FALL 2012 MATH 8230 (VECTOR BUNDLES) LECTURE NOTES

1. Definitions: vector bundles and structure groups

A vector bundle over a topological space M (or "with base space M") is, essentially, family of vector spaces continuously parametrized by M. (I'm using the letter M to denote the base space of the vector bundle as a concession to the fact that in most of the applications we'll be interested in the base space will be a smooth manifold; however for basic definitions and results it there is no need to restrict to this case.) One way of making this precise is as follows:

Definition 1.1. A *(real, rank-k) vector bundle* over a topological space M is a continuous map $\pi\colon E \to M$ where E is a topological space such that, for all $m \in M$:

(i) the "fiber" $E_m := \pi^{-1}(\{m\})$ is equipped with the structure of a vector space over $\mathbb{R}$.
(ii) There is an neighborhood $U \subset M$ of m and a "local trivialization" $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ which, for each $x \in U$, maps the fiber E_x to $\{x\} \times \mathbb{R}^k$ by a *linear* isomorphism.

Remark 1.2. In practice one says things like, "Let E be a vector bundle over M" instead of "Let $\pi\colon E \to M$" be a vector bundle over M. But one should keep in mind that, even so, the "vector bundle E" refers not just to the topological space but also to the projection map and the local trivializations.

Example 1.3. There is one very obvious example (for any M and any k): Simply take $E = M \times \mathbb{R}^k$ and let π be the projection. For item (ii) in Definition 1.1 one can just set $U = M$ (for any $m \in M$), so that $\pi^{-1}(M) = M \times \mathbb{R}^k$, and let $\Phi\colon \pi^{-1}(M) \to M \times \mathbb{R}^k$ be the identity map.

Appropriately, this bundle is called the "trivial bundle of rank k over M."

Definition 1.4.
- Let $\pi_E\colon E \to M$ and $\pi_F\colon F \to M$ be two vector bundles over M. An *isomorphism* from E to F is a homeomorphism $\Psi\colon E \to F$ such that for each $m \in M$ Ψ restricts to E_m as a linear map (and hence a linear isomorphism, since Ψ is a homeomorphism) from E_m to F_m.
- The vector bundle $\pi\colon E \to M$ is called *trivial* it is isomorphic to the trivial rank-k bundle over M for some k.

Exercise 1.5. Suppose that $\pi_E\colon E \to M$ and $\pi_F\colon F \to M$ are two vector bundles and $\Psi\colon E \to F$ is a continuous map that maps each fiber E_m to F_m by a linear isomorphism. Prove that Ψ is an isomorphism. (The nontrivial part of this is showing that Ψ^{-1} is continuous. Hint: By restricting to appropriate local trivializations, you can reduce to the case where Ψ is a continuous map of the trivial bundle $U \times \mathbb{R}^k$ to itself which restricts to each $\{x\} \times \mathbb{R}^k$ as a linear automorphism, and you should be able to show then that the fact that Ψ is continuous implies that Ψ^{-1} is also continuous.)

At first glance one might not be sure whether there exist any nontrivial bundles. A first example of one is given by the following.

Example 1.6. Let E denote the topological space which can be expressed as a quotient in either of the following two equivalent ways:

$$E = \frac{[0,1] \times \mathbb{R}}{(1,t) \sim (0,-t)} = \frac{\mathbb{R} \times \mathbb{R}}{(s+1,t) \sim (s,-t)}.$$

You should recognize this topological space as an (open, i.e. with no boundary) Möbius strip. Where we identify the circle S^1 with $\mathbb{R}/\mathbb{Z}$, there is a continuous map $\pi\colon E \to S^1$ defined by $\pi([s,t]) = [s]$. It is not hard to see that this is a rank-1 vector bundle over S^1. Indeed, using the second expression for E above, for any $[s_0] \in S^1$ (where $s_0 \in \mathbb{R}$), we have $\pi^{-1}([s_0]) = \{[s_0,t] | t \in \mathbb{R}\}$, which is naturally identified with $\mathbb{R}$ and so inherits the vector space structure of $\mathbb{R}$. To construct a local trivialization around s_0, let $U = \{[s] | s_0 - 1/2 < s < s_0 + 1/2\}$, and then define $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}$ by $\Phi([(s,t)]) = ([s],t)$ for $s_0 - 1/2 < s < s_0 + 1/2$. This is easily seen to be a homeomorphism (it is well-defined and injective because s is taken from an open interval of length only one) which satisfies the requirements of a local trivialization.

There are various ways of seeing that this "Möbius bundle" is not trivial. For one thing, if it were trivial then the open Möbius strip would be homeomorphic to the open cylinder, which it is not, though showing this is not exactly straightforward (but perhaps already familiar to you). There is also a somewhat easier argument (which is best phrased in terms of the language of the upcoming Definition 1.10) in which one starts from a system of local trivializations for the Möbius bundle and derives a contradiction from the way that a hypothetical isomorphism to the trivial bundle would have to act with respect to the transition functions for the local trivializations.

Example 1.7. As one learns in an introductory smooth manifolds course, for any smooth manifold M there is a naturally associated vector bundle with rank equal to $\dim M$, namely the *tangent bundle* $\pi\colon TM \to M$, whose fiber at a point $m \in M$ is the tangent space T_mM. (Thus, loosely speaking, T_mM consists of all possible velocity vectors of curves passing through m. For a review of this see any introductory text on smooth manifolds, e.g. [Lee, Chapter 3], or [U1, Section 3.1]).

Exercise 1.8. Show that $TS^1 \to S^1$ is trivial, but that the Möbius bundle over S^1 is not.

Remark 1.9. In fact, every rank-1 vector bundle over S^1 is either trivial or isomorphic to the Möbius bundle, as we should be able to prove within the next few weeks.

In general, a smooth manifold M such that TM is trivial is called *parallelizable*. The example of S^1 should not be considered representative, as most manifolds are not parallelizable—the only spheres with this property are S^0, S^1, S^3, and S^7, as was proven by Kervaire and Bott-Milnor in the late 1950s.

I'd now like to give an equivalent definition of a vector bundle, which will have some useful generalizations when we wish to speak of vector bundles carrying some additional structure.

Definition 1.10. A *(real, rank-k) vector bundle* over a topological space M is a continuous map $\pi\colon E \to M$ where E is a topological space such that there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of M and a collection of "local trivializations" $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that

(i) Each Φ_α is a homeomorphism, which for every $x \in U_\alpha$ maps $\pi^{-1}(\{x\})$ to $\{x\} \times \mathbb{R}^k$.

(ii) For each $\alpha, \beta \in A$, the map $\Phi_\beta \circ \Phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form
$$\Phi_\beta \circ \Phi_\alpha^{-1}(x,v) = (x, g_{\alpha\beta}(x)v)$$
where for every $x \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(x)\colon \mathbb{R}^k \to \mathbb{R}^k$ is a *linear* map.

This definition is clearly closely related to Definition 1.1; in particular it should be clear that a vector bundle in the sense of Definition 1.1 gives a vector bundle in the sense of Definition 1.10 (just take the open cover to consist of all the sets U in Definition 1.1 obtained for various values of $m \in M$). Conversely, given a vector bundle in the sense of Definition 1.10 and $m \in M$, one can choose α so that $x \in U_\alpha$ and then Φ_α is an obvious candidate for the local trivialization around m

required in Definition 1.1(ii). But perhaps at this point one notices a difference between the two definitions, namely that in Definition 1.1 I assumed that the fibers E_m were all vector spaces and that the local trivializations were fiberwise linear, whereas I didn't say anything about a vector space structure on the fibers in Definition 1.10. This discrepancy is resolved by the following proposition, whose proof generalizes in important ways and so should be paid close attention.

Proposition 1.11. *Given a vector bundle in the sense of Definition 1.10, for each $m \in M$ there is a unique vector space structure on the fiber $E_m = \pi^{-1}(\{m\})$ such that for every $\alpha \in A$ with $m \in U_\alpha$ the local trivialization restricts as a linear isomorphism $\Phi_\alpha|_{E_m} : E_m \to \{m\} \times \mathbb{R}^k$.*

Proof. For any $v, w \in E_m$ and $c \in \mathbb{R}$, our job is to define $v + w, cv \in E_m$ in such a way that the vector space axioms are satisfied and such that each of the Φ_α restrict to E_m as a linear isomorphism to $\mathbb{R}^k$.

If we fix an α such that $m \in U_\alpha$, there is one and only one way of trying to do this: If $v \in E_m$ we have $\Phi_\alpha(v) = (m, v_\alpha)$ for some unique $v_\alpha \in \mathbb{R}^k$, and if Φ_α is to be a linear isomorphism we *must* define addition $+_\alpha$ and scalar multiplication $\cdot_\alpha$ by

$$v +_\alpha w = \Phi_\alpha^{-1}(m, v_\alpha + w_\alpha) \qquad c \cdot_\alpha v = \Phi_\alpha^{-1}(m, cv_\alpha).$$

These are well-defined algebraic operations on E_m which inherit the vector space axioms from $\{m\} \times \mathbb{R}^k$ because we have defined the operations by forcing the bijection $\Phi_\alpha|_{E_m} : E_m \to \{m\} \times \mathbb{R}^k$ to intertwine them with the standard operations on $\{m\} \times \mathbb{R}^k$.

What remains to be checked is that these operations are independent of the choice of α with $m \in U_\alpha$; here we use a fact that we haven't used before, namely that the "transition functions" $g_{\alpha\beta}(m)$ from Definition 1.10(ii) are linear. So choose $\alpha, \beta \in A$ such that $m \in U_\alpha \cap U_\beta$; we are to show that for any $v, w \in E_m$ and $c \in \mathbb{R}$ we have $v +_\alpha w = v +_\beta w$ and $c \cdot_\alpha v = c \cdot_\beta v$.

Note first that, where as before $v_\alpha \in \mathbb{R}^k$ is defined by the property that $\Phi_\alpha(v) = (m, v_\alpha)$ and likewise for v_β, we have

$$\begin{aligned}(m, v_\beta) = \Phi_\beta(v) &= (\Phi_\beta \circ \Phi_\alpha^{-1})\big(\Phi_\alpha(v)\big) \\ &= (\Phi_\beta \circ \Phi_\alpha^{-1})(x, v_\alpha) = (x, g_{\alpha\beta}(m)v_\alpha).\end{aligned}$$

Thus $v_\beta = g_{\alpha\beta}(m)v_\alpha$, and likewise $w_\beta = g_{\alpha\beta}(m)w_\alpha$.

Therefore,

$$\begin{aligned}v +_\beta w &= \Phi_\beta^{-1}(m, v_\beta + w_\beta) = \Phi_\beta^{-1}(m, g_{\alpha\beta}(m)v_\alpha + g_{\alpha\beta}(m)w_\alpha) \\ &= \Phi_\beta^{-1}(m, g_{\alpha\beta}(m)(v_\alpha + w_\alpha)) = \Phi_\beta^{-1}\left(\Phi_\beta \circ \Phi_\alpha^{-1}(m, v_\alpha + w_\alpha)\right) = \Phi_\alpha^{-1}(m, v_\alpha + w_\alpha) \\ &= v +_\alpha w,\end{aligned}$$

where in the third equality we have used the fact that $g_{\alpha\beta}(m)$ is linear. In exactly the same way one shows that $c \cdot_\alpha v = c \cdot_\beta v$. Thus the operations $+_\alpha$ and $\cdot_\alpha$ are in fact independent of the local trivialization Φ_α used to define them, and so give well-defined vector space operations on E_m satisfying the required properties. □

This result gets at an important idea which I want to introduce now—we'll come back to it in more detail later. Note first of all that one could imagine a version of Definition 1.10 which is exactly the same except that in the final line one eliminates the requirement that the $g_{\alpha\beta}(m)$ are linear and just requires them to be homeomorphisms from $\mathbb{R}^k$ to itself. (For what it's worth, the name for the object defined in this way is a "fiber bundle with fiber $\mathbb{R}^k$.") Proposition 1.11 shows that by requiring the transition functions of a fiber bundle with fiber $\mathbb{R}^k$ to be not just arbitrary homeomorphisms but rather linear isomorphisms (in other words, to belong to the group of those homeomorphisms which preserve the vector space structure of $\mathbb{R}^k$), we obtain a natural vector space structure on the fibers.

Working now within the class of vector bundles (so that the transition functions $g_{\alpha\beta}$ are automatically linear), we could restrict the transition maps further, and ask that they preserve some additional structure on $\mathbb{R}^k$.

Definition 1.12. Let $G \leq GL(k;\mathbb{R})$ be any subgroup. A vector bundle $\pi\colon E \to M$ is said to *have structure group G* if the transition functions $g_{\alpha\beta}(m)$ from Definition 1.10(ii) all belong to the group G. More generally, we say that the structure group of E *reduces to G* if E is isomorphic to a vector bundle which has structure group G.

Exercise 1.13. Suppose that $\pi\colon E \to M$ has structure group $\{e\}$ (where e is the $k \times k$ identity matrix). Prove that E is trivial.

Many familiar subgroups G of $GL(k;\mathbb{R})$ can be characterized as consisting of linear maps which preserve some linear-algebraic structure $\mathscr{S}_G$ on $\mathbb{R}^k$. Consistently with the last few paragraphs, there is a general principle that *if E has structure group G, then the fibers of E can be naturally regarded as carrying the structure $\mathscr{S}_G$.* More specifically, exactly as in the proof of Proposition 1.11 one can use the local trivializations Φ_α to impose the structure $\mathscr{S}_G$ on the fibers, and the fact that the transition functions preserve $\mathscr{S}_G$ implies that the imposed structure is independent of which local trivialization is used.

Example 1.14. Let $G = GL^+(k;\mathbb{R}) := \{A \in GL(k;\mathbb{R}) | \det A > 0\}$. $GL^+(k;\mathbb{R})$ can be regarded as the group of linear maps which preserve the standard orientation of $\mathbb{R}^k$.[1] Thus if E has structure group $GL^+(k;\mathbb{R})$ then the fibers of E can naturally be regarded as *oriented* vector spaces, in which case we say that the bundle is oriented. (Note that, of course, regardless of the structure group, the fibers of E can all perfectly well be given orientations—the issue here is whether orientations can be assigned to the fibers in a consistent, continuous way.) One can show that the structure group of the Möbius bundle does *not* reduce to $GL^+(k;\mathbb{R})$

Example 1.15. Let $G = O(k) := \{A \in GL(k;\mathbb{R}) | (\forall v, w \in \mathbb{R}^k)((Av)\cdot(Aw) = v \cdot w)\}$ be the group of orthogonal matrices. As the above expression indicates, $O(k)$ can be described as the group of matrices which preserves the standard dot product $\cdot$ on $\mathbb{R}^k$, and so the general principle says that a bundle with structure group $O(k)$ can be given a continuous family of inner products (such a continuous family of inner products is sometimes called a Euclidean or Riemannian or orthogonal structure on E). In fact, for *any* vector bundle, at least over a reasonable base space M (*e.g.* any manifold—where as usual manifolds are required to be Hausdorff and second countable and hence paracompact—or any cell complex), the structure group reduces to $O(k)$. On one level can be seen as a consequence of the fact that $GL(k;\mathbb{R})$ deformation retracts to $O(k)$. Alternately one can use a partition of unity to first construct a continuous family of inner products on the fibers, and then construct new local trivializations with orthogonal transition maps by modifying the original local trivializations so that they send the inner product on the fibers to the standard inner product.

Example 1.16. Suppose that $k = 2n$. Then $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n = \mathbb{R}^n \times (i\mathbb{R}^n)$. We can think of "multiplication by i" in $\mathbb{C}^n$ as a linear automorphism of $\mathbb{R}^{2n}$ given in terms of

[1] There are various ways of saying what an orientation on a vector space V is; one way is to say that an orientation is equivalent to specifying a distinction between "positive" and "negative" ordered bases for V—in the standard orientation for $\mathbb{R}^3$ the bases (e_1, e_2, e_3) and (e_2, e_3, e_1) are positive while (e_2, e_1, e_3) is negative, since the right-hand rule is satisfied by the first two but violated by the third. A more sophisticated way of saying this is that an orientation of a k-dimensional vector space V is equivalent to specifying a positive direction in the one-dimensional vector space $\Lambda^k V$, or (as is convenient for working with differential forms) in $\Lambda^k V^*$.

the standard basis by the block matrix $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Now if $A \in GL(2n;\mathbb{R})$ is any linear isomorphism of $\mathbb{R}^{2n}$, if we now think of $\mathbb{R}^{2n} = \mathbb{C}^n$ as a vector space over $\mathbb{C}$ we can ask whether or not A continues to be a ($\mathbb{C}$)-linear map of $\mathbb{C}$. It is easy to see that this holds if and only if A commutes with multiplication by i. Thus inside $GL(2n;\mathbb{R})$ we have the subgroup

$$GL(n;\mathbb{C}) = \{A \in GL(2n;\mathbb{R}) | AJ_0 = J_0 A\}.$$

If E has structure group $GL(n;\mathbb{C})$, then the fibers of E can naturally be seen as *complex* vector spaces, giving E what is called a "complex structure." The construction is just as before—we just need to say how to "multiply by i" in the fibers E_m, and we do this by directly transporting the corresponding operation via the local trivializations; this is independent of which trivialization we use precisely because the transition maps are complex linear. It is rather common to find this sort of structure—the tangent bundles to complex manifolds (i.e. manifolds defined just like smooth manifolds but with the transition maps between coordinate charts required to be holomorphic maps between open subsets in $\mathbb{C}^n$) and to symplectic manifolds all have structure group which reduces to $GL(n;\mathbb{C})$.

2. Toward the classification of vector bundles over a space

Before too long, will introduce characteristic classes as a method for studying vector bundles. For the moment, we will set our sights toward a more ambitious goal, namely a complete classification of the (finite-type, at least) vector bundles over a given (paracompact) space, up to isomorphism. As occurs in most areas of mathematics, the full classification problem turns out to be somewhat unmanageable, but very satisfactory partial results can be obtained. To get at the methods underlying these partial results, we will first precisely reformulate the classification problem into a problem about homotopy classes of maps from the given space to certain auxiliary spaces (Grassmannians). Of course, although fully classifying homotopy classes of maps is typically a hard problem (for instance homotopy classes of maps between spheres are still not completely understood in spite of 80 years of intensive effort), the methods of algebraic topology provide a great deal of insight—in our case they will give us characteristic classes.

For this to work we need to impose at least a modest hypothesis on the base spaces of our fiber bundles. The following assumption is in force throughout the rest of this section:

Assumption 2.1. *The topological space M is Hausdorff and* paracompact*, or equivalently, M is Hausdorff and for every open cover $\{U_\alpha\}_{\alpha\in A}$ of X there is a continuous partition of unity subordinate to $\{U_\alpha\}$* (i.e.*, continuous functions $\chi_\alpha \colon M \to [0,1]$ such that $supp(\chi_\alpha) \subset U_\alpha$, such that each point in m has a neighborhood on which all but finitely many χ_α are identically zero, and such that $\sum_\alpha \chi_\alpha(m) = 1$ for all m).*

Note that paracompact Hausdorff spaces include all manifolds (provided that, as is customary, the definition of a manifold includes a second-countability assumption), and all CW complexes.

2.1. Pre-classifying maps.

Out of a rank-k vector bundle $\pi\colon E \to M$ we will eventually construct something called a classifying map from M to a Grassmannian (to be defined later). Before that, we introduce the following notion.

Definition 2.2. Let $\pi\colon E \to M$ be a rank-k vector bundle. A *pre-classifying map* for E is a continuous map $F\colon E \to \mathbb{R}^N$ (for some natural number N), such that for each fiber $E_m = \pi^{-1}(\{m\})$ of E the restriction $F|_{E_m}\colon E_m \to \mathbb{R}^N$ is linear and injective for all $m \in M$.

Of course, if $E = M \times \mathbb{R}^k$ were the trivial bundle, then letting $F\colon E \to \mathbb{R}^k$ be the projection onto the second factor would give a pre-classifying map. Conversely, if there exists a pre-classifying map $F\colon E \to \mathbb{R}^k$ (so that the parameters k and N in Definition 2.2 are the same) then $(\pi, F)\colon E \to M \times \mathbb{R}^k$ would give an isomorphism between E and the trivial bundle. However, by allowing N to be larger than k we can accommodate nontrivial bundles into Definition 2.2:

Example 2.3. Let $E \to S^1$ be the Möbius bundle, so we may identify $E = \frac{[0,1]\times\mathbb{R}}{(1,t)\sim(0,-t)}$. Now define

$$F\colon E \to \mathbb{R}^2$$
$$[s,t] \mapsto (\cos(\pi s)t, \sin(\pi s)t)$$

(Note that $F([1,t]) = (-t,0) = F([0,-t])$, so this map is well-defined on E). Then F is clearly a pre-classifying map. Note that the image $F(E_s)$ of the fiber over $s \in S^1$ is the line of slope $\tan(\pi s)$—in particular the image of the fiber rotates by 180 degrees as s varies through S^1. We will see later that this specific rotational behavior of the image of the fiber under the pre-classifying map can be used to distinguish the Möbius bundle from the trivial bundle.

For the simplest-to-state-and-prove results we will need to restrict to a certain class of vector bundle:

Definition 2.4. A vector bundle $\pi\colon E \to M$ is said to be of *finite type* if there is a collection of local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ as in Definition 1.10, such that $\{U_\alpha\}$ is a *finite* open cover of M.

As should be obvious, if M is compact then any vector bundle over M is of finite type. Even in the noncompact case, in my experience it is unusual to run across non-finite-type vector bundles, though of course such can be constructed. By working with maps to $\mathbb{R}^\infty$ instead of to $\mathbb{R}^N$ it is usually possible to do without finite-type assumptions, as is detailed in [MS, Chapter 5], but in any case it seems best to understand the finite-type case first.

Proposition 2.5. *If $\pi\colon E \to M$ is a finite-type vector bundle over a paracompact Hausdorff space M then for some N there exists a classifying map $F\colon E \to \mathbb{R}^N$.*

Proof. By the finite-type assumption there are open subsets $U_1, \ldots, U_T$ with $M = U_1 \cup \cdots \cup U_T$ and local trivializations $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$. We can write $\Phi_i(e) = (\pi(e), \phi_i(e))$ for some $\phi_i\colon \pi^{-1}(U_i) \to \mathbb{R}^k$ which restricts to each fiber E_m for $m \in U_i$ as a linear isomorphism.

Since M is paracompact there is a partition of unity $\{\xi_i\}_{i=1}^T$ subordinate to the open cover $\{U_i\}$. Since for each i χ_i is identically zero on an open subset containing $M \setminus U_i$, for each i the function $e \mapsto \chi_i(\pi(e))\phi_i(e)$ (initially defined as a map from $\pi^{-1}(U_i)$ to $\mathbb{R}^k$) extends by zero to give a continuous function $E \to \mathbb{R}^k$. With this continuous extension understood, let $N = kT$ and define $F\colon E \to \mathbb{R}^N$ by

$$F(e) = \big(\chi_1(\pi(e))\phi_1(e), \ldots, \chi_T(\pi(e))\phi_T(e)\big).$$

The continuity of π and the χ_i and ϕ_i obviously implies that F is continuous. For $m \in M$, $F|_{E_m}$ is given by $e \mapsto (\chi_1(m)\phi_1(e), \ldots, \chi_T(m)\phi_T(e))$, which is linear since for all i either $\chi_i(m) = 0$ or $\phi_i|_{E_m}$ is linear. If i is such that $\chi_i(m) \neq 0$ (and of course i exists since $\sum \chi_i = 1$) then if $e \in E_m$ with $F(e) = 0$ we would have $\phi_i(e) = 0$ and hence $e = 0$ since $\phi_i|_{E_m}\colon E_m \to \mathbb{R}^k$ is a linear isomorphism. Thus $F|_{E_m}$ is linear and injective for all m, and so F is a pre-classifying map. □

While pre-classifying maps $F\colon E \to \mathbb{R}^N$ exist for all finite-type vector bundles E over M (where as always we assume M is paracompact Hausdorff), they are clearly not unique: for example if $A \in GL(N;\mathbb{R})$ we could obviously replace F by $A \circ F$. Also, the number N is not uniquely determined. In particular, if $N < N'$ we have an obvious "stabilization" map

$$\begin{aligned} I_{N,N'} \colon \mathbb{R}^N &\to \mathbb{R}^{N'} \\ \vec{x} &\mapsto (\vec{x}, \vec{0}), \end{aligned}$$

and if $F \colon E \to \mathbb{R}^N$ is a pre-classifying map for E then clearly so is $I_{N,N'} \circ F \colon E \to \mathbb{R}^{N'}$.

Definition 2.6.

- If $F_0, F_1 \colon E \to \mathbb{R}^N$ are two pre-classifying maps, we say that F_0 and F_1 are *isotopic* if they are homotopic through pre-classifying maps, *i.e.* if there is a continuous map $F \colon [0,1] \times E \to \mathbb{R}^N$ such that for all t the map $e \mapsto F(t,e)$ is a pre-classifying map for E, and such that for $i = 0,1$ the given pre-classifying maps F_i are given by $F_i(m) = F(i,m)$.
- If $F_0 \colon E \to \mathbb{R}^{N_0}$ and $F_1 \colon E \to \mathbb{R}^{N_1}$ are two pre-classifying maps, we say that F_0 and F_1 are *stably isotopic* if there is $N' \geq \max\{N_0, N_1\}$ such that the pre-classifying maps $I_{N_0,N'} \circ F_0$ and $I_{N_1,N'} \circ F_1$ are isotopic.

It should be clear that both isotopy and stable isotopy are equivalence relations.

Exercise 2.7. Give, with proof, an example of a pair of pre-classifying maps $F_0, F_1 \colon E \to \mathbb{R}^N$ for the same vector bundle E (and with the same value of N) which are not isotopic.

Theorem 2.8. *Let $F_0 \colon E \to \mathbb{R}^{N_0}$ and $F_1 \colon E \to \mathbb{R}^{N_1}$ be two pre-classifying maps for the same vector bundle. Then F_0 and F_1 are stably isotopic.*

Proof. Without loss of generality let us assume that $N_0 \geq N_1$. Then let $N' = N_0 + N_1$. First of all note that F_1 is stably isotopic to the map $F_1' \colon E \to \mathbb{R}^{N'}$ defined by $F_1'(e) = (\vec{0}_{N_0}, F_1(e))$ (where $\vec{0}_{N_0}$ denotes the 0-vector in $\mathbb{R}^{N_0}$; we will use similar notation below). Indeed,

$$(t,e) \mapsto \big((1-t)F_1(e), \vec{0}_{N_0-N_1}, tF_1(e)\big)$$

gives an isotopy from $I_{N_1,N'} \circ F_1$ to F_1'. But the map $F' \colon [0,1] \times E \to \mathbb{R}^{N'}$ defined by $F'(e) = ((1-t)F_0(e), tF_1(e))$ gives an isotopy from $I_{N_0,N'} \circ F_0$ to F_1'. Thus (by the transitivity of the isotopy relation) $I_{N_0,N'} \circ F_0$ is isotopic to $I_{N_1,N'} \circ F_1$, and so F_0 and F_1 are stably isotopic. □

Summing up, any finite-type vector bundle over M has pre-classifying maps, and any two such maps (for a given vector bundle) are stably isotopic. This begins to suggest an approach to distinguishing one vector bundle from another: we should construct pre-classifying maps for both of them and then (somehow) show that these are *not* stably isotopic. (It might be instructive to try to do this for the Möbius and trivial bundles over S^1.) To do this in general will require us to introduce Grassmannians and classifying maps.

2.2. **The Grassmannian and the tautological vector bundle.** As was already sort of suggested in Example 2.3, the real information in a pre-classifying map $F \colon E \to \mathbb{R}^N$ lies in the way that the k-dimensional subspaces $F(E_m)$ vary with m. Accordingly, we introduce the following auxiliary set:

Definition 2.9. The *Grassmannian* of k-planes in $\mathbb{R}^N$ is the set

$$Gr_k(\mathbb{R}^N) = \{V \leq \mathbb{R}^N \mid \dim V = k\}.$$

Then any pre-classifying map $F\colon E \to \mathbb{R}^N$, where E is a rank-k vector bundle over M, gives rise to a function

$$\begin{aligned} f\colon M &\to Gr_k(\mathbb{R}^N) \\ m &\mapsto F(E_m) \end{aligned}$$

This map f is an example of what we will later call a classifying map, though our formal definition will be a bit different.

The plan is to study the map f using the methods of algebraic topology; one obvious prerequisite to doing this is making the Grassmannian $Gr_k(\mathbb{R}^N)$ into a topological space rather than just a set. There are various equivalent definitions for a topology on $Gr_k(\mathbb{R}^N)$ in the literature; for definiteness we will use the following. First introduce the *Stiefel manifold*

$$\tilde{V}_k(\mathbb{R}^N) = \left\{(\vec{v}_1, \ldots, \vec{v}_k) \in (\mathbb{R}^N)^k \,\middle|\, \{\vec{v}_1, \ldots, \vec{v}_k\} \text{ is a linearly independent set}\right\}.$$

Of course $\tilde{V}_k(\mathbb{R}^N)$ has a natural topology, as it is an open subset of $(\mathbb{R}^N)^k$. Moreover there is an obvious map $p\colon \tilde{V}_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ defined by setting $p(\vec{v}_1, \ldots, \vec{v}_k) = span\{\vec{v}_1, \ldots, \vec{v}_k\}$. We define the topology on $Gr_k(\mathbb{R}^N)$ by requiring p to be a quotient map, *i.e.* by saying that $U \subset Gr_k(\mathbb{R}^N)$ is open if and only if $p^{-1}(U)$ is open.

Exercise 2.10. In the literature it is somewhat more common to see the Stiefel manifold defined as the subset of $\tilde{V}_k(\mathbb{R}^N)$ given by

$$V_k(\mathbb{R}^N) = \left\{(\vec{v}_1, \ldots, \vec{v}_k) \in (\mathbb{R}^N)^k \,\middle|\, \vec{v}_i \cdot \vec{v}_j = \delta_{ij}\right\}$$

(*i.e.* as the set of *orthonormal* k-tuples of vectors in $\mathbb{R}^N$). Prove that we would obtain the same topology on $Gr_k(\mathbb{R}^N)$ as in the previous paragraph if we instead defined the topology by requiring $p|_{V_k(\mathbb{R}^N)}$ to be a quotient map.

Note that since $V_k(\mathbb{R}^N)$ is compact (it is a closed subset of $(S^{N-1})^k$), Exercise 2.10 shows that $Gr_k(\mathbb{R}^N)$ is compact. One can show that $Gr_k(\mathbb{R}^N)$ is a smooth manifold (indeed this is a popular example in differential topology textbooks, see *e.g.* [Lee, Chapter 1]), but at least for now we won't need this fact.

Exercise 2.11. Let $F\colon E \to \mathbb{R}^N$ be a pre-classifying map for a rank-k vector bundle $\pi\colon E \to M$, and define $f\colon M \to Gr_k(\mathbb{R}^N)$ by $f(m) = F(E_m)$. Prove that f is continuous. (It's probably slightly easier to do this using our initial definition for the topology on $Gr_k(\mathbb{R}^N)$ rather than the one from Exercise 2.10.)

There is a very important vector bundle over $Gr_k(\mathbb{R}^N)$: the *tautological bundle*[2] $\gamma^k(\mathbb{R}^N)$. We define

$$\gamma^k(\mathbb{R}^N) = \left\{(V, \vec{v}) \in Gr_k(\mathbb{R}^N) \times \mathbb{R}^N | \vec{x} \in V\right\},$$

with the bundle projection $\pi\colon \gamma^k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ defined by projection onto the first factor. Thus for $V \in Gr_k(\mathbb{R}^N)$, (*i.e.* for V a k-dimensional subspace of $\mathbb{R}^N$), $\pi^{-1}(V) = \{V\} \times V$—the fiber over V is (basically) the subspace V itself.

Proposition 2.12. *$\pi\colon \gamma^k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ is a rank-k vector bundle.*

Proof. The topology on $\gamma^k(\mathbb{R}^N)$ is the subspace topology coming from the product topology on $Gr_k(\mathbb{R}^N) \times \mathbb{R}^N$ (where we topologize $Gr_k(\mathbb{R}^N)$ as before). So the projection $\pi\colon \gamma^k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ is continuous, and its fibers have obvious vector space structures. It remains to construct local trivializations.

[2] In [MS] and some other references this is called the "canonical bundle." I prefer to avoid this name in order to avoid conflict with some terminology from algebraic geometry.

Endow $\mathbb{R}^N$ with its standard inner product, and let $V_0 \in Gr_k(\mathbb{R}^N)$; we will construct a local trivialization around V_0. The open set U over which the trivialization is defined will be

$$U = \{V \in Gr_k(\mathbb{R}^N) | V \cap V_0^\perp = \{0\}\}.$$

We should first check that U is open—by the definition of the quotient topology this is equivalent to the statement that $p^{-1}(U)$ is open where $p\colon \tilde{V}_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ is the projection. Now if $\{\vec{v}_1^\perp, \ldots, \vec{v}_{N-k}^\perp\}$ is an orthonormal basis for $V_0^\perp$, then an element $(\vec{w}_1, \ldots, \vec{w}_k)$ belongs to $p^{-1}(U)$ if and only if $\{\vec{v}_1^\perp, \ldots, \vec{v}_{N-k}^\perp, \vec{w}_1, \ldots, \vec{w}_k\}$ is a basis for $\mathbb{R}^N$, *i.e.* if and only if the $N \times N$ matrix whose columns are $\vec{v}_1^\perp, \ldots, \vec{v}_{N-k}^\perp, \vec{w}_1, \ldots, \vec{w}_k$ has nonzero determinant. This is clearly an open condition on $(\vec{w}_1, \ldots, \vec{w}_k)$, proving that $p^{-1}(U) \subset \tilde{V}_k(\mathbb{R}^N)$ is open, and hence that $U \subset Gr_k(\mathbb{R}^N)$ is open.

We now construct a local trvialization for $\gamma^k(\mathbb{R}^N)$ over U. Choose an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for V_0. Then where $\Pi\colon \mathbb{R}^N \to V_0$ is the orthogonal projection, Π is given by the formula $\Pi(\vec{v}) = \sum_{i=1}^k (\vec{v} \cdot \vec{v}_i)\vec{v}_i$, and since $\ker \Pi = V_0^\perp$, for any $V \in U$ the restriction $\Pi|_V\colon V \to V_0$ is an isomorphism. Define

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$$

by

$$\Phi(V, \vec{v}) = (V, \vec{v} \cdot \vec{v}_1, \ldots, \vec{v} \cdot \vec{v}_k).$$

This is clearly continuous and fiberwise linear, and for each $V \in U$ restricts to $\pi^{-1}(\{V\})$ as a linear isomorphism since $\Pi|_V$ is a linear isomorphism. To complete the proof we need to show that $\Phi^{-1}\colon U \times \mathbb{R}^k \to \pi^{-1}(U)$ is continuous. Now $\Phi^{-1}(V, c_1, \ldots, c_k) = \left(V, \sum_{i=1}^k c_i (\Pi|_V)^{-1}(\vec{v}_i)\right)$, so the desired continuity will follow if we show that, for each i, the map $\phi_i\colon U \to \mathbb{R}^N$ defined by $\phi_i(V) = (\Pi|_V)^{-1}(v_i)$ is continuous.

Now where as before $p\colon \tilde{V}_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ is the quotient projection, let $\tilde{\phi}_i = \phi_i \circ p$. It follows from the definition of the quotient topology that ϕ_i is continuous if and only if $\tilde{\phi}_i\colon p^{-1}(U) \to \mathbb{R}^N$ is continuous. Denote a general element of $p^{-1}(U)$ by $W = (\vec{w}_1, \ldots, \vec{w}_k)$. Let $A(W)$ denote the $k \times k$ matrix representing $\Pi|_{p(W)}\colon p(W) \to V_0$ in terms of the basis $\{\vec{w}_1, \ldots, \vec{w}_k\}$ for W and our fixed orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for V_0. Then $A(W)_{ij} = \vec{v}_i \cdot \vec{w}_j$, and so the map $W \mapsto A(W)$ is continuous. Moreover $A(W)$ is invertible for all $W \in p^{-1}(U)$, so since inversion is a continuous operation on invertible matrices it follows that $W \mapsto A(W)^{-1}$ is continuous as a function of W. We then have, where $\{\vec{v}_1^\perp, \ldots, \vec{v}_{N-k}^\perp\}$ is the orthonormal basis for $V_0^\perp$ from earlier,

$$\begin{aligned}\tilde{\phi}_i(W) = (\Pi|_{p(W)})^{-1}(v_i) &= \sum_{j=1}^k (A(W)^{-1})_{ji}\vec{w}_j \\ &= \vec{v}_i + \sum_{j=1}^k (A(W)^{-1})_{ji} \sum_{m=1}^{N-k} (\vec{w}_j \cdot \vec{v}_m^\perp)\vec{v}_m^\perp,\end{aligned}$$

where in the last line we have split up $\sum_{j=1}^k (A(W)^{-1})_{ji}\vec{w}_j$ into its components with respect to the orthogonal direct sum $\mathbb{R}^N = V_0 \oplus V_0^\perp$. This formula for $\tilde{\phi}_i(W)$ makes clear that $\tilde{\phi}_i$ is continuous as a function of W, and hence that Φ^{-1} is continuous, completing the proof. □

2.3. **Pullbacks and the main classification result.** Here we describe a general construction which in particular will serve to link the constructions from the previous two sections. Let $\pi\colon V \to Y$ be a vector bundle, and let $f\colon X \to Y$ be a continuous map. We then define, first as a toplogical space,

$$f^*V = \left\{(x, v) \in X \times V | \pi(v) = f(x)\right\}$$

where of course the topology is given by the subspace topology induced by the product topology on $X \times V$. Using π_V and π_X to denote the projections from (subspaces of) $X \times V$ to V and X, we have a commutative diagram

$$\begin{tikzcd} f^*V \arrow[r, "\pi_V"] \arrow[d, "\pi_X"'] & V \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} ,$$

where all maps involved are continuous.

The fiber $(\pi_X)^{-1}(\{x\})$ of f^*V over a point $x \in X$ is just $\{x\} \times V_{f(x)}$; thus the fibers of $f^*V \to X$ inherit vector space structures from those of $\pi\colon V \to Y$. $f^*V \to X$ also inherits local trivializations from $V \to Y$: if $W \subset Y$ is an open set and if $\Phi\colon \pi^{-1}(W) \to W \times \mathbb{R}^k$ is a local trivialization, say given by $\Phi(v) = (\pi(v), \phi(v))$, then we obtain a local trivialization for f^*V over $\pi_X^{-1}(f^{-1}(W))$ by the formula $\Psi((x, v)) = (x, \phi(v))$. Thus f^*V is a vector bundle. Since if $\{W_\alpha\}_{\alpha\in A}$ is an open cover of Y then $\{f^{-1}(W_\alpha)\}_{\alpha\in A}$ is a cover of X, clearly f^*V will be of finite type if V is of finite type. Moreover the transition maps $g_{\alpha\beta}$ for a collection of local trivializations for V pull back in an obvious way to transition maps for a collection of local trivializations for π^*V, in view of which if V has structure group $G \leq GL(k;\mathbb{R})$ then so does f^*V. In particular if V is trivial then so is f^*V.

As a special case, if $A \subset Y$ is some subset (viewed as a topological space with the subspace topology), then where $i\colon A \to Y$ is the inclusion for any vector bundle $\pi\colon V \to Y$ we can form the bundle $i^*V \to A$. This is the same as (or, strictly speaking, isomorphic to by a rather obvious isomorphism) the "restriction of V to A," sometimes written $V|_A$ and defined simply by setting $V|_A = \pi^{-1}(A)$ and having the bundle projection be the restriction $\pi|_{\pi^{-1}(A)}\colon \pi^{-1}(A) \to A$.

In particular one could have A be a single point. Of course every vector bundle over a point is trivial, so i^*V would in this case be trivial, though of course the original vector bundle $V \to Y$ typically would not be. This shows that the implications two paragraphs above (*e.g.* that if V is trivial then so is f^*V) usually cannot be reversed—in general, the bundle f^*V is "more trivial" than V.

Proposition 2.13. *Given a pair of continuous maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *and a vector bundle* $\pi\colon V \to Z$ *there is an isomorphism of vector bundles*

$$f^*(g^*V) \cong (g \circ f)^*V.$$

Proof. We have

$$(g \circ f)^*V = \{(x, v) \in X \times V | g(f(x)) = \pi(V)\}$$

while

$$\begin{aligned} f^*(g^*V) &= \{(w, x) \in (g^*V) \times X | f(x) = \pi_Y(w)\} \\ &= \big\{\big(x, (y, v)\big) \in X \times (Y \times V) | g(y) = \pi(v), f(x) = y\big\} \\ &= \big\{\big(x, (f(x), v)\big) | g(f(x)) = \pi(v)\big\}. \end{aligned}$$

The map $(x, v) \mapsto (x, (f(x), v))$ is then obviously a bundle isomorphism from $(g \circ f)^*V$ to $f^*(g^*V)$. □

Remark 2.14. In categorical language, Proposition 2.13 shows that there is a contravariant functor $Vect_k$ from the category of topological spaces to the category of sets which, for any topological space X, has $Vect_k(X)$ equal to the set of isomorphism classes of rank-k vector bundles. To any continuous function $f\colon X \to Y$ (*i.e.* any morphism in) this functor assigns the function $f^*\colon Vect_k(Y) \to Vect_k(X)$ defined by $[E] \mapsto [f^*E]$ where we denote the isomorphism class of a vector bundle E by $[E]$.

We now return to the topic of pre-classifying maps. Let $\pi\colon E \to X$ be a rank-k vector bundle. Recall that to any pre-classifying map $F\colon E \to \mathbb{R}^N$ we may associate the map $f : M \to Gr_k(\mathbb{R}^N)$ defined by $f(m) = F(E_m)$; for the moment we will call this map f the "induced map" of F. By Exercise 2.11, f is continuous.

Proposition 2.15. *If $F\colon E \to \mathbb{R}^N$ is a pre-classifying map with induced map $f : M \to Gr_k(\mathbb{R}^N)$ then E is isomorphic to $f^*\gamma^k(\mathbb{R}^N)$. Conversely, if $f : M \to Gr_k(\mathbb{R}^N)$ is a continuous map such that E is isomorphic to $f^*\gamma^k(\mathbb{R}^N)$, then there is a pre-classifying map $F\colon E \to \mathbb{R}^N$ whose induced map is f.*

Proof. For a continuous map $f : M \to Gr_k(\mathbb{R}^N)$, we have

$$f^*\gamma^k(\mathbb{R}^N) = \left\{(m,(V,\vec{v})) \in M \times (Gr_k(\mathbb{R}^N) \times \mathbb{R}^N) | \vec{v} \in V, f(m) = V\right\}.$$

In particular the fiber of $f^*\gamma^k(\mathbb{R}^N)$ over $m \in M$ consists of all points of the form $(m,(f(m),\vec{v}))$ where $\vec{v} \in f(m)$.

Suppose that f is the induced map of some pre-classifying map $F\colon E \to \mathbb{R}^N$, where $\pi\colon E \to M$ is a rank-k vector bundle. Define $G\colon E \to f^*\gamma^k(\mathbb{R}^N)$ by

$$G(e) = \big(\pi(e),(f(\pi(e)),F(e))\big).$$

Evidently G is continuous, and since by definition F maps the fiber E_m by a linear isomorphism to $f(m)$ for each $m \in M$, we see that G restricts to each E_m as a linear isomorphism to $\big(f^*\gamma^k(\mathbb{R}^N)\big)_m$. So by Exercise 1.5, G is an isomorphism.

Conversely, suppose that $G\colon E \to f^*\gamma^k(\mathbb{R}^N)$ is an isomorphism of bundles over M. In view of the formula for $f^*\gamma^k(\mathbb{R}^N)$ above, G necessarily takes the form

$$G(e) = (\pi(e),(f(\pi(e)),F(e)))$$

for some continuous map $F\colon E \to \mathbb{R}^N$. For each $m \in M$, G restricts to E_m as the map $e \mapsto (m,(f(m)),F(e))$, and the fact that this is an isomorphism to $\big(f^*\gamma^k(\mathbb{R}^N)\big)_m$ amounts to the statement that F maps E_m isomorphically to the k-dimensional subspace $f(m) \le \mathbb{R}^N$. But this precisely means that $F\colon E \to \mathbb{R}^N$ is a pre-classifying map for E. □

This motivates the following important definition:

Definition 2.16. Let $\pi\colon E \to M$ be a rank-k vector bundle. A *classifying map* for E is a continuous map $f : M \to Gr_k(\mathbb{R}^N)$ (for some N) such that E is isomorphic to $f^*\gamma^k(\mathbb{R}^N)$.

Corollary 2.17. *If E is a finite-type rank-k vector bundle over a paracompact Hausdorff space M then there exists a classifying map for E.*

Proof. There is a pre-classifying map for E by Proposition 2.5, and the induced map of this pre-classifying map is a classifying map by Proposition 2.15. □

Remark 2.18. It also follows from Proposition 2.15 that, conversely, if E admits a (pre-)classifying map then E must be finite-type, since $\gamma^k(\mathbb{R}^N)$ is finite-type and pullbacks of finite-type vector bundles are finite-type.

We now address the uniqueness of classifying maps; the situation quite closely corresponds to that for pre-classifying maps as described in Theorem 2.8. Corresponding to the stabilization maps $I_{N,N'}\colon \mathbb{R}^N \to \mathbb{R}^{N'}$ for $N \le N'$ discussed earlier (recall that these are simply given by $\vec{x} \mapsto (\vec{x}, \vec{0})$), we have a map

$$\begin{aligned} i_{N,N'}\colon Gr_k(\mathbb{R}^N) &\to Gr_k(\mathbb{R}^{N'}) \\ V &\mapsto V \times \{\vec{0}\}. \end{aligned}$$

It is trivial to check that $i_{N,N'}$ is continuous. Moreover, if $F\colon E \to \mathbb{R}^N$ is a pre-classifying map for E which induces the map $f\colon M \to Gr_k(\mathbb{R}^N)$, it follows directly from the definitions that the pre-classifying map $I_{N,N'} \circ F\colon E \to \mathbb{R}^{N'}$ induces the map $i_{N,N'} \circ F\colon M \to Gr_k(\mathbb{R}^{N'})$.

Proposition 2.19. *For $N' \ge N$ there is an isomorphism of vector bundles $\gamma^k(\mathbb{R}^N) \cong i^*_{N,N'}\gamma^k(\mathbb{R}^{N'})$.*

Proof. It's not hard to check this directly from the definition of the pullback. Alternately, we can observe that the map $F\colon \gamma^k(\mathbb{R}^N) \to \mathbb{R}^{N'}$ defined by $F(V, \vec{v}) = I_{N,N'}(\vec{v})$ is a pre-classifying map for $\gamma^k(\mathbb{R}^N)$ which induces the map $i_{N,N'}\colon Gr_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^{N'})$, and so the proposition follows from Proposition 2.15. □

Analogously to the relation of stable isotopy for pre-classifying maps defined before Theorem 2.8, we make the following definition.

Definition 2.20. Two maps $f_0\colon M \to Gr_k(\mathbb{R}^{N_0})$ and $f_1\colon Gr_k(\mathbb{R}^{N_1})$ are *stably homotopic* if there is $N' \ge \max\{N_0, N_1\}$ such that $i_{N_0,N'} \circ f_0$ is homotopic to $i_{N_1,N'} \circ f_1$.

Theorem 2.21. *Let M be a paracompact Hausdorff space. Then there is a well-defined, surjective map*

$$\Phi\colon \left\{ \begin{array}{c} \textit{Isomorphism classes of finite-type,} \\ \textit{rank-}k\textit{ vector bundles over } M \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Stable homotopy classes of} \\ \textit{maps } f\colon M \to Gr_k(\mathbb{R}^N) \end{array} \right\}$$

given by, where $[E]$ denotes the isomorphism class of a vector bundle E, setting $\Phi([E])$ equal to the stable homotopy class of any classifying map for E.

Proof. Any finite-type rank-k vector bundle admits a classifying map by Corollary 2.15. If f_0, f_1 are two classifying maps for E, then by the second half of Proposition 2.15 there exist pre-classifying maps $F_0\colon E \to \mathbb{R}^{N_0}$ and $F_1\colon E \to \mathbb{R}^{N_1}$ inducing f_0 and f_1, respectively. By Theorem 2.8 there is $N' \ge \max\{N_0, N_1\}$ and an isotopy $F'\colon [0,1] \times E \to \mathbb{R}^{N'}$ from $I_{N_0,N'} \circ F_0$ to $I_{N_1,N'} \circ F_1$. Writing $F'_t = F'(t, \cdot)$ (so by the definition of isotopy each F'_t is a pre-classifying map), for $t \in [0,1]$ let $f'_t\colon M \to Gr_k(\mathbb{R}^{N'})$ be the map induced by F'_t, and define $f\colon [0,1] \times M \to Gr_k(\mathbb{R}^N)$. (It is straightforward to see that f is continuous, by the same method as in Exercise 2.11). Then $f'(0, \cdot) = i_{N_0,N'} \circ f_0$ and $f'(1, \cdot) = i_{N_1,N'} \circ f_1$. Thus $i_{N_0,N'} \circ f_0$ and $i_{N_1,N'} \circ f_1$ are homotopic. This proves that the stable homotopy class of a classifying map for a finite-type vector bundle depends only on the vector bundle. Moreover, it is obvious from Definition 2.16 that isomorphic vector bundles have the same (sets of) classifying maps. This completes the proof that Φ is well-defined.

