimes\Omega^{p,0}\xrightarrow{\overline{\partial}_{p,A}{}^{[1]}}\Omega^{0,2}\otimes\Omega^{p,0}\xrightarrow{\overline{\partial}_{p,A}{}^{[2]}}\ldots,$$

can be seen from the commutative diagram

$$\begin{tikzcd}
\Omega^{p,q} \arrow[rr, "\overline{\partial}"] & & \Omega^{p,q+1} \\
\Omega^{0,q} \otimes_A \Omega^{p,0} \arrow[u, "\wedge"] \arrow[rr, "{\overline{\partial}_{p,A}}^{[q]}"] & & \Omega^{0,q+1} \otimes_A \Omega^{p,0}, \arrow[u, "\wedge"']
\end{tikzcd}$$

where the maps $\wedge$ are isomorphisms by the factorisability condition. □

An obvious question to ask is how the Dolbeault cohomology is related to the de Rham cohomology. The next remark gives a partial answer to this.

Remark 7.27 We construct a spectral sequence using the Dolbeault complex, called the Hodge to de Rham spectral sequence or the Frölicher spectral sequence. Begin with the spectral sequence of a filtration in Example 4.52. Consider the complex $\Omega^n = \oplus_{p,q:p+q=n}\Omega^{p,q}$, with the usual d, and give it a filtration

$$F^m\Omega = \bigoplus_{p,q:p\geq m} \Omega^{p,q}.$$

Taking the quotient $F^m\Omega/F^{m+1}\Omega$ gives the cochain complex

$$\Omega^{m,0} \xrightarrow{\bar\partial} \Omega^{m,1} \xrightarrow{\bar\partial} \Omega^{m,2} \xrightarrow{\bar\partial} \dots$$

Then there is a spectral sequence with first page $E_1^{p,q} = \mathrm{H}^{p,q}(A, \overline{\partial})$ which converges to the de Rham cohomology. ◇

Example 7.28 For the noncommutative sphere $\mathbb{C}_q[S^2]$ with its standard differential structure in Proposition 2.35, the only nonvanishing Dolbeault cohomologies are

$$\mathrm{H}^{0,0}(\mathbb{C}_q[S^2], \overline{\partial}) = \mathbb{C}, \quad \mathrm{H}^{1,1}(\mathbb{C}_q[S^2], \overline{\partial}) = \mathbb{C}.$$

This result depends on using isomorphisms of holomorphic modules which preserve the $\overline{\partial}$-operators; $\Omega^{0,0} \cong E_0$ and $\Omega^{1,0} \cong E_2$. Proposition 7.36 calculates the holomorphic sheaf cohomology H^0 and Proposition 7.39 calculates H^1 for both these homomorphic modules. Then Corollary 7.26 shows that this gives the Dolbeault cohomology. We put these values in the first page of the Frölicher spectral sequence,

$$\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{C} & 0 \\
0 & \mathbb{C} & 0 & 0 \\
\hline
0 & 0 & 0 & 0
\end{array}\qquad \text{Page one}$$

Considering the differentials we see that the sequence converges at page one, and summing over the diagonals gives the isomorphisms $\mathrm{H}^{0,0}(\mathbb{C}_q[S^2]) \cong \mathrm{H}^0(\mathbb{C}_q[S^2])$, $\mathrm{H}^{0,1}(\mathbb{C}_q[S^2])\oplus\mathrm{H}^{1,0}(\mathbb{C}_q[S^2]) \cong \mathrm{H}^1(\mathbb{C}_q[S^2])$ and finally $\mathrm{H}^{1,1}(\mathbb{C}_q[S^2]) \cong \mathrm{H}^2(\mathbb{C}_q[S^2])$. Of course, this gives an alternative method for calculating $\mathrm{H}^n(\mathbb{C}_q[S^2])$. ◇

We next revisit Example 3.115 on the vanishing ideal to give a construction of a holomorphic structure on a quotient module.

Example 7.29 Let $(E, \overline{\partial}_E)$ be a holomorphic left A-module which is also fgp, and suppose that $e \in E$ obeys $\overline{\partial}_E e = 0$. Now suppose that $e \in E$ has a nontrivial vanishing right ideal $\mathrm{Van}(e) \subseteq A$. We take the dual right $\overline{\partial}$-connection ${}_{E^\flat}\overline{\partial} : E^\flat \to E^\flat \otimes_A \Omega^{0,1}$ from Proposition 7.18. Then for all $\alpha \in E^\flat$,

$$\overline{\partial}\mathrm{ev}(e \otimes \alpha) = (\mathrm{id} \otimes \mathrm{ev})(\overline{\partial}_E(e) \otimes \alpha) + (\mathrm{ev} \otimes \mathrm{id})(e \otimes {}_{E^\flat}\overline{\partial}(\alpha)).$$

The first term on the right-hand side vanishes, so $\overline{\partial}\mathrm{Van}(e) \subseteq \mathrm{Van}(e) \otimes_A \Omega^{0,1}$. Thus $\mathrm{Van}(e)$ has a right $\overline{\partial}$-connection and, as $\overline{\partial}^2 = 0$, this has zero holomorphic curvature. Using the obvious right connection $\overline{\partial} : A \to A \otimes_A \Omega^{0,1}$ on A, if $\Omega^{0,1}$ and $\Omega^{0,2}$ are flat then we deduce that the quotient $A/\mathrm{Van}(e)$ also has a right connection induced by $\overline{\partial}$, $\overline{\partial}[a] = [1] \otimes \overline{\partial}a \in A/\mathrm{Van}(e) \otimes_A \Omega^{0,1}$, where [ ] denotes an equivalence class. This makes $A/\mathrm{Van}(e)$ into a holomorphic bundle. ◇

We apply this to line bundles on $\mathbb{C}_q[SU_2]$ in Example 7.41, where the vanishing ideal is not all of A. First we cover the case where the vanishing ideal equals A.

Proposition 7.30 *Let $(L, \overline{\partial}_L)$ be a holomorphic left A-line module, and suppose there are holomorphic sections $s_1, \ldots, s_n$ of L such that $\mathrm{Van}(\{s_1, \ldots, s_n\}) = A$. Then there are $\beta_1, \ldots, \beta_n \in \mathrm{Hom}_A(L, A)$ which together with $s_1, \ldots, s_n$ give dual bases for L.*

Proof As $1 \in \mathrm{Van}(\{s_1, \ldots, s_n\}) = A$, by definition there are $\alpha_i \in L^\flat$ such that $\mathrm{ev}(s_i \otimes \alpha_i) = 1$ (summing over i). Now for any $e \in L$, $\mathrm{ev}(s_i \otimes \alpha_i).e = 1.e = e$. By Proposition 3.97, we can use

$$\mathrm{ev} \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{coev}^{-1} : L \otimes_A L^\flat \otimes_A L \to L$$

to write $s_i.\mathrm{coev}^{-1}(\alpha_i \otimes e) = e$. Now set $\beta_i(e) = \mathrm{coev}^{-1}(\alpha_i \otimes e)$. □

When the condition $\mathrm{Van}\{s_i\} = A$ holds, we say that the holomorphic sections $\{s_i\}$ of L are 'not simultaneously vanishing'. Now in the classical case where $A = C^\infty(M)$ and M is a complex manifold with holomorphic line bundle L, suppose that holomorphic sections $\{s_i\}$ do not simultaneously vanish at any point $x \in M$. As L may be nontrivial, we do not have a way to attach a well defined complex value to each of the $s_i(x) \in L_x$, but we can attach a value to their ratios. The value $[s_1(x), \ldots, s_n(x)]$, denoting the equivalence class of the vector $(s_1(x), \ldots, s_n(x)) \in \mathbb{C}^n$ in the complex projective space $\mathbb{CP}^{n-1}$, is left unchanged by applying transition functions between patches as these are just multiplication by nonzero complex functions in the case of line bundles. This is well defined since the sections do not simultaneously vanish and gives us a function $[\underline{s}] : M \to \mathbb{CP}^{n-1}$, which is holomorphic given that the sections are. However, this map need not be very interesting without a further condition: a holomorphic line bundle is called *very ample* if it has enough holomorphic sections to form an injective map $[\underline{s}] : M \to \mathbb{CP}^{n-1}$.

In §3.1 we saw a way to deal with noncommutative projective spaces by using projection matrices. If we have holomorphic sections s_i of the holomorphic line module L satisfying the condition in Proposition 7.30 then we have a dual basis β_i, say, and thus a projection matrix. We saw in Lemma 3.42 that, under the assumption that A is a local C^*-algebra, we can convert any projection to a hermitian projection by multiplying on the right. This is the same as leaving the s_i alone but changing the β_i in the formula $P_{ij} = \mathrm{ev}(s_i \otimes \beta_j)$. As the β_i were chosen fairly randomly anyway, we shall assume that we can choose them to make P hermitian. Next, there is a map from the universal Grassmannian algebra $\mathrm{G}_n^{\mathrm{nc}}$ (see §3.1) to A, sending the generator a_{ij} of $\mathrm{G}_n^{\mathrm{nc}}$ to $P_{ij} \in A$. As injective maps of spaces roughly correspond to surjective maps of algebras, we could define a noncommutative holomorphic line A-module $(L, \overline{\partial}_L)$ to be *very ample* if there are holomorphic sections $s_1, \ldots, s_n$ such that $\mathrm{Van}(\{s_1, \ldots, s_n\}) = A$ and there is an associated hermitian projection matrix P with entries generating the algebra A. On the other hand, we do not have an obvious noncommutative analogue of the classical map $M \to \mathbb{CP}^{n-1}$ which is central in Serre's GAGA programme.

7.3 Holomorphic Vector Fields

The classical Newlander–Nirenberg integrability condition is given in terms of vector fields and here we obtain a noncommutative version in that form. We use right vector fields $\mathfrak{X} = \mathfrak{X}^R$ as in Chap. 6, defined as the right A-module maps from Ω^1 to A. Throughout this section, we suppose that $(A, \Omega, \mathrm{d}, \wedge, *, J)$ is a $*$-differential calculus with almost complex structure and that Ω^1 is fgp as a right A-module. As $\Omega^1 = \Omega^{0,1} \oplus \Omega^{1,0}$, we deduce from Theorem 3.7 that $\Omega^{0,1}$ and $\Omega^{1,0}$ are right fgp.

Definition 7.31 Given an almost complex structure J on forms, define $J : \mathfrak{X} \to \mathfrak{X}$ by $J(v) = v \circ J$ and $\mathfrak{X}^{1,0}, \mathfrak{X}^{0,1}$ respectively as the $\pm\mathrm{i}$ eigenspaces of J.

Note the evaluation map vanishes on $\mathfrak{X}^{1,0} \otimes_A \Omega^{0,1}$ and $\mathfrak{X}^{0,1} \otimes_A \Omega^{1,0}$. Remembering the subset $\Lambda^2\mathfrak{X}$ of 'antisymmetric' tensor products of vector fields at the beginning of §2.7 and the Lie bracket $[\![\ ,\ ]\!]_R$ in (2.27) and (6.14), we can present the noncommutative integrability condition for vector fields as follows.

Proposition 7.32 *Suppose that the integrability condition in Lemma 7.2 holds for an almost complex structure J on forms. Then $[\![u, v]\!]_R \in \mathfrak{X}^{1,0}$ for all $u \otimes v \in \Lambda^2\mathfrak{X}^{1,0}$ (summation implicit). Conversely, if both Ω^1 and Ω^2 are finitely generated projective as right A-modules and if $[\![u, v]\!]_R \in \mathfrak{X}^{1,0}$ for all $u \otimes v \in \Lambda^2\mathfrak{X}^{1,0}$ then the integrability condition on forms holds.*

Proof Suppose that the integrability condition holds. We need to show $[\![u, v]\!]_R(\xi) = 0$ for all $\xi \in \Omega^{0,1}$. Let $z \in \Omega^1 \otimes_A \Omega^1$ be chosen such that $\wedge z = \mathrm{d}\xi$, then

$$[\![u, v]\!]_R(\xi) = \mathrm{ev}(u \otimes \mathrm{d}v(\xi)) + \mathrm{ev}^{\langle 2\rangle}(u \otimes v \otimes z).$$

Now, $v(\xi) = 0$ so we only have to show that $\mathrm{ev}^{\langle 2\rangle}(u \otimes v \otimes z) = 0$. As $\xi \in \Omega^{0,1}$, Lemma 7.2 states that $\mathrm{d}\xi \in \Omega^{0,2} \oplus \Omega^{1,1}$. Thus we can choose to have z in the sum of $\Omega^{0,1} \otimes_A \Omega^{0,1}$, $\Omega^{1,0} \otimes_A \Omega^{0,1}$ and $\Omega^{0,1} \otimes_A \Omega^{1,0}$, so $\mathrm{ev}^{\langle 2\rangle}(u \otimes v \otimes z) = 0$.

Conversely if the stated condition on the bracket holds, given $\xi \in \Omega^{0,1}$ we need to show that $\mathrm{d}\xi \in \Omega^{0,2} \oplus \Omega^{1,1}$. Suppose that $\mathrm{d}\xi$ has a nonzero $\Omega^{2,0}$ component. As Ω^2 is fgp as a right module, there is an $\alpha \in \mathrm{Hom}_A(\Omega^2, A)$ such that $\alpha(\mathrm{d}\xi) \neq 0$ and vanishing on $\Omega^{0,2} \oplus \Omega^{1,1}$ (using the projection to $\Omega^{2,0}$ to arrange the last part). As $\Omega^1 = \Omega^{0,1} \oplus \Omega^{1,0}$, we can make dual bases $\xi_i \otimes u_i \in \Omega^1 \otimes_A \mathfrak{X}$ for Ω^1 by merging bases for $\Omega^{0,1}$ and $\Omega^{1,0}$. Then the element

$$u \otimes v = \alpha(\xi_i \wedge \xi_j)u_j \otimes u_i \in \Lambda^2\mathfrak{X}$$

given in Proposition 6.26 lies in $\Lambda^2\mathfrak{X}^{1,0}$ and $[\![u, v]\!]_R(\xi) = \alpha(\mathrm{d}\xi) \neq 0$, contradicting our assumption that $[\![u, v]\!]_R \in \mathfrak{X}^{1,0}$. We conclude that $\mathrm{d}\xi \in \Omega^{0,2} \oplus \Omega^{1,1}$. □

Next, given local complex coordinates $z_1, \ldots, z_n$ on a classical complex manifold, we can define a holomorphic vector field as having the form

$$v = f_1(z_1, \ldots, z_n)\frac{\partial}{\partial z_1} + \cdots + f_n(z_1, \ldots, z_n)\frac{\partial}{\partial z_n},$$

where the $f_i(z_1, \ldots, z_n)$ are complex-valued holomorphic functions. Differentiating a holomorphic function $a(z_1, \ldots, z_n)$ along a holomorphic vector field gives another holomorphic function, and we would like this rather obvious property to be true in noncommutative geometry. Thus, in the quantum case we require that a vector field $v \in \mathfrak{X}^{1,0}$, if it is to be holomorphic, should obey $\overline{\partial}\mathrm{ev}(v \otimes \partial a) = 0$ for all $a \in A$ with $\overline{\partial} a = 0$. Next, we suppose a right $\overline{\partial}$-operator $\clubsuit : \Omega^{1,0} \to \Omega^{1,0} \otimes_A \Omega^{0,1}$ making $\Omega^{1,0}$ into a holomorphic right A-module, with dual left $\overline{\partial}$-operator $\clubsuit : \mathfrak{X}^{1,0} \to \Omega^{0,1} \otimes_A \mathfrak{X}^{1,0}$. Then Proposition 7.18 tells us that

$$\overline{\partial}\mathrm{ev}(v \otimes \partial a) = (\mathrm{id} \otimes \mathrm{ev})(\clubsuit v \otimes \partial a) + (\mathrm{ev} \otimes \mathrm{id})(v \otimes \clubsuit\partial a).$$

Thus we see that $\clubsuit v = 0$ gives a satisfactory definition of a holomorphic vector field provided we can find $\clubsuit$ such that $\overline{\partial} a = 0$ implies $\clubsuit\partial a = 0$ for all $a \in A$. Fortunately, we have such a connection in a reasonable case as follows.

Proposition 7.33 *Given a factorisable integrable almost complex structure, the connection*

$$\clubsuit : \Omega^{p,0} \to \Omega^{p,0} \otimes_A \Omega^{0,1}, \quad \clubsuit = (-1)^p \Theta^{p001}\,\overline{\partial}, \quad \tilde{\sigma} = (-1)^p \Theta^{p001} \circ \wedge$$

makes $\Omega^{p,0}$ into a right holomorphic A-bimodule such that $\overline{\partial} a = 0$ implies $\clubsuit\partial a = 0$ for all $a \in A$.

Proof The proof for a holomorphic left module is similar to that of Lemma 7.24, taking care with the signs. For the bimodule part, $\sigma(\overline{\partial} a \otimes \xi)$ is given by

$$\clubsuit(a.\xi) - a.\clubsuit(\xi) = (-1)^p \Theta^{p001}\overline{\partial}(a.\xi) - (-1)^p a \Theta^{p001}(\overline{\partial}\xi) = (-1)^p \Theta^{p001}(\overline{\partial} a \wedge \xi)$$

for all $\xi \in \Omega^{p,0}$. Using $\partial\overline{\partial} + \overline{\partial}\partial = 0 : A \to \Omega^2$ from Lemma 7.2, if $\overline{\partial} a = 0$ then

$$\clubsuit(\partial a) = -\Theta^{1001}(\overline{\partial}\partial a) = \Theta^{1001}(\partial\overline{\partial} a) = 0. \qquad \square$$

We now fix the assumptions and connection in Proposition 7.33 and define the subspace $\mathfrak{X}_{\mathrm{hol}} \subseteq \chi^{1,0}$ of *holomorphic vector fields* as $v \in \mathfrak{X}^{1,0}$ with $\clubsuit v = 0$.

Theorem 7.34 *Suppose that the calculus is factorisable and that Ω^1 is fgp as a right A-module. If $u \otimes v \in \Lambda^2 \mathfrak{X}_{\mathrm{hol}}$ (summation implicit) then $[\![u, v]\!]_R \in \mathfrak{X}_{\mathrm{hol}}$.*

Proof First we show that $(\mathrm{id} \otimes \mathrm{ev})(\clubsuit[\![u, v]\!]_R \otimes \xi) = 0$ for all $\xi \in \Omega^{1,0}$ by demonstrating the equality

$$\overline{\partial}([\![u, v]\!]_R(\xi)) = (\mathrm{ev} \otimes \mathrm{id})([\![u, v]\!]_R \otimes \clubsuit\xi). \tag{7.10}$$

We rewrite (6.14) as

$$[\![u, v]\!]_R(\xi) = \mathrm{ev}\big(u \otimes \big(\mathrm{ev}(v \otimes \kappa_1)\kappa_2 + \partial \mathrm{ev}(v \otimes \xi)\big)\big),$$

where $\partial\xi = \kappa_1 \wedge \kappa_2$ with $\kappa_1, \kappa_2 \in \Omega^{1,0}$ (sum implicit). Then, as $\clubsuit u = 0$,

$$\overline{\partial}([\![u, v]\!]_R(\xi)) = (\mathrm{ev} \otimes \mathrm{id})\big(u \otimes \clubsuit\big(\mathrm{ev}(v \otimes \kappa_1)\kappa_2 + \partial \mathrm{ev}(v \otimes \xi)\big)\big). \tag{7.11}$$

If we write $\clubsuit\xi = \tau_1 \otimes \tau_2 \in \Omega^{1,0} \otimes_A \Omega^{0,1}$ (sum implicit) then, as $\clubsuit v = 0$,

$$\begin{aligned}\clubsuit\partial \mathrm{ev}(v \otimes \xi) &= -\Theta^{1001}\overline{\partial}\partial \mathrm{ev}(v \otimes \xi) = \Theta^{1001}\partial\overline{\partial} \mathrm{ev}(v \otimes \xi) \\ &= \Theta^{1001}\partial(\mathrm{ev}(v \otimes \tau_1).\tau_2) = \partial \mathrm{ev}(v \otimes \tau_1) \otimes \tau_2 + \mathrm{ev}(v \otimes \tau_1).\Theta^{1001}\partial\tau_2,\end{aligned}$$

$$\clubsuit(\mathrm{ev}(v\otimes\kappa_1)\kappa_2) = -\Theta^{1001}\big(\overline{\partial}\mathrm{ev}(v\otimes\kappa_1)\wedge\kappa_2+\mathrm{ev}(v\otimes\kappa_1).\overline{\partial}\kappa_2\big)$$
$$= -\Theta^{1001}\big(\mathrm{ev}(v\otimes\beta_1).\beta_2\wedge\kappa_2+\mathrm{ev}(v\otimes\kappa_1).\overline{\partial}\kappa_2\big)$$

where $\clubsuit\kappa_1=\beta_1\otimes\beta_2$ (sum implicit). By (7.11), the LHS of (7.10) is

$$(\mathrm{ev}^{\langle 2\rangle}\otimes\mathrm{id})\big(u\otimes v\otimes\big(-\beta_1\otimes\Theta^{1001}(\beta_2\wedge\kappa_2)-\kappa_1\otimes\Theta^{1001}\overline{\partial}\kappa_2+\tau_1\otimes\Theta^{1001}\partial\tau_2\big)\big)$$
$$+(\mathrm{ev}\otimes\mathrm{id})\big(u\otimes\partial\mathrm{ev}(v\otimes\tau_1)\otimes\tau_2\big)$$

whereas the RHS of (7.10) becomes

$$(\mathrm{ev}\otimes\mathrm{id})\big(u\otimes\big(\mathrm{ev}(v\otimes\alpha_1)\alpha_2+\partial\mathrm{ev}(v\otimes\tau_1)\big)\otimes\tau_2\big),$$

where $\partial\tau_1=\alpha_1\wedge\alpha_2$ for $\alpha_1,\alpha_2\in\Omega^{1,0}$ (sum implicit). Hence to verify (7.10), we need to show the vanishing of

$$(\mathrm{ev}^{\langle 2\rangle}\otimes\mathrm{id})\big(u\otimes v\otimes\big(-\beta_1\otimes\Theta^{1001}(\beta_2\wedge\kappa_2)-\kappa_1\otimes\Theta^{1001}\overline{\partial}\kappa_2$$
$$+\tau_1\otimes\Theta^{1001}\partial\tau_2-\alpha_1\otimes\alpha_2\otimes\tau_2\big)\big)$$

and, as $u\otimes v$ is antisymmetric, this is implied by the vanishing of

$$-\beta_1\wedge\Theta^{1001}(\beta_2\wedge\kappa_2)-\kappa_1\wedge\Theta^{1001}\overline{\partial}\kappa_2+\tau_1\wedge\Theta^{1001}\partial\tau_2-\alpha_1\wedge\alpha_2\otimes\tau_2$$

as an element of $\Omega^{2,0}\otimes_A\Omega^{0,1}$. By the factorisation condition, this is the same as requiring the vanishing of

$$-\beta_1\wedge\beta_2\wedge\kappa_2-\kappa_1\wedge\overline{\partial}\kappa_2+\tau_1\wedge\partial\tau_2-\alpha_1\wedge\alpha_2\wedge\tau_2\in\Omega^{2,1}$$

which holds given previous expressions as

$$\overline{\partial}\kappa_1\wedge\kappa_2-\kappa_1\wedge\overline{\partial}\kappa_2+\tau_1\wedge\partial\tau_2-\partial\tau_1\wedge\tau_2=\overline{\partial}(\kappa_1\wedge\kappa_2)-\partial(\tau_1\wedge\tau_2)$$
$$=\overline{\partial}(\partial\xi)-\partial(-\overline{\partial}\xi)=0.$$

This proves (7.10). From $(\mathrm{id}\otimes\mathrm{ev})(\clubsuit[\![u,v]\!]_R\otimes\xi)=0$ and the fgp property, we conclude that $\clubsuit[\![u,v]\!]_R=0$. □

As in Chap. 6, we use $\mathbb{C}_q[S^2]$ to illustrate calculations with vector fields.

Example 7.35 On $\mathbb{C}_q[S^2]$, the right $\overline{\partial}$-operator $\clubsuit:\Omega^{1,0}\to\Omega^{1,0}\otimes_A\Omega^{0,1}$ is

$$\clubsuit(e^+p)=\Theta^{1001}(e^+\wedge\overline{\partial}(p))=e^+D_1D_1'\otimes D_2'D_2\overline{\partial}(p)$$

for $|p| = -2$, using the $D_1 \otimes D_2 = d \otimes a - qb \otimes c$ notation of Proposition 2.36. Similarly to Example 6.31, we introduce a basis f_+ and f_- for vector fields on $\mathbb{C}_q[S^2]$ dual to e^+ and e^- respectively. Then the dual $\overline{\partial}$-operator on $\mathfrak{X}^{1,0}$ is characterised by

$$(\mathrm{ev} \otimes \mathrm{id})(kf_+ \otimes \clubsuit(e^+ p)) + (\mathrm{id} \otimes \mathrm{ev})(\clubsuit(kf_+) \otimes e^+ p) = \overline{\partial} k.p + k.\overline{\partial} p$$

for $|k| = 2$, giving

$$\clubsuit(kf_+) = \overline{\partial} k.D_1 D_1' \otimes D_2' D_2 f_+.$$

Thus, the $\overline{\partial}$-operator on $\mathfrak{X}^{1,0}$ can be viewed as the usual $\overline{\partial}$-operator on E_{-2} and then from Proposition 7.36 applied to this we see that

$$a^2 f_+, \quad acf_+, \quad c^2 f_+.$$

is a linear basis for the holomorphic vector fields. We take

$$q^4 a^2 f_+ \otimes c^2 f_+ - c^2 f_+ \otimes a^2 f_+ \in \Lambda^2 \mathfrak{X}_{\mathrm{hol}}$$

and calculate the Lie bracket by evaluating this against $\xi = e^+ p \in \Omega^{1,0}$ for $p \in \mathbb{C}_q[SU_2]$ with $|p| = -2$. As $\partial \xi = 0$, we have

$$\begin{aligned}
&[\![q^4 a^2 f_+ \otimes c^2 f_+ - c^2 f_+ \otimes a^2 f_+]\!]_R(\xi) \\
&\qquad = \mathrm{ev}(q^4 a^2 f_+ \otimes \partial \mathrm{ev}(c^2 f_+ \otimes \xi)) - \mathrm{ev}(c^2 f_+ \otimes \partial \mathrm{ev}(a^2 f_+ \otimes \xi)) \\
&\qquad = \mathrm{ev}(q^4 a^2 f_+ \otimes \partial(c^2 p)) - \mathrm{ev}(c^2 f_+ \otimes \partial(a^2 p)) \\
&\qquad = q^4 \mathrm{ev}(a^2 f_+ \otimes \partial(c^2)) p - \mathrm{ev}(c^2 f_+ \otimes \partial(a^2)) p
\end{aligned}$$

as the terms with ∂p cancel. Using $\partial(c^2) = [2]_{q^2} e^+ dc$ and $\partial(a^2) = q[2]_{q^2} e^+ ab$,

$$q^4 [2]_{q^2} a^2 dcp - q[2]_{q^2} c^2 abp = q^4 [2]_{q^2} a^2 dcp - q^3 [2]_{q^2} acbcp = q^4 [2]_{q^2} acp$$

so we deduce that $[\![q^4 a^2 f_+ \otimes c^2 f_+ - c^2 f_+ \otimes a^2 f_+]\!]_R = q^4 [2]_{q^2} acf_+$. ◇

7.4 The Borel–Weil–Bott Theorem and Other Topics

We conclude with some miscellaneous results in the noncommutative holomorphic theory, centred on examples.

7.4.1 Positive Line Bundles and the Borel–Weil–Bott Theorem

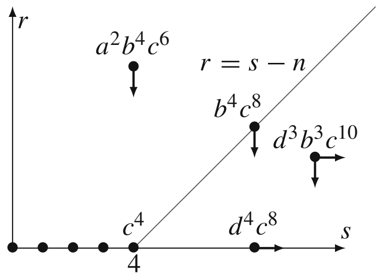
We will not attempt to provide the general theory of positive line bundles nor the full Borel–Weil–Bott Theorem here, but we explore these for the q-sphere $\mathbb{C}_q[S^2]$ with its complex structure going back to Proposition 2.35. We know from Example 7.17 that the line modules $E_{\pm n}$ described in Example 5.51 as grade $\pm n$ components of $\mathbb{C}_q[SU_2]$ are holomorphic. Now we ask whether they have holomorphic sections.

Proposition 7.36 *Let q be generic. The holomorphic module $(E_{-n}, \overline{\partial}_{E_{-n}})$ on $\mathbb{C}_q[S^2]$ has $\mathrm{H}^0(E_{-n}, \overline{\partial}_{E_{-n}}) = 0$ for $n < 0$ and is otherwise $n+1$-dimensional, with basis $\{a^n, a^{n-1}c, \ldots, ac^{n-1}, c^n\}$.*

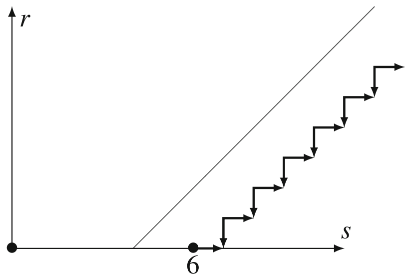
Proof From the differential calculus in Exercise 2.32 we have a formula for $\overline{\partial}_{E_{-n}}$,

$$\overline{\partial}_{E_{-n}}(a^m b^r c^s) = q^{1+s-r}[r]_{q^2} a^{m+1} b^{r-1} c^s e^-,$$
$$\overline{\partial}_{E_{-n}}(d^m b^r c^s) = q^{s-r}\big(q^{2-2m}[m+r]_{q^2} d^{n-1} b^r c^{s+1} + q[r]_{q^2} d^{m-1} b^{r-1} c^s\big)e^-,$$
$$\overline{\partial}_{E_{-n}}(b^r c^s) = q^{1-r+s}[r]_{q^2} a b^{r-1} c^s e^-,$$

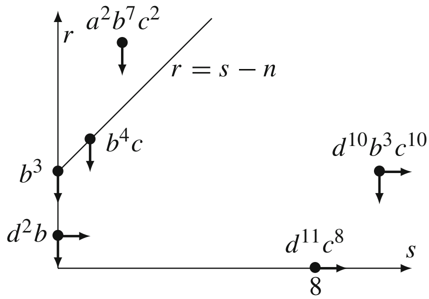
where $m \geq 1$ and $r, s \geq 0$. For generic q, $[r]_{q^2} = 0$ only when $r = 0$ and $[m + r]_{q^2} \neq 0$ given our range of values. We have the following picture for the $\overline{\partial}_{E_{-n}}$:



This plots the value (r, s) for a monomial $b^r c^s$, with appropriate a or d multiplied to make grade n. The diagonal line $r = s - n$ is where there is no a or d factor. The little arrows have length 1 and indicate the monomial location after applying $\overline{\partial}_{E_{-n}}$. We show the case $n = 4$ and the 5 points here with no arrows indicate that $\overline{\partial}_{E_{-n}}$ vanishes on the corresponding monomials $\{a^4, a^3c, a^2c^2, ac^3, c^4\}$. Observe that for a linear combination of terms to give zero under the $\overline{\partial}_{E_{-n}}$, every point entering with nonzero coefficient must have the ends of its arrows meeting another arrow end to be cancelled. Apart from the $m + 1$ points with no arrows, there is only one other type of possibility, leading to the following 'staircase' shown here starting at d^2c^6,



This staircase is infinite because if it is truncated then it would have an unmatched end of an arrow. Thus it cannot correspond to an element of the vector space with the given basis. This proves the cohomology for $n \geq 0$. Now consider the $n < 0$ case, shown here with $n = -3$.



We can see that there are no points without arrows, hence there is no way to make a finite combination of basis vectors which is in the kernel of $\overline{\partial}_{E_{-n}}$. □

To connect this to standard notations in algebraic geometry, we consider $\mathcal{O}(n)$ to be E_{-n} and recall that classically the sheaf $\mathcal{O}(n)$ has no holomorphic sections for $n < 0$. Also classically, the notion of the positivity of a line bundle is defined in terms of the positivity of an inner product provided by the curvature. We make a similar definition for certain of holomorphic line modules in noncommutative geometry. Recall from Corollary 4.20 that for a line module L with a left extendable connection, if the curvature is a right module map then it is of the form $\omega_L \otimes : L \to \Omega^2 \otimes_A L$. If L is also holomorphic for the connection, then $\omega_L \in \Omega^{1,1}$ by Proposition 7.16.

Definition 7.37 Suppose that the integrable almost complex structure on A is factorisable and that a line module L with left connection has curvature given by

$\omega_L \otimes : L \to \Omega^2 \otimes_A L$, where $\omega_L \in \Omega^{1,1}$ is central. We say that L is *positive* if

$$\mathfrak{X}^{1,0} \otimes_A \overline{\mathfrak{X}^{1,0}} \xrightarrow{\mathrm{id}\otimes\kappa_L\otimes\mathrm{id}} \mathfrak{X}^{1,0} \otimes_A \Omega^{1,0} \otimes_A \overline{\Omega^{1,0}} \otimes_A \overline{\mathfrak{X}^{1,0}}$$
$$\xrightarrow{\mathrm{id}\otimes\Upsilon^{-1}} \mathfrak{X}^{1,0} \otimes_A \Omega^{1,0} \otimes_A \overline{\mathfrak{X}^{1,0} \otimes_A \Omega^{1,0}} \xrightarrow{\mathrm{ev}\otimes\overline{\mathrm{ev}}} A \otimes_A \overline{A} \xrightarrow{\cdot(\mathrm{id}\otimes\star^{-1})} A$$

gives a semi-inner product A-module according to Definition 4.77. Here κ_L is the image of the curvature ω_L under the map

$$\Omega^{1,1} \xrightarrow{\Theta^{1001}} \Omega^{1,0} \otimes_A \Omega^{0,1} \xrightarrow{\mathrm{id}\otimes\star} \Omega^{1,0} \otimes_A \overline{\Omega^{1,0}}$$

and commutes with elements of A.

We check that this is reasonable on the q-sphere.

Example 7.38 Classically, the line bundles $\mathcal{O}(n)$ are positive precisely for $n \geq 0$. We show that this is also the case for our line modules E_{-n} on $\mathbb{C}_q[S^2]$. From Example 5.51, we know that the curvature on E_{-n} is $R(f) = \omega_n \otimes f$, where $\omega_n = q^3[-n]_{q^2} e_+ \wedge e_-$. We obtain κ_n as in Definition 7.37 from the ω_n by

$$\omega_n \mapsto q^3[-n]_{q^2} e_+ D_1 D_1' \otimes D_2' D_2 e_- \mapsto -q^4[-n]_{q^2} e_+ D_1 D_1' \otimes \overline{e_+ (D_2' D_2)^*},$$

where $D_1 \otimes D_2 = d \otimes a - qb \otimes c$ and D' is another copy of D. To check positivity, we take $v \in \mathfrak{X}^{1,0}$ and apply the composition in Definition 7.37 to obtain

$$v \otimes \overline{v} \longmapsto -q^4[-n]_{q^2} \mathrm{ev}(v \otimes e_+ D_1 D_1')(\mathrm{ev}(v \otimes e_+ (D_2' D_2)^*))^*.$$

Now set $v = r f_+$, where f_+ is the dual basis element to e^+ in $\Omega^1(\mathbb{C}_q[SU_2])$ and $r \in \mathbb{C}_q[SU_2]$ has grade 2. Since $D_1 D_2 = 1$, we obtain

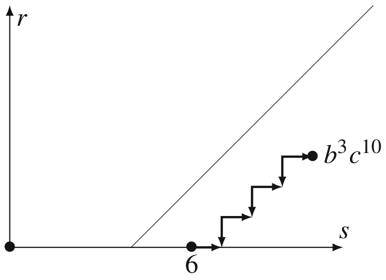
$$v \otimes \overline{v} \longmapsto -q^4[-n]_{q^2} r r^*.$$

Hence if $q \neq 0$ is real, we have a semi-inner product precisely when $n \geq 0$. ◇

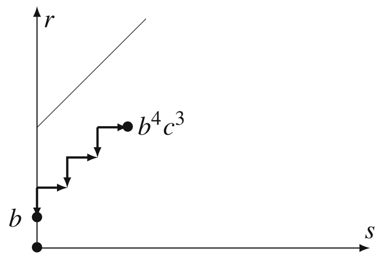
Classically, Kodaira's vanishing and embedding theorems deal with the consequences of positivity for holomorphic line bundles. Establishing reasonably general noncommutative analogues would be a significant part of implementing a noncommutative version of Serre's GAGA, but is beyond our scope. Here we simply verify that another well-known classical principle holds for our q-sphere example, namely Serre duality. To do this we need the H^1 version of Proposition 7.36.

Proposition 7.39 *For the holomorphic module $(E_{-n}, \overline{\partial}_{E_{-n}})$ on $\mathbb{C}_q[S^2]$ with generic q, $\mathrm{H}^1(E_{-n}, \overline{\partial}_{E_{-n}}) = 0$ for $n > -2$ and for $n \leq -2$ it is $-n-1$-dimensional, with basis $\{[b^{-n-2}e^-], [db^{-n-3}e^-], \ldots, [d^{-n-2}e^-]\}$.*

Proof We refer to the diagrams in Proposition 7.36, but this time we look at the image of $\overline{\partial}_{E_{-n}}$, so we consider the ends of the arrows, again only considering the $b^r c^s$ part of the basis (multiply by a's and d's as required). Every $b^r c^s$ with $r \geq s - n - 1$ is at the head of a single vertical arrow, and so is in the image. The two interesting cases are finite staircases entirely below the diagonal $r = s - n$. The first starts at c^s for $s \geq 0$, and looks like



In particular, this shows that every $b^r c^s$ (multiply by a's and d's as required) for $n \geq 0$ with $r < s - n - 1$ is the single terminus of a sequence of arrows, i.e., that it is in the image of $\overline{\partial}_{E_{-n}}$. In fact, the only times where the basis elements are not in the image are covered by our second case of interest, which only occurs when $n \leq -2$.



This sequence of arrows has two termini, so there is a nontrivial linear combination of b and b^4c^3 (add a's and d's as required) in the image, i.e., the cohomology class of b^4c^3 is a multiple of the cohomology class of b. Looking at all the possible staircases gives the result. □

To describe Serre duality, we first remind the reader about the cup product in cohomology described in the noncommutative case in Proposition 4.49. Writing E_{-n} as $\mathcal{O}(n)$ for clarity, this implies a cup product

$$\cup : \mathrm{H}^0(\mathcal{O}(n)) \otimes \mathrm{H}^1(\mathcal{O}(-2-n)) \to \mathrm{H}^1(\mathcal{O}(n) \otimes_A \mathcal{O}(-2-n)) = \mathrm{H}^1(\mathcal{O}(-2)) \cong \mathbb{C}.$$

The classical statement of Serre duality for $\mathbb{CP}^1$ is that this product gives a nondegenerate dual pairing between $\mathrm{H}^0(\mathcal{O}(n))$ and $\mathrm{H}^1(\mathcal{O}(-2-n))$ for all $n \geq 0$. From Propositions 7.36 and 7.39 for $\mathbb{C}_q[S^2]$, both these vector spaces are isomorphic to $\mathbb{C}^{n+1}$, so this is certainly possible on dimensional grounds. We now check Serre duality at least for $0 < q < 1$ by explicit calculation.

Proposition 7.40 *On $\mathbb{C}_q[S^2]$ with $0 < q < 1$ and $\mathcal{O}(n) = E_{-n}$, the cup product $\cup : \mathrm{H}^0(\mathcal{O}(n)) \otimes \mathrm{H}^1(\mathcal{O}(-2-n)) \to \mathrm{H}^1(\mathcal{O}(-2))$ is a nondegenerate dual pairing.*

Proof By previous results, we take a basis $\{[a^n], [ca^{n-1}], \dots, [c^n]\}$ of $\mathrm{H}^0(\mathcal{O}(n))$ and similarly a basis of $\mathrm{H}^1(\mathcal{O}(-2-n))$ given by $\{[d^n e^-], [d^{n-1}be^-], \dots, [b^n e^-]\}$. For the standard basis of $\mathbb{C}_q[SU_2]$, the only basis elements which do not give zero in $\mathrm{H}^1(\mathcal{O}(-2))$ are $[b^s c^s e^-]$. Thus the only possible nonzero pairings between the given bases of the cohomology are

$$[c^{n-r}a^r] \cup [d^r b^{n-r} e^-] = [c^{n-r}a^r d^r b^{n-r} e^-],$$

for $0 \le r \le n$, and if these are all nonzero then Serre duality is proven.

To make explicit the isomorphism $\mathrm{H}^1(\mathcal{O}(-2)) \cong \mathbb{C}$ we use

$$\overline{\partial}_{E_2}(db^{s+1}c^s) = q^{-1}\big([s+2]_{q^2} b^{s+1}c^{s+1} + q[s+1]_{q^2} b^s c^s\big)e^-,$$

and from this we get (up to normalisation, and not very surprisingly) the isomorphism as $[xe^-] \mapsto \int x$, where $\int$ is the Haar integral from Example 2.21. Thus we need to show that $\int c^{n-r}a^r d^r b^{n-r} \neq 0$. To do this, we assume that $0 < q < 1$ and use the 'Jackson integral' expression for the Haar integral of a polynomial $p(x)$,

$$\int p(x) = (1-q^2)\Big(p(1) + q^2 p(q^2) + q^4 p(q^4) + q^6 p(q^6) + \dots\Big),$$

where $x = -q^{-1}bc$. We also have

$$a^r d^r = (1+q^{-1}bc)(1+q^{-3}bc)\dots(1+q^{1-2r}bc) = (1-x)(1-q^{-2}x)\dots(1-q^{2-2r}x),$$

and setting $f(x)$ to be x^{n-r} times this product gives $\int f(x) > 0$. □

The main content of the classical Borel–Weil–Bott theorem is that the irreducible representations of a compact connected Lie group G are given by the holomorphic sections of line bundles over G/T, where T is the maximal torus of G. These line bundles are given by the weights of the representation. In addition, similar results are true about the higher cohomology of the line bundles. For $G = SU_2$, the maximal torus is the diagonal unitaries and G/T is the 2-sphere. In our q-sphere example, the group representation in question becomes a left coaction of $\mathbb{C}_q[SU_2]$ on $\mathrm{H}^0(\mathcal{O}(n))$ and $\mathrm{H}^1(\mathcal{O}(n))$. From Proposition 7.36, the nonzero holomorphic sections of the bundles have basis $\{a^n, a^{n-1}c, \dots, c^n\}$ and the left coaction is just given by application of the coproduct $\Delta a = a \otimes a + b \otimes c$, etc., on these basis elements. From Proposition 7.39, the nonzero first cohomology has $\{[b^{-n-2}e^-], [db^{-n-3}e^-], \dots, [d^{-n-2}e^-]\}$ and as e^- is left-invariant, the left coaction is just given by the coproduct on the elements $b^{-n-2}, \cdots, d^{-n-2}$. For example,

$$\Delta_L[dbe^-] = ca \otimes [b^2e^-] + (da + q^{-1}cb) \otimes [dbe^-] + db \otimes [d^2e^-].$$

The last topic we wish to explore on our q-spheres is the vanishing ideal from Definition 3.11 and the long exact sequences of holomorphic sheaf cohomology.

Example 7.41 On $A = \mathbb{C}_q[S^2]$ with generic q, let $e \in E_{-n}$ for some $n > 0$ be a nonzero section with $\overline{\partial} e = 0$ (this is possible as $\mathrm{H}^0(E_{-n}, \overline{\partial}_{E_{-n}}) \neq 0$). From Example 7.29, we have a short exact sequence of right holomorphic modules

$$0 \longrightarrow \mathrm{Van}(e) \longrightarrow \mathbb{C}_q[S^2] \longrightarrow \mathbb{C}_q[S^2]/\mathrm{Van}(e) \longrightarrow 0.$$

Swapping sides in Proposition 7.20, we have a long exact cohomology sequence (omitting the decoration on the $\overline{\partial}$-operators)

$$\mathrm{H}^0(\mathrm{Van}(e), \overline{\partial}) \longrightarrow \mathrm{H}^0(\mathbb{C}_q[S^2], \overline{\partial}) \longrightarrow \mathrm{H}^0(\mathbb{C}_q[S^2]/\mathrm{Van}(e), \overline{\partial}) \longrightarrow$$
$$\mathrm{H}^1(\mathrm{Van}(e), \overline{\partial}) \longrightarrow \mathrm{H}^1(\mathbb{C}_q[S^2], \overline{\partial}) \longrightarrow \mathrm{H}^1(\mathbb{C}_q[S^2]/\mathrm{Van}(e), \overline{\partial}) \longrightarrow 0.$$

From Example 7.28, we can fill in $\mathrm{H}^0(\mathbb{C}_q[S^2], \overline{\partial}) = \mathbb{C}$ and $\mathrm{H}^1(\mathbb{C}_q[S^2], \overline{\partial}) = 0$ and deduce that $\mathrm{H}^1(\mathbb{C}_q[S^2]/\mathrm{Van}(e), \overline{\partial}) = 0$.

If e is nowhere vanishing (i.e., there is an $\alpha \in E_{-n}{}^\flat$ such that $\mathrm{ev}(e \otimes \alpha) = 1 \in A$) then by Proposition 3.12, we must have $E_{-n} = A.e \oplus K$ where $K = \{v \in E : \alpha(v) = 0\}$. However, the dual of E_{-n} is E_n, so saying $\alpha(v) = 0$ means that there is a product of $\alpha \in E_n$ and $e \in E_{-n}$ which is zero in $\mathbb{C}_q[SU_2]$. Now we quote the fact that the latter has no zero divisors (i.e., if two things multiply to give zero then one must be zero) to deduce that $K = 0$. Hence $E_{-n} = A.e$, which is not possible as E_{-n} is not trivial. Thus e cannot be nowhere vanishing, i.e., $1 \notin \mathrm{Van}(e)$ and $\mathrm{H}^0(\mathrm{Van}(e), \overline{\partial}) = 0$ as the only holomorphic elements of A are constants. Then we have the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{H}^0(\mathbb{C}_q[S^2]/\mathrm{Van}(e), \overline{\partial}) \longrightarrow \mathrm{H}^1(\mathrm{Van}(e), \overline{\partial}) \longrightarrow 0$$

and deduce that $\mathrm{H}^0(\mathbb{C}_q[S^2]/\mathrm{Van}(e), \overline{\partial}) \neq 0$. ◇

7.4.2 A Representation of a Noncommutative Complex Plane

The quantum plane $A = \mathbb{C}_q[\mathbb{C}^2]$ was described just before Proposition 2.13, but here we write it with generators $z, \bar{z}$ and relations $z\bar{z} = q\bar{z}z$, with q real and $*$-operation $z^* = \bar{z}$. This extends to a $*$-differential calculus with a real parameter p, which we take with the restriction $0 < p$ and $p \neq 1$, and relations

$$z\mathrm{d}z = p(\mathrm{d}z)z, \quad z\mathrm{d}\bar{z} = q(\mathrm{d}\bar{z})z, \quad \mathrm{d}z \wedge \mathrm{d}\bar{z} = -q\mathrm{d}\bar{z} \wedge \mathrm{d}z, \quad \mathrm{d}z \wedge \mathrm{d}z = \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{z} = 0.$$

We take $\Omega^{1,0}$, $\Omega^{0,1}$ to be generated over A by $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ respectively so as to form the Dolbeault complex of a factorisable integrable almost complex structure with $\mathrm{d} = \overline{\partial} + \partial$. We take right-handed partial derivatives $\partial_{\bar{z}} : A \to A$ and $\partial_z : A \to A$ defined by $\overline{\partial} f = \mathrm{d}\bar{z}.\partial_{\bar{z}} f$ and $\partial f = \mathrm{d}z.\partial_z f$.

Next we define an A-module $E = \mathbb{C}\mathbb{Z} \otimes C_c((0, \infty))$ by left action

$$z{\triangleright}(s^n \otimes g(r)) = q^{-(n+1)/2} s^{n+1} \otimes rg(r), \quad \bar{z}{\triangleright}(s^n \otimes g(r)) = q^{-n/2} s^{n-1} \otimes rg(r),$$

where $\mathbb{C}\mathbb{Z}$ has basis s^n for $n \in \mathbb{Z}$ and $g \in C_c((0, \infty))$ is a function on $(0, \infty)$ with compact support. If we write $\underline{g}(r) = \sum s^n \otimes g_n(r)$ as the column vector transpose of $(\cdots, g_2(r), g_1(r), g_0(r), g_{-1}(r), g_{-2}(r), \cdots)$ then

$$z{\triangleright} = r \begin{pmatrix} & & \vdots & & \\ 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 & 0 \\ .\,0 & 0 & 0 & 1 & 0\ . \\ 0 & 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & \end{pmatrix}, \quad \bar{z}{\triangleright} = r \begin{pmatrix} & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 & 0 \\ .\ 0 & q^{-1/2} & 0 & 0 & 0\,. \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} & 0 \\ & & \vdots & & \end{pmatrix}$$

acting as a matrix product on column vectors. There is also a hermitian inner product $\langle\ ,\ \rangle : \overline{E} \otimes E \to C_c((0, \infty))$ given by

$$\langle \overline{\underline{h}(r)}, \underline{g}(r) \rangle = \sum_n h_n(r)^* g_n(r),$$

which descends to a hermitian inner product $\langle\ ,\ \rangle : \overline{E} \otimes_A E \to C_c((0, \infty))$ since the above matrices for z and $\bar{z}$ are conjugate transposes.

We write a connection $\nabla_E : E \to \Omega^1 \otimes_A E$ as

$$\nabla_E e = \mathrm{d}z \otimes \partial_{z,E} e + \mathrm{d}\bar{z} \otimes \partial_{\bar{z},E} e,$$

in terms of linear maps $\partial_{z,E} : E \to E$ and $\partial_{\bar{z},E} : E \to E$ obeying certain commutation relations between $z, \bar{z}$ and $\partial_{z,E}, \partial_{\bar{z},E}$ as operators on E. To find these, we deduce from the Leibniz rule for ∇_E that

$$\begin{aligned} \mathrm{d}z \otimes \partial_{z,E}(z.e) + \mathrm{d}\bar{z} \otimes \partial_{\bar{z},E}(z.e) = \nabla_E(z.e) &= \mathrm{d}z \otimes e + z.\nabla_E(e) \\ &= \mathrm{d}z \otimes e + z.\mathrm{d}z \otimes \partial_{z,E}(e) + z.\mathrm{d}\bar{z} \otimes \partial_{\bar{z},E}(e) \\ &= \mathrm{d}z \otimes e + \mathrm{d}z \otimes pz.\partial_{z,E}(e) + \mathrm{d}\bar{z} \otimes qz.\partial_{\bar{z},E}(e), \end{aligned}$$

and similarly

$$\mathrm{d}z \otimes \partial_{z,E}(\bar{z}.e) + \mathrm{d}\bar{z} \otimes \partial_{\bar{z},E}(\bar{z}.e)$$
$$= \mathrm{d}\bar{z} \otimes e + \mathrm{d}z \otimes q^{-1}\bar{z}.\partial_{z,E}(e) + \mathrm{d}\bar{z} \otimes p^{-1}\bar{z}.\partial_{\bar{z},E}(e).$$

These identities require us to impose commutation relations for $\partial_{\bar{z},E}$ and $\partial_{z,E}$,

$$\partial_{\bar{z},E} z - qz\partial_{\bar{z},E} = 0, \quad \partial_{\bar{z},E}\bar{z} - p^{-1}\bar{z}\partial_{\bar{z},E} = \mathrm{id},$$
$$\partial_{z,E} z - pz\partial_{z,E} = \mathrm{id}, \quad \partial_{z,E}\bar{z} - q^{-1}\bar{z}\partial_{z,E} = 0.$$

It is not hard to solve these, for example, we can take $\partial_{\bar{z},E}\underline{g}(r)$ to be

$$\frac{p}{(1-p)r}\left(p^{-1/2}\begin{pmatrix} & & \vdots & & \\ 0 & pq & 0 & 0 & 0 \\ 0 & 0 & (pq)^{1/2} & 0 & 0 \\ .\ 0 & 0 & 0 & 1 & 0\ . \\ 0 & 0 & 0 & 0 & (pq)^{-1/2} \\ 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & \end{pmatrix}\underline{g}(p^{-1/2}r) - \begin{pmatrix} & & \vdots & & \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 & 0 \\ .\ 0 & 0 & 0 & 1 & 0\ . \\ 0 & 0 & 0 & 0 & q^{-1/2} \\ 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & \end{pmatrix}\underline{g}(r)\right)$$

and $\partial_{z,E}\underline{g}(r)$ to be

$$\frac{1}{(p-1)r}\left(\begin{pmatrix} & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 \\ pq & 0 & 0 & 0 & 0 \\ .\ 0 & (pq)^{1/2} & 0 & 0 & 0\ . \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (pq)^{-1/2} & 0 \\ & & \vdots & & \end{pmatrix}\underline{g}(p^{1/2}r) - \begin{pmatrix} & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 \\ .\ 0 & q^{1/2} & 0 & 0 & 0\ . \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1/2} & 0 \\ & & \vdots & & \end{pmatrix}\underline{g}(r)\right)$$

The operator $\partial_{\bar{z},E}$ gives a holomorphic structure on E. For the above example, if $\partial_{\bar{z},E}\underline{g}(r) = 0$ then $g_n(p^{-1/2}r) = p^{-n/2}g_n(r)$ for all $n \in \mathbb{Z}$, which cannot happen for a function of compact support on $(0, \infty)$ unless it is zero. So zero is the only holomorphic section (an outcome which would be different if we had chosen polynomials or Laurent polynomials). This example also has ∇_E flat,

$$R_E = \mathrm{d}z \wedge \mathrm{d}\bar{z} \otimes (q^{-1}\partial_{z,E}\partial_{\bar{z},E} - \partial_{\bar{z},E}\partial_{z,E}) = 0.$$

Finally, we show that $\mathrm{H}^2(E, \nabla_E) \neq 0$ for the above connection. To do this we will construct a linear map $\tilde{\phi} : \mathrm{H}^2(E, \nabla_E) \to \mathbb{C}$ by starting with a linear map $\phi : E \to \mathbb{C}$ and the formula $\tilde{\phi}[\mathrm{d}z \wedge \mathrm{d}\bar{z}.e] = \phi(e)$. For $\tilde{\phi}$ to be well defined on the cohomology, we need $\phi(\partial_{z,E}\underline{g}(r)) = \phi(\partial_{\bar{z},E}\underline{g}(r)) = 0$ for all $\underline{g}(r) \in E$, or

$$\phi(s^{n+1}r^{-1}(p^{n/2}g_n(p^{-1/2}r) - g_n(r))) = \phi(s^{n-1}r^{-1}(p^{n/2}g_n(p^{1/2}r) - g_n(r))) = 0$$

for all $g_n(r) \in C_c((0, \infty))$. Writing $f(r) = r^{-1}g_n(r)$, this becomes

$$\phi(s^{n+1}(p^{(n-1)/2}f(p^{-1/2}r) - f(r))) = \phi(s^{n-1}(p^{(n+1)/2}f(p^{1/2}r) - f(r))) = 0,$$

and so for all n and f,

$$\phi(s^n f(r)) = p^{(n+2)/2}\phi(s^n f(p^{1/2}r)) = p^{(2-n)/2}\phi(s^n f(p^{1/2}r)).$$

We deduce that $\phi(s^n f(r)) = 0$ for all f unless $n = 0$. We suppose that the remaining $\phi(s^0 f(r))$ is given by an integral

$$\phi(s^0 f(r)) = \int_0^\infty f(r)\mu(r)\mathrm{d}r,$$

with respect to a density times Lebesgue measure, in which case we require

$$\int_0^\infty f(r)\mu(r)\mathrm{d}r = p\int_0^\infty f(p^{1/2}r)\mu(r)\mathrm{d}r.$$

An obvious way to achieve this (after changing variables on the left-hand side) is to take μ such that $\mu(p^{1/2}r) = p^{1/2}\mu(r)$, for example $\mu(r) = r$. We see that we can construct $\tilde{\phi}$ with nonzero values and hence that $\mathrm{H}^2(E, \nabla_E) \neq 0$.

Exercises for Chap. 7

E7.1 Consider the canonical $\Omega(\mathbb{Z}_3)$ for the Cayley graph calculus for $\mathcal{C} = \{1, 2\} \subset \mathbb{Z}_3$ as in E3.8 (so the e_1, e_2 are closed and anticommute). Define a left module map $J : \Omega^1 \to \Omega^1$ by $J(e_1) = \alpha e_1 + \beta e_2$ and $J(e_2) = \gamma e_1 + \delta e_2$, for $\alpha, \beta, \gamma, \delta \in A$ and require that $J^2 = -\mathrm{id}$. Show that if J is a bimodule map then $\beta = \gamma = 0$. Show that if J also extends to Ω^2 as a derivation then $\alpha + R_1(\delta)$ is a multiple of the identity. Show that if J is also integrable then either both α, δ are constant or $R_1(\delta) = -\alpha$. Show that if J also commutes with $*$ then $R_1(\delta) = -\alpha$, giving this and $\alpha^2 = 1$, $\beta = \gamma = 0$ as the conditions for an integrable almost complex structure.

E7.2 Let $A = \mathbb{C}(Q_8)$ where the quaternion group $Q_8 = \{g_{\pm 1}, g_{\pm i}, g_{\pm j}, g_{\pm k}\}$ has product given by the quaternions, e.g., $g_k g_j = g_{kj} = g_{-i}$. Let $\mathcal{C}^{1,0} = \{g_i, g_j\}$, $\mathcal{C}^{0,1} = \{g_{-i}, g_{-j}\}$ and $\mathcal{C} = \mathcal{C}^{1,0} \cup \mathcal{C}^{0,1}$ with the corresponding first-order calculi $(\Omega^{1,0}, \partial)$, $(\Omega^{0,1}, \bar{\partial})$ and (Ω^1, d). Writing $e_{\pm i} = e_{g_{\pm i}}$ for brevity for the associated left-invariant 1-forms, define the exterior algebra $\Omega(Q_8)$ by

$$\sum_{a,b \in \mathcal{C}: ab = x} e_a \wedge e_b = 0, \quad \mathrm{d}e_a = \sum_{b \in \mathcal{C}} (e_b \wedge e_a + e_a \wedge e_b)$$

for all $x \in Q_8$ (this is a further quotient of the inner calculus in Corollary 1.54). Write down the bimodule relations for Ω^1 and show that we have to impose the additional relations

$$e_{-i} \wedge e_{-i} + e_{-j} \wedge e_{-j}, \quad e_i \wedge e_i + e_j \wedge e_j, \quad e_i \wedge e_j, \quad e_{-i} \wedge e_{-j},$$
$$e_j \wedge e_{-i} + e_{-j} \wedge e_i, \quad e_{-i} \wedge e_j + e_i \wedge e_{-j}, \quad e_{-j} \wedge e_{-i}, \quad e_j \wedge e_i$$

to have an integrable almost complex structure. Show on the resulting calculus that the factorisation condition in Definition 7.21 does not hold.

E7.3 For the maximal prolongation calculus on $A = \mathbb{C}(S_4)$ in E2.10, show that the subsets $\mathcal{C}^{1,0} = \{(01), (23)\}$ and $\mathcal{C}^{0,1} = \{(12), (30)\}$ of $\mathcal{C}$ define $\Omega^{1,0}$ and $\Omega^{0,1}$ respectively satisfying all the conditions for an integrable almost complex structure except for the $*$-condition. Now use the exotic star operation given by the bar category in E2.10 to show that we get an integrable almost complex structure in the bar category, and find a nonconstant holomorphic $a \in A$. [The latter is not possible for Ω^1 connected with the usual $*$-structure on $\mathbb{C}(S_4)$.]

E7.4 Consider the quantum plane $\mathbb{C}_q[\mathbb{C}^2]$ with generators $z, \bar{z}$ and the calculus described in §7.4.2 with $p = 1$, and similarly $\mathbb{C}_{q^4}[\mathbb{C}^2]$ with generators $w, \bar{w}$. Show that there is a $*$-algebra map $\psi : \mathbb{C}_{q^4}[\mathbb{C}^2] \to \mathbb{C}_q[\mathbb{C}^2]$ defined by $\psi(w) = z^2$ and $\psi(\bar{w}) = \bar{z}^2$. Now localise this by adjoining inverses of all the generators to get the algebras $A = \mathbb{C}_q[\mathbb{C}^2][z^{-1}, \bar{z}^{-1}]$ and $B = \mathbb{C}_{q^4}[\mathbb{C}^2][w^{-1}, \bar{w}^{-1}]$. [The reader may recognise here the construction of a noncommutative Riemann surface for the square root function on the punctured complex plane, since as a $*$-algebra $\mathbb{C}[\mathbb{C}^2]$ corresponds to the variety $\mathbb{C}$.] Show that the formula

$$\bar{\partial}_{\psi A}(\bar{z}^n z^m) = \frac{n}{2} \mathrm{d}\bar{w} \otimes \bar{z}^{n-2} z^m$$

defines a holomorphic structure on the left B-module ${}_\psi A$ given by applying ψ and left multiplication in A. What should B be if we want a higher power $*$-algebra map $\psi(w) = z^n$ for $n > 2$?

E7.5 Let A be a real coquasitriangular Hopf $*$-algebra and recall the definition of $A \bowtie A$ in Example 2.111. Suppose that A has a bicovariant $*$-calculus extended to Ω_A as a super $*$-Hopf algebra. Define $\Omega_{A \bowtie A} = \Omega_A \bowtie \Omega_A$ by an identical construction as in Example 2.111 but with $\mathcal{R}$ extended as zero on forms. Show that this provides an integrable almost complex structure on $A \bowtie A$, giving the differentials and wedge product in terms of those for A. [You may also like to show that $\Omega_{A\bowtie A}$ satisfies the factorisation condition in Definition 7.21 one way round; the other way is harder but also true.]

Notes for Chap. 7

There are many excellent works on classical complex manifolds and their relation to algebraic geometry, for example [128, 145, 320]. In noncommutative algebraic geometry, there is a large literature in terms of categories of modules and quasi-coherent sheaves, see Stafford & Van den Bergh's survey article [307], which highlights that many results of classical algebraic geometry generalise to that setting. Our approach in this chapter is somewhat different and based on [38] by Smith and the first author, which in turn built on and generalised the $\mathbb{C}_q[S^2]$ example from [216] by the second author, in particular the bigraded exterior algebra or 'Dolbeault double complex' introduced there as an application of frame bundles. Another work proceeding from the double complex as in [216] to complex structures was [159] by Khalkali, Landi & van Suijklekom. Several papers containing interesting examples include [61, 81, 96, 97, 99, 160, 165, 216, 282, 318].

The integrability condition in §7.1 is a straightforward generalisation of the differential form version of the classical Newlander–Nirenberg integrability condition in [267]. The n-dimensional noncommutative torus C^*-algebra $\mathbb{C}_\theta[\mathbb{T}^n]$ was described in [287] and now provides the key Example 7.7. The $*$-differential calculus on $\mathbb{C}_\theta[\mathbb{C}^n]$ defined in Example 7.8 is taken from [244]. Example 7.11 is motivated by the work by Polishchuk & Schwarz on holomorphic structures on noncommutative tori [282], but stated from our differential calculus point of view. The action of $SL(2,\mathbb{Z})$ by automorphisms of $\mathbb{C}_\theta[\mathbb{T}^2]$ was given in [55, 322]. Stereographic projection was described in antiquity by Ptolemy [303]. For the algebraic method of localisation as well as the fact on zero divisors needed in Example 7.41, see Goodearl & Warfield [125] in general and [134].

§7.2 includes aspects of the Koszul–Malgrange theorem, see [145] for the classical case. Classically, if the conditions for Hodge theory are satisfied then the Frölicher spectral sequence for a complex manifold converges at the first page (see [320]). The factorisability condition for noncommutative complex calculi in Definition 7.21 was introduced in [137, 270]. For Serre's famous work *Géométrie algébrique et géométrie analytique* see [300]. Various results such as Kodaira's embedding theorem and Chow's Theorem underlie it.

For the classical Borel–Weil–Bott theorem, see [49, 98]. An early version of the theorem for $\mathbb{C}_q[S^2]$ in terms of holomorphic sections was in [216], while our exposition of the cohomological version in §7.4.1 is new. A different proof of the H^0 part of this can also be found in [159]. A Borel–Weil–Bott theorem for higher-dimensional quantum Grassmannians is in [260]. For a view of Serre duality from the point of noncommutative algebraic geometry, see [328]. The Jackson integral used in the proof of Proposition 7.40 can be found in [146]. For the existence and uniqueness proofs for the Haar measure, see [326], and later [95] under less restrictive conditions, showing that it is positive but not giving an explicit formula.

While we have focussed on the q-sphere as q-deformed $\mathbb{CP}^1$, the same approach works for $\mathbb{CP}^{n-1}$ as a quantum homogeneous space for $\mathbb{C}_q[SU_n]$. The quantum principal bundle for this was given in [252], with faithful flatness of $\mathbb{C}_q[SU_n]$ as a module over $\mathbb{C}_q[\mathbb{CP}^{n-1}]$ established in [261]. The full picture was obtained by Ó Buachalla in [270, 271] as a calculus on $\mathbb{C}_q[SU_n]$ which restricts to the Heckenberger & Kolb calculus [136] on quantum complex projective space in just the same way that Woronowicz's 3D calculus $\mathbb{C}_q[SU_2]$ restricts (by taking the weakly horizontal invariant forms) to a calculus on $\mathbb{C}_q[S^2]$. In fact, Heckenberger & Kolb gave complex structures more generally on quantum group analogues of irreducible generalised flag manifolds [137, 138].

Various authors have examined quantisations of complex projective space from other points of view [53, 96, 97, 159–161]. Of these, [53] introduced quantised projective spaces by cocycle twist in the sense of §9.6.3. The differential calculus and path integral on the quantum plane were investigated in [61, 66, 323]. The q-Heisenberg algebra is described in [166], among other places.

The exercises are small new examples aside from the general result in E7.5, which uses a super/exterior algebra version of the double cross product $A{\bowtie}A$ of a coquasitriangular Hopf algebra with itself from [201]. When $A = \mathbb{C}_q[SU_2]$, the double cross product is a version of the q-Lorentz group (which is a kind of complexification). Some related results are in [13] and the appendix of [219].

Chapter 8
Quantum Riemannian Structures

We have now met all the ingredients required for noncommutative Riemannian geometry and in this chapter we bring them together and compute basic examples. It is possible to read this chapter directly after Chap. 1 with the intervening chapters as reference.

First and foremost, after we have fixed an algebra A in the role of 'functions' on a possibly noncommutative space and an exterior algebra (Ω, d) on it, we need a quantum metric

$$g \in \Omega^1 \otimes_A \Omega^1.$$

This was already explored in Chap. 1, where we saw that we need it to be central when we multiply from either side using the bimodule structure of Ω^1, nondegenerate in the sense of an inverse bimodule map $(\,,\,) : \Omega^1 \otimes_A \Omega^1 \to A$. We also typically ask for it to be *quantum symmetric* in the sense

$$\wedge(g) = 0 \tag{8.1}$$

for the exterior product $\wedge : \Omega^1 \otimes_A \Omega^1 \to \Omega^2$. If we do not impose quantum symmetry then we will say that g is a *generalised metric*. We are usually interested in the case working over $\mathbb{C}$ and will say that a metric or generalised metric is 'real' if

$$g^\dagger = g, \tag{8.2}$$

where $\dagger = \mathrm{flip}(* \otimes *)$ applies $*$ and then swaps tensor factors. We adopt the superscript notation $g^\dagger = \dagger \circ g$. We have already seen, but one can check, that $\dagger$ is well defined as an antilinear map on $\Omega^1 \otimes_A \Omega^1$, indeed $(\omega \otimes a\eta)^\dagger = \eta^* a^* \otimes \omega^* = \eta^* \otimes a^*\omega^* = (\omega a \otimes \eta)^\dagger$. This reality condition on g implies that the metric coefficients would be hermitian in any basis where the basis elements are hermitian

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and central, hence real in the classical limit if the quantum symmetry condition reduces to the classical symmetry. The reality requirement (8.2) is equivalent in terms of the inverse metric inner product $(\ ,\ )$ to $(\omega, \eta)^* = (\eta^*, \omega^*)$, as we explained in Chap. 1, or more formally,

$$* \circ (\ ,\ ) = (\ ,\ ) \circ \dagger. \tag{8.3}$$

We also define

$$\underline{\dim} = (\ ,\ )(g) \tag{8.4}$$

as the 'quantum metric dimension' of the Riemannian geometry. It is a self-adjoint element of A in the sense that $\underline{\dim}^* = \underline{\dim}$ when the metric obeys its reality condition and is typically a constant (we will see that it is in the kernel of d in nice cases). Classically it would be the dimension of the manifold. We can similarly 'quantum metric trace' any right-module operator on Ω^1, as we saw in Corollary 4.17 for the curvature when it is a right module map.

One can already begin to do some noncommutative geometry at this level in the form of an induced Laplace–Beltrami operator (as we saw in Chap. 1) but for a fuller picture we need a quantum Levi-Civita connection. We will develop this particularly in a bimodule connection approach but also relate back to frame bundles and other constructions that we have seen. Compared to usual texts in Riemannian geometry, the main features are that we work with differential forms rather than vector fields and we do everything at an abstract hence 'coordinate free' algebraic level that does not assume a local trivialisation. Tensor calculus means tensor products over A as the classical analogue of pointwise tensor products, and we will shall be explicit about this. Our theory will include a constructive approach to the Ricci tensor, although it is not the last word from an abstract point of view. Some applications to quantum spacetime and hence to the modelling of quantum gravity effects are in Chap. 9.

§8.3 considers wave operators or Laplace–Beltrami operators as a separate topic that bypasses the full quantum Riemannian geometry. §8.4 considers hermitian inner products on modules. This is similar to the use of Hilbert C^*-modules in §4.5.2. For some purposes when working with hermitian inner products (such as for metric compatible connections), it is no longer necessary to have central metrics. Hermitian inner products are used in §8.5 on spectral triples (through the involvement of Hilbert spaces) and in §8.6 on Chern connections and complex structures. Spectral triples were famously introduced by Connes as an axiomatisation of the notion of a Dirac operator and have formed an important part of the study of noncommutative geometry from the C^* and operator algebra approach. Adjacent to this larger body of work, we will focus on the problem of the 'geometric realisation' of spectral triples using differential calculi and connections in the manner of the classical construction of the Dirac operator.

8.1 Bimodule Quantum Levi-Civita Connections

Let (A, Ω, d) be an exterior algebra in the sense of Chap. 1, specified at least to degree 2. We have already met the notion of a connection on a general A-module in Chap. 3 but now we focus exclusively on connections $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ on Ω^1. As usual, a left connection obeys a left Leibniz rule

$$\nabla(a\eta) = \mathrm{d}a \otimes \eta + a\nabla\eta, \quad a \in A,\ \eta \in \Omega^1$$

and has curvature R_∇ and torsion T_∇ given by left A-module maps

$$R_\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1, \quad R_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla, \tag{8.5}$$

$$T_\nabla : \Omega^1 \to \Omega^2, \quad T_\nabla = \wedge\nabla - \mathrm{d}, \tag{8.6}$$

where $\wedge : \Omega^1 \otimes_A \Omega^1 \to \Omega^2$ is the exterior product. We have met both formulae before in Chap. 3 and then again in Chap. 5. The concept of a connection itself requires only Ω^1, while curvature and torsion require Ω^2 in the role, classically, of defining curvature and torsion on antisymmetric combinations of vector fields. In Chap. 5, coming from quantum frame bundles, we were also led to introduce a new tensor built from a metric and a connection, the *cotorsion*, defined as

$$\mathrm{co}T_\nabla \in \Omega^2 \otimes_A \Omega^1, \quad \mathrm{co}T_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)g. \tag{8.7}$$

This arises naturally in the frame bundle picture as the torsion of the induced connection on a dual model in which the roles of 1-forms and vector fields are interchanged. The cotorsion also appears as a derivative in (4.9). We have already seen most of the following in §4.1 but we give a fresh self-contained proof starting only with ∇. The combination $\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla$ appeared as $\nabla^{[1]}$ in (4.2).

Lemma 8.1 *Let (A, Ω, d) be an algebra with exterior algebra and ∇ a left connection on Ω^1. Then $\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla : \Omega^1 \otimes_A \Omega^1 \to \Omega^2 \otimes_A \Omega^1$ is a well-defined map and T_∇, R_∇ are left-module maps. Moreover, the curvature and in the metric case the cotorsion obey*

$$\wedge R_\nabla = \mathrm{d}T_\nabla - (\mathrm{id} \wedge T_\nabla)\nabla, \quad \wedge\mathrm{co}T_\nabla = \mathrm{d}(\wedge g) - (\mathrm{id} \wedge T_\nabla)g$$

(the first of these is called the 1st Bianchi identity).

Proof For the first part, calling the map X, we have $X(\omega \otimes a\eta) = \mathrm{d}\omega \otimes a\eta - \omega \wedge \nabla(a\eta) = (\mathrm{d}\omega)a \otimes \eta - \omega \wedge a \otimes \nabla\eta - \omega \wedge \mathrm{d}a \otimes \eta = X(\omega a \otimes \eta)$, so the map descends. $R_\nabla(a\omega) = X(a\nabla\omega + \mathrm{d}a \otimes \omega) = \mathrm{d}a \wedge \nabla\omega + aX(\nabla\omega) - \mathrm{d}a \wedge \nabla\omega = aR_\nabla(\omega)$ by the left connection property and the Leibniz rule for d. These two properties also give $T_\nabla(a\omega) = \mathrm{d}a \wedge \omega + a \wedge \nabla\omega - \mathrm{d}a \wedge \omega - a\mathrm{d}\omega = aT_\nabla\omega$.

For the last part, we adopt the shorthand $\nabla\xi = \xi^1 \otimes \xi^2$ (sum of such terms understood) then

$$\begin{aligned}\wedge R_\nabla(\xi) &= (\mathrm{d}\xi^1) \wedge \xi^2 - (\xi^1 \wedge \xi^{21}) \wedge \xi^{22} = \mathrm{d}(\xi^1 \wedge \xi^2) + \xi^1 \wedge (\mathrm{d}\xi^2 - \xi^{21} \wedge \xi^{22})\\ &= \mathrm{d}^2\xi + \mathrm{d}T_\nabla(\xi) - \xi^1 \wedge T_\nabla(\xi^2) = \mathrm{d}T_\nabla(\xi) - (\mathrm{id} \wedge T_\nabla)\nabla\xi\end{aligned}$$

for all $\xi \in \Omega^1$. We used $\mathrm{d}^2 = 0$ and associativity of the wedge product. The proof for the cotorsion is similar, $\wedge\mathrm{co}T_\nabla = (\mathrm{d}g^1) \wedge g^2 - g^1 \wedge \mathrm{d}g^2 - g^1 \wedge T_\nabla(g^2)$ on writing $g = g^1 \otimes g^2$ (sum understood), which we recognise as stated. □

The above ingredients provide for a natural weak version of Riemannian geometry (both classical and quantum) which we have seen arise naturally in the frame bundle picture of Chap. 5 as follows.

Definition 8.2 A left *weak quantum Levi-Civita connection* (WQLC) means a torsion free and cotorsion free left connection ∇.

The cotorsion identity in Lemma 8.1 implies in this case that although $\wedge(g)$ may not be zero (if the metric is not quantum symmetric), it is a closed 2-form. In the $*$-differential algebra case, we also have a more general theory of left connections and hermitian-metric compatible connections which we defer to §8.4 below.

To have a theory closer to the classical one, we next recall from §3.4 that a left connection is a *bimodule connection* if it also obeys a twisted right Leibniz rule $\nabla(\eta a) = (\nabla\eta)a + \sigma(\eta \otimes \mathrm{d}a)$ for a bimodule map $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ known as the *generalised braiding*. Such a map, if it exists, is uniquely determined by

$$\sigma(\eta \otimes \mathrm{d}a) = \mathrm{d}a \otimes \eta + \nabla[\eta, a] - [\nabla\eta, a],$$

so this is a property of a left connection as to whether it is in fact a bimodule connection for a given right action. We know from Theorem 3.78 that the tensor product of two bundles with such connections gets a connection via the Leibniz rule and σ. Hence a bimodule connection ∇ extends canonically to $\Omega^1 \otimes_A \Omega^1$ via σ as

$$\nabla g = (\nabla \otimes \mathrm{id})g + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)g, \quad g \in \Omega^1 \otimes_A \Omega^1, \tag{8.8}$$

so that $\nabla g = 0$ makes sense. We say that a bimodule connection is *quantum metric compatible* if $\nabla g = 0$.

Definition 8.3 A *quantum Levi-Civita connection* (QLC) for a given generalised metric g means a left bimodule connection ∇ on Ω^1 which is torsion free in the sense that $\wedge\nabla = \mathrm{d}$ and quantum metric compatible in the sense that $\nabla g = 0$.

There is an obvious sense in which an inverse metric $(\,,\,)$ can be defined to be compatible with a bimodule connection,

$$\mathrm{d}(\xi,\eta) = (\mathrm{id}\otimes(\ ,\ ))(\nabla\xi\otimes\eta + (\sigma\otimes\mathrm{id})(\xi\otimes\nabla\eta)). \tag{8.9}$$

To show that metric-compatibility and the inverse-metric compatibility are equivalent, we need one more idea which we repeat from (4.10). We say that $(\,,\,)$ and g commute with σ if

$$((\,,\,)\otimes\mathrm{id}) = (\mathrm{id}\otimes(\,,\,))(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma),\quad \mathrm{id}\otimes g = (\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(g\otimes\mathrm{id}) \tag{8.10}$$

respectively. From the discussion around (4.10), we know that if σ is invertible then the first of the equations in (8.10) holds if and only if the second holds.

Lemma 8.4 *Suppose we have a bimodule connection on Ω^1 where σ is invertible. Then ∇ is quantum metric compatible with g if and only if it is compatible with the quantum inverse metric $(\,,\,)$. In this case* $\mathrm{d}(\underline{\dim}) = 0$ *(in which case $\underline{\dim}$ is constant if the calculus is connected).*

Proof First we note from the discussion around (3.33) that if g is metric compatible then g also commutes with σ. As σ is invertible, this also implies that the inverse metric commutes with σ. The same conclusion holds if we begin with the inverse metric being compatible, so we can always assume commutation with σ in proving the result in either direction. Now begin by assuming that $(\,,\,)$ is compatible, write $g = g^1\otimes g^2$ (sum understood) and apply ∇ to $\xi = (\xi, g^1)g^2$ to get

$$\nabla\xi = \mathrm{d}(\xi, g^1)\otimes g^2 + (\xi, g^1)\nabla g^2.$$

Assuming compatibility with the inverse metric (8.9), we obtain

$$0 = \nabla\xi - (\xi, g^1)\nabla g^2 - (\mathrm{id}\otimes(,))(\nabla\xi\otimes g^1 + (\sigma\otimes\mathrm{id})(\xi\otimes\nabla g^1))\otimes g^2.$$

By the definition of the metric, the first and third terms cancel, giving

$$0 = -(\xi, g^1)\nabla g^2 - (\mathrm{id}\otimes(,))(\sigma\otimes\mathrm{id})(\xi\otimes\nabla g^1)\otimes g^2.$$

The last term here is

$$(\mathrm{id}\otimes(\,,\,))(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\mathrm{id}\otimes\sigma^{-1})(\xi\otimes\nabla g^1)\otimes g^2$$

and as the function $(\ ,\ )$ is preserved by the connection it commutes with σ, giving the result

$$0 = -(\xi, g^1)\nabla g^2 - ((\ ,\ )\otimes\mathrm{id})(\xi\otimes\sigma^{-1}\nabla g^1)\otimes g^2.$$

As this is true for all ξ we have $g^1\nabla g^2+\sigma^{-1}\nabla g^1\otimes g^2=0$, as required. Similarly we can prove the reverse implication. Differentiating the equation for the quantum dimension gives the last result. □

Although we will usually focus on the case of central metrics, the theory if we do not need an inverse $(\ ,\ )$ (or at least not one that descends to $\otimes_A$) is still of interest and the notion of QLC still makes sense.

Example 8.5 We take $A=\mathbb{C}_q[S^1]$ in Example 1.11 with left-invariant basis $e^1=t^{-1}\mathrm{d}t$ and general forms of metric $g=ae^1\otimes e^1$ and connection $\nabla e^1=\beta e^1\otimes e^1$, where $a,\beta\in A$. The latter is always a bimodule connection with

$$\sigma(e^1\otimes e^1)=(1+(1-q)\beta)e^1\otimes e^1$$

and using the tensor product connection and the formula for d in Example 1.11,

$$\nabla(ae^1\otimes e^1)=\big(\frac{a_1-a}{q-1}+a(\beta+\beta_1(1+(1-q)\beta))\big)e^1\otimes e^1\otimes e^1,$$

where $\beta_1=\beta(qt)$ denotes $\beta(t)$ with t replaced by qt and similarly for a_1. To solve this for Laurent polynomials β and $a\neq 0$ in the case $q\neq 0$ and generic (not a root of unity), we begin by noting that a_1 must be divisible by a, which requires that $a\propto t^m$ for some m (consider a a power of t times some product of linear factors to deduce this). So up to a normalisation we have an integer m choice of metric $g=t^m e_1\otimes e_1$ possibly admitting a QLC. For the latter, we need to solve

$$[m]_q+\beta+\beta_1(1+(1-q)\beta)=0,$$

where $[m]_q=(1-q^m)/(1-q)$. Considering the highest and lowest powers of t in β shows that β must be a constant and indeed that there are two solutions,

$$\text{(i)}:\quad \nabla e_1=-[\frac{m}{2}]_q e_1\otimes e_1,\quad \text{(ii)}:\quad \nabla e_1=\big(\frac{2}{q-1}+[\frac{m}{2}]_q\big)e_1\otimes e_1,$$

where we choose a square root of q if m is odd. Here $T_\nabla=0$ if we take the canonical calculus with $(e^1)^2=0$, so for each metric we have two QLCs, but note that only one of these has a classical $q\to 1$ limit. On the other hand, unless $q^2=1$, the metric is not central so cannot be invertible in the full sense of an inverse metric above. (We can still define a bimodule map $\Omega^1\otimes\Omega^1\to A$ inverse to g in the required sense but not descending to $\otimes_A$, namely $(e^1,e^1)=1/a_{-1}$ where $a_{-1}(t)=a(q^{-1}t)$.) See also Exercise E8.1 for $A=\mathbb{C}(\mathbb{Z})$ for similar results in a slightly more general context. ⋄

Next, for a bimodule connection, we have a condition that is weaker than torsion free, namely that ∇ is *torsion compatible* in the sense that

$$\wedge(\mathrm{id}+\sigma)=0, \tag{8.11}$$

which we derived just after Lemma 3.72 as the condition for the torsion T_∇ to be a bimodule map. When this holds, it follows that

$$(\wedge \otimes \mathrm{id})\nabla_{\Omega^1\otimes\Omega^1} = \wedge\nabla \otimes \mathrm{id} + (\wedge\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla) + T_\nabla \otimes \mathrm{id},$$

where we used the subscript $\Omega^1 \otimes \Omega^1$ to denote the connection on the tensor product to avoid confusion, and we used the definition of T_∇. Hence

$$\mathrm{co}T_\nabla + (T_\nabla \otimes \mathrm{id})g = (\wedge \otimes \mathrm{id})\nabla g, \quad R_\nabla + (T_\nabla \otimes \mathrm{id})\nabla = (\wedge \otimes \mathrm{id})\nabla^2, \tag{8.12}$$

where $\nabla^2 := \nabla_{\Omega^1\otimes\Omega^1} \circ \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$. The geometric content of our definitions becomes clearer in the bimodule torsion-compatible case. Particularly, zero is a bimodule map so the above formulae apply in the case of a torsion free bimodule connection with

$$\mathrm{co}T_\nabla = (\wedge \otimes \mathrm{id})\nabla g, \quad R_\nabla = (\wedge \otimes \mathrm{id})\nabla^2. \tag{8.13}$$

We see that in this case, cotorsion free is weaker than $\nabla g = 0$ itself (which is why we have called such connections 'weak quantum Levi-Civita') and also that there is a slightly stronger notion than curvature vanishing, namely $\nabla^2 = 0$.

A further ingredient of Riemannian geometry is the *geometric Laplace–Beltrami operator* defined by $\Delta = (\,,\,)\nabla\mathrm{d}$ on functions and in the bimodule connection case by $\Delta = (\,,\,)_{12}\nabla^2$ on 1-forms. We use the notation that $(\,,\,)_{12} = (\,,\,) \otimes \mathrm{id}$ or $\sigma_{12} = \sigma \otimes \mathrm{id}$ denotes the operator applied to the first and second tensor terms, and the identity elsewhere. These Δ have a reasonable Leibniz rule as 2nd order operators in the sense of Chap. 1 but with respect to a new inner product $\frac{1}{2}(\,,\,)(\mathrm{id} + \sigma)$.

Lemma 8.6 *For a bimodule connection on Ω^1, the geometric Laplace–Beltrami operators obey*

$$\Delta(ab) = (\Delta a)b + a\Delta b + (\,,\,)(\mathrm{id} + \sigma)(\mathrm{d}a \otimes \mathrm{d}b),$$

$$\Delta(a\omega) = (\Delta a)\omega + a\Delta\omega + (\,,\,)_{12}(\mathrm{id} + \sigma)_{12}(\mathrm{d}a \otimes \nabla\omega)$$

for all $a, b \in A$, $\omega \in \Omega^1$.

Proof Using the properties of a bimodule connection and the inverse metric,

$$\Delta(ab) = (\,,\,)(\nabla(a\mathrm{d}b + (\mathrm{d}a)b) = (\,,\,)(\mathrm{d}a \otimes \mathrm{d}b + a\nabla\mathrm{d}b + \sigma(\mathrm{d}a \otimes \mathrm{d}b) + (\nabla\mathrm{d}a)b),$$

which gives the 2nd order derivation rule stated. Similarly for the second case. □

Next, in the $*$-differential calculus case we need conditions on a bimodule connection ∇ which classically in a hermitian basis would imply real Christoffel symbols. We formulate this as

$$\sigma\circ\dagger\circ\nabla=\nabla* \tag{8.14}$$

and we say in this case that such a connection is *$*$-preserving*. We have already seen in §3.4.3 how the abstract bar-category theory in §2.8 applies to connections and (8.14) follows from this. In some cases, this is too strong and we need a weaker notion

$$\dagger\circ\sigma=\sigma^{-1}\circ\dagger \tag{8.15}$$

for which we say a connection is *$*$-compatible*. Again, this has a bar-categorical origin which we have translated in concrete terms.

Lemma 8.7 *Let ∇ be $*$-preserving. Then*

(1) *∇ is $*$-compatible;*
(2) *if the quantum metric is real then $(\Delta a)^*=\bar{\Delta}(a^*)$, where $\bar{\Delta}:=(\ ,\ )\sigma^{-1}\nabla\mathrm{d}$;*
(3) *if ∇ is torsion compatible then $[T_\nabla,*]=0$.*

Proof We give a more elementary proof of (1) than in §3.4.3,

$$\begin{aligned}\sigma\dagger\sigma(\xi\otimes\mathrm{d}a)&=\sigma\dagger\nabla(\xi.a)-\sigma\dagger(\nabla(\xi).a)=\sigma\dagger\nabla(\xi.a)-a^*\sigma\dagger\nabla(\xi)\\&=\nabla(a^*\xi^*)-a^*\nabla(\xi^*)=\mathrm{d}a^*\otimes\xi^*.\end{aligned}$$

Part (2) is straightforward from the definitions. For (3),

$$\wedge\nabla(\xi^*)-\mathrm{d}\xi^*=\wedge\sigma\dagger\nabla\xi-(\mathrm{d}\xi)^*=-\wedge\dagger\nabla\xi-(\mathrm{d}\xi)^*=*T_\nabla(\xi).\qquad\square$$

We see from this and our earlier result that if $(\ ,\ )\circ\sigma=(\ ,\ )$ then our geometric Laplace–Beltrami operator on functions obeys the usual 2nd order rule in §1.3 with respect to the original $(\ ,\ )$ and commutes with $*$.

Finally, we can similarly ask for the Riemann curvature to be a bimodule map, which we call *Riemann compatible*. By Lemma 3.72, this comes down to

$$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla)\sigma(\mathrm{id}\otimes\mathrm{d})=(\mathrm{id}\wedge\sigma)(\nabla\otimes\mathrm{d}):\Omega^1\otimes A\to\Omega^2\otimes_A\Omega^1 \tag{8.16}$$

or in the torsion-compatible case to

$$(\wedge\otimes\mathrm{id})\nabla_{\Omega^1\otimes\Omega^1}\sigma(\mathrm{id}\otimes\mathrm{d})-(T_\nabla\otimes\mathrm{id})\sigma(\mathrm{id}\otimes\mathrm{d})=(\mathrm{id}\wedge\sigma)(\nabla\otimes\mathrm{d}).$$

In the Riemann-compatible case we can define *quantum metric traces*

$$(\mathrm{id} \otimes (\ ,\ ))(R_\nabla^n \otimes \mathrm{id})g \in \Omega^{2n},$$

where $R_\nabla^n : \Omega^1 \to \Omega^{2n} \otimes \Omega^1$ means to repeatedly apply R_∇ and wedge the resulting 2-form outputs. For even n, this is the loose equivalent of the Pontryagin class in classical geometry except that there the trace would not depend on a choice of metric.

It is also useful in this context to be able to swap orders of 2-forms and 1-forms, which is a further notion that σ is extendable to $\sigma : \Omega^1 \otimes_A \Omega^n \to \Omega^n \otimes_A \Omega^1$ by

$$(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = \sigma(\mathrm{id} \otimes \wedge) : \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1 \to \Omega^2 \otimes_A \Omega^1$$

for $n = 2$ with corresponding equations for $n > 2$. This is a special case of Definition 4.10 and combined with Riemann compatibility says that $(\Omega^1, \nabla, \sigma)$ is an object of the DG-category ${}_A\mathcal{G}_A$ as in Theorem 4.11. This, while true classically, turns out to be quite restrictive in the noncommutative case but when it holds it includes a formula from Lemma 4.12 for the differential of σ, namely

$$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\sigma = (\mathrm{id} \wedge \sigma)(\nabla \otimes \mathrm{id}) + \sigma(\mathrm{id} \otimes \mathrm{d}) : \Omega^1 \otimes_A \Omega^1 \to \Omega^2 \otimes_A \Omega^1.$$

We also have in this context that the Riemann tensor is quantum antisymmetric in the sense of Corollary 4.16, which classically would refer to the last two indices of the Riemann tensor (antisymmetry of the first two is already expressed by the curvature being 2-form-valued). Most of this was already covered in Chap. 4.

Lemma 8.8 *If $(\Omega^1, \nabla, \sigma)$ is an object of ${}_A\mathcal{G}_A$ and metric compatible then the curvature traces are closed, the quantum antisymmetry of R_∇ holds and, in the $*$-algebra case with ∇ $*$-preserving, $R_\nabla \circ * = \dagger\, \sigma^{-1} R_\nabla$.*

Proof The first parts are a special case of Corollary 4.17 and Corollary 4.16 respectively and we omit direct proofs. In the $*$-preserving case, the Riemann compatibility and extendability of σ imply that

$$R_\nabla(\xi^*) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\sigma\, \dagger\, \nabla\xi = (\mathrm{id} \wedge \sigma)(\nabla \otimes \mathrm{id})\, \dagger\, \nabla\xi + \sigma(\mathrm{id} \otimes \mathrm{d})\, \dagger\, \nabla\xi.$$

Setting $\nabla\xi = \xi^1 \otimes \xi^2$ (sum understood) and writing our conditions more explicitly, including using extendability, leads to the last result. □

A different compatibility of a bimodule connection ∇ with wedge products, which should not be confused with the above, is if the tensor product connection on $\Omega^1 \otimes_A \Omega^1$ (the same as we used to define ∇g) descends to a bimodule connection on Ω^2 (and higher). The necessary and sufficient condition for ∇ to extend to a connection on Ω^2 is that

$$(\mathrm{id} \otimes \wedge)\nabla_{\Omega^1 \otimes \Omega^1}|_{\ker \wedge} = 0 \tag{8.17}$$

(i.e., vanishes on $\ker\wedge\subseteq\Omega^1\otimes_A\Omega^1$). If this is satisfied then for $\omega\in\Omega^2$, we define $\nabla\omega=(\mathrm{id}\otimes\wedge)\nabla_{\Omega^1\otimes\Omega^1}z$ for any representative $z\in\Omega^1\otimes_A\Omega^1$ with $\wedge z=\omega$. Then

$$\nabla(\xi\wedge\eta)=(\nabla\xi)\wedge\eta+\sigma(\xi\otimes\eta^1)\wedge\eta^2 \tag{8.18}$$

for $\nabla\eta=\eta^1\otimes\eta^2$ (sum understood), and is a bimodule connection with

$$\sigma(\xi\wedge\eta\otimes\zeta)=(\mathrm{id}\otimes\wedge)(\sigma\otimes\mathrm{id})(\xi\otimes\sigma(\eta\otimes\zeta)). \tag{8.19}$$

To summarise, we see that the various conditions we want to consider in noncommutative Riemannian geometry have full strength versions that match up with classical Riemannian geometry and weaker versions that we are sometimes forced to use due to obstructions in the noncommutative case. This is summarised in the following table

Structure	Full condition	Weaker condition
Connection	$\nabla(\eta a)=(\nabla\eta)a+\sigma(\eta\otimes\mathrm{d}a)$ & left connection bimodule connection	$\nabla(a\eta)=\mathrm{d}a\otimes\eta+a\nabla\eta$ left connection
Metric	$(\nabla\otimes\mathrm{id})g+(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\nabla)g=0$ metric compatible	$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla)g=0$ cotorsion free
Torsion	$\wedge\nabla-\mathrm{d}=0$ torsion free	$\wedge(\mathrm{id}+\sigma)=0$ torsion compatible
Curvature	$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla)\nabla=0$ flat	$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\nabla)\sigma(\mathrm{id}\otimes\mathrm{d})$ $=(\mathrm{id}\otimes\wedge\sigma)(\nabla\otimes\mathrm{d})$ Riemann compatible
Star	$\sigma\circ\dagger\circ\nabla=\nabla*$ $*$-preserving	$\dagger\circ\sigma=\sigma^{-1}\circ\dagger$ $*$-compatible

We conclude with a 'constructive definition' of the Ricci tensor in the absence of a more conceptual approach. This should not be seen as the last word on the topic but does give 'reasonable' answers in several of our examples. The key new ingredient is a lifting map $i:\Omega^2\to\Omega^1\otimes_A\Omega^1$ defined as a bimodule map obeying $\wedge\circ i=\mathrm{id}$. In classical geometry, this would be the map that writes a 2-form as an antisymmetric tensor product of 1-forms. In noncommutative geometry, however, we do not necessarily know what 'antisymmetric' means and i becomes additional data. In the $*$-calculus case, we require where possible that the map is 'real' in the sense that

$$\dagger\circ i=-i\circ * \tag{8.20}$$

although this is optional in that it is not an end in itself. We then define

$$\mathrm{Ricci}=((\ ,\ )\otimes\mathrm{id})(\mathrm{id}\otimes i\otimes\mathrm{id})(\mathrm{id}\otimes R_\nabla)(g) \tag{8.21}$$

as determined by i. Finally, we choose i, if possible, such that Ricci is quantum symmetric and real in the same sense as the metric, namely

$$\wedge\,(\mathrm{Ricci}) = 0, \quad \mathrm{Ricci}^{\dagger} = \mathrm{Ricci}, \tag{8.22}$$

where the latter is in the $*$-calculus case. This process is not guaranteed to work and even if it does, it may not be unique, i.e., there may be a moduli space of Ricci tensors by this construction according to the possible i. In short, Ricci becomes subordinate to the additional data i. We define the *Ricci scalar* by

$$S = (\ ,\ )\mathrm{Ricci} \tag{8.23}$$

again subordinate to the choice of i. By construction we will have $S^* = S$ if we have achieved the reality property of Ricci. The next result gives a formula for $\mathrm{Ricci}^{\dagger}$ which is similar to Ricci but contracted oppositely.

Proposition 8.9 *Suppose that $(\Omega^1, \nabla, \sigma)$ is in ${}_A\mathcal{G}_A$, that the metric is real as in (8.2) and is preserved by the connection, and that the connection is $*$-preserving and that (8.20) holds. Then*

$$\mathrm{Ricci}^{\dagger} = (\mathrm{id} \otimes (\ ,\ ))(\mathrm{id} \otimes i \otimes \mathrm{id})(\mathrm{id} \otimes R_{\nabla})g.$$

Proof As the metric is preserved by the connection, we have

$$(R_{\nabla} \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes R_{\nabla}))g = 0,$$

by Corollary 4.16. From this, it follows that

$$\mathrm{Ricci} = -((\ ,\ ) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\sigma^{-1} R_{\nabla} \otimes \mathrm{id})g.$$

Setting $g = g^1 \otimes g^2$ (sum understood) and using the reality of g in (8.2), and Lemma 8.8, we have

$$\begin{aligned}\mathrm{Ricci} &= -((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\sigma^{-1} R_{\nabla} * g^2 \otimes * g^1)\\ &= -((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\sigma^{-1} \dagger \sigma^{-1} R_{\nabla} g^2 \otimes * g^1)\\ &= -((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\dagger R_{\nabla} g^2 \otimes * g^1).\end{aligned}$$

Using the shorthand $R_{\nabla} g^2 = \omega \otimes \xi$ (sum of such terms) and $i(\omega) = \omega^1 \otimes \omega^2$ (sum of such terms),

$$\begin{aligned}\mathrm{Ricci} &= -((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(*\xi \otimes *\omega \otimes *g^1) = ((\ ,\ ) \otimes \mathrm{id})(*\xi \otimes \dagger i\omega \otimes *g^1)\\ &= ((\ ,\ ) \otimes \mathrm{id})(*\xi \otimes *\omega^2 \otimes *\omega^1 \otimes *g^1) = \dagger(g^1 \otimes \omega^1(\omega^2, \xi)). \qquad \square\end{aligned}$$

To understand our conventions, we pause now to examine these ideas for the case of a classical manifold in terms of usual tensor calculus.

Example 8.10 (Classical Case) Clearly, if M is a smooth Riemannian manifold then all the above applies with $A = C^\infty(M)$. We have already seen this for the curvature and torsion in Examples 3.20 and 3.29 and for the former,

$$R_\nabla \mathrm{d}x^\rho = \frac{1}{2}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes [\nabla_\mu, \nabla_\nu]\mathrm{d}x^\rho = -\frac{1}{2}R^\rho{}_{\kappa\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathrm{d}x^\kappa$$

in terms of the usual Riemann curvature tensor in standard conventions. Vanishing cotorsion in the torsion free case comes down to $g_{\mu\nu;\rho} - g_{\rho\nu;\mu} = 0$ as in Corollary 5.70, whereas full metric compatibility is $g_{\mu\nu;\rho} = 0$. The classical lift is

$$i(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \frac{1}{2}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu)$$

and applying this in (8.21) gives our Ricci as

$$\begin{aligned}\mathrm{Ricci} &= -\frac{1}{2}g_{\lambda\rho}R^\rho{}_{\kappa\mu\nu}\big((\ ,\ )\otimes \mathrm{id}\big)\big(\mathrm{d}x^\lambda \otimes i(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)\otimes \mathrm{d}x^\kappa\big)\\ &= -\frac{1}{4}g_{\lambda\rho}R^\rho{}_{\kappa\mu\nu}\big((\mathrm{d}x^\lambda, \mathrm{d}x^\mu)\mathrm{d}x^\nu \otimes \mathrm{d}x^\kappa - (\mathrm{d}x^\lambda, \mathrm{d}x^\nu)\mathrm{d}x^\mu \otimes \mathrm{d}x^\kappa\big)\\ &= -\frac{1}{4}g_{\lambda\rho}R^\rho{}_{\kappa\mu\nu}\big(g^{\lambda\mu}\mathrm{d}x^\nu \otimes \mathrm{d}x^\kappa - g^{\lambda\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\kappa\big)\\ &= -\frac{1}{4}R^\rho{}_{\kappa\rho\nu}\mathrm{d}x^\nu \otimes \mathrm{d}x^\kappa + \frac{1}{4}R^\rho{}_{\kappa\mu\rho}\mathrm{d}x^\mu \otimes \mathrm{d}x^\kappa = -\frac{1}{2}R_{\kappa\nu}\mathrm{d}x^\nu \otimes \mathrm{d}x^\kappa\end{aligned}$$

according to the standard convention for the Ricci tensor $R_{\kappa\nu} = R^\rho{}_{\kappa\rho\nu}$. ◇

We will also be interested in the opposite extreme from the classical case, namely where Ω is an inner calculus as in Chap. 1, i.e., with $\theta \in \Omega^1$ such that $\mathrm{d} = [\theta,\ \}$.

Proposition 8.11 *Let A be an algebra with an inner differential structure Ω^1, and for parts (2) and (3) we assume further that the same θ extends to an inner Ω^2.*

(1) *Bimodule connections (∇, σ) are in 1-1 correspondence with pairs (σ, α) of bimodule maps*

$$\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1, \quad \alpha : \Omega^1 \to \Omega^1 \otimes_A \Omega^1,$$

where the connection, curvature and torsion are

$$\begin{aligned}&\nabla = \theta \otimes (\,) + \alpha - \sigma_\theta; \quad \sigma_\theta = \sigma((\,)\otimes\theta),\\ &T_\nabla = -\wedge((\,)\otimes\theta + \sigma_\theta - \alpha),\\ &R_\nabla = \theta^2 \otimes (\,) + (\wedge \otimes \mathrm{id})\tilde{R}_\nabla; \quad \tilde{R}_\nabla = -(\mathrm{id}\otimes(\alpha - \sigma_\theta))(\alpha - \sigma_\theta).\end{aligned}$$

(2) ∇ *is torsion free if and only if* $\wedge\alpha = 0$ *and* $\wedge\sigma = -\wedge$, *and also cotorsion free if and only if*

$$\theta \wedge g - (\wedge \otimes \mathrm{id})(\mathrm{id} \otimes (\alpha - \sigma_\theta))g = 0.$$

It is metric compatible if and only if

$$\theta \otimes g + (\alpha \otimes \mathrm{id})g + \sigma_{12}(\mathrm{id} \otimes (\alpha - \sigma_\theta))g = 0.$$

(3) *In the* $*$*-algebra case over* $\mathbb{C}$ *with* $\theta^* = -\theta$, ∇ *is* $*$*-preserving if and only if*

$$\dagger \circ \sigma = \sigma^{-1} \circ \dagger, \quad \sigma \circ \dagger \circ \alpha = \alpha * .$$

(4) *Let* $\tilde{\alpha} = (\ ,\ )(\theta \otimes \theta + \alpha\theta) \in A$. *The geometric Laplacians on* $a \in A$ *and* $\omega \in \Omega^1$ *are*

$$\begin{aligned}
\Delta a &= -(\ ,\ )(\mathrm{id} + \sigma)(\mathrm{d}a \otimes \theta) + [\tilde{\alpha}, a],\\
\Delta\omega &= (\ ,\ )_{12}(\mathrm{id} + \sigma)_{12}(\theta \otimes (\alpha - \sigma_\theta)\omega) + ((\ ,\ )\alpha \otimes \mathrm{id})(\alpha - \sigma_\theta)\omega\\
&\quad - (\ ,\ )_{12}\sigma_{12}\tilde{R}_\nabla\omega + \tilde{\alpha}\omega.
\end{aligned}$$

Proof (1) Let σ, α be bimodule maps and ∇ defined from them as stated. Then

$$\begin{aligned}
\nabla(a\omega) &= \theta \otimes a\omega - \sigma(a\omega \otimes \theta) + \alpha(a\omega)\\
&= \theta a \otimes \omega - a\sigma(\omega \otimes \theta) + a\alpha(\omega) = \mathrm{d}a \otimes \omega + a\nabla\omega,\\
\nabla(\omega a) &= \theta \otimes \omega a - \sigma(\omega \otimes a\theta) + \alpha(\omega)a = (\nabla\omega)a + \sigma(\omega \otimes \mathrm{d}a)
\end{aligned}$$

for all $a \in A$ and $\omega \in \Omega^1$. Hence we have a bimodule connection. Conversely, let (∇, σ) be a bimodule connection. Then σ is a bimodule map and $\nabla^0\omega = \theta \otimes \omega - \sigma(\omega \otimes \theta)$ is a connection by the above (with $\alpha = 0$). Hence $\nabla - \nabla^0$ is a bimodule map, which we take as α. We then compute

$$T_\nabla\omega = \wedge\nabla\omega - \mathrm{d}\omega = \theta \wedge \omega - \wedge\sigma(\omega \otimes \theta) + \wedge\alpha\omega - \theta \wedge \omega - \omega \wedge \theta,$$

which gives the result as stated. For the curvature, we first do the simpler $\alpha = 0$ case

$$\begin{aligned}
R_{\nabla^0}\omega &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla^0)(\theta \otimes \omega - \sigma(\omega \otimes \theta))\\
&= \mathrm{d}\theta \otimes \omega - (\mathrm{d} \otimes \mathrm{id})\sigma(\omega \otimes \theta) - \theta^2 \otimes \omega + \theta \wedge \sigma(\omega \otimes \theta)\\
&\quad + \mathrm{id} \wedge \nabla^0\sigma(\omega \otimes \theta)
\end{aligned}$$

$$= \theta^2 \otimes \omega - \sigma_1 \wedge \theta \otimes \sigma_2 + \mathrm{id} \wedge \nabla^0 \sigma(\omega \otimes \theta)$$
$$= \theta^2 \otimes \omega - (\wedge \otimes \mathrm{id})\sigma_{23}\sigma_{12}(\omega \otimes \theta \otimes \theta),$$

where $\sigma(\omega \otimes \theta) = \sigma_1 \otimes \sigma_2$ is a shorthand notation (sum understood). Then

$$\begin{aligned} R_\nabla \omega &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)(\theta \otimes \omega - \sigma(\omega \otimes \theta) + \alpha\omega) \\ &= R_{\nabla^0}\omega + (\mathrm{d} \otimes \mathrm{id})\alpha\omega - \alpha_1 \wedge (\theta \otimes \alpha_2 - \sigma(\alpha_2 \otimes \theta) + \alpha\alpha_2) \\ &\quad - \theta \wedge \alpha\omega + \sigma_1 \wedge \alpha\sigma_2, \end{aligned}$$

where $\alpha\omega = \alpha_1 \otimes \alpha_2$ is another shorthand notation (sum understood). Using the form of d and cancelling two terms, we arrive at the expression stated in terms of

$$\tilde{R}_\nabla \omega = -\sigma_{23}\sigma_{12}(\omega \otimes \theta \otimes \theta) + (\sigma_{23}(\alpha \otimes \mathrm{id}) + (\mathrm{id} \otimes \alpha)\sigma)(\omega \otimes \theta) - (\mathrm{id} \otimes \alpha)\alpha.$$

This in turn can be written compactly as stated in terms of σ_θ.

(2) If $\wedge(\mathrm{id} + \sigma) = 0$ and $\wedge\alpha = 0$ hold then we see from part (2) that $T_\nabla = 0$. Conversely, torsion free is a special case of torsion-compatibility so (8.11) holds, in which case we need also $\wedge\alpha = 0$. Metric compatibility has the form stated on putting the form of ∇ into $(\nabla \otimes \mathrm{id} + \sigma_{12}(\mathrm{id} \otimes \nabla))g$ and cancelling two terms, and wedge of this gives the cotorsion-freeness equation.

(3) The requirement to be $*$-preserving implies $*$-compatible, hence the condition on σ. In this case it becomes the condition on α.