If $f\colon M \to Gr_k(\mathbb{R}^N)$ is any continuous map, $[f^*\gamma^k(\mathbb{R}^N)]$ is sent by Φ to the stable homotopy class of f. Thus Φ is surjective. □

This suffices to justify the following strategy for distinguishing vector bundles from each other: construct classifying maps for each of them, and prove that the classifying maps are not stably homotopic. Here is a simple example:

Example 2.22. Let E be the Möbius bundle over S^1, and let $E' = S^1 \times \mathbb{R}$ be the trivial rank-1 vector bundle. Note that since we are dealing here with rank-1 bundles the relevant Grassmannians

$Gr_k(\mathbb{R}^N)$ are already familiar to you from algebraic topology: they are the real projective spaces $\mathbb{R}P^{N-1}$.

The trivial bundle $E' = S^1 \times \mathbb{R}$ admits the projection to the second factor as a pre-classifying map. The induced classifying map is then the constant map to $\mathbb{R}P^0$ (which stabilizes under the maps $i_{1,N}$ to constant maps to $\mathbb{R}P^{N-1}$).

Meanwhile in Example 2.3 we constructed a pre-classifying map $F\colon E \to \mathbb{R}^2$ for the Möbius bundle, given by the formula $F([s,t]) = (\cos(\pi s)t, \sin(\pi s)t)$ (for $0 \le s \le 1$). The induced classifying map is then the map $f : S^1 \to \mathbb{R}P^1$ defined by $f([s]) = [\cos(\pi s) : \sin(\pi s)]$ where we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and use standard homogeneous coordinates on $\mathbb{R}P^1$. So to prove the nontriviality of the Möbius bundle it suffices to show that f is not stably homotopic to the constant map, *i.e.* that for all $N \ge 2$ the map $i_{2,N} \circ f : S^1 \to \mathbb{R}P^{N-1}$ given by $i_{2,N} \circ f([s]) = [\cos(\pi s) : \sin(\pi s) : 0 : \cdots : 0]$ is not homotopic to a constant.

In fact, as you likely learned in algebraic topology, the above map $i_{2,N} \circ f$ represents a generator for the fundamental group of $\pi_1(\mathbb{R}P^{N-1})$. Viewing $\mathbb{R}P^{N-1}$ as the quotient of S^{N-1} by the antipodal map, so that we have a two-to-one cover $q\colon S^{N-1} \to \mathbb{R}P^{N-1}$, if $i_{2,N} \circ f$ were homotopic to a constant then it would lift to a closed curve in S^{N-1}, whereas in fact it lifts to the non-closed path $s \mapsto (\cos(\pi s), \sin(\pi s), 0, \dots, 0)$.

Thus the classifying map f for the Möbius bundle is not stably homotopic to a constant, demonstrating that Theorem 2.21 can be used to distinguish the Möbius bundle from the trivial bundle.

In fact, at least in principle, Theorem 2.21 could be used to distinguish *any* two nontrivial bundles, as the following shows.

Theorem 2.23. *The map Φ from Theorem 2.21 is injective. Thus any finite-type rank-k vector bundle over a paracompact Hausdorff space M has a classifying map $f : M \to Gr_k(\mathbb{R}^N)$ (and so the bundle is isomorphic to $f^*\gamma^k(\mathbb{R}^N)$), and two such bundles are isomorphic* if and only if *their classifying maps are stably homotopic.*

Proof. This quickly follows from the following fundamental result, whose proof will be delayed to Section 2.4.

Theorem 2.24. *Let X be a paracompact Hausdorff space, $\pi\colon E \to Y$ a vector bundle, and $h_0, h_1\colon X \to Y$ two homotopic continuous maps. Then h_0^*E is isomorphic to h_1^*V.*

To prove Theorem 2.23 assuming Theorem 2.24, we must show that if $E_0, E_1 \to M$ are vector bundles with stably homotopic classifying maps, then E_0 and E_1 are isomorphic. Let $f_0\colon M \to \mathbb{R}^{N_0}$ and $f_1\colon M \to \mathbb{R}^{N_1}$ be classifying maps for E_0 and E_1 respectively, and choose $N' \ge \max\{N_0, N_1\}$ so that $i_{N_0,N'} \circ f_0$ and $i_{N_1,N'} \circ f_1$ are homotopic. Then using Propositions 2.19 and 2.15 and Theorem 2.24 we obtain the following chain of isomorphisms:

$$\begin{aligned} E_0 &\cong f_0^*\gamma^k(\mathbb{R}^{N_0}) \cong f_0^* i_{N_0,N'}^* \gamma^k(\mathbb{R}^{N'}) \cong (i_{N_0,N'} \circ f_0)^*\gamma^k(\mathbb{R}^{N'}) \cong (i_{N_1,N'} \circ f_1)^*\gamma^k(\mathbb{R}^{N'}) \\ &\cong f_1^* i_{N_1,N'}^* \gamma^k(\mathbb{R}^{N'}) \cong f_1^*\gamma^k(\mathbb{R}^{N_1}) \cong E_1. \end{aligned}$$

□

2.4. **Homotopy invariance of pullbacks.**

In this section we will complete the reformulation of the classification problem for finite-type vector bundles over a paracompact Hausdorff space by proving Theorem 2.24, which asserts that if $h_0, h_1\colon X \to Y$ are homotopic maps where X is paracompact and Hausdorff, then for any vector bundle E over Y the pullbacks h_0^*E and h_1^*E are isomorphic vector bundles over X.

Let $H: [0,1]\times X \to Y$ be a homotopy from h_0 to h_1, and for $t \in [0,1]$ define $i_t: X \to [0,1]\times X$ by $i_t(x) = (t,x)$. In particular $h_0 = H \circ i_0$ and $h_1 = H \circ i_1$. So we have a vector bundle $H^*E \to [0,1]\times X$ and, in view of Proposition 2.13, the theorem amounts to the statement that $i_0^*(H^*E)$ is isomorphic to $i_1^*(H^*E)$. (In effect, by replacing E with H^*E we have reduced to the case where $Y = [0,1]\times X$ and h_0, h_1 are the inclusions i_0, i_1.)

First we prove:

Lemma 2.25. *For any $x \in X$ there is a neighborhood U_x such that the restriction $(H^*E)|_{[0,1]\times U_x}$ is trivial.*

(As usual, the restriction $(H^*E)|_{[0,1]\times U_x}$ means the pullback of H^*E by the inclusion of $[0,1]\times U_x$ into $[0,1]\times X$—in other words if $\pi_{H^*E}: H^*E \to [0,1]\times X$ is the bundle projection, the total space of $(H^*E)|_{[0,1]\times U_x}$ is just $\pi_{H^*E}^{-1}([0,1]\times U_x)$ and the bundle projection is the restriction of π_{H^*E} to this total space.)

Proof. By the definition of a vector bundle and of the product topology, for any $s \in [0,1]$ there is an interval I_s around s (with I_s relatively open in $[0,1]$), a neighborhood U^s of x in X, and a local trivialization $\Phi_s: (H^*E)|_{I_s\times U^s} \to (I_s\times U^s)\times\mathbb{R}^k$. The relatively open intervals I_s cover $[0,1]$, so by the compactness of $[0,1]$ there are $s_1,\ldots,s_m$ such that $[0,1] = I_{s_1}\cup\cdots\cup I_{s_m}$. Without loss of generality we may assume that none of the I_{s_i} is contained in another I_{s_j}, and that the s_i are ordered so that the left endpoints of the I_{s_i} are in increasing order. So necessarily $0 \in I_{s_1}$ and $1 \in I_{s_m}$, and $I_{s_i}\cap I_{s_{i+1}} = \varnothing$ for each i. If for $i = 1,\ldots,m-1$ we choose $t_i \in I_{s_i}\cap I_{s_{i+1}}$, and set $t_0 = 0$ and $t_m = 1$, then for each $i = 1,\ldots,m$ we will have $[t_{i-1},t_i] \in I_{s_i}$, with the intervals $[t_{i-1},t_i]$ covering $[0,1]$.

Let $U = U^{s_1}\cap\cdots\cap U^{s_m}$, so U is an open neighborhood of x in X. For $i = 1,\ldots,m$, by restricting the local trivialization Φ_{s_i} from earlier, we obtain a trivialization (still denoted Φ_{s_i}) of $(H^*E)|_{[t_{i-1},t_i]\times U}$. For each $i = 1,\ldots,m-1$, we have a map $\Psi_i: \{t_i\}\times U \to \mathbb{R}^k \to \{t_i\}\times U\times\mathbb{R}^k$ given by $\Psi = \Phi_{s_i}\circ\Phi_{s_{i+1}}^{-1}$. Ψ_i is a homeomorphism which is compatible with the projection to $\{t_i\}\times U$ and with the vector space structure on the fibers, so it has the form $\Psi_i(t_i,z,v) = (t_i,z,\psi_i(z)v)$ for some continuous map $\psi_i: U \to GL(k;\mathbb{R})$.

We can now define a trivialization of H^*E over $[0,1]\times U$. Let us write the local trivializations $\Phi_{s_i}: (H^*E)|_{[t_{i-1},t_i]\times U} \to ([t_{i-1},t_i]\times U)\times\mathbb{R}^k$ as $\Phi_{s_i}(e) = (\pi_{H^*E}(e),\phi_i(e))$ where $\phi_i: (H^*E)|_{[t_{i-1},t_i]\times U} \to \mathbb{R}^k$ restricts to each fiber as a linear isomorphism.

Then define $\Phi: (H^*E)|_{[0,1]\times U} \to [0,1]\times U\times\mathbb{R}^k$ by declaring its restriction to $(H^*E)_{(t,z)}$ where $z \in U$ and $t \in [t_{i-1},t_i]$ to be given by $e \mapsto (\pi_{H^*E}(e),\psi_1(z)\cdots\psi_{i-1}(z)\phi_i(e))$ (for $i = 1$ this should be interpreted as just $e \mapsto (\pi_{H^*E}(e),\phi_1(e))$). While this appears to be multiply defined at the values t_j (as there is a prescription both for $i = j$ and $i = j+1$), the definition of the ψ_i ensures that both prescriptions give the same value, in view of which it easily follows that Φ is a trivialization of H^*E over $[0,1]\times U$. □

The following would be obvious under rather modest hypotheses on X, *e.g.* if we assumed that X was Lindelöf, but is needed for the general paracompact Hausdorff case.

Lemma 2.26. *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a paracompact Hausdorff space X. Then there is a countable open cover $\{V_n\}_{n=1}^\infty$ of X such that each V_n is a disjoint union of sets each of which is contained in some U_α.*

Proof. The lemma is obvious if A is finite (or indeed even if A is countable) so we assume that A is infinite. Let $\{\chi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to $\{U_\alpha\}$ (and recall that partitions of unity are by definition locally finite). For each nonempty finite subset $S \subset A$ let

$$W_S = \{x \in X | (\alpha \in S,\ \beta \notin S \Rightarrow \chi_\alpha(x) > \chi_\beta(x))\}.$$

By the local finiteness condition, for each $x \in X$ there is a neighborhood U of X and a finite subset $T \subset A$ such that $\phi_\gamma|_U = 0$ unless $\gamma \in U$. Without loss of generality (by adding extraneous elements of A to T) we may assume that $T \setminus S \neq \varnothing$. In this case $W_S \cap U = \varnothing$ unless $S \subset T$, and if $S \subset T$ then

$$W_S \cap U = \bigcap_{\alpha \in S, \beta \in T \setminus S} (\chi_\alpha - \chi_\beta)^{-1}((0,\infty))$$

which is a finite intersection of open sets and so is open. So since X is covered by such open sets U it follows that W_S is open for all finite subsets S.

Also, if $\alpha \in S$ then $W_S \subset U_\alpha$ since $supp(\chi_\alpha) \subset U_\alpha$. The W_S cover X as S varies through finite subsets of T: for instance if $x \in X$ we will have $x \in W_S$ if S is the set of indices α such that $\chi_\alpha(x) > 0$.

Furthermore, if $\#(S) = \#(S')$, then $W_S \cap W_{S'} = \varnothing$, since if $\#(S) = \#(S')$ then there exist $\alpha \in S \setminus S'$ and $\beta \in S' \setminus S$, and if $x \in W_S$ then $\chi_\alpha(x) > \chi_\beta(x)$, while if $x \in W_{S'}$ the reverse inequality holds.

So for any positive integer n let $V_n = \cup_{S:\#(S)=n} W_S$. Then each V_n is indeed a disjoint union of sets each of which is contained in some U_α, and so $\{V_n\}_{n=1}^\infty$ is the desired cover. $\square$

Combining the above two lemmas immediately gives:

Corollary 2.27. *Let $H\colon [0,1] \times X \to Y$ be a continuous map where X is a paracompact Hausdorff space, and let $E \to Y$ be a vector bundle. Then there is a countable open cover $\{V_n\}_{n=1}^\infty$ of X such that each restriction $(H^*E)|_{[0,1]\times V_n}$ is trivial.*

Proof. Lemma 2.25 gives an open cover $\{U_x\}_{x\in X}$ of X such that each $(H^*E)|_{[0,1]\times U_x}$ is trivial. Associate to the cover $\{U_x\}$ the countable cover $\{V_n\}_{n=1}^\infty$ as in Lemma 2.26. So for each n, V_n is a *disjoint* union $V_n = \sqcup_{\beta \in B_n} W_\beta$ so that for all $\beta \in B_n$ there is $x \in X$ with $W_\beta \subset U_x$. So we have trivializations of $(H^*E)|_{[0,1]\times W_\beta}$ for each $\beta \in B_n$, and taking the union of these trivializations gives a trivialization for $(H^*E)|_{[0,1]\times V_n}$. $\square$

Let us now finish the proof of Theorem 2.27. Let $\{V_n\}_{n=1}^\infty$ be an open cover as in Corollary 2.27. Since X is paracompact and Hausdorff there is a partition of unity $\{\chi_n\}_{n=1}^\infty$ subordinate to $\{V_n\}_{n=1}^\infty$. We will inductively use the following proposition:

Proposition 2.28. *For a continuous function $f : X \to [0,1]$ define $i_f : X \to [0,1] \times X$ by $i_f(x) = (f(x), x)$. Suppose that $j \in \mathbb{Z}_+$ and that f and $f + \chi_j$ are both continuous functions from X to $[0,1]$. Then there is an isomorphism of vector bundles $i_f^*(H^*E) \cong i_{f+\chi_j}^*(H^*E)$ which is the identity on all fibers over points x where $\psi_j(x) = 0$.*

Proof. Let $W_j = X \setminus supp(\chi_j)$. The restrictions of the two bundles to W_j are then identical, since $i_f(x) = i_{f+\chi_j}(x)$ for all $x \in W_j$, and so over W_j we will just use the identity as our bundle isomorphism. So since $V_j \cup W_j = X$ it suffices to construct an isomorphism $i_f^*(H^*E)|_{V_j} \to i_{f+\chi_j}^*(H^*E)|_{V_j}$ which is the identity over $V_j \cap W_j$.

But this is straightforward given that h^*E is trivial over $[0,1] \times V_j$. Let $\Psi\colon (H^*E)|_{[0,1]\times V_j} \to [0,1] \times V_j \times \mathbb{R}^k$ be a trivialization, say given by $\Psi(e) = (\pi_{H^*E}(e), \psi(e))$, and define $\Phi\colon i_f^*(H^*E)|_{V_j} \to i_{f+\chi_j}^*(H^*E)|_{V_j}$ by, for $x \in V_j$ and $e \in (H^*E)_{(f(x),x)}$,

$$\Phi(x, e) = \left(x, \Psi^{-1}(f(x) + \chi_j(x), x, \psi(e))\right).$$

This is easily seen to have the desired properties. (Intuitively, the trivialization Ψ identifies all fibers over $[0,1] \times V_j$ with $\mathbb{R}^k$, and we are using this identification to identify the fiber over $(f(x), x)$ with the fiber over $(f(x) + \chi_j(x), x)$.) $\square$

The above proposition gives us, for each $m \in \mathbb{Z}_+$, an isomorphism

$$\Psi_m \colon i^*_{\sum_{j=1}^{m-1} \chi_j}(H^*E) \to i^*_{\sum_{j=1}^{m} \chi_j}(H^*E).$$

By the local finiteness of the partition of unity $\{\chi_j\}$, X is covered by open sets U_M such that $\sum_{j=1}^{M} \chi_j(x) = 1$ for each $x \in U_M$. In particular we have, for all $N \geq M$,

$$\left(i^*_{\sum_{j=1}^{N} \chi_j}(H^*E)\right)|_{U_M} = \left(i_1^*(H^*E)\right)|_{U_M}.$$

We may then define the "infinite composition" $\Psi\colon i_0^*(H^*E) \to i_1^*(H^*E)$ by setting $\Psi|_{\pi^{-1}_{i_0^*H^*E}(U_M)}$ equal to $\Psi_N \circ \cdots \circ \Psi_1$ for any $N \geq M$. This is a globally defined map which restricts over each U_M to the finite composition $\Psi_M \circ \cdots \circ \Psi_1$, which is a bundle isomorphism.

Therefore $\Psi\colon i_0^*(H^*E) \to i_1^*(H^*E)$ is a bundle isomorphism, proving that $h_0^*E \cong i_0^*(H^*E)$ and $h_1^*E \cong i_1^*(H^*E)$ are isomorphic.

Remark 2.29. There is a somewhat more intuitive proof of the homotopy invariance theorem in the case that X and Y are smooth manifolds, $E \to Y$ is a smooth vector bundle (*i.e.*, E is a smooth manifold and the local trivializations are diffeomorphisms), and $H\colon [0,1] \times X \to Y$ is a smooth map (so that $H^*E \to [0,1] \times X$ is also a smooth vector bundle). In this case, as we'll probably discuss more later, one can use a partition of unity to construct a *connection* on the vector bundle H^*E, which is to say a splitting of the tangent bundle $T(H^*E) = T^v \oplus T^h$ where the "vertical bundle" T^v is the kernel of the derivative of the projection map to $[0,1] \times X$ and the "horizontal bundle" T^h is complementary to T^v and is appropriately compatible with the vector bundle structure. There will then be a unique vector field V on the smooth manifold H^*E which lies in T^h and projects down to $[0,1] \times X$ as the vector field ∂_t in the $[0,1]$ direction, and an isomorphism from h_0^*E to h_1^*E can be obtained by taking the time-one flow of the vector field V.

Remark 2.30. Directly from Theorem 2.24 (and without appealing to any classification results) one can infer that any vector bundle $E \to M$ where M is paracompact, Hausdorff and *contractible* is trivial. Indeed, if $f\colon M \to \{pt\}$ is a homotopy equivalence with homotopy inverse $\{pt\}$, then $g \circ f$ is homotopic to the identity and so by Theorem 2.24 E is isomorphic to $(g \circ f)^*E \cong f^*(g^*E)$. But g^*E is a vector bundle over a single point and so is obviously trivial, and so $f^*(g^*E)$ is trivial.

More generally, if X and Y are two paracompact spaces, a similar argument shows that a homotopy equivalence $f\colon X \to Y$ induces a bijection $f^*\colon Vect_k(Y) \to Vect_k(X)$ of isomorphism classes of vector bundles. This is consistent with the classification theorem 2.23, as f induces a bijection $[Y, Gr_k(\mathbb{R}^N)] \to [X, Gr_k(\mathbb{R}^N)]$, by precomposition. (Here we use the standard notation $[A,B]$ for the set of homotopy classes of maps from A to B.)

2.5. **Variants and generalizations of the main classification theorem.**

2.5.1. *The infinite Grassmannian Gr_k.* One can in fact obtain a version of Theorem 2.23 which is at least superficially simpler in that it avoids any reference to stabilization, by working with $\mathbb{R}^\infty$ rather than $\mathbb{R}^N$ for (varying) large N. Here $\mathbb{R}^\infty$ is the direct *sum* of infinitely many copies of $\mathbb{R}$, *i.e.*, it is the space of sequences $\{x_i\}_{i=1}^\infty$ where each $x_i \in \mathbb{R}$ and all but finitely many x_i are equal to zero. So for any integer N we have inclusions $I_{N,\infty}\colon \mathbb{R}^N \to \mathbb{R}^\infty$ defined as before by $\vec{x} \mapsto (\vec{x}, \vec{0})$ (though now the zero-vector is infinite-dimensional), and $\mathbb{R}^\infty$ is the union of the images of the $I_{N,\infty}$. The topology on $\mathbb{R}^\infty$ is the "direct limit" topology, defined by saying that $U \subset \mathbb{R}^\infty$ is open if and only if $I^{-1}_{N,\infty}(U)$ is open for all N—thus this is the finest topology on $\mathbb{R}^\infty$ consistent with the $I_{N,\infty}$ all being continuous, and $I_{N,\infty}$ embeds $\mathbb{R}^N$ homeomorphically as a subspace of $\mathbb{R}^\infty$.

Similarly, we define
$$Gr_k = \{V \leq \mathbb{R}^\infty | \dim V = k\},$$
so that there are maps $i_{N,\infty}\colon Gr_k(\mathbb{R}^N) \to Gr_k$ defined by $V \mapsto V \times \{\vec{0}\}$. Since any basis for a finite-dimensional subspace of $\mathbb{R}^\infty$ has all of its elements in some $I_{N,\infty}(\mathbb{R}^N)$, Gr_k is the union of the $i_{N,\infty}(Gr_k(\mathbb{R}^N))$, and we define a topology on Gr_k by saying that $U \subset Gr_k$ is open if and only if $i_{N,\infty}^{-1}(U)$ is open for all N.

As with the finite-dimensional Grassmannians there is a tautological vector bundle $\gamma^k \to Gr_k$, whose fiber over the subspace $V \leq \mathbb{R}^\infty$ is naturally identified with that subspace. A pre-classifying map $F\colon E \to \mathbb{R}^N$ for a vector bundle $E \to M$ with induced map $f\colon M \to Gr_k(\mathbb{R}^N)$ gives rise (with obvious extensions of the definitions) to the pre-classifying map $I_{N,\infty} \circ F\colon E \to \mathbb{R}^\infty$ with induced map $i_{N,\infty} \circ f$. Clearly if the pre-classifying maps $F_0\colon E \to \mathbb{R}^{N_0}$ and $F_1\colon E \to \mathbb{R}^{N_1}$ are stably isotopic then $I_{N_0,\infty} \circ F_0$ and $I_{N_1,\infty} \circ F_1$ are isotopic and the induced maps $i_{N_0,\infty} \circ f_0$ and $i_{N_1,\infty} \circ f_1$ are homotopic. (Note that we no longer have to say "stably" here.)

Define $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ and $\beta\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by setting the jth element of $\alpha(\{x_i\}_{i=1}^\infty)$ equal to x_i if $j = 2i+1$ and to 0 if j is even, and by setting the jth element of $\beta(\{x_i\}_{i=1}^\infty)$ equal to x_i if $j = 2i$ and to 0 if j is odd (so $\alpha(x_1, x_2, x_3, \ldots) = (0, 0, x_1, 0, x_2, 0, x_3, 0, \ldots)$ and $\beta(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots)$). It is easy to see that α and β are both continuous maps. Moreover, if $F_0, F_1\colon E \to \mathbb{R}^\infty$ are pre-classifying maps, then there are "straight-line isotopies" from F_0 to $\alpha \circ F_0$, from $\alpha \circ F_0$ to $\beta \circ F_1$, and from $\beta \circ F_1$ to F_1. (Here by a straight-line isotopy from a pre-classifying map G_0 to a pre-classifying map G_1 we mean one given by $G(t, e) = (1-t)G_0(e) + tG_1(e)$. This is indeed an isotopy provided that it is injective on each fiber, which holds if $G_1(e)$ is never a negative multiple of $G_0(e)$ for nonzero e—this is easily seen to hold for each of the three pairs $(G_0, G_1) = (F_0, \alpha \circ F_0), (\alpha \circ F_0, \beta \circ F_1)$, or $(\beta \circ F_1, F_1)$ listed above since in each case for all nonzero e the set of indices j such that $G_0(e)_j \neq 0$ is different from the corresponding set for $G_1(e)$). This shows that *any* two pre-classifying maps $F_0, F_1\colon E \to \mathbb{R}^\infty$ are isotopic (this slightly generalizes the previous paragraph, which established this result for pre-classifying maps of the form $I_{N,\infty} \circ F$).

Just as in the case (Proposition 2.15) of pre-classifying maps to $\mathbb{R}^N$ for finite N, if $F\colon E \to \mathbb{R}^\infty$ is a pre-classifying map with induced map $f\colon M \to Gr_k$, then E is isomorphic to $f^*\gamma^k$. We have just shown that any two pre-classifying maps to $\mathbb{R}^\infty$ are isotopic, and so their associated induced maps are homotopic. Conversely, given two homotopic maps $f_0, f_1\colon M \to Gr_k$, by Theorem 2.24 the bundles $f_0^*\gamma^k$ and $f_1^*\gamma^k$ are isomorphic.

Moreover, one can show that for *any* rank-k vector bundle $E \to M$, not necessarily of finite type, where M is paracompact and Hausdorff, there exists a pre-classifying map $F\colon E \to \mathbb{R}^\infty$ (this is [MS, Theorem 5.6]; it can be deduced from Lemma 2.26 using the same method as in our earlier construction of pre-classifying maps in the finite-type case—details are left to the reader). Consequently one obtains:

Theorem 2.31. *For any paracompact Hausdorff space M, there is a one-to-one correspondence between isomorphism classes of rank-k vector bundles over M and homotopy classes of maps $f\colon M \to Gr_k$, according to which (the homotopy class of) $f\colon M \to Gr_k$ corresponds to (the isomorphism class of) $f^*\gamma^k$.*

Accordingly Gr_k is described as a "classifying space for rank-k vector bundles" (or, alternatively, a classifying space for the group $O(k)$, as all for all such bundles the structure group reduces to $O(k)$).

2.5.2. *Classifying bundles with additional structure.* The discussion in this section can be modified in a mostly-straightforward way to accommodate bundles with certain extra structures, provided that we replace the Grassmannian $Gr_k(\mathbb{R}^N)$ with another space depending on the structure.

The most important case of this is that of *complex* vector bundles $\pi\colon E \to M$, *i.e.* rank-$2n$ (real) vector bundles with structure group $GL(n;\mathbb{C}) \le GL(2n;\mathbb{R})$. (See Example 1.16.) Given a collection of local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ for E whose transition functions $g_{\alpha\beta}(m)$ belong to $GL(n;\mathbb{C})$, one can endow each fiber E_m of E with the structure of a *complex* vector space by demanding that the restriction of each Φ_α to E_m be a complex-linear isomorphism to $\{m\} \times \mathbb{C}^n$. (Equivalently, one could define a rank-n complex vector bundle as a vector bundle in which the fibers carry complex vector space structures such that the local trivializations are complex linear.)

If $\pi\colon E \to M$ is a (finite-type) complex vector bundle of rank n (and hence in particular a real vector bundle of rank $2n$) and if we form the pre-classifying map $F\colon M \to \mathbb{R}^{2nT} = \mathbb{C}^{nT}$ exactly as in the proof of Proposition 2.5 we note that F restricts to each fiber E_m as a complex-linear map. So we can define a *complex pre-classifying map* to be a continuous map $F\colon E \to \mathbb{C}^N$ which restricts to each fiber as a complex-linear injection. As in the real case we have stabilization maps $I^{\mathbb{C}}_{N,N'}\colon \mathbb{C}^N \to \mathbb{C}^{N'}$, and a replication of the proof of Theorem 2.8 shows that two complex pre-classifying maps for the same complex vector bundle are stably isotopic (with the isotopy passing only through complex pre-classifying maps).

Now define the *complex* Grassmannian $Gr_n(\mathbb{C}^N)$ to be the set of n-dimensional complex vector subspaces of $\mathbb{C}^N$. Where $\tilde{V}_n(\mathbb{C}^N)$ is the space of tuples $(\vec{v}_1,\ldots,\vec{v}_n) \in (\mathbb{C}^N)^n$ which are linearly independent over $\mathbb{C}$, we have a projection $p\colon \tilde{V}_n(\mathbb{C}^N) \to Gr_n(\mathbb{C}^N)$ taking $(\vec{v}_1,\ldots,\vec{v}_n)$ to $span\{\vec{v}_1,\ldots,\vec{v}_n\}$, and so we topologize $Gr_n(\mathbb{C}^N)$ by using the quotient topology associated to p. As in the real case, a pre-classifying map $F\colon M \to \mathbb{C}^N$ induces a classifying map $f: M \to Gr_n(\mathbb{C}^N)$ defined by $f(m) = F(E_m)$, and a stable isotopy of pre-classifying maps induces a stable homotopy of induced maps f. Moreover, there is a (complex) tautological bundle $\gamma^n(\mathbb{C}^N) \to Gr_n(\mathbb{C}^N)$ (with fiber over $V \in Gr_n(\mathbb{C}^N)$ naturally identified with the complex vector space V), and if the pre-classifying map $F\colon E \to \mathbb{C}^N$ induces the map $f: M \to Gr_n(\mathbb{C}^N)$ then E is isomorphic as a complex vector bundle to $f^*\gamma^n(\mathbb{C}^N)$, while conversely just as in Proposition 2.15 if $f: M \to Gr_n(\mathbb{C}^N)$ then one can construct a complex pre-classifying map for the complex vector bundle[3] $f^*\gamma^n(\mathbb{C}^N)$ which induces the map f. Moreover the proof of Theorem 2.24 generalizes straightforwardly to show that the pullbacks of a complex vector bundle by two homotopic maps are isomorphic as complex vector bundles. Consequently we obtain the following direct generalization of Theorem 2.23.

Theorem 2.32. *Any complex finite-type rank-n vector bundle E over a paracompact Hausdorff space M has a classifying map $f: M \to Gr_n(\mathbb{C}^N)$, in which case E is isomorphic as a complex vector bundle to $f^*\gamma^n(\mathbb{C}^N)$. Moreover, two such bundles are isomorphic as complex vector bundles if and only if their classifying maps are stably homotopic.*

While this result exactly parallels Theorem 2.23, note that the spaces involved are somewhat different: for instance the homology groups of $Gr_1(\mathbb{R}^N) = \mathbb{R}P^{N-1}$ are, as you know from algebraic topology, rather different from the homology groups of $Gr_1(\mathbb{C}^N) = \mathbb{C}P^{N-1}$. Just as in Section 2.5.1, if one instead uses the Grassmannian $Gr_n^{\mathbb{C}}$ of complex n-dimensional subspaces of $\mathbb{C}^\infty$ one may obtain that isomorphism classes of rank-n complex vector bundles over a paracompact Hausdorff space M are in one-to-one correspondence with homotopy classes of maps from M to $Gr_n^{\mathbb{C}}$.

[3]It is complex because, as noted in the discussion of pullbacks, the pullback of a bundle with structure group G has structure group G

Aside from complex vector bundles, another case of some note is that of *oriented* vector bundles $\pi: E \to M$, in which the structure group is the group $GL^+(k;\mathbb{R})$ of matrices with positive determinant. In this case the targets for classifying maps may be taken to be the "oriented Grassmannians" $\widetilde{Gr}_k(\mathbb{R}^N)$, whose elements are pairs $(V, \mathfrak{o})$ where V is a k-dimensional subspace of $\mathbb{R}^N$ and $\mathfrak{o}$ is an orientation for V. The projection $p: \tilde{V}_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ factors through $\widetilde{Gr}_k(\mathbb{R}^N)$, with $(\vec{v}_1, \ldots, \vec{v}_k)$ mapping to the pair consisting of the vector space $span\{\vec{v}_1, \ldots, \vec{v}_k\}$ together with the orientation obtained by declaring $(\vec{v}_1, \ldots, \vec{v}_k)$ to be a positively oriented basis; in particular this latter projection allows us to topologize $\widetilde{Gr}_k(\mathbb{R}^N)$ with the quotient topology.

If E is an oriented vector bundle, with the fiber over m having orientation $\mathfrak{o}_m$, then a pre-classifying map $F: E \to \mathbb{R}^N$ gives rise to a map $M \to \widetilde{Gr}_k(\mathbb{R}^N)$ which sends m to the pair $(F(E_m), F_*\mathfrak{o}_m)$, where we push forward orientations in the obvious way (a positively-oriented basis for the pushforward is the image under F of a positively-oriented basis for E_m). From here, an exact replication of the arguments from earlier allows one to set up a one-to-one correspondence between isomorphism classes of finite-type oriented bundles and stable homotopy classes of maps to $\widetilde{Gr}_k(\mathbb{R}^N)$.

Moreover, just as in the unoriented case, one can form the oriented Grassmannian $\widetilde{Gr}_k$ of oriented k-planes in $\mathbb{R}^\infty$; it is topologized by the quotient topology induced by the projection $\tilde{p}: \tilde{V}_k(\mathbb{R}^\infty) \to \widetilde{Gr}_k$ which sends a k-tuple of linearly independent vectors to its span together with the orientation induced by declaring the k-tuple to be a positive basis. We have a two-to-one cover $q: \widetilde{Gr}_k \to Gr_k$ given by $q(V, \mathfrak{o}) = V$, and the pullback $\tilde{\gamma}^k = q^*\gamma^k$ of the tautological bundle is naturally oriented. (Its fiber over a point $(V, \mathfrak{o})$ of $\widetilde{Gr}_k$ is V, and the orientation on this fiber is just $\mathfrak{o}$.) The general classification theorem for oriented vector bundles is then:

Theorem 2.33. *For any paracompact Hausdorff space M, there is a one-to-one correspondence between isomorphism classes of rank-k oriented vector bundles over M and homotopy classes of maps $\tilde{f}: M \to \widetilde{Gr}_k$, according to which (the homotopy class of) $\tilde{f}: M \to \widetilde{Gr}_k$ corresponds to (the isomorphism class of) $\tilde{f}^*\tilde{\gamma}^k$.*

A general vector bundle $\pi: E \to M$ is called *orientable* if there exists an oriented vector bundle $\pi': E' \to M$ such that E is vector-bundle-isomorphic to E'. In this case, if $\tilde{f}': M \to \widetilde{Gr}_k$ has the property that E' is isomorphic as an oriented bundle to $(\tilde{f}')^*\tilde{\gamma}^k$, then where $q: \widetilde{Gr}_k \to Gr_k$ is the two-to-one cover mentioned above $q \circ \tilde{f}': M \to Gr_k$ is a classifying map for E' and hence is homotopic to any classifying map $f: M \to Gr_k$ for E. Conversely, if a classifying map $f: M \to Gr_k$ for E is homotopic to a map f' such that there exists $\tilde{f}': M \to \widetilde{Gr}_k$ with $f' = q \circ \tilde{f}'$, then E is isomorphic to $(f')^*\gamma^k = (\tilde{f}')^*\tilde{\gamma}^k$ and so is orientable. This shows that a vector bundle is orientable if and only if it has a classifying map which is homotopic to a map $f': M \to Gr_k$ which lifts to a map $\tilde{f}': M \to \widetilde{Gr}_k$.

Exercise 2.34. a) When $k = 1$, prove that the space $\widetilde{Gr}_1$ is homeomorphic to the "infinite-dimensional sphere"

$$S^\infty = \{\{x_i\}_{i=1}^\infty \in \mathbb{R}^\infty | \sum x_i^2 = 1\}.$$

b) Prove that S^∞ is contractible.

c) Prove that any orientable rank-1 vector bundle over a paracompact Hausdorff space is trivial.

For the following exercise we first recall some linear algebra: if V is a finite-dimensional vector space over $\mathbb{R}$ and p is a positive integer we denote by $\Lambda^p V^*$ the space of p-linear alternating forms $\omega\colon V^p \to \mathbb{R}$, *i.e.* maps satisfying, for $v_1, \dots, v_p, w \in V$ and $c \in \mathbb{R}$,

$$\omega(v_1, \dots, v_{l-1}, cv_l + w, v_{l+1}, \dots, v_p) = c\omega(v_1, \dots, v_l, \dots, v_p) + \omega(v_1, \dots, w, \dots, v_p) \quad (1 \le l \le p)$$

and

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_p) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_p) \quad (1 \le i < j <\le p)$$

It is a standard fact that $\Lambda^p V^*$ is a vector space of dimension $\binom{\dim V}{p}$, and in particular of dimension one if $p = \dim V$. Moreover a generator δ for $\Lambda^{\dim V} V^*$ is obtained by choosing a basis $(e_1, \dots, e_{\dim V})$ for V and setting $\delta(v_1, \dots, v_{\dim V})$ equal to the determinant of the square matrix whose jth column is consists of the coefficients of v_j in terms of the basis (e_i).

A linear map $A\colon V \to W$ between two finite-dimensional vector spaces induces a (linear) pullback map $A^*\colon \Lambda^p W^* \to \Lambda^p V^*$ by the formula

$$(A^*\omega)(v_1, \dots, v_p) = \omega(Av_1, \dots, Av_p).$$

In the case that $V = W$ and $p = \dim V$, so that A^* is a linear endomorphism of a one-dimensional vector space, A^* is given by multiplication by $\det A$.

Exercise 2.35. Let $\pi\colon E \to M$ be a rank-k vector bundle, with local trivializations $\Phi_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$. For $m \in U_\alpha$ denote the restriction of Φ_α^{-1} to $\{m\} \times \mathbb{R}^k$ by $\psi_\alpha(m)$, so for each $m \in U_\alpha$, $\psi_\alpha(m)$ is a linear isomorphism from $\{m\} \times \mathbb{R}^k$ to E_m.

(a) Define

$$\Lambda^k E^* = \bigcup_{m \in M} \{m\} \times \Lambda^k E_m^*.$$

Prove that there is a vector bundle structure on $\Lambda^k E^*$ having local trivializations over U_α given by $(m, \omega) \mapsto (m, \psi_\alpha(m)^*\omega)$. (Note that in order to do this problem you in particular have to put a topology on $\Lambda^k E^*$ and prove that various maps are continuous with respect to this topology.) If the transition functions for the original bundle are given by $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k; \mathbb{R})$, what are the transition functions for $\Lambda^k E^*$?

(b) If M is paracompact and Hausdorff prove that a rank-k vector bundle over M is orientable if and only if $\Lambda^k E^*$ is trivial.

3. Introduction to cohomology

3.1. **Some recollections about homology.**

In MATH 8200 you learned about the *singular homology* groups of a space X; to set up notation and pave the way for some generalizations let us recall the construction. For $k \in \mathbb{Z}_{\ge 0}$ one has the k-simplex

$$\Delta^k = \{(t_1, \dots, t_k) \in \mathbb{R}^k | t_i \ge 0, \sum t_i \le 1\}$$
$$= \left\{ \sum_{i=0}^k t_i v_i^k \middle| \sum_{i=0}^k t_i = 1 \right\}.$$

where v_0^k is the zero-vector and v_i^k is the ith standard basis vector in $\mathbb{R}^k$ for $1 \le i \le k$. For $j = 0, \dots, k+1$ there are face maps

$$\phi_j\colon \Delta^k \to \Delta^{k+1}$$

defined by

$$\phi_j\left(\sum_{i=0}^k t_i v_i^k\right) = \sum_{i=0}^{j-1} t_i v_i^{k+1} + \sum_{i=j}^k t_i v_{i+1}^{k+1}$$

which embed Δ^k as part of the boundary of Δ^{k+1}.

The singular chain complex of X is defined by letting $S_k(X)$ be the free abelian group (*i.e.* free $\mathbb{Z}$-module) generated by all continuous maps $\sigma\colon \Delta^k \to X$; in other words

$$S_k(X) = \left\{ \sum_{i=1}^{N} n_i\sigma_i \,\middle|\, n_i \in \mathbb{Z}, \sigma_i \in C(\Delta^k, X) \right\}.$$

The singular boundary operator is the unique homomorphism $\partial\colon S_{k+1}(X) \to S_k(X)$ whose values on the generators σ for $S_{k+1}(X)$ (*i.e.* on continuous maps $\sigma\colon \Delta^{k+1} \to X$) is given by the formula

$$\partial\sigma = \sum_{j=0}^{k+1}(-1)^j(\sigma \circ \phi_j).$$

One then shows that $\partial \circ \partial = 0$, and the kth (singular) homology of X (with coefficients in $\mathbb{Z}$) is defined by

$$H_k(X;\mathbb{Z}) = \frac{\ker(\partial\colon S_k(X) \to S_{k-1}(X))}{Im(\partial\colon S_{k+1}(X) \to S_k(X))}.$$

For a subspace $A \subset X$, we can view $S_k(A)$ as a subgroup of $S_k(X)$, and form the relative singular chain group $S_k(X,A) := \frac{S_k(X)}{S_k(A)}$. The fact that ∂ maps $S_{k+1}(A)$ to $S_k(A)$ implies that ∂ descends to a map $\bar{\partial}\colon S_{k+1}(X,A) \to S_k(X,A)$, and we have the relative homology

$$H_k(X,A;\mathbb{Z}) = \frac{\ker(\bar{\partial}\colon S_k(X,A) \to S_{k-1}(X,A))}{Im(\bar{\partial}\colon S_{k+1}(X,A) \to S_k(X,A))}.$$

For any abelian group R one can form the *relative singular homology of* (X,A) *with coefficients in* R as follows. Since R is a $\mathbb{Z}$-module, we can form $S_k(X,A;R) = S_k(X,A) \otimes R$, and then the boundary operator $\partial\colon S_{k+1}(X,A) \to S_k(X,A)$ naturally induces a map $\partial \otimes 1\colon S_{k+1}(X,A;R) \to S_k(X,A;R)$ which again squares to zero, and so we obtain the relative homology with coefficients in R

$$H_k(X,A;R) = \frac{\ker(\bar{\partial}\colon S_k(X,A;R) \to S_{k-1}(X,A;R))}{Im(\bar{\partial}\colon S_{k+1}(X,A;R) \to S_k(X,A;R))}.$$

As a special case we have the absolute homology with coefficients in R, $H_k(X;R) := H_k(X,\varnothing;R)$.

In the case that R is a commutative ring with unity, $S_k(X;R)$ and $S_k(X,A;R)$ are naturally R-modules, and the boundary operators on the respective complexes are R-module homomorphisms. Hence in this case the homology with coefficients in R is also an R-module.

The basic results and computational tools about singular homology from 8200, such as the Mayer-Vietoris sequence, the long exact sequence of a pair, invariance under homotopy equivalence, and excision, all extend to homology with coefficients in R in a straightforward way provided that one works with the same coefficients throughout. There exists a general formula (called the universal coefficient theorem for homology—we won't need or precisely formulate this, though it can be found in any algebraic topology book and we will discuss the corresponding result for cohomology in more detail) which relates the homology with coefficients in R to the ordinary ($\mathbb{Z}$-coeffcient) homology. In particular the $\mathbb{Z}$-coefficient homology determines the homology with coefficients in any abelian group. (If R is a field of characteristic zero then one has the simple formula $H_k(X,A;R) \cong H_k(X,A;\mathbb{Z}) \otimes R$, but in general the relationship is more complicated and involves $H_{n-1}(X,A;\mathbb{Z})$, similarly to the situation that we'll soon see for cohomology.)

As you learned in 8200, a useful way of computing the singular homology of a space X (at least with coefficients in $\mathbb{Z}$) involves first showing that X is homeomorphic (or even just homotopy equivalent) to a *cell complex* (a.k.a. CW complex). Without fully rehashing the definition, recall that a cell complex X consists of the images of various maps ("k-cells") $f_\alpha\colon D^k \to X$ where D^k is the closed unit disk in $\mathbb{R}^k$ (with k varying), with X equal to the disjoint union of the embedded images of the interiors of D^k under the f_α, while $f_\alpha(\partial D^k)$ is contained in a union of cells of dimension less than k. One can then form a chain complex $(C_*^{cell}(X), \partial^{cell})$, with $C_k^{cell}(X)$ equal to the free abelian group having one generator for each k-cell, and the cellular boundary operator $\partial^{cell}\colon C_{k+1}^{cell}(X) \to C_k^{cell}(X)$ is computed by the prescription on [H, p. 140].

An important theorem in algebraic topology asserts that $H_*(X;\mathbb{Z})$ is isomorphic to the homology of the cellular chain complex $(C_*^{cell}(X), \partial^{cell})$. This is very useful, since many spaces have cell decompositions which aren't too complicated (in particular any compact smooth manifold is homeomorphic to a cell complex with only finitely many cells, so the cellular chain complex is finitely-generated) and have homology which can be effectively computed, whereas the singular chain complex itself is usually unmanageably large (typically uncountably-generated). This theorem extends to the case of homology with coefficients: for any abelian group R we can form the chain complex $(C_*^{cell}(X)\otimes R, \partial^{cell}\otimes 1)$, and the homology of this chain complex is isomorphic to $H_*(X;R)$.

Example 3.1. Let us recall the standard cell decomposition for $\mathbb{R}P^n$, which we view for the purposes of this example as the quotient of the sphere $S^n = \left\{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \middle| \sum_{i=0}^n x_i^2 = 1\right\}$ by the equivalence relation which identifies $\vec{x}$ with $-\vec{x}$.

The 0-dimensional unit disk D^0 is just a one-point space $\{0\}$ (and $\partial D^0 = \varnothing$ and $int(D^0) = D^0$), and we define a 0-cell $f_0\colon D^0 \to \mathbb{R}P^n$ by $f_0(0) = [(1,0,\ldots,0)]$. For $1 \le k \le n$ define $f_k\colon D^k \to \mathbb{R}P^n$ by

$$f_k(x_0,\ldots,x_{k-1}) = \left[\left(x_0,\ldots,x_{k-1},\sqrt{1-\sum_{i=0}^{k-1} x_i^2},0,\ldots,0\right)\right]$$

An element $[(y_0,\ldots,y_n)] \in \mathbb{R}P^n$ lies in the image of $f_k(int(D^k))$ for a unique value of k, namely the largest k such that $y_k \neq 0$. Moreover for each k, $f_k|_{int(D^k)}$ is a homeomorphism to its image, while $f_k|_{\partial D^k}$ maps ∂D^k in two-to-one fashion to $\cup_{j=0}^{k-1} f_j(int(D^j))$. From this one can see that $f_0,\ldots,f_n$ give a cell decomposition for the space $\mathbb{R}P^n$.

Thus the cellular chain complex is, as an abelian group, given by $C_k^{cell}(X) = \mathbb{Z}\langle f_k\rangle$ (*i.e.* the free $\mathbb{Z}$-module with generator f_k) for $0 \le k \le n$ and $C_k^{cell}(X) = 0$ otherwise. As is shown in [H, Example 2.42], the cellular boundary operator is given by

$$\partial^{cell} f_k = \begin{cases} 2f_{k-1} & 1 \le k \le n \text{ and } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

So $\ker \partial^{cell}$ is generated by $f_0, f_1, f_3, f_5, \ldots, f_n$ if n is odd and by $f_0, f_1, f_3, f_5, \ldots, f_{n-1}$ if n is even, while $Im(\partial^{cell})$ is generated by $2f_1, 2f_3, \ldots, 2f_{n-2}$ is n is odd and $2f_1, 2f_3, \ldots, 2f_{n-1}$ if n is even. So we obtain

$$H_k(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd and } 1 \le k < n \\ \mathbb{Z} & \text{if } k = 0 \text{ or if } (k = n \text{ and } n \text{ is odd}) \\ 0 & \text{otherwise} \end{cases}$$

Things become rather simpler if we instead work with coefficients in the ring $R = \mathbb{Z}/2\mathbb{Z}$. Then $C_k^{cell}(\mathbb{R}P^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is the free $\mathbb{Z}/2\mathbb{Z}$-module generated by f_k for $0 \le k \le n$ and 0 otherwise, and the boundary operator $\partial^{cell} \otimes 1$ over $\mathbb{Z}/2\mathbb{Z}$ is just 0 since in $\mathbb{Z}/2\mathbb{Z}$ one has $2 = 0$. Of course, for a chain complex with zero boundary operator the homology is just isomorphic to the chain complex, so we have

$$H_k(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

In particular for even numbers k from 2 to n, $\mathbb{R}P^n$ has trivial kth homology over $\mathbb{Z}$ but non-trivial kth homology over $Z/2\mathbb{Z}$. If one knows the universal coefficient theorem for homology one could use it to compute $H_*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ from $H_*(\mathbb{R}P^n;\mathbb{Z})$, but it seems easier to directly use the coefficient-extended cellular boundary operator.

Exercise 3.2. For every integer $m \ge 3$, compute $H_*(\mathbb{R}P^n;\mathbb{Z}/m\mathbb{Z})$. Also, compute $H_*(\mathbb{R}P^n;\mathbb{Q})$.

3.2. **Cohomology and the universal coefficient theorem.** The singular cohomology of a space X with coefficients in $\mathbb{Z}$ will be the (co)homology of the *dual complex* of the singular chain complex of X. Presently we will work out some of the algebra relating to this.

Definition 3.3. Let R be a commutative ring with unity.

(i) A *chain complex of R-modules* $(C_\bullet, \partial)$ is a sequence $\{C_k\}_{k\in\mathbb{Z}}$ where each C_k is an R-module, together with R-module homomorphisms $\partial_k\colon C_k \to C_{k-1}$ such that $\partial_{k-1}\circ\partial_k = 0$. For $k\in\mathbb{Z}$, the kth *homology* of $(C_\bullet,\partial)$ is $H_k(C_\bullet) = \frac{\ker\partial_k}{Im\partial_{k+1}}$.