(4) For the geometric Laplacians as in Lemma 8.6,

$$\begin{aligned} (\,,\,)\nabla \mathrm{d}a &= (\,,\,)(\theta \otimes \mathrm{d}a - \sigma(\mathrm{d}a \otimes \theta)) + (\,,\,)\alpha(\mathrm{d}a) \\ &= -(\,,\,)(\mathrm{id} + \sigma)(\mathrm{d}a \otimes \theta) + (\theta, \mathrm{d}a) + (\mathrm{d}a, \theta) + [(\,,\,)\alpha(\theta), a] \\ &= -(\,,\,)(\mathrm{id} + \sigma)(\mathrm{d}a \otimes \theta) + [(\theta, \theta) + (\,,\,)\alpha(\theta), a] \end{aligned}$$

for all $a \in A$ since $(\theta, \theta)a = (\theta a, \theta) + (\theta, \mathrm{d}a) = a(\theta, \theta) + (\mathrm{d}a, \theta) + (\theta, \mathrm{d}a)$. Finally,

$$\Delta\omega = (\,,\,)_{12}\,(\nabla \otimes \mathrm{id} + \sigma_{12}(\mathrm{id} \otimes \nabla))\,\nabla\omega \tag{8.24}$$

can be broken down in our case as

$$\begin{aligned} \Delta\omega &= (\,,\,)_{12}(\nabla \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))(\theta \otimes \omega - \sigma(\omega \otimes \theta) + \alpha\omega) \\ &= (\,,\,)_{12}[\theta \otimes \theta \otimes \omega - \theta \otimes \sigma(\omega \otimes \theta) + \sigma_{12}\sigma_{23}(\sigma(\omega \otimes \theta) \otimes \theta - \theta \otimes \omega \otimes \theta) \\ &\quad + \alpha(\theta) \otimes \omega - (\alpha \otimes \mathrm{id})\sigma(\omega \otimes \theta) + \theta \otimes \alpha\omega + (\alpha \otimes \mathrm{id})\alpha\omega \\ &\quad + \sigma_{12}(\theta \otimes \alpha\omega - (\mathrm{id} \otimes \alpha)\sigma(\omega \otimes \theta) - \sigma_{23}(\alpha\omega \otimes \theta) + (\mathrm{id} \otimes \alpha)\alpha\omega)] \end{aligned}$$

after cancellations. We then recognise three of the α terms and the $\sigma_{12}\sigma_{23}\sigma_{12}$ term as the four terms of $-(\ ,\ )_{12}\sigma_{12}\tilde{R}_\nabla\omega$ and we adopt the stated notations. □

This result illustrates some of the rigidity of noncommutative geometry compared to the classical case: classically σ would be the flip map and all the information for a connection would be in the tensor α, but at the extreme of an inner calculus it is the other way around and much of the information is in the 'generalised braiding' σ, indeed all of it when $\alpha = 0$. Also note that the Laplacian on functions starts with Δ_θ for the averaged inner product $\frac{1}{2}(\ ,\ )(\mathrm{id}+\sigma)$ that also featured in Lemma 8.6. The Laplacian on 1-forms also features this averaged inner product as well as the lifted and contracted curvature $\tilde{R}_\nabla\omega$. It may also be of interest to ask for σ to obey the braid relations over $\otimes_A$ and this turns out to be related to zero curvature.

Corollary 8.12 *In the setting of an inner calculus as above and a torsion-compatible connection with $\alpha = 0$, if $\sigma(\theta\otimes\theta) = \theta\otimes\theta$, σ obeys the braid relations and the characteristic of the field is not 2 then $R_\nabla = 0$.*

Proof If σ preserves $\theta\otimes\theta$ (which implies that $\theta^2 = 0$ in the torsion compatible case) and the braid relations $\sigma_{23}\sigma_{12}\sigma_{23} = \sigma_{12}\sigma_{23}\sigma_{12}$ hold, then

$$\begin{aligned} R_\nabla\omega &= -(\wedge\otimes\mathrm{id})\sigma_{23}\sigma_{12}(\omega\otimes\theta\otimes\theta)\\ &= -(\wedge\otimes\mathrm{id})\sigma_{23}\sigma_{12}\sigma_{23}(\omega\otimes\theta\otimes\theta)\\ &= -(\wedge\otimes\mathrm{id})\sigma_{12}\sigma_{23}\sigma_{12}(\omega\otimes\theta\otimes\theta)\\ &= (\wedge\otimes\mathrm{id})\sigma_{23}\sigma_{12}(\omega\otimes\theta\otimes\theta) = -R_\nabla\omega, \end{aligned}$$

where we used (8.11). □

We conclude with an easy example that can be done by hand before we turn to general constructions.

Example 8.13 Let $A = M_2(\mathbb{C})$ with the maximal prolongation $\Omega = M_2(\mathbb{C}).\mathbb{C}[s,t]$ of the 2D calculus as in Example 1.37. We recall that $\mathrm{d}a = [E_{12}, a]s + [E_{21}, a]t$ and s, t are central and this is an inner calculus with $\theta = E_{12}s + E_{21}t$ and a $*$-differential calculus with hermitian conjugation of matrices and $s^* = -t$. A central metric has to have constant coefficients in the s, t basis and up to normalisation there is a unique choice which obeys the quantum symmetry and reality properties. This is nondegenerate, namely

$$g = s\otimes t - t\otimes s,\quad (t,s) = -(s,t) = 1,\quad (s,s) = (t,t) = 0.$$

One can easily check that

$$\nabla s = 2E_{12}s\otimes s + 2E_{21}t\otimes s,\quad \nabla t = 2E_{21}t\otimes t + 2E_{12}s\otimes t,\quad \sigma = -\mathrm{flip}$$

on the generators is the unique left bimodule quantum-Levi Civita connection for this metric. Here $\nabla = 2\theta\otimes$ on the generators is of the form in Proposition 8.11 with $\alpha = 0$ and from this one sees readily that the connection is flat and $*$-preserving. The geometric Laplacian is

$$\begin{aligned}\Delta a &= (\ ,\ )\nabla([E_{12}, a]s + [E_{21}, a]t)\\ &= (\ ,\ )(\mathrm{d}[E_{12}, a] \otimes s + [E_{12}, a]\nabla s + \mathrm{d}[E_{21}, a] \otimes t + [E_{21}, a]\nabla t)\\ &= [E_{21}, [E_{12}, a]] - [E_{12}, [E_{21}, a]] + 2[E_{12}, a]E_{21} - 2[E_{21}, a]E_{12}\\ &= [[E_{21}, E_{12}], a] + 2[E_{12}, a]E_{21} - 2[E_{21}, a]E_{12}\\ &= \{E_{22} - E_{11}, a\} + 2E_{12}aE_{21} - 2E_{21}aE_{12}\end{aligned}$$

for $a \in M_2(\mathbb{C})$, which works out as $\Delta E_{12} = \Delta E_{21} = \Delta 1 = 0$ as the only eigenmodes, and $\Delta E_{11} = -2$, $\Delta E_{22} = 2$ in the algebra. By contrast,

$$\Delta_\theta a = 2(\theta, \mathrm{d}a) = 2(E_{22}-E_{11})a+2E_{12}aE_{21}-2E_{21}aE_{12} = \Delta a+[E_{22}-E_{11}, a],$$

from Proposition 1.18, has eigenvalues $\pm 2, 0$ while also not fully diagonalisable.

If we add the further relations $s^2 = t^2 = 0$ as in Proposition 1.38 then there is a much richer moduli of QLCs for the metric $g = s \otimes t - t \otimes s$, including some with curvature. The largest component of the moduli space here has the map $\alpha = 0$ and 4-parameter generalised braiding and resulting connection

$$\sigma = \begin{pmatrix} \mu-1 & \alpha & -\alpha & -\frac{\alpha\nu}{\beta}\\ \beta & \nu & -\nu-1 & \alpha-\frac{\nu(\mu+\nu)}{\beta}\\ \beta-\frac{\mu(\mu+\nu)}{\alpha} & -\mu-1 & \mu & \alpha\\ -\frac{\beta\mu}{\alpha} & -\beta & \beta & \nu-1\end{pmatrix};\quad \mu, \nu, \alpha, \beta \in \mathbb{C},$$

$$\begin{aligned}\nabla s &= 2\theta \otimes s - (\alpha E_{12} + \nu E_{21})g - (\mu E_{12} + \beta E_{21})s \otimes s\\ &\quad + \Big(\frac{\nu}{\beta}(\alpha E_{12} + (\mu + \nu)E_{21}) - \alpha E_{21}\Big)t \otimes t,\\ \nabla t &= 2\theta \otimes t + (\mu E_{12} + \beta E_{21})g - (\alpha E_{12} + \nu E_{21})t \otimes t\\ &\quad + \Big(\frac{\mu}{\alpha}(\beta E_{21} + (\mu + \nu)E_{12}) - \beta E_{12}\Big)s \otimes s,\end{aligned}$$

where the conventions are such that the second row gives the coefficients of $\sigma(s\otimes t)$ in basis order $s\otimes s$, $s\otimes t$, $t\otimes s$, $t\otimes t$. Note that now $\alpha \in \mathbb{C}$ denotes a parameter. This ∇ is $*$-preserving by Proposition 8.11 when $\nu = -\bar{\mu}$ and $\beta = -\bar{\alpha}$ and includes our previous point when $\mu, \nu, \alpha, \beta \to 0$ in a suitable way.

Also, with this reduced Ω^2 we have more general quantum symmetric metrics,

$$g = cs \otimes s + \bar{c}t \otimes t + b(s \otimes t - t \otimes s)$$

with $c \in \mathbb{C}$ and $b \in \mathbb{R}$ for the real case. Each of these typically has a rich moduli of QLCs, leading to a large overall moduli space of quantum Riemannian geometries on $M_2(\mathbb{C})$ with this calculus. For example, $g = s \otimes s + t \otimes t$ in Exercise E8.3 has principal component of the moduli of QLCs the map $\alpha = 0$ and a 3-parameter σ. $\diamond$

8.2 More Examples of Bimodule Riemannian Geometries

Here we use bases of 1-forms to write explicit formulae for ideas which we previously discussed in a basis-free fashion. The existence of the basis $\{e^i\}$ corresponds to the assumption that Ω^1 is finitely generated projective as a left module as in §3.1, and the uniqueness of the coefficients of the basis elements in the following formulae corresponds to Ω^1 being left-parallelisable as in Definition 1.2. Without the latter we would have to insert a projection matrix in various places (this generality is discussed in Chap. 3) so to keep things simple here we proceed under the assumption that Ω^1 is left-parallelisable. To fix conventions, we write basis 1-forms e^i with indices *up*, which has not been our preference in most of the book where we have tended to use lower indices where possible as upper ones clash with powers. This is needed to fit conventions in physics and we combine this with Einstein's summation convention where repeated up-down pairs of indices are to be summed. Thus the defining formulae for 'partial derivatives' from Chap. 1 and left connections in terms of Christoffel symbols from §3.2 now appear as,

$$e^i a = C^i{}_j(a)e^j, \quad \mathrm{d}a = (\partial_i a)e^i, \quad \nabla(e^i) = -\Gamma^i{}_{jk}e^j \otimes e^k \tag{8.25}$$

for all $a \in A$ in our coordinate algebra. If e^i and a commute (e.g. if a is an element of the field $\Bbbk$, which we refer to loosely as a constant) then $C^i{}_j(a) = a\delta^i{}_j$. For a bimodule connection we write σ as

$$\sigma(e^i \otimes e^j) = \sigma^{ij}{}_{mn}e^m \otimes e^n \tag{8.26}$$

with coefficients determined from $\Gamma^i{}_{jk}$ and $C^i{}_j$ and such that σ extends as a bimodule map, which will depend on $\Gamma^i{}_{jk}$ as not every left connection is necessarily a bimodule connection. We next suppose that there is a central metric $g = g_{ij}e^i \otimes e^j$ and define the inverse-metric tensor as $g^{ij} = (e^i, e^j)$. This is inverse in the sense that

$$g_{ij}C^i{}_n(g^{jk}) = \delta^k{}_n, \quad C^k{}_p(g_{ij})g^{pi} = \delta^k{}_j \tag{8.27}$$

while centrality of g comes down to

$$ag_{ij} = g_{qs}C^q{}_i(C^s{}_j(a)) \tag{8.28}$$

for all $a \in A$. We give one detailed calculation of converting tensor product notation to index notation and leave the rest to the reader. Namely, the equation for metric compatibility $\nabla g = 0$ is

$$\mathrm{d}g_{ij} \otimes e^i \otimes e^j = g_{ij} \Gamma^i{}_{pk} e^p \otimes e^k \otimes e^j + g_{ij} \sigma(e^i \otimes \Gamma^j{}_{pk} e^p) \otimes e^k,$$
$$(\partial_r g_{ij}) e^r \otimes e^i \otimes e^j = g_{ij} \Gamma^i{}_{pk} e^p \otimes e^k \otimes e^j + g_{ij} C^i{}_q(\Gamma^j{}_{pk}) \sigma^{qp}{}_{rm} e^r \otimes e^m \otimes e^k,$$

so on re-indexing and taking coefficients of the basis elements we get the equation

$$\partial_r g_{mn} = g_{in} \Gamma^i{}_{rm} + g_{ij} C^i{}_q(\Gamma^j{}_{pn}) \sigma^{qp}{}_{rm}. \tag{8.29}$$

The torsion and cotorsion are given by

$$T_\nabla(e^i) = -\Gamma^i{}_{jk} e^j \wedge e^k - \mathrm{d}e^i, \tag{8.30}$$

$$\mathrm{co}T_\nabla = (\partial_r g_{iq}) e^r \wedge e^i \otimes e^q + g_{iq} \mathrm{d}e^i \otimes e^q + g_{ij} C^i{}_n(\Gamma^j{}_{pq}) e^n \wedge e^p \otimes e^q. \tag{8.31}$$

Over $\mathbb{C}$, if $(e^i)^* = -e^i$ for all i (or also if $(e^i)^* = e^i$ for all i), the reality for g holds if and only if

$$g_{ij} = C^p{}_i(C^q{}_j(g_{qp}{}^*)). \tag{8.32}$$

From $\Delta = (\,,\,)\nabla \mathrm{d}$ we get the formula for the geometric Laplacian on A

$$\Delta a = (\partial_j \partial_i a) g^{ji} - (\partial_i a) \Gamma^i{}_{mn} g^{mn}. \tag{8.33}$$

This completes our summary of the main formulae of the last section in tensor terms as far as possible.

If we suppose that the calculus is inner then from Proposition 8.11 we write

$$\theta = \theta_i e^i, \quad \alpha(e^i) = \alpha^i{}_{mn} e^m \otimes e^n, \quad \sigma_\theta(e^i) = C^i{}_k(\theta_j) \sigma^{kj}{}_{mn} e^m \otimes e^n,$$

giving the Christoffel symbols

$$\Gamma^i{}_{mn} = C^i{}_k(\theta_j) \sigma^{kj}{}_{mn} - \alpha^i{}_{mn} - \theta_m \delta^i{}_n.$$

The curvature and other quantities become

$$R_\nabla(e^i) = \theta \wedge \theta \otimes e^i - \gamma^i{}_{mn} C^m{}_r(\gamma^n{}_{kp}) e^r \wedge e^k \otimes e^p,$$
$$\Delta(a) = -(\partial_i a) C^i{}_k(\theta_j)(g^{kj} + \sigma^{kj}{}_{mn} g^{mn}) + [\tilde{\alpha}, a],$$

$$\Delta(e^i) = \theta_j C^j{}_p(\gamma^i{}_{mn})(g^{pm} + \sigma^{pm}{}_{rs}g^{rs})e^n + \gamma^i{}_{mn}\alpha^m{}_{pq}g^{pq}e^n$$
$$+ \gamma^i{}_{ma}C^m{}_r(\gamma^a{}_{pn})\sigma^{rp}{}_{st}g^{st}e^n + \tilde{\alpha}e^i,$$

where

$$\gamma^i{}_{mn} := \alpha^i{}_{mn} - C^i{}_k(\theta_j)\sigma^{kj}{}_{mn}, \quad \tilde{\alpha} := \theta_i(C^i{}_m(\theta_n) + \alpha^i{}_{mn})g^{mn}.$$

For comparison, the inner element Laplacians from Proposition 1.18 are

$${}_\theta\Delta(a) = -2(\partial_i a)C^i{}_k(\theta_j)g^{kj}, \quad \Delta_\theta(a) = 2\theta_i C^i{}_k(\partial_j a)g^{kj}.$$

8.2.1 Riemannian Geometry with Grassmann Exterior Algebra

We continue in the case of Ω^1 left-parallelisable with basis $\{e^i\}$, denoting the vector space it spans over the field by Λ^1. In this section, we consider the special case where $\mathrm{d}e^i = 0$ and the exterior algebra has the usual Grassmann relations with $(e^i)^2 = \{e^i, e^j\} = 0$. Then the condition for torsion free becomes $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ and cotorsion free becomes

$$\partial_r g_{pq} + g_{ij}C^i{}_r(\Gamma^j{}_{pq}) = \partial_p g_{rq} + g_{ij}C^i{}_p(\Gamma^j{}_{rq}).$$

We further assume that the metric coefficients g_{ij} and g^{ij} and the Christoffel symbols are constants, so the partial derivatives in this equation vanish and then torsion and cotorsion free amount to $\Gamma_{ijk} := g_{im}\Gamma^m{}_{jk}$ being totally symmetric.

In the inner case, we take a slightly different approach and assume that the tensors $\sigma^{ij}{}_{mn}$, $\alpha^i{}_{mn}$ are constants (which implies that $\Gamma^i{}_{jk}$ are also constants if θ has constant coefficients). Then ensuring that σ, α extend as bimodule maps depends on the representation theory of the algebra. For example, in the group algebra case $A = \Bbbk G$ of §1.6.2, the bimodule relations on Ω^1 take the form $e^i u = u\rho(u)^i{}_j e^j$ for all $u \in G$ for some right action (with matrices ρ) on Λ^1 and σ, α extend as bimodule maps precisely when they commute with all $\rho(u)$ (with the diagonal action on tensor products), i.e., when they are G-module maps. The flip map given on the basis by $\mathrm{flip}(e^i \otimes e^j) = e^j \otimes e^i$ will be an example that extends as a bimodule map in this case. The idea therefore is to solve for σ, α on Λ^1 using representation theory and then impose the torsion and metric compatibility conditions on these tensors as above to construct the noncommutative Riemannian geometry.

To follow this through concretely, we describe the results for $A = \mathbb{C}S_3$, the group algebra of the permutation group. Recall that the group here has generators u, v and an inner calculus in Example 1.50 with Grassmann algebra basis e^u, e^v, e^{uv}, e^{vu} and $-3\theta = e^{uv} + e^{vu}$. An associated element e^w is defined by $-3\theta = e^u + e^v + e^w$ associated to $w = uvu = vuv$, which also shows that θ has constant coefficients. We have $\mathrm{d}e^i = 0$ and we also found a metric g with constant coefficients in this basis, so we are in the setting above. Moreover, both the calculus and the metric have a natural invariance under the adjoint action of the group, generated by

$$P = \mathrm{Ad}_w: \quad e^u \leftrightarrow e^v, \quad e^{uv} \leftrightarrow e^{vu}$$
$$Q = \mathrm{Ad}_{uv}: \quad e^u \mapsto e^v \mapsto e^w \mapsto e^u$$

with e^{uv}, e^{vu} invariant in the latter case. We are therefore particularly interested in Ad-invariant connections.

Example 8.14 Let $A = \mathbb{C}S_3$ with the 4D calculus and metric in Example 1.50.

(1) There is a 1-parameter space of Ad-invariant bimodule WQLCs with $\sigma = \text{flip}$,

$$\nabla e^u = \lambda(\begin{pmatrix} e^u & e^v \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^u \\ e^v \end{pmatrix} + 3(e^v \otimes \theta + \theta \otimes e^v) + e^{uv} \otimes e^{vu} + e^{vu} \otimes e^{uv}),$$
$$\nabla e^v = \lambda(\begin{pmatrix} e^u & e^v \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^u \\ e^v \end{pmatrix} + 3(e^u \otimes \theta + \theta \otimes e^u) + e^{uv} \otimes e^{vu} + e^{vu} \otimes e^{uv}),$$
$$\nabla e^{uv} = \lambda((e^{vu} - e^{uv}) \otimes e^{uv} + e^{uv} \otimes e^{vu}),$$
$$\nabla e^{vu} = \lambda((e^{uv} - e^{vu}) \otimes e^{vu} + e^{vu} \otimes e^{uv})$$

with Riemann curvature

$$R_\nabla e^u = \lambda^2(e^{uv}e^{vu} \otimes (e^{vu} - e^{uv}) + 3(e^u e^v + (e^u - e^v)\theta) \otimes (e^v - e^w)),$$
$$R_\nabla e^v = \lambda^2(e^{uv}e^{vu} \otimes (e^{vu} - e^{uv}) + 3(e^u e^v + (e^u - e^v)\theta) \otimes (e^w - e^u)),$$
$$R_\nabla e^{uv} = \lambda^2 e^{uv}e^{vu} \otimes (2e^{vu} - e^{uv}), \quad R_\nabla e^{vu} = -\lambda^2 e^{uv}e^{vu} \otimes (2e^{uv} - e^{vu})$$

and Ricci $= 0$. The connection is $*$-preserving for $e^{i*} = -e^i$ if $\lambda \in \mathrm{i}\mathbb{R}$. Moreover, $\Delta = \Delta_\theta = g^{ij}\partial_i\partial_j$ on functions as in Example 1.50, while on 1-forms we find

Eigenvalues of Δ	Eigenvectors in Ω^1
$3\lambda^2$	$\{e^i\}$
$2 + 3\lambda^2$	$\{uv, vu\}\{e^u - e^v, e^v - e^w\}$
$\frac{4}{3} + 2\lambda + 3\lambda^2$	$\{u, v, w\}\theta, u(e^v - e^w), v(e^w - e^u), w(e^u - e^v)$
$\frac{4}{3} - 2\lambda + 3\lambda^2$	$\{u, v, w\}(e^{uv} - e^{vu}), u(e^u - \frac{2}{3}e^{uv}), v(e^v - \frac{2}{3}e^{uv}), w(e^w - \frac{2}{3}e^{uv})$
$2 \pm \sqrt{3}\lambda + 3\lambda^2$	$uv(e^{uv} + (1 \pm \sqrt{3})e^{vu}), vu((1 \pm \sqrt{3})e^{uv} + e^{vu})$

(2) The full moduli of Ad-invariant bimodule WQLCs is 2-parameter for σ and 1-parameter for α. Only $\lambda = 0$ in the above family is fully metric compatible. The full moduli dropping Ad-invariance is 12-parameter for σ and 4-parameter for α.

Proof We first define the matrices $\phi = \rho(u)$, $\psi = \rho(v)$ for the bimodule structure in Example 1.50, which come out as

$$\phi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & -2 & 1 & 1 \end{pmatrix}.$$

We then solve $[(\phi\otimes\phi), \tau] = 0$, $[(\psi\otimes\psi), \tau] = 0$ where $(\phi\otimes\phi)^{ij}{}_{mn} = \phi^i{}_m\phi^j{}_n$ and $\tau^{ij}{}_{mn} = \sigma^{ij}{}_{mn} - \delta^i{}_n\delta^j{}_m$ are regarded as 16×16 matrices for ij, mn as multi-indices. Similarly, we require $\phi\alpha = \alpha(\phi\otimes\phi)$ as matrices where α is regarded as 4×16. In each case we suppose α and τ are symmetric in their last two indices. By our analysis above, this gives our moduli of torsion free connections as 28-dimensional and 6-dimensional, respectively. The former are conveniently parametrised by the 15 values of $\tau^{ii}{}_{jj}$ other than $\tau^{33}{}_{11}$ (say) and a further 13 values. When we further demand the cotorsion equations that $\tau.\theta$ and α are totally symmetric in their three indices, these get cut down to 12 and 4-dimensional spaces respectively (in principle we need only the combination $\Gamma = \tau.\theta - \alpha$ totally symmetric, but this does not lead to further solutions). The 12-dimensional moduli space is conveniently parametrised by $\tau^{ii}{}_{jj}$ except for the four values $\tau^{33}{}_{11}$, $\tau^{44}{}_{11}$, $\tau^{44}{}_{22}$, $\tau^{44}{}_{33}$.

For full metric compatibility we need only look within this torsion-free cotorsion-free moduli space. This further condition then becomes

$$\tau^{ij}{}_{mn}\Gamma_{ijk} + \Gamma_{nmk} + \Gamma_{kmn} = 0 \tag{8.34}$$

and writing Γ in terms of τ, we see that this is quadratic in τ. To have a manageable size, we first impose invariance under P, Q (we commute with the matrices for P, Q in the same way as for ϕ, ψ above), i.e., look for Ad-invariant connections to give a 2-parameter moduli for τ and 1-parameter for α of torsion free cotorsion free invariant connections. Setting $\tau = 0$ focusses on this 1-dimensional moduli space of α for WQLCs as stated in (1) but we do not solve (8.34) unless $\lambda = 0$. Searching with τ in its full 2-parameter space yields no solutions to (8.34) either.

We now study this 1-dimensional parameter space further. In fact, the parameter enters as a scaling of α which we separate off, so we write $\nabla e^i = \lambda\alpha(e^i)$ where now α denotes the displayed right-hand side of ∇ without the λ factor. Note that while the equations for α are tensorial, the connection is not, for example $\nabla(ue^i) = \mathrm{d}u\otimes e^i + u\lambda\alpha(e^i)$. As an illustrative check, we verify that this indeed gives a bimodule connection. Thus for example, with $\lambda = 1$,

$$\begin{aligned}\nabla(e^u v) &= \nabla(v(e^{uv} - e^v)) = \mathrm{d}v\otimes(e^{uv} - e^v) + v\nabla(e^{uv} - e^v)\\ &= \sigma(e^u\otimes \mathrm{d}v) + v\Big(-e^{uv}\otimes e^{uv} - 2e^u\otimes e^u + e^v\otimes e^v\\ &\qquad - e^v\otimes e^u - e^u\otimes e^v - 3e^u\otimes\theta - 3\theta\otimes e^u\Big),\end{aligned}$$

which we want to equal $\sigma(e^u \otimes dv) + (\nabla e^u)v$. Here σ = flip on the basis means $\sigma(e^u \otimes dv) = \sigma(e^u \otimes v dv) = -\sigma(e^u \otimes e^v)v = -e^v \otimes e^u v = ve^v \otimes (e^{uv} - e^v) = dv \otimes (e^{uv} - e^v)$ and meanwhile

$$\begin{aligned}(\nabla e^u)v = v\Big(&-(e^{uv} - e^v) \otimes (e^{uv} - e^v) + 2e^v \otimes e^v - e^v \otimes (e^{uv} - e^v)\\ &-(e^{uv} - e^v) \otimes e^v - 3e^v \otimes \theta - 3\theta \otimes e^v - 6e^v \otimes e^v\\ &+(e^u - e^v) \otimes (e^w - e^v) + (e^w - e^v) \otimes (e^u - e^v)\Big)\end{aligned}$$

using the form of ∇e^u, the bimodule relations from Example 1.50 and $\theta v = v\theta + ve^v$. Expanding out $-3\theta = e^{uv} + e^{vu}$ and $e^w = e^{uv} + e^{vu} - e^u - e^v$, this equates to the previous expression from $\nabla(e^{uv} - e^v)$. One can similarly check this for u and for the other three basis elements of Λ^1 as well as weak metric compatibility for the metric in Example 1.50.

We then compute R_∇ from the connection, and from this we find Ricci = 0 defined with $i(e^i \wedge e^j) = (e^i \otimes e^j - e^j \otimes e^i)/2$. Finally, $\tau = 0$ means that $\Delta = \Delta_\theta$ on functions, where we already know from Chap. 1 that u, v, w have eigenvalue 4/3 and uv, vu eigenvalue 2, while 1 of course has eigenvalue 0. We denote these eigenvalues by μ_a when acting on $a = e, u, v, w, uv, vu$ in that order in S_3. On 1-forms we find $\Delta e^i = 3\lambda^2 e^i$ and then from Lemma 8.6 we have

$$\Delta(ae^i) = a((\mu_a + 3\lambda^2)e^i + 2d_a^k g_{km}\alpha_i{}^{mj} e^j)$$

for the Laplacian on a basis element of $\Omega^1(S_3)$, where $\partial_i a = d_a^i a$ as in Example 1.50. We are then able to compute its eigensystem as stated. □

In short, the quantum Levi-Civita connection for this model is unique and as expected has $\nabla e^i = 0$, but there is also a larger moduli of cotorsion free connections with curvature but Ricci flat.

Example 8.15 $U(su_2)$ with its 4D calculus and central metric $g = dx^i \otimes dx^i + \lambda^2 \theta \otimes \theta$ in Example 1.45 has no rotationally invariant weak bimodule QLC connection with constant coefficient α, σ other than the QLC $\nabla dx^i = \nabla\theta = 0$.

Proof We look among constant coefficients in our basis $\{dx^1, dx^2, dx^3, \theta\}$ (since the metric has constant coefficients). Then from an analysis of the possible invariant tensors, and since we want a symmetric output so as to be torsion free, we need

$$\alpha(dx^i) = A(dx^i \otimes \theta + \theta \otimes dx^i), \quad \alpha(\theta) = B dx^i \otimes dx^i + C\theta \otimes \theta$$

as θ is invariant. For this to be a bimodule map we need $\alpha([\theta, x^i]) = [\alpha(\theta), x^i]$, which tells us $A = C - B\lambda^2$ but $\alpha([dx^i, x^j]) = [\alpha(dx^i), x^j]$ forces $A = B = C = 0$. Thus, there are no nonzero invariant bimodule maps α. Similarly, imposing torsion free gives

$$\sigma(\mathrm{d}x^i\otimes\mathrm{d}x^j) = a\mathrm{d}x^i\otimes\mathrm{d}x^j+(1+a)\mathrm{d}x^j\otimes\mathrm{d}x^i, \quad \sigma(\theta\otimes\theta) = b\mathrm{d}x^i\otimes\mathrm{d}x^i+c\theta\otimes\theta,$$

$$\sigma(\mathrm{d}x^i\otimes\theta) = f\mathrm{d}x^i\otimes\theta+(1+f)\theta\otimes\mathrm{d}x^i, \quad \sigma(\theta\otimes\mathrm{d}x^i) = h\theta\otimes\mathrm{d}x^i+(1+h)\mathrm{d}x^i\otimes\theta.$$

We then check that σ is a bimodule map if and only if $b=0$, $c=1+2a$ and $f=h=a$. For example, $[\sigma(\mathrm{d}x^i\otimes\theta), x^j] = \sigma([\mathrm{d}x^i\otimes\theta, x^j])$ tells us that $c=1+2f$, $b=0$ and $f=a$. Thus, by Proposition 8.11, we have only the 1-parameter family of torsion free bimodule connections

$$\nabla\mathrm{d}x^i = -a(\mathrm{d}x^i\otimes\theta+\theta\otimes\mathrm{d}x^i), \quad \nabla\theta = -2a\theta\otimes\theta$$

of this invariant form. However, one can check directly that only $a=0$ is cotorsion free (and metric compatible). □

For our final example, we consider a model with a 3-parameter moduli space of quantum metrics, which turn out again to have constant coefficients in the right basis.

Example 8.16 We consider the noncommutative torus $\mathbb{C}_\theta[T^2]$ with relations $vu = quv$ and calculus in Example 1.36, where $q=e^{\mathrm{i}\theta}$ and θ is a real parameter which we now take to be an irrational multiple of 2π. In this case only the constants are central in the algebra. If ∇ is a left bimodule connection, then

$$\sigma(e^i\otimes e^1) = e^1\otimes e^i + [u,\nabla e^i]u^{-1}$$

and similarly for e^2 and v. As σ is a bimodule map we see that $[u,\nabla e^i]u^{-1}$ must have constant coefficients in our $\{e^1,e^2\}$ basis, and similarly for $[v,\nabla e^i]v^{-1}$, and deduce from this that the Christoffel symbols in this basis must themselves be constants, and hence that σ is just the flip map on the basis elements. For a general bimodule torsion free connection we therefore write

$$\nabla e^i = h^i{}_1 e^1\otimes e^1 + h^i{}_2 e^2\otimes e^2 + h^i{}_3(e^1\otimes e^2+e^2\otimes e^1)$$

for constants $h^i{}_j$. A general real quantum symmetric metric has the form $g=c_1e^1\otimes e^1+c_2e^2\otimes e^2+c_3(e^1\otimes e^2+e^2\otimes e^1)$ for constant real coefficients c_i since it has to be central and with $c_1c_2\neq c_3^2$ to be invertible. The cotorsion is

$$\begin{aligned}\mathrm{co}T_\nabla = {} & c_2(h^2{}_1e^1\wedge e^2\otimes e^1 + h^2{}_3e^1\wedge e^2\otimes e^2)\\ & + c_1(h^1{}_2e^2\wedge e^1\otimes e^2 + h^1{}_3e^2\wedge e^1\otimes e^1)\\ & + c_3(h^2{}_2e^2\wedge e^1\otimes e^2 + h^2{}_3e^2\wedge e^1\otimes e^1\\ & \quad + h^1{}_1e^1\wedge e^2\otimes e^1 + h^1{}_3e^1\wedge e^2\otimes e^2)\end{aligned}$$

so we obtain a bimodule WQLC if and only if

$$c_2 h^2{}_1 = c_1 h^1{}_3 + c_3(h^2{}_3 - h^1{}_1), \quad c_2 h^2{}_3 = c_1 h^1{}_2 + c_3(h^2{}_2 - h^1{}_3),$$

which is a 4-parameter moduli space. It corresponds to the lowered Christoffel symbols being totally symmetric. For example, if $c_3 \neq 0$ then we can solve for $h^1{}_1$ and $h^2{}_2$ in terms of the other variables, in which case

$$R_\nabla(e^i) = \rho^i{}_j e^1 \wedge e^2 \otimes e^j; \quad \rho = S\begin{pmatrix} c_3 & c_2 \\ -c_1 & -c_3 \end{pmatrix}; \quad S := \frac{h^1{}_2 h^2{}_1 - h^1{}_3 h^2{}_3}{c_3}.$$

If we lift in the obvious way by $i(e^1 \wedge e^2) = \frac{1}{2}(e^1 \otimes e^2 - e^2 \otimes e^1)$ then we find

$$\mathrm{Ricci} = \frac{S}{2} g$$

and that S is indeed the Ricci scalar curvature. If $c_3 = 0$ then we can solve for $h^1{}_2, h^2{}_1$ in terms of $h^1{}_3, h^2{}_3$ (and independently of $h^1{}_1, h^2{}_2$). In this case, we have the same form of Ricci tensor but with

$$S = \frac{c_1 h^1{}_3(h^1{}_3 - h^2{}_2) + c_2 h^2{}_3(h^2{}_3 - h^1{}_1)}{c_1 c_2}.$$

By contrast, the only full QLC for a nondegenerate quantum symmetric metric is the less interesting $\nabla e^i = 0$ with zero curvature. If we allow $\det(g) = 0$ then there can be solutions, for example if $c_1 c_2 = c_3^2$ and $c_3 \neq 0$ then there is a 3-parameter family of QLCs where ∇e^1 is arbitrary (so $h^1{}_i \in \mathbb{C}$ are free) and $\nabla e^2 = -\frac{c_1}{c_3} \nabla e^1$.

If we relax quantum symmetry then there are nondegenerate metrics with full QLCs. For example, $g = c_1 e^1 \otimes e^1 + c_4(e^1 \otimes e^2 - e^2 \otimes e^1)$ for $c_1, c_4 \neq 0$ has a 1-parameter family of QLCs $\nabla e^1 = 0$, $\nabla e^2 = \lambda e^1 \otimes e^1$ for $\lambda = 0$. ◇

8.2.2 Riemannian Geometry of Graphs and Finite Groups

Here we look at general quantum metrics in the case of Cayley graphs where the vertex set is a group G and arrows are right translation by $c \in \mathcal{C}$, a finite Ad-stable subset not containing the group identity and closed under inversion. The quantum metric here was already studied in Chap. 1 and we recall that it has the form

$$g = \sum_{c \in \mathcal{C}} g_c e^c \otimes e^{c^{-1}}$$

as dictated by being central. The functions g_c should nowhere vanish for nondegeneracy and should be real-valued for 'reality'. If we want to impose quantum symmetry for the canonical exterior algebra, this would be $g_c = g_{c^{-1}}$ for all $c \in \mathcal{C}$. Also note that the calculus is inner with $\theta = \sum_a e^a$, so Proposition 8.11 applies and we can work with ∇ in terms of σ, α. If we write

$$\sigma(e^a \otimes e^b) = \sum_{c,d \in \mathcal{C}} \sigma^{a,b}{}_{c,d} e^c \otimes e^d$$

in terms of a tensor on the group (the coefficients are functions) then σ being a bimodule map requires that $\sigma^{a,b}{}_{c,d}$ is zero unless $cd = ab$ in the group. We similarly have $\alpha(e^a) = \sum_{b,c \in \mathcal{C}} \alpha^a{}_{b,c} e^b \otimes e^c$ with $\alpha^a{}_{b,c} = 0$ unless $a = bc$. The bimodule conditions on α often force it to vanish so we concentrate on this case. We do not really need G to be finite provided we work with the left-invariant basis and some algebra $C(G)$ stable under right translation by $a \in \mathcal{C}$.

Lemma 8.17 *Let $\Omega^1(G)$ on a group G be given by $\mathcal{C} \subseteq G \setminus \{e\}$ and let $\alpha = 0$.*

(1) *Torsion freeness/compatibility (they are equivalent here) is*

$$e^a \wedge e^b + \sum_{c,d \in \mathcal{C}} \sigma^{a,b}{}_{c,d} e^c \wedge e^d = 0$$

for all $a, b \in \mathcal{C}$.

(2) *Metric compatibility is*

$$\sum_{c,m,n \in \mathcal{C}} g_c R_c(\sigma^{c^{-1},n}{}_{m,r}) \sigma^{c,m}{}_{p,q} = \delta_{r^{-1},q} R_p(g_q)$$

for all $r, p, q \in \mathcal{C}$.

(3) *Assuming vanishing torsion, vanishing of cotorsion is*

$$\sum_{a \in \mathcal{C}} R_a(g_c) e^a \wedge e^c + \sum_{a,b,d \in \mathcal{C}} g_d R_d(\sigma^{d^{-1},b}{}_{a,c^{-1}}) e^d \wedge e^a = 0$$

for all $c \in \mathcal{C}$.

(4) *The reality of the connection comes down to*

$$\sum_{c,d \in \mathcal{C}} R_{cd}(\sigma^{d^{-1},c^{-1}}{}_{p,q}) \overline{\sigma^{a,b}{}_{c,d}} = \delta_{p,b^{-1}} \delta_{q,a^{-1}}$$

for all $a, b, p, q \in \mathcal{C}$.

Proof (1) We know from Proposition 8.11 with $\alpha = 0$ that vanishing of torsion is equivalent to torsion-compatibility, which is the condition stated. (2) For metric compatibility, the condition in Proposition 8.11 using our notations and omitting commas in the tensor indices for brevity, is

$$\sum_{m,n} e^m \otimes g_n e^n \otimes e^{n^{-1}} = \sum_{b,c,p,q} \sigma\left(g_c e^c \otimes \sigma^{c^{-1}b}{}_{pq} e^p\right) \otimes e^q.$$

We expand out the second σ and move all coefficients to the left (this introduces right translations). We then match indices so that we can equate coefficients in our basis of $\Omega^{1\otimes_A 3}$ to obtain the equation stated on the coefficients. (3) This is similarly immediate from the expression for cotorsion in Proposition 8.11. (4) We need

$$\sigma((\sigma(e^a \otimes e^b))^\dagger) = \sigma\Big(\sum_{c,d} e^{d^{-1}} \otimes e^{c^{-1}} \overline{\sigma^{ab}{}_{cd}}\Big)$$
$$= \sum_{c,d,p,q} \sigma^{d^{-1}c^{-1}}{}_{pq} e^p \otimes e^q \overline{\sigma^{ab}{}_{cd}} = e^{b^{-1}} \otimes e^{a^{-1}}$$

and we use the commutation relations to bring the functions together so that we can equate basis coefficients. □

One can also look for connections where σ obeys the braid relations with tensor products over the algebra, but note that when written in terms of $\sigma^{a,b}{}_{c,d}$, the form of these equations is modified from the standard braid relations as the entries are functions and do not typically commute with the basic 1-forms. In fact, the lemma also holds for left-covariant calculi, i.e. until now we have not used that $\mathcal{C}$ is Ad-stable. However, the latter has the merit of a canonical 'Woronowicz' $\Omega(G)$ as in Theorem 1.56. Then the σ part of the torsion free condition is $(\mathrm{id}-\Psi)(\mathrm{id}+\sigma) = 0$, where the crossed module braiding $\Psi(e^a \otimes e^b) = e^{aba^{-1}} \otimes e^a$ on the basis is extended as a bimodule map. Moreover, we have a canonical lift and hence Ricci tensor in our constructive approach.

Lemma 8.18 *For finite Ad-stable $\mathcal{C}$ and the canonical $\Omega(G)$, there is a canonical bicovariant lift i given on left-invariant 2-forms by*

$$i : \Lambda^2 \to \Lambda^1 \otimes \Lambda^1, \quad i(v) = \tilde{v} - \frac{\mathrm{id} + \Psi + \cdots + \Psi^{n-1}}{n} \tilde{v},$$

where n is the order of the crossed module braiding Ψ from (1.8) and $\tilde{v}$ is a representative of v in $\Lambda^1 \otimes \Lambda^1$.

Proof We extend i to general 2-forms as a bimodule map, recalling that we can identify $\Omega^1 \otimes_{C(G)} \Omega^1 = C(G) \otimes (\Lambda^1)^{\otimes 2}$ and $\Omega^2 = C(G) \otimes \Lambda^2$, where $\Lambda^2 = (\Lambda^1)^{\otimes 2}/\ker(\mathrm{id} - \Psi)$. But Ψ permutes the basis vectors $\{e^a \otimes e^b \mid a, b \in \mathcal{C}\}$ of $(\Lambda^1)^{\otimes 2}$ and hence has some finite order n, say, and we let $f(\Psi) = \mathrm{id} + \Psi + \cdots + \Psi^{n-1}$. Then $(\Psi - \mathrm{id}) f(\Psi) = 0$, which implies $\pi \circ i = \mathrm{id}$, and $f(1) = n$, which implies that if $\wedge\tilde{v} = 0$, i.e., $\Psi\tilde{v} = \tilde{v}$ then the right-hand side of the expression for

i vanishes, hence i is well defined. The same method applies to any Hopf algebra with a bicovariant calculus and a suitable polynomial in the braiding. $\square$

The intrinsic choice of metric is the Euclidean one (1.9) where $g_c = 1$. In the bicovariant case, we can consider the canonical left 'Maurer–Cartan' connection with $\nabla e^c = 0$ and $\sigma = \Psi$ and the right 'Maurer–Cartan' connection

$$\nabla e^c = \theta \otimes e^c - \sum_b e^b \otimes e^{b^{-1}cb} = \theta \otimes e^c - \Psi^{-1}(e^c \otimes \theta),$$

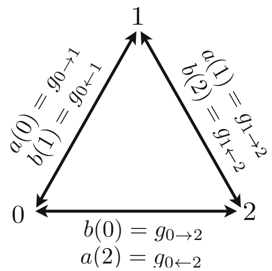
$$\sigma(e^a \otimes e^b) = e^b \otimes e^{b^{-1}ab} = \Psi^{-1}(e^a \otimes e^b)$$

(see Example 3.74), where we recognised expressions in terms of the crossed module braiding Ψ; both connections are compatible with the Euclidean metric. Both also have $R_\nabla = 0$ and typically have nonzero torsion (unless the braiding is trivial, as when the group is abelian, in which case $\nabla e^c = 0$). For actual QLCs or WQLCs for nonabelian groups or when the metric is more general, we will have to work harder. We content ourselves with three geometrically motivated examples.

Example 8.19 (Riemannian Geometry of a Triangle Graph) We take $G = \mathbb{Z}_3$ with the universal $\Omega^1(\mathbb{Z}_3)$ and a left-invariant basis e^1, e^2 corresponding to $\mathcal{C} = \{1, 2\} \subset \mathbb{Z}_3$, where $e^1 = \omega_{0\to1} + \omega_{1\to2} + \omega_{2\to0}$ and $e^2 = \omega_{0\to2} + \omega_{1\to0} + \omega_{2\to1}$ from a graph point of view. The relations are $e^1 f = R_1(f)e^1$ and $e^2 f = R_2(f)e^2$ and $\mathrm{d}f = (R_1(f) - f)e^1 + (R_2(f) - f)e^2$ for all $f \in \mathbb{C}(\mathbb{Z}_3)$, where $R_1\delta_i = \delta_{i-1}$ on δ-functions, etc. To be central, a quantum metric therefore takes the form

$$g = ae^1 \otimes e^2 + be^2 \otimes e^1$$

for nowhere-zero functions $a, b \in \mathbb{C}(\mathbb{Z}_3)$. These correspond to 'directed edge weights' $a(0) = g_{0\to1}$, $b(1) = g_{1\to0}$ etc., from the graph calculus point of view according to



The canonical $*$-structure is $e^{1*} = -e^2$ which for a real metric needs a and b to be real-valued. It is natural to focus on the edge-symmetric case where the weights are the same in the two directions, for which we need $b = R_2(a)$. So our more geometric moduli of central metrics has just three 'lengths' as the three values of a.

We set

$$\rho = \frac{R_2 a}{a},$$

which is 1 if and only if a is constant, i.e., the triangle is equilateral. The inverse metric is then $(e^1, e^2) = 1/a$, $(e^2, e^1) = 1/R_2(a)$ and zero on the diagonals. The canonical exterior algebra from the finite group construction is the Grassmann algebra on the e^i (since the group is abelian) with dimensions 1 : 2 : 1 and top form $\mathrm{Vol} = e_1 \wedge e_2$ (this is antihermitian, so the geometric normalisation would be i times this). If we take this calculus then quantum symmetry would require $a = b$ which in the symmetric edge length case would force us to $a = b$ constant. We would like to keep general edge lengths on our triangle, so we do *not* suppose that the metric is quantum symmetric for this calculus but *do* suppose that it is edge-symmetric.

Next, for the maps α and σ in Proposition 8.11 to be bimodule maps, they should respect the total $\mathbb{Z}_3$-degrees of the e^i and therefore have the form

$$\alpha(e^1) = a_1 e^2 \otimes e^2,\ \alpha(e^2) = a_2 e^1 \otimes e^1,\ \sigma(e^1 \otimes e^1) = c_1 e^1 \otimes e^1,\ \sigma(e^2 \otimes e^2) = c_2 e^2 \otimes e^2,$$

$$\sigma(e^1 \otimes e^2) = d_{11} e^1 \otimes e^2 + d_{12} e^2 \otimes e^1, \quad \sigma(e^2 \otimes e^1) = d_{21} e^1 \otimes e^2 + d_{22} e^2 \otimes e^1.$$

We then examine the metric compatibility condition $\theta \otimes g = \sigma_{12}(\mathrm{id} \otimes (\sigma_\theta - \alpha))g - (\alpha \otimes \mathrm{id})g$ in Proposition 8.11 where $\theta = e^1 + e^2$ and $\sigma_\theta(e^i) = \sigma(e^i \otimes \theta)$. Moving all coefficients to the left, this is

$$\begin{aligned}
&R_1(a)e^1 \otimes e^1 \otimes e^2 + a e^1 \otimes e^2 \otimes e^1 + R_2(a)e^2 \otimes e^1 \otimes e^2 + R_1(a)e^2 \otimes e^2 \otimes e^1 \\
&\quad = a R_1(c_2)\sigma(e^1 \otimes e^2) \otimes e^2 + a c_1 R_1(d_{21}) e^1 \otimes e^1 \otimes e^2 + a R_1(d_{22})\sigma(e^1 \otimes e^2) \otimes e^1 \\
&\qquad + R_2(a)R_2(c_1)\sigma(e^2 \otimes e^1) \otimes e^1 + R_2(a)R_2(d_{11})\sigma(e^2 \otimes e^1) \otimes e^2 \\
&\qquad + R_2(a)R_2(d_{12})c_2 e^2 \otimes e^2 \otimes e^1 - a R_1(a_2)c_1 e^1 \otimes e^1 \otimes e^1 \\
&\qquad - R_2(a)R_2(a_1)c_2 e^2 \otimes e^2 \otimes e^2 - a a_1 e^2 \otimes e^2 \otimes e^2 - R_2(a)a_2 e^1 \otimes e^1 \otimes e^1
\end{aligned}$$

which gives us 7 independent equations

$$R_1(d_{22})d_{12} = -\rho R_2(c_1)d_{22}, \quad R_2(d_{11})d_{21} = -\rho^{-1} d_{11} R_1(c_2),$$

$$d_{12} R_1(c_2) = \rho, \quad d_{21} R_2(c_1) = \rho^{-1}, \quad R_2(d_{11})d_{22} = 0,$$

$$R_2(a_1)c_2 = -\rho^{-1} a_1, \quad R_1(a_2)c_1 = -\rho a_2.$$

We have analysed metric compatibility ahead of torsion-freeness so that the above can also be used with other choices of exterior algebra. For our Grassmann case, torsion freeness, which by Proposition 8.11 means $\wedge\alpha = 0$ and $\wedge\sigma = -\wedge$, amounts to the restriction $d_{12} = 1 + d_{11}, d_{21} = 1 + d_{22}$. Regarding d_{11}, d_{22} as the variables, metric compatibility as above tells us that $R_2(d_{11}) = -d_{11}/((1+d_{11})(1+d_{22}))$ (and a similar equation for d_{22}). Since $R_2(d_{11})d_{22} = 0$, this simplifies to $R_2(d_{11}) = -d_{11}/(1+d_{11})$, which forces $d_{11} = 0$ or $d_{11} = -2$.

(i) Let $d_{11} = 0$, in which case $d_{12} = 1$, $c_2 = R_2(\rho)$ and $R_1(d_{22}) = -d_{22}/(1+d_{22})$. The latter has one solution with $d_{22} = 0$ hence $d_{21} = 1$, $c_1 = R_1\rho^{-1}$ from which we deduce that $a_1 = a_2 = 0$. This gives us the unique QLC in this context

$$\nabla e^1 = (1 - R_1\rho^{-1})e^1 \otimes e^1, \quad \nabla e^2 = (1 - R_2\rho)e^2 \otimes e^2,$$

$$\sigma(e^i \otimes e^j) = \left\{ \begin{matrix} 1 & i \neq j \\ R_1\rho^{-1} & i = j = 1 \\ R_2\rho & i = j = 2 \end{matrix} \right\} e^j \otimes e^i.$$

It is clear from the form of σ that this is $*$-preserving when ρ is real.

(ii) Still with $d_{11} = 0$ and $c_2 = R_2(\rho)$, we can take $d_{22} = -2$, then $d_{21} = -1$ and $c_1 = -R_1(\rho^{-1})$. Then $a_1 = 0$ and a_2 is a function obeying $R_1(a_2) = a_2/R_2(\rho)$, which is solved by $a_2 = \lambda R_1(a)$ for a free parameter λ, giving a family of QLCs

$$\begin{aligned}
&\nabla e^1 = (1 + R_1\rho^{-1})e^1 \otimes e^1, \\
&\nabla e^2 = (1 - R_2\rho)e^2 \otimes e^2 + 2(e^1 \otimes e^2 + e^2 \otimes e^1) + \lambda R_1 a e^1 \otimes e^1, \\
&\sigma(e^1 \otimes e^1) = -R_1\rho^{-1}e^1 \otimes e^1, \quad \sigma(e^2 \otimes e^2) = R_2\rho e^2 \otimes e^2, \\
&\sigma(e^2 \otimes e^1) = -e^1 \otimes e^2 - 2e^2 \otimes e^1
\end{aligned}$$

and flip for the remaining case of σ.

(iii) Let $d_{11} = -2$ then $d_{12} = -1$ and $d_{22} = 0$, hence $d_{21} = 1$, $c_2 = -R_2(\rho)$ and $c_1 = R_1\rho^{-1}$. Then $a_2 = 0$ and a_1 is a function obeying $R_2(a_1) = R_1(\rho)a_1$ again solved as $a_1 = \lambda R_1(a)$ for a free parameter λ. The connection and generalised braiding are similar to case (ii) with $1 \leftrightarrow 2$ and $\rho \leftrightarrow \rho^{-1}$.

Of these solutions, case (i) is the more natural since one has $\nabla e^i = 0$ in the equilateral case where a is constant. In case (i) the geometric wave operator is

$$\Delta f = (\ ,\ )\nabla \mathrm{d} f = \frac{1}{a}\partial_1\partial_2 f + \frac{1}{R_2(a)}\partial_2\partial_1 f = \frac{3}{a}(1 + \rho^{-1})(f - \mathrm{av}(f))),$$

where $\partial_i = R_i - \mathrm{id}$ and av is the average value. This coincides with the inner element Laplacian ${}_\theta\Delta f = -\frac{2}{a}\partial_1 f - \frac{2}{R_2(a)}\partial_2 f$ if a is constant. The curvature computes as

$$R_\nabla(e^1) = -\partial_1(\rho^{-1})\mathrm{Vol} \otimes e^1, \quad R_\nabla(e^2) = \partial_2(\rho)\mathrm{Vol} \otimes e^2$$

which with the canonical lift $i(\mathrm{Vol}) = \frac{1}{2}(e^1 \otimes e^2 - e^2 \otimes e^1)$ and contraction gives

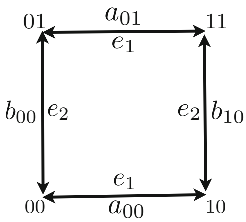
$$\mathrm{Ricci} = \frac{1}{2}\big(\partial_1(\rho)e^1 \otimes e^2 + \partial_2(\rho^{-1})e^2 \otimes e^1\big). \qquad \diamond$$

The simpler case of $\mathbb{Z}_2$ is in Exercise E8.4 while the general case of $\mathbb{Z}_n, \mathbb{Z}$ their 2-dimensional calculi and the Euclidean metric is in E8.6 (the latter can also be solved for general edge-symmetric metrics). The above triangle example is in fact a warm up to the square case, where the proofs will include fewer details.

Example 8.20 (Riemannian Geometry of a Square Graph) We take $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with its canonical 2D calculus as in Example 1.61, now with upper indices on the basic forms e^1, e^2 and now calling the nowhere vanishing metric coefficient functions a, b, so that a quantum metric, in order to be central, has the form

$$g = ae^1 \otimes e^1 + be^2 \otimes e^2.$$

This is real for the canonical $*$ structure $e^{i*} = -e^i$ precisely when a, b are real-valued. The diagram in Example 1.61 explained how the coefficient values correspond to directed edge weights and as in the preceding example we focus on the symmetric edge length case where the weight assigned to an edge does not depend on the direction of the arrow, which means $\partial_1 a = \partial_2 b = 0$. Here $\partial_i = R_i - \mathrm{id}$ and R_1, R_2 are translation by 10, 01 respectively in a compact notation for the elements of the group. So in this case the metric has four independent parameters $a_{00}, a_{01}, b_{00}, b_{10}$ as 'edge lengths' attached to edges according to



We will say that the metric has Euclidean signature if $a, b > 0$ at all points and Minkowski signature if, say, $a < 0, b > 0$ at all points (so that e^1 would be the time direction in this case).

As for the preceding example, we use Proposition 8.11 to construct bimodule connections. It is immediate that there can be no nonzero bimodule map α (so we are free in what follows to use α for something else) while σ to be a bimodule map must have the form

$$\sigma(e^1 \otimes e^1) = Pe^1 \otimes e^1 + Be^2 \otimes e^2, \quad \sigma(e^2 \otimes e^2) = Qe^2 \otimes e^2 + Ae^1 \otimes e^1,$$
$$\sigma(e^1 \otimes e^2) = \alpha e^2 \otimes e^1 + \gamma e^1 \otimes e^2, \quad \sigma(e^2 \otimes e^1) = \beta e^1 \otimes e^2 + \delta e^2 \otimes e^1$$

for some functional parameters $P, Q, A, B, \alpha, \beta, \gamma, \delta$. Assuming torsion compatibility (8.11) gives $\gamma = \alpha - 1$ and $\delta = \beta - 1$. Under the assumption that σ is invertible, we write the equations for metric compatibility as

$$\sigma^{-1}(\theta \otimes g^1) \otimes g^2 = g^1 \otimes \sigma(g^2 \otimes \theta),$$

where $g = g^1 \otimes g^2$, which comes down to

$$\begin{pmatrix} QR_1a & -AR_2b \\ -BR_1a & PR_2b \end{pmatrix} = (PQ - BA) \begin{pmatrix} aR_1P & a(R_1\alpha - 1) \\ b(R_2\beta - 1) & bR_2Q \end{pmatrix},$$

$$\begin{pmatrix} (\beta-1)R_1b & -\beta R_2a \\ -\alpha R_1b & (\alpha-1)R_2a \end{pmatrix} = (1 - \alpha - \beta) \begin{pmatrix} aR_1B & aR_1\alpha \\ bR_2\beta & bR_2A \end{pmatrix}$$

as well as invertibility of the factors in front of the matrices. We look at two cases.

(i) In the 'rectangular' case where a, b are constant, one finds $\alpha = \beta = 1$, $A = B = 0$ and P, Q are invertible functions obeying $PR_1P = QR_2Q = 1$ i.e., a 4-parameter moduli of QLCs given by $P = (p_1, p_2, p_1^{-1}, p_2^{-1})$ and $Q = (q_1, q_1^{-1}, q_2, q_2^{-1})$ for nonzero constants p_i, q_i, where we list the values on the points in order 00, 01, 10, 11. The element $\theta = e^1 + e^2$ makes the calculus inner with the result that connections and curvature take the form

$$\nabla e^1 = (1 - P)e^1 \otimes e^1, \quad \nabla e^2 = (1 - Q)e^2 \otimes e^2,$$

$$R_\nabla e^1 = (\partial_2 P)\mathrm{Vol} \otimes e^1, \quad R_\nabla e^2 = -(\partial_1 Q)\mathrm{Vol} \otimes e^2,$$

which one can think of as applying Exercise E8.4 to each $\mathbb{Z}_2$ edge of the square. The connection is $*$-preserving when $\sigma \circ \dagger \circ \sigma = \dagger$, which comes down to $P^*P = Q^*Q = 1$ or $|p_i| = |q_i| = 1$. The Ricci tensor is defined by a lifting map i, for which in our case there is a canonical choice $i(\mathrm{Vol}) = \frac{1}{2}(e^1 \otimes e^2 - e^2 \otimes e^1)$ given the Grassmann nature of the exterior algebra. Then

$$\mathrm{Ricci} = \frac{1}{2}\left((\partial_2 P^{-1})e^2 \otimes e^1 - (\partial_1 Q^{-1})e^1 \otimes e^2\right), \quad S = 0.$$

This Ricci tensor obeys the same reality condition as the metric when $\partial_2 P^{-1} = \partial_1 Q^{-1}$ and is quantum symmetric when $\partial_2 P^{-1} = -\partial_1 Q^{-1}$, so both are possible only when the full curvature vanishes, i.e., when $p_1 = p_2$ and $q_1 = q_2$, which is a 2-parameter moduli space of flat QLCs.

(ii) Of principal interest is the generic case where *neither* a, b are constant. This is more involved but can similarly be solved, resulting in a 1-parameter moduli space of QLCs with

$$\sigma(e^1 \otimes e^1) = -Q^{-1}e^1 \otimes e^1 + \frac{b(R_2\beta - 1)}{a}e^2 \otimes e^2,$$
$$\sigma(e^2 \otimes e^2) = Qe^2 \otimes e^2 + \frac{a(R_1\alpha - 1)}{b}e^1 \otimes e^1,$$
$$\sigma(e^1 \otimes e^2) = \alpha e^2 \otimes e^1 + (\alpha - 1)e^1 \otimes e^2,$$
$$\sigma(e^2 \otimes e^1) = \beta e^1 \otimes e^2 + (\beta - 1)e^2 \otimes e^1,$$

and now Q, α, β are functions on the group defined as

$$Q = (q, q^{-1}, q^{-1}, q) = q^\chi, \quad \alpha = (\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}}), \quad \beta = (1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1)$$

when we use the same order of points as above. Here q is the free nonzero parameter and $\chi = (1, -1, -1, 1)$ or $\chi(i, j) = (-1)^{i+j}$ is a function on $\mathbb{Z}_2 \times \mathbb{Z}_2$. As a matrix $\sigma^{i_1 i_2}{}_{j_1 j_2}$, where the multi-indices are in order $11, 12, 21, 22$, we have

$$\sigma = \begin{pmatrix} -Q^{-1} & 0 & 0 & \frac{b(R_2\beta-1)}{a} \\ 0 & \alpha - 1 & \alpha & 0 \\ 0 & \beta & \beta - 1 & 0 \\ \frac{a(R_1\alpha-1)}{b} & 0 & 0 & Q \end{pmatrix}.$$

This has a full '8-vertex-like' form normally associated with quantum integrable systems but here arising naturally out of nothing but the quantum Riemannian geometry of a square graph. What we have in our case is a field of such generalised braiding matrices because the entries here are functions on the group. The eigenvalues are $-1, \alpha\beta, -Q^{-1}, Q$ as functions on the group.

The connection is then given as $\nabla\omega = \theta \otimes \omega - \sigma(\omega \otimes \theta)$ for any 1-form ω, so that

$$\nabla e^1 = (1 + Q^{-1})e^1 \otimes e^1 + (1 - \alpha)(e^1 \otimes e^2 + e^2 \otimes e^1) - \frac{b}{a}(R_2\beta - 1)e^2 \otimes e^2,$$

$$\nabla e^2 = -\frac{a}{b}(R_1\alpha - 1)e^1 \otimes e^1 + (1 - \beta)(e^1 \otimes e^2 + e^2 \otimes e^1) + (1 - Q)e^2 \otimes e^2,$$

which leads to the geometric Laplacian $\Delta f = (\ ,\ )\nabla((\partial_i f)e^i)$ as

$$\begin{aligned}\Delta f &= -\frac{2}{a}\partial_1 f - \frac{2}{b}\partial_2 f + \partial_i f(\ ,\ )\nabla e^i \\ &= -\left(\frac{R_2\beta - Q^{-1}}{a}\right)\partial_1 f - \left(\frac{R_1\alpha + Q}{b}\right)\partial_2 f\end{aligned}$$

using our formula for ∇, the connection property, and $\partial_i^2 = -2\partial_i$. This can only coincide with Δ_θ when $\nabla e^i = 0$, which is possible as part of the exceptional solutions but not within the class above. Also note that the Laplacian has an expected symmetry swapping the roles of a, b but also needs one to swap $Q, -Q^{-1}$, implying that we cannot just work with $q = 1$, say, if we want to preserve this symmetry of the square graph. Apart from the zero eigenvalue, there are typically two positive eigenvalues and one that can be either sign and is zero in the constant case, but there are regions in the metric parameter space in which this is not the case, notably where some of the eigenvalues become complex.

Next, the curvatures are given by

$$\begin{aligned}R_\nabla e^1 &= \big(Q^{-1}R_1\alpha - Q\alpha + (1-\alpha)(R_1\beta - 1) + \frac{R_2 a}{a}(R_2\beta - 1)(R_2R_1\alpha - 1)\big)\mathrm{Vol}\otimes e^1 \\ &\quad + \big(Q^{-1}(1-\alpha) + \alpha(R_2\alpha - 1) + \frac{Q^{-1}R_1 b}{a}(\beta^{-1} - 1)) + \frac{b}{a}(R_2\beta - 1)R_2\beta\big)\mathrm{Vol}\otimes e^2,\end{aligned}$$

where $\mathrm{Vol} = e^1 \wedge e^2$, and a similar formula for $R_\nabla e^2$ interchanging e^1, e^2; R_1, R_2; α, β; a, b and $Q, -Q^{-1}$ (so that Vol also changes sign). One can discern contributions from $q \neq 1$ and from a, b nonconstant. The 'reality' condition for the connection to be $*$-preserving comes down $|q| = 1$.

If we write $R_\nabla e^i = \rho_{ij}\mathrm{Vol}\otimes e^j$ and use the canonical symmetric lift of Vol then

$$\mathrm{Ricci} = ((\ ,\ )\otimes \mathrm{id})(\mathrm{id}\otimes i\otimes \mathrm{id})(\mathrm{id}\otimes R_\nabla)(g) = \frac{1}{2}\begin{pmatrix} -R_2\rho_{21} & -R_2\rho_{22} \\ R_1\rho_{11} & R_1\rho_{12}\end{pmatrix}$$

as the matrix of coefficients on the left in our tensor product basis. Applying $(\ ,\ )$ again, we have scalar curvature

$$S = \frac{1}{2}\Big(-\frac{R_2\rho_{21}}{a} + \frac{R_1\rho_{12}}{b}\Big)$$

which is invariant under the interchange above.

The most general Ricci curvature for the canonical i and general q, a, b is more complicated but for $q = 1$, say, it has values

$$\mathrm{Ricci}_{q=1} = \frac{1}{2}\begin{pmatrix} \frac{1}{b}(-\frac{\partial_2 a}{\alpha} + \chi\frac{\partial_1 b}{\beta}) & -\frac{\partial_1 b}{b}(\alpha + \frac{1}{\alpha} - \chi - 2) \\ -\frac{\partial_2 a}{a}(\beta + \frac{1}{\beta} - \chi - 2) & \frac{1}{a}(-\frac{\partial_2 a}{\alpha} + \chi\frac{\partial_1 b}{\beta}) \end{pmatrix}$$

for the matrix of coefficients. This is generically neither quantum symmetric nor real in the sense of the metric, so again does not fully meet our criteria for a Ricci tensor. Although it is possible to improve this by modifying i in the spirit of our general scheme, this produces rather involved results leading us to continue with the canonical i. In that case the scalar curvature comes out as

$$S = -\frac{1}{4ab}\Big((3 + q + (1-q)\chi)\frac{\partial_2 a}{\alpha} + (1 - q^{-1} - (3 + q^{-1})\chi)\frac{\partial_1 b}{\beta}\Big).$$

Finally, it is not obvious what measure we should use to integrate either of these, but if we take the measure $\mu = |ab| = ab$ (we assume for the sake of discussion that the a, b are positive edge lengths, i.e., the theory has Euclidean signature) and sum over $\mathbb{Z}_2 \times \mathbb{Z}_2$ then we have

$$\int S = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu S = (a_{00} - a_{01})^2(\frac{1}{a_{00}} + \frac{1}{a_{01}}) + (b_{00} - b_{10})^2(\frac{1}{b_{00}} + \frac{1}{b_{10}})$$

independently of q. We consider this action as some kind of energy of the metric configuration. In the Euclidean signature case it has a 'bathtub' shape with minimum at a, b constant. If we took other measures such as $\mu = 1$ or $\mu = \sqrt{|g|} = \sqrt{|ab|}$ then we would not have invariance under q so the action would not depend only on the metric but on the choice of ∇. Quantum gravity of course involves 'functional integration' over the space of metrics, which would now entail a 4-dimensional integral over our space of metrics. Note that the generic solution which we focussed on reduces at constant a, b to a 1-parameter line within our larger 4-parameter moduli in part (i). There is in principle also an intermediate moduli space for the case of just one of a, b constant. ◇

For our final example, we take a nonabelian group so that the canonical exterior algebra is not just of Grassmann form. As the algebra is much more complicated, we will focus on its natural Euclidean metric and Ad-invariant connections as its intrinsic geometry.

Example 8.21 (Riemannian Geometry of the Permutation Group) We fix $G = S_3$ with its 3D calculus given by 2-cycles as in Example 1.60 and its Euclidean metric, which is left-invariant. In this case, we search among left-invariant connections so that the Christoffel symbols in the $\{e^u, e^v, e^w\}$ basis are constants like the metric coefficients. Clearly there are no S_3-grade-preserving maps $\Lambda^1 \to \Lambda^1 \otimes \Lambda^1$, so we have to take $\alpha = 0$. Similarly, one can analyse the possible σ that are Ad-invariant, and this gives the 5-parameter matrix in Example 3.76. The associated connections are controlled by

$$\sigma_\theta(e^u) = ae^u \otimes e^u + ee^u \otimes (e^v + e^w) + d(e^v + e^w) \otimes e^u \\ + b(e^v \otimes e^v + e^w \otimes e^w) + c(e^v \otimes e^w + e^w \otimes e^v),$$

and cyclic permutations of this. We are particularly interested in torsion free connections, which using the relations of the calculus is easily found to amount to $c - 1 = e = d - 1$, while similarly, the condition to be cotorsion free is $c - 1 = b = d - 1$, so we have a 2-parameter space of WQLCs (which are both). Writing $\lambda = c - a - 2$ and $\mu = -c$, we have

$$\nabla e^u = (3 + \lambda)e^u \otimes e^u + (1 + \mu)\theta \otimes \theta - \Psi^{-1}(e^u \otimes \theta),$$

where $\Psi^{-1}(e^a \otimes e^b) = e^b \otimes e^{b^{-1}ab}$, as the 2-parameter moduli. Note that this moduli of WQLCs is by construction invariant under cyclic rotations $e^u \to e^v \to e^w \to e^u$ and indeed has no intersection with a different 2-parameter family of WQLCs coming from the frame bundle approach in Example 5.83 other than the unique regular point $(a, b, c, d, e) = (\frac{5}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ or $\lambda = -3$, $\mu = -\frac{2}{3}$. Recall that regularity was an additional condition that R_∇ comes from a 'Lie algebra-valued' 2-form F_α at the level of the frame bundle in Example 5.83. Within our present 2-parameter family, this intersection point is the unique connection for which $\nabla\theta = 0$.

The associated generalised braiding and curvature for our 2-parameter moduli of cyclically invariant WQLCs are

$$\sigma(e^a \otimes e^a) = -(1+\lambda)e^a \otimes e^a - (1+\mu)g, \quad \sigma(e^a \otimes e^b) = -e^a \otimes e^b - \mu \sum_{cd=ab} e^c \otimes e^d$$

for $a \neq b$ and

$$\begin{aligned} R_\nabla e^u &= -(3 + \lambda)\big((1 + \mu)(e^u \wedge \theta \otimes \theta + \theta \wedge g) + e^v \wedge e^u \otimes e^v + e^w \wedge e^u \otimes e^w\big) \\ &\quad + (2 + \lambda)\mathrm{d}e^u \otimes e^u - \mathrm{d}e^v \otimes e^w - \mathrm{d}e^w \otimes e^v \\ &= -(3 + \lambda)\big(\mu \mathrm{d}e^u \otimes e^u + (2(1 + \mu)e^u e^v - \mu e^v e^u) \otimes e^v \\ &\quad + (2(1 + \mu)e^u e^w - \mu e^w e^u) \otimes e^w\big) - (\mathrm{d} \otimes \mathrm{id})\Psi^{-1}(e^u \otimes \theta). \end{aligned}$$

For the Ricci tensor, we use the canonical i in Lemma 8.18, which in our case is

$$i(e^a \wedge e^b) = e^a \otimes e^b - \frac{1}{3} \sum_{cd=ab} e^c \otimes e^d$$

for all $a \neq b$, because $ab = uv$ or $ab = vu$ and in each case Ψ has order 3 permuting among the three vectors $\{e^c \otimes e^d \mid cd = ab\}$. This gives

$$\text{Ricci} = -\frac{2}{3}((1 + 2(3 + \lambda))g + \frac{2}{3}(1 - \mu(3 + \lambda))\theta \otimes \theta, \quad S = -(3 + \lambda)(2 + \mu),$$

which means that this choice of i gives Ricci with the same quantum symmetry property as the metric as desired. We also see that there is a 1-parameter moduli of 'Einstein connections' where

$$\mu = \frac{1}{3 + \lambda}, \quad \text{Ricci} = -\frac{2}{3}(1 + 2(3 + \lambda))g, \quad S = -2(1 + 2(3 + \lambda)),$$

which does not include the special point of intersection with the frame bundle approach, where

$$\text{Ricci} = \frac{2}{3}(-g + \theta \otimes \theta), \quad S = 0.$$

The extra $\theta \otimes \theta$ term here is forced by the regularity requirement as a purely 'quantum' feature that does not have a classical analogue.

Finally, the geometric Laplacian for our present 2-parameter moduli of cyclically invariant WQLCs is easily found as

$$\Delta f = (\,,\,)\nabla \mathrm{d} f = \sum_{a \in \mathcal{C}} \partial_a f(\,,\,)\nabla e^a + \sum_{a \in \mathcal{C}} (\partial_a)^2 f = -\frac{1}{2}(3 + \lambda + 3\mu)\Delta_\theta f,$$

where $\Delta_\theta f = (2\theta, \mathrm{d} f) = -2(\mathrm{d} f, \theta) = {}_\theta\Delta f$ (as all elements of $\mathcal{C}$ square to the identity in this example) is the canonical Laplacian for an inner calculus in Proposition 1.18. Thus we can have $\Delta = \Delta_\theta$, for example at the special point of intersection with the frame bundle approach. For comparison, if we look at the 2-parameter moduli of WQLCs in the frame bundle approach in Example 5.83, the Laplacian using the ∇ there is

$$\Delta f = {}_\theta\Delta f - \sum_a (3\beta_a + 1)\partial_a f,$$

where β_a parametrised our space with the relation $\sum_a \beta_a = -1$. In that moduli, only the special regular point has $\Delta = \Delta_\theta$. $\diamond$

8.2.3 *The Riemannian Structure of q-Deformed Examples*

Two standard q-Riemannian geometries have already been constructed in depth in Chap. 5 using quantum frame bundles, namely on the q-sphere $\mathbb{C}_q[S^2]$ and the quantum group $\mathbb{C}_q[SU_2]$, up to but not including the Ricci tensor. Here we complete their structure and tie up some loose ends.

For $A = \mathbb{C}_q[S^2]$ with its standard 2D differential structure, the quantum metric was found in Proposition 2.36 and the quantum Levi-Civita connection and its curvature in Example 5.79. The quantum metric dimension is the symmetric q^2-integer $(2)_{q^2} = q^2 + q^{-2}$ and using (5.39) we can add to this the quantum trace

$$(\mathrm{id} \otimes (\ ,\ ))(R_\nabla \otimes \mathrm{id})g = (q^4 - 1)(2)_q(2)_{q^2}\mathrm{Vol},$$

where $\mathrm{Vol} = e^+ \wedge e^-$ is the (non-geometrically normalised) volume form. We see that when $q^4 \neq 1$, we have a top curvature class not visible in the classical limit where it would be zero. Note that to compute the quantum dimension and trace it pays to work with $e^\pm$ and extend $(\ ,\ )$ and other maps to $\mathbb{C}_q[SU_2]$, in which case we can effectively write the metric as $-q^2 e^+ \otimes e^- - e^- \otimes e^+$ suppressing the $D, \tilde{D}$ factors in the correct expressions. This example has the nicest possible properties in our theory, including ∇ $*$-preserving when q is real and $\Omega^1 \in {}_A\mathcal{G}_A$. We have already noted the quantum anti-symmetry of the Riemann curvature in Example 5.79 (v).

For the Ricci tensor, we now take a lift map of the form $i(\mathrm{Vol}) = \alpha g_{+-} + \beta g_{-+}$ using the notation of Proposition 2.36 and with constant coefficients α, β which we will fix. We fix the overall normalisation by $\wedge i(\mathrm{Vol}) = \mathrm{Vol}$ so as to split wedge, which from the proof of Proposition 2.36 comes out as $\alpha - \beta = -q^{-2}$. We now similarly compute the trace on the left of the Riemann curvature in Example 5.79 lifted by i, which comes out as

$$\mathrm{Ricci} = -(2)_q \beta g_{+-} + q^4 (2)_q \alpha g_{-+}.$$

Asking for Ricci to have the same quantum symmetry as g, namely $\wedge(\mathrm{Ricci}) = 0$, then gives us $q^4\alpha + \beta = 0$ thereby fixing α, β. The result is

$$i(\mathrm{Vol}) = \frac{1}{(2)_{q^2}}(-q^{-4} g_{+-} + g_{-+}), \quad \mathrm{Ricci} = -\frac{(2)_q}{(2)_{q^2}} g, \quad S = (\ ,\ )(\mathrm{Ricci}) = -(2)_q$$

in our conventions. The geometric Laplacian was already given in Example 5.79.

Similarly, the q-Riemannian geometry of $A = \mathbb{C}_q[SU_2]$ with the 4D calculus from Example 2.59 (but with geometric normalisation of d with λ^{-1} times the differentials listed there) was computed in Proposition 5.85 using the frame bundle approach, for the metric and inverse metric in Proposition 2.60. From the latter, we easily find the quantum dimension

$$\underline{\dim} = (\ ,\ )(g) = (2)_q^2.$$

The connection is not, however, a bimodule connection, so much of our more advanced theory does not apply in this example. We can still write down the geometric Laplacian for which it turns out that $(\ ,\ )\nabla = 0$ on the basis vectors

e^b, e^c, e^z, θ, so that $\Delta f = (\partial_i \partial_j f) g^{ij}$ in our basis. This is therefore the same, $\Delta = {}_\theta\Delta$, as studied in Proposition 2.62 after allowing for our different up-down conventions on the indices.

For the Ricci tensor, we proceed to find a suitable map i using our general approach that works for any bicovariant calculus, in our case we take the canonical 'Woronowicz' exterior algebra. Namely, as in Lemma 8.18, we now turn to the minimal polynomial of the braiding Ψ on $\Lambda^1 \otimes \Lambda^1$. Its eigenvalues from Example 2.77 are $1, -q^{-2}, -q^2$, with the 1-eigenvalue subspace the kernel of $\wedge : \Lambda^1 \otimes \Lambda^1 \to \Lambda^2$. The $-q^{\pm 2}$-eigenvalue subspaces $e^{\pm}$ are 3-dimensional and are comodules under Δ_R since Ψ is covariant. One can check that Ψ is fully diagonalisable and hence its minimal polynomial is

$$(\Psi - \mathrm{id}) f(\Psi) = 0, \quad f(\Psi) = (\Psi + q^2)(\Psi + q^{-2})$$

so that the canonical i in our case is

$$i(v) = \tilde{v} - \frac{(\Psi + q^2)(\Psi + q^{-2})}{(2)_q^2} \tilde{v}.$$

This is well defined and does the job by the same arguments as in the proof of Lemma 8.18 whenever the exterior algebra relations are given via a diagonalisable braiding, as is the case here. The canonical i in our case then computes as

$$\begin{aligned}
i(e^{ac}) &= (2)_q^{-1}(q e^a \otimes e^c - q^{-1} e^c \otimes e^a),\\
i(e^{bd} - \lambda e^{ab}) &= (2)_q^{-1} q (e^b \otimes e^d - e^d \otimes e^b - \lambda e^a \otimes e^b),\\
i(e^{ad} - q^{-2} e^{cb}) &= (2)_q^{-1} q^{-1} (q^2 e^a \otimes e^d - e^d \otimes e^a + e^b \otimes e^c - e^c \otimes e^b - \lambda e^a \otimes e^a),\\
i(e^{ab}) &= (2)_q^{-1}(q^{-1} e^a \otimes e^b - q e^b \otimes e^a),\\
i(e^{cd}) &= (2)_q^{-1} q^{-1} (e^c \otimes e^d - e^d \otimes e^c + \lambda e^c \otimes e^a),\\
i(e^{ad} + e^{cb}) &= (2)_q^{-1} q^{-1} (e^a \otimes e^d - q^2 e^d \otimes e^a + e^c \otimes e^b - e^b \otimes e^c + \lambda e^a \otimes e^a),
\end{aligned}$$

where we use the shorthand $e^{ac} = e^a \wedge e^c$, etc. and $\lambda = 1 - q^{-2}$. We will not cover the theory here but there is an associated decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ into 'self-dual' and 'anti-self dual' subspaces

$$\Lambda^2_+ = \mathrm{span}\{e^{ac},\ e^{bd} - \lambda e^{ab},\ e^{ad} - q^{-2} e^{cb}\}, \quad \Lambda^2_- = \mathrm{span}\{e^{ab},\ e^{cd},\ e^{ad} + e^{cb}\}.$$

One can check that i splits the wedge product as dictated by the canonical construction along the lines of Lemma 8.18.

The curvature for the above metric and connection was also computed in Proposition 5.85 and contracting with the above (,) after lifting by i, we obtain

$$\mathrm{Ricci} = \frac{2}{q(4)_q}\Big(g + \frac{q^3}{(2)_q}\theta \otimes \theta\Big), \quad S = \frac{2(3)_q}{q(4)_q},$$

i.e., an 'Einstein space' up to a shift by $\theta \otimes \theta$. Here

$$i(e^b \wedge e^z) = (2)_q^{-1} q((\lambda - q^{-2})e^a \otimes e^b + e^b \otimes e^a + e^d \otimes e^b - e^b \otimes e^d),$$
$$i(e^c \wedge e^z) = (2)_q^{-1} q^{-1}((q^{-2} - \lambda))e^c \otimes e^a - e^a \otimes e^c + e^d \otimes e^c - e^c \otimes e^d),$$
$$i(e^c \wedge e^b) = (2)_q^{-2} 2(e^c \otimes e^b - e^b \otimes e^c - \frac{1}{2}q^2\lambda(e^a \otimes e^d + e^d \otimes e^a) + \lambda e^a \otimes e^a)$$

for the lift of the relevant outputs of R_∇. One also has a QLC for the open q-disk, see Exercise E8.7 using methods of §8.4.

8.3 Wave Operator Quantisation of $C^\infty(M) \rtimes \mathbb{R}$

This section explores a shortcut to the full noncommutative Riemannian geometry, namely instead of trying to construct the quantum Levi-Civita connection, we can go straight to a proposed Laplace–Beltrami or wave operator Δ guided by other principles and the correct classical limit. This is a bit like Connes' approach in §8.5 where one goes straight to a proposed Dirac operator, except that we will not be guided by axioms but rather a point of view on the origin of wave operators in the first place out of noncommutative geometry. In fact, we have already seen for $\Omega^1(U(su_2))$ in Example 1.45 and $\Omega^1(\mathbb{C}_q[SU_2])$ in Proposition 2.62 that one is sometimes forced by symmetries and a 'quantum anomaly' for differential structures to use a calculus of higher dimension than expected classically (in both examples the calculus is 4D not 3D). In both cases, we define partial derivatives by $\mathrm{d}f = (\partial_i f)e^i + \frac{\lambda}{2}(\Delta f)\theta$ where the e^i have a classical limit and there is an extra direction which in those examples is the inner element θ and which has a natural noncommutative 3D Laplacian Δ as its 'partial derivative'. We also saw this phenomenon in Proposition 1.22 going the other way from any 2nd order operator Δ on a differential algebra $(A, \Omega^1, \mathrm{d})$ with respect to an 'inverse metric' (,) to a central extension generalised differential calculus $(\widetilde{\Omega}, \tilde{\mathrm{d}})$ on A. Here we adjoined an extra dimension θ' with partial derivative $\lambda\Delta/2$ in the expansion of $\tilde{\mathrm{d}}$.

Motivated by these examples and the general idea that the Laplacian is the partial derivative of an extra dimension in the quantum differential calculus, we describe in this section a particular application to the quantisation of any static 'spacetime' of the form $M \times \mathbb{R}$ where M will be the spatial manifold and $\mathbb{R}$ is 'time'. We look at specific applications in Chap. 9, while the general theory here could also have other geometrical applications. The idea is that (M, g) is Riemannian and we use a Laplacian on M to construct a generalised calculus of one dimension higher on M with additional direction θ'. We then semidirect product by a noncommutative 'time' direction as a quantisation of $M \times \mathbb{R}$, extending both the algebra and the calculus, and read back the partial derivative for θ' here as a quantum wave operator on this. We finally quotient out θ' to have a noncommutative calculus of classical dimension on our quantization of $M \times \mathbb{R}$ and a wave operator as a remnant of the above process.

Thus, let (M, g) be a Riemannian manifold with coordinate algebra $C^\infty(M)$ and classical exterior algebra $(\Omega(M), \mathrm{d})$. We let $(\ ,\ )$ be the inverse of the classical metric with Levi-Civita connection $\widehat{\nabla}$ and Δ a classical 2nd order operator with respect to it. We will also need an extension of Δ to 1-forms such that

$$\Delta(f\omega) = (\Delta f)\omega + f\Delta\omega + 2\widehat{\nabla}_{\mathrm{d}f}\omega, \tag{8.35}$$

$$\Delta((\omega, \eta)) = (\Delta\omega, \eta) + (\omega, \Delta\eta) + 2(\widehat{\nabla}\omega, \widehat{\nabla}\eta), \tag{8.36}$$

$$[\Delta, \mathrm{d}]f = \mathrm{Ricci}_\Delta(\mathrm{d}f) \tag{8.37}$$

for all $f \in C^\infty(M)$ and $\omega, \eta \in \Omega^1(M)$ and for some tensorial operator which we have denoted Ricci_Δ (this can be thought of as an element of $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$ evaluated against its second leg). Here the inverse metric is extended to tensor products in the obvious way. In our case, we consider 2nd order operators of the form

$$\Delta f = \Delta_{LB} f + \zeta(f), \quad \Delta\omega = \Delta_{LB}\omega + \widehat{\nabla}_\zeta \omega \tag{8.38}$$

for all $f \in C^\infty(M)$ and $\omega \in \Omega^1(M)$, where Δ_{LB} is the Laplace–Beltrami operator and ζ any classical vector field on M. Then the above holds with

$$\mathrm{Ricci}_\Delta = \mathrm{Ricci} + \widehat{\nabla}_\zeta - \mathcal{L}_\zeta,$$

where we view Ricci as an operator and $\mathcal{L}_\zeta$ is the Lie derivative along ζ (see (3.12)). We will later fix ζ in terms of a functional parameter below, but for the moment it is unspecified. We let $\zeta^\sharp$ be the 1-form corresponding to ζ via the metric. A further ingredient from the classical geometry on M will be a conformal vector field τ. This leaves the metric invariant up to scale and obeys the conformal Killing equation

$$\widehat{\nabla}_\mu \tau_\nu + \widehat{\nabla}_\nu \tau_\mu = (1+\alpha) g_{\mu\nu}\ ; \quad 1 + \alpha = \frac{2}{n}\mathrm{div}(\tau) \tag{8.39}$$

in local coordinates, where $\tau_\mu = g_{\mu\nu}\tau^\nu$.

Lemma 8.22 *Let M be a Riemannian manifold, $\Delta = \Delta_{LB} + \zeta$ and τ a conformal Killing vector field. Then*

$$[\Delta, \tau] f = (1+\alpha)\Delta f - \frac{1}{2}(n-2)(\mathrm{d}\alpha, \mathrm{d}f) - (\mathcal{L}_\tau \zeta^\sharp, \mathrm{d}f)$$

for all $f \in C^\infty(M)$ and τ acting on functions as a derivation.

Proof We focus on $\zeta = 0$, i.e. we prove the identity

$$[\Delta_{LB}, \tau]f = (1+\alpha)\Delta_{LB} f - \frac{1}{2}(n-2)(\mathrm{d}\alpha, \mathrm{d}f),$$

from which the general case easily follows. We apply $\widehat{\nabla}^\mu = g^{\mu\gamma}\widehat{\nabla}_\gamma$ to both sides of the conformal Killing equation (summation understood) to give $\Delta_{LB}\tau_\nu = -\widehat{\nabla}^\mu\widehat{\nabla}_\nu\tau_\mu + \widehat{\nabla}_\nu\alpha = -R^\mu{}_{\nu\mu}{}^\gamma\tau_\gamma - \widehat{\nabla}_\nu\mathrm{div}(\tau) - \widehat{\nabla}_\nu\alpha = -\mathrm{Ricci}_\nu{}^\gamma\tau_\gamma - (\frac{n-2}{2})\widehat{\nabla}_\nu\alpha$. Next we compute

$$\begin{aligned}
\Delta_{LB}\tau f &= g^{\mu\nu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu(\tau_\gamma\widehat{\nabla}^\gamma f) = g^{\mu\nu}\widehat{\nabla}_\mu((\widehat{\nabla}_\nu\tau_\gamma)(\widehat{\nabla}^\gamma f)) + g^{\mu\nu}\widehat{\nabla}_\mu(\tau_\gamma\widehat{\nabla}_\nu\widehat{\nabla}^\gamma f)\\
&= g^{\mu\nu}(\widehat{\nabla}_\mu\widehat{\nabla}_\nu\tau_\gamma)\widehat{\nabla}^\gamma f + g^{\mu\nu}(\widehat{\nabla}_\nu\tau_\gamma)(\widehat{\nabla}_\mu\widehat{\nabla}^\gamma f) + 2g^{\mu\nu}(\widehat{\nabla}_\mu\tau_\gamma)(\widehat{\nabla}_\nu\widehat{\nabla}^\gamma f)\\
&\quad + g^{\mu\nu}\tau_\gamma\widehat{\nabla}_\mu\widehat{\nabla}_\nu\widehat{\nabla}^\gamma f\\
&= (\Delta_{LB}\tau_\gamma)\widehat{\nabla}^\gamma f + (1+\alpha)\Delta_{LB} f + g^{\mu\nu}\tau_\gamma\widehat{\nabla}_\mu\widehat{\nabla}^\gamma\widehat{\nabla}_\nu f\\
&= (\Delta_{LB}\tau_\gamma)\widehat{\nabla}^\gamma f + (1+\alpha)\Delta_{LB} f + g^{\mu\nu}\tau_\gamma\widehat{\nabla}^\gamma\widehat{\nabla}_\mu\widehat{\nabla}_\nu f + g^{\mu\nu}\tau_\gamma R_\mu{}^\gamma{}_\nu{}^\delta\widehat{\nabla}_\delta f\\
&= (\Delta_{LB}\tau_\gamma)\widehat{\nabla}^\gamma f + (1+\alpha)\Delta_{LB} f + \tau\Delta_{LB} f + \tau_\mu\mathrm{Ricci}^{\mu\nu}\widehat{\nabla}_\nu f
\end{aligned}$$

for all f. We use the Leibniz rule and that our local basis covariant derivatives commute when acting on functions. We then combine these two observations. □

We will use these results. Since we will have both classical and quantum structures *we henceforth write* d_M for the classical differential on $\Omega(M)$ and Δ_M for a 2nd order operator with respect to a classical inverse metric (,). By Proposition 1.22, we have a generalised calculus $\widetilde{\Omega}^1$ with commutation relations and differential

$$[f, \omega] = \lambda(\mathrm{d}_M f, \omega)\theta', \quad [f, \theta'] = 0, \quad \mathrm{d}f = \mathrm{d}_M f + \frac{1}{2}\lambda(\Delta_M f)\theta' \tag{8.40}$$

for all $f \in \mathbb{C}^\infty(M)$, $\omega \in \Omega^1(M)$. Next, suppose that τ is a vector field on M and let $A = C^\infty(M) \rtimes \mathbb{R}$ be 'noncommutative spacetime' where we adjoin t by relations

$$[f, t] = \lambda\tau(f)$$

for all $f \in C^\infty(M)$. At least when M is compact, one can exponentiate the action as well as complete to an operator algebra, although we shall not do so here.

Theorem 8.23 *Let M be a Riemannian manifold equipped with a vector field ζ, $\beta \in C^\infty(M)$ and τ a conformal Killing vector field. Then the calculus $\widetilde{\Omega}^1$ on M*

extends to a possibly generalised calculus $(\Omega^1(C^\infty(M) \rtimes \mathbb{R}), \mathrm{d})$ *by*

$$[\omega, t] = \lambda(\mathcal{L}_\tau - \mathrm{id})\omega - \frac{1}{4}\lambda^2(n-2)(\mathrm{d}_M\alpha, \omega)\theta' - \frac{1}{2}\lambda^2(\mathcal{L}_\tau\zeta^\sharp, \omega)\theta',$$
$$[\theta', t] = \alpha\lambda\theta', \quad [f, \mathrm{d}t] = \lambda\mathrm{d}f, \quad [\mathrm{d}t, t] = \beta\lambda\theta' - \lambda\mathrm{d}t$$

for all $\omega \in \Omega^1(M)$, $f \in C^\infty(M)$. *Here* $n = \dim(M)$ *and* $\alpha = \frac{2}{n}\mathrm{div}(\tau) - 1$.

Proof That τ is a conformal Killing vector field can be written in terms of the inverse metric as

$$\tau((\omega, \eta)) = (\mathcal{L}_\tau\omega, \eta) + (\omega, \mathcal{L}_\tau\eta) - (1+\alpha)(\omega, \eta), \tag{8.41}$$

which is the form we shall use. We have to verify the various Jacobi identities concerning the extension by t, $\mathrm{d}t$. Thus

$$\begin{aligned}
&\lambda^{-2}\left([[\omega, t], f] + [[t, f], \omega] + [[f, \omega], t]\right)\\
&= \lambda^{-1}[\mathcal{L}_\tau(\omega) - \omega - \frac{1}{4}\lambda(n-2)(\mathrm{d}_M\alpha, \omega)\theta' - \frac{1}{2}\lambda(\mathcal{L}_\tau\zeta^\sharp, \omega)\theta', f]\\
&\quad + \lambda^{-1}[\omega, \tau(f)] - \lambda^{-1}[(\omega, \mathrm{d}_M f)\theta', t]\\
&= (\mathcal{L}_\tau(\omega) - \omega, \mathrm{d}_M f)\theta' + (\omega, \mathrm{d}_M\tau(f))\theta' - \tau((\omega, \mathrm{d}_M f))\theta' - \alpha(\omega, \mathrm{d}_M f)\theta' = 0
\end{aligned}$$

by (8.41) with $\eta = \mathrm{d}_M f$. Also, $[[\theta', t], f] + [[t, f], \theta'] + [[f, \theta'], t] = 0$ as each term is zero, while

$$[[\mathrm{d}t, f], g] + [[f, g], \mathrm{d}t] + [[g, \mathrm{d}t], f] = -\lambda[\mathrm{d}f, g] + \lambda[\mathrm{d}g, f] = 0$$

by symmetry of (,) and θ' central in $\tilde{\Omega}^1$. Next,

$$\begin{aligned}
&\lambda^{-1}\left([[\mathrm{d}t, t], f] + [[t, f], \mathrm{d}t] + [[f, \mathrm{d}t], t]\right)\\
&\quad = [\beta\theta' - \mathrm{d}t, f] - [\tau(f), \mathrm{d}t] + [\mathrm{d}f, t] = \lambda\mathrm{d}f - \lambda\mathrm{d}\tau(f) + [\mathrm{d}f, t]\\
&\quad = \lambda\mathrm{d}f - \lambda\mathrm{d}\tau(f) + [\mathrm{d}_M f + \frac{1}{2}\lambda\Delta_M f\theta', t]\\
&\quad = -\lambda(\mathrm{d}\tau(f) - \mathrm{d}f) + \lambda(\mathrm{d}\tau(f) - \mathrm{d}f) - \frac{1}{4}\lambda^2(n-2)(\mathrm{d}_M\alpha, \mathrm{d}_M f)\theta'\\
&\qquad - \frac{1}{2}\lambda^2(\mathcal{L}_\tau\zeta^\sharp, \mathrm{d}_M f)\theta' + \frac{1}{2}\alpha\lambda^2\Delta_M f\theta' + \frac{1}{2}\lambda^2\tau(\Delta_M f)\theta'\\
&\quad = \frac{1}{2}\lambda^2\big(-\Delta_M\tau(f) + (1+\alpha)\Delta_M f + \tau(\Delta_M f) - \frac{1}{2}(n-2)(\mathrm{d}_M\alpha, \mathrm{d}_M f)\\
&\qquad - (\mathcal{L}_\tau\zeta^\sharp, \mathrm{d}_M f)\big)\theta'\\
&\quad = 0
\end{aligned}$$

by the property of conformal Killing vectors in Lemma 8.22. Finally, there is an easy check that $\mathrm{d}[f, t] = [\mathrm{d}f, t] + [f, \mathrm{d}t]$. Note that we do not claim surjectivity of the calculus, however this appears to be true in practice. $\square$

If we restrict to exact quantum differentials then $\Omega^1(C^\infty(M) \rtimes \mathbb{R})$ can be presented more palatably as generated by these and t, $C^\infty(M)$ with relations

$$\begin{aligned}
&[f, h] = 0, \quad [f, t] = \lambda\tau(f), \quad [\mathrm{d}f, h] = \lambda(\mathrm{d}_M f, \mathrm{d}_M h)\theta', \\
&[\theta', f] = 0, \quad [\theta', t] = \alpha\lambda\theta', \quad [\mathrm{d}f, t] = \lambda(\mathrm{d}\tau(f) - \mathrm{d}f), \\
&[f, \mathrm{d}t] = \lambda\mathrm{d}f, \quad [\mathrm{d}t, t] = \beta\lambda\theta' - \lambda\mathrm{d}t
\end{aligned} \tag{8.42}$$

for all $f, h \in C^\infty(M)$. One could take these relations as a definition of the calculus and verify the Jacobi identities, one of which would rapidly lead back to the conformal Killing equation (8.41). Theorem 8.23 shows that the construction is properly defined with respect to the structure of the manifold M by virtue of being built with classical geometric objects and it is only in this identification that the choice of ζ is visible. The case of constant $\alpha = -1$ is that of a Killing vector field but we will see that the constant conformal case $\alpha = 1$ is also of interest.

Proposition 8.24 *Suppose that $\mu, \nu \in C^\infty(M)$ obey the first-order differential equations*

$$\tau(\mu) = \beta - (1+\alpha)\mu, \quad \tau(\nu) = \mu - \alpha\nu.$$

Then $\Omega^1(C^\infty(M) \rtimes \mathbb{R})$ on a normal-ordered element $f(t) = \sum f_n t^n$, where $f_n \in C^\infty(M)$ (i.e., keeping the t-dependence to the right), obeys

$$\mathrm{d}f(t) = \mathrm{d}_M f(t) + \frac{\lambda}{2}(\Delta_M f)(t + \lambda\alpha)\theta' + (\partial_0 f(t))\mathrm{d}t + \lambda(\Delta_0 f(t))\theta',$$

$$\theta' f(t) = f(t + \lambda\alpha)\theta', \quad \partial_0 f(t) = \frac{f(t) - f(t-\lambda)}{\lambda},$$

$$\Delta_0 f(t) = \frac{\nu f(t+\lambda\alpha) + \mu f(t - \lambda(\frac{\beta}{\mu} - \alpha)) - (\nu+\mu) f(t + \lambda(\alpha - \frac{\beta}{\nu+\mu}))}{\lambda^2}$$

and we also have

$$[\mathrm{d}t, f(t)] = -\lambda\mathrm{d}f(t) + \lambda(\mu+\nu)\Big(\frac{f(t+\lambda\alpha) - f(t + \lambda(\alpha - \frac{\beta}{\mu+\nu}))}{\lambda}\Big)\theta'.$$

Proof The behaviour on functions only on M is already covered in Theorem 8.23. For a function purely of t, we prove the result at least for polynomials by induction as follows. Assume $[\mathrm{d}t, t^n] = p_n\mathrm{d}t + q_n\theta'$. Then using the commutation relations,

$$p_n = (t-\lambda)p_{n-1} - \lambda t^{n-1}, \quad q_n = (t-\lambda)q_{n-1} + \lambda\beta(t+\lambda\alpha)^{n-1},$$

which are solved by

$$[\mathrm{d}t, t^n] = \Big((t-\lambda)^n - t^n\Big)\,\mathrm{d}t + \mu\Big((t+\lambda\alpha)^n - \big(t-\lambda(\frac{\beta}{\mu}-\alpha)\big)^n\Big)\theta' \tag{8.43}$$

provided μ obeys the relation stated. The proof for the p_n is more elementary and left for the reader, while for q_n we have $q_1 = \lambda\mu(\alpha + \frac{\beta}{\mu} - \alpha) = \lambda\beta$ as required, and

$$\begin{aligned}
&(t-\lambda)q_{n-1} + \lambda\beta(t+\lambda\alpha)^{n-1}\\
&= \mu(t-\lambda(1+\frac{\tau(\mu)}{\mu}))\Big((t+\lambda\alpha)^{n-1} - (t-\lambda(\frac{\beta}{\mu}-\alpha))^{n-1}\Big) + \lambda\beta(t+\lambda\alpha)^{n-1}\\
&= q_n
\end{aligned}$$

taking account of the commutation relation $t\mu = \mu t - \lambda\tau(\mu)$. A further similar induction on $\mathrm{d}t^n = t\mathrm{d}t^{n-1} + [\mathrm{d}t, t^{n-1}] + t^{n-1}\mathrm{d}t$ provides the stated formulae as $\mathrm{d}f = \partial_0 f + \lambda\Delta_0 f$. Now suppose that $f = f(\ ,t)$ where the dependence on t is kept to the right and combine the two cases via the Leibniz rule for the general $\mathrm{d}f(t)$. Note that with regard to the t-dependence, $\theta'.(\Delta_M f)(t) = (\Delta_M f)(t+\lambda\alpha).\theta'$ when our basic 1-forms are ordered to the right using the stated commutation relation. Similarly, we deduce from (8.43) and the commutation $[\mathrm{d}t, f] = -\lambda\mathrm{d}f$ for a function on M that in general for a normal ordered function,

$$\begin{aligned}
[\mathrm{d}t, f(t)] = -\lambda\mathrm{d}_M f(t) - \frac{1}{2}\lambda^2\theta'\Delta_M f(t) - \lambda\partial_0 f(t)\mathrm{d}t&\\
+\lambda\mu\Big(\frac{f(t+\lambda\alpha) - f(t-\lambda(\frac{\beta}{\mu}-\alpha))}{\lambda}\Big)\theta'&,
\end{aligned}$$

which we can then write as stated. Note that there are also commutation relations between other differentials and functions. □

For example, if $\tau(\alpha) = \tau(\beta) = 0$ then $\mu = \beta/(1+\alpha)$ is killed by τ and solves the μ equation. Similarly, $\nu = \mu/\alpha$ is killed by τ and solves the ν equation. In this case $\nu + \mu = \beta/\alpha$ and

$$\Delta_0 f = \beta\,\frac{f(t+\alpha\lambda) + \alpha f(t-\lambda) - (1+\alpha)f(t)}{\lambda^2\alpha(1+\alpha)}.$$

If, moreover, $\alpha = 1$ then we have Δ_0 as $\frac{\beta}{2}$ times the standard symmetric finite difference Laplacian, while in the limit for $\alpha \to -1$ we have

$$\Delta_0 f \overset{\alpha\to -1}{\longrightarrow} \frac{\beta}{\lambda}\left(\partial_0 f - \dot f(t-\lambda)\right),$$

where $\dot f$ denotes the usual derivative in t, which is more readily seen to tend to $\frac{\beta}{2}\ddot f$ as $\lambda \to 0$. We see that the construction of Theorem 8.23 involves a differential calculus in the extra 'time' direction of the finite-difference type that exists in noncommutative geometry even in one variable, as we have seen a version of in Example 1.10. Note also that $[t^n, \mathrm{d}t]$ and $\mathrm{d}t^n$ and hence Δ_0 do not depend on the freedom in choices for μ, ν as the inductive relations do not depend on them, we only require these in order to have finite-difference type formulae; indeed they might only exist locally so as to have such a 'finite difference' picture differently in different coordinate patches.

Corollary 8.25 *Working in the calculus $\Omega^1(C^\infty(M) \rtimes \mathbb{R})$, we define the* induced wave operator $\Box$ *on* $C^\infty(M) \rtimes \mathbb{R}$ *by* $\mathrm{d}f = \mathrm{d}_M f + (\partial_0 f)\mathrm{d}t + \frac{\lambda}{2}(\Box f)\theta'$.

(1) $\Box f(t) = (\Delta_M f)(t + \lambda\alpha) + 2\Delta_0 f(t)$ *on normal ordered* $f(t) = \sum f_n t^n$.
(2) *The classical limit is* $\lim_{\lambda\to 0} \Box f = \Delta_M f + \beta\ddot{f}$.
(3) *If* $\zeta^\sharp = -\frac{1}{2}\beta^{-1}\mathrm{d}\beta$ *on* M *then this classical limit is the Laplace–Beltrami operator on* $M \times \mathbb{R}$ *for the static metric*

$$g_{M\times\mathbb{R}} = \beta^{-1}\mathrm{d}t \otimes_{C^\infty(M)} \mathrm{d}t + g.$$

Proof The shift by $\lambda\alpha$ in (1) is from the normal ordering $f(t) = \sum_n f_n t^n$. The classical limit (2) is a delicate computation assuming a Taylor expansion of f about t but being careful about the normal ordering. Assuming μ, ν locally, we find

$$\lambda^2\Delta_0 f = \lambda^2 \sum_n f_n \binom{n}{2}\Big(\nu(\alpha^2 - \tau(\alpha)) + \mu((\alpha - \frac{\beta}{\mu})^2 - \tau(\alpha - \frac{\beta}{\mu}))$$
$$- (\nu+\mu)((\alpha - \frac{\beta}{\nu+\mu})^2 - \tau(\alpha - \frac{\beta}{\nu+\mu}))\Big)t^{n-2} + O(\lambda^3) = \lambda^2\frac{\beta}{2}\ddot{f} + O(\lambda^3)$$

using that τ is a derivation and the defining equations for $\tau(\mu), \tau(\nu)$. The τ terms here arise from normal ordering of the different positions of g in the expansion

$$(t+\lambda g)^n = t^n + \lambda n g t^{n-1} - \frac{n(n-1)}{2}\lambda^2\tau(g)t^{n-2} + \binom{n}{2}\lambda^2 g^2 t^{n-2} + O(\lambda^3).$$

One can view this as the first terms of a noncommutative binomial identity for the action of a vector field on a function. One can also derive the limit from the next lemma, but we have given the more direct proof. Part (3) is some elementary classical differential geometry on M; for a metric that splits as shown for some function $\beta \in C^\infty(M)$ in the 'time' direction, the Levi-Civita connection is easily computed and $\widehat{\nabla}\omega$ for the product metric acquires an extra term from $\mathrm{d}\beta$ which then enters into the corresponding Laplace–Beltrami operator,

$$\Box f = \beta\ddot{f} + \Delta_{LB} - \frac{1}{2}\beta^{-1}(\mathrm{d}\beta, \mathrm{d}f). \qquad \Box$$

This corollary makes good our philosophy that the 'extra dimension' in the extended cotangent bundle expresses the Laplacian, but now on the noncommutative version of $M \times \mathbb{R}$. The following lemma provides more information about the 'time' derivative component of the wave operator.

Lemma 8.26

$$\Delta_0 1 = \Delta_0 t = 0, \quad \Delta_0 t^2 = \beta, \quad \Delta_0 t^3 = 3\beta t + \lambda((\alpha-1)\beta - 2\tau(\beta));$$
$$\Delta_0 t^n = \sum_{i=0}^{n-1} (\partial_0 t^{n-1-i})\beta(t+\lambda\alpha)^i,$$
$$[\mathrm{d}t, t^n] = -(\partial_0 t^n)\mathrm{d}t + \lambda(\Delta_0 t^{n+1} - t\Delta_0 t^n)\theta'.$$

Proof The first three cases are already contained in the proof of part (2) of Corollary 8.25 as there are no powers higher than λ^2 analysed there. For t^3 we note that for any $f \in C^\infty(M)$,

$$(t+\lambda f)^3 = t^3 + \lambda(t^2 f + tft + ft^2) + \lambda^2(f^2 t + ftf + tf^2) + \lambda^3 f^3$$
$$= t^3 + 3\lambda f t^2 + 3\lambda^2 f^2 t + \lambda^3 f^3 - 3\lambda^2(\tau f)t + \lambda^3\tau^2 f - \lambda^3 f\tau f - \lambda^3\tau(f^2).$$

We know in computing $\Delta_0 t^3$ from the expression in Proposition 8.24 that the order $1, \lambda$ terms do not contribute while the order λ^3 terms give us the classical contribution $3\beta t$. It remains to add up terms at order λ^3, which contribute to $\Delta_0 t^3$ some $\lambda\times$

$$\nu(\alpha^3 + \tau^2\alpha - 3\alpha\tau\alpha) + \mu\Big((\alpha - \frac{\beta}{\mu})^3 + \tau^2(\alpha - \frac{\beta}{\mu}) - 3(\alpha - \frac{\beta}{\mu})\tau(\alpha - \frac{\beta}{\mu})\Big)$$
$$- (\mu+\nu)\Big((\alpha - \frac{\beta}{\mu+\nu})^3 + \tau^2(\alpha - \frac{\beta}{\mu+\nu}) - 3(\alpha - \frac{\beta}{\mu+\nu})\tau(\alpha - \frac{\beta}{\mu+\nu})\Big)$$
$$= 3\alpha\frac{\beta^2}{\mu} - 3\alpha\frac{\beta^2}{\mu+\nu} - \frac{\beta^3}{\mu^2} + \frac{\beta^3}{(\mu+\nu)^2} - \mu\tau^2(\frac{\beta}{\mu}) + (\mu+\nu)\tau^2(\frac{\beta}{\mu+\nu})$$
$$+ 3(\alpha\mu - \beta)\tau(\frac{\beta}{\mu}) - 3(\alpha(\mu+\nu) - \beta)\tau(\frac{\beta}{\mu+\nu}),$$

which eventually simplifies to the result stated on repeated use of the relations

$$\tau(\frac{\beta}{\mu}) = \frac{\tau\beta}{\mu} - \frac{\beta}{\mu}(\frac{\beta}{\mu} - (1+\alpha)), \quad \tau(\frac{\beta}{\mu+\nu}) = \frac{\tau\beta}{\mu+\nu} - \frac{\beta}{\mu+\nu}(\frac{\beta}{\mu} - \alpha).$$

We omit the details in view of the general formula from which the final result can more easily be obtained. For the general formula, we use the Leibniz rule and commutation relations to find

$$\begin{aligned} \mathrm{d}t^n &= (\mathrm{d}t^{n-1})t + t^{n-1}\mathrm{d}t = (\partial_0 t^{n-1})(\mathrm{d}t)t + \lambda\Delta_0 t^{n-1}\theta' t + t^{n-1}\mathrm{d}t \\ &= (\partial_0 t^{n-1})t\mathrm{d}t + \partial_0 t^{n-1}\lambda(\beta\theta' - \mathrm{d}t) + \lambda\Delta_0 t^{n-1}(t+\lambda\alpha)\theta' + t^{n-1}\mathrm{d}t. \end{aligned}$$

Comparing with $\mathrm{d}t^n = (\partial_0 t^n)\mathrm{d}t + \lambda\Delta_0 t^n\theta'$, we deduce

$$\partial_0 t^n = (\partial_0 t^{n-1})(t-\lambda) + t^{n-1}, \quad \Delta_0 t^n = (\Delta_0 t^{n-1})(t+\lambda\alpha) + \partial_0 t^{n-1}\beta.$$

The second of these provides the induction step easily solved to provide the result stated. We then use the formula $\mathrm{d}t^{n+1} = t\mathrm{d}t^n + [\mathrm{d}t, t^n] + t^n\mathrm{d}t$ for the different induction in Proposition 8.24 now as a way to recover this $[\mathrm{d}t, t^n]$ from Δ_0. □

We also have expressions to low order:

Lemma 8.27 *To low deformation order,*

$$\begin{aligned} \Delta_0 e^{i\omega t} &= -\frac{\omega^2}{2}\Big(\beta + \frac{i\lambda\omega}{3}(-2\tau(\beta) + (\alpha-1)\beta) \\ &- \frac{\lambda^2\omega^2}{12}\Big(3\tau^2(\beta) + 3\tau(\beta) - 2\tau(\beta)\alpha + (1-\alpha+\alpha^2-3\tau(\alpha))\beta\Big) + O((\lambda\omega)^3)\Big)e^{i\omega t} \\ e^{i\omega(t+\lambda\alpha)} &= \Big(1 + i\lambda\omega\alpha - \frac{\lambda^2\omega^2}{2}(\alpha^2 - \tau(\alpha)) + O((\lambda\omega)^3)\Big)e^{i\omega t}. \end{aligned}$$

Proof First expand the exponential as a power series and use Lemma 8.26 to apply Δ_0 to each power of t. Next use the expression for $(t+\lambda\alpha)^i$ in the proof of Corollary 8.25 and neglect terms higher than λ^2, and then the formula for ∂_0 in Proposition 8.24. Finally the commutation relations in (8.42) are used to order the t's to the right, which to second order in λ gives

$$[t^n, f] = -\lambda n\tau(f)t^{n-1} + \frac{1}{2}\lambda^2 n(n-1)\tau(\tau(f))t^{n-2} + O(\lambda^3).$$

The second formula comes from using the expression for $(t+\lambda f)^n$ used above in the power series for the exponential. □

One could go on and consider elements of noncommutative geometry with this calculus of one-higher dimension than classically on $C^\infty(M) \rtimes \mathbb{R}$. However, in other approaches we would not have the extra dimension and would see $\Box$ as an additional operator different from the differential calculus (it is only with the extra dimension that it becomes a partial derivative). This classical-dimensional calculus in our approach here is just given by setting $\theta' = 0$.

Corollary 8.28 *Setting $\theta' = 0$ in Theorem 8.23 gives a cross product calculus $\Omega(M) \rtimes \Omega_\lambda(t)$ on $C^\infty(M) \rtimes \mathbb{R}$ with*

$$[\omega, f] = 0, \quad [\omega, t] = \lambda(\mathcal{L}_\tau - \mathrm{id})\omega, \quad [f, \mathrm{d}t] = \lambda \mathrm{d}f, \quad [t, \mathrm{d}t] = \lambda \mathrm{d}t,$$
$$\{\omega, \eta\} = 0, \quad (\mathrm{d}t)^2 = 0, \quad \{\omega, \mathrm{d}t\} = -\lambda \mathrm{d}\omega$$

for all $\omega, \eta \in \Omega^1(M)$ and $f \in C^\infty(M)$ with $\mathrm{d}f(t) = \mathrm{d}_M f(t) + (\partial_0 f)(t)\mathrm{d}t$ for normal ordered $f(t) = \sum f_n t^n$, where ∂_0 is the finite-difference derivative as above. The calculus is inner with $\theta = -\mathrm{d}t/\lambda$ and $*$ extends to a $*$-differential calculus.

Proof We simply set $\theta' = 0$ in Theorem 8.23 to leave these relations. The classical $\Omega(M)$ appears as a subalgebra and has classical differentials, the $t, \mathrm{d}t$ generate a subalgebra $\Omega_\lambda(t)$ which is actually a super-Hopf algebra by results in Chap. 2 and has a standard finite-difference form where we have already given ∂_0 in Proposition 8.24. Finally, there are cross relations as in Proposition 1.13 with $-t$ in the role of x there. Also, unlike the calculus in the theorem, this one is clearly surjective and inner with θ as shown. Also, $C^\infty(M) \rtimes \mathbb{R}$ if we allow complex functions is a $*$-algebra with real-valued elements of $C^\infty(M)$ self-adjoint and $t^* = t$, provided $\lambda^* = -\lambda$ and $\tau(f^*) = \tau(f)^*$ for all complex f on M. This in turn holds as τ is a real vector field, and we can take this as the definition on complexified f. Finally, these $*$-relations extend by commutation with d, for example $[\mathrm{d}f, t]^* = -\lambda(\mathrm{d}(\tau(f) - f))^* = -\lambda \mathrm{d}(\tau(f^*) - f^*) = [t^*, \mathrm{d}(f^*)]$. □

In terms of exact quantum differentials the exterior algebra and relations d are

$$[\mathrm{d}f, h] = 0, \quad [\mathrm{d}f, t] = \lambda \mathrm{d}(\tau(f) - f), \quad [f, \mathrm{d}t] = \lambda \mathrm{d}f, \quad [t, \mathrm{d}t] = \lambda \mathrm{d}t$$
$$\{\mathrm{d}f, \mathrm{d}h\} = (\mathrm{d}t)^2 = \{\mathrm{d}f, \mathrm{d}t\} = 0$$

for all $f, h \in C^\infty(M)$. This is the actual quantum calculus we propose for our model but its origin as a quotient of a one-higher-dimensional calculus implies a quantum wave operator $\square$ on Corollary 8.25 and Proposition 8.26 which in the upstairs calculus is a higher derivative but which now is just an additional operator which in the limit is the Laplace–Beltrami operator for our static metric. The idea is that this is an example of the kind of mechanism by which wave operators in our actual spacetime could emerge from purely algebraic considerations in an underlying noncommutative geometry not immediately visible to us. We will discuss more about physics in the next chapter. We close with a general analysis of the simplest case.

Example 8.29 We let $M = \mathbb{R}$ and for coordinate x we take $g_M = \mathrm{d}x \otimes \mathrm{d}x$, and take $\beta(x)$ to be an arbitrary function. We have a static metric of the form

$$g_{M\times\mathbb{R}} = \mathrm{d}x \otimes \mathrm{d}x + \beta^{-1}(x)\mathrm{d}t \otimes \mathrm{d}t$$

and we set $\tau = \tau(x)\frac{\partial}{\partial x}$ for another arbitrary function $\tau(x)$ controlling the commutation relation of the form $[x, t] = \lambda\tau(x)$. Writing prime for derivative, we have $\alpha = 2\tau' - 1$ and the equations for μ, ν come down to

$$\mu' = \frac{\beta - 2\tau'\mu}{\tau}, \quad \nu' = \frac{\mu + (1 - 2\tau')\nu}{\tau},$$

which can be solved up to constants of integration as

$$\mu = \frac{1}{\tau^2}\int \beta\tau\mathrm{d}x, \quad \nu = -\mu + \frac{h}{\tau}\int \frac{\beta}{h}\mathrm{d}x; \quad h(x) = \frac{\mathrm{e}^{\int\frac{\mathrm{d}x}{\tau}}}{\tau},$$

where h solves $(h\tau)' = h$. The commutation relations in the extended calculus are

$$[\mathrm{d}x, x] = \lambda\theta', \quad [\theta', x] = 0, \quad [\theta', t] = \lambda(2\tau' - 1)\theta',$$
$$[\mathrm{d}x, t] = \lambda(\tau' - 1)\mathrm{d}x, \quad [x, \mathrm{d}t] = \lambda\mathrm{d}x, \quad [\mathrm{d}t, t] = \lambda\beta\theta' - \lambda\mathrm{d}t$$

and we now set $\theta' = 0$ to obtain the classical-dimensional calculus on our quantised 2D ‘spacetime’. The above then become

$$[\mathrm{d}x, x] = 0, \quad [\mathrm{d}x, t] = \lambda(\tau' - 1)\mathrm{d}x, \quad [x, \mathrm{d}t] = \lambda\mathrm{d}x, \quad [t, \mathrm{d}t] = \lambda\mathrm{d}t.$$

Applying d gives the maximal prolongation as the usual exterior algebra on $\mathrm{d}x, \mathrm{d}t$. Using the bimodule commutation relations, one finds that

$$e^1 = h\mathrm{d}x, \quad e^2 = \tau h\mathrm{d}t - te^1$$

together comprise a central basis of 1-forms. We then have

$$e^1 \wedge e^1 = 0, \quad e^1 \wedge e^2 + e^2 \wedge e^1 = 0, \quad e^2 \wedge e^2 = -\lambda\tau h^2\mathrm{d}x \wedge \mathrm{d}t = -\lambda e^1 \wedge e^2$$

and $*$-structure $(e^1)^* = e^1$, $(e^2)^* = (\mathrm{d}t)\tau h - e^1 t = e^2 - \lambda e^1$, where $x^* = x$ and $t^* = t$ and we assume our functions are real. Comparing with the bicrossproduct model in Example 1.43, we set $r = \tau h$ and find exactly this model

$$[r, t] = \lambda r, \quad [r, \mathrm{d}r] = 0, \quad [t, \mathrm{d}r] = 0, \quad [r, \mathrm{d}t] = \lambda\mathrm{d}r, \quad [t, \mathrm{d}t] = \lambda\mathrm{d}t$$

with its standard 2D calculus (of the second type). So the 2D quantum differential calculus can always be put into the bicrossproduct form after a change of variables. Its intrinsic noncommutative geometry is deferred to §9.4, while in the present context the 2D calculus itself does not see the function β and we do not actually quantise the associated static metric. Rather, we obtain a quantum wave operator given on $\psi(t)=\sum_n \psi_n(x)t^n$ by

$$\Box\psi=\sum_n \psi_n''(x)(t+\lambda\alpha)^n+2\Delta_0\psi,\quad \Delta_0\psi=\sum_n \psi_n(x)\sum_{i=0}^{n-1}(\partial_0 t^{n-1-i})\beta(t+\lambda\alpha)^i,$$

where ∂_0 is the finite difference in Proposition 8.24 and we also see the variable finite difference picture of Δ_0 given there.

To give a concrete example, we take the anti-de Sitter metric

$$g_{M\times\mathbb{R}}=\mathrm{d}x\otimes\mathrm{d}x-\mathrm{e}^{-\delta x}\mathrm{d}t\otimes\mathrm{d}t;\quad \beta=\mathrm{e}^{\delta x}.$$

(i) If we take $\tau=-n/\delta$ in terms of a dimensionless constant n then $\alpha=-1$ (the case of an actual Killing vector field) and

$$[t,x]=\lambda\frac{n}{\delta},\quad [x,\mathrm{d}x]=0,\quad [t,\mathrm{d}x]=[x,\mathrm{d}t]=\lambda\mathrm{d}x,\quad [t,\mathrm{d}t]=\lambda\mathrm{d}t \tag{8.44}$$

is the Heisenberg algebra with a particular 2D differential calculus in the family described in Example 1.14. We take $h=-\frac{\delta}{n}\mathrm{e}^{-\frac{\delta x}{n}}$ on choosing a constant of integration so that the change of variables to standard bicrossproduct form is $r=\mathrm{e}^{-\frac{\delta x}{n}}$. Meanwhile, for the quantum wave operator for the anti-de Sitter metric, choosing constants of integration, we have

$$\mu=-\frac{\mathrm{e}^{\delta x}}{n},\quad \nu=\frac{\mathrm{e}^{\delta x}}{n(n+1)},$$
$$\Box\psi=\frac{\partial^2}{\partial x^2}\psi(t-\lambda)$$
$$+\frac{2\mathrm{e}^{\delta x}}{n(n+1)\lambda^2}\left(n\psi(t+\lambda n)+\psi(t-\lambda)-(n+1)\psi(t+\lambda(n-1))\right),$$

which has the form of a weighted lattice double differential in the time direction times the expected de Sitter factor $\beta(x)$. Here $n=1$ would be the usual lattice double differential. The dimensionless constant n does not need to be an integer.

(ii) If we take $\tau=x$ then $\alpha=1$ and the 2D calculus is already in standard bicrossproduct form since we can take $h=1$ on choice of a constant of integration so that $r=x$. For our anti-de Sitter choice of β, choosing the constant of integration appropriately, we can take

$$\mu=\frac{(\delta x-1)\mathrm{e}^{\delta x}}{\delta^2x^2},\quad \nu=\frac{\mathrm{e}^{\delta x}}{\delta^2x^2},$$

giving a more complicated Δ_0 for the second part of the wave operator, namely.

$$\Delta_0\psi(t) = \frac{e^{\delta x}}{\lambda^2\delta^2x^2}\Big(\psi(t+\lambda) + (\delta x - 1)\psi(t - \lambda\frac{\delta^2x^2 + 1 - \delta x}{\delta x - 1}) - \delta x\psi(t + \lambda(1 - \delta x))\Big).$$

One can compute for example that $\Delta_0 t^2 = e^{\delta x}$, in agreement with Lemma 8.26. ◇

8.4 Hermitian Riemannian Geometry

This section only applies in the $*$-differential case over $\mathbb{C}$. Instead of working with the metric $g \in \Omega^1 \otimes_A \Omega^1$ and inverse $(,) : \Omega^1 \otimes_A \Omega^1 \to A$, we can work with the equivalent *hermitian metric* $(\star \otimes \mathrm{id})g$ (which we shall call $\langle , \rangle^{-1}$) and its inverse $\langle , \rangle = (,)(\mathrm{id} \otimes \star^{-1})$, where

$$\langle , \rangle^{-1} \in \overline{\Omega^1} \otimes_A \Omega^1, \quad \langle , \rangle : \Omega^1 \otimes_A \overline{\Omega^1} \to A. \tag{8.45}$$

We recall from §2.8 that $\overline{\Omega^1}$ is the conjugate bimodule with the same abelian group for addition but a conjugate structure for the left and right A actions, namely $a.\overline{\xi} = \overline{\xi.a^*}$ and $\overline{\xi}.a = \overline{a^*.\xi}$. Here we shall not assume that the hermitian metric arises from the construction above, but deal with it in its own right. All matters concerning $*$ in the previous section are actually clearer in this language using the conversion stated, but unless the connection is a $*$-preserving bimodule connection, the hermitian Riemannian theory is in principle a different generalisation.

Also, for the most part, we will work more generally with a 'hermitian inner product' $\langle , \rangle : \Omega^1 \otimes \overline{\Omega^1} \to A$ rather than insisting that it descends to $\otimes_A$. There are in fact two definitions of such hermitian structures on modules, depending on which side gives a conjugate linear map, and this side cannot be swapped without assuming extra structure. Our choice of side here fits better with left-covariant derivatives, whereas in §4.5 we chose the other side as it gave the standard form of a Hilbert C^*-module. Having chosen a side, we find that for many purposes we only have to work with one-sided modules rather than bimodules, and this is the approach which we will take. For example, we can comfortably deal with noncentral metrics.

We will describe hermitian inner products using bases and matrices, as we did for connections with Christoffel symbols in §3.2, and we will recover the familiar g^{ij} and g_{ij} matrices of classical Riemannian geometry. This requires the use of *finitely generated projective (fgp)* modules and dual bases as discussed in §3.1. From there we recall the definition of the right dual $E^\flat = {}_A\mathrm{Hom}(E, A)$ of a left A-module E, and the evaluation bimodule map $\mathrm{ev}_E : E \otimes E^\flat \to A$. Although we start with a general definition of hermitian inner products in terms of the bar categories in §2.8, we will rapidly specialise it to more familiar language.

Definition 8.30 A nondegenerate left hermitian structure on a left A-module E is an invertible right module map $\mathrm{G} : \overline{E} \to E^\flat$ such that $\langle\ ,\ \rangle = \mathrm{ev}_E(\mathrm{id} \otimes \mathrm{G}) : E \otimes \overline{E} \to A$ obeys the hermitian property that

$$E \otimes \overline{E} \xrightarrow{\mathrm{bb}\otimes\mathrm{id}} \overline{\overline{E}} \otimes \overline{E} \xrightarrow{\Upsilon^{-1}} \overline{E \otimes \overline{E}} \xrightarrow{\overline{\langle\, ,\,\rangle}} \bar{A} \xrightarrow{*^{-1}} A$$

composes to $\langle\ ,\ \rangle$. The latter is the associated *hermitian inner product*.

If we write $\langle e, \bar{f}\rangle = \mathrm{ev}(e \otimes \mathrm{G}(\bar{f}))$ for $e, f \in E$ and compare with the composition

$$e \otimes \bar{f} \longmapsto \bar{\bar{e}} \otimes \bar{f} \longmapsto \overline{f \otimes \bar{e}} \longmapsto \overline{\langle f, \bar{e}\rangle} \longmapsto \langle f, \bar{e}\rangle^*$$

displayed in Definition 8.30, we see that equality of the two comes down to the familiar equation $\langle e, \bar{f}\rangle = \langle f, \bar{e}\rangle^*$. Various formulae for $\langle\ ,\ \rangle$ are virtually automatic in this approach. For example, since ev_E is both a right A-module map and a left A-module map, we have respectively and for all $a \in A$,

$$\langle e, \overline{a.f}\rangle = \langle e, \bar{f}.a^*\rangle = \langle e, \bar{f}\rangle a^*, \quad \langle a.e, \bar{f}\rangle = a\langle e, \bar{f}\rangle.$$

Next we make the assumption that E is finitely generated projective as a left module. By fixing dual bases $\{e^i\}$ and $\{e_i\}$ we can describe the inner product by matrices $g^{ij} = \langle e^i, \overline{e^j}\rangle$. The hermitian property is then equivalent to $g^{ij*} = g^{ji}$. We will sum over repeated indices.

Proposition 8.31 *Suppose that E is fgp as a left A-module, with dual bases or coevaluation element $e_i \otimes e^i \in E^\flat \otimes E$ and that* G *as in Definition 8.30 is a nondegenerate left hermitian structure on E.*

(1) G *can be recovered from* $\langle\ ,\ \rangle$ *by* $\mathrm{G}(\overline{e^i}) = e_j.g^{ji}$.
(2) *There exist g_{ij} such that* $\mathrm{G}^{-1}(e_i) = \overline{g_{ij}.e^j}$ *and* $g_{ij}.\mathrm{ev}(e^j \otimes e_k) = g_{ik}$. *Then*

$$g^{ij}g_{jk} = \mathrm{ev}(e^i \otimes e_k), \quad g_{ij}g^{jk} = \mathrm{ev}(e^k \otimes e_i)^*, \quad g^*_{iq} = g_{qi}.$$

Proof (1) If we define G as stated then we recover the inner product as we should,

$$\begin{aligned}\mathrm{ev}(e^i \otimes \mathrm{G}(\overline{e^j})) &= \mathrm{ev}(e^i \otimes e_k.g^{kj}) = \mathrm{ev}(e^i \otimes e_k)\langle e^k, \overline{e^j}\rangle \\ &= \langle \mathrm{ev}(e^i \otimes e_k)e^k, \overline{e^j}\rangle = \langle e^i, \overline{e^j}\rangle.\end{aligned}$$

(2) We can always find h_{ij} such that $\mathrm{G}^{-1}(\overline{e^i}) = e_j.h_{ij}$. But then $h_{ij}\mathrm{ev}(e^j \otimes e_k) = g_{ik}$ will do equally well for the first required property and also obey the second required property. Then, as G is invertible,

$$\overline{e^i} = \mathrm{G}^{-1}(\mathrm{G}(\overline{e^i})) = \mathrm{G}^{-1}(e_j.g^{ji}) = \mathrm{G}^{-1}(e_j).g^{ji} = \overline{g_{jk}.e^k}.g^{ji} = \overline{g^{ij}g_{jk}.e^k},$$

and applying e_n to both sides of $e^i = g^{ij} g_{jk}.e^k$, we obtain the first of the displayed identities. Also

$$e_i = \mathrm{G}(\mathrm{G}^{-1}(e_i)) = \mathrm{G}(\overline{g_{ij}.e^j}) = \mathrm{G}(\overline{e^j}).g^*_{ij} = e_k.g^{kj} g^*_{ij},$$

applying both sides to e^p gives $\mathrm{ev}(e^p \otimes e_i) = g^{pj} g^*_{ij}$, and applying $*$ gives the second identity. Finally,

$$g_{ni} = g_{nk}\mathrm{ev}(e^k \otimes e_i) = g_{nk} g^{kj} g^*_{ij} = \mathrm{ev}(e^j \otimes e_n)^* g^*_{ij} = (g_{ij}\mathrm{ev}(e^j \otimes e_n))^* = g^*_{in}.$$

□

Note that the matrix g_{nk} is not necessarily the inverse matrix to g^{nk}, but this is also true classically; in classical differential geometry, the inverse matrix property is only true over a single coordinate chart chosen to trivialise the bundle and if we were to work globally then we would need a projection matrix P as we now explain in the general possibly noncommutative setting above. Namely, we let $P_{ij} = \mathrm{ev}(e^i \otimes e_j)$ and regard

$$P = (P_{ij}), \quad g = (g^{ij}), \quad \tilde{g} = (g_{ij})$$

as A-valued matrices so that we can use matrix notation for products and hermitian conjugation $\dagger$ (which in turn is in the same spirit as our use of this symbol for $*$ on tensor products in §8.1). Here $P^2 = P$ from the definition of an fgp module, while the results in Proposition 8.31 and just before it can be summarised as

$$g^\dagger = g, \quad \tilde{g}^\dagger = \tilde{g}, \quad g\tilde{g} = P, \quad \tilde{g}P = \tilde{g}, \quad Pg = g. \tag{8.46}$$

While not automatic, we may be able to choose our dual bases such that $P^\dagger = P$. This is always possible, for example, if A is a local C^*-algebra in view of Lemma 3.42.

In classical differential geometry, every finite-dimensional locally trivial $\mathbb{R}$ or $\mathbb{C}$ vector bundle can be given an inner product. This is not always true in the quantum case as we saw in Example 4.37, where there is no central metric on the original closed disk algebra—to find a central metric, we had to invert w to get a different algebra. However, if we consider left hermitian structures then we have the following existence result. We recall that by 'positive' in a C^*-algebra A, we mean a sum of elements of the form a^*a for $a \in A$.

Proposition 8.32 *Suppose that E is a left fgp A-module where A is a local C^*-algebra and that dual bases $e_i \otimes e^i \in E^\flat \otimes E$ are chosen such that $P^\dagger = P$. Then there is a hermitian inner product $\langle\, , \rangle : E \otimes \overline{E} \to A$ such that $\langle x, \overline{e^i}\rangle e^i = x$ and $\langle x, \overline{x}\rangle \geq 0$ for $x \in E$ and $\langle x, \overline{x}\rangle = 0$ only when $x = 0$. Moreover, the associated* G *is invertible.*

Proof We suppose that $P_{ij} = \mathrm{ev}(e^i \otimes e_j)$ is a hermitian projection matrix. Now define a right module map $\mathrm{G} : \overline{E} \to E^\flat$ by $\mathrm{G}(\overline{a_i e^i}) = \mathrm{G}(\overline{e^i} a_i^*) = e_i a_i^*$. This is well defined as $\mathrm{G}(\overline{P_{ji} e^i}) = e_i P_{ji}^* = e_i P_{ij} = e_j = \mathrm{G}(\overline{e^j})$. The associated inner product has $\langle e^i, \overline{e^j} \rangle = P_{ij}$, which is a hermitian matrix, leading to $\langle x, \overline{y} \rangle^* = \langle y, \overline{x} \rangle$ for all $x, y \in E$. Also observe that $\langle x, \overline{e^i} \rangle e^i = \mathrm{ev}(x \otimes e_i) e^i = x$ for all $x \in E$. Hence taking the inner product of this with $\overline{x}$ gives

$$\langle x, \overline{x} \rangle = \langle x, \overline{e^i} \rangle \langle e^i, \overline{x} \rangle = \langle x, \overline{e^i} \rangle \langle x, \overline{e^i} \rangle^* \geq 0$$

being a sum over i of positive terms. Further, if $\langle x, \overline{x} \rangle = 0$ then every $\langle x, \overline{e^i} \rangle = 0$, so we deduce that $x = 0$. The inverse of G is given by $\mathrm{G}^{-1}(e_i) = \overline{e^i}$. □

For geometry, we are also interested in the condition for a left connection to preserve a left hermitian structure, which we state in terms of Christoffel symbols. These were defined in (3.8) by $\nabla_E e^i = -\Gamma^i{}_k \otimes e^k$ and it will be convenient to view them as a matrix $(\Gamma)_{ij} = \Gamma^i{}_j$ as we did just after (3.8). This is very similar to the definition stated for Hilbert C^*-modules in (4.38), though for the other sided versions of inner products and connections.

Definition 8.33 A left connection $\nabla_E : E \to \Omega^1 \otimes_A E$ preserves the hermitian inner product $\langle\ ,\ \rangle : E \otimes \overline{E} \to A$ if

$$\mathrm{d}\langle\ ,\ \rangle = (\mathrm{id} \otimes \langle\ ,\ \rangle)(\nabla_E \otimes \mathrm{id}) + (\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{\nabla}_E) : E \otimes \overline{E} \to \Omega^1,$$

where $\tilde{\nabla}_E : \overline{E} \to \overline{E} \otimes_A \Omega^1$ is the right connection constructed from ∇_E by $\tilde{\nabla}_E(\overline{e}) = \overline{f} \otimes \kappa^*$, where $\nabla_E(e) = \kappa \otimes f$. When E is left fgp, this takes the following explicit form in terms of Christoffel symbols, which we also write in the compact notation of (8.46), namely

$$\mathrm{d}\langle e^i, \overline{e^j} \rangle = -\Gamma^i{}_k \langle e^k, \overline{e^j} \rangle - \langle e^i, \overline{e^k} \rangle (\Gamma^j{}_k)^*; \quad \mathrm{d}g = -\Gamma g - g\Gamma^\dagger.$$

This completes the left module theory. Next we turn to the case where E is an A-bimodule and consider Definition 8.30, but now with G a bimodule map, in which case we say that we have a *left bimodule hermitian structure* on E. This time $\langle\ ,\ \rangle$ necessarily descends to a *hermitian metric inner product* $\langle\ ,\ \rangle = \mathrm{ev}_E(\mathrm{id} \otimes \mathrm{G}) : E \otimes_A \overline{E} \to A$. If E is also left fgp then we can use the coevaluation map $\mathrm{coev}_E : A \to E^\flat \otimes_A E$ to define a *hermitian metric* or 'inverse inner product' $\langle\ ,\ \rangle^{-1} : A \to \overline{E} \otimes_A E$ by $\langle\ ,\ \rangle^{-1} = (\mathrm{G}^{-1} \otimes \mathrm{id})\mathrm{coev}_E$, obeying

$$(\mathrm{id} \otimes \langle\ ,\ \rangle)(\langle\ ,\ \rangle^{-1} \otimes \mathrm{id}) = \mathrm{id} : \overline{E} \to \overline{E}, \quad (\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \langle\ ,\ \rangle^{-1}) = \mathrm{id} : E \to E.$$

Here $\langle\ ,\ \rangle^{-1}$ can be regarded as a central element $\langle\ ,\ \rangle^{-1}(1) \in \overline{E} \otimes_A E$.

For an example, we recall the construction of a line module L and its dual in Proposition 3.98. These take the form

$$L = \{b.\underline{w}^T \subset \mathrm{Row}^n(A) \mid b \in B_k\}, \quad L^\flat = \{\underline{v}.b \subset \mathrm{Col}^n(A) \mid b \in B_{k^{-1}}\},$$

where B is a K-graded unital $*$-algebra with $B_e = A$, $B_k{}^* \subseteq B_{k^{-1}}$ and such that the product $B_k \otimes_A B_{k^{-1}} \to A$ is surjective for all k. Here K is a discrete group and the above line module is specified by a choice of $k \in K$ and column vectors $\underline{v} \in \mathrm{Col}^n(B_k)$, $\underline{w} \in \mathrm{Col}^n(B_{k^{-1}})$ such that $\underline{w}^T\underline{v} = \sum_i w_i v_i = 1$. Line modules are always bimodules and in the above form this is just given by left and right multiplication of $b \in B_{k^{\pm 1}}$ by A. Moreover, the above line modules came with a canonical dual basis such that $P = \underline{v}\underline{w}^T \in M_n(A)$.

Corollary 8.34 *Let A be a local C^*-algebra. In the line module construction of Proposition 3.98, if $w_i = v_i^*$ for all i then $\mathrm{G}(\overline{\underline{y}^T}) = \underline{y}^*$ (applying $*$ to each entry) defines a nondegenerate positive left bimodule hermitian structure and a nondegenerate hermitian metric on L.*

Proof Under the assumption, $P^*_{ji} = w_i^* v_j^* = v_i w_j = P_{ij}$ is hermitian. Hence Proposition 8.32 applies and we have a positive hermitian inner product with G invertible. Applying that construction, G comes out as $\mathrm{G}(\overline{b\underline{w}^T}) = (b\underline{w})^* = \underline{v}b^*$ (where $*$ acts on each entry of a vector) and we check that it is a bimodule map,

$$\mathrm{G}(a\triangleright\overline{b\underline{w}^T}) = \mathrm{G}(\overline{(b\underline{w}^T)\triangleleft a^*}) = \mathrm{G}(\overline{ba^*\underline{w}^T}) = \underline{v}(ba^*)^* = \underline{v}ab^* = a\triangleright(\underline{v}b^*),$$

$$\mathrm{G}(\overline{b\underline{w}^T}\triangleleft a) = \mathrm{G}(\overline{a^*\triangleright(b\underline{w}^T)}) = \mathrm{G}(\overline{a^*b\underline{w}^T}) = \underline{v}(a^*b)^* = \underline{v}b^*a = (\underline{v}b^*)\triangleleft a. \qquad \square$$

In the presence of a nondegenerate hermitian metric on a line module, we have an addition to Corollary 3.94 about the unital invertible Frölich map Φ_L.

Corollary 8.35 *If A is a $*$-algebra and L is a line module with nondegenerate hermitian metric inner product, then $\Phi_L : \mathcal{Z}(A) \to \mathcal{Z}(A)$ is a $*$-algebra map.*

Proof A nondegenerate hermitian metric on L implies an A-bimodule isomorphism between $\overline{L}$ and $L^\flat$, so we have $\Phi_{\overline{L}} = \Phi_{L^\flat}$. Now $L \otimes_A L^\flat$ is isomorphic to A as a bimodule, so by the result on tensor products in Corollary 3.94, we have $\Phi_{L^\flat} = \Phi_L{}^{-1}$. Then by Proposition 3.95, $(\Phi_L{}^{-1}(z^*))^* = \Phi_L{}^{-1}(z)$. $\square$

Also recall that Proposition 5.27 constructed from a line module L an integer graded algebra $T_{\mathbb{Z}}(L)$ from tensor products of $L, L^\flat$ with grades $1, -1$ respectively and product defined by iterated evaluation or inverse of coevaluation.

Proposition 8.36 *Suppose A is a $*$-algebra with a line module L equipped with a hermitian metric inner product $\langle\,,\rangle : L \otimes_A \overline{L} \to A$ given by a bimodule isomorphism $\mathrm{G} : \overline{L} \to L^\flat$. Then $T_{\mathbb{Z}}(L)$ is a $*$-algebra with $*$-operation*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_m \longmapsto x_m^* \otimes \cdots \otimes x_2^* \cdots \otimes x_1^*,$$

where each $x_i \in L$ or $L^\flat$ with $*$-operation such that $\overline{(\ )^*}$ is given by

$$L \xrightarrow{\mathrm{bb}} \overline{\overline{L}} \xrightarrow{\overline{\mathrm{G}}} \overline{L^\flat} \quad \text{or} \quad L^\flat \xrightarrow{\mathrm{G}^{-1}} \overline{L}.$$

Proof The main part of the proof is to show that $(xy)^* = y^*x^*$ and $(yx)^* = x^*y^*$ for $x \in L$ and $y \in L^\flat$. This is done by taking an explicit dual basis and the matrix notation of Proposition 8.31 for the metric G. As the product is an A-bimodule map, we need only check the required relations on the basis elements $x = e^i$ and $y = e_k$, and applying the $*$-operation to these gives

$$e^{i*} = e_j g^{ji}, \quad {e_k}^* = g_{kq} e^q.$$

(i) Using $P_{qj} = \mathrm{ev}(e^q \otimes e_j)$, we have immediately that

$$y^* x^* = g_{kq} e^q e_j g^{ji} = g_{kq} P_{qj} g^{ji} = g_{kq} g^{qi} = (P_{ik})^* = (e^i e_k)^* = (xy)^*.$$

(ii) To check $(yx)^* = x^*y^*$, we set $a = x^*y^*$ so that by definition $\mathrm{coev}(a) = x^* \otimes y^* \in L^\flat \otimes_A L$. Hence, given a basis element $e^n \in L$, we have by Proposition 3.97 that $e^n.a = \mathrm{ev}(e^n \otimes x^*).y^*$. Applying $*$ to this and rearranging gives

$$\begin{aligned} a^* e_m &= y.\mathrm{ev}(e^n \otimes x^*)^* g_{nm} = y.\mathrm{ev}(g_{mn} e^n \otimes x^*)^* \\ &= y.\mathrm{ev}({e_m}^* \otimes x^*)^* = y.\mathrm{ev}(x \otimes e_m), \end{aligned}$$

where we used part (i) for the last step. Now

$$\mathrm{ev}(e^p \otimes a^* e_m) = \mathrm{ev}(e^p \otimes y)\mathrm{ev}(x \otimes e_m) = \mathrm{ev}\big(\mathrm{ev}(e^p \otimes y)x \otimes e_m\big),$$

which implies $e^p a^* = \mathrm{ev}(e^p \otimes y)x = e^p \mathrm{coev}^{-1}(y \otimes x)$, as required. □

We conclude the section with some examples of hermitian metric compatible connections. There will be further examples in §8.6 on Chern connections.

Example 8.37 Topologists are used to having a definite idea of the isomorphism class of a bundle, and especially of whether it is nontrivial or not. It is not always clear, however, how this arises if the continuum geometry is the limit of a discrete or finite version. This is broadly analogous to statistical mechanics where finite systems often do not give clear phase transitions; only by passing to a limit as the size of the system tends to infinity does the discontinuity in a physical quantity or its derivative characterising a phase transition become obvious. An actual physical system may have a finite character but in a thermodynamic context, for example, the number of atoms (of the order of Avogadro's number 6.022×10^{23}) may be large enough that the system can be treated as a continuum (the thermodynamic limit). Now in our context of a differential version of sheaf cohomology in Chap. 4, we use a flat covariant derivative to define what it means for a section to be 'locally constant'. Hence it makes sense to look for some kind of 'energy' that depends on the covariant derivative of a section and measures how far if deviates from being 'locally constant', and we might then hope for a finite system of size N that topological distinctions of the continuum system could be separated by an increasing energy barrier as $N \to \infty$.

We illustrate this idea on the discrete Möbius bundle in Example 3.86, where each bundle $E = \mathrm{M\ddot{o}b}_N$ is free, being a copy of $A = \mathbb{C}(\mathbb{Z}_N)$ as a bimodule, but is equipped with a flat connection and a $\star$-operation. We observe first that there is a standard hermitian metric on $\mathrm{M\ddot{o}b}_N$ with $\langle s, \overline{t}\rangle = \sum_{x\in\mathbb{Z}_N} s_x {t_x}^* \delta_x \in A$, if $s = \sum_{x\in\mathbb{Z}_N} s_x\delta_x$ and similarly for t. Then (with a sum over x understood),

$$
\begin{aligned}
&(\mathrm{id}\otimes\langle\,,\,\rangle)(\nabla_{\mathrm{M\ddot{o}b}_N} s\otimes\overline{t})\\
&\quad = (s_{x+1} - \mathrm{e}^{\mathrm{i}\pi/N} s_x){t_{x+1}}^* e_1\delta_{x+1} + (s_{x-1} - \mathrm{e}^{-\mathrm{i}\pi/N} s_x){t_{x-1}}^* e_{-1}\delta_{x-1}\\
&\quad = (s_x - \mathrm{e}^{\mathrm{i}\pi/N} s_{x-1}){t_x}^* e_1\delta_x + (s_x - \mathrm{e}^{-\mathrm{i}\pi/N} s_{x+1}){t_x}^* e_{-1}\delta_x,
\end{aligned}
$$

and for the right connection on the conjugate module,

$$
\begin{aligned}
&(\langle\,,\,\rangle\otimes\mathrm{id})(s\otimes\tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}\overline{t})\\
&\quad = \langle s, \overline{\delta_{x+1}}\rangle((t_{x+1} - \mathrm{e}^{\mathrm{i}\pi/N} t_x)e_1)^* + \langle s, \overline{\delta_{x-1}}\rangle((t_{x-1} - \mathrm{e}^{-\mathrm{i}\pi/N} t_x)e_{-1})^*\\
&\quad = -s_{x+1}\delta_{x+1}e_{-1}({t_{x+1}}^* - \mathrm{e}^{-\mathrm{i}\pi/N}{t_x}^*) - s_{x-1}\delta_{x-1}e_1({t_{x-1}}^* - \mathrm{e}^{\mathrm{i}\pi/N}{t_x}^*)\\
&\quad = -s_{x+1}({t_{x+1}}^* - \mathrm{e}^{-\mathrm{i}\pi/N}{t_x}^*)e_{-1}\delta_x - s_{x-1}({t_{x-1}}^* - \mathrm{e}^{\mathrm{i}\pi/N}{t_x}^*)e_1\delta_x,
\end{aligned}
$$

from which we see that the connection $\nabla_{\mathrm{M\ddot{o}b}_N}$ preserves the hermitian structure in the sense of Definition 8.33,

$$
\begin{aligned}
&(\mathrm{id}\otimes\langle\,,\,\rangle)(\nabla_{\mathrm{M\ddot{o}b}_N} s\otimes\overline{t}) + (\langle\,,\,\rangle\otimes\mathrm{id})(s\otimes\tilde{\nabla}_{\overline{\mathrm{M\ddot{o}b}_N}}\overline{t})\\
&\qquad = (s_x - \mathrm{e}^{-\mathrm{i}\pi/N} s_{x+1}){t_x}^* e_{-1}\delta_x - s_{x+1}({t_{x+1}}^* - \mathrm{e}^{-\mathrm{i}\pi/N}{t_x}^*)e_{-1}\delta_x\\
&\qquad\quad + (s_x - \mathrm{e}^{\mathrm{i}\pi/N} s_{x-1}){t_x}^* e_1\delta_x - s_{x-1}({t_{x-1}}^* - \mathrm{e}^{\mathrm{i}\pi/N}{t_x}^*)e_1\delta_x\\
&\qquad = s_x{t_x}^*(e_{-1}\delta_x - e_{-1}\delta_{x-1} + e_1\delta_x - e_1\delta_{x+1})\\
&\qquad = s_x{t_x}^*((\delta_{x+1} - \delta_x)e_{-1} + (\delta_{x-1} - \delta_x)e_1) = \mathrm{d}\langle s, \overline{t}\rangle.
\end{aligned}
$$

We also give Ω^1 a hermitian metric by $\langle e_\pm, \overline{e_\pm}\rangle = 1$ and $\langle e_\pm, \overline{e_\mp}\rangle = 0$ and we define a normalised integral $\int : A \to \mathbb{C}$ by $\int \delta_x = 1/N$ for all x, where $\int 1 = 1$.

Using the tensor product inner product, we now define the 'energy' of a section by

$$
\begin{aligned}
E(s) &= N^2 \int \langle \nabla_{\mathrm{M\ddot{o}b}_N}(s), \overline{\nabla_{\mathrm{M\ddot{o}b}_N}(s)} \rangle \\
&= N^2 \int \langle\ ,\ \rangle(\mathrm{id} \otimes \langle\ ,\ \rangle \otimes \mathrm{id})(\nabla_{\mathrm{M\ddot{o}b}_N}(s) \otimes \Upsilon \overline{\nabla_{\mathrm{M\ddot{o}b}_N}(s)}) \\
&= N \sum_{x \in \mathbb{Z}_N} \Big(|s_{x+1} - e^{i\pi/N} s_x|^2 + |s_{x-1} - e^{-i\pi/N} s_x|^2 \Big).
\end{aligned}
$$

This is the square of the L^2 norm of the derivative of s. We have N^2 because the finite difference derivative is taken over a distance proportional to $1/N$ on embedding $\mathbb{Z}_N$ in the unit circle with its usual metric, and to get the classical limit for each derivative we divide by the step length. Now, the classical Möbius bundle is nontrivial because every section vanishes at some point. In our finite case we can have nonvanishing sections but we look at their energy. Thus, let $r \in \mathrm{M\ddot{o}b}_N$ be a real section with respect to $\star$, which as explained in Example 3.86 means $r = \sum_{x \in \mathbb{Z}_N} r_x e^{xi\pi/N} \delta_x$ with $r_x \in \mathbb{R}$. We control the normalisation by requiring norm 1 at each point, which in our language means $\langle r, \bar{r} \rangle = \sum_x r_x^2 \delta_x = 1$, the constant function, and amounts to $r_x \in \{1, -1\}$. Meanwhile, the energy of a real section is

$$
E(r) = 2N \Big(\sum_{x=0}^{N-2} |r_{x+1} - r_x|^2 + |r_0 + r_{N-1}|^2 \Big),
$$

where the last term comes from the wrap around in the formula for E. We see that real sections of constant norm 1 have energy at least $8N$, which tends to infinity as the number of points increases. ◇

Example 8.38 We have already covered the regular quantum Riemannian geometry of the permutation group $G = S_3$ with its 3D calculus and the Euclidean metric in Example 8.21 from Example 1.60 and its Euclidean metric. Here we consider the same 5-parameter class of invariant bimodule connections ∇ on $E = \Omega^1$ as described in terms of Christoffel symbols in Example 3.76 with parameters a, b, c, d, e, but now with the Euclidean metric viewed as a central hermitian metric with matrix g^{ij} taken as a positive multiple of the identity matrix in the basis $\{e^u, e^v, e^w\}$. A little calculation using the matrix formulae in Definition 8.33 shows that our class of ∇ preserve the hermitian metric if and only if a, c, d are real and $e = b^*$. We analyse this further.

(1) ∇ preserves the hermitian metric and is torsion free if and only if a, c are real, $c = d = e + 1 = b + 1$, i.e., a 2-parameter moduli space of such connections. We use the condition for torsion-free in Example 3.76.

(2) ∇ preserves the hermitian metric and is $*$-compatible if and only if it is one of the following discrete moduli of possibilities, where $\pm$ and $\pm'$ are independent signs:

i) $b = e = c = 0, a = \pm 1, d = \pm' 1$;
ii) $b = e = d = 0, a = \pm 1, c = \pm' 1$;
iii) $a = d = \pm\frac{1}{3}, b = e = c = \mp\frac{2}{3}$;
iv) $a = c = \pm\frac{1}{3}, b = e = d = \mp\frac{2}{3}$.

Here $*$-compatibility as in Definition 3.85 was analysed for ∇ in Proposition 3.87.

(3) ∇ cannot be hermitian metric compatible, torsion free and $*$-compatible. $\diamond$

Example 8.39 Here we consider a left-hermitian Riemannian metric for the 3D calculus on $\mathbb{C}_q[SU_2]$ where Ω^1 is viewed only as a left module as in Definition 8.30. This calculus has no obvious central quantum metric and hence also no corresponding central hermitian metric as in the full bimodule theory. We use the class of left-invariant bimodule connections in Example 3.77 with parameters $\alpha_\pm, \beta_\pm, \mu, \nu, \gamma$. We define a hermitian metric by $\langle e^i, \overline{e^j}\rangle = g^{ij}$, where $i, j \in \{+, 0, -\}$, and set $g^{ij} = 0$ if $i \neq j$ and $g^{ii} \in \mathbb{R} \setminus \{0\}$ for some real constants. Here $\mathrm{d}g = 0$ so the connection preserves the inner product as in Definition 8.33 if and only if $\Gamma g = -g\Gamma^\dagger$, i.e.,

$$\begin{pmatrix} g^{++}\alpha_+ e^0 & g^{00}\beta_+ e^+ & 0 \\ g^{++}\mu e^- & g^{00}\gamma e^0 & g^{--}\nu e^+ \\ 0 & g^{00}\beta_- e^- & g^{--}\alpha_- e^0 \end{pmatrix} = -\begin{pmatrix} g^{++}\alpha_+ e^0 & g^{00}\beta_+ e^+ & 0 \\ g^{++}\mu e^- & g^{00}\gamma e^0 & g^{--}\nu e^+ \\ 0 & g^{00}\beta_- e^- & g^{--}\alpha_- e^0 \end{pmatrix}^\dagger$$

from which hermitian metric compatibility comes down to

$$\alpha_+, \alpha_-, \gamma \in \mathbb{R}, \quad g^{00}\beta_+ = q g^{++}\mu^*, \quad g^{00}\beta_- = q^{-1} g^{--}\nu^*.$$

Conditions for this class of ∇ to be torsion-compatible are in Example 3.77 and $*$-compatible in Example 3.89. Adding these constraints leads to a 2-component 1-parameter moduli space given by $\gamma = 0$ or $\gamma = \frac{2}{q^2-1}$ and $\alpha_+ \in \mathbb{R}$ with

$$\alpha_- = \frac{-\alpha_+}{1 + (1-q^2)\alpha_+}, \quad \beta_+ = q^4\alpha_+ - (1+q^2), \quad \beta_- = \frac{\beta_+\alpha_-}{q^4\alpha_+},$$
$$\nu = q\beta_- \frac{g^{00}}{g^{--}}, \quad \mu = q^{-1}\beta_+ \frac{g^{00}}{g^{++}}$$

and in this case the torsion is $T_\nabla(e^\pm) = 0$ and $T_\nabla(e^0) = (\nu - q^2\mu - q^3)e^+ \wedge e^-$.

Setting $T_\nabla = 0$ gives a quadratic equation for α_+ with positive discriminant, giving *four* hermitian metric-compatible torsion free $*$-compatible connections. Of these, there is a *unique* one with classical limit as $q \to 1$, namely $\gamma = 0$ and α_+ a certain function of q, g^{++}, g^{--}. The limit $q \to 1$ has $\alpha_+ = 2 - \frac{g^{++}g^{--}}{g^{00}(g^{++}+g^{--})}$, giving the classical Levi-Civita connection for this metric on SU_2, which is $*$-preserving. But our solutions for real $q \neq 1$ are not $*$-preserving and there are no hermitian metric-compatible $*$-preserving torsion free solutions in this case. $\diamond$

We close with a more detailed discussion of the relation between a hermitian metric inner product $\langle\ ,\ \rangle$ and the Riemannian metric inner product $(\ ,\ )$ as in (8.45), noting that so far in this section we have worked more generally with the former defined on $\Omega^1 \otimes \overline{\Omega^1}$ rather than on $\Omega^1 \otimes_A \overline{\Omega^1}$.

Proposition 8.40 *Given a hermitian inner product $\langle\, ,\rangle : \Omega^1 \otimes \overline{\Omega^1} \to A$, we define a bimodule inner product $(\, ,) : \Omega^1 \otimes \Omega^1 \to A$ by $(\tau, \xi) = \langle \tau, \overline{\xi^*}\rangle$. Then a $*$-preserving left bimodule connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ preserves the hermitian inner product (see Definition 8.33) if and only if*

$$\mathrm{d}(\tau, \xi) = (\mathrm{id} \otimes (\, ,))(\nabla\tau \otimes \xi) + ((\, ,) \otimes \mathrm{id})(\tau \otimes \sigma^{-1}\nabla\xi).$$

If $\langle\, ,\rangle$ (or equivalently $(\, ,)$) descends $\otimes_A$ then the above condition is equivalent to $(\ ,\ )$ metric compatible in the usual sense (8.9).

Proof The condition for $\langle\ ,\ \rangle$ to be preserved is

$$\mathrm{d}\langle\tau, \overline{\xi^*}\rangle = (\mathrm{id} \otimes \langle\, ,\ \rangle)(\nabla\tau \otimes \overline{\xi^*}) + (\langle\, ,\ \rangle \otimes \mathrm{id})(\tau \otimes \tilde{\nabla}(\overline{\xi^*}))$$

for $\tau, \xi \in \Omega^1$. The left-hand side is $\mathrm{d}(\tau, \xi)$ and if we write $\nabla(\xi^*) = \eta \otimes \kappa$ (sum of such terms) then $\tilde{\nabla}(\overline{\xi^*}) = \overline{\kappa} \otimes \eta^*$ and the condition is

$$\mathrm{d}(\tau, \xi) = (\mathrm{id} \otimes (\, ,))(\nabla\tau \otimes \xi) + (\tau, \kappa^*)\eta^*.$$

That ∇ is $*$-preserving is (8.14), which we write as

$$\kappa^* \otimes \eta^* = \dagger\,\nabla(\xi^*) = \dagger\,\dagger\,\sigma^{-1}\nabla\xi = \sigma^{-1}\nabla\xi,$$

giving the condition in the $(\ ,\ )$ form stated. Finally, if $(\, ,)$ descends to $\otimes_A$ then

$$0 = \mathrm{d}(\tau.a, \xi) - \mathrm{d}(\tau, a.\xi) = (\mathrm{id}\otimes(\, ,))(\sigma(\tau\otimes\mathrm{d}a)\otimes\xi) - ((\, ,)\otimes\mathrm{id})(\tau\otimes\sigma^{-1}(\mathrm{d}a\otimes\xi))$$

for all $a \in A$, giving the metric compatibility in the inner product form (8.9). $\square$

It can be easier to find hermitian metric compatible connections, and the above says that if they are $*$-preserving then we can convert them afterwards to ordinary metric compatible connections. An example is provided in Exercise E8.7 where we use this method to find a QLC for the q-deformed open disk. However, the nicest case of a central metric g with inverse $(\ ,\ )$ may not always be possible and noncentral g are also of interest (as in Example 8.5 and Exercise E8.1). Similarly, nondegenerate $(\ ,\ )$ that do not descend to $\otimes_A$ can also be of interest as a different generalisation, in which case the condition stated in Proposition 8.40 is the appropriate compatibility.

8.5 Geometric Realisation of Spectral Triples

Dirac developed an approach to relativistic quantum theory using a first-order differential operator taking values in the spinor bundle on a manifold rather than the 2nd order Klein–Gordon operator. The original Dirac equation on $\mathbb{R}^{3+1}$ space time used spinor fields written as 4-dimensional column matrices with entries complex-valued functions on space time. The Dirac operator $\not{D}$ is, for a spinor field ψ,

$$\not{D}\psi = \gamma^\mu \frac{\partial \psi}{\partial x^\mu},$$

where we sum over repeated indices and γ^μ are certain 4×4 complex matrices. The relation between the Dirac operator and the Klein–Gordon wave operator $\Box$ is that $\not{D}^2\psi = \Box\psi = g^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu \partial x^\nu}$, where $g^{\mu\nu}$ is the spacetime metric (on Minkowski spacetime this would be a diagonal matrix with entries $1, -1, -1, -1$). For this to work we need $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. The algebra generated by these matrices γ^μ is known as a *Clifford algebra* and can be generalised to other vector spaces with inner products. Then the matrices γ^μ can be thought of as giving a representation of the abstract Clifford algebra on the spinors. In what follows we are going to think of the Clifford algebra here more geometrically as generated by a left action (which we call the *Clifford action*) of the cotangent bundle on the spinor bundle, denoted $\mathrm{d}x^\mu \triangleright \psi = \gamma^\mu \psi$ (but generating the Clifford algebra, not the exterior algebra). Then the Dirac operator can be expressed simply as

$$\not{D} = \triangleright \circ \nabla,$$

where $\nabla\psi = \mathrm{d}x^\mu \otimes \frac{\partial \psi}{\partial x^\mu}$ is the trivial connection on the spinor bundle. This point of view is convenient first to generalise to any Riemannian or pseudo-Riemannian manifold and then to the quantum case in our approach.

Meanwhile, the Dirac operator is the starting point of a very different 'Connes approach' to noncommutative geometry, where the axioms of a *spectral triple* capture its key properties in a way that makes sense in the noncommutative case. We focus on the algebraic side of these axioms, omitting the functional analysis. Connes' books provide more details on the latter and the larger picture.

Definition 8.41 A real spectral triple in dimension n modulo 8 consists of:

(1a) A Hilbert space $\mathcal{S}$, with inner product $\langle\!\langle\, , \rangle\!\rangle$ conjugate linear in the first argument, with a faithful representation of the $*$-algebra A on $\mathcal{S}$ such that

$$\langle\!\langle a^*.\psi, \phi \rangle\!\rangle = \langle\!\langle \psi, a.\phi \rangle\!\rangle$$

for all $a \in A$ and $\phi, \psi \in \mathcal{S}$.

(1b) Hermitian linear operators γ, $\not{D}$ and a conjugate linear operator $\mathcal{J}$ on $\mathcal{S}$ obeying $\langle\!\langle \mathcal{J}\psi, \mathcal{J}\phi \rangle\!\rangle = \langle\!\langle \phi, \psi \rangle\!\rangle$. (For odd dimension we omit γ.)

(2) $\mathcal{J}^2 = \epsilon$, $\mathcal{J}\gamma = \epsilon''\gamma\mathcal{J}$, $\gamma^2 = 1$, $[\gamma, a] = 0$.

(3) $\not{D}\gamma = (-1)^{n-1}\gamma\not{D}$.

(4) $[a, \mathcal{J}b\mathcal{J}^{-1}] = 0$ for all $a, b \in A$.

(5) $\mathcal{J}\not{D} = \epsilon'\not{D}\mathcal{J}$.

(6) $[[\not{D}, a], \mathcal{J}b\mathcal{J}^{-1}] = 0$ for all $a, b \in A$.

The signs $\epsilon, \epsilon', \epsilon''$ in $\{+1, -1\}$ are given by the table

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Note that the γ here for an even spectral triple should not be confused with the Clifford algebra matrices γ^μ. The latter do not enter directly in Connes' formulation in which $\not{D}$ is a starting point in its own right.

8.5.1 *Construction of Spectral Triples from Connections*

Our goal here is to build up noncommutative Dirac operators more along the lines of Dirac's original construction, starting with a connection on a vector bundle of 'spinors' and in our case a Clifford action as above (we do not really need a Clifford algebra as such). In nice cases, this may give us the algebraic side of a spectral triple but in other cases we will be guided by the different layers of the noncommutative Riemannian geometry elsewhere in the book rather than aiming 'top down' for some particular axioms. Moreover, to best describe the inner product and the antilinear map $\mathcal{J}$, we will use the language of bar categories from §2.8. We begin with a $*$-algebra A with a $*$-differential calculus $(\Omega, \mathrm{d}, \wedge)$ and a left A-module $\mathcal{S}$, which we call the spinor bundle. We fix n mod 8 and the corresponding signs in the table.

Lemma 8.42 *Let $\mathcal{S}$ be a left A-module, $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ an antilinear map and when n is even $\gamma : \mathcal{S} \to \mathcal{S}$ a linear map, satisfying axioms (2) and (4) of a spectral triple* *in Definition 8.41. Then $\mathcal{S}$ is an A-bimodule and γ a bimodule map with respect to a right action $\psi.a = \mathcal{J}a^*\mathcal{J}^{-1}\psi$ for all $\psi \in \mathcal{S}$ and $a \in A$.*

Proof The two actions commute by (4), and γ is a left module map by (2). Also,

$$\gamma\mathcal{J}a^*\mathcal{J}^{-1} = \epsilon''\mathcal{J}\gamma a^*\mathcal{J}^{-1} = \epsilon''\mathcal{J}a^*\gamma\mathcal{J}^{-1} = \mathcal{J}a^*\mathcal{J}^{-1}\gamma$$

shows that $[\gamma, \mathcal{J}a^*\mathcal{J}^{-1}] = 0$, and as a result γ is a bimodule map. Note that we do not need $\gamma^2 = 1$ in this result but we will assume it later. $\square$

The right action here amounts to a particular form of bimodule relations. We next need a bimodule connection $\nabla_\mathcal{S} : \mathcal{S} \to \Omega^1 \otimes_A \mathcal{S}$ on our spinor bundle and a *Clifford action* $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$ with certain properties which we elaborate.

Lemma 8.43 *Let $\mathcal{S}, \mathcal{J}, \gamma$ be as in Lemma 8.42 and $(\nabla_\mathcal{S}, \sigma_\mathcal{S})$ be a left bimodule connection on $\mathcal{S}$. Suppose that $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$ is a left module map and define $\not{D} = \triangleright \circ \nabla_\mathcal{S} : \mathcal{S} \to \mathcal{S}$. Then*

(i) $[\not{D}, a]\phi = \mathrm{d}a\triangleright\phi$.
(ii) *$\triangleright$ is a bimodule map if and only if axiom (6) of a spectral triple holds.*
(iii) *$\epsilon'\mathcal{J}[\not{D}, a^*]\mathcal{J}^{-1}\phi = \triangleright(\sigma_\mathcal{S}(\phi \otimes \mathrm{d}a))$ if and only if axiom (5) holds.*
(iv) *For even n, if in addition $\gamma^2 = \mathrm{id}$ with*

$$\nabla_\mathcal{S}\gamma = (\mathrm{id} \otimes \gamma)\nabla_\mathcal{S}, \quad \gamma \circ \triangleright = -\triangleright \circ (\mathrm{id} \otimes \gamma), \quad \mathcal{J} \circ \gamma = \epsilon''\gamma \circ \mathcal{J},$$

then all conditions for γ in axioms (2)–(6) hold.

Proof For (i), $\triangleright \circ \nabla_\mathcal{S}(a.\phi) = \triangleright(\mathrm{d}a \otimes \phi + a.\nabla_\mathcal{S}\phi) = \mathrm{d}a\triangleright\phi + a.(\triangleright\nabla_\mathcal{S}\phi)$. (ii) is immediate. For (iii),

$$\begin{aligned}\not{D}(\phi.a) &= \not{D}(\mathcal{J}a^*\mathcal{J}^{-1}\phi) = \epsilon'\mathcal{J}\not{D}a^*\mathcal{J}^{-1}\phi = \epsilon'\mathcal{J}[\not{D}, a^*]\mathcal{J}^{-1}\phi + \epsilon'\mathcal{J}a^*\not{D}\mathcal{J}^{-1}\phi \\ &= \epsilon'\mathcal{J}[\not{D}, a^*]\mathcal{J}^{-1}\phi + \mathcal{J}a^*\mathcal{J}^{-1}\not{D}\phi = \epsilon'\mathcal{J}[\not{D}, a^*]\mathcal{J}^{-1}\phi + (\not{D}\phi).a.\end{aligned}$$

For (iv), $\gamma\not{D}(\phi) = \gamma \circ \triangleright \circ \nabla_\mathcal{S}\phi = -\triangleright \circ (\mathrm{id} \otimes \gamma)\nabla_\mathcal{S}\phi = -\triangleright \circ \nabla_\mathcal{S}\gamma\phi = -\not{D}(\phi)$. $\square$

To make $\mathcal{J}$ into a linear map we define $j : \mathcal{S} \to \overline{\mathcal{S}}$ by $j(\phi) = \overline{\mathcal{J}\phi}$ (using the language of §2.8 to render an antilinear map as linear to a conjugate object). In the setting of Lemma 8.42 we have that j is a bimodule map since

$$\begin{aligned}&j(a.\phi) = \overline{\mathcal{J}(a.\phi)} = \overline{\mathcal{J}a\mathcal{J}^{-1}\mathcal{J}\phi} = \overline{\mathcal{J}(\phi).a^*} = a.\overline{\mathcal{J}(\phi)} = a.j(\phi),\\ &j(\phi.a) = j(\mathcal{J}a^*\mathcal{J}^{-1}\phi) = \overline{\mathcal{J}^2a^*\mathcal{J}^{-1}\phi} = \epsilon\overline{a^*\mathcal{J}^{-1}\phi} = \epsilon\overline{\mathcal{J}^{-1}\phi}.a = \overline{\mathcal{J}\phi}.a = j(\phi).a,\end{aligned}$$

for $a \in A$ and $\phi \in \mathcal{S}$. Now we are ready to turn this around and re-express the axioms for the real spectral triple parts (2)–(6) in terms of modules and connections. We fix signs according to n mod 8 in the table and we define $\star : \Omega^1 \to \overline{\Omega^1}$ as $*$ viewed more categorically as a bimodule map.

Theorem 8.44 *Let A be a $*$-differential algebra. Suppose that*

(i) $(\mathcal{S}, \nabla_{\mathcal{S}}, \sigma_{\mathcal{S}})$ *is a left bimodule connection with* $\sigma_{\mathcal{S}}$ *invertible;*
(ii) $j : \mathcal{S} \to \overline{\mathcal{S}}$ *is a bimodule map obeying* $\overline{j}\, j = \epsilon\, \mathrm{bb}$ *and* $(\mathrm{id} \otimes j)\nabla_{\mathcal{S}} = \nabla_{\overline{\mathcal{S}}}$*;*
(iii) $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$ *is a bimodule map obeying* $j \circ \triangleright = \epsilon' \,\overline{\triangleright \sigma_{\mathcal{S}}}\, \Upsilon^{-1}(\star \otimes j)$.

Then $\not{D} = \triangleright \circ \nabla_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ *satisfies axioms (2)–(6) for a spectral triple when n is odd. If there is also a bimodule map* $\gamma : \mathcal{S} \to \mathcal{S}$ *intertwining* $\nabla_{\mathcal{S}}$ *and obeying*

$$\gamma^2 = \mathrm{id}, \quad \gamma \circ \triangleright = -\triangleright \circ (\mathrm{id} \otimes \gamma) : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}, \quad j \circ \gamma = \epsilon'' \,\overline{\gamma} \circ j : \mathcal{S} \to \overline{\mathcal{S}}$$

then $(\not{D}, \mathcal{J}, \gamma)$ *satisfies the axioms (2)–(6) for a spectral triple when n is even.*

Proof Construct a left bimodule connection $\nabla_{\overline{\mathcal{S}}}$ on $\overline{\mathcal{S}}$ by

$$\nabla_{\overline{\mathcal{S}}}(\overline{\phi}) = (\star^{-1} \otimes \mathrm{id})\Upsilon\, \overline{\sigma_{\mathcal{S}}{}^{-1}\nabla_{\mathcal{S}}\phi}.$$

The condition for j to intertwine the connections is

$$(\mathrm{id} \otimes j)\nabla_{\mathcal{S}}\phi = \nabla_{\overline{\mathcal{S}}}\, j(\phi) = \nabla_{\overline{\mathcal{S}}}(\overline{\mathcal{J}\phi}) = (\star^{-1} \otimes \mathrm{id})\Upsilon\, \overline{\sigma_{\mathcal{S}}{}^{-1}\nabla_{\mathcal{S}}(\mathcal{J}\phi)}.$$

Then

$$\overline{\not{D}\mathcal{J}\phi} = \overline{\triangleright \nabla_{\mathcal{S}}(\mathcal{J}\phi)} = \overline{\triangleright \sigma_{\mathcal{S}}}\, \Upsilon^{-1}(\star \otimes j)\nabla_{\mathcal{S}}\phi,$$
$$\overline{\mathcal{J}\not{D}\phi} = j\, \not{D}\phi = j \circ \triangleright \circ \nabla_{\mathcal{S}}\phi.$$

Hence for axiom (5), we need $j \circ \triangleright = \epsilon' \,\overline{\triangleright \sigma_{\mathcal{S}}}\, \Upsilon^{-1}(\star \otimes j)$ as stated, or

$$\mathcal{J}(\xi \triangleright \phi) = \epsilon' \triangleright \circ\, \sigma_{\mathcal{S}}(\mathcal{J}\phi \otimes \xi^*) \tag{8.47}$$

for $\xi \in \Omega^1$. The construction here can also be applied for other signs than in the table and other choices as to whether to include γ or not. □

Note that if we have an actual spectral triple then, as $\mathcal{J}$ is antilinear, we can define a bilinear map $(\!(\,,\,)\!) : \mathcal{S} \otimes \mathcal{S} \to \mathbb{C}$ from the Hilbert space inner product $\langle\!\langle\,,\,\rangle\!\rangle$ in axiom (1a) by $(\!(\phi, \psi)\!) = \langle\!\langle \mathcal{J}\phi, \psi \rangle\!\rangle$. From the axioms, we have

$$\langle\!\langle a^*\phi, \psi \rangle\!\rangle = \epsilon (\!(\mathcal{J}a^*\phi, \psi)\!) = \epsilon (\!(\mathcal{J}a^*\mathcal{J}^{-1}\mathcal{J}\phi, \psi)\!) = \epsilon (\!((\mathcal{J}\phi).a, \psi)\!),$$
$$\langle\!\langle a^*\phi, \psi \rangle\!\rangle = \langle\!\langle \phi, a\psi \rangle\!\rangle = \epsilon (\!(\mathcal{J}\phi, a\psi)\!),$$

so $(\!(\,,\,)\!)$ descends to $(\!(\,,\,)\!) : \mathcal{S} \otimes_A \mathcal{S} \to \mathbb{C}$. The hermitian property of $\langle\!\langle\,,\,\rangle\!\rangle$ and the isometry condition for $\mathcal{J}$ in axiom (1b) now appear respectively as

$$(\!(\psi, \phi)\!)^* = \epsilon (\!(\mathcal{J}\phi, \mathcal{J}\psi)\!), \quad (\!(\psi, \phi)\!) = \epsilon (\!(\phi, \psi)\!) \tag{8.48}$$

and the remaining hermitian properties in axiom (1) for $\not{D}, \gamma$ now appear as

$$\epsilon'(\!(\not{D}\psi, \phi)\!) = (\!(\psi, \not{D}\phi)\!), \quad \epsilon''(\!(\gamma\psi, \phi)\!) = (\!(\psi, \gamma\phi)\!). \tag{8.49}$$

We will also have recourse to a more categorical notation for Hilbert space inner products where we set $\langle\!\langle \phi, \psi \rangle\!\rangle = \langle\!\langle \overline{\phi}, \psi \rangle\!\rangle$, so that $\langle\!\langle\ ,\ \rangle\!\rangle : \overline{\mathcal{S}} \otimes \mathcal{S} \to \mathbb{C}$ is a linear map.

Finally, we cannot discuss the Dirac operator without going back to its historical origins and showing that it squares to the Laplacian, with some modifications. Classically, the action of Ω^1 on $\mathcal{S}$ is part of an action of the Clifford algebra, and one way to realise the Clifford algebra is as an associative product © on the vector space of Ω reducing to the usual module action if one of the elements is in $A = \Omega^0$ and to $\xi © \eta = (\xi, \eta) + \xi \wedge \eta$ if $\xi, \eta \in \Omega^1$, where $(\ ,\ )$ is the Riemannian inverse metric on Ω^1. This motivates the following noncommutative version.

Proposition 8.45 *Let $(\mathcal{S}, \nabla_{\mathcal{S}}, \sigma_{\mathcal{S}})$ be a left bimodule connection and $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$ a left module map (as in Theorem 8.44) and suppose that (i) $(\ ,\ )$ is an inverse metric on Ω^1 and that $\triangleright$ extends to a left-module map $\triangleright : \Omega^2 \otimes_A \mathcal{S} \to \mathcal{S}$ such that*

$$\varphi(\xi \triangleright (\eta \triangleright s)) = \kappa(\xi, \eta)s + (\xi \wedge \eta) \triangleright s$$

for some invertible left A-bimodule map $\varphi : \mathcal{S} \to \mathcal{S}$ and constant κ and (ii) (∇, σ) is a torsion compatible left bimodule connection on Ω^1. Then $\not{D} = \triangleright \circ \nabla_{\mathcal{S}}$ obeys

$$\varphi \circ \not{D}^2 = \kappa \Delta_{\mathcal{S}} + \triangleright \circ R_{\mathcal{S}} + (T_\nabla \triangleright \mathrm{id})\nabla_{\mathcal{S}} + \varphi \circ \triangleright \circ \boldsymbol{\nabla}(\triangleright) \circ \nabla_{\mathcal{S}},$$

where $\Delta_{\mathcal{S}} = (\ ,\ )_{12} \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}}$ is the geometric Laplacian on spinors.

Proof Begin with

$$\not{D}^2 = \triangleright \circ \nabla_{\mathcal{S}} \circ \triangleright \circ \nabla_{\mathcal{S}} = \triangleright \circ \boldsymbol{\nabla}(\triangleright) \circ \nabla_{\mathcal{S}} + \triangleright \circ (\mathrm{id} \otimes \triangleright) \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}}$$

using the formula (4.1) in §4.1 for $\boldsymbol{\nabla}(\triangleright)$. Now by our assumption on the Clifford action,

$$\begin{aligned}\varphi \circ \triangleright \circ (\mathrm{id} \otimes \triangleright) \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}} &= \kappa((\ ,\ ) \otimes \mathrm{id}) \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}} + \triangleright (\wedge \otimes \mathrm{id}) \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}} \\ &= \kappa((\ ,\ ) \otimes \mathrm{id}) \nabla_{\Omega^1 \otimes \mathcal{S}} \nabla_{\mathcal{S}} + \triangleright (\wedge \nabla \otimes \mathrm{id}) \nabla_{\mathcal{S}} + \triangleright (\wedge \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla_{\mathcal{S}}) \nabla_{\mathcal{S}},\end{aligned}$$

where we expanded out $\nabla_{\Omega^1 \otimes \mathcal{S}}$. If ∇ has torsion T_∇ a bimodule map then $\wedge(\mathrm{id} + \sigma) = 0$ allows us to replace σ to give a part of the curvature and we can also replace $\wedge \nabla = \mathrm{d} + T_\nabla$ to give the other part of the curvature. Hence this expression becomes the last three terms of the stated result. □

Here φ, κ provide some flexibility needed when we construct examples while $\Delta_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ is the natural Laplace–Beltrami operator on spinors in the same form as our previous geometric Laplacians on A and Ω^1 (as studied in Lemma 8.6). The nicest case is when ∇ is torsion free, in which case $T_{\nabla} = 0$ in $\not{D}^2$, and when the Clifford action intertwines the connections, in which case $\boldsymbol{\nabla}(\triangleright) = 0$. This happens classically since the Clifford algebra is defined by the metric and the connections are metric compatible. In this case we recover the classical Lichnerowicz formula.

8.5.2 Examples of Geometric Spectral Triples

Here we give a finite and two q-deformation examples of the preceding construction.

Example 8.46 ($n = 2$ Spectral Triple on $M_2(\mathbb{C})$) We use the calculus in Example 1.37 for $A = M_2(\mathbb{C})$, where $\Omega = M_2(\mathbb{C}).\mathbb{C}[s, t]$ with generating 1-forms s, t and optionally the relations $s^2 = t^2 = 0$ if one wants an exterior algebra more like a 2-manifold (what follows works either way).

Now we take another copy of the same bimodule as $\mathcal{S} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) = \mathcal{S}_+ \oplus \mathcal{S}_-$ where now we denote the central basis over $M_2(\mathbb{C})$ by $e = \mathrm{id} \oplus 0$ and $f = 0 \oplus \mathrm{id}$. We take the Hilbert space inner product and Dirac operator

$$\langle\!\langle \overline{xe + yf}, ve + wf \rangle\!\rangle = \mathrm{Tr}(x^* v + y^* w), \tag{8.50}$$

$$\not{D}(xe + yf) = [E_{12}, y]e + [E_{21}, x]f. \tag{8.51}$$

It is easy to see that the latter is hermitian in the usual sense, which we write as

$$\langle\!\langle \overline{\not{D}(xe + yf)}, ve + wf \rangle\!\rangle = \langle\!\langle \overline{xe + yf}, \not{D}(ve + wf) \rangle\!\rangle.$$

Next, we define $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ by $\mathcal{J}(xe + yf) = -y^* e + x^* f$, giving $\epsilon = -1$. To see that $\mathcal{J}$ is an isometry, we calculate

$$\langle\!\langle \overline{\mathcal{J}(ve + wf)}, \mathcal{J}(xe + yf) \rangle\!\rangle = \langle\!\langle \overline{-w^* e + v^* f}, -y^* e + x^* f \rangle\!\rangle = \mathrm{Tr}(wy^* + vx^*),$$

which we recognise as $\langle\!\langle \overline{xe + yf}, ve + wf \rangle\!\rangle$. Comparing $\mathcal{J}\not{D}$ and $\not{D}\mathcal{J}$, we see that $\epsilon' = 1$. Finally setting $\gamma(xe+yf) = xe - yf$ with $\epsilon'' = -1$ satisfies axioms (1)–(6) for a dimension $n = 2$ mod 8 algebraic real spectral triple.

For the geometric realisation of this spectral triple, we calculate

$$\mathcal{J} a^* \mathcal{J}^{-1}(xe + yf) = xae + yaf = (xe + yf)a,$$

so $\mathcal{S}$ is indeed a right A-module by the matrix product as we said we would take for the bimodule structure of $\mathcal{S}$. We also define

$$\nabla_{\mathcal{S}} e = \nabla_{\mathcal{S}} f = 0, \quad \sigma_{\mathcal{S}}(f \otimes s) = s \otimes f, \quad \sigma_{\mathcal{S}}(e \otimes t) = t \otimes e$$

and similarly $\sigma =$ flip on the other two cases for a bimodule connection. This has zero curvature. We then define the Clifford action

$$s{\triangleright} f = e,\ t{\triangleright} e = f,\ s{\triangleright} e = t{\triangleright} f = 0; \quad (as+bt){\triangleright}(xe+yf) = aye+bxf \tag{8.52}$$

for all $a, b \in M_2(\mathbb{C})$ from which we find ${\triangleright}\nabla_{\mathcal{S}}(xe + ys) = \not{D}(xe + yf)$, as desired.

Clearly, we must also set $\gamma(e) = e$, $\gamma(f) = -f$ and $j(e) = \overline{f}$ and $j(f) = -\overline{e}$. We check that $\overline{j}\, j(e) = \overline{j(f)} = -\overline{\overline{e}}$ and similarly for f, as required for $\epsilon = -1$ and that

$$\epsilon' \overline{{\triangleright}\sigma_{\mathcal{S}}}\, \Upsilon^{-1}(\star \otimes j)(t \otimes e) = -\epsilon' \overline{{\triangleright}\sigma_{\mathcal{S}}(f \otimes s)} = -\epsilon' \overline{s{\triangleright} f} = -\epsilon' \overline{e} = j(f) = j(t{\triangleright} e)$$

with $\epsilon' = 1$, and similarly for the other cases. Finally,

$$(\mathrm{id} \otimes j)\nabla_{\mathcal{S}}(e) = \nabla_{\overline{\mathcal{S}}}\, j(e) = (\star^{-1} \otimes \mathrm{id})\Upsilon \overline{\sigma^{-1}\nabla(f)} = 0$$

and similarly for f, as $\nabla_{\mathcal{S}} = 0$ on the generators. Hence $\mathbf{\nabla}(j) = 0$, where the left-hand side being a left module map means that it is sufficient to show that it vanishes on the generators. The conditions involving γ are easily checked with $\epsilon'' = -1$ so Theorem 8.44 applies to construct a spectral triple, the one we began with.

We next check the conditions for Proposition 8.45 with respect to the above Clifford action and the metric $g = s \otimes t - t \otimes s$ from Example 8.13. The latter has inverse metric $(t, s) = 1 = -(s, t)$ on the basis and the flat quantum Levi-Civita connection $\nabla s = 2\theta \otimes s$, $\nabla t = 2\theta \otimes t$ from Example 8.13. The only nonzero iterated Clifford actions among the generators are $s{\triangleright}(t{\triangleright} e) = e$, $t{\triangleright}(s{\triangleright} f) = f$ and we take $(s \wedge t){\triangleright} e = \frac{e}{2}$, $(s \wedge t){\triangleright} f = -\frac{f}{2}$ and also $s^2{\triangleright} = 0$ and $t^2{\triangleright} = 0$ if we work in the maximal prolongation. This then works with

$$\varphi(e) = e, \quad \varphi(f) = -f, \quad \kappa = -\frac{1}{2}. \tag{8.53}$$

For example, $\varphi(s{\triangleright}(t{\triangleright} e)) = \varphi(e) = e = \kappa(s, t)e + (s \wedge t){\triangleright} e$ and $\varphi(t{\triangleright}(s{\triangleright} e)) = 0 = -\frac{e}{2} + \frac{e}{2} = \kappa(t, s)e + (t \wedge s){\triangleright} e$. Similarly for f.

We next illustrate that Proposition 8.45 indeed gives $\not{D}^2$ as it must. Here

$$\mathbf{\nabla}({\triangleright})(t \otimes e) = \nabla_{\mathcal{S}}(t{\triangleright} e) - (\mathrm{id} \otimes {\triangleright})\nabla_{\Omega^1 \otimes \mathcal{S}}(t \otimes e) = -2\theta \otimes f,$$
$$\mathbf{\nabla}({\triangleright})(s \otimes f) = -2\theta \otimes e,$$

with the other cases zero. This implies that

$${\triangleright}\mathbf{\nabla}({\triangleright})\nabla_{\mathcal{S}}(xe + yf) = -2([E_{21}, x]E_{12}e + [E_{12}, y]E_{21})f.$$

Because $\nabla_{\mathcal{S}} = 0$ on the basis, one also has

$$\Delta_{\mathcal{S}}(xe + yf) = (\,,\,)_{12}\nabla_{\Omega^1\otimes\mathcal{S}}\nabla_{\mathcal{S}}(xe + yf) = (\Delta x)e + (\Delta y)f,$$

where Δ is the scalar Laplacian in Example 8.13. Combining together, we have

$$\begin{aligned}
&\kappa\Delta_{\mathcal{S}}(xe) + \varphi{\triangleright}\boldsymbol{\nabla}(\triangleright)\nabla_{\mathcal{S}}(xe)\\
&\quad = (-\frac{1}{2}[[E_{21}, E_{12}], x] - [E_{12}, x]E_{21} + [E_{21}, x]E_{12})e - 2[E_{21}, x]E_{12}e\\
&\quad = [E_{12}, [E_{21}, x]]e = \varphi(\not{D}^2(xe))
\end{aligned}$$

for all $x \in M_2(\mathbb{C})$ and similarly on yf.

Finally, we can write $\mathcal{S} = M_2(\mathbb{C}) \otimes \mathbb{C}^2$, i.e. $xe + yf$ as the column vector transpose of $(x, y)^T$, in which notation

$$\langle\langle \overline{\begin{pmatrix} x \\ y \end{pmatrix}}, \begin{pmatrix} v \\ w \end{pmatrix} \rangle\rangle = \begin{pmatrix} x \\ y \end{pmatrix}^{\dagger} \begin{pmatrix} v \\ w \end{pmatrix}, \quad \not{D} = \begin{pmatrix} 0 & \partial_s \\ \partial_t & 0 \end{pmatrix},$$

where $\partial_s = [E_{12},\ ]$ and $\partial_t = [E_{21},\ ]$ are the partial derivatives for our s, t basis and $\dagger$ denotes $*$ and transpose. We also have, in terms of Pauli matrices,

$$\gamma = \sigma_3, \quad \mathcal{J}\begin{pmatrix} x \\ y \end{pmatrix} = (C\begin{pmatrix} x \\ y \end{pmatrix})^*; \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma_2.$$

We can make $\not{D}$ look a bit more like a classical Dirac operator as follows. Here σ_i also generate the usual Clifford algebra for $\mathbb{R}^2$ with its Euclidean metric, so $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ for $i = 1, 2$, and

$$\gamma = -\mathrm{i}\sigma_1\sigma_2, \quad \not{D} = \frac{1}{2}(\sigma_1[\sigma_1,\ ] - \sigma_2[\sigma_2,\ ]),$$

where the 'partial derivatives' $[\sigma_i,\ ]$ act on the $M_2(\mathbb{C})$-valued spinor entries. $\diamond$

We can also have more geometric examples where $\boldsymbol{\nabla}(\triangleright) = 0$ as in our next example, the q-sphere.

Example 8.47 (Geometric Dirac Operator on the q-Sphere) We take $A = \mathbb{C}_q[S^2]$ with q real and its usual calculus $\Omega^1 = \mathcal{E}_2 \oplus \mathcal{E}_{-2}$ as in Examples 2.35 and 5.79. Here $E_{\pm 2}$ are charge ± 2 q-monopole sections constructed as the degree ∓ 2 sub-bimodules of $\mathbb{C}_q[SU_2]$ for a certain grading, or the grade 0 subspaces of $\mathbb{C}_q[SU_2]e^{\pm}$, where $e^{\pm}$ have grade ± 2 and commute with A. The $e^{\pm}$ are part of a basis for the 3D calculus on $\mathbb{C}_q[SU_2]$ along with e^0, and we write $\mathrm{d}f = \partial_+ f e^+ + \partial_- f e^- + \partial_0 a e^0$ for $f \in A$. We know from Example 5.79 that the q-monopole 'spin connection' on the Hopf fibration quantum principal bundle induces the QLC ∇ on Ω^1.

For the spinor bundle, we similarly have $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = E_{\pm 1}$ is similarly the grade ∓ 1 sub-bimodule of $\mathbb{C}_q[SU_2]$, or the grade 0 subspaces of $\mathbb{C}_q[SU_2]f^\pm$, where $f^\pm$ have grade ± 1 and commute with A. This time, the q-monopole 'spin connection' induces $\nabla_{\mathcal{S}} : \mathcal{S} \to \Omega^1 \otimes_A \mathcal{S}$ on spinors, namely

$$\nabla_{\mathcal{S}}(xf^+ + yf^-) = (\partial_+ xe^+ + \partial_- xe^-).\tilde{D}.f^+ + (\partial_+ ye^+ + \partial_- ye^-).D.f^-,$$
$$\sigma_{\mathcal{S}}((xf^+ + yf^-) \otimes fe^\pm) = xfe^\pm.\tilde{D}.f^+ + yfe^\pm.D.f^-$$

by Example 5.51 for each $\mathcal{S}_\pm$, where $D = d \otimes a - qb \otimes c$ and $\tilde{D} = a \otimes d - q^{-1} b \otimes c$ and $|x| = -1$, $|y| = 1$. For the Clifford action $\triangleright$ of Ω^1, we adopt a canonical form

$$fe^+ \triangleright yf^- = \alpha fyf^+, \quad ge^- \triangleright xf^+ = \beta gxf^-$$

with nonzero parameters $\alpha, \beta \in \mathbb{C}$, where f, g have grade ∓ 2 respectively and the other combinations are zero. Then $\not{D} = \triangleright \circ \nabla_{\mathcal{S}}$ comes out as

$$\not{D}(xf^+ + yf^-) = \alpha q^{-1} \partial_+ yf^+ + \beta q \partial_- xf^-.$$

The maps γ and $\mathcal{J}$ are given by $\gamma = \pm\mathrm{id}$ on $\mathcal{S}_\pm$ and $\mathcal{J}(xf^\pm) = \pm\delta^{\pm 1} x^* f^\mp$ for some real parameter δ, giving $\epsilon = -1$ and $\epsilon'' = -1$. It is nontrivial but straightforward to check that the connection intertwines j. Meanwhile, since $e^{\pm *} = -q^{\mp 1} e^\mp$,

$$\triangleright \circ \sigma_{\mathcal{S}}(\mathcal{J}(xf^+) \otimes (ye^-)^*) = -\alpha\delta q^{-2} x^* y^* f^+,$$
$$\triangleright \circ \sigma_{\mathcal{S}}(\mathcal{J}(xf^-) \otimes (ye^+)^*) = \beta\delta^{-1} q^2 x^* y^* f^-$$

so that the condition (8.47) with $\epsilon' = 1$ becomes

$$\mathcal{J}(ye^- \triangleright xf^+) = -\delta^{-1} \beta^* (yx)^* f^+, \quad \mathcal{J}(ye^+ \triangleright xf^-) = \delta\alpha^* (yx)^* f^-,$$

which comes down to $\delta^2 = q^2 \beta / \alpha^*$; we let δ be the positive square root of this. By Theorem 8.44, the axioms (2)–(6) of a real spectral triple are satisfied with $n = 2$.

Heading towards the remaining axioms (1), we next define a positive hermitian inner product

$$\langle\ ,\ \rangle : \overline{\mathcal{S}} \otimes_A \mathcal{S} \to A, \quad \langle \overline{x_+ f^+ + x_- f^-}, y_+ f^+ + y_- f^- \rangle = x_+{}^* y_+ + \mu x_-{}^* y_-$$

for a constant $\mu > 0$. For a $\mathbb{C}$-valued inner product, we use the Haar integral from Example 2.21 to define

$$\langle\langle\ ,\ \rangle\rangle = \int \langle\ ,\ \rangle : \overline{\mathcal{S}} \otimes_A \mathcal{S} \to \mathbb{C}.$$

For xf^+ and yf^- of grade zero, under the assumption $\mu = q\delta^{-2}$, we have

$$\langle \overline{xf^+}, D\!\!\!/(yf^-)\rangle - \langle \overline{D\!\!\!/(xf^+)}, yf^-\rangle = \alpha q^{-1}(x^*\partial_+ y - q(\partial_- x)^* y).$$

Multiplying this by $\mathrm{Vol} = e^+ \wedge e^-$ and writing its integral as the cohomology class of the result (see Example 4.36), we find that its integral vanishes and hence that $\langle\!\langle \overline{xf^+}, D\!\!\!/(yf^-)\rangle\!\rangle = \langle\!\langle \overline{D\!\!\!/(xf^+)}, yf^-\rangle\!\rangle$, i.e., $D\!\!\!/$ is hermitian. We used that

$$\mathrm{d}(x^* y e^-) \sim \mathrm{d}(x^* y) \wedge e^- \sim (x^*\partial_+ y - q(\partial_- x)^* y)e^+ \wedge e^-,$$

where $\sim$ means to drop expressions involving e^0, so the RHS is exact in $\Omega^2(\mathbb{C}_q[S^2])$.

On the other hand, we know from Example 2.21 that $\int : \mathbb{C}_q[SU_2] \to \mathbb{C}$ is a twisted trace as $\int fg = \int \varsigma(g) f$ for all f, g in $\mathbb{C}_q[SU_2]$, where ς is an algebra automorphism

$$\varsigma(a^i b^j c^k d^l) = q^{2(l-i)} a^i b^j c^k d^l$$

which preserves grading and obeys $\varsigma(f^*) = (\varsigma^{-1}(f))^*$. Now we can show

$$\langle\!\langle \overline{\mathcal{J}(x.f^\pm)}, \mathcal{J}(y.f^\pm)\rangle\!\rangle = q^{\pm 1}\langle\!\langle \overline{\varsigma^{-1}(y).f^\pm}, x.f^\pm\rangle\!\rangle,$$

so for $q \neq 1$, $\mathcal{J}$ obeys a twisted version of the isometry condition, taking us slightly outside Connes' framework.

To summarise the values of the parameters, for conditions (2)–(6) we needed $\delta^2 = q^2\beta/\alpha^*$ and for $D\!\!\!/$ to be hermitian we needed $\mu = q\delta^{-2}$ as the inner product parameter. By rescaling $f^\pm$, we can without loss of generality set $\alpha = 1$ and then have only one free parameter $\beta > 0$, with $\delta = \sqrt{\beta} q$ and $\mu = \beta^{-1} q^{-1}$. Thus, we have a 1-parameter moduli space of Dirac operators by our construction but with $\mathcal{J}$ only a twisted isometry when $q \neq 1$.

Now we check the conditions for Proposition 8.45. Applying the Ω^1 action twice gives the only nonzero cases

$$ae^- {\triangleright} (fe^+ {\triangleright} yf^-) = \alpha\beta afyf^-, \quad fe^+ {\triangleright} (ae^- {\triangleright} xf^+) = \alpha\beta faxf^+.$$

Using the inner product

$$(e^- \tilde{D}_1 \tilde{D}_1', \tilde{D}_2' \tilde{D}_2 e^+) = 1, \quad (e^+ D_1 D_1', D_2' D_2 e^-) = q^2$$

(with other values zero) from Proposition 2.36, these conditions are

$$\kappa q^{-2} afyf^- + q^{-2} af(e^- \wedge e^+) {\triangleright} yf^- = \varphi(\alpha\beta afyf^-),$$
$$\kappa q^{-2} afxf^+ + q^{-2} af(e^- \wedge e^+) {\triangleright} xf^+ = 0,$$

$$\kappa q^4 faxf^+ + q^2 fa(e^+ \wedge e^-){\triangleright}xf^+ = \varphi(\alpha\beta faxf^+),$$
$$\kappa q^4 fayf^- + q^2 fa(e^+ \wedge e^-){\triangleright}yf^- = 0.$$

These can readily be solved by assuming that φ acts as a constant on each $\mathcal{S}_\pm$, then

$$\mathrm{Vol}{\triangleright}(xf^+) = q^{-2}\kappa xf^+, \quad \mathrm{Vol}{\triangleright}(yf^-) = -\kappa q^2 yf^-,$$

$$\kappa = \frac{q^{-1}}{(2)_{q^2}}\alpha\beta, \quad \varphi(xf^+) = qxf^+, \quad \varphi(yf^-) = q^{-1}yf^-,$$

where $\mathrm{Vol} = e^+ \wedge e^-$. Thus, $\mathrm{Vol}{\triangleright}s = q^{2|s|}s$ and $\varphi(s) = q^{-|s|}s$ on sections when viewed as subspaces in $\mathbb{C}_q[SU_2]$ of appropriate grade $|s|$. As the connections on $\mathcal{S}_\pm = E_{\pm 1}$ are the standard ones from Example 5.51, we know the curvature

$$R_{\mathcal{S}}(xf^+ + yf^-) = q^3[1]_{q^2}\mathrm{Vol} \otimes xf^+ + q^3[-1]_{q^2}\mathrm{Vol} \otimes yf^-,$$

giving ${\triangleright}R_{\mathcal{S}} = \kappa q^2 \varphi^{-1}$. One can also show that $\mathbf{\nabla}(\triangleright) = 0$, which is expected as both connections come from the same connection on the quantum principal bundle. $T_\nabla = 0$ since we used the QLC on Ω^1. Hence Proposition 8.45 applies and tells us that

$$\not{D}^2 = \kappa\varphi^{-1} \circ \left(\Delta_{\mathcal{S}} + q^2\varphi^{-1}\right)$$

as our q-Lichnerowicz formula. ◇

Finally, we consider a Dirac operator on the q-disk, which is in some respects similar to the q-sphere, but with an added complication of a restriction on the domain.

Example 8.48 (Dirac Operator on the q-Disk) We let $A = \mathbb{C}_q[D]$ with q real, as generated by $z, \bar{z}$ with calculus and integral in Examples 3.40 and 4.37. We define the spinor bimodule $\mathcal{S}$ as the free left module with basis $\{s, \bar{s}\}$ and relations

$$s.a = q^{|a|}a.s, \quad \bar{s}.a = q^{|a|}a.\bar{s},$$

for homogenous $a \in \mathbb{C}_q[D]$, and we define a flat bimodule connection on $\mathcal{S}$ by

$$\nabla_{\mathcal{S}}s = \nabla_{\mathcal{S}}\bar{s} = 0, \quad \sigma_{\mathcal{S}}(s \otimes \mathrm{d}a) = q^{|a|}\mathrm{d}a \otimes s, \quad \sigma_{\mathcal{S}}(\bar{s} \otimes \mathrm{d}a) = q^{|a|}\mathrm{d}a \otimes \bar{s}.$$

We specify a Clifford action with nonzero parameters α, β by

$$\mathrm{d}z{\triangleright}\bar{s} = \alpha ws, \quad \mathrm{d}\bar{z}{\triangleright}s = \beta w\bar{s}, \quad \mathrm{d}z{\triangleright}s = 0, \quad \mathrm{d}\bar{z}{\triangleright}\bar{s} = 0.$$

The resulting Dirac operator is

$$\not{D}(a.s) = \beta(\partial_{\bar{z}}a)w\bar{s}, \quad \not{D}(a.\bar{s}) = \alpha(\partial_z a)ws,$$

where $\mathrm{d}a = (\partial_z a)\mathrm{d}z + (\partial_{\bar{z}}a)\mathrm{d}\bar{z}$. Set $\gamma(s) = s$, $\gamma(\bar{s}) = -\bar{s}$, $\mathcal{J}(s) = \delta\bar{s}$ and $\mathcal{J}(\bar{s}) = -\delta^{-1}s$ for a nonzero real parameter δ, giving $\epsilon = -1$ and $\epsilon'' = -1$. The connection preserves j as ∇_S on the generators is zero. Next,

$$\triangleright \circ \sigma_S(\mathcal{J}(as) \otimes \mathrm{d}z) = \delta q q^{|a^*|}a^*\alpha ws, \quad \mathcal{J}(\mathrm{d}\bar{z}\triangleright as) = -\beta^*\delta^{-1}q^{|a^*|}a^*ws$$

for homogeneous a, and similarly for $\bar{s}$ and the other choice of $\mathrm{d}z, \mathrm{d}\bar{z}$. Then (8.47) holds with $\epsilon' = 1$ if

$$\delta^2 = -q^{-1}\alpha^{-1}\beta^*.$$

We let δ be the positive square root of this. Then Theorem 8.44 tells us that axioms (2)–(6) of a real spectral triple hold with $n = 2$.

Heading towards axioms (1), we now define a positive hermitian inner product

$$\langle\ ,\ \rangle : \overline{\mathcal{S}} \otimes_A \mathcal{S} \to A, \quad \langle\overline{sa + \bar{s}b}, sc + \bar{s}d\rangle = a^*wc + \mu b^*wd$$

for some $\mu > 0$ and for all $a, b, c, d \in A$ (this is positive as $w = 1 - \bar{z}z \geq 0$). To see if $\not{D}$ is hermitian we calculate

$$\langle\overline{\not{D}(a.\bar{s})}, b.s\rangle = \alpha^* q^{|b|-|\partial_z a|}w(\partial_z a)^*bw,$$
$$\langle\overline{a.\bar{s}}, \not{D}(b.s)\rangle = \mu\beta q^{|a|-|\partial_{\bar{z}}b|}wa^*(\partial_{\bar{z}}b)w.$$

Defining the complex-valued inner product $\langle\langle\ ,\ \rangle\rangle$ by the integral of $\langle\ ,\ \rangle$, we find

$$\langle\langle\overline{\not{D}(a.\bar{s})}, b.s\rangle\rangle = \alpha^*\int w(\partial_z a)^*bw, \quad \langle\langle\overline{a.\bar{s}}, \not{D}(b.s)\rangle\rangle = \mu\beta\int wa^*(\partial_{\bar{z}}b)w$$

as the integral vanishes unless the grade is zero. Since

$$\begin{aligned}\mathrm{d}(a^*b) &= ((\partial_z a)\mathrm{d}z + (\partial_{\bar{z}}a)\mathrm{d}\bar{z})^*b + a^*((\partial_z b)\mathrm{d}z + (\partial_{\bar{z}}b)\mathrm{d}\bar{z}) \\ &= (\mathrm{d}\bar{z}(\partial_z a)^* + \mathrm{d}z(\partial_{\bar{z}}a)^*)b + a^*((\partial_z b)\mathrm{d}z + (\partial_{\bar{z}}b)\mathrm{d}\bar{z}),\end{aligned}$$

we see that $\partial_{\bar{z}}(a^*b) = q^{2(|b|-|\partial_z a|)}(\partial_z a)^*b + a^*\partial_{\bar{z}}b$. If $\mu\beta = -\alpha^*$, i.e., setting $\mu = q^{-1}\delta^{-2}$, we have

$$\langle\langle\overline{\not{D}(a.\bar{s})}, b.s\rangle\rangle - \langle\langle\overline{a.\bar{s}}, \not{D}(b.s)\rangle\rangle = \alpha^*\int w(\partial_{\bar{z}}(a^*b))w.$$

Hence to show that $\not{D}$ is hermitian, we need $\int w(\partial_{\bar{z}}a)w = 0$ for all a with the only possible nonzero cases being when $|a| = -1$. We set $a = \bar{z}w^m$ for some $m \geq 1$, and then $(\partial_{\bar z}a)\mathrm{d}\bar z = \mathrm{d}\bar z w^m + \bar z(\partial_{\bar z}(w^m))\mathrm{d}\bar z$, so $\partial_{\bar z}a = w^m + \bar z\partial_{\bar z}w^m$. Hence

$$q^{-2m}\partial_{\bar z}a = [m+1]_{q^{-2}}w^m - [m]_{q^{-2}}w^{m-1}$$

by (3.19), in which case the formula for the integral in Example 4.37 gives the result. Hence $\not{D}$ is hermitian. Note that the argument would fail if $a = \bar z$, in line with the domain of $\not{D}$ classically excluding functions that do not vanish on the boundary.

The condition that $\mathcal{J}$ is an isometry would require equality of the integrals of

$$\langle\overline{\mathcal{J}(as)}, \mathcal{J}(bs)\rangle = \mu\delta^2 awb^* = q^{-1}awb^*, \quad \langle\overline{bs}, as\rangle = q^{-|a|-|b|}b^*wa.$$

Both integrate to zero unless $|a| = |b|$, and then using the twisted trace property $\int ab = \int \varsigma(b)a$ for the algebra automorphism $\varsigma(b) = q^{2|b|}b$, we find instead that

$$\langle\langle\overline{\mathcal{J}(as)}, \mathcal{J}(bs)\rangle\rangle = q^{-1}\langle\langle\overline{\varsigma^{-1}(b)s}, as\rangle\rangle$$

for all $a, b \in \mathbb{C}_q[D]$. Finally, we can rescale the generators $s, \bar s$ so that we can without loss of generality assume $\alpha = 1$, giving a 1-parameter family of Dirac operators by our construction with $\delta = \sqrt{-q^{-1}\beta}$, $\mu = -\beta^{-1}$ but with $\mathcal{J}$ a twisted isometry when $q \neq 1$. This is similar to our result for the q-sphere. ◇

8.5.3 A Dirac Operator for Endomorphism Calculi

More generally, we can use the geometric format $\not{D} = \triangleright \circ \nabla_{\mathcal{S}}$ to construct a candidate 'geometric Dirac operator' whenever we have a natural bundle map

$$\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S},$$

where $(\mathcal{S}, \nabla_{\mathcal{S}})$ is a bundle with a left connection and $(\Omega^1, \nabla, \sigma)$ is (say) a QLC or bimodule WQLC. We ask for $\triangleright$ to intertwine the tensor product connection and the connection on $\mathcal{S}$. For example, in the frame bundle approach of §5.6, we can have a connection on the quantum principal bundle which induces a natural connection on both Ω^1 and $\mathcal{S}$. We will not do so, but one can impose further 'Clifford relations', such as the extension condition in Proposition 8.45, useful when we consider $\not{D}^2$.

Here we look at a class of differential structures Ω^1 where there is an intrinsic choice of $\triangleright$ to consider. These are examples with $\Omega^1 = A\otimes\mathrm{End}(W)$, where the basic 1-forms Λ^1 are an endomorphism space of a group or Hopf algebra representation W, such as Example 1.49 on a group algebra, Example 1.44 for the enveloping algebra of a Lie algebra and Corollary 2.57 for any coquasitriangular Hopf algebra (in this case W is a matrix corepresentation). In all these cases it is natural to define

$\mathcal{S} = A \otimes W$ as the spinor bundle and 'Dirac operator' $\not{D}\colon \mathcal{S} \to \mathcal{S}$ by

$$\mathcal{S} \xrightarrow{\nabla} \Omega^1 \otimes_A \mathcal{S} = (A \otimes \mathrm{End}(W)) \otimes_A A \otimes W = A \otimes \mathrm{End}(W) \otimes W \xrightarrow{\triangleright} A \otimes W = \mathcal{S}.$$

Here we focus on parallelisable examples, but the same ideas extend to the frame bundle setting of Chap. 5. Thus, we fix a basis $e_\alpha^\beta = e_\alpha \otimes f^\beta$ of $\mathrm{End}(W)$ and a basis $\{e_\alpha\}$ of W. We fix conventions by

$$\mathrm{d}f = (\partial^\alpha{}_\beta f) e_\alpha{}^\beta, \quad \psi^\alpha e_\alpha \in \mathcal{S}, \quad \nabla_{\mathcal{S}} e_\gamma = A^\alpha{}_\beta{}^\delta{}_\gamma e_\alpha{}^\beta \otimes e_\delta$$

for partial derivatives associated to the calculus and 'gauge field' associated to the connection. Then clearly

$$(\not{D}\psi)^\alpha = \partial^\alpha{}_\beta \psi^\beta + \psi^\beta A^\alpha{}_\gamma{}^\gamma{}_\beta.$$

More generally, we can have $\Omega^1 = A \otimes \Lambda^1$ and $\Lambda^1 \subseteq \mathrm{End}(W)$ by an inclusion map γ. If $\{e^a\}$ is a basis of Λ^1 then $\gamma^a = \gamma(e^a)$ are matrices and define a canonical Clifford action on $\mathcal{S} = A \otimes W$. We also set

$$\nabla e^a = -\Gamma^a{}_{bc} e^b \otimes e^c, \quad \nabla_{\mathcal{S}} e_\gamma = A_a{}^\delta{}_\gamma e^a \otimes e_\delta$$

for Christoffel symbols and connection coefficients valued in the algebra A. Letting ∂_a be the partial derivatives for the basis e^a, the above more generally becomes

$$(\not{D}\psi)^\alpha = \gamma^{a\alpha}{}_\beta \partial_a \psi^\beta + \psi^\beta \gamma^{a\alpha}{}_\delta A_a{}^\delta{}_\beta,$$

which looks more like standard formulae in the commutative case with γ-matrices.

Example 8.49 We consider a group algebra $A = \mathbb{C}G$ with a representation $\rho(x) = \rho(x)^\alpha{}_\beta e_\alpha{}^\beta$ or $\rho(x).e_\beta = \rho(x)^\alpha{}_\beta e_\alpha$ in our current conventions. We take the calculus in Example 1.49 with basis elements $e^a = a^{-1}\mathrm{d}a = \rho(a-e)$ of $\Lambda^1 \subseteq \mathrm{End}(W)$ for some subset $a \in \mathcal{C} \subseteq G \setminus \{e\}$. We also make the tautological choice $\gamma^a = \gamma(e^a) = \rho(a-e)$ or $e^a \triangleright e_\beta = \rho(a-e)^\alpha{}_\beta e_\alpha$ in our conventions. We assume a connection ∇ on $\Omega^1 = A.\Lambda^1$ and another connection $\nabla_{\mathcal{S}}$ on $\mathcal{S} = A.W$ and ask that $\mathbf{\nabla}(\triangleright) = 0$, i.e., $(\mathrm{id} \otimes \triangleright)\nabla_{\Omega^1 \otimes \mathcal{S}}(e^a \otimes e_\beta) = \nabla(e^a \triangleright e_\beta)$. For simplicity, we assume that ∇ has $\sigma = $ flip on the generators. In this case, a short computation gives the condition as

$$\sum_c \Gamma^a{}_{bc} \gamma^c = [\gamma^a, A_b] + \sum_c \gamma^c [e^a, A_b]_c$$

for all $a, b \in \mathcal{C}$, where $[e^a, A_b] = [e^a, A_b]_c e^c$ is the commutator in Ω^1 of e^a with the gauge field coefficient functions $A_b{}^\alpha{}_\beta$, which depends on ρ. In the simplest case of $\Lambda^1 = \mathrm{End}(W)$ and the $e_\alpha{}^\beta$ basis in place of $e_a = \rho(a-e)^\alpha{}_\beta e_\alpha{}^\beta$, we have $\partial^\alpha{}_\beta x = x\rho(x-e)^\alpha{}_\beta$ and $e_\alpha{}^\beta x = x e_\alpha{}^\gamma \rho(x)^\beta{}_\gamma$ for all $x \in G$.

For a concrete example, we can consider a Dirac operator on $\mathbb{C}S_3$ with its natural 4D calculus in Example 1.50 defined by basis set $\mathcal{C} = \{u, v, uv, vu\}$. We have a natural invariant metric and a unique QLC with $\Gamma = 0$ in Example 8.14. In this case an obvious solution for $\nabla(\triangleright) = 0$ is $A = 0$, which gives $(\not{D}\psi)^\alpha = \partial^\alpha{}_\beta\psi^\beta$. Since the $\partial^\alpha{}_\beta$ as above are diagonalised on each $x \in S_3$, the eigenfunctions of $\not{D}$ are linear combinations of eigenfunction of the form $x\psi$ where ψ is an eigenvector of $\rho(x - e)$. This gives eigenvalues and eigenvectors diagonalising $\not{D}$ as

$$0:\ e\begin{pmatrix}1\\0\end{pmatrix},\ e\begin{pmatrix}0\\1\end{pmatrix},\ u\begin{pmatrix}1\\0\end{pmatrix},\ v\begin{pmatrix}1\\\sqrt{3}\end{pmatrix},\ uvu\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$$

$$-2:\ u\begin{pmatrix}0\\1\end{pmatrix},\ v\begin{pmatrix}-\sqrt{3}\\1\end{pmatrix},\ uvu\begin{pmatrix}\sqrt{3}\\1\end{pmatrix};\quad \frac{-3\pm \mathrm{i}\sqrt{3}}{2}:\ uv\begin{pmatrix}\mp\mathrm{i}\\1\end{pmatrix},\ vu\begin{pmatrix}\pm\mathrm{i}\\1\end{pmatrix}.\quad \diamond$$

Similarly to the above, but at the level of enveloping algebra, we can consider $U(\mathfrak{g})$ as a quantum space with calculus induced by $\rho : \mathfrak{g} \to \mathrm{End}(W)$.

Example 8.50 We take $U(su_2)$ with its natural 4D calculus in Example 1.45, which is an example of a full matrix calculus $\Lambda^1 = \mathrm{End}(W)$ where $W = \mathbb{C}^2$. We take a basis $\mathrm{d}x^i = \rho(x^i) = -\mathrm{i}\lambda\sigma_i$ the Pauli matrices and θ the identity matrix. Thus the natural γ-matrix embedding of our basis is $\gamma^i = \gamma(\mathrm{d}x^i) = -\lambda\sigma_i$ for $i = 1, 2, 3$ and $\gamma^0 = \gamma(\theta)$ the identity matrix. We recall that $\mathrm{d}f = (\partial_i f)\mathrm{d}x^i + (\partial_0 f)\theta$ defines mutually commuting partial derivatives, where $\partial_0 = \sqrt{1 - \lambda^2\sum_i \partial_i^2} - 1$. The only difference is that we are being more careful about using upper or lower indices. We take the zero connection on the basic forms for the Euclidean metric in Example 8.15. As with the preceding example, one can check that only $A = 0$ is compatible with this. Let $D = -\mathrm{i}\sigma^i\partial_i$ then $D^2 = -\lambda^2\sum_i \partial_i^2$ and

$$\not{D} = -\mathrm{i}\sigma^i\partial_i + \partial_0 = D + \sqrt{1 + D^2} - 1$$

is the natural Dirac operator. The geometric normalisation would divide this by λ to have the right classical limit. For its eigenfunctions it is then sufficient to find eigenfunctions of D, resulting in modified eigenfunctions for $\not{D}$. These and solutions to related equations can be solved in a formal manner in terms of plane waves $\psi = \int_{SU_2} \mathrm{d}\mu_p e^{\mathrm{i}p\cdot x}\psi_p$, where $\psi_p \in W$ and μ is the Haar measure, albeit we will not make this more precise here. The same remark applies to the Laplacian in Example 1.45, which will be revisited in Chap. 9. $\diamond$

Finally, we give a truly quantum group example. In this case the natural connection is given by the frame bundle approach of §5.6 as both the connection on Ω^1 and on the spinor bundle are induced from a spin connection on the frame bundle.