(ii) A *cochain complex of R-modules* $(C^\bullet,\delta)$ is a sequence $\{C^k\}_{k\in\mathbb{Z}}$ where each C^k is an R-module, together with R-module homomorphisms $\delta_k\colon C^k \to C^{k+1}$ such that $\delta_{k+1}\circ\delta_k = 0$. We denote $C^* = \oplus_{k\in\mathbb{Z}} C^k$. For $k\in\mathbb{Z}$, the kth *cohomology* of $(C^\bullet,\delta)$ is $H^k(C^\bullet) = \frac{\ker\delta_k}{Im\delta_{k-1}}$.

(iii) Let $(C_\bullet,\partial)$ be a chain complex of R-modules and let S be another R-module. The *dual cochain complex of* $(C_\bullet,\partial)$ *with coefficients in S*, denoted $(C_S^\bullet,\delta)$, is given by setting $C_S^k = Hom_R(C_k,S)$[4] and defining $\delta_k\colon C_S^k \to C_S^{k+1}$ by, for each $\alpha\in C_S^k$ and $c\in C_{k+1}$,

$$(\delta_k\alpha)(c) = \alpha(\partial_{k+1}c).$$

So in case $S = R$, the (co)boundary operator δ on $C_R^\bullet$ is just the transpose of the boundary operator ∂ on $C_\bullet$.

Exercise 3.4. Where $R=\mathbb{Z}$, compute, $H^k(C_S^\bullet)$ in the cases where $C_\bullet$ is the cellular chain complex for $\mathbb{R}P^n$ from Example 3.1 and $S = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}$.

Shortly, we will prove the universal coefficient theorem for cohomology, which will show in particular that (provided at least that R is a PID, as is pretty much always true in applications—indeed for the most part for our purposes R will be either $\mathbb{Z}$ or a field) and the C_k are free R-modules, the cohomology $H^*(C_S^\bullet) := \oplus_k H^k(C_S^\bullet)$ is determined up to graded R-module isomorphism by the homology $H_*(C_\bullet) := \oplus_k H_k(C_\bullet)$. In view of this it might seem at first that there wouldn't be much point in discussing the cohomology of a space instead of the homology, since the latter already determines the former. One significant part of the importance of the singular cohomology $H^*(X;R)$ of a space X is that, unlike the homology, it possesses a natural *ring* structure, not just an R-module structure, and this ring structure gives information that cannot be seen in the homology. Another feature of the cohomology that is useful for our purposes here is that, whereas for homology a continuous map $f: X\to Y$ induces a chain map $f_*\colon S_*(X)\to S_*(Y)$ and hence an induced map on homology $H_*(X)\to H_*(Y)$, for cohomology taking the transpose of f_* gives a cochain map (*i.e.* a map that commutes with the coboundary operator δ) $f^*\colon S^*(Y)\to S^*(X)$ and hence an induced map on cohomology

[4]Here for any R-modules M and N we denote $Hom_R(M,N)$ the R-module consisting of R-module homomorphisms from M to N.

$H^*(Y;R) \to H^*(X;R)$. Note that this induced map "goes in the same direction" as the natural pullback operation on vector bundles. One way of constructing the characteristic classes $a(E)$ of a vector bundle $E \to M$ is by identifying certain special cohomology classes $a \in H^*(Gr_k(\mathbb{R}^N);R)$ and setting $a(E) = f^*a$ where $f : M \to Gr_k(\mathbb{R}^N)$ is a classifying map for E.

The rest of the discussion in this subsection will be purely algebraic. Let R be a PID. Recall that, by definition, an R-module S is said to be *free* if S is isomorphic as an R-module to a direct sum of some collection of copies of R. We will require the important algebraic fact that, since R is a PID, any submodule of a free R-module is free (for a proof see, *e.g.*, [La, III.7.1]).

We assume throughout the following that we are given a chain complex $(C_\bullet, \partial)$ with the property that each of the R-modules C_k is a free R-module (for instance this applies with $(C_\bullet, \partial)$ equal to the singular or the cellular chain complex of some space, with coefficients in R). Our goal is to describe the cohomology of the cochain complex $C_S^\bullet$ in terms of the homology of $C_\bullet$.

Since by definition $C_S^k = Hom_R(C_k, S)$, a first guess might be that we should likewise have $H^k(C_S^\bullet) \cong Hom_R(H_k(C_\bullet), S)$. We'll see that, in general, this guess in incorrect (as you should have found in Exercise 3.4, since $Hom_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\}$); however there is a natural "evaluation" map $e\colon H^k(C_S^\bullet) \to Hom_R(H_k(C_\bullet), S)$, defined as follows.

Note that if $\alpha \in C_S^k$ with $\delta_k \alpha = 0$, and if $c \in C_k$ with $\partial_k c = 0$, then for any $\beta \in C_S^{k-1}$ and any $d \in C_{k+1}$ we have

$$\begin{aligned}(\alpha + \delta_{k-1}\beta)(c + \partial_{k+1} b) &= \alpha(c) + (\delta_k \alpha)(b) + \beta(\partial_k c + \partial_k \partial_{k+1} d) \\ &= \alpha(c)\end{aligned}$$

by the assumptions on α and c and the fact that $\partial_k \partial_{k+1} = 0$. In other words, the value $\alpha(c) \in S$ depends only on the cohomology class $[\alpha] \in H^k(C^\bullet)$ of α and the homology class $[c] \in H_k(C_\bullet)$ of c. Thus we have a well-defined map

$$\begin{aligned} e\colon H^k(C_S^\bullet) &\to Hom_R(H_k(C_\bullet), S) \\ [\alpha] &\mapsto ([c] \mapsto \alpha(c)) \end{aligned} \tag{1}$$

which is clearly an R-module homomorphism.

Lemma 3.5. *The map e in (1) is surjective. Indeed, there is an R-module homomorphism $\zeta\colon Hom_R(H_k(C_\bullet), S) \to H^k(C_S^\bullet)$ such that $e \circ \zeta$ is the identity on $Hom_R(H_k(C_\bullet), S)$*

Proof. For all integers i let us denote $H_i = H_i(C_\bullet)$, $Z_i = \ker(\partial_i\colon C_i \to C_{i-1})$, and $B_i = Im(\partial_{i+1}\colon C_{i+1} \to C_i)$. Thus $H_i = Z_i/B_i$. Let $\pi\colon Z_k \to H_k$ denote the quotient projection.

Let $\psi \in Hom_R(H_k, S)$, so that $\psi \circ \pi \in Hom_R(Z_k, S)$. Our intention is to construct an element $\alpha \in C^k = Hom_R(C_k, S)$ such that $\alpha|_{Z_k} = \psi \circ \pi$. Note that if we can do this, then since $\pi(B_k) = \{0\}$ we would have $\alpha \circ \partial_{k+1} = 0$, *i.e.* $\delta_k \alpha = 0$, so that α would represent a cohomology class $[\alpha] \in H^k(C^\bullet)$; moreover it would follow directly from the fact that $\alpha|_{Z_k} = \psi \circ \pi$ and the definition of e that $e([\alpha]) = \psi$. So we have effectively reduced the problem to one of systematically extending the R-module homomorphism $\psi \circ \pi\colon Z_k \to S$ to a module homomorphism $\alpha\colon C_k \to S$.

To do this, note that we have a short exact sequence

$$0 \longrightarrow Z_k \longrightarrow C_k \xrightarrow{\partial_k} B_{k-1} \longrightarrow 0 \tag{2}$$

where the first map is the inclusion. Now because R is a PID and B_{k-1} is a submodule of the free R-module C_{k-1}, B_{k-1} is a free R-module. Thus B_{k-1} has a basis $\{b_\alpha\}_{\alpha \in A}$ (*i.e.* B_{k-1} is the internal direct sum $\oplus_{\alpha \in A} R b_\alpha$), and we can use this basis to "split" the sequence (2). Namely, choose arbitrarily elements $c_\alpha \in C_k$ with $\partial_k c_\alpha = b_\alpha$, and define $q\colon B_{k-1} \to C_k$ to be the unique R-module homomorphism such that $q(b_\alpha) = c_\alpha$ for all α. Thus $\partial_k \circ q = 1_{B_{k-1}}$. Moreover for any

$c \in C_k$ we have $c - q\partial_k c \in \ker \partial_k = Z_k$, and if $c \in Z_k$ then $c - q\partial_k c = c$. Consequently if we define $\alpha \colon C_k \to S$ by

$$\alpha(c) = (\psi \circ \pi)\big(c - q\partial_k c\big),$$

then $\alpha|_{Z_k} = \psi \circ \pi$, as desired.

The above immediately yields the desired map $\zeta \colon Hom_R(H_k(C_\bullet), S) \to H^k(C_S^\bullet)$: an explicit formula is given by

$$\zeta(\psi) = [\psi \circ \pi \circ (1_{C_k} - q \circ \partial_k)].$$

□

Corollary 3.6. *There is an isomorphism*

$$H^k(C_S^\bullet) \cong \ker(e) \oplus Hom_R(H_k(C_\bullet), S)$$

Proof. Given the map ζ from Lemma 3.5 the idea is rather similar to the end of the proof of that lemma: the map

$$[\alpha] \mapsto ([\alpha] - \zeta(e([\alpha])), e([\alpha]))$$

is the desired isomorphism (with inverse $(x, y) \mapsto x + \zeta(y)$). □

So to compute $H^k(C_S^\bullet)$ it remains to determine the kernel of the map e from (1). An element $\alpha \in \ker(\delta_k)$ has the property that $e([\alpha]) = 0$ if and only if it holds that $\alpha(c) = 0$ for all $c \in Z_k$. (Actually, for any $\alpha \in C_S^k$ such that $\alpha|_{Z_k} = 0$ we automatically have $\alpha \in \ker(\delta_k)$, since $\delta_k \alpha = \alpha \circ \partial_{k+1}$ vanishes provided that $\alpha|_{B_k} = 0$, and $B_k \leq Z_k$.)

In particular it holds that $\ker(e) = 0$ if and only if every $\alpha \in C_S^k$ with $\alpha|_{Z_k} = 0$ has the form $\alpha = \delta_{k-1}\beta$ (*i.e.*, $\alpha = \beta \circ \partial_k$) for some $\beta \in C_S^{k-1}$. So given $\alpha \in C_S^k$ with $\alpha|_{Z_k} = 0$ we consider the problem of constructing β with $\alpha = \beta \circ \partial_k$. Now there is a unique R-module homomorphism $\beta_0 \colon B_{k-1} \to S$ such that $\beta_0(\partial_k c) = \alpha(c)$ for all $c \in C_k$ (this is well-defined since if $\partial_k c = \partial_k c'$ then $c - c' \in Z_k$ so that $\alpha(c - c') = 0$). So the problem is now one of extending $\beta_0 \colon B_{k-1} \to S$ to an R-module homomorphism $\beta \colon C_{k-1} \to S$. Actually, if we can just extend β_0 to $\beta_1 \in Hom_R(Z_{k-1}, S)$ then this will suffice by an argument similar to one used in the proof of Lemma 3.5: we have an exact sequence

$$0 \longrightarrow Z_{k-1} \longrightarrow C_{k-1} \xrightarrow{\partial_{k-1}} B_{k-2} \longrightarrow 0$$

where B_{k-2} is free, so there is $q \colon B_{k-2} \to C_{k-1}$ with $\partial_{k-1} \circ q = 1_{B_{k-2}}$, and then any $\beta_1 \in Hom_R(Z_{k-1}, S)$ extends to an element $\beta \in Hom_R(C_{k-1}, S)$ by the formula $\beta(c) = \beta_1(c - q\partial_{k-1}c)$. In particular this argument shows that $\ker(e) = 0$ if every R-module homomorphism $B_{k-1} \to S$ extends to an R-module homomorphism $Z_{k-1} \to S$.

This holds if R is a field, but when R is not a field it may not hold: for instance R, S, and Z_{k-1} could all be $\mathbb{Z}$, and B_{k-1} could be $2\mathbb{Z}$. Then there is an R-module homomorphism $B_{k-1} \to S$ which sends 2 to 1, and this does not extend to an R-module homomorphism $Z_{k-1} \to S$.

More precisely, we have the following:

Proposition 3.7. *Where $i \colon B_{k-1} \to Z_{k-1}$ is the inclusion and $i^* \colon Hom_R(Z_{k-1}, S) \to Hom_R(B_{k-1}, S)$ is the restriction map ($(i^*\beta)(b) = \beta(i(b))$), we have*

$$\ker(e) \cong \frac{Hom_R(B_{k-1}, S)}{Im(i^* \colon Hom_R(Z_{k-1}, S) \to Hom_R(B_{k-1}, S))}.$$

Proof. We have

$$\ker(e) = \frac{\{\alpha \in C_S^k | \alpha|_{Z_k} = 0\}}{\{\beta \circ \partial_k | \beta \in C_S^{k-1}\}}. \tag{3}$$

Define an R-module homomorphism

$$\Phi\colon \{\alpha \in C_S^k | \alpha|_{Z_k} = 0\} \to \frac{Hom_R(B_{k-1}, S)}{Im(i^*\colon Hom_R(Z_{k-1}, S) \to Hom_R(B_{k-1}, S))}$$

by declaring $\Phi(\alpha)$ to be the coset of the unique element $\beta_0 \in Hom_R(B_{k-1}, S)$ such that $\alpha = \beta_0 \circ \partial_k$ (as noted earlier the fact that $\alpha|_{Z_k} = 0$ implies that such β_0 exists; its uniqueness follows from the fact that ∂_k maps C_k surjectively to B_{k-1}). It suffices to show that Φ is surjective and that $\ker(\Phi) = \{\beta \circ \partial_k | \beta \in C_S^{k-1}\}$.

Certainly Φ is surjective, as for any $\beta_0 \in Hom_R(B_{k-1}, S)$ the element $\beta_0 \circ \partial_k \in C_S^k$ has $\beta \circ \partial_k|_{Z_k} = 0$ and $\Phi(\beta \circ \partial_k) = [\beta]$.

If $\alpha = \beta \circ \partial_k$ where $\beta \in C_S^{k-1}$, then the unique element $\beta_0 \in Hom_R(B_{k-1}, S)$ associated to α is just the restriction of β and so belongs to $Im(i^*)$. Thus in this case $\Phi(\alpha) = 0$, proving that $\{\beta \circ \partial_k | \beta \in C_S^{k-1}\} \leq \ker(\Phi)$.

Conversely, if $\Phi(\alpha) = 0$ then there is $\beta_1 \in Hom_R(Z_{k-1}, S)$ such that $\alpha = \beta_1 \circ \partial_k$. As noted a couple of paragraphs above the lemma, any element of $Hom_R(Z_{k-1}, S)$ extends to an R-module homomorphism $C_{k-1} \to S$ (because B_{k-2} is a free R-module), so choosing $\beta \in Hom_R(C_{k-1}, S)$ such that $\beta|_{Z_{k-1}} = \beta_1$ we have $\alpha = \beta \circ \partial_k$.

Thus by (3) Φ descends to an isomorphism from $\ker(e)$ to $\frac{Hom_R(B_{k-1},S)}{Im(i^*)}$. □

We have now identified $\ker(e)$ with the cokernel of the restriction map $i^*\colon Hom_R(Z_{k-1}, S) \to Hom_R(B_{k-1}, S)$; hence by Corollary 3.6 we have $H^k(C_S^\bullet) \cong coker(i^*) \oplus Hom_R(H_k(C_\bullet), S)$. This actually might seem to be contrary to what I earlier said that we'd show, namely that $H^k(C_S^\bullet)$ should only depend on the homology $H_*(C_\bullet)$—the map i^* is a map between modules that will be different for different chain complexes having the same homology. However the cokernel of i^* turns out to depend only on $H_{k-1}(C_\bullet)$ and S, as we will now see.

Definition 3.8. Let M be an R-module. A *short free resolution* of M is a short exact sequence of R-modules

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\pi} M \longrightarrow 0$$

where F and G are free R-modules.

The relevant example for us is that

$$0 \longrightarrow B_{k-1} \xrightarrow{i} Z_{k-1} \xrightarrow{\pi} H_{k-1}(C_\bullet) \longrightarrow 0$$

is a free resolution of $H_{k-1}(C.)$, where i is the inclusion and π is the quotient projection. Of course B_{k-1} and Z_{k-1} are free by virtue of being submodules of the free R-module C_{k-1} (using that R is a PID).

The exact sequence

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\pi} M \longrightarrow 0$$

gives rise, for any R-module S, to a dual sequence

$$0 \longrightarrow Hom_R(M, S) \xrightarrow{\pi^*} Hom_R(G, S) \xrightarrow{\phi^*} Hom_R(F, S) \longrightarrow 0$$

However this latter sequence is *not exact*—indeed we essentially noted earlier that there are cases where ϕ^* is not surjective.[5] What is true is the following.

[5]As you'll show in Exercise 3.13, it is true that the rest of the sequence $0 \longrightarrow Hom_R(M, S) \xrightarrow{\pi^*} Hom_R(G, S) \xrightarrow{\phi^*} Hom_R(F, S)$ is exact—$Hom_R(\cdot, S)$ is a "left exact functor".

Definition-Theorem 3.9. *Let M and S be R-modules and let*

$$0 \longrightarrow F_0 \xrightarrow{\phi_0} G_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

and

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} G_1 \xrightarrow{\pi_1} M \longrightarrow 0$$

be two short free resolutions of M. Then the R-modules

$$\frac{Hom_R(F_0,S)}{Im(\phi_0^*\colon Hom_R(G_0,S) \to Hom_R(F_0,S))} \quad \textit{and} \quad \frac{Hom_R(F_1,S)}{Im(\phi_1^*\colon Hom_R(G_1,S) \to Hom_R(F_1,S))}$$

are isomorphic. Consequently we ***define*** *$Ext_R(M,S)$ as the cokernel of the map $\phi^*\colon Hom_R(G,S) \to Hom_R(F,S)$ where*

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\pi} M \longrightarrow 0$$

is any short free resolution of M.

Corollary 3.10 (Universal Coefficient Theorem for Cohomology)**.** *Let $(C_\bullet, \partial)$ be a chain complex of R-modules where R is a PID such that C_k is a free R-module for all $k \in \mathbb{Z}$. Then for any R-module S and any $k \in \mathbb{Z}$ we have*

$$H^k(C_S^\bullet) \cong Hom_R(H_k(C_\bullet),S) \oplus Ext_R(H_{k-1}(C_\bullet),S).$$

Proof. Indeed, given Definition-Theorem 3.9, this follows directly from Corollary 3.6 and Proposition 3.7. □

Proof of Definition-Theorem 3.9.

Lemma 3.11. *There are R-module homomorphisms $f\colon F_0 \to F_1$ and $g\colon G_0 \to G_1$ such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & F_0 \arrow[r, "\phi_0"] \arrow[d, "f"] & G_0 \arrow[r, "\pi_0"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "\phi_1"] & G_1 \arrow[r, "\pi_1"] & M \arrow[r] & 0 \end{tikzcd} \tag{4}$$

is commutative. Moreover if $f'\colon F_0 \to F_1$ and $g'\colon G_0 \to G_1$ are two other R-module homomorphisms making (4) commutative, then there is $K \in Hom_R(G_0,F_1)$ such that

$$f - f' = K \circ \phi_0 \quad \textit{and} \quad g - g' = \phi_1 \circ K.$$

Proof of Lemma 3.11. Let $\{b_0^\beta\}_{\beta\in B}$ be a basis for the free R-module G_0. By the surjectivity of π_1, for each β we may choose $b_1^\beta \in G_1$ such that $\pi_1(b_1^\beta) = \pi_0(b_0^\beta)$, and then if we let $g\colon G_0 \to G_1$ be the unique R-module homomorphism such that $g(b_0^\beta) = b_1^\beta$ then the right square of (4) will commute.

Given this g, if $\{a_0^\alpha\}_{\alpha\in A}$ is a basis for the free R-module F_0, then for each α we have $g(\phi_0 a_0^\alpha) \in \ker \pi_1$, and so there is (in fact a unique) $a_1^\alpha \in F_1$ such that $\phi_1(a_1^\alpha) = g(\phi_0 a_0^\alpha)$. Then letting $f\colon F_0 \to F_1$ be the unique R-module homomorphism such that $f(a_0^\alpha) = a_1^\alpha$ for each α results in (4) commuting.

As for the uniqueness statement, if (f,g) and (f',g') are two pairs of maps making (4) commute, then in particular $\pi_1 \circ (g-g') = 0$, so that $Im(g-g') \leq Im\phi_1$. So we may define $K\colon G_0 \to F_1$ by, for $b \in G_0$, setting Kb equal to the (unique, since ϕ_1 is injective) element $a \in F_1$ such that $\phi_1(a) = (g-g')(b)$. K is easily seen to define a module homomorphism since the other maps involved in the diagram are module homomorphisms, and it is immediate from the definition that $g - g' = \phi_1 \circ K$. Meanwhile we have

$$\phi_1 \circ (f - f') = (g - g') \circ \phi_0 = \phi_1 \circ K \circ \phi_0.$$

Since ϕ_1 is injective this immediately implies that $f - f' = K \circ \phi_0$. □

Now the map $f: F_0 \to F_1$ in (4) induces an adjoint map $f^*: Hom_R(F_1, S) \to Hom_R(F_0, S)$, namely $f^*\alpha = \alpha \circ f$ for $\alpha \in Hom_R(F_1, S)$. If $\alpha \in Im(\phi_1^*)$, say $\alpha = \phi_1^*\beta = \beta \circ \phi_1$ where $\beta \in Hom_R(G_1, S)$, then

$$f^*\alpha = \beta \circ \phi_1 \circ f = \beta \circ g \circ \phi_0 = \phi_0^*(\beta \circ g).$$

Thus f^* maps $Im(\phi_1^*)$ to $Im(\phi_0^*)$, and so f^* descends to a map $\underline{f}^*: coker(\phi_1^*) \to coker(\phi_0^*)$ (which we will eventually show is an isomorphism, proving the theorem).

For another choice of maps f', g' making (4) commute, for all $\alpha \in Hom_R(F_1, S)$ we have

$$f^*\alpha - f'^*\alpha = \alpha \circ (f - f') = \alpha \circ K \circ \phi_0 \in Im(\phi_0^*).$$

Hence the induced maps $\underline{f}^*: coker(\phi_1^*) \to coker(\phi_0^*)$ and $\underline{f'}^*: coker(\phi_1^*) \to coker(\phi_0^*)$ are equal to each other. In other words, the map $\underline{f}^*: coker(\phi_1^*) \to coker(\phi_0^*)$ induced by a pair of maps f and g that make (4) commute is independent of the particular choice of maps f and g.

Reversing the roles of the two resolutions of M, we obtain R-module homomorphisms p and q making the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "\phi_1"] \arrow[d, "p"] & G_1 \arrow[r, "\pi_1"] \arrow[d, "q"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
0 \arrow[r] & F_0 \arrow[r, "\phi_0"] & G_0 \arrow[r, "\pi_0"] & M \arrow[r] & 0
\end{tikzcd}$$

commute, and hence we get a map $\underline{p}^*: coker(\phi_0^*) \to coker(\phi_1^*)$ which is independent of the particular maps p and q. We then also have commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & F_0 \arrow[r, "\phi_0"] \arrow[d, "p\circ f"] & G_0 \arrow[r, "\pi_0"] \arrow[d, "q\circ g"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
0 \arrow[r] & F_0 \arrow[r, "\phi_0"] & G_0 \arrow[r, "\pi_0"] & M \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "\phi_1"] \arrow[d, "f\circ p"] & G_1 \arrow[r, "\pi_1"] \arrow[d, "g\circ q"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
0 \arrow[r] & F_1 \arrow[r, "\phi_1"] & G_1 \arrow[r, "\pi_1"] & M \arrow[r] & 0
\end{tikzcd}$$

But the above two diagrams would also commute if the vertical maps were replaced by the identity map, and by the independence property discussed above it hence follows that $\underline{(p \circ f)}^* = 1_{coker(\phi_0^*)}$ and $\underline{(f \circ p)}^* = 1_{coker(\phi_1^*)}$. But by the standard functoriality property of adjoints, we have $\underline{p}^* \circ \underline{f}^* = \underline{(f \circ p)}^*$ and $\underline{f}^* \circ \underline{p}^* = \underline{(p \circ f)}^*$. This proves that $\underline{f}^*$ and $\underline{p}^*$ are inverses to each other, and so define isomorphisms between $coker(\phi_0^*)$ and $coker(\phi_1^*)$. □

The universal coefficient theorem implies in particular that, to compute the cohomology of the dual complex to some chain complex S_*, it suffices to compute the cohomology of any other chain complex C_* whose homology is the same as that of S_*. In particular, since the cellular homology of a cell complex X is isomorphic its singular homology, the singular cohomology of X can be computed by computing the cohomology of the dual complex of the cellular chain complex. Thus in Exercise 3.4 you in fact computed the singular cohomology groups $H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$, $H^*(\mathbb{R}P^n;\mathbb{Q})$, and $H^*(\mathbb{R}P^n;\mathbb{Z})$—and in principle you could even have done this without first computing the (cellular or singular) homology.

Exercise 3.12. (a) Prove that if F is a free R-module and S is any R-module then $Ext_R(F,S)=0$.
(b) For all integers $m,n\geq 2$ compute $Ext_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z})$.

Exercise 3.13. Let

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\pi} P \longrightarrow 0$$

be a short exact sequence of R-modules, and let S be another R-module. Prove that the sequence

$$0 \longrightarrow Hom_R(P,S) \xrightarrow{\pi^*} Hom_R(N,S) \xrightarrow{\phi^*} Hom_R(M,S)$$

is exact.

Exercise 3.14. A short exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\pi} P \longrightarrow 0 \tag{5}$$

is said to *split* if there is $q\in Hom_R(P,N)$ such that $\pi\circ q\colon P\to P$ is the identity.
(a) Prove that if P is a free R-module then the sequence (5) splits.
(b) Prove that, if the sequence (5) splits, then the sequence

$$0 \longrightarrow Hom_R(P,S) \xrightarrow{\pi^*} Hom_R(N,S) \xrightarrow{\phi^*} Hom_R(M,S) \longrightarrow 0$$

is exact.

3.3. **Basic properties of singular cohomology.** As discussed earlier, if X is any space and R is an abelian group, the *singular cohomology* of X, $H^*(X;R)$, is by definition the cohomology of the *singular cochain complex* $S^\bullet(X;R)$, *i.e.* the dual complex with coefficients in R of the singular chain complex of $\mathbb{Z}$-modules $S_\bullet(X)$[6] Moreover if $A\subset X$ is a subspace, the relative singular homology with coefficients in R, $H^*(X,A;R)$, is defined as the cohomology of the dual complex with coefficients in R of the relative singular chain complex $S_\bullet(X,A)=\frac{S_\bullet(X)}{S_\bullet(A)}$.

Since $\mathbb{Z}$ is a PID and each $S_k(X)$ is, by definition, a free $\mathbb{Z}$-module, the universal coefficient theorem applies to give isomorphisms

$$H^k(X;R)\cong Hom_{\mathbb{Z}}(H_k(X;\mathbb{Z}),R)\oplus Ext_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}),R).$$

Alternately, if the coefficient group R is a PID, then in accordance with the most recent footnote we have an isomorphism

$$H^k(X;R)\cong Hom_R(H_k(X;R),R)\oplus Ext_R(H_{k-1}(X;R),R).$$

This latter formulation is perhaps most convenient when R is a field, since in that case the Ext_R group automatically vanishes and so the cohomology with coefficients in R is just the dual of the homology with coefficients in R.

[6]Here $S_\bullet(X)$ is considered as a chain complex of $\mathbb{Z}$-modules, and $\mathbb{Z}$ and R are playing the role of the groups that were denoted by, respectively, R and S in the notation of the last section. (I used slightly different notation in class, which avoided the conflicting roles played by R.) If R is a ring, it would be equivalent to say that $S^\bullet(X;R)$ is the dual complex with coefficients in R of $S_\bullet(X)\otimes R$, which in the notation of the last section would mean taking both R and S equal to R.

It might not be immediately clear that the universal coefficient theorem applies to the relative singular cohomology, since $S_k(X,A) = \frac{S_k(X)}{S_k(A)}$ is defined as a quotient of free $\mathbb{Z}$-modules, and of course such quotients are not always free. However, $S_k(X,A)$ is a free $\mathbb{Z}$-module: it is isomorphic to the free $\mathbb{Z}$-module generated by those continuous maps $\sigma\colon \Delta^k \to X$ such that $\sigma(\Delta^k) \not\subset A$, as you should be able to convince yourself. Consequently the universal coefficient theorem does apply to give an isomorphism

$$H^k(X,A;R) \cong Hom_{\mathbb{Z}}(H_k(X,A;\mathbb{Z}),R) \oplus Ext_{\mathbb{Z}}(H_{k-1}(X,A;\mathbb{Z}),R) \tag{6}$$

Recall that if $f: (X,A) \to (Y,B)$ is a map of pairs (*i.e.*, $f: X \to Y$ is a continuous map, $A \subset X$, $B \subset Y$, and $f(A) \subset B$) then there is (for all k) an induced map $f_*\colon S_k(X,A) \to S_k(Y,B)$ which descends from the map $f_*\colon S_k(X) \to S_k(Y)$ defined on simplices $\sigma\colon \Delta^k \to X$ by the obvious formula $f_*\sigma = f \circ \sigma$. Such induced maps f_* are functorial with respect to composition in the sense that $(g\circ f)_* = g_*\circ f_*$ for $f\colon (X,A) \to (Y,B)$ and $g\colon (Y,B) \to (Z,C)$, and moreover they are chain maps: $f_* \circ \partial^{(X,A)} = \partial^{(Y,B)} \circ f_*$. Moreover if $f_0, f_1\colon (X,A) \to (Y,B)$ are homotopic as maps of pairs (*i.e.* there is a homotopy $F\colon [0,1] \times X \to Y$ from f_0 to f_1 such that $F([0,1] \times A) \subset B$), then there is a *chain homotopy* from $(f_0)_*$ to $(f_1)_*$, *i.e.*, for all k a map $K\colon S_k(X,A) \to S_{k+1}(Y,B)$ such that $(f_0)_* - (f_1)_* = K \circ \partial^{(X,A)} + \partial^{(Y,B)} \circ K$.

As you likely learned in 8200, this implies that a map of pairs $(X,A) \to (Y,B)$ induces a map $H_*(X,A;\mathbb{Z}) \to H_*(Y,B;\mathbb{Z})$, and moreover that homotopic maps induce the same map on homology. We have the corresponding result for cohomology, with the direction of the maps reversed:

Proposition 3.15. *Let R be any abelian group. To any map of pairs $f\colon (X,A) \to (Y,B)$ there corresponds (for all k) a map $f^*\colon H^k(Y,B;R) \to H^k(X,A;R)$, obeying the functoriality property $(g \circ f)^* = f^* \circ g^*$ and the property that if f_0 and f_1 are homotopic through maps of pairs then $f_0^* = f_1^*$. Moreover the identity map $1_{(X,A)}\colon (X,A) \to (X,A)$ has the property that $1^*_{(X,A)}$ is the identity on $H^k(X,A;R) \to H^k(X,A;R)$.*

Proof. Define $\tilde{f}^*\colon S^k(Y,B;R) \to S^k(X,A;R)$ by $\tilde{f}^*\alpha = \alpha \circ f_*$ where f_* is the map on singular chain complexes described above the proposition. We have, generally

$$\widetilde{g\circ f}^*\alpha = \alpha\circ(g\circ f) = (\alpha\circ g)\circ f = \tilde{f}^*\tilde{g}^*\alpha,$$

and

$$\tilde{f}^*\delta^{(Y,B)}\alpha = (\alpha\circ\partial^{(Y,B)}\circ f_* = \alpha\circ f_*\circ\partial^{(X,A)} = \delta^{(X,A)}\tilde{f}^*\alpha.$$

The latter identity implies that $\tilde{f}^*$ maps $\ker\delta^{(Y,B)}$ to $\ker\delta^{(X,A)}$ and $Im\delta^{(Y,B)}$ to $Im\delta^{(X,A)}$ and so descends to a map $f^*\colon H^k(Y,B;R) \to H^k(X,A;R)$ for all k, and the relation $\widetilde{g\circ f}^*\alpha = \tilde{f}^*(\tilde{g}^*\alpha)$ directly implies that $(g\circ f)^* = f^*\circ g^*$. Clearly the map induced by the identity is the identity.

It remains to prove the homotopy invariance property. Given homotopic maps $f_0, f_1\colon (X,A) \to (Y,B)$ we have $K\colon S_k(X,A) \to S_{k+1}(Y,B)$ with $(f_0)_* - (f_1)_* = K\circ\partial^{(X,A)} + \partial^{(Y,B)}\circ K$. We then see that, for $\beta \in S^k(Y,B;R)$,

$$\tilde{f}_0^*\beta - \tilde{f}_1^*\beta = \beta\circ\big((f_0)_* - (f_1)_*\big) = \beta\circ K\circ\partial^{(X,A)} + \beta\circ\partial^{(Y,B)}\circ K = \delta^{(X,A)}(\beta\circ K) + (\delta^{(Y,B)}\beta)\circ K.$$

So if β is a cocycle in $S^k(Y,B;R)$ (*i.e.* an element of $\ker\delta^{(Y,B)}$), then $\tilde{f}_0^*\beta - \tilde{f}_1^*\beta$ is a coboundary in $S^k(X,A;R)$ (*i.e.* an element of $Im\delta^{(X,A)}$). Consequently where $[\beta]$ denotes the cohomology class of β we will have $f_0^*[\beta] = f_1^*[\beta]$, proving that indeed $f_0^* = f_1^*$. □

Remark 3.16. Of course, as the special case where $A = B = \varnothing$, Proposition 3.15 includes the statement that a continuous map $f: X \to Y$ induces a homomorphism of absolute cohomology groups $f^*\colon H^k(Y;R) \to H^k(X;R)$, satisfying all of the indicated properties.

Corollary 3.17. *If $f: (X,A) \to (Y,B)$ is a homotopy equivalence*[7] *then for all coefficient groups R and all k the induced map $f^*: H^k(Y,B;R) \to H^k(X,A;R)$ is an isomorphism.*

Proof. Indeed, Proposition 3.15 immediately implies that, if g is a homotopy inverse for f (as in the footnote) then $f^* \circ g^*$ and $g^* \circ f^*$ are the respective identities. □

Remark 3.18. The universal coefficient theorem, together with the corresponding result for homology (which you presumably learned in 8200) would be enough to prove that two homotopy equivalent spaces have isomorphic cohomology. However the isomorphism in the universal coefficient theorem is not generally compatible with the maps f^*, and so one needs the homotopy invariance statement in Proposition 3.15 to obtain the more specific statement that the induced map of a homotopy equivalence is an isomorphism. (Anyway, Proposition 3.15 doesn't use the universal coefficient theorem, and is easier to prove.)

Exercise 3.19. Assume that R is a nontrivial commutative ring with unity, so that we have a ring morphism $\epsilon: \mathbb{Z} \to R$ obtained by sending $1 \in \mathbb{Z}$ to the multiplicative identity in R.

For each path component P of a space X define an element $\alpha_P \in S^0(X;R)$ as follows. Since Δ^0 is a one-point set, a general element of $S_0(X)$ has the form

$$c = \sum_{x \in X} n_x x$$

where only finitely many of the n_x are nonzero. Accordingly define

$$\alpha_P\left(\sum_{x \in X} n_x x\right) = \sum_{x \in P} \epsilon(n_x).$$

(a) Prove that α_P is a cocycle (*i.e.* that $\delta\alpha_P = 0$), and that the cohomology class $[\alpha_P]$ is a nontrivial element of $H^0(X;R)$.

(b) Assuming that X has just finitely many path components $P_1, \ldots, P_m$, prove that

$$H^0(X;R) = \left\{ \sum_{i=1}^m r_i[\alpha_{P_i}] \,\middle|\, r_1, \ldots, r_m \in R \right\}.$$

(c) For any space X let $\pi_0(X)$ denote the set of path components of X. Prove that (regardless of whether $\pi_0(X)$ is finite) for any abelian group R, $H^0(X;R)$ is isomorphic to the abelian group of functions from $\pi_0(X)$ to R, by a map which sends a cohomology class $[\alpha]$ to the function which assigns to each path component $P \subset X$ the value $\alpha(p)$ where p is some point in P (viewed as a zero-simplex).

Recall that if $(C_\bullet, \partial_C)$ and $(D_\bullet, \partial_D)$ are chain complexes a *chain map* $f: C_\bullet \to D_\bullet$ is a collection of homomorphisms $f: C_k \to D_k$ for all k such that $f \circ \partial_C = \partial_D \circ f$. In this case f maps $\ker \partial_C$ to $\ker \partial_D$ and $Im\partial_C$ to $Im\partial_D$ and hence induces homomorphisms $f_*: H_k(C_\bullet) \to H_k(D_\bullet)$ for all k. A *short exact sequence of chain complexes* is by definition a sequence

$$0 \longrightarrow C_\bullet \xrightarrow{f} D_\bullet \xrightarrow{g} E_\bullet \longrightarrow 0$$

where f and g are chain maps such that each of the sequences

$$0 \longrightarrow C_k \xrightarrow{f} D_k \xrightarrow{g} E_k \longrightarrow 0$$

[7] *i.e.*, if there is $g: (Y,B) \to (X,A)$ such that $f \circ g$ and $g \circ f$ are homotopic to the respective identities through maps of pairs

is exact. A basic fact from elementary homological algebra (presumably familiar from 8200) is that *a short exact sequence of chain complexes induces a long exact sequence of homology groups*: for certain homomorphisms $d\colon H_{j+1}(E_\bullet) \to H_j(C_\bullet)$ we have an exact sequence

$$\cdots \xrightarrow{g_*} H_{k+1}(E_\bullet) \xrightarrow{d} H_k(C_\bullet) \xrightarrow{f_*} H_k(D_\bullet) \xrightarrow{g_*} H_k(E_\bullet) \xrightarrow{d} H_{k-1}(C_\bullet) \xrightarrow{f_*} \cdots$$

For instance, if X is a topological space and $A \subset X$, essentially by definition we have a short exact sequence of chain complexes

$$0 \longrightarrow S_\bullet(A) \xrightarrow{i} S_\bullet(X) \xrightarrow{\pi} S_\bullet(X,A) \longrightarrow 0 \tag{7}$$

where i is the inclusion and π is the projection. Hence we get the long exact sequence for relative homology:

$$\cdots \xrightarrow{\pi_*} H_{k+1}(X,A;\mathbb{Z}) \xrightarrow{d} H_k(A;\mathbb{Z}) \xrightarrow{i_*} H_k(X;\mathbb{Z}) \xrightarrow{\pi_*} H_k(X,A;\mathbb{Z}) \xrightarrow{d} H_{k-1}(A;\mathbb{Z}) \cdots$$

Meanwhile, for any abelian group R we can dualize (7) to obtain the sequence

$$0 \longrightarrow S^\bullet(X,A;R) \xrightarrow{\pi^*} S^\bullet(X;R) \xrightarrow{i^*} S^\bullet(A;R) \longrightarrow 0$$

The exactness of this sequence follows from Exercises 3.13 and 3.14 together with the observation made at the start of this subsection that the groups $S_k(X,A) = \frac{S_k(X)}{S_k(A)}$ are free $\mathbb{Z}$-modules. Thus we have a short exact sequence of cochain complexes (under the obvious definition), which hence[8] induces a long exact sequence of cohomology groups:

$$\cdots \xrightarrow{i^*} H^{k-1}(A;R) \xrightarrow{d} H^k(X,A;R) \xrightarrow{\pi^*} H^k(X;R) \xrightarrow{i^*} H^k(A;R) \xrightarrow{d} H^{k+1}(X,A;R) \cdots \tag{8}$$

So the exact sequence for the cohomology of a pair is just like that for homology except that all of the maps go in the opposite direction. Note that the map $i^*\colon H^k(X;R) \to H^k(A;R)$ is just the same as the map induced by the inclusion $\iota\colon A \to X$, as can be seen by inspecting the definitions. At the level of cochains it is easy to understand both of the maps $i^*\colon S^k(X;R) \to S^k(A;R)$ and $\pi^*\colon S^k(X,A;R) \to S^k(X;R)$. We have $S^k(X;R) = Hom_{\mathbb{Z}}(S_k(X),R)$, and for a homomorphism $\alpha\colon S_k(X) \to R$, $i^*\alpha\colon S_k(A) \to R$ is just given by restricting α to $S_k(A)$ (which, since it consists of formal linear combinations of simplices in A, is obviously contained in $S_k(X)$). As for π^*, note that $S^k(X,A;R) = Hom_{\mathbb{Z}}\left(\frac{S_k(X)}{S_k(A)},R\right)$ may be naturally identified with those elements of $S^k(X;R) = Hom_{\mathbb{Z}}(S_k(X),R)$ which vanish on all simplices which are contained in A, and π^* is precisely this identification.

Exercise 3.20. Prove that the following are equivalent, for any nontrivial abelian group R:

(i) $H^0(X,A;R) = \{0\}$.
(ii) The map $i^*\colon H^0(X;R) \to H^0(A;R)$ is injective.
(iii) Every path component of X intersects A.

Example 3.21. Let us compute, for all $n \geq 1$, the relative cohomology of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ with coefficients in an arbitrary abelian group R by using the long exact sequence (8). (We'll soon see that this example is rather important in the theory of characteristic classes.) First we

[8] Since, abstractly, a cochain complex can be turned into a chain complex, and vice versa, by negating the grading k, it follows formally from the corresponding result for homology that a short exact sequence of cochain complexes induces a long exact sequence on cohomology, with the "connecting homomorphisms" d increasing grading by 1 rather than decreasing it.

should compute the absolute cohomologies. $\mathbb{R}^n$ is homotopy equivalent to a point (and hence has the same cohomology), and so there are several easy ways of seeing that therefore we have

$$H^k(\mathbb{R}^n;R) \cong \begin{cases} R & k=0 \\ 0 & \text{otherwise} \end{cases}$$

(for instance one can use the corresponding computation for homology and the universal coefficient theorem, since $Hom(\mathbb{Z},R) \cong R$; alternatively, one could use the cellular cochain complex for a single point). Similarly $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}, which is a cell complex with one 0-cell, one $(n-1)$-cell, and trivial boundary operator (or, if $n = 1$, two 0-cells and trivial boundary operator). So cellular cohomology establishes

$$H^k(\mathbb{R}^n \setminus \{0\};R) \cong \begin{cases} R & k=0,n-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n>1 \text{ and } H^k(\mathbb{R}^1 \setminus \{0\};R) \cong \begin{cases} R\oplus R & k=0 \\ 0 & \text{otherwise} \end{cases}$$

Part of the exact sequence (8) reads

$$H^{k-1}(\mathbb{R}^n;R) \longrightarrow H^{k-1}(\mathbb{R}^n \setminus \{0\};R) \longrightarrow H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \longrightarrow H^k(\mathbb{R}^n;R) .$$

If $k \geq 2$ the two outermost terms are zero, and so $H^{k-1}(\mathbb{R}^n \setminus \{0\};R) \longrightarrow H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R)$ is an isomorphism. Thus

$$\text{For } k \geq 2,\ H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \cong \begin{cases} R & k=n \\ 0 & \text{otherwise} \end{cases}$$

In fact we'll see that the above formula holds for all k, but the lower values require particular attention. The first nontrivial part of (8) is

(9)

$$H^0(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \longrightarrow H^0(\mathbb{R}^n;R) \xrightarrow{i^*} H^0(\mathbb{R}^n \setminus \{0\};R) \longrightarrow H^1(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \longrightarrow 0$$

(the 0 is $H^1(\mathbb{R}^n;R)$). Of course $\mathbb{R}^n$ is path-connected, while $\mathbb{R}^n \setminus \{0\}$ is path-connected if $n > 1$ and has two path components if $n = 1$. So from the characterization of H^0 in Exercise 3.19(c), using the fact that i^* is the map induced by inclusion, if $n > 1$ then $H^0(\mathbb{R}^n;R)$ and $H^0(\mathbb{R}^n \setminus \{0\};R)$ may both be identified with R in such a way that i^* acts by the identity, while if $n = 1$ then $H^0(\mathbb{R}^n;R) \cong R$ and $H^0(\mathbb{R}^n \setminus \{0\};R) \cong R \oplus R$, with i^* acting as the diagonal map $r \mapsto (r,r)$.

So in either case i^* is injective, so that $H^0(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) = 0$ (as indeed also follows from Exercise 3.20). Moreover the exactness of (9) implies that $H^1(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R)$ is isomorphic to the cokernel of i^*; if $n > 1$ then i^* is an isomorphism and so $H^1(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \cong 0$, while if $n = 1$ (so i^* is the map $r \mapsto (r,r)$) then $H^1(\mathbb{R}^1,\mathbb{R}^1 \setminus \{0\};R) \cong R$. This completes the proof that, for all k,n,

$$H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};R) \cong \begin{cases} R & k=n \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

3.3.1. *The Mayer-Vietoris sequence for relative cohomology.* Let X be a space, with $X = U \cup V$ where U and V are open. Denote by $S_k^{U,V}(X)$ the free abelian group generated by k-simplices $\sigma\colon \Delta^k \to X$ with the property that either $Im(\sigma) \subset U$ or $Im(\sigma) \subset V$. The boundary operator for the singular chain complex $S_\bullet(X)$ maps $S_k^{U,V}(X)$ to $S_{k-1}^{U,V}(X)$, since if σ has image in U or V then so does the boundary of σ. Thus we obtain a chain complex $S_\bullet^{U,V}(X)$, with an inclusion $j\colon S_\bullet^{U,V}(X) \to S_\bullet(X)$ (which is a chain map and hence induces a map on homology).

A key lemma (see for instance [H, Proposition 2.21]) from 8200 asserts that the induced map on homology $j_*\colon H_k(S_\bullet^{U,V}(X)) \to H_k(X;\mathbb{Z})$ is an isomorphism. More specifically, there is a "subdivision map" $s\colon S_\bullet(X) \to S_\bullet^{U,V}(X)$ which is a chain map such that $s \circ j\colon S_\bullet^{U,V}(X) \to S_\bullet^{U,V}(X)$

is the identity while $j \circ s\colon S_\bullet(X) \to S_\bullet(X)$ is chain homotopic to the identity, *i.e.* there are maps $K\colon S_k(X) \to S_{k+1}(X)$ such that $j \circ s - 1 = \partial \circ K + K \circ \partial$. Dualizing, for any abelian group R we have cochain maps $j^*\colon Hom(S_\bullet(X),R) \to Hom(S_\bullet^{U,V}(X),R)$ and $s^*\colon Hom(S_\bullet^{U,V}(X),R) \to Hom(S_\bullet(X),R)$ such that $(s\circ j)^* = j^*\circ s^*$ is the identity and there are maps $K^*\colon Hom(S_{k+1}(X),R) \to Hom(S_k(X),R)$ such that $s^* \circ j^* - 1 = K^* \circ \delta + \delta \circ K^*$ where $\delta = \partial^*$ is the coboundary operator. As in Proposition 3.15, it follows that the induced maps on cohomology:

$$s^*\colon H^k(Hom(S_\bullet^{U,V}(X),R)) \to H^k(X;R) \qquad j^*\colon H^k(X;R) \to H^k(Hom(S_\bullet^{U,V}(X),R))$$

are inverses to each other. In particular, for any space X with open cover $X = U \cup V$ and any abelian group R and integer k,

$$j^*\colon H^k(X;R) \to H^k(Hom(S_\bullet^{U,V}(X),R)) \quad \text{is an isomorphism.} \tag{11}$$

There is also a relative version of this statement: suppose that $A \subset X$ is a subspace, so that $U \cap A$ and $V \cap A$ form an open cover of A, and so we have a chain complex $S_\bullet^{U\cap A,V\cap A}(A)$ which is both a subcomplex of $S_\bullet(A)$ and a subcomplex of $S_\bullet^{U,V}(X)$, with the inclusion $S_\bullet^{U\cap A,V\cap A}(A) \to S_\bullet(A)$ being given by the restriction of $j\colon S_\bullet^{U,V}(X) \to S_\bullet(X)$. By (11), $j^*\colon H^k(A;R) \to H^k(Hom(S_\bullet^{U\cap A,V\cap A}(A),R))$ is an isomorphism for every abelian group R and integer k. Meanwhile we have the quotient complexes

$$S_\bullet^{U,V}(X,A) := \frac{S_\bullet^{U,V}(X)}{S_\bullet^{U\cap A,V\cap A}(A)} \qquad S_\bullet(X,A) = \frac{S_\bullet(X)}{S_\bullet(A)},$$

and j descends to a chain map (still denoted j) from $S_\bullet^{U,V}(X,A)$ to $S_\bullet(X,A)$.