Example 8.51 We consider a matrix form of differential calculus on a coquasitriangular Hopf algebra A as in Corollary 2.57. Here A has generators $\{t^\alpha{}_\beta\}$ and $\Lambda^1 \cong \mathrm{End}(W)$, say, by an identification γ. We regard $P = A \otimes A$ as a trivial 'frame bundle' as in §5.6.2 and in this context a spin connection was given by a gauge field $\alpha : \Lambda^1 \to \Omega^1$. If we let $\{e_\alpha\}$ be a basis of W with the canonical right coaction $\Delta_R e_\beta = e_\alpha \otimes t^\alpha{}_\beta$ then $\mathcal{S} = A \otimes W$ acquires the covariant derivative

$$\nabla e_\gamma = -\alpha(\varpi \pi_\epsilon S^{-1} t^\delta{}_\gamma) \otimes e_\delta$$

according to the theory of tensor bundles in Example 5.55. In this case

$$(\not{D}\psi)^\alpha = \partial^\alpha{}_\beta \psi^\beta - \psi^\beta \alpha(\pi_\epsilon S^{-1} t^\gamma{}_\beta)^\alpha{}_\gamma,$$

where we apply γ to convert the value of α to an element of $A \otimes \mathrm{End}(W)$.

To be concrete, we consider $\mathbb{C}_q[SL_2]$ with its standard bicovariant 4D calculus in Example 2.59 with basis e^a, e^b, e^c, e^d so that $\gamma(e^a) = e_1{}^1, \gamma(e^b) = e_1{}^2$ etc. The latter is the matrix with 1 in the (1,2) position. If we took more usual linear combinations e^x, e^y, e^z, θ, where e^x, e^y are linear combinations of e^b, e^c, then the γ would deform the usual Pauli and identity matrices. We have a canonical invariant 'Killing metric' and spin-connection $\alpha : \Lambda^1 \to \Omega^1$ in Proposition 5.85, where $\alpha_a = \alpha(e_a)$, etc., inducing a weak quantum Levi-Civita connection on Ω^1. The specific form of this spin connection gives us

$$\alpha(\pi_\epsilon S^{-1} a) = \alpha(\pi_\epsilon d) = -\frac{1}{(4)_q} e_z, \quad \alpha(\pi_\epsilon S^{-1} b) = -q^{-1} \alpha(\pi_\epsilon b) = -\frac{q^{-1}}{(2)_{q^2}} e_c,$$

$$\alpha(\pi_\epsilon S^{-1} c) = -q\alpha(\pi_\epsilon c) = -\frac{q}{(2)_{q^2}} e_b, \quad \alpha(\pi_\epsilon S^{-1} d) = \alpha(\pi_\epsilon a) = \frac{q^2}{(4)_q} e_z.$$

We then convert to the spinor basis via γ as above and compute the matrix $\alpha(\pi_\epsilon S^{-1} t^\gamma{}_\beta)^\alpha{}_\gamma$. This turns out to be diagonal with equal values, giving

$$\alpha(\pi_\epsilon S^{-1} t^1{}_1)^1{}_1 + \alpha(\pi_\epsilon S^{-1} t^2{}_1)^1{}_2 = \alpha(\pi_\epsilon S^{-1} t^1{}_2)^2{}_1 + \alpha(\pi_\epsilon S^{-1} t^2{}_2)^2{}_2 = -\frac{(3)_q}{(4)_q},$$

$$(\not{D}\psi)^\alpha = \partial^\alpha{}_\beta \psi^\beta - \frac{(3)_q}{(4)_q} \psi^\alpha,$$

where d and hence $\partial^\alpha{}_\beta$ are in the geometric normalisation (recall that this was the canonical normalisation divided by $\lambda = 1 - q^{-2}$). With rather more work, using Fourier transform methods, one finds that the first term of the Dirac operator is fully diagonalised with eigenvalues for every positive half-integer spin j,

$$(i) \quad q^{j+1}(j)_q, \quad (ii) \quad -q^{-j}(j+1)_q,$$

where the second case applies for $j > 0$. ◇

8.6 Hermitian Metrics and Chern Connections

This section combines the theory of noncommutative complex structures from Chap. 7, and in particular the idea of holomorphic modules from §7.2, with the theory in the last section of modules with hermitian metrics. Classically, suppose that we have a holomorphic bundle with specified holomorphic $\overline{\partial}$-connection, which also has a hermitian inner product. Then there is a unique ∂-connection such that the sum of the ∂-connection and the holomorphic $\overline{\partial}$-connection gives an ordinary connection, the *Chern connection*, which preserves the hermitian metric. Moreover, the Chern connection has curvature mapping with values in the $\Omega^{1,1}$ part of the exterior algebra rather than all of Ω^2. In this section, we shall prove that the same works in a noncommutative context. We begin with a classical example.

Example 8.52 (Classical Chern Connection) Following from Example 7.13, we construct a hermitian metric on the classical tautological line bundle L on $\mathbb{CP}^n$. We work with restricted homogeneous coordinates $(z_0, z_1, \dots, z_n)$ where $|z_0|^2 + \cdots + |z_n|^2 = 1$. The fibre of L at $(z_0, z_1, \dots, z_n)$ then consists of elements of the form $\lambda(z_0, z_1, \dots, z_n)$ for $\lambda \in \mathbb{C}$ and we have a well-defined hermitian metric

$$\langle \lambda(z_0, z_1, \dots, z_n), \overline{\mu(z_0, z_1, \dots, z_n)} \rangle = \lambda\mu^*.$$

The fibre of the dual $L^\flat$ consists of column vectors $\mu(z_0^*, \cdots, z_n^*)^T$ for $\mu \in \mathbb{C}$, with pointwise evaluation given by matrix multiplication. This leads to a metric map $G : \overline{L} \to L^\flat$ given by a restriction of the conjugate transpose map from $\overline{\mathrm{Row}^{n+1}(A)}$ to $\mathrm{Col}^{n+1}(A)$, where $A = C^\infty(\mathbb{CP}^n)$.

To construct the connection, we work with the chart with $z_0 \neq 0$ and take coordinates $(1, z_1, \dots, z_n)$. The holomorphic $\overline{\partial}$-connection is defined by,

$$\overline{\partial}_L(f(1, z_1, \dots, z_n)) = \overline{\partial} f \otimes (1, z_1, \dots, z_n)$$

for $f \in A$. We construct a ∂-connection ∂_L such that $\partial_L + \overline{\partial}_L$ preserves the inner product by considering the differential of

$$\langle f(1, z_1, \dots, z_n), \overline{g(1, z_1, \dots, z_n)} \rangle = fg^*(1 + |z_1|^2 + \cdots + |z_n|^2)$$

for $g \in A$. We need the equation $\partial_L(\overline{x}) = \eta^* \otimes \overline{y}$ if $\overline{\partial}_L(x) = \eta \otimes y$ (sum of such terms understood), and also $\partial(1, z_1, \dots, z_n) = \xi \otimes (1, z_1, \dots, z_n)$. Then

$$\begin{aligned} &((\partial f)g^* + f\xi g^* + f(\overline{\partial} g)^*)\langle (1, z_1, \dots, z_n), \overline{(1, z_1, \dots, z_n)} \rangle \\ &= ((\partial f)g^* + f\partial(g^*))(1 + |z_1|^2 + \cdots + |z_n|^2) + fg^*\partial(1 + |z_1|^2 + \cdots + |z_n|^2), \end{aligned}$$

and using $(\overline{\partial} g)^* = \partial(g^*)$, we obtain

$$\partial_L(1, z_1, \ldots, z_n) = \partial(\log_{\mathrm{e}}(1 + |z_1|^2 + \cdots + |z_n|^2)) \otimes (1, z_1, \ldots, z_n).$$

Here $\partial_L + \overline{\partial}_L$ is the Chern connection on L. ◇

In the noncommutative theory, Definition 7.14 specified what we mean by a holomorphic left module E in terms of a $\overline{\partial}$-operator $\overline{\partial}_E : E \to \Omega^{0,1} \otimes_A E$. We now have the following existence and uniqueness result.

Theorem 8.53 (Chern Connection) *Let (A, Ω, d) be a $*$-algebra with integrable almost complex structure and $(E, \overline{\partial}_E)$ an fgp holomorphic left A-module with a left hermitian structure $\langle\, , \rangle : E \otimes \overline{E} \to A$. There is a unique connection $\nabla_E : E \to \Omega^1 \otimes_A E$, the* Chern connection*, compatible with the hermitian metric and such that $(\pi^{0,1} \otimes \mathrm{id})\nabla_E = \overline{\partial}_E$. Moreover, the $\Omega^{0,2}$, $\Omega^{2,0}$ components of its curvature vanish.*

Proof We choose dual bases and adopt the matrix notation in Definition 8.33. We write the Christoffel symbols of the desired connection as $\Gamma = \Gamma_+ + \Gamma_-$, where $\Gamma_+ \in M_n(\Omega^{1,0})$, $\Gamma_- \in M_n(\Omega^{0,1})$ and n is the size of the dual bases for E. The specified $\overline{\partial}_E$ determines Γ_-. The condition for metric compatibility from Definition 8.33 is $\mathrm{d}g = -\Gamma g - g\Gamma^\dagger$ and applying $\pi^{1,0}$ to this gives $\partial g = -\Gamma_+ g - g\pi^{1,0}(\Gamma^\dagger) = -\Gamma_+ g - g{\Gamma_-}^\dagger$. Given the condition $\Gamma_+ P = \Gamma_+$, where $P = g\tilde{g}$, this has a unique solution

$$-\Gamma_+ = \partial g.\tilde{g} + g{\Gamma_-}^\dagger\tilde{g}. \tag{8.54}$$

By Proposition 3.23, we also have to check that $\Gamma = P\Gamma - (\mathrm{d}P)P$, which is verified using the identities in (8.46).

The $\Omega^{0,2}$ component of the curvature vanishes because $\overline{\partial}_E$ has zero holomorphic curvature. For vanishing of the $\Omega^{2,0}$ component, by Proposition 3.23, we show that the $\Omega^{2,0}$ component of $(\mathrm{d}\Gamma + \Gamma \wedge \Gamma).P$ vanishes. We calculate

$$\partial\Gamma_+ = \partial g \wedge \partial\tilde{g} - \partial g \wedge {\Gamma_-}^\dagger\,\tilde{g} - g(\partial{\Gamma_-}^\dagger)\,\tilde{g} + g\,{\Gamma_-}^\dagger \wedge \partial\tilde{g},$$

$$\Gamma_+ \wedge \Gamma_+ = \partial g.\tilde{g} \wedge \partial g.\tilde{g} + \partial g \wedge {\Gamma_-}^\dagger\,\tilde{g} + g{\Gamma_-}^\dagger\,\tilde{g} \wedge \partial g.\tilde{g} + g{\Gamma_-}^\dagger \wedge {\Gamma_-}^\dagger\,\tilde{g},$$

so that

$$\begin{aligned}\partial\Gamma_+ + \Gamma_+ \wedge \Gamma_+ &= (\partial g + g{\Gamma_-}^\dagger) \wedge (\partial\tilde{g} + \tilde{g}.\partial g.\tilde{g}) - g(\partial{\Gamma_-}^\dagger)\tilde{g} + g\,{\Gamma_-}^\dagger \wedge {\Gamma_-}^\dagger\,\tilde{g}\\ &= (\partial g + g{\Gamma_-}^\dagger) \wedge (\partial\tilde{g} + \tilde{g}.\partial g.\tilde{g}) - gP^\dagger(\overline{\partial}\Gamma_- + \Gamma_- \wedge \Gamma_-)^\dagger\,\tilde{g},\end{aligned}$$

and the last bracket vanishes as the holomorphic curvature of the holomorphic connection vanishes. As

$$(\partial\tilde{g} + \tilde{g}.\partial g.\tilde{g}).P = (\partial P^\dagger).\tilde{g}, \quad (\partial g + g{\Gamma_-}^\dagger).P^\dagger = \partial g + g{\Gamma_-}^\dagger$$

(using $\Gamma_- = P\Gamma_- - (\overline{\partial}P)P$ for the second of these), the $\Omega^{2,0}$ component is

$$(\partial\Gamma_+ + \Gamma_+ \wedge \Gamma_+).P = (\partial\Gamma_+ + \Gamma_+ \wedge \Gamma_+).P^2 = (\partial g + g\Gamma_-{}^\dagger).P^\dagger \wedge (\partial P^\dagger).\tilde{g}g\tilde{g},$$

where $Q = P^\dagger = \tilde{g}g$ obeys $Q^2 = Q$. Differentiating this gives $\partial Q.Q = (1 - Q).\partial Q$, so $Q.\partial Q.Q = 0$. □

We next look at the question of when the Chern connection is a bimodule connection in the case where we have a left bimodule hermitian structure. To do this, it is convenient to start with a more categorical alternative to the proof of the preceding theorem. Recall that $\overline{\partial}_E$ in the fgp case has a corresponding right ∂-connection $\tilde{\partial}_E : \overline{E} \to \overline{E} \otimes_A \Omega^{1,0}$ defined by $\tilde{\partial}_E(\overline{e}) = \overline{f} \otimes \kappa^*$ if $\overline{\partial}_E e = \kappa \otimes f$ (sum understood). Also recall that if $\overline{\partial}_E$ makes E a left holomorphic bimodule (see Definition 7.14) then it is a bimodule connection with respect to $(\Omega^{0,1}, \overline{\partial})$.

Proposition 8.54 *Suppose in Theorem 8.53 that E is a bimodule and $\langle\, , \,\rangle : E \otimes_A \overline{E} \to A$ is a nondegenerate left bimodule hermitian structure.*

(1) *The diagram in Fig. 8.1 defines a left ∂-connection ∂_E in terms of the connection $\tilde{\partial}_E$ and the Chern connection ∇_E obeys $\partial_E = (\pi^{1,0} \otimes \mathrm{id})\nabla_E$.*
(2) *If additionally $(E, \overline{\partial}_E, \sigma_E^{0,1})$ is a left holomorphic bimodule then ∂_E is a left bimodule connection $(E, \partial_E, \sigma_E^{1,0})$ with respect to $(\Omega^{1,0}, \partial)$ and ∇_E is a bimodule connection with $\sigma_E = \sigma_E^{1,0}(\mathrm{id} \otimes \pi^{1,0}) + \sigma_E^{0,1}(\mathrm{id} \otimes \pi^{0,1})$.*

Proof (1) Note that the expression on the right-hand side of Fig. 8.1 only depends on $\langle\, , \,\rangle^{-1} \in \overline{E} \otimes_A E$ (with emphasis on $\otimes_A$). It is easy to see that this diagram defines a left ∂-connection. Next we show that $\partial_E + \overline{\partial}_E$ preserves the metric, and is therefore the Chern connection ∇_E by the uniqueness result in Theorem 8.53. Applying $\mathrm{id} \otimes \langle -, \overline{c}\rangle$ for $\overline{c} \in \overline{E}$ to the result of the diagram shows that

$$\partial\langle e, \overline{c}\rangle = (\mathrm{id} \otimes \langle\, , \,\rangle)(\partial_E e \otimes \overline{c}) + (\langle\, , \,\rangle \otimes \mathrm{id})(e \otimes \tilde{\partial}_E \overline{c}),$$

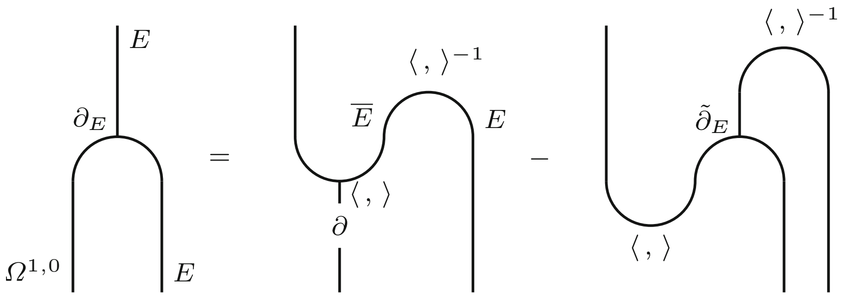
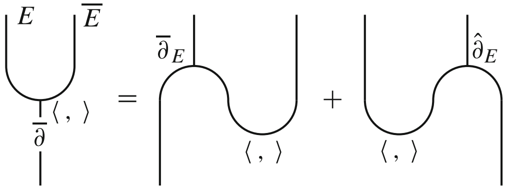


Fig. 8.1 Diagrammatic definition of the ∂_E part of the Chern connection

which is half (the $\Omega^{1,0}$ part) of metric compatibility. For the $\Omega^{0,1}$ part we need



where $\hat{\partial}_E : \overline{E} \to \overline{E} \otimes_A \Omega^{0,1}$ is the right $\overline{\partial}$-covariant derivative defined by $\hat{\partial}_E(\overline{c}) = \overline{g} \otimes \xi^*$ if we write $\partial_E c = \xi \otimes g$ (sum of such terms) for any $c \in E$. We have

$$(\overline{\partial}\langle e, \overline{c}\rangle)^* = \partial\langle c, \overline{e}\rangle = (\mathrm{id} \otimes \langle\ ,\ \rangle)(\partial_E c \otimes \overline{e}) + (\langle\ ,\ \rangle \otimes \mathrm{id})(c \otimes \tilde{\partial}_E(\overline{e}))$$
$$= \xi\langle g, \overline{e}\rangle + \langle c, \overline{f}\rangle\kappa^*,$$

where $\tilde{\partial}_E(\overline{e}) = \overline{f} \otimes \kappa^*$. Hence on applying $*$ again,

$$\overline{\partial}\langle e, \overline{c}\rangle = \langle g, \overline{e}\rangle^*\xi^* + \kappa\langle c, \overline{f}\rangle^* = \langle e, \overline{g}\rangle\xi^* + \kappa\langle f, \overline{c}\rangle$$
$$= (\langle\ ,\ \rangle \otimes \mathrm{id})(e \otimes \hat{\partial}_E(\overline{c})) + (\mathrm{id} \otimes \langle\ ,\ \rangle)(\overline{\partial}_E e \otimes \overline{c}),$$

as required.

(2) For the bimodule structures, set $\langle\ ,\ \rangle^{-1}(1) = \overline{c} \otimes g$ (sum of such terms and no relation to the previous shorthand notation just above),

$$\partial_E(e.a) = \partial\langle e \otimes a\overline{c}\rangle \otimes g - (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes \tilde{\partial}_E(a\overline{c}) \otimes g)$$
$$+ (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes \tilde{\partial}_E(a\overline{c}) \otimes g) - (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes a\tilde{\partial}_E(\overline{c}) \otimes g).$$

As $a.\overline{c} \otimes g = \overline{c} \otimes g.a \in \overline{E} \otimes_A E$, we have

$$\partial_E(e.a) = \partial\langle e \otimes \overline{c}\rangle \otimes ga - (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes \tilde{\partial}_E(\overline{c}) \otimes ga)$$
$$+ (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes \tilde{\partial}_E(a\overline{c}) \otimes g)$$
$$- (\langle\ ,\ \rangle \otimes \mathrm{id} \otimes \mathrm{id})(e \otimes a\tilde{\partial}_E(\overline{c}) \otimes g),$$
$$\partial_E(e.a) - \partial_E(e).a = (\langle\ ,\ \rangle \otimes \mathrm{id})\big(e \otimes (\tilde{\partial}_E(\overline{c.a^*}) - a\tilde{\partial}_E(\overline{c}))\big) \otimes g.$$

Now $\overline{\partial}_E(c.a^*) = \overline{\partial}_E(c).a^* + \sigma_E^{0,1}(c \otimes \overline{\partial}(a^*))$, and if we put $\overline{\partial}_E(c) = \kappa \otimes f$ and $\sigma_E^{0,1}(c \otimes \overline{\partial}(a^*)) = \xi \otimes k$ as shorthand (sum understood) then

$$\tilde{\partial}_E(\overline{c.a^*}) - a\tilde{\partial}(\overline{c}) = \overline{f.a^*} \otimes \kappa^* + \overline{k} \otimes \xi^* - a\overline{f} \otimes \kappa^* = \overline{k} \otimes \xi^*,$$
$$\partial_E(e.a) - \partial_E(e).a = \langle e, \overline{k}\rangle.\xi^* \otimes g.$$

Hence in general if $e \in E, \eta \in \Omega^{1,0}$ and $\sigma_E^{0,1}(c \otimes \eta^*) = \xi \otimes k$ is our shorthand (sum of such terms) then $\sigma_E^{1,0}(e \otimes \eta) = \langle e, \overline{k}\rangle \xi^* \otimes g$. On the other hand,

$$(\mathrm{id} \otimes \star^{-1})\Upsilon\, \overline{\sigma_E^{0,1}}\, \Upsilon^{-1}(\star \otimes \mathrm{id})(\partial a \otimes \overline{c}) = \overline{k} \otimes \xi^*,$$

where $\star : \Omega^1 \to \overline{\Omega^1}$ is the bimodule map $\star(\eta) = \overline{\eta^*}$ so that

$$\begin{aligned}\sigma_E^{1,0}&(e \otimes \eta)\\ &= (\langle\ ,\ \rangle \otimes \mathrm{id})\big(\mathrm{id} \otimes (\mathrm{id} \otimes \star^{-1})\Upsilon\, \overline{\sigma_E^{0,1}}\, \Upsilon^{-1}(\star \otimes \mathrm{id}) \otimes \mathrm{id}\big)(e \otimes \eta \otimes \langle\ ,\ \rangle^{-1}).\end{aligned}$$

This is a composition of bimodule maps, and therefore a bimodule map. □

We follow this construction through in our three running examples.

Example 8.55 Let $A = M_2(\mathbb{C})$ with its 2D calculus and exterior algebra in Example 1.37 and complex structure in Example 7.25. In terms of central generators s, t we have graded commutators $\partial = [E_{12}s, -\}$ and $\overline{\partial} = [E_{21}t, -\}$ and $\Omega^{n,m} = M_2(\mathbb{C})s^n t^m$. From Example 7.25, $E = \Omega^{1,0}$ is a holomorphic bundle with

$$\overline{\partial}_E(as) = \{E_{21}, a\}t \otimes s \in \Omega^{0,1} \otimes_A E$$

for $a \in M_2(\mathbb{C})$. As $\overline{\partial}_E(s) = 2E_{21}t \otimes s$, we have the Christoffel symbol $\Gamma_{-11} = -2E_{21}t$. We take the inner product on $\Omega^{1,0}$ as

$$\langle bs, \overline{as}\rangle = ba^*,$$

so that g is a 1×1 matrix with the single element $g^{11} = 1$. Then by Theorem 8.53, $\Gamma_{+11} = -(-2E_{21}t)^* = -2E_{12}s$, resulting in

$$\nabla_E(s) = 2E_{12}s \otimes s + 2E_{21}t \otimes s.$$

We obtain a bimodule covariant derivative, as

$$\nabla_E(s.a) - (\nabla_E s).a = \mathrm{d}a \otimes s + [a, \nabla_E(s)] = -\mathrm{d}a \otimes s,$$

which extends to the map $\sigma_E(a.s \otimes \xi) = -a.\xi \otimes s$. This is the restriction to $\Omega^{1,0}$ of the QLC on Ω^1 in Example 8.13. ◇

Example 8.56 We find Chern connections on two $A = \mathbb{C}_q[S^2]$ modules with its standard $*$-differential and complex structure.

(1) We take $E = \Omega^{1,0}$. By Lemma 7.24 and $\overline{\partial}(xe^+) = \overline{\partial}x \wedge e^+$, we write a holomorphic structure on E as

$$\overline{\partial}_E(xe^+) = \overline{\partial}x.\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2 e^+.$$

We also take the hermitian inner product and metric

$$\langle xe^+, \overline{ye^+}\rangle_E = xy^*, \quad \langle\, ,\,\rangle_E^{-1} = \overline{\tilde{D}_1'^*\tilde{D}_1^* e^+} \otimes \tilde{D}_2'\tilde{D}_2 e^+.$$

Using this and $\overline{\partial}_E(\tilde{D}_1'^*\tilde{D}_1^* e^+) = \overline{\partial}(\tilde{D}_1'^*\tilde{D}_1^*).\tilde{D}_1''\tilde{D}_1''' \otimes \tilde{D}_2'''\tilde{D}_2'' e^+$, the induced right connection $\tilde{\partial}_E$ on $\overline{E}$ obeys

$$(\tilde{\partial}_E \otimes \mathrm{id})\langle\, ,\,\rangle_E^{-1} = \overline{\tilde{D}_2'''\tilde{D}_2'' e^+} \otimes (\overline{\partial}(\tilde{D}_1'^*\tilde{D}_1^*).\tilde{D}_1''\tilde{D}_1''')^* \otimes \tilde{D}_2'\tilde{D}_2 e^+$$

and Fig. 8.1 then gives

$$\partial_E(xe^+) = (\partial x).\tilde{D}_1\tilde{D}_1' \otimes \tilde{D}_2'\tilde{D}_2 e^+.$$

Putting these together, we obtain that the Chern connection is exactly the $\Omega^{1,0}$ part of the QLC in Proposition 5.79 as explicitly given in (6.3) in Example 6.5.

(2) We take $E = \mathcal{S}_+$ as in Example 8.47 with holomorphic structure

$$\overline{\partial}_{\mathcal{S}_+} : \mathcal{S}_+ \to \Omega^{0,1} \otimes_A \mathcal{S}_+, \quad \overline{\partial}_{\mathcal{S}_+}(xf^+) = (\overline{\partial}x)\tilde{D}_1 \otimes \tilde{D}_2 f^+.$$

Our usual convention for the hermitian inner product would have the conjugate on the other side from Example 8.47, but we swap sides by using commutativity of

$$\begin{tikzcd} \overline{\mathcal{S}} \otimes_A \mathcal{S} \arrow[r, "\langle\, ,\,\rangle"] & A \\ \mathcal{S} \otimes_A \overline{\mathcal{S}} \arrow[u, "j\otimes j^{-1}"] \arrow[ur, "\langle\, ,\,\rangle"'] & \end{tikzcd} \tag{8.55}$$

The new inner product is

$$\langle xf^+, \overline{yf^+}\rangle = \langle \overline{\mathcal{J}(xf^+)}, \mathcal{J}^{-1}(yf^+)\rangle = \langle \delta\overline{x^* f^-}, \delta y^* f^-\rangle = \delta^2\mu xy^*.$$

For our present purposes, we absorb the $\delta^2\mu$ factor into the normalisation of the inner product, in which case $\langle\, ,\,\rangle^{-1}(1) = \overline{\tilde{D}_1^* f^+} \otimes \tilde{D}_2 f^+$ and Proposition 8.54 gives

$$\begin{aligned}\partial_{\mathcal{S}_+}(xf^+) &= \partial\langle xf^+, \overline{\tilde{D}_1^* f^+}\rangle \otimes \tilde{D}_2 f^+ \\ &\quad - (\langle\, ,\,\rangle \otimes \mathrm{id} \otimes \mathrm{id})(xf^+ \otimes \tilde{\partial}_{S_+}(\overline{\tilde{D}_1^* f^+}) \otimes \tilde{D}_2 f^+).\end{aligned}$$

Using $\overline{\partial}_{\mathcal{S}_+}(\tilde{D}_1^* f^+) = (\overline{\partial}\tilde{D}_1^*)\tilde{D}'_1 \otimes \tilde{D}'_2 f^+$, where $\tilde{D}'$ is an independent copy of $\tilde{D}$ (similarly for further primes later), this comes out as

$$\partial_{\mathcal{S}_+}(xf^+) = (\partial x)\tilde{D}_1 \otimes \tilde{D}_2 f^+.$$

Then the Chern connection is the connection on $\mathcal{S}_+$ used in Example 8.47. $\diamond$

Example 8.57 We let A be the algebra for the open quantum disk (with w inverted) in Example 4.37 with its central quantum symmetric metric (4.24) on Ω^1. Recall that this tends to the classical hyperbolic metric as $q \to 1$.

(1) We take $E = \Omega^{1,0}$ generated by $\mathrm{d}z$, and give it a holomorphic structure in the form of a $\overline{\partial}$-connection

$$\overline{\partial}_E : \Omega^{1,0} \xrightarrow{\overline{\partial}} \Omega^{1,1} \xrightarrow{\wedge^{-1}} \Omega^{0,1} \otimes_A \Omega^{1,0}.$$

One can check that $\overline{\partial}_E(a.\xi) = \wedge^{-1}(\overline{\partial}a \wedge \xi + a.\overline{\partial}\xi) = \overline{\partial}a \otimes \xi + a.\overline{\partial}_E(\xi)$ for $a \in A$, $\xi \in \Omega^{1,0}$, as required. Moreover, its curvature maps to $\Omega^{0,2} \otimes_A \Omega^{1,0}$, and hence must vanish. We also have $\overline{\partial}_E(\mathrm{d}z) = 0$, so $\Gamma_- = 0$. Restricting the metric to E, $\langle\ ,\ \rangle^{-1} = w^{-2}q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z$ means that g is a 1×1 matrix with the single element

$$g^{11} = \langle \mathrm{d}z, \overline{\mathrm{d}z}\rangle = w^2 q^2.$$

Then $\Gamma_{+11} = -\partial(w^2)w^{-2} = \bar{z}\mathrm{d}z[2]_{q^{-2}}w^{-1}$ implies that the Chern connection associated to this hermitian metric is

$$\nabla_E \mathrm{d}z = -\bar{z}\mathrm{d}z[2]_{q^{-2}}w^{-1} \otimes \mathrm{d}z.$$

This is the $\Omega^{1,0}$ part of the QLC for the q-disk obtained in Exercise E8.7.

(2) Next we take the sub-bimodule $E = S_+$ of the spinor bundle as in Example 8.48, generated by s. We define $\overline{\partial}_{\mathcal{S}_+}s = 0$ or $\Gamma_{-11} = 0$ with zero holomorphic curvature. Using (8.55) to switch sides of the previous spinor inner product gives

$$\langle as, \overline{bs}\rangle = \langle \overline{\mathcal{J}(as)}, \mathcal{J}^{-1}(bs)\rangle = \delta^2\langle \overline{\bar{s}a^*}, \bar{s}b^*\rangle = \delta^2 \mu a w b^*,$$

from which $\langle s, \bar{s}\rangle = \delta^2\mu w$ or $g^{11} = \delta^2\mu w$. Hence $\Gamma_{+11} = -\partial(w)w^{-1} = \bar{z}.\mathrm{d}z.w^{-1}$ is the remaining Christoffel symbol of the Chern connection. The result is exactly the spinor connection $\nabla_{\mathcal{S}_+}$ used in Example 8.48. $\diamond$

Exercises for Chap. 8

E8.1 Show that a left connection on the directed graph calculus as in §1.4 for the infinite string $\cdots \to -1 \to 0 \to 1 \to 2 \to \cdots$ is always a bimodule connection and has the form $\nabla\omega_i = \omega_{i-1} \otimes \omega_i + \beta^i \omega_i \otimes \omega_{i+1}$ for arbitrary $\beta^i \in \mathbb{k}$, where $\omega_i = \omega_{i \to i+1}$. Show that any $g \in \Omega^1 \otimes_A \Omega^1$ (which will necessarily not be a full quantum metric since it will be noncentral and hence not bimodule invertible) has the form $g = \sum_i g^i \omega_i \otimes \omega_{i+1}$ for $g^i \in \mathbb{k}$ and that there is a 1-parameter moduli of metric compatible connections in the sense that $\nabla g = 0$ for any such metric with all $g^i \neq 0$. The same result holds for an open string $0 \to 1 \cdots \to n-1$ with nonapplicable terms in ∇ and g dropped. For a closed string $0 \to 1 \to \cdots \to n-1 \to 0$ (identifying the point n with 0), show that if n is odd then there are two ∇ solving $\nabla g = 0$ for any g while for n even there is a one-parameter moduli as before but a constraint $(g^0 \cdots g^{n-2})^2 = (g^1 \cdots g^{n-1})^2$ on the metric. [Fourier transform as in E2.5 can be used to transfer the closed string results here to $c_q[S^1]$, see also Example 8.5 for $\mathbb{C}_q[S^1]$.]

E8.2 Suppose that Ω^1 has a central basis $\{s^i\}$ over an algebra A with trivial centre, so that the coefficients tensors of $\alpha(s^i) = \alpha^i{}_{mn} s^m \otimes s^n$ and $\sigma(s^i \otimes s^j) = \sigma^{ij}{}_{mn} s^m \otimes s^n$ as well as of $g = g_{ij} s^i \otimes s^j$ are all constants (this is the same context as in exercise E3.9 with $E = \Omega^1$). Suppose that the calculus is inner with coefficients θ_i in $\theta = \theta_i s^i$ linearly independent of each other and of 1.

(i) Show from Proposition 8.11 that $\nabla g = 0$ holds if and only if

$$g_{an}\alpha^a{}_{im} + \sigma^{ac}{}_{im} g_{ab} \alpha^b{}_{cn} = 0, \quad \sigma^{ac}{}_{im} g_{ab} \sigma^{bj}{}_{cn} = \delta^j{}_i g_{mn} \qquad (*)$$

for all i, j, m, n (and sum over repeated indices).

(ii) Use (i) with $s^1 = s, s^2 = t$ to show that the QLC shown in Example 8.13 for $M_2(\mathbb{C})$ with the maximal prolongation exterior algebra calculus and metric $g = s \otimes t - t \otimes s$ is unique.

(iii) Now add relations $s^2 = t^2 = 0$ for the reduced $\Omega(M_2(\mathbb{C}))$ and show that the 4-parameter σ and $\alpha = 0$ claimed in Example 8.13 are indeed QLCs for this metric. [Its full moduli of QLCs has more components.]

E8.3 Using the reduced calculus $\Omega(M_2(\mathbb{C}))$ as in the preceding E8.2(iii) but now with $g = s \otimes s + t \otimes t$, show that (i) there is a 3-parameter moduli of $*$-preserving QLCs with $\alpha = 0$ and containing the curved connection

$$\nabla s = 2E_{21} t \otimes s, \quad \nabla t = 2E_{12} s \otimes t, \quad \sigma(s^i \otimes s^j) = (-1)^{i-j} s^j \otimes s^i$$

and (ii) there is a 4-parameter moduli of QLCs with $\sigma = -\mathrm{flip}$ and

$$\nabla s = 2\theta \otimes s + \mu s \otimes s + \alpha(s \otimes t - t \otimes s) + \beta t \otimes t,$$
$$\nabla t = 2\theta \otimes t + \alpha s \otimes s - \beta(s \otimes t - t \otimes s) + \nu t \otimes t$$

for $\mu, \nu, \alpha, \beta \in \mathbb{C}$, $*$-preserving when $\nu = \bar{\mu}$ and $\beta = \bar{\alpha}$. [The full moduli space of QLCs for this metric again has several more components.]

E8.4 Let $\Omega(\mathbb{Z}_2)$ be the canonical exterior algebra on $\mathbb{Z}_2$ for $\mathcal{C} = \{1\} \subset \mathbb{Z}_2$ with left-invariant form $e^1 = \omega_{0\to 1} + \omega_{1\to 0}$ and metric $g = ae^1 \otimes e^1$, where $a \in \mathbb{C}(\mathbb{Z}_2)$ corresponds to directed edge weights $a(0) = g_{0\to 1}$ and $a(1) = g_{0\leftarrow 1}$. Show that there exists a QLC for the metric if and only if $a(1) = \pm a(0)$ and that in this case there is a 1-parameter moduli of them. Which of these is $*$-preserving? What happens if we use the universal exterior algebra instead of $\Omega^2 = 0$?

E8.5 Repeat the triangle graph Example 8.19 for $A = \mathbb{C}(\mathbb{Z}_3)$ with the same universal $\Omega^1(\mathbb{Z}_3)$ but now with $\Omega^2(\mathbb{Z}_3)$ from E5.7 with relations $e^1 \wedge e^2 = e^2 \wedge e^1 = 0$ in degree 2. The first-order calculus is inner, but not the higher order calculus. Consequently, Proposition 8.11 gives the general form of ∇ but not the torsion condition. Solving directly for torsion-freeness, show that this time among real metrics only the constant Euclidean metric admits a QLC, which is then uniquely given by $\nabla e^1 = 2\theta \otimes e^1 - e^2 \otimes e^2$, $\nabla e^2 = 2\theta \otimes e^2 - e^1 \otimes e^1$ and $\sigma = -\mathrm{flip}$ on the generators and which is $*$-preserving.

E8.6 For the canonical $\Omega(\mathbb{Z})$ for the bidirected line graph calculus $\cdots - 1 \leftrightarrow 0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots$ and the Euclidean metric (so constant line lengths), show that there is a 1-parameter moduli of QLCs of the form

$$\nabla e^+ = be^+ \otimes e^+, \quad \nabla e^- = b(e^+ \otimes e^- + e^- \otimes e^+);$$
$$\sigma(e^\pm \otimes e^-) = e^- \otimes e^\pm, \quad \sigma(e^+ \otimes e^+) = (1-b)e^+ \otimes e^+,$$
$$\sigma(e^- \otimes e^+) = (1-b)e^+ \otimes e^- - be^- \otimes e^+,$$

where $b(2m) = 1 + q$, $b(2m+1) = 1 + q^{-1}$ on the vertex set and q is a constant, and another family with $e^\pm$ swapped. Show that only $\nabla e^\pm = 0$ is $*$-preserving. [The same applies for any n-polygon for even n while only $q = \pm 1$ are allowed for n odd and $n > 3$. For $n = 3$, see Example 8.19.]

E8.7 For the central quantum metric (4.24) in Example 4.37 for the open disk algebra $\mathbb{C}_q[D]$ with w inverted, apply $(* \otimes \mathrm{id})$ to define $\langle\, ,\, \rangle^{-1} \in \overline{\Omega^1} \otimes_A \Omega^1$ and find the corresponding hermitian metric. Write the corresponding matrices g and $\tilde{g}$ and find the equations for hermitian metric compatibility and torsion freeness of a left connection ∇ on Ω^1 in terms of the Christoffel symbols in $\nabla e^i = -\Gamma^i{}_k \otimes e^k$, where $e^1 = \mathrm{d}z$, $e^2 = \mathrm{d}\bar{z}$. Also find a formula for σ (assuming that it exists) and the condition for ∇ to be $*$-preserving.

Show that

$$\Gamma^1{}_1 = q^{-4}(1+q^2)w^{-1}\bar{z}\mathrm{d}z,\quad \Gamma^2{}_2 = q^4(1+q^2)w^{-1}z\mathrm{d}\bar{z},\quad \Gamma^1{}_2 = \Gamma^2{}_1 = 0$$

solve your conditions and provide a $*$-preserving hermitian metric compatible torsion free connection. [Proposition 8.40 then implies that it is also a QLC for the original quantum metric (4.24).]

E8.8 In the framework of the general analysis of Example 8.29, find the quantum wave operator on the Heisenberg algebra given by a constant $\tau(x) = 1/\delta$ and for the metric given by $\beta = (\delta x)^m$.

E8.9 On $M_2(\mathbb{C})$, use the same 2D calculus, metric and Levi-Civita bimodule connection (with $\sigma = -$flip on the generators) as in Example 8.13. Also use the 2D spinor bundle with central basis e, f and Clifford action as in Example 8.46 but now take connection $\nabla_{\mathcal{S}} e = \theta \otimes e$ and $\nabla_{\mathcal{S}} f = \theta \otimes f$. Show that this leads to $\not{D}(xe + uf) = E_{12}ue + E_{21}xf$ and directly verify the Lichnerowicz formula for $\not{D}^2$ in Proposition 8.45. Show that $\sigma_{\mathcal{S}} = 0$ and that $\not{D}$ cannot be part of a spectral triple via Theorem 8.44.

E8.10 Let $\Omega^1(G) = \mathbb{C}(G).\Lambda^1$ be a finite-group bicovariant calculus for $\mathcal{C} \subseteq G\setminus\{e\}$ with 'spin connection' $\alpha : \Lambda^1 \to \Omega^1$ for the tensor product frame bundle in §5.6.2. Let $\mathcal{S} = \mathbb{C}(G).V$ be a 'spinor bundle' on G associated to a representation ρ of G on V with basis $\{s^i\}$ and associated Clifford action $e^a{\triangleright}s^i = s^j\gamma^a{}_{ji}$, where $\gamma^a = \rho(a^{-1} - e)$ for all $a \in \mathcal{C}$. Write down $\nabla_{\mathcal{S}}$ similarly to E5.8 and show that $\not{D} = \gamma^a(\partial_a - \alpha_{ba}\rho(b^{-1} - e))$, where $\alpha(e^b) = \alpha_{ba}e^a$. Show for $G = S_3$ and V its 2-dimensional irreducible representation that $\sum_a \gamma^a = -3$ and $\gamma^a\gamma^b + \gamma^b\gamma^a + 2(\gamma^a + \gamma^b) + 3 = 3\delta_{a,b}$ for all $a, b \in \mathcal{C} = \{u, v, w\}$ and $\not{D} = \gamma^a\partial_a - 3$ for the WQLC in Example 5.83.

Notes for Chap. 8

The basics of what we now know as *Riemannian geometry* first appeared in Riemann's habilitation thesis of 1854 [290]. The notation of differential forms, including their antisymmetry, seems to have been first used by Cartan [72] in 1899. Reformulating the former in terms of the latter is the basis of our approach in the quantum case, as summarised in §8.1. A key step here was the notion of *cotorsion* as introduced by the second author in [207] as a weak version of metric compatibility that makes sense for any left connection ∇. Bimodule connections were introduced in [105, 106, 259] and the ability to write full metric compatibility in this case as $\nabla g = 0$ leads to the richer theory in the main part of §8.1, as developed in [31, 32, 222] by the present authors. The hermitian version in §8.4 appeared in [28] along with notions such as $*$-preserving and $*$-compatible for a bimodule connection. Proposition 8.11 on connections for inner calculi is from [222]. By contrast to the general curvature, Ricci curvature is less well understood in noncommutative geometry and we have followed a 'working definition' based on a lifting map $i : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ introduced in [285] and [210] (from where Lemma 8.18 is largely taken) and explored further in later works such as [32]. Classically, the Ricci tensor is key to Einstein's 1915 field equations for gravity [109].

§8.2 collects and extends some of the simplest examples from works of the authors. Example 8.14 on $\mathbb{C}S_3$ is from [239], Example 8.21 on $\mathbb{C}(S_3)$ is from [31, 210] and Example 8.20 on $\mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is from [227]. Example 8.19 on $\mathbb{C}(\mathbb{Z}_3)$ is new. A unique real QLC for any metric on the bidirected integer lattice is in [228].

The 'wave operator approach' in §8.3 based on constructing a quantum Laplacian is due to the second author [220]. Theorem 8.23 and Corollary 8.25 quantising any static spacetime with spatial part admitting a conformal Killing vector field is taken from here, where it was used to quantise the Schwarzschild black hole as we will see in Chap. 9. By contrast, Dirac in [101] took the 'square root' of the Laplacian, the Dirac operator $\not{D}$, in order to formulate fermions in relativistic quantum theory. The axiomatisation of this is at the heart of Connes' approach to noncommutative geometry in the notion of a *spectral triple* [88], with a deep connection to KO-homology. In §8.5, we give only the barest introduction from the more algebraic side, omitting most of the functional analytic aspects needed for a genuine appreciation of Connes' axioms, such as the theory of unbounded, Fredholm and compact operators. Instead, we focus on the constructive 'geometric realisation' of Dirac operators by noncommutative steps that follow the classical geometry, using tools from earlier chapters of the book. Classically, one defines the Dirac operator via a spin connection which induces both the Levi-Civita connection and a connection on the spinor bundle. This was achieved by the second author for $\mathbb{C}(S_3)$ in [210], for $\mathbb{C}_q[SL_2]$ as in Example 8.51 in [211] and for $\mathbb{C}_q[S^2]$, basically Example 8.47, in [216], all as applications of the quantum frame bundle theory of [207]. Our formulation in Theorem 8.44 is taken from [35] and comes closer to a Connes spectral triple as it includes an antilinear operator $\mathcal{J}$ made possible by use of the theory of bimodule connections and conjugates from [31]. Another scheme for constructing noncommutative 'Dirac operators' from left connections was proposed in [180] but without the conjugation map $\mathcal{J}$ needed for a spectral triple. The full 1-parameter family of algebraic spectral triples on the standard q-sphere in Example 8.47 is also taken from [35], albeit the $\mathcal{J}$ operator turns out to go slightly beyond Connes' axioms. The same applies to the q-disk in Example 8.48. Note that earlier work [93] on Dirac operators on the q-sphere also required modification of Connes' axioms, differing from our approach in replacing the bimodule condition (4) by the commutator being a compact operator, whereas we instead replace the isometry property of $\mathcal{J}$ by a twisted isometry. Many other spectral triples and near-spectral triples of interest can be found in the literature, including [73] on $\mathbb{C}_q[SU_{l+1}]$ and its extension to other quantum groups in [265]. A classification of spectral triples on matrix algebras is in [15]. Some recent work bridging the algebraic side to some analysis using Fourier theory on quantum groups is [1], which obtained the eigenvalues for the $\not{D}$ on $\mathbb{C}_q[SU_2]$ quoted in Example 8.51. The theory of topological insulators is a practical application of Dirac operators and noncommutative index theory [51, 52, 296]. It is also of interest to extract metric information on state spaces [288, 289]. Ricci curvature can also be seen in the spectral triple point of view, see [115] for a recent work.

Chern or hermitian connections appeared in [77]; our quantisation of these in §8.6 is from [35], building on the bimodule connection and $*$-conjugation theory in [31] and the theory of noncommutative complex structures in [38]. This gave us a different approach to the q-monopole connection on the q-sphere that was used for the Dirac operator as well as in earlier chapters. A different but related approach that leads more exactly to Connes' spectral triples on q-flag varieties in general is Ó Buachalla's formulation of integrable almost complex structures (see the notes to Chap. 7) and a q-version of the Lefschetz decomposition to formulate a Dirac operator and Hodge structure in [272].

The exercises in this chapter are largely a collection of new basic examples, starting with the moduli of metric compatible connections on $\mathbb{Z}$, $\mathbb{Z}_n$ and a finite line graph in E8.1 and including a QLC for the q-disk in E8.9. However, the 'Dirac operator' on S_3 in E8.10 is from [210] as part of a general Hopf algebra construction.

Chapter 9
Quantum Spacetime

Here we will discuss some applications of quantum Riemannian geometry specifically to quantum spacetime and potential physical effects. The idea is that spacetime, due to quantum gravity effects, is better described as a noncommutative or 'quantum' geometry than a classical one and that this has mathematical implications even if we do not know quantum gravity itself. We focus for the most part on the Majid–Ruegg or bicrossproduct model where spatial coordinates x^i mutually commute and

$$[x^i, t] = \mathrm{i}\lambda_P x^i$$

which, since its introduction in the 1990s, has had several physical aspects and predictions associated to it. We also cover the quantum black-hole in the Laplacian approach (its actual bimodule noncommutative Riemannian geometry is thought to be nonassociative). Here λ_P is expected to be of order the Planck time, around 10^{-44} seconds. Our goal is to illustrate some of the arguments that can be made and which could be made in other models using the methods of the book. It should be stressed that any physical interpretation inevitably involves assumptions to connect the mathematics to the proposed physics which, in the absence of a fundamental theory, have to be posited rather than deduced. We will highlight these for the three illustrative models that we have chosen. The chapter concludes with a section on the semiclassicalisation of quantum Riemannian geometry as a new paradigm which we call 'classical quantum gravity' and which bears the same relation to quantum gravity as classical mechanics does to quantum mechanics, the difference being that the deformation parameter is the Planck scale not Planck's constant.

It should be said for mathematicians that we will mostly avoid technical issues either by working with formal power series or by building our algebras on classical components. A more complete treatment would need significant functional analysis, which is beyond our scope. Moreover, we will usually focus actual calculations on the polynomial level or use the same ideas to make finite-dimensional models.

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9.1 The Quantum Spacetime Hypothesis

Here we put forward the radical view that modern theoretical physics as it was during most of the twentieth century rests on an unfounded and ultimately misguided continuum hypothesis. We may not know what quantum gravity is, but we do know that this hypothesis is suspect. The reasons are quite fundamental and are an immediate consequences of any attempt to combine quantum theory as we currently know it with gravity as we currently know it.

Firstly, gravity. The main idea here is that matter causes spacetime to curve and this curvature causes geodesics to bend, and this is what we call gravity. The mathematics goes back to Riemann and Gauss, well before Einstein, but the successful physical interpretation as gravity in these terms goes to Einstein's 1915 theory of 'general relativity' or GR. Probably the best-known implication of this theory is that if you squeeze enough matter into a given volume of space then it curves spacetime so much that it forms a black-hole from which even light cannot escape. It is believed that most large stars after they supernova at the end of their life cycle end up as extremely dense neutron stars or as black-holes. The important thing for us is that as more matter falls into a black-hole, it gets bigger according to the formula

$$r_{BH} = \frac{G_N M}{c^2}, \tag{9.1}$$

where the length scale r_{BH} in the simplest case of a spherically symmetric black-hole is such that $2r_{BH}$ is the value of the standard Schwarzschild 'radius' coordinate at the event horizon. In particular, it is believed that most galaxies, including our own, have a large black-hole at their centre grown in this way. The constant G_N is 'Newton's constant', c is the speed of light and M is the mass of the black-hole.

Secondly, quantum theory has at its core the idea that light and other waves can be viewed as quanta of energy ('photons' in the case of light) and these have energy proportional to their frequency ω or inversely proportional to their wavelength λ,

$$E = h\omega = \frac{hc}{\lambda},$$

where h is Planck's constant. It was in fact Einstein who found a proof for this in the form of an explanation of the 'photoelectric effect', for which work he received a Nobel prize in 1921. This energy also curves space according to the conversion $E = mc^2$ between the energy E of a particle and its effective mass m (coming from the theory of 'special relativity' as formulated by Einstein but also known to contemporaneous mathematicians such as Poincaré). Even massive particles are waves with a wavelength (called the Compton wavelength) given by the formula $\lambda = \frac{h}{mc}$ on eliminating E in the above observations. It is convenient to write this in terms of the reduced wavelength $\lambda\!\!\!^{-} = \frac{\lambda}{2\pi}$ and reduced Planck's constant $\hbar = \frac{h}{2\pi}$ as

$$\lambda\!\!\!^{-} = \frac{\hbar}{mc}. \tag{9.2}$$

Here m should be understood loosely as mass-energy to cover both the case of massless particles such as light and massive particles.

Now let us think about what geometry means in operational terms. Why do we think that 'geometry' is out there? The answer is that we can 'see' it or rather probe its structure using particles or waves moving in the geometry, either directly through geodesics or by interacting with other structures located in spacetime. Thus, radio waves, light, electron microscopes, particle accelerators such as the Large Hadron Collider, which can be thought of as a 'proton microscope', all probe space and spacetime with finer and finer resolution due to the smaller and smaller associated wavelength. This in turn needs greater and greater mass energies. As we try to probe at smaller and smaller length scales, we therefore move down the left slope in Fig. 9.1 until we come to the point where the mass-energy of the probe particle begins to curve and distort the very spacetime geometry we are trying to observe, eventually forming a black-hole. This is the point where (9.1) and (9.2) intersect, i.e., where the Compton wavelength of the rest particle is of the same order as the Schwarzschild radius of a black-hole of the same mass. If we set $\lambda = r_{BH}$ and $M = m$ then the former is

$$\lambda_P = \sqrt{\frac{G_N \hbar}{c^3}} \sim 10^{-35}\ \mathrm{m} \sim 10^{-5}\ \mathrm{g} \sim 10^{-44}\ \mathrm{s},$$

where the first number is in metres, the second equivalently via (9.2) in grams and the third equivalently the light-travel time in seconds. Any of these forms is called the 'Planck scale'. So *resolution of spacetime below this scale is intrinsi-*

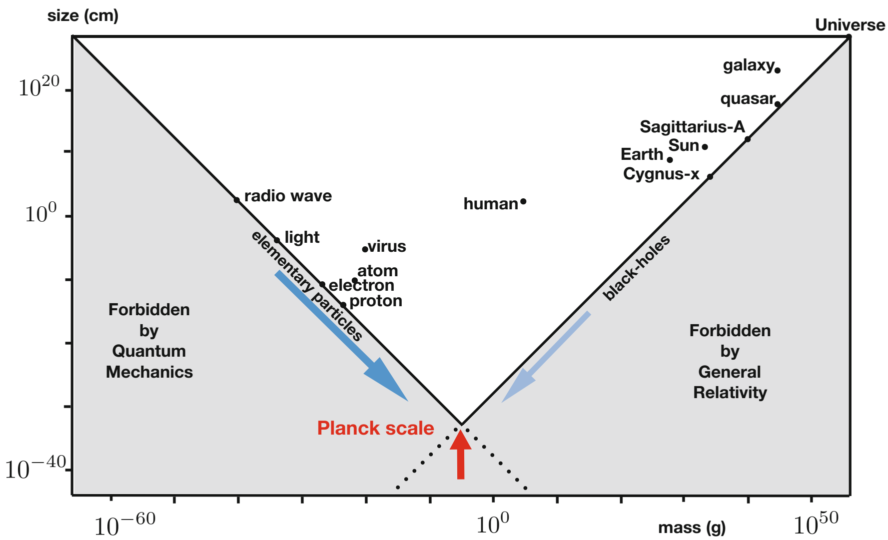

Fig. 9.1 Log-log plot of objects in the Universe showing the Planck scale as the point where quantum theory and general relativity intersect

cally impossible. Therefore the continuum hypothesis that spacetime is infinitely divisible, which was the cornerstone of geometry and physics since the invention of differential calculus by Newton and Leibniz, can have no basis in measurement and hence has no place in science as we currently know it if we apply the principle of Occam's razor. Rather, one should view the foundation of theoretical physics on a spacetime continuum as a convenient assumption that was fine for both general relativity and quantum theory as far as they go separately but no more than that. From the point of view of Example 1.43, this continuum hypothesis is just a particular type of differential structure where differentials $\mathrm{d}x$ commute with functions, which in turn underlay the picture of these in the work of Newton and Leibniz as infinitesimal displacements.

Of course, just because the continuum assumption has no scientific basis does not mean that it is not true by accident. However, if one tries to develop quantum theory and general relativity together to see what emerges when these paradigms of physics 'collide' then one finds currently insurmountable problems, many of which can be traced back to the continuum assumption. To explain this, we need to explain a little quantum field theory. Firstly, classical fields such as a scalar field ϕ obey the spacetime Klein–Gordon wave equation

$$\Box\phi = \frac{m^2c^2}{\hbar^2}\phi$$

in our conventions from which, if one writes $\phi = e^{\frac{\mathrm{i}mc^2t}{\hbar}}\psi(x,t)$ where ψ is slowly varying, one can derive that ψ approximately obeys the Schrödinger equation of quantum mechanics. More generally, one can solve the wave equation in flat spacetime in terms of plane waves $\phi_{(\omega,k)} = e^{\mathrm{i}\omega t}e^{\mathrm{i}k.x}$ labelled by (ω, k) as a vector in a 'momentum space' copy of Minkowski space Fourier dual to spacetime and with the Klein–Gordon wave equation as the constraint $\frac{\omega^2}{c^2} - k^2 = \frac{m^2c^2}{\hbar^2}$. A general solution has the form

$$\phi(x,t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega(k)}}(\phi_k e^{\mathrm{i}k.x}e^{-\mathrm{i}\omega t} + \overline{\phi_k}e^{-\mathrm{i}k.x}e^{\mathrm{i}\omega t}); \quad \omega(k) = c\sqrt{\frac{m^2c^2}{\hbar^2} + k^2}$$

for Fourier coefficients ϕ_k. Massless particles such as photons are not amenable to the Schrödinger equation limit of quantum mechanics but both have the same origin and ontological status as classical fields on spacetime. The wave function ψ in the Schrödinger limit describes a single quantum particle, while for light the wave equation (more usually expressed in terms of largely equivalent electromagnetic field equations) describes a particular electromagnetic wave. Quantum field theory extends this paradigm to describe many particles at once on replacing the usual Fourier expansion of a classical field ϕ by an operator-valued field

$$\Phi(x,t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega(k)}}(a_k e^{\mathrm{i}k.x}e^{-\mathrm{i}\omega t} + a_k^\dagger e^{-\mathrm{i}k.x}e^{\mathrm{i}\omega t}),$$

where $[a_k, a^\dagger_{k'}] = (2\pi)^3\delta^3(k-k')$ describe commuting quantum harmonic oscillators at each momentum. These quantum harmonic oscillators replace the classical field Fourier coefficients and allow for a quantum mechanical treatment of each momentum mode. In the quantum mechanics of a single oscillator we represent the algebra $[a, a^\dagger] = 1$ on a Hilbert space and a given state in this Hilbert space has an expected value of the operator energy $H = \hbar\frac{\omega}{2}(aa^\dagger + a^\dagger a) = \hbar\omega(a^\dagger a + \frac{1}{2})$. In the lowest energy or 'vacuum state' $|0\rangle$ one has $a|0\rangle = 0$ and hence $\langle 0|H|0\rangle = \hbar\frac{\omega}{2}$. Other states are created by application of $a^\dagger$ to the vacuum and have higher energy using the above commutation rules. The notation used here is that the Hilbert space inner product of two vectors $|\psi\rangle, |\psi'\rangle$ is denoted $\langle\psi|\psi'\rangle$. What this means is that the minimum vacuum energy of a quantum field is an integral of $1/2$ over all momenta k. If we believe in a spacetime continuum then we should allow all possible k which implies an infinite 'zero point energy'. One fix of course is to accept the above arguments and limit wavelengths to above the Plank scale or $|k|$ to below the Planck energy-momentum scale. However, simply doing this implies an average zero point energy of Planck scale order (this is 10^{94} grams per cubic cm). Such a vast zero point energy would cause the universe to contract in a way that simply is not observed. One could always redefine the energy as zero in the vacuum but astronomical data suggests that there *is* in fact an apparent uniform 'dark energy' density. Its apparent value is around 10^{-29} grams per cubic centimetre, so naive theory and experiment are out by a factor of 10^{120}. This 'dark energy problem' is one of many issues of which the resolution should be provided by an actual theory of quantum gravity.

Incidentally, when the large hadron collider was turned on there were misguided fears that micro black-holes could be produced and some scientists pointed out that they would just evaporate. This evaporation is an effect not so much of quantum gravity but of quantum fields in a curved classical spacetime whereby even black-holes radiate a small amount of energy due to quantum effects. Bekenstein and Hawking showed that this radiation has a spectrum similar to a black body of temperature T inversely proportional to the mass of the black-hole. This link between gravity and thermodynamics and ultimately between gravity and information theory and entropy is also something that should be resolved by an actual theory of quantum gravity. We should point out that as far as black-hole evaporation is concerned, we would be moving down the right slope of Fig. 9.1, but as we approach the Planck mass, so that the black-hole becomes the same size as an elementary particle, we do not know what actually happens. There is no reason to think, as many assume, that it simply shrinks out of existence (that would assume the continuum hypothesis) and quite possibly the black-hole enters a hybrid quantum-gravity state in which the tendency to evaporate is balanced by the principle that a smaller mass particle occupies more space. The existence and properties of such remnants would again need to be resolved by an actual theory of quantum gravity.

It goes without saying that the creation of the universe in a 'hot big bang' is again something that we can only extrapolate back to the Planck time because time, like space, cannot be resolved below this scale. Hence before 10^{-44} seconds or so, we have no idea as to the actual physics, and hence no idea as to how the universe was actually created, without a theory of quantum gravity. Similarly, the singularity at the centre of a black-hole where the curvature spikes to infinity is again predicated on the continuum assumption. As a particle approaches to within the Planck length of the singularity, we do not actually know what happens after that without a theory of quantum gravity. For the particular purposes of the book we will cut through all of this ignorance with the following working hypothesis:

(*Quantum spacetime hypothesis*) Spacetime is better modelled by noncommutative or 'quantum' spacetime in which spacetime coordinates do not commute as an expression of quantum gravity effects.

The idea is that whatever quantum gravity is, in the classical limit it maps onto general relativity and conversely as we approach that limit but do not yet reach it, it is better modelled by something that corrects general relativity. Since these corrections are quantum gravity effects, we can guess that they could be expressed as noncommutativity of the spacetime coordinates. This is expressed in Fig. 9.2. The weakest version of this would say that this should hold at least at order λ_P, which is not actually quantum Riemannian geometry but its semiclassical counterpart 'Poisson Riemannian geometry' (which we cover in §9.6). But given the power and success of quantum theory methods in the form of operator algebras, it seems quite plausible that this also holds more strongly to all orders in λ_P as a particular sector of the various physical processes that constitute the full quantum gravity. If so then the mathematical framework of the present book, in so far as it is mathematically natural, could well apply in the context of model-building. Model building is a time-honoured approach in theoretical physics in which one takes a reasonable mathematical framework and attempts to construct physical models without yet having the ultimate 'theory of everything'. One could even argue that this is the *only* way to do science in that all science applies within certain physical ranges and within the constraints of certain mathematical structures.

As far as actual evidence for quantum spacetime, there are a few hints. One is 3D quantum gravity as defined by the quantisation of the Einstein–Hilbert action and which, unlike the 4D case, is quite well understood in the regime of point particle sources coupled through to their quantum effects. Gravity here in 3D is

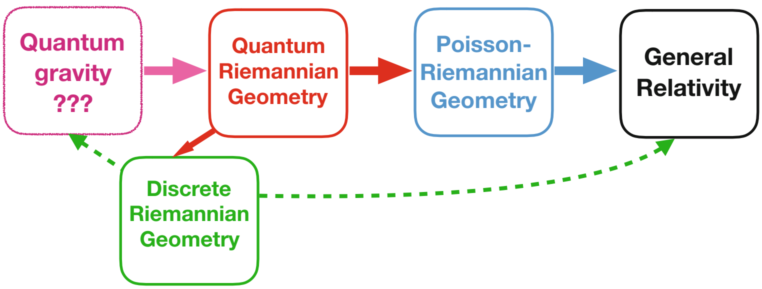

Fig. 9.2 The quantum spacetime hypothesis says that quantum gravity factors through quantum Riemannian geometry on its way to the limit of classical gravity. Another limit is discrete geometry

topological and integrable, and in this context some of the quantum groups that we will discuss can be seen to arise and to effectively act on quantum spacetimes such as $U(su_2)$ in Example 1.45, as we will see in §9.2.3. Another more general rationale is *quantum Born reciprocity*. This is the idea that in classical mechanics it is well known that one has a symplectic or at least a Poisson manifold but the polarisation into which part is position and which part is momentum is to some extent arbitrary. In particular, one might be able to reverse the roles of position and momentum. Now, curvature of spacetime, which means gravity, corresponds to the momenta, which appear naturally as covariant derivatives, noncommuting. If we believe there is nothing unique in quantum gravity about which variables are actually space and which are momentum then one should equally well have noncommutativity of spacetime coordinates or in some sense curvature in momentum space. These ideas can be explored relatively precisely in terms of Hopf algebra duality and led in the 1980s to bicrossproduct quantum groups as one of the first general classes of noncommutative and noncocommutative Hopf algebras, associated to Lie group factorisations (the other main class emerging in the 1980s was Drinfeld's q-deformation quantum groups motivated from integrable systems). Later on, these bicrossproduct quantum groups turned out to be natural candidates for quantum Poincaré groups for quantum spacetimes, as we explain shortly. Thus, while the possibility of quantum spacetime was certainly speculated upon since the start of quantum mechanics, it is only with the arrival of quantum group examples and at a similar time of the development of noncommutative geometry that the idea could be meaningfully realised. In what follows, we normally set $c = 1$ by measuring time in appropriate units.

9.2 Bicrossproduct Models and Variable Speed of Light

There are many points of view on the bicrossproduct model spacetime and here we describe the original one as a quantisation of flat space. Note that the algebra alone is like specifying the 'topological space' in some sense, which could be $\mathbb{R}^n$ or a part of $\mathbb{R}^n$ if we invert some generators etc., but does not address its geometry. For that one needs to decide its differential structure or have some other point of view. One of these is symmetries. If a quantum group can reasonably be said to deform the Poincaré group on flat spacetime and acts on an algebra with the right number of generators and reasonable relations then one could consider the algebra as an analogue of Minkowski space.

This can happen as an example of the following general construction. The starting point is a factorisation of a group X into subgroups G, M such that $X = GM = MG$ and $G \cap M = \{e\}$, where e is the identity. It means every element of X can be uniquely expressed as a normal ordered product of elements in G, M. In this situation, define a left action $\triangleright$ of M on G and a right action $\triangleleft$ of G on M by the equation

$$su = (s\triangleright u)(s\triangleleft u) \tag{9.3}$$

for all $u \in G$ and $s \in M$. These actions obey

$$
\begin{aligned}
&s{\triangleleft}e = s, \quad e{\triangleright}u = u, \quad s{\triangleright}e = e, \quad e{\triangleleft}u = e,\\
&(s{\triangleleft}u){\triangleleft}v = s{\triangleleft}(uv), \quad s{\triangleright}(t{\triangleright}u) = (st){\triangleright}u,\\
&s{\triangleright}(uv) = (s{\triangleright}u)((s{\triangleleft}u){\triangleright}v), \quad (st){\triangleleft}u = (s{\triangleleft}(t{\triangleright}u))(t{\triangleleft}u)
\end{aligned} \tag{9.4}
$$

for all $u, v \in G$, $s, t \in M$. A pair of groups (G, M) equipped with such actions is said to be a 'matched pair'. One can then define a 'double cross product group' $G{\bowtie}M$ with product

$$(u, s).(v, t) = (u(s{\triangleright}v), (s{\triangleleft}v)t) \tag{9.5}$$

and with G, M as subgroups. Since it is built on the direct product space, the bigger group factorises into these subgroups, and in fact one recovers X in this way. These notions have been known for finite groups since the 1910s but in a Lie group setting one has the similar notion of a 'local factorisation' $X \approx GM$ and a corresponding double cross sum $\mathfrak{g}{\bowtie}\mathfrak{m}$ of Lie algebras which is more recent. Then the differential version of (9.4) become a pair of coupled first-order differential equations for families of vector fields α_ξ on M and β_ϕ on G labelled by $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{m}$, respectively. Here $\alpha_\xi(s)$ is the differential of $s{\triangleleft}v$ with respect to v at the identity in the direction $\xi \in \mathfrak{g}$, and similarly $\beta_\phi(v)$ is the differential of $s{\triangleright}v$. We write these vector fields in terms of Lie algebra-valued functions $A_\xi \in C^\infty(M, \mathfrak{m})$ and $B_\phi \in C^\infty(G, \mathfrak{g})$ according to left and right translation from the tangent space at the identity,

$$\alpha_\xi(s) = L_{s*}(A_\xi(s)), \quad \beta_\phi(u) = R_{u*}(B_\phi(u)). \tag{9.6}$$

In these terms, the matched pair equations become

$$A_\xi(st) = \mathrm{Ad}_t^{-1}(A_{t\triangleright\xi}(s)) + A_\xi(t), \quad A_\xi(e) = 0, \tag{9.7}$$

$$B_\phi(uv) = B_\phi(u) + \mathrm{Ad}_u(B_{\phi\triangleleft u}(v)), \quad B_\phi(e) = 0 \tag{9.8}$$

along with auxiliary data a pair of linear actions $\triangleright$ of M on $\mathfrak{g}$ and $\triangleleft$ of G on $\mathfrak{m}$ exponentiating Lie algebra actions $\triangleright, \triangleleft$ of $\mathfrak{m}, \mathfrak{g}$, respectively. Finally, (9.7) becomes a pair of differential equations if we let u, t be infinitesimal, i.e. elements $\eta \in \mathfrak{g}$, $\psi \in \mathfrak{m}$ say of the Lie algebras. Then

$$\psi^L(A_\xi)(t) = \mathrm{Ad}_{t^{-1}}(\psi{\triangleleft}(t{\triangleright}\xi)), \quad \eta^R(B_\phi)(u) = \mathrm{Ad}_u((\phi{\triangleleft}u){\triangleright}\eta), \tag{9.9}$$

where ψ^L is the left-invariant vector field on the Lie group M generated by ψ and η^R the right-invariant vector field on G generated by η. Note that this implies

$$\psi^L(A_\xi)(e) = \psi{\triangleleft}\xi, \quad \eta^R(B_\phi)(e) = \phi{\triangleright}\eta, \tag{9.10}$$

which shows how the auxiliary data are determined. This allows one to construct the factorisation by solving a pair of cross-coupled nonlinear PDEs. In practice, we will construct the factorisation directly but this nonlinearity is the origin of the singularities we will typically encounter when the groups are noncompact.

We also note that the Lie algebra splitting data alone implies a factorisation of the enveloping algebra $U(\mathfrak{g}{\triangleright\!\!\!\triangleleft}\mathfrak{m}) = U(\mathfrak{g}){\triangleright\!\!\!\triangleleft}U(\mathfrak{m})$ by a pair of Hopf algebra actions.

Lemma 9.1 (Classical Factorisation A-Model) *In a Lie group factorisation, the group $X = GM$ acts from the right on the set of M by $s{\triangleleft}u$ and $s{\triangleleft}t = st$, and hence from the left on $C^\infty(M)$. Here $u \in G$ and $s, t \in M$. In the local factorisation case, we obtain an action of $U(\mathfrak{g}{\triangleright\!\!\!\triangleleft}\mathfrak{m})$ on $C^\infty(M)$ where $\mathfrak{g}$ acts by the vector fields α_ξ and $\mathfrak{m}$ acts by right-invariant vector fields.*

Proof At the group level, each subgroup acts separately on M and $(t{\triangleleft}s){\triangleleft}u = (ts){\triangleleft}u = (t{\triangleleft}(s{\triangleright}u))(s{\triangleleft}u) = t{\triangleleft}((s{\triangleright}u)(s{\triangleleft}u)) = t{\triangleleft}(su)$ using one of the matched pair identities. Hence we have a right action of the group X on the set of M, which then implies an action on the algebra $C^\infty(M)$ in the Lie group case. We also have a sister 'factorisation B-model' where X acts from the left on $C^\infty(G)$ and for which the action on the group level is $u{\triangleright}v = uv$ and $s{\triangleright}v$. □

If suitable algebraic versions exist then we can replace $C^\infty(M)$ here by a Hopf algebra $\mathbb{C}[M]$ generated by some representative functions and work more algebraically. The bicrossproduct construction proceeds by now 'semidualising' $U(\mathfrak{g}){\triangleright\!\!\!\triangleleft}U(\mathfrak{m})$, in which we replace one of the factor Hopf algebras, $U(\mathfrak{m})$ say, by essentially its dual, $\mathbb{C}[M]$. We use the action of $U(\mathfrak{g})$ on $\mathbb{C}[M]$ in Lemma 9.1 (i.e., the vector fields α_ξ) to make a semidirect product algebra $\mathbb{C}[M]{>\!\!\!\triangleleft}U(\mathfrak{g})$ and we use the other half of the matched pair data to simultaneously semidirect coproduct the coalgebra to give a coalgebra $\mathbb{C}[M]{\blacktriangleright\!\!<}U(\mathfrak{g})$. It turns out that these then fit together to give a Hopf algebra, the *bicrossproduct quantum group* $\mathbb{C}[M]{\blacktriangleright\!\!\triangleleft}U(\mathfrak{g})$ associated to the same factorisation data. This supposes at the algebraic level that there is a right coaction $\Delta_R : \mathfrak{g} \to \mathfrak{g} \otimes \mathbb{C}[M]$ dual to the left action of M on $\mathfrak{g}$ exponentiating the Lie algebra action of $\mathfrak{m}$ on $\mathfrak{g}$. Let us denote the latter group action by a matrix defined by $m{\triangleright}e_j = \rho^i{}_j(m)e_i$ (sum over j) for a basis $\{e_i\}$ of $\mathfrak{g}$. The idea is that in nice cases we can regard these matrix entries as functions $\rho^i{}_j \in \mathbb{C}[M]$ and the coaction is then $\Delta_R e_j = e_i \otimes \rho^i{}_j$. The structure of $\mathbb{C}[M]{\blacktriangleright\!\!\triangleleft}U(\mathfrak{g})$ is then

$$[e_i, f] = \alpha_{e_i}(f), \quad \Delta e_j = e_i \otimes \rho^i{}_j + 1 \otimes e_j \tag{9.11}$$

for $f \in \mathbb{C}[M]$ together with the algebra relations of $U(\mathfrak{g})$, the algebra relations of $\mathbb{C}[M]$ and the coproduct of the latter as a sub-Hopf algebra.

Lemma 9.2 (Bicrossproduct A-Model) *The above quantum group $\mathbb{C}[M]{\blacktriangleright\!\!\triangleleft}U(\mathfrak{g})$ acts from the right on the 'quantum spacetime' $U(\mathfrak{m})$ by the right Lie algebra action of $\mathfrak{g}$ on $\mathfrak{m}$ extended to $U(\mathfrak{m})$ by acting as skew derivations*

$$(x^{i_1}\cdots x^{i_m}){\triangleleft}e_j = x^{i_1}\cdots x^{i_{m-1}}(x^{i_m}{\triangleleft}e_j) + ((x^{i_1}\cdots x^{i_{m-1}}){\triangleleft}e_i)(x^{i_m}{\triangleleft}\rho^i{}_j),$$

where $f \in \mathbb{C}[M]$ *acts on* $U(\mathfrak{m})$ *by*

$$x^i \triangleleft f = x^i f(e) + f(x^i)$$

with $e \in M$ *the group identity. We assume that* $f \in \mathbb{C}[M]$ *differentiates to have a value* $f(x^i)$ *on a basis* $\{x^i\}$ *of* $\mathfrak{m}$ *viewed as infinitesimally close to the group identity.*

Proof Any Hopf algebra acts on its dual by evaluation against the coproduct. In our case $f(x) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\mathrm{e}^{tx}) = \langle f, x\rangle$ is more geometrically the left-invariant vector field generated by x applied to $f \in \mathbb{C}[M]$ and evaluated at e, and is part of the duality between $\mathbb{C}[M]$ and $U(\mathfrak{m})$. Then $x \triangleleft f = (f \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = xf(e) + 1\langle f, x\rangle$ for all $x \in \mathfrak{m}$. Meanwhile, the left action of $\mathfrak{g}$ on $\mathfrak{m}$ extends by the general theory of bicrossproducts to $U(\mathfrak{m})$ as a subalgebra of the bicrossproduct quantum group, the coproduct of which then dictates the form on products of elements of $\mathfrak{m}$. One can check that these actions obey the relations of the bicrossproduct, $x \triangleleft [e_i, f] = f(e)x \triangleleft e_i + f(x \triangleleft e_i) - (xf(e) + f(x)) \triangleleft e_i = f(x \triangleleft e_i) = \alpha_{e_i}(f)(x) + x\alpha_{e_i}(f)(e)$ by definition of the vector fields α_{e_i}. The last term vanishes as $e \triangleleft u = e$ for all $u \in G$. There is also a sister 'bicrossproduct B-model' where we semidualise the other Hopf algebra factor of $U(\mathfrak{g}) \bowtie U(\mathfrak{m})$ resulting, in a suitably algebraic context, in a quantum group $U(\mathfrak{m}) \triangleright\!\blacktriangleleft \mathbb{C}[G]$ acting from the left on $U(\mathfrak{g})$. $\square$

We think of $\mathfrak{g}$ here as generating rotations when acting on $U(\mathfrak{m})$ regarded as quantum spacetime. The latter has coordinates $x^i \in \mathfrak{m}$ and we think of the generators of the dual or 'Fourier conjugate' $\mathbb{C}[M]$ as 'momentum' acting by evaluation against the coproduct or 'differentiation' on $U(\mathfrak{m})$. In this way, we think of the bicrossproduct as a Poincaré quantum group for the quantum spacetime. Compared to the classical factorisation model in Lemma 9.1, this represents a swap of the roles $\mathbb{C}[M]$ and $U(\mathfrak{m})$, which is the promised quantum Born reciprocity implemented by semidualisation. Note that if M is nonabelian then the process takes a model with (in some sense) a curved but classical space acted upon by noncommutative momentum algebra $U(\mathfrak{m})$ to a model with a quantum but in some sense 'flat' spacetime (insofar as it has an additive coproduct structure), namely $U(\mathfrak{m})$ as quantisation of $\mathbb{C}[\mathfrak{m}^*]$. This is therefore a kind of quantum-gravity duality and is the reason that quantum Born reciprocity was proposed as an idea for toy models of quantum gravity. Also note that the sister B-model bicrossproduct quantum group is dual to $\mathbb{C}[M] \blacktriangleright\!\triangleleft U(\mathfrak{g})$ and has the roles of position and momentum played by $U(\mathfrak{g})$, $\mathbb{C}[G]$, while the sister factorisation B-model mentioned at the end of the proof of Lemma 9.1 has these swapped. Thus we actually have four models for each factorisation datum.

These constructions may be a lot for the reader to take in, therefore our approach will be to give a simple worked example in detail so as to make the above concrete. This will not be the very simplest nontrivial example, which comes from solving the matched pair equations for $(\mathbb{R}, \mathbb{R}, \triangleright, \triangleleft)$ and leads to the self-dual 'Planck scale Hopf algebra' $\mathbb{C}[\mathbb{R}]\blacktriangleright\!\!\triangleleft\mathbb{C}[\mathbb{R}]$ acting on $\mathbb{C}[\mathbb{R}]$ as a curved deformation of quantum mechanics (a slightly different interpretation from the one we are discussing). Our algebraic treatment also circumvents nonlinear global issues in the exponentiation of the actions that would be needed for a full treatment of the quantum groups as Hopf–von Neumann or Hopf C^*-algebras. We also will not have room to give details of the most general picture even at the algebraic level, whereby the factorising object can itself be a quantum group to start with. Here, if $H = H_1 \bowtie H_2$ is a Hopf algebra factorisation then it results in a matched pair of actions with the help of which H acts from the left on the algebra H_2^*. And if the latter can be made into a Hopf algebra compatibly with the matched pair actions then there is a bicrossproduct quantum group $H_2^* \blacktriangleright\!\!\triangleleft H_1$ acting from the right on H_2, as well as two sister models with the two factors swapped. When applied to $U_q(so_{1,3}) = U_q(su_2) \bowtie U_q(su_2^*)$, where su_2^* is the dual Lie bialgebra to su_2, these covariant systems turn out to describe a key part of the structure of 3D quantum gravity with cosmological constant from two different points of view. We will say a little more about this in §9.2.3

9.2.1 Classical Data for the 2D Model and Planckian Bound

We now turn to a family of examples that come from factorising $SO_{n,m}$, focussing our exposition on the 2D case. We start with the classical model so as to look carefully at the global issues as well. The first remark in the 2D case is that for a convenient description of the global picture we work not with $SO_{2,1}$ exactly but its double cover $X = SL_2(\mathbb{R})$. This has a local factorisation $SL_2(\mathbb{R}) \approx SO_{1,1}^+ .(\mathbb{R} \rtimes \mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} q & qz \\ 0 & \frac{1}{q} \end{pmatrix};$$

$$|c| < |a|, \quad C = \cosh(\frac{1}{2}\theta), \quad S = \sinh(\frac{1}{2}\theta), \quad q \neq 0,$$

where θ, q, z are real (the $\frac{1}{2}$ is for the physical interpretation as a part of the double cover of $SO_{2,1}$). The factorisation is valid for the domain on the left as shown, which includes the identity. The group $\mathbb{R} \rtimes \mathbb{R}$ displayed is that of upper triangular matrices of determinant 1 and has two components. One could equally work with a component where $q > 0$ and allow the full $SO_{1,1}$ in the factorisation. We have θ, q, z determined by

$$S = \frac{c}{a}C, \quad q = \frac{a}{C}, \quad z = \frac{ab - cd}{q^2}; \quad C = \frac{1}{\sqrt{1 - \frac{c^2}{a^2}}}.$$

This cannot be a global decomposition for topological reasons, which one can also see as encountering singularities when solving the matched pair equations. This is a typical feature when there are noncompact directions.

Next, we focus on the component connected to the identity where $a > 0$ by writing $q = e^{\frac{\lambda_P}{2}p^0}$, $z = \lambda_P p^1$ for real parameters p^μ. Here λ_P is a fixed positive normalisation constant and we use upper indices. We now take group elements in the wrong order and refactorise:

$$\begin{pmatrix} e^{\frac{\lambda_P}{2}p^0} & \lambda_P p^1 e^{\frac{\lambda_P}{2}p^0} \\ 0 & e^{-\frac{\lambda_P}{2}p^0} \end{pmatrix} \begin{pmatrix} C & S \\ S & C \end{pmatrix} = \begin{pmatrix} (C + S\lambda_P p^1)e^{\frac{\lambda_P}{2}p^0} & (S + C\lambda_P p^1)e^{\frac{\lambda_P}{2}p^0} \\ Se^{-\frac{\lambda_P}{2}p^0} & Ce^{-\frac{\lambda_P}{2}p^0} \end{pmatrix}$$
$$= \begin{pmatrix} C' & S' \\ S' & C' \end{pmatrix} \begin{pmatrix} e^{\frac{\lambda_P}{2}p'^0} & \lambda_P p'^1 e^{\frac{\lambda_P}{2}p'^0} \\ 0 & e^{-\frac{\lambda_P}{2}p'^0} \end{pmatrix}$$

where $S' = \sinh(\frac{1}{2}\theta')$, $C' = \cosh(\frac{1}{2}\theta')$ for Lorentz boost value θ' and (p'^0, p'^1) denotes the new momentum value. This is only possible if $|c'| < |a'|$ for the entries of the matrix before we refactorised, which translates as

$$\big(C + S(\lambda_P p^1 + e^{-\lambda_P p^0})\big)\big(C + S(\lambda_P p^1 - e^{-\lambda_P p^0})\big) > 0.$$

Moreover, we are limiting attention to the case where $a' > 0$ so that we stay in the component connected to the identity (the general case is similar), i.e., $C + S\lambda_P p^1 > 0$. These two restrictions together are equivalent to the one condition

$$C + S\lambda_P p^1 > |S|e^{-\lambda_P p^0}. \tag{9.12}$$

Then we see that according to (9.3),

$$p^{0\prime} = p^0 \triangleleft \theta = p^0 + \frac{1}{\lambda_P} \ln\big((C + S\lambda_P p^1)^2 - S^2 e^{-2\lambda_P p^0})\big), \tag{9.13}$$

$$p^{1\prime} = p^1 \triangleleft \theta = \frac{(C + S\lambda_P p^1)(S + C\lambda_P p^1) - SCe^{-2\lambda_P p^0}}{\lambda_P \big((C + S\lambda_P p^1)^2 - S^2 e^{-2\lambda_P p^0}\big)}, \tag{9.14}$$

$$\theta' = (p^0, p^1) \triangleright \theta = 2\operatorname{arcsinh}\Big(\frac{Se^{-\lambda_P p^0}}{\sqrt{(C + S\lambda_P p^1)^2 - S^2 e^{-2\lambda_P p^0}}}\Big). \tag{9.15}$$

Figure 9.3 shows orbits under $\triangleleft$ in (p^0, p^1) space. One can check from the expressions above that these orbits are lines of constant values of

$$||p||^2_{\lambda_P} = (p^1)^2 e^{\lambda_P p^0} - \frac{2}{\lambda_P^2}\Big(\cosh(\lambda_P p^0) - 1\Big) = e^{\lambda_P p^0}\Big((p^1)^2 - (\frac{1 - e^{-\lambda_P p^0}}{\lambda_P})^2\Big). \tag{9.16}$$

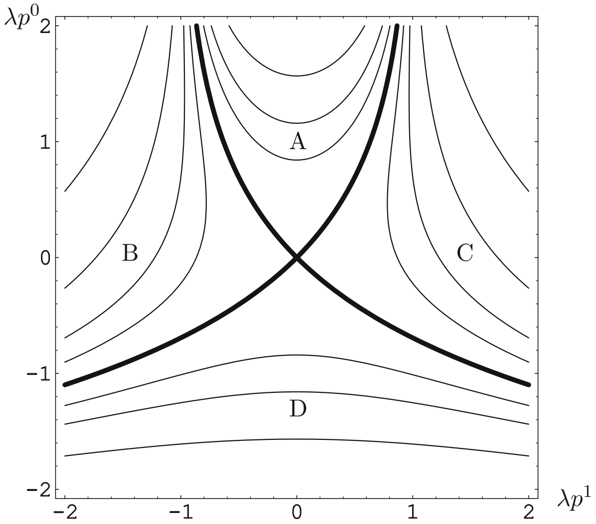

Fig. 9.3 Deformed orbits under $SO_{1,1}$ in the bicrossproduct model momentum group. Increasing θ moves anticlockwise along an orbit in regions A,D and clockwise in regions B,C

This vanishes on the bold 'deformed light cone' separating the regions in Fig. 9.3. The expression is also invariant under inversion in the curved momentum group $\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}$. If we are near to the origin (the identity in the momentum group) then the orbits look at they do classically, and indeed one has

$$(p^0, p^1){\triangleleft}\theta = (p^0, p^1)\begin{pmatrix}\cosh(\theta) & \sinh(\theta)\\ \sinh(\theta) & \cosh(\theta)\end{pmatrix} + O(\lambda_P). \tag{9.17}$$

But for larger values we see from the figure that the physical 'upper mass-shell' orbits in region A are compressed so that $|p^1| < \lambda_P^{-1}$. This remarkable Planck scale bound on the spatial momentum has nothing to do with the Poincaré quantum group, which we have not constructed yet, but rather comes from the nonlinear nature of the factorisation equations. Note also that group inversion takes us from region A to region D, which has a very different character (as in the next proposition). This suggests a potential breakdown of a classical CPT symmetry that normally replaces particles by antiparticles and reverses momentum.

Proposition 9.3

(1) *The actions $\triangleright, \triangleleft$ are defined for all θ if and only if (p^0, p^1) lies in the upper mass shell (region A).*
(2) *For any other (p^0, p^1), there exists a finite boost θ_c that sends $p^0 \to -\infty$, after which $\triangleright$ breaks down.*

(3) *For any θ, there is a critical curve (not intersecting region A) such that the action by (p^0, p^1) approaching it sends $\theta \to \pm\infty$, after which $\triangleleft$ breaks down.*

Proof We analyse the situation for the two cases $S > 0$ and $S < 0$; if $S = 0$ then the condition (9.12) always holds. Considering the first case, to lie in regions A, C means $\lambda_P p^1 + 1 - \mathrm{e}^{-\lambda_P p^0} \geq 0$. Hence

$$C + S(\lambda_P p^1 - \mathrm{e}^{-\lambda_P p^0}) = (C - S) + S(\lambda_P p^1 + 1 - \mathrm{e}^{-\lambda_P p^0}) > 0$$

so the condition holds. But conversely, strictly inside regions B, D means that $\mathrm{e}^{-\lambda_P p^0} - \lambda_P p^1 > 1$ and $C + S(\lambda_P p^1 - \mathrm{e}^{-\lambda_P p^0}) = 0$ has a solution $\theta_c > 0$ according to $\coth(\frac{\theta_c}{2}) = \mathrm{e}^{-\lambda_P p^0} - \lambda_P p^1$. We also note that our assumption $C + S\lambda_P p^1 > 0$ holds here and for all smaller θ. As $\theta \to \theta_c$ from below, the denominator or argument of log in the actions tends to zero and the transformed $p^{0\prime} \to -\infty$. If $S < 0$ then $\lambda_P p^1 + \mathrm{e}^{-\lambda_P p^0} - 1 \leq 0$ in regions A, B means that $C + S(\lambda_P p^1 - \mathrm{e}^{-\lambda_P p^0}) > 0$ and (9.12) holds as before. Conversely, to be strictly inside regions C, D means $\lambda_P p^1 + \mathrm{e}^{-\lambda_P p^0} > 1$ and hence $-\coth\frac{\theta_c}{2} = \lambda_P p^1 + \mathrm{e}^{-\lambda_P p^0}$ has a solution with $\theta_c < 0$, where the denominators or argument of log again $\to 0$ from above as $\theta \to \theta_c$ from above. In particular, this means that the only region which has nonsingular actions for all θ and hence all signs of S is A. For (3), the same expression vanishing in the denominator of the action on θ in (9.15) means that there is a curve in the (p^0, p^1)-space such that $\theta = \theta_c$, the points of which send θ to $\pm\infty$. $\square$

Thus, the fuller story is that any point outside region A is sent to infinite negative p^0 by a finite boost. There is a similar story for any θ and certain momenta outside region A sending θ to an infinite value. In fact the actions $\triangleright, \triangleleft$ break down where the factorisation itself breaks down. Using this data, we have a locally defined covariant system consisting of $X = SL_2(\mathbb{R})$ acting from the left on $C^\infty(\mathbb{R}{\bowtie}\mathbb{R})$ where $SL_2(\mathbb{R})$ acts locally from the right on $\mathbb{R}{\bowtie}\mathbb{R}$ (the classical factorisation model). This is an interesting model with singularities, as we have seen above. Its semidual will be a quantum group acting on $U(\mathbb{R}{\bowtie}\mathbb{R})$ and we turn to this next.

9.2.2 The Flat Bicrossproduct Model and Its Wave Operator

In the above, the p^μ parametrised the momentum group. We now think of them abstractly as functions on it (having a numerical value at each group element). As such they commute and enjoy the coproduct

$$\Delta\begin{pmatrix} \mathrm{e}^{\frac{\lambda_P}{2}p^0} & \lambda_P p^1 \mathrm{e}^{\frac{\lambda_P}{2}p^0} \\ 0 & \mathrm{e}^{-\frac{\lambda_P}{2}p^0} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\frac{\lambda_P}{2}p^0} & \lambda_P p^1 \mathrm{e}^{\frac{\lambda_P}{2}p^0} \\ 0 & \mathrm{e}^{-\frac{\lambda_P}{2}p^0} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{e}^{\frac{\lambda_P}{2}p^0} & \lambda_P p^1 \mathrm{e}^{\frac{\lambda_P}{2}p^0} \\ 0 & \mathrm{e}^{-\frac{\lambda_P}{2}p^0} \end{pmatrix}$$

where matrix multiplication is understood. This gives us

$$[p^0, p^1] = 0, \quad \Delta p^0 = p^0 \otimes 1 + 1 \otimes p^0, \quad \Delta p^1 = p^1 \otimes \mathrm{e}^{-\lambda_P p^0} + 1 \otimes p^1$$
$$S(p^0, p^1) = (-p^0, -p^1 \mathrm{e}^{\lambda_P p^0}) \tag{9.18}$$

as the coordinate Hopf algebra $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}]$ corresponding to our nonabelian momentum group and its group inversion.

We similarly regard (9.13) and (9.14) as $\theta{\triangleright}p^0$ and $\theta{\triangleright}p^1$ for the induced left action of $SO^+_{1,1}$ on $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}]$ and consider θ infinitesimal, i.e., we differentiate by $\frac{\partial}{\partial\theta}|_0$, which is all we need for the algebraic part of the bicrossproduct Hopf algebra (the full operator algebra structure needs the global data). Thus, denoting by N the Lie algebra generator conjugate to θ,

$$N{\triangleright}p^0 = -\mathrm{i}\frac{\partial p^{0\prime}}{\partial\theta}|_0 = -\mathrm{i}p^1, \quad N{\triangleright}p^1 = -\mathrm{i}\frac{\partial p^{1\prime}}{\partial\theta}|_0 = -\frac{\mathrm{i}}{2}\Big(\frac{1-\mathrm{e}^{-2\lambda_P p^0}}{\lambda_P} - \lambda_P (p^1)^2\Big).$$

A cross product by this left action gives the relations

$$[N, p^0] = -\mathrm{i}p^1, \quad [N, p^1] = -\frac{\mathrm{i}}{2}\Big(\frac{1-\mathrm{e}^{-2\lambda_P p^0}}{\lambda_P} - \lambda_P (p^1)^2\Big).$$

Similarly differentiating the action (9.15) *on* θ at $\theta = 0$ gives the action of an element of $\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}$ on N as $(p^0, p^1){\triangleright}N = \mathrm{e}^{-\lambda_P p^0}N$, which we view equivalently as a right coaction Δ_R of the group coordinate algebra in algebraic terms,

$$\Delta_R N = N \otimes \mathrm{e}^{-\lambda_P p^0},$$

which yields the right cross coproduct by (9.11) and corresponding antipode

$$\Delta N = N \otimes \mathrm{e}^{-\lambda_P p^0} + 1 \otimes N, \quad SN = -N\mathrm{e}^{\lambda_P p^0}.$$

Along with (9.18), this completes the structure of the bicrossproduct Poincaré quantum group $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}]{\blacktriangleright\!\!\!\lhd}U(so_{1,1})$. As $\lambda_P \to 0$, we obtain the 2D Poincaré algebra $U(\mathrm{poinc}_{1,1})$ with its usual additive coproduct. Moreover, the deformed norm (9.16) was invariant under θ and is therefore central (a Casimir) in the bicrossproduct.

Next, although the deformed expression for the Casimir suggests a deformed energy-momentum or 'mass-shell' equation, such a claim is meaningless until we establish the quantum spacetime on which the Poincaré quantum group acts and plane waves in it. This is because expressions such as (9.16) depend only on the algebra and can look however one wants depending on which algebra generators we chose to call p^μ. The general theory of bicrossproducts provides the required action on the enveloping algebra $U(\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R})$ of the Lie algebra of the momentum group. We start with a concrete realisation of the latter, namely

$$t = \mathrm{i}\begin{pmatrix} -\frac{\lambda_P}{2} & 0 \\ 0 & \frac{\lambda_P}{2} \end{pmatrix}, \quad x = \mathrm{i}\begin{pmatrix} 0 & \lambda_P \\ 0 & 0 \end{pmatrix}; \quad [x, t] = \mathrm{i}\lambda_P x.$$

This exponentiates to a representation of the group and comparing with our standard matrix parametrised by (p^0, p^1), we see that

$$\langle \mathrm{e}^{\frac{\lambda_P p^0}{2}}, t\rangle = -\mathrm{i}\frac{\lambda_P}{2}, \quad \langle \lambda_P p^1 \mathrm{e}^{\frac{\lambda_P p^0}{2}}, x\rangle = \mathrm{i}\lambda_P$$

as respectively the (1,1)-entry of t and the (1,2)-entry of x. From these and the zero entries, one finds that $\langle p^0, t\rangle = -\mathrm{i}$, $\langle p^1, x\rangle = \mathrm{i}$ and zero otherwise for the Hopf algebra pairing between the generators of $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}]$ and $U(\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R})$. The geometrically natural generators here would be $p_0 = -p^0$ and $p_1 = p^1$ so that p_μ are i times a dual basis to t, x. The right action of $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R}]$ is then given by evaluation against the left output of the coproduct of $U(\mathbb{R}{\bowtie\!\!\!\lhd}\mathbb{R})$, and one finds easily that ${\triangleleft} p^0 = -\mathrm{i}\partial/\partial t$, ${\triangleleft} p^1 = \mathrm{i}\partial/\partial x$ when acting on functions of t, x provided we normal order to keep t to the left. We also have a right action of $SO^+_{1,1}$ on the momentum group which we previously worked out near the identity as (9.17) and which we now see as a Lorentz transformation on the Lie algebra. On the basis elements, this becomes

$$(t, x){\triangleleft}\theta = (t, x)\begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}; \quad t{\triangleleft}N = -\mathrm{i}x, \quad x{\triangleleft}N = -\mathrm{i}t$$

according to dualisation of the action on the p^i, or by more geometric arguments. This action of the Poincaré quantum group then leads naturally to plane waves

$$\phi_{\omega,k} = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx} = \mathrm{e}^{-\mathrm{i}k\mathrm{e}^{\lambda_P\omega}x}\mathrm{e}^{\mathrm{i}\omega t}; \quad \phi_{\omega,k}{\triangleleft}p^0 = \omega\phi_{\omega,k}, \quad \phi_{\omega,k}{\triangleleft}p^1 = k\phi_{\omega,k}.$$

One can extend this to a $*$-algebra with x, t, p^μ, N invariant under $*$ and the Poincaré quantum group a Hopf $*$-algebra acting 'unitarily' in the sense of a right action version of (2.1) in Chap. 2.

There is a similar story for higher-dimensional models, replacing p^1, x by 3-vectors $\vec{p}$ and $\vec{x}$ respectively in the 4D case. There is a local factorisation $SO_{4,1} \approx SO_{3,1}.(\mathbb{R}^3{\bowtie\!\!\!\lhd}\mathbb{R})$ which leads to a Poincaré quantum group $\mathbb{C}[\mathbb{R}^3{\bowtie\!\!\!\lhd}\mathbb{R}]{\blacktriangleright\!\!\lhd} U(so_{3,1})$. We now have commuting translation generators p^μ, rotations M_i and boosts N_i.

$$[p^\mu, p^\nu] = 0, \quad [M_i, M_j] = \mathrm{i}\epsilon_{ij}{}^k M_k, \quad [N_i, N_j] = -\mathrm{i}\epsilon_{ij}{}^k M_k,$$

$$[M_i, N_j] = \mathrm{i}\epsilon_{ij}{}^k N_k, \quad [M_i, p^0] = 0, \quad [M_i, p^j] = \mathrm{i}\epsilon_i{}^j{}_k p^k, \quad [N_i, p^0] = -\mathrm{i}p^i,$$

as usual, and the modified relations and coproduct

$$[N_i, p^j] = -\frac{\mathrm{i}}{2}\delta^i_j\Big(\frac{1-\mathrm{e}^{-2\lambda_P p^0}}{\lambda_P} + \lambda_P \vec{p}^{\,2}\Big) + \mathrm{i}\lambda_P p^i p^j,$$

$$\Delta N_i = N_i \otimes \mathrm{e}^{-\lambda_P p^0} + 1 \otimes N_i + \lambda_P \epsilon_{ijk} M_j \otimes p^k, \quad \Delta p^i = p^i \otimes \mathrm{e}^{-\lambda_P p^0} + 1 \otimes p^i$$

along with the usual additive coproducts on p^0, M_i. The deformed Minkowski norm/Casimir now has the same form as (9.16) with the same picture as in Fig. 9.3 except that now the horizontal axis is any one of the p^i (there is a suppressed rotational symmetry among them). We likewise have a quantum spacetime $[x^i, t] = \mathrm{i}\lambda_P x^i$. Then $\triangleleft p^0 = -\mathrm{i}\partial/\partial t$ and $\triangleleft p^i = \mathrm{i}\partial/\partial x^i$ on normal ordered functions with eigenfunctions the plane waves $\phi_{\omega,\vec{k}} = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}$ and eigenvalues (ω, k^i), and the standard $*$-algebra structures. We also have

$$t\triangleleft N^i = -\mathrm{i}x^i, \quad x^i \triangleleft N^j = -\mathrm{i}\delta_{ij} t, \quad t\triangleleft M_i = 0, \quad x^i \triangleleft M_j = -\mathrm{i}\epsilon_{ijk} x^k$$

for the rest of the action on the quantum spacetime.

To complete the picture, we will justify these 'quantum plane waves' geometrically as diagonalising the partial derivatives for a quantum differential structure. In fact we are in the situation of a 'quantum anomaly' where there is no 4D quantum Poincaré-invariant quantum differential calculus; one must make an extension to a 5D one. We construct this and an induced wave operator using Theorem 8.23 from the general theory in Chap. 8. Here $M = \mathbb{R}^3$ with its Euclidean metric g and we take

$$\tau = x^i \frac{\partial}{\partial x^i}$$

as the conformal Killing vector field (sum over i). This has $\alpha = 1$. We are interested in quantising Minkowski spacetime so we take $\beta = -1$ for the static metric to be the Minkowski one. Then our algebra is $C^\infty(\mathbb{R}^3) \rtimes\!\!\!\triangleleft \mathbb{R}$ with commutation relations $[f, t] = \mathrm{i}\lambda_P x^i \frac{\partial}{\partial x^i} f$ for any smooth f. For plane waves we assume that this extends from polynomials in t to exponentials as above. The resulting differential structure has

$$[\mathrm{d}x^i, x^j] = \mathrm{i}\lambda_P \delta^{ij}\theta', \quad [\mathrm{d}x^i, t] = 0, \quad [\theta', x^i] = 0, \quad [\theta', t] = \mathrm{i}\lambda_P \theta',$$
$$[x^i, \mathrm{d}t] = \mathrm{i}\lambda_P \mathrm{d}x^i, \quad [t, \mathrm{d}t] = \mathrm{i}\lambda_P(\theta' + \mathrm{d}t) \tag{9.19}$$

which is inner with $\theta = \theta' + \mathrm{d}t$ so that $[\ ,\theta] = \mathrm{i}\lambda_P \mathrm{d}$ and has a flat quantum metric

$$g = \sum_i \mathrm{d}x^i \otimes \mathrm{d}x^i - \mathrm{d}t \otimes \mathrm{d}t + \theta \otimes \theta \tag{9.20}$$

which is nondegenerate. It is easy to see that $\nabla \mathrm{d}x^i = \nabla \mathrm{d}t = \nabla\theta' = 0$ and σ the flip on the generators is a quantum Levi-Civita connection with the natural exterior algebra where $\mathrm{d}x^\mu, \theta'$ anticommute.

Proposition 9.4 *The above 5D differential calculus (9.19) on the quantum spacetime $[x^i, t] = \mathrm{i}\lambda_P x^i$ is covariant under the 4D bicrossproduct Poincaré quantum group acting as above from the right along with $\theta'{\triangleleft}p^\mu = 0, \theta'{\triangleleft}M_i = 0, \theta'{\triangleleft}N^i = \mathrm{i}\mathrm{d}x^i$. The flat quantum metric is invariant.*

Proof We show that the action extends, commuting with d, so for example $(\mathrm{d}x^i){\triangleleft}N^j = \mathrm{d}(x^i{\triangleleft}N^j) = -\mathrm{i}\delta_{ij}\mathrm{d}t$, and in a way that is compatible with the relations (9.19), which also forces the action on θ' as stated. For example, using the coproduct to determine the action on products

$$
\begin{aligned}
&((\mathrm{d}x^i)t - t(\mathrm{d}x^i)){\triangleleft}N^k\\
&\quad= (\mathrm{d}x^i{\triangleleft}N^k)(t{\triangleleft}\mathrm{e}^{-\lambda_P p^0}) + (\mathrm{d}x^i)(t{\triangleleft}N^k) + \lambda_P\epsilon_{kmn}(\mathrm{d}x^i{\triangleleft}M_m)(t{\triangleleft}p^n)\\
&\qquad - (t{\triangleleft}N^k)(\mathrm{d}x^i{\triangleleft}\mathrm{e}^{-\lambda_P p^0}) - t(\mathrm{d}x^i{\triangleleft}N^k) - \lambda_P\epsilon_{kmn}(t{\triangleleft}M_m)(\mathrm{d}x^i{\triangleleft}p^n)\\
&\quad= -\mathrm{i}\delta_{ik}(\mathrm{d}t)(t+\mathrm{i}\lambda_P) - \mathrm{i}(\mathrm{d}x^i)x^k + \mathrm{i}x^k\mathrm{d}x^i + \mathrm{i}\delta_{ik}t\mathrm{d}t\\
&\quad= \mathrm{i}\delta_{ik}[t, \mathrm{d}t] + \lambda_P\delta_{ik}\mathrm{d}t - \mathrm{i}[\mathrm{d}x^i, x^j] = 0
\end{aligned}
$$

which is compatible with the relation $[\mathrm{d}x^i, t] = 0$. We used some of the other relations for the last equality. Similarly,

$$
\begin{aligned}
&((\mathrm{d}x^i)x^j - x^j(\mathrm{d}x^i)){\triangleleft}N^k\\
&= (\mathrm{d}x^i{\triangleleft}N^k)(x^j{\triangleleft}\mathrm{e}^{-\lambda_P p^0}) + (\mathrm{d}x^i)(x^j{\triangleleft}N^k) + \lambda_P\epsilon_{kmn}(\mathrm{d}x^i{\triangleleft}M_m)(x^j{\triangleleft}p^n)\\
&\quad - (x^j{\triangleleft}N^k)(\mathrm{d}x^i{\triangleleft}\mathrm{e}^{-\lambda_P p^0}) - x^j(\mathrm{d}x^i{\triangleleft}N^k) - \lambda_P\epsilon_{kmn}(x^j{\triangleleft}M_m)(\mathrm{d}x^i{\triangleleft}p^n)\\
&= -\mathrm{i}\delta_{ik}[\mathrm{d}t, x^j] - \mathrm{i}\delta_{jk}[\mathrm{d}x^i, t] + \lambda_P\epsilon_{kmj}\epsilon_{imp}\mathrm{d}x^p = -\lambda_P\mathrm{d}x^k\delta_{ij} = \mathrm{i}\lambda_P\delta_{ij}\theta'{\triangleleft}N^k
\end{aligned}
$$

which is compatible with the relation $[\mathrm{d}x^i, x^j] = \mathrm{i}\lambda_P\delta_{ij}\theta'$. In this way, one can check that the set of relations are preserved by the action of N^k, and more simply by the action of the other generators. Invariance of the metric is then immediate. □

We can also solve for the auxiliary functions μ, ν in Proposition 8.23,

$$\mu = \nu = -\frac{1}{2}$$

giving the induced wave operator in the time direction in the theory there as

$$2\Delta_0 f(t) = \frac{1}{\lambda_P^2}(f(t+\mathrm{i}\lambda_P) + f(t-\mathrm{i}\lambda_P) - 2f(t)) = -(\partial_0^2 f)(t+\mathrm{i}\lambda_P)$$

on $f(t)$. Combining with the spatial Laplacian, the full quantum wave operator is

$$\Box\psi = \Big(((\frac{\partial}{\partial x^i})^2 - \partial_0^2)\psi\Big)(t + \mathrm{i}\lambda_P); \quad \partial_0 f(t) = \frac{f(t) - f(t - \mathrm{i}\lambda_P)}{\mathrm{i}\lambda_P} \tag{9.21}$$

on normal ordered functions $\psi = \sum_n \psi_n(x)t^n$ with t to the right. From the point of view purely of the differential calculus, the more obvious quantum plane waves here are right-time ordered waves

$$\psi_{\vec{k},\omega} = \mathrm{e}^{-\mathrm{i}\vec{k}\cdot x}\mathrm{e}^{\mathrm{i}\omega t}; \quad \partial_i \psi_{\vec{k},\omega} = -\mathrm{i}k_i \psi_{\vec{k},\omega}, \quad \partial_0 \psi_{\vec{k},\omega} = \mathrm{i}\frac{\mathrm{e}^{\omega\lambda_P} - 1}{\lambda_P}\psi_{\vec{k},\omega},$$

$$\Box\psi_{\vec{k},\omega} = \mathrm{e}^{-\lambda_P\omega}\Big(-\vec{k}^2 + \Big(\frac{\mathrm{e}^{\lambda_P\omega} - 1}{\lambda_P}\Big)^2\Big)\psi_{\vec{k},\omega}, \tag{9.22}$$

while on our original left-time ordered plane waves we have, using (9.16)

$$\partial_i \phi_{\omega,\vec{k}} = -\mathrm{i}k_i \mathrm{e}^{\lambda_P\omega}\phi_{\omega,\vec{k}}, \quad \partial_0 \phi_{\omega,\vec{k}} = \mathrm{i}\frac{\mathrm{e}^{\omega\lambda_P} - 1}{\lambda_P}\phi_{\omega,\vec{k}},$$

$$\Box\phi_{\omega,\vec{k}} = \mathrm{e}^{\lambda_P\omega}\Big(-\vec{k}^2 + \Big(\frac{\mathrm{e}^{-\lambda_P\omega} - 1}{\lambda_P}\Big)^2\Big)\phi_{\omega,\vec{k}} = -\phi_{\omega,\vec{k}}\triangleleft||p||^2_{\lambda_P}. \tag{9.23}$$

Since either of these form a basis, we see from (9.23) that the induced wave operator is indeed the action of the Casimir element of $\mathbb{C}[\mathbb{R}^3{>\!\!\!\triangleleft}\mathbb{R}]$ as in (9.16) when viewed in the momentum sector of the quantum Poincaré group.

Finally, for the analysis of an experiment, we might assume the identification of left-ordered quantum plane waves $\phi_{\omega,\vec{k}}$ with classical waves that a detector might register. In that case we might plausibly argue that the speed for such waves can be computed as usual by $|\frac{\partial\omega}{\partial k}| = \mathrm{e}^{\lambda_P\omega}$ in units where 1 is the usual speed of light and $\Box = 0$ in (9.23) makes ω a function of k^i. If we identified physical waves with the right-ordered $\psi_{\vec{k},\omega}$, these have eigenvalue the same as $\phi_{-\omega,\vec{k}}$ and we would have $|\frac{\partial\omega}{\partial k}| = \mathrm{e}^{-\lambda_P\omega}$. Either way, the prediction is that the speed of light depends on energy. What is remarkable is that even if $\lambda_P \sim 10^{-44}s$ (the Planck time scale), this prediction could in principle be tested, for example using γ-ray bursts. These are known in some cases to travel cosmological distances before arriving here, and have a spread of energies from 0.1-100 MeV. According to the above, the relative time delay Δ_T on travelling distance L for energies ω, $\omega + \Delta_\omega$ is

$$\Delta_T \sim \lambda_P \Delta_\omega \frac{L}{c} \sim 10^{-44}\mathrm{s} \times 100\mathrm{MeV} \times 10^{10}\mathrm{y} \sim 1 \text{ ms}$$

which is in principle observable by statistical analysis of a large number of bursts correlated with distance (determined for example by using the Hubble telescope to lock in on the host galaxy of each burst). Although the above is only one of a class of predictions, it is striking that Planck scale effects are now in principle within experimental reach. We also see that the precise prediction depends on our ordering assumption for which of the waves is 'physical' and that positive and negative frequencies behave differently. Whichever wave we use, only one of $\omega < 0$ or $\omega > 0$ will land us in the better behaved region A of Fig. 9.3.

Remark 9.5 From the point of view of physics, we are not quite done with our analysis as one usually prefers a left action of the Poincaré algebra, not a right action. One way is to use the B-model versions of Lemmas 9.1 and 9.2. Alternatively, albeit less canonically, one can use the antipode of a quantum group to convert right actions to left ones. Here, if a Hopf algebra H right acts covariantly on an algebra A then $h{\triangleright}a = a{\triangleleft}Sh$ is a left action of H covariantly on A^{op}, *the opposite algebra* with reversed product $a \cdot_{op} b = ba$. We can apply this in our case to A the quantum spacetime algebra. We also redefine $t \mapsto -t$ so that the opposite algebra keeps the same commutation relations $[x^i, t] = \mathrm{i}\lambda_P x^i$ and we also redefine $x^i \mapsto -x^i$. On our 2D model, one then obtains again

$$p^0{\triangleright}\phi_{\omega,k} = \omega\phi_{\omega,k}, \quad p^1{\triangleright}\phi_{\omega,k} = k\phi_{\omega,k}$$

or $p^0{\triangleright} = -\mathrm{i}\partial/\partial t$ and $p^1{\triangleright} = \mathrm{i}\partial/\partial x$ on left-time ordered functions when working with the new generators. For example,

$$\begin{aligned} p^1{\triangleright}(\mathrm{e}^{\mathrm{i}\omega t^{new}} \cdot_{op} \mathrm{e}^{-\mathrm{i}kx^{new}}) &= -(\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}\omega t}){\triangleleft}p^1\mathrm{e}^{\lambda_P p^0} = -(\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}k\mathrm{e}^{\omega\lambda_P}x}){\triangleleft}p^1\mathrm{e}^{\lambda_P p^0} \\ &= k\mathrm{e}^{\omega\lambda_P}(\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}k\mathrm{e}^{\omega\lambda_P}x}){\triangleleft}\mathrm{e}^{\lambda_P p^0} = k\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}k\mathrm{e}^{\omega\lambda_P}x} = k\mathrm{e}^{\mathrm{i}\omega t^{new}} \cdot_{op} \mathrm{e}^{-\mathrm{i}kx^{new}} \end{aligned}$$

computed in terms of our previous generators t, x and our previous right actions. ⋄

9.2.3 *The Spin Model and 3D Quantum Gravity*

Here we look at a model which is in the bicrossproduct family but in a relatively trivial way in that ${\triangleleft}$ is trivial and $X = SU_2{\bowtie}SU_2$ is an ordinary cross product by the remaining left action ${\triangleright} = \mathrm{Ad}$ given by group conjugation. The B-model version of Lemma 9.1 tells us that this group acts on another copy of SU_2 by $(u, s){\triangleright}t = ustu^{-1}$ and hence on $C^\infty(SU_2)$ as our classical but curved factorisation model. In fact, X is isomorphic to $SU_2 \times SU_2$ so that the classical model here is equivalent to a classical particle on SU_2 with commuting left and right translation, but we are thinking of it differently as a curved version of the isometry group $\mathbb{R}^3{\rtimes}SU_2$ acting on $C^\infty(\mathbb{R}^3)$.

On the other side of quantum Born reciprocity, we have the semidualisation in the B-model version of Lemma 9.2 with Poincaré quantum group $U(su_2){\triangleright\!\!\!<}\mathbb{C}[SU_2]$ acting from the left on the quantum spacetime $U(su_2)$ with relations

$$[x^\mu, x^\nu] = 2\mathrm{i}\lambda_P\epsilon^{\mu\nu}{}_\rho x^\rho. \tag{9.24}$$

For the rest of this section, we will write the ϵ tensor with all indices down since this makes no difference as we will be using a Euclidean metric. This is a well-known algebra which we studied already in Chap. 1 and now we see that it also has a natural Poincaré quantum group. To describe the latter in more detail, we take $\mathbb{C}[SU_2]$ with its matrix of commuting generators $t^\alpha{}_\beta$ dually paired with generators J_i of $U(su_2)$ by $\langle t^\alpha{}_\beta, J_i\rangle = \frac{1}{2}\sigma_i{}^\alpha{}_\beta$, where the σ_i are the standard Pauli matrices. We have a right action of $U(su_2)$ by evaluation against the left adjoint coaction $\mathrm{Ad}_L a = a_{(1)}Sa_{(3)} \otimes a_{(2)}$, which on the generators works out as

$$t^\alpha{}_\beta \triangleleft J_i = t^\gamma{}_\delta \langle t^\alpha{}_\gamma S t^\delta{}_\beta, J_i\rangle = \frac{1}{2}\left(\sigma_i{}^\alpha{}_\gamma t^\gamma{}_\beta - t^\alpha{}_\gamma \sigma_i{}^\gamma{}_\beta\right)$$

and gives right cross relations and tensor product coproduct

$$[t^\alpha{}_\beta, J_i] = \frac{1}{2}(\sigma_i{}^\alpha{}_\gamma t^\gamma{}_\beta - t^\alpha{}_\gamma\sigma_i{}^\gamma{}_\beta), \quad [J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k$$
$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i, \quad \Delta t^\alpha{}_\beta = t^\alpha{}_\gamma \otimes t^\gamma{}_\beta.$$

We now change variables to P_0, P_i, $i = 1, 2, 3$ defined via Pauli matrices by

$$t^\alpha{}_\beta = P_0\delta^\alpha{}_\beta + \mathrm{i}\lambda_P P_i\sigma_i{}^\alpha{}_\beta = \begin{pmatrix} P_0 + \mathrm{i}\lambda_P P_3 & \mathrm{i}\lambda_P(P_1 - \mathrm{i}P_2) \\ \mathrm{i}\lambda_P(P_1 + \mathrm{i}P_2) & P_0 - \mathrm{i}\lambda_P P_3 \end{pmatrix}.$$

The structure in terms of the new generators is with P_0 central and

$$P_0^2 + \lambda_P^2\vec{P}^2 = 1, \quad [P_i, P_j] = 0, \quad [P_i, J_j] = \mathrm{i}\epsilon_{ijk}P_k,$$
$$\Delta P_0 = P_0 \otimes P_0 - \lambda_P^2 P_i \otimes P_i, \quad \Delta P_i = P_i \otimes P_0 + P_0 \otimes P_i - \lambda_P\epsilon_{ijk}P_j \otimes P_k,$$

for $i = 1, 2, 3$. The $\det(t) = 1$ relation appears now as the 3-sphere relation for $SU_2 \subset \mathbb{R}^4$ and $\vec{P} = (P_1, P_2, P_3)$ is a local coordinate for a region of SU_2 containing the group identity, valid for $|\vec{P}| \le 1/\lambda_P$ and $P_0 = \sqrt{1 - \lambda_P^2\vec{P}^2}$. There is another parametrisation covering the lower half with $P_0 \le 0$. In either case, we see that SU_2 as momentum space for this model is a curved version of $\mathbb{R}^3$ with the latter recovered in the limit $\lambda_P \to 0$. Note that the two regions above are not open sets and one should really use their interior as patches along with a third patch around the equator, but for most purposes we can proceed by continuity from either region. The action of the quantum group on the x^μ is

$$J_i \triangleright x^\mu = \mathrm{i}\epsilon_{i\mu\nu}x^\nu, \quad P_i \triangleright x^\mu = -\mathrm{i}\delta_{i\mu}, \quad P_0 \triangleright x^\mu = x^\mu,$$

where the $t^\alpha{}_\beta \triangleright x^\mu = (\mathrm{id} \otimes t^\alpha{}_\beta)(x^\mu \otimes 1 + 1 \otimes x^\mu) = \delta^\alpha{}_\beta x^\mu + \lambda_P\sigma_\mu{}^\alpha{}_\beta$ and where we use a similar pairing as for the pairing with J_i but with an extra $2\lambda_P$ factor due to our normalisation in the commutation relations. This then translates into the action of the P_0, P_i above. Equivalently, the pairing translates to $\langle P_i, x^\mu\rangle = -\mathrm{i}\delta_{i\mu}$, from which it is also clear what the adjoint action of J_i on the x^μ is. The $\vec{P}$ are a good coordinate system for each hemisphere of the momentum group SU_2 but they are not the physical momentum, depending on what we mean by 'plane waves'. For this we prefer new coordinates p_i valid for the open set $|\vec{p}| < \pi/\lambda_P$ and related to the P_i where they overlap by

$$P_i = p_i \frac{\sin(\lambda_P |\vec{p}|)}{\lambda_P |\vec{p}|}, \quad P_0 = \cos(\lambda_P |\vec{p}|). \tag{9.25}$$

Now $e^{i\lambda_P \vec{p}\cdot\vec{\sigma}} \in SU_2$ is the corresponding group element. In principle, we still need a patch around the south pole $-1 \in SU_2$ but we can usually deal with this by continuity, with all directions extended to $|\vec{p}| = \pi/\lambda_P$ leading to this point.

Next, we have already studied the natural 4D calculus on this algebra in Example 1.45 as well as its quantum Riemannian geometry with metric and a flat quantum Levi-Civita connection in Example 8.15. There is an extra direction θ forced by a quantum anomaly for differentials and we recall the calculus and flat metric, but now with upper indices and a differently normalised θ,

$$[\mathrm{d}x^\mu, x^\nu] = i\lambda_P \epsilon_{\mu\nu\rho}\mathrm{d}x^\rho + i\lambda_P \delta_{\mu\nu}\theta, \quad [x^\mu, \theta] = i\lambda_P \mathrm{d}x^\mu,$$

$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu + i\frac{\lambda_P}{2}(\Delta f)\theta, \quad \Delta = \frac{2}{\lambda_P^2}\left(\sqrt{1+\lambda_P^2\vec{\partial}^2} - 1\right),$$

$$g = \mathrm{d}x^\mu \otimes \mathrm{d}x^\mu + \theta \otimes \theta.$$

This is an example of the wave operator approach and similar to the bicrossproduct model, except that θ is not central but behaves more like $\mathrm{d}t$ there. The expression for the Laplacian here is formal but can be expanded to apply on polynomials and as such obtained inductively.

Proposition 9.6 *The 4D calculus $\Omega(U(su_2))$ in Example 1.45 is covariant under $U(su_2) \triangleright\!\!\!< \mathbb{C}[SU_2]$ acting from the left, where $P_0 \triangleright \theta = \theta$ and $P_i \triangleright \theta = J_i \triangleright \theta = 0$ and $P_i \triangleright = -i\partial_i$, $P_0 \triangleright = \mathrm{id} + \frac{1}{2}\lambda_P^2 \Delta$ on functions of the x^i. Moreover, the above flat quantum metric is invariant.*

Proof It is straightforward to check that the relations are stable under the action. Thus, dropping $P_k \triangleright \mathrm{d}x^\mu = \mathrm{d}(P_k \triangleright x^\mu) = -i\delta_{k\mu}\mathrm{d}1 = 0$, we have

$$\begin{aligned} P_k \triangleright ((\mathrm{d}x^\mu)x^\nu - x^\nu \mathrm{d}x^\mu) &= (\mathrm{d}x^\mu) P_k \triangleright x^\nu - (P_k \triangleright x^\nu)\mathrm{d}x^\mu = 0 \\ &= P_k \triangleright (i\lambda_P \epsilon_{\mu\nu\rho}\mathrm{d}x^\rho + i\lambda_P \delta_{\mu\nu}\theta) \end{aligned}$$

provided $P_k \triangleright \theta = 0$. In this case $P_0 \triangleright \theta = \theta$ and

$$P_k \triangleright (x^\mu \theta - \theta x^\mu) = (P_k \triangleright x^\mu)\theta - \theta P_k \triangleright x^\mu = 0 = P_k \triangleright (i\lambda_P \mathrm{d}x^\mu).$$

The J_k act as classically by infinitesimal rotations and the commutation relations are clearly rotationally invariant (as one can check directly if in doubt). That the P_i

act as shown on the generator is clear. On products,

$$\begin{aligned}
\mathrm{d}(f(x)x^\nu) &= \partial_\mu(fx^\nu)\mathrm{d}x^\mu + \frac{1}{2}\mathrm{i}\lambda_P\Delta(fx^\nu)\theta \\
&= ((\partial_\mu f)\mathrm{d}x^\mu + \frac{1}{2}\mathrm{i}\lambda_P(\Delta f)\theta)x^\nu + f\mathrm{d}x^\nu \\
&= (\partial_\mu f)x^\nu\mathrm{d}x^\mu + \mathrm{i}\lambda_P(\partial_\mu f)(\epsilon_{\mu\nu\rho}\mathrm{d}x^\rho + \delta_{\mu\nu}\theta) \\
&\quad + \frac{1}{2}\mathrm{i}\lambda_P(\Delta f)x^\nu\theta + \frac{1}{2}\lambda_P^2(\Delta f)\mathrm{d}x^\nu + f\mathrm{d}x^\nu,
\end{aligned}$$

from which we conclude that

$$\partial_\mu(f(x)x^\nu) = (\partial_\mu f)x^\nu + \mathrm{i}\lambda_P(\partial_\rho f)\epsilon_{\rho\nu\mu} + (f + \frac{1}{2}\lambda_P^2\Delta f)\delta_{\mu\nu}, \tag{9.26}$$

which is the same derivation rule as $\mathrm{i}P_i\triangleright$ for the stated coproduct since

$$P_i\triangleright(f(x)x^\nu) = (P_i\triangleright f)x^\nu + (P_0\triangleright f)(-\mathrm{i}\delta_{i\nu}) - \lambda_P\epsilon_{im\mu}(P_m\triangleright f)(-\mathrm{i}\delta_{\mu\nu}).$$

We also have

$$\Delta(f(x)x^\nu) = (\Delta f)x^\nu + 2\partial_\nu f, \tag{9.27}$$

which has the same derivation rule as $(P_0 - 1)2/\lambda_P^2$ for the stated coproduct. Since the $P_0\triangleright$, $P_i\triangleright$ agree with the stated operators on the generators and extend to products in the same way, these formulae hold in general. □

Motivated by both the Lie algebra coordinate system and the differential calculus, we now suppose that the spacetime algebra can be extended to include exponentials and define plane waves $\psi_{\vec{k}} = e^{\mathrm{i}\vec{k}\cdot\vec{x}}$. Unlike §9.2.2, we do not have any obvious 'normal ordering' by which to plausibly identify 'physical' plane waves but our choice now at least has the merit of preserving rotational symmetry. From the modified Leibniz rules (9.26)–(9.27), one can show that

$$\partial_i\psi_{\vec{k}} = \mathrm{i}k_i\frac{\sin(\lambda_P|\vec{k}|)}{\lambda_P|\vec{k}|}\psi_{\vec{k}}, \quad \Delta\psi_{\vec{k}} = \frac{2}{\lambda_P^2}(\cos(\lambda_P|\vec{k}|) - 1)\psi_{\vec{k}}$$

for the Laplacian in Example 1.45 at least at some formal level with $|\vec{k}| < \pi/2\lambda_P$. It then follows that

$$p_i\triangleright\psi_{\vec{k}} = k_i\psi_{\vec{k}},$$

which motivates these p_i coordinates as momentum.

For further physical insight, we can look at the quantum plane waves $\psi_{\vec{k}}$ in an operator representation of the quantum spacetime algebra. To do this we let $V^{(j)}$ be the vector space which carries a unitary irreducible representation of spin $j \in \frac{1}{2}\mathbb{Z}_+$, generated by states $| \, j, m\rangle$, with $m = -j, \dots, j$ such that

$$x^{\pm} \mid j, m\rangle = \lambda_P \sqrt{(j \mp m)(j \pm m + 1)} \mid j, m \pm 1\rangle, \quad x^3 \mid j, m\rangle = 2\lambda_P m \mid j, m\rangle,$$

where $x^{\pm} = (x^1 \pm \mathrm{i}x^2)/2$. Next, for each fixed spin j representation we look for normalised states $|j, \theta, \varphi\rangle$ parametrised by $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$, such that

$$\langle j, \theta, \varphi \mid x^1 | j, \theta, \varphi\rangle = r \sin\theta \cos\varphi,$$
$$\langle j, \theta, \varphi \mid x^2 | j, \theta, \varphi\rangle = r \sin\theta \sin\varphi,$$
$$\langle j, \theta, \varphi \mid x^3 | j, \theta, \varphi\rangle = r \cos\theta,$$

where r is some constant (independent of θ, φ) which we do not fix. Rather, in the space of such states and possible $r \ge 0$, we seek to minimise the normalised variance

$$\delta = \frac{\langle x \cdot x\rangle - \langle x\rangle \cdot \langle x\rangle}{\langle x\rangle \cdot \langle x\rangle},$$

where $\langle\ \rangle = \langle j, \theta, \varphi| \ |j, \theta, \varphi\rangle$ is the expectation value in our state and we regard $\langle x^\mu\rangle$ as a classical vector in the dot product. Thus we seek states which are 'closest to classical'. This is a constrained problem and leads to

$$|j, \theta, \varphi\rangle = \sum_{k=1}^{2j+1} 2^{-j} \sqrt{\binom{2j}{k-1}} (1+\cos\theta)^{\frac{j-k+1}{2}} (1-\cos\theta)^{\frac{k-1}{2}} \mathrm{e}^{\mathrm{i}(k-1)\varphi} \mid j, j-k+1\rangle$$

obeying $\langle j, \theta, \varphi | j, \theta, \varphi\rangle = 1$ and the above properties with

$$r = \sqrt{\langle x\rangle \cdot \langle x\rangle} = \lambda_P j, \quad \delta = j^{-1}. \tag{9.28}$$

We see that in these states the 'true radius' $|\langle x\rangle|$ is $\lambda_P j$. The square root of the Casimir does not give this true radius since it also contains the uncertainty expressed in the variance of the position operators, but the error of order δ vanishes as $j \to \infty$. Thus the larger the representation, the more the geometry resembles the classical case. One also has

$$|\langle j, \theta, \varphi | j, \theta', \varphi'\rangle|^2 = \Big(\frac{1}{2}(1 + \cos(\mathrm{angle}(\theta, \varphi | \theta', \varphi')))\Big)^{2j},$$

where the angle is the classical angle between vectors in directions (θ, φ) and (θ', φ') in polar coordinates. In such a coherent state, our quantum plane waves have an expectation value which we can think of as a classical 'shadow' in polar coordinates. For example,

$$\langle\frac{1}{2},\theta,\varphi|e^{i\vec{k}\cdot x}|\frac{1}{2},\theta,\varphi\rangle=\cos(\lambda_P|\vec{k}|)+i\frac{\vec{k}\cdot\langle x\rangle}{\lambda_P|\vec{k}|}\sin(\lambda_P|\vec{k}|),$$

which should be compared with the value of the classical wave $e^{i\vec{k}\cdot\langle x\rangle}$. The approximation gets better as $j\to\infty$ and $\lambda_P\to 0$ with fixed radius $r=\lambda_P j$.

We envisage many applications of the above model due to the importance of angular momentum and spin in many areas of physics. And we have already remarked in Example 1.45 that if we think of x^i as 'space' then the extra direction induced by $\theta=dt$ suggests an 'algebraic origin of time'. We conclude now with a very brief flavour of how the same algebra arises in 3D quantum gravity with point sources now as 'spacetime', with λ_P of order the Planck scale. For this we should take a real form closer to $so_{2,1}$ but for convenience we stick to the simpler su_2 case, i.e., we discuss so-called 'Euclideanised' quantum gravity.

The story here starts with the Chern–Simons action

$$\mathcal{S}=\int_{\Sigma\times\mathbb{R}}B(\alpha\wedge d\alpha+\frac{1}{3}\alpha\wedge[\alpha,\alpha])$$

for a gauge field α valued in a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant symmetric bilinear form B applied to the two sides of the wedge product. For the basic theory, we take $\mathfrak{g}=so_3{\triangleright\!\!<}\mathbb{R}^3$ and use the generalised framing formalism of §5.6 with $G=SU_2{\triangleright\!\!<}\mathbb{R}^3$ as the double cover of the affine frame transformations of a 3-manifold of the form $\Sigma\times\mathbb{R}$ as shown. Limiting ourselves here to the local picture, we can write the connection as $\alpha=-i(\omega^i J_i+e^i p_i)$ where

$$[J_i,J_j]=i\epsilon_{ij}{}^k J_k,\quad [p_i,J_j]=i\epsilon_{ij}{}^k p_k,$$
$$[p_i,p_j]=0,\quad B(J_i,p_j)=B(p_i,J_j)=-\delta_{ij}$$

and zero for other components of B. The ω^i, e^i are locally 1-forms and can be viewed respectively as coefficients of a spin connection ω on the universal cover of the usual frame bundle and of a local dreibein parallelisation $\{e^i\}$. In these terms the action becomes, up to a boundary term $\int d(e^i\wedge\omega^i)$ which we discard,

$$\mathcal{S}=2\int_{\Sigma\times\mathbb{R}}\left(e^i\wedge d\omega^i+\frac{1}{2}\epsilon_{ijk}e^i\wedge\omega^j\wedge\omega^k\right)=2\int_{\Sigma\times\mathbb{R}}e^i\wedge R^i,$$

where $R_\omega=d\omega+\omega\wedge\omega=-iR^iJ_i$ is the curvature of the spin connection part of α. Up to the factor 2, we obtain the Einstein–Cartan action in 3D. The equations of motion from varying the spin connection gives

$$de^i+\epsilon_{ijk}\omega^j\wedge e^k=0$$

or $D_\omega e=0$, which we saw in §5.6 is equivalent to zero torsion for the induced connection, i.e., forces us to the Levi-Civita connection for the metric $g=\sum_i e^i\otimes$

e^i. Assuming the dreibein is invertible, we can solve for ω and insert this back into the action to eventually obtain the Einstein–Hilbert action. In the absence of sources, the equations of motion for this imply zero curvature (the gravitational field in 3D is not dynamical), which is also clear directly from the Einstein–Cartan form. From the Chern–Simons point of view, the equations of motion locally are zero curvature of α or $d\alpha + \alpha \wedge \alpha = 0$ and give the torsion and zero curvature of ω when the $\mathfrak{g}$ value is decomposed in terms of su_2 and $\mathbb{R}^3$. In summary, the basic Chern–Simons theory in this context describes a metric-compatible torsion-free connection on $\Omega^1(\Sigma \times \mathbb{R})$ governed by Einstein's equation in 3D without cosmological constant and with zero curvature in the absence of sources.

To account for point sources, we take the above but delete a finite number n of marked points from Σ. The idea is that classically these gravitational sources couple through Einstein's equations to produce curvature singularities at these points, but away from these we still have zero curvature. In this case the connection is determined up to gauge transformations by the holonomies and these in turn are determined by defects at the marked points and by the topology of Σ, which we take to be genus γ. Without going into all the details, the defect data at each puncture i, which physically amounts to a mass and spin, can be re-expressed as an element M_i of a conjugacy class C_i of G, for the holonomy around the puncture, while the topology is expressed as holonomies $A_1, \cdots, A_\gamma, B_1, \cdots, B_\gamma$ respectively around and through handles in the surface. All the holonomies are defined with respect to some arbitrary base point, hence defined up to conjugation. The connection α is then largely determined by an extended phase space

$$\tilde{P} = G^{2\gamma} \times \prod C_i$$

while the actual phase space is a subset of this modulo global conjugation, namely

$$P = \left\{(A_\gamma, B_\gamma, \cdots A_1, B_1, M_i) \in \tilde{P} \;\middle|\; [A_\gamma, B_\gamma]^{-1} \cdots [A_1, B_1]^{-1} \prod M_i = 1\right\}/G,$$

where G acts by conjugation. The reader may wonder where in this moduli space is the location of our n marked points at any given time. The answer is that the physics is diffeomorphism invariant, so to a large extent the particle locations themselves are not significant and not directly visible. Nevertheless, there is still a kind of 'model spacetime' for the physical particle associated to each puncture, which we will be interested in. Finally, moduli spaces of solutions of equations of motion generally acquire induced Poisson structures, and in the present case this can be written in terms of the product of Poisson–Lie group structures on G and on the conjugacy classes C_i. It is beyond our scope to go here into the theory of Poisson–Lie groups; suffice it to say they quantise to quantum groups.

Indeed, quantisation of a Chern–Simons theory generally results in a topological quantum field theory and this in turn is controlled largely by representations of a quantum group deforming the enveloping algebra of the gauge group. Here this turns out to be our above quantum group $H = U(su_2) \bowtie \mathbb{C}[SU_2]$, which deforms the enveloping algebra of $\mathfrak{g} = su_2 \triangleright\!\!< \mathbb{R}^3$. For the physical states, we mainly need to construct the invariant subspace in the tensor product of representations of many copies of H in line with the above structure of the phase space P. If we focus on just one copy related to a single point source then we need a representation of $U(su_2) \triangleright\!\!< \mathbb{C}[SU_2]$ associated to a particle of a given mass and spin, which one can do on the quantum spacetime $U(su_2)$ with the above as its Poincaré quantum group. This sketches how the spin model quantum spacetime ends up as a local part of 3D quantum gravity with point sources.

All of the above can be repeated with a cosmological constant term in the gravitational action. We again can cast this in Einstein–Cartan form as a Chern–Simons theory but for a modified bilinear form B. This time the quantisation is controlled by the quantum group $H = U_q(su_2) \bowtie \mathbb{C}_q[SU_2]^{op}$ (the quantum double of $U_q(su_2)$) now acting on $U_q(su_2)$ as quantum spacetime. This is a factorisation of Hopf algebras and semidualisation gives a quantum-Born reciprocal model $U_q(su_2)^{cop} \blacktriangleright\!\!\triangleleft U_q(su_2)$ which q-deforms the enveloping algebra of the Lie algebra of $SU_2 \bowtie SU_2$ that we began the section with, acting on a version of $\mathbb{C}_q[SU_2]$ (in fact this quantum group is isomorphic to a tensor product, acting by left and right translations). In §9.6 we will mention the notion of module-algebra twists where two covariant systems are 'categorically equivalent' up to an algebraic cocycle. The above two systems related by quantum Born reciprocity are equivalent in this sense at some algebraic level. It turns out in the limit that $U(su_2) \triangleright\!\!< \mathbb{C}[SU_2]$ acting on $U(su_2)$ as above is then twist equivalent to the 3D version of the bicrossproduct model $\mathbb{C}[\mathbb{R}^2 \rtimes \mathbb{R}] \blacktriangleright\!\!\triangleleft U(su_2)$ acting on $U(\mathbb{R}^2 \rtimes \mathbb{R})$ as in §9.2.2.

9.3 The Quantum Black-Hole Wave Operator

We continue in the wave operator approach of §8.3 but now applied to (M, g) a general spherically symmetric Riemannian manifold so as to obtain a quantum wave operator on $A = C^\infty(M) \rtimes \mathbb{R}$ quantising a static spacetime $M \times \mathbb{R}$ with (M, g) as spatial part. This will include the Schwarzschild black-hole as an example. The first subsection merely covers ordinary classical spherically symmetric metrics and associated static spacetimes. However, we do so as an illustration of our more algebraic way of doing Riemannian geometry, both in terms of the connection formalism and in terms of how we handle polar coordinates.

9.3.1 Algebraic Methods and Polar Coordinates

By (M, g) spherically symmetric we mean that there are coordinates where the metric appears in the form

$$g = h(r)^2 \mathrm{d}r + r^2 \mathrm{d}\Omega^2 = h(r)^2 \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}z^i \otimes \mathrm{d}z^i,$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2$ in standard polar coordinates with angles θ, φ and $r = \sqrt{x^i x^i}$, and we sum over repeated indices unless indicated otherwise. We assume that the manifold looks locally like $\mathbb{R}^3$, possibly with some points deleted. The second expression is a more algebraic way of doing polar coordinates in terms of overdetermined sphere coordinates

$$z^i = \frac{x^i}{r}, \quad z^i z^i = 1, \quad \mathrm{d}z^i = \frac{1}{r}(\mathrm{d}x^i - \frac{x^i}{r}\mathrm{d}r); \quad \mathrm{d}x^i = r\mathrm{d}z^i + z^i\mathrm{d}r, \quad z^i\mathrm{d}z^i = 0$$

summing repeated $i = 1, 2, 3$. Only z^1, z^2, say, should be regarded as actual coordinates, with z^3 determined up to sign depending on one of two patches of the sphere. This works for spheres in any dimensions in a parallel way. The use of these coordinates corresponds to our description of Ω^1 for the classical sphere as a rank 2 projective module (see Chap. 3) with $r\mathrm{d}z^i = \mathrm{d}x^i - \frac{x^i}{r}\mathrm{d}r$ the projection of $\mathrm{d}x^i$. The inverse metric is

$$(\mathrm{d}r, \mathrm{d}r) = \frac{1}{h(r)^2}, \quad (\mathrm{d}r, \mathrm{d}z^i) = 0, \quad (\mathrm{d}z^i, \mathrm{d}z^j) = \frac{1}{r^2}e^{ij}, \tag{9.29}$$

where $e^{ij} = \delta^{ij} - z^i z^j$ is the relevant projection matrix idempotent.

Lemma 9.7 *The (trivial) Levi-Civita connection for $\mathbb{R}^3$ can be written in polar coordinates as (sum over j)*

$$\widehat{\nabla}\mathrm{d}z^i = -z^i\mathrm{d}z^j \otimes \mathrm{d}z^j - \frac{1}{r}(\mathrm{d}r \otimes \mathrm{d}z^i + \mathrm{d}z^i \otimes \mathrm{d}r), \quad \widehat{\nabla}\mathrm{d}r = r\mathrm{d}z^i \otimes \mathrm{d}z^i.$$

Proof For the trivial connection $\widehat{\nabla}\mathrm{d}x^i = 0$ we compute

$$\widehat{\nabla}(r\mathrm{d}r) = \mathrm{d}r \otimes \mathrm{d}r + r\widehat{\nabla}(\mathrm{d}r) = \widehat{\nabla}(x^i\mathrm{d}x^i) = \mathrm{d}x^i \otimes \mathrm{d}x^i = g$$

for the flat metric (where $h(r) = 1$), from which we conclude $\widehat{\nabla}(\mathrm{d}r)$ as stated. Also

$$\widehat{\nabla}\mathrm{d}z^i = \widehat{\nabla}(\frac{1}{r}(\mathrm{d}x^i - z^i\mathrm{d}r)) = -\frac{\mathrm{d}r}{r^2} \otimes \mathrm{d}x^i - \frac{z^i}{r}\widehat{\nabla}\mathrm{d}r - \mathrm{d}(\frac{z^i}{r}) \otimes \mathrm{d}r,$$

which simplifies to the expression stated. One can verify directly that these expressions give a (flat) torsion free and metric compatible connection, as they must. The first is immediate from the symmetric form of $\widehat{\nabla}$ while

$$\widehat{\nabla}(r^2\mathrm{d}\Omega^2) = 2r\mathrm{d}r \otimes \mathrm{d}z^i \otimes \mathrm{d}z^i + r^2\widehat{\nabla}(\mathrm{d}z^i \otimes \mathrm{d}z^i) = -r\mathrm{d}z^i \otimes (\mathrm{d}z^i \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}z^i)$$

using $z^i \mathrm{d}z^i = 0$. This is precisely compensated by $\widehat{\nabla}(\mathrm{d}r \otimes \mathrm{d}r)$ to give $\widehat{\nabla} g = 0$ as it should. Similarly

$$R_{\widehat{\nabla}} \mathrm{d}r = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \widehat{\nabla})(\mathrm{d}z^i \otimes r\mathrm{d}z^i) = \mathrm{d}(\mathrm{d}z^i) - \mathrm{d}z^i \widehat{\nabla}(r\mathrm{d}z^i) = 0$$

as both terms vanish. Similarly

$$\begin{aligned} R_{\widehat{\nabla}} \mathrm{d}z^i &= -(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \widehat{\nabla})(z^i \mathrm{d}z^j \otimes \mathrm{d}z^j + \frac{1}{r}(\mathrm{d}r \otimes \mathrm{d}z^i + \mathrm{d}z^i \otimes \mathrm{d}r)) \\ &= -\mathrm{d}z^i \wedge \mathrm{d}z^j \otimes \mathrm{d}z^j + \frac{\mathrm{d}r}{r^2} \wedge (\mathrm{d}z^i \otimes \mathrm{d}r) + z^i \mathrm{d}z^j \widehat{\nabla} \mathrm{d}z^j + \frac{\mathrm{d}r}{r} \widehat{\nabla} \mathrm{d}z^i + \frac{\mathrm{d}z^i}{r} \widehat{\nabla} \mathrm{d}r \end{aligned}$$

vanishes on putting in the values of $\widehat{\nabla}$ and cancelling. Of course, we already know this, but these polar coordinate computations are a model for the next result. □

Now looking carefully at the preceding check of how the trivial $\widehat{\nabla}$ gets to be torsion free and metric compatible in our 'radial/tangential' framework, one can see that the proof extends to give the Levi-Civita connection in the general spherically symmetric case.

Proposition 9.8 *Let*

$$g = h(r)^2 \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}z^i \otimes \mathrm{d}z^i$$

for a function $h(r)$, which we assume to be invertible. Then

$$\begin{aligned} \widehat{\nabla} \mathrm{d}z^i &= -z^i \mathrm{d}z^j \otimes \mathrm{d}z^j - \frac{1}{r}(\mathrm{d}r \otimes \mathrm{d}z^i + \mathrm{d}z^i \otimes \mathrm{d}r), \\ \widehat{\nabla} \mathrm{d}r &= \frac{r}{h(r)^2} \mathrm{d}z^i \otimes \mathrm{d}z^i - \frac{h'(r)}{h(r)} \mathrm{d}r \otimes \mathrm{d}r \end{aligned}$$

is torsion free, metric compatible and has curvature

$$\begin{aligned} R_{\widehat{\nabla}} \mathrm{d}z^i &= -\mathrm{d}z^i \Big(\Big(1 - \frac{1}{h(r)^2}\Big) \mathrm{d}z^j \otimes \mathrm{d}z^j + \frac{h'(r)}{h(r)r} \mathrm{d}r \otimes \mathrm{d}r \Big), \\ R_{\widehat{\nabla}}(\mathrm{d}r) &= -\frac{h'(r)r}{h(r)^3} \mathrm{d}r \mathrm{d}z^i \otimes \mathrm{d}z^i . \end{aligned}$$

Proof Because the connection on 'angular' forms is unchanged, this part remains torsion free. For the radial part, clearly $\wedge \widehat{\nabla} \mathrm{d}r = 0$ so the torsion on $\mathrm{d}r$ also vanishes. We have to check metric compatibility and we note that

$$\widehat{\nabla}(h(r) r \mathrm{d}r) = \frac{1}{h(r)} g, \quad \widehat{\nabla}(h(r) \mathrm{d}r) = \frac{r}{h(r)} \mathrm{d}z^i \otimes \mathrm{d}z^i \tag{9.30}$$

similarly to Lemma 9.7. The computation of $\widehat{\nabla}(r^2 \mathrm{d}\Omega^2)$ is unchanged and now clearly killed in just the same way by $\widehat{\nabla}(h(r)\mathrm{d}r \otimes h(r)\mathrm{d}r)$. It remains to compute the curvature. The only difference for $R_{\widehat{\nabla}}(\mathrm{d}z^i)$ compared to the direct calculation in the proof of Lemma 9.7 is the form of $\widehat{\nabla}(r\mathrm{d}r)$, so this time

$$R_{\widehat{\nabla}}\mathrm{d}z^i = -\frac{\mathrm{d}z^i}{r^2}\eta + \frac{\mathrm{d}z^i}{r^2}\Big(\frac{1}{h(r)^2}r^2\mathrm{d}z^j \otimes \mathrm{d}z^j + (1-\frac{h'(r)}{h(r)}r)\mathrm{d}r \otimes \mathrm{d}r\Big)$$

giving the result as stated. Similarly

$$R_{\widehat{\nabla}}(h(r)\mathrm{d}r) = (\mathrm{d}\otimes \mathrm{id} - \mathrm{id}\wedge \widehat{\nabla})(\frac{1}{h(r)}\mathrm{d}z^i \otimes r\mathrm{d}z^i) = \mathrm{d}(\frac{1}{h(r)})\mathrm{d}z^i \otimes r\mathrm{d}z^i$$

giving the stated result. □

We also compute the Ricci tensor in our algebraic approach (which we recall in our conventions reduces in the classical case to $-\frac{1}{2}$ times the usual Ricci tensor).

Corollary 9.9 *The standard antisymmetric lifting $i : \Omega^2 \to \Omega^1 \otimes_{C^\infty(M)} \Omega^1$ and trace applied to the curvature in Proposition 9.8 gives*

$$\mathrm{Ricci} = -\frac{1}{2}\Big(r\frac{h'(r)}{h(r)^3} + \Big(1 - \frac{1}{h(r)^2}\Big)\Big)\mathrm{d}z^j \otimes \mathrm{d}z^j - \frac{h'(r)}{h(r)r}\mathrm{d}r \otimes \mathrm{d}r.$$

In particular, Ricci $\propto g$ *(an Einstein space) if and only if* $rh'(r) = h(r)(h(r)^2 - 1)$.

Proof We obtain Ricci by lifting the 2-form output and contracting the curvature in Proposition 9.8, but the result is the same as setting all terms involving f to zero in our later 4D theorem, so we omit the details. Hence for an Einstein space we need

$$\frac{h'(r)}{h(r)^3} + \frac{1}{r}\Big(1 - \frac{1}{h(r)^2}\Big) = 2\frac{h'(r)}{h(r)^3}. \qquad \square$$

Over $\mathbb{R}$, the equation is solved by $h(r) = 1/\sqrt{1 + Kr^2}$ where K is a parameter. Then Ricci $= Kg$. Hence this is a space of constant curvature. For $K < 0$ (in our nonstandard conventions) it is essentially S^3, while for $K > 0$ it is hyperbolic 3-space. In both cases the removal of $r = 0$ is not required other than for use of our polar coordinates. Note that we cannot solve Ricci $= 0$ unless $h(r) = 1$, which is the flat case in Lemma 9.7.

Next, we consider Killing vector fields needed later. Because we prefer differential forms, we map a vector field τ to a 1-form via the metric. In these terms, a conformal Killing 1-form is required to obey $(\mathrm{id} + \mathrm{flip})\widehat{\nabla}\tau \propto g$ as in §8.3.

Corollary 9.10 *The metric in Proposition 9.8 has conformal Killing forms τ and three Killing forms τ_i (these need not be linearly independent),*

$$\tau = h(r)r\mathrm{d}r, \quad \tau_i = \epsilon_{ijk}r^2 z^j \mathrm{d}z^k,$$

where ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$.

Proof That τ is a conformal Killing form is immediate from the first equation of (9.30), which indeed says that $\widehat{\nabla}\tau = g/h(r)$. If one tries a more general form $\tau = f(r)h(r)r\mathrm{d}r$ then one can deduce that $f'(r) = 0$. The τ_i correspond to the action of the group of rotations in 3-dimensions and one easily computes

$$\widehat{\nabla}\tau_i = \epsilon_{ijk}(r^2\mathrm{d}z^j \otimes \mathrm{d}z^k + z^j r(\mathrm{d}r \otimes \mathrm{d}z^k - \mathrm{d}z^k \otimes \mathrm{d}r)),$$

which is manifestly antisymmetric. Hence its symmetrisation vanishes and we have a Killing 1-form. □

One can work equally well with vector fields and the inverse metric. Some natural vector fields for our 'polar coordinates' are

$$\rho = r\frac{\partial}{\partial r}, \quad e_i = r\frac{\partial}{\partial x^i} - z^i\rho; \quad z^i e_i = 0,$$

where ρ acts as the degree operator so $\rho(x^i) = x^i$ and the e_i are not independent as shown. They are partial derivatives associated to the $\mathrm{d}z^i$ in the sense that

$$\mathrm{d}f = \frac{\partial f}{\partial r}\mathrm{d}r + e_i(f)\mathrm{d}z^i$$

for all functions $f \in C^\infty(M)$. Their corresponding 1-forms are

$$g(\rho) = h(r)^2 r\mathrm{d}r, \quad g(e_i) = r^2 e^{ij}\mathrm{d}z^j = r^2\mathrm{d}z^i;$$
$$g^{-1}(\mathrm{d}r) = \frac{1}{h(r)^2 r}\rho, \quad g^{-1}(\mathrm{d}z^i) = \frac{1}{r^2}e_i.$$

Then, for example, the corresponding conformal Killing vector fields in Corollary 9.10 are

$$\tau = \frac{1}{h(r)}\rho, \quad \tau_i = \frac{\epsilon_{ijk}}{h(r)^2}z^j e_k. \tag{9.31}$$

Finally, we can go one step further with the above as the spatial part of a radially-symmetric static spacetime geometry on $M \times \mathbb{R}$. Now there is an additional 1-form $\mathrm{d}t$ and we still keep everything commutative.

Proposition 9.11 *Consider a spherically static spacetime metric of the form*

$$g = -f(r)^2\mathrm{d}t \otimes \mathrm{d}t + h(r)^2\mathrm{d}r \otimes \mathrm{d}r + r^2\mathrm{d}z^i \otimes \mathrm{d}z^i$$

for invertible functions $h(r)$, $f(r)$. Then $\widehat{\nabla}\mathrm{d}z^i$ as before and

$$\widehat{\nabla}\mathrm{d}r = \frac{r}{h(r)^2}\mathrm{d}z^i \otimes \mathrm{d}z^i - \frac{h'(r)}{h(r)}\mathrm{d}r \otimes \mathrm{d}r - \frac{f'(r)f(r)}{h(r)^2}\mathrm{d}t \otimes \mathrm{d}t,$$
$$\widehat{\nabla}\mathrm{d}t = -\frac{f'(r)}{f(r)}(\mathrm{d}t \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}t)$$

is torsion free, metric compatible and has curvature

$$R_{\widehat{\nabla}}\mathrm{d}z^i = -\mathrm{d}z^i\Big(\Big(1 - \frac{1}{h(r)^2}\Big)\mathrm{d}z^j \otimes \mathrm{d}z^j + \frac{h'(r)}{h(r)r}\mathrm{d}r \otimes \mathrm{d}r + \frac{f'(r)f(r)}{h(r)^2 r}\mathrm{d}t \otimes \mathrm{d}t\Big),$$
$$R_{\widehat{\nabla}}\mathrm{d}r = -\frac{h'(r)r}{h(r)^3}\mathrm{d}r\mathrm{d}z^i \otimes \mathrm{d}z^i + \frac{f(r)}{h(r)^3}\big(f'(r)h'(r) - f''(r)h(r)\big)\,\mathrm{d}r\mathrm{d}t \otimes \mathrm{d}t,$$
$$R_{\widehat{\nabla}}\mathrm{d}t = \frac{1}{f(r)h(r)}\big(f'(r)h'(r) - f''(r)h(r)\big)\,\mathrm{d}r\mathrm{d}t \otimes \mathrm{d}r + \frac{f'(r)r}{f(r)h(r)^2}\mathrm{d}t\mathrm{d}z^i \otimes \mathrm{d}z^i.$$

Proof The torsion on $\mathrm{d}r$ continues to vanish as $(\mathrm{d}t)^2 = 0$ and vanishes on $\mathrm{d}t$ by $\{\mathrm{d}t, \mathrm{d}r\} = 0$. For metric compatibility, we write the connection in the form

$$\widehat{\nabla}(h(r)\mathrm{d}r) = \frac{r}{h(r)}\mathrm{d}z^i \otimes \mathrm{d}z^i - \frac{f'(r)f(r)}{h(r)}\mathrm{d}t \otimes \mathrm{d}t, \quad \widehat{\nabla}(f(r)\mathrm{d}t) = -f'(r)\mathrm{d}t \otimes \mathrm{d}r.$$

Then compared to the previous case, $\widehat{\nabla}(h(r)\mathrm{d}r \otimes h(r)\mathrm{d}r)$ acquires an extra term

$$-\frac{f'(r)f(r)}{h(r)}(\mathrm{d}t \otimes \mathrm{d}t \otimes h(r)\mathrm{d}r + \mathrm{d}t \otimes h(r)\mathrm{d}r \otimes \mathrm{d}t),$$

which is exactly cancelled by $\widehat{\nabla}(-f(r)\mathrm{d}t \otimes f(r)\mathrm{d}t)$. Hence the connection remains metric compatible. As $\widehat{\nabla}(\mathrm{d}z^i)$ is unchanged, $R_{\widehat{\nabla}}(\mathrm{d}z^i)$ in the previous computation is affected only through $\widehat{\nabla}(r\mathrm{d}r)$, which acquires an extra $-(f'(r)f(r)/h(r)^2)r\mathrm{d}t\otimes\mathrm{d}t$, giving the additional contribution stated. Similarly, in the previous computation of $R_{\widehat{\nabla}}(h(r)\mathrm{d}r)$, the change in $\widehat{\nabla}(h(r)\mathrm{d}r)$ gives an additional contribution

$$(\mathrm{d}\otimes\mathrm{id}-\mathrm{id}\wedge\widehat{\nabla})\Big(-\frac{f'(r)f(r)}{h(r)}\mathrm{d}t\otimes\mathrm{d}t\Big) = -\mathrm{d}\Big(\frac{f'(r)f(r)}{h(r)}\Big)\mathrm{d}t\otimes\mathrm{d}t + \frac{f'(r)f(r)}{h(r)}\mathrm{d}t\wedge\widehat{\nabla}\mathrm{d}t$$

giving the additional contribution stated. Finally, we compute

$$\begin{aligned} R_{\widehat{\nabla}}(f(r)\mathrm{d}t) &= (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \widehat{\nabla})(-f'(r)\mathrm{d}t \otimes \mathrm{d}r) \\ &= -\mathrm{d}(f'(r))\mathrm{d}t \otimes \mathrm{d}r + f'(r)\mathrm{d}t \wedge \Big(\frac{r}{h^2}\mathrm{d}z^i \otimes \mathrm{d}z^i - \frac{h'(r)}{h(r)}\mathrm{d}r \otimes \mathrm{d}r\Big) \end{aligned}$$

as $(\mathrm{d}t)^2 = 0$. This gives the result stated. □

Theorem 9.12 *The standard antisymmetric lifting $i : \Omega^2 \to \Omega^1 \otimes_{C^\infty(M)} \Omega^1$ and trace applied to the curvature in Proposition 9.11 gives*

$$
\begin{aligned}
\mathrm{Ricci} = &\frac{1}{2}\Big(\frac{f'(r)r}{f(r)h(r)^2} - \frac{h'(r)r}{h(r)^3} - \Big(1 - \frac{1}{h(r)^2}\Big)\Big)\mathrm{d}z^j \otimes \mathrm{d}z^j\\
&- \Big(\frac{1}{2f(r)h(r)}\Big(f'(r)h'(r) - f''(r)h(r)\Big) + \frac{h'(r)}{h(r)r}\Big)\mathrm{d}r \otimes \mathrm{d}r\\
&+ \Big(\frac{f(r)}{2h(r)^3}\Big(f'(r)h'(r) - f''(r)h(r)\Big) - \frac{f'(r)f(r)}{h(r)^2 r}\Big)\mathrm{d}t \otimes \mathrm{d}t.
\end{aligned}
$$

In particular, Ricci $= 0$ *if* $h(r) = 1/f(r)$ *and* $r\frac{\mathrm{d}}{\mathrm{d}r}f(r)^2 = 1 - f(r)^2$.

Proof We use the standard antisymmetric lift of a 2-form, so for example $i(\mathrm{d}r \wedge \mathrm{d}t) = \frac{1}{2}(\mathrm{d}r \otimes \mathrm{d}t - \mathrm{d}t \otimes \mathrm{d}r)$. We then take a trace of $(i \otimes \mathrm{id})R_{\widehat{\nabla}}$ as an operator mapping to the first tensor factor (say) of its output, to give Ricci in the remaining two tensor factors. When doing this, clearly $R_{\widehat{\nabla}}\mathrm{d}r \propto \mathrm{d}r \wedge \mathrm{d}t \otimes \mathrm{d}t$ will contribute $\frac{1}{2}\mathrm{d}t \otimes \mathrm{d}t$ to the trace as only the first term of the lift will contribute. Similarly for the contribution from $R_{\widehat{\nabla}}\mathrm{d}t$. For $R_{\widehat{\nabla}}\mathrm{d}z^i$, where a term is of the form $\mathrm{d}z^i X$ and X does not involve $\{\mathrm{d}z^j\}$ in its first tensor factor, we will similarly have X for the contribution to the trace from $\mathrm{d}z^i \to \frac{1}{2}\mathrm{d}z^i \otimes X$, because the projective module has rank 2 and the operation is as a multiple of the identity. For a term in $R_{\widehat{\nabla}}(\mathrm{d}z^i)$ of the form $\mathrm{d}z^i \wedge \mathrm{d}z^j \otimes \mathrm{d}z^j$, we will again have $\mathrm{d}z^j \otimes \mathrm{d}z^j$ for the same reason but also the trace of

$$
\mathrm{d}z^i \mapsto -\frac{1}{2}\mathrm{d}z^j \otimes \mathrm{d}z^i \otimes \mathrm{d}z^j
$$

from the antisymmetrisation. This will contribute $-\frac{1}{2}\mathrm{d}z^j \otimes \mathrm{d}z^j$, giving a total contribution from such a term in $R_{\widehat{\nabla}}\mathrm{d}z^i$ of $\frac{1}{2}\mathrm{d}z^j \otimes \mathrm{d}z^j$. With these observations, we see without further computation that

$$
\begin{aligned}
\mathrm{Ricci} = &-\Big(\frac{1}{2}\Big(1 - \frac{1}{h(r)^2}\Big)\mathrm{d}z^j \otimes \mathrm{d}z^j + \frac{h'(r)}{h(r)r}\mathrm{d}r \otimes \mathrm{d}r + \frac{f'(r)f(r)}{h(r)^2 r}\mathrm{d}t \otimes \mathrm{d}t\Big)\\
&- \frac{h'(r)r}{2h(r)^3}\mathrm{d}z^i \otimes \mathrm{d}z^i + \frac{f(r)}{2h(r)^3}\Big(f'(r)h'(r) - f''(r)h(r)\Big)\mathrm{d}t \otimes \mathrm{d}t\\
&- \frac{1}{2f(r)h(r)}\Big(f'(r)h'(r) - f''(r)h(r)\Big)\mathrm{d}r \otimes \mathrm{d}r + \frac{f'(r)r}{2f(r)h(r)^2}\mathrm{d}z^i \otimes \mathrm{d}z^i,
\end{aligned}
$$

which then combines as stated. Note that if Ricci $= 0$ then combining the $\mathrm{d}t \otimes \mathrm{d}t$ and $\mathrm{d}r \otimes \mathrm{d}r$ equations, we deduce that $f'/f + h'/h = (fh)'/(fh) = 0$ which over $\mathbb{R}$ implies that $h \propto 1/f$. $\square$

Over $\mathbb{R}$, the equation is solved by $f(r) = \sqrt{1 - \frac{\gamma}{r}}$ where γ is a parameter, this being the Schwarzschild metric for a black-hole with event horizon at $r = \gamma$. As usual, our natural algebraic conventions in defining Ricci differ by $-\frac{1}{2}$ from the more usual ones but this does not affect Ricci flatness of course. Also note that it was convenient (and conventional in physics) but not essential to work with f, h—one can work just as well throughout with $f^2(r)$ and $h^2(r)$ as the functions of interest. All formulae can be reworked in terms of these without square roots and one should do so for a fully algebraic treatment.

Finally, we compute the associated classical spacetime wave operator in the setting of Proposition 9.11, for later reference. The inverse spacetime metric is

$$(\mathrm{d}t, \mathrm{d}t) = -\frac{1}{f^2(r)}, \quad (\mathrm{d}t, \mathrm{d}r) = (\mathrm{d}t, \mathrm{d}z^i) = 0, \quad (\mathrm{d}z^i, \mathrm{d}z^j) = \frac{1}{r^2}e^{ij}$$

and the spacetime wave operator on functions is $\Box = (\,,\,)\widehat{\nabla}\mathrm{d}$ for the spacetime connection and exterior derivative.

Corollary 9.13 *The classical spacetime wave operator associated to the metric in Proposition 9.11 is*

$$\Box = -\frac{1}{f^2}\frac{\partial^2}{\partial t^2} + \frac{1}{h^2}\Big(\frac{2}{r} - \frac{h'}{h} + \frac{f'}{f}\Big)\frac{\partial}{\partial r} + \frac{1}{h^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}e_i e_i$$

(where we sum over i).

Proof We compute

$$\begin{aligned}\Box\psi &= (\,,\,)\widehat{\nabla}\mathrm{d}\psi = (\,,\,)\widehat{\nabla}((\frac{\partial}{\partial t}\psi)\mathrm{d}t + (\frac{\partial}{\partial r}\psi)\mathrm{d}r + (e_i\psi)\mathrm{d}z^i)\\ &= (\,,\,)((\frac{\partial^2}{\partial t^2}\psi)\mathrm{d}t\otimes\mathrm{d}t + \frac{\partial}{\partial r}\psi)\widehat{\nabla}\mathrm{d}r + (\frac{\partial^2}{\partial r^2}\psi)\mathrm{d}r\otimes\mathrm{d}r + (e_j e_i\psi)\mathrm{d}z^j\otimes\mathrm{d}z^i)\\ &= \Big(-\frac{1}{f^2}\frac{\partial^2}{\partial t^2} + \frac{1}{h^2}(\frac{2}{r} - \frac{h'}{h} + \frac{f'}{f})\frac{\partial}{\partial r} + \frac{1}{h^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}e_i e_i\Big)\psi\end{aligned}$$

on a general function ψ. We dropped terms in the output of $\widehat{\nabla}$ that are immediately killed by the block-diagonal form of the inverse metric. $\square$

Doing the same computation for the 3-geometry in Proposition 9.8 involves the same steps but without any of the terms involving f and gives

$$\Delta_{LB} = \frac{1}{h^2}(\frac{2}{r} - \frac{h'}{h})\frac{\partial}{\partial r} + \frac{1}{h^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}e_i e_i. \tag{9.32}$$

Note that the spatial part of $\Box$ differs from this by an extra $\frac{f'}{fh^2}\frac{\partial}{\partial r}$ and hence

$$\Box = \beta\frac{\partial^2}{\partial t^2} + \Delta, \quad \Delta = \Delta_{LB} - \frac{1}{2}\beta^{-1}g^{-1}(\mathrm{d}\beta),$$

where $\beta = -1/f^2$, in accord with the general picture for this kind of metric explained in the proof of Corollary 8.25. Here $\beta^{-1}\mathrm{d}\beta = -\frac{2f'}{f}\mathrm{d}r = -2g(\frac{f'}{fh^2}\frac{\partial}{\partial r})$ using the metric g to convert a vector field to a 1-form.

9.3.2 *Quantum Wave Operators on Spherically Symmetric Static Spacetimes*

We are now ready to apply the machinery of §8.3 to spherically symmetric static spacetimes. Thus we let (M, g) be a classical Riemannian manifold as in Proposition 9.11. To avoid confusion with the quantum versions, *we now denote the classical exterior derivative and Laplacian etc.,* by d_M, Δ_M respectively, unless this is clear from context. From Corollary 9.10, we have two types of conformal Killing vector and we mainly consider the first one and its divergence measure

$$\tau = \frac{r}{h(r)}\frac{\partial}{\partial r}, \quad \alpha = \frac{2}{h(r)} - 1. \tag{9.33}$$

Thus the free function h in the 3-geometry metric is now encoded in α as well as in the inverse 3-metric $(,)$. We will not construct the full noncommutative 4-geometry itself, only its quantum differential calculus and an associated quantum wave operator. From the classical wave operator in Corollary 9.13 compared with Corollary 8.25, we express the free function $f(r)$ in the static metric in Proposition 9.11 and Theorem 9.12 as the free parameter

$$\beta = -\frac{1}{f(r)^2} \tag{9.34}$$

of our quantum calculus. The functions μ, ν there are now generically given by

$$\mu(r) = -\frac{1}{r^2}\int^r \frac{h(r')}{f(r')^2}r'\mathrm{d}r',$$

$$\nu(r) = e^{\int_1^r \frac{h(r')-2}{r'}\mathrm{d}r'}\int^r e^{-\int_1^{r'}\frac{h(r'')-2}{r''}\mathrm{d}r''}\frac{h(r')\mu(r')}{r'}\mathrm{d}r'$$

and provide the time part $2\Delta_0$ of the wave operator in the 'finite difference' form in Proposition 8.24. It remains the case that Lemma 8.26 is a better route for its actual calculation. Then the quantum wave operator according to Corollary 8.25 and Corollary 9.13 is

$$\Box\psi(t) = 2\Delta_0\psi(t) + \Big(\Big(\frac{1}{h^2}(\frac{2}{r} - \frac{h'}{h} + \frac{f'}{f})\frac{\partial}{\partial r} + \frac{1}{h^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}e_i e_i\Big)\psi\Big)(t + \lambda\alpha) \tag{9.35}$$

on normal ordered functions $\psi = \sum_n \psi_n t^n$. The bracketed expression is the spatial Laplacian Δ_M. Working through the constructions of §8.3, the quantum differential algebra associated to the classical geometry now takes the form:

Proposition 9.14 *The quantum calculus $\Omega^1(C^\infty(M) \rtimes \mathbb{R})$ quantising the classical picture in Proposition 9.11 with respect to radial conformal Killing vector (9.33) has relations*

$$[x^i, x^j] = 0, \quad [x^i, t] = \frac{\lambda}{h}x^i, \quad [\mathrm{d}z^i, x^j] = \frac{\lambda}{r}e^{ij}\theta', \quad [\mathrm{d}r, x^i] = \frac{\lambda}{h^2}\frac{x^i}{r}\theta',$$

$$[\theta', x^i] = 0, \quad [\mathrm{d}z^i, t] = -\lambda\mathrm{d}z^i, \quad [\theta', t] = \lambda(\frac{2}{h} - 1)\theta', \quad [x^i, \mathrm{d}t] = \lambda\mathrm{d}x^i,$$

$$[\mathrm{d}t, t] = \beta\lambda\theta' - \lambda\mathrm{d}t, \quad [\mathrm{d}r, t] = \lambda(\mathrm{d}(\frac{r}{h}) - \mathrm{d}r), \quad \mathrm{d}g(r) = g'(r)\mathrm{d}r + \frac{\lambda}{2h^2}g''(r)\theta'$$

for any function $g(r)$.

Proof This is by application of Theorem 8.23 but can conveniently be found using the quantum commutation relations (8.42). We use the same 1-forms $\mathrm{d}z^i$ as in §9.3.1 but these can also be viewed as the angular projection of the quantum differentials,

$$\mathrm{d}z^i = \frac{1}{r}(\mathrm{d}x^i - \frac{x^i}{r^2}x^j\mathrm{d}x^j) = \frac{1}{r}\mathrm{d}x^i - \frac{x^i}{r^2}x^j\mathrm{d}x^j = \frac{1}{r}e^{ik}\mathrm{d}x^k,$$

so we need not distinguish between their classical and quantum counterparts. To see this we use d from Proposition 1.22 and Δ_M given below (9.32) to compute

$$\mathrm{d}x^i = \mathrm{d}_M x^i + \frac{\lambda}{2h^2}\Big(\frac{2}{r}(1-h^2) - \frac{h'}{h} + \frac{f'}{f}\Big)\theta'.$$

One can then deduce properties of $\mathrm{d}z^i$ noting that $[e^{ij}, t] = 0$. We similarly have

$$\mathrm{d}r = \mathrm{d}_M r + \frac{\lambda}{2h^2}\Big(\frac{2}{r} - \frac{h'}{h} + \frac{f'}{f}\Big)\theta'$$

and hence $\mathrm{d}g = g'\mathrm{d}_M r + \frac{\lambda}{2h^2}((\frac{2}{r} - \frac{h'}{h} + \frac{f'}{f})g' + g'')\theta'$ comes out as stated. $\square$

One similarly has useful relations such as

$$z^i \mathrm{d}z^i = [\mathrm{d}z^i, r] = [\mathrm{d}r, \frac{x^i}{r}] = 0, \quad [\mathrm{d}r, g(r)] = \frac{\lambda g'(r)}{h^2}\theta', \quad [\mathrm{d}r, x^i] = \lambda \frac{x^i}{rh^2}\theta',$$

$$\mathrm{d}z^i = \frac{1}{r}(\mathrm{d}x^i - \frac{x^i}{r}\mathrm{d}r + \frac{\lambda x^i}{h^2r^2}\theta'), \quad [g(r), t] = \lambda \frac{r}{h} g'(r), \quad [g(r), \mathrm{d}t] = \lambda \mathrm{d}g(r)$$

on any function $g(r)$, where the last is mentioned for completeness (it also applies to any function g on M). Among our various relations, we have a closed subalgebra of $\mathrm{d}r, \mathrm{d}t, \theta'$ and functions of r, t. Of interest is the form of Δ_0. This is too complicated in general but we have given it in Lemma 8.27 to order λ^3. Putting in the particular form of β, α, τ in our case we find immediately that

$$\begin{aligned}
\Delta_0 \mathrm{e}^{\mathrm{i}\omega t} &= -\frac{\omega^2}{2}\Big(\beta - \frac{2\mathrm{i}\lambda\omega}{3h}((h-1)\beta + r\beta') \\
&- \frac{\lambda^2\omega^2}{12h^3}\left(3r^2h\beta'' - 3r^2\beta'h' + rh\beta'(5h-1) + (6rh' + 3h^3 - 6h^2 + 4h)\beta\right) + \cdots\Big)\mathrm{e}^{\mathrm{i}\omega t}, \\
\mathrm{e}^{\mathrm{i}\omega(t+\lambda\alpha)} &= \Big(1 + \mathrm{i}\lambda\omega\alpha - \frac{\lambda^2\omega^2}{2}\Big(\frac{2rh'}{h^3} + \Big(1 - \frac{2}{h}\Big)^2\Big) + \cdots\Big)\mathrm{e}^{\mathrm{i}\omega t}
\end{aligned}$$

to $O((\lambda\omega)^3)$ terms.

Example 9.15 (Quantum Black-Hole) We now specialise to the case of a Schwarzschild black-hole of mass M where, from Theorem 9.12, we have classically

$$f = \sqrt{1 - \frac{\gamma}{r}}, \quad h = \frac{1}{f}, \quad \tau = fr\frac{\partial}{\partial r}, \quad \alpha = 2f - 1, \quad \beta = -\frac{1}{(1 - \frac{\gamma}{r})},$$

with $\gamma = 2GM$. Here h, τ, α are associated to the 3-geometry and $\beta = -1/f^2$ enters the construction of the calculus and expresses the time component of the metric. We work in units where $c = 1$. For these specific functions, some of the formulae in Proposition 9.14 simplify, for example

$$[\mathrm{d}r, t] = \frac{\lambda}{2}(\sqrt{h} - \frac{1}{\sqrt{h}})^2\mathrm{d}r - \frac{\lambda^2\gamma^2}{8r^3}h\theta'.$$

In principle, we can put our specific structure functions into the theory in §8.3 to find the quantum wave operator. We have

$$\Delta_0 \mathrm{e}^{\mathrm{i}\omega t} = -\frac{\omega^2}{2}\Big(1 - \frac{2\mathrm{i}\lambda\omega}{3}\Big(1 - \frac{1}{\sqrt{1 - \frac{\gamma}{r}}}\Big) + \cdots\Big)\beta \mathrm{e}^{\mathrm{i}\omega t} + O(\lambda^2),$$

$$(\Delta_M \psi(r)\mathrm{e}^{\mathrm{i}\omega t})(t + \lambda\alpha) = (1 + \mathrm{i}\lambda\omega\alpha + \cdots)\Big(1 - \frac{\gamma}{r}\Big)\Big(\frac{(2 - \frac{\gamma}{r})}{r(1 - \frac{\gamma}{r})}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\Big)\psi(r)\mathrm{e}^{\mathrm{i}\omega t}.$$

Dividing the expression for Δ_0 by the $1 + i\lambda\omega\alpha + \cdots$ factor, we write the radial equation in $\Box(\psi(r)e^{i\omega t}) = 0$ as

$$D(\omega, r)\psi(r) + (1 - \frac{\gamma}{r})\Big(\frac{(2 - \frac{\gamma}{r})}{r(1 - \frac{\gamma}{r})}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\Big)\psi(r) = 0$$

for a mode of frequency ω. We are interested in order $\lambda\omega$ but include the next order as well for comparison, and we set $\lambda = i\lambda_P$. Then,

$$D(\omega, r) = \frac{\omega^2}{1 - \frac{\gamma}{r}}(1 + \lambda_P\omega f_1 + \lambda_P^2\omega^2 f_2 + \cdots);$$

$$f_1(r) = 2\sqrt{1 - \frac{\gamma}{r}} - \frac{1}{3} - \frac{2}{3\sqrt{1 - \frac{\gamma}{r}}},$$

$$f_2(r) = \frac{1}{24}\Big(45\Big(1 - \frac{\gamma}{r}\Big) - 18\sqrt{1 - \frac{\gamma}{r}} - 28 + \frac{6}{\sqrt{1 - \frac{\gamma}{r}}} + \frac{9}{1 - \frac{\gamma}{r}}\Big).$$

The functions f_1, f_2 weighted by their contribution are plotted dashed in Fig. 9.4 and limit to $1, 7/12$ respectively as they must since for large r the quantum geometry becomes that of the flat spacetime bicrossproduct model in §9.2.2 on right-ordered waves, so that

$$D(\omega, \infty) = \Big(\frac{1 - e^{\lambda_P}}{\lambda_P}\Big)^2 = \omega^2(1 + \lambda_P\omega + \frac{7}{12}\lambda_P^2\omega^2 + \cdots).$$

For our current analysis, we think of $D(\omega, r)$ as the square of the effective frequency for the radial modes. At large r, the effect for positive ω is to increase the effective frequency so that the radial wavelength is reduced, in keeping with the effective speed of light being reduced as in the flat bicrossproduct model for right-ordered

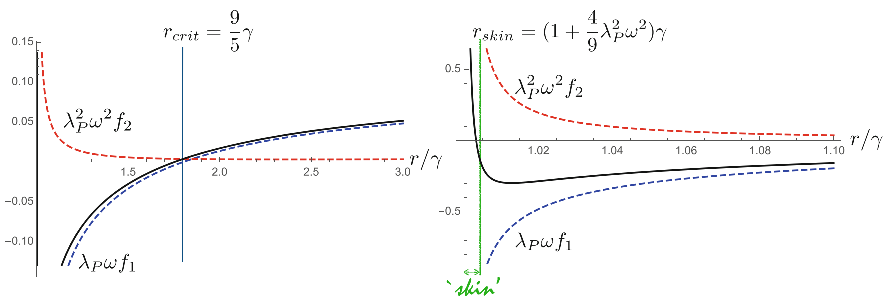

Fig. 9.4 Low order terms $\lambda_P\omega f_1 + \lambda_P^2\omega^2 f_2$ (solid) in quantum correction factor to redshift at $\lambda_P\omega = 0.1$, and a close-up showing the 'quantum skin' approaching or below which the approximation breaks down. The separate terms shown dashed

waves. All corrections enter via $\lambda_P\omega$ and hence are frequency dependent, including corrections to the speed of light. As we approach the event horizon, the $1-\frac{\gamma}{r}$ in the leading denominator of $D(\omega,r)$ and its inverse in the radial term are the source of gravitational redshift as light propagates from a given radius to infinity (or the source of the increasing frequency it must have been at its radius of origin in order to have a given frequency far out). Compared to this classical behaviour, we have the quantum correction factor $1+\lambda_P\omega f_1+\lambda_P^2\omega^2 f_2+\cdots$ which as we approach the horizon depends more strongly on r, implying a frequency dependent change to the amount of gravitational redshift. In particular, f_1 decreases to $f_1=0$ at

$$r_{crit}=\frac{9}{5}\gamma$$

so just inside twice the horizon the first-order quantum correction *vanishes* and indeed starts to decrease the usual redshift factor as we continue down to about

$$r_{skin}=(1+\frac{4}{9}\lambda_P^2\omega^2)\gamma.$$

This is the point where the first-order correction is $\lambda_P\omega f_1\approx -1$ (approximated for $\lambda_P\omega << 1$) completely cancelling the classical term. The second-order contribution is $\lambda_P^2\omega^2 f_2(r)\approx\frac{27}{32}$ for this radius and decays more rapidly than $|\lambda_P\omega f_1(r)|$ at larger r. Thus our first-order analysis is broadly valid as long as we are well above this 'quantum skin' radius just above the event horizon. Below this, the second and presumably higher order corrections start to dominate.

In summary, above r_{crit} the radial frequency is increased and becomes radius-independent, corresponding to a slowing of the speed of light for positive ω. For $r_{skin}<r<r_{crit}$ the effective radial frequency is instead *decreased* for positive ω, which implies among other things a lower than classical gravitational redshift factor. This effect renders $D(\omega,r)$ to about 84% of its classical value as we approach the skin radius based on first and second-order contributions, *even at low frequencies*. The classical value also blows up as we approach the horizon but we have a finite value at the skin radius at least up to first and second order corrections, with

$$\lim_{\omega\to 0}D(\omega,r_{skin})\approx\frac{2}{\lambda_P^2}.$$

It seems likely that higher order corrections also contribute and indeed our analysis cannot determine the exact behaviour approaching or below this distance above the horizon. In particular, there is no evidence that the overall correction factor turns negative inside this region (rendering our radial modes unstable and nonoscillatory) as would have been the case if we had looked only at f_1. We also cannot tell if the apparently finite redshift from the skin extends down to the horizon itself, but neither is the classical phenomenon of infinite redshift from right at the horizon valid, as modes cannot be treated classically inside the skin. As in §9.2.2, first-order effects are reversed for negative frequencies, a potential violation of CPT symmetry. ◇

We also have a quantisation by the Killing vector τ_3 (say) in Corollary 9.10. Here

$$\tau_3 = \frac{z^1 e_2 - z^2 e_1}{h(r)^2}, \quad \alpha = -1$$

and we may take h, β as desired, e.g. for the Schwarzschild metric. This of course breaks the rotational symmetry in the quantisation but provides further examples of the more general theory in §8.3, see exercise E9.3.

9.4 Curved Quantum Geometry of the 2D Bicrossproduct Model

In §9.2.2, we considered the bicrossproduct spacetime algebra A with relations $[x^i, x^j] = 0$, $[x^i, t] = \lambda x^i$ and its standard differential calculus as an analogue of flat Minkowski spacetime from a wave operator and Poincaré quantum group point of view. We saw in equation (9.20) and Proposition 9.4 that the flat quantum geometry and Poincaré invariance more properly applies to an extension of the differential calculus by an extra direction θ'. In the present section we find the quantum metric and a quantum Levi-Civita connection for the standard differential algebra *without* making any extension. This turns out to be far from flat and we will see that its intrinsic quantum Riemannian geometry contains within it either a strong gravitational source or an expanding universe, depending on a choice of sign.

The differential calculus we use is the representative $\beta = 1$ case of the family

$$[x^i, \mathrm{d}x^j] = 0, \quad [x^i, \mathrm{d}t] = \lambda\beta \mathrm{d}x^i, \quad [t, \mathrm{d}x^i] = \lambda(\beta - 1)\mathrm{d}x^i, \quad [t, \mathrm{d}t] = \lambda\beta \mathrm{d}t$$

for a constant parameter β. This β calculus is in the family covered by Theorem 8.23 and Corollary 8.28 and is essentially one of two translation-invariant choices in Example 1.43 on the quantum spacetime algebra in 2D, and hence also in higher dimensions if we want to treat the x^i symmetrically so as to preserve rotational invariance. The other choice will be covered in §9.5.

9.4.1 Emergence of the Bicrossproduct Model Quantum Metric

We first note that the commutation relations for differentials imply that

$$[f(x), t] = \lambda \sum_i x^i \frac{\partial}{\partial x^i} f, \quad [g(t), x^i] = x^i (g(t - \lambda) - g(t))$$

for all functions $f(x)$, $g(t)$. It follows that $f(x)$ is central if and only if it has scaling degree 0, for example rational functions such as x^1/x^2 etc. will be degree 0. For $g(t)$ to be central we need that g is periodic in imaginary time. But $|\lambda|$ is the Planck scale so such functions have no classical analogue and no limit as $\lambda \to 0$. We therefore exclude them from algebra. We do want to allow smooth functions of the x^i, polynomials in t and reasonable functions in t such as exponentials $e^{\imath\omega t}$, but with real ω or at least with an imaginary component that is below the Planck frequency. Then the centre of the algebra is just the degree zero functions of the x^i. In 2D, this means the centre is just multiples of the identity. We already know for $n = 2$ that there is a unique form of 'real' quantum metric. A more general result is as follows.

Lemma 9.16 *For $\lambda \neq 0$ and $n > 2$ there are no 'real' quantum-symmetric metrics.*

Proof We consider a metric of the arbitrary form

$$g = \sum_{i,j} a_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + \sum_i b_i(\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) + c\mathrm{d}t \otimes \mathrm{d}t$$

where the coefficients obey $a_{ij} = a_{ji}$ (they are all elements of A) and where we have assumed 'quantum symmetry' in the form $\wedge(g) = 0$. Then using the Leibniz rule, and the bimodule relations, we find (summations understood)

$$\begin{aligned}
[g, t] &= [a_{ij}, t]\mathrm{d}x^i \otimes \mathrm{d}x^j + ([b_i, t] - \lambda b_i)(\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) \\
&\quad + ([c, t] - 2\lambda c)\mathrm{d}t \otimes \mathrm{d}t, \\
[g, x^k] &= [a_{ij}, x^k]\mathrm{d}x^i \otimes \mathrm{d}x^j - \lambda b_i(\mathrm{d}x^i \otimes \mathrm{d}x^k + \mathrm{d}x^k \otimes \mathrm{d}x^i) \\
&\quad + [b_i, x^k](\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i) - \lambda c(\mathrm{d}x^k \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^k) \\
&\quad + [c, x^k]\mathrm{d}t \otimes \mathrm{d}t.
\end{aligned}$$

If we now use that $\mathrm{d}x^i$, $\mathrm{d}t$ form a basis over A then g central amounts to

$$[b_i, t] = \lambda b_i, \quad [c, t] = 2\lambda c, \quad [a_{ij}, x^j] = \lambda b_i, \quad [a_{ij}, t] = [a_{ij}, x^k] = [b_i, x^k] = 0$$

for all i, j and $k \neq i, j$, and

$$[b_k, x^k] = \lambda c, \quad [c, x^k] = 0$$

for all k. If $n > 2$ then we can find $k \neq i$ for any i and hence $[b_i, x^k] = 0$ tells us that b_i is a function only of x. Then the $[b_k, x^k]$ relation tells us that $[b_k, x^k] = 0 = \lambda c$, so if $\lambda \neq 0$ then we conclude that $c = 0$. Similarly for any i, we can take $k \neq i$ and $[a_{ii}, x^k] = 0$ tells us that a_{ii} is a function of the x^j only. Then $[a_{kk}, x^k] = 0 = \lambda b_k$ tells us that $b_k = 0$ for all k. □

If a metric is not central then we know from §1.3 that we cannot have an inverse $(\,,\,)$ that is well defined as a map on $\Omega^1 \otimes_A \Omega^1$. We can still ask for something weaker such as centrality over the subalgebra of functions of r, t, which would be of interest in purely spherically symmetric computations. For this, we choose radial coordinate $r = \sqrt{\sum_i (x^i)^2}$, which we assume invertible, and 'angular' coordinates $z^i = x^i/r$ as in §9.3.1. The $\mathrm{d}r$ and z^i are central and one can compute that

$$v := r\mathrm{d}t - t\mathrm{d}r, \quad [v, t] = [v, r] = 0, \quad [v, z^i] = -\lambda r\mathrm{d}z^i, \quad [\mathrm{d}z^i, t] = -\lambda\mathrm{d}z^i,$$
$$[\mathrm{d}z^i, z^j] = [\mathrm{d}z^i, r] = \{\mathrm{d}z^i, \mathrm{d}z^j\} = \{\mathrm{d}z^i, \mathrm{d}r\} = \{\mathrm{d}z^i, \mathrm{d}t\} = 0$$

if we work with $\mathrm{d}r, v$ and any $n-2$ of the $\mathrm{d}z^i$ as basis of Ω^1. Working in this basis we look for a rotationally symmetric form of the metric where the coefficients depend only on r, t and where the spatial sector is rotationally invariant. This takes us to a tensor of the form

$$g = c(r,t)\mathrm{d}z^i \otimes \mathrm{d}z^i + a(r,t)\mathrm{d}r \otimes \mathrm{d}r + b(r,t)(v^* \otimes v + \lambda(\mathrm{d}r \otimes v - v \otimes \mathrm{d}r)),$$

where we have adjusted $v \otimes v$ to something that has wedge zero so as to have quantum symmetry.

Corollary 9.17 *For $n > 2$, rotationally invariant quantum symmetric g in the sense of the form above that are 'real' and commute with r, t require a, b to be real constants and $c = r^2$ up to an overall normalisation.*

Proof By considering $[g, r] = 0$, we see that all coefficients commute with r since all basis elements do. As we are looking only among functions of r, t, we deduce that they have no t dependence. Next, by considering $[g, t] = 0$ we see that $[a, t] = [b, t] = 0$ since they are coefficients of terms that commute with t, and the other terms give multiples of themselves. Hence a, b are constants. We also see that $[c, t] = \lambda 2c$ which fixes this as r^2 up to a constant. Note that if we had posited terms such as $b_i(r,t)(v \otimes \mathrm{d}z^i + \mathrm{d}z^i \otimes v)$ and $c_i(r,t)(\mathrm{d}r \otimes \mathrm{d}z^i + \mathrm{d}z^i \otimes \mathrm{d}r)$ then the coefficients here also would have to have no t dependence by the same arguments as above, and in this case such terms cannot be rotationally invariant by the same arguments as classically. Finally, we impose the metric 'reality' requirement which in our cases fixes the coefficients as real. (The arguments for the $\mathrm{d}r, v$ sector are as in Example 1.43.) □

This is also the unique form for $n = 2$ but without the $\mathrm{d}z^i \otimes \mathrm{d}z^i$ term. For $n > 2$, it is the best we can do in terms of a class of quantum metrics that preserve the spatial rotational symmetry and remain as central as possible. We now explore the classical limits of these metrics, i.e. setting $\lambda = 0$ in the above. This is the class of classical metrics that can arise from our quantum spacetimes under our assumptions, namely

$$g = r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) + br^2\mathrm{d}t^2 + \mathrm{d}r^2(a + bt^2) - 2brt\mathrm{d}r\mathrm{d}t$$

in coordinates t, r, θ, φ. One may easily compute the Levi-Civita connection from this and find the classical Ricci curvature $R_{\mu\nu}$ and scalar curvature S as

$$R_{\mu\nu} = \frac{1}{a}\begin{pmatrix} -6b & \frac{6bt}{r} & 0 & 0 \\ \frac{6bt}{r} & -\frac{2(3bt^2+a)}{r^2} & 0 & 0 \\ 0 & 0 & a-3 & 0 \\ 0 & 0 & 0 & (a-3)\sin^2(\theta) \end{pmatrix}, \quad S = \frac{2(a-7)}{ar^2}.$$

We see that we have a curvature singularity at $r = 0$ along the t-axis, although no scalar curvature if $a = 7$. From this one may compute the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{S}{2}g_{\mu\nu}$ and ask when it matches that of a possible matter source. For the latter, we consider a perfect fluid where the stress energy tensor has the form

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu, \tag{9.36}$$

where $u_\mu = g_{\mu\nu}u^\nu$ is the normalised index-lowered 4-velocity of the fluid with $g^{\mu\nu}u_\mu u_\nu = -1$, p is the pressure and ρ is the energy density.

Proposition 9.18 *The Einstein curvature of the above metric matches a perfect fluid in the sense $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ if and only if one of the following holds.*

(1) $b < 0$, $a = 1$, $p = \frac{1}{2\pi G_N r^2}$, $\rho = 0$. *Here* $u^\alpha = (1/(r\sqrt{-b}), 0, 0, 0)$.
(2) $b > 0$, $a = -3$, $p = -\frac{1}{6\pi G_N r^2}$, $\rho = -2p$. *Here* $u^\alpha = (t/r, 1, 0, 0)/\sqrt{3}$.

Proof The form of the energy-momentum tensor means that $G_{\mu\nu} - sg_{\mu\nu}$ has to be a rank 1 matrix for some $s = 8\pi G_N p$. We can solve for this by asking for the determinant of this combination to vanish. This results in three choices for s, namely $4/(ar^2)$, $(5-a)/(ar^2)$ and $(1-a)/(ar^2)$. The first case gives a rank 1 matrix (i.e., a product of a row and a column vector) when $a = 1, -3$ and the other two cases need $a = 1$ and $a = -3$ respectively, which reduces them to the first case. For the sign of b, remember that ab is negative for g_{ij} to have signature $- + + +$ (take the determinant of g_{ij} to see this). We then identify u, ρ. □

The case (1) is some kind of gravitational source at $r = 0$ and is more easily understood if we change variables to $\tilde{t} = t/(r\sqrt{-b})$ and r. Then the $a = 1$ metric looks like

$$g = -b^2 r^4 \mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2,$$

which is of the form of a static metric with flat spatial part. Note that b has dimensions of length^{-2}. One can go further and also define $\tilde{r} = 1/(br)$ then

$$g = \frac{1}{b^2\tilde{r}^4}(-\mathrm{d}t^2 + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2)$$

is a conformal scaling of flat spacetime. Using this, one can solve for the null geodesics (as straight lines) and convert back to our original coordinates.

The case (2), by contrast, makes more sense in coordinates $\tilde{t} = r\sqrt{3}$ and $\tilde{r} = t/(r\sqrt{b})$. Then the $a = -3$ metric looks like

$$g = -\mathrm{d}\tilde{t}^2 + (\frac{b\tilde{t}^2}{3})^2\mathrm{d}\tilde{r}^2 + \frac{\tilde{t}^2}{3}\mathrm{d}\Omega^2,$$

which is some kind of expanding universe with an initial Ricci singularity at $\tilde{t} = 0$, albeit not of the usual form of a single time-dependent scale factor applied to a spatial metric. This case has 'quintessence ratio' $w_Q := \frac{p}{\rho} = -\frac{1}{2}$ of interest in cosmology. One can similarly invert $\tilde{t}$ to render this metric a conformal rescaling of the flat one.

To describe the geodesics more fully, we have to solve the geodesic equations

$$\ddot{x}^\mu = -\Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho,$$

where dot means derivative with respect to an affine parameter τ. The Christoffel symbols here are easily computed from the metric and we just describe the resulting equations. In fact, the θ equation says that motion is in a plane of constant θ and the φ equation that $\dot{\varphi}r^2$ is a constant (this is conservation of angular momentum). The radial and time geodesic equations are

$$a\ddot{r} = r\dot{\varphi}^2 + 2\frac{b}{r}(r\dot{t} - t\dot{r})^2, \quad a\ddot{t} = t\dot{\varphi}^2 - \frac{2a\dot{r}}{r^2}(r\dot{t} - t\dot{r}) + \frac{2bt}{r^2}(r\dot{t} - t\dot{r})^2.$$

From these we find

$$\frac{\mathrm{d}(r\dot{t} - t\dot{r})}{\mathrm{d}\tau} = r\ddot{t} - t\ddot{r} = -\frac{2\dot{r}}{r}(r\dot{t} - t\dot{r}), \tag{9.37}$$

which implies that $r^2(r\dot{t} - t\dot{r}) = M$ is a constant of motion. We also note that $r^{-4}M = r^{-2}(r\dot{t} - t\dot{r}) = \frac{\mathrm{d}}{\mathrm{d}\tau}(t/r)$ so that

$$t = r\Big(\int \frac{M}{r^4}\mathrm{d}\tau + c\Big), \tag{9.38}$$

where c is a constant of integration. Because the angular behaviour is straightforward, we now focus on the radial-time sector by setting $\theta = \pi/2$ and $\varphi = 0$ say, or equivalently we focus on the 2D case where the metric is the same as above except that we omit the angular part. In this context, we also allow $r < 0$ as it is a coordinate not a radius and we are free to normalise a, which we set as $a = 1$ when $b < 0$ and $a = -1$ when $b > 0$.

The length square of the velocity is a further constraint and dropping the angular momentum term, this is

$$a\dot{r}^2 = s - \frac{b}{r^4}M^2,$$

where $s = 0$ for null geodesics and $s = -1$ for timelike. It is always possible here to have $\dot{r}^2 \geq 0$ for r sufficiently small and hence solve for τ as

$$\tau = \pm \int \frac{\mathrm{d}r}{\sqrt{\frac{s}{a} - \frac{b}{ar^4}M^2}}, \tag{9.39}$$

where $\frac{\mathrm{d}\tau}{\mathrm{d}r}$ comes from (9.38). For example, for $s = -1$ and $b < 0$, we have

$$t = r\Big(c \pm M \int \frac{\mathrm{d}r}{r^2\sqrt{-bM^2 - r^4}}\Big)$$

depending on the branch of the τ equation. We start with a geodesic for the positive branch. This is then solved as an elliptic function,

$$t(r) = rc - \frac{M}{D^2}\sqrt{1 - (\frac{r}{D})^4} - \frac{M}{D^3} rE(\frac{r}{D}); \quad E(x) = \int_0^x \frac{u^2}{\sqrt{1-u^4}}\mathrm{d}u$$

for all $|r| \leq D := (-bM^2)^{\frac{1}{4}}$, parametrised by M, c. The motion is shown on the left in Fig. 9.5 starting at $r = -D$ and is necessarily bounded in the region $|r| \leq D$ (for the solutions to exist in view of (9.38)). The proper time to a given r is

$$\tau(r) = D\gamma + \int_0^r \frac{s^2}{\sqrt{D^4 - s^4}}\mathrm{d}s = D\gamma + DE(\frac{r}{D})$$

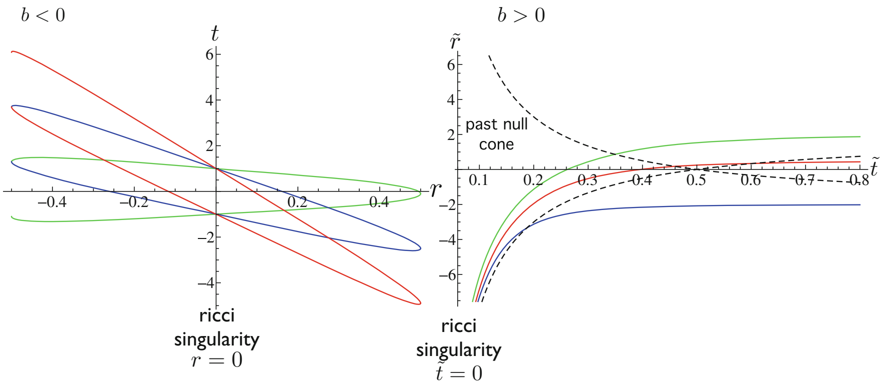

Fig. 9.5 Geodesics for the intrinsic geometry of the algebra $[r, t] = \lambda r$ as 2D spacetime in its classical limit show either (left) a very strong gravitational source at the origin or (right) an expanding universe, depending on the sign of a real parameter

so $D\gamma$ is the proper time from the left boundary at $r = -D$ to $r = 0$, where

$$\gamma = E(1) = \sqrt{\pi}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \approx 0.59907.$$

Thus our geodesic segment starts at proper time $\tau = -D\gamma$ at $(-D, -Dc - \frac{M}{D^2}\gamma)$ and ends at $\tau = D\gamma$ at $(D, Dc - \frac{M}{D^2}\gamma)$. At half way it passes through $r = 0$ and one of two points $P_\pm = (0, \pm\frac{1}{\sqrt{-b}})$ on the t axis depending on the sign of M, in our case P_-. As the geodesic segment approaches the $r = D$ boundary, the value of

$$\ddot{r} = 2b\frac{M^2}{r^5}$$

from the above is finite and retains its negative sign while $\dot{r} \to 0$ in finite proper time. It follows that motion bounces off the boundary and continues with the reversed sign of the branch of the square root so that the radius is now decreasing with proper time. This starts a new geodesic segment which passes through P_+, so in particular we have a timelike geodesic from P_- to P_+. If we allow negative r then the geodesic goes on until $r = -D$, where the geodesic bounces off the boundary and continues back with the positive branch, and so on. In doing so there is precession depending on the value of c, see Fig. 9.5. What this describes is a particle that seeks to move away from $r = 0$ but gets sucked back into the Ricci singularity there at later time. Moreover, as $|M|$ increases, a particle from P_- reaches further out in radius, the geodesic becomes flatter and closer to 45 degrees, but eventually turns back to arrive along an increasingly parallel line through P_+. In the limit $|M| \to \infty$ one has null geodesics as parallel lines through $P_\pm$ except that we understand the lower geodesic now as a photon being unable to escape and turning back at infinite radius to the upper geodesic. In other words, this is something like the inside of a black-hole with even light being unable to escape and the interior region stretched out to infinity.

One can similarly compute the timelike geodesics for $b > 0$ and these are shown on the right in Fig. 9.5. For positive $\tilde{t}$, geodesics start arbitrarily close to $\tilde{t} = 0$ and $\tilde{r} = \mp\infty$ (according to the sign of M) and asymptote to a constant value in a finite proper time from any finite point (but the proper time from $\tilde{r} = \mp\infty$ is infinite). The limit of infinite $|M|$ gives null geodesics (a null cone through a point is shown dashed in the figure; they are hyperbolae $\tilde{r} = \text{constant} \mp \frac{1}{\sqrt{b\tilde{t}}}$). There is no particle horizon in the sense of a limited range above and below a point through which pass all geodesics from $\tilde{t} = 0$ that pass through its backward light cone.

9.4.2 Quantum Connections for the Bicrossproduct Model

Having explored the classical meaning of the quantum metric for the 2D bicrossproduct model with its standard calculus, we now turn to the construction of 'quantum Levi-Civita' connection for it. We also find the larger moduli of metric compatible connections possibly with torsion but under a particular ansatz

$$\begin{aligned}\nabla \mathrm{d}r &= \frac{1}{r}(\alpha v \otimes v + \beta v \otimes \mathrm{d}r + \gamma \mathrm{d}r \otimes v + \delta \mathrm{d}r \otimes \mathrm{d}r),\\ \nabla v &= \frac{1}{r}(\alpha' v \otimes v + \beta' v \otimes \mathrm{d}r + \gamma' \mathrm{d}r \otimes v + \delta' \mathrm{d}r \otimes \mathrm{d}r)\end{aligned} \tag{9.40}$$

for some constants $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$. If it exists then the generalised braiding is determined as

$$\begin{aligned}\sigma(v \otimes v) &= (1 + \lambda\alpha')v \otimes v + \lambda\beta' v \otimes \mathrm{d}r + \lambda\gamma' \mathrm{d}r \otimes v + \lambda\delta' \mathrm{d}r \otimes \mathrm{d}r,\\ \sigma(\mathrm{d}r \otimes v) &= (1 + \lambda\beta)v \otimes \mathrm{d}r + \lambda\alpha v \otimes v + \lambda\gamma \mathrm{d}r \otimes v + \lambda\delta \mathrm{d}r \otimes \mathrm{d}r,\\ \sigma(x \otimes \mathrm{d}r) &= \mathrm{d}r \otimes x,\end{aligned} \tag{9.41}$$

as is the torsion

$$T_\nabla(\mathrm{d}r) = \frac{1}{r}(\lambda\alpha + \beta - \gamma)v \wedge \mathrm{d}r, \quad T_\nabla(v) = \frac{1}{r}(\lambda\alpha' + \beta' - \gamma' + 2)v \wedge \mathrm{d}r.$$

To see where this ansatz comes from, note first of all that the classical Levi-Civita covariant derivative is of this form,

$$\nabla_0(\mathrm{d}r) = \frac{2b}{r} v \otimes v, \quad \nabla_0(v) = -\frac{2}{r} v \otimes \mathrm{d}r.$$

Now suppose that $\nabla = \nabla_0 + \lambda\nabla_1 + O(\lambda^2)$ and determine σ, if it exists, by

$$\sigma(\omega \otimes \mathrm{d}a) = \mathrm{d}a \otimes \omega + [a, \nabla\omega] + \nabla([\omega, a])$$

for all ω, where to $O(\lambda)$ it is enough to calculate this using ∇_0. Next suppose that ∇ is $*$-preserving in the sense of Chap. 8. If we let $\nabla_0(\xi^*) = \eta_0 \otimes \zeta_0$ and $\nabla_1(\xi^*) = \eta_1 \otimes \zeta_1$ (sum of terms) and remember that $\lambda^* = -\lambda$ then this becomes

$$\nabla_0(\xi) + \lambda\nabla_1(\xi) = \sigma(\zeta_0^* \otimes \eta_0^*) - \lambda\sigma(\zeta_1^* \otimes \eta_1^*) = \sigma(\zeta_0^* \otimes \eta_0^*) - \lambda\eta_1^* \otimes \zeta_1^*$$

to $O(\lambda)$ as σ is just the flip map to $O(\lambda^0)$. Finally, our basic 1-forms of interest have $\xi^* = \xi$ to $O(\lambda^0)$, giving

$$\eta_1 \otimes \zeta_1 + \eta_1^* \otimes \zeta_1^* = \frac{\sigma(\zeta_0^* \otimes \eta_0^*) - \nabla_0(\xi)}{\lambda}.$$

This provides a way to compute part of ∇_1 knowing only ∇_0. The missing part can be fixed by imposing that the torsion vanishes and metric compatibility in the hermitian framework of §8.4. Without going into details, the resulting ∇_1 and hence ∇ to first order can be found in our case and indeed has the form of our ansatz. It is also possible to find the quantum Levi-Civita connection to order λ using formulae in §9.6.

Now adopting our ansatz (9.40) and (9.41), we impose that the connection is $*$-preserving. This amounts to

$$\begin{aligned}
\alpha' &= \frac{1}{\lambda\bar{\alpha}}(\alpha - \bar{\alpha} - \lambda\bar{\beta}\alpha + \lambda^2|\alpha|^2),\\
\beta' &= \frac{1}{\lambda\bar{\alpha}}(\beta - \bar{\beta} + \lambda(\bar{\alpha} - |\beta|^2) + \lambda^2\bar{\alpha}\beta),\\
\gamma' &= \frac{1}{\lambda\bar{\alpha}}(\gamma - \bar{\gamma} + \lambda(\bar{\alpha} - \bar{\beta}\gamma) + \lambda^2\bar{\alpha}\gamma),\\
\delta' &= \frac{1}{\lambda\bar{\alpha}}(\delta - \bar{\delta} + \lambda(\bar{\gamma} - \bar{\beta}(\delta - 1)) + \lambda^2\bar{\alpha}(\delta - 1)).
\end{aligned} \tag{9.42}$$

Hence we only need solve for the four variables $\alpha, \beta, \gamma, \delta$, with the primed variables determined. Now using the generalised braiding and the known metric, metric compatibility comes down to 6 nonlinear equations for the 4 complex variables $\alpha, \beta, \gamma, \delta$, which we solve for generic parameter values. Here we only state the solution under a simplifying assumption

$$\alpha, \delta \in \mathbb{R}, \quad \beta, \gamma \in \mathrm{i}\mathbb{R} \tag{9.43}$$

which necessarily holds at order λ (so this is again a motivation). This is a tedious computation and we omit the proof.

Proposition 9.19 *The space of $*$-preserving metric compatible connections of the form (9.43) consists of* (1) *a conic*

$$\beta(\beta + \frac{2}{\lambda}) + \alpha(1 + \lambda\beta) + \frac{\alpha^2}{b}(1 + b\lambda^2) = 0; \quad \gamma = -\frac{\lambda}{2}\alpha, \quad \delta = -\frac{\lambda}{2}\beta,$$

where α is real and β is imaginary and (2) *a line with a real parameter δ,*

$$\alpha = \frac{b}{1 + b\lambda^2}, \quad \beta = -\frac{2}{\lambda}, \quad \gamma = \frac{[2]_\lambda}{2\lambda} - \frac{\delta}{\lambda},$$

and intersecting the conic at $\delta = 1$. Here $[n]_\lambda = \frac{n+b\lambda^2}{1+b\lambda^2}$.

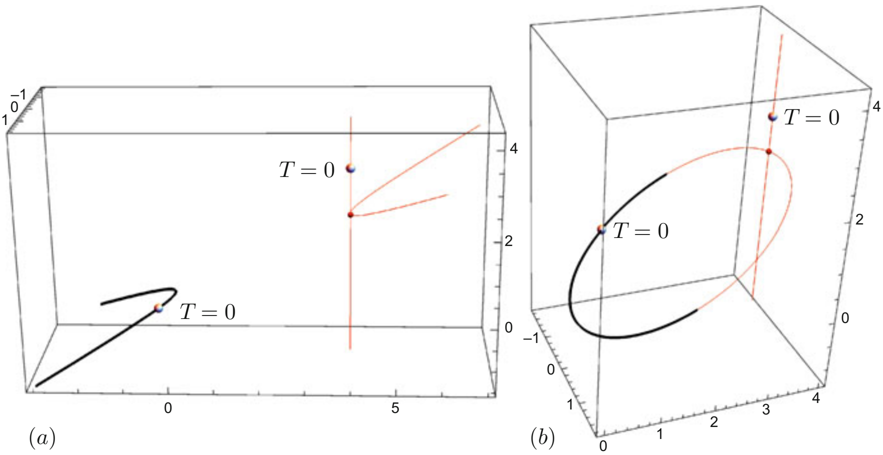

Fig. 9.6 Space of $*$-preserving metric compatible connections at (**a**) $b > 0$ and (**b**) $b < 0$. The bold black curves are the branch of the square-root with classical limit as $\lambda \to 0$, the other part of the conic and intersecting line are nonperturbative. In each case there is a unique torsion free or 'Levi-Civita' point as marked. The axes are α, $-\mathrm{i}\beta$ horizontally and δ vertically

This is depicted in Fig. 9.6. The conic part (1) also has the novel property

$$\gamma = -\frac{\lambda\alpha}{2}, \quad \delta = -\frac{\lambda\beta}{2}, \quad \gamma' = -\frac{\lambda\alpha'}{2}, \quad \delta' = -\frac{\lambda\beta'}{2}$$

whereby the connection is 'decomposable' in the sense that

$$\nabla \mathrm{d}r = \frac{1}{r}(v - \frac{\lambda \mathrm{d}r}{2}) \otimes (\alpha v + \beta \mathrm{d}r), \quad \nabla v = \frac{1}{r}(v - \frac{\lambda \mathrm{d}r}{2}) \otimes (\alpha' v + \beta' \mathrm{d}r).$$

The line part (2) of the moduli space does not have this feature and hence is 'entangled' in this sense.

As shown in the figure, each part of the moduli space has a unique 'quantum Levi-Civita' point where $T_\nabla = 0$. On the conic this is

$$\alpha = \frac{8b}{4 + 7b\lambda^2}, \quad \beta = -\frac{12b\lambda}{4 + 7b\lambda^2}$$

along with decomposability and (9.42) to determine the rest. This comes out as

$$\begin{aligned} \nabla \mathrm{d}r &= \frac{1}{r}(v - \frac{\lambda \mathrm{d}r}{2}) \otimes \Big(\frac{8b}{4 + 7b\lambda^2} v - \frac{12b\lambda}{4 + 7b\lambda^2} \mathrm{d}r\Big), \\ \nabla v &= -\frac{1}{r}(v - \frac{\lambda \mathrm{d}r}{2}) \otimes \Big(\frac{8(1 + b\lambda^2)}{4 + 7b\lambda^2} \mathrm{d}r + \frac{4b\lambda}{4 + 7b\lambda^2} v\Big) \end{aligned} \tag{9.44}$$

for the QLC and has a classical limit as $\lambda \to 0$. The QLC for the line part of the moduli space has

$$\alpha = \frac{b}{1+b\lambda^2}, \quad \beta = -\frac{2}{\lambda}, \quad \gamma = -\frac{[2]_\lambda}{\lambda}, \quad \delta = \frac{3[2]_\lambda}{2}$$

along with (9.42) and has no classical limit $\lambda \to 0$.

Next, for a general connection of the form that we have been considering, the Riemann curvature has the form

$$R_\nabla(\mathrm{d}r) = -\frac{1}{r^2} v \wedge \mathrm{d}r \otimes (c_1 v + c_2 \mathrm{d}r), \quad R_\nabla(v) = -\frac{1}{r^2} v \wedge \mathrm{d}r \otimes (c_3 v + c_4 \mathrm{d}r);$$
$$c_1 = \alpha\left(\lambda\alpha' + \beta\lambda + \gamma' - \delta + 1\right) + \gamma\left(\beta - \alpha'\right),$$
$$c_2 = \alpha\lambda\beta' + \alpha\delta' + \beta^2\lambda - \gamma\beta' + \beta,$$
$$c_3 = \lambda\left(\alpha'\right)^2 + \alpha' + \beta'(\alpha\lambda + \gamma) - \alpha\delta',$$
$$c_4 = \beta'\left(\lambda\alpha' + \beta\lambda - \gamma' + \delta + 1\right) + \delta'\left(\alpha' - \beta\right).$$

If (9.43) holds then we find similarly that

$$c_1, c_4 \in \mathbb{R}, \quad c_2, c_3 \in \mathrm{i}\mathbb{R}. \tag{9.45}$$

Following our approach to the Ricci tensor, we now need a lifting map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ subject to the axioms in §8.1 and which we will choose in such a way that the contracted Ricci tensor after lifting the left output of Riemann has the same quantum symmetry and reality properties as the metric. Without giving all the details, there turns out to be a *unique* such i, namely,

$$\begin{aligned} i(v \wedge \mathrm{d}r) = {} & \frac{1}{2}(v \otimes \mathrm{d}r - \mathrm{d}r \otimes v) \\ & + \frac{bc_3\lambda^2 + bc_4\lambda + c_1\lambda + c_2}{2\left(1+b\lambda^2\right)(c_4 - \lambda(c_2 - c_3 + c_1\lambda))} v \otimes (v - \lambda \mathrm{d}r) \\ & - \frac{(c_3 - \lambda c_1)(c_4 + \lambda(c_3 - (c_2 + \lambda c_1)(2 + b\lambda^2)))}{2(c_1 + bc_3\lambda)(c_4 - \lambda(c_2 - c_3 + c_1\lambda))} \mathrm{d}r \otimes \mathrm{d}r\,, \end{aligned}$$

resulting in

$$\begin{aligned} \mathrm{Ricci} = {} & -\frac{\left(1+b\lambda^2\right)(c_2c_3 - c_1c_4)}{2r^2(c_4 - \lambda(c_2 - c_3 + c_1\lambda))}\Big(v^* \otimes v + \lambda(\mathrm{d}r \otimes v - v^* \otimes \mathrm{d}r) \\ & + \frac{\lambda\left(\left(1+b\lambda^2\right)(c_1\lambda - c_3) + c_2\left(2+b\lambda^2\right)\right) - c_4}{c_1 + c_3 b\lambda} \mathrm{d}r \otimes \mathrm{d}r\Big) \end{aligned}$$

provided the c_i are such that the denominators do not vanish. This Ricci is also central, just like the metric. We also have the Ricci scalar

$$S = (\ ,\ )\mathrm{Ricci} = -\frac{(c_2c_3 - c_1c_4)\left(c_1(1+b\lambda^2)^2 + b(-c_4 + c_2\lambda(2+b\lambda^2))\right)}{2r^2b(c_1+bc_3\lambda)(c_4 - \lambda(c_2 - c_3 + c_1\lambda))}.$$

For the above conic family of metric compatible $*$-preserving connections, we find

$$\mathrm{Ricci} = \frac{2-k}{r^2(k-1)2\alpha\lambda^2}g; \quad k = \pm\sqrt{4 - \frac{\alpha^2\lambda^2}{b}\left(4+3b\lambda^2\right)},$$

where $+$ corresponds to the part of the conic with classical limit. This obeys

$$\mathrm{Ricci} - \frac{S}{(\ ,\ )(g)}g = 0$$

for the entire conic family, in line with the vanishing of Einstein classically in 2D. Specifically, the unique QLC in the conic has

$$\mathrm{Ricci} = \left(\frac{4+7b\lambda^2}{4-9b\lambda^2}\right)\frac{g}{r^2}. \tag{9.46}$$

The other, nonclassical, QLC (in the line) has a more complicated form of Ricci not proportional to the quantum metric, which should not surprise us as vanishing of the Einstein tensor in 2D classically depends on the symmetries of Riemann and these would be very different when we are far from the classical point.

Finally, we compute the quantum geometric wave operator $\square = (\ ,\ )\nabla \mathrm{d}$. As usual, we define partials by $\mathrm{d}\psi(x,t) = (\partial_r\psi)\mathrm{d}r + (\partial_v\psi)v$, where now ∂_r, ∂_v are derivations of the noncommutative algebra since $\mathrm{d}r$, v are central. They obey

$$\partial_v f(r) = 0, \quad \partial_v f(t) = r^{-1}\partial_0 f, \quad \partial_r f(r) = f', \quad \partial_r f(t) = -r^{-1}t\partial_0 f$$

as easily found using the derivation rule, the values on r, t and the relations $tr^{-1} = r^{-1}(t+\lambda)$ in the algebra. Here $\partial_0 f(t) = \lambda^{-1}(f(t+\lambda) - f(t))$ is a finite difference. In these terms, for any normal ordered $\psi(r,t)$ in the algebra and $(\ ,\ )$ given already back in Example 1.43, our unique QLC with classical limit gives

$$\square\psi(r,t) = \frac{(8-6b\lambda^2)\partial_r\psi - \lambda(8+4b\lambda^2)\partial_v\psi}{4+7b\lambda^2}r^{-1} + \frac{\partial_r^2\psi + \lambda\partial_v\partial_r\psi}{1+b\lambda^2} + b^{-1}\partial_v^2\psi.$$

This quantises the Laplacian for the classical metric studied in §9.4.1, where now

$$\Box_0=\frac{2}{r}\partial_r+\partial_r^2+b^{-1}\partial_v^2;\quad \partial_v f=\frac{1}{r}f_{,t},\quad \partial_r f=f_{,r}+\frac{t}{r}f_{,t}$$

in terms of usual partials in the r, t directions. The classical solutions of the massless wave equation are

$$\psi^{\pm}_{\omega}(t,r)=e^{i\frac{\omega t}{r}}e^{\pm i\frac{\omega}{r\sqrt{-b}}}$$

for $b<0$ and with parallel results for $b>0$; the quantum solutions deform these.