Proposition 3.22. *For all R and k the induced map on cohomology $j^*\colon H^k(X,A;R) \to H^k(Hom(S_\bullet^{U,V}(X,A),R)$ is an isomorphism.*

Proof. Let $i\colon A \to X$ be the inclusion. Dualizing the short exact sequences corresponding to the inclusions $i_*\colon S_\bullet(A) \to S_\bullet(X)$ and $i_*\colon S_\bullet^{U\cap A,V\cap A}(A) \to S_\bullet^{U,V}(X)$ gives a commutative diagram of short exact sequences of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & Hom(S_\bullet(X,A),R) \arrow[r, "\pi^*"] \arrow[d, "j^*"] & Hom(S_\bullet(X),R) \arrow[r, "i^*"] \arrow[d, "j^*"] & Hom(S_\bullet(A),R) \arrow[r] \arrow[d, "j^*"] & 0, \\
0 \arrow[r] & Hom(S_\bullet^{U,V}(X,A),R) \arrow[r, "\pi^*"] & Hom(S_\bullet^{U,V}(X),R) \arrow[r, "i^*"] & Hom(S_\bullet^{U\cap A,V\cap A}(A),R) \arrow[r] & 0
\end{tikzcd}$$

where all maps (both horizontal and vertical) are (co)chain maps. (The exactness of the top row has already been discussed. As for the bottom row, its exactness follows from Exercise 3.14 once we observe that $S_k^{U,V}(X,A)$ is a free $\mathbb{Z}$-module for all k: specifically, $S_k^{U,V}(X,A)$ is isomorphic to the free $\mathbb{Z}$-module generated by those simplices $\sigma\colon \Delta^k \to X$ such that $Im(\sigma)$ is not contained in A but is contained either in U or in V.) Both the top and the bottom row have associated long exact sequences in cohomology, and the facts that the above diagram commutes and the vertical maps are chain maps implies that the following diagram commutes (check this yourself if you haven't seen it):

$$\begin{tikzcd}
H^{k-1}(X;R) \arrow[r, "i^*"] \arrow[d, "j^*"] & H^{k-1}(A;R) \arrow[r, "d"] \arrow[d, "j^*"] & H^k(X,A;R) \arrow[r, "\pi^*"] \arrow[d, "j^*"] & H^k(X;R) \arrow[r, "i^*"] \arrow[d, "j^*"] & H^k(A;R) \arrow[d, "j^*"] \\
H^{k-1}(Hom(S_\bullet^{U,V}(X),R)) \arrow[r, "i^*"] & H^{k-1}(Hom(S_\bullet^{U\cap A,V\cap A}(A),R)) \arrow[r, "d"] & H^k(Hom(S_\bullet^{U,V}(X,A),R)) \arrow[r, "\pi^*"] & H^k(Hom(S_\bullet^{U,V}(X),R)) \arrow[r, "i^*"] & H^k(Hom(S_\bullet^{U\cap A,V\cap A}(A),R))
\end{tikzcd}$$

Now the rows above are both exact and the diagram commutes, and the outer four maps j_* are all isomorphisms by (11). Hence the five-lemma implies that the innermost map $j^*\colon H^k(X,A;R) \to H^k(Hom(S_\bullet^{U,V}(X,A),R)$ is also an isomorphism. $\square$

Continuing to assume that we have an open cover $X = U \cup V$ and an arbitrary subspace $A \subset X$, the relative Mayer–Vietoris sequence is a consequence of the following short exact sequence of chain complexes:

(12)

$$0 \longrightarrow S_\bullet(U \cap V, (U \cap V) \cap A) \xrightarrow{f} S_\bullet(U, U \cap A) \oplus S_\bullet(V, V \cap A) \xrightarrow{g} S_\bullet^{U,V}(X,A) \longrightarrow 0$$

Here the map f is induced by the map $\tilde{f}\colon S_\bullet(U \cap V) \to S_\bullet(U) \oplus S_\bullet(V)$ which sends a simplex $\sigma\colon \Delta^k \to U \cap V$ to the pair $(\sigma, -\sigma) \in S_\bullet(U) \oplus S_\bullet(V)$. Meanwhile $g\colon S_\bullet(U, U \cap A) \oplus S_\bullet(V, V \cap A) \to S_\bullet^{U,V}(X,A)$ is induced by the map $\tilde{g}\colon S_\bullet(U) \oplus S_\bullet(V) \to S_\bullet^{U,V}(X)$ defined by, for simplices $\sigma\colon \Delta^k \to U$ and $\tau\colon \Delta^k \to (V)$, $\tilde{g}(\sigma,\tau) = \sigma + \tau$. Since $\tilde{f}$ and $\tilde{g}$ both preserve the property of mapping into A, they do indeed descend to maps f and g.

It is not hard to see that (12) is indeed exact. An element $\sum n_i \sigma_i \in S_\bullet(U \cap V)$ (with the $\sigma_i\colon \Delta^k \to U \cap V$ all distinct and n_i all nonzero) which is mapped by $\tilde{f}$ to an element of $S_\bullet(U \cap A) \oplus S_\bullet(V \cap A)$ would have all of its σ_i contained in A and would hence belong to $S_\bullet(U \cap V \cap A)$; hence the induced map $f\colon S_\bullet(U \cap V, (U \cap V) \cap A) \to S_\bullet(U, U \cap A) \oplus S_\bullet(V, V \cap A)$ is injective. It is obvious that $g \circ f = 0$, *i.e.* that $Im(f) \leq \ker(g)$ (this is just the equation $\sigma - \sigma = 0$). For the reverse inclusion, note first that any element of $S_\bullet(U, U \cap A) \oplus S_\bullet(V, V \cap A)$ can be uniquely represented by an element of $S_\bullet(U) \oplus S_\bullet(V)$ of the form $(\sum n_i \sigma_i, \sum m_j \tau_j)$, where the integers n_i and m_j are all nonzero, where each $\sigma_i\colon \Delta^k \to U$ and each $\tau_j\colon \Delta^k \to V$ has image not contained in A, and where the σ_i are all distinct from one another and the τ_j are all distinct from one another. This element of $S_\bullet(U, U \cap A) \oplus S_\bullet(V, V \cap A)$ lies in the kernel of g if and only if

$$\tilde{g}\left(\sum n_i \sigma_i, \sum m_j \tau_j\right) = \sum n_i \sigma_i + \sum m_j \tau_j \in S_\bullet^{U \cap A, V \cap A}(A).$$

But since the σ_i and τ_j all have image not contained in A, this can occur only if $\sum n_i \sigma_i + \sum m_j \tau_j = 0$. So we must have

$$\sum m_j \tau_j = -\sum n_i \sigma_i \in S_\bullet(U) \cap S_\bullet(V).$$

But evidently $S_\bullet(U) \cap S_\bullet(V) = S_\bullet(U \cap V)$, and so $(\sum n_i \sigma_i, \sum m_j \tau_j) = \tilde{f}(\sum n_i \sigma_i) \in Im(\tilde{f})$, and so the equivalence class of this element lies in the image of f.

Finally, the definition of $S_\bullet^{U,V}(X)$ makes it obvious that $\tilde{g}\colon S_\bullet(U) \oplus S_\bullet(V) \to S_\bullet^{U,V}(X)$ is surjective–essentially by definition, an element $x \in S_\bullet^{U,V}(X)$ has the form $x = \sigma + \tau$ where σ is a linear combination of simplices in U and τ is a linear combination of simplices in V, and in this case we have $x = \tilde{g}(\sigma,\tau)$. Since $\tilde{g}$ is surjective, its induced map g on the relative chain complex must also be surjective.

So (12) is indeed a short exact sequence of chain complexes. Taking the resulting long exact sequence on homology and using the analogue of Proposition 12 for homology gives the "Mayer-Vietoris sequence for relative homology," which at least in the absolute case where $A = \varnothing$ should be familiar. We will instead need the version of this sequence for relative cohomology. Since (as noted in the proof of Proposition 3.22) for each k the chain group $S_k^{U,V}(X,A)$ is a free abelian group, by Exercise 3.14 we have for any abelian group R a short exact sequence of cochain complexes[9]

$$0 \to Hom(S_\bullet^{U,V}(X,A),R) \xrightarrow{g^*} Hom(S_\bullet(U,U\cap A),R) \oplus Hom(S_\bullet(V,V\cap A),R) \xrightarrow{f^*} Hom(S_\bullet(U\cap V,(U\cap V)\cap A),R) \to$$

Combining the long exact sequence on cohomology associated to this short exact sequence of cochain complexes with the isomorphism $j^*\colon H^k(X,A;R) \to H^k(Hom(S_\bullet^{U,V}(X,A),R))$ from Proposition 3.22 we obtain:

Theorem 3.23 (Mayer-Vietoris sequence for relative cohomology). *Given a space X with an open cover $X = U \cup V$, a subspace $A \subset X$, and an abelian group R, there is a long exact sequence*

$$\begin{tikzcd}
H^{k+1}(X,A;R) \arrow[r] & \cdots & \\
H^k(X,A;R) \arrow[r] & H^k(U,U\cap A;R)\oplus H^k(V,V\cap A;R) \arrow[r] & H^k(U\cap V,(U\cap V)\cap A;R) \arrow[llu] \\
& \cdots \arrow[r] & H^{k-1}(U\cap V,(U\cap V)\cap A;R) \arrow[llu]
\end{tikzcd}$$

It is sometimes important to know what the horizontal maps in the above sequence are, so here is an exercise:

Exercise 3.24. Using j^* to identify $H^k(X,A;R)$ with $H^k(Hom(S_\bullet^{U,V}(X,A),R))$, prove that, according to our construction of the Mayer-Vietoris sequence as the long exact sequence associated to the dual exact sequence of (12),

- The map $H^k(X,A;R) \to H^k(U,U\cap A;R)\oplus H^k(V,V\cap A;R)$ is given by
$$x \mapsto (i_U^* x, i_V^* x),$$
where $i_U\colon (U,U\cap A) \to (X,A)$ and $i_V\colon (V,V\cap A)\to(X,A)$ are the inclusions.
- The map $H^k(U,U\cap A;R)\oplus H^k(V,V\cap A;R) \to H^k(U\cap V,(U\cap V)\cap A;R)$ is given by
$$(y,z) \mapsto j_U^* y - j_V^* z,$$
where $j_U\colon (U\cap V, U\cap V\cap A) \to (U,U\cap A)$ and $j_V\colon (U\cap V,U\cap V\cap A)\to (V,V\cap A)$ are the inclusions.

Given this identification of the maps, let us quickly note one consequence of the exactness of the sequence in Theorem 3.23: Suppose we have classes $y \in H^k(U,U\cap A;R)$ and $z\in H^k(V,V\cap A;R)$ which restrict to the same class in $H^k(U\cap V,U\cap V\cap A;R)$ (*i.e.*, $j_U^* y = j_V^* z$). Then there is a class $x \in H^k(X,A;R)$ which restricts both to y and to z (*i.e.*, $i_U^* x = y$ and $i_V^* x = z$—note that the converse to this statement is obvious: if x restricts both to y and to z then y and z must have the same restriction to $U\cap V$). In general such a class x is not uniquely determined by y and z; rather, given one such class x_0, all others differ from x_0 by an element of the image of the map $H^{k-1}(U\cap V,(U\cap V)\cap A;R)\to H^k(X,A;R)$. (One does however have uniqueness of x in the special case that $H^{k-1}(U\cap V,(U\cap V)\cap A;R) = 0$.)

[9]In the middle term we are making use of the standard isomorphism $Hom(N\oplus P,R)\cong Hom(N,R)\oplus Hom(P,R)$, given by sending a map $N\oplus P\to R$ to the pair consisting of its restriction to N and its restriction to P.

4. The Thom and Euler classes

We now know enough about cohomology to begin the construction of our first example of a characteristic class, the Euler class, for certain vector bundles.

Consider a rank-k vector bundle $\pi\colon E \to M$. There is a distinguished copy of M, the *zero-section* 0_E, inside E: define $s_0\colon M \to E$ by sending a point $m \in M$ to the zero element in the fiber $E_m = \pi^{-1}(\{m\})$ of E over m. If $U \subset M$ is an open subset with local trivialization $\Phi\colon E|_U \to U \times \mathbb{R}^k$, we have $\Phi \circ s_0(m) = (m, 0)$ for all $m \in U$ (since Φ restricts to each fiber E_m as a linear map to $\{m\} \times \mathbb{R}^k$). So since Φ is a homomorphism s_0 restricts to U is a continuous map; thus $s_0\colon M \to E$ is continuous since M is covered by open sets U of the sort just described. Similarly one sees that the image $0_E := s_0(M)$ is a closed subset of E, and that s_0 maps M homeomorphically to 0_E.

Now define

$$E^0 = E \setminus 0_E,$$

i.e., E^0 is the *complement* of the zero section in E. For $A \subset M$ we denote $E|_A = \pi^{-1}(A)$ and $E^0|_A = E|_A \cap E^0$, and for $m \in M$ we denote $E_m = \pi^{-1}(\{m\})$ and $E^0_m = E_m \cap E^0$.

It turns out to be worthwhile to investigate the *relative cohomology* $H^*(E, E^0; R)$ for various coefficient rings R.[10] Specifically we will work either with $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$. *In the case when $R = \mathbb{Z}$ we will additionally assume that E is an oriented vector bundle*, but we will not need this assumption when $R = \mathbb{Z}/2\mathbb{Z}$.

For each $m \in M$ the inclusion $i_m\colon E_m \to E$ of the fiber over m maps E^0_m, and so there is an induced map

$$i^*_M\colon H^*(E, E^0; R) \to H^*(E_m, E^0_m; R).$$

Now (E_m, E^0_m) is homeomorphic (as a pair) to $(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$, and we have computed the relative cohomology of $(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ in Example 3.21: $H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; R)$ is isomorphic to R, while for $j \neq k$ $H^j(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; R)$ is trivial.

Let us choose once and for all (independently of the vector bundle E) a generator ω of $H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; R)$. Of course, if $R = \mathbb{Z}/2\mathbb{Z}$ there is no choice here since there is only one generator (indeed only one nonzero element). If $R = \mathbb{Z}$ then there are two possibilities (since both 1 and -1 generate the abelian group $\mathbb{Z}$), so we need to choose one or the other. We could of course just say that we are doing so arbitrarily, though this would lead to a sign ambiguity in later definitions. For definiteness, we can note that the simplex $\sigma\colon \Delta^k \to \mathbb{R}^k$ obtained by translating the standard simplex in $\mathbb{R}^k$ by a small amount (say by the vector $(-\epsilon, \ldots, -\epsilon)$ for small $\epsilon > 0$) so that the origin is contained in the interior defines a relative cycle in $(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ (since its boundary maps to $\mathbb{R}^k \setminus \{0\}$) and it is not too difficult to see that the relative homology class $[\sigma]$ of this cycle generates $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; \mathbb{Z})$ (check that the boundary of σ generates the reduced homology of $\mathbb{R}^k \setminus \{0\}$, and use the long exact sequence of the pair). Then let $\omega \in H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; \mathbb{Z})$ be the unique element which evaluates as 1 on $[\sigma]$.

Return now to our vector bundle $\pi\colon E \to M$, which is arbitrary if $R = \mathbb{Z}/2\mathbb{Z}$ but is assumed oriented if $R = \mathbb{Z}$. For any $m \in M$ we may choose an open neighborhood U of m and a local trivialization $\Phi\colon E|_U \to U \times \mathbb{R}^k$, say given by $\Phi(e) = (\pi(e), \phi(e))$ where $\phi\colon E|_U \to \mathbb{R}^k$. If $R = \mathbb{Z}$ we assume this local trivialization to be compatible with the orientation that is assumed on E. We thus have a (linear) homeomorphism of pairs $\phi \circ i_m\colon (E_m, E^0_m) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$.

Define

$$\mu_m = (\phi \circ i_m)^* \omega \in H^k(E_m, E^0_m; R).$$

[10] Perhaps a more obvious cohomology to consider would be $H^*(E, 0_E; R)$, but since E deformation retracts to 0_E by the map $(t, e) \mapsto te$, we have $H^*(E, 0_E; R) = 0$.

We claim that μ_m is independent of this choice of trivialization Φ. If $R=\mathbb{Z}/2\mathbb{Z}$ this is obvious, since μ_m may be alternatively characterized as the unique nonzero element of $H^k(E_m,E_m^0;\mathbb{Z}/2\mathbb{Z})$. Suppose instead that $R=\mathbb{Z}$. If $\Phi'\colon\, e\mapsto(\pi(e),\phi'(e))$ is another choice of local trivialization defined over an open set containing m, then since we are requiring the local trivializations to be orientation-compatible there is $g\in GL^+(k;\mathbb{R})$ such that $(\phi'\circ i_m)\circ(\phi\circ i_m)^{-1}=g$, *i.e.*, $\phi'\circ i_m = g\circ(\phi\circ i_m)$. But then since $GL^+(k;\mathbb{R})$ is path-connected, there is a path $\gamma\colon\,[0,1]\to GL^+(k;\mathbb{R})$ such that $\gamma(0)=g$ and $\gamma(1)$ is the identity. Then $(t,e)\mapsto\gamma(t)\circ(\phi\circ i_m)$ gives a homotopy of maps of pairs $(E_m,E_m^0)\to(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$. So indeed $(\phi\circ i_m)^*\omega=(\phi'\circ i_m)^*\omega$.

Thus to each point m in the base M of the vector bundle $\pi\colon\, E\to M$ we have assigned a distinguished element $\mu_m\in H^k(E_m,E_m^0;R)$. (When $R=\mathbb{Z}$ this depends on the orientation of E: indeed it is not hard to check that reversing the orientation of E would reverse the sign of μ_m.)

Definition 4.1. If $\pi\colon\, E\to M$ is a rank-k vector bundle (assumed oriented if $R=\mathbb{Z}$), a **Thom class** for E with coefficients in R is a relative cohomology class $\tau\in H^k(E,E^0;R)$ such that, for all $m\in M$,

$$i_m^*\tau=\mu_m$$

where $i_m\colon\, E_m\to E$ is the inclusion of the fiber over m.

Our next goal is to show that such classes exist and are unique in some generality. We start with the following.

Lemma 4.2. *Suppose M is a paracompact Hausdorff space which is contractible. If $\pi\colon\, E\to M$ is any rank-k vector bundle then there exists a unique Thom class for E with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and if additionally E is oriented then there exists a unique Thom class for E with coefficients in $\mathbb{Z}$.*

Proof. As noted in Remark 2.30, the homotopy invariance of pullbacks of vector bundles over paracompact Hausdorff spaces implies that E is trivial, and we may choose a trivialization $\Psi\colon\, E\to M\times\mathbb{R}^k$ (so Ψ is a homeomorphism which is a linear isomorphism on each fiber).

If E is oriented, I claim that we may choose the trivialization Ψ to be compatible with the orientation (incidentally, the argument to follow does not depend on the topological hypotheses on M, but only on the fact that E is trivial). Indeed, let $\{\Phi_\alpha\colon\, E|_{U_\alpha}\to U_\alpha\times\mathbb{R}^k\}$ be a collection of orientation-compatible local trivializations (so the transition functions $\Phi_\beta\colon\,\Phi_\alpha^{-1}$ are given by $(m,v)\mapsto(m,g_{\alpha\beta}(m)v)$ for continuous functions $g_{\alpha\beta}\colon\, U_\alpha\cap U_\beta\to GL^+(k;\mathbb{R})$). Given an initial global trivialization Ψ for E which may not be orientation-compatible, For each α we may define a function $f_\alpha\colon\, U_\alpha\to GL(k;\mathbb{R})$ by the property that $\Psi\circ\Phi_\alpha^{-1}\colon\, U_\alpha\times\mathbb{R}^k\to U_\alpha\times\mathbb{R}^k$ is given by $(m,v)\mapsto(m,f_\alpha(m)v)$. The fact that $\det g_{\alpha\beta}(m)>0$ implies that, if $m\in U_\alpha\cap U_\beta$, then $\det f_\alpha(m)$ and $\det f_\beta(m)$ have the same sign. So there is a continuous function $\delta\colon\, M\to\{1,-1\}$ which restricts to each U_α as $m\mapsto\frac{\det f_\alpha(m)}{|\det f_\alpha(m)|}$. Where $\{e_1,\dots,e_k\}$ is the standard basis for $\mathbb{R}^k$, define $A\colon\, M\to GL(k;\mathbb{R})$ by $A(m)e_1=\delta(m)e_1$ and $A(m)e_i=e_i$ for $i\geq 2$. (So in particular $\det A(m)=\delta(m)$.) If the initial trivialization Ψ is given by $e\mapsto(\pi(e),\psi(e))$, the new trivialization $\Psi'\colon\, E\to M\times\mathbb{R}^k$ given by $\Psi'(e)=(\pi(e),A(\pi(e))\psi(e))$ will be compatible with the orientation on E given by the Φ_α.

Having arranged Ψ to be compatible with the orientation (in the case that E is oriented), let $p_2\colon\, M\times\mathbb{R}^k$ be the projection. Then for each $m\in M$, we have by definition

$$\mu_m=(p_2\circ\Psi\circ i_m)^*\omega=i_m^*(p_2\circ\Psi)^*\omega.$$

But this precisely says that $\tau=(p_2\circ\Psi)^*\omega$ is a Thom class, proving existence (for either coefficient group $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$). As for uniqueness, just note that the fact that M is contractible implies that $p_2\colon\,(M\times\mathbb{R}^k,M\times(\mathbb{R}^k\setminus\{0\}))\to(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$ is a homotopy equivalence of pairs, so since

$\Psi\colon (E,E^0)\to(M\times\mathbb{R}^k, M\times(\mathbb{R}^k\setminus\{0\}))$ is a homeomorphism, $p_2\circ\Psi$ is a homotopy equivalence of pairs. Meanwhile for each $m\in M$, $p_2\circ\Psi\circ i_m\colon (E_m,E^0_m)\to(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$ is also a homotopy equivalence of pairs (indeed a homeomorphism), so homotopy-inverting $p_2\circ\Psi$ shows that $i_m\colon (E_m,E^0_m)\to(E,E^0)$ is a homotopy equivalence of pairs. So for any m and any coefficient ring R there can only be one class $\tau\in H^k(E,E^0;R)$ which restricts to (E_m,E^0_m) as μ_m, which is (more than) enough to prove uniqueness of the Thom class. □

Our plan now is to make use of the Mayer–Vietoris sequence to obtain Thom classes for vector bundles over a fairly general class of spaces. In fact this can be done in rather more generality than we will work in (see [MS, Chapter 10]), though doing so requires more machinery.

Definition 4.3. A *good cover* $\{U_\alpha\}_{\alpha\in J}$ of a topological space X is an open cover with the property that for each finite subset $\{\alpha_1,\ldots,\alpha_m\}\subset J$ the intersection $U_{\alpha_1}\cap\cdots\cap U_{\alpha_m}$ is either empty or contractible.

Theorem 4.4. *Let M be a Hausdorff space in which every open subset is paracompact, and assume that M has a finite good cover. If $\pi\colon E\to M$ is any rank-k vector bundle then there exists a unique Thom class for E with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and if additionally E is oriented then there exists a unique Thom class for E with coefficients in $\mathbb{Z}$. Moreover we have*

$$H^j(E,E^0;R)=0$$

for all coefficient rings R and all $j<k$.

Remark 4.5. It is a fact that any smooth manifold has a good cover (see [BT, Theorem 5.1]), as does any simplicial complex (this is easier—just construct the open sets in the cover from the stars of the various vertices). If the manifold or simplicial complex is compact then it of course follows that it has a finite good cover (so this applies to $Gr_k(\mathbb{R}^N)$, though not to the infinite Grassmannian Gr_k—once we know more about the topology of these spaces we will however be able to work around this problem). Manifolds and CW complexes (and hence in particular simplicial complexes) also have the property that all of their open subsets are paracompact (this property is called "hereditary paracompactness" in the literature). Of course in the case of manifolds this just follows from the fact that manifolds are paracompact (see *e.g.* [Lee, Chapter 2]) since open subsets of manifolds are manifolds; for CW complexes see [FP, Section 1.3].

Proof of Theorem 4.4. The proof is by induction on the (minimal) number r of open sets required to form a good cover of the space M. For the case that M admits a good cover by just one open set, of course that open set must be M itself and so M is contractible. So the statement about the Thom class has been proven as Lemma 4.2. As for the statement about $H^j(E,E^0;R)$ for $j<k$, let $\Psi\colon E\to M\times\mathbb{R}^k$ be a trivialization of E, which exists since M is contractible, paracompact, and Hausdorff. Then where $p_2\colon M\times\mathbb{R}^k\to\mathbb{R}^k$ is the projection, as noted in the proof of Lemma 4.2 $\pi_2\circ\Psi\colon (E,E^0)\to(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$ is a homotopy equivalence of pairs. So since $H^j(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\};R)=0$ for $j<k$ and for all coefficient rings R by Example 3.21, the same holds for $H^j(E,E^0;R)$.

So we may assume inductively that the theorem holds for vector bundles over all spaces that admit open covers by at most a open sets, where $a\geq 1$. Now suppose that we have a good open cover

$$M=U_1\cup\cdots\cup U_a\cup V$$

by $a+1$ open sets, and let $U=U_1\cup\cdots\cup U_a$. Now V is contractible, so the theorem holds for bundles over V. Since $U=U_1\cup\cdots\cup U_a$ is a good open cover of U by a open sets, the theorem holds for bundles over U. Also, we have

$$U\cap V=(U_1\cap V)\cup\cdots\cup(U_a\cap V),$$

which is a good open cover of $U \cap V$ by a open sets, so the theorem holds for bundles over $U \cap V$.

We apply the Mayer–Vietoris sequence for relative cohomology to the pair (E, E^0), writing $E = (E|_U) \cup (E|_V)$ and noting that $(E|_U) \cap (E|_V) = E|_{U \cap V}$. For $j < k$ and any coefficient ring R, part of this sequence reads

$$H^{j-1}(E|_{U\cap V}, E^0|_{U\cap V}; R) \longrightarrow H^j(E, E^0; R) \longrightarrow H^j(E|_U, E^0|_U; R) \oplus H^j(E|_V, E^0|_V; R)$$

and by induction the outer two terms are zero, so $H^j(E, E^0; R) = 0$. This proves that the second statement of the inductive hypothesis holds for M; it remains to prove the statement about the Thom class.

So we specialize to coefficients $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$, and if $R = \mathbb{Z}$ we assume that E is oriented. Another part of the Mayer–Vietoris sequence reads

(13)
$$H^{k-1}(E|_{U\cap V}, E^0|_{U\cap V}; R) \xrightarrow{\delta} H^k(E, E^0; R) \xrightarrow{f} H^k(E|_U, E^0|_U; R) \oplus H^k(E|_V, E^0|_V; R) \xrightarrow{g} H^k(E|_{U\cap V}, E^0|_{U\cap V}; R)$$

By induction we have unique Thom classes $\tau_U, \tau_V, \tau_{U\cap V}$ for $E|_U$, $E|_V$, and $E|_{U\cap V}$, respectively. Recall from Exercise 3.24 that $f(x) = (x|_U, x|_V)$ and $g(y,z) = y|_{U\cap V} - z|_{U\cap V}$, where the meaning of our restriction notation should be obvious (really we should write $x|_{E|_U}$, $y|_{E|_{U\cap V}}$, etc.).

If $m \in U \cap V$, then since τ_U is a Thom class for $E|_U$ we have

$$i_m^*(\tau_U|_{U\cap V}) = i_m^* \tau_U = \mu_m$$

and likewise since τ_V is a Thom class for $E|_V$ we have $i_m^*(\tau_V|_{U\cap V}) = \mu_m$. This means that both $\tau_U|_{U\cap V}$ and $\tau_V|_{U\cap V}$ are Thom classes for $E|_{U\cap V}$. But by induction there is only one Thom class for $E|_{U\cap V}$, so we must have $\tau_U|_{U\cap V} = \tau_V|_{U\cap V}$, *i.e.* $g(\tau_U, \tau_V) = 0$. So by the exactness of (13) there is $\tau \in H^k(E, E^0; R)$ such that $\tau|_U = \tau_U$ and $\tau|_V = \tau_V$. If $m \in M$, then either $m \in U$ in which case $i_m^*\tau = i_m^*(\tau|_U) = \mu_m$, or else $m \in V$ in which case $i_m^*\tau = i_m^*(\tau|_V) = \mu_m$. This proves that τ is a Thom class for E. For uniqueness, if σ were another Thom class for E then by the same argument as before $\sigma|_U$ would be a Thom class for $E|_U$ and $\sigma|_V$ would be a Thom class for $E|_V$. But by induction the Thom classes of $E|_U$ and $E|_V$ are unique, so $\sigma|_U = \tau|_U$ and $\sigma|_V = \tau|_V$, *i.e.* $f(\sigma) = f(\tau)$. Now again by induction, $H^{k-1}(E|_{U\cap V}, E^0|_{U\cap V}; R) = 0$, so the exactness of (13) shows that f is injective and so $\sigma = \tau$. This proves that the Thom class is unique, completing the induction and hence the proof. □

We can now define our first example of a characteristic class. Recall the zero-section $s_0 \colon M \to E$; we can view this as map of pairs $(M, \varnothing) \to (E, E^0)$ so that there is an induced map $s_0^* \colon H^*(E, E^0; R) \to H^*(M; R)$.

Definition 4.6. Given a vector bundle $\pi \colon E \to M$ with Thom class τ (with coefficients in $R = \mathbb{Z}/2\mathbb{Z}$, or in $R = \mathbb{Z}$ if E is oriented), the *Euler class* of E is the cohomology class

$$e(E) = s_0^* \tau \in H^k(M; R).$$

Remark 4.7. It is sometimes useful to note that the map of pairs $s_0 \colon (M, \varnothing) \to (E, E^0)$ factors as $(M, \varnothing) \to (E, \varnothing) \to (E, E^0)$, where the first map is a homotopy equivalence and the second map is the inclusion. So we have induced maps $H^k(E, E^0; R) \to H^k(E; R) \to H^k(M; R)$, where the first map is the map that appears in the cohomology long exact sequence of the pair (E, E^0) and the second map is an isomorphism; the Euler class is the image of the Thom class under this composition. In particular the Euler class is zero if and only if the Thom class is in the kernel of the map $H^k(E, E^0; R) \to H^k(E; R)$ that appears in the long exact sequence of the pair.

Remark 4.8. In principle one could imagine a bundle having more than one Thom class, in which case the Euler class as defined above would depend on the Thom class. Since Theorem

4.4 includes a uniqueness statement for the Thom class, this issue does not arise for bundles over suitably nice spaces (including Grassmannians of k-planes in $\mathbb{R}^N$). We will shortly see how to canonically remove the possible ambiguity at least for finite-type bundles over paracompact Hausdorff spaces by using classifying maps. (In fact this isn't really an issue, since for vector bundles over paracompact Hausdorff spaces there always exists a unique Thom class with appropriate coefficients, though we won't show this in full generality.)

Example 4.9. For the trivial bundle $E_{triv} = M \times \mathbb{R}^k$ over a space M, a Thom class is given by $\tau = p_2^*\omega \in H^k(M \times \mathbb{R}^k, M \times (\mathbb{R}^k \setminus \{0\}); R)$ where p_2 is the projection. Now $p_2 \circ s_0 \colon M \to \mathbb{R}^k$ is the constant map (to $\vec{0}$), so $(p_2 \circ s_0)^*$ acts trivially on H^j for $j > 0$. Thus

$$e(E_{triv}) = s^* p_2^* \omega = (p_2 \circ s_0)^* \omega = 0.$$

Thus, contrapositively, if a vector bundle $\pi \colon E \to M$ has nonzero Euler class then E must be nontrivial. Actually more is true.

Definition 4.10. A *section* of a vector bundle $\pi \colon E \to M$ is a continuous map $s \colon M \to E$ such that $\pi \circ s = 1_M$.

In other words a section s assigns to each $m \in M$ an element $s(m)$ of the vector space E_m, in a continuous fashion. We will say that a section s is *nonvanishing* if, for every $m \in M$, $s(m)$ is a *nonzero* element of E_m.

Proposition 4.11. *Let $\pi \colon E \to M$ be any rank-k vector bundle with a nonvanishing section. Then if E has a Thom class with coefficients in R, the Euler class $e(E) \in H^k(M; R)$ is zero.*

Proof. If $s \colon M \to E$ is a nonvanishing section then the map $S \colon [0,1] \times M \to E$ defined by $S(t,m) = ts(m)$ defines a homotopy from the zero-section s_0 to s. Hence the maps $s_0^* \colon H^k(E, E^0; R) \to H^k(M; R)$ and $s^* \colon H^k(E, E^0; R) \to H^k(M; R)$ are equal, and so where τ is a Thom class for E we have $e(E) = s^*\tau$.

The assumption that s is nonvanishing amounts to the statement that the image of s is in fact contained in E^0, so (via the composition of maps of pairs $(M, \varnothing) \to (E^0, E^0) \to (E, E^0)$) the map $s^* \colon H^k(E, E^0; R) \to H^k(M; R)$ splits as a composition

$$H^k(E, E^0; R) \longrightarrow H^k(E^0, E^0; R) \longrightarrow H^k(M; R)$$

But the group $H^k(E^0, E^0; R)$ is obviously zero (use the long exact sequence of the pair (E^0, E^0), or just note that $H^k(E^0, E^0; R)$ is the cohomology of the zero cochain complex). So the map $s^* \colon H^k(E, E^0; R) \to H^k(M; R)$ must be zero, and in particular $e(E) = s^*\tau = 0$. □

Now a trivial bundle $M \times \mathbb{R}^k$ obviously has nonvanishing sections, namely $m \mapsto (m, \vec{v})$ for any nonzero vector $\vec{v} \in \mathbb{R}^k$. But the converse is not true: starting with any rank-k vector bundle $E \to M$, as we will discuss in detail later one can form the "direct sum" $E \oplus \underline{\mathbb{R}}$ with the trivial rank-one bundle over M, and one can obtain nonvanishing sections of the rank-$k+1$ bundle $E \oplus \underline{\mathbb{R}} \to M$ by mapping into the second summand. It will follow from something that we will do later that if E has nonvanishing Euler class *over* $\mathbb{Z}/2\mathbb{Z}$ then $E \oplus \underline{\mathbb{R}}$ is nontrivial (because it has a nontrivial Stiefel–Whitney class; this statement is false for Euler classes over $\mathbb{Z}$, as can be seen for instance when E is the tangent bundle to an even-dimensional sphere). For vector bundles V over paracompact Hausdorff spaces, one can use a Euclidean structure on V to show that V has a nonvanishing section if and only if V is isomorphic to a direct sum $E \oplus \underline{\mathbb{R}}$ (take E equal to the orthogonal complement of the nonvanishing section).

Here are some examples where the Euler class is nontrivial:

Example 4.12. Let $\pi\colon E \to S^1$ be the Möbius bundle. As usual we identify S^1 with $\mathbb{R}/\mathbb{Z}$, and $E = [0,1] \times \mathbb{R}/(1,t) \sim (0,-t)$. Now the bundle E is not orientable, so we only have Thom and Euler classes over $\mathbb{Z}/2\mathbb{Z}$, not over $\mathbb{Z}$; we will now find them.

As with any vector bundle, as a topological space E is homotopy equivalent to the base of the bundle, which in this case is S^1. Thus $H^k(E;\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $k = 0,1$ and is trivial otherwise. Meanwhile, the complement E^0 of the zero section is also homotopy equivalent to S^1: the map

$$x \mapsto \begin{cases} [2x,1] & 0 \le x \le 1/2 \\ [2x-1,-1] & 1/2 \le x \le 1 \end{cases}$$

embeds S^1 into E^0; if we denote by A the image of this embedding then the map $[s,t] \mapsto [s, \frac{t}{|t|}]$ gives a deformation retract of E^0 to A. (You can think of A as the boundary of a copy of the closed Möbius strip inside E.)

In particular E^0, like E, is path-connected, so the restriction map $H^0(E;\mathbb{Z}/2\mathbb{Z}) \to H^0(E^0;\mathbb{Z}/2Z)$ is an isomorphism. Consequently the map $H^1(E,E^0;\mathbb{Z}/2\mathbb{Z}) \to H^1(E;\mathbb{Z}/2\mathbb{Z})$ is injective. In fact it's an isomorphism—to see this, note that the Thom class τ_E of E (which exists by Theorem 4.4) is a nontrivial element of $H^1(E,E^0;\mathbb{Z}/2\mathbb{Z})$ (it restricts nontrivially to the fibers), so $H^1(E,E^0;\mathbb{Z}/2\mathbb{Z})$ has at least two elements, while $H^1(E;\mathbb{Z}/2\mathbb{Z})$ has exactly two elements, so an injective homomorphism from the former to the latter has to be an isomorphism. So we infer that $H^1(E,E^0;\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, with τ_E the unique nonzero element.

Thus the canonical map $H^1(E,E^0;\mathbb{Z}/2\mathbb{Z}) \to H^1(E;\mathbb{Z}/2\mathbb{Z})$ (which is the map induced by the inclusion $(E,\varnothing) \hookrightarrow (E,E^0)$) sends the Thom class τ_E to the generator of $H^1(E;\mathbb{Z}/2\mathbb{Z})$. The Euler class is the image of τ_E under the composition

$$H^1(E,E^0;\mathbb{Z}/2\mathbb{Z}) \to H^1(E;\mathbb{Z}/2\mathbb{Z}) \to H^1(S^1;\mathbb{Z}/2\mathbb{Z})$$

where the second map is induced by the inclusion s_0 of the zero section and is an isomorphism since E deformation retracts to its zero section. This proves that

For the Möbius bundle $\pi\colon E \to S^1$, $e(E)$ is the generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$.

Example 4.13. The simplest nontrivial example of an *oriented* bundle is probably the *tautological complex line bundle* $E = \gamma^1(\mathbb{C}^2)$ over $\mathbb{C}P^1 = Gr_1(\mathbb{C}^2)$.[11] Thus

$$E = \{(V,\vec{v}) \in Gr_1(\mathbb{C}^2) \times \mathbb{C}^2 | \vec{v} \in V\}$$

and $E^0 \subset E$ consists of those $(V,\vec{v})$ where $\vec{v} \neq 0$. Now a one-dimensional complex vector space is uniquely determined by a single nonzero element in it, in view of which the projection $p_2\colon E^0 \to \mathbb{C}^2 \setminus \{\vec{0}\}$ is a homeomorphism.

As usual, E is homotopy equivalent to the base $\mathbb{C}P^1$ of the bundle, which of course is homeomorphic to S^2. Meanwhile, E^0 is, by the previous paragraph, homotopy equivalent to S^3. In particular $H^1(E^0;\mathbb{Z}) = H^2(E^0;\mathbb{Z}) = 0$, so part of the cohomology long exact sequence of the pair reads

$$0 \longrightarrow H^2(E,E^0;\mathbb{Z}) \longrightarrow H^2(E;\mathbb{Z}) \longrightarrow 0$$

[11]Incidentally, any complex vector bundle is naturally oriented bundle: in terms of structure groups the point here is that $GL(n;\mathbb{C}) \le GL^+(2n;\mathbb{R})$, *i.e.* that any real $(2n)\times(2n)$ matrix that commutes with the matrix J_0 that represents multiplication by i has positive determinant. More geometrically, given a finite-dimensional complex vector space V, there is a natural orientation induced on the underlying real vector space of V: if $\{e_1,\dots,e_n\}$ is a basis for V over $\mathbb{C}$, use the orientation on V with respect to which $\{e_1,\dots,e_n,ie_1,\dots,ie_n\}$ is a positive basis. This can be seen to be independent of the choice of complex basis by virtue of the fact that the set of complex bases for V is connected (optional exercise: prove this, perhaps by first using Gram-Schmidt to reduce to the case of orthonormal bases and then using induction on the dimension).

As in the previous example, whatever the Thom class $\tau_E \in H^2(E,E^0;\mathbb{Z})$ is, it must be nontrivial since it restricts nontrivially to fibers, so by the above exact sequence τ_E maps to a nontrivial element under $H^2(E,E^0;\mathbb{Z}) \longrightarrow H^2(E;\mathbb{Z})$.

Hence by Remark 4.7 the Euler class $e(E)$ is a nontrivial element of $H^2(\mathbb{C}P^1;\mathbb{Z})$. In fact, with a little more care we can see that $e(E)$ generates $H^2(\mathbb{C}P^1;\mathbb{Z})$. Namely, since $H^2(E;\mathbb{Z}) \cong H^2(\mathbb{C}P^1;\mathbb{Z}) \cong \mathbb{Z}$ and since we have seen that the canonical map $H^2(E,E^0;\mathbb{Z}) \to H^2(E;\mathbb{Z})$ is an isomorphism, it follows that $H^2(E,E^0;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Now the Thom class τ_E is characterized by the fact that, for any $m \in \mathbb{C}P^1$, τ_E is mapped to the generator μ_m of $H^2(E_m,E_m^0;\mathbb{Z})$ under restriction. Since $H^2(E,E^0;\mathbb{Z}) \cong \mathbb{Z}$, τ_E is n times a generator of $H^2(E,E^0;\mathbb{Z})$ for some $n \geq 1$; but then this other generator would need to map under restriction to $\frac{1}{n}\mu_m$, which is only possible of $n=1$ since we are working over $\mathbb{Z}$. So τ_E generates $H^2(E,E^{;}\mathbb{Z})$, and so since $e(E) = s_0^*\tau_E$ and we have seen that $s_0^*\colon H^2(E,E^0;\mathbb{Z}) \to H^2(\mathbb{C}P^1;\mathbb{Z})$ is an isomorphism this proves that $e(E)$ is a generator for $H^2(\mathbb{C}P^1;\mathbb{Z})$.

Exercise 4.14. Viewing S^2 as a subset of $\mathbb{R}^3$ in the usual way, the *tangent bundle* TS^2 is given as a topological space as
$$TS^2 = \{(\vec{v},\vec{x}) \in S^2 \times \mathbb{R}^3 | \vec{v}\cdot\vec{x} = 0\}$$
where $\cdot$ is the standard dot product, with the bundle[12] projection $\pi\colon TS^2 \to S^2$ given by the projection $(\vec{v},\vec{x}) \mapsto \vec{v}$. The standard orientation on TS^2 is given by saying that a basis $\{\vec{x},\vec{y}\}$ for $T_{\vec{v}}S^2 = \pi^{-1}(\{\vec{v}\})$ is a positive basis if and only if $\{\vec{v},\vec{x},\vec{y}\}$ is a positive basis for $\mathbb{R}^3$.

(a) Prove that the complement of the zero section $(TS^2)^0$ is homotopy equivalent to $\{(\vec{v},\vec{x} \in S^2 \times S^2 | \vec{v}\cdot\vec{x} = 0\}$, which in turn is homeomorphic to the group $SO(3)$ of orthogonal 3×3 matrices with determinant one.

(b) It is a fact that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. Given this fact, prove that the Euler class with coefficients in $\mathbb{Z}$ of the oriented vector bundle TS^2 is nonzero, while the Euler class with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is zero.

Exercise 4.15. If $\phi\colon R \to S$ is any homomorphism of A-modules and if $C_\bullet$ is any chain complex of A-modules, there is an induced map of cochain complexes $\phi_*\colon C_R^\bullet \to C_S^\bullet$ (where as before $C_R^\bullet = Hom_A(C_\bullet,R)$) defined by $\phi_*\alpha = \phi\circ\alpha$.

(a) Prove that ϕ_* is a cochain map (*i.e.* it commutes appropriately with the coboundary operators δ on the cochain complexes).

(b) Let $\pi\colon E \to M$ be an oriented vector bundle with $\mathbb{Z}$-coefficient Thom class $\tau^{\mathbb{Z}} \in H^k(E,E^0;\mathbb{Z})$. Where in the notation of part (a) we set $A = R = \mathbb{Z}$, $S = \mathbb{Z}/2\mathbb{Z}$, $\phi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient, and $C_\bullet$ is the relative singular chain complex $S_*(E,E^0)$, by part (a) ϕ_* induces a map $\underline{\phi}_*\colon H^k(E,E^0;\mathbb{Z}) \to H^k(E,E^0;\mathbb{Z}/2\mathbb{Z})$. Prove that $\underline{\phi}_*\tau^{\mathbb{Z}}$ is a Thom class of E with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Remark 4.16. Exercise 4.15 suggests a definition for the Thom class of an oriented bundle $E \to M$ with coefficients in an arbitrary ring with unity R: let $\phi\colon \mathbb{Z} \to R$ be the additive group homomorphism which maps 1 to the multiplicative identity in R, and put $\tau^R = \underline{\phi}_*\tau^{\mathbb{Z}} \in H^k(E,E^0;R)$. The resulting Euler class $e^R = s_0^*\tau^R$ with coefficients in R likewise has $e^R = \underline{\phi}_*e^{\mathbb{Z}} \in H^k(M;R)$.

As should be expected of a characteristic class, the Euler class is functorial with respect to pullbacks of bundles; indeed we will see that this functoriality is inherited from a corresponding property for the Thom class. In general, as I'll leave to you to check, if $f\colon X \to Y$ is a continuous

[12] It's not too hard to prove that this is an oriented vector bundle, but I'm not asking you to do so.

map and $\pi_W\colon W \to Y$ is a vector bundle, then a vector bundle $\pi_V\colon V \to X$ is isomorphic to the pullback f^*W if and only if there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi_V"'] & W \arrow[d, "\pi_W"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that for each $x \in X$, $\tilde{f}$ maps V_x by a linear isomorphism to $W_{f(x)}$. (If V and W are oriented, then for an oriented-bundle isomorphism $V \cong f^*W$ one should additionally ask for the restriction of $\tilde{f}$ to each fiber to respect orientation.) In particular in this case we have a map of pairs $\tilde{f}\colon (V, V^0) \to (W, W^0)$.

Proposition 4.17. *In the above situation, if $\tau_W \in H^k(W, W^0; R)$ is a Thom class for W with Euler class $e(W) \in H^k(Y; R)$, then $\tilde{f}^*\tau_W \in H^k(V, V^0; R)$ is a Thom class with Euler class $f^*e(W) \in H^k(X; R)$.*

Proof. If $m \in X$, let $U \subset Y$ be an open neighborhood of $f(m)$ which admits a local (oriented, if applicable) trivialization $\Phi\colon W|_U \to U \times \mathbb{R}^k$, given in coordinates by $w \mapsto (\pi_W(w), \phi(w))$. Then the map $\Psi\colon V|_{f^{-1}(U)} \to f^{-1}(U) \times \mathbb{R}^k$ defined by $\Psi(v) = (\pi_V(V), \phi(\tilde{f}(v)))$ is a local trivialization for U around m, and so the distinguished generator $\mu_m \in H^k(V_m, V_m^0; R)$ is given by

$$\mu_m = i_m^*(\phi \circ \tilde{f})^*\omega = (\tilde{f} \circ i_m)^*(\phi^*\omega)$$

where $i_m\colon V_m \to V$ is the inclusion Meanwhile where $i_{f(m)}\colon W_{f(m)} \to W$ is the inclusion the distinguished generator $\mu_{f(m)} \in H^k(W_{f(m)}, W^0_{f(m)}; R)$ is given by $\mu_{f(m)} = i^*_{f(m)}(\phi^*\omega)$.

Let us write $\tilde{f}_m\colon V_m \to W_{f(m)}$ for the restriction of $\tilde{f}$, so that we obviously have

$$i_{f(m)} \circ \tilde{f}_m = \tilde{f} \circ i_m.$$

Thus

$$\mu_m = (i_{f(m)} \circ \tilde{f}_m)^*(\phi^*\omega) = \tilde{f}_m^* i^*_{f(m)} \phi^*\omega = \tilde{f}_m^* \mu_{f(m)}.$$

Now the Thom class τ_W has $i^*_{f(m)}\tau_W = \mu_{f(m)}$ for all $m \in X$, so we see that

$$\begin{aligned} i_m^*(\tilde{f}^*\tau_W) &= (\tilde{f} \circ i_m)^*\tau_W = (i_{f(m)} \circ \tilde{f}_m)^*\tau_W = \tilde{f}_m^* i^*_{f(m)}\tau_W \\ &= \tilde{f}_m^*\mu_{f(m)} = \mu_m, \end{aligned}$$

confirming that $\tilde{f}^*\tau_W$ is a Thom class for V.

Where s_0^V and s_0^W are the zero-sections of V and W respectively we evidently have $s_0^W \circ f = \tilde{f} \circ s_0^V$. Hence the Euler classes are related by

$$e(V) = (s_0^V)^*(\tilde{f}^*\tau_W) = f^*(s_0^W)^*\tau_W = f^*e(W).$$

□

Let us return to the subject of (pre-)classifying maps from Section 2. If $\pi\colon E \to M$ is a finite-type, rank-k vector bundle over a paracompact Hausdorff space, Proposition 2.5 showed that there is a pre-classifying map $F\colon E \to \mathbb{R}^N$. This map induces a classifying map $f\colon M \to Gr_k(\mathbb{R}^N)$

sending m to $F(E_m)$; to rephrase this into the language of Proposition 4.17, we may define a map $\tilde{f}\colon E \to \gamma^k(\mathbb{R}^N)$ by $\tilde{f}(e) = (f(\pi(e)), F(e))$ and then we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \gamma^k(\mathbb{R}^N) \arrow[d] \\ M \arrow[r, "f"] & Gr_k(\mathbb{R}^N) \end{tikzcd}$$

with $\tilde{f}$ mapping fibers isomorphically to fibers. Now since $Gr_k(\mathbb{R}^N)$ is paracompact and Hausdorff and admits a finite good cover, Theorem 4.4 shows that there is a unique Thom class $\tau_{k,N} \in H^k(\gamma^k(\mathbb{R}^N), \gamma^k(\mathbb{R}^N)_0; \mathbb{Z}/2\mathbb{Z})$, with Euler class $e_{k,N} \in H^k(Gr_k(\mathbb{R}^N); \mathbb{Z}/2\mathbb{Z})$. Hence by Proposition 4.17, $\tilde{f}^*\tau_{k,N} \in H^k(E, E^0; \mathbb{Z}/2\mathbb{Z})$ is a Thom class for E, with Euler class $f^*e_{k,N} \in H^k(E, E^0; \mathbb{R}^N)$.