9.5 Bertotti–Robinson Quantum Spacetimes

Here we look at the same quantum spacetime algebra A as in the preceding section, namely the familiar bicrossproduct model spacetime $[x^i,t]=\lambda x^i$, but now with the other main choice of translation-invariant calculus in Example 1.43. The analysis there was for the 2D case but if we want a form of calculus that works in all dimensions and treats the x^i symmetrically so as to allow rotational invariance then there is a unique form 1-parameter form. Denoting the parameter by α, this is

$$[t,dx^i]=-\lambda dx^i,\quad [t,dt]=\lambda\alpha dt. \tag{9.47}$$

9.5.1 *Emergence of the Bertotti–Robinson Quantum Metric*

The quantum metric in the 2D case was in Example 1.43, while in general we let

$$g=\sum_{i,j}^{n-1}a_{ij}dx^i\otimes dx^j+\sum_{i}^{n-1}b_i(dx^i\otimes dt+dt\otimes dx^i)+c\,dt\otimes dt,$$

where the coefficients a_{ij},b_i,c are all elements in the quantum spacetime algebra and obey $a_{ij}=a_{ji}$. This form is dictated by quantum symmetry in the form $\wedge(g)=0$. Using the Leibniz rule and the relation (9.47), we have

$$\begin{aligned}[g,t]=&\sum_{i,j}^{n-1}([a_{ij},t]+2\lambda a_{ij})dx^i\otimes dx^j\\&+\sum_{i}^{n-1}([b_i,t]-\lambda(\alpha-1)b_i)(dx^i\otimes dt+dt\otimes dx^i)+([c,t]-2\lambda\alpha c)dt\otimes dt,\end{aligned}$$

$$[g, x^k] = \sum_{i,j}^{n-1} [a_{ij}, x^k] \mathrm{d}x^i \otimes \mathrm{d}x^j + \sum_{i}^{n-1} [b_i, x^k](\mathrm{d}x^i \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^i)$$
$$+ [c, x^k]\mathrm{d}t \otimes \mathrm{d}t.$$

Hence g central amounts to

$$[a_{ij}, t] = -2\lambda a_{ij}, \quad [b_i, t] = \lambda(\alpha - 1)b_i, \quad [c, t] = 2\lambda\alpha c$$

for all i, j, k, and all coefficients commute with the x^k. By similar remarks as in §9.4.1, this means that the coefficients have no time dependence and degrees $-2, \alpha - 1, 2\alpha$ respectively. Thus there is a large moduli of quantum metrics for this differential calculus; we just have to make sure that the coefficients are homogeneous functions of the space coordinates of the appropriate degree. If we look among spherically symmetric quantum metrics then, using our radius r and overdetermined angular variables $z^i = x^i/r$ as before, we have the unique form

$$g = \delta^{-1} \sum_i \mathrm{d}z^i \otimes \mathrm{d}z^i + ar^{-2}\mathrm{d}r \otimes \mathrm{d}r + br^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + cr^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t \tag{9.48}$$

for $\delta, a, b, c \in \mathbb{R}$. Here $\delta > 0$ could be normalised to $\delta = 1$ but we have refrained from this as it has dimensions of inverse square length. This quantum metric is manifestly quantum symmetric and 'real' in the sense of Chap. 8 given that r commutes with $\mathrm{d}x^i$, $\mathrm{d}t$ in this calculus.

To explore the meaning of this quantum metric, we now look at the classical limit

$$g = \delta^{-1}\mathrm{d}\Omega^2 + ar^{-2}\mathrm{d}r \otimes \mathrm{d}r + br^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + cr^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t \tag{9.49}$$

with $b^2 - ac > 0$ for a Minkowski signature. We define another inverse square length

$$\bar{\delta} = \frac{c\alpha^2}{b^2 - ac}$$

and a routine computation gives the Einstein tensor and Ricci scalar as

$$G = -\frac{1}{2}(n-2)(n-3)\delta g + ((n-3)\delta - \bar{\delta})\delta^{-1}\mathrm{d}\Omega^2, \quad S = (n-2)(n-3)\delta + 2\bar{\delta}$$

which we read as zero for the Einstein tensor when $n = 2$, as there is then no angular term. One can show that the metric is conformally flat for $n < 4$, while for $n = 4$ it is conformally flat when $\delta + \bar{\delta} = 0$.

Our first observation is that this G can never match a perfect fluid other than the vacuum energy case of $\rho = -p$ given by $(n-3)\delta = \bar{\delta}$. This is because the one-upper index Einstein tensor $\underline{G}$ is diagonal in our coordinate basis with eigenvalues

$$-\frac{1}{2}(n-2)(n-3)\delta, \quad -\bar{\delta} - \frac{1}{2}(n-3)(n-4)\delta,$$

where the first eigenspace is spanned by the t, r directions and the other eigenspace is spanned by the angular directions. Now if the two eigenvalues of $\underline{G}$ are distinct then we cannot have $\underline{G} = 8\pi G_N(p\mathrm{id} + (p+\rho)U \otimes u)$ for a timelike 1-form u and associated vector field U because this would require u to have only one nonzero entry (since otherwise $U \otimes u$ would have off-diagonals) and in that case adding $U \otimes u$ can only change the eigenvalue in a 1-dimensional subspace, contradicting the equality of the eigenvalues in the r, t subspace.

On the other hand, it can be matched to a Maxwell field electromagnetic curvature

$$F = q\sqrt{b^2 - acr}^{\alpha-1}(\mathrm{d}t \otimes \mathrm{d}r - \mathrm{d}r \otimes \mathrm{d}t) \tag{9.50}$$

when viewed as a tensor product of 1-forms. Its stress energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{F^2}{4}g_{\mu\nu})$$

works out as

$$T = -\frac{q^2}{4\pi}(\frac{g}{2} - \delta^{-1}\mathrm{d}\Omega^2)$$

after a short computation. Here $F^2 = -2q^2$ so that when present (i.e., when $q \neq 0$), the electromagnetic field type is non-null. Comparing with the Einstein tensor, we see that we obey Einstein's equation with cosmological constant Λ if we set

$$\Lambda = \frac{1}{2}\left((n-3)^2\delta + \bar{\delta}\right), \quad q^2 = \frac{1}{2G_N}\left((n-3)\delta - \bar{\delta}\right),$$

which entails $\bar{\delta} \leq (n-3)\delta$, with the case of equality being the vacuum energy solution already noted. For $n = 4$, we see that the cosmological constant vanishes exactly in the conformally flat case $\bar{\delta} = -\delta$, while the Maxwell field strength vanishes exactly in the vacuum energy case $\bar{\delta} = \delta$.

We now show that all cases of our classical metric as we vary α are locally equivalent to a standard 'Bertotti–Robinson' metric up to a change of variables, with in fact only the two real parameters $\delta, \bar{\delta}$. We treat the different signs of $\bar{\delta}$ separately.

(i) If $\bar\delta > 0$ then this implies $c, a + \frac{\alpha^2}{\bar\delta}, \alpha^2 > 0$. We define a change of variables

$$t' = \frac{\alpha}{\sqrt{\bar\delta}} \ln r, \quad r' = \sqrt{c}t - \frac{\sqrt{a + \frac{\alpha^2}{\bar\delta}}}{\alpha r^\alpha}$$

when $b > 0$ and the opposite sign in the 2nd term of r' when $b < 0$. Then our metric becomes

$$g = \delta^{-1}\mathrm{d}\Omega^2 + \mathrm{e}^{2t'\sqrt{\bar\delta}}\mathrm{d}r'^2 - \mathrm{d}t'^2, \tag{9.51}$$

which is a known form of the Bertotti–Robinson metric. Indeed, comparing to 2D de Sitter spacetime dS_2 in a standard ‘flat slicing’

$$g_{dS_2} = \mathrm{e}^{2t\sqrt{\bar\delta}}\mathrm{d}x^2 - \mathrm{d}t^2,$$

the metric is that of a part of $S^{n-2} \times dS_2$ with respective curvature scales $\delta, \bar\delta$.

(ii) If $\bar\delta = 0$ and $\alpha^2 > 0$ then we have $c = 0$ and we use a different change of variables: if $a > 0$ say,

$$r' = \alpha r^\alpha - \frac{1}{\alpha r^\alpha} + \frac{2b}{a}t, \quad t' = \alpha r^\alpha + \frac{1}{\alpha r^\alpha} - \frac{2b}{a}t,$$
$$g = \delta^{-1}\mathrm{d}\Omega^2 + \frac{a}{4\alpha^2}\left(\mathrm{d}r'^2 - \mathrm{d}t'^2\right).$$

If $a < 0$ then we use the same but swap the roles of t', r'. If $\alpha = 0$ then we have a different change of variables given by linear combinations of $\ln r, t$, with a similar conclusion. In all cases, the metric is that of a part of $S^{n-2} \times \mathbb{R}^2$ with sphere curvature scale δ.

(iii) If $\bar\delta < 0$ then $c, a + \frac{\alpha^2}{\bar\delta} < 0 < \alpha^2$ and we define

$$r' = \frac{\alpha}{\sqrt{-\bar\delta}} \ln r, \quad t' = \sqrt{-c}t + \frac{\sqrt{-a - \frac{\alpha^2}{\bar\delta}}}{\alpha r^\alpha}$$

when $b > 0$ and the opposite sign in the second term of t' when $b < 0$. Then our metric becomes

$$g = \delta^{-1}\mathrm{d}\Omega^2 - \mathrm{e}^{2r'\sqrt{-\bar\delta}}\mathrm{d}t'^2 + \mathrm{d}r'^2. \tag{9.52}$$

This should be compared with the metric of 2D anti-de Sitter space AdS_2, a part of which in certain coordinates can be written as

$$g_{AdS_2} = -\mathrm{e}^{2v\sqrt{-\bar\delta}}\mathrm{d}t^2 + \mathrm{d}v^2.$$

We see that the metric is that of a part of $S^{n-2} \times AdS_2$ with respective curvature scales $\delta, \bar{\delta}$.

Note that if we drop the spherical symmetry assumption, i.e., we just ask for classical metrics that are limits of quantum metrics, then we have the allowed form

$$g = h + r^{-1}(\eta \otimes \mathrm{d}r + \mathrm{d}r \otimes \eta) + r^{\alpha}(\zeta \otimes \mathrm{d}t + \mathrm{d}t \otimes \zeta)$$
$$+ ar^{-2}\mathrm{d}r \otimes \mathrm{d}r + br^{\alpha-1}(\mathrm{d}r \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}r) + cr^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t$$

to meet the degree requirements for quantisation, where $h = h_{ij}(z)\mathrm{d}z^i \otimes \mathrm{d}z^j$ is a general metric (not necessarily the usual round one) on S^{n-2}, a, b, c are now functions on S^{n-2}, and η, ζ are further possible 1-forms on S^{n-2}.

Classically, there are more general metrics of a similar form where we replace S^{n-2} in (9.49) by a Riemannian manifold Σ. For $n = 4$, we take Σ a Riemann surface with metric h_Σ and set $\delta = S_\Sigma/2$, where S_Σ is the Ricci scalar curvature of Σ. If we keep a, b, c constant then our above classical calculations go through in the same way and the Einstein tensor suggestively matches the stress-energy of a Maxwell field (9.50) with Λ, q^2 given by the same formulae as before but typically now varying on Σ as δ varies. The constant case $H^2 \times dS_2$ or $H^2 \times AdS_2$, where $\Sigma = H^2$ is the hyperboloid with constant curvature scale $\delta < 0$, completes the standard Bertotti–Robinson family.

9.5.2 The Quantum Connection for the Bertotti–Robinson Model

We start with the $n = 2$ case with $\bar{\delta} > 0$ so we are considering 'quantum de Sitter space', leaving out the $\mathrm{d}z^i \otimes \mathrm{d}z^i$ term from the quantum metric (9.48). The anti-de Sitter case where $\bar{\delta} < 0$ can be handled similarly. In the classical limit in the preceding section, we used a change of variables (9.51) to convert this to de Sitter spacetime for some scale $\bar{\delta} = c\alpha^2/(b^2 - ac)$.

Since r commutes with both $\mathrm{d}t$ and $\mathrm{d}r$, the change of variable we used classically to derive (9.51) works just as well in the quantum case as long as we allow suitable functions of r. Working in the quantum algebra in this case, we let

$$T = \frac{\alpha}{\sqrt{\bar{\delta}}}\ln r, \quad R = \sqrt{c}t - \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{\alpha r^{\alpha}}, \quad g = \mathrm{e}^{2T\sqrt{\bar{\delta}}}\mathrm{d}R \otimes \mathrm{d}R - \mathrm{d}T \otimes \mathrm{d}T,$$

$$\mathrm{d}T = \frac{\alpha}{r\sqrt{\bar{\delta}}}\mathrm{d}r, \quad \mathrm{d}R = \sqrt{c}\mathrm{d}t + \frac{\sqrt{a + \frac{\alpha^2}{\bar{\delta}}}}{r^{\alpha+1}}\mathrm{d}r,$$

changing variables in just the same way as classically. In terms of the new variables, the spacetime commutation relations become

$$[T, R] = [\frac{\alpha}{\sqrt{\delta}} \ln r, \sqrt{c}t] = \lambda'; \quad \lambda' = \lambda\sqrt{b^2 - ac}. \tag{9.53}$$

In other words, T, R are a canonical conjugate pair with Heisenberg relations between them, for a modified parameter λ'. Similarly, using the relations of the α family calculus, we find

$$[R, \mathrm{d}T] = [\sqrt{c}t, \frac{\alpha}{r\sqrt{\delta}}\mathrm{d}r] = \frac{\alpha\sqrt{c}}{\sqrt{\delta}}[t, \frac{\mathrm{d}r}{r}] = 0,$$

$$[R, \mathrm{d}R] = [\sqrt{c}t, \sqrt{c}\mathrm{d}t + \frac{\sqrt{a + \frac{\alpha^2}{\delta}}}{r^{\alpha+1}}\mathrm{d}r] = \lambda c\alpha\mathrm{d}t + \lambda\sqrt{a + \frac{\alpha^2}{\delta}}\frac{\alpha}{r^{\alpha+1}}\mathrm{d}r = \lambda'\sqrt{\delta}\mathrm{d}R$$

and more obviously $[T, \mathrm{d}T] = 0$, $[T, \mathrm{d}R] = 0$. So we have a closed algebra generated by R, T and their differentials, which we now adopt. This is a slightly different model at the algebraic level but has the merits of the spacetime algebra being the standard Heisenberg algebra along with a certain nonstandard quantum differential calculus. We also have $R^* = R$ and $T^* = T$ as our change of variables involved only real coefficients and we suppose as in quantum mechanics that we can extend our Heisenberg algebra so as to include exponentials of T. We can check our calculations by seeing that g is indeed central,

$$[R, g] = [R, e^{2T\sqrt{\delta}}]\mathrm{d}R \otimes \mathrm{d}R + e^{2T\sqrt{\delta}}([R, \mathrm{d}R] \otimes \mathrm{d}R + \mathrm{d}R \otimes [R, \mathrm{d}R]) = 0$$

by the Heisenberg relations for the 1st term and the $[R, \mathrm{d}R]$ relations for the 2nd term.

Proposition 9.20 *The de Sitter quantum spacetime model as above has QLC*

$$\nabla \mathrm{d}R = -\sqrt{\delta}(\mathrm{d}R \otimes \mathrm{d}T + \mathrm{d}T \otimes \mathrm{d}R), \quad \nabla \mathrm{d}T = -\sqrt{\delta}e^{2T\sqrt{\delta}}\mathrm{d}R \otimes \mathrm{d}R.$$

Proof The formula is modelled on the classical one on the generators and we check that this extends as a left quantum connection in the sense that $\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega$ for all 1-forms ω and all elements f of our quantum algebra. Torsion-freeness holds in the sense that $\wedge\nabla\mathrm{d}R = \wedge\nabla\mathrm{d}T = 0$ under the wedge product (here $\mathrm{d}T, \mathrm{d}R$ anticommute as usual from the fact that the $\mathrm{d}t, \mathrm{d}r$ do). We let σ be the 'flip' map on $\mathrm{d}R, \mathrm{d}T$. These values of σ are determined from ∇ by the formula stated, we just have to check that it is well defined when extended 'strongly tensorially' as a bimodule map i.e., commuting with multiplication by elements of the quantum coordinate algebra from either side. It will not simply be a flip on general 1-forms.

For example, using the commutation relations for the calculus,

$$\sigma(\mathrm{d}R.R\otimes\mathrm{d}R)=\sigma(\mathrm{d}R\otimes R\mathrm{d}R)=\sigma(\mathrm{d}R\otimes\mathrm{d}R.R)+\lambda'\sqrt{\bar{\delta}}\sigma(\mathrm{d}R\otimes\mathrm{d}R)$$
$$=\sigma(\mathrm{d}R\otimes\mathrm{d}R)R+\lambda'\sqrt{\bar{\delta}}\sigma(\mathrm{d}R\otimes\mathrm{d}R)=\mathrm{d}R\otimes\mathrm{d}R(R+\lambda'\sqrt{\bar{\delta}}).$$

We also have ∇ 'real' in the sense of Chap. 8. Finally, metric compatibility now makes sense and we compute

$$\nabla g=\nabla(\mathrm{e}^{2T\sqrt{\delta}}\mathrm{d}R)\otimes\mathrm{d}R-\sqrt{\bar{\delta}}\sigma(\mathrm{e}^{2T\sqrt{\delta}}\mathrm{d}R\otimes\mathrm{d}R)\otimes\mathrm{d}T$$
$$-\sqrt{\bar{\delta}}\sigma(\mathrm{e}^{2T\sqrt{\bar{\delta}}}\mathrm{d}R\otimes\mathrm{d}T)\otimes\mathrm{d}R$$
$$-\nabla\mathrm{d}T\otimes\mathrm{d}T+\sqrt{\bar{\delta}}\sigma(\mathrm{d}T\otimes\mathrm{e}^{2T\sqrt{\bar{\delta}}}\mathrm{d}R)\otimes\mathrm{d}R=0.$$

Here the value of ∇ on the second tensor factor has been inserted and σ is used to bring its left output to the far left. When the value of ∇ on the first tensor factor is also inserted and the rules for σ are used, all the terms cancel. □

The curvature R_∇ as a 2-form-valued operator on 1-forms can also be computed and we find

$$R_\nabla\mathrm{d}R=\bar{\delta}\mathrm{d}R\wedge\mathrm{d}T\otimes\mathrm{d}T,\quad R_\nabla\mathrm{d}T=\bar{\delta}\mathrm{e}^{2T\sqrt{\bar{\delta}}}\mathrm{d}R\wedge\mathrm{d}T\otimes\mathrm{d}R.$$

Lifting the 2-forms to antisymmetric tensors and tracing, one then gets Ricci $=\bar{\delta}g$ when normalised in a way that matches the classical conventions. For this, the inverse metric is $(\mathrm{d}R,\mathrm{d}R)=\mathrm{e}^{-2T\sqrt{\bar{\delta}}}$, $(\mathrm{d}T,\mathrm{d}T)=-1$ extended as a bimodule map. These calculations for 'quantum de Sitter geometry' would be much harder in the r,t algebra variables but in the R,T variables, which are very close to classical, we see that they follow the classical form provided we are careful about some of the orderings.

The general case in $n\geq 4$ or 'quantum Bertotti–Robinson space' is not really any different. In the quantum case we do not want to work with angles but work with $z^i=\frac{x^i}{r}$. These commute with r,t and, in the α calculus, so do their differentials $\mathrm{d}z^i=\omega^i/r$, as we saw in §9.3.1. It is also true, again for the α calculus, that the z^i commute with $\mathrm{d}r,\mathrm{d}t$. Hence they describe an entirely classical S^{n-2} which commutes with R,T and their differentials as well. After our change of variables, the quantum metric in §9.3.1 becomes

$$g=\delta^{-1}\mathrm{d}\Omega^2+\mathrm{e}^{2T\sqrt{\delta}}\mathrm{d}R\otimes\mathrm{d}R-\mathrm{d}T\otimes\mathrm{d}T$$

much as before. Now, because the z^i and their differentials decouple from the R,T sector as explained, one can show that the quantum Levi-Civita connection is given by that on the S^{n-2}, which is the same as classically, namely

$$\nabla \mathrm{d}z^i = -z^i \delta^{-1} \mathrm{d}\Omega^2,$$

and the connection in the R, T sector already obtained above. In principle, there could also be other exotic quantum Levi-Civita connections with no classical limit as we saw in §9.4.2. We remark that these formulae look deceptively like their classical counterparts because the angular sector remains classical and decouples while in the radial-time sector the coefficients of the quantum metric in our basis involve only r (or T in the new variables) which, in the α calculus, commutes with itself and both differentials. As long as functions of t (or R) do not enter, we do not see the noncommutation relations. The same applies to the particular Maxwell field (9.50), so Einstein's equation holds at the quantum level if we take the same formulae as before for the stress-energy tensor and the definition of the Einstein tensor.

Finally, we find the quantum-geometric wave operator $\square = (,)\nabla \mathrm{d}$. From the above we have immediately that $\square T = (,)\nabla \mathrm{d}T = -\sqrt{\delta}$ and $\square R = 0$ while for a general normal-ordered function $\psi(T, R)$ with $T's$ to the left, we have

$$\mathrm{d}\psi = (\frac{\partial \psi}{\partial T})\mathrm{d}T + (\partial^1 \psi)\mathrm{d}R; \quad \partial^1 \psi(R) = \frac{\psi(R) - \psi(R - \lambda' \sqrt{\delta})}{\lambda' \sqrt{\delta}}.$$

In these terms, we find

$$\begin{aligned} \square \psi = (,)\nabla \mathrm{d}\psi &= -\sqrt{\delta}\frac{\partial \psi}{\partial T} - \frac{\partial^2 \psi}{\partial T^2} + ((\partial^1)^2 \psi)\mathrm{e}^{-2T\sqrt{\delta}} \\ &= -\sqrt{\delta}\frac{\partial \psi}{\partial T} - \frac{\partial^2 \psi}{\partial T^2} + \mathrm{e}^{-2T\sqrt{\delta}} \Delta \psi \end{aligned}$$

where

$$\Delta \psi(T, R) = \frac{\psi(T, R + 2\lambda' \sqrt{\delta}) - 2\psi(T, R - \lambda' \sqrt{\delta}) + \psi(T, R)}{\left(\lambda' \sqrt{\delta}\right)^2}.$$

We see that the quantum-geometric wave operator when acting on normal-ordered quantum wave functions has the classical form except that the double derivative in the R direction is a finite difference double derivative. This is an identical situation to the standard Minkowski spacetime bicrossproduct model in §9.2.2 except that there the time derivative became a finite difference and there was no full quantum Riemannian geometry as we have here. We therefore have a variable speed of light prediction under similar assumptions about the interpretation, but this time on a curved space. In both cases, the fact that the correction is order λ not order λ^2 can be traced to the normal ordering and is absent if we identify classical and quantum functions by a more symmetric Weyl ordering (as we do implicitly in the next section). We have focussed on the 2D case but the same conclusions hold for the

Bertotti–Robinson quantum metric on $S^{n-1} \times dS_2$ keeping the angular coordinates to the left along with T. Again, only the double R-derivative deforms, namely to the above Δ on normal-ordered functions.

9.6 Poisson–Riemannian Geometry and Nonassociativity

In this section, we consider the semiclassicalisation of the concepts of quantum Riemannian geometry that we have studied in this book, i.e., we suppose the coordinate algebra and the noncommutative geometry has a classical limit as we set a deformation parameter λ to zero and we ask what is the equivalent classical 'quantisation data' that controls the quantum Riemannian geometry at first order in λ. This can be a first step to solving the full quantisation problem, and it can be a paradigm for quantum gravity effects where λ is so small that we can forget λ^2 for most situations. It also solves a conceptual problem of what exactly it is that we quantised. As far as physics is concerned, this remains an effective description of quantum gravity effects (among other applications of the mathematics) in the form of a self-contained paradigm that could be called *classical quantum gravity*. As such, it should be rather more amenable than the full quantum gravity of which it is expected to be a small but particularly relevant part.

The first layer of this is of course the Poisson structure, known since the early days of quantum mechanics as the initial data for 'quantising' functions $C^\infty(M)$ on a manifold M (originally 'phase space' but now spacetime) to a noncommutative algebra A. Here we suppose that A has a parameter λ such that the case $\lambda = 0$ is just $C^\infty(M)$ and that

$$a \bullet b = ab + O(\lambda),$$

where we denote the $C^\infty(M)$ product by juxtaposition and the A product by $\bullet$. We assume all expressions can be expanded in λ and equated order by order. In this case,

$$a \bullet b - b \bullet a = \lambda\{a, b\} + O(\lambda^2)$$

for all $a, b \in C^\infty(M)$ defines a map $\{\ ,\ \}$ and the assumption of an associative algebra quickly leads to this being a Lie bracket (i.e., antisymmetric and satisfies the Jacobi identity, making $C^\infty(M)$ into a Lie algebra) and to the Hamiltonian vector fields $\hat{a} := \{a,\ \}$ being derivations on $C^\infty(M)$. Such a structure on $C^\infty(M)$ is called a *Poisson bracket*. In the other direction, we may start with $C^\infty(M)$ equipped with a Poisson bracket and the '*quantisation problem*' is to find an algebra A over the ring $\mathbb{C}[[\lambda]]$ of formal power series such that $A = C^\infty(M)[[\lambda]]$ as a vector space and the above holds. A formal solution was obtained in the 1990s by Fedosov in the symplectic case (where the tensor underlying the Poisson bracket is invertible) and later by Kontsevich in general. The second layer is to find a differential structure on A as in Chap. 1 in the form of a bimodule Ω^1_A with potentially different left and right products by A. One can analyse the data for such a quantum differential calculus in just the same way, defining some map ∇ by

$$a\bullet(\mathrm{d}b)-(\mathrm{d}b)\bullet a=\lambda\nabla_{\hat{a}}\mathrm{d}b+O(\lambda^2)$$

for all a, b. The assumption of a bimodule, which in turn is part of associativity of the full Ω_A, and the Leibniz rule for d, requires at order λ that

$$\nabla_{\hat{a}}(b\mathrm{d}c)=\{a,b\}\mathrm{d}c+b\nabla_{\hat{a}}\mathrm{d}c,\quad \mathrm{d}\{a,b\}=\nabla_{\hat{a}}\mathrm{d}b-\nabla_{\hat{b}}\mathrm{d}a \tag{9.54}$$

(these follow easily from $[a, b\bullet\mathrm{d}c]=[a,b]\bullet\mathrm{d}c+b\bullet[a,\mathrm{d}c]$ and $\mathrm{d}[a,b]=[\mathrm{d}a,b]+[a,\mathrm{d}b]$). The first condition of (9.54) says that ∇ is a covariant derivative along Hamiltonian vector fields $\hat{a}$ and the second is an additional 'Poisson-compatibility'. Finally, the associativity of left and right actions on a bimodule gives

$$R_\nabla(\hat{a},\hat{b}):=\nabla_{\hat{a}}\nabla_{\hat{b}}-\nabla_{\hat{b}}\nabla_{\hat{a}}-\nabla_{\widehat{\{a,b\}}}=0$$

(this follows from the Jacobi identity $[a,[\mathrm{d}b,c]]+[\mathrm{d}b,[c,a]]+[c,[a,\mathrm{d}b]]=0$). So a flat Poisson-compatible partially-defined connection is what we strictly need.

We are going to make two variations of this, however. First of all, we are *not* going to require zero curvature because the effect of curvature is visible only at order λ^2, so we do not really need this in the order λ theory. If there is curvature then it will not be possible to have an associative differential calculus of classical dimension on A, but this is actually a situation that we have encountered many times in the book (we called it the 'quantum anomaly for differential structures' at the start of §8.3). We can either absorb this into a higher-dimensional associative differential structure or we can live with nonassociative differentials at order λ^2. Strictly speaking, the same applies to the Poisson bracket where obeying the Jacobi identity is not strictly needed if we are willing to have A itself nonassociative at order λ^2. Secondly, for simplicity, we are going to make the assumption that $\nabla_{\hat{a}}$ is indeed the restriction of an actual connection ∇. This will allow us to speak more freely of geometric concepts such as the contorsion tensor (otherwise one needs a less familiar theory of contravariant or Lie–Rinehart connections). In fact, if the Poisson tensor in these coordinates is $\omega^{\mu\nu}$ then in many formulae we can arrange to use only the combination $\nabla^\mu:=\omega^{\mu\nu}\nabla_\nu$ rather than the full covariant derivatives ∇_μ themselves. This means that our data has redundant degrees of freedom that do not affect the differential structure, a situation not unfamiliar from other situations such as gauge theory. There is also the small matter of extending from Ω^1_A analysed above to forms of all degree, but this will turn out to impose no further conditions.

The third layer of the problem is the construction of a quantum metric and the natural data for this will be a classical metric g on M. As one might guess, centrality of the quantum metric at $O(\lambda)$ requires metric compatibility $\nabla g=0$ at least along

Hamiltonian vector fields. We will suppose this in full and to avoid confusion with the classical Levi-Civita connection of g, we write the latter as $\widehat{\nabla}$. We then let S be the contorsion tensor of ∇ whereby $\widehat{\nabla} = \nabla + S$. It is well known in general relativity that a metric compatible connection is determined by its torsion tensor $T^\alpha{}_{\beta\gamma}$ or equivalently its cotorsion tensor $S^\alpha{}_{\beta\gamma}$ characterised as antisymmetric in its outer indices when all indices are lowered. Hence under our simplifying assumption the data for ∇ can be thought of as T or S. In this case, Poisson compatibility of ∇ can be written as

$$\widehat{\nabla}_\gamma \omega^{\alpha\beta} + S^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + S^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0. \tag{9.55}$$

The fourth layer of the problem is more specialised as the quantisation data for a bimodule quantum Levi-Civita connection. The condition for this in the case of the canonical first-order quantisation of the metric is

$$\widehat{\nabla}_\rho \mathcal{R}_{\mu\nu} + S^\beta{}_{\alpha\nu} H^\alpha{}_{\beta\rho\mu} - S^\beta{}_{\alpha\mu} H^\alpha{}_{\beta\rho\nu} = 0 \tag{9.56}$$

(which is easily adapted with an additional term for any other quantisation of the metric). Here the curvature R of ∇ was combined with the contorsion to define

$$H^\alpha{}_{\beta\mu\nu} := g_{\beta\gamma}\omega^{\gamma\rho}\left(\nabla_\rho S^\alpha{}_{\mu\nu} + R^\alpha{}_{\nu\mu\rho}\right), \quad \mathcal{R}_{\mu\nu} := \frac{1}{2}\left(H^\alpha{}_{\alpha\mu\nu} - H^\alpha{}_{\alpha\nu\mu}\right). \tag{9.57}$$

The latter is called the *generalised Ricci 2-form* associated to our classical data. In summary, the field equations of Poisson–Riemannian geometry come down to

(1) A metric $g_{\mu\nu}$ and an antisymmetric bivector $\omega^{\mu\nu}$ typically obeying the Poisson bracket Jacobi identity;
(2) A metric compatible connection ∇ at least along Hamiltonian vector fields;
(3) Poisson-compatibility of ∇ given in the fully defined case by (9.55);
(4) The optional quantum Levi-Civita condition (9.56).

In what follows, we give a brief introduction to how this data implies the lowest order quantisation and we provide a couple of examples.

9.6.1 Semiquantisation Constructions

Semiquantisation means formally that we work over the ring $\mathbb{C}[\lambda]/\langle\lambda^2\rangle$ rather than over power series in λ. We will present this as quantisation to order λ, meaning we work to errors $O(\lambda^2)$. This is not of course quantisation itself but a semiclassical theory exhibited as a first step on the road to quantisation. We denote by A the semiquantised algebra in this sense, built on the vector space $C^\infty(M)$ extended as explained over the ring. We will work in a local coordinate basis for the manifold M so that

$$\{f, h\} = \omega^{\alpha\beta}(\partial_\alpha f)\partial_\beta h, \quad \nabla_\nu \mathrm{d}x^\mu = -\Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\rho,$$

where Γ are the Christoffel symbols of our (partial) linear connection ∇. We adopt standard conventions where

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}, \quad R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\delta\beta,\gamma} - \Gamma^\alpha{}_{\gamma\beta,\delta} + \Gamma^\kappa{}_{\delta\beta}\Gamma^\alpha{}_{\gamma\kappa} - \Gamma^\kappa{}_{\gamma\beta}\Gamma^\alpha{}_{\delta\kappa}$$

are the torsion and Riemann curvature tensors, and we use subscript comma to denote partial differentiation and semicolon for covariant derivative ∇. This agrees with our more abstract treatment of connections in Chap. 3. When we have a metric $g_{\mu\nu}$, we similarly define the Christoffel symbols $\widehat{\Gamma}^\mu{}_{\nu\rho}$ of the Levi-Civita connection $\widehat{\nabla}$ and we also use the metric to raise or lower indices. If ∇ is also metric compatible then its contorsion tensor S defined by

$$\Gamma^\alpha{}_{\beta\gamma} = \widehat{\Gamma}^\alpha{}_{\beta\gamma} + S^\alpha{}_{\beta\gamma}$$

obeys $S_{\alpha\beta\gamma} = -S_{\gamma\beta\alpha}$. Such a tensor is equivalent to specifying the torsion via

$$S^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(T_{\delta\beta\gamma} + T_{\beta\delta\gamma} + T_{\gamma\delta\beta}), \quad T^\alpha{}_{\beta\gamma} = S^\alpha{}_{\beta\gamma} - S^\alpha{}_{\gamma\beta}. \tag{9.58}$$

In this metric compatible case, $R_{\alpha\beta\gamma\delta}$ is antisymmetric in the first pair of indices (as well as the second).

Lemma 9.21 ∇ *is Poisson compatible if and only if*

$$\nabla_\gamma\omega^{\alpha\beta} + T^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + T^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0.$$

Proof Poisson-compatibility from its definition means

$$\mathrm{d}(\omega^{\alpha\beta}) - \omega^{\delta\beta}\nabla_\delta(\mathrm{d}x^\alpha) - \omega^{\alpha\delta}\nabla_\delta(\mathrm{d}x^\beta) = 0,$$

which in terms of Christoffel symbols is

$$\omega^{\alpha\beta}{}_{,\gamma} + \omega^{\delta\beta}\Gamma^\alpha{}_{\delta\gamma} + \omega^{\alpha\delta}\Gamma^\beta{}_{\delta\gamma} = 0.$$

We write the expression on the left as

$$\omega^{\alpha\beta}{}_{,\gamma} + \omega^{\delta\beta}\Gamma^\alpha{}_{\gamma\delta} + \omega^{\alpha\delta}\Gamma^\beta{}_{\gamma\delta} + \omega^{\delta\beta}T^\alpha{}_{\delta\gamma} + \omega^{\alpha\delta}T^\beta{}_{\delta\gamma}$$

and recognise the first three terms as the covariant derivative. □

This is then equivalent to (9.55) in terms of S when we have a metric. We also normally want ω to be a Poisson tensor, even though this is not strictly needed at order λ, which requires the condition

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu}\omega^{\beta\gamma}{}_{,\mu} = 0$$

or equivalently, given Poisson-compatibility,

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu}\omega^{\beta\nu}T^{\gamma}{}_{\mu\nu} = 0. \tag{9.59}$$

We now start to construct using the above data. Given the Poisson tensor and the condition in Lemma 9.21 we 'quantise' the product of functions y, z with each other and with 1-forms η by,

$$y \bullet z = yz + \frac{\lambda}{2}\{y, z\}, \quad y \bullet \eta = y\eta + \frac{\lambda}{2}\omega^{ab}y_{,a}\nabla_b\eta, \quad \eta \bullet y = \eta y - \frac{\lambda}{2}\omega^{ab}y_{,a}\nabla_b\eta \tag{9.60}$$

to order λ, which implies commutation relations

$$[x^\alpha, \eta]_1 = \lambda\omega^{\alpha\beta}\nabla_\beta\eta \tag{9.61}$$

to order λ for the 1-forms Ω^1_A. Here (9.60) is the symmetric or 'Weyl ordering' quantisation (there are other ways to match classical and quantum products to get the same commutators).

Proposition 9.22 *Given a Poisson-compatible connection as in Lemma 9.21, the semiquantised Ω^1_A extends to an exterior algebra Ω_A to order λ where the exterior derivative* d *is taken as undeformed on the underlying vector spaces, and the deformed wedge product is*

$$\mathrm{d}x^\alpha \wedge_1 \mathrm{d}x^\beta = \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta + \frac{\lambda}{2}\omega^{\gamma\delta}\nabla_\gamma \mathrm{d}x^\alpha \wedge \nabla_\delta \mathrm{d}x^\beta + \lambda H^{\alpha\beta}$$

(the general form is below) giving the anticommutation relations

$$\{\mathrm{d}x^\alpha, \mathrm{d}x^\beta\}_1 = \lambda\omega^{\gamma\delta}\Gamma^\alpha{}_{\gamma\mu}\Gamma^\beta{}_{\delta\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + 2\lambda H^{\alpha\beta},$$

where

$$H^{\alpha\beta} = -\frac{1}{2}H^{\alpha\beta}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

Proof This takes a lot of work, including use of the Bianchi identities for connections with torsion (all of which we omit). However, it is also possible to derive the first displayed formula by applying d to the expression for $x^\alpha \bullet \mathrm{d}x^\beta$ from (9.60) and then rearranging indices of the Christoffel symbols. Similarly applying d to (9.61) gives the displayed anti-commutation relations after some tensor calculus and use of Poisson-compatibility. (The calculus is the maximal prolongation of the first-order calculus.) For the general deformed wedge product, there are similarly two parts, one part $\wedge_Q$ is functorial and the other is a 'quantum correction' involving H,

$$\xi \wedge_Q \eta = \xi \wedge \eta + \frac{\lambda}{2}\omega^{\alpha\beta}\nabla_\alpha \xi \wedge \nabla_\beta \eta,$$

$$\xi \wedge_1 \eta = \xi \wedge_Q \eta + \lambda(-1)^{|\xi|+1} H^{\alpha\beta} \wedge i_\alpha \xi \wedge i_\beta \eta$$

for $\xi, \eta \in \Omega(M)$, where i_α denotes interior product along $\partial/\partial x^\alpha$. $\square$

Note that because the product by functions is modified, the quantum tensor product over the quantum algebra A, which we denote $\otimes_1$, is not the usual tensor product over $C^\infty(M)$, which we denote $\otimes_0$. It is characterised by

$$\eta \otimes_1 y \bullet \zeta = \eta \bullet y \otimes_1 \zeta$$

for all functions y and any η, ζ. There are no problems with associativity to first order in λ and we can write multiple tensor products such as $\Omega^1_A \otimes_1 \Omega^1_A \otimes_1 \Omega^1_A$ without brackets. (Things are not so simple at the next order λ^2.) The quantum and classical tensor products are in fact identified by a natural transformation q to order λ best understood as part of a monoidal functor

$$Q : \text{Vect bundles with connection over } M\} \to \{A\text{-bimodules with connection}\}$$

to order λ. This associates to a classical vector bundle with sections E and connection ∇_E a A-bimodule $Q(E)$ to order λ which is E as a vector space and has actions

$$y \bullet e = ye + \frac{\lambda}{2}\omega^{\alpha\beta} y_{,\alpha} \nabla_{E\beta} e, \quad e \bullet y = ey - \frac{\lambda}{2}\omega^{\alpha\beta} y_{,\alpha} \nabla_{E\beta} e. \tag{9.62}$$

In particular, (9.60) says that $\Omega^1_A = Q(\Omega^1)$ as bimodules. Given a bundle map $T : E \to F$, we have a left A-module map $Q(T) : Q(E) \to Q(F)$ to order λ by

$$Q(T) = T + \frac{\lambda}{2}\omega^{\alpha\beta}\nabla_{F_\alpha} \circ \boldsymbol{\nabla}_\beta(T),$$

where $\boldsymbol{\nabla}_\beta(T) = \nabla_{F\beta} \circ T - T \circ \nabla_{E\beta}$. We are mainly interested in the case where morphisms on the vector bundles are bundle maps commuting with the connections (i.e., $\boldsymbol{\nabla}_\beta(T) = 0$), in which case we have $Q(T) = T$ as an A-bimodule map to order λ. Associated to Q are order λ natural isomorphisms

$$q_{E,F} : Q(E)\otimes_1 Q(F) \to Q(E\otimes_0 F),\ \ q_{E,F}(e\otimes_1 f) = e\otimes_0 f + \frac{\lambda}{2}\omega^{\alpha\beta}\nabla_{E\alpha} e\otimes_0 \nabla_{F\beta} f$$

between the tensor products for all classical pairs (E, ∇_E), (F, ∇_F). Using the special case $q_{\Omega^1,E}$ for the classical connection (Ω^1, ∇), we define a left A-bimodule connection $\nabla_{Q(E)}$ on $Q(E)$ by

$$\nabla_{Q(E)} = q^{-1}_{\Omega^1,E}\nabla_E - \frac{\lambda}{2}\omega^{\alpha\beta}\mathrm{d}x^\gamma \otimes_1 [\nabla_{E\gamma}, \nabla_{E\beta}]\nabla_{E\alpha},$$

$$\sigma_{Q(E)}(e\otimes_1 \xi) = \xi \otimes_1 e + \lambda\omega^{\alpha\beta}\nabla_\beta\xi \otimes_1 \nabla_{E\alpha} e + \lambda\omega^{\alpha\beta}\xi_\beta \mathrm{d}x^\gamma \otimes_1 [\nabla_{E\gamma}, \nabla_{E\alpha}]e$$

for all $e \in E$ and $\xi \in \Omega^1$. This completes our brief sketch of the order λ monoidal functor (Q, q) between the bimodule connection categories. It is not in our scope to give details, but using (Q, q) followed by a further quantum correction is the basis of most of our constructions, as we saw for the wedge product. As an application of this method, we give a construction of a quantum Riemannian metric.

Corollary 9.23 *Let g be a classical metric on M and ∇ be both Poisson and metric compatible. Then*

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\left(\mathcal{R}_{\mu\nu} + \omega^{\alpha\beta}\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu}\right)\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu,$$

$$(\mathrm{d}x^\mu, \mathrm{d}x^\nu)_1 = g^{\mu\nu} - \frac{\lambda}{2}\left(\mathcal{R}^{\mu\nu} - \omega^{\alpha\beta}g^{\eta\zeta}\Gamma^\mu{}_{\alpha\eta}\Gamma^\nu{}_{\beta\zeta}\right)$$

is a natural quantum metric and its inverse to order λ, where $\mathcal{R}_{\mu\nu}$ is in (9.57).

Proof First, the functorial construction using Q above gives

$$g_Q = q^{-1}_{\Omega^1,\Omega^1}(g) = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu,$$

and will necessarily be central to order λ as the λ term of g_Q commutes to $O(\lambda)$ and

$$\begin{aligned}[x^\kappa, g_Q]_1 &= [x^\kappa, g_{\mu\nu}\mathrm{d}x^\mu]_1 \otimes_1 \mathrm{d}x^\nu + g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 [x^\kappa, \mathrm{d}x^\nu]_1\\ &= \lambda\omega^{\kappa\delta}\nabla_\delta(g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu)\end{aligned}$$

to order λ. On the other hand, we find

$$\wedge_1 g_Q = \lambda\mathcal{R};\quad \mathcal{R} = g_{\alpha\beta}H^{\alpha\beta},$$

so we obtain quantum symmetry by a further 'quantum correction' as stated. To check the expression for the inverse quantum metric to order λ, one needs metric compatibility of ∇ in the form $g_{\gamma\nu,\beta} = \Gamma_{\gamma\beta\nu} + \Gamma_{\nu\beta\gamma}$. □

Note that one can write $g_1 = \mathrm{d}x^\mu \otimes_1 \tilde{g}_{\mu\nu} \bullet \mathrm{d}x^\nu$ with coefficients in the middle, in which case the commutation relations imply the quantum correction

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2} h_{\mu\nu}; \quad h_{\mu\nu} = \mathcal{R}_{\mu\nu} + \omega^{\alpha\beta}(\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu} + \Gamma^\gamma{}_{\alpha\mu} g_{\gamma\nu,\beta}) = -h_{\nu\mu}$$

to order λ. Moreover, $(\mathrm{d}x^\mu, \mathrm{d}x^\nu) = \tilde{g}^{\mu\nu}$ is the A-valued matrix inverse to $\tilde{g}_{\mu\nu}$. We now turn to the quantum Levi-Civita connection for the above quantum metric. Again, this is beyond our scope and we provide only an outline. Recall that $\widehat{\Gamma}^\rho{}_{\mu\nu}$ are the Christoffel symbols of the classical Levi-Civita connection $\widehat{\nabla}$.

Theorem 9.24 *There is a unique quantum connection $\nabla_1 \mathrm{d}x^\rho = -\Gamma_1{}^\rho{}_{\mu\nu} \mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu$ to order λ which to this order is quantum torsion free and for which the symmetric part in the last two factors of $\nabla_1 g_1$ vanishes, namely*

$$\Gamma_1{}^\rho{}_{\mu\nu} = \widehat{\Gamma}^\rho{}_{\mu\nu} + \frac{\lambda}{2}\omega^{\alpha\beta}\big(\widehat{\Gamma}^\rho{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu} - \widehat{\Gamma}^\rho{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu} + \widehat{\Gamma}^\rho{}_{\alpha\kappa}(R^\kappa{}_{\nu\mu\beta} + \nabla_\beta S^\kappa{}_{\mu\nu})\big).$$

This ∇_1 is a QLC to order λ if and only if (9.56) holds.

Proof The calculations are rather long with the result that here we only sketch some of the ideas. Applying the semiquantisation functor to Ω^1, the classical connection ∇ gets quantised as a map $\nabla_Q : \Omega^1_A \to \Omega^1_A \otimes_1 \Omega^1_A$ together with $\sigma_Q : \Omega^1_A \otimes_1 \Omega^1_A \to \Omega^1_A \otimes_1 \Omega^1_A$ to make it a bimodule connection to order λ. In indices, the connection comes out as

$$\nabla_Q \mathrm{d}x^\rho = -\big(\Gamma^\rho{}_{\mu\nu} + \frac{\lambda}{2}\omega^{\alpha\beta}(\Gamma^\rho{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu} - \Gamma^\rho{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu} - \Gamma^\rho{}_{\beta\kappa}R^\kappa{}_{\nu\mu\alpha})\big)\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu.$$

We also functorially quantise the contorsion to a quantum one $Q(S)$ and also to allow a further $O(\lambda)$ adjustment by a classical tensor K, i.e., we search for a quantum Levi-Civita connection in the form

$$\nabla_1 = \nabla_Q + q^{-1}Q(S) + \lambda K.$$

This ansatz is then close enough to uniquely determine K and the result is the expression stated when we just ask for the symmetric part of $\nabla_1 g_1$ to vanish. The remainder of $\nabla_1 g_1$ is then the left-hand side of (9.56). Interestingly, as illustrated in §9.4.2, being $*$-preserving is actually enough to fix K and gives the same answer when this condition holds. There is also a generalised braiding $\sigma_1 : \Omega^1_A \otimes_1 \Omega^1_A \to \Omega^1_A \otimes_1 \Omega^1_A$ given by

$$\begin{aligned}\sigma_1(\mathrm{d}x^\alpha \otimes_1 \mathrm{d}x^\beta) &= \sigma_Q(\mathrm{d}x^\alpha \otimes_1 \mathrm{d}x^\beta) + \lambda\omega^{\beta\mu}(\nabla_\mu S)(\mathrm{d}x^\alpha)\\ &= \mathrm{d}x^\beta \otimes_1 \mathrm{d}x^\alpha + \lambda\left(\omega^{\mu\nu}\Gamma^\alpha{}_{\mu\gamma}\Gamma^\beta{}_{\nu\delta} - \omega^{\mu\beta}(R^\alpha{}_{\gamma\delta\mu} + S^\alpha{}_{\delta\gamma;\mu})\right)\mathrm{d}x^\delta \otimes_1 \mathrm{d}x^\gamma. \quad \square\end{aligned}$$

To round off the theory and relate to earlier themes in this chapter, similar but involved calculations give the order λ quantum-geometric wave operator arising from the above as

$$\Box_1\psi := (\,,\,)_1\nabla_1 \mathrm{d}\psi = \Box\psi + \frac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha})(\widehat{\nabla}_\beta \mathrm{d}\psi)_\gamma \tag{9.63}$$

for $\psi \in C^\infty(M)$, where ; and Ric refer to ∇ and $S^\gamma = S^\gamma{}_{\alpha\beta}g^{\alpha\beta}$. In Poisson–Riemannian geometry there is from the start an identification of classical and quantum functions as built essentially on the same vector space.

9.6.2 Some Solutions of the PRG Equations

Clearly, we can take any of our quantum models of Chap. 8 and §9.4 and §9.5 deforming a commutative algebra and 'differentiate' to get the underlying solutions of the Poisson–Riemannian geometry or PRG equations with a flat ∇. Note that our construction of an order λ quantum metric g_1 was not unique and we may not necessarily land on this exactly; if not then the condition for a QLC to order λ will need an extra term reflecting the difference in quantum metrics.

Example 9.25 (Bicrossproduct Model with Curved Metric) We take g the curved metric in §9.4.1. The 2D bicrossproduct model quantum spacetime and its $\beta = 1$ calculus quantise the Poisson tensor and flat connection

$$\omega^{10} = r = -\omega^{01}, \quad \Gamma^0{}_{10} = r^{-1} = -\Gamma^0{}_{10};$$

$$\nabla \mathrm{d}r = 0, \quad \nabla \mathrm{d}t = r^{-1}(\mathrm{d}t \otimes \mathrm{d}r - \mathrm{d}r \otimes \mathrm{d}t).$$

Letting $v = r\mathrm{d}t - t\mathrm{d}r$, we have $\nabla \mathrm{d}r = \nabla v = 0$ so a pair of central 1-forms $v, \mathrm{d}r$ at least at first order. Next we take $g = \mathrm{d}r \otimes \mathrm{d}r + bv \otimes v$ where b is a nonzero real parameter which clearly has inverse $(\mathrm{d}r, \mathrm{d}r) = 1$, $(v, v) = b^{-1}$, $(\mathrm{d}r, v) = (v, \mathrm{d}r) = 0$, as the unique form of classical metric for which $\nabla g = 0$ for the above Poisson-compatible connection. The Levi-Civita connection for g is $\widehat{\nabla} v = -\frac{2v}{r} \otimes \mathrm{d}r$, $\widehat{\nabla}\mathrm{d}r = \frac{2bv}{r} \otimes v$ as studied in §9.4.1. In tensor terms for our coordinate basis $x^0 = t, x^1 = r$, the metric tensor and Levi-Civita connection are

$$g_{\mu\nu} = \begin{pmatrix} br^2 & -brt \\ -brt & 1+bt^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1+bt^2}{br^2} & \frac{t}{r} \\ \frac{t}{r} & 1 \end{pmatrix},$$

$$\widehat{\Gamma}^0{}_{\mu\nu} = \begin{pmatrix} -2bt & r^{-1}(1+2bt^2), \\ r^{-1}(1+2bt^2) & -2r^{-2}t(1+bt^2) \end{pmatrix}, \quad \widehat{\Gamma}^1{}_{\mu\nu} = \begin{pmatrix} -2br & 2bt \\ 2bt & -2br^{-1}t^2 \end{pmatrix}.$$

The contorsion, generalised Ricci 2-form and quantum metric correction are then

$$S^{\kappa}{}_{\alpha\beta} = 2b\epsilon_{\alpha\mu}x^{\mu}\epsilon_{\beta\nu}g^{\kappa\nu}, \quad S^{\mu} = 2\frac{x^{\mu}}{r^2}, \quad \mathcal{R}_{\mu\nu} = -br\epsilon_{\mu\nu}, \quad h_{\mu\nu} = -3br\epsilon_{\mu\nu},$$

where $\epsilon_{01} = 1$ is antisymmetric. The condition (9.56) holds and the quantum Levi-Civita connection at first order comes out as

$$\nabla_1 \mathrm{d}r = -\widehat{\Gamma}^{1}{}_{\mu\nu}\mathrm{d}x^{\mu} \otimes_1 \mathrm{d}x^{\nu} - 2\lambda\frac{b}{r}v \otimes_1 \mathrm{d}r,$$
$$\nabla_1 \mathrm{d}t = -\widehat{\Gamma}^{0}{}_{\mu\nu}\mathrm{d}x^{\mu} \otimes_1 \mathrm{d}x^{\nu} - \frac{\lambda}{2r^2}\mathrm{d}r \otimes_1 \mathrm{d}r - 2\lambda\frac{b}{r}v \otimes_1 \mathrm{d}t.$$

After a somewhat careful analysis, one can show that these formulae as well as g_1 agree with the leading order part of the quantum model in §9.4.2. This is an example where the Laplacian *is* deformed at order λ. ◇

Next, we already found the 4D quantum geometry on $U(su_2)$ in Example 8.15 which quotients to a 3D calculus on the fuzzy sphere, there being no rotationally invariant 2D calculus on the latter. An alternative is the following.

Example 9.26 (Nonassociative Calculus on the Fuzzy Sphere) Viewed as $\mathbb{C}P^1$, this is a Kahler manifold and as such it has a symplectic structure inverting to a canonical and covariantly constant Poisson tensor ω. This means that we can solve the Poisson-compatibility in Lemma 9.21 by $T = 0$, i.e., we can take $\nabla = \widehat{\nabla}$, the Levi-Civita connection with contorsion $S = 0$. This will, however, have curvature so the differential calculus will fail to be associative at order λ^2. To see what this model entails in detail, we work with the angular generators z^i, $i = 1, 2, 3$, and the relation $\sum_i (z^i)^2 = 1$ as in §9.3.1. We designate coordinates $z^{\mu} = z^1, z^2$ but could equally well work in another patch, and we take $\omega^{\mu\nu} = \epsilon^{\mu\nu}z^3$ (in terms of the totally antisymmetric tensor with $\epsilon^{12} = 1$) as the inverse of the volume form. Together with the Levi-Civita connection coefficients $\Gamma^{\mu}{}_{\alpha\beta} = \widehat{\Gamma}^{\mu}{}_{\alpha\beta} = z^{\mu}g_{\alpha\beta}$, this gives us a particular 'fuzzy sphere' with differential calculus

$$[z^i, z^j]_{\bullet} = \lambda\epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j]_{\bullet} = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n$$

to order λ. These are initially computed for $i = 1, 2$ but must hold in this form for $i = 1, 2, 3$ by rotational symmetry of both the Poisson bracket and the Levi-Civita connection. Note that the totally antisymmetric tensor with $\epsilon_{123} = 1$ is taken with the same values when some of the indices are raised. One also finds from the algebra that $z^m \bullet \mathrm{d}z^m = 0$ (sum over $m = 1, 2, 3$) at order λ on differentiating the radius 1 relation. Here Ω^1 is a projective module with $\mathrm{d}z^i$ as a redundant set of generators and one relation. We also have

$$\{\mathrm{d}z^i, \mathrm{d}z^j\}_{\bullet} = \lambda(3z^i z^j - \delta_{ij})\mathrm{Vol}$$

to order λ in terms of the top form Vol characterised by $\mathrm{d}z^i \wedge \mathrm{d}z^j = \epsilon^{ij}{}_k z^k \mathrm{Vol}$. These relations can also be derived by applying d to the bimodule relations, i.e., we work in the maximal prolongation. The classical sphere metric is $g = \sum_{i=1}^3 \mathrm{d}z^i \otimes \mathrm{d}z^i$ as we know from §9.3.1, from which one can extract $g_{\mu\nu}$. Similarly, the classical inverse metric and metric inner product are

$$g^{\mu\nu} = \delta_{\mu\nu} - z^\mu z^\nu, \quad (\mathrm{d}z^i, \mathrm{d}z^j) = \delta_{ij} - z^i z^j = g^{ij}$$

derived for $\mu, \nu = 1, 2$, but extended as the second equality for $i, j = 1, 2, 3$. The quantum metric correction and inverse quantum metric to lowest order are then

$$h_{\mu\nu} = \frac{(2 - (z^3)^2)}{(z^3)^3}\epsilon_{\mu\nu}, \quad (\mathrm{d}z^i, \mathrm{d}z^j)_1 = g^{ij} + \frac{\lambda}{2}\epsilon_{ijk} z^k,$$

where we extended to $i, j = 1, 2, 3$ with $g^{ij} = \delta_{ij} - z^i z^j$. Here we use the totally antisymmetric tensor $\epsilon_{\mu\nu}$ with $\epsilon_{12} = 1$. The quantum connection to lowest order is

$$\begin{aligned}\nabla_1 \mathrm{d}z^\mu &= -z^\mu \bullet g_1\\ &= -\widehat{\Gamma}^\mu{}_{\alpha\beta} \mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2(z^3)^2}\Big(\epsilon_{3\alpha\beta} z^\mu z^3 \mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \epsilon^\mu{}_{\nu 3} \mathrm{d}z^3 \otimes_1 \mathrm{d}z^\nu\Big)\end{aligned}$$

for $\mu = 1, 2$. ◇

9.6.3 *Quantisation by Twisting*

Here we look not at Poisson–Riemannian geometry exactly but at an earlier 'twisting' construction which can be used to control nonassociativity when this is present, for example when the Poisson-compatible connection has curvature. First we give the general construction, which we have already touched upon at various points in the book. Let H be a Hopf algebra and $F \in H^{\otimes 2}$ be invertible (with inverse written $F^{-1} \in H^{\otimes 2}$) and obey the *cocycle condition*

$$(\epsilon \otimes \mathrm{id})F = 1, \quad F_{12}(\Delta \otimes \mathrm{id})F = F_{23}(\mathrm{id} \otimes \Delta)F, \tag{9.64}$$

where Δ is the coproduct of H. Without the second part of the cocycle condition one says that F is a *cochain* and in this case we separately demand that $(\mathrm{id} \otimes \epsilon)F = 1$ holds. In the cochain case there is a 'quasi-Hopf algebra' (the 'Drinfeld twist') H^F with a certain 'quasi-coproduct'

$$\Delta_F = F(\Delta(\ ))F^{-1}$$

and unchanged product. If F is a cocycle then H^F is another Hopf algebra. In either case, the category ${}_H\mathcal{M}$ of H-modules is monoidally equivalent in the sense of §2.4 to that of H^F-modules. As part of this, there are natural isomorphisms $c_{V,W} : V \otimes W \to V \otimes^F W$ of the tensor product of V, W as H-modules to their tensor product as H^F-modules, given by $c(v \otimes w) = F{\triangleright}(v \otimes w)$. This implies that if $A \in {}_H\mathcal{M}$ is an algebra of which the product $\bullet$ is H-covariant then there is a new twisted algebra structure on the vector space A. We denote the new algebra by A_F and its new product by $\bullet$, where

$$a \bullet b = \cdot\, F^{-1}{\triangleright}(a \otimes b)$$

is H^F-covariant. The algebra A_F is now 'quasiassociative' in the sense that

$$a \bullet (b \bullet c) = \bullet(\bullet \otimes \mathrm{id})(\Phi^{-1}{\triangleright}(a \otimes b \otimes c)); \quad \Phi = F_{23}(\mathrm{id} \otimes \Delta F)((\Delta \otimes \mathrm{id})F^{-1})F_{12}^{-1},$$

where $\Phi \in H^{\otimes 3}$ acts on $A^{\otimes 3}$. This is called a *module algebra twist* and can be used to construct nonassociative algebras such as the octonions and render them 'associative up to isomorphism' in the above sense. We have also seen it as a special case of a trivial principal bundle in §5.2.1. Now, suppose that A has an H-covariant exterior algebra $\Omega(A)$. Then we can apply the above to this algebra, giving $\Omega(A)_F = \Omega(A_F)$ as an exterior algebra on A_F, with d undeformed. For example,

$$a \bullet \mathrm{d}b = (F^{-1}{\triangleright}a)(\mathrm{d}F^{-2}{\triangleright}b), \quad \omega \wedge_\bullet \eta = (F^{-1}{\triangleright}\omega) \wedge (F^{-2}{\triangleright}\eta),$$

where we abuse the notation and write the inverse of F as $F^{-1} \otimes F^{-2}$ (sum of such terms understood). The assumed covariance means that the action on $\mathrm{d}b \in \Omega^1$ is d of the action on $b \in A$. Part of this is that Ω^1 as a bimodule twists to a bimodule over A_F (in a quasi-sense with Φ) and that tensor products over A become equivalent to tensor products over A_F. It is a nice check to see this directly.

Lemma 9.27 *There is a well-defined invertible map*

$$c : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_{A_F} \Omega^1, \ \ c(\omega \otimes_A \eta) = \otimes_{A_F} c_{\Omega^1,\Omega^1}(\omega \otimes \eta) = F^1{\triangleright}\omega \otimes_{A_F} F^2{\triangleright}\eta,$$

where $F = F^1 \otimes F^2$ (sum of terms understood).

Proof First we compute

$$\begin{aligned} c(\omega a \otimes_A \eta) &= (F^1{}_{(1)}{\triangleright}\omega)(F^1{}_{(2)}{\triangleright}a) \otimes_{A_F} F^2{\triangleright}\eta \\ &= (F'^1 F^1{}_{(1)}{\triangleright}\omega) \bullet (F'^2 F^1{}_{(2)}{\triangleright}a) \otimes_{A_F} F^2{\triangleright}\eta \\ &= (\Phi^{-1} F^1{\triangleright}\omega) \bullet (\Phi^{-2} F'^1 F^2{}_{(1)}{\triangleright}a) \otimes_{A_F} \Phi^{-3} F'^2 F^2{}_{(2)}{\triangleright}\eta, \end{aligned}$$

where the last equality is the definition of Φ, F' is another copy of F and we write $\Phi^{-1} = \Phi^{-1} \otimes \Phi^{-2} \otimes \Phi^{-3}$ (sum of terms understood). Using the definition of $\otimes_{A_F}$ and inserting Φ to do the required rebracketing, we obtain

$$c(\omega a \otimes_A \eta) = F^1{\triangleright}\omega \otimes_{A_F} (F'^1 F^2{}_{(1)}{\triangleright} a) \bullet (F'^2 F^2{}_{(2)}{\triangleright}\eta)$$
$$= F^1{\triangleright}\omega \otimes_{A_F} (F^2{}_{(1)}{\triangleright} a)(F^2{}_{(2)}{\triangleright}\eta) = c(\omega \otimes_A a\eta).$$

Hence c is well defined. It is clear that it is invertible as c_{Ω^1,Ω^1} and F are. □

We have included Φ in the proof but there is a general principle that one can prove the simpler cocycle case first and then insert Φ just as we can omit the associator when working in a monoidal category and insert it afterwards (the coherence theorem for monoidal categories). Moreover, proofs such as the above are automatic from the functorial nature of the construction and indeed Lemma 9.27 is just an example of a general categorical equivalence between H-covariant A-bimodules with its tensor product over A and H^F-covariant A_F-bimodules with tensor product over A_F.

In fact, because there is an equivalence of categories between ${}_H\mathcal{M}$ and ${}_{H^F}\mathcal{M}$ in the first place, *any* construction defined by covariant maps and equations as commuting diagrams in one implies the same commuting diagrams in the other by application of the functor. In particular, we have the following twisting construction. The twisted geometry on (A_F, Ω_F) could be nonassociative even if we start with an associative one on (A, Ω). For a quantum metric, the notions of quantum symmetry and centrality still make sense as does nondegeneracy with more care.

Proposition 9.28

(1) *If* $g = g^1 \otimes g^2 \in \Omega^1 \otimes_A \Omega^1$ *is an* H*-invariant quantum metric on* H*-covariant* (A, Ω^1) *and* F *a cochain then this twists to a quantum metric*

$$g_F = c(g) = F^1{\triangleright} g^1 \otimes_{A_F} F^2{\triangleright} g^2.$$

(2) *If* (∇, σ) *is an* H*-covariant bimodule connection on* Ω^1 *(say; similar remarks apply in general) then this twists to a bimodule connection*

$$\nabla^F = c \circ \nabla, \quad \sigma_F = c \circ \sigma \circ c^{-1}.$$

(3) *If* ∇ *is torsion free/cotorsion free/metric compatible then so is* ∇^F.

Proof We give some elements explicitly and leave the rest to the reader in view of our general remarks. It is clear that if g is killed by $\wedge$ then g_F is killed by the twisted version as the F^{-1}, F cancel. If g is invariant then

$$g_F \bullet a = (F^1{\triangleright} g^1 \otimes_{A_F} F^2{\triangleright} g^2) \bullet a = \Phi^1 F^1{\triangleright} g^1 \otimes_{A_F} ((\Phi^2 F^2{\triangleright} g^2) \bullet (\Phi^3{\triangleright} a))$$
$$= \Phi^1 F^1{\triangleright} g^1 \otimes_{A_F} (F^{-1}\Phi^2 F^2{\triangleright} g^2)(F^{-2}\Phi^3{\triangleright} a)$$

$$= F^1 F^{-1}{}_{(1)}\triangleright g^1 \otimes_{A_F} (F^2{}_{(1)} F^{-1}{}_{(2)}\triangleright g^2)(F^2{}_{(2)} F^{-2}\triangleright a)$$
$$= c\big(F^{-1}{}_{(1)}\triangleright g^1 \otimes_A (F^{-1}{}_{(2)}\triangleright g^2)(F^{-2}\triangleright a)\big) = c\big((F^{-1}\triangleright g)(F^{-2}\triangleright a)\big) = c(ga),$$

where the 4th equality is the definition of Φ (or the cocycle condition in the cocycle case) and the last equality is that g is invariant for the action of H. We similarly find $a \bullet g_F = c(ag)$, hence g_F is central for the $\bullet$ product exactly when g is in the original geometry. One can also consider the twisting of $(\ ,\ )$ for the nondegeneracy condition. Next we check that ∇^F obeys the left $\bullet$-Leibniz rule,

$$\begin{aligned}
\nabla^F(a \bullet \xi) &= c(\nabla((F^{-1}\triangleright a)(F^{-2}\triangleright \xi)))\\
&= c\big(\mathrm{d}(F^{-1}\triangleright a) \otimes_A (F^{-2}\triangleright\xi) + (F^{-1}\triangleright a)\nabla(F^{-2}\triangleright\xi)\big)\\
&= c\big((F^{-1}\triangleright \mathrm{d}a) \otimes_A (F^{-2}\triangleright\xi) + (F^{-1}\triangleright a)\nabla(F^{-2}\triangleright\xi)\big)\\
&= \mathrm{d}a \otimes_{A_F} \xi + c\big((F^{-1}\triangleright a)\nabla(F^{-2}\triangleright\xi)\big),\\
a \bullet \nabla^F(\xi) &= a \bullet c(\nabla(\xi)) = a \bullet ((F^1\triangleright\eta) \otimes_{A_F} (F^2\triangleright\kappa))\\
&= (\Phi^{-1}\triangleright a) \bullet (\Phi^{-2}F^1\triangleright\eta) \otimes_{A_F} (\Phi^{-3}F^2\triangleright\kappa)\\
&= (F^{-1}\Phi^{-1}\triangleright a)(F^{-2}\Phi^{-2}F^1\triangleright\eta) \otimes_{A_F} (\Phi^{-3}F^2\triangleright\kappa)\\
&= (F^1{}_{(1)}F^{-1}\triangleright a)(F^1{}_{(2)}F^{-2}{}_{(1)}\triangleright\eta) \otimes_{A_F} (F^2F^{-2}{}_{(2)}\triangleright\kappa)\\
&= c\big((F^{-1}\triangleright a)(F^{-2}{}_{(1)}\triangleright\eta) \otimes_{A_F} (F^{-2}{}_{(2)}\triangleright\kappa)\big),
\end{aligned}$$

where $\nabla\xi = \eta \otimes \kappa$ (summation implicit). Then invariance of ∇ completes the proof. The explicit formula for the generalised braiding is

$$\sigma_F(\omega \otimes_{A_F} \eta) = c \circ \sigma(F^{-1}\triangleright\omega \otimes_A F^{-2}\triangleright\eta)$$

and one can check that this works. The last part can similarly be checked explicitly by the same methods, at least in the cocycle case (and suitable insertions of Φ in the cochain case). □

In particular, we could start with a classical manifold with $A = C^\infty(M)$ and a Lie group with Lie algebra $\mathfrak{g}$ acting on it by vector fields X_ξ for all $\xi \in \mathfrak{g}$. This extends by the Lie derivative $\mathcal{L}_{X_\xi}$ to an action on $\Omega(M)$ which in turn extends to an action of $U(\mathfrak{g})$ that commutes with the exterior derivative. Now suppose that $F_\lambda \in U(\mathfrak{g})^{\otimes 2}$ is a cochain. This could be set up as a formal power series in λ with values in $U(\mathfrak{g})^{\otimes 2}$, but since in physics we want λ to have an actual value, we think of it as a parameter and assume that relevant series can be made sense of. We suppose that F_λ is invertible and that $F_\lambda = 1 \otimes 1 - \lambda\mathfrak{f} + O(\lambda^2)$, where $\mathfrak{f} = \mathfrak{f}^1 \otimes \mathfrak{f}^2 \in \mathfrak{g} \otimes \mathfrak{g}$ (summation understood). We can then use the above to 'quantise'

$C^\infty(M)$ and $\Omega(M)$ to $A_\lambda = C^\infty_{F_\lambda}(M)$, $\Omega_{F_\lambda}(M) = \Omega(A_\lambda)$. The first-order part of this corresponds to

$$\omega^{\mu\nu} = X^\mu_{\mathfrak{f}^1} X^\nu_{\mathfrak{f}^2} - X^\nu_{\mathfrak{f}^1} X^\mu_{\mathfrak{f}^2}\,, \quad \omega^{\rho\alpha}\Gamma^\mu{}_{\alpha\nu} = X^\rho_{\mathfrak{f}^2} X^\mu_{\mathfrak{f}^1,\nu} - X^\rho_{\mathfrak{f}^1} X^\mu_{\mathfrak{f}^2,\nu}, \tag{9.65}$$

where the second expression only determines part of ∇ unless we are in the symplectic case where ω is invertible. For everything to be associative, we could take F_λ to be a cocycle, but it could be that it is not a cocycle and yet the algebra $C^\infty_{F_\lambda}(M)$ happens to remain associative at least to order λ^2. For example, if $F_\lambda = 1 - \lambda\mathfrak{f} + \frac{\lambda^2}{2}\mathfrak{f}^2 + O(\lambda^3)$ (such as $F_\lambda = e^{-\lambda\mathfrak{f}}$) then associativity of the coordinate algebra holds to order λ^2 if and only if $\varphi^{\mu\nu\rho} = 0$ where we define the 3-tensor

$$\varphi^{\mu\nu\rho} := X^\mu_{\phi^1} X^\nu_{\phi^2} X^\rho_{\phi^3}\,; \quad \phi = \phi^1 \otimes \phi^2 \otimes \phi^3 = 2[\mathfrak{f}_{23}, \mathfrak{f}_{12}] + [\mathfrak{f}_{13}, \mathfrak{f}_{12}] + [\mathfrak{f}_{23}, \mathfrak{f}_{13}] \in \mathfrak{g}^{\otimes 3}$$

(summation understood for ϕ and with $\mathfrak{f}_{12} = \mathfrak{f} \otimes 1$, etc.). This follows from $\Phi = 1 + \frac{\lambda^2}{2}\phi + O(\lambda^3)$ after a short computation. On the other hand, $\Omega_{F_\lambda}(M)$ could still be nonassociative at order λ^2 so that the implied partial connection ∇ has curvature.

Example 9.29 To recover the nonassociative fuzzy sphere in Example 9.26 as a twist, we start with $U(so_{1,3})$ with generators and relations

$$[M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, N_j] = -\epsilon_{ijk} M_k$$

acting on the classical sphere generators z^i as,

$$M_i \triangleright z^j = \epsilon_{ijk} z^k, \quad N_i \triangleright z^j = z^i z^j - \delta_{ij}.$$

This is the action of $so_{1,3}$ on the 'sphere at infinity'. The cochain we need is then

$$F = 1 - \lambda\mathfrak{f} + \frac{\lambda^2}{2}\mathfrak{f}^2 + \cdots, \quad \mathfrak{f} = \frac{1}{2} M_i \otimes N_i,$$

where we do not need to commit ourselves to the higher $O(\lambda^3)$ terms. One can check that the twisted algebra remains associative to order λ^2, so we have a Poisson structure, but the differential calculus is nonassociative at order λ^2 so the induced ∇ has curvature. Specifically, if we start with the classical algebra and exterior algebra on the sphere, the deformed products are

$$z^i \bullet z^j = (F^{-1} \triangleright z^i)(F^{-2} \triangleright z^j) = z^i z^j + \frac{\lambda}{2}\epsilon_{ijk} z^k,$$

$$z^i \bullet dz^j = (F^{-1} \triangleright z^i) dF^{-2} \triangleright z^j = z^i dz^j + \frac{\lambda}{2} z^j \epsilon_{imn} z^m dz^n,$$

$$dz^j \bullet z^i = (F^{-1} \triangleright dz^j) F^{-2} \triangleright z^i = (dz^j) z^i - \frac{\lambda}{2} z^i \epsilon_{jmn} z^m dz^n - \frac{\lambda}{2}\epsilon_{ijm} dz^m$$

to order λ, giving relations

$$[z^i, \mathrm{d}z^j]_\bullet = \frac{\lambda}{2}((z^i\epsilon_{jmn} + z^j\epsilon_{imn})z^m\mathrm{d}z^n + \epsilon_{ijm}\mathrm{d}z^m) = \lambda z^j\epsilon_{imn}z^m\mathrm{d}z^n$$

in agreement with Example 9.26. For the last step, we let $w^i = \epsilon^i{}_{jk}z^j\mathrm{d}z^k$ and note that classically $z^iw^j\epsilon_{ijk} = -\mathrm{d}z^k$ and hence $z^iw^j - z^jw^i = -\epsilon_{ijk}\mathrm{d}z^k$, from the differential of the sphere relation. We also have

$$\mathrm{d}z^i \bullet \mathrm{d}z^j = (\mathrm{d}F^{-1}\triangleright z^i) \wedge (\mathrm{d}F^{-2}\triangleright z^j) = \mathrm{d}z^i \wedge \mathrm{d}z^j + \frac{\lambda}{2}(3z^iz^j - \delta_{ij})\mathrm{Vol}$$

to order λ using the action of $\mathfrak{f}$ and the definition of Vol, which gives the correct anticommutator. Most of the products themselves are asymmetrically quantised and would match those of Poisson–Riemannian geometry if we replaced $\mathfrak{f}$ by its antisymmetric part. The Poisson-compatible connection in this example happens to be the classical Levi-Civita connection and we see that this is induced by twisting. ◇

If, moreover, M has a classical metric g with classical Levi-Civita connection $\widehat{\nabla}$ which is covariant under the classical symmetry then by Proposition 9.28, we have twist-quantised versions

$$\begin{aligned}
g_\lambda &= \mathcal{L}_{F_\lambda^1}(g_{\mu\nu}\mathrm{d}x^\mu) \otimes_{A_\lambda} \mathrm{d}F_\lambda^2\triangleright x^\nu \\
&= g_{\mu\nu}\mathrm{d}x^\mu \otimes_{A_\lambda} \mathrm{d}x^\nu - \lambda\big(X^\mu_{\mathfrak{f}^1}g_{\alpha\nu,\mu} + g_{\mu\nu}X^\mu_{\mathfrak{f}^1,\alpha}\big)X^\nu_{\mathfrak{f}^2,\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + O(\lambda^2), \\
\nabla_\lambda(\mathrm{d}x^\rho) &= -\mathcal{L}_{F_\lambda^1}(\widehat{\Gamma}^\rho{}_{\mu\nu}\mathrm{d}x^\mu) \otimes_{A_\lambda} \mathrm{d}F_\lambda^2\triangleright x^\nu \\
&= -\widehat{\Gamma}^\rho{}_{\mu\nu}\mathrm{d}x^\mu \otimes_{A_\lambda} \mathrm{d}x^\nu + \lambda\big(X^\mu_{\mathfrak{f}^1}\widehat{\Gamma}^\rho{}_{\alpha\nu,\mu} + \widehat{\Gamma}^\rho{}_{\mu\nu}X^\mu_{\mathfrak{f}^1,\alpha}\big)X^\nu_{\mathfrak{f}^2,\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + O(\lambda^2), \\
\sigma_\lambda(\mathrm{d}x^\mu \otimes_{A_\lambda} \mathrm{d}x^\nu) &= \mathrm{d}(F_\lambda^1F_\lambda^{-2}\triangleright x^\nu) \otimes_{A_\lambda} \mathrm{d}(F_\lambda^2F_\lambda^{-1}\triangleright x^\mu) \\
&= \mathrm{d}x^\nu \otimes_{A_\lambda} \mathrm{d}x^\mu - \lambda(X^\nu_{\mathfrak{f}^1,\alpha}X^\mu_{\mathfrak{f}^2,\beta} - X^\nu_{\mathfrak{f}^2,\alpha}X^\mu_{\mathfrak{f}^1,\beta})\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + O(\lambda^2),
\end{aligned}$$

where we expanded to order λ in the same spirit as Poisson–Riemannian geometry.

These formulae for twist-quantisation differ from Corollary 9.23 and Theorem 9.24 because the setting is a little different, but we see that they do also generate quantum Riemannian geometries at order λ, although not necessarily the same ones as in Poisson–Riemannian geometry even if the Poisson tensor and Poisson connection are the same (because quantum metrics for a given quantum differential algebra are typically not unique). The twist-quantised geometry is characterised by covariance under $U(\mathfrak{g})^F$, a triangular quantum group with category of representations symmetric. Its involutive braiding is given by the action of $F_{21}F^{-1}$ followed by flip and its inverse provides σ_λ. The constriction requires that the classical data is $\mathfrak{g}$-covariant, so is much more restrictive than Poisson–Riemannian geometry. Nevertheless, our outline of the latter also involved a kind of functorial equivalence and one can, roughly speaking, think of that as a 'micro-local' version of twist-quantisation.

In its basic form, twisting cannot, for example, be used to directly quantise the spherical geometry in Example 9.29 as the classical sphere metric is not Lorentz-invariant (it is only invariant up to conformal rescaling). Although twisting is more restrictive in this respect, it is slightly more general in that $\mathfrak{f}$ need not be antisymmetric (in principle, one could extend Poisson–Riemannian geometry to cover this too).

Example 9.30 (Moyal Plane) The most famous example of twist-quantisation is the Heisenberg algebra $[x, t]_\bullet = \lambda$ regarded as quantum spacetime. We start with $\mathbb{R}^2$ acting on itself by translations and formally define

$$F_\lambda = \mathrm{e}^{-\lambda \frac{\partial}{\partial x}\otimes\frac{\partial}{\partial t}} = \mathrm{e}^{-\lambda\mathfrak{f}}, \quad \mathfrak{f} = \frac{\partial}{\partial x}\otimes\frac{\partial}{\partial t}$$

to give $x\bullet t = xt+\lambda$ and other products of the generators undeformed. Since $\mathrm{d}1 = 0$, all products of differentials of generators with generators are also undeformed, so $\mathrm{d}x, \mathrm{d}t$ after twisting are central and behave as classically. The translation-invariant metric $g = -\mathrm{d}t\otimes\mathrm{d}t + \mathrm{d}x\otimes\mathrm{d}x$ (say) is likewise undeformed as is the QLC $\nabla\mathrm{d}x = \nabla\mathrm{d}t = 0$ and σ = flip on the basic 1-forms. There are still quantum aspects, but these come from the noncommutativity of the algebra alone. The symmetry H^F is the same Hopf algebra $U(\mathbb{R}^2)$ but now with a triangular structure.

A variant is $\mathbb{R}^2$ acting by the degree operator $x\partial/\partial x$ and translation in t, then

$$F_\lambda = \mathrm{e}^{-\lambda x \frac{\partial}{\partial x}\otimes\frac{\partial}{\partial t}} = \mathrm{e}^{-\lambda\mathfrak{f}}, \quad \mathfrak{f} = x\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial t},$$
$$[x, t]_\bullet = \lambda x, \quad [\mathrm{d}x, t]_\bullet = \lambda\mathrm{d}x$$

with other commutators zero (the only deformed products among the generators and basic 1-forms are $x\bullet t = xt + \lambda x$ and $(\mathrm{d}x)\bullet t = (\mathrm{d}x)t + \lambda\mathrm{d}x = (t+\lambda)\mathrm{d}x$). This is the bicrossproduct model quantum spacetime with $\beta = 0$ in the classification of differential calculi in Example 1.43 (b). The natural invariant metric is (say)

$$g = -\mathrm{d}t\otimes\mathrm{d}t + bx^{-2}\mathrm{d}x\otimes\mathrm{d}x$$

for b a nonzero real parameter, and we can apply Proposition 9.28 (extending the algebra to include x^{-1} for this). The quantisation by twisting has the same form but viewed now in the quantum algebra. The classical Levi-Civita connection is

$$\nabla\mathrm{d}t = 0, \quad \nabla\mathrm{d}x = x^{-1}\mathrm{d}x\otimes\mathrm{d}x$$

and again has the same form in the quantum algebra. Here $\mathrm{d}t$, $x^{-1}\mathrm{d}x = x^{-1} \bullet \mathrm{d}x$ are a central basis and a quantum metric to be central has to have constant coefficients in this basis much as in Example 1.43. On the other hand, this model at our formal level is the same as the Heisenberg algebra model; if we write $x = \mathrm{e}^y$ then $\mathrm{d}y = x^{-1} \bullet \mathrm{d}x$ and $[y, t]_\bullet = \lambda$ with central $\mathrm{d}y$, $\mathrm{d}t$ reproduces the other relations. ◇

Twisting is also an easy source of examples for fully nonassociative geometry with A itself nonassociative. When F_λ is not cocycle, this nonassociativity is strictly controlled by the action of Φ_λ obtained from F_λ as we have explained in the general theory. This nonassociativity is typical for a nonabelian symmetry group where different Lie algebra directions in $\mathfrak{f}$ do not commute.

Example 9.31 (Nonassociative Axisymmetric Fuzzy $\mathbb{R}^3$ and Fuzzy Sphere) We take $H = U(su_2)$ acting on $C^\infty(S^2)$ with its standard metric, by angular momentum rotation fields M_i acting on the three classical sphere generators z^i as in Example 9.29, and an axisymmetric quantisation with cochain

$$F = \mathrm{e}^{-\lambda \mathfrak{f}}, \quad \mathfrak{f} = \epsilon_{3ij} M_i \otimes M_j.$$

Here the axis is north-south; any other axis will have correspondingly rotated formulae. We do not yet impose the sphere relation, i.e., we act on $C^\infty(\mathbb{R}^3)$ and compute

$$\mathfrak{f}\triangleright(z^i \otimes z^j) = \epsilon_{3mn}\epsilon_{mip}\epsilon_{njq} z^p \otimes z^q = \epsilon_{ijk} z^3 \otimes z^k + \delta_{i3}\epsilon_{jmn} z^m \otimes z^n.$$

This gives us $z^i \bullet z^j$, $z^i \bullet \mathrm{d}z^j$, $(\mathrm{d}z^i) \bullet z^j$ with result

$$[z^i, z^j]_\bullet = 2\lambda\epsilon_{ijk} z^3 z^k,$$
$$[z^i, \mathrm{d}z^j]_\bullet = \lambda\big(\epsilon_{ijk}\mathrm{d}(z^3 z^k) + (\delta_{i3}\delta_{jk} + \delta_{j3}\delta_{ik})\epsilon_{kmn} z^m \mathrm{d}z^n\big)$$

to $O(\lambda^2)$. Compared to the angular-momentum space algebra Example 1.45, there is an extra z^3 dependence and a different form of the symmetric term in the calculus. The exterior algebra also deforms, as

$$(\mathrm{d}z^i)^{\bullet 2} = \lambda\delta_{i3}\epsilon_{3mn}\mathrm{d}z^m \wedge \mathrm{d}z^n, \quad \{\mathrm{d}z^i, \mathrm{d}z^j\}_\bullet = \lambda(\delta_{i3}\delta_{jk} + \delta_{j3}\delta_{ik})\epsilon_{kmn}\mathrm{d}z^m \wedge \mathrm{d}z^n$$

to $O(\lambda^2)$, while the metric deforms to

$$g_\lambda = \mathrm{d}z^i \otimes_{A_\lambda} \mathrm{d}z^i - \lambda\epsilon_{3ij}\mathrm{d}z^i \otimes \mathrm{d}z^j + O(\lambda^2),$$

where summation of repeated indices is understood. One can check that this quantum metric is central to $O(\lambda^2)$, as it should be to all orders by Proposition 9.28. So far, we have a nonassociative axisymmetric 'fuzzy $\mathbb{R}^3$' version of Example 9.29. Its quantum Levi-Civita connection by twisting is still zero on $\mathrm{d}z^i$.

Next, the action of M_i is compatible with the sphere relation $\sum_i (z^i)^2 = 1$ and hence the deformed products and metric are given by the same formulae as above just with this additional relation added classically (so $\mathrm{d}z^i$ are no longer independent but have a relation $z^i \mathrm{d}z^i = 0$ with summation understood). Then the above becomes an axisymmetric fuzzy sphere version of Example 9.26 along the lines of Example 9.29 but now including a quantum metric. With the sphere relation, the classical Levi-Civita connection is $\widehat{\nabla}\mathrm{d}z^i = -z^i g = -z^i \mathrm{d}z^j \otimes \mathrm{d}z^j$ which twists to

$$
\begin{aligned}
\nabla_\lambda \mathrm{d}z^i &= -F^1 \triangleright (z^i \mathrm{d}z^j) \otimes F^2 \triangleright \mathrm{d}z^j \\
&= -z^i \mathrm{d}z^j \otimes \mathrm{d}z^j + \lambda \left((\mathfrak{f}^1 \triangleright z^i) \mathrm{d}z^j \otimes \mathrm{d}\mathfrak{f}^2 \triangleright z^j + z^i \mathfrak{f}^1 \triangleright \mathrm{d}z^j \otimes \mathrm{d}\mathfrak{f}^2 \triangleright z^j \right) \\
&= -z^i g_\lambda + \lambda \mathrm{d}z^j \otimes (\mathfrak{f}^1 \triangleright z^i) \mathrm{d}\mathfrak{f}^2 \triangleright z^j \\
&= -z^i g_\lambda + \lambda \mathrm{d}z^j \otimes \left(\epsilon_{ijk} z^3 \mathrm{d}z^k + \delta_{i3} \epsilon_{jmk} z^m \mathrm{d}z^k \right) \\
&= -z^i \bullet g_\lambda + \lambda (\epsilon_{ijk} z^3 - \epsilon_{mjk} \delta_{i3} z^m) \mathrm{d}z^j \otimes \mathrm{d}z^k
\end{aligned}
$$

to $O(\lambda^2)$, where we used $M_j \triangleright g = 0$ to replace $z^i g_\lambda$ by $z^i \bullet g_\lambda$ at this order. On the sphere in a patch where z^3 is invertible, we can regard z^1, z^2 as coordinates. As well as working at the algebraic level to $O(\lambda^2)$, we can also obtain the underlying geometry from (9.65) using angular coordinates θ, φ, which relate to our previous functions by our usual $z^1 = \sin\theta\, \cos\varphi$, $z^2 = \sin\theta\, \sin\varphi$ and $z^3 = \cos\theta$, the vector fields corresponding to the M_i are

$$
X_{M_1} = \cot\theta\, \cos\varphi \frac{\partial}{\partial\varphi} + \sin\varphi \frac{\partial}{\partial\theta}, \quad X_{M_2} = \cot\theta\, \sin\varphi \frac{\partial}{\partial\varphi} - \cos\varphi \frac{\partial}{\partial\theta},
$$
$$
X_{M_3} = -\frac{\partial}{\partial\varphi}.
$$

By an abuse of notation, we will use θ, φ as indices, so the coordinates x^μ are $x^\theta = \theta$ and $x^\varphi = \varphi$ (this avoids confusion with the original 1,2,3 indices). The only nonzero components of the metric are $g_{\theta\theta} = 1$ and $g_{\varphi\varphi} = \sin^2\theta$, giving the only nonzero components of the Levi-Civita Christoffel symbols as

$$
\widehat{\Gamma}^\theta{}_{\varphi\varphi} = -\cos\theta\, \sin\theta, \quad \widehat{\Gamma}^\varphi{}_{\varphi\theta} = \widehat{\Gamma}^\varphi{}_{\theta\varphi} = \cot\theta.
$$

From (9.65), the antisymmetric bivector ω and quantising connection are

$$
\begin{aligned}
\omega^{\theta\varphi} &= 2X^\theta_{\mathfrak{f}^1} X^\varphi_{\mathfrak{f}^2} = 2X^\theta_{M_1} X^\varphi_{M_2} - 2X^\varphi_{M_1} X^\theta_{M_2} = 2\cot\theta, \\
\omega^{\rho\alpha} \Gamma^\mu{}_{\alpha\nu} &= 2X^\rho_{\mathfrak{f}^2} X^\mu_{\mathfrak{f}^1,\nu} = 2X^\rho_{M_2} X^\mu_{M_1,\nu} - 2X^\rho_{M_1} X^\mu_{M_2,\nu}
\end{aligned}
$$

and we get the only nonzero cases of $\Gamma^a{}_{bc}$ as

$$\Gamma^\theta{}_{\varphi\varphi} = -\tan\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = \frac{1}{\cos\theta\ \sin\theta}, \quad \Gamma^\varphi{}_{\theta\varphi} = \cot\theta.$$

The quantum metric and connection given by twisting-quantisation are then

$$\begin{aligned} g_\lambda &= \mathrm{d}\theta \otimes_{A_\lambda} \mathrm{d}\theta + \sin^2\theta\, \mathrm{d}\varphi \otimes_{A_\lambda} \mathrm{d}\varphi + \lambda \cot\theta (\mathrm{d}\theta \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}\theta), \\ \nabla_\lambda \mathrm{d}\theta &= \sin\theta \cos\theta \mathrm{d}\varphi \otimes_{A_\lambda} \mathrm{d}\varphi + \lambda \cot^2\theta\, \mathrm{d}\theta \otimes \mathrm{d}\varphi - \lambda \mathrm{d}\varphi \otimes \mathrm{d}\theta, \\ \nabla_\lambda \mathrm{d}\varphi &= -\cot\theta\, \mathrm{d}\theta \otimes_{A_\lambda} \mathrm{d}\varphi - \cot\theta\, \mathrm{d}\varphi \otimes_{A_\lambda} \mathrm{d}\theta \\ &\quad + \lambda \csc^4\theta (\mathrm{d}\theta \otimes \mathrm{d}\theta - \sin^2\theta\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi) \end{aligned}$$

which look rather different from the expressions in terms of the z^i given earlier, as should be expected due to the tensor product over A_λ.

Moreover, the nonvanishing components of the curvature and torsion tensors of the Poisson-compatible connection are then

$$R^\varphi{}_{\theta\theta\varphi} = -1 = -R^\varphi{}_{\theta\varphi\theta}\ , \quad T^\varphi{}_{\varphi\theta} = \tan\theta = -T^\varphi{}_{\theta\varphi},$$

and the only nonzero components of the contorsion tensor S are

$$S^\varphi{}_{\varphi\theta} = \tan\theta, \quad S^\theta{}_{\varphi\varphi} = -\sin^2\theta \tan\theta.$$

The only nonzero covariant derivatives of S are then

$$\nabla_\theta S^\theta{}_{\varphi\varphi} = -\tan^2\theta,\ \nabla_\varphi S^\theta{}_{\theta\varphi} = \tan^2\theta,\ \nabla_\theta S^\varphi{}_{\varphi\theta} = \sec^2\theta,\ \nabla_\varphi S^\varphi{}_{\theta\theta} = -\sec^2\theta.$$

Hence from (9.57) we have

$$\begin{aligned} H^\theta{}_{\theta\theta\varphi} &= \tan\theta, \quad H^\varphi{}_{\theta\theta\theta} = \cot\theta\, (\sec^2\theta + 1), \\ H^\theta{}_{\varphi\varphi\varphi} &= \tan\theta\ \sin^2\theta, \quad H^\varphi{}_{\varphi\varphi\theta} = -\tan\theta \left(1 + \cos^2\theta\right) \end{aligned}$$

and the antisymmetric tensor $\mathcal{R}_{\mu\nu}$ is given by

$$\mathcal{R}_{\theta\varphi} = \frac{1}{2}(H^\theta{}_{\theta\theta\varphi} - H^\varphi{}_{\varphi\varphi\theta}) = \tan\theta(1 + \frac{1}{2}\cos^2\theta).$$

One can check that $\nabla g = 0$ and (9.55) hold, so there is also a quantum metric g_1 at order λ by Corollary 9.23 in Poisson–Riemannian geometry, different from the leading deformation g_λ obtained by twisting. In fact (9.56) does not hold and would need to be adapted to allow for this difference. $\diamond$

Finally, although not the context of our semiclassical discussion, there is also a dual version of twisting, *cotwisting*, which allows us to apply the same ideas in a more algebraic context as in other chapters. Here the cotwisting quantum group H coacts on an initial algebra A, the 2-cochain $F : H\otimes H \to \Bbbk$ should be convolution-invertible in the same manner as for $\mathcal{R}$ in Proposition 2.52 and $F(h\otimes 1) = F(1\otimes h) = \epsilon(h)$ for all $h\in H$. It may or may not also obey the 2-cocycle condition

$$F(h_{(1)}\otimes g_{(1)})F(h_{(2)}g_{(2)}\otimes f) = F(g_{(1)}\otimes f_{(1)})F(h\otimes g_{(2)}f_{(2)}) \tag{9.66}$$

for all $h, g, f \in H$. The cotwisted Hopf algebra H^F in the cocycle case has a conjugated product in a convolution sense

$$h\bullet g = F(h_{(1)}\otimes g_{(1)})h_{(2)}g_{(2)}F^{-1}(h_{(3)}\otimes g_{(3)}) \tag{9.67}$$

and in the cochain case forms a coquasi-Hopf algebra dual to Drinfeld's notion of a quasi-Hopf algebra. The comodule algebra cotwist, as in the module case, works from either the left or the right and we focus on the former, so $\Delta_L a = a_{(\bar 1)}\otimes a_{(\bar\infty)}$ is a left coaction (the right comodule version would be more immediately dual to the left module version above.) We assume that $\Omega(A)$ is a left H-covariant calculus as well. Then the cotwisted A^F, $\Omega(A^F) = \Omega(A)^F$ proceed as before with

$$a\bullet b = F(a_{(\bar 1)}\otimes b_{(\bar 1)})a_{(\bar\infty)}b_{(\bar\infty)},\quad a\bullet \mathrm{d}b = F(a_{(\bar 1)}\otimes b_{(\bar 1)})a_{(\bar\infty)}\mathrm{d}b_{(\bar\infty)} \tag{9.68}$$

etc., (d or not on either element) and is quasiassociative in the category of H^F-comodules, so

$$\begin{aligned}
&(a\bullet b)\bullet c = (\partial F)^{-1}(a_{(\bar 1)}\otimes b_{(\bar 1)}\otimes c_{(\bar 1)})a_{(\bar\infty)}\bullet(b_{(\bar\infty)}\bullet c_{(\bar\infty)});\\
&(\partial F)^{-1}(h\otimes g\otimes f)\\
&\quad = F(h_{(1)}\otimes g_{(1)})F(h_{(2)}g_{(2)}\otimes f_{(1)})F^{-1}(h_{(3)}\otimes g_{(3)}f_{(2)})F^{-1}(g_{(4)}\otimes f_{(3)}),
\end{aligned}$$

which similarly provides the associator in the category of H^F-comodules. The dual version of Lemma 9.27 and Proposition 9.28 similarly applies, so

$$g^F = F^{-1}(g^1{}_{(\bar 1)}\otimes g^2{}_{(\bar 1)})g^1{}_{(\bar\infty)}\otimes_{A^F} g^2{}_{(\bar\infty)} \tag{9.69}$$

if the metric is $g^1\otimes_A g^2$. Similarly for ∇^F. Exercises E9.7–E9.9 provide examples.

Exercises for Chap. 9

E9.1 Suppose that a differential algebra has generators x^μ and a basis of 1-forms e^i with commutation relations $[e^i, x^\mu] = C^{\mu i}{}_j e^j$ for constants $C^{\mu i}{}_j$. Show that α, σ defined by constant tensors $\alpha^i{}_{mn}$ and $\sigma^{ij}{}_{mn}$ in Proposition 8.11 are bimodule maps if and only if

$$\alpha^i{}_{ma}C^{\mu a}{}_n + \alpha^i{}_{an}C^{\mu a}{}_m = C^{\mu i}{}_a \alpha^a{}_{mn},$$
$$\sigma^{ij}{}_{ma}C^{\mu a}{}_n + \sigma^{ij}{}_{an}C^{\mu a}{}_m = C^{\mu i}{}_a \sigma^{aj}{}_{mn} + C^{\mu j}{}_a \sigma^{ia}{}_{mn}$$

for all μ, i, j, m, n. Using this, show for the 2D bicrossproduct model quantum spacetime with its 3D calculus in §9.2.2 (see (9.19)) that there is a 2-parameter moduli of torsion-free bimodule connections. Show for the metric $g = \mathrm{d}r \otimes \mathrm{d}r - \mathrm{d}t \otimes \mathrm{d}t + \theta \otimes \theta$ that there is a 1-parameter moduli of WQLCs and that among them only $\nabla \mathrm{d}r = \nabla \mathrm{d}t = 0$ is a QLC.

E9.2 Compute the structure of the discrete bicrossproduct 'spacetime' model coming from a factorisation of the symmetric group $S_4 = S_3 \bowtie \mathbb{Z}_4$ to give a Hopf algebra $H = \mathbb{C}(\mathbb{Z}_4) \blacktriangleright\!\!\triangleleft \mathbb{C}S_3$ acting on $A = \mathbb{C}\mathbb{Z}_4 = \mathbb{C}[s]/\langle s^4 - 1\rangle$, using the algebraic methods in Lemma 9.2 but now adapted to the finite group case. Show that the 3D calculus $\Omega(\mathbb{C}\mathbb{Z}_4)$ from exercise E2.5 defined by anticommuting 1-form basis e^i and $e^i s = q^i s e^i$, $\mathrm{d}s = s \sum_i (q^i - 1)e^i$ is H-covariant and a $*$-calculus, where $q = \mathrm{i}$ (this is isomorphic to the complete graph calculus $\Omega(\hat{\mathbb{Z}}_4)$). Show that $g = e^1 \otimes e^3 + e^2 \otimes e^2 + e^3 \otimes e^1$ is a central H-invariant quantum metric and that $\nabla e^i = 0$ and $\sigma = $ flip on the basis is a QLC for it.

E9.3 Determine the wave-operator quantisation of a static spherically symmetric metric as in Proposition 9.11 using the Killing vector τ_3 in Corollary 9.10. Compute the commutation relations in r, z^i, t variables for the differential algebra from (8.42) and show that

$$\mu(r, \varphi) = -\frac{h(r)^2}{f(r)^2}\varphi, \quad \nu(r, \varphi) = \frac{1 + h(r)^2\varphi}{f(r)^2}$$

solve the equations for the auxiliary functions in Proposition 8.24. Use these to describe the Δ_0 part of the wave operator.

E9.4 $\mathbb{C}\mathbb{P}^2$ has a natural Poisson tensor ω associated to its complex structure and a natural connection ∇ (the Levi-Civita connection of the Fubini–Study metric) which we describe in real coordinates x^μ in the coordinate patch $[(1, z^1, z^2)] \in \mathbb{C}\mathbb{P}^2$ where $z^1 = x^1 + \mathrm{i}x^3$ and $z^2 = x^2 + \mathrm{i}x^4$. We set $t^2 = (1 + |z^1|^2 + |z^2|^2)^{-1}$ and use a 'signed mod 4' rule where $x^\mu = -x^{\mu+4}$ and we let $\kappa_{\mu\nu} = +1/-1$ if $\mu - \nu$ is an even/odd multiple of 4 and zero otherwise. Then the connection coefficients, Fubini–Study metric and Poisson tensor are

$$\Gamma^{\mu}{}_{\nu\rho} = -t^2(x^\rho\kappa_{\mu\nu} + x^\nu\kappa_{\mu\rho} + \kappa_{\mu+2,\rho}x^{\nu+2} + \kappa_{\mu+2,\nu}x^{\rho+2}),$$
$$g_{\mu\nu} = 2t^2\kappa_{\mu\nu} - 2t^4(x^\mu x^\nu + x^{\mu+2}x^{\nu+2}),$$
$$\omega^{\mu\nu} = \frac{1}{2}t^{-2}(\kappa_{\mu,\nu+2} + x^\mu x^{\nu+2} - x^{\mu+2}x^\nu)$$

and obey the field equations for Poisson–Riemannian geometry in §9.6. Compute the semiquantization $x^\mu \bullet x^\nu$ and show that

$$[z^a, z^b]_\bullet = 0, \quad [z^a, \bar{z}^b]_\bullet = \mathrm{i}\lambda t^{-2}(\delta_{ab} + z^a\bar{z}^b)$$

for $a, b = 1, 2$. The reader can also find with rather more effort that

$$[z^a, \mathrm{d}z^b]_\bullet = 0,$$
$$[z^a, \mathrm{d}\bar{z}^b]_\bullet = \mathrm{i}\lambda t^{-2}\Big((\delta_{ab} + z^a\bar{z}^b)\frac{z^1\mathrm{d}\bar{z}^1 + z^2\mathrm{d}\bar{z}^2}{1 + |z^1|^2 + |z^2|^2} + z^a\mathrm{d}\bar{z}^b\Big).$$

E9.5 Consider de Sitter space dS_2 with coordinates $t = x^0$ and $x = x^1$, metric $g = \mathrm{e}^{2\mu t}\mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}t \otimes \mathrm{d}t$ for $\mu \in \mathbb{R}$ and its Levi-Civita connection

$$\widehat{\nabla}\mathrm{d}x = -\mu(\mathrm{d}x \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x), \quad \widehat{\nabla}\mathrm{d}t = -\mu \mathrm{e}^{2\mu t}\mathrm{d}x \otimes \mathrm{d}x,$$

and fix a Poisson tensor by $\omega^{01} = 1$. Show that there is a unique solution for a Poisson connection ∇ solving the field equations of Poisson–Riemannian geometry summarised at the end of the preamble to §9.6, with the only nonzero Christoffel $\Gamma^1{}_{01} = \mu$ and only nonzero contorsion tensor components $S^1{}_{10} = -\mu$ and $S^0{}_{11} = -\mu\mathrm{e}^{2\mu t}$. Show that it has zero curvature. [Hence there is a unique quantisation at semiclassical order. In fact the quantum de Sitter space dS_2 described in §9.5.2 semiclassicalises to the data given here, with t quantising to T and x to R.]

E9.6 (Jordanian twist quantisation of de Sitter spacetime.) Working in the slicing coordinates for de Sitter spacetime as in exercise E9.5, show that $\tau = \frac{1}{\mu}\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x}$ are Killing vectors for the de Sitter metric $g = \mathrm{e}^{2\mu t}\mathrm{d}x^{\otimes 2} - \mathrm{d}t^{\otimes 2}$ and give an action of the Lie algebra b_+ according to $[\tau, \frac{\partial}{\partial x}] = \frac{\partial}{\partial x}$. Letting $L = \ln(1 + \lambda\frac{\partial}{\partial x})$ and $F = \mathrm{e}^{\tau \otimes L}$, and proceeding formally, use the identity $\mathrm{e}^A B \mathrm{e}^{-A} = \mathrm{e}^{[A,\ ]}B$ with $A = \tau \otimes L$ and $B = \frac{\partial}{\partial x} \otimes 1$ to show that

$$\mathrm{e}^{\tau\otimes L}(\partial_x \otimes 1)\mathrm{e}^{-\tau\otimes L} = \partial_x \otimes (1 + \lambda\partial_x).$$

Hence or otherwise, show that F is formally a 2-cocycle. Show that the 1-sided twist quantisation by this recovers the radial-time, i.e. de Sitter sector, of the Bertotti–Robinson quantisation with the α-calculus in §9.5.2. [This also applies to the whole model as the S^{n-2} factor was not quantised.]

E9.7 Show that the q-noncommutative torus $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$ as in exercises E1.5 and E5.10 with its given exterior algebra is the comodule-algebra cotwist of $A = \mathbb{C}_q[\mathbb{T}^2] = \mathbb{C}_q[S^1] \otimes \mathbb{C}_q[S^1]$, where $F(s^{a_0}t^{a_1}, s^{b_0}t^{b_1}) = e^{i\theta a_1 b_0}$ is a 2-cocycle on $H = \mathbb{C}[\mathbb{T}^2] = \mathbb{C}\mathbb{Z}^2 = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ coacting by $\Delta_L(u^m v^n) = s^m t^n \otimes \otimes u^m v^n$. Use this to find a metric and four QLCs for $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$ by cotwisting the Euclidean metric on $\mathbb{C}_q[\mathbb{T}^2]$ defined by $m = 0$ and two QLCs in Example 8.5 for each circle.

E9.8 (Geometry of the quaternions $\mathbb{H}$.) Reinterpret Example 5.19 to show that the reduced torus $A = c_{-1}[\mathbb{T}^2] = \mathbb{R}[u, v]/\langle u^2 - 1, v^2 - 1\rangle$ (as in E9.7 but with calculus given by $q = -1$) comodule-algebra cotwists by a cocycle $F(a_0 a_1, b_0 b_1) = (-1)^{a_0 b_0 + a_0 b_1 + a_1 b_1}$ on $H = \mathbb{R}\mathbb{Z}_2^2$ (with basis elements written as $a_0 a_1 \in \mathbb{Z}_2^2$) to the quaternions $\mathbb{H}$ and a 2D calculus $\Omega(\mathbb{H})$, where

$$(\mathrm{d}i)i = -i\mathrm{d}i, \quad (\mathrm{d}i)j = -j\mathrm{d}i, \quad (\mathrm{d}j)i = -i\mathrm{d}j, \quad (\mathrm{d}j)j = -j\mathrm{d}j,$$
$$\mathrm{d}i \wedge \mathrm{d}i = \mathrm{d}j \wedge \mathrm{d}j = 0, \quad \mathrm{d}i \wedge \mathrm{d}j = \mathrm{d}j \wedge \mathrm{d}i.$$

Show that the Euclidean metric on $c_{-1}[\mathbb{T}^2] \cong \mathbb{R}(\mathbb{Z}_2^2)$ with its square calculus cotwists to the quantum metric $g_{\mathbb{H}} = \mathrm{d}i \otimes \mathrm{d}i + \mathrm{d}j \otimes \mathrm{d}j$ and that its four suitable QLCs by exercise E8.4 for each $\mathbb{Z}_2$ cotwist to four QLCs

$$\nabla \mathrm{d}i = \begin{cases} i\mathrm{d}i \otimes \mathrm{d}i, & \sigma(\mathrm{d}i \otimes \mathrm{d}i) = \mathrm{d}i \otimes \mathrm{d}i \\ 0, & \sigma(\mathrm{d}i \otimes \mathrm{d}i) = -\mathrm{d}i \otimes \mathrm{d}i \end{cases}$$

with similarly two choices for $\nabla \mathrm{d}j$ (and $\sigma = -$flip on mixed $\mathrm{d}i, \mathrm{d}j$). Show that $s = k\mathrm{d}i$ and $t = k\mathrm{d}j$ are a central basis with $s^2 = t^2 = 0$, $s \wedge t = t \wedge s$ and that the calculus is inner with $\theta = \frac{1}{2}k\mathrm{d}k$. [You can also use exercise E8.2 to find the general quantum metrics and moduli of QLCs on $\mathbb{H}$.]

E9.9 (Geometry of the octonions $\mathbb{O}$.) Similarly to E9.8, $A = c_{-1}[\mathbb{T}^3]$ over $\mathbb{R}$ comodule algebra cotwists by the cochain

$$F(a_0 a_1 a_2, b_0 b_1 b_2) = (-1)^{\sum_{i \le j} a_i b_j + a_0 b_1 b_2 + b_0 a_1 b_2 + b_0 b_1 a_2}$$

on $H = \mathbb{R}\mathbb{Z}_2^3$ to the octonion algebra in the Albuquerque–Majid form with basis $\{e_{\vec{a}}\}$ for $\vec{a} = a_0 a_1 a_2 \in \mathbb{Z}_2^3$. This has $e_{\vec{a}} \bullet (e_{\vec{b}} \bullet e_{\vec{c}}) = \pm(e_{\vec{a}} \bullet e_{\vec{b}}) \bullet e_{\vec{c}}$, $e_{\vec{a}} \bullet e_{\vec{b}} = \pm e_{\vec{b}} \bullet e_{\vec{a}}$ and $e_{\vec{a}}^2 = \pm 1$ with the $-$ signs if and only if the vector indices involved are linearly independent over $\mathbb{Z}_2$ (so $e_{\vec{a}}^2 = -1$ if and only if $\vec{a} \neq 0$). Describe the resulting quasiassociative exterior algebra $\Omega(\mathbb{O})$ and its quantum Riemannian geometry over $\mathbb{R}$ coming from the standard 3D calculus and Euclidean metric on $c_{-1}[\mathbb{T}^3] \cong \mathbb{R}(\mathbb{Z}_2^3)$ with its cube calculus.