Example 4.18. For $N \le N'$ we have an inclusion map $i_{N,N'}\colon Gr_k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^{N'})$ defined by $V \mapsto V \times \{\vec{0}\}$, and we observed in Proposition 2.19 that this is a classifying map for $\gamma^k(\mathbb{R}^N)$; the lift $\tilde{i}_{N,N'}\colon \gamma^k(\mathbb{R}^N) \to \gamma^k(\mathbb{R}^{N'})$ is given by $(V, \vec{v}) \mapsto \big(V \times \{\vec{0}\}, (\vec{v}, \vec{0})\big)$. Hence $\tilde{i}^*_{N,N'}\tau_{k,N'}$ is a Thom class with coefficients in $\mathbb{Z}/2\mathbb{Z}$ for $\gamma^k(\mathbb{R}^N)$, with Euler class $i^*_{N,N'}e_{k,N'}$. Now by Theorem 4.4 there is a *unique* Thom class for $\gamma^k(\mathbb{R}^N)$, which we have denoted $\tau_{k,N}$. So it must hold that

$$\tilde{i}^*_{N,N'}\tau_{k,N'} = \tau_{k,N} \qquad i^*_{N,N'}e_{k,N'} = e_{k,N} \tag{14}$$

Now suppose that, where again $\pi\colon E \to M$ is a finite-type bundle over a paracompact Hausdorff space, we have two pre-classifying maps $F_0\colon E \to \mathbb{R}^{N_0}$ and $F_1\colon E \to \mathbb{R}^{N_1}$, inducing classifying maps $f_i\colon M \to Gr_k(\mathbb{R}^{N_i})$ and lifts $\tilde{f}_i\colon E \to \gamma^k(\mathbb{R}^{N_i})$. Then by Theorem 2.8 there is $N \ge \max\{N_0, N_1\}$ such that $I_{N_0,N} \circ F$ and $I_{N_1,N} \circ F$ are homotopic through pre-classifying maps; hence the maps $\tilde{i}_{N_0,N} \circ \tilde{f}_0$ and $\tilde{i}_{N_1,N} \circ \tilde{f}$ will be homotopic as maps of pairs $(E, E^0) \to (\gamma^k(\mathbb{R}^N), \gamma^k(\mathbb{R}^N)_0)$, and $i_{N_0,N} \circ f_0$ and $i_{N_1,N} \circ f_1$ are homotopic as maps $M \to Gr_k(\mathbb{R}^N)$. Hence we have, using (14),

$$\tilde{f}_0^*\tau_{k,N_0} = \tilde{f}_0^*\tilde{i}^*_{N_0,N}\tau_{k,N} = (\tilde{i}_{N_0,N} \circ \tilde{f}_0)^*\tau_{k,N} = (\tilde{i}_{N_1,N} \circ \tilde{f}_1)^*\tau_{k,N} = \tilde{f}_1^*\tau_{k,N_1}$$

and similarly

$$f_0^*e_{k,N_0} = f_1^*e_{k,N_1}.$$

If Theorem 4.4 applies to M, then by Proposition 4.17 we could already have said that $\tilde{f}_0^*\tau_{k,N_0}$ and $\tilde{f}_1^*\tau_{k,N_1}$ are equal to the unique Thom class for E provided by that Theorem. However, when Theorem 4.4 does not apply to M, we can now define *the* Thom class τ_E of E to be $\tilde{f}^*\tau_{k,N}$ for any map $\tilde{f}\colon E \to \gamma^k(\mathbb{R}^N)$ obtained as above from a pre-classifying map $F\colon E \to \mathbb{R}^N$ as above, since what we have just done shows that this class τ_E is independent of the choice of pre-classifying map. Likewise, we can define *the* Euler class of E to be $e(E) = f^*e_{k,N}$ for any classifying map $f\colon M \to Gr_k(\mathbb{R}^N)$. Since if $g\colon X \to M$ is a continuous map and $f\colon M \to Gr_k(\mathbb{R}^N)$ is a classifying map for E then $f \circ g$ is a classifying map for g^*E, it follows that the Euler class so defined satisfies the important naturality property

$$e(g^*E) = g^*e(E).$$

5. The cup product and the Leray–Hirsch theorem

While the universal coefficient theorem shows that, for purely algebraic reasons, the cohomology groups (with arbitrary coefficients) of a space X are explicitly determined by the homology with integer coefficients, an important fact about cohomology is that, provided that one works with coefficients in a ring, it also carries a ring structure, and this ring structure is not determined by the homology. So cohomology is a stronger invariant than homology: there are many examples of pairs of spaces with isomorphic homology groups but nonisomorphic cohomology rings ($S^2 \times S^2$ and the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ is one example of such a pair).

To describe the multiplicative operation (called the *cup product*) for this ring structure, we first introduce the maps between simplices upon which it is based. For notational simplicity, we will regard all of the standard simplices Δ^k as subsets of the same space $\mathbb{R}^\infty$. We denote by v_0 the zero-vector in $\mathbb{R}^\infty$ and $\{v_1, \dots, v_k, \dots\}$ the standard basis vectors, so that the standard k-simplex is

$$\Delta^k = \left\{ \sum_{i=0}^{k} t_i v_i \,\middle|\, t_i \geq 0, \sum_{i=0}^{k} t_1 = 1 \right\}$$

As alluded to at the start of Section 3, we have *face maps*

$$\phi_j^k \colon \Delta^k \to \Delta^{k+1} \quad (0 \leq j \leq k+1)$$
$$\sum_{i=0}^{k} t_i v_i \mapsto \sum_{i=0}^{j-1} t_i v_i + \sum_{i=j}^{k} t_i v_{i+1}$$

(so ϕ_j^k embeds Δ^k as that boundary face of Δ^{k+1} which does not contain the jth vertex of Δ^{k+1}). Of course, the boundary operator is defined by setting $\partial\sigma = \sum(-1)^j \sigma \circ \phi_j$ for a map $\sigma \colon \Delta^k \to X$.

We now introduce new maps

$$\mathfrak{f}_{p,p+q} \colon \Delta^p \to \Delta^{p+q}$$
$$\sum_{i=0}^{p} t_i v_i \mapsto \sum_{i=0}^{p} t_i v_i$$

and

$$\mathfrak{b}_{q,p+q} \colon \Delta^q \to \Delta^{p+q}$$
$$\sum_{i=0}^{q} t_i v_i \mapsto \sum_{i=p}^{p+q} t_{i-p} v_i$$

Thus $\mathfrak{f}_{p,p+q}$ embeds Δ^p into Δ^{p+q} as the "front p-face," while $\mathfrak{b}_{q,p+q}$ embeds Δ^q into Δ^{p+q} as the "back q-face." The images of these two maps intersect in a single point, namely v_p (which is both the last vertex in the front p-face and the first vertex in the back q-face).

The following identities relating these maps is left as a recommended exercise to the reader (none of them should be difficult):

Proposition 5.1. *The face maps ϕ_j^k, front maps $\mathfrak{f}_{p,p+q}$, and back maps $\mathfrak{b}_{q,p+q}$ obey the following:*

$$\begin{array}{ll}
\mathfrak{f}_{p,p} = 1_{\Delta^p} & \mathfrak{b}_{q,q} = 1_{\Delta^q} \\
\mathfrak{f}_{p,p+1} = \phi_{p+1}^p & \mathfrak{b}_{q,q+1} = \phi_0^q \\
\mathfrak{f}_{p+q,p+q+r} \circ \mathfrak{f}_{p,p+q} = \mathfrak{f}_{p,p+q+r} & \textit{as maps } \Delta^p \to \Delta^{p+q+r} \\
\mathfrak{b}_{q+r,p+q+r} \circ \mathfrak{b}_{r,q+r} = \mathfrak{b}_{r,p+q+r} & \textit{as maps } \Delta^r \to \Delta^{p+q+r} \\
\mathfrak{f}_{p+q,p+q+r} \circ \mathfrak{b}_{q,p+q} = \mathfrak{b}_{q+r,p+q+r} \circ \mathfrak{f}_{q,q+r} & \textit{as maps } \Delta^q \to \Delta^{p+q+r}
\end{array}$$

$$\phi_j^{p+q} \circ \mathfrak{f}_{p,p+q} = \begin{cases} \mathfrak{f}_{p+1,p+q+1} \circ \phi_j^p & 0 \leq j \leq p+1 \\ \mathfrak{f}_{p,p+q+1} & p+1 \leq j \leq p+q+1 \end{cases}$$

$$\phi_j^{p+q} \circ \mathfrak{b}_{q,p+q} = \begin{cases} \mathfrak{b}_{q,p+q+1} & 0 \le j \le p \\ \mathfrak{b}_{q+1,p+q+1} \circ \phi_{j-p}^{q} & p \le j \le p+q+1 \end{cases}$$

We now define the cup product on the singular cochain complex $S_\bullet(X;R) = Hom(S_\bullet(X);R)$ where R *is an arbitrary commutative ring with unity*. This takes the form of a map

$$\cup\colon\ S^p(X;R) \times S^q(X;R) \to S^{p+q}(X;R)$$
$$(\alpha, \beta) \mapsto \alpha \cup \beta$$

To define $\alpha \cup \beta$ we have to give the value (in R) of $(\alpha \cup \beta)(\sigma)$ where $\sigma\colon\ \Delta^{p+q} \to X$ is an arbitrary continuous map. This is given by the following formula:

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma \circ \mathfrak{f}_{p,p+q})\beta(\sigma \circ \mathfrak{b}_{q,p+q}).$$

(This makes sense, since $\sigma \circ \mathfrak{f}_{p,p+q}$ and $\sigma \circ \mathfrak{b}_{q,p+q}$ are maps into X with domains Δ^p and Δ^q respectively, so $\alpha(\sigma \circ \mathfrak{f}_{p,p+q})$ and $\beta(\sigma \circ \mathfrak{b}_{q,p+q})$ are elements of R, which can then be multiplied together since we have assumed that R is a ring and not just an abelian group.)

It should be obvious that we have distributive laws $\alpha \cup (\beta_0 + \beta_1) = \alpha \cup \beta_0 + \alpha \cup \beta_1$ and $(\alpha_0 + \alpha_1) \cup \beta = \alpha_0 \cup \beta + \alpha_1 \cup \beta$, essentially inherited from the distributive laws in R. Moreover:

Proposition 5.2. $\cup$ *is associative: for* $\alpha \in S^p(X;R)$*,* $\beta \in S^q(X;R)$*,* $\gamma \in S^r(X;R)$ *we have*

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma).$$

Proof. This follows fairly quickly from the third, fourth, and fifth lines of Proposition 5.1: for any map $\sigma\colon\ \Delta^{p+q+r} \to X$ we have

$$\begin{aligned}\big((\alpha \cup \beta) \cup \gamma\big)(\sigma) &= (\alpha \cup \beta)(\sigma \circ \mathfrak{f}_{p+q,p+q+r})\gamma(\sigma \circ \mathfrak{b}_{r,p+q+r}) \\ &= \alpha(\sigma \circ \mathfrak{f}_{p+q,p+q+r} \circ \mathfrak{f}_{p,p+q})\beta(\sigma \circ \mathfrak{f}_{p+q,p+q+r} \circ \mathfrak{b}_{q,p+q+r})\gamma(\sigma \circ \mathfrak{b}_{r,p+q+r}) \\ &= \alpha(\sigma \circ \mathfrak{f}_{p,p+q+r})\beta(\sigma \circ \mathfrak{b}_{q+r,p+q+r} \circ \mathfrak{f}_{q,q+r})\gamma(\sigma \circ \mathfrak{b}_{q+r,p+q+r} \circ \mathfrak{b}_{r,q+r}) \\ &= \alpha(\sigma \circ \mathfrak{f}_{p,p+q+r}) \cdot (\beta \cup \gamma)(\sigma \circ \mathfrak{b}_{q+r,p+q+r}) \\ &= \big(\alpha \cup (\beta \cup \gamma)\big)(\sigma)\end{aligned}$$

□

The above shows that $\cup$ makes $S^*(X;R) = \oplus_{p=0}^{\infty} S^p(X;R)$ into a *graded ring* (*i.e.* a ring A which decomposes as a direct sum as $A = \oplus_{p\in\mathbb{N}} A^p$ where $A^p \cdot A^q \subset A^{p+q}$). Moreover since we assume that R has a multiplicative identity there is also a multiplicative identity in $S^*(X;R)$, which we will denote by $\tilde{1}$: namely the element $\tilde{1} \in S^0(X;R)$ defined by the property that for each 0-simplex $\sigma\colon\ \Delta^0 \to X$ (*i.e.*, each point in X) we have $\tilde{1}(\sigma) = 1$ (so for a general 0-chain $\sum n_i x_i$ we have $\tilde{1}(\sum n_i x_i) = \sum n_i$). The fact that $\tilde{1} \cup \alpha = \alpha \cup \tilde{1} = \alpha$ for any $\alpha \in S^p(X;R)$ follows directly from the first line of Proposition 5.1. Since $\tilde{1}$ assigns the same value to every point (and in particular assigns the same value to any two points in the same path component), it follows as in Exercise 3.19 that

$$\delta \tilde{1} = 0. \tag{15}$$

So we have made $S^*(X;R)$ into a graded ring with unity. (It is not commutative in any sense, as you can easily check.) This is not what was promised earlier, of course—in order to make the *cohomology* into a graded ring using $\cup$ we need to show that $\cup$ interacts appropriately with the coboundary operator $\delta\colon\ S^k(X;R) \to S^{k+1}(X;R)$. In fact we have the following "Leibniz rule":

Proposition 5.3. *For* $\alpha \in S^p(X;R)$ *and* $\beta \in S^q(X;R)$*,*

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^p \alpha \cup (\delta\beta).$$

Proof. We need to show that both sides evaluate in the same way on an arbitrary continuous map $\sigma\colon \Delta^{p+q+1} \to X$. We have:

$$\begin{aligned}
\big(\delta(\alpha \cup \beta)\big)(\sigma) &= (\alpha \cup \beta)(\partial \sigma) = \sum_{j=0}^{p+q+1} (-1)^j (\alpha \cup \beta)(\sigma \circ \phi_j^{p+q}) \\
&= \sum_{j=0}^{p+q+1} (-1)^j \alpha(\sigma \circ \phi_j^{p+q} \circ \mathfrak{f}_{p,p+q}) \beta(\sigma \circ \phi_j^{p+q} \circ \mathfrak{b}_{q,p+q}) \\
&= \sum_{j=0}^{p} (-1)^j \alpha(\sigma \circ \mathfrak{f}_{p+1,p+q+1} \circ \phi_j^p) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) + \sum_{j=p+1}^{p+q+1} (-1)^j \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta(\sigma \circ \mathfrak{b}_{p+1,p+q+1} \circ \phi_{j-p}^q) \\
&= \alpha\left(\sum_{j=0}^{p} (-1)^j (\sigma \circ \mathfrak{f}_{p+1,p+q+1}) \circ \phi_j^p\right) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) \\
&\qquad + (-1)^p \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta\left(\sum_{k=1}^{q+1} (-1)^k (\sigma \circ \mathfrak{b}_{q+1,p+q+1}) \circ \phi_k^q\right)
\end{aligned}$$

Now we have

$$\partial(\sigma \circ \mathfrak{f}_{p+1,p+q+1}) = \sum_{j=0}^{p+1} (-1)^j (\sigma \circ \mathfrak{f}_{p+1,p+q+1}) \circ \phi_j^p$$

and

$$\partial(\sigma \circ \mathfrak{b}_{q+1,p+q+1}) = \sum_{k=0}^{q+1} (-1)^k (\sigma \circ \mathfrak{b}_{q+1,p+q+1}) \circ \phi_k^q$$

(so these each differ from expressions above by one term). We thus obtain:

$$\begin{aligned}
\big(\delta(\alpha \cup \beta)\big)(\sigma) &= \alpha(\partial(\sigma \circ \mathfrak{f}_{p+1,p+q+1})) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) + (-1)^p \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta(\partial(\sigma \circ \mathfrak{b}_{q+1,p+q+1})) \\
&\qquad - (-1)^{p+1} \alpha(\sigma \circ \mathfrak{f}_{p+1,p+q+1} \circ \phi_{p+1}^p) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) - (-1)^p \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta(\sigma \circ \mathfrak{b}_{q+1,p+q+1} \circ \phi_0^q) \\
&= (\delta\alpha)(\sigma \circ \mathfrak{f}_{p+1,p+q+1}) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) + (-1)^p \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) (\delta\beta)(\sigma \circ \mathfrak{b}_{q+1,p+q+1}) \\
&\qquad - (-1)^{p+1} \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) - (-1)^p \alpha(\sigma \circ \mathfrak{f}_{p,p+q+1}) \beta(\sigma \circ \mathfrak{b}_{q,p+q+1}) \\
&= \big((\delta\alpha) \cup \beta + (-1)^p \alpha \cup (\delta\beta)\big)(\sigma)
\end{aligned}$$

where in the second-to-last line we have used lines 2-4 of Proposition 5.1. □

Corollary 5.4. *The map* $\cup\colon S^p(X;R) \times S^q(X;R) \to S^{p+q}(X;R)$ *gives rise to a well-defined map* $\cup\colon H^p(X;R) \times H^q(X;R) \to) H^{p+q}(X;R)$ *by means of the formula* $[\alpha] \cup [\beta] = [\alpha \cup \beta]$ *for cocycles* α *and* β*. This operation makes* $H^*(X;R) = \oplus_{k=0}^{\infty} H^k(X;R)$ *into a graded ring, with multiplicative identity given by* $1 := [\tilde{1}]$.

Proof. If $\delta\alpha = \delta\beta = 0$ then $\delta(\alpha \cup \beta) = 0 \cup \beta + (-1)^p \alpha \cup 0 = 0$ by (5.2), so whenever α and β are cocycles $\alpha \cup \beta$ is also a cocycle.

To show that $\cup$ is well-defined as an operation on cohomology we must show that, for any two cocycles $\alpha \in S^p(X;R)$ and $\beta \in S^q(X;R)$ the cohomology class $[\alpha \cup \beta]$ of $\alpha \cup \beta$ depends only on the cohomology classes of α and β. Indeed by Proposition 5.2 we have, for $\eta \in S^{p-1}(X;R)$ and $\zeta \in S^{q-1}(X;R)$, $\delta(\alpha \cup \zeta) = (-1)^p \alpha \cup \delta\zeta$ (since $\delta\alpha = 0$), and $\delta(\eta \cup (\beta + \delta\zeta)) = (\delta\eta) \cup (\beta\delta\zeta)$ (since $\delta(\beta + \delta\zeta) = 0$). Hence

$$(\alpha + \delta\eta) \cup (\beta + \delta\zeta) = \alpha \cup \beta + \delta\big((-1)^p \alpha \cup \zeta + \eta \cup (\beta + \delta\zeta)\big)$$

and so $(\alpha + \delta\eta) \cup (\beta + \delta\zeta)$ represents the same cohomology classas $\alpha \cup \beta$. So indeed $\cup$ is well-defined on cohomology.

Given that $\cup$ is well-defined, the ring axioms for $H^*(X;R)$ are straightforwardly inherited from those of $S^*(X;R)$. In view of (15), we have a well-defined cohomology class $1 = [\tilde{1}]$, which is a multiplicative identity in $H^*(X;R)$ since $\tilde{1}$ is a multiplicative identity in $S^*(X;R)$. □

Remark 5.5. One can show ([H, Theorem 3.14]) that $H^*(X;R)$ is a *graded-commutative* ring in the sense that for $x \in H^p(X;R)$ and $y \in H^q(X;R)$ one has $x \cup y = (-1)^{pq} y \cup x$. This is perhaps somewhat surprising since there is not a corresponding identity on $S^*(X;R)$—the most that one can say is that if α and β are cocycles then $\alpha \cup \beta - (-1)^{pq}\beta \cup \alpha$ is a coboundary.[13]

Remark 5.6. The same discussion as above goes through without change for the relative cohomology $H^*(X,A;R)$ where $A \subset X$, except that there is generally no unit when $A \neq \emptyset$. Indeed recall that the relative singular cochain complex $S^*(X,A : R)$ can be regarded as the subcomplex consisting of those cochains on X which vanish on all simplices contained in A, and this condition is clearly preserved by the cup product operation (but is not obeyed by $\tilde{1} \in S^*(X;R)$).

In fact a stronger statement is true: if one has a subsets $A \subset X$ and cochains $\alpha, \beta \in S^*(X;R)$ such that β vanishes on all simplices in A, then $\alpha \cup \beta$ also vanishes on all simplices in A. As a result we have a well-defined cup product operation

$$\cup\colon H^*(X;R) \times H^*(X,A;R) \to H^*(X,A;R).$$

This makes $H^*(X,A;R)$ into a module over the ring $H^*(X;R)$.

As may be intuitively obvious, the ring structure on $H^*(X;R)$ behaves well with respect to the homomorphisms induced by continuous maps.

Proposition 5.7. *For any continuous map of pairs $f : (X,A) \to (Y,B)$ the induced map $f^*\colon H^*(Y,B;R) \to H^*(X,A;R)$ is a ring homomorphism. If $A = B = \emptyset$ then $f^*(1) = 1$.*

Proof. For $\alpha \in S^p(Y;R)$, $\beta \in S^q(Y;R)$, and $\sigma\colon \Delta^{p+q} \to X$ we have

$$\begin{aligned}(f^*(\alpha \cup \beta))(\sigma) &= (\alpha \cup \beta)(f \circ \sigma) = \alpha((f \circ \sigma) \circ \mathfrak{f}_{p,p+q})\beta((f \circ \sigma) \circ \mathfrak{b}_{q,p+q}) \\ &= \alpha(f \circ (\sigma \circ \mathfrak{f}_{p,p+q}))\beta(f \circ (\sigma \circ \mathfrak{b}_{q,p+q})) = (f^*\alpha)(\sigma \circ \mathfrak{f}_{p,p+q}) \cdot (f^*\beta)(\sigma \circ \mathfrak{b}_{q,p+q}) = \big((f^*\alpha) \cup (f^*\beta)\big)(\sigma).\end{aligned}$$

Thus we have (even on cochain level) $f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$, so the induced map $f^*\colon H^*(Y,B;R) \to H^*(X,A;R)$ is a ring homomorphism.

As for the units, it is obvious from the definitions that $f^*\tilde{1} = \tilde{1}$, and so upon passing to cohomology we have $f^*1 = 1$. □

Exercise 5.8. Identify the torus with $T = \mathbb{R}^2/\mathbb{Z}^2$, and define one-cycles $\gamma_1, \gamma_2\colon \Delta^1 \to T$ by $\gamma_1(t) = [(t,0)]$ and $\gamma_2(t) = [(0,t)]$. As you learned in 8200, $H_2(T;\mathbb{Z}) \cong \mathbb{Z}$, and the homology classes $[\gamma_1], [\gamma_2]$ form a basis for $H_1(T,\mathbb{Z})$. So by the universal coefficient theorem there are $\alpha, \beta \in H^1(T;\mathbb{Z})$ such that $\alpha([\gamma_1]) = \beta([\gamma_2]) = 1$ and $\alpha([\gamma_2]) = \beta([\gamma_1]) = 0$.

(a) Prove directly from the definition of the cup product that $\alpha \cup \alpha = \beta \cup \beta = 0$ while $\alpha \cup \beta \neq 0$. (Suggestion: Construct a singular two-cycle which generates $H_2(T;\mathbb{Z})$ by dividing the square into two triangles.)

(b) Prove that every continuous map $f : S^2 \to T$ has degree zero (*i.e.* induces the zero map on H_2; hint: functoriality of the cup product).

[13] In fact, there are certain cohomology operations called Steenrod squares whose nontriviality reflects the impossibility of creating a cochain-level definition of the cup product which is exactly graded-commutative. You may be familiar with de Rham cohomology (which for a smooth manifold X is isomorphic to $H^*(X;\mathbb{R})$) in which the multiplication is graded commutative on cochain level—however nothing like this can be constructed using coefficients in $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

Exercise 5.9. Let Σ denote the compact surface of genus two. Prove directly from the definition of the cup product that the bilinear pairing $\cup\colon H^1(\Sigma;\mathbb{Z})\times H^1(\Sigma;\mathbb{Z})\to H^2(\Sigma;\mathbb{Z})$ is *nondegenerate*, *i.e.* that for any nonzero $\alpha\in H^1(\Sigma;\mathbb{Z})$ there is $\beta\in H^1(\Sigma;\mathbb{Z})$ such that $\alpha\cup\beta\neq 0$. Deduce from this that, where T is the torus, every continuous map $g\colon T\to\Sigma$ has degree zero.

(More generally, a similar argument shows that if $g<h$ then all maps from the genus-g surface to the genus-h surface have degree zero.)

5.1. **Fiber bundles and their cohomology.** A rank-k vector bundle $E\to M$ can be thought of as a family of copies of $\mathbb{R}^k$ parametrized by M. There is a generalization of this where $\mathbb{R}^k$ is replaced by an arbitrary topological space:

Definition 5.10. If M and F are topological spaces, a *fiber bundle over M with fiber F* consists of a continuous map $\pi\colon P\to M$ where P is another topological space, such that each $m\in M$ has a neighborhood U and a "local trivialization" $\Phi\colon \pi^{-1}(U)\to U\times F$ where Φ is a homeomorphism which map each "fiber" $P_x:=\pi^{-1}(\{x\})$ to $\{x\}\times F$.

Thus a rank-k vector bundle is a particular kind of fiber bundle with fiber $\mathbb{R}^k$ (for the general fiber bundle over $\mathbb{R}^k$ one would not require the local trivializations to respect a linear structure).

We will find it useful to consider a relative version of the above definition:

Definition 5.11. If M,F,G are topological spaces with $G\subset F$, a *fiber bundle pair* over M with fiber (F,G) is a pair of topological spaces (P,Q) where $Q\subset P$ together with a fiber bundle $\pi\colon P\to M$ such that the local trivializations $\Phi\colon \pi^{-1}(U)\to U\times F$ for P map $\pi^{-1}(U)\cap Q$ homeomorphically to $U\times G$.

For example of $\pi\colon E\to M$ is a rank-k vector bundle and E^0 is the complement of the zero-section, then the local trivializations for E map $\pi^{-1}(U)\cap E^0$ homeomorphically to $U\times(\mathbb{R}^k\setminus\{0\})$. Thus (E,E^0) is a fiber bundle pair with fiber $(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$.

Remark 5.12. If M is a contractible paracompact Hausdorff space, then every fiber bundle pair (P,Q) over M with fiber (F,G) is trivial, *i.e.* there is a homeomorphism $\Psi\colon P\to M\times F$ which maps Q to $M\times G$ and each fiber (P_x,Q_x) homeomorphically to $(\{x\}\times F,\{x\}\times G)$. The proof of this proceeds exactly as in Section 2.4: one can define the pullback of a fiber bundle in the same way as the pullback of a vector bundle, and the proof of Theorem 2.24 extends to show that the pullbacks of a fiber bundle by two homotopic maps are isomorphic, from which the conclusion follows just as in Proposition 2.30.

The Leray–Hirsch theorem, stated below, will determine the cohomology of fiber bundle pairs that satisfy the following property:

Definition 5.13. Let R be a commutative ring with unity. We say that the fiber bundle pair (P,Q) over M satisfies the *Leray–Hirsch property* over R if there exist cohomology classes $f_j\in H^{n_j}(P,Q;R)$ (with j ranging over some index set J) such that, for each $m\in M$, where $i_m\colon(P_m,Q_m)\to(P,Q)$ denotes the fiber over m,

$$\{i_m^*f_j\}_{j\in J}\text{ is a basis for the }R\text{-module }H^*(P_m,Q_m;R).$$

Example 5.14. Assuming that $H^*(F,G;R)$ is a free R-module, say generated by $x_j\in H^{n_j}(F,G;R)$ $(j\in J)$, the trivial bundle $(M\times F,M\times G)$ over M satisfies the Leray–Hirsch property over R: where $\pi_2\colon(M\times F,M\times G)\to(F,G)$ is the projection we can just set $f_j=\pi_2^*x_j$. (Of course, $H^*(F,G;R)$ will automatically be a free R-module if R is a field.)

Example 5.15. If $\pi\colon E\to M$ is a finite-type rank-k vector bundle over a paracompact Hausdorff space and if either $R=\mathbb{Z}$ and E is oriented or else $R=\mathbb{Z}/2\mathbb{Z}$, then the fiber bundle pair (E,E^0)

over M satisfies the Leray–Hirsch property over R. Indeed, in this case the bundle E has a Thom class $\tau \in H^k(E, E^0; R)$ by results of Section 4. By Example 3.21, $H^*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}; R)$ is a free rank-one R-module generated by the element denoted μ_m in Section 4, and the definition of a Thom class is precisely that $i_m^* \tau = \mu_m$ for all m.

Example 5.16. Consider the quotient map $\pi\colon \mathbb{C}^{n+1} \setminus \{\vec{0}\} \to \mathbb{C}P^n$ where $n \geq 1$; this is easily seen to be a fiber bundle with fiber $\mathbb{C} \setminus \{0\}$. (So we are letting $P = \mathbb{C}^{n+1} \setminus \{\vec{0}\}$ and $Q = \varnothing$ in this example.) If $m \in M$ then $H^1(P_m; R) \cong R$, while since $\mathbb{C}^{n+1} \setminus \{\vec{0}\}$ is homotopy equivalent to S^{2n+1}, $H^1(P; R) = \varnothing$. So for all m the restriction $i_m^*\colon H^*(P; R) \to H^*(P_m; R)$ fails to be surjective, in view of which P does *not* satisfy the Leray–Hirsch property.

Example 5.17. Let $\pi\colon E \to S^1$ denote the Möbius bundle and consider the fiber bundle pair (E, E^0) but now with coefficients in $\mathbb{Z}$. As in Example 4.12, both E and E^0 are homotopy equivalent to S^1. It is not hard to check that the inclusion maps a generator for $H_1(E^0; \mathbb{Z})$ to two times a generator of $H_1(E; \mathbb{Z})$ (the former generator can be represented by the boundary of the closed Möbius strip and the latter generator can be represented by the zero section). Taking adjoints we see that, under appropriate identifications of $H^1(E; \mathbb{Z})$ and $H^1(E^0; \mathbb{Z})$ with $\mathbb{Z}$, $i^*\colon H^1(E; \mathbb{Z}) \to H^1(E^0; \mathbb{Z})$ is given by multiplication by 2. But then consideration of the long exact sequence of the pair shows that $H^1(E, E^0; \mathbb{Z})$ must be isomorphic to $\ker(i^*)$, and hence is zero. So since the relative cohomology $H^1(\mathbb{R}, \mathbb{R} \setminus \{0\}; \mathbb{Z})$ is nonzero it follows that (E, E^0) must not satisfy the Leray–Hirsch property over $\mathbb{Z}$ (even though it does satisfy it over $\mathbb{Z}/2\mathbb{Z}$).

Suppose that the fiber bundle pair (P, Q) over M satisfies the Leray–Hirsch property with coefficients in R, with classes $f_j \in H^{n_j}(F, G; R)$ as in Definition 5.13. Now we have a ring structure on $H^*(M; R)$; we can then form the graded (left) $H^*(M; R)$-module $H^*(M; R)[\{x_j\}]$ where we set the grading of each generator x_j equal to n_j (*i.e.* to the grading of f_j). Thus a general element of $H^*(M; R)[\{x_j\}]$ is given by a sum $\sum_j c_j x_j$ where $c_j \in H^*(M; R)$ and only finitely many c_j are nonzero, and such an element has grading k provided that $c_j \in H^{k-n_j}(M; R)$.

We can moreover define a map

$$\begin{aligned} \Phi\colon H^*(M; R)[\{x_j\}] &\to H^*(P, Q; R) \\ \sum_j c_j x_j &\mapsto \sum_j (\pi^* c_j) \cup f_j \end{aligned} \tag{16}$$

(this makes sense, since we have $\pi^* c_j \in H^*(P; R)$ and $f_j \in H^*(P, Q; R)$, so as explained in the last section they have a well-defined cup product in $H^*(P, Q; R)$, making $H^*(P, Q; R)$ into a $H^*(P; R)$-module). The ring homomorphism $\pi^*\colon H^*(M; R) \to H^*(P; R)$ induces a natural $H^*(M; R)$-module structure on $H^*(P, Q; R)$, and it is easy to see that Φ is a $H^*(M; R)$-module homomorphism. (It is not a ring homomorphism, since we have not even made its domain into a ring.)

Theorem 5.18 (Leray–Hirsch)**.** *Let M be a Hausdorff space in which every open subset is paracompact, and assume that M has a finite good cover. Let (P, Q) be a fiber bundle pair over M which satisfies the Leray–Hirsch property over R. Then the map $\Phi\colon H^*(M; R)[\{x_j\}] \to H^*(P, Q; R)$ of (16) is an* ***isomorphism*** *of $H^*(M; R)$-modules.*

Remark 5.19. In fact, this result holds for fiber bundles over rather more general spaces B than those mentioned in the theorem; see for instance [H, Theorem 4D.8].

Remark 5.20. As a special case, in view of Example 5.14 we see that if M is as in Theorem 5.18 and if F is a space whose cohomology is a free R-module then $H^*(M \times F; R)$ is isomorphic to the tensor product of $H^*(M; R)$ and $H^*(F; R)$: elements of $H^(M \times F; R)$ can be written uniquely as finite sums $\sum \pi_1^* c_i \cup \pi_2^* f_i$ where $c_i \in H^*(M;R)$ and $f_i \in H^*(F;R)$. This is (a special case of) the *Künneth theorem* for cohomology.

Proof of Theorem 5.18. Similarly to the proof of Theorem 4.4, we use induction on the number of sets required to form a good cover of M.

If M has a good cover by just one open set, then of course this open set is all of M and so M is contractible. So the cohomology of M is isomorphic to that of a point, and so is a rank-1 R-module generated by $1 \in H^0(M;R)$. So in this case the map Φ is just the R-module homomorphism $\Phi\colon R[\{x_j\}] \to H^*(P,Q;R)$ which sends x_j to $\pi^*1 \cup f_j = f_j$. Given that Φ has this form, the Leray–Hirsch condition amounts to the statement that, for each $m \in M$, the composition $i_m^* \circ \Phi\colon R[\{x_j\}] \to H^*(P_m,Q_m;R)$ is an R-module isomorphism.

Meanwhile by Remark 5.12, the bundle pair (P,Q) is trivial, so there is a homeomorphism $\Psi\colon (P,Q) \to (M \times F, M \times G)$ mapping (P_m,Q_m) to $(\{m\} \times F, \{m\} \times G)$. Since M is contractible, the map $(M \times F, M \times G) \to (\{m\} \times F, \{m\} \times G)$ which is the identity on the second factor is a homotopy equivalence, in view of which the inclusion $i_m\colon (P_m,Q_m) \to (P,Q)$ is also a homotopy equivalence. So the fact that $i_m^* \circ \Phi\colon H^*(M;R) \to H^*(P_m,Q_m;R)$ is an isomorphism implies that Φ is an isomorphism. This proves the result in the case that M admits a good open cover by just one set.

Now assume the theorem to be proven whenever the base of the bundle admits a good open cover by at most a open sets, where $a \geq 1$, and suppose that M admits a good open cover by $a+1$ sets—say this good open cover is

$$M = U_1 \cup \cdots \cup U_a \cup V.$$

As in the proof of Theorem 4.4, write $U = U_1 \cup \cdots \cup U_a$, and where $\pi\colon P \to M$ is the bundle projection for any $W \subset M$ write $P|_W = \pi^{-1}(W)$ and $Q|_W = P|_W \cap Q$. We observe that the inductive hypothesis implies that the conclusion of the theorem holds for the fiber bundle pairs over U, $(P|_V, Q|_V)$ over V, and $(P|_{U\cap V}, Q|_{U\cap V})$ over $U \cap V$ (in the last case this follows from the observation that $U \cap V = (U_1 \cap V) \cup \cdots \cup (U_a \cap V)$ gives a good cover of $U \cap V$ by a sets).

The Leray–Hirsch property gives a set of classes $f_j \in H^*(P,Q;R)$ which restrict to a basis for the cohomology of the fibers; these classes restrict to sets of classes $f_j^U \in H^*(P|_U, Q|_U;R)$, $f_j^V \in H^*(P|_V, Q|_V;R)$, and $f_j^{U\cap V} \in H^*(P|_{U\cap V}, Q|_{U\cap V};R)$ as in the Leray–Hirsch property. So by the inductive hypothesis the corresponding maps $\Phi^U\colon H^*(U;R)[\{x_j\}] \to H^*(P|_U, Q|_U;R)$, $\Phi^V\colon H^*(V;R)[\{x_j\}] \to H^*(P|_V, Q|_V;R)$, and $\Phi^{U\cap V}\colon H^*(U\cap V;R)[\{x_j\}] \to H^*(P|_{U\cap V}, Q|_{U\cap V};R)$ (given for example by $\Phi^U(\sum c_j x_j) = \sum (\pi|_{P|_U})^* c_j \cup f_j^U$) are all isomorphisms.

From the naturality of the construction of the Mayer–Vietoris sequence, for each k the following diagram of Mayer–Vietoris sequences commutes:

(17)

$$\begin{tikzcd}
H^{k-1}(P|_U;R) \oplus H^{k-1}(P|_V;R) \arrow[r] & H^{k-1}(P|_{U\cap V};R) \arrow[r] & H^k(P;R) \arrow[r] & H^k(P|_U;R) \oplus H^k(P|_V;R) \arrow[r] & \cdots \\
H^{k-1}(U;R) \oplus H^{k-1}(V;R) \arrow[r] \arrow[u, "\pi^*\oplus\pi^*"] & H^{k-1}(U\cap V;R) \arrow[r] \arrow[u, "\pi^*"] & H^k(M;R) \arrow[r] \arrow[u, "\pi^*"] & H^k(U;R) \oplus H^k(V;R) \arrow[r] \arrow[u, "\pi^*\oplus\pi^*"] & \cdots
\end{tikzcd}$$

For any subset W introduce the notation $\widehat{H}^*(W)$ for the graded $H^*(M;R)$-module $\widehat{H}^*(W) = H^*(W;R)[\{x_j\}]$. Note that this module is isomorphic to a direct sum of $\#\{x_j\}$-many copies of $H^*(W;R)$, each shifted in degree by the degree of the corresponding x_j. We can trivially extend the Mayer–Vietoris sequence associated to the decomposition $M = U \cap V$ to a sequence

$$\widehat{H}^{k-1}(U\cap V) \longrightarrow \widehat{H}^k(M) \longrightarrow \widehat{H}^k(U) \oplus \widehat{H}^k(V) \longrightarrow \widehat{H}^k(U\cap V)$$

(with each map sending x_j to x_j and otherwise acting as before); this is a direct sum of exact sequences and therefore is exact. We then have a diagram

(18)
$$\begin{tikzcd}
H^{k-1}(P|_U,Q|_U;R)\oplus H^{k-1}(P|_V,Q|_V;R) \arrow[r] & H^{k-1}(P|_{U\cap V},Q|_{U\cap V};R) \arrow[r] & H^k(P,Q;R) \arrow[r] & H^k(P|_U,Q|_U;R)\oplus H^k(P|_V,Q|_V;R) \arrow[r] & H^k(P|_{U\cap V},Q|_{U\cap V};R) \\
\widehat{H}^{k-1}(U;R)\oplus \widehat{H}^{k-1}(V;R) \arrow[r] \arrow[u, "\Phi^U\oplus\Phi^V"'] & \widehat{H}^{k-1}(U\cap V;R) \arrow[r] \arrow[u, "\Phi^{U\cap V}"'] & \widehat{H}^k(M;R) \arrow[r] \arrow[u, "\Phi"'] & \widehat{H}^k(U;R)\oplus H^k(V;R) \arrow[r] \arrow[u, "\Phi^U\oplus\Phi^V"'] & \widehat{H}^k(U\cap V) \arrow[u, "\Phi^{U\cap V}"']
\end{tikzcd}$$

Using the facts that the classes f_j, f_j^U, f_j^V, and $f_j^{U\cap V}$ are intertwined by the inclusion maps that appear in the top line above, together with the fact that (17) commutes, one can see from a little diagram chasing (left as a recommended exercise to the reader) that (18) also commutes. By the inductive hypothesis, the outer four vertical maps in (18) are all isomorphisms, and so the five lemma shows that Φ is an isomorphism. This completes the induction and hence the proof. $\square$

Specializing to the case of Example 5.15, we have:

Corollary 5.21 (Thom Isomorphism Theorem)**.** *Let M be a Hausdorff space in which every open subset is paracompact and which admits a finite good cover. Let $\pi\colon E\to M$ be a vector bundle with Thom class $\tau_E\in H^k(E,E^0;R)$ where either $R=\mathbb{Z}$ and E is oriented or $\mathbb{R}=\mathbb{Z}/2\mathbb{Z}$. Then for all integers j the map*

$$\begin{aligned}\Phi\colon H^j(M;R)&\to H^{j+k}(E,E^0;R)\\ c&\mapsto(\pi^*c)\cup\tau_E\end{aligned}$$

is an isomorphism.

Of course this extends Theorem 4.4, which showed that $H^{j+k}(E,E^0;R)=0$ for $j<0$, and produced a nonzero element (namely the Thom class) in $H^k(E,E^0;R)$. Note that the Thom class τ_E is the image of the unit 1 under the Thom isomorphism Φ.

Recall that the Euler class of E is defined by $e(E)=(j\circ s_0)^*\tau_E$ where $s_0\colon M\to E$ is the zero section and $j\colon(E,\varnothing)\to(E,E^0)$ (in the past we have just written s_0 instead of $j\circ s_0$, but for clarity here it is better to record the two steps of the inclusion of M into (E,E^0) separately). Now we have isomorphisms $s_0^*\colon H^*(E;R)\to H^*(M;R)$ (since s_0 is a homotopy equivalence), and $\Phi\colon H^{*-k}(M;R)\to H^*(E,E^0;R)$ (the Thom isomorphism). At the same time, where $i\colon E^0\to E$ is the inclusion, we have the long exact sequence of the pair

$$\cdots\longrightarrow H^{j-1}(E^0;R)\longrightarrow H^j(E,E^0;R)\xrightarrow{j^*}H^j(E;R)\xrightarrow{i^*}H^j(E,E^0;R)\longrightarrow\cdots$$

By means of the isomorphisms s_0^* and Φ, this yields an sequence

$$\cdots\longrightarrow H^{j-1}(E^0;R)\longrightarrow H^{j-k}(M;R)\xrightarrow{s_0^*\circ j^*\circ\Phi}H^j(M;R)\xrightarrow{i^*\circ(s_0^*)^{-1}}H^j(E^0;R)\longrightarrow\cdots\tag{19}$$

Now $s_0\colon M\to E$ has homotopy inverse given by the bundle projection $\pi\colon E\to M$, so $(s_0^*)^{-1}=\pi^*$, and $i^*\circ(s_0^*)^{-1}=(\pi\circ i)^*$. So the map $H^j(M;R)\to H^j(E^0;R)$ is just the map induced by the restriction of the bundle projection π to E^0.

Meanwhile for $c\in H^{j-k}(M;R)$ we have

$$s_0^*\circ j^*\circ\Phi(c)=(j\circ s_0)^*\big((\pi^*c)\cup\tau_E\big)=(\pi\circ j\circ s_0)^*c\cup(j\circ s_0)^*\tau_E.$$

But $\pi\circ j\circ s_0$ is the identity on M, and $(j\circ s_0)^*\tau_E=e(E)$ by definition. So the second map in (19) is simply the map $\cdot\cup e(E)\colon H^{j-k}(M;R)\to H^j(M;R)$ given by $c\mapsto c\cup e(E)$, *i.e.* by *cup product* with the Euler class.

Summing up, we have proven:

Corollary 5.22. *Let M be a Hausdorff space in which every open subset is paracompact and which admits a finite good cover. Let $\pi\colon E \to M$ be a vector bundle with Euler class $e \in H^k(M;R)$ where either $R = \mathbb{Z}$ and E is oriented or $\mathbb{R} = \mathbb{Z}/2\mathbb{Z}$. Then there is an exact sequence (the* ***Gysin sequence****)*

$$\cdots \longrightarrow H^{j-1}(E^0;R) \longrightarrow H^{j-k}(M;R) \xrightarrow{\cdot\cup e} H^j(M;R) \xrightarrow{\pi^*} H^j(E^0;R) \longrightarrow H^{j-k+1}(M;R) \xrightarrow{\cdot\cup e} \cdots$$

where E^0 is the complement of the zero-section in E.

Using the Gysin sequence, we can determine the cohomology rings of $\mathbb{R}P^n$ (over $\mathbb{Z}/2\mathbb{Z}$) and of $\mathbb{C}P^n$ (over $\mathbb{Z}$):

Example 5.23. Let $\pi\colon E \to \mathbb{R}P^n$ be the tautological rank-one line bundle over $\mathbb{R}P^n = Gr_1(\mathbb{R}^{n+1})$. We then have

$$E^0 = \{(V, \vec{v}) \in \mathbb{R}P^n \times (\mathbb{R}^{n+1} \setminus \{\vec{0}\}) | \vec{v} \in V\} \cong \mathbb{R}^{n+1} \setminus \{\vec{0}\}$$

(the projection onto the second factor is a homeomorphism, since a one-dimensional subspace of $\mathbb{R}^{n+1}$ is uniquely determined by a single nonzero vector in it). Let $e \in H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ be the Euler class of E. For $1 \le j \le n-1$ we have $H^j(E^0;\mathbb{Z}/2\mathbb{Z}) = 0$ since E^0 deformation retracts to S^n, so part of the Gysin sequence reads

$$0 \longrightarrow H^j(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cdot\cup e} H^{j+1}(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$$

Here both $H^j(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ and $H^{j+1}(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (again we are assuming $1 \le j \le n-1$), so the fact that $\cdot\cup e\colon H^j(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \to H^{j+1}(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ is injective implies that it is an isomorphism.

It readily follows by induction that, for $1 \le k \le n$, the k-fold cup product $e^{\cup k} \in H^k(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ is the unique nonzero element of $H^k(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$. Of course $H^0(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ has its unique nonzero element given by the unit 1, while for $k \notin \{0,\dots,n\}$, $H^k(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = 0$. Thus, *as a graded ring*

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = \frac{(\mathbb{Z}/2\mathbb{Z})[e]}{\langle e^{n+1}\rangle}$$

where $e \in H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ is the Euler class of the tautological line bundle.

Example 5.24. Now let $\pi\colon E \to \mathbb{C}P^n$ be the tautological *complex* line bundle over $\mathbb{C}P^n = Gr_1(\mathbb{C}^{n+1})$; since any complex vector bundle is naturally oriented (and in our case the rank of E considered as a real vector bundle is 2), we have an Euler class $e \in H^2(\mathbb{C}P^n;\mathbb{Z})$. Now E^0 is homeomorphic to $C^{n+1} \setminus \{\vec{0}\}$, which is homotopy equivalent to S^{2n+1}. In particular $H^j(E^0;\mathbb{Z})$ for $1 \le j \le 2n$. So part of the Gysin sequence reads

$$0 \longrightarrow H^{j-2}(\mathbb{C}P^n;\mathbb{Z}) \xrightarrow{\cdot\cup e} H^j(\mathbb{C}P^n;\mathbb{Z}) \longrightarrow 0$$

for all $2 \le j \le 2n$. In other words, for $2 \le j \le 2n$, cup product with e defines an isomorphism $H^{j-2}(\mathbb{C}P^n;\mathbb{Z}) \to H^j(\mathbb{C}P^n;\mathbb{Z})$.

Now you likely learned in 8200 that $\mathbb{C}P^n$ has a cell decomposition consisting precisely of one $2k$ cell for $1 \le k \le n$. (This cell $f_k\colon D^{2k} \to \mathbb{C}P^n$ is given by

$$f_k(z_1,\dots,z_k) = [1 - \sum_{i=1}^{k} |z_i|^2 : z_1 : z_2 : \cdots : z_k : 0 : \cdots : 0]$$

where we identify D^{2k} with the closed unit disk in $\mathbb{C}^k$ and we use homogeneous coordinates on $\mathbb{C}P^n$. So the image of f_k is the standard copy of $\mathbb{C}P^k$ in $\mathbb{C}P^n$, and the restriction of f_k to the boundary gives the Hopf map $S^{2k-1} \to \mathbb{C}P^{k-1}$.) By grading considerations, the cellular boundary and coboundary operators must vanish, and so $H^j(\mathbb{C}P^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $0 \le j \le 2n$ and j is even, and is zero otherwise.

So just as in the case of $\mathbb{R}P^n$, it follows by induction that, for $1 \le k \le n$, the k-fold cup product $e^{\cup k}$ generates $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$. Thus

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \frac{\mathbb{Z}[e]}{\langle e^{n+1}\rangle}$$

where $e \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is the Euler class of the tautological complex line bundle over $\mathbb{C}P^n$.

6. Stiefel–Whitney and Chern classes

We are almost finally ready to define the Stiefel–Whitney and Chern classes of a (respectively, real or complex) vector bundle; before we do this we need to introduce the notion of the *projectivization* of a vector bundle.

Let $\pi\colon E \to M$ be a rank-k vector bundle, and as before let E^0 denote the complement of the zero section in E. We can then define the topological space

$$\mathbb{P}(E) = \frac{E^0}{v \sim \lambda v \text{ for } v \in E^0, \lambda \in \mathbb{R}\setminus\{0\}}$$

where of course we are using the vector space structures on the fibers of E when we write "λv." Since v and λv are always in the same fiber for $v \in E, \lambda \in \mathbb{R}$, the map $\pi\colon E \to M$ descends to a map $\underline{\pi}\colon \mathbb{P}(E) \to M$ defined by $\underline{\pi}([v]) = \pi(v)$. It follows immediately from the definition of the quotient topology that $\underline{\pi}$ is continuous.

Suppose that $\Phi\colon E|_U \to U \times \mathbb{R}^k$ is a local trivialization for the vector bundle E, given in coordinates by $\Phi(v) = (\pi(v), \phi(v))$. Since ϕ restricts to the fiber E_m as a linear isomorphism to $\mathbb{R}^k$ for each $m \in U$, we have an induced map

$$\begin{aligned}\Psi\colon \mathbb{P}(E)|_U &\to U \times \mathbb{R}P^{k-1}\\ [v] &\mapsto (\pi(v), [\phi(v)])\end{aligned}$$

where we write $\mathbb{P}(E)|_U = \underline{\pi}^{-1}(U)$. It is not hard to see that this map is a homeomorphism: First, the fact that Φ is bijective readily implies that Ψ is also bijective. Now let $p_1\colon E^0|_U \to \mathbb{P}(E)|_U$ and $p_2\colon U \times (\mathbb{R}^k \setminus \{\vec{0}\}) \to U \times \mathbb{R}P^{k-1}$ be the quotient projections. Then for any set $W \subset \mathbb{P}(E)|_U$ note that $p_2^{-1}(\Psi(W)) = \Phi(p_1^{-1}(W))$. So since $\Phi|_{E^0|_U}$ maps $E^0|_U$ homeomorphically to $U \times (\mathbb{R}^k \setminus \{\vec{0}\})$ it follows from the definition of the quotient topology that $\Psi(W)$ is open if and only if W is, proving that Ψ is a homeomorphism.

So since Ψ maps each fiber $\mathbb{P}(E)_m$ to $\{m\} \times \mathbb{R}P^{k-1}$, it follows that $\underline{\pi}\colon \mathbb{P}(E) \to M$ **is a fiber bundle with fiber** $\mathbb{R}P^{k-1}$ (and with local trivializations given by the maps Ψ). This fiber bundle is called the (real) *projectivization* of E

Similarly, if $\pi\colon E \to M$ is a rank-k *complex* vector bundle, then we obtain the *complex projectivization of* E, given by

$$\mathbb{P}_{\mathbb{C}}(E) = \frac{E^0}{v \sim \lambda v \text{ for } \lambda \in \mathbb{C}\setminus\{0\}},$$

with the obvious projection induced from π; this is a fiber bundle with fiber $\mathbb{C}P^{k-1}$.

6.1. **Stiefel–Whitney classes.** Recall from earlier that, if $\pi\colon E \to M$ is a vector bundle, a *pre-classifying map* for E is a continuous map $F\colon E \to \mathbb{R}^N$ (for some N) which restricts to each fiber E_m as a linear injection. Such a map evidently gives rise to a "projectivized pre-classifying map"

$$\begin{aligned} \mathbb{P}(F)\colon \mathbb{P}(E) &\to \mathbb{R}P^{N-1} \\ [v] &\mapsto [F(v)] \end{aligned}$$

Proposition 6.1. *Let $F\colon E \to \mathbb{R}^N$ be a pre-classifying map, inducing the projectivized pre-classifying map $\mathbb{P}(F)\colon \mathbb{P}(E) \to \mathbb{R}P^{N-1}$. Define*

$$x_F = (\mathbb{P}(F))^* e(\gamma^1(\mathbb{R}^N)) \in H^1(\mathbb{P}(E); \mathbb{Z}/2\mathbb{Z}).$$

Then for each $m \in M$, where $i_m\colon \mathbb{P}(E_m) \hookrightarrow \mathbb{P}(E)$ is the inclusion of the fiber, we have

$$H^*(\mathbb{P}(E_m); \mathbb{Z}/2\mathbb{Z}) = span_{\mathbb{Z}/2\mathbb{Z}}\{i_m^* 1, i_m^* x_F, \ldots, i_m^* x_F^{k-1}\}.$$

Proof. This follows quickly from the following lemma:

Lemma 6.2. *Let A: $\mathbb{R}^k \to \mathbb{R}^N$ be an injective linear map and define $\underline{A}$: $\mathbb{R}P^{k-1} \to \mathbb{R}P^{N-1}$ by $\underline{A}([v]) = [Av]$. Then $\underline{A}^* e(\gamma^1(\mathbb{R}^N))$ generates $H^1(\mathbb{R}P^{k-1}; \mathbb{Z}/2\mathbb{Z})$.*

Proof of Lemma 6.2. We have a commutative diagram

$$\begin{tikzcd} \gamma^1(\mathbb{R}^k) \arrow[r, "\tilde{A}"] \arrow[d] & \gamma^1(\mathbb{R}^N) \arrow[d] \\ \mathbb{R}P^{k-1} \arrow[r, "\underline{A}"] & \mathbb{R}P^{N-1} \end{tikzcd}$$

where $\tilde{A}$ is defined by $\tilde{A}(V, \vec{v}) = (\underline{A}V, A\vec{v})$ (recalling that $\gamma^1(\mathbb{R}^k)$ consists of pairs $(V, \vec{v}) \in \mathbb{R}P^{k-1} \times \mathbb{R}^k$ with $\vec{v} \in V$). The map $\tilde{A}$ sends the fiber over a point $V \in \mathbb{R}P^{k-1}$ by a linear isomorphism to the fiber over $\underline{A}V$. So by Proposition 4.17, we have $e(\gamma^1(\mathbb{R}^k)) = \underline{A}^* e(\gamma^1(\mathbb{R}^N))$. By Example 5.23, $e(\gamma^1(\mathbb{R}^k))$ generates $H^1(\mathbb{R}P^{k-1}; \mathbb{Z}/2\mathbb{Z})$. $\square$

Given this lemma, note that for any $m \in M$, $\mathbb{P}(F) \circ i_m\colon \mathbb{P}(E_m) \to \mathbb{R}P^{N-1}$ is the projectivization of an injective linear map from the k-dimensional real vector space E_m to $\mathbb{R}^N$. So by Lemma 6.2, the class $(\mathbb{P}(F) \circ i_m)^* e(\gamma^1(\mathbb{R}^N)) = i_m^* x_F$ generates $H^1(\mathbb{P}(E_m); \mathbb{Z}/2\mathbb{Z})$ (where $\mathbb{P}(E_m)$ is homeomorphic to $\mathbb{R}P^{k-1}$). Then by Example 5.23, $H^*(\mathbb{P}(E_m); \mathbb{Z}/2\mathbb{Z})$ is the $(\mathbb{Z}/2\mathbb{Z})$-span of the classes $1, i_m^* x_F, \ldots, (i_m^* x_F)^{k-1}$. Since i_m^* is a unital ring homomorphism, Proposition 6.1 follows. $\square$

Proposition 6.3. *If $F_0\colon E \to \mathbb{R}^{N_0}$ and $F_1\colon E \to \mathbb{R}^{N_1}$ are two pre-classifying maps for the same vector bundle then $x_{F_0} = x_{F_1}$ (where x_{F_j} are defined as in Proposition 6.1).*

Proof. By Theorem 2.8, there is $N \geq \max\{N_0, N_1\}$ such that the pre-classifying maps $I_{N_0,N'} \circ F_0$ and $I_{N_1,N'} \circ F_1$ are isotopic (*i.e.*, homotopic through pre-classifying maps), where $I_{N_j,N'}(\vec{v}) = (\vec{v}, \vec{0})$. Where $i_{N_j,N'}\colon \mathbb{R}P^{N_j-1} \to \mathbb{R}P^{N-1}$ is defined by $i_{N_j,N'}([v]) = [(v, \vec{0})]$, we evidently have, for $j = 0, 1$,

$$\mathbb{P}(I_{N_j,N'} \circ F) = i_{N_j,N'} \circ \mathbb{P}(F).$$

Moreover the isotopy from $I_{N_0,N'} \circ F_0$ to $I_{N_1,N'} \circ F_1$ gives rise to a homotopy from $i_{N_0,N'} \circ \mathbb{P}(F)$ to $i_{N_1,N'} \circ \mathbb{P}(F)$. Now as noted in (14), we have $i_{N_j,N'}^* e(\gamma^1(\mathbb{R}^N)) = e(\gamma^1(\mathbb{R}^{N_j}))$. So

$$\begin{aligned} x_{F_0} &= \mathbb{P}(F_0)^* e(\gamma^1(\mathbb{R}^{N_0})) = \mathbb{P}(F_0)^* i_{N_0,N'}^* e(\gamma^1(\mathbb{R}^{N'})) = (i_{N_0,N'} \circ \mathbb{P}(F_0))^* e(\gamma^1(\mathbb{R}^{N'})) \\ &= (i_{N_1,N'} \circ \mathbb{P}(F_1))^* e(\gamma^1(\mathbb{R}^{N'})) = \mathbb{P}(F_1)^* i_{N_1,N'}^* e(\gamma^1(\mathbb{R}^{N'})) = x_{F_1} \end{aligned}$$

$\square$

Definition-Theorem 6.4. *Let $\pi\colon E \to M$ be a finite-type rank-k vector bundle over any paracompact Hausdorff space M such that the Leray–Hirsch Theorem 5.18 holds for fiber bundles over M. Then there are unique classes $w_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ $(0 \le i \le k)$ (the **Stiefel–Whitney classes** of E) such that $w_0(E) = 1$ and, for any pre-classifying map $F\colon E \to \mathbb{R}^N$, the class $x_F \in H^1(\mathbb{P}(F);\mathbb{Z}/2\mathbb{Z})$ obeys the equation*

$$\sum_{i=0}^{k} \underline{\pi}^* w_i(E) \cup x_F^{k-i} = 0$$

Proof. The bundle E admits a pre-classifying map F by Proposition 2.5, and the class x_F is independent of the choice of F by Proposition 6.3. Theorem 5.18 and Proposition 6.1 then combine to show that any class $y \in H^k(\mathbb{P}(E);\mathbb{Z}/2\mathbb{Z})$ can be written uniquely in the form

$$y = \sum_{i=1}^{k} \pi^* c_i \cup x_F^{k-i}$$

where $c_i \in H^i(E;\mathbb{Z}/2\mathbb{Z})$. In particular this applies to the class $y = -x_F^k$, and then the Stiefel–Whitney classes are given by $w_0(E) = 1$ and $w_i(E) = c_i$. □

Example 6.5. Suppose that $\pi\colon E \to M$ is a trivial rank-k vector bundle. Then the second component of a trivialization $\Phi\colon E \to M \times \mathbb{R}^k$ gives a pre-classifying map $F\colon E \to \mathbb{R}^k$ (which is an isomorphism on each fiber, not just a linear injection as usual). Now $x_F = \mathbb{P}(F)^* e(\gamma^1(\mathbb{R}^k))$, and of course we have $e(\gamma^1(\mathbb{R}^k))^k = 0$ since $H^k(\mathbb{R}P^{k-1};\mathbb{Z}/2\mathbb{Z}) = 0$. Hence $x_F^k = 0$. Thus the Stiefel–Whitney classes $w_i(E)$ are equal to 0 for $i \ge 1$.

Example 6.6. Let $\pi\colon E \to \mathbb{R}P^n$ be the tautological line bundle over $\mathbb{R}P^n$, so $E = \{(V,\vec{v}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} | \vec{v} \in V\}$. Then

$$\mathbb{P}(E) = \{(V,[\vec{v}]) \in \mathbb{R}P^n \times \mathbb{R}P^n | [\vec{v}] = V\}$$

is the diagonal in the product $\mathbb{R}P^n \times \mathbb{R}P^n$; the fiber bundle projection $\underline{\pi}\colon \mathbb{P}(E) \to \mathbb{R}P^n$ just sends (V,V) to V and is a homeomorphism (this should make sense: $\mathbb{P}(E)$ is an $\mathbb{R}P^0$-bundle, and $\mathbb{R}P^0$ is a single point).

There is a pre-classifying map $F\colon E \to \mathbb{R}^{n+1}$ defined by $F(V,\vec{v}) = \vec{v}$, and this projectivizes to the map $\mathbb{P}(F)\colon \mathbb{P}(E) \to \mathbb{R}P^n$ defined by $\mathbb{P}(F)(V,V) = V$. In other words, using $\underline{\pi}$ to identify $\mathbb{P}(E)$ with $\mathbb{R}P^n$, $\mathbb{P}(F)$ is just the identity.

So continuing to use $\underline{\pi}$ to identify $\mathbb{P}(E)$ with $\mathbb{R}P^n$, the Stiefel–Whitney classes $w_0(E) \in H^0(M;\mathbb{Z}/2\mathbb{Z})$ and $w_1(E) \in H^1(M;\mathbb{Z}/2\mathbb{Z})$ are determined by the properties that $w_0(E) = 1$ and

$$w_0(E) \cup e(\gamma^1(\mathbb{R}^{n+1})) + w_1(E) \cup 1 = 0$$

Thus

$$w_1(E) = e(\gamma^1(\mathbb{R}^{n+1}))$$

(there is no need for a sign as we are working over $\mathbb{Z}/2\mathbb{Z}$).

The above example may seem slightly silly, but it turns out that this one special case, together with some general formal properties proven below, is enough to determine all of the Stiefel–Whitney classes of all finite-type bundles over paracompact Hausdorff spaces.

Definition 6.7. The *total Stiefel–Whitney class* of a rank-k vector bundle as in Definition–Theorem 6.4 is the class

$$w(E) = w_0(E) + w_1(E) + \cdots + w_k(E) \in H^*(M;\mathbb{Z}/2\mathbb{Z})$$

We will see later that the total Stiefel–Whitney class is useful for keeping track of information about how bundles behave under direct sum. Note that the trivial bundle has total Stiefel–Whitney class $w(E) = 1$.

Proposition 6.8. *Let $f: X \to M$ be a continuous map and let $\pi_E: E \to M$ and $\pi_V: V \to X$ be vector bundles as in Definition–Theorem 6.4 such that there is a map $\tilde{f}: V \to E$ making the diagram*

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi_V"] & E \arrow[d, "\pi_E"] \\ X \arrow[r, "f"] & M \end{tikzcd}$$

commute, with $\tilde{f}|_{V_x}$ a linear isomorphism to $E_{f(x)}$. Then, for all i,

$$w_i(V) = f^* w_i(E)$$

In particular

(i) *$w_i(E)$ depends only on the isomorphism type of E.*
(ii) *The Stiefel–Whitney classes satisfy the naturality property*

$$w_i(f^*E) = f^* w_i(E)$$

Proof. The assumption implies that there is a projectivized map $\mathbb{P}(\tilde{f}): \mathbb{P}(V) \to \mathbb{P}(E)$ leading to a commutative diagram

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r, "\mathbb{P}(\tilde{f})"] \arrow[d, "\underline{\pi}_V"] & \mathbb{P}(E) \arrow[d, "\underline{\pi}_E"] \\ X \arrow[r, "f"] & M \end{tikzcd}$$

If $F: E \to \mathbb{R}^N$ is a pre-classifying map, then $F \circ \tilde{f}: V \to \mathbb{R}^N$ is also a pre-classifying map, with projectivization $\mathbb{P}(F) \circ \mathbb{P}(\tilde{f})$, so that by definition (and by the functoriality of pullbacks) we have

$$x_{F \circ \tilde{f}} = \mathbb{P}(\tilde{f})^* x_F.$$

Hence

$$\begin{aligned} 0 = \mathbb{P}(\tilde{f})^* \left(\sum_{k=0}^{n} \underline{\pi}_E^* w_i(E) \cup x_F^{k-i} \right) &= \sum_{k=0}^{n} (\mathbb{P}(\tilde{f})^* \underline{\pi}_E^* w_i(E)) \cup x_{F\circ\tilde{f}}^{k-i} \\ = \sum_{k=0}^{n} (\pi_V^* f^* w_i(E)) \cup x_{F\circ\tilde{f}}^{k-i} & \end{aligned}$$

So by the uniqueness part of Definition–Theorem 6.4, we must have $w_i(V) = f^* w_i(E)$.

As for the two statements at the end, (i) is the special case in which f is the identity, and (ii) is immediate, since $V = f^*E$ fits into a commutative diagram of the form described in the proposition. □

In particular, it follows that, if $f: M \to Gr_k(\mathbb{R}^N)$ is a classifying map for a vector bundle $\pi: E \to M$ then $w_i(E) = f^* w_i(\gamma^k(\mathbb{R}^N))$. This suggests another way of defining Stiefel–Whitney classes: first define them just for the tautological bundles $\gamma^k(\mathbb{R}^N) \to Gr_k(\mathbb{R}^N)$ as before, and then define them for an arbitrary finite-type vector bundle $E \to M$ over any paracompact Hausdorff space M by setting $w_i(E) = f^* w_i(\gamma^k(\mathbb{R}^N))$ for any classifying map f. This conveniently gets around the issue that we have only proven the Leray–Hirsch theorem for a limited class of spaces, since this limited class does include the Grassmannians. Indeed, if one (correctly) assumes the Leray–Hirsch theorem and hence the construction of Stiefel–Whitney classes for the infinite Grassmannian $Gr_k = Gr_k(\mathbb{R}^\infty)$, then one could drop the finite-type assumption on E and define $w_i(E) = f^*w_i(\gamma^k)$ where $\gamma^k \to Gr_k$ is the tautological bundle and $f: M \to Gr_k$ is any classifying map.

Exercise 6.9. Let $\pi: E \to S^1$ be any rank-k vector bundle. Cover S^1 by the two open sets

$$U_0 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1,\ y > -1/2\} \qquad U_1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1,\ y < 1/2\}$$

By Remark 2.30, since U_0 and U_1 are contractible there are trivializations $\Phi_0: E|_{U_0} \to U_0 \times \mathbb{R}^k$ and $\Phi: E|_{U_1} \to U_1 \times \mathbb{R}^k$. The map $\Phi_1 \circ \Phi_0^{-1}: (U_0 \times U_1) \times \mathbb{R}^k \to (U_0 \cap U_1) \times \mathbb{R}^k$ is given by $(x,v) \mapsto (x, g_{01}(x)v)$ for some map $g_{01}: U_0 \cap U_1 \to GL(k;\mathbb{R})$. Note that both $U_0 \cap U_1$ and $GL(k;\mathbb{R})$ have two path components.

(a) Prove that if g_{01} maps the two path components of $U_0 \cap U_1$ to the same path component of $GL(k;\mathbb{R})$, then E is trivial.

(b) Prove that if g_{01} maps the two path compenents of $U_0 \cap U_1$ to different path components of $GL(k;\mathbb{R})$, then $w_1(E) \neq 0$. (Suggestion: first show that any two vector bundles satisfying the hypothesis are isomorphic, and then prove the statement for just one such bundle—the Whitney product formula (Theorem 6.16 below) might help for latter.)

(c) Now let $\pi: V \to M$ be any vector bundle, where M is assumed paracompact, Hausdorff, and locally path-connected. Prove that V is orientable if and only if, for every continuous $\gamma: S^1 \to M$, γ^*V is orientable. (Hint: A bundle is orientable if and only if its classifying maps lift to maps to the oriented Grassmannian.)

(d) Where $\pi: V \to M$ is as in part (c), prove that V is orientable if and only if $w_1(V) = 0$.

6.2. **Chern classes.** A very similar construction, when combined with Example 5.24 rather than 5.23, gives rise to the *Chern classes* $c_i(E) \in H^{2i}(M;\mathbb{Z})$ of a finite-type rank-k complex vector bundle $\pi: E \to M$ over a space M to which the Leray–Hirsch theorem applies. Recall that such a bundle admits a "complex pre-classifying map" $F: E \to \mathbb{C}^N$ (which is continuous and complex-linear on each fiber); this projectivizes to a map $\mathbb{P}_{\mathbb{C}}(F): \mathbb{P}_{\mathbb{C}}(E) \to \mathbb{C}P^{N-1}$, where $\underline{\pi}_{\mathbb{C}}: \mathbb{P}_{\mathbb{C}}(E) \to M$ is the complex projectivization of E. We summarize the construction of the Chern classes in the following; the proofs of the various statements are left to the reader (they are directly analogous to the corresponding statements for Stiefel–Whitney classes, proven in the last subsection):

Definition-Theorem 6.10. *Under the above assumptions, the class $z_F \in H^2(\mathbb{P}_{\mathbb{C}}(E);\mathbb{Z})$ defined by $z_F = -\mathbb{P}_{\mathbb{C}}(F)^*e(\gamma^1(\mathbb{C}^N))$ is independent of the choice of complex pre-classifying map F and has the property that, for each $m \in M$, one has*

$$H^*(\mathbb{P}_{\mathbb{C}}(E_m);\mathbb{Z}) = span_{\mathbb{Z}}\{i_m^*1, i_m^*z_F, \dots, i_m^*z_F^{k-1}\}$$

Consequently by the Leray–Hirsch theorem there are unique cohomology classes (the ***Chern classes*** *of E) $c_i(E) \in H^{2i}(M;\mathbb{Z})$ with $c_0(E) = 1$ and*

$$\sum_{i=0}^{k} \underline{\pi}_{\mathbb{C}}^* c_i(E) \cup z_F^{k-i} = 0$$

The classes $c_i(E)$ depend only on the complex-isomorphism type of E and obey the naturality property $c_i(f^*E) = f^*c_i(E)$ for a continuous map $f: X \to M$.

Note the negative sign in the definition of z_F; it is included so as to avoid a negative sign in the conclusion of the following example:

Example 6.11. Let $\pi\colon E \to \mathbb{C}P^n$ be the tautological complex line bundle over $\mathbb{C}P^n$. Then just as in Example 6.6 one has

$$\mathbb{P}_{\mathbb{C}}(E) = \{(V, [\vec{v}]) \in \mathbb{C}P^n \times \mathbb{C}P^n | [\vec{v}] = V\},$$

i.e. $\mathbb{P}_{\mathbb{C}}(E)$ is the diagonal in the product $\mathbb{C}P^n \times \mathbb{C}P^n$, with the bundle projection given by projection onto the first factor. There is a complex pre-classifying map $F\colon E \to \mathbb{C}^{n+1}$ given by projection to the second factor, and the projectivization of this pre-classifying map is again the map $(V,V) \mapsto V$ from $\mathbb{P}_{\mathbb{C}}(E)$ to $\mathbb{C}P^n$. So, using the projection to identify $\mathbb{P}_{\mathbb{C}}(E)$ with $\mathbb{C}P^n$, the Chern classes $c_0(E)$ and $c_1(E)$ are characterized by $c_0(E) = 1$ and

$$1 \cup (-e(E)) + c_1(E) \cup 1 = 0,$$

i.e.,

$$c_1(\gamma^1(\mathbb{C}^n)) = e(\gamma^1(\mathbb{C}^n)) \in H^2(\mathbb{C}P^n; \mathbb{Z})$$

As with Stiefel–Whitney classes, we define the total Chern class of a rank-k complex vector bundle as $c(E) = c_0(E) + \cdots + c_k(E) \in H^*(M; \mathbb{Z})$.

One could alternately define the Chern classes by first just defining the Chern classes $c_i(\gamma^k(\mathbb{C}^N)) \in H^{2i}(Gr_k(\mathbb{C}^N); \mathbb{Z})$ of the tautological bundles and then putting $c_i(E) = f^* c_i(\gamma^k(\mathbb{C}^N))$ for any complex classifying map $f\colon M \to Gr_k(\mathbb{C}^N)$.

Exercise 6.12. For each $d \in \mathbb{Z}$ define a complex line bundle $\pi\colon E_d \to \mathbb{C}P^1$ by

$$E_d = \left\{ \left([z_0 : z_1], (w_0, w_1) \right) \in \mathbb{C}P^1 \times \mathbb{C}^2 | z_1^d w_0 = z_0^d w_1 \right\}$$

if $d \geq 0$ and

$$E_d = \left\{ \left([z_0 : z_1], (w_0, w_1) \right) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \bar{z}_1^{-d} w_0 = \bar{z}_0^{-d} w_1 \right\}$$

if $d < 0$ (with the bundle projection π given by projection to the first factor).

(a) Determine the first Chern classes $c_1(E_d)$, and in particular show that for each $c \in H^2(\mathbb{C}P^1; \mathbb{Z})$ there is a unique value of d with $c_1(E_d) = c$.

(b) Prove that every complex line bundle $\pi\colon E \to \mathbb{C}P^1$ is isomorphic to E_d for some d. (Hint: By Remark 2.30 the restrictions of E to small neighborhoods of the northern and southern hemispheres are trivial. The transition map $\Phi_1 \circ \Phi_0^{-1}$ associated to a pair of trivializations over neighborhoods of the northern and southern hemispheres gives a map from a neighborhood of the equator to $GL(1; \mathbb{C}) = \mathbb{C} \setminus \{0\}$. The correct value of d is determined by the homotopy class of this map.)

6.3. **Whitney Sums.** We now give details for a construction alluded to earlier of the "direct sum" (sometimes called "Whitney sum") of two vector bundles. Let $\pi_V\colon V \to M$ and $\pi_W\colon W \to M$ be vector bundles of ranks k and l, respectively, over the same space M. Define

$$V \oplus W = \{(v, w) \in V \times W | \pi_V(v) = \pi_W(w)\}.$$

We have an obvious continuous map $\pi_{V\oplus W}\colon V \oplus W \to M$ given as the composition of the projection to V with $\pi_V\colon V \to M$ (or equivalently as the composition of the projection to W with π_W). M is covered by open sets U with local trivializations

$$\begin{aligned} \Phi_V\colon V|_U &\to U \times \mathbb{R}^k \\ v &\mapsto (\pi_V(v), \phi_V(v)) \end{aligned} \qquad\qquad \begin{aligned} \Phi_W\colon W|_U &\to U \times \mathbb{R}^l \\ f &\mapsto (\pi_W(w), \phi_W(w)) \end{aligned}$$

which gives rise to a continuous bijection $\Phi\colon \pi_{V\oplus W}^{-1}(U) \to U \times \mathbb{R}^{k+l}$ defined by $\Phi(v, w) = (\pi_V(v), \phi_V(v), \phi_W(w))$. Each fiber $(V \oplus W)_m = \pi_{V\oplus W}^{-1}(\{m\})$ is in obvious bijection with the vector space $V_m \oplus W_m$, and if $m \in U$ then Φ maps each $(V \oplus W)_m$ by a linear isomorphism to $\mathbb{R}^{k+l}$. We have $\Phi^{-1}(m, v, w) = (\Phi_V^{-1}(m, v), \Phi_W^{-1}(m, w))$, so Φ^{-1} is continuous.

This suffices to prove that the maps $\Phi\colon (V \oplus W)|_U \to U \times \mathbb{R}^{k+l}$ form a system of local trivializations, thus making $V \oplus W$ into a rank-$(k+l)$ vector bundle. Evidently if V and W are complex vector bundles then so too is $V \oplus W$.

Example 6.13. An important context in which direct sums of bundles naturally appear is the following. Suppose that M is a smooth manifold, and suppose that N is an embedded submanifold of M. There are then three natural vector bundles over N: one has the tangent bundle TN; the restriction $TM|_N$ of the tangent bundle of M to N; and the *normal bundle* $\nu_{N,M}$, whose fiber at $n \in N$ can be naturally identified with $\frac{T_nM}{T_nN}$ (see for instance [U2, Example 2.4]). It is fairly easy to see from the definitions that there is an isomorphism of vector bundles

$$TM|_N \cong TN \oplus \nu_{N,M}.$$

Somewhat more generally, if $g\colon N \to M$ is an immersion, one can still form the normal bundle $\nu_g \to N$ to g with fiber over n isomorphic to $\frac{T_{g(n)}M}{g_*T_nN}$: just use the fact that the restriction of an immersion to a sufficiently small open set is an embedding and use such small open sets to construct local trivializations. In this case one has a direct sum decomposition

$$g^*TM \cong TN \oplus \nu_g.$$

We will soon see that, using these direct sum splittings, Stiefel–Whitney classes can be used to give obstructions to the existence of certain classes of immersions or embeddings.

We will now work toward establishing formulas that represent the characteristic classes of the direct sum $V \oplus W$ of two vector bundles V and W to the characteristic classes of V and W. The following lemma, which is of independent interest, turns out to be relevant:

Proposition 6.14. *Let R be a commutative ring with unity, let X be a space, and let $i_1\colon U_1 \to X$, $i_2\colon U_2 \to X$ be inclusions of open sets, with $X = U_1 \cup U_2$. Suppose that $c_1, c_2 \in H^*(X;R)$ be two classes such that $i_1^*c_1 = 0$ and $i_2^*c_2 = 0$. Then $c_1 \cup c_2 = 0$.*

Proof. For $j = 1,2$ consider the long exact sequence of the pair (X, U_j), part of which reads

$$H^*(X,U_j;R) \xrightarrow{\pi_j^*} H^*(X;R) \xrightarrow{i_j^*} H^*(U_j;R)$$

So the assumption on c_j implies that $c_j \in Im(\pi_j^*\colon H^*(X,U_j;R) \to H^*(X;R))$, *i.e.*, that the cohomology class c_j can be represented can be represented by a cocycle $\alpha_j \in S^*(X;R)$ which vanishes on all simplicies with image contained in U_j.

But then, as is obvious from the definition of the cup product, the cocycle $\alpha_1 \cup \alpha_2$ vanishes on all simplices whose image is contained either in U_1 or in U_2, *i.e.* the cocycle $\alpha_1 \cup \alpha_2$ evaluates trivially on all elements of the subcomplex $S_\bullet^{U_1,U_2}(X)$ described at the start of Section 3.3.1. By (11), this implies that the cohomology class $c_1 \cup c_2 = [\alpha_1 \cup \alpha_2]$ is zero. □

Exercise 6.15. If R is a commutative ring with unity, the R-**cuplength** of a space X, denoted $cl(X;R)$, is the supremum of all positive integers k such that there are classes $c_i \in H^{n_i}(X;R)$ $(1 \le i \le k)$, with each $n_i > 0$, such that $c_1 \cup \cdots \cup c_k \neq 0$. Prove that any open cover $X = U_1 \cup \cdots U_l$ of X by *contractible* open sets must have cardinality $l \ge cl(X;R) + 1$. Show also that there is an open cover of $\mathbb{R}P^n$ by exactly $l = cl(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) + 1$ many contractible open sets.

Recall that the total Stiefel-Whitney class of a rank-k vector bundle $\pi\colon E \to M$ is the (inhomogeneous) cohomology class $w(E) = \sum_{i=0}^k w_i(E) \in H^*(M;\mathbb{Z}/2\mathbb{Z})$. (Of course, as always, $w_0(E) = 1$.)

Theorem 6.16 (Whitney Product Formula). *Let $\pi_V\colon V \to M$ and $\pi_W\colon W \to M$ be two (finite-type) vector bundles over the paracompact Hausdorff space M. Then we have*

$$w(V \oplus W) = w(V)w(W).$$

Proof. The maps $v \mapsto (v,0)$ and $w \mapsto (0,w)$ projectivize to embeddings $j_V\colon \mathbb{P}(V) \to \mathbb{P}(V \oplus W)$ and $j_W\colon \mathbb{P}(W) \to \mathbb{P}(V \oplus W)$, respectively; we use these embeddings to identify $\mathbb{P}(V)$ and $\mathbb{P}(W)$ as subsets of $\mathbb{P}(V \oplus W)$. Define open sets

$$U_V = \{[(v,w)] \in \mathbb{P}(V \oplus W) | v \neq 0\} \qquad U_W = \{[(v,w)] \in \{(V \oplus W) | w \neq 0\}$$

Note that $\mathbb{P}(V \oplus V) = U_V \cup U_W$. Also, the homotopy $(t,[(v,w)]) \mapsto [(v,tw)]$ gives a deformation retraction of U_V to $\mathbb{P}(V)$, and likewise $(t,[v,w]) \mapsto [(tv,w)]$ gives a deformation retraction of U_W to $\mathbb{P}(W)$. So where $i_V\colon U_V \to \mathbb{P}(V \oplus W)$ and $i_W\colon U_W \to \mathbb{P}(V \oplus W)$ are the inclusions, for any class $c \in H^*(\mathbb{P}(V \oplus W); \mathbb{Z}/2\mathbb{Z})$ we have $i_V^* c = 0$ if and only if $j_V^* c = 0$, and $i_W^* c = 0$ if and only if $j_W^* c = 0$. In particular it follows from Proposition 6.14 that

$$\text{If } j_V^* c_1 = j_W^* c_2 = 0, \text{ then } c_1 \cup c_2 = 0 \tag{20}$$

Now let $F_0\colon V \to \mathbb{R}^{N_0}$ and $F_1\colon W \to \mathbb{R}^{N_1}$ be two pre-classifying maps. We then obtain a pre-classifying map $F\colon V \oplus W \to \mathbb{R}^{N_0+N_1}$ by setting $F(v,w) = (F_0(v), F_1(w))$. Note that the compositions $\mathbb{P}(F) \circ j_V\colon \mathbb{P}(V) \to \mathbb{R}P^{N_0+N_1-1}$ and $\mathbb{P}(F) \circ j_W\colon \mathbb{P}(W) \to \mathbb{R}P^{N_0+N_1-1}$ are projectivized pre-classifying maps for V an W.

Now let $x = \mathbb{P}(F)^* e(\gamma^1(\mathbb{R}^{N_0+N_1}))$. Where $\underline{\pi}_V\colon \mathbb{P}(V) \to M$, $\underline{\pi}_W\colon \mathbb{P}(W) \to M$, and $\underline{\pi}\colon \mathbb{P}(V \oplus W) \to M$ are the bundle projections, we hence have $\underline{\pi} \circ j_V = \underline{\pi}_V$, $\underline{\pi} \circ j_W = \underline{\pi}_W$, and, by the definition of the Stiefel–Whitney classes

$$\sum_{i=0}^{k} \underline{\pi}_V^* w_i(V) \cup j_V^* x^{k-i} = 0 \in H^*(V; \mathbb{Z}/2\mathbb{Z}) \tag{21}$$

and

$$\sum_{i=0}^{l} \underline{\pi}_W^* w_i(W) \cup j_W^* x^{l-i} = 0 \in H^*(\mathbb{P}(W); \mathbb{Z}/2\mathbb{Z}) \tag{22}$$

(here we denote the rank of V by k and the rank of W by l).

Now define

$$c_V = \sum_{i=0}^{k} \underline{\pi}^* w_i(V) \cup x^{k-i} \in H^*(\mathbb{P}(V \oplus W); \mathbb{Z}/2\mathbb{Z})$$

and

$$c_W = \sum_{i=0}^{l} \underline{\pi}^* w_i(W) \cup x^{l-i} \in H^*(\mathbb{P}(V \oplus W); \mathbb{Z}/2\mathbb{Z})$$

From (21) and (22) we see that $j_V^* c_V = j_W^* c_W = 0$. Hence by (20) $c_V \cup c_W = 0$, *i.e.*,

$$\begin{aligned} 0 &= \left(\sum_{i=0}^{k} \underline{\pi}^* w_i(V) \cup x^{k-i} \right) \cup \left(\sum_{i=0}^{l} \underline{\pi}^* w_i(W) \cup x^{l-i} \right) \\ &= \sum_{m=0}^{k+l} \underline{\pi}^* \left(\sum_{i=0}^{m} w_i(V) \cup w_{m-i}(W) \right) x^{k+l-m} \end{aligned}$$

(we have used Remark 5.5 and the fact that we are working over $\mathbb{Z}/2\mathbb{Z}$ to commute x with some of the $w_j(W)$ in the last equation). The coefficient on x^{k+l} above is $w_0(V) \cup w_0(W) = 1$, so by the definition of the Stiefel–Whitney classes this shows that

$$w_m(V \oplus W) = \sum_{i=0}^{m} w_i(V) \cup w_{m-i}(W),$$

which is more concisely expressed by the formula for the total Stiefel–Whitney classes

$$w(V \oplus W) = w(V) \cup w(W)$$

□

Similarly for the total Chern classes:

Theorem 6.17. *Let $\pi_V : V \to M$ and $\pi_W : W \to M$ be two finite-type complex vector bundles over the paracompact Hausdorff space M. Then their total Chern classes $c(V) = \sum c_i(V)$, $c(W) = \sum c_i(W)$ obey*

$$c(V \oplus W) = c(V) \cup c(W)$$

The proof of Theorem 6.17 is exactly the same as that of Theorem 6.16 and so is left to the reader—just replace real projectivizations and the classes x_F of Definition-Theorem 6.4 by complex projectivizations and the classes z_F of Definition-Theorem 6.10. (All of the relevant cup products are of even-degree classes, so they are still commutative by Remark 5.5.)

The Whitney product formula is especially helpful for gaining information about subbundles of vector bundles. First we define these:

Definition 6.18. Let $\pi : E \to M$ be a rank-n real vector bundle. A *rank-k subbundle* is a subset $V \subset E$ such that there is an open cover $\{U_\alpha\}$ of M and a collection of local trivializations $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ such that $E|_{U_\alpha} \cap V = \Phi_\alpha^{-1}(U_\alpha \times \mathbb{R}^k \times \{\vec{0}\})$.

Obviously V is in this case a vector bundle in its own right, with local trivializations given by the $\Phi_\alpha|_{E|_{U_\alpha} \cap V}$.

Proposition 6.19. *If M is a paracompact Hausdorff space and $\pi : E \to M$ is a vector bundle over M, then for any subbundle $V \subset E$ there is another subbundle $W \subset E$ such that $E \cong V \oplus W$.*

Proof. (Sketch) The main point is that E admits an orthogonal structure (*i.e.* a continuously varying family of inner products on the fibers E_m). Indeed, if

$$\begin{aligned} \Phi_\alpha : E|_{U_\alpha} &\to U_\alpha \times \mathbb{R}^n \\ e &\mapsto (\pi(e), \phi_\alpha(e)) \end{aligned}$$

is a system of local trivializations, since M is paracompact and Hausdorff we may find a partition of unity $\{\chi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ of M. Then for $m \in M$ and $e_1, e_2 \in E_m$ define

$$\langle e_1, e_2 \rangle = \sum_\alpha \chi_\alpha(m) \phi_\alpha(e_1) \cdot \phi_\alpha(e_2)$$

where $\cdot$ denotes the standard dot product on $\mathbb{R}^n$. Since a convex combination of inner products is still an inner product, $\langle \cdot, \cdot \rangle$ defines an inner product on each fiber E_m.

We accordingly define W to be the orthogonal complement to V with respect to $\langle \cdot, \cdot \rangle$:

$$W = \{e \in E | \langle e, v \rangle = 0 \text{ for all } v \in V_{\pi(e)}\}$$

It is not difficult to check (see [MS, Theorem 3.3] for details) that W is a subbundle of E, and that $E \cong V \oplus W$. □

One way of constructing a subbundle is the following. Suppose that we have sections $s_1, \ldots, s_k \colon M \to E$ (where E is a rank-n vector bundle) with the property that, for each $m \in M$, the elements $s_1(m), \ldots, s_k(m)$ are linearly independent. Define

$$V = \left\{ \sum_{i=1}^{k} c_i s_i(m) \middle| c_i \in \mathbb{R}, m \in M \right\}$$

Let us show that V is a subbundle. If $m \in M$ choose a local trivialization $\Phi \colon E|_U \to U \times \mathbb{R}^n$ (given by $\Phi(e) = (\pi(e), \phi(e))$). Let A_m be the orthogonal complement of $\{\phi(s_1(m)), \ldots, \phi(s_k(m))\}$ in $\mathbb{R}^n$. By continuity, since $A_m \oplus span\{\phi(s_1(m)), \ldots, \phi(s_k(m))\} = \mathbb{R}^n$, we will have

$$A_m \oplus span\{\phi(s_1(x)), \ldots, \phi(s_k(x))\} = \mathbb{R}^n \tag{23}$$

for all x in some neighborhood $O \subset U$ of m. Moreover the map $p_x \colon \mathbb{R}^n \to \mathbb{R}^k$ given by composing the projection to $span\{\phi(s_1(x)), \ldots, \phi(s_k(x))\}$ determined by the direct sum decomposition (23) with the map $\sum c_i s_i(x) \mapsto (c_1, \ldots, c_k)$ is continuous as a function of x. Likewise the function $q_x \colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ defined by composing the projection onto A_m determined by (23) with some fixed isomorphism from A_m to $\mathbb{R}^{n-k}$ is continuous in x. Therefore we can define a new local trivialization Φ' for E over the neighborhood O of m by

$$\Phi'(e) = (\pi(e), p_x(\phi(e)), q_x(\phi(e)))$$

This trivialization maps $V \cap E|_O$ to $O \times \mathbb{R}^k$. So since $E|_M$ is covered by the domains of such trivializations it follows that V is a subbundle of E.

Moreover it is obvious that in this situation V is a trivial bundle: a global trivialization $\Psi \colon V \to M \times \mathbb{R}^k$ is given by

$$\Psi\left(\sum_{i=1}^{k} c_i s_i(m) \right) = (m, c_1, \ldots, c_k)$$

Consequently we obtain the following consequence of the Whitney product formula, which may shed some additional light on the meaning of the Stiefel–Whitney classes.

Corollary 6.20. *Let M be a paracompact Hausdorff space. Suppose that the rank-n vector bundle $\pi \colon E \to M$ admits sections $s_1, \ldots, s_k \colon M \to E$ such that $\{s_1(m), \ldots, s_k(m)\}$ is linearly independent for every m. Then*

$$w_{n-k+1}(E) = \cdots\cdots = w_{n-1}(E) = w_n(E) = 0$$

Proof. We have a trivial, rank-k subbundle $V \subset E$ by the above discussion (V is the span of the sections s_i). By Proposition 6.19 we may write $E \cong V \oplus W$ for some rank-$(n-k)$ vector bundle W. Now since V is trivial, the its total Stiefel–Whitney class is given by $w(V) = 1$. So

$$w(E) = w(V) \cup w(W) = w(W) = 1 + w_1(W) + \cdots + w_{n-k}(W),$$

as the rank of W is only $n - k$. □

The special case where $k = 1$ shows that a rank-n vector bundle E with a nonvanishing section must have $w_n(E) = 0$. Of course we earlier saw in Proposition 4.11 that such a bundle also has $e(E) = 0$; Theorem 6.24 below will show that this is not a coincidence.

The same argument shows that if a rank-n *complex* vector bundle $\pi \colon E \to M$ has sections $s_1, \ldots, s_k$ which are everywhere linearly independent *over* $\mathbb{C}$, then we must have $c_{n-k+1}(E) = \cdots = c_n(E) = 0$.

Exercise 6.21. (a) Let M be a paracompact Hausdorff space and let $\pi\colon E \to M$ be a vector bundle with projectivization $\underline{\pi}\colon \mathbb{P}(E) \to M$. Prove that there is a rank-1 subbundle $V \subset \underline{\pi}^*E$. (Hint: Your V should be somewhat similar to the tautological line bundle over $\mathbb{R}P^n$.)

(b) Prove the *splitting principle*: If M is a paracompact Hausdorff space and $\pi\colon E \to M$ is a rank-k vector bundle then there is a space N, a map $p\colon N \to M$ such that $p^*\colon H^*(M;\mathbb{Z}/2\mathbb{Z}) \to H^*(N;\mathbb{Z}/2\mathbb{Z})$ is injective, and rank-1 vector bundles $V_1,\ldots,V_k$ over N such that $p^*E = V_1 \oplus \cdots \oplus V_k$. (Throughout your proof you may freely assume without verifying it that all spaces that you construct are paracompact and Hausdorff and have the property that the Leray–Hirsch theorem applies to fiber bundles over them.)

Exercise 6.21 (and the fact that the space N in (b) can be taken to be paracompact and Hausdorff) allows one to show that the Stiefel–Whitney classes of vector bundles over paracompact Hausdorff spaces are uniquely characterized by the following axioms (each of which we have verified based on our construction):

- For each finite-type rank-k vector bundle $\pi\colon E \to M$ over any paracompact Hausdorff space M we have classes $w_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ for $0 \le i \le k$, such that $w_0(E) = 1$. We write $w(E) = w_0(E) + w_1(E) + \cdots + w_k(E)$.
- **(Naturality)** If $f: X \to Y$ is a continuous map between paracompact Hausdorff spaces, then for any vector bundle $\pi\colon E \to Y$
$$w(f^*E) = f^*w(E)$$
- **(Whitney Product)** If V and W are two vector bundles over M then
$$w(V \oplus W) = w(V) \cup w(W)$$
- **(Normalization)** For the tautological line bundle $E = \gamma^1(\mathbb{R}^n)$ over $\mathbb{R}P^{n-1}$, $w_1(E) = e(E)$.

Indeed the naturality and normalization axioms suffice to determine $w(V)$ for any line bundle $V \to M$, since such a bundle is isomorphic to $f^*\gamma^1(\mathbb{R}^n)$ for some n and some $f: M \to \mathbb{R}P^{n-1}$. So the Whitney Product axiom determines $w(E)$ whenever E splits as a direct sum of line bundles. Although not every vector bundle splits as a direct sum of line bundles (see Exercise 6.26), if $p\colon N \to M$ is as in Exercise 6.21(b) then $p^*w(E) = w(p^*E)$ is determined by the axioms, and hence $w(E)$ is as well since p^* is injective.

Again, a similar set of axioms may be used to characterize the Chern classes of complex vector bundles.

Of course, this argument proves uniqueness of the Stiefel–Whitney classes subject to the above axioms, but does not by itself prove existence (since, for instance, a vector bundle might split as a sum of line bundles in many different ways)—for existence we needed an explicit construction such as the one given earlier in this section based on the Leray–Hirsch theorem. A different construction of the Stiefel–Whitney classes is given in [MS] using Steenrod squares; the uniqueness argument shows that the Stiefel–Whitney classes defined there are the same as ours.

6.4. **Relationships between the characteristic classes.** If $\pi\colon E \to M$ is a rank-k complex vector bundle we now have three types of characteristic classes for E: the Euler class $e(E) \in H^{2k}(M;\mathbb{Z})$; the Stiefel–Whitney classes $w_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$; and the Chern classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$. As we will see, these are not completely independent of each other. We begin by comparing the w_i and c_i. Of course these classes live in different cohomology groups (with $\mathbb{Z}/2\mathbb{Z}$ coefficients for one and $\mathbb{Z}$ coefficients for the other); however as discussed in Exercise 4.15 there is a natural map $H^j(M;\mathbb{Z}) \to H^j(M;\mathbb{Z}/2\mathbb{Z})$ induced by the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. In general, for a class $x \in H^j(M;\mathbb{Z})$, we will write $\underline{x}$ for the image of x under this map, so

$\underline{x} \in H^j(M;\mathbb{Z}/2\mathbb{Z})$. So in particular the classes w_{2i} and $\underline{c_i(E)}$ both belong to the cohomology group $H^{2i}(M;\mathbb{Z}/2\mathbb{Z})$.

To relate these, we first prove the following:

Lemma 6.22. *For any integer $N \geq 1$ define $p\colon \mathbb{R}P^{2N-1} \to \mathbb{C}P^{N-1}$ to be the map which sends a one-dimensional real subspace V of $\mathbb{R}^{2N} = \mathbb{C}^N$ to the one-dimensional complex subspace of $\mathbb{C}^N$ spanned by any nonzero element of V. Then we have*

$$e(\gamma^1(\mathbb{R}^{2N})) \cup e(\gamma^1(\mathbb{R}^{2N})) = p^*\underline{e(\gamma^1(\mathbb{C}^N))}$$

Proof. If $N = 1$ both sides are zero (they belong to $H^2(\mathbb{R}P^1;\mathbb{Z}/2\mathbb{Z})$) so we assume $N \geq 2$.

It is not difficult (and left to the reader) to show that the map p is a fiber bundle with fiber $\mathbb{R}P^1$ (the fiber over $W \leq \mathbb{C}^N$ consists of the one-real-dimensional subspaces of the two-real-dimensional subspace W of $\mathbb{C}^N$, which is a copy of $\mathbb{R}P^1$). Moreover this fiber bundle satisfies the Leray–Hirsch property: the class $e := e(\gamma^1(\mathbb{R}^{2N}))$ restricts to each $\mathbb{R}P^1$-fiber as the Euler class of the tautological line bundle over (that copy of) $\mathbb{R}P^1$, so the cohomology of the fiber over a point $m \in \mathbb{C}P^{N-1}$ is the $\mathbb{Z}/2\mathbb{Z}$ module generated by i_m^*1 and i_m^*e.

So by the Leray–Hirsch theorem we can write

$$e \cup e = p^*y_2 \cup 1 + p^*y_1 \cup e$$

for some classes $y_1 \in H^1(\mathbb{C}P^{N-1};\mathbb{Z}/2\mathbb{Z})$ and $y_2 \in H^2(\mathbb{C}P^{N-1};\mathbb{Z}/2\mathbb{Z})$. But of course $H^1(\mathbb{C}P^{N-1};\mathbb{Z}/2\mathbb{Z}) = 0$, so $y_1 = 0$. Meanwhile $e \cup e \neq 0$ by Example 5.23 (since $N \geq 2$), while the only nonzero element of $H^2(\mathbb{C}P^{N-1};\mathbb{Z}/2\mathbb{Z})$ is $\underline{e(\gamma^1(\mathbb{C}^N))}$ by (the $\mathbb{Z}/2\mathbb{Z}$-coefficient version of) Example 5.24. So we indeed have

$$e \cup e = p^*\underline{e(\gamma^1(\mathbb{C}^N))}$$

□

This leads to the following relationship between the Stiefel–Whitney and Chern classes:

Theorem 6.23. *If $\pi\colon E \to M$ is a rank-k complex vector bundle with Stiefel–Whitney classes $w_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ and mod 2 Chern classes $\underline{c_i(E)} \in H^{2i}(M;\mathbb{Z}/2\mathbb{Z})$ then we have*

$$w_{2i}(E) = \underline{c_i(E)} \qquad w_{2i+1}(E) = 0$$

for all natural numbers i.

Proof. Let $F\colon E \to \mathbb{C}^N$ be a complex pre-classifying map. Of course, viewing $\mathbb{C}^N$ as $\mathbb{R}^{2N}$, we can also view F as an ordinary (real) pre-classifying map to $\mathbb{R}^{2N}$.