E9.10 (Quantum Hamiltonian flow.) Given a differential algebra A and a bimodule map lift $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ splitting $\wedge$, as used in §2.7 and Chap. 8, we have a bimodule map 'interior product' $i^R_\psi : \Omega^2 \to \Omega^1$ by applying i and evaluation of $\psi \in \mathfrak{X}^R$ on the first factor of the result. We say that $\omega \in \Omega^2$ is 'nondegenerate' if $i^R_{()}\omega : \mathfrak{X}^R \to \Omega^1$ is invertible and in this case we define X_h as the 'Hamiltonian vector field' associated to $h \in A$ by $i^R_{X_h}\omega = \mathrm{d}h$. Define an associated 'quantum hamiltonian flow' by $\dot{a} = X_h(\mathrm{d}a)$ in analogy with classical mechanics. Calculate the quantum Hamiltonian flow on $M_2(\mathbb{C})$ with its standard 2D $\Omega(M_2)$ in Proposition 1.38, $\omega = \mathrm{i}s \wedge t$ (so that $\omega^* = \omega$ in the calculus) and lift $i(\omega) = \frac{1}{2}(s \otimes t + t \otimes s)$, showing that

$$X_h(\sigma_1) = 2[h, \sigma_2]\sigma_3, \quad X_h(\sigma_2) = -2[h, \sigma_1]\sigma_3, \quad X_h(\sigma_3) = 4\mathrm{i}(h\sigma_3 + h_3 1),$$

where $h = h_0 1 + h_i \sigma_i$ in the Pauli matrix basis (sum over i and $1 \in M_2(\mathbb{C})$). Writing $a(t) = a_0 1 + a_i(t)\sigma_i$, show that $h = \frac{\sigma_3}{4}$ induces simple harmonic motion in the plane $a_3 = 0$ with a_0 constant, oscillating between σ_1, σ_2.

Notes for Chap. 9

Quantum spacetime is a relatively new topic with the result that rather than attempting any kind of survey of a changing field, we have instead focussed on a few established models that relate to themes in the book, taken mainly from works of the authors. As far as quantum spacetime is concerned, the work of Snyder [305] is often cited as an early model but in fact Snyder was not proposing a closed algebra of spacetime but rather that $[x^\mu, x^\nu] = M^{\mu\nu}$ are generators of so_5, thereby merging the two. One of the first works in modern times specifically to consider quantum spacetime in the sense of noncommutative geometry as a model of quantum gravity was the second author's PhD thesis [187–189, 193]. The original focus here was noncommutative and noncommutative Hopf algebras as 'quantum phase space'—so both position and momentum potentially 'quantum' with the main examples, the bicrossproduct quantum groups, thought of primarily as classical position space but an enveloping algebra as 'noncommutative momentum space'. These works also introduced the notion of quantum Born or position-momentum reciprocity (in the form of observable-state duality) as a key principle for quantum gravity, in which one could equally well think of the enveloping algebra as noncommutative position and the momentum as classical (but typically curved). This emergence of one of the two main classes of quantum groups from ideas for quantum gravity complemented the Drinfeld–Jimbo quantum groups that emerged from integrable systems at a similar time. Figure 9.1 is taken from [187]. References [187, 193] included the bicrossproduct $U(su_2){\triangleright\!\!\!\blacktriangleleft}\mathbb{C}[H^3]$ coming from a factorisation of $SL_2(\mathbb{C})$ in present notations, albeit in a Hopf–von Neumann algebra form (the book [202] gives a fully algebraic treatment). C^*-algebra versions are also known [317].

Another early argument for quantum spacetime was [190], demonstrating that quantisation can regularise infinities in quantum field theory. The paper [235] later introduced the Majid–Ruegg spacetime $[x^i, t] = \lambda x^i$ as a module algebra for the bicrossproduct quantum group $U(so_{1,3}){\triangleright\!\!\blacktriangleleft}\mathbb{C}[H^4]$ as its Poincaré quantum group, which was a slightly different role of these quantum groups. At the time, Lukierski *et al.* [183] had contracted $U_q(so_{3,2})$ to obtain a certain 'κ-Poincaré' Hopf algebra but had been looking for its action on functions on classical spacetime; the paper [235] showed that the bicrossproduct quantum group was nontrivially isomorphic to this and provided instead a canonical action on the above quantum spacetime, thereby solving the problem. One of the first applications was [6] by Amelino-Camelia and the second author, using a normal-ordering hypothesis for plane waves on this quantum spacetime to justify what had otherwise been speculation that the speed of light should become energy dependent as testable by γ-ray bursts. Also around this time, Doplicher *et al.* [102] made a variation of Snyder's model in which they closed the algebra by setting $[x^\mu, x^\nu] = \theta^{\mu\nu}$ (say) but $[\theta^{\mu\nu}, X^\rho] = 0$ rather than the relations of so_5. This was of interest in string theory [298] among other contexts, with $\theta^{\mu\nu} \in \mathbb{C}.1$ where it becomes the Heisenberg algebra. Another model from this era and which we have only touched upon at the end of Chap. 2 was q-Minkowski space introduced by the second author in [192] as the algebra of 2×2 hermitian matrices. This has a central time direction in contrast to the bicrossproduct family. Independently, Wess *et al.* proposed the same algebra from the point of view of covariance under a q-Lorentz-Weyl group [71] and the associated Poincaré–Weyl quantum group [274]. The full picture here was explained in [194, 229] by the theory of bosonisation of Hopf algebras in braided categories, see [202]. The q-Lorentz group from a different quantum double or q-Iwasawa factorisation point of view was in [281]. Other works included a Poincaré quantum group-induced wave operator on both q-Euclidean and q-Minkowski spaces but with formulae somewhat less accessible than those of the bicrossproduct model. The q-Wick rotation relating the two by transmutation in [201] was an early example of a module-algebra twist. That the q-Lorentz group is a twist of two copies of $U_q(su_2)$ (or the q-Euclidean rotation group) was also in [201]. We refer to [202] for more details and references to the rather extensive literature from this era.

Meanwhile, in the quantum gravity literature, 't Hooft in [141] was one of the first to propose the angular momentum algebra or 'spin model' spacetime as likely arising in 3D quantum gravity. This was taken further in [17] by Batista and the second author with the introduction of its noncommutative differential calculus and covariance under the quantum double $U(su_2)\ltimes\mathbb{C}[SU_2]$, as covered in §9.2.3. The coherent states $|j, \theta, \varphi\rangle$ are derived there. Later works which we have not been able to cover are [117, 215] and [116] on 3D quantum gravity. See also [236] for an overview of quantum Born reciprocity and the relation between different quantum spacetimes in the 3D case, including the q-deformed models in the context of a cosmological constant. The Poisson-level theory alluded to in §9.2.3 can be found in [249] while the quantum theory is treated in [3], in both cases as part of a much larger literature that includes loop quantum gravity and spin foams. Recently in [231], it was shown that the twisting from q-Euclidean to q-Lorentz groups has a remnant as $q \to 1$ in the form of a twisting of the 3D bicrossproduct model $U(su_2) \triangleright\!\blacktriangleleft \mathbb{C}[H^3]$ acting on $U(h_3)$ to the 3D spin model $U(su_2) \rtimes \mathbb{C}[SU_2]$ acting on $U(su_2)$, which also proved that the former is formally quasitriangular.

The quantisation of spacetimes with spatial geometry admitting a conformal Killing vector as in §8.3 is due to the second author [220] as is the spherically symmetric case including the black-hole in §9.3. The solution of the quantum wave equation for the black-hole in [220] was done nonperturbatively but under a crude 'minimal coupling' simplification; we have preferred now to stick to the actual model but only compute perturbatively to second order in λ. This cannot then fully explore the effects at the horizon but what we find is consistent with the 'quantum skin' and frequency-dependent time dilation phenomena that were found before. The work [223] looks at the same theory to quantise Newtonian gravity through a wave-operator with the gravitational potential expressed in the choice of the functional parameter β. If a quantum mechanical particle arises from the Newtonian limit of a solution of a noncommutative spacetime then this work predicts a decoupling of the rest mass, the passive gravitational mass and of the effective inertial mass as these approach the Planck mass. Reference [223] also proposed that dark energy could arise as a quantum correction to zero and gave some indication of this in the Newtonian limit.

§9.4 is taken from [32] by the authors as probably the first nontrivially curved quantum Riemannian geometry model that can be fully solved and contains a bit of physics (a strong gravitational source or an expanding cosmology). The proof of Proposition 9.19 can be found here. In higher dimensions, there is still a preferred classical metric in the spherically symmetric case but the quantum metric is part of a more general theory where the metric need not be central. §9.5 on the quantum Bertotti–Robinson model (which includes AdS_2 and dS_2) is taken from [237] by Tao and the second author. This work also predicts the need for a cosmological constant and/or Maxwell field as coming out of quantum geometry.

Most of §9.6 is taken from [33] by the authors as the first attempt to systematically semiclassicalise noncommutative Riemannian geometry at least within our bimodule approach. A fuller proof of Proposition 9.22 can be found there. Some of the tensor calculus formulae, notably for the quantum connection in Theorem 9.24 and the quantum Laplacian in equation (9.63) are from a sequel [119] by Fritz and the second author. Previous work on the Poisson-level data of a Lie–Rinehart or contravariant connection behind quantum vector bundles in general was in [144] and on the data for a differential calculus specifically was looked at in [27, 29, 135]. Among results that we have not had room to cover, [119] solves the generic spherically symmetric case and [240] looks at differentials on Poisson–Lie groups.

§9.6.3 on twisting to construct and control quasi-associative noncommutative geometry is due to the present authors [30]. The cochain F in the theory was first introduced by Drinfeld in [104] as a 'gauge equivalence' among quasi-Hopf algebras by conjugation of the coproduct. A super version was used in [27] to cotwist bicovariant exterior algebras as a solution to the often-encountered quantum anomaly for differentials. (In more algebraic works the cotwist version as at the end of the chapter tends to be more precise but the content is the same.) The cocycle condition, which ensures that a Hopf algebra (co)twists into itself appeared in early works of the second author, such as [131]. More relevant to us and not to be confused with such 'Drinfeld twists' is the one-sided (co)module algebra (co)twist of an algebra on which a quantum group acts. An early work here was [201] using this to one-sided twist q-Euclidean space to q-Minkowski (and Drinfeld-twist the q-Euclidean quantum group to the q-Lorentz one). That covariant quantum differential structures also twist in such a process appeared in [230] by R. Oeckl and the second author. In the cochain case, one-sided (co)twists were used by Albuquerque and the second author in [2] to obtain the octonions as quasi-algebras, i.e. associative when viewed in a suitable monoidal category (see E9.9) and extended to nonassociative differentials by one-sided (co)twists in [217].

Proposition 9.28 on the one-sided cochain twist of metrics and bimodule connections is from [30]. Drinfeld's specific cochain F which twists $U(\mathfrak{g})$ for general semisimple $\mathfrak{g}$ as a natural quasi-Hopf algebra into the quantum group $U_q(\mathfrak{g})$ can also be used in reverse to construct nonassociative Lie group geometries as explained in [30], and Example 9.31 is in this line. Twisting was extended to quantum principal bundles in the sense of Chap. 5 in [10], which is also a general review. It can also be extended to twist noncovariant maps through the use of internal hom [14]. Example 9.29, which is a cochain twist at least to order λ^2, is from [29].

In [19] it is shown that generalising the group factorisations in §9.2 to the case of one factor not a subgroup gives a construction of a nontrivially associated monoidal category and associated algebras, which can also be related to a twist. The works [162, 163] introduced a more truly nonassociative concept of a *Hopf quasigroup*. These go beyond the quasiassociative examples from twisting, are not related to Drinfeld's quasi-Hopf algebras and include the enveloping algebra $U(\mathcal{L})$ of a Mal'tsev algebra $\mathcal{L}$. They open the way to examples of more truly nonassociative geometry, such as the enveloping algebra of the 7-dimensional Mal'tsev algebra of a 7-sphere regarded as a 7-dimensional nonassociative version of Example 1.45.

There remains much more that we have not been able to cover and cannot even begin to cover. Among these is Connes' approach to the standard model of elementary particle physics as defined by a Dirac operator or spectral triple on a noncommutatively-extended spacetime $C^\infty(\mathbb{R}^{1,3}) \otimes A^{\mathrm{f.d.}}$, where $A^{\mathrm{f.d.}}$ is a finite-dimensional but noncommutative algebra such as $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, see [74, 85] for some major works. It would be interesting to look at bimodule quantum Riemannian geometry in this context with a view to arriving at particularly geometric Dirac operators on the finite-dimensional part. Also not covered is an approach to classical and Riemannian geometry as a central extension of the classical exterior algebra within the context of noncommutative exterior algebras [221, 224]. This is an example of a concrete mechanism for precisely how classical Riemannian geometry could emerge, or *why* there is gravity, out of little more than the quantum spacetime hypothesis and the axioms of a noncommutative differential structure (mainly the Leibniz rule). This can also be used to construct bimodule quantum Riemannian geometries. Reference [221] also showed that the exterior algebra of a classical Riemannian manifold together with the Hodge codifferential is a Batalin–Vilkovisky algebra, a concept more often associated with BRST quantisation of gauge symmetries in quantum field theory. Some of these ideas together with more truly nonassociative examples should lead into modern categorical 'higher geometry' and categorical deformation theory. Obviously too, for physics, we would like to have a quantum Einstein tensor and quantum stress energy tensor (the latter arises in quantum field theory anyway and one would not have to take its vacuum expectation value), so as to have a quantum Einstein's equation.

Most of the exercises are new, but E9.4 is from [34], which also covers the $\mathbb{CP}^n$ case, and E9.8 and E9.9 are from [217], which also characterises the cochains F for $\mathbb{H}$ and $\mathbb{O}$ in terms of self-duality under Fourier transform. The 'Jordanian twist' cocycle in $U_q(b_+)^{\otimes 2}$ used in E9.6 goes back to [273] and was applied to obtain the bicrossproduct model spacetime in [48], but our application to obtain the quantum de Sitter geometry of [237] as a twist is new. E9.10 is new but there are some related semiclassical remarks about the concept at the end of [34].

Solutions

Solutions to Exercises for Chap. 1

E1.1 Since the space of elements of the form $a\mathrm{d}b$ for $a, b \in A$ spans the calculus (Ω^1, d) by the surjectivity axiom, a morphism $\phi : \Omega^1 \to \Omega^{1'}$ must be of the form $\phi(a\mathrm{d}b) = a\phi(\mathrm{d}b) = a\mathrm{d}'b$ and hence unique, and surjective as such elements span $(\Omega^{1'}, \mathrm{d}')$. Hence we have a commutative diagram

$$\begin{tikzcd} & \Omega^1_{\mathrm{uni}} \arrow[dl, "\pi_N"'] \arrow[dr, "\pi_{N'}"] & \\ \Omega^1 \arrow[rr, "\phi"] & & \Omega^{1'} \end{tikzcd}$$

and as $\phi\pi_N(n) = 0$ for $n \in N$, we have $\pi_{N'}(n) = 0$, i.e., $n \in N'$.

E1.2 It is easy to see that $\phi_* : \Omega^1_{\mathrm{uni}}(A) \to \Omega^1_{\mathrm{uni}}(B)$ is given by $\phi \otimes \phi$ and that we have a commutative diagram

$$\begin{tikzcd} \Omega^1_{\mathrm{uni}}(A) \arrow[r, "\phi\otimes\phi"] \arrow[d, "\pi_{N_A}"] & \Omega^1_{\mathrm{uni}}(B) \arrow[d, "\pi_{N_B}"] \\ \Omega^1_A \arrow[r, "\phi_*"] & \Omega^1_B \end{tikzcd}$$

By applying this to $n \in N_A$, we see that we must have $(\phi \otimes \phi)(n) \in N_B$.

E1.3 Begin by showing that Ω^1_A is an A-bimodule. If $a, a', a'' \in A$ then

$$a''(a\mathrm{d}a') = a''a\mathrm{d}a' \in \Omega^1_A, \quad (a\mathrm{d}a')a'' = a\mathrm{d}(a'a'') - aa'\mathrm{d}a'' \in \Omega^1_A.$$

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The map $\mathrm{d}|_A : A \to \Omega^1_B$ has image in Ω^1_A and is a derivation, so we have a calculus on A. The inclusion ι is differentiable with $\iota_* : \Omega^1_A \subseteq \Omega^1_B$ given by inclusion. Finally, if $\Omega^{1'}_A$ is another calculus with the stated property and $\phi : \Omega^1_A \to \Omega^{1'}_A$ is not an isomorphism then it must have a kernel, as it is surjective by E1.1. But

$$\begin{tikzcd} \Omega^1_A \arrow[rr, "\phi"] \arrow[dr, "\iota_*"'] & & \Omega^{1'}_A \arrow[dl, "\iota_*"] \\ & \Omega^1_B & \end{tikzcd}$$

is commutative (since it can be filled in with commuting triangles with a common vertex A and arrows $\mathrm{d}_A, \mathrm{d}'_A, \mathrm{d}_B\iota$), giving a contradiction as ι_* are inclusions.

E1.4 As all elements of I are set to zero in B, we must have $I\Omega^1_A$, $\Omega^1_A I$ and $\mathrm{d}I$ set to zero in Ω^1_B, so the given quotient is by the smallest ideal possible. The canonical map is given by taking the equivalence class [ ] in the quotient and needs to obey $[a].\mathrm{d}[a'] = [a.\mathrm{d}a'] \in \Omega^1_B$ for $a, a' \in A$, to be differentiable. We need to check that the operations of multiplication and derivative are well defined under this quotient, e.g. for the above to depend only on the equivalence classes $[a], [a'] \in B$. We apply the formula to $a + x$ and $a' + x'$ for any $x, x' \in I$, to get

$$[a+x].\mathrm{d}[a'+x'] = [(a+x).\mathrm{d}(a'+x')] = [a.\mathrm{d}a'] + [x.\mathrm{d}a'] + [(a+x).\mathrm{d}x'] = [a.\mathrm{d}a']$$

as $x.\mathrm{d}a' \in I\Omega^1_A$ and $\mathrm{d}x' \in \mathrm{d}I$.

E1.5 Applying d to the degree 1 relations gives the degree 2 relations except when $q^2 = 1$, in which case we impose $(\mathrm{d}u)^2 = (\mathrm{d}v)^2 = 0$ in any case. (i) It is easy to see that we have a $*$-calculus when $q^* = q$. We also have $\mathrm{d}u^m = [m]_q u^m \mathrm{d}u$ from Example 1.11, and similarly for $\mathrm{d}v^m$. The elements $e_1 = u^{-1}\mathrm{d}u$ and $e_2 = v^{-1}\mathrm{d}v$ obey $e_1 u = que_1$, $e_1 v = ve_1$, $e_2 u = ue_2$ and $e_2 v = qve_2$ so that the calculus of $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$ is inner with $\theta = (e_1 + e_2)/(q-1)$ (not to be confused with the parameter θ). (ii) If $q = e^{\mathrm{i}m\theta}$ then $t = v^{-m}e_1 = v^{-m}u^{-1}\mathrm{d}u$, $s = u^m e_2 = u^m v^{-1}\mathrm{d}v$ are central using the above relations. (iii) When q is a primitive n-th root of unity we have $[n]_q = 0$ so that the calculus is compatible with the additional relations $u^n = v^n = 1$ in this case, which is the reduced torus $c_{q,\theta}[\mathbb{T}^2]$. For the matrix realisation, it is easy to check that $VU = \omega UV$ and that $c_{q,\theta}[\mathbb{T}^2] \cong M_n(\mathbb{C})$ as an algebra by ϕ. The calculus is transferred to the latter by $\mathrm{d}U = UVt$, $\mathrm{d}V = VU^{-1}s$ and $\theta = (Vt + U^{-1}s)/(q-1)$ which for $n = 2$ with $s = 1 \oplus 0$, $t = 0 \oplus 1$ is $\theta = -\frac{1}{2}((E_{11} - E_{22}) \oplus (E_{12} + E_{21}))$. This is the point $(0, -1, 1)$ in our classification of 2D calculi on $M_2(\mathbb{C})$.

E1.6 Here $J_i = x_i/(2\lambda_P)$ obey the usual relations $[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k$ with $\rho(\sum_i J_i^2) = j(j+1)$ in the spin j representation ρ. Hence

$$\rho(\sum x_i^2) = \lambda_P^2 2j(2j+2) = \lambda_P^2(n-1)(n+1) = \lambda_P^2(n^2-1) = 1-\lambda_P^2$$

using $\lambda_P = 1/n$, as required for ρ to descend from the fuzzy sphere $\mathbb{C}_\lambda[S^2]$ to M_n. This is not the only relation in the kernel of ρ (i.e., not the only identity among the generators in the spin j representation). If $a \in U(su_2)$ then $\mathrm{d}a = [\theta, a] = \frac{n}{2\mathrm{i}}s_i[x_i, a]$ so that the partial derivatives for the s_i basis are $\frac{n}{2\mathrm{i}}[x_i, \ ]$ (i.e., proportional to the spin 1 representation or adjoint action). Hence if $a \in \ker\rho$ then $\rho([x_i, a]) = [\rho(x_i), \rho(a)] = 0$ and hence $\mathrm{d}a = 0$. So $\Omega^1(U(su_2))$ in Example 1.46 descends further to $\Omega^1(c_\lambda[S^2])$. The higher order calculus is not inner but has the form $\Omega(U(su_2)) = U(su_2).\Lambda$ as an algebra factorisation, where Λ is the Grassmann algebra of the s_i with a certain d. Hence this also descends to $\Omega(c_\lambda[S^2])$.

For $n = 2$, the matrix representation is $\rho(x_i) = \rho(J_i) = \frac{1}{2}\sigma_i$ (in terms of the Pauli matrices) and so $c_\lambda[S^2]$ has generators x_i and the stronger relations $x_i x_j = \frac{1}{4}\delta_{ij} + \frac{\mathrm{i}}{2}\epsilon_{ijk}x_k$. For $n = 3$, we use the vector representation $\rho(J_i) = L_i$ where $(L_i)_{jk} = -\mathrm{i}\epsilon_{ijk}$ and $\rho(x_i) = \frac{2}{3}L_i$. The L_i obey, summing over repeated indices,

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k, \quad \sum_i L_i^2 = 2, \quad L_iL_jL_k = \mathrm{i}\epsilon_{ijk} + \delta_{ij}L_k + \mathrm{i}\epsilon_{ikm}L_mL_j$$

as one can check by direct computation. The latter implies for example that $L_1L_2L_1 = 0$ etc., and shows that any degree 3 or higher products of the L_i can be reduced to degree 2 or less. Hence the stated relations hold for $c_\lambda[S^2]$ and there are no further relations as the algebra is now 9-dimensional with basis $1, x_i$ and $a_{ij}x_ix_j$ for a basis of symmetric traceless matrices a_{ij}. Finally, these reduced fuzzy sphere calculi transfer over to 3D calculi $\Omega(M_n(\mathbb{C}))$ in such a way that ρ is a diffeomorphism. For $n = 2$, we identify $\rho_*(s_1) = \mathrm{i}(t_2 + t_3)$, $\rho_*(s_2) = -t_2 + t_3$ and $\rho_*(s_3) = \mathrm{i}t_1$ in terms of $t_1 = 1 \oplus 0 \oplus 0$, $t_2 = 0 \oplus 1 \oplus 0$, $t_3 = 0 \oplus 0 \oplus 1$ where $1 \in M_2(\mathbb{C})$. Then $\rho_*(\theta) = \frac{1}{2\mathrm{i}}\sigma_i\rho_*(s_i) = \frac{1}{2}(E_{11} - E_{22})t_1 + E_{12}t_2 + E_{21}t_3$ is the canonical form for the 3D calculus in Example 1.8, which is the universal calculus at degree 1. However, the Grassmann algebra for the s_i transfers under ρ_* to $\{t_i, t_j\} = 0$, so is a quotient at higher degree.

E1.7 Part (i) is a straight verification of the stated diffeomorphism as instructed, the critical part being that conjugation of θ makes the $(1,2)$-entry of both matrix parts vanish. For (ii), conjugate θ for general $(1, r, s)$ by a general unit determinant matrix

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and require $a(a+br) = -b(b-as) = 0$ for the resulting $(1,2)$-entries to vanish. Here if either a or b vanishes then both do, which contradicts Φ being invertible. Thus we must have $s = \frac{b}{a}$ and $r = -\frac{a}{b}$, so $rs = -1$ as before.

E1.8 For the differential of functions, the standard formulae in Proposition 1.24 give

$$\mathrm{d}\delta_x = -\omega_{x\to y} - \omega_{x\to z}, \quad \mathrm{d}\delta_y = \omega_{x\to y} - \omega_{y\to z}, \quad \mathrm{d}\delta_z = \omega_{x\to z} + \omega_{y\to z}.$$

It follows that $\mathrm{H}^0_{\mathrm{dR}}(A) = \mathbb{C}$, as expected since the undirected graph is connected. There is only one possible basis 2-form, $\omega_{x\to y}\wedge\omega_{y\to z}$. Following Proposition 1.40, as there is an arrow $x \to z$, we do not set this 2-form to zero in the maximal prolongation. It is convenient to write $\omega_{p\to q} = \delta_p \mathrm{d}\delta_q$ for arrows $p \to q$, and then the proof of Proposition 1.40 contains the equation

$$\mathrm{d}\omega_{p\to q} = \sum_{a\to p\to q} \omega_{a\to p}\wedge\omega_{p\to q} + \sum_{p\to q\to c} \omega_{p\to q}\wedge\omega_{q\to c} - \sum_{p\to b\to q} \omega_{p\to b}\wedge\omega_{b\to q}$$

where we sum over all possible arrows. In particular, we find

$$\mathrm{d}\omega_{x\to y} = \omega_{x\to y}\wedge\omega_{y\to z}, \quad \mathrm{d}\omega_{y\to z} = \omega_{x\to y}\wedge\omega_{y\to z}, \quad \mathrm{d}\omega_{x\to z} = -\omega_{x\to y}\wedge\omega_{y\to z}.$$

We see that $\mathrm{d} : \Omega^1 \to \Omega^2$ is a surjective map from a 3D space to a 1D space, so its kernel is 2D, which is the same as the dimension of the image of $\mathrm{d} : \Omega^0 \to \Omega^1$. As $\Omega^i = 0$ for $i > 2$ (there are no paths of length 3) we find $\mathrm{H}^i_{\mathrm{dR}}(A) = 0$ for $i > 0$.

E1.9 From the β-calculus commutation relations with β real, $u = x^{\beta-1}\mathrm{d}x$ and $v = x^{\beta-1}(x\mathrm{d}t - \beta t\mathrm{d}x)$ are central and obey $u^* = u$, $v^* = v + \lambda\beta(\beta-2)u$. As the exterior algebra for $\mathrm{d}x$, $\mathrm{d}t$ form a Grassmann algebra, one has $u\wedge u = 0$, $v^*\wedge v + \lambda\beta(u\wedge v - v^*\wedge u) = 0$ and $u\wedge v + v^*\wedge u = 0$. The middle relation here is best proven by first finding $v\wedge v^*$. Hence a general form of quantum metric which is central, quantum symmetric and real is

$$g = au\otimes u + v^*\otimes v + \lambda\beta(u\otimes v - v^*\otimes u) - c(u\otimes v + v^*\otimes u)$$

for constants a, c (as our algebra has a trivial centre). Setting $v' = v - cu$, this is

$$g = (a - c^2)u\otimes u + v'^*\otimes v' + \lambda\beta(u\otimes v' - v'^*\otimes u),$$

which we can think of as a linear shift $t' = t + \frac{c}{\beta}$ and so that $v' = x^{\beta-1}(x\mathrm{d}t' - \beta t'\mathrm{d}x)$. Up to such a change of variables (now dropping all primes), and up to an overall normalisation, a general quantum metric for this calculus therefore has the form

$$g = x^{2\beta-2}\left(\pm \mathrm{d}x\otimes \mathrm{d}x + b(w^*\otimes w + \lambda\beta(\mathrm{d}x\otimes w - w^*\otimes \mathrm{d}x))\right)$$

for some real $b\neq 0$, where $w = x\mathrm{d}t - \beta t\mathrm{d}x$ and $w^* = w + \lambda\beta(\beta-2)\mathrm{d}x$. [Note that this derivation includes the case $\beta = 1$ which we had omitted in the text.]

E1.10 Clearly, any multiple of θ is a solution since $\theta^* = -\theta$ and $\mathrm{d}\theta = 0 = \theta\wedge\theta$. Given this, let $\phi = \alpha + \theta$ so that we are solving for $\phi\in\Omega^1$ with $\phi^* = -\phi$ and $\mathrm{d}\phi - \mathrm{d}\theta + \phi\wedge\phi + \theta\wedge\theta - \phi\wedge\theta - \theta\wedge\phi = 0$, which reduces to $\phi\wedge\phi = 0$. Here $\phi = \mu\theta$ for any real μ is one solution, as is fe_u as $e_u\wedge e_u = 0$ and $(fe_u)^* = e_u^* f^* = -e_u f^* = -R_u(f^*)e_u = -fe_u$ if we assume $f^* = R_u(f)$. For the last case, we use the relations of $\Omega(S_3)$ with its 2-cycles calculus to similarly check that $\phi\wedge\phi = 0$. Use that $R_a(\delta_x) = \delta_{xa^{-1}} = \delta_{xa}$ for all $x\in S_3$ and $a = u, v, w$.

Solutions to Exercises for Chap. 2

E2.1 We have $(h_{(1)}\triangleright f)(h_{(2)}\triangleright g) = h_{(1)}f(Sh_{(2)})h_{(3)}g(Sh_{(4)}) = h\triangleright(fh)$ and $h\triangleright 1 = h_{(1)}1(Sh_{(2)}) = \epsilon(h)$ for all $h, g, f\in H$, so we get a left module algebra using the definitions from §2.1. For a right comodule coalgebra (C,∇,ϵ) with coaction Δ_R we need the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta_R"'] \arrow[d, "\Delta"'] & C\otimes H \arrow[d, "\Delta\otimes\mathrm{id}"'] \\ C\otimes C \arrow[r, "\Delta_R"] & C\otimes C\otimes H \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta_R"'] \arrow[d, "\epsilon"'] & C\otimes H \arrow[d, "\epsilon\otimes\mathrm{id}"'] \\ \Bbbk \arrow[r, "\Delta_R"] & \Bbbk\otimes H \end{tikzcd}$$

to commute. This is easily verified in our case of $H = C$ and $\Delta_R = \mathrm{Ad}_R$.

For the Hopf algebra $U_q(b_+)$ or its quotient $u_q(b_+)$ (the formulae are the same),

$$t\triangleright(t^n x^m) = t^{n+1}x^m t^{-1} = q^{-m}t^n x^m,$$
$$x\triangleright(t^n x^m) = xt^n x^m t^{-1} + t^n x^m S(x) = q^n t^n x^{m+1}t^{-1} - t^n x^{m+1}t^{-1}$$
$$= q^{-m-1}(q^n-1)t^{n-1}x^{m+1},$$
$$\mathrm{Ad}_R(t) = t\otimes(St)t = t\otimes 1,$$
$$\mathrm{Ad}_R(x) = t\otimes(Sx)t + x\otimes(S1)t + 1\otimes(S1)x = (1-t)\otimes x + x\otimes t$$

using $\Delta(\mathrm{id}\otimes\Delta)t = t\otimes t\otimes t$, $\Delta(\mathrm{id}\otimes\Delta)x = x\otimes t\otimes t + 1\otimes x\otimes t + 1\otimes 1\otimes x$.

E2.2 Let $\{\alpha, \beta, \gamma, \delta\}$ denote the dual basis to the basis $\{1, t, x, tx\}$ for the Sweedler-Taft algebra $u_{-1}(b_+)$ (in the stated order). We use the duality between product and coproduct in the form $\langle\theta\phi, h\rangle = \langle\theta, h_{(1)}\rangle\langle\phi, h_{(2)}\rangle$ for θ, ϕ in the dual to show $\alpha^2 = \alpha$, $\beta^2 = \beta$ and $\alpha\beta = \beta\alpha = 0$. Also $\epsilon = \alpha + \beta = 1$ in the dual Hopf algebra. We map these to $u_{-1}(b_+)$ itself by $\alpha + \beta \mapsto 1, \alpha - \beta \mapsto t$. Another calculation we need is

$$\langle(\alpha-\beta)(\gamma+\delta), x\rangle = \langle(\alpha-\beta), x\rangle\langle(\gamma+\delta), t\rangle + \langle(\alpha-\beta), 1\rangle\langle(\gamma+\delta), x\rangle = 1$$

and similar calculations give $(\alpha - \beta)(\gamma + \delta) = \gamma - \delta$ and $(\alpha - \beta)(\gamma - \delta) = \gamma + \delta$. Information on the coproduct is given by

$$\begin{aligned}
1 &= \langle\delta, tx\rangle = \langle\Delta\delta, t\otimes x\rangle = \langle\Delta\delta, tx\otimes 1\rangle = \langle\Delta\delta, 1\otimes tx\rangle = -\langle\Delta\delta, x\otimes t\rangle,\\
0 &= \langle\delta, x\rangle = \langle\Delta\delta, 1\otimes x\rangle = \langle\Delta\delta, x\otimes 1\rangle = \langle\Delta\delta, t\otimes tx\rangle = -\langle\Delta\delta, tx\otimes t\rangle,\\
0 &= \langle\gamma, tx\rangle = \langle\Delta\gamma, t\otimes x\rangle = \langle\Delta\gamma, tx\otimes 1\rangle = \langle\Delta\gamma, 1\otimes tx\rangle = -\langle\Delta\gamma, x\otimes t\rangle,\\
1 &= \langle\gamma, x\rangle = \langle\Delta\gamma, 1\otimes x\rangle = \langle\Delta\gamma, x\otimes 1\rangle = \langle\Delta\gamma, t\otimes tx\rangle = -\langle\Delta\gamma, tx\otimes t\rangle,
\end{aligned}$$

from which we read off and then rewrite as

$$\Delta\gamma = \alpha\otimes\gamma + \gamma\otimes\alpha + \beta\otimes\delta - \delta\otimes\beta, \quad \Delta\delta = \delta\otimes\alpha + \alpha\otimes\delta + \beta\otimes\gamma - \gamma\otimes\beta,$$

$$\Delta(\gamma+\delta) = (\alpha+\beta)\otimes(\gamma+\delta) + (\gamma+\delta)\otimes(\alpha-\beta),$$

$$\Delta(\gamma-\delta) = (\alpha-\beta)\otimes(\gamma-\delta) + (\gamma-\delta)\otimes(\alpha+\beta).$$

Comparing this with the $u_{-1}(b_+)$ coproduct, we need to complete our map by $\gamma + \delta \mapsto x$, $\gamma - \delta \mapsto tx$, which one can then check is an isomorphism.

E2.3 First we calculate Δx^n for $0 \le n < r$ as

$$\Delta x^n = (x\otimes t + 1\otimes x)^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (x\otimes t)^m (1\otimes x)^{n-m}$$

by $A = x\otimes t$ and $B = 1\otimes x$ in the q-binomial formula Lemma 2.15. Hence

$$\Delta(t^s x^n) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (t^s x^m \otimes t^{m+s} x^{n-m})$$

for $0 \le s < r$. Definition 2.18 requires $(\int\otimes\mathrm{id})\Delta = 1\int$, so in particular,

$$1\int(t^s x^{r-1}) = \sum_{m=0}^{r-1} \begin{bmatrix} r-1 \\ m \end{bmatrix}_q \left(\int t^s x^m\right) t^{m+s} x^{r-1-m}.$$

By linear independence, we have $\int(t^s x^m) = 0$ unless $m = r - 1$ and $m + s$ is a multiple of r. Given the first condition, the second is $r - 1 + s = 0$ mod r or $s = 1$ as $0 \le s < r$. Hence $\int(tx^{r-1})$ is arbitrary and all other $\int(t^s x^m) = 0$.

E2.4 Let J be a nontrivial right ideal and write $j \in J$ as $j = \alpha(1-t) + \beta x + \gamma xt$ for coefficients $\alpha, \beta, \gamma \in \mathbb{C}$. Then

$$j(1+t) = (\beta+\gamma)(x+xt) \in J, \quad jx = \alpha(x+xt) \in J.$$

In case (1), we suppose that $x + xt \notin J$ in which case any $j \in J$ must have $\gamma = -\beta$ and $\alpha = 0$, so $J = \mathrm{span}_{\mathbb{k}}\{x - xt\}$. In case (2), we suppose that $x + xt \in J$, so either $J = \mathrm{span}_{\mathbb{k}}\{x + xt\}$ or the proper ideal J is 2D. In the latter case, we can subtract a multiple of $x + xt$ so that a general element $j \in J$ has the form $j = \alpha(1-t) + \beta x$ for some $\alpha, \beta \in \mathbb{C}$. Given such an element, if $\alpha = 0$ then $J = \mathrm{span}_{\mathbb{k}}\{x, xt\}$ and if $\alpha \neq 0$ then $J = \mathrm{span}_{\mathbb{k}}\{x + xt, 1 - t + \mu x\}$ for some $\mu = \beta/\alpha$.

Next, we found $\mathrm{Ad}_R t$, $\mathrm{Ad}_R x$ in E2.1 and from $\Delta(\mathrm{id}\otimes\Delta)$ there, we find

$$\Delta(\mathrm{id}\otimes\Delta)(xt) = xt\otimes 1\otimes 1 + t\otimes xt\otimes 1 + t\otimes t\otimes xt,$$

$$\mathrm{Ad}_R(xt) = xt\otimes t - (t-1)\otimes x,$$

$$\mathrm{Ad}_R(x+xt) = (x+xt)\otimes t - 2(t-1)\otimes x, \quad \mathrm{Ad}_R(x-xt) = (x-xt)\otimes t.$$

Hence if $x + xt \in J$ then we must also have $t - 1 \in J$, so $\langle x + xt, 1 - t\rangle$ is the only possibility for an Ad_R-stable J in case (2). We also get $\langle x - xt\rangle$ in case (1).

For the calculus associated to $J = \langle x + xt, 1 - t + \mu x\rangle$, clearly $[t-1] = \mu[x]$ and $[xt] = -[x]$. We also have $[x^2] = [0]$ since $x^2 = 0$. Hence $\mathrm{d}h = h_{(1)}\varpi\pi_\epsilon(h_{(2)})$ and $\omega h = h_{(1)}(\omega\triangleleft h_{(2)})$ tell us that

$$\mathrm{d}t = t\varpi\pi_\epsilon(t) = \mu t\omega, \quad \mathrm{d}x = x\varpi\pi_\epsilon(t) + \varpi\pi_\epsilon(x) = (1+\mu x)\omega,$$

$$\omega t = -t\omega, \quad \omega x = x(\omega\triangleleft t) + \omega\triangleleft x = -x\omega.$$

E2.5 Here the Fourier dual of $\mathbb{Z}_r$ is another copy, which we will denote $\hat{\mathbb{Z}}_r$ for clarity, so strictly speaking the isomorphism we have in mind is the Fourier transform $\mathbb{C}\mathbb{Z}_r \cong \mathbb{C}(\hat{\mathbb{Z}}_r)$. This is given by evaluation of the character, which comes down to $t^i(j) = q^{ij}$, where $j \in \hat{\mathbb{Z}}^r$ and $\{t^i\}$ is a basis of $\mathbb{C}\mathbb{Z}_r$. Thus, $t^i \mapsto \sum_j q^{ij}\delta_j$, where δ_j is the delta-function at $j \in \hat{\mathbb{Z}}_r$, with inverse $\delta_j \mapsto \frac{1}{r}\sum_i q^{-ij}t^i$. Next, we let the calculus on $\mathbb{C}(\hat{\mathbb{Z}}_r)$ be given by invariant forms e_a, $a \in \mathcal{C}$, and relations $e_a f = R_a(f)e_a$ where $R_a(f)(j) = f(j+a)$ and $\mathrm{d}f = \sum_a (R_a(f) - f)e_a$. So in terms of t, we have the differential calculus

$$e_a t = q^a t e_a, \quad \mathrm{d}t = t\sum_a (q^a - 1)e_a, \quad \{e_a, e_b\} = 0, \quad \mathrm{d}e_a = 0.$$

We also know that the calculus is inner with $\theta = \sum_a e_a$. The case $\mathcal{C} = \{1\}$ has just one basic form e_1 with $e_1 t = qte_1$ and $\mathrm{d}t = t(q-1)e_1$ so that $e_1 = \frac{1}{q-1}t^{-1}\mathrm{d}t$ and $\Omega^2 = 0$. The first-order calculus is the same as Example 1.11 with in our case a particular normalisation for the basic forms and $t^r = 1$.

E2.6 From $\mathcal{Q}(a \otimes b) = \mathcal{R}(b_{(1)} \otimes a_{(1)})\mathcal{R}(a_{(2)} \otimes b_{(2)})$, we have in our case $\mathcal{Q}(t^m \otimes t^n) = q^{2mn}$. We have a single basis element $f^1 = t$ of $\mathcal{L}$ and letting e_1 be dual to this as a basis of $\mathcal{L}^*$, we have $\varpi(t^m - 1) = \mathcal{Q}(t^m - 1 \otimes (\ ))|_{\mathbb{C}.t} = (q^{2m} - 1)e_1$ in Proposition 2.55. This is surjective so $\Lambda^1 = \mathcal{L}^*$. The left crossed module structure on $\mathcal{L}$ is $t^m {\triangleright} t = q^{2m}t$ and $\Delta_L t = 1 \otimes t$ which dualises to e_1 invariant under Δ_R and $e_1 {\triangleleft} t = q^2 e_1$. The latter implies commutation relations $e_1 t = q^2 te_1$ which with $\theta = e_1$ gives $\mathrm{d}t^m = (q^{2m} - 1)t^m e_1$ (or one can compute this from $\mathrm{d}a = a_{(1)}\varpi\pi_\epsilon a_{(2)}$).

Because the coaction is trivial, the braiding on two copies of Λ^1 is just the flip map, so $\Lambda^2 = 0$ in the canonical construction. As $t^{-1}\mathrm{d}t = (q^2 - 1)e_1$ is both left and right invariant, the super-coproduct is $\Delta_*(t^{-1}\mathrm{d}t) = t^{-1}\mathrm{d}t \otimes 1 + 1 \otimes t^{-1}\mathrm{d}t$ or $\Delta_* \mathrm{d}t = \mathrm{d}t \otimes t + t \otimes \mathrm{d}t$ where we write Δ_* to avoid confusion with Δ in degree 0. If q is a primitive r-th root of unity (with r odd) we see that $\mathrm{d}t^r = 0$, so our calculus $\Omega(\mathbb{C}_{q^2}[S^1])$ is not connected. However, since t^r has zero derivative, we set $t^r = 1$, giving a connected calculus $\Omega(c_{q^2}(S^1))$ on the quotient.

The quotient Hopf algebra now is $A = \mathbb{C}\mathbb{Z}_r$ and inherits a coquasitriangular structure with quantum Killing form sending t^m to $\mathcal{Q}(t^m \otimes (\ ))$, which as a function on $\mathbb{C}(\mathbb{Z}_r)$ has value q^{2jm} at j. Hence $\mathcal{Q}$ as a map is the Fourier isomorphism in the preceding exercise but for q^2 in place of q. It is therefore invertible, so the Hopf algebra is factorisable. By Proposition 2.56, bicovariant Ω^1 on A are in 1-1 correspondence with ideals $\mathcal{J} \subseteq H^+$. Since H^+ is functions on the set $\mathbb{Z}_r$ that vanish at $0 \in \mathbb{Z}_r$, its ideals are simply characterised by subsets $\mathcal{C} \subseteq \mathbb{Z}_r \setminus \{0\}$ on which all elements of the ideal vanish. This is the same as the classification of bicovariant calculi on H (and hence on A) from the Cayley graph point of view; the resulting calculi have the form in exercise E2.5 with q replaced by q^2 (since q is odd, q^2 is some other root of unity).

E2.7 Objects W of the category of $\mathbb{Z}_r$-graded spaces have a direct sum decomposition $W = \oplus_{s \in \mathbb{Z}_r} W_s$ according to grade $|w| = x$ for $w \in W_x$. We define objects $\Bbbk_x$ as the vector space $\Bbbk$ with grade $x \in \mathbb{Z}_r$. For (V, λ_V) in the centre, each morphism $\lambda_{V,\Bbbk_x} : V \otimes \Bbbk_x \to \Bbbk_x \otimes V$ defines $\alpha_x : V \to V$ by $\lambda_{V,\Bbbk_x}(v \otimes 1) = 1 \otimes \alpha_x(v)$ which must also be a morphism, i.e., preserve grades, since $\lambda_{V,\Bbbk_x}$ does. As $\Bbbk_0$ is the unit object, we have $\alpha_0 = \mathrm{id}$. The product in $\Bbbk$ gives a morphism $\mu : \Bbbk_x \otimes \Bbbk_y \to \Bbbk_{x+y}$, and by the functoriality of λ with respect to the morphism μ and the behaviour of λ_V with respect to tensor products, we see that $\alpha_x \circ \alpha_y = \alpha_{x+y}$. Hence α is an action of $\mathbb{Z}_r$ on V. For a general object W and $w \in W$ with $|w| = x$, we define a morphism

$\iota_w : \Bbbk_x \to W$ by $\iota_w(1) = w$. Then we have a commutative diagram

$$\begin{array}{ccc} V \otimes \Bbbk_x & \xrightarrow{\mathrm{id}\otimes\iota_w} & V \otimes W \\ \downarrow{\scriptstyle \lambda_{V,\Bbbk_x}} & & \downarrow{\scriptstyle \lambda_{V,W}} \\ \Bbbk_x \otimes V & \xrightarrow{\iota_w\otimes\mathrm{id}} & W \otimes V \end{array}$$

which tells us that $\lambda_{V,W}(v \otimes w) = w \otimes \alpha_{|w|}(v)$. In this way, objects of the centre can be identified with vector spaces which are both $\mathbb{Z}_r$-graded and have a grade-preserving $\mathbb{Z}_r$-action (i.e., a $\mathbb{Z}_r$ crossed module in the present case).

Next we fix the field to be $\mathbb{C}$ and Fourier transform this $\mathbb{Z}_r$-action to a coaction of $\mathbb{C}[\hat{\mathbb{Z}}_r]$ and hence to another $\mathbb{Z}_r$-grading (by $\hat{\mathbb{Z}}_r$). Letting q be a primitive r-th root of unity, define $P_n = \frac{1}{r}\sum_{i\in\mathbb{Z}_r} q^{-ni}\alpha_i : V \to V$ and note that $P_n P_m = \delta_{n,m} P_n$, so we have commuting projections on V defining a second $\mathbb{Z}_r$-grading on V. As the new grading projections commute with our first $\mathbb{Z}_r$-grading, we have a $\mathbb{Z}_r^2$-grading. As $\alpha_1 P_n = q^n P_n$, if we write the new grading as $\|v\| \in \mathbb{Z}_r$ for $v \in V$ then $\lambda_{V,W}(v \otimes w) = q^{|w|\,\|v\|} w \otimes v$. We now view the double grading as a right $\mathbb{C}_q\mathbb{Z}_r^2$-coaction by $\Delta_R(v) = v \otimes s^{|v|}t^{\|v\|}$. From Proposition 2.52, the braiding on right comodules of the coquasitriangular Hopf algebra $\mathbb{C}_q\mathbb{Z}_r^2$ is $\Psi_{V,W}(v\otimes w) = w_{(\bar{0})} \otimes v_{(\bar{0})}\mathcal{R}(v_{(\bar{1})} \otimes w_{(\bar{1})})$, where $\mathcal{R}$ is to be determined. The preceding formula for $\lambda_{V,W}$ also provides the braiding between (V, λ_V) and (W, λ_W) as objects of the centre, which gives $\mathcal{R}$ as stated in the question. Thus the centre can be identified with the braided category of comodules of $\mathbb{C}_q\mathbb{Z}_r^2$ with this coquasitriangular structure.

To check that the latter is factorisable, the quantum Killing form is

$$\mathcal{Q}(s^{m_1}t^{m_2}\otimes s^{n_1}t^{n_2}) = \mathcal{R}(s^{n_1}t^{n_2}\otimes s^{m_1}t^{m_2})\mathcal{R}(s^{m_1}t^{m_2}\otimes s^{n_1}t^{n_2}) = q^{m_2 n_1 + n_2 m_1},$$

which is equivalent to $\mathcal{Q} = \sum_{m_1,m_2,n_1,n_2} q^{m_2n_1+n_2m_1}\delta_{s^{m_1}t^{m_2}} \otimes \delta_{s^{n_1}t^{n_2}} \in H$ where $H = \mathbb{C}(\mathbb{Z}_r \times \mathbb{Z}_r)$ is the dual Hopf algebra. We can also view it as a map

$$\mathcal{Q} : A \to H, \quad \mathcal{Q}(s^{m_1}t^{m_2}) = \sum_{n_1,n_2} q^{m_2n_1+n_2m_1}\delta_{s^{n_1}t^{n_2}}$$

by evaluating an element of A on the first H factor. This is again a isomorphism, in fact of Hopf algebras, just recovering the Fourier isomorphism $\mathbb{C}\mathbb{Z}_r^2 \cong \mathbb{C}(\hat{\mathbb{Z}}_r^2)$ much as in exercise E2.5 but with an extra swap of s, t. [In general, $\mathcal{Q}$ is a morphism of braided-dual Hopf algebras and the categorial or braided dual involves the opposite product (and coproduct) to the usual one.]

E2.8 (i) As $\Psi(x\otimes x)=qx\otimes x$ from Definition 2.67, $[n,\Psi]:V^{\otimes n}\to V^{\otimes n}$ is the q-integer $[n]_q$ times the identity, which in turn gives $[n,\Psi]!:V^{\otimes n}\to V^{\otimes n}$ as $[n]_q[n-1]_q\dots[1]_q$ times the identity. Then $[n,\Psi]!$ is a bijection for $n<r$ and $[n,\Psi]!=0$ for $r\geq 0$, so $B_+(V)=T_+V/\oplus_n\ker[n,\Psi]!$ in Corollary 2.72 gives $T_+V=\mathbb{C}[x]$ modulo the relation $x^r=0$. From the discussion preceding Proposition 2.78, we take $\varpi:B_+(V)\to\Lambda^1=V$ to be the projection to the degree one component. Hence the differential calculus in the braided category is given by Proposition 2.78 as $\Omega=B_+(V)\underline{\otimes}\Lambda$, where $\Lambda=T\Lambda^1/\langle\mathrm{image}(\mathrm{id}+\Psi_{\Lambda^1,\Lambda^1})\rangle$. Since $r>2$ (so $q\neq -1$), we see that the image here is all of $\Lambda^1\otimes\Lambda^1$, so $\Omega^0=B_+(V)$ and $\Omega^1=B_+(V)\underline{\otimes}V$, with all others zero. To find d, we use the braided coproduct from Proposition 2.71. The braided tensor product $\underline{\otimes}$ here means that $(\mathrm{d}x)x=(1\otimes\varpi x)(x\otimes 1)=q(x\otimes\varpi x)=qx\mathrm{d}x$ which also implies that $\mathrm{d}x^n=[n]_qx^{n-1}\mathrm{d}x$ similarly to Example 1.11 as far as the algebra is concerned (and in agreement with the general definition of d and of the braided binomials). Here $\Omega(B_+(V))=\Omega(c_q[\mathbb{C}])$ is a super-braided Hopf algebra with coproduct $\underline{\Delta}\mathrm{d}x=\mathrm{d}x\otimes 1+1\otimes\mathrm{d}x$.

(ii) The grading viewed as a right coaction is $\Delta_Rx=x\otimes t$ and $\Delta_R(\mathrm{d}x)=\mathrm{d}x\otimes t$, and the exercise says to extend this nontrivially to a coaction of the entire super-Hopf algebra $\Omega(c_q[S^1])$. The latter from exercise E2.6 has $\Delta_*\mathrm{d}t=\mathrm{d}t\otimes t+t\otimes\mathrm{d}t$. We do not have a super-coquasitriangular structure to automatically provide an action but we take the right action stated, then proceed as similarly to the usual bosonisation as in Example 2.65. For example, $\Delta_{R*}((\mathrm{d}x)^2)=(\mathrm{d}x\otimes t+x\otimes\mathrm{d}t)(\mathrm{d}x\otimes t+x\otimes\mathrm{d}t)=(\mathrm{d}x)^2\otimes t^2+(\mathrm{d}x)x\otimes t\mathrm{d}t-x\mathrm{d}x\otimes(\mathrm{d}t)t+x^2\otimes(\mathrm{d}t)^2=0$ using the graded product. On the action side, we have, for example, $x^2{\triangleleft}\mathrm{d}t=(x{\triangleleft}\mathrm{d}t)(x{\triangleleft}t)+(x{\triangleleft}t)(x{\triangleleft}\mathrm{d}t)=(q^{-1}-1)(\mathrm{d}x)q^{-1}x+q^{-1}x(q^{-1}-1)\mathrm{d}x=0$ by the relations of the calculus. The right handed cross product $\Omega(c_q[S^1]){\triangleright\!\!\!<}\Omega(c_q[\mathbb{C}])$ then has cross relations

$$xt=t(x{\triangleleft}t)=q^{-1}tx,\quad (\mathrm{d}x)t=t(\mathrm{d}x{\triangleleft}t)=t\mathrm{d}x,$$
$$x\mathrm{d}t=\mathrm{d}t(x{\triangleleft}t)+t(x{\triangleleft}\mathrm{d}t)=q^{-1}(\mathrm{d}t)x+(q^{-1}-1)t\mathrm{d}x$$

along with the expected $(\mathrm{d}x)\mathrm{d}t=-(\mathrm{d}t)(\mathrm{d}x{\triangleleft}t)+t(\mathrm{d}x{\triangleleft}\mathrm{d}t)=-(\mathrm{d}t)\mathrm{d}x$ and the separate $(\mathrm{d}t)t=qt\mathrm{d}t$, $(\mathrm{d}t)^2=0$ and $(\mathrm{d}x)x=qx\mathrm{d}x$, $(\mathrm{d}x)^2$ relations. [You may also like to compute the super right cross coproduct making this into a super-Hopf algebra, which expresses that this calculus on $u_{q^{-1}}(b_+)$ is bicovariant.] One can also use this calculus for $q=-1$, where it corresponds to the ideal $\{x-xt\}$ in E2.4.

E2.9 We have the same $\mathcal{R}$ as in exercise E2.6 and proceed similarly, this time with $e_\pm$ dual bases to $t^{\pm1}$ resulting in $e_\pm{\triangleleft}t=q^{\pm2}e_\pm$, $\varpi(t^m-1)=(q^{2m}-1)e_++(q^{-2m}-1)e_-$, $\theta=e_++e_-$ and $\mathrm{d}t^m=t^m\varpi\pi_\epsilon t^m$; if we replace q^2 by q then we get the stated calculus on $\mathbb{C}[S^1]$ (which one can verify directly to be a bicovariant calculus and to have the $*$-calculus properties in the two cases).

For the associated right quantum Lie algebra by Theorem 2.85, the adjoint coaction on Λ^1 and hence on Λ^{1*} are trivial for $H=\mathbb{C}_q\mathbb{Z}$, so $[\ ,\ ]_R=0$ and $\sigma_R=$ flip. The corresponding right braided Lie algebra (by a right version of Corollary 2.91) is $[x,y]=x$ for all $x,y\in\mathcal{L}$ and Ψ is the flip.

This is also the structure of the trivial right IP quandle for $\mathcal{C}=\{\pm1\}\subset\mathbb{Z}$ in Example 2.86. Indeed, we check, that the algebra map $\mathbb{C}[S^1]\to\mathbb{C}(\mathbb{Z})$ sending t to the function $f_t(i)=q^i$ is differentiable for the graph calculus $\Omega(\mathbb{Z})$ of the bidirected *integer lattice*. This is just the 1D case of Example 1.62 and has translation invariant (and wedge-anticommuting) basis $e_\pm$ with relations

$$e_\pm f=R^{\pm1}(f)e_\pm,\ \mathrm{d}f=\partial^+fe_++\partial^-fe_-;\quad \partial^\pm=R^{\pm1}-\mathrm{id},\ R^{\pm1}(f)(i)=f(i\pm1).$$

In this calculus, $R^{\pm1}(f_t)(i)=q^{i\pm1}=q^{\pm1}f_t(i)$ and $(\partial^\pm f_t)(i)=q^{i\pm1}-q^i=(q^{\pm1}-1)f_t(i)$ so that $e_\pm f_t=q^{\pm1}f_te_\pm$ and $\mathrm{d}f_t=f_t((q-1)e_-+(q^{-1}-1)e_-)$ matches the 2D calculus on $\mathbb{C}[S^1]$ stated in the question. At q a primitive r-th root of unity, we quotient by $t^r=1$ to recover the isomorphism in E2.5 for $\mathcal{C}=\{\pm1\}$.

E2.10 The definition of $*$-calculus comes down to $\star:\Omega\to\overline{\Omega}$ defined by $\xi\mapsto\overline{\xi^*}$ being a map of DGAs, where $\overline{\Omega}$ has derivative $\overline{\mathrm{d}}$ and product

$$\overline{\Omega^n}\otimes_A\overline{\Omega^m}\stackrel{(-1)^{nm}\Upsilon^{-1}}{\longrightarrow}\overline{\Omega^m\otimes_A\Omega^n}\stackrel{\wedge}{\longrightarrow}\overline{\Omega^{n+m}}.$$

This means that we have two commutative diagrams

$$\begin{tikzcd}\Omega^n \arrow[r, "\mathrm{d}"'] \arrow[d, "\star"'] & \Omega^{n+1} \arrow[d, "\star"'] \\ \overline{\Omega^n} \arrow[r, "\overline{\mathrm{d}}"] & \overline{\Omega^{n+1}}\end{tikzcd}\qquad\begin{tikzcd}\Omega^n\otimes_A\Omega^m \arrow[r, "\wedge"'] \arrow[d, "\star\otimes\star"'] & \Omega^{n+m} \arrow[d, "\star"'] \\ \overline{\Omega^n}\otimes_A\overline{\Omega^m} \arrow[r, "\wedge"] & \overline{\Omega^{n+m}}\end{tikzcd}$$

replacing the more usual $\mathrm{d}(\xi^*)=(\mathrm{d}\xi)^*$ and $(\xi\wedge\eta)^*=(-1)^{|\xi||\eta|}\eta^*\wedge\xi^*$. We use these diagrams as the definition of a $*$-DGA in any bar category, given $(\Omega,\mathrm{d},\wedge)$ a DGA in the category.

For our exotic example, the group automorphism $z{\triangleright}$ extends to a differentiable map if we define $z{\triangleright}e_a=e_{z\triangleright a}$, noting that $\mathcal{C}$ is closed under $z{\triangleright}$. The relations of the maximal prolongation as in Proposition 1.53 are preserved by $z{\triangleright}$, so $\mathbb{Z}_4$ acts on Ω by maps of DGAs. Now define $\star e_{(i,i+1)}=\overline{z{\triangleright}e^*_{(i,i+1)}}=-\overline{z{\triangleright}e_{(i,i+1)}}=-\overline{e_{(i+1,i+2)}}$ (mod 4), and check that

$$\begin{aligned}\overline{\mathrm{d}}\star(\xi)&=\overline{\mathrm{d}(z{\triangleright}\xi^*)}=\overline{z{\triangleright}\mathrm{d}(\xi^*)}=\overline{z{\triangleright}(\mathrm{d}\xi)^*}=\star\mathrm{d}\xi,\\ \wedge(\star\xi\otimes\star\eta)&=\overline{z{\triangleright}(\xi^*)}\wedge\overline{z{\triangleright}(\eta^*)}=(-1)^{|\xi||\eta|}\overline{(z{\triangleright}(\eta^*))\wedge(z{\triangleright}(\xi^*))}\\ &=(-1)^{|\xi||\eta|}\overline{z{\triangleright}(\eta^*\wedge\xi^*)}=\overline{z{\triangleright}(\xi\wedge\eta)^*}=\star(\xi\wedge\eta).\end{aligned}$$

We have $\star\star e_{(i,i+1)} = \overline{\overline{e_{(i+2,i+3)}}}$ rather than just $\overline{\overline{e_{(i,i+1)}}}$, but that is allowed by the more general setting of a bar category as compared to a usual $*$-operation.

Solutions to Exercises for Chap. 3

E3.1 First note that multiplication by $a_3 \in A$ is the identity on E, and that multiplication by $a_4 \in A$ is the identity on F. Then given $\phi \in {}_A\mathrm{Hom}(E, A) = E^\flat$ we have $a_3.\phi(a_3) = \phi(a_3.a_3) = \phi(a_3)$ from which one can see conversely that $\phi(a_3) \in E$. Also, $\phi(e) = \phi(e.a_3) = e.\phi(a_3)$ (viewing $e \in A$ and $a_3 \in E$), so we can identify ϕ as given by an element $\phi(a_3) \in E$. Similarly, we identify $\psi \in {}_A\mathrm{Hom}(F, A) = F^\flat$ with $\psi(a_4) \in F$ as $\psi(f) = f.\psi(a_4)$.

Now define $\mathrm{coev}_E(a_1) = a_3 \otimes a_3 \in E^\flat \otimes_A E$ and check that

$$(\mathrm{ev} \otimes \mathrm{id})(e \otimes \mathrm{coev}_E(a_1)) = (ea_3)a_3 = e$$

for all $e \in E$. Thus the 1×1 projection matrix is $P_E = (a_3)$. Similarly, define $\mathrm{coev}_F(a_1) = a_2 \otimes a_2 \in F^\flat \otimes_A F$ and check that

$$(\mathrm{ev} \otimes \mathrm{id})(f \otimes \mathrm{coev}_E(a_1)) = (fa_2)a_2 = fa_4 = f$$

for all $f \in F$. Thus the 1×1 projection matrix is $P_F = (a_4)$. The map of multiplication by a_4 from A to A is a projection to A, and its kernel is the complementary module $F^\perp$ which is spanned by $a_1 - a_4$, $a_3 - a_0$ and $a_5 - a_2$.

E3.2 Corresponding to the dual bases $a_3 \otimes a_3$, we work with a 1×1 Christoffel symbol matrix $\Gamma = (\xi)$, and as $P = (a_3)$ from Proposition 3.23, we obtain the equations $\xi.a_3 = \xi$ and $(\mathrm{d}a_3)a_3 = (a_3 - 1)\xi$. If we set $\xi = \sum_i a_i \otimes v_i \in \Omega^1 = A \otimes V$ then

$$\xi.a_3 = \sum_i a_{3i} \otimes v_i \triangleleft a_3 = \xi, \quad \mathrm{d}a_3.a_3 = a_3 \otimes \zeta(a_3) \triangleleft a_3 = \sum_i (a_{3i} - a_i) \otimes v_i.$$

From the first equation, we read off $v_1 = v_2 = v_4 = v_5 = 0$ and $v_0 \triangleleft a_3 = v_0$ and $v_3 \triangleleft a_3 = v_3$. Then the second equation requires $\zeta(a_3)\triangleleft a_3 = 0$, which is satisfied. Thus the most general connection is given by arbitrary $p, q \in \mathbb{C}$ as

$$\Gamma = \big(a_0 \otimes (p, 0) + a_3 \otimes (q, 0)\big).$$

E3.3 Begin with

$$\begin{aligned}\nabla^L(\xi{\triangleleft}h) &= -\,\varpi\pi_\epsilon S^{-1}(\xi{\triangleleft}h)_{(\bar 1)}\otimes(\xi{\triangleleft}h)_{(\bar 0)}\\ &= -\,\varpi\pi_\epsilon((S^{-1}h_{(3)})(S^{-1}\xi_{(\bar 1)})h_{(1)})\otimes\xi_{(\bar 0)}{\triangleleft}h_{(2)}\\ &= -\,(\varpi\pi_\epsilon S^{-1}h_{(3)}){\triangleleft}(S^{-1}\xi_{(\bar 1)})h_{(1)}\otimes\xi_{(\bar 0)}{\triangleleft}h_{(2)}\\ &\quad -(\varpi\pi_\epsilon S^{-1}\xi_{(\bar 1)}){\triangleleft}h_{(1)}\otimes\xi_{(\bar 0)}{\triangleleft}h_{(2)}-\varpi\pi_\epsilon h_{(1)}\otimes\xi{\triangleleft}h_{(2)}\end{aligned}$$

for $\xi\in\Lambda^1$ and $h\in H$, from which we find that

$$\begin{aligned}(\nabla^L(\xi{\triangleleft}Sh_{(1)})){\triangleleft}h_{(2)} &= -\,(\varpi\pi_\epsilon h){\triangleleft}S^{-1}\xi_{(\bar 1)}\otimes\xi_{(\bar 0)}-\varpi\pi_\epsilon S^{-1}\xi_{(\bar 1)}\otimes\xi_{(\bar 0)}\epsilon h\\ &\quad -(\varpi\pi_\epsilon Sh_{(2)}){\triangleleft}h_{(3)}\otimes\xi{\triangleleft}(Sh_{(1)})h_{(4)}\\ &= -\,(\varpi\pi_\epsilon h){\triangleleft}S^{-1}\xi_{(\bar 1)}\otimes\xi_{(\bar 0)}+(\nabla^L\xi)\epsilon h\\ &\quad +\varpi\pi_\epsilon h_{(2)}\otimes\xi{\triangleleft}(Sh_{(1)})h_{(3)}.\end{aligned}$$

Using this, the displayed formula for σ in Proposition 3.73 reduces to

$$\sigma^L(\xi\otimes\varpi\pi_\epsilon h)=(\varpi\pi_\epsilon h){\triangleleft}S^{-1}\xi_{(\bar 1)}\otimes\xi_{(\bar 0)}$$

which is indeed the inverse right crossed module braiding Ψ^{-1}, as required.

E3.4 Rearrange the relation to read $ww^*+1=q^2(w^*w+1)$. If w is represented as a bounded operator on a Hilbert space, then ww^*+1 is a positive operator with spectral radius $|w|^2+1$, where $|w|$ is the operator norm of w. On the other hand, w^*w+1 also has spectral radius $|w|^2+1$ and as $q^2\neq 1$, this contradicts the relation.

E3.5 The s_i are central and anticommute so it is immediate that $\int(\omega\wedge\eta)=(-1)^{|\omega||\eta|}\int(\eta\wedge\omega)$ on different degrees. We check that this is closed. Thus

$$\int \mathrm{d}(as_is_j)=\int((\partial_k a)s_ks_is_j)-\frac{1}{2}\int a(\epsilon_{imn}s_ms_ns_j-\epsilon_{jmn}s_is_ms_n)=\mathrm{Tr}(\partial_k a)\epsilon_{kij}=0$$

by the graded Leibniz rule. The second term in the middle vanishes as the double ϵ expression is proportional to δ_{ij} while $\partial_k=\frac{1}{2\lambda}[x_k,\ ]$ in the algebra and hence vanishes under the trace. (The form of ∂_k is immediate from the form of θ and the s_k being central.) We also define $\int_\theta$ and check that

$$\int_\theta \mathrm{d}(as_i)=\int\theta\mathrm{d}(as_i)=-\int\mathrm{d}(\theta as_i)+\int(\mathrm{d}\theta)as_i=\int\theta^2as_i=\frac{1}{2\lambda}\mathrm{Tr}(x_ia)$$

using the formula for θ^2. This is not always zero, e.g. $\sum\mathrm{Tr}(x_ix_i)$ is not zero if Tr is the trace in an irreducible matrix representation as $x\cdot x$ is represented as a nonzero multiple of the identity. We also have, using the cycle and closure properties of $\int$,

$$\int_\theta (\eta a - a\eta) = \int \theta\eta a - \int \theta a\eta = -\int [\theta, a]\eta = -\int (\mathrm{d}a)\eta = \int a\mathrm{d}\eta.$$

E3.6 We have

$$\int (ae_1 \wedge e_2 u^n v^m) = q^{n+m} \int (au^n v^m e_1 \wedge e_2) = q^{n+m} \int (au^n v^m)$$

and as $\int$ is a trace on the algebra,

$$\int (ae_1 \wedge e_2 u^n v^m) = q^{n+m} \int (u^n v^m ae_1 \wedge e_2)$$

so that we have a twisted 2-cycle with twisting function $\varsigma(u^n v^m) = q^{n+m} u^n v^m$. A quick calculation shows that $\mathrm{d}(u^n v^m) = u^n v^m ([n]_q e_1 + [m]_q e_2)$, so the associated twisted cyclic cocycle $\phi(u^a v^b, u^n v^m, u^r v^s)$ is

$$\begin{aligned}
&\int u^a v^b \mathrm{d}(u^n v^m) \wedge \mathrm{d}(u^r v^s) \\
&\quad = \int u^a v^b u^n v^m ([n]_q e_1 + [m]_q e_2) \wedge u^r v^s ([r]_q e_1 + [s]_q e_2) \\
&\quad = \int u^a v^b u^n v^m u^r v^s (q^r [n]_q e_1 + q^s [m]_q e_2) \wedge ([r]_q e_1 + [s]_q e_2) \\
&\quad = \mathrm{e}^{\mathrm{i}\theta(rm+bn+br)} \int u^{a+b+r} v^{b+m+s} (q^r [n]_q [s]_q - q^s [m]_q [r]_q) e_1 \wedge e_2 \\
&\quad = \mathrm{e}^{\mathrm{i}\theta(rm+bn+br)} (q^r [n]_q [s]_q - q^s [m]_q [r]_q) \delta_{a+b+r,0} \delta_{b+m+s,0}.
\end{aligned}$$

E3.7 For a general graph, the Leibniz rule for a left connection is

$$\delta_{k,p} \nabla \omega_{p\to q} = \mathrm{d}\delta_k \otimes \omega_{p\to q} + \delta_k \nabla \omega_{p\to q}$$

which we rewrite (summing over all existing edges obeying the conditions) as

$$(\delta_{k,p} - \delta_k) \nabla \omega_{p\to q} = \delta_{k,p} \sum_{r\to p} \omega_{r\to p} \otimes \omega_{p\to q} - \sum_{k\to p} \omega_{k\to p} \otimes \omega_{p\to q}.$$

In our case, $\Omega^1 \otimes_A \Omega^1$ is 1D with basis $\omega_{x\to y} \otimes \omega_{y\to z}$, so the condition is

$$(\delta_{k,p} - \delta_{k,x}) \nabla \omega_{p\to q} = (\delta_{k,p} - \delta_{k,x}) \delta_{q,z} \delta_{p,y} \omega_{x\to y} \otimes \omega_{y\to z}$$

and this reduces to the one condition $\nabla\omega_{y\to z} = \omega_{x\to y}\otimes\omega_{y\to z}$ and arbitrary constants $a, b \in \mathbb{C}$ for the other values,

$$\nabla\omega_{x\to y} = a\omega_{x\to y}\otimes\omega_{y\to z}, \quad \nabla\omega_{x\to z} = b\omega_{x\to y}\otimes\omega_{y\to z}.$$

We have $R_\nabla = 0$ since Ω^2 has basis $\omega_{x\to y}\wedge\omega_{y\to z}$ so that $\Omega^2\otimes_A\Omega^1 = 0$, while

$$T_\nabla\omega_{x\to y} = (a-1)\omega_{x\to y}\wedge\omega_{y\to z}, \quad T_\nabla\omega_{x\to z} = (b+1)\omega_{x\to y}\wedge\omega_{y\to z}$$

and $T_\nabla\omega_{y\to z} = 0$. For a general graph bimodule connection we want

$$\delta_{k,q}\nabla\omega_{p\to q} = \sigma(\omega_{p\to q}\otimes \mathrm{d}\delta_k) + (\nabla\omega_{p\to q})\delta_k$$

which in our case is

$$\begin{aligned}(\delta_{k,q}-\delta_{k,z})\nabla\omega_{p\to q} &= \delta_{p,x}\delta_{q,y}\sigma(\omega_{x\to y}\otimes\mathrm{d}\delta_k)\\ &= \delta_{p,x}\delta_{q,y}(\delta_{k,z}-\delta_{k,q})\sigma(\omega_{x\to y}\otimes\omega_{y\to z}).\end{aligned}$$

We get no condition for $q = z$, while for $q = y$ (the only alternative) we get

$$\sigma(\omega_{x\to y}\otimes\omega_{y\to z}) = -\nabla\omega_{x\to y} = -a\omega_{x\to y}\otimes\omega_{y\to z}$$

so that all left connections are bimodule connections.

E3.8 We work with basic 1-form indices mod 3 and set $\nabla e_i = -\Gamma^i{}_{jr}e_j\otimes e_r$. Then

$$\begin{aligned}\sigma(e_i\otimes\mathrm{d}\delta_k) = \nabla(e_i\delta_k) - (\nabla e_i)\delta_k &= \nabla(\delta_{k-i}e_i) - (\nabla e_i)\delta_k\\ &= \mathrm{d}\delta_{k-i}\otimes e_i + \delta_{k-i}\nabla e_i - (\nabla e_i)\delta_k\end{aligned}$$

so

$$\sum_a(\delta_{k-i-a}-\delta_{k-i})(\sigma(e_i\otimes e_a) - e_a\otimes e_i) = \Gamma^i{}_{jr}(\delta_{k-j-r}-\delta_{k-i})e_j\otimes e_r.$$

Since $i+a\neq i$ and $j+r\neq i$ we have, multiplying by δ_{k-i-a} and summing over k,

$$\sigma(e_i\otimes e_a) = e_a\otimes e_i + \sum_{j+r=i+a}\Gamma^i{}_{jr}e_j\otimes e_r.$$

This gives a bimodule map, hence all left connections are bimodule connections. Moreover, since $\mathrm{d}e_a = 0$, the curvature and torsion are

$$R_\nabla(e_i) = -\mathrm{d}\Gamma^i{}_{st} \wedge e_s \otimes e_t - \Gamma^i{}_{jr} R_{-j}(\Gamma^r{}_{st}) e_j \wedge e_s \otimes e_t,$$
$$T_\nabla(e_i) = -\Gamma^i{}_{jr} e_j \wedge e_r = (\Gamma^i{}_{21} - \Gamma^i{}_{12}) e_1 \wedge e_2.$$

E3.9 For central $s^i \in \Omega^1$ and $a \in A$, we have $a\sigma(e^\mu \otimes s^i) = \sigma(e^\mu \otimes s^i)a$ as σ is a bimodule map. Hence $\sigma^{\mu i}{}_{j\nu}$ are central, so in $\Bbbk$ under our centre assumption. Also

$$\sigma(e^\mu \otimes \mathrm{d}a) = \nabla(e^\mu.a) - (\nabla e^\mu).a = \mathrm{d}a \otimes e^\mu - a\Gamma^\mu{}_{i\nu} s^i \otimes e^\nu + \Gamma^\mu{}_{i\nu} a s^i \otimes e^\nu$$

and substituting for $\mathrm{d}a = [\theta, a]$ gives

$$[\theta_i, a]\sigma(e^\mu \otimes s^i) = [\theta_i, a]s^i \otimes e^\mu - a\Gamma^\mu{}_{i\nu} s^i \otimes e^\nu + \Gamma^\mu{}_{i\nu} a s^i \otimes e^\nu$$

or $[\Gamma^\mu{}_{i\nu} - \theta_j \sigma^{\mu j}{}_{i\nu} + \theta_i \delta_{\mu\nu}, a] = 0$ for all $a \in A$. This gives $\Gamma^\mu{}_{i\nu}$ as stated for some $\alpha^\mu{}_{i\nu} \in \Bbbk$ by the centre assumption. Then

$$R_\nabla e^\mu = -\mathrm{d}\Gamma^\mu{}_{i\nu} \wedge s^i \otimes e^\nu + \Gamma^\mu{}_{i\nu} s^i \wedge \nabla e^\nu$$

evaluates as stated in the question.

E3.10 Setting $A = M_2(\Bbbk)$ in E3.9, the condition for zero curvature is

$$0 = [\theta_1, \Gamma^1{}_{21}] + \Gamma^1{}_{11}\Gamma^1{}_{21} + [\theta_2, \Gamma^1{}_{11}] + \Gamma^1{}_{21}\Gamma^1{}_{11}$$

which, setting $\Gamma^1{}_{i1} = \theta_j c_i{}^j + \gamma_i$ for some $c_i{}^j, \gamma_i \in \Bbbk$, becomes

$$0 = [\theta_1, \theta_2](c_2{}^2 - c_1{}^1) + c_1{}^r c_2{}^s(\theta_r\theta_s + \theta_s\theta_r) + 2\gamma_2\theta_r c_1{}^r + 2\gamma_1\theta_s c_2{}^s + 2\gamma_1\gamma_2.$$

In our case, $\theta_1 = E_{12}$ and $\theta_2 = E_{21}$ so that $\theta_r\theta_s + \theta_s\theta_r = 1$ if $r \neq s$ and zero otherwise. Hence our condition is

$$0 = [\theta_1, \theta_2](c_2{}^2 - c_1{}^1) + c_1{}^1 c_2{}^2 + c_1{}^2 c_2{}^1 + 2\theta_r(\gamma_2 c_1{}^r + \gamma_1 c_2{}^r) + 2\gamma_1\gamma_2,$$

which is equivalent to the four conditions stated in the question since the matrices $[\theta_1, \theta_2], \theta_1, \theta_2, I_2$ are linearly independent.

These are solved (assuming a square root and that $\Bbbk$ is not of characteristic 2) by

$$c_1{}^1 = \sqrt{-\gamma_1\gamma_2}, \quad c_2{}^2 = c_1{}^1, \quad c_1{}^2 = -\frac{c_1{}^1\gamma_1}{\gamma_2}, \quad c_2{}^1 = -\frac{c_1{}^1\gamma_2}{\gamma_1}$$

if both γ_i are nonzero. If $\gamma_1 \neq 0, \gamma_2 = 0$ then ${c_2}^2 = {c_1}^1 = {c_2}^1 = 0$ and ${c_1}^2$ is free. If $\gamma_1 = 0, \gamma_2 \neq 0$ then ${c_2}^2 = {c_1}^1 = {c_1}^2 = 0$ and ${c_2}^1$ is free. If $\gamma_1 = \gamma_2 = 0$ then we need only ${c_2}^2 = {c_1}^1$ and ${c_1}^2{c_2}^1 = -({c_1}^1)^2$. These are all 2-parameter components.

Solutions to Exercises for Chap. 4

E4.1 From (3.32), we have for e_u (similarly for e_v and e_w)

$$\nabla e_u = -e_v \otimes e_w - e_w \otimes e_v, \quad R_\nabla e_u = e_u \wedge e_v \otimes e_v + e_u \wedge e_w \otimes e_w.$$

The first Bianchi identity $\wedge R_\nabla e_u = 0$ follows immediately from the relations of $\Omega(S_3)$ in Example 1.60. The second Bianchi identity $(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \wedge \nabla)R_\nabla = (\mathrm{id} \wedge R_\nabla)\nabla$ can similarly be checked on e_u. Next, the 4-cycle properties hold on basic forms again by the relations of $\Omega(S_3)$. For example, swapping $e_u \wedge e_v \wedge e_u$ with e_w gives a minus sign ($e_u \wedge e_v \wedge e_w = e_w \wedge e_v \wedge e_u = -e_w \wedge e_u \wedge e_w = -e_u \wedge e_w \wedge e_u$ and its two cyclic permutations are a convenient basis of 3-forms) while the basis $e_u \wedge e_v, e_v \wedge e_u, e_v \wedge e_w, e_w \wedge e_v$ in degree 2 mutually commute. Also note that a basic top form is central and zero unless the total degree given by the product of the basis labels is the group identity e. Then for the general case (where we sum over labels belonging to our declared bases),

$$\int(\omega \wedge \rho) = \int \omega_a e_a \wedge \rho_{bcd} e_b \wedge e_c \wedge e_d = \int \omega_a R_a(\rho_{bcd}) e_a \wedge e_b \wedge e_c \wedge e_d$$
$$= -\int R_{bcd}(\omega_a)\rho_{bcd} e_b \wedge e_c \wedge e_d \wedge e_a = -\int \rho_{bcd} e_b \wedge e_c \wedge e_d \wedge \omega_a e_a = -\int \rho \wedge \eta$$

for degrees 1,3 (and similarly for degrees 3,1), and

$$\int(\omega \wedge \rho) = \int \omega_{ab} e_a \wedge e_b \wedge \rho_{cd} e_c \wedge e_d = \int \omega_{ab} R_{ab}(\rho_{cd}) e_a \wedge e_b \wedge e_c \wedge e_d$$
$$= \int R_{cd}(\omega_{ab})\rho_{cd} e_c \wedge e_d \wedge e_a \wedge e_b = \int \rho_{cd} e_c \wedge e_d \wedge \omega_{ab} e_a \wedge e_b = \int \rho \wedge \eta$$

for degrees 2,2. We have $\int \mathrm{d}\omega = \int[\theta, \omega\} = 0$ (where we use the graded-commutator) since the calculus is inner by $\theta = e_u + e_v + e_w$. By explicit calculation, the $\otimes e_u$ component of $(\mathrm{id} \wedge R_\nabla)R_\nabla(e_u)$ vanishes, so $\int \mathrm{Tr} R_\nabla^2 = 0$.

E4.2 By Example 4.30 with $X = G = \mathbb{Z}_4$, differentiability of the action needs an arrow $x \to xa$ in the graph on X for every $a \in \mathcal{C}_H$ such that $x \neq ax$. (i) If $\mathcal{C}_H = \{z^3\}$ then this is not the case as there is no arrow $1 \to z^3$ in the calculus on $P = \Bbbk(X)$. (ii) By contrast, $\mathcal{C}_H = \{z\}$ satisfies the conditions and we find $\varDelta_{L*}$ from formula (4.17). The stated answer then follows from

$$\varDelta_{L*}e_b = \sum_{x\in\mathbb{Z}_4} \varDelta_{L*}\omega_{x\to bx} = \sum_{g,x\in\mathbb{Z}_4} \Big(\sum_{a\in\mathcal{C}_H} \delta_g h_a \otimes \delta_{g^{-1}x}\delta_{b,a} + \delta_g \otimes \omega_{g^{-1}x\to g^{-1}bx}\Big)$$

for $b \in \mathcal{C}_P$. This extends to the maximal prolongation by Lemma 4.29.

E4.3 For s and t generators of the first and second circle algebras respectively, we have the right coaction $\varDelta_R(u^i v^j) = u^i v^j \otimes s^i t^j$. Its differential gives

$$\varDelta_{R*}(\mathrm{d}u.u) = \mathrm{d}u.u \otimes s^2 \otimes 1 + u^2 \otimes \mathrm{d}s.s \otimes 1,$$
$$q\varDelta_{R*}(u.\mathrm{d}u) = qu.\mathrm{d}u \otimes s^2 \otimes 1 + qu^2 \otimes s.\mathrm{d}s \otimes 1,$$
$$\varDelta_{R*}(\mathrm{d}v.u) = \mathrm{d}v.u \otimes s \otimes t + vu \otimes s \otimes \mathrm{d}t,$$
$$e^{i\theta}\varDelta_{R*}(u.\mathrm{d}v) = e^{i\theta}u.\mathrm{d}v \otimes s \otimes t + e^{i\theta}uv \otimes s \otimes \mathrm{d}t,$$

sending the relations $\mathrm{d}u.u - qu.\mathrm{d}u$ and $\mathrm{d}v.u - e^{i\theta}u.\mathrm{d}v$ to zero modulo the relations.

E4.4 In the general case we have

$$(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(e^i\otimes s^j\otimes s^k) = \sigma^{ij}{}_{mn}s^m\otimes\sigma(e^n\otimes s^k) = \sigma^{ij}{}_{mn}\sigma^{nk}{}_{pq}s^m\otimes s^p\otimes e^q.$$

For extendability in the case of $M_2(\mathbb{C})$, we require that $\wedge$ applied to this is zero when $j = k$, i.e.,

$$0 = \sigma^{1j}{}_{m1}\sigma^{1j}{}_{p1}s^m \wedge s^p$$

and the condition for this is $0 = \sigma^{1j}{}_{11}\sigma^{1j}{}_{21} + \sigma^{1j}{}_{21}\sigma^{1j}{}_{11}$. We also require the same for $e^1 \otimes s^1 \otimes s^2 - e^1 \otimes s^2 \otimes s^1$, i.e.,

$$0 = \sigma^{11}{}_{m1}\sigma^{12}{}_{p1}s^m \wedge s^p - \sigma^{12}{}_{m1}\sigma^{11}{}_{p1}s^m \wedge s^p$$

and the condition for this is

$$0 = \sigma^{11}{}_{11}\sigma^{12}{}_{21} - \sigma^{12}{}_{11}\sigma^{11}{}_{21} + \sigma^{11}{}_{21}\sigma^{12}{}_{11} - \sigma^{12}{}_{21}\sigma^{11}{}_{11},$$

which is automatically satisfied. In terms of the notations of E3.10, the condition for extendability becomes

$$(c_1{}^1 + 1)c_2{}^1 = c_1{}^2(c_2{}^2 + 1) = 0.$$

The condition for the curvature to be a bimodule map can be read off from the formula for the curvature in the solution of E3.10, as

$$[\theta_1, \theta_2](c_2{}^2 - c_1{}^1) + c_1{}^1c_2{}^2 + c_1{}^2c_2{}^1 + 2\theta_r(\gamma_2 c_1{}^r + \gamma_1 c_2{}^r) + 2\gamma_1\gamma_2 \in \Bbbk$$

which by linear independence of the matrices amounts to

$$c_2{}^2 = c_1{}^1, \quad \gamma_2 c_1{}^1 + \gamma_1 c_2{}^1 = \gamma_2 c_1{}^2 + \gamma_1 c_2{}^2 = 0.$$

There are thus three classes of connections giving objects in ${}_A\mathcal{G}_A$, namely

$$\begin{aligned}
&\text{(i)} \quad c_1{}^1 = c_2{}^2 = -1, \quad \gamma_1 c_2{}^1 = \gamma_2, \quad \gamma_2 c_1{}^2 = \gamma_1,\\
&\text{(ii)} \quad c_1{}^1 = c_2{}^2 = c_2{}^1 = c_1{}^2 = 0,\\
&\text{(iii)} \quad c_1{}^1 = c_2{}^2 \notin \{0, -1\}, \quad c_2{}^1 = c_1{}^2 = 0, \quad \gamma_1 = \gamma_2 = 0.
\end{aligned}$$

The curvature is $R_\nabla e^1 = -\frac{1}{2}(c_1{}^1 c_2{}^2 + c_1{}^2 c_2{}^1 + 2\gamma_1\gamma_2) s \wedge t \otimes e^1$ and can be nonzero (e.g. case (ii) with $\gamma_1\gamma_2 \neq 0$).

E4.5 We first check the claim about the ideal $\langle x - xt\rangle$. Setting $e_1 = \varpi(x)$ and $e_2 = \varpi(t-1)$, one has

$$\mathrm{Ad}_R e_2 = e_2 \otimes 1, \quad \mathrm{Ad}_R e_1 = e_1 \otimes t - e_2 \otimes x,$$
$$e_2 \triangleleft t = -e_2, \quad e_2 \triangleleft x = -2e_1, \quad e_1 \triangleleft t = e_1, \quad e_1 \triangleleft x = 0,$$

with differentials $\mathrm{d}h = h_{(1)}\varpi\pi_\epsilon(h_{(2)})$ and relations $\omega h = h_{(1)}(\omega \triangleleft h_{(2)})$,

$$\mathrm{d}t = t\varpi\pi_\epsilon(t) = te_2, \quad \mathrm{d}x = x\varpi\pi_\epsilon(t) + \varpi\pi_\epsilon(x) = xe_2 + e_1, \quad \mathrm{d}(xt) = te_1,$$
$$e_2 t = -te_2, \quad e_1 t = te_1, \quad e_1 x = x(e_1 \triangleleft t) + e_1 \triangleleft x = xe_1,$$
$$e_2 x = x(e_2 \triangleleft t) + e_2 \triangleleft x = -xe_2 - 2e_1.$$

For the higher order calculus the braiding $\Psi(v \otimes w) = \sum w_{(\bar{0})} \otimes v \triangleleft w_{(\bar{1})}$ from Definition 2.22 is computed from $\mathrm{Ad}_R, \triangleleft$ and the resulting relations from $\ker(\mathrm{id} - \Psi)$ and the differentials from $\mathrm{d}\varpi\pi_\epsilon(h) = -\varpi\pi_\epsilon(h_{(1)}) \wedge \varpi\pi_\epsilon(h_{(2)})$ are

$$\Psi(e_i \otimes e_2) = e_2 \otimes e_i, \quad \Psi(e_1 \otimes e_1) = e_1 \otimes e_1,$$
$$\Psi(e_2 \otimes e_1) = -e_1 \otimes e_2 + 2e_2 \otimes e_1, \quad e_1 \wedge e_1 = e_2 \wedge e_2 = 0,$$
$$e_1 \wedge e_2 = e_2 \wedge e_1, \quad \mathrm{d}e_2 = -e_2 \wedge e_2 = 0, \quad \mathrm{d}e_1 = -e_1 \wedge e_2.$$

The cohomologies $\mathrm{H}^i_{\mathrm{dR}}(H) = \mathrm{H}^i(H; \Omega)$ are easily computed. Here $\mathrm{H}^0(H; \Omega) = \mathbb{C}$ (the calculus is connected) while further values $\mathrm{d}(te_1) = \mathrm{d}(xe_1) = \mathrm{d}(te_2) = 0$ and

$$\mathrm{d}(xte_1) = -xte_1 \wedge e_2, \quad \mathrm{d}(xe_2) = e_1 \wedge e_2, \quad \mathrm{d}(xte_2) = te_1 \wedge e_2$$

give

$$\mathrm{H}^1(H;\Omega) = \frac{\mathrm{sp}_{\mathbb{C}}\{te_1, te_2, xe_2 + e_1, e_2, xe_1\}}{\mathrm{sp}_{\mathbb{C}}\{te_1, te_2, xe_2 + e_1\}} \cong \mathbb{C}^2$$

with basis $\{[xe_1], [e_2]\}$ and $\mathrm{H}^2(H;\Omega) \cong \mathbb{C}$ with basis $\{[xe_1 \wedge e_2]\}$. Also,

$$\Delta_L[xe_1] = x \otimes [te_1] + 1 \otimes [xe_1] = 1 \otimes [xe_1], \quad \Delta_L[xe_1 \wedge e_2] = 1 \otimes [xe_1 \wedge e_2]$$

so is trivial on all the cohomology. From the data above we can also read off the cohomology of the invariant forms as $\mathrm{H}^0(\Lambda) \cong \mathbb{C}$, $\mathrm{H}^1(\Lambda) \cong \mathbb{C}$ and $\mathrm{H}^2(\Lambda) = 0$.

E4.6 Given the calculations of the cohomology in exercise E4.5, the second page of the van Est spectral sequence is

$$\begin{tikzcd}[column sep=small]
0 & 0 & 0 & 0 & 0 & \text{Page 2} \\
\mathrm{H}_c^0(H;\mathbb{C}) \arrow[drr] & \mathrm{H}_c^1(H;\mathbb{C}) \arrow[drr] & \mathrm{H}_c^2(H;\mathbb{C}) \arrow[drr] & \mathrm{H}_c^3(H;\mathbb{C}) \arrow[dr] & \mathrm{H}_c^4(H;\mathbb{C}) & {} \\
\mathrm{H}_c^0(H;\mathbb{C}^2) \arrow[drr] & \mathrm{H}_c^1(H;\mathbb{C}^2) \arrow[drr] & \mathrm{H}_c^2(H;\mathbb{C}^2) \arrow[drr] & \mathrm{H}_c^3(H;\mathbb{C}^2) \arrow[dr] & \mathrm{H}_c^4(H;\mathbb{C}^2) & {} \\
\mathrm{H}_c^0(H;\mathbb{C}) & \mathrm{H}_c^1(H;\mathbb{C}) & \mathrm{H}_c^2(H;\mathbb{C}) & \mathrm{H}_c^3(H;\mathbb{C}) & \mathrm{H}_c^4(H;\mathbb{C}) & {}
\end{tikzcd}$$

for $\mathbb{C}$ with the trivial coaction, and this converges to a limit which has $\mathbb{C}, \mathbb{C}, 0, \ldots$ summing along the diagonals. As the lowest leftmost term (position (0, 0)) is left unchanged in the limit, we must have $\mathrm{H}_c^0(H;\mathbb{C}) \cong \mathbb{C}$. On the assumption that $\mathrm{H}_c^1(H;\mathbb{C}) \cong 0$ then on the third page the (0, 1) entry must be $\mathbb{C}$. We also know that as the diagonal sum in the limit of the third diagonal is zero then on the third page the (2, 0) entry must be zero, as there is no way to cancel this entry later. As a result, the cohomology for the page two differential

$$0 \longrightarrow \mathrm{H}_c^0(H;\mathbb{C}^2) \cong \mathbb{C}^2 \longrightarrow \mathrm{H}_c^2(H;\mathbb{C}) \longrightarrow 0$$

must give $\mathbb{C}$ in the second and 0 in the third position. The only way this can happen is if $\mathrm{H}_c^2(H;\mathbb{C}) \cong \mathbb{C}$ and the differential from $\mathrm{H}_c^0(H;\mathbb{C}^2)$ to $\mathrm{H}_c^2(H;\mathbb{C})$ is surjective.

It is instructive to verify this result by calculation of $\mathrm{H}_c(H;\mathbb{C})$ directly. First for $\bar{\mathrm{d}}: G^0=\mathbb{C}\to G^1=H\otimes\mathbb{C}$, we have $\bar{\mathrm{d}}1=1\otimes 1-1\otimes 1$ so $\mathrm{H}^0_c(H;\mathbb{C})\cong\mathbb{C}$. Next for $\bar{\mathrm{d}}: G^1=H\otimes\mathbb{C}\to G^2=H\otimes H\otimes\mathbb{C}$,

$$\bar{\mathrm{d}}(x\otimes 1)=x\otimes 1\otimes 1-x\otimes t\otimes 1,\ \ \bar{\mathrm{d}}(t\otimes 1)=1\otimes t\otimes 1-t\otimes t\otimes 1+t\otimes 1\otimes 1,$$
$$\bar{\mathrm{d}}(xt\otimes 1)=1\otimes xt\otimes 1-t\otimes xt\otimes 1,\ \ \bar{\mathrm{d}}(1\otimes 1)=1\otimes 1\otimes 1,$$

which is injective, so indeed $\mathrm{H}^1_c(H;\mathbb{C})\cong 0$. Finally, a nontrivial calculation shows that the kernel of $\bar{\mathrm{d}}: G^2=H\otimes H\otimes\mathbb{C}\to G^3=H\otimes H\otimes H\otimes\mathbb{C}$ has basis

$$1\otimes1\otimes1,\quad x\otimes xt\otimes1,\quad (t-1)\otimes xt\otimes1,\quad x\otimes(t-1)\otimes1,\quad (t-1)\otimes(t-1)\otimes1,$$

verifying the spectral sequence calculation.

E4.7 We identify $H^{\otimes n}\otimes F$ and $C(G^n,F)$ by sending $\delta_{k_1}\otimes\dots\delta_{k_n}\otimes f$ to the function $\phi(g_1,\dots,g_n)=\delta_{k_1,g_1}\dots\delta_{k_n,g_n}f$. We define $\mathrm{d}\phi$ by using the formula for $\bar{\mathrm{d}}$ in the statement of Proposition 4.55 applied to $\delta_{k_1}\otimes\dots\delta_{k_n}\otimes f$ and then evaluated against $(g_1,\dots,g_{n+1})$. We then use the fact that the coproduct in $\Bbbk(G)$ is dual to the product in G, and recover the formula for $\mathrm{d}\phi$ stated in the question.

E4.8 Remembering that $\mathrm{d}x^3$ anticommutes with $\mathrm{d}Q$ and $\mathrm{d}Q^*$ and $\mathrm{d}x^3\wedge\mathrm{d}x^3=0$,

$$\begin{aligned}&\mathrm{d}P\wedge\mathrm{d}P\wedge\mathrm{d}P\wedge\mathrm{d}P.P\\ &=\frac{1}{16}\begin{pmatrix}\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^* & 4\mathrm{d}x^3\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\\ -4\mathrm{d}x^3\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^* & \mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\end{pmatrix}.P\\ &=\frac{1}{32}\left(\begin{matrix}(1+x^3)\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*+4\mathrm{d}x^3\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q.Q^*\\ -4(1+x^3)\mathrm{d}x^3\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*+\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q.Q^*\end{matrix}\right.\\ &\qquad\left.\begin{matrix}\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*.Q+4(1-x^3)\mathrm{d}x^3\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\\ -4\mathrm{d}x^3\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*.Q+(1-x^3)\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q\end{matrix}\right)\end{aligned}$$

gives the first stated formula on taking the 2×2 matrix trace. Next, in the product $\mathrm{d}Q^*\wedge\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q$ within the trace, move the $\mathrm{d}Q$ to the front introducing a minus sign. Using $\mathrm{d}Q.Q^*+Q.\mathrm{d}Q^*=-2x^3\mathrm{d}x^3$ times the identity matrix and applying d to $x^3\mathrm{d}Q\wedge\mathrm{d}Q^*\wedge\mathrm{d}Q.Q^*$ then gives the second stated formula. For the last part, use

$$\mathrm{d}Q\wedge\mathrm{d}Q^*=2\begin{pmatrix}\mathrm{d}x^1\wedge\mathrm{d}x^5+\mathrm{d}x^2\wedge\mathrm{d}x^4 & 2r^{-1}\mathrm{d}x^1\wedge\mathrm{d}x^2\\ 2r\mathrm{d}x^4\wedge\mathrm{d}x^5 & -(\mathrm{d}x^1\wedge\mathrm{d}x^5+\mathrm{d}x^2\wedge\mathrm{d}x^4)\end{pmatrix}.$$

E4.9 For the first parts, direct computation gives

$$\begin{aligned} R_\nabla(e_1) &= \mathrm{d}(xte_1)\otimes e_2 - xte_1\wedge\nabla e_2\\ &= (te_1\wedge e_1 - xte_1\wedge e_2 + xte_1\wedge e_2)\otimes e_2 = 0,\\ R_\nabla(e_2) &= -\mathrm{d}e_2\otimes e_2 + e_2\wedge\nabla e_2 = 0,\\ (\mathrm{id}\otimes\phi)\nabla e_1 - \nabla\phi e_1 &= -\nabla(te_2) = -te_2\otimes e_2 + te_2\otimes e_2 = 0.\end{aligned}$$

The zeroth cohomologies on K (generated by e_2) and Ω^1 are calculated by

$$\nabla(ae_2) = (\mathrm{d}a - ae_2)\otimes e_2,\quad \nabla(be_1 + ce_2) = \mathrm{d}b\otimes e_1 + (bxte_1 + \mathrm{d}c - ce_2)\otimes e_2$$

leading to $\mathrm{H}^0(K)\cong\Bbbk$ with basis $[te_2]$ and $\mathrm{H}^0(\Omega^1)\cong\Bbbk$ with basis $[te_2]$. The quotient connection on Ω^1/E is $\nabla[e_1] = 0$ and hence $\nabla[ae_1] = \mathrm{d}a\otimes[e_1]$, so $\mathrm{H}^0(\Omega^1/K)\cong\Bbbk$ with basis $[e_1]$. As a result, the beginning of the cohomology long exact sequence is

$$\mathrm{H}^0(K)\cong\Bbbk\xrightarrow{\cong}\mathrm{H}^0(\Omega^1)\cong\Bbbk\longrightarrow\mathrm{H}^0(\Omega^1/K)\cong\Bbbk\longrightarrow\mathrm{H}^1(K)\longrightarrow\dots,$$

so the second map is zero and the third must be nonzero.

E4.10 For $c\in A$, we have $\langle\overline{a}, a'\rangle c = a^*a'c = \langle\overline{a}, a'c\rangle$ and $c\langle\overline{a}, a'\rangle = ca^*a' = (ac^*)^*a' = \langle\overline{ac^*}, a'\rangle = \langle c\overline{a}, a'\rangle$, so $\langle\ ,\ \rangle$ is a bimodule map. We also have

$$\langle\overline{a}, b{\triangleright}a'\rangle = \langle\overline{a}, \pi(b)a'\rangle = a^*\pi(b)a' = (\pi(b)^*a)^*a' = \langle\overline{b^*{\triangleright}a}, a'\rangle = \langle\overline{a}b, a'\rangle$$

for $b\in B$, so this descends to $\otimes_B$. For $a\in\mathbb{C}[t,t^{-1}]$, we have $\langle\overline{a}, a\rangle = a^*a$ which is just the pointwise norm squared of a as a function from $a: S^1\to\mathbb{C}$, so we get positivity. The right connection property holds as

$$\tilde\nabla(a'a) = \mathrm{d}(a'a) = (\mathrm{d}a')a + a'\mathrm{d}a\in{}_\pi\Omega^1_A\cong{}_\pi A\otimes_A\Omega^1_A.$$

To have a bimodule connection, we compute

$$\begin{aligned}\tilde\sigma(\mathrm{d}z\otimes 1) &= \tilde\nabla(z{\triangleright}1) - z{\triangleright}\tilde\nabla(1) = \tilde\nabla(t) = 1\otimes\mathrm{d}t,\\ z{\triangleright}\tilde\sigma(\mathrm{d}z\otimes 1) &= z{\triangleright}1\otimes\mathrm{d}t = t\otimes\mathrm{d}t = 1\otimes t\mathrm{d}t,\\ \tilde\sigma(z\mathrm{d}z\otimes 1) &= q^{-2}\tilde\sigma((\mathrm{d}z)z\otimes 1) = q^{-2}\tilde\sigma(\mathrm{d}z\otimes z{\triangleright}1) = q^{-2}\tilde\sigma(\mathrm{d}z\otimes t) = q^{-2}\otimes(\mathrm{d}t)t,\end{aligned}$$

so we require $t\mathrm{d}t = q^{-2}(\mathrm{d}t)t$ for $\tilde\sigma$ to be a bimodule map. The other relations are similarly without further constraints. For the inner product,

$$\begin{aligned}(\langle\ ,\ \rangle\otimes\mathrm{id})(\overline{a}\otimes 1\otimes\mathrm{d}a') + (\mathrm{id}\otimes\langle\ ,\ \rangle)(\mathrm{d}a^*\otimes\overline{1}\otimes a') &= a^*\mathrm{d}a' + (\mathrm{d}a^*)a'\\ &= \mathrm{d}(a^*a') = \mathrm{d}\langle\overline{a}, a'\rangle.\end{aligned}$$

Solutions to Exercises for Chap. 5

E5.1 The first part follows from the close relation between ω and $\omega^\sharp$. Now let $\omega^\sharp$ be a strong connection map. Since $(*S)^2 = \mathrm{id}$ in a Hopf $*$-algebra, it is clear that

$$\omega^\sharp_{\mathrm{new}}(h)^\dagger = \frac{1}{2}(\omega^\sharp(h)^\dagger + \omega^\sharp(*Sh)) = \frac{1}{2}(\omega^\sharp(*Sh) + \omega^\sharp(*S*Sh)^\dagger) = \omega^\sharp_{\mathrm{new}}(*Sh).$$

We have to check that it remains a strong connection map. Thus, $\cdot\omega^\sharp_{\mathrm{new}}(h) = \frac{1}{2}(\epsilon(h) + \overline{\epsilon(*Sh)}) = 1\epsilon(h)$, where $\cdot(*\otimes *)\mathrm{flip} = *\cdot$ by the involution property. We also have $\omega^\sharp_{\mathrm{new}}(1) = \frac{1}{2}(1\otimes 1 + (1\otimes 1)^\dagger) = 1\otimes 1$ so it remains unital. For the bicomodule properties, we need only prove these for the second part of $\omega^\sharp_{\mathrm{new}}$. Using the shorthand $\omega^\sharp(h) = h^{(1)}\otimes h^{(2)}$ and applying Δ_L to the first factor of this,

$$\begin{aligned}
&S^{-1}((*Sh)^{(2)*}{}_{\bar{(1)}}) \otimes (*Sh)^{(2)*}{}_{\bar{(0)}} \otimes (*Sh)^{(1)*} \\
&\quad = S^{-1}({(*Sh)^{(2)}}_{\bar{(1)}}{}^*) \otimes {(*Sh)^{(2)}}_{\bar{(0)}}{}^* \otimes (*Sh)^{(1)*} \\
&\quad = S^{-1}((*Sh)_{(2)})^*) \otimes ((*Sh)_{(1)})^{(2)*} \otimes ((*Sh)_{(1)})^{(1)*} \\
&\quad = S^{-1}(Sh_{(1)}) \otimes (*Sh_{(2)})^{(2)*} \otimes (*Sh_{(2)})^{(1)*} = h_{(1)} \otimes (*\otimes *)\mathrm{flip}\,\omega^\sharp(*Sh_{(2)}),
\end{aligned}$$

where we used the right covariance of $\omega^\sharp$ and that the right coaction on P is a $*$-algebra map. There is a similar but easier computation for the right covariance using left covariance of $\omega^\sharp$.

E5.2 Using the usual linear basis, we see that an element of grade $(1, 1)$ can be written as $p(bc)a$ for some polynomial p, and similarly an element of grade $(-1, -1)$ can be written as $ds(bc)$ for some polynomial s. For a strong grading we would have to be able to find p, s such that $1 = p(bc)ads(bc) = p(bc)(1 + q^{-1}bc)s(bc)$. This is not possible as $1 + q^{-1}bc$ does not have inverse a polynomial in bc.

E5.3 For any Hopf algebra right coacting on itself by the coproduct, it is easy to check that $\omega^\sharp(h) = Sh_{(1)} \otimes h_{(2)}$ satisfies (5.9) and so gives a strong universal connection. In our case, this is $\omega^\sharp(\delta_{z^j}) = \sum_i \delta_{z^{-i}} \otimes \delta_{z^{j-i}}$. For the nonuniversal case, we require the short exact sequence of left P-modules in Definition 5.39, but in the present case $P\Omega^1_A P = 0$ as $A = \mathbb{k}.1$ so that we need $\mathrm{ver} : \Omega^1_P \to P \otimes \Lambda^1_H$ to be an isomorphism. This is not the case, e.g., $\mathrm{ver}(e_2) = 0$.

E5.4 (i) is a direct calculation starting from $e_a = \sum_i \delta_{z^{i-a}} \mathrm{d}\delta_{z^i}$. (ii) The comodule condition reduces to $\Phi(\delta_t) + R_{z^2}(\Phi(\delta_t)) = 1$, and then $\Phi^{-1}(\delta_t) = \delta_{z^0} + \delta_{z^1}$. (iii) From $\omega^\sharp$, one finds $\omega(E_1) = (\delta_{z^1} + \delta_{z^3})e_1 + e_2 + (\delta_{z^2} + \delta_{z^0})e_3$. (iv) $\nabla\delta_{z^1} = (e_1 + e_3) \otimes (\delta_{z^1} - \delta_{z^0})$. [This is a special case of Example 5.49 with $N = 2$ and $x_0 = 2$, $x_1 = 3$ for the choice of trivialisation.]

E5.5 We use torus generators s, t and $e_1 = s^{-1}\mathrm{d}s$, $e_2 = t^{-1}\mathrm{d}t$. Using $\langle\ \rangle$ to denote the bimodule generated by a list of elements, the nonzero values for $m = 2$ and $m = 1$ and hence the kernel are

$$\mathrm{ver}^{0,2}(e^u \wedge e^v) = 1 \otimes e_1 \wedge e_2, \quad \ker \mathrm{ver}^{0,2} = \langle e^w \wedge e^u, e^w \wedge e^v \rangle_{\mathbb{C}\mathrm{Hg}},$$
$$\mathrm{ver}^{0,1}(e^u) = 1 \otimes e_1, \quad \mathrm{ver}^{0,1}(e^v) = 1 \otimes e_2, \quad \ker \mathrm{ver}^{0,1} = \langle e^w \rangle_{\mathbb{C}\mathrm{Hg}}.$$

Thus the exact sequence applies and we have a differential fibration.