Now there is a quotient map $p_E\colon \mathbb{P}(E) \to \mathbb{P}_{\mathbb{C}}(E)$, since the equivalence relation on E^0 defining $\mathbb{P}_{\mathbb{C}}(E)$ is a refinement of (*i.e.*, contains) the equivalence relation on E^0 defining $\mathbb{P}(E)$, and clearly $\underline{\pi}_{\mathbb{C}} \circ p_E = \underline{\pi}\colon \mathbb{P}(E) \to M$. Moreover we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{\mathbb{P}(F)} & \mathbb{R}P^{2N-1} \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}_{\mathbb{C}}(E) & \xrightarrow{\mathbb{P}_{\mathbb{C}}(F)} & \mathbb{C}P^{N-1} \end{array}$$

Using straightforward functoriality properties of the mod 2 reduction process of Exercise 4.15 (in particular the fact that it defines a ring homomorphism, by virtue of the fact that $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a ring homomorphism), we see from the above diagram that the classes $x_F = \mathbb{P}(E)^*e(\gamma^1(\mathbb{R}^{2N}))$ from Lemma 6.2 and $\underline{z}_F = \mathbb{P}_{\mathbb{C}}(E)^*\underline{e(\gamma^1(\mathbb{C}^N))}$ from Definition-Theorem 6.10 obey

$$x_F \cup x_F = p_E^*\underline{z}_F$$

Now the definition of the Chern classes gives

$$\sum_{i=0}^{k} \underline{\pi}_{\mathbb{C}}^{*} \underline{c_i(E)} \cup z_F^{k-i} = 0$$

Applying p_E^* to this equation gives, since $\underline{\pi}_{\mathbb{C}} \circ p_E = \underline{\pi}$,

$$\sum_{i=0}^{k} \underline{\pi}^{*} \underline{c_i(E)} \cup x_F^{2k-2i} = 0$$

So since the $w_j(E)$ are the *unique* classes with $w_0(E) = 1$ satisfying an equation

$$\sum_{j=0}^{2k} \underline{\pi}^{*} w_j(E) \cup x_F^{2k-j}$$

the result follows. □

Now we turn to the Euler class:

Theorem 6.24. *Let $\pi\colon E \to M$ be a finite-type rank-k vector bundle over a paracompact Hausdorff space. Then the (mod 2) Euler class $e(E)$ is given by*

$$e(E) = w_k(E) \in H^k(M; \mathbb{Z}/2\mathbb{Z})$$

Proof. By pulling back by classifying maps and using the naturality of w_1 and e we can assume that $M = Gr_k(\mathbb{R}^N)$—in particular we can assume that the Leray–Hirsch theorem applies to bundles over M.

Let $\underline{\mathbb{R}}$ be the trivial rank-1 bundle over M, and form the Whitney sum $E \oplus \underline{\mathbb{R}}$. So $E \oplus \underline{\mathbb{R}}$ may be identified with $E \times \mathbb{R}$, with the bundle projection $\pi_{E\oplus\underline{\mathbb{R}}}\colon E \oplus \underline{\mathbb{R}} \to M$ just given by projection to E followed by π. The fiber of $E \oplus \underline{\mathbb{R}}$ over a point m is just $E_m \oplus \mathbb{R}$. There is a special *non-vanishing* section $s_1\colon M \to E \oplus \underline{\mathbb{R}}$, defined by setting $s_1(m)$ equal to the element $(0, 1)$ of $(E \oplus \underline{\mathbb{R}})_m = E_m \oplus \mathbb{R}$.

Now form the projectivization $\underline{\pi}_{E\oplus\underline{\mathbb{R}}}\colon \mathbb{P}(E \oplus \underline{\mathbb{R}}) \to M$. A convenient thing about $\mathbb{P}(E \oplus \underline{\mathbb{R}})$ for our purposes is that it contains both a copy of $\mathbb{P}(E)$ (embedded by the map $[e] \mapsto [(e, 0)]$) and a copy of M (embedded by the map $m \mapsto [s_1(m)]$ where $s_1\colon M \to E \oplus \underline{\mathbb{R}}$ is defined in the last paragraph). For the rest of the proof we use these embeddings to regard $\mathbb{P}(E)$ and M as closed subsets of $\mathbb{P}(E \oplus \underline{\mathbb{R}})$; note that these closed subsets are disjoint.

In fact, the whole bundle E embeds into $\mathbb{P}(E \oplus \underline{\mathbb{R}})$, via the map $j\colon E \to \mathbb{P}(E \oplus \underline{\mathbb{R}})$ defined by

$$j(e) = [(e, 1)]$$

We see that

$$j(E) = \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus \mathbb{P}(E) \qquad j(E^0) = \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus (\mathbb{P}(E) \cup M)$$

Now the Excision Theorem from 8200 implies that the inclusion of pairs

$$(\mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus \mathbb{P}(E), \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus (\mathbb{P}(E) \cup M)) \hookrightarrow (\mathbb{P}(E \oplus \underline{\mathbb{R}}), \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus M)$$

induces a chain homotopy equivalence of relative singular chain complexes, and hence an isomorphism on cohomology. So since $j\colon (E, E^0) \to (\mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus \mathbb{P}(E), \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus (\mathbb{P}(E) \cup M))$ is a homeomorphism of pairs it follows that

(24) $\quad j^*\colon H^*(\mathbb{P}(E \oplus \underline{\mathbb{R}}), \mathbb{P}(E \oplus \underline{\mathbb{R}}) \setminus M; \mathbb{Z}/2\mathbb{Z}) \to H^*(E, E^0; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism

In particular, $H^{2k}(\mathbb{P}(E\oplus\underline{\mathbb{R}}),\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M;\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$, with generator $(j^*)^{-1}(\tau)$ where τ is the Thom class of E. Moreover for any $m\in M$ the inclusions of fibers give a commutative diagram

$$\begin{tikzcd} H^*(\mathbb{P}(E\oplus\underline{\mathbb{R}}),\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "j^*"] \arrow[d] & H^*(E,E^0;\mathbb{Z}/2\mathbb{Z}) \arrow[d] \\ H^*(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m,\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^*(E_m,E_m^0;\mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

where the bottom row is again an isomorphism by excision. Consequently $(j^*)^{-1}(\tau)$ restricts to each $(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m,\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\})$ as a generator for $H^{2k}(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m,\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z})$

Note also that $\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M$ deformation retracts to $\mathbb{P}(E)$ by the homotopy $(t,[(e,z)])\mapsto[(e,tz)]$. In particular a class $c\in H^*(\mathbb{P}(E\oplus\underline{\mathbb{R}});\mathbb{Z}/2\mathbb{Z})$ restricts to zero in $\mathbb{P}(E)$ if and only if it restricts to zero in $\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M$. Restricting this deformation retraction to the fiber over $m\in M$ gives a deformation retraction of $\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\}$ to $\mathbb{P}(E)_m\cong\mathbb{R}P^{k-1}$. In particular $H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z})=0$.

Let $F_0\colon E\to\mathbb{R}^N$ be a pre-classifying map for E. Define $F\colon E\oplus\underline{\mathbb{R}}\to\mathbb{R}^{N+1}$ by $F(e,z)=(F_0(e),z)$; clearly F is a pre-classifying map for $E\oplus\underline{\mathbb{R}}$, and it also restricts to E as another pre-classifying map for E. As in Definition-Theorem 6.10, define

$$x=\mathbb{P}(F)^*e(\gamma^1(\mathbb{R}^{N+1}))\in H^2(\mathbb{P}(E\oplus\underline{\mathbb{R}});\mathbb{Z}/2\mathbb{Z})$$

Note that $\mathbb{P}(F)$ restricts to our copy $M\subset\mathbb{P}(E\oplus\underline{\mathbb{R}})$ as the constant map to $[(\vec{0},1)]$, and so

$$x|_M=0.$$

Also, where $\underline{\pi}\colon\mathbb{P}(E)\to M$ is the bundle projection, the Stiefel–Whitney classes $w_i(E)\in H^i(M;\mathbb{Z}/2\mathbb{Z})$ $(1\le i\le k)$ are by definition the *unique* classes in $H^i(M;\mathbb{Z}/2\mathbb{Z})$ such that

$$x^k|_{\mathbb{P}(E)}+\sum_{i=1}^k\underline{\pi}^*w_i(E)\cup x^{k-1}|_{\mathbb{P}(E)}=0$$

Now consider the commutative diagram of long exact sequences of pairs

$$\begin{tikzcd} H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}}),\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "p^*"] \arrow[d, "i_m^*"'] & H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}});\mathbb{Z}/2\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "i_m^*"'] & H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M;\mathbb{Z}/2\mathbb{Z}) \arrow[d] \\ H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m,\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

where p^* is induced by the inclusion $p\colon(\mathbb{P}(E\oplus\underline{\mathbb{R}}),\varnothing)\to(\mathbb{P}(E\oplus\underline{\mathbb{R}}),\mathbb{P}(E\oplus\underline{\mathbb{R}})\setminus M)$. As noted earlier, the group on the bottom right is zero—it may be identified with $H^k(\mathbb{R}P^{k-1};\mathbb{Z}/2\mathbb{Z})$—so the first map on the bottom is surjective. The group in the bottom middle is isomorphic to $H^{2k}(\mathbb{R}P^k;\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$. Now we saw earlier that $i_m^*(j^*)^{-1}\tau$ generates $H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m,\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\setminus\{[0,1]\};\mathbb{Z}/2\mathbb{Z})$, so its image under the bottom left map must generate $H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m;\mathbb{Z}/2\mathbb{Z})$. So since the diagram commutes, $p^*(j^*)^{-1}\tau\in H^{2k}(\mathbb{P}(E\oplus\underline{\mathbb{R}});\mathbb{Z})$ restricts to each fiber $\mathbb{P}(E\oplus\underline{\mathbb{R}})_m$ as a generator for $H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}})_m;\mathbb{Z}/2\mathbb{Z})$.

By the Leray–Hirsch theorem the general element of $H^k(\mathbb{P}(E\oplus\underline{\mathbb{R}});\mathbb{Z}/2\mathbb{Z})$ takes the form $\sum_{j=0}^k\underline{\pi}_{E\oplus\underline{\mathbb{R}}}^*y_j\cup x^{k-j}$ where $y_j\in H^j(M;\mathbb{Z}/2\mathbb{Z})$. Such a class restricts to each fiber $\mathbb{P}(E\oplus\underline{\mathbb{R}})_m\cong$

$\mathbb{R}P^k$ as a generator if and only if $y_0 = 1$. So we can write

$$p^*(j^*)^{-1}\tau = x^k + \sum_{j=1}^{k} \underline{\pi}^*_{E\oplus\underline{\mathbb{R}}} y_j \cup x^{k-j}$$

But the exactness of our sequence implies that $i^* p^*(j^*)^{-1}\tau = 0 \in H^k(\mathbb{P}(E); \mathbb{Z}/2\mathbb{Z})$. As noted above, this forces $y_j = w_j(E)$ by the definition of the Steifel–Whitney classes of E. Now we have a commutative diagram

$$\begin{tikzcd}
& (\mathbb{P}(E\oplus\underline{\mathbb{R}}), \varnothing) \arrow[dr, "p"] & \\
(M, \varnothing) \arrow[ur] \arrow[dr, "s_0"'] & & (\mathbb{P}(E\oplus\underline{\mathbb{R}}), \mathbb{P}(E\oplus\underline{\mathbb{R}})|_M) \\
& (E, E^0) \arrow[ur, "j"'] &
\end{tikzcd}$$

where the unlabeled map is the inclusion of M into $\mathbb{P}(E\oplus\underline{\mathbb{R}})$. By definition $e(E) = s_0^*\tau$, and so we obtain

$$e(E) = \left(p^*(j^*)^{-1}\tau\right)|_M = \left.\left(\sum_{j=0}^{k} \underline{\pi}^*_{E\oplus\underline{\mathbb{R}}} w_j(E) x^{k-j}\right)\right|_M$$

But as noted earlier $x|_M = 0$, so the only term that contributes is that where $j = k$, and the fiber bundle projection $\underline{\pi}_{E\oplus\underline{\mathbb{R}}}$ restricts to $M \subset \mathbb{P}(E\oplus\underline{\mathbb{R}})$ as the identity; thus we indeed have $e(E) = w_k(E)$.

□

There is a similar result for the $\mathbb{Z}$-coefficient Euler class and the Chern class, whose proof we omit:

Theorem 6.25. *Let $\pi\colon E \to M$ be a finite-type rank-k complex vector bundle over a paracompact Hausdorff space. Then the (oriented) Euler class $e(E)$ is given by*

$$e(E) = c_k(E) \in H^{2k}(M; \mathbb{Z})$$

The proof of this is almost identical to that of Theorem 6.24, just replacing real projectivizations by complex projectivizations everywhere. The only real difference comes near the end of the proof: since for Theorem 6.25 we are working with coefficients in $\mathbb{Z}$ and not $\mathbb{Z}/2\mathbb{Z}$, there are two different generators for the top-degree cohomology of a fiber of a $\mathbb{C}P^k$-bundle, and so the argument that we gave to say that the class denoted y_0 in the above proof was equal to 1 only applies in the Chern class case to show that $y_0 = \pm 1$. From this one can see just as in the rest of the proof that $e(E) = \pm c_k(E)$ with the sign $\pm$ determined by the property that our distinguished generator $\omega \in H^{2k}(\mathbb{C}^k, \mathbb{C}^k \setminus \{0\}; \mathbb{Z})$ (chosen long ago in the definition of the Thom class) is sent by the the composition

$$H^{2k}(\mathbb{C}^k, \mathbb{C}^k \setminus \{0\}; \mathbb{Z}) \to H^{2k}(\mathbb{C}P^k, \mathbb{C}P^k \setminus \mathbb{C}P^{k-1}; \mathbb{Z}) \to H^{2k}(\mathbb{C}P^k; \mathbb{Z})$$

(where the first map is the excision isomorphism) to

$$\pm\left(-e(\gamma^1(\mathbb{C}^k))\right)^k$$

In particular this sign depends only on k, not on the bundle or on the base space. Example 6.11 shows that the sign is positive when $k = 1$, and then one can prove that it is positive for all k

by appealing to product formulas for Whitney sums both for the Euler class ([MS, 9.6]) and for the Chern class.

Exercise 6.26. Let $\pi\colon E \to \mathbb{C}P^1$ be the tautological complex line bundle, but regard E as a real rank-2 vector bundle. Prove that there does not exist a rank-1 subbundle $V \subset E$. (Hint: Show that if V existed then $w_1(V)$ would be a nonzero element of $H^1(\mathbb{C}P^1;\mathbb{Z}/2\mathbb{Z})$.)

7. Immersions and embeddings

We can now start to put Stiefel–Whitney classes to work in the study of smooth manifolds. Recall that a basic example of a vector bundle is the tangent bundle TM of a smooth manifold M. If $\phi\colon U \to \mathbb{R}^m$ is a smooth coordinate chart for M and $p \in U$, the tangent space T_pM is the vector space with basis $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_m}\Big|_p \right\}$. This then gives rise to a local trivialization $\Phi\colon TM|_U \to U \times \mathbb{R}^m$, defined by

$$\Phi\left(\sum_{i=1}^m c_i \frac{\partial}{\partial x_i}\Big|_p\right) = (p, c_1, \dots, c_m).$$

(The transition maps between two such local trivializations are given by differentiating the smooth transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ of the atlas for m, and hence are continuous (indeed smooth), and so we get a well-defined topology on TM by declaring such trivializations Φ to be homeomorphisms, and with this topology the projection $\pi\colon TM \to M$ defines a vector bundle.)

If M and N are smooth manifolds and if $f\colon M \to N$ is a smooth map then for each $p \in M$ we have a linear map $f_*\colon T_pM \to T_{f(p)}N$ obtained by differentiating f. Recall that a smooth map $f\colon M \to N$ is called an *immersion* if for every $p \in M$ the derivative $f_*\colon T_pM \to T_{f(p)}N$ is injective. Using the inverse function theorem, one can show that if $f\colon M \to N$ is an immersion, then for each $p \in M$ there is a coordinate chart $(x_1, \dots, x_m)\colon U \to \mathbb{R}^m$ for M around p and a coordinate chart $(y_1, \dots, y_n)\colon V \to \mathbb{R}^n$ for N around $f(p)$ in terms of which f takes the form $f(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$. (In particular $U \subset f^{-1}(V)$)

We have a local trivialization for TN around $f(p)$ associated to the coordinates $(y_1, \dots, y_n)$; pulling this back gives a local trivialization $\Psi\colon (f^*TN)|_U \to U \times \mathbb{R}^n$ for the bundle f^*TN around $p \in M$. At each point $q \in M$, f_* defines an injection $T_qM \to (f^*TN)_q$, and in fact we see that f_* identifies $TM|_U$ with $\Psi^{-1}(U \times \mathbb{R}^m \times \{\vec{0}\})$. So f_* identifies TM with a *rank-m subbundle* of the rank-n vector bundle $TN|_M$. Combined with Proposition 6.19 (which applies since all smooth manifolds are paracompact and Hausdorff) this proves:

Proposition 7.1. *If $f\colon M \to N$ is an immersion between smooth manifolds M and N with* $\dim M = m$ *and* $\dim N = n$*, then there is a rank-$(n-m)$ vector bundle $\nu_f \to M$ (called the* normal bundle of *f) such that*

$$f^*TN \cong TM \oplus \nu_f$$

Consider the following problem:

Problem 7.2. *Given a smooth n-manifold N, a smooth m-manifold M, and a smooth map $f_0\colon M \to N$, determine whether there is an immersion $f\colon M \to N$ which is homotopic to f.*

There is an obvious necessary condition here, namely that $n \geq m$. Less trivially, Whitney showed in 1936 that the desired immersion always exists if $n \geq 2m$ (in fact, what he really showed is that immersions form a residual subset of the space of smooth maps from M to N with respect to a certain topology, so that in particular a random perturbation of f_0 should be expected to be an immersion—see [Hi, Section 3.2]).

Proposition 7.1 and the Whitney product formula have implications for Problem 7.2. Indeed, suppose that an immersion $f: M \to N$ exists which is homotopic to f_0. Then where w denotes total Stiefel–Whitney class we have

$$f_0^* w(TN) = f^* w(TN) = w(f^*TN) = w(TM \oplus \nu_f) = w(TM) \cup w(\nu_f).$$

Here is a useful observation:

Proposition 7.3. *Let M be a topological space such that there exists $m \in \mathbb{Z}_+$ with $H^i(M;\mathbb{Z}/2\mathbb{Z}) = 0$ for all $i > m$, and let $\pi: E \to M$ be a vector bundle with total Stiefel–Whitney class $w(E)$. Then there is a unique class $\overline{w}(E) = \overline{w}_0(E) + \cdots + \overline{w}_m(E)$, where $\overline{w}_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$, such that $\overline{w}(E) \cup w(E) = 1$.*

(The classes $\overline{w}_i(E)$ are called the *dual Stiefel-Whitney classes* of E.)

Proof. We show by induction on k that there is a unique collection of classes $\overline{w}_i(E) \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ for $0 \le i \le k$ such that, for all $i \le k$, the grading-i part of $(\overline{w}_0(E) + \cdots + \overline{w}_k(E)) \cup w(E)$ is 1 for $i = 0$ and 0 otherwise.

For $k = 0$ this is obvious: since $w_0(E) = 1$ the grading-0 part of $\overline{w}_0(E) \cup w(E)$ is just $\overline{w}_0(E)$, and so the unique $\overline{w}_0(E)$ satisfying the requirement is $\overline{w}_0(E) = 1$.

Assume the statement for $k-1$ where $k \ge 1$, so that we have identified elements $\overline{w}_0(E), \dots, \overline{w}_{k-1}(E)$. Now for an arbitrary collection of elements $x_i \in H^i(M;\mathbb{Z}/2\mathbb{Z})$, for all $i \le k-1$ the grading i part of $(x_0 + \cdots + x_{k-1} + x_k) \cup w(E)$ is the same as the grading-i part of $(x_0 + \cdots + x_{k-1}) \cup w(E)$. Thus the requirement will be satisfied for all $i \le k-1$ if and only if we set $x_i = \overline{w}_i(E)$ for $i \le k-1$. Meanwhile the grading k part of $(\overline{w}_0(E) + \cdots + \overline{w}_{k-1}(E) + x_k) \cup w(E)$ is given by

$$x_k + \sum_{j=1}^{k} \overline{w}_{k-j}(E) \cup w_j(E)$$

so that the requirement for $i = k$ is satisfied if and only if we put $x_k = \overline{w}_k(E) := \sum_{j=1}^{k} \overline{w}_{k-j}(E) \cup w_j(E)$.

This completes the induction; applying this in particular when $k = m$ gives a class $\overline{w}(E)$ with $\overline{w}(E) \cup w(E) = 1$ (since the degree-i part is automatically zero for $i > m$ by the assumption on M).

□

It is a general fact that a smooth m-dimensional manifold is homotopy equivalent to a cell complex all of whose cells have dimension at most m, so $H^i(M;\mathbb{Z}/2\mathbb{Z}) = 0$ for $i > m$; thus Proposition 7.3 applies to our manifold M.

Corollary 7.4. *Suppose that $f_0: M \to N$ is a smooth map which is homotopic to some immersion $f: M \to N$. Then the normal bundle $\nu_f \to M$ must have total Stiefel–Whitney class given by*

$$w(\nu_f) = \overline{w}(TM) \cup f_0^* w(TN).$$

Proof. By the Whitney product formula we have

$$f_0^* w(TN) = w(f^*TN) = w(TM) \cup w(\nu_f).$$

Multiplying both sides on the left by $\overline{w}(TM)$ immediately gives the result. □

Now we are taking M, N, and f_0 is given, so let us regard $w(TN)$, $w(TM)$, and the map $f_0^*: H^*(N;\mathbb{Z}/2\mathbb{Z}) \to H^*(M;\mathbb{Z}/2\mathbb{Z})$ as known (in actual examples it may or may not be easy to compute them). Then the class $\overline{w}(TM) \cup f_0^* w(TN)$ is known, and so the normal bundle of any hypothetical immersion homotopic to f must have its total Stiefel–Whitney classes equal to this known cohomology class. Moreover the rank of the vector bundle $\nu_f \to M$ is $n-m$. So if we can show that there does not exist any rank-$(n-m)$ vector bundle with total Stiefel–Whitney class equal to $\overline{w}(TM) \cup f_0^* w(TN)$ then it will follow that there does not exist any immersion homotopic to f_0.

One conceivable reason why there might not exist such a vector bundle is that the total Stiefel–Whitney class of a rank-$(n-m)$ vector bundle is a sum of terms of grading at most $n-m$. Thus if, for some $i > n-m$, the grading-i part of $\overline{w}(TM) \cup f_0^* w(TN)$ is nonzero, then there is no immersion homotopic to f_0. If $n \geq 2m$, of course, this issue does not arise, since then whenever $i > n-m$ we will have $i > m$ and so $H^i(M; \mathbb{Z}/2\mathbb{Z}) = 0$. This is consistent with Whitney's theorem that when $n \geq 2m$ every smooth map from M to N is homotopic to an immersion. But if $n < 2m$ then we obtain the constraint

$$\left(\overline{w}(TM) \cup f_0^* w(TN)\right)_{n-m+1} = \cdots = \left(\overline{w}(TM) \cup f_0^* w(TN)\right)_m = 0$$

which must be satisfied in order for there to exist an immersion homotopic to f_0. (Here for a general cohomology class y we denote by $(y)_i$ its grading-i part.) So if we compute these classes and show that one of them is nonzero then we will have proven that no such immersion exists. We will do just that in some concrete cases later on, but first we explain how one can get some slightly sharper constraints on embeddings than we have on immersions.

7.1. **Embeddings and the normal Euler class.** Recall that a smooth map $f \colon M \to N$ is an *embedding* if it is an immersion and moreover is a homeomorphism to its image $f(M)$. So in this case $f(M)$ is a submanifold of N; we may as well assume that M already is a submanifold of N and that our embedding is just the inclusion of M into N. So since pullback by inclusion is the same as restriction, we have a splitting

$$TN|_M = TM \oplus \nu_{M,N}$$

where $\nu_{M,N}$ is a rank-$(n-m)$ vector bundle over M. The following is an important and standard result in basic smooth manifold theory:

Theorem 7.5. *(Tubular Neighborhood Theorem,* [Lee, 10.17]*) If $M \subset N$ is any submanifold then there is a neighborhood $U \subset N$ of M and a diffeomorphism $T \colon \nu_{M,N} \to U$ which maps the zero section of $0_M \subset \nu_{M,N}$ to $M \subset U$ by the identity.*

At the same time that Whitney proved that any m-manifold immerses in to any n-manifold if $n \geq 2m$, he also proved that any m-manifold embeds in any n-manifold if $n \geq 2m+1$ (with image which is closed as a subset). Consistently with this, we will now see that there is an additional constraint on the characteristic classes of the manifolds in the case of a closed embedding, going just one step beyond those that arise for immersions that we described above.

Theorem 7.6. *Let $M \subset N$ be a submanifold which is closed as a subset of N. Let $m = \dim M$ and $n = \dim N$. Then there is a class $c \in H^{n-m}(N; \mathbb{Z}/2\mathbb{Z})$ such that $c|_M = e(\nu_{M,N}) \in H^{n-m}(M; \mathbb{Z}/2\mathbb{Z})$. If additionally $\nu_{M,N}$ is oriented, with oriented Euler class $e_{\mathbb{Z}} \in H^{n-m}(M; \mathbb{Z})$, then there is $c_{\mathbb{Z}} \in H^{n-m}(M; \mathbb{Z})$ with $c_{\mathbb{Z}}|_M = e_{\mathbb{Z}}$.*

Proof. Consider the following commutative diagram of maps of pairs, where T is as in Theorem 7.5, $s_0\colon M \to \nu_{M,N}$ is the zero-section and the other maps are inclusions:

$$\begin{tikzcd} & (N,\varnothing) \arrow[r, "i"] & (N, N\setminus M) \\ (M,\varnothing) \arrow[ur] \arrow[dr, "s_0"'] & & \\ & (\nu_{M,N}, \nu^0_{M,N}) \arrow[r, "T"] & (U, U\setminus M) \arrow[uu, "j"'] \end{tikzcd}$$

Where $\tau \in H^{n-m}(\nu_{M,N}, \nu^0_{M,N}; \mathbb{Z}/2\mathbb{Z})$ is the $(\mathbb{Z}/2\mathbb{Z})$-Thom class, we have by definition $e(\nu_{M,N}) = s_0^*\tau$. Now by excision the map $j^*\colon H^*(N, N\setminus M; \mathbb{Z}/2\mathbb{Z}) \to H^*(U, U\setminus M; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism (check this—we are using here that $M \subset N$ is closed). Of course T induces an isomorphism on cohomology since it is a homeomorphism of pairs. So the class $c = i^*(j^*)^{-1}(T^*)^{-1}\tau \in H^{n-m}(N; \mathbb{Z}/2\mathbb{Z})$ restricts to M as $s_0^*\tau = e(\nu_{M,N})$.

The argument in the $\mathbb{Z}$-coefficient case when $\nu_{M,N}$ is oriented is the same. □

Remark 7.7. The proof of Theorem 7.6 canonically associates to any closed m-dimensional submanifold $M \subset N$ a cohomology class $c^M \in H^{n-m}(n; \mathbb{Z}/2\mathbb{Z})$; if M and N are, additionally, oriented then we obtain a natural orientation on $\nu_{M,N}$ and so we get an integer-coefficient class $c^M_{\mathbb{Z}}$. When N is compact this can be understood in terms of *Poincaré duality* which gives an isomorphism between $H_m(N; \mathbb{Z}/2\mathbb{Z})$ and $H^{n-m}(N; \mathbb{Z}/2\mathbb{Z})$ and, if N is oriented, an isomorphism between $H_m(N; \mathbb{Z})$ and $H^{n-m}(N; \mathbb{Z})$: the submanifold M determines a degree-m homology class $[M]$ in N (always over $\mathbb{Z}/2\mathbb{Z}$, and if M is oriented also over $\mathbb{Z}$), and the class c_M that we have constructed turns out to be the image of $[M]$ under the Poincaré duality isomorphism. Note that c_M, as constructed in the proof, lies in the image of the map $H^{n-m}(N, N-M) \to H^{n-m}(N)$, and hence vanishes on restriction to $N\setminus M$. So c_M evaluates as zero on cycles in N which are disjoint from M; more generally, the evaluation of c_M on an $(n-m)$-cycle P can be interpreted in terms of the number of intersections of P with M.

This quickly leads to an obstruction to embedding some manifolds in others:

Corollary 7.8. *Suppose that the m-dimensional smooth manifold M is a closed submanifold of the $(m+1)$-dimensional smooth manifold N. Then where $i\colon M \to N$ is the inclusion,*

$$w_1(TM) \in Im(i^*\colon H^1(N; \mathbb{Z}/2\mathbb{Z}) \to H^1(M; \mathbb{Z}/2\mathbb{Z}))$$

Proof. If $\nu_{M,N}$ is the normal bundle, which by the assumption on dimensions has rank one, we see from the Whitney product formula that

$$w(TN)|_M = w(TM) \cup (1 + w_1(\nu_{M,N}))$$

By Theorems 7.6 and 6.24 we can write $w_1(\nu_{M,N}) = c|_M$ for some $c \in H^1(N; \mathbb{Z}/2\mathbb{Z})$. So equating degree-one terms gives

$$w_1(TN)|_M = w_1(TM) + c|_M,$$

i.e. $w_1(TM) = i^*(w_1(TN) - c)$. □

Here is a striking consequence:

Corollary 7.9. *Let M be any m-dimensional smooth manifold such that TM is not orientable. If M embeds as a closed submanifold of some $(m+1)$-dimensional manifold N, then $H^1(N; \mathbb{Z}/2\mathbb{Z}) \neq 0$. In particular M does not embed as a closed submanifold of $\mathbb{R}^{m+1}$.*

Proof. According to Exercise 6.9, $w_1(TM) \neq 0$. So by Corollary 7.8 the map $i^*\colon H^1(N;\mathbb{Z}/2\mathbb{Z}) \to H^1(M;\mathbb{Z}/2\mathbb{Z})$ has nontrivial image, and in particular $H^1(N;\mathbb{Z}/2\mathbb{Z}) \neq 0$. □

A smooth manifold is called orientable if its tangent bundle is orientable; this coincides with any other notion of orientability that you may have learned. In particular Corollary 7.9 applies to the nonorientable surfaces such as the Klein bottle and the real projective plane, proving that they do not embed in $\mathbb{R}^3$, a famous result that you have probably heard of before. Higher-dimensional examples of nonorientable manifolds include the even-dimensional real projective spaces $\mathbb{R}P^{2k}$; we will soon see that for some values of k one can prove a much stronger result.

Now a nonorientable manifold M does embed as a closed codimension-one submanifold of *some* smooth manifolds N, notably $N = M \times \mathbb{R}$ or $N = M \times S^1$. And of course the compact orientable surfaces embed in $\mathbb{R}^3$, and S^m embeds in $\mathbb{R}^{m+1}$. This begins to illustrate that the problem of which manifolds embed into which others depends in a somewhat subtle way on the topology both of the source and of the target, rather than just depending on the dimension.

7.2. **The tangent bundle of real projective space.** We will now compute the Stiefel–Whitney classes of the tangent bundle to $\mathbb{R}P^m$. Using the arguments described above, this will quickly allow us to get some restrictions on embeddings and immersions of $\mathbb{R}P^m$ into Euclidean space.

Let $\gamma \to \mathbb{R}P^m$ denote the tautological line bundle. (So $\gamma = \gamma^1(\mathbb{R}^{m+1})$ in previous notation.) We thus have $w(\gamma) = 1 + e$, where $e = e(\gamma)$ generates $H^*(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z})$ as a unital ring (more specifically, by Example 5.23 we have $H^*(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) = \frac{\mathbb{Z}/2\mathbb{Z}[e]}{\langle e^{m+1}\rangle}$). We intend to compute $w(T\mathbb{R}P^m)$; to do this we will relate $T\mathbb{R}P^m$ to γ.

The starting geometric observation is that, viewing an element $\ell \in \mathbb{R}P^m$ as a line through the origin in $\mathbb{R}^{m+1}$, there is a natural identification of the tangent space $T_\ell\mathbb{R}P^m$ with the vector space $Hom(\ell, \ell^\perp)$ of linear maps from ℓ to its orthogonal complement. To describe this identification, we will associate to each $A \in Hom(\ell,\ell^\perp)$ a curve $c_A\colon \mathbb{R} \to \mathbb{R}P^m$ with $c_A(0) = \ell$. Then A will correspond to the tangent vector $c_A'(0) \in T_\ell\mathbb{R}P^m$.

Specifically, we let

$$c_A(t) = \{\vec{v} + tA\vec{v} | \vec{v} \in \ell\}$$

It is clear that $c_A(t)$ is a one-dimensional subspace of $\mathbb{R}^{m+1}$ for all A and t (since A has image in $\ell^\perp$ we never have $tA\vec{v} = -\vec{v}$ for nonzero $\vec{v} \in \ell$). Let us study c_A in terms of a (conveniently-chosen) coordinate chart for $\mathbb{R}P^m$. Choose an orthonormal basis $\{e_1, \ldots, e_{m+1}\}$ for $\mathbb{R}^{m+1}$ such that e_{m+1} generates the given one-dimensional subspace ℓ. $Hom(\ell,\ell^\perp)$ is then identified with $\mathbb{R}^m$ by associating to $A \in Hom(\ell,\ell^\perp)$ the tuple $(A_1,\ldots,A_m)$ where $Ae_{m+1} = \sum_{i=1}^m A_i e_i$. One choice of local coordinate patch around ℓ is given by letting the domain be $U = \{[x_1 : \ldots : x_{m+1}] | x_{m+1} \neq 0\}$ and defining $\phi\colon U \to \mathbb{R}^m$ by $\phi([x_1 : \cdots : x_{m+1}]) = \left(\frac{x_1}{x_{m+1}}, \frac{x_2}{x_{m+1}}, \ldots, \frac{x_m}{x_{m+1}}\right)$. We see then that

$$\phi(c_{(A_1,\ldots,A_m)}(t)) = \phi\left([tA_1 : \cdots : tA_m : 1]\right) = (tA_1, \ldots, tA_m).$$

So

$$\phi_*(c'_{(A_1,\ldots,A_m)}(0)) = (A_1, \ldots, A_m).$$

This proves both that the curve c_A is smooth for each A, and that the map $A \mapsto c_A'(0)$ gives an isomorphism from $Hom(\ell,\ell^\perp)$ to $T_\ell\mathbb{R}P^m$.

Recall that $\gamma = \{(\ell,\vec{v}) \in \mathbb{R}P^m \times \mathbb{R}^{m+1} | \vec{v} \in \ell\}$. Define likewise

$$\gamma^\perp = \{(\ell,\vec{v}) \in \mathbb{R}P^m \times \mathbb{R}^{m+1} | \vec{v} \in \ell^\perp\}$$

One can construct local trivializations for $\gamma^\perp$ in much the same way that they were constructed for γ (or for a quicker argument, note that there is a homeomorphism $\mathbb{R}P^m \to Gr_m(R^{m+1})$ which sends ℓ to $\ell^\perp$, and $\gamma^\perp$ is the pullback of $\gamma^m(\mathbb{R}^{m+1})$ by this homeomorphism). Now in general if V and W are two vector bundles over the same space M it is straightforward to construct a vector bundle structure on

$$Hom(V,W) = \{(x,A)|A \in Hom(V_x, W_x)\}$$

(over an open set U on which $V|_U$ is identified by a local trivialization with $U \times \mathbb{R}^k$ and $W|_U$ is identified with $U \times \mathbb{R}^l$, the restriction $Hom(V,W)|_U$ is identified with $U \times \mathbb{R}^{l,k}$ where $\mathbb{R}^{l,k}$ denotes the space of $l \times k$ real matrices).

Considering the case where $V = \gamma$ and $W = \gamma^\perp$, the construction described two paragraphs ago gives a map

$$\begin{aligned} \Phi\colon\ Hom(\gamma,\gamma^\perp) &\to T\mathbb{R}P^m \\ (\ell,A) &\mapsto c_A'(0) \end{aligned}$$

and it follows directly from the construction that Φ is a bundle isomorphism. Thus

$$T\mathbb{R}P^m \cong Hom(\gamma,\gamma^\perp) \tag{25}$$

This still may not seem like enough to determine the Stiefel–Whitney classes, but after some further manipulation we will see that it is. For any positive integer k let $\underline{\mathbb{R}}^k$ be the trivial rank-k vector bundle over $\mathbb{R}P^m$. Note that

$$\gamma^\perp \oplus \gamma = \{(\ell,\vec{v}) \in \mathbb{R}P^m \times \mathbb{R}^{m+1} | \vec{v} \in \ell^\perp \oplus \ell\} = \mathbb{R}P^m \times \mathbb{R}^{m+1} = \underline{\mathbb{R}}^{m+1}.$$

Moreover it is straightforward to see that, very generally for any vector bundles V, W_1, W_2 over the same space, we have $Hom(V, W_1 \oplus W_2) \cong Hom(V,W_1) \oplus Hom(V,W_2)$. So we see that

$$\begin{aligned} Hom(\gamma,\gamma^\perp) \oplus Hom(\gamma,\gamma) &\cong Hom(\gamma, \underline{\mathbb{R}}^{m+1}) \\ &\cong Hom(\gamma,\underline{\mathbb{R}})^{\oplus(m+1)} \end{aligned} \tag{26}$$

If V is a real vector bundle over a paracompact Hausdorff space, one can choose an orthogonal structure $\langle\cdot,\cdot\rangle$ on V, and then the map which sends a vector $\vec{v} \in V_x$ to the linear functional on V_x defined by $\vec{w} \mapsto \langle \vec{v}, \vec{w}\rangle$ defines an isomorphism from V to $Hom(V,\underline{\mathbb{R}})$. So (26) gives

$$Hom(\gamma,\gamma^\perp) \oplus Hom(\gamma,\gamma) \cong \gamma^{\oplus(m+1)}$$

But the bundle $Hom(\gamma,\gamma)$ is trivial: the map which sends the element $(\ell,t) \in \mathbb{R}P^m \times \mathbb{R} = \underline{\mathbb{R}}$ to the endomorphism of γ_ℓ given by multiplication by t is an isomorphism. Thus we have:

Proposition 7.10.

$$T\mathbb{R}P^m \oplus \underline{\mathbb{R}} \cong \gamma^{\oplus(m+1)}$$

Consequently

$$w(T\mathbb{R}P^m) = (1+e)^{m+1} = \sum_{i-0}^{m} \binom{m+1}{i} e^i$$

Proof. The first line has just been proven in view of (25). For the second, since $w(\underline{\mathbb{R}}) = 1$, applications of the Whitney product formula give

$$w(T\mathbb{R}P^m) = w(T\mathbb{R}P^M) \cup w(\underline{\mathbb{R}}) = w(T\mathbb{R}P^m \oplus \underline{\mathbb{R}}) = w(\gamma^{\oplus(m+1)}) = w(\gamma)^{m+1}$$

and so the first equality follows since $w(\gamma) = 1 + e$. The second equality follows from the binomial formula and the fact that $e^{m+1} = 0$. □

Corollary 7.11. *We have $w(T\mathbb{R}P^m) = 1$ if and only if, for some integer j, $m = 2^j - 1$.*

Proof. For any $x \in H^*(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z})$ we have $(1+x)^2 = 1+x^2$ (as we are working in an algebra over $\mathbb{Z}/2\mathbb{Z}$, so that $2x = 0$). From this it follows by induction on j that

$$(1+e)^{2^j} = 1+e^{2^j}.$$

If $m = 2^j - 1$ we thus have

$$w(T\mathbb{R}P^m) = (1+e)^{2^j} = 1+e^{2^j} = 1+e^{m+1} = 1$$

(as $e^{m+1} = 0$), proving the backward implication.

For the forward implication, if m is not of the form $2^j - 1$ then there is an odd integer r such that $r > 1$ and $m+1 = 2^j r$ for some $j \in \mathbb{N}$. Then

$$w(T\mathbb{R}P^m) = (1+e)^{2^j r} = (1+e^{2^j})^r = \sum_{i=0}^{r} \binom{r}{i} e^{2^j i}$$

In particular

$$w_{2^j}(T\mathbb{R}P^m) = re^{2^j} = e^{2^j} \neq 0$$

where the second equality follows from r being odd (as we are working over $\mathbb{Z}/2\mathbb{Z}$) and the reason that $e^{2^j} \neq 0$ is that the fact that $r > 1$ implies that $2^j = \frac{m+1}{r} < m+1$.

□

We consider the problem of embedding or immersing $\mathbb{R}P^m$ into $\mathbb{R}^n$ for various values of m and n. Whitney showed in 1944 that any smooth m-manifold may be embedded in $\mathbb{R}^{2m}$ and (for $m > 1$) immersed into $\mathbb{R}^{2m-1}$. For certain values of m this is best possible:

Theorem 7.12. *Let $f\colon \mathbb{R}P^{2^j} \to \mathbb{R}^n$ be an immersion. Then $n \geq 2^{j+1} - 1$. If f is an embedding then $n \geq 2^{j+1}$.*

Proof. We have a splitting $f^*T\mathbb{R}^n \cong T\mathbb{R}P^{2^j} \oplus \nu_f$ where $\nu_f \to \mathbb{R}P^{2^j}$ has rank $n - 2^j$. Of course $w(f^*T\mathbb{R}^n) = 1$ since $T\mathbb{R}^n$ is trivial. Meanwhile

$$\begin{aligned} w(T\mathbb{R}P^{2^j}) &= (1+e)^{2^j+1} = (1+e)^{2^j}(1+e) = (1+e^{2^j})(1+e) \\ &= 1+e+e^{2^j} \end{aligned}$$

So we have

$$(1+e+e^{2^j}) \cup w(\nu_f) = 1$$

Evaluating grading-2^j terms shows that

$$e^{2^j} + e \cup w_{2^j-1}(\nu_f) + w_{2^j}(\nu_f) = 0$$

So one or both of $w_{2^j-1}(\nu_f)$ and $w_{2^j}(\nu_f)$ must be nonzero. But since ν_f has rank $n-2^j$, we have $w_i(\nu_f) = 0$ for $i > n-2^j$. So we must have $2^j - 1 \leq n - 2^j$, *i.e.* $n \geq 2^{j+1} - 1$.

If f is an embedding then $w_{n-2^j}(\nu_f)$ is the restriction of a class in $H^*(\mathbb{R}^n; \mathbb{Z}/2\mathbb{Z})$ by Theorems 7.6 and 6.24, and hence is zero—thus when f is an embedding $w_i(\nu_f) = 0$ for $i > n - 2^j - 1$. So in this case $2^j - 1 \leq n - 2^j - 1$, *i.e.* $n \geq 2^{j+1}$.

□

There is an interesting connection between the tangent bundle to $\mathbb{R}P^n$ and algebra. If V is a vector space over a field K, a *division algebra structure* on V is a bilinear map

$$\begin{aligned} p\colon V \times V &\to V \\ (v,w) &\mapsto v \cdot w \end{aligned}$$

with no zero-divisors, *i.e.*, $v \cdot w \neq 0$ whenever $v \neq 0$ and $w \neq 0$. We take $K = \mathbb{R}$ and assume V to be finite-dimensional, so we may as well set $V = \mathbb{R}^n$ for some n.

Proposition 7.13. *Suppose that* $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is a division algebra structure. Then the vector bundle* $T\mathbb{R}P^{n-1} \cong Hom(\gamma, \gamma^{\perp})$ *is trivial.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. The map $v \mapsto v \cdot e_n$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with trivial kernel, so it has an inverse, say A. (Thus $(Av) \cdot e_n = v$ for all v.) Then for all nonzero $v \in V$, $\{(Av) \cdot e_1, \ldots (Av) \cdot e_n\}$ is a basis for $\mathbb{R}^n$, with last entry equal to v.

For $\ell \in \mathbb{R}P^{n-1}$ let Π_ℓ denote the orthogonal projection $\mathbb{R}^{n+1} \to \ell^{\perp}$. Then if $v \in \ell \setminus \{\vec{0}\}$ we see that $\{\Pi_\ell((Av) \cdot e_1), \ldots, \Pi_\ell((Av) \cdot e_{n-1})\}$ is a basis for $\ell^{\perp}$. So for $i = 1, \ldots, n-1$ and $\ell \in \mathbb{R}P^{n-1}$ we may define a linear map $s_i(\ell)\colon \ell \to \ell^{\perp}$ by $s_i(\ell)(v) = \Pi_\ell((Av) \cdot e_i)$; the maps $s_1(\ell), \ldots, s_{n-1}(\ell)$ are linearly independent elements of $Hom(\ell, \ell^{\perp})$ (and hence form a basis for it).

We have thus constructed a set of $n-1$ sections $s_i\colon \mathbb{R}P^{n-1} \to Hom(\gamma, \gamma^{\perp})$ (they are easily checked to be continuous) which are linearly independent at every $\ell \in \mathbb{R}P^{n-1}$. Since the rank of $Hom(\gamma, \gamma^{\perp})$ is only $n-1$, it follows from the discussion above Corollary 6.20 that $Hom(\gamma, \gamma^{\perp})$ is trivial. □

Corollary 7.14. *If there exists a division algebra structure on* $\mathbb{R}^n$ *then* $n = 2^j$ *for some* j.

Proof. By what have just proven, $T\mathbb{R}P^{n-1}$ would have to be trivial, and hence we would have $w(T\mathbb{R}P^{n-1}) = 1$. So the conclusion follows from Corollary 7.11. □

For $n = 1, 2, 4$ there are familiar division algebra structures on $\mathbb{R}^n$ given by multiplication in the real numbers, the complex numbers, and the quaternions, respectively. A division algebra structure on $\mathbb{R}^8$, called octonion multiplication, was discovered in the 1840s by Cayley and Graves. It turns out that this is a complete list—it follows from work of Bott and Milnor that for $n \notin \{1, 2, 4, 8\}$ the tangent bundle $T\mathbb{R}P^{n-1}$ is nontrivial (indeed TS^{n-1}, which is the pullback of $T\mathbb{R}P^{n-1}$ by the covering map, is nontrivial), though when n is a power of two this is not detected by Stiefel–Whitney classes.

8. Orientations and the fundamental class

Throughout this section we will work with coefficients in a ring R which is either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. Recall from Example 3.21 the following calculation of relative cohomology groups:

$$H^i(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R) = \begin{cases} R & i = m \\ 0 & \text{otherwise} \end{cases}$$

In Section 4 we chose once and for all a generator $\omega \in H^m(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R)$ (of course when $R = \mathbb{Z}/2\mathbb{Z}$ ω is just the unique nonzero element, but when $R = \mathbb{Z}$ there were two possible choices of generator, and we chose one).

Considering instead relative homology, either by using a similar argument to Example 3.21 or by using the conclusion of Example 3.21 together with the Universal Coefficient Theorem, one sees that we likewise have

$$H_i(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R) = \begin{cases} R & i = m \\ 0 & \text{otherwise} \end{cases}$$

Moreover the Universal Coefficient Theorem shows that the evaluation map gives an isomorphism from $H^m(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R)$ to $Hom_R(H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R), R) \cong Hom_R(R, R)$. In particular there is a unique $\mathfrak{o}_{\mathbb{R}^m} \in H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R)$ such that $\langle \omega, \mathfrak{o}_{\mathbb{R}^m} \rangle = 1$, and then this element $\mathfrak{o}_{\mathbb{R}^m}$

generates $H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{\vec{0}\}; R)$. (As is essentially indicated at the start of Section 4, $\mathfrak{o}_{\mathbb{R}^m}$ is represented by the singular chain that is obtained by translating the standard m-simplex $\Delta^m \subset \mathbb{R}^m$ slightly so that the origin lies in its interior.)

Now suppose that U, V are open subsets of $\mathbb{R}^m$ such that $\vec{0} \in U \subset V$, and let $j^{UV}\colon U \to V$ be the inclusion. By the Excision Theorem, the induced map on relative homology $j^{UV}_*\colon H_*(U, U \setminus \{\vec{0}\}; R) \to H_*(V, V \setminus \{\vec{0}\}; R)$ is an isomorphism.

Applying this with $V = \mathbb{R}^m$ shows that, for any open subset $U \subset \mathbb{R}^m$ with $\vec{0} \in U$, we have

$$H_i(U, U \setminus \{\vec{0}\}; R) = \begin{cases} R & i = m \\ 0 & \text{otherwise} \end{cases}$$

with $H_m(U, U \setminus \{\vec{0}\}; R)$ generated by the unique element $\mathfrak{o}_U$ such that $j^{U\mathbb{R}^m}_* \mathfrak{o}_U = \mathfrak{o}_{\mathbb{R}^m}$.