E5.6 $\{\delta_x e_b\}$ for $x \in X$ and $b \in \mathcal{D}$ is a basis of the weakly horizontal forms. Hence summing over G, the elements

$$|G|\mathrm{av}(\delta_x e_b) = \sum_{g\in G} \delta_{xg} e_{g^{-1}bg} \qquad (*)$$

span the weakly horizontal invariant forms. In the quotient, we take $[y] \in X/G$ so that A is spanned by

$$\delta_{[y]} = \sum_{g\in G} \delta_{yg}, \quad \mathrm{d}\delta_{[y]} = \sum_{g\in G,\ a\in\mathcal{C}_X} \big(\delta_{yga^{-1}} - \delta_{yg}\big)e_a = \sum_{g\in G,\ a\in\mathcal{D}} \big(\delta_{yga^{-1}} - \delta_{yg}\big)e_a$$

and compute for $[x] \neq [y]$ that

$$\delta_{[x]}\mathrm{d}\delta_{[y]} = \sum_{g\in G,\ a\in\mathcal{D}} \delta_{[x],[yga^{-1}g^{-1}]}\delta_{yga^{-1}}e_a.$$

If $\delta_{[x],[yga^{-1}g^{-1}]} \neq 0$ then $b = gag^{-1} \in \mathcal{D}$ and is such that $[x] = [yb^{-1}]$; we sum over all possibilities for b with $\delta_{g^{-1}bg,a}$ to get

$$\begin{aligned}\delta_{[x]}\mathrm{d}\delta_{[y]} &= \sum_{b\in\mathcal{D}:[x]=[yb^{-1}]} \sum_{g\in G,\ a\in\mathcal{D}} \delta_{g^{-1}bg,a}\delta_{yga^{-1}}e_a \\ &= \sum_{b\in\mathcal{D}:[x]=[yb^{-1}]} \sum_{g\in G} \delta_{yb^{-1}g}e_{g^{-1}bg} = |G| \sum_{b\in\mathcal{D}:[x]=[yb^{-1}]} \mathrm{av}(\delta_{yb^{-1}}e_b).\end{aligned}$$

Now set $x = yc^{-1}$ for some $c \in \mathcal{D}$ and note that $[yb^{-1}] = [yc^{-1}]$ is the same as $bc^{-1} \in G$ to obtain

$$\sum_{b\in\mathcal{D}:bc^{-1}\in G} \mathrm{av}(\delta_{xcb^{-1}}e_b), \quad \sum_{g\in G}\delta_{xg}\sum_{a\in\mathcal{D}} e_a \qquad (**)$$

for all $c \in \mathcal{D}$ and $x \in X$ as spanning Ω^1_A (the second expression is $-\delta_{[x]}\mathrm{d}\delta_{[x]}$). The bundle is strong if and only if the collection in (**) spans the collection in (*). If $\mathcal{D}\mathcal{D}^{-1} \cap G = \{e\}$ then this is obviously true. If

$\mathcal{D}\mathcal{D}^{-1} \cap G \neq \{e\}$, choose $c \neq d \in \mathcal{D}$ so that $dc^{-1} \in G$ and consider

$$\sum_{b\in\mathcal{D}:bc^{-1}\in G} \mathrm{av}(\delta_{xcb^{-1}}e_b) = \mathrm{av}(\delta_x e_c) + \mathrm{av}(\delta_{xcd^{-1}}e_d) + \dots .$$

If the second term was a multiple of $\mathrm{av}(\delta_x e_c)$ then we would have to have $\delta_{xcd^{-1}}e_d = \delta_{xg}e_{g^{-1}cg}$ for some $g \in G$, but that would imply $c = d$. There is no linear combination of the elements in (**) which will give $\mathrm{av}(\delta_x e_c)$, in fact each term is the average over a different orbit. So in this case the bundle is not strong.

In Example 5.49, we have $\mathcal{D} = \{1, N-1, N+1, 2N-1\}$ and (writing additively) $(\mathcal{D}-\mathcal{D}) \cap G = \{0, N\}$, so this bundle is not strong. In Example 5.64, $\mathcal{D}$ consists of 2-cycles of the form $(i, n+1)$ for $1 \leq i \leq n$ and G consists of all permutations which do not move $n+1$. If follows that $\mathcal{D}\mathcal{D}^{-1} \cap G = \{e\}$, so this bundle is strong.

E5.7 Here $G = \{e, u\}$ and $X = S_3$. Using equation (*) in E5.6 and averaging xe_w and xe_v gives e_1, e_2 as stated in the question. The inherited relations in degree 1 can be identified as the Cayley graph calculus on $\Bbbk(\mathbb{Z}_3)$ for the subset $\{1, 2\} \subset \mathbb{Z}_3$ (this is the only 2D calculus on $\Bbbk(\mathbb{Z}_3)$, the universal one). The exterior algebra relations for $\Omega(S_3)$ are in Example 1.60 and applying $e_v\wedge$ to the $e_u \wedge e_v + e_v \wedge e_w + e_w \wedge e_u$ relation and a similar process for $\wedge e_w$ and the $e_v \wedge e_u$ relation gives

$$0 = e_v \wedge e_w \wedge e_v + e_v \wedge e_u \wedge e_w = e_v \wedge e_w \wedge e_v - e_w \wedge e_v \wedge e_w.$$

Substituting in our previous quadratic wedge relations gives the relations of $\Lambda^1_{\mathcal{D}}$ as well as $e_1 \wedge e_2 = e_2 \wedge e_1 = 0$ and $e_1 \wedge e_1 \wedge e_1 = e_2 \wedge e_2 \wedge e_2$. From the definition of x, we have $\mathrm{d}x = (1-2x)(e_u + e_v + e_w) = (1-2x)\theta$. We then compute $\mathrm{d}e_1, \mathrm{d}e_2$ from $\mathrm{d}e_v, \mathrm{d}e_w$ in Example 1.60 using the $\Omega(S_3)$ relations.

We adopt these as the higher order relations of a certain $\Omega(\mathbb{Z}_3)$ (not the canonical Grassmann algebra on the e_i). Then d on general elements of $\Omega^1(\mathbb{Z}_3)$, $\Omega^2(\mathbb{Z}_3)$ is

$$\mathrm{d}\big(ae_1 + be_2\big) = (R_1a + a - b)e_1 \wedge e_1 + (R_2b + b - a)e_2 \wedge e_2,$$
$$\mathrm{d}\big(ae_1 \wedge e_1 + be_2 \wedge e_2\big) = (R_1a - a + R_2b - b)e_1 \wedge e_1 \wedge e_1$$

for $a, b \in \Bbbk(\mathbb{Z}_3)$. If $\Bbbk$ is not of characteristic 2 then one computes that $\mathrm{d} : \Omega^1 \to \Omega^2$ has rank 4 and nullity 2, and that $\mathrm{d} : \Omega^2 \to \Omega^3$ has rank 2 and nullity 4, giving the stated $\mathrm{H}_{\mathrm{dR}}(\mathbb{Z}_3)$ for this calculus. Now taking the Leray–Serre spectral sequence at page 2 (using $1, 1, 0 \dots$ for the dimensions of $\mathrm{H}_{\mathrm{dR}}(S_2)$), we see that there are no possible cancellations and $\mathrm{H}_{\mathrm{dR}}(S_3)$ then follows by summing along diagonals.

E5.8 Here $H = \mathbb{C}(S_3)$ is again taken with its standard 2-cycles calculus $\Omega(S_3)$ where Λ^1_H has basis e_a for $a \in \mathcal{C} = \{u, , v, w\}$. By Example 5.55 for tensor product bundles, connections are given by $\alpha : \Lambda^1_H \to \Omega^1(M)$ or three 1-forms $\alpha_a = \alpha(e_a) \in \Omega^1(M)$. The curvature is similarly given by $F(\alpha)_a = F(\alpha)(\delta_a) = \mathrm{d}\alpha_a + \alpha(\varpi\pi_\epsilon\delta_{a(1)}) \wedge \alpha(\varpi\pi_\epsilon\delta_{a(2)})$ where $\varpi(\delta_a) = e_a$ if $a \in \{u, v, w\}$ and is zero for other $a \neq e$, as explained in Example 2.29. If $a \in \{u, v, w\}$ then this gives

$$F(\alpha)_a = \mathrm{d}\alpha_a - (\alpha_u + \alpha_v + \alpha_w) \wedge \alpha_a - \alpha_a \wedge (\alpha_u + \alpha_v + \alpha_w) = \mathrm{d}\alpha_a$$

since 1-forms anticommute in our classical $\Omega(M)$ (here $\pi_\epsilon(\delta_e) = \delta_e - 1 = -\sum_{x\neq e}\delta_x$ gives the quadratic expressions, which then cancel). There is a potential contribution $\sum_{bc=a}\alpha_b \wedge \alpha_c$ from $\Delta\delta_a$ but in our case there are no $b, c \in \mathcal{C}$ obeying the condition in the sum when $a \in \mathcal{C}$. In addition, there is an extended part of the curvature $F(\alpha)_a = F(\alpha)(\delta_a)$ for all $a \notin \mathcal{C} \cup \{e\}$ to complete the values of $F(\alpha)$ on all H^+. Here $\sum_{bc=a}\alpha_b \wedge \alpha_c$ can be nonzero and we find

$$F(\alpha)_{uv} = -F(\alpha)_{vu} = \alpha_u \wedge \alpha_v + \alpha_v \wedge \alpha_w + \alpha_w \wedge \alpha_u$$

(so regular α are connections for which this is zero).

Next let $V = \mathbb{C}^2$ with basis s_i and $s = a_i s_i \in C^\infty(M) \otimes V$ (sum over i) be a section of the associated bundle. Here $\mathbb{C}(S_3)$ coacts by $\Delta_R s_j = s_i \sum_{x\in S_3}\rho(x)_{ij} \otimes \delta_x$ where ρ is a matrix representation. The covariant derivative is $\nabla s = (\mathrm{d}a_i)s_i - a_j\sum_x \alpha(\varpi\pi_\epsilon\delta_{x^{-1}})s_i\rho(x)_{ij}$ or in physics notation (where a_i stands for $s = a_i s_i$),

$$\nabla a_i = \mathrm{d}a_i - \sum_{a\in\mathcal{C}} \rho(a^{-1} - e)_{ij} a_j \alpha_a.$$

($\{a - e \,|\, a \in \mathcal{C}\}$ is a dual basis of Λ^{1*}_H and generates a quantum Lie algebra for another point of view.) In our case $a = u, v, w$ are their own inverse and ∇ can be evaluated further for any specific ρ, for example the one in Example 1.48.

E5.9 (i) We start with the universal calculus, constructing a right-handed Hopf–Galois extension and strong connection on it. H^{op} based on the vector space H with the same coproduct as H but reversed product has antipode $S^{\mathrm{op}} = S^{-1}$. Writing the product in $H^{op} \otimes P$ in terms of that of H, we have

$$\begin{aligned}(\Delta_L p).(\Delta_L q) &= (S^{-1}p_{(\bar{1})} \otimes p_{(\bar{0})})(S^{-1}q_{(\bar{1})} \otimes q_{(\bar{0})}) = (S^{-1}q_{(\bar{1})})(S^{-1}p_{(\bar{1})}) \otimes p_{(\bar{0})}q_{(\bar{0})}\\ &= S^{-1}(p_{(\bar{1})}q_{(\bar{1})}) \otimes p_{(\bar{0})}q_{(\bar{0})} = \Delta_L(pq)\end{aligned}$$

so P is an $H^{\rm op}$-comodule algebra. Now define $\mathrm{ver}^\sharp_R : P\otimes_A P\to H\otimes P$ by $\mathrm{ver}^\sharp_R(p\otimes q)=S^{-1}p_{(\bar 1)}\otimes p_{(\bar 0)}q$. The conditions for a strong connection map as a unital bicomodule map (see Lemma 5.8) are symmetric, so we take the same $\omega^\sharp : H\to P\otimes P$ on the right bundle. Writing $\omega^\sharp(h)=h^{(1)}\otimes h^{(2)}$ and using (5.9),

$$\mathrm{ver}^\sharp_R\,\omega^\sharp(h)=\mathrm{ver}^\sharp_R(h^{(1)}\otimes h^{(2)})=S^{-1}h^{(1)}{}_{(1)}\otimes h^{(1)}{}_{(\bar 0)}h^{(2)}=S^{-1}Sh_{(1)}\otimes h_{(2)}{}^{(1)}h_{(2)}{}^{(2)},$$

which collapses to $h\otimes 1$ as required. By a straightforward right-handed version of Lemma 5.8, we then have a bundle and a strong connection on it.

(ii) We obtain a regular quantum principal bundle if the Δ_L and $\mathrm{ver}^\sharp$ descend to quotient differential structures such that the 'right handed' version of the exact sequence condition holds. Here a right quantum principal bundle means a Hopf algebra H_R with a bicovariant calculus left coacting by $\Delta_L p=p_{(\bar 1)}\otimes p_{(\bar\infty)}$ on a differential algebra P with ver defined so that

$$0\to P\Omega^1_A P\to \Omega^1_P\xrightarrow{\mathrm{ver}^R}\Lambda^1_R\otimes P\to 0$$

is exact. Here we need a right factorisation $\Omega^1_{H_R}=\Lambda^1_R.H_R$ by $\mathrm{d}g=((\mathrm{d}g_{(1)})Sg_{(2)(2)}).g_{(3)}$ and ver^R is the component of Δ_{L*} that maps $\Omega^1_P\to\Omega^1_{H_R}\otimes^{H_R}P=\Lambda^1_R.H_R\otimes^{H_R}P=\Lambda^1_R\otimes P$. This comes down to

$$(\mathrm{d}p_{(\bar 1)})q_{(\bar 1)}\otimes p_{(\bar\infty)}q_{(\bar\infty)}=((\mathrm{d}p_{(\bar 1)(1)})Sp_{(\bar 1)(2)}).(p_{(\bar 1)(3)}q_{(\bar 1)})\otimes p_{(\bar\infty)}q_{(\bar\infty)}$$

giving $\mathrm{ver}^R((\mathrm{d}p)q)=(\mathrm{d}p_{(\bar 1)(1)})Sp_{(\bar 1)(2)}\otimes p_{(\bar\infty)}q$. This is a right P-module map and associated bundles in this theory will canonically be right A-modules. From this point of view, Δ_L and ver descending amounts to the former being differentiable (both parts of Δ_{L*} are well defined).

In our case, $H_R=H^{\rm op}$ and we define $\Omega^1_{H^{\rm op}}$ to be Ω^1_H as a vector space, with the same differential d and coactions but opposite product (in the strongly bicovariant case we similarly take the graded-opposite wedge products). Also recall that the commuting left and right coactions both applied to $g\mathrm{d}h\in\Omega^1_H$ gives $g_{(1)}h_{(1)}\otimes g_{(2)}\mathrm{d}h_{(2)}\otimes h_{(3)}h_{(3)}$. Differentiability of the coaction requires that

$$\begin{aligned}\Delta_{L*}((\mathrm{d}p)q)&=(\mathrm{d}S^{-1}p_{(\bar 1)})\cdot^{\rm op}S^{-1}q_{(\bar 1)}\otimes p_{(\bar 0)}q_{(\bar 0)}+(S^{-1}p_{(\bar 1)})\cdot^{\rm op}(S^{-1}q_{(\bar 1)})\otimes(\mathrm{d}p_{(\bar 0)})q\\&=S^{-1}q_{(\bar 1)}\mathrm{d}S^{-1}p_{(\bar 1)}\otimes p_{(\bar 0)}q_{(\bar 0)}+S^{-1}(p_{(\bar 1)}q_{(\bar 1)})\otimes(\mathrm{d}p_{(\bar 0)})q_{(\bar 0)}\end{aligned}$$

is a well-defined map $\Delta_{L*}:\Omega^1_P\to(\Omega^1_{H^{\rm op}}\otimes P)\oplus(H^{\rm op}\otimes\Omega^1_P)$. The second term is the coaction $\Delta_L((\mathrm{d}p)q)$ which we see is well defined as $\Delta_R((\mathrm{d}p)q)$ is. In the first term, we insert $(S^{-1}p_{(\bar 1)(3)})p_{(\bar 1)(2)}$ pairs and use the Leibniz rule to write this as

$$
\begin{aligned}
&(S^{-1}q_{(\bar 1)})(S^{-1}p_{(\bar 1)(3)})p_{(\bar 1)(2)}\mathrm{d}S^{-1}p_{(\bar 1)(1)}\otimes p_{(\bar 0)}q_{(\bar 0)}\\
&\quad=-S^{-1}(p_{(\bar 1)(3)}q_{(\bar 1)})(\mathrm{d}p_{(\bar 1)(2)})S^{-1}p_{(\bar 1)(1)}\otimes p_{(\bar 0)}q_{(\bar 0)}\\
&\quad=-S^{-1}(p_{(\bar 1)(3)}q_{(\bar 1)(3)})(\mathrm{d}p_{(\bar 1)(2)})q_{(\bar 1)(2)}S^{-1}(p_{(\bar 1)(1)}q_{(\bar 1)(1)})\otimes p_{(\bar 0)}q_{(\bar 0)},
\end{aligned}
$$

where we similarly inserted $q_{(\bar 1)(2)}S^{-1}q_{(\bar 1)(1)}$. On the other hand, $\Delta_{R*}((\mathrm{d}p)q)=p_{(\bar 0)}q_{(\bar 0)}\otimes(\mathrm{d}p_{(\bar 1)})q_{(\bar 1)}+(\mathrm{d}p_{(\bar 0)})q_{(\bar 0)}\otimes p_{(\bar 1)}q_{(\bar 1)}$ is well defined as P is assumed to be a usual bundle via H. The left and right coactions applied to the first term tell us that

$$p_{(\bar 0)}q_{(\bar 0)}\otimes p_{(\bar 1)(1)}q_{(\bar 1)(1)}\otimes(\mathrm{d}p_{(\bar 1)(2)})q_{(\bar 1)(2)}\otimes p_{(\bar 1)(3)}q_{(\bar 1)(3)}$$

is well defined. Now apply S^{-1} on two factors and multiply up to obtain minus our expression for the first term of $\Delta_{L*}((\mathrm{d}p)q)$, so this too is well defined on Ω^1_P and has the same kernel as the first term of $\Delta_{R*}((\mathrm{d}p)q)$. The same then applies to

$$
\begin{aligned}
\mathrm{ver}^R((\mathrm{d}p)q)&=(\mathrm{d}p_{(\bar 1)(1)})\cdot_{\rm op}S^{\rm op}p_{(\bar 1)(2)}\otimes p_{(\bar\infty)}q\\
&=S^{-1}p_{(\bar 1)(2)}\mathrm{d}p_{(\bar 1)(1)}\otimes p_{(\bar\infty)}q=S^{-2}p_{(\bar 1)(1)}\mathrm{d}S^{-1}p_{(\bar 1)(2)}\otimes p_{(\bar 0)}q.
\end{aligned}
$$

It remains to show that ver^R is surjective. To do this, let $h\in H^+$ and $\omega^\sharp(h)=h^{(1)}\otimes h^{(2)}\in P\otimes P$ as in part (i). Using (5.9), we have

$$
\begin{aligned}
\mathrm{ver}^R((\mathrm{d}h^{(1)})h^{(2)})&=S^{-2}{h^{(1)}}_{(\bar 1)(1)}\mathrm{d}S^{-1}{h^{(1)}}_{(\bar 1)(2)}\otimes{h^{(1)}}_{(\bar 0)}h^{(2)}\\
&=S^{-2}((Sh_{(1)})_{(1)})\mathrm{d}S^{-1}((Sh_{(1)})_{(2)})\otimes{h_{(2)}}^{(1)}{h_{(2)}}^{(2)}\\
&=S^{-2}((Sh)_{(1)})\mathrm{d}S^{-1}((Sh)_{(2)})\otimes 1=S^{-1}h_{(2)}\mathrm{d}h_{(1)}\otimes 1
\end{aligned}
$$

so we can recover any element of Λ^1_R defined as the image of the right Maurer–Cartan form. On H the right Maurer–Cartan form is $\varpi_R(h)=(\mathrm{d}h_{(1)})Sh_{(2)}$ for $h\in H^+$ and gives a natural factorisation $\Omega^1_H=\Lambda^1_R.H$. When we compute the same thing on $H_R=H^{\rm op}$, we indeed have $\varpi^{\rm op}(h)=(\mathrm{d}h_{(1)})\cdot_{\rm op}S^{\rm op}h_{(2)}=(S^{-1}h_{(2)})\mathrm{d}h_{(1)}$ [and $=-\varpi_R(S^{-1}h)$ so Λ^1_R is the same either way].

E5.10 (i) The base is $A=\mathbb{C}_q[S^1]=\mathbb{C}[u,u^{-1}]$ and the map $\mathrm{ver}^\sharp(u^mv^n\otimes_A u^iv^j)=u^mv^nu^iv^j\otimes t^j=e^{\mathrm{i}\theta ni}u^{m+i}v^{n+j}\otimes t^j$ is clearly a bijection $P\otimes_A P\to P\otimes H$, so we have a Hopf–Galois extension. It is trivial with $\Phi:H\to P$ given by $\Phi(t^m)=v^m$ making $P\cong\mathbb{C}_q[S^1]{>\!\!\!\triangleleft}\mathbb{C}_q[S^1]$, where the two copies have generators $u^{\pm 1}$, $t^{\pm 1}$ and $t \triangleright u = vuv^{-1} = e^{i\theta}u$, and the cocycle is trivial. It is differentiable with $\Phi_*(dt) = dv$ since $(dv)v = qv dv$ just as for t. Moreover, Δ_R is differentiable, which includes that ver is well defined. For example, $\mathrm{ver}((dv)u) = \mathrm{ver}(d(vu) - v du) = \mathrm{ver}(d(e^{i\theta}uv)) = e^{i\theta}uv \otimes t^{-1}dt = \mathrm{ver}(e^{i\theta}u dv)$ while $\mathrm{ver}(d(v^2)) = v^2 \otimes t^{-2}dt^2 = v^2 \otimes [2]_q t^{-1}dt = v^2 \otimes (1+q)t^{-1}dt = \mathrm{ver}((q+1)v dv) = \mathrm{ver}((dv)v + v dv)$ as also required (and similarly for the general case). The bundle is clearly strong given the commutation relations of the calculus on P.

(ii) Ω^1 on A is trivially parallelised by the 1-bein $e_1 = u^{-1}du$. To express this, we set $V = \mathbb{C}c$ with basis c of grade 1 and $\theta(c) = vu^{-1}du \in \Omega^1 \subset P\Omega^1$ is then equivariant. We can also denote 1-form by θ. The left module map $s_\theta : (P \otimes V)^H = \mathbb{C}_{q,\theta}[\mathbb{T}^2]_{-1} \otimes c \cong \Omega^1$ is just $s_\theta(u^m v^{-1} \otimes c) = u^{m-1}du$ so we have a framing (this is essentially the same as for a tensor product bundle in Corollary 5.81). We similarly let $\theta^* = f(du)v^{-1}$ for some invertible element $f \in A$ (suppressing $\otimes c$ needed to make it V-valued with total grade 0). The metric is then $g = f du \otimes u^{-1}du = q^{-1}u^{-1}f du \otimes du$. This is not central, and need not be in the framing approach.

(iii) We omit the check that $\omega(t^{-1}dt) = v^{-1}dv + \alpha du \in \Omega^1_P$ has the properties of a connection form for any $\alpha \in A$ (it is similar to the tensor product case in Example 5.55). The horizontal projection is $(\mathrm{id}-\Pi_\omega)(du) = du$ and $(\mathrm{id}-\Pi_\omega)(dv) = -v\alpha du$. For the induced covariant derivative on Ω^1, du corresponds to $uv^{-1} \in \mathbb{C}_{q,\theta}[\mathbb{T}^2]_{-1}$ on which

$$
\begin{aligned}
(\mathrm{id} - \Pi_\omega)d(uv^{-1}) &= (\mathrm{id} - \Pi_\omega)((du)v^{-1} - q^{-1}uv^{-2}dv) = (du)v^{-1} + q^{-1}uv^{-1}\alpha du \\
&= (1 + q^{-1}e^{-i\theta}u\tilde{\alpha})(du)v^{-1} = (1 + q^{-1}e^{-i\theta}u\tilde{\alpha})q^{-1}u^{-1}(du)uv^{-1},
\end{aligned}
$$

where $\tilde{\alpha}(u) := \alpha(e^{-i\theta}u)$ and translating to

$$
\nabla du = (1 + q^{-1}e^{-i\theta}u\tilde{\alpha})q^{-1}u^{-1}du \otimes du.
$$

This calculation is invertible and we can find a connection form ω to land on any ∇. [We will see in Example 8.5 that up to normalisation only $g = u^m e_1 \otimes e_1$ for some integer m (so $f \propto u^{m-1}$) admits metric compatible ∇, as found there.]

Solutions to Exercises for Chap. 6

E6.1 Recall that the bimodule actions on $f \in \mathfrak{X}$ are $(a.f)(\xi) = a.f(\xi)$ and $(f.a)(\xi) = f(a.\xi)$ for all $\xi \in \Omega^1$ and $a \in A$. Then

$$\delta_r.\mathrm{ev}(f_{p\leftarrow q} \otimes \omega_{r\to s}) = \delta_{q,r}\delta_{p,s}\delta_{p,r}\delta_p = \mathrm{ev}((\delta_r.f_{p\leftarrow q}) \otimes \omega_{r\to s})$$

so that $\delta_r.f_{p\leftarrow q} = \delta_{p,r} f_{p\leftarrow q}$, and similarly $f_{p\leftarrow q}.\delta_r = \delta_{q,r} f_{p\leftarrow q}$. The dual bases are $\mathrm{coev} = \omega_{x\to y} \otimes f_{y\leftarrow x} + \omega_{x\to z} \otimes f_{z\leftarrow x} + \omega_{y\to z} \otimes f_{z\leftarrow y}$. For the divergence,

$$\tilde\nabla(f_{z\leftarrow x}) = -f_{z\leftarrow x} \otimes \omega_{x\to y} - f_{z\leftarrow x} \otimes \omega_{x\to z}, \quad \mathrm{div}(f_{z\leftarrow x}) = -1,$$
$$\tilde\nabla(f_{y\leftarrow x}) = -f_{y\leftarrow x} \otimes \omega_{x\to y} - f_{y\leftarrow x} \otimes \omega_{x\to z}, \quad \mathrm{div}(f_{z\leftarrow x}) = -1,$$
$$\tilde\nabla(f_{z\leftarrow y}) = -f_{z\leftarrow y} \otimes \omega_{y\to z}, \quad \mathrm{div}(f_{z\leftarrow x}) = -1.$$

Any other right connection would have to be $\tilde\nabla$ plus a right module map $K : \mathfrak{X} \to \mathfrak{X} \otimes_A \Omega^1$. As no arrows end in x, we see that necessarily $K(f_{z\leftarrow x}) = 0$, so we cannot change $\mathrm{div}(f_{z\leftarrow x}) = -1$.

E6.2 There are two basis 1-forms, $e^1$and e^2, and one basis 2-form $\omega^1 = e^1 \wedge e^2 = -e^2 \wedge e^1$. As $\mathrm{d}e^1 = \mathrm{d}e^2 = 0$, we have all the $d^i{}_k$ coefficients in (6.18) zero. As $e^i \wedge e^i = 0$, we have $c^{11}{}_1 = c^{22}{}_1 = 0$ and the antisymmetric wedge product gives $c^{12}{}_1 = -c^{21}{}_1 = 1$. Now take a dual basis $\{f_1, f_2\}$ to $\{e^1, e^2\}$. By (6.19), the relations for the left invariant part of $\mathcal{D}_H$ are just $f_1 \bullet f_2 - f_2 \bullet f_1 = 0$.

E6.3 In Example 3.40, the kernel of $\wedge$ is spanned as a right submodule of $\Omega^1 \otimes_A \Omega^1$ by $\mathrm{d}z \otimes \mathrm{d}z$, $\mathrm{d}\bar z \otimes \mathrm{d}\bar z$ and $\mathrm{d}z \otimes \mathrm{d}\bar z + q^{-2}\mathrm{d}\bar z \otimes \mathrm{d}z$. Now

$$\mathrm{ev}(\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id})\big((v_z \otimes v_{\bar z} - q^{-2} v_{\bar z} \otimes v_z) \otimes (\mathrm{d}z \otimes \mathrm{d}\bar z + q^{-2}\mathrm{d}\bar z \otimes \mathrm{d}z)\big)$$
$$= \mathrm{ev}(q^{-2} v_z \otimes \mathrm{d}z - q^{-2} v_{\bar z} \otimes \mathrm{d}\bar z) = 0$$

and similarly for the two easier cases, so $v_z \otimes v_{\bar z} - q^{-2} v_{\bar z} \otimes v_z$ is in $\Lambda^2\mathfrak{X}$. We now apply $[\![-,-]\!]_R$ to it and evaluate on $\mathrm{d}z$ and $\mathrm{d}\bar z$. As applying d to $\mathrm{d}z$ and $\mathrm{d}\bar z$ gives zero, we have a simplified formula

$$[\![v_z \otimes v_{\bar z} - q^{-2} v_{\bar z} \otimes v_z]\!]_R(\mathrm{d}z) = -q^{-2} v_{\bar z}(\mathrm{d}(1)) = 0$$

and similarly evaluating against $\mathrm{d}\bar z$ gives zero, so the bracket vanishes.

E6.4 Define a linear map $\Phi : A.\Bbbk\langle f_i\rangle \to T\mathfrak{X}_\bullet$ by inserting $\bullet$, e.g. $\Phi(a.f_{i_1} f_{i_2}) = a f_{i_1} \bullet f_{i_2}$. If we give $A.\Bbbk\langle f_i\rangle$ a grade by counting the number of f_is, then we see that the quotient map from $A.\Bbbk\langle f_i\rangle_n$ to the nth filtration of $T\mathfrak{X}_\bullet$ quotiented by the $n-1$th filtration (which is just $\mathfrak{X}^{\otimes n}$) is a 1-1 correspondence. We can then show by induction on the grades that $\Phi : A.\Bbbk\langle f_i\rangle \to T\mathfrak{X}_\bullet$ is a 1-1 correspondence, and is thus invertible. We have given the formula for Φ but the inverse requires a little more work. For the first grade, $\Phi^{-1}(v^i f_i) = v^i f_i$. For the second, we use $v \otimes w = v \bullet w - \heartsuit_v w$, so for $v = v^i f_i$ and $w = w^j f_j$,

$$\begin{aligned} v \otimes w &= v^i \bullet f_i \bullet w^j \bullet f_j - \heartsuit_v w \\ &= v^i \bullet (f_i w^j) \bullet f_j + v^i \bullet \tilde{\partial}_i(w^j) \bullet f_j - \heartsuit_v w \\ &= v^i C_i{}^k(w^j) \bullet f_k \bullet f_j + v^i \tilde{\partial}_i(w^j) \bullet f_j - v^i \tilde{\partial}_i(w^j) f_j - v^i f_i(w^j \Gamma^k{}_j) f_k \\ &= v^i C_i{}^k(w^j) \bullet f_k \bullet f_j + v^i \tilde{\partial}_i(w^j) \bullet f_j - v^i \tilde{\partial}_i(w^j) f_j - v^i C_i{}^p(w^j) f_p(\Gamma^k{}_j) f_k \end{aligned}$$

so $\Phi^{-1}(v \otimes w) = v^i C_i{}^k(w^j) \otimes f_k \bullet f_j - v^i C_i{}^k(w^j) f_k(\Gamma^s{}_j) \otimes f_s$. The relation on the $T\mathfrak{X}_\bullet$ algebra is

$$f_i \bullet a = f_i a + \partial_i(a) = C_i{}^k(a) f_k + \partial_i(a) = C_i{}^k(a) \bullet f_k + \tilde{\partial}_i(a).$$

E6.5 If we have $\nabla^L e_a = -\sum_{b,c \in \mathcal{C}} \Gamma^a{}_{bc} e_b \otimes e_c$ then $f_b \triangleright e_a = -\sum_{c \in \mathcal{C}} \Gamma^a{}_{bc} e_c$. From (3.31), we have for x, y, z all different, and the resulting action,

$$\Gamma^x{}_{xx} = -1, \quad \Gamma^x{}_{yz} = 0, \quad \Gamma^x{}_{yx} = 0, \quad \Gamma^x{}_{xy} = 1, \quad \Gamma^y{}_{xx} = 1;$$
$$f_u \triangleright e_u = e_u - e_v - e_w, \quad f_u \triangleright e_v = -e_u, \quad f_u \triangleright e_w = -e_u$$

and similarly for f_v, f_w. We write the action of f_u, f_v, f_w respectively as matrices

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

acting on column vectors of the coefficients in the basis in u, v, w order. One can then check that the relations hold by matrix multiplication.

E6.6 This arises because the left connection in Lemma 6.16 is a right $T\mathfrak{X}_\bullet$-module map (this is then quotiented to $\mathcal{D}_A$). Right multiplication by an element of $\mathcal{D}_A$ on $\Omega \otimes_A \mathcal{D}_A$ is then a cochain map, and so induces a map on the cohomology.

Solutions to Exercises for Chap. 7

E7.1 For the bimodule map property, comparing for all $a \in A$,

$$J(e_1.a) = R_1(a) J e_1 = R_1(a)(\alpha e_1 + \beta e_2),$$
$$(J e_1).a = (\alpha e_1 + \beta e_2) a = R_1(a) \alpha e_1 + R_2(a) \beta e_2$$

requires $\beta = 0$. Similarly considering e_2 requires $\gamma = 0$ and hence $\alpha^2 = \delta^2 = -1$ for $J^2 = -\mathrm{id}$. If J is well defined on Ω^2 then we have the vanishing of

$$Je_1 \wedge e_2 + e_1 \wedge Je_2 + Je_2 \wedge e_1 + e_2 \wedge Je_1 = (\alpha + R_1(\delta) - \delta - R_2(\alpha))e_1 \wedge e_2$$

showing that $\alpha + R_1(\delta)$ is a multiple of the identity. Writing $\mathrm{d}f = \partial^1 f e_1 + \partial^2 f e_2$ and applying Lemma 7.3 with $\mathrm{d}e_i = 0$, we have

$$\begin{aligned} 0 = &\ (1 - J \wedge J)\mathrm{d}Js_1 - J\mathrm{d}s_1 = (1 - J \wedge J)\mathrm{d}(\alpha e_1) \\ = &\ (1 - J \wedge J)(-(\partial^2\alpha)e_1 \wedge e_2) = -(\partial^2\alpha)(e_1 \wedge e_2 - Je_1 \wedge Je_2) \\ = &\ -(\partial^2\alpha)(1 - \alpha R_1(\delta))e_1 \wedge e_2, \\ 0 = &\ (1 - J \wedge J)\mathrm{d}Js_2 - J\mathrm{d}s_2 = (\partial^1\delta)(1 - \alpha R_1(\delta))e_1 \wedge e_2. \end{aligned}$$

Now using $J^2 = -\mathrm{id}$, we have $(\partial^1\delta)(\alpha + R_1(\delta)) = (\partial^2\alpha)(\alpha + R_1(\delta)) = 0$, so that either both α, δ are constant or $R_1(\delta) = -\alpha$. For the $*$-operation we have, using α, δ imaginary,

$$\begin{aligned} J(e_1{}^*) &= -J(e_2) = -\delta e_2, \quad (Je_1)^* = -(\alpha e_1)^* = -e_2\alpha^* = e_2\alpha = R_2(\alpha)e_2, \\ J(e_2{}^*) &= -J(e_1) = -\alpha e_2, \quad (Je_2)^* = -(\delta e_2)^* = -e_1\delta^* = e_1\delta = R_1(\delta)e_1. \end{aligned}$$

E7.2 The relations are

$$\begin{aligned} (x = g_1) &\quad e_{-i} \wedge e_i + e_{-j} \wedge e_j + e_i \wedge e_{-i} + e_j \wedge e_{-j} \\ (x = g_{-1}) &\quad e_{-i} \wedge e_{-i} + e_{-j} \wedge e_{-j} + e_i \wedge e_i + e_j \wedge e_j \\ (x = g_k) &\quad e_i \wedge e_j + e_{-i} \wedge e_{-j} + e_j \wedge e_{-i} + e_{-j} \wedge e_i \\ (x = g_{-k}) &\quad e_{-i} \wedge e_j + e_i \wedge e_{-j} + e_{-j} \wedge e_{-i} + e_j \wedge e_i \end{aligned}$$

but only the first case lives in a single bigrade $\Omega^{1,1}$. For the rest, we have to impose the extra relations

$$\begin{aligned} (x = g_{-1}) &\quad e_{-i} \wedge e_{-i} + e_{-j} \wedge e_{-j}, \quad e_i \wedge e_i + e_j \wedge e_j \\ (x = g_k) &\quad e_i \wedge e_j, \quad e_{-i} \wedge e_{-j}, \quad e_j \wedge e_{-i} + e_{-j} \wedge e_i \\ (x = g_{-k}) &\quad e_{-i} \wedge e_j + e_i \wedge e_{-j}, \quad e_{-j} \wedge e_{-i}, \quad e_j \wedge e_i \end{aligned}$$

in order to have a well-defined bigrading. However, the image of $\wedge : \Omega^{1,0} \otimes_A \Omega^{0,1} \to \Omega^{1,1}$ contains $e_{-i} \wedge e_i + e_{-j} \wedge e_j$ but not $e_{-i} \wedge e_i$, so factorisability does not hold.

E7.3 The relations of the maximal prolongation are $e_{(i,i+1)} \wedge e_{(i+2,i+3)} + e_{(i+2,i+3)} \wedge e_{(i,i+1)} = 0$ and $e_{(i,i+1)} \wedge e_{(i-1,i)} = e_{(i,i+1)} \wedge e_{(i+1,i+2)} = 0$ for all $i \in \{0, 1, 2, 3\}$ mod 4. As the first relations lie entirely in either $\Omega^{2,0}$ or $\Omega^{0,2}$ (depending on the value of i) and the second ones lie entirely in $\Omega^{1,1}$, we have a consistent bigrading. In the equation for d in Proposition 1.53, the last term is zero in this example (as the product of two 2-cycles is never a 2-cycle), from which integrability in the limited sense of conditions (4)–(5) of Lemma 7.2 holds. However, for $\xi \in \Omega^{1,0}$ we have $\xi^* \in \Omega^{1,0}$ rather than $\Omega^{0,1}$ (i.e., J does not commute with $*$), so we do not get an integrable almost complex structure for the usual star operation.

For the star operation of exercise E2.10, we have $\star e_{(i,i+1)} = -\overline{e_{(i+1,i+2)}}$ (mod 4), so $\star\Omega^{1,0} \subseteq \overline{\Omega^{0,1}}$ and $\star\Omega^{0,1} \subseteq \overline{\Omega^{1,0}}$. (Although we never formulated the condition for a complex structure in a bar category, this is equivalent in the case of a standard $*$-structure and generalises without further problems.) Then the holomorphic functions $a \in \mathbb{C}(S_4)$ are precisely those for which $a(g) = a(g(0, 1)) = a(g(2, 3))$ for all $g \in S_4$, so we might choose $a = \delta_e + \delta_{(0,1)} + \delta_{(2,3)} + \delta_{(0,1)(2,3)}$.

E7.4 We first check that the relations are preserved by ψ,

$$\psi(w\bar{w}) = \psi(w)\psi(\bar{w}) = z^2\bar{z}^2 = q^4\bar{z}^2z^2 = q^4\psi(\bar{w})\psi(w) = \psi(q^4\bar{w}w).$$

Recall that the left B-action on ${}_\psi A$ is given by $b{\triangleright}a = \psi(b)a$. The monomials $\bar{z}^n z^m$ for $n, m \in \mathbb{Z}$ form a basis for the algebra A and we have the $\bar\partial$-Leibniz rule,

$$\begin{aligned}\bar\partial_{{}_\psi A}(w^r{\triangleright}\bar{z}^n z^m) &= q^{2nr}\bar\partial_{{}_\psi A}(\bar{z}^n z^{m+2r})\\ &= q^{2nr}\frac{n}{2}\mathrm{d}\bar{w} \otimes \bar{z}^{n-2}z^{m+2r} = w^r\bar\partial_{{}_\psi A}(\bar{z}^n z^m),\\ \bar\partial_{{}_\psi A}(\bar{w}^r{\triangleright}\bar{z}^n z^m) &= \bar\partial_{{}_\psi A}(\bar{z}^{n+2r}z^m) = (\frac{n}{2} + r)\mathrm{d}\bar{w} \otimes \bar{z}^{n+2r-2}z^m\\ &= (\frac{n}{2} + r)\bar{w}^r\mathrm{d}\bar{w} \otimes \bar{z}^{n-2}z^m = \bar\partial\bar{w}^r \otimes \bar{z}^n z^m + \bar{w}^r\bar\partial_{{}_\psi A}(\bar{z}^n z^m).\end{aligned}$$

Now note that as $\Omega_B^{0,2} = 0$, the holomorphic curvature must vanish and by Definition 7.14 we get a holomorphic left B-module. The analogue of ψ in power r is $\psi_r : \mathbb{C}_{q^{r^2}}[\mathbb{C}^2] \to \mathbb{C}_q[\mathbb{C}^2]$ defined by $\psi_r(w) = z^r$ and $\psi_r(\bar{z}) = \bar{z}^r$.

E7.5 The product on $\Omega_A \bowtie \Omega_A$ is

$$(\xi \otimes \eta) \wedge (\tau \otimes \omega) = (-1)^{|\eta|\,|\tau|}\xi \wedge \tau_{(2)} \otimes \eta_{(2)} \wedge \omega\, \mathcal{R}(S\eta_{(1)} \otimes \tau_{(1)})\mathcal{R}(\eta_{(3)} \otimes \tau_{(3)}),$$

and as $\mathcal{R}$ is extended as zero on forms, we only require to take instances of the super coproducts where $\eta_{(1)}\otimes\eta_{(2)}\otimes\tau_{(3)} \in A\otimes\Omega_A\otimes A$, i.e., the A-bicomodule structure given by application of the left and right coactions of the bicovariant calculus Ω_A. The $*$-operation and differential are

$$(\xi \otimes \eta)^* = (-1)^{|\xi||\eta|}\eta^* \otimes \xi^*, \quad \mathrm{d}(\xi \otimes \eta) = \mathrm{d}\xi \otimes \eta + (-1)^{|\xi|}\xi \otimes \mathrm{d}\eta$$

and the usual derivation properties of d on $\Omega_{A\bowtie A}$ hold because d on Ω_A is a bicomodule map. We define $\Omega^{p,q}_{A\bowtie A}$ to be the subspace $\Omega^p_A \otimes \Omega^q_A \subseteq \Omega_A \bowtie \Omega_A$, and then from the $*$-operation and product above, we see that $(\Omega^{p,q}_{A\bowtie A})^* = \Omega^{q,p}_{A\bowtie A}$ and $\Omega^{p,q}_{A\bowtie A} \wedge \Omega^{p',q'}_{A\bowtie A} = \Omega^{p+p',q+q'}_{A\bowtie A}$, so we have an almost complex structure. The formula for the differential tells us that $\mathrm{d}\Omega^{p,q}_{A\bowtie A} \subseteq \Omega^{p+1,q}_{A\bowtie A} \oplus \Omega^{p,q+1}_{A\bowtie A}$, so we have integrability. Finally, we note that $\wedge : \Omega^{p,0}_{A\bowtie A} \otimes_{A\bowtie A} \Omega^{0,q}_{A\bowtie A} \to \Omega^{p,q}_{A\bowtie A}$ is a bimodule map with inverse $(\xi \otimes \eta) \mapsto (\xi \otimes 1) \otimes (1 \otimes \eta)$. [In the example of $A = \mathbb{C}_q[SU_2]$, one can think of $A\bowtie A$ as a q-deformation of $SL_2(\mathbb{C})$, i.e. as its complexification, although this point of view is only valid when $q \neq 1$.]

Solutions to Exercises for Chap. 8

E8.1 Here A is the algebra of functions on the vertices and the differential forms are associated to the edges. Using the shorthand $\omega_i = \omega_{i\to i+1}$, a general element of $\Omega^1 \otimes_A \Omega^1$ has to have the form $\sum_i \alpha(i)\omega_i \otimes \omega_{i+1} = \alpha\theta \otimes \theta$ for some coefficient function $\alpha \in A$. A connection specifies such elements $\nabla\omega_i = \alpha^i\theta \otimes \theta$ for functions α^i. The left connection property requires

$$f(i)\alpha^i\theta\otimes\theta = \nabla(f\omega_i) = \mathrm{d}f\otimes\omega_i + f\alpha^i\theta\otimes\theta = (f(i)-f(i-1))\omega_{i-1}\otimes\omega_i + f\alpha^i\theta\otimes\theta.$$

If $f = \delta_j$ with $j \neq i, i-1$ then we find $\alpha^i(j) = 0$. With $j = i-1$, we find $\alpha^i(i-1) = 1$ and with $j = i$, we find $\omega_{i-1} \otimes \omega_i + \alpha^i(i)\omega_i \otimes \omega_{i+1} = \omega_{i-1} \otimes \omega_i + \alpha^i(i)\omega_i \otimes \omega_{i+1}$, which is empty. Hence a left connection has the form stated in the question and direct computation shows that this is a bimodule connection with $\sigma(\omega_i \otimes \omega_{i+1}) = -\beta^i\omega_i \otimes \omega_{i+1}$. Writing also $g = \sum g^i\omega_i \otimes \omega_{i+1}$ where $g^i \neq 0 \in \Bbbk$,

$$\nabla g = \sum_i g^i\omega_{i-1} \otimes \omega_i \otimes \omega_{i+1} + g^i\sigma(\omega^i\beta^{i+1}\omega_{i+1}) \otimes \omega_{i+1}$$

and given the form of σ, $\nabla g = 0$ is $g^{i+1} = g^i\beta^i\beta^{i+1}$ for all i. We can set $\beta^0 \neq 0$ freely and then uniquely determine β^i for positive and negative i for any fixed g. Hence there is a 1-parameter moduli of metric compatible connections. If there are endpoints say $0 \to 1 \cdots \to n-1 \to n$ then the same analysis applies except to drop any terms with ω_i for $i < 0$ or $i \geq n$. This means that $\nabla\omega_0 = \beta^0\omega_0 \otimes \omega_1$ and $\nabla\omega_{n-1} = \omega_{n-2} \otimes \omega_{n-1}$. The metric has the form $g^0\omega_0 \otimes \omega_1 + \cdots + g^{n-2}\omega_{n-2} \otimes \omega_{n-1}$ with no weight attached to ω_{n-1} entering, and we solve starting with $\beta^1 = \frac{g^1}{g^0\beta^0}$ through to $\beta^{n-2} = \frac{g^{n-2}}{g^{n-3}\beta^{n-3}}$. For a closed string $0 \to \cdots \to n-1 \to 0$ (identifying the endpoints of the open string), we have an extra term $+g^{n-1}\omega_{n-1}\otimes\omega_0$ in the metric and the cyclic form

$$\nabla\omega^0 = \omega_{n-1}\otimes\omega_0+\beta^0\omega_0\otimes\omega_1, \quad \nabla\omega^{n-1} = \omega_{n-2}\otimes\omega_{n-1}+\beta^{n-1}\omega_{n-1}\otimes\omega_0.$$

The extra terms in ∇g produce the two additional conditions

$$\beta^{n-1} = \frac{g^{n-1}}{g^{n-2}\beta^{n-2}}, \quad g^0 = g^{n-1}\beta^{n-1}\beta^0.$$

If n is odd then this allows us to solve for $\pm\beta^0$, so we have two metric compatible connections for a free choice of metric coefficients. If n is even then the choice of β^0 remains free but we need $(g^0\cdots g^{n-2})^2 = (g^1\cdots g^{n-1})^2$ for a solution.

E8.2 (i) We write out the condition in Proposition 8.11(2), applying the maps as specified and extracting the coefficient of $s^i\otimes s^m\otimes s^n$ to obtain

$$\theta_i g_{mn} + g_{an}\alpha^a{}_{im} + \sigma^{ac}{}_{im}g_{ab}(\alpha^b{}_{cn} - \sigma^{bd}{}_{cn}\theta_d) = 0.$$

Under our assumptions, the constant terms and the terms involving the $\{\theta_i\}$ equate separately to the two equations stated as (*).

(ii) For $A = M_2(\mathbb{C})$, we take the maximal prolongation as at the start of Example 8.13. The kernel of the wedge product is spanned by $s\otimes t - t\otimes s$, so a quantum symmetric metric has the form $g_{ij} = \epsilon_{ij}$ up to an overall constant. Similarly, to be torsion free by Proposition 8.11(2), we need $\alpha^i{}_{jk} = \alpha^i\epsilon_{jk}$ and $\sigma^{ij}{}_{mn} + \delta^i{}_m\delta^j{}_n = \tau^{ij}\epsilon_{mn}$ for some constant coefficients α^i, τ^{ij}. Inserting these and $g_{ij} = \epsilon_{ij}$ into the metric compatibility equation rapidly gives $\alpha^i = 0$ and $\tau^{ij} = \epsilon_{ij}$ as the unique QLC for this metric. Here $\epsilon_{12} = -\epsilon_{21} = 1$ as usual.

(iii) When we add the relations $s^2 = t^2 = 0$, the kernel now includes $s\otimes s$ and $t\otimes t$ allowing a more general 3-parameter antisymmetric-or-diagonal form for a central tensor to be quantum symmetric. Thus, for a torsion-free connection, we have a 6-parameter $\alpha^i{}_{mn}$ where each $\alpha^i{}_{..}$ is quantum symmetric, and a 12-parameter $\sigma^{ij}{}_{mn} = -\delta^i_m\delta^j_n + \tau^{ij}{}_{mn}$ where each $\tau^{ij}{}_{..}$ is quantum symmetric. The quantum metric can also have this 3-parameter quantum symmetric form as stated in Example 8.13. Fixing $g_{ij} = \epsilon_{ij}$ and the allowed form of the α, σ tensors, equations (*) leads to a quadratic system for the moduli of QLCs. It is easy enough to check that this is solved by $\alpha^i{}_{mn} = 0$ and $\sigma^{ij}{}_{mn}$ as displayed in Example 8.13, where ij and mn label rows and columns in order 11, 12, 21, 22 and $s^1 = s, s^2 = t$. [With more work, one finds that it is a principal component of the moduli space.]

E8.3 We proceed as in E8.3(iii) with the reduced calculus on $\Omega(M_2(\mathbb{C}))$ with $s^2 = t^2 = 0$, but this time we fix $g_{ij} = \delta_{ij}$. (i) One can verify with $\alpha = 0$ that

$$\sigma = \begin{pmatrix} 1-\mu & \rho & -\rho & -\nu \\ \rho + \frac{\mu(\mu+\nu-2)}{\rho} & -\mu & \mu - 1 & -\rho \\ \rho & \nu - 1 & -\nu & -\rho - \frac{\nu(\mu+\nu-2)}{\rho} \\ -\mu & \rho & -\rho & 1-\nu \end{pmatrix}; \quad \mu, \nu, \rho \in \mathbb{C}$$

is a 3-parameter space of solutions of the quadratic system. This gives

$$\nabla s = 2E_{21} t \otimes s + \Big(\mu E_{12} - (\rho + (\mu+\nu-2)\frac{\mu}{\rho})E_{21}\Big) s \otimes s$$
$$+ (\mu E_{21} - \rho E_{12})(s \otimes t - t \otimes s) + (\nu E_{12} + \rho E_{21}) t \otimes t,$$
$$\nabla t = 2E_{12} s \otimes t + (\mu E_{21} - \rho E_{12}) s \otimes s - (\nu E_{12} + \rho E_{21})(s \otimes t - t \otimes s)$$
$$+ \Big(\nu E_{21} + (\rho + (\mu+\nu-2)\frac{\nu}{\rho})E_{12}\Big) t \otimes t$$

and is $*$-preserving by $\sigma \dagger \sigma = \dagger$ when $\bar\rho = -\rho$, $\nu = \bar\mu$. The limit $\mu, \nu, \rho \to 0$ with $\mu, \nu \sim \rho^2$ gives the QLC stated in the question. [There is also a subset $\nu = -\rho^2/\mu$ admitting an additional 1-parameter $\alpha(s) = \lambda(-\frac{\mu}{\rho} s \otimes s + s \otimes t - t \otimes s + \frac{\rho}{\mu} t \otimes t)$ and $\alpha(t) = -\frac{\rho}{\mu}\alpha(s)$ added to the above ∇ and still giving a QLC.] (ii) Similarly with $\tau = 0$ or $\sigma = -$flip on the generators, one can solve the quadratic system with 4 parameters for the map α, leading to ∇ as stated in the question. We use $\sigma \circ \dagger \circ \alpha = \alpha \circ *$ for the map α to find which connections are $*$-preserving.

E8.4 This is the geometry of the complete graph on two points, but writing the vertices as $\mathbb{Z}_2$ gives a convenient basic 1-form e^1 with $e^1 f = \bar f e^1$, where $\bar f$ swaps the two values of $f \in \mathbb{C}(\mathbb{Z}_2)$ (not to be confused with pointwise complex conjugation, which we will denote by f^*) and $\mathrm{d}f = (\bar f - f)e^1$. A quantum metric means $g = a e^1 \otimes e^1$ with $a \in \mathbb{C}(\mathbb{Z}_2)$ nonvanishing. From the graph point of view, the two edge lengths are the same if a is a constant or $\rho = \bar a / a$ has value 1, but we do not assume this. In the $*$-algebra setting with $(e^1)^* = -e^1$, the metric is real if a is real-valued.

Next, any connection is $\nabla e^1 = be^1 \otimes e^1$ for some $b \in \mathbb{C}(\mathbb{Z}_2)$ hence $\nabla(fe^1) = \mathrm{d}f \otimes e^1 + f\nabla e^1 = (\bar{f} - f(1-b))e^1 \otimes e^1$. This is a bimodule connection with

$$\begin{aligned}(f-\bar{f})\sigma(e^1 \otimes e^1) &= \sigma(e^1 \otimes (\bar{f}-f)e^1)\\ &= \nabla(e^1 f) - (\nabla e^1)f = \nabla(\bar{f}e^1) - (\nabla e^1)f\\ &= (f - \bar{f}(1-b) - bf)e^1 \otimes e^1 = (f-\bar{f})(1-b)e^1 \otimes e^1\end{aligned}$$

so $\sigma(e^1 \otimes e^1) = (1-b)e^1 \otimes e^1$. This is $*$-preserving when $(1-b^*)(1-b) = 1$.

A metric compatible bimodule connection then needs

$$\begin{aligned}\nabla g &= (\bar{a} - a(1-b))e^1 \otimes e^1 \otimes e^1 + \sigma(ae^1 \otimes be^1) \otimes e^1\\ &= (\bar{a} - a(1-b) + a\bar{b}(1-b))e^1 \otimes e^1 \otimes e^1 = 0,\end{aligned}$$

which is $\bar{a} = a(1-\bar{b})(1-b)$. This has no solutions unless $\rho = \pm 1$, when

$$b(0) = 1 + q, \quad b(1) = 1 + \rho q^{-1}$$

for any nonzero constant q. Such a connection is $*$-preserving exactly when $|q| = 1$. This is the moduli of QLCs for the canonical $\Omega(\mathbb{Z}_2)$ where $\Omega^2 = 0$.

For the universal exterior algebra, applying d to $\mathrm{d}f$ tells us that $0 = 2(f-\bar{f})(e^1)^2 + (\bar{f} - f)\mathrm{d}e^1$ for all f, which tells us that $\mathrm{d}e^1 = 2(e^1)^2$. Applying d to the bimodule relations gives us nothing more, so this is the universal calculus in degree 2 (similarly in higher degrees). In this case, torsion is zero when $\wedge\nabla(\mathrm{d}f) = \wedge\nabla((\bar{f}-f)e^1) = ((f-\bar{f}) - (\bar{f}-f)(1-b))(e^1)^2$ for all f, which forces $b = 2$, so there is a unique torsion free connection $\nabla(fe^1) = (\bar{f}+f)e^1 \otimes e^1$ and $\sigma(e^1 \otimes e^1) = -e^1 \otimes e^1$. This is metric compatible only with $\rho = 1$, i.e., the constant a case, and is $*$-preserving as $q = 1$ in the above. The metric is not quantum symmetric for the universal exterior algebra here.

E8.5 Here $\Omega^1(\mathbb{Z}_3)$ is the universal calculus in terms of left-invariant 1-forms e^i as in Example 8.19. The universal exterior algebra is free on these with $\mathrm{d}e^1 = 2(e^1)^2 - (e^2)^2 + \{e^1, e^2\}$, $\mathrm{d}e^2 = 2(e^2)^2 - (e^1)^2 + \{e^1, e^2\}$ (using Proposition 1.53) and we quotient this by $e^1e^2 = e^2e^1 = 0$ (as in a subcalculus of $\Omega(S_3)$ in exercise E5.7). Torsion compatibility gives $c_1 = c_2 = -1$ and then from the formula for ∇ in Proposition 8.11 we find the condition $a_1 = a_2 = -1$ for torsion freeness. This time, any metric $ae_1 \otimes e_2 + be_2 \otimes e_1$ is quantum symmetric but now the formula for metric-compatibility in Proposition 8.11 requires that $a = b$ by looking at the $e^i \otimes e^i \otimes e^i$ coefficients. The other coefficients give $R_1(d_{ii}) = -\rho d_{ii}$,

$$R_1(d_{22})d_{11} = \rho - R_2(\rho) = \frac{1}{\rho} - R_1(\frac{1}{\rho}), \quad d_{21} = -\frac{a}{R_2(a)}, \quad d_{12} = -\frac{a}{R_1(a)},$$

where $\rho = R_1(a)/a$. From the first equation, we deduce $d_{11} = d_{22} = 0$. This shows that ρ is a constant and, since a is real-valued, that in fact $\rho = 1$. Hence a must be constant and $d_{12} = d_{21} = -1$. This gives $\sigma = -\mathrm{flip}$ and ∇ as stated on the generators. This is $*$-preserving by the conditions in Proposition 8.11.

E8.6 Having done the $\mathbb{Z}_2$ analysis in detail in exercise E8.4, we limit ourselves to analysing the stated form of ∇ and σ for some function b. We first compute

$$\begin{aligned}\nabla(e^+ f) &= (\nabla e^+)f + \sigma(e^+ \otimes \partial_+ f e^+ + \partial_- f e^-)\\ &= R_{+2}(f)be^+ \otimes e^+ + (1-b)R_+(\partial_+ f)e^+ \otimes e^+ + R_+(\partial_- f)e^- \otimes e^+,\\ \nabla(R_+ f e^+) &= \partial_+(R_+ f)e^+ \otimes e^+ + \partial_-(R_+ f)e^- \otimes e^+ + R_+(f)be^+ \otimes e^+,\end{aligned}$$

where $R_+(f)(i) = f(i+1)$ and $e^\pm f = R_\pm f e^\pm$ in the graph calculus in a left-invariant basis. The two expressions are equal so we have a well-defined bimodule connection without further constrains on b. The same for $\nabla(e^- f)$. [With more work, one can show that torsion free bimodule connections take the stated form for any function b or a similar one with $e^\pm$ swapped.] Next we require metric compatibility

$$\begin{aligned}&\nabla(e^+ \otimes e^- + e^- \otimes e^+)\\ &\quad = be^+ \otimes e^+ \otimes e^- + R_+(b)\sigma(e^+ \otimes e^+) \otimes e^- + R_+(b)\sigma(e^+ \otimes e^-) \otimes e^+\\ &\qquad + be^+ \otimes e^- \otimes e^+ + be^- \otimes e^+ \otimes e^+ + R_-(b)\sigma(e^- \otimes e^+) \otimes e^+ = 0,\end{aligned}$$

which comes down to $b + R_+(b)(1-b) = 0 = b + R_-(b)(1-b)$ and $R_+ b + b - bR_- b = 0$. These require $R_+ b = R_- b$ and $R_+ b = \frac{b}{b-1}$, which can be expressed as stated. The $*$-preserving condition is from Proposition 8.11 and forces both families to the unique connection where $b = 0$.

E8.7 We define $\langle\,,\,\rangle^{-1} \in \overline{\Omega^1} \otimes_A \Omega^1$ from the Riemannian metric in Example 4.37,

$$\langle\,,\,\rangle^{-1} = (\star \otimes \mathrm{id})\big(w^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z)\big) = w^{-2}(\overline{\mathrm{d}\bar{z}} \otimes \mathrm{d}\bar{z} + q^{-2}\overline{\mathrm{d}z} \otimes \mathrm{d}z)$$

and by using $(\mathrm{id} \otimes \langle,\rangle)(\langle\,,\,\rangle^{-1} \otimes \mathrm{id}) = \mathrm{id} : \overline{\Omega^1} \to \overline{\Omega^1}$, we see that

$$\langle \mathrm{d}z, \overline{\mathrm{d}z}\rangle = q^2 w^2, \quad \langle \mathrm{d}\bar{z}, \overline{\mathrm{d}\bar{z}}\rangle = w^2, \quad \langle \mathrm{d}z, \overline{\mathrm{d}\bar{z}}\rangle = \langle \mathrm{d}\bar{z}, \overline{\mathrm{d}z}\rangle = 0.$$

In the notation of Proposition 8.31 with $E = \Omega^1$, we fix dual bases $e^1 = \mathrm{d}z$, $e^2 = \mathrm{d}\bar{z}$ and $\mathrm{ev}(e^i \otimes e_j) = \delta_{ij}$. Since e^1, e^2 freely generate Ω^1, we have $P = I_2$ and

$$g = \begin{pmatrix} q^2 w^2 & 0 \\ 0 & w^2 \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} q^{-2} w^{-2} & 0 \\ 0 & w^{-2} \end{pmatrix}.$$

Using Christoffel symbols in matrix form, the condition for metric compatibility in Definition 8.33 is $\mathrm{d}g = -\Gamma g - g\Gamma^\dagger$. Substituting the entries of g gives

$$\Gamma^2{}_1 = -q^{-2}w^2\Gamma^1{}_2{}^*w^{-2}, \quad -\mathrm{d}w^2 = \Gamma^1{}_1 w^2 + w^2\Gamma^1{}_1{}^* = \Gamma^2{}_2 w^2 + w^2\Gamma^2{}_2{}^*.$$

Using (3.19), the second if these is

$$(1+q^2)w^{-1}(z\mathrm{d}\bar{z} + \bar{z}\mathrm{d}z) = w^{-2}\Gamma^1{}_1 w^2 + \Gamma^1{}_1{}^* = w^{-2}\Gamma^2{}_2 w^2 + \Gamma^2{}_2{}^*.$$

If we set $\Gamma^i{}_i = \alpha_i w^{-1}\bar{z}\mathrm{d}z + \beta_i w^{-1}z\mathrm{d}\bar{z}$ for constants α_i, β_i for $i = 1, 2$ then this is satisfied if $q^4\alpha_i + q^{-4}\beta_i{}^* = 1 + q^2$. Next, torsion freeness $\Gamma^i{}_k \wedge e^k = 0$ implies

$$\Gamma^1{}_2 = q^2\beta_1 w^{-1}z\mathrm{d}z + \gamma_1\mathrm{d}\bar{z}, \quad \Gamma^2{}_1 = q^{-2}\alpha_2 w^{-1}\bar{z}\mathrm{d}\bar{z} + \mathrm{d}z\gamma_2$$

for $\gamma_1, \gamma_2 \in A$, in which case our first condition for metric compatibility becomes

$$\begin{aligned} q^{-2}\alpha_2 w^{-1}\bar{z}\mathrm{d}\bar{z} + \mathrm{d}z\gamma_2 &= -q^{-2}w^2(q^2\beta_1 w^{-1}z\mathrm{d}z + \gamma_1\mathrm{d}\bar{z})^* w^{-2} \\ &= -q^{-2}w^2(q^2\beta_1{}^*\mathrm{d}\bar{z}\bar{z}w^{-1} + \mathrm{d}z\gamma_1{}^*)w^{-2} \end{aligned}$$

so $\gamma_2 = -q^{-2}w^2\gamma_1{}^*w^{-2}$ and $\alpha_2 = -q^{-6}\beta_1{}^*$. Next, if σ exists, it is given by

$$\begin{aligned} \sigma(\mathrm{d}z \otimes \mathrm{d}z) &= q^2\mathrm{d}z \otimes \mathrm{d}z - q^2[z, \Gamma^1{}_k] \otimes e^k, \\ \sigma(\mathrm{d}z \otimes \mathrm{d}\bar{z}) &= q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z - q^{-2}[\bar{z}, \Gamma^1{}_k] \otimes e^k, \\ \sigma(\mathrm{d}\bar{z} \otimes \mathrm{d}\bar{z}) &= q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}\bar{z} - q^{-2}[\bar{z}, \Gamma^2{}_k] \otimes e^k, \\ \sigma(\mathrm{d}\bar{z} \otimes \mathrm{d}z) &= q^2\mathrm{d}z \otimes \mathrm{d}\bar{z} - q^2[z, \Gamma^2{}_k] \otimes e^k \end{aligned}$$

and in that case, using $e^{i*} = e^{i+1}$ (indices mod 2), the $*$-preservation condition is

$$\sigma(e^k \otimes \Gamma^{i+1}{}_{k+1}{}^*) = \Gamma^i{}_k \otimes e^k.$$

We now check that the above conditions are satisfied by $\alpha_1 = q^{-4}(1+q^2)$, $\beta_2 = q^4(1+q^2)$ and $\alpha_2 = \beta_1 = \gamma_1 = \gamma_2 = 0$ from the Γ stated in the question. Thus

$$\sigma(\mathrm{d}z \otimes \mathrm{d}z) = q^{-2}\mathrm{d}z \otimes \mathrm{d}z, \quad \sigma(\mathrm{d}z \otimes \mathrm{d}\bar{z}) = q^{-2}\mathrm{d}\bar{z} \otimes \mathrm{d}z,$$
$$\sigma(\mathrm{d}\bar{z} \otimes \mathrm{d}\bar{z}) = q^2\mathrm{d}\bar{z} \otimes \mathrm{d}\bar{z}, \quad \sigma(\mathrm{d}\bar{z} \otimes \mathrm{d}z) = q^2\mathrm{d}z \otimes \mathrm{d}\bar{z}.$$

A brief check shows that this is a bimodule map. We then write down ∇ itself (half of it is displayed in part (1) of Example 8.57) and check that it is $*$-preserving.

E8.8 Setting $\tau = 1/\delta$ and $\beta = (\delta x)^m$, we have the Heisenberg algebra $[x,t] = \frac{\lambda}{\delta}$ with differential relations as in (8.44) in Example 8.29 but now

$$h = \mathrm{e}^{\delta x}, \quad \mu = \frac{(\delta x)^{m+1}}{m+1}, \quad \nu = -m! \sum_{k=0}^{m+1} \frac{(\delta x)^k}{k!} = -m! \mathrm{e}^{\delta x}_{(m+1)},$$

where we indicate a truncated exponential to order $m+1$. Then $\mu + \nu = -m! \mathrm{e}^{\delta x}_{(m)}$ and $\Box\psi = \frac{\partial^2}{\partial x^2}\psi(t-\lambda) + 2\Delta_0\psi$ on normal ordered functions $\psi(x,t)$, where Proposition 8.24 tells us that for any $f(t)$,

$$\lambda^2 \Delta_0 f(t) = -m! \mathrm{e}^{\delta x}_{(m+1)} f(t-\lambda) + \frac{(\delta x)^{m+1}}{m+1} f(t - \lambda - \lambda \frac{(m+1)}{\delta x})$$
$$+ m! e^{\delta x}_{(m)} f(t - \lambda + \lambda \frac{(\delta x)^m}{m! e^{\delta x}_{(m)}}).$$

E8.9 The set up is the same as Example 8.46 but with a new connection $\nabla_{\mathcal{S}}$ as stated. From the Clifford action (8.52), we find the Dirac operator and its square

$$\not{D}(xe) = \triangleright(\mathrm{d}x \otimes e + x\nabla_{\mathcal{S}} e) = [E_{21}, x]f + xE_{21}f = E_{21}xf, \quad \not{D}(uf) = E_{12}ue,$$
$$\not{D}^2(xe + uf) = E_{11}xe + E_{22}uf.$$

The Clifford action also solves the condition $\mathbf{\nabla}(\triangleright) = 0$, for example

$$(\mathrm{id} \otimes \triangleright)\nabla_{\Omega^1 \otimes \mathcal{S}}(s \otimes f) = (\mathrm{id} \otimes \triangleright)(2\theta \otimes s \otimes f + \sigma(s \otimes \theta) \otimes f)$$
$$= 2\theta \otimes s\triangleright f - \theta \otimes s\triangleright f = \theta \otimes e = \nabla_{\mathcal{S}}(s\triangleright f).$$

The connection $\nabla_{\mathcal{S}}$ has curvature

$$R_{\mathcal{S}} = \mathrm{d}\theta \otimes \mathrm{id} - \theta \wedge \theta \otimes \mathrm{id} = \theta \wedge \theta \otimes \mathrm{id} = (E_{12}E_{21} + E_{21}E_{12}) s \wedge t \otimes \mathrm{id} = s \wedge t \otimes \mathrm{id}$$

on generators and the spinor Laplacian $\Delta_{\mathcal{S}}$ is

$$(\, , \,)_{12}\nabla_{\Omega^1 \otimes \mathcal{S}}\nabla_{\mathcal{S}}(xe) = (\, , \,)_{12}\nabla_{\Omega^1 \otimes \mathcal{S}}(\mathrm{d}x \otimes e + x\theta \otimes e) = (\, , \,)_{12}\nabla_{\Omega^1 \otimes \mathcal{S}}(\theta \otimes xe)$$
$$= (E_{22} - E_{11})xe.$$

In Proposition 8.45, the Lichnerowicz formula for vanishing torsion and $\mathbf{\nabla}(\triangleright) = 0$ is just $\varphi \circ \not{D}^2 = \kappa\Delta_{\mathcal{S}} + \triangleright \circ R_{\mathcal{S}}$, which is immediate using ϕ, κ in (8.53).

On the other hand, although $\nabla_{\mathcal{S}}$ is a bimodule connection with e, f taken as central, it has $\sigma_{\mathcal{S}}(e \otimes \mathrm{d}a) = \nabla_{\mathcal{S}}(e.a) - (\nabla_{\mathcal{S}} e).a = \nabla_{\mathcal{S}}(ae) - (\nabla_{\mathcal{S}} e).a = \mathrm{d}a \otimes e + a\theta \otimes e - \theta \otimes e.a = 0$. Then the condition in Lemma 8.43(iii) cannot hold since its left hand side does not vanish, hence axiom (5) of a spectral triple cannot hold.

E8.10 We saw in exercise E5.8 that a connection on a trivial bundle with finite group fibre induces a covariant derivative $\nabla_{\mathcal{S}} s = (\mathrm{d}a_i) \otimes s^i - a_j \alpha_a \otimes s^i \rho(a^{-1} - e)_{ij}$ where $s = a_i s^i$ and $\alpha_a = \alpha(e^a)$. Writing $\mathrm{d}f = \partial_a f e^a$ and $\alpha_b = \alpha_{ba} e^a$, we apply the 'Clifford action' of Ω^1 by matrices γ^a to the output of $\nabla_{\mathcal{S}}$ to obtain $\not{D} s = \partial_a a_i s^j \gamma^a{}_{ji} - a_i \alpha_{ba} s^j \gamma^a{}_{jk} \rho(b^{-1} - e)_{ki}$, which is the operator stated in the question on sections written as vectors with entries a_i.

In our case, $A = \mathbb{C}(G)$ for another copy of G, so we have a tautological choice $\gamma^a = \rho(a^{-1} - e)$. For S_3, $u + v + w$ and $uv + vu$ are Casimirs in the group algebra with values $0, -1$ respectively as in Example 1.51, giving the γ identities stated in the question. [In fact the γ^a here are the same matrices as e^a in the dual model calculus on $\mathbb{C}S_3$ in Example 1.50.] The natural WQLC α for S_3 from the frame bundle theory in Example 5.83 has $\alpha_{ab} = \delta_{ab} - \frac{1}{3}$, so

$$\not{D} = \gamma^a \partial_a - \gamma^a \delta_{ab} \gamma^b + \frac{1}{3} (\sum_a \gamma^a)(\sum_a \gamma^a) = \gamma^a \partial_a - 3$$
$$= -\frac{1}{2} \begin{pmatrix} 3(\partial_v + \partial_w) + 6 & \sqrt{3}(\partial_w - \partial_v) \\ \sqrt{3}(\partial_w - \partial v) & \partial_v + \partial_w + 4\partial_u + 6 \end{pmatrix},$$

where $(\gamma^a)^2 = -2\gamma^a$. [The explicit matrix was not asked for but we give it for ρ as in Example 1.50. Also note that we are not claiming that this $\not{D}$ fits neatly into Connes' axioms, for which there could be a more relevant choice for γ^a.]

Solutions to Exercises for Chap. 9

E9.1 The stated tensor equations follow immediately by requiring bimodule maps. In the example, the exterior algebra has $e^1 = \mathrm{d}t, e^2 = \mathrm{d}r, e^3 = \theta'$ anticommuting so $\alpha^i{}_{mn}$ and $\tau^{ij}{}_{mn} = \sigma^{ij}{}_{mn} - \delta^i{}_n \delta^j{}_m$ need to be antisymmetric in m, n for the connection in Proposition 8.11 to be torsion free. Assuming this, we can then solve the alpha bimodule map equation to find only $\alpha = 0$ (this is also easy enough to do directly, showing that $\alpha(\theta') = 0$ by considering its commutation relations, then $\alpha(\mathrm{d}r)$ and $\alpha(\mathrm{d}t)$). The tensor equation in terms of τ has the same form as for σ and solving it in the torsion free case gives a 2-parameter family

$$\sigma(e^i\otimes e^j)=-\mu e^i\otimes e^j+(1-\mu)e^j\otimes e^i+s^{ij}\nu(\mathrm{d}t\otimes\theta'+\theta'\otimes\mathrm{d}t+\mathrm{d}r\otimes\mathrm{d}r+\theta'\otimes\theta')$$

for free parameters μ, ν and the notation $s^{22}=s^{13}=s^{31}=1=-s^{11}$ with all other $s^{ij}=0$. The corresponding torsion-free bimodule connection is

$$\nabla e^i=\mu(e^i\otimes\theta+\theta\otimes e^i)-\nu\delta_{i,3}(\mathrm{d}t\otimes\theta'+\theta'\otimes\mathrm{d}t+\mathrm{d}r\otimes\mathrm{d}r+\theta'\otimes\theta'),$$

which is then cotorsion free when $\mu+\nu=0$ and metric compatible when $\mu=\nu=0$.

E9.2 S_4 is generated by S_3 and the additional cycle $s=(1234)=uvz$ where $z=(34)$ is an additional simple reflection obeying $z^2=e$, $vzv=zvz$. Using the latter relations, one finds $s^4=e$, $su=vs$, and $sv=vus^2$. The last two imply $s{\triangleright}u=v$, $s{\triangleleft}u=s$, $s{\triangleright}v=vu$, $s{\triangleleft}v=s^2$ for the double cross product. The axioms of a matched pair determine the remaining actions and one finds that ${\triangleleft}u$ fixes e,s and flips s^2,s^3 while ${\triangleleft}v$ fixes e,s^3 and flips s,s^2. There is a corresponding left action on $\delta_i:=\delta_{s^i}\in\mathbb{C}(\mathbb{Z}_4)$ whereby $u{\triangleright}$ swaps δ_2,δ_3 etc., which defines the cross relations of $H=\mathbb{C}(\mathbb{Z}_4){\blacktriangleright\!\!\triangleleft}\mathbb{C}S_3$ (so that $u\delta_2=\delta_3u$, $v\delta_1=\delta_2v$ etc. according to the action). The right coaction dualising $s^i{\triangleright}$ is $\Delta_R u=\sum_i s^i{\triangleright}u\otimes\delta_i$ and the coproduct of H is then the right handed cross coproduct where $\mathbb{C}(\mathbb{Z}_4)$ is a sub-Hopf algebra while $\Delta u=(1\otimes u_{(1)(\bar 1)})\otimes(u_{(1)(\bar 0)}\otimes u_{(2)})=\sum_i s^i{\triangleright}u\otimes\delta_i u$. Similarly for $\Delta_R v$, to give

$$\Delta u=u\otimes\delta_0u+v\otimes\delta_1u+vu\otimes\delta_2+uv\otimes\delta_3u,$$
$$\Delta v=v\otimes\delta_0v+vu\otimes\delta_1v+uv\otimes\delta_2v+u\otimes\delta_3v.$$

According to the general bicrossproduct theory, this H right acts on the group algebra $\mathbb{C}\mathbb{Z}_4$ with S_3 acting by $s^i{\triangleleft}u$, $s^i{\triangleleft}v$ extended linearly and $\mathbb{C}(\mathbb{Z}_4)$ acting by evaluation against the left output of the coproduct $\Delta s^i=s^i\otimes s^i$ (i.e., by 'translations' in a Hopf algebra sense), which comes out as $s^i{\triangleleft}\delta_j=\delta_{ij}s^i$.

The calculus $\Omega(\mathbb{C}\mathbb{Z}_4)$ stated in the question is equivalent to the canonical $\Omega(\hat{\mathbb{Z}}_4)$ under $\mathbb{C}\mathbb{Z}_4\cong\mathbb{C}(\hat{\mathbb{Z}}_4)$ (see exercise E2.5), where we take $\mathcal{C}=\{1,2,3\}$ (which is the universal one in degree 1) with basic forms e^i for $i=1,2,3$. By construction, the e^i are invariant for the action of $\mathbb{C}(\mathbb{Z}_4)$ so $e^i{\triangleleft}\delta_j=\epsilon(\delta_j)e^i=\delta_{j,0}e^i$. The action of S_3 on the e^i is obtained from solving $(\mathrm{d}s^i){\triangleleft}u=\mathrm{d}(s^i{\triangleleft}u)$ for $e^i{\triangleleft}u$, and comes out as

$$e^i{\triangleleft}u=\phi^i{}_je^j,\quad e^i{\triangleleft}v=\overline{\phi^i{}_j}e^j=\phi^j{}_ie^j;\quad \phi=\frac12\begin{pmatrix}1&1+\mathrm{i}&-\mathrm{i}\\1-\mathrm{i}&0&1+\mathrm{i}\\\mathrm{i}&1-\mathrm{i}&1\end{pmatrix}.$$

One can check that the commutation relations with s are compatible, for example applying $\triangleleft u$ to $e^1 s = qse^1$ using Δu above requires $(e^1 \triangleleft v)s = qse^1 \triangleleft u$ and this holds. The calculus on $\mathbb{C}\mathbb{Z}_4$ is inner with $\theta = e^1 + e^2 + e^3$ invariant under the S_3 and in the Hopf sense via the counit, the whole of H. There is also a $*$-structure according to the general theory, with $s^* = s^{-1}$, $u^* = u$ and $v^* = v$ giving a $*$-calculus.

The stated quantum metric commutes with s and is invariant under the above action of H due to $(\phi^1{}_a\phi^3{}_b + \phi^2{}_a\phi^2{}_b + \phi^3{}_a\phi^1_b)e^a \otimes e^b = g$. The map $\sigma(e^i \otimes e^j) = e^j \otimes e^i$ automatically extends as a bimodule map and together with $\alpha = 0$ gives $\nabla e^i = 0$, which is clearly then a QLC. [We do not claim it is unique.]

E9.3 From (9.31), we have Killing vector $\tau_3 = \frac{\epsilon_{3jk}}{h(r)^2} z_j e_k = \frac{1}{h(r)^2}\frac{\partial}{\partial\varphi}$, where $e_k(\mathrm{d}r) = 0$ and $e_k(\mathrm{d}z^i) = \delta_{ik}$. Then the commutation relations for the induced 5D calculus in (8.42) reduce to

$$[r, t] = [r, z^i] = [z^i, z^j] = [r, \theta'] = [z^i, \theta'] = [\mathrm{d}r, t] = [\mathrm{d}r, z^i] = [\mathrm{d}z^i, r] = 0,$$

$$[z^i, t] = -\frac{\lambda}{h(r)^2}\epsilon_{ij3} z^j, \quad [r, \mathrm{d}t] = \lambda \mathrm{d}r, \quad [t, \theta'] = \lambda\theta',$$

$$[\mathrm{d}z^i, z^j] = \frac{\lambda}{r^2} e_{ij}\theta', \ \ [\mathrm{d}z^i, t] = -\lambda\epsilon_{ij3}\mathrm{d}(\frac{z^j}{h(r)^2}) - \lambda \mathrm{d}z^i, \ \ [t,\ \mathrm{d}t] = \lambda(\frac{\theta'}{f(r)^2} + \mathrm{d}t),$$

since $\alpha = -1$ for a Killing vector and we set $\beta = -1/f(r)^2$ for the wave operator in Corollary 8.25 to match the metric in Proposition 9.11. Some solutions for $\tau_3(\mu) = \beta$ and then $\tau_3(\nu) = \mu + \nu$ are as stated in the question, on making choices for constants of integration. Substituting into the formula for Δ_0 in Proposition 8.24 gives

$$\Delta_0 g(t) = \frac{(1 + h(r)^2\varphi)g(t - \lambda) - h(r)^2\varphi g\big(t - \lambda(1 + \frac{1}{h(r)^2\varphi})\big) - g(t)}{\lambda^2 f(r)^2}$$

with an apparent coordinate singularity at $\varphi = 0$. But taking into account the commutator $[\varphi, t] = \frac{\lambda}{h(r)^2}$, one can check for example that $\Delta_0 t^2 = -1/f(r)^2 = \beta$, as it should. By Proposition 8.24, we also have

$$[g(t), \mathrm{d}t] = \lambda \mathrm{d}g + \Big(\frac{g(t) - g(t - \lambda)}{f(r)^2}\Big)\theta'$$

in the quantum calculus. The wave operator can be studied further, for example with $f(r) = h(r) = 1$ for flat spacetime.

E9.4 From the Poisson tensor, we compute

$$x^\mu \bullet x^\nu = x^\mu x^\nu + \frac{\lambda}{2}\omega^{\mu\nu} = x^\mu x^\nu + \frac{\lambda}{4}t^{-2}(\kappa_{\mu,\nu+2} + x^\mu x^{\nu+2} - x^{\mu+2}x^\nu),$$

$$\mathrm{i}x^{a+2} \bullet x^b = \mathrm{i}x^{a+2}x^b + \mathrm{i}\frac{\lambda}{4}t^{-2}(\kappa_{a+2,b+2} + x^{a+2}x^{b+2} - x^{a+4}x^b),$$

$$\mathrm{i}x^a \bullet x^{b+2} = \mathrm{i}x^a x^{b+2} + \mathrm{i}\frac{\lambda}{4}t^{-2}(\kappa_{a,b+4} + x^a x^{b+4} - x^{a+2}x^{b+2}),$$

$$-x^{a+2} \bullet x^{b+2} = -x^{a+2}x^{b+2} - \frac{\lambda}{4}t^{-2}(\kappa_{a+2,b+4} + x^{a+2}x^{b+4} - x^{a+4}x^{b+2}).$$

On adding, the κ's cancel and we obtain

$$\begin{aligned}(x^a + \mathrm{i}x^{a+2}) \bullet (x^b + \mathrm{i}x^{b+2}) &= (x^a + \mathrm{i}x^{a+2})(x^b + \mathrm{i}x^{b+2})\\ &\quad + \frac{\lambda}{4}t^{-2}(x^a x^{b+2} - x^{a+2}x^b - x^{a+2}x^{b+4} + x^{a+4}x^{b+2})\\ &\quad + \mathrm{i}\frac{\lambda}{4}t^{-2}(x^{a+2}x^{b+2} - x^{a+4}x^b + x^a x^{b+4} - x^{a+2}x^{b+2})\end{aligned}$$

and then the $O(\lambda)$ parts cancel by the signed mod 4 rule, so we obtain $z^1 \bullet z^2 = z^1 z^2$ and their commutator vanishes. If we substitute $\bar{z}^b = x^b - \mathrm{i}x^{b+2}$ in the above then some terms do not cancel, giving the other formula stated. We also have

$$\begin{aligned}x^\mu \bullet \mathrm{d}x^\nu &= x^\mu \mathrm{d}x^\nu + \frac{\lambda}{2}\omega^{\mu\rho}\nabla_\rho(\mathrm{d}x^\nu)\\ &= x^\mu \mathrm{d}x^\nu + \frac{\lambda}{2}\omega^{\mu\rho}t^2(x^\gamma \kappa_{\nu\rho} + x^\rho \kappa_{\nu,\gamma} + \kappa_{\nu+2,\gamma}x^{\rho+2} + \kappa_{\nu+2,\rho}x^{\gamma+2})\mathrm{d}x^\gamma,\end{aligned}$$

which can in principle be used to compute the $z, \mathrm{d}z$ and $z, \mathrm{d}\bar{z}$ commutation relations.

E9.5 We use the convention that α, α' are different elements of the index set $\{0, 1\}$. Poisson compatibility in Lemma 9.21 reduces to $\omega^{\gamma\beta}\Gamma^\alpha{}_{\gamma\delta} + \omega^{\alpha\gamma}\Gamma^\beta{}_{\gamma\delta} = 0$ (summing over γ) which for $\omega^{\gamma\beta}$ as given becomes $\Gamma^\alpha{}_{\beta'\delta} = (-1)^{\alpha+\beta}\Gamma^\beta{}_{\alpha'\delta}$ or equivalently $\Gamma^\alpha{}_{\beta\delta} = -(-1)^{\alpha+\beta}\Gamma^{\beta'}{}_{\alpha'\delta}$. If $\alpha \neq \beta$ then this is automatically satisfied while if $\alpha = \beta$ then we get

$$\Gamma^\alpha{}_{\alpha\delta} = -\Gamma^{\alpha'}{}_{\alpha'\delta}. \qquad (\dagger)$$

Metric compatibility (summing over κ) is $g_{\alpha\beta,\gamma} - \Gamma^\kappa{}_{\gamma\alpha} g_{\kappa\beta} - \Gamma^\kappa{}_{\gamma\beta} g_{\alpha\kappa} = 0$ as usual which, since the metric is diagonal, gives two cases as $\beta = \alpha, \alpha'$. Thus,

$$\text{(i)}\quad g_{\alpha\alpha,\gamma} = 2\Gamma^\alpha{}_{\gamma\alpha} g_{\alpha\alpha}; \qquad \text{(ii)}\quad \Gamma^{\alpha'}{}_{\gamma\alpha} g_{\alpha'\alpha'} + \Gamma^\alpha{}_{\gamma\alpha'} g_{\alpha\alpha} = 0.$$

The case (i) combined with (†) tells us that

$$\Gamma^0{}_{00} = \Gamma^0{}_{10} = \Gamma^1{}_{11} = 0, \quad \Gamma^1{}_{01} = \mu, \quad \Gamma^1{}_{10} = \Gamma^0{}_{01} = 0$$

after which case (ii) fixes the remaining Christoffel symbols as

$$\Gamma^1{}_{00} = \Gamma^0{}_{11} = 0.$$

The torsion is computed from the Christoffel symbols as $T^1{}_{01} = -T^1{}_{10} = \mu$ with all other components zero, so the downstairs version of torsion is $T_{101} = -T_{110} = e^{2\mu x^0}\mu$. The contorsion $S^\alpha{}_{\beta\gamma}$ immediately follows from (9.58) on noting that the metric is diagonal. Because the Christoffel symbols are constants, the curvature is

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^1{}_{\delta\beta}\Gamma^\alpha{}_{\gamma 1} - \Gamma^1{}_{\gamma\beta}\Gamma^\alpha{}_{\delta 1} = 0.$$

It remains to check the last of the Poisson–Riemannian field equations, namely the condition (9.56) for quantisation of the Levi-Civita connection. We need the covariant derivative

$$\nabla_\rho S^\alpha{}_{\delta\nu} = S^\alpha{}_{\delta\nu,\rho} + \Gamma^\alpha{}_{\rho\kappa} S^\kappa{}_{\delta\nu} - \Gamma^\kappa{}_{\rho\delta} S^\alpha{}_{\kappa\nu} - \Gamma^\kappa{}_{\rho\nu} S^\alpha{}_{\delta\kappa}$$

which immediately gives $\nabla_1 S^\alpha{}_{\delta\nu} = 0$, while

$$\begin{aligned}\nabla_0 S^\alpha{}_{\delta\nu} &= S^\alpha{}_{\delta\nu,0} + \Gamma^\alpha{}_{01} S^1{}_{\delta\nu} - \Gamma^1{}_{0\delta} S^\alpha{}_{1\nu} - \Gamma^1{}_{0\nu} S^\alpha{}_{\delta 1}\\ &= S^\alpha{}_{\delta\nu,0} + \mu(\delta_{\alpha,1} - \delta_{\delta,1} - \delta_{\nu,1}) S^\alpha{}_{\delta\nu} = 0,\end{aligned}$$

where the only nontrivial case is $\nabla_0 S^0{}_{11} = S^0{}_{11,0} - 2\mu S^0{}_{11} = 0$ from the form of $S^0{}_{11}$. Hence $H^\alpha{}_{\beta\delta\nu} = \mathcal{R}_{\mu\nu} = 0$ from (9.57), so condition (9.56) holds.

E9.6 Note that $x = x^1$ and $t = x^0$ should not be confused with the original x, t coordinates in §9.5.2 as rather they correspond to R, T there. We write $\partial_t = \partial/\partial t$ etc., and start by computing Lie derivatives $\mathcal{L}_\tau \mathrm{d}x = -\mathrm{d}x$ and

$$\mathcal{L}_\tau(e^{2t\mu}\mathrm{d}x) = \tau(e^{2\mu t})\mathrm{d}x + e^{2\mu t}\mathcal{L}_\tau \mathrm{d}x = 2e^{2\mu t}\mathrm{d}x - e^{2\mu t}\mathrm{d}x = e^{2\mu t}\mathrm{d}x,$$

so that $\mathcal{L}_\tau(\mathrm{e}^{2\mu t}\mathrm{d}x\otimes\mathrm{d}x)=0$ and τ is a Killing vector, as more obviously is ∂_x. We also have $[\tau,\partial_x]=\partial_x$, hence an action of $b_+=\mathbb{R}{>\!\!\triangleleft}\mathbb{R}$ on de Sitter space by isometries. We let $F=\mathrm{e}^{\tau\otimes L}$ and use coproducts $\Delta\tau=\tau\otimes 1+1\otimes\tau$ and $\Delta\partial_x=\partial_x\otimes 1+1\otimes\partial_x$ for $U(b_+)$ and the identity that $[\tau\otimes L,\partial_x\otimes 1]=\partial_x\otimes L$, $[\tau\otimes L,[\tau\otimes L,\partial_x\otimes 1]]=\partial_x\otimes L^2$ etc., hence by a well-known identity for conjugation,

$$\mathrm{e}^{\tau\otimes L}(\partial_x\otimes 1)\mathrm{e}^{-\tau\otimes L}=\mathrm{e}^{[\tau\otimes L,\ ]}(\partial_x\otimes 1)=\partial_x\otimes e^L=\partial_x\otimes(1+\lambda\partial_x).$$

Because Δ is an algebra map, we have

$$\begin{aligned}
F_{23}(\mathrm{id}\otimes\Delta)\mathrm{e}^{\tau\otimes L}&=\mathrm{e}^{1\otimes\tau\otimes L}\mathrm{e}^{\tau\otimes\ln(1\otimes 1+\lambda\partial_x\otimes 1+1\otimes\lambda\partial_x)}\\
&=\mathrm{e}^{\tau\otimes\ln(1\otimes 1+\lambda\mathrm{e}^{\tau\otimes L}(\partial_x\otimes 1)\mathrm{e}^{-\tau\otimes L}+1\otimes\lambda\partial_x)}\mathrm{e}^{1\otimes\tau\otimes L}\\
&=\mathrm{e}^{\tau\otimes\ln(1\otimes 1+\lambda\partial_x\otimes 1+\lambda^2\partial_x\otimes\partial_x+1\otimes\lambda\partial_x)}\mathrm{e}^{1\otimes\tau\otimes L}\\
&=\mathrm{e}^{\tau\otimes\ln((1\otimes 1+\lambda\partial_x\otimes 1)(1\otimes 1+1\otimes\lambda\partial_x))}\mathrm{e}^{1\otimes\tau\otimes L}\\
&=\mathrm{e}^{\tau\otimes(\ln(1+\lambda\partial_x)\otimes 1+1\otimes\ln(1+\lambda\partial_x))}\mathrm{e}^{1\otimes\tau\otimes L}=\mathrm{e}^{\tau\otimes L\otimes 1+\tau\otimes 1\otimes L}\mathrm{e}^{1\otimes\tau\otimes L}\\
&=\mathrm{e}^{\tau\otimes L\otimes 1}\mathrm{e}^{\tau\otimes 1\otimes L}\mathrm{e}^{1\otimes\tau\otimes L}=\mathrm{e}^{\tau\otimes L\otimes 1}\mathrm{e}^{\tau\otimes 1\otimes L+1\otimes\tau\otimes L}=F_{12}(\Delta\otimes\mathrm{id})\mathrm{e}^{\tau\otimes L}.
\end{aligned}$$

We now use this cocycle and action of b_+ to quantise de Sitter space by a 1-sided twist. We have

$$x\bullet t=\cdot\mathrm{e}^{-\tau\otimes L}(x\otimes t)=xt,\quad t\bullet x=\cdot\mathrm{e}^{-\tau\otimes L}(t\otimes x)=tx-\frac{\lambda}{\mu},$$
$$x\bullet x=x^2+\lambda x,\quad x\bullet\mathrm{d}x=x\mathrm{d}x,\quad \mathrm{d}x\bullet x=\mathrm{d}x.x+\lambda\mathrm{d}x,$$
$$t\bullet\mathrm{d}x=t\mathrm{d}x=\mathrm{d}x.t=\mathrm{d}x\bullet t,\quad x\bullet\mathrm{d}t=x\mathrm{d}t=\mathrm{d}t.x=\mathrm{d}t\bullet x$$

and more obviously $t\bullet t=t^2$, $t\bullet\mathrm{d}t=t\mathrm{d}t=\mathrm{d}t.t=\mathrm{d}t\bullet t$. These are the correct commutation relations setting $\lambda=-\lambda'\mu$, $\mu=\sqrt{\delta}$ and changing to T,R symbols for t,x. The quantum metric computed in the same way is not deformed and looks the same in terms of bullet products since $t\bullet\mathrm{d}x$ is not deformed. On the other hand, $x\bullet x$ *is* deformed, which means that operations on the underlying vector space of functions, even if not deformed, can appear deformed in terms of algebra elements x with the quantised product. This is relevant to the Laplacian for the model in §9.5.2.

E9.7 We start with $H = \mathbb{C}\mathbb{Z}^2 = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ (the classical algebraic torus) coacting on the same algebra but viewed geometrically as $A = \mathbb{C}_q[S^1] \otimes \mathbb{C}_q[S^1]$ with generators $u^{\pm 1}, v^{\pm 1}$ and calculus with relations $\mathrm{d}u.u = qu\mathrm{d}u$, $\mathrm{d}v.u = qv\mathrm{d}v$ and $[u, v] = [\mathrm{d}u, v] = [\mathrm{d}v, u] = 0$ (while $\mathrm{d}u$, $\mathrm{d}v$ anticommute as part of the super tensor product of the exterior algebra). Twisting gives $v \bullet u = F(t, s)vu = e^{i\theta} vu = e^{i\theta} uv = e^{i\theta} F(s, t)uv = e^{i\theta} u \bullet v$ and similarly

$$\mathrm{d}v \bullet u = e^{i\theta} \mathrm{d}v.u = e^{i\theta} u\mathrm{d}v = e^{i\theta} u \bullet \mathrm{d}v,$$

$$\mathrm{d}v \bullet \mathrm{d}u = e^{i\theta} \mathrm{d}v \wedge \mathrm{d}u = -e^{i\theta} \mathrm{d}u \wedge \mathrm{d}v = -e^{i\theta} \mathrm{d}u \bullet \mathrm{d}v$$

etc., give the calculus of $\mathbb{C}_{q,\theta}[\mathbb{T}^2]$. The basis elements $e^1 = u^{-1}\mathrm{d}u$, $e^2 = v^{-1}\mathrm{d}v$ are invariant and look the same, $u^{-1} \bullet \mathrm{d}u = u^{-1}\mathrm{d}u$, etc. Hence $g = \sum_i e^i \otimes e^i = u^{-1} \bullet \mathrm{d}u \otimes u^{-1} \bullet \mathrm{d}u + v^{-1} \bullet \mathrm{d}v \otimes v^{-1} \bullet \mathrm{d}v$ looks the same and has $\nabla e_i = \beta_i e^i \otimes e^i$ similarly expressed. Here each $\beta_i = 0$ or $2/(q-1)$ in Example 8.5, so there are four QLCs which each twist. There are two choices for $\mathrm{d}u$,

$$\nabla \mathrm{d}u = \begin{cases} q^{-1} \\ \frac{q^{-1}+1}{q-1} \end{cases} \quad u^{-1} \bullet \mathrm{d}u \otimes \mathrm{d}u$$

and similarly two choices for $\nabla \mathrm{d}v$. The $\sigma(e^i \otimes e^i) = \pm e^i \otimes e^i$ respectively for the two cases and flip between e^1, e^2 and the latter do look different after twisting, e.g. $\sigma_F(\mathrm{d}u \otimes \mathrm{d}v) = \sigma_F(F(s, t)\mathrm{d}u \otimes \mathrm{d}v) = F^{-1}(t, s)\mathrm{d}v \otimes \mathrm{d}u = e^{-i\theta}\mathrm{d}v \otimes \mathrm{d}u$ and similarly $\sigma_F(\mathrm{d}v \otimes \mathrm{d}u) = e^{i\theta}\mathrm{d}v \otimes \mathrm{d}u$.

If q and $e^{i\theta}$ are r-th roots of unity then F descends to H and we can twist in the same way to get $c_{q,\theta}[\mathbb{T}^2]$ with its calculus in exercise E1.5 from $H = \mathbb{C}\mathbb{Z}_r^2$ and $A = c_q[S^1] \otimes c_q[S^1]$. [In both cases, when q is a power of $e^{i\theta}$, one has a central basis and can explore the full moduli of quantum Riemannian geometries using E8.2.]

E9.8 Comparing (5.16) in the case $A = \mathbb{k}$ there with the cotwisting formulae in §9.6.3, it is clear that this coincides with a Hopf algebra H left coacting on itself and a 1-sided cotwist by $\chi : H \otimes H \to \mathbb{k}$. In our case, we work over $\mathbb{k} = \mathbb{R}$ and set $F = \chi$ in Example 5.19. We regard one copy of the Hopf algebra geometrically as $A = c_{-1}[\mathbb{T}^2]$ with generators u, v and calculus as in E9.7 reduced at $q = -1$. The same Hopf algebra is also regarded as $H = \mathbb{R}\mathbb{Z}_2^2$ coacting as in E9.7 according to the tautological $\mathbb{Z}_2^2$-grading of A for which we write $s^{a_0}t^{a_1} \in H$ as $a_0a_1 \in \mathbb{Z}_2^2$. The cocycle is different, so we are not simply quotienting E9.7.

Under the cotwist, we identify $i = v$, $j = u$ and then $k = uv = vu = \chi(01, 10)vu = v \bullet u = ij$, $k = uv = -\chi(10, 01)uv = -u \bullet v = -ji$ and $-1 = -u^2 = \chi(10, 10)u^2 = u \bullet u = i^2$ etc., (a product of quaternions indicates that the quaternion product is understood). Now similarly $\mathrm{d}v \bullet u = (\mathrm{d}v)u = u\mathrm{d}v = -u \bullet \mathrm{d}v$, $\mathrm{d}v \bullet v = -(\mathrm{d}v)v = v\mathrm{d}v = -v \bullet \mathrm{d}v$ and $\mathrm{d}v \bullet \mathrm{d}u = \mathrm{d}v \wedge \mathrm{d}u = -\mathrm{d}u \wedge \mathrm{d}v = \mathrm{d}u \bullet \mathrm{d}v$ etc. gives the 2D quaternion calculus as stated. [It can be related to a 2D calculus on $M_2(\mathbb{C})$ compatible with the natural embedding $\mathbb{H} \subset M_2(\mathbb{C})$ via Pauli matrices.] For reference, we also have that

$$\mathrm{d}k = i\mathrm{d}j - j\mathrm{d}i, \quad i\mathrm{d}k = (\mathrm{d}k)i, \quad j\mathrm{d}k = (\mathrm{d}k)j, \quad k\mathrm{d}k = i\mathrm{d}i + j\mathrm{d}j = -(\mathrm{d}k)k,$$

$$\mathrm{d}k \wedge \mathrm{d}k = 0, \quad \mathrm{d}i \wedge \mathrm{d}k = -\mathrm{d}k \wedge \mathrm{d}i, \quad \mathrm{d}j \wedge \mathrm{d}k = -\mathrm{d}k \wedge \mathrm{d}j$$

which breaks the cyclic symmetry of the quaternions. Next, the 1-forms $e^1 = u\mathrm{d}u, e^2 = v\mathrm{d}v$ are invariant and hence undeformed but look slightly differently as $e^1 = -j\mathrm{d}j = j^{-1}\mathrm{d}j, e^2 = -i\mathrm{d}i = i^{-1}\mathrm{d}i$. Hence the Euclidean metric on $c_{-1}[\mathbb{T}^2] \cong \mathbb{R}(\mathbb{Z}_2^2)$ up to a normalisation now looks like

$$g_{\mathbb{H}} = \mathrm{d}i \otimes \mathrm{d}i + \mathrm{d}j \otimes \mathrm{d}j.$$

Next, we look only for QLCs on $\mathbb{Z}_2^2$ with constant coefficients so that they cotwist. For each $\mathbb{Z}_2$, we know from exercise E8.4 that there are two as cases $b = 0, 2$ which in our terms is $\nabla e^i = 0$ (with $\sigma = $ flip) or $\nabla e^i = -e^i \otimes e^i$ (with $\sigma(e^i \otimes e^i) = -e^i \otimes e^i$) and $\sigma = $ flip on e^1, e^2. To see this, if we denote the basis in E8.4 by E^1 to avoid confusion then $\mathrm{d}u = -2uE^1$ as we know from the $r = 2$ case of E2.5. So our present $e^1 = -2E^1$ with result that the $b = 2$ case in E8.4 converts to $\nabla e^1 = -e^1 \otimes e^1$. For the same reason, the inner generator is $\theta = -\frac{1}{2}(e^1 + e^2)$ in present terms. We then cotwist $\nabla \mathrm{d}u = -u\mathrm{d}u \otimes \mathrm{d}u$ or $\nabla \mathrm{d}u = 0$ to obtain the same element written as $u \bullet \mathrm{d}u \otimes \mathrm{d}u$ or 0. We can make these choices independently for each $\mathbb{Z}_2$, so there are four QLCs for the Euclidean metric that cotwist to provide the four QLCs on $\mathbb{H}$ as stated. Note that cotwisting σ now gives $-$flip between the two different copies since the relevant value of one of the F before or after is -1 and this is needed if one wants to check directly that ∇ is a bimodule connection. It is also a nice check to see directly that $\nabla g_H = 0$. E.g., for the upper connection choices, $\nabla(\mathrm{d}i \otimes \mathrm{d}i) = i\mathrm{d}i \otimes \mathrm{d}i \otimes \mathrm{d}i + \sigma(\mathrm{d}i \otimes i\mathrm{d}i) \otimes \mathrm{d}i = i\mathrm{d}i \otimes \mathrm{d}i \otimes \mathrm{d}i - i\sigma(\mathrm{d}i \otimes \mathrm{d}i) \otimes \mathrm{d}i = 0$ for each term of the metric separately.

E9.9 The construction follows the same pattern as exercise E9.8, with $H = \mathbb{R}\mathbb{Z}_2^3$ and $A = c_{-1}[\mathbb{T}^3] = c_{-1}[S^1]^{\otimes 3}$ over $\mathbb{R}$, where the notation indicates that A has the calculus $\mathrm{d}u.u = -u\mathrm{d}u, \mathrm{d}v.v = -v\mathrm{d}v, \mathrm{d}w.w = -w\mathrm{d}w$ for commuting generators with $u^2 = v^2 = w^2 = 1$. We also adopt a notation $e_{\vec{a}} = u^{a_0}v^{a_1}w^{a_2}$ and regard $a_0a_1a_2 \in \mathbb{Z}_2^3$ as a vector. This time F is not a cocycle but its coboundary in the relevant sense is $(-1)^{\det(\vec{a},\vec{b},\vec{c})}$. Since

$F(\vec{a},\vec{a}) = -1$ for $\vec{a} \neq 0$, we have $e_{\vec{a}} \bullet e_{\vec{a}} = -1$. By construction, the calculus deforms in the same way, so $\mathrm{d}e_{\vec{a}} \bullet (e_{\vec{b}} \bullet e_{\vec{c}}) = \pm(\mathrm{d}e_{\vec{a}} \bullet e_{\vec{b}}) \bullet e_{\vec{c}}$ and $\mathrm{d}e_{\vec{a}} \bullet e_{\vec{b}} = \pm e_{\vec{b}} \bullet \mathrm{d}e_{\vec{a}}$ etc. with the same sign rule based on linear independence (with all cases of d's on the different factors) on top of the sign rule for $c_{-1}[\mathbb{T}^3]$ where $\mathrm{d}u$ anticommutes with any monomial involving u and commutes with the others, etc. For the 3D calculus, we focus on $l = e_{100} = u$, $j = e_{010} = v$, $i = e_{001} = w$, say, and $e^1 = u\mathrm{d}u = -l\mathrm{d}l$, then $\mathrm{d}i.j = -j\mathrm{d}i$ but $\mathrm{d}i.k = k\mathrm{d}i$ as we saw already for the quaternions and any pair from i, j, l behave similarly. Moreover, $e^2 = v\mathrm{d}v = -j\mathrm{d}j$ and $e^3 = w\mathrm{d}w = -i\mathrm{d}i$, where the use of the octonion elements indicates the octonion product $\bullet$ is understood. The Euclidean metric on $c_{-1}[\mathbb{T}^3]$ up to normalisation is then

$$g_{\mathbb{O}} = \mathrm{d}i \otimes \mathrm{d}i + \mathrm{d}j \otimes \mathrm{d}j + \mathrm{d}l \otimes \mathrm{d}l.$$

This time, we have $\nabla\mathrm{d}i, \nabla\mathrm{d}j, \nabla\mathrm{d}l$ with the same binary choices as for $\nabla\mathrm{d}i, \nabla\mathrm{d}j$ in the quaternion case. If one wants to check this directly, that $\nabla g_{\mathbb{O}} = 0$ holds proceeds as before for each term. For σ, we first check $(\mathrm{d}i \otimes \mathrm{d}j).l = -\mathrm{d}i \otimes (\mathrm{d}j.l) = \mathrm{d}i \otimes (l\mathrm{d}j) = -(\mathrm{d}i.l) \otimes \mathrm{d}j = (l.\mathrm{d}i) \otimes \mathrm{d}j = -l(\mathrm{d}i \otimes \mathrm{d}j)$. The same with i, j swapped then means that both cases $\sigma = \pm\mathrm{flip}$ are bimodule maps.

E9.10 The general framework is a corollary of the way the space of noncommutative vector fields $\mathfrak{X}^R$ was defined in §2.7. [One might also impose $\mathrm{d}\omega = 0$ for a 'quantum symplectic form' as part of a further theory beyond our scope here, for example to prove a Liouville theorem whereby $\mathcal{L}_X\omega = 0$ along Hamiltonian vector fields.] If Ω^1 has a central basis $\{s^i\}$ for $i = 1, \cdots, n$, let $\{f^i\}$ be a dual basis, $\mathrm{d}a = \partial_i a s^i$ (summation understood) and $i(\omega) = \omega_{ij} s^i \otimes s^j$ say, where $\omega_{ij} \in A$. If ω^{ij} is the inverse in $M_n(A)$ then it is immediate that $X_h = (\partial_i h)\omega^{ij} f^j$ obeys

$$i^R_{X_h}\omega = (\partial_i h)\omega^{ij}\langle f^j \omega_{mk}, s^m\rangle s^k = (\partial_i h)\omega^{ij}\omega_{jk}s^k = \mathrm{d}h$$

as required and $X_h(a) = (\partial_i h)\omega^{ij}\partial_j a$.

In the case of $M_2(\mathbb{C})$, we have $\mathrm{d}h = [E_{12}, h]s + [E_{21}, h]t$ showing the partial derivatives and $\omega_{ij} = \frac{\mathrm{i}}{2}(\sigma_1)_{ij}$ from the stated form of the lift i in our s, t basis, where σ_1 is the Pauli matrix with 1 on the off-diagonals. Hence $X_h = \frac{2}{\mathrm{i}}([E_{12}, h]f^t + [E_{21}, h]f^s)$ for the dual basis f^s, f^t and

$$X_h(a) = \frac{2}{\mathrm{i}}([E_{12}, h][E_{21}, a] + [E_{21}, h][E_{12}, a])$$

for all $a \in M_2(\mathbb{C})$. Clearly, $X_h(1) = 0$ and a little calculation gives the other values stated in the question. In particular, if $h = \frac{\sigma_3}{4}$ then we find

$$X_{\frac{\sigma_3}{4}}(\sigma_1) = -\sigma_2, \quad X_{\frac{\sigma_3}{4}}(\sigma_2) = \sigma_1, \quad X_{\frac{\sigma_3}{4}}(\sigma_3) = 2\mathrm{i}.$$

If we write $a(t) = a_0(t)1 + a_i(t)\sigma_i \in M_2(\mathbb{C})$ (sum over i and 1 the identity matrix) then the flow is $\dot{a}_3 = 0$, $\dot{a}_0 = 2\mathrm{i}a_3$, $\dot{a}_1 = a_2$, $\dot{a}_2 = -a_1$ with motion as stated.

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Index

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