If now $\vec{0} \subset U \subset V \subset \mathbb{R}^m$, since we have $j^{U\mathbb{R}^m} = j^{V\mathbb{R}^m} \circ j^{UV}$, we obtain

$$j^{V\mathbb{R}^m}_* \mathfrak{o}_V = \mathfrak{o}_{\mathbb{R}^m} = j^{U\mathbb{R}^m}_* \mathfrak{o}_U = j^{V\mathbb{R}^m}_* j^{UV}_* \mathfrak{o}_U$$

and so, since $j^{V\mathbb{R}^m}_*$ is an isomorphism

$$j^{UV}_* \mathfrak{o}_U = \mathfrak{o}_V \tag{27}$$

In fact, the classes $\mathfrak{o}_U$ behave consistently under more general maps than inclusions:

Proposition 8.1. *Let U and V be open subsets of $\mathbb{R}^m$ with $\vec{0} \in U \cap V$, and let $f\colon U \to V$ be a smooth injective map such that $f(\vec{0}) = \vec{0}$, so that we have an induced map $f_*\colon H_*(U, U \setminus \{\vec{0}\}; R) \to H_*(V, V \setminus \{0\}; R)$. Assume moreover that the derivative of f at $\vec{0}$ obeys $(df)_{\vec{0}} \in GL^+_m(\mathbb{R})$. Then*

$$f_* \mathfrak{o}_U = \mathfrak{o}_V$$

Proof. Since $j^{V\mathbb{R}^m}_* \mathfrak{o}_V = \mathfrak{o}_{\mathbb{R}^m}$ and since $j^{V\mathbb{R}^m}_*$ is injective, by replacing f with $j^{V\mathbb{R}^m} \circ f$ we may as well assume that $V = \mathbb{R}^m$.

For $r > 0$ let $B^m(r) = \{\vec{x} \in \mathbb{R}^m | \|\vec{x}\| < r\}$ be the ball of radius r around the origin. Choose r so that $\overline{B^m(r)} \subset U$, and abbreviate $A = (df)_{\vec{0}}$. Then Taylor's Theorem shows that there is a constant C so that, whenever $\|\vec{x}\| \le r$, we have

$$\|f(\vec{x}) - A\vec{x}\| \le C\|\vec{x}\|^2$$

Consequently (using that A is invertible to obtain a lower bound $\|A\vec{x}\| \ge c\|\vec{x}\|$ for some $c > 0$) we may choose $\epsilon > 0$ so that

$$\|f(\vec{x}) - A\vec{x}\| \le \frac{1}{3}\|A\vec{x}\| \text{ whenever } \|x\| \le 3\epsilon \tag{28}$$

Now choose a smooth function $\eta\colon \mathbb{R} \to [0,1]$ such that $\eta(s) = 0$ for $s \le 2\epsilon$ and $\eta(s) = 1$ for $s \ge 3\epsilon$, and define $g\colon U \to \mathbb{R}^m$ by

$$g(\vec{x}) = A\vec{x} + \eta(\|\vec{x}\|)(f(\vec{x}) - A\vec{x})$$

Thus $g(\vec{x}) = A\vec{x}$ for $\|x\| \le 2\epsilon$ and $g(\vec{x}) = f(\vec{x})$ for $\|\vec{x}\| \ge 3\epsilon$. Also, if we define

$$\begin{aligned} G(t, \vec{x}) &= (1-t)g(\vec{x}) + tf(\vec{x}) \\ &= (1-t)A\vec{x} + (1-t)\eta(\|\vec{x}\|)(f(\vec{x}) - A\vec{x}) + tf(\vec{x}) \\ &= A\vec{x} + \big(t + (1-t)\eta(\|\vec{x}\|)\big)(f(\vec{x}) - A\vec{x}) \end{aligned}$$

then G gives a homotopy from g to f, with $G(t, \vec{x}) = f(\vec{x})$ for $\|\vec{x}\| \ge 3\epsilon$. Further, since $0 \le t + (1-t)\eta(s) \le 1$ for all $s \in \mathbb{R}$, $t \in [0,1]$ we see that, for $\|\vec{x}\| \le 3\epsilon$, $\|G(t, \vec{x}) - A\vec{x}\| \le \frac{1}{3}\|A\vec{x}\|$ and so $\|G(t,\vec{x})\| \geq \frac{2}{3}\|A\vec{x}\|$. These facts together imply that $G(t,\vec{x}) = 0$ only when $\vec{x} = 0$. Thus G gives a homotopy from g to f through maps *of pairs* $(U, U\setminus\{\vec{0}\}) \to (\mathbb{R}^m, \mathbb{R}^m\setminus\{\vec{0}\})$. Consequently

$$g_*\mathfrak{o}_U = f_*\mathfrak{o}_U \tag{29}$$

Now since $A \in GL_m^+(\mathbb{R})$, which is path-connected, there exists a continuous path $\gamma\colon [\epsilon, 2\epsilon] \to GL_m^+(\mathbb{R})$ such that $\gamma(\epsilon)$ is the identity and $\gamma(2\epsilon) = A$. Define a map $H\colon [\epsilon, 2\epsilon] \times U \to \mathbb{R}^m$ by

$$H(\delta,\vec{x}) = \begin{cases} \gamma(\delta)\vec{x} & \|\vec{x}\| \leq \delta \\ \gamma(\|\vec{x}\|)\vec{x} & \delta \leq \|\vec{x}\| \leq 2\epsilon \\ g(\vec{x}) & \|\vec{x}\| \geq 2\epsilon \end{cases}$$

Since when $\|\vec{x}\| = 2\epsilon$ we have $\gamma(\|\vec{x}\|) = A$ and $g(\vec{x}) = A\vec{x}$, H is a continuous map. Clearly $H(\delta,\vec{x}) = \vec{0}$ only when $\vec{x} = 0$, since the same statement holds for $g(\vec{x})$ and since each $\gamma(s)$ is an invertible matrix. So where $h(\vec{x}) = H(\epsilon,\vec{x})$, H is a homotopy from h to g through maps of pairs $(U, U\setminus\{\vec{0}\}) \to (\mathbb{R}^m, \mathbb{R}^m\setminus\{\vec{0}\})$. So

$$h_*\mathfrak{o}_U = g_*\mathfrak{o}_U \tag{30}$$

The map $h\colon U \to \mathbb{R}^m$ has the property that $h(\vec{x}) = \vec{x}$ for $\|\vec{x}\| \leq \epsilon$, *i.e.*, $h \circ j^{B^m(\epsilon)U} = j^{B^m(\epsilon)\mathbb{R}^m}$. Thus

$$h_* j^{B^m(\epsilon)U}\mathfrak{o}_{B^m(\epsilon)} = j_*^{B^m(\epsilon)\mathbb{R}^m}\mathfrak{o}_{B^m(\epsilon)} = \mathfrak{o}_{\mathbb{R}^m}$$

But $j_*^{B^m(\epsilon)U}\mathfrak{o}_{B^m(\epsilon)} = \mathfrak{o}_U$, so this shows that $h_*\mathfrak{o}_U = \mathfrak{o}_{\mathbb{R}^m}$. Combined with (29) and (30), this completes the proof (as in the first paragraph we reduced to the case that $V = \mathbb{R}^m$). $\square$

Proposition 8.1 will allow us to identify an analogue of the classes $\mathfrak{o}_U$ in the homology $H_m(M, M\setminus\{p\};\mathbb{Z})$ where M is an oriented m-dimensional smooth manifold and $p \in M$. (As we will see, there is also a version with $\mathbb{Z}/2\mathbb{Z}$ coefficients, which is easier to construct and does not require M to be oriented.)

Recall that, in our conventions, an **orientation** of a smooth manifold M is an orientation (in the sense of vector bundles) of its tangent bundle TM. Suppose that M is a smooth oriented m-dimensional manifold. We can then choose a smooth atlas $\{\phi_\alpha\colon U_\alpha \to \mathbb{R}^m\}$ of coordinate charts for m—thus the $U_\alpha \subset M$ are open and each ϕ_α is a homeomorphism to its image (which is open in $\mathbb{R}^m$) and the transition functions $\phi_\beta \circ \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth. For simplicity let us suppose that the U_α are all connected.

Now the ϕ_α determine local trivializations $\Phi_\alpha\colon TM|_{U_\alpha} \to U_\alpha \times \mathbb{R}^m$: if $p \in U_\alpha$ then we have a natural identification of $T_{\phi_\alpha(p)}\mathbb{R}^m$ with $\mathbb{R}^m$, and in terms of this identification Φ_α is given by $\Phi_\alpha(v) = \big(p, (\phi_\alpha)_*(v)\big)$ for $v \in T_pM$ where $(\phi_\alpha)_*\colon T_pM \to T_{\phi_\alpha(p)}\mathbb{R}^m \cong \mathbb{R}^m$ is the derivative of ϕ_α at p. The orientation of TM gives us an orientation of each T_pM, and for all $p \in U_\alpha$, $(\phi_\alpha)_*\colon T_pM \to \mathbb{R}^m$ is a linear isomorphism, so it is either orientation-preserving or orientation-reversing (where we use the standard orientation on $\mathbb{R}^m$). We claim that (since U_α is connected) either $(\phi_\alpha)_*$ is orientation-preserving for all $p \in U_\alpha$ or else $(\phi_\alpha)_*$ is orientation-reversing for all $p \in U_\alpha$. Indeed, around any $p \in U_\alpha$ there is some local trivialization $\Psi\colon TM|_V \to V \times \mathbb{R}^m$ which is consistent with the orientation on TM. We have a transition function $\Psi\circ\Phi_\alpha^{-1}\colon (U_\alpha \cap V)\times\mathbb{R}^m \to (U_\alpha \cap V) \times \mathbb{R}^m$, given by $(q,\vec{v}) \mapsto (q, g(q)\vec{v})$ where $g\colon U_\alpha \cap V \to GL_m(\mathbb{R})$ is continuous. Then, for any $q \in U_\alpha \cap V$, $(\phi_\alpha)_*\colon T_pM \to \mathbb{R}^m$ is orientation-preserving if and only if $\det(g(q)) > 0$ and orientation-reversing if and only if $\det(g(q)) < 0$. From this it follows that the subsets of U_α on which $(\phi_\alpha)_*$ is, respectively, orientation-preserving or orientation-reversing are both open. So since U_α is connected one of these sets is empty and the other is all of U_α, confirming the claim.

This almost suffices to prove:

Proposition 8.2. *Let M be an oriented smooth m-manifold. Then there is an atlas $\{\psi_\alpha\colon U_\alpha \to \mathbb{R}^m\}$ for M such that, for each α and each $p \in U_\alpha$, the map $(\psi_\alpha)_*\colon T_pM \to T_{\psi_\alpha(p)}\mathbb{R}^m \cong \mathbb{R}^m$ is orientation-preserving (with respect to the given orientation on T_pM and the standard orientation on $\mathbb{R}^m$).*

Proof. Start with an atlas $\{\phi_\alpha\colon U_\alpha \to \mathbb{R}^m\}$ where each U_α is connected, as in the discussion above the statement of the proposition. Then for each α, either $(\phi_\alpha)_*$ is orientation preserving at all $p \in U_\alpha$, or else $(\phi_\alpha)_*$ is orientation-reversing at all $p \in U_\alpha$. In the first case, let $\psi_\alpha = \phi_\alpha$. In the second case, choose some $m \times m$ matrix B with $\det B = -1$ and define $\psi_\alpha(q) = B\phi_\alpha(q)$. Since the composition of two orientation-reversing maps is orientation-preserving, $(\psi_\alpha)_*$ will be orientation-preserving at all $p \in U_\alpha$ in either case. □

Definition 8.3. Let M be an oriented smooth manifold and let $p \in M$. An *oriented chart around* p is a diffeomorphism $\phi\colon U \to \phi(U)$ where $U \subset M$ and $\phi(U) \subset \mathbb{R}^m$ are open subsets such that $\phi(p) = \vec{0}$ and $\phi_*\colon T_pM \to \mathbb{R}^m$ is orientation-preserving.

It follows from Proposition 8.2 that oriented charts around p always exist: given an atlas $\{\psi_\alpha\colon U_\alpha \to \mathbb{R}^m\}$ as in Proposition 8.2 choose α such that $p \in U_\alpha$ and define $\phi\colon U_\alpha \to \phi(U) \subset \mathbb{R}^m$ by $\phi(q) = \psi_\alpha(q) - \psi_\alpha(p)$.

Of course, oriented charts are far from unique; however, if $\psi\colon V \to \psi(V)$ is another oriented chart around p we can consider the transition function $\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$. Evidently we have $\psi \circ \phi^{-1}(\vec{0}) = \vec{0}$, and moreover by the chain rule $d(\psi \circ \phi^{-1})_{\vec{0}}$ is the composition

$$\mathbb{R}^m \xrightarrow{\phi_*^{-1}} T_pM \xrightarrow{\psi_*} \mathbb{R}^m$$

where both maps are orientation-preserving. Thus $d(\psi \circ \phi^{-1})_{\vec{0}}$ is an orientation-preserving linear operator on $\mathbb{R}^m$, *i.e.* it belongs to $GL_m^+(\mathbb{R})$. This makes Proposition 8.1 relevant.

Theorem 8.4. *Let M be a smooth oriented m-manifold and let $p \in M$. Then*

$$H_i(M, M \setminus \{p\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = m \\ 0 & \textit{otherwise} \end{cases}$$

and there is a generator $\mathfrak{o}_{M,p}$ for $H_m(M, M \setminus \{p\}; \mathbb{Z})$ which is uniquely specified by the following prescription: Choose any oriented chart $\phi\colon U \to \phi(U) \subset \mathbb{R}^m$ around p, and let

$$\mathfrak{o}_{M,p} = j_*^{UM}(\phi^{-1})_*\mathfrak{o}_{\phi(U)}$$

where $j^{UM}\colon U \to M$ is the inclusion and $\mathfrak{o}_{\phi(U)} \in H_m(\phi(U), \phi(U) \setminus \{\vec{0}\}; \mathbb{Z})$ is the class described at the start of this section.

Proof. If $\phi\colon U \to \phi(U)$ is an oriented chart around p, we have a sequence of maps

$$H_*(\phi(U), \phi(U) \setminus \{\vec{0}\}; \mathbb{Z}) \xrightarrow{(\phi^{-1})_*} H_*(U, U \setminus \{p\}; \mathbb{Z}) \xrightarrow{j_*^{UM}} H_*(M, M \setminus \{p\}; \mathbb{Z})$$

where the first map is an isomorphism because ϕ^{-1} is a homeomorphism of pairs and the second map is an isomorphism by excision. So it follows from our earlier work in $\mathbb{R}^m$ that

$$H_i(M, M \setminus \{p\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = m \\ 0 & \text{otherwise} \end{cases}$$

with $H_m(M, M \setminus \{p\}; \mathbb{Z})$ generated by the class

$$\mathfrak{o}_{M,p}(\phi, U) := j_*^{UM}(\phi^{-1})_*\mathfrak{o}_{\phi(U)}.$$

It remains only to prove that this class is independent of the choice of oriented chart around p.

First we observe that if $W \subset U$ is an open subset containing p, then we have

$$\begin{aligned}\mathfrak{o}_{M,p}(\phi|_W, W) &= j_*^{WM}(\phi^{-1}|_{\phi(W)})_*\mathfrak{o}_{\phi(W)} = (j^{UM} \circ \phi^{-1} \circ j^{\phi(W)\phi(U)})_*\mathfrak{o}_{\phi(W)}\\ &= j_*^{UM}(\phi^{-1})_*\mathfrak{o}_{\phi(U)} = \mathfrak{o}_{M,p}(\phi, U),\end{aligned}$$

where we have used (27) to see that $j_*^{\phi(W)\phi(U)}\mathfrak{o}_{\phi(W)} = \mathfrak{o}_{\phi(U)}$.

Now suppose that $\phi\colon U \to \phi(U)$ and $\psi\colon V \to \psi(V)$ are two different oriented charts around p. We must show that $\mathfrak{o}_{M,p}(U,\phi) = \mathfrak{o}_{M,p}(V,\psi)$. By the observation that we have just made, we have

$$\mathfrak{o}_{M,p}(\phi, U) = \mathfrak{o}_{M,p}(\phi|_{U\cap V}, U\cap V) \qquad \mathfrak{o}_{M,p}(\psi, V) = \mathfrak{o}_{M,p}(\psi|_{U\cap V}, U\cap V)$$

But applying Proposition 8.1 to $\psi \circ \phi^{-1}\colon \phi(U\cap V) \to \psi(U\cap V)$ gives

$$\begin{aligned}\mathfrak{o}_{M,p}(\psi|_{U\cap V}, U\cap V) &= j_*^{U\cap V\,M}((\psi|_{U\cap V})^{-1})_*\mathfrak{o}_{\psi(U\cap V)}\\ &= j_*^{U\cap V}((\psi|_{U\cap V})^{-1})_*(\psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1})_*\mathfrak{o}_{\phi(U\cap V)}\\ &= j_*^{U\cap V\,M}((\phi|_{U\cap V})^{-1})_*\mathfrak{o}_{\phi(U\cap V)} = \mathfrak{o}_{M,p}(\phi|_{U\cap V}, U\cap V)\end{aligned}$$

So indeed we have $\mathfrak{o}_{M,p}(U,\phi) = \mathfrak{o}_{M,p}(V,\psi)$ for any two oriented charts $\phi\colon U \to \phi(U)$ and $\psi\colon V \to \psi(V)$ around p. □

As alluded to earlier there is an (easier) analogue to Theorem 8.4 for $\mathbb{Z}/2\mathbb{Z}$ coefficients, which does not require an orientation for M. (Consistently with earlier terminology, if M is a not-necessarily-oriented smooth manifold and $p \in M$ a *chart around* p is a diffeomorphism $\phi\colon U \to \phi(U)$ where $U \subset M$ and $\phi(U) \subset \mathbb{R}^m$ are open subsets and $\phi(p) = \vec{0}$.)

Theorem 8.5. *Let M be a smooth m-manifold and let $p \in M$. Then*

$$H_i(M, M\setminus\{p\}; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = m\\ 0 & \textit{otherwise}\end{cases}$$

and there is a generator $\mathfrak{o}_{M,p}$ for $H_m(M, M\setminus\{p\}; \mathbb{Z})$ which is uniquely specified by the following prescription: Choose any chart $\phi\colon U \to \phi(U) \subset \mathbb{R}^m$ around p, and let

$$\mathfrak{o}_{M,p} = j_*^{UM}(\phi^{-1})_*\mathfrak{o}_{\phi(U)}$$

where $j^{UM}\colon U \to M$ is the inclusion and $\mathfrak{o}_{\phi(U)} \in H_m(\phi(U), \phi(U)\setminus\{\vec{0}\}; \mathbb{Z}/2\mathbb{Z})$ is the class described at the start of this section.

Proof. For any chart $\phi\colon U \to \phi(U)$ around p the composition

$$j_*^{UM} \circ \phi_*^{-1}\colon H_*(\phi(U), \phi(U)\setminus\{\vec{0}\}; \mathbb{Z}/2\mathbb{Z}) \to H_*(M, M\setminus\{p\}; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism by excision, so the formula for $H_*(M, M\setminus\{p\}; \mathbb{Z}/2\mathbb{Z})$ follows from the calculations at the start of the section. Furthermore $\mathfrak{o}_{M,p} = j_*^{UM}(\phi^{-1})_*\mathfrak{o}_{\phi(U)}$ is the unique nonzero element of $H_*(M, M\setminus\{p\}; \mathbb{Z}/2\mathbb{Z})$, so it must be independent of the choice of chart $\phi\colon U \to \phi(U)$ (since any other chart $\psi\colon V \to \psi(V)$ would also have $j_*^{VM}(\psi^{-1})_*\mathfrak{o}_{\psi(V)} \neq 0$). □

Now we consider the effect of replacing the singleton $\{p\}$ by a more general compact subset $K \subset M$.

Where $R = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, define

$$H_*(M|_K) = H_*(M, M\setminus K; R)$$

(this is sometimes called the *local homology of M at K*). Note that if $L \subset K \subset M$ with L and K compact, then the inclusion $i^{KL}\colon (M, M\setminus K) \subset (M, M\setminus L)$ induces a map

$$i_*^{KL}\colon H_*(M|_K) \to H_*(M|_L).$$

In particular this applies with $L=\{p\}$ where $p\in K$ (in which case we will write i^{Kp} instead of $i^{K\{p\}}$).

Definition 8.6. Let $R=\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$, and let M be a smooth m-manifold, which we assume to be oriented if $R=\mathbb{Z}$. Let $K\subset M$ be compact. A *fundamental class for M at K* with coefficients in R is a class $\mathfrak{o}_{M,K}\in H_m(M|_K)$ such that for all $p\in K$ we have

$$i_*^{Kp}\mathfrak{o}_{M,K}=\mathfrak{o}_{M,p}$$

Of course, for any nonempty K, if $\mathfrak{o}_{M,K}$ exists it must be nonzero, since $\mathfrak{o}_{M,p}\neq 0$ for all $p\in K$. Actually, the case in which we are most interested is where M is a compact smooth m-manifold and $K=M$. Then the local homology $H_*(M|_M)$ of M at M is just the absolute homology $H_*(M;R)$, and a fundamental class $\mathfrak{o}_{M,M}$ for M at M would give a nonzero element of $H_m(M;R)$ (more commonly this element is just denoted $[M]$). The reason for considering the more general situation is that in order to construct the fundamental class we will need to patch together constructions made in various coordinate charts, and the coordinate charts are noncompact even if M is compact.

Theorem 8.7. *Let $R=\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$, let M be a smooth m-manifold which is oriented if $R=\mathbb{Z}$. Then there exists a unique fundamental class $\mathfrak{o}_{M,K}$ for M at K. Moreover, for $i>m$, $H_i(M|_K)=0$.*

Proof. We divide the proof (which is somewhat similar to but more complicated than our earilier proof of Theorem 4.4 on the existence of Thom classes) into a series of eight claims.

Claim 8.8. If $L\subset K\subset M$ with K and L compact and if $\mathfrak{o}_{M,K}$ is a fundamental class for M at K then $i_*^{KL}\mathfrak{o}_{M,K}$ is a fundamental class for M at L.

Proof of Claim 8.8. For $p\in L$ the various inclusions obviously obey $i^{Kp}=i^{Lp}\circ i^{KL}$, and therefore

$$i_*^{Lp}(i_*^{KL}\mathfrak{o}_{M,K})=(i^{Lp}\circ i^{KL})_*\mathfrak{o}_{M,K}=i_*^{Kp}\mathfrak{o}_{M,K}=\mathfrak{o}_{M,p},$$

so $i_*^{KL}\mathfrak{o}_{M,K}$ indeed satisfies the defining property for a fundamental class for M at L. □

Claim 8.9. If K_1 and K_2 are compact subsets of M such that Theorem 8.7 holds for the pairs $(M,K_1),(M,K_2)$, and $(M,K_1\cap K_2)$, then Theorem 8.7 also holds for $(M,K_1\cup K_2)$.

Proof of Claim 8.9. Just like in Theorem 4.4 we use the Mayer–Vietoris sequence (but now for relative homology instead of relative cohomology). In view of De Morgan's laws and the general definition $H_*(M|_K)=H_*(M,M\setminus K;R)$, part of the Mayer–Vietoris sequence reads
(31)

$$H_{i+1}(M|_{K_1\cap K_2})\longrightarrow H_i(M|_{K_1\cup K_2})\xrightarrow{(i_*^{K_1\cup K_2,K_1},\,i_*^{K_1\cup K_2,K_2})}H_i(M|_{K_1})\oplus H_i(M|_{K_2})\xrightarrow{i_*^{K_1,K_1\cap K_2}-i_*^{K_2,K_1\cap K_2}}H_i(M|_{K_1\cap K_2})$$

Suppose first that $i>m$. Then by assumption the first and third terms in the above sequence are zero, whence so is the second term. This proves the part of Theorem 4.4 stating that $H_i(M|_{K_1\cup K_2})=0$ for $i>m$, so we now turn to the uniqueness and existence of the fundamental class.

Set $i=m$ in the above Mayer–Vietoris sequence. Then in particular the first term vanishes and so the map

$$(i_*^{K_1\cup K_2,K_1},i_*^{K_1\cup K_2,K_2})\colon H_m(M|_{K_1\cup K_2})\to H_m(M|_{K_1})\oplus H_m(M|_{K_2}) \tag{32}$$

is injective. Suppose now that $c\in H_m(M|_{K_1\cup K_2})$ is a fundamental class for M at $K_1\cup K_2$. Now by assumption there are *unique* fundamental classes $\mathfrak{o}_{M,K_j}\in H_m(M|_{K_j})$ for $j=1,2$, so by Claim 1 it must hold that $i_*^{K_1\cup K_2,K_j}c=\mathfrak{o}_{M,K_j}$ for $j=1,2$. But the injectivity of (32) shows that there can be at most one class c obeying this property. This proves the uniqueness of any fundamental class for M at $K_1 \cup K_2$.

As for existence, by assumption there is a unique fundamental class $\mathfrak{o}_{M,K_1\cap K_2}$ for M at $K_1 \cap K_2$. So in view of Claim 8.8 we have $i_*^{K_j,K_1\cap K_2}\mathfrak{o}_{M,K_j} = \mathfrak{o}_{M,K_1\cap K_2}$ for $j = 1,2$. Therefore $(\mathfrak{o}_{M,K_1}, \mathfrak{o}_{M,K_2})$ lies in the kernel of the last map in the Mayer–Vietoris sequence 31, and hence in the image of the map (32). So there is a class $\mathfrak{o}_{M,K_1\cup K_2} \in H_m(M|_{K_1\cup K_2})$ such that $i_*^{K_1\cup K_2,K_j}\mathfrak{o}_{M,K_1\cup K_2} = \mathfrak{o}_{M,K_j}$ for $j = 1,2$. To see that this is indeed the desired fundamental class, just note that if $p \in K_1 \cup K_2$ then $p \in K_j$ for some $j \in \{1,2\}$ and so

$$i_*^{K_1\cup K_2,p}\mathfrak{o}_{M,K_1\cup K_2} = i_*^{K_jp} i_*^{K_1\cup K_2,K_j}\mathfrak{o}_{M,K_1\cup K_2} = i_*^{K_jp}\mathfrak{o}_{M,K_j} = \mathfrak{o}_{M,p},$$

as desired.

□

Claim 8.10. Let $K_1,\dots,K_r$ be compact subsets of M such that Theorem 8.7 holds for the pair $(M, K_{j_1} \cap \cdots \cap K_{j_l})$ for each intersection $K_{j_1} \cap \cdots \cap K_{j_l}$ of a subcollection of the K_j. Then Theorem 8.7 also holds for $(M, K_1 \cup \cdots \cup K_r)$.

Proof of Claim 8.10. We use induction on r together with Claim 8.9. For $r = 1$ the statement is trivial. Assuming the statement for r and supposing that $K = K_1 \cup \cdots K_{r+1}$ is a union of compact sets such that the theorem holds for all intersections of subcollections of the K_j, let us write $L = K_1 \cup \cdots \cup K_r$. The theorem is known to hold for K_{r+1} (by assumption), for L (by inductive hypothesis), and also (again by the inductive hypothesis) for $L \cap K_{r+1} = (K_1 \cap K_{r+1}) \cap \cdots \cap (K_r \cap K_{r+1})$. So by Claim 8.9 it holds for $K = L \cup K_{r+1}$ as well. □

Claim 8.11. Suppose that $K \subset U \subset M$ with K compact and U open, so that in particular U is also a smooth m-manifold containing the compact set K. Then Theorem 8.7 holds for (U,K) if and only if Theorem 8.7 holds for (M,K).

Proof of Claim 8.11. It follows directly from excision that the inclusion $j^{UM}\colon (U, U \setminus K) \to (M, M \setminus K)$ induces an isomorphism on homology, so in particular $H_i(U|_K) = 0$ for $i > m$ if and only if $H_i(M|_K) = 0$ for $i > m$.

For $p \in U$, let $\psi\colon V \to \psi(V)$ be a chart (oriented if $R = \mathbb{Z}$) around p; replacing V by its intersection with U, if necessary, we may assume that $V \subset U$. We then have, essentially by definition,

$$\mathfrak{o}_{U,p} = j_*^{VU}(\psi^{-1})_*\mathfrak{o}_{\psi(V)} \qquad \mathfrak{o}_{M,p} = j_*^{VM}(\psi^{-1})_*\mathfrak{o}_{\psi(V)}$$

Since $j^{VM} = j^{UM} \circ j^{VU}$ it follows directly that $j_*^{UM}\mathfrak{o}_{U,p} = \mathfrak{o}_{M,p}$ for all $p \in U$.

In view of the commutative diagram

$$\begin{tikzcd} H_*(U|_K) \arrow[r, "j_*^{UM}"] \arrow[d, "i_*^{Kp}"'] & H_*(M|_K) \arrow[d, "i_*^{Kp}"] \\ H_*(U|_{\{p\}}) \arrow[r, "j_*^{UM}"] & H_*(M|_{\{p\}}) \end{tikzcd}$$

in which the bottom line sends $\mathfrak{o}_{U,p}$ to $\mathfrak{o}_{M,p}$ for all $p \in K \subset U$, it follows directly from the definition of a fundamental class that $\mathfrak{o}_{U,K}$ is a fundamental class for U at K if and only if $j_*^{UM}\mathfrak{o}_{U,K}$ is a fundamental class for M at K. This obviously proves the forward implication of the claim, and it also proves the backward implication in view of the fact that the top line of the above commutative diagram is an isomorphism. □

Claim 8.12. It suffices to prove Theorem 8.7 in the special case that $M = \mathbb{R}^m$.

Proof of Claim 8.12. Assume given an arbitrary pair $K \subset M$ as in the statement of the theorem. For each $p \in K$ let $\phi_p \colon U_p \to \phi_p(U_p)$ be a coordinate chart (taken from an oriented atlas as in Proposition 8.2 if $R = \mathbb{Z}$) around p, and choose a neighborhood V_p of p such that $\overline{V_p} \subset U_p$. We then have an open cover $\{V_p\}_{p \in K}$ of K, so for some $p_1, \ldots, p_r \in K$ we have $K \subset V_{p_1} \cup \cdots \cup V_{p_r}$. Write $K_i = \overline{V_{p_i}} \cap K$. Then $K = K_1 \cup \cdots \cup K_r$, where each K_i is a compact set contained in the domain of a coordinate chart $\phi_{p_i} \colon U_{p_i} \to \phi_{p_i}(U_{p_i})$. Of course, each intersection $K_{i_1} \cap \cdots \cap K_{i_l}$ is also contained in $U_{p_{i_1}}$.

Therefore, in view of Claim 8.10, it suffices to prove Theorem 8.7 in the case where the compact set K is contained in the domain of a coordinate chart $\phi \colon U \to \phi(U)$ (which may be taken to be an oriented coordinate chart in the case that $R = \mathbb{Z}$). Furthermore, Claim 8.11 shows that in this situation the theorem holds for the pair (M, K) if and only if the theorem holds for (U, K).

It follows directly from the definitions (and a brief diagram chase that is left to the reader) that the theorem holds for (U, K) if and only if it holds for the pair $(\phi(U), \phi(K))$, where now $\phi(U)$ is an open subset of $\mathbb{R}^m$. But then another application of Claim 8.11 shows that if the theorem holds for $(\mathbb{R}^m, \phi(K))$ then it also holds for $(\phi(U), \phi(K))$. So this indeed reduces us to the case that $M = \mathbb{R}^m$. □

Claim 8.13. Theorem 8.7 holds when $M = \mathbb{R}^m$ and K is a rectangular prism, *i.e.* a (possibly empty) set of the form $\{(x_1, \ldots, x_m) \in \mathbb{R}^m | (\forall i)(a_i \le x_i \le b_i)\}$ where $a_i, b_i \in \mathbb{R}$ are arbitrary.

Proof of Claim 8.13. The case where K is empty is trivial, since then $H_*(M|_K) = H_*(M, M \setminus M; R) = \{0\}$, and 0 serves as a fundamental class since there are no $p \in K$ on which to check the required property. So we assume $K \neq \varnothing$.

Let B be an open ball which contains K, centered at some point $p_0 \in K$. Let $\phi_{p_0} \colon \mathbb{R}^m \setminus \{p_0\} \to \mathbb{R}^m \setminus B$ be the map which is the identity outside of B and which sends each point $x \in B \setminus \{p_0\}$ to the point on the boundary of B which lies on the ray from p_0 to x. (So ϕ_{p_0} is the terminal map of a deformation retraction of $\mathbb{R}^m \setminus \{p_0\}$ to $\mathbb{R}^n \setminus B$.)

For any $p \in K$ let $h_p \colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism which is the identity on $\mathbb{R}^n \setminus B$ and which, for each point q on the boundary of B, sends the line segment from p_0 to q to the line segment from p to q. (So in particular $h(p) = q$, and we can and do arrange that h_p is differentiable away from the boundary of B and that $(dh)_{p_0} \in GL_m^+(\mathbb{R})$). It then follows from Proposition 8.1 and the definition of $\mathfrak{o}_{M,p}$ in Theorems 8.4,8.5 that

$$(h_p)_* \mathfrak{o}_{\mathbb{R}^m, p_0} = \mathfrak{o}_{\mathbb{R}^m, p}$$

(strictly speaking to apply Proposition 8.1 we should replace h_p by a function which is smooth eveywhere and not just away from the boundary of B, but since we can find such a map which is homotopic to h_p through maps of pairs $(\mathbb{R}^m, \mathbb{R}^m \setminus \{p_0\}) \to (\mathbb{R}^m, \mathbb{R}^m \setminus \{p\})$ the conclusion applies to h_p as well, as homotopic maps induce the same maps on homology.) Define $\phi_p = \phi_{p_0} \circ h_p^{-1}$. Thus ϕ_p retracts $\mathbb{R}^m \setminus \{p\}$ to $\mathbb{R}^m \setminus B$ by pushing points outward along line segments from p to the boundary of B.

Regard ϕ_p as having range $\mathbb{R}^m \setminus K \supset \mathbb{R}^m \setminus B$. Where $i^{Kp} \colon \mathbb{R}^m \setminus K \to \mathbb{R}^n \setminus \{p\}$ denotes the inclusion, $i^{Kp} \circ \phi_p \colon \mathbb{R}^m \setminus \{p\} \to \mathbb{R}^m \setminus \{p\}$ is homotopic to the identity, with a homotopy given by a family of maps which push points of $\mathbb{R}^m \setminus \{p\}$ just part of the way along segments from p to the boundary of B. Likewise $\phi_p \circ i^{Kp} \colon \mathbb{R}^m \setminus K \to \mathbb{R}^m \setminus K$ is homotopic to the identity: since K is convex, once a line segment from p to the boundary of B exits K, it remains outside of K (otherwise a line segment from p to another point of K would pass through $\mathbb{R}^m \setminus K$), so the homotopy given by progressively pushing points of $\mathbb{R}^m \setminus K$ outward along such line segments preserves the set $\mathbb{R}^m \setminus K$.

Thus all of the maps $\phi_p\colon \mathbb{R}^m \setminus \{p\} \to \mathbb{R}^m \setminus K$ are homotopy equivalences, and so induce isomorphisms $H_*(\mathbb{R}^m \setminus \{p\}; R) \to H_*(\mathbb{R}^m \setminus K; R)$. Since $H_i(\mathbb{R}^m) = \{0\}$ for $i \geq 1$, the long exact sequences of the pair show that the boundary maps $\partial_p\colon H_i(\mathbb{R}^m|_{\{p\}}) \to H_{i-1}(\mathbb{R}^m \setminus \{p\}; R)$ and $\partial_K\colon H_i(\mathbb{R}^m|_K) \to H_{i-1}(\mathbb{R}^m \setminus K; R)$ are isomorphisms for $i \geq 2$ and injective for $i = 1$. In particular for $i > m$ we have $H_i(\mathbb{R}^m|_K) \cong H_i(\mathbb{R}^m|_{\{p_0\}}) = \{0\}$, proving part of the theorem.

For $i = m$, we can then define (for each $p \in K$) an isomorphism $\bar{\phi}_p\colon H_m(\mathbb{R}^m|_{\{p\}}) \to H_m(\mathbb{R}^m|_K)$ by $\bar{\phi}_p(x) = \partial_K^{-1}(\phi_p)_*(\partial_p x)$.[14]

Define
$$\mathfrak{o}_{\mathbb{R}^m,K} = \bar{\phi}_{p_0}\mathfrak{o}_{\mathbb{R}^m,p_0} \in H_m(\mathbb{R}^m|_K).$$
Thus
$$\partial_K \mathfrak{o}_{\mathbb{R}^m,K} = (\phi_{p_0})_* \partial_{p_0}\mathfrak{o}_{\mathbb{R}^m,p_0}.$$
Now for any $p \in K$ the maps on homology induced by the inclusion $i^{Kp}\colon (\mathbb{R}^m, \mathbb{R}^m \setminus K) \to (\mathbb{R}^m, \mathbb{R}^m \setminus \{p\})$ commute with the maps appearing in the long exact sequences of the pairs $(\mathbb{R}^m, \mathbb{R}^m \setminus K)$ and $(\mathbb{R}^m, \mathbb{R}^m \setminus \{p\})$, and a similar statement applies to the maps on homology induced by $h_p^{-1}\colon (\mathbb{R}^m, \mathbb{R}^m \setminus \{p\}) \to (\mathbb{R}^m, \mathbb{R}^m \setminus \{p_0\})$. Therefore
$$\begin{aligned}
\partial_p i_*^{Kp}\mathfrak{o}_{\mathbb{R}^m,K} &= i_*^{Kp}\partial_K \mathfrak{o}_{\mathbb{R}^m,K} = i_*^{Kp}(\phi_{p_0})_*(\partial_{p_0}\mathfrak{o}_{\mathbb{R}^m,p_0}) \\
&= i_*^{Kp}(\phi_{p_0})_*(\partial_{p_0} h_{p*}^{-1}\mathfrak{o}_{\mathbb{R}^m,p}) \\
&= i_*^{Kp}(\phi_{p_0} \circ h_p^{-1})_* \partial_p \mathfrak{o}_{\mathbb{R}^m,p} \\
&= i_*^{Kp}(\phi_p)_* \partial_p \mathfrak{o}_{\mathbb{R}^m,p} \\
&= \partial_p \mathfrak{o}_{\mathbb{R}^m,p}
\end{aligned}$$
where in the last line we have used that ϕ_p and i^{Kp} are homotopy inverses. Since ∂_p is injective it follows that $i_*^{Kp}\mathfrak{o}_{\mathbb{R}^m,K} = \mathfrak{o}_{\mathbb{R}^m,p}$ for all $p \in K$, *i.e.* that $\mathfrak{o}_{\mathbb{R}^m,K}$ is a fundamental class for $\mathbb{R}^m$ at K.

As for uniqueness, since for all $p \in K$ we have isomorphisms $H_m(\mathbb{R}^m|_K) \cong H_m(\mathbb{R}^m|_{\{p\}}) \cong R$ where $R = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$, we just note that either of our possible rings R has the property that any nontrivial homomorphism $R \to R$ is injective. (So the nontrivial element $\mathfrak{o}_{M,p}$ cannot have more than one preimage under i_*^{Kp}.) ☐

Claim 8.14. Theorem 8.7 holds when $M = \mathbb{R}^m$ and K is a union of finitely many rectangular prisms.

Proof. This follows directly from Claims 8.13 and 8.10, together with the fact that an intersection of finitely many rectangular prisms is a (possibly empty) rectangular prism. ☐

Claim 8.15. Theorem 8.7 holds when $M = \mathbb{R}^m$ and K is an arbitrary compact subset of $\mathbb{R}^m$.

Proof. Existence of the fundamental class is easy: let L be a large rectangular prism that contains K, so that we have a fundamental class $\mathfrak{o}_{\mathbb{R}^m,L}$ for $\mathbb{R}^m$ at L by Claim 8.13. Then by Claim 8.8, $i_*^{LK}\mathfrak{o}_{\mathbb{R}^m,L}$ is a fundamental class for $\mathbb{R}^m$ at K.

For the other statements, the key point is that for any chain in $\mathbb{R}^m \setminus K$ we can find a finite union of rectangular prisms L disjoint from the given chain.

[14]A slightly different formulation is required when $m = 1$—for instance one can work with reduced homology, or just observe quite directly that $(\phi_p)_*$ maps the image of ∂_p isomorphically to the image of ∂_K and define again $\bar{\phi}_p = \partial_K^{-1} \circ (\phi_p)_* \circ \partial_p$ with $(\phi_p)_*$ now regarded as a map $Im(\partial_p) \to Im(\partial_K)$.

More specifically, suppose that $x \in H_i(M|_K)$. There is then a chain $c \in S_i(\mathbb{R}^m;R)$ which descends to a relative cycle in $S_i(\mathbb{R}^m, \mathbb{R}^m \setminus K;R) = S_i(\mathbb{R}^m)/S_i(\mathbb{R}^m \setminus K)$ whose homology class is x. That c is a relative cycle means that the boundary ∂c is given by $\partial c = \sum_j n_j\sigma_j$ for some $n_j \in \mathbb{R}$ and $\sigma_j \colon \Delta^{i-1} \to \mathbb{R}^m \setminus K$.

Since the various (finitely many) $Im(\sigma_j)$ are compact, around any point $p \in K$ we can find a neighborhood V_p whose closure $\bar{V}_p$ is a rectangular prism which is disjoint from $\cup_j Im(\sigma_j)$. The collection $\{V_p\}_{p\in K}$ then has a finite subcover $\{V_{p_1}, \dots, V_{p_r}\}$. Let $L = \bar{V}_{p_1} \cup \cdots \cup \bar{V}_{p_r}$. Then $K \subset L$ and L is a compact subset of $\mathbb{R}^m$ which is disjoint from each $Im(\sigma_j)$, and by Claim 8.14, Theorem 8.7 applies to the pair $(\mathbb{R}^m, L)$.

Since L is disjoint from each $Im(c_i)$, our chain c descends to a relative cycle in $S_i(\mathbb{R}^m, \mathbb{R}^m \setminus L;R)$, representing a class $[c] \in H_i(M|_L)$. But since c, when considered as a relative cycle in $(\mathbb{R}^m, \mathbb{R}^m \setminus K)$, was chosen to be a representative of the given class x, we have $i_*^{LK}[c] = x$.

If $i > m$, then by Claim 8.14 we have $H_i(M|_L) = \{0\}$, so that $[c] = 0$ and so $x = 0$. Since x was an arbitrary element of $H_i(M|_K)$, this completes the proof that $H_i(M|_K) = \{0\}$ for $i > m$.

Finally we prove the uniqueness of the fundamental class of M at K. Suppose that $\mathfrak{o}_{\mathbb{R}^m,K}, \mathfrak{o}'_{\mathbb{R}^m,K}$ are two fundamental classes of M at K, and let $x = \mathfrak{o}_{\mathbb{R}^m,K} - \mathfrak{o}'_{\mathbb{R}^m,K}$. Then $x \in H_m(M|_K)$ has the property that $i_*^{Kp}x = 0$ for all $p \in K$; we will be done if we show that $x = 0$. Let L and c be as above, so that in particular $i_*^{LK}[c] = x$. Note also that, by the construction of L, for each point q of L there is a rectangular prism contained in L which contains both q and some point of K. Now if $p \in K$, then $i_*^{Lp}[c] = i_*^{Kp}i_*^{LK}[c] = 0$. We claim that, more generally, if $q \in L$ then $i_*^{Lq}[c] = 0$. Indeed, choose $p \in K$ and a rectangular prism $P \subset L$ such that $p, q \in P$. Since $p \in K$, $i_*^{Lp}[c] = i_*^{Pp}(i_*^{LP}[c]) = 0$. Also, as noted in the proof of Claim 8.13, $i_*^{Pp} \colon H_m(\mathbb{R}^m|_P) \to H_m(\mathbb{R}^m|_{\{p\}})$ and $i_*^{Pq} \colon H_m(\mathbb{R}^m|_P) \to H_m(\mathbb{R}^m|_{\{q\}})$ are isomorphisms. So the fact that $i_*^{Pp}(i_*^{LP}[c]) = 0$ implies that $i_*^{Lq}[c] = i_*^{Pq}(i_*^{LP}[c]) = 0$.

Thus $[c] \in H_m(\mathbb{R}^m|_L)$ has the property that $i_*^{Lq}[c] = 0$ for all $q \in L$. But this implies that, where $\mathfrak{o}_{\mathbb{R}^m,L}$ is a fundamental class for $\mathbb{R}^m$ at L, $\mathfrak{o}_{\mathbb{R}^m,L} + [c]$ is also a fundamental class for $\mathbb{R}^m$ at L. So by the uniqueness part of Theorem 8.7 applied to $(\mathbb{R}^m, L)$, $[c] = 0$. So since $i_*^{LK}[c] = x$, $x = 0$ and $\mathfrak{o}_{\mathbb{R}^m,K} = \mathfrak{o}'_{\mathbb{R}^m,K}$. □

The theorem now follows from Claims 8.12 and 8.15. □

8.1. **Manifolds with boundary.** For any positive integer m we define the left half-space

$$H^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m | x_1 \le 0\},$$

endowed with its subspace topology inherited from $\mathbb{R}^m$. Recall that a smooth m-dimensional manifold with boundary is a second countable Hausdorff space M equipped with an atlas $\{\phi_\alpha \colon U_\alpha \to \phi_\alpha(U_\alpha)\}$ where each ϕ_α is a homeomorphism from an open subset $U_\alpha \subset M$ to an open subset $\phi_\alpha(U_\alpha) \subset H^m$, where $M = \cup_\alpha U_\alpha$, and where the transition functions $\phi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth. (Here a function $f : U \to V$ where $U, V \subset H^m$ are open subsets is said to be smooth provided that there is an open subset $U' \subset \mathbb{R}^m$ such that $U \subset U'$ and a smooth function $f' \colon U' \to \mathbb{R}^m$ such that $f'|_U = f$.)

Note that, contrary to what one might imagine on grammatical grounds, a manifold with boundary is *not* a type of manifold. Rather, a manifold is a type of manifold with boundary.

We denote

$$\partial H^m = \{(x_1, \dots, x_m) \in H^m | x_1 = 0\}$$

If M is a smooth m-manifold with boundary (with atlas $\{\phi_\alpha\}$ as above) let

$$\partial M = \{x \in M | (\exists \alpha)(\phi_\alpha(x) \in \partial H^m)\}$$

One can see that a homeomorphism between two open subsets $U, V \subset H^m$ maps any point $p \in U \cap \partial H^m$ to ∂H^m (for instance, look at the local homology $H_*(U, U \setminus \{p\}; R)$). Therefore we equivalently have

$$\partial M = \{x \in M | (\forall \alpha)(x \in U_\alpha \Rightarrow \phi_\alpha(x) \in \partial H^m)\}.$$

From this it is easy to see that ∂M is a closed subset of M, and that the maps $\phi_\alpha|_{U_\alpha \cap \partial M} \colon U_\alpha \cap \partial M \to \partial H^m \cong \mathbb{R}^{m-1}$ comprise an atlas which makes ∂M into a smooth $(m-1)$ manifold (without boundary). The subset $int(M) = M \setminus \partial M$ is open in M and is an m-manifold (without boundary).

Smooth maps, diffeomorphisms, and tangent bundles of manifolds with boundary are defined in the obvious way, straightforwardly generalizing the case of manifolds without boundary.

We quote the following standard result:

Theorem 8.16 (Collar neighborhood theorem)**.** [Lee, Theorem 9.25] *If M is a smooth manifold with boundary then there is a diffeomorphism $\Phi \colon (-1, 0] \times \partial M \to U$ where $U \subset M$ is an open neighborhood of ∂M, such that $\Phi(0, x) = x$ for all $x \in \partial M$.*

There is another noteworthy manifold without boundary associated to a manifold with boundary M, called the *double* of M and denoted DM. To construct it, choose a collar neighborhood $\Phi \colon (-1, 0] \times \partial M \to U$ as described above, and also let M^+ denote an additional copy of M, with the general point $x \in M$ corresponding to the point $x^+ \in M^+$. Now define

$$DM = \frac{int(M) \coprod int(M^+) \coprod (-1, 1) \times \partial M}{\begin{array}{c} \text{For } t \in (-1, 0), x \in \partial M, (t, x) \sim \Phi(t, x) \\ \text{For } t \in (0, 1), x \in \partial M, (t, x) \sim \Phi(-t, x)^+ \end{array}}$$

Since $int(M)$, $int(M^+)$, and $(-1, 1) \times \partial M$ are all smooth manifolds and the equivalence relation glues open sets of these smooth manifolds together by diffeomorphisms, DM is a smooth manifold. M and M^+ both appear as closed subsets of DM, with $DM = M \cup M^+$ and $M \cap M^+ = \partial M$.

One can think of DM visually as follows: imagine that there is a mirror along the boundary ∂M of M; then DM is the union of M together with its image M^+ as seen in the mirror.

If M is an oriented manifold with boundary, then one naturally obtains an orientation on ∂M by requiring a collar neighborhood $\Phi \colon (-1, 0] \times \partial M \to U \subset M$ to be orientation-preserving—this is called the "outer normal first" convention, since it means that ∂M is oriented in such a way that, at a point $x \in \partial M$, an oriented basis for $T_x M$ is given by $\{e_1, \ldots, e_m\}$ where e_1 is a tangent vector in $T_x M$ that points outward with respect to ∂M and $\{e_2, \ldots, e_m\}$ is an oriented basis for $T_x \partial M$.

Still assuming M oriented, we can orient the double $DM = M \cup M^+$ by having the orientation on M coincide with the given orientation, while having the orientation on the other copy M^+ of M be opposite to the given orientation on M (corresponding to the fact that the map $(t, x) \mapsto \Phi(-t, x)^+$ by which M^+ is glued to $(-1, 1) \times \partial M$ is orientation-reversing with respect to the given orientation on M).

Now let $R = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ and assume that our smooth m-manifold with boundary M is *compact*, and if $R = \mathbb{Z}$ also assume that M is oriented. Then DM is a compact smooth m-manifold (without boundary), and ∂M is a compact smooth $(m-1)$-manifold (without boundary), and if $R = \mathbb{Z}$ these manifolds are oriented as well. Hence Theorem 8.7 gives fundamental classes

$$[DM] := \mathfrak{o}_{DM,DM} \in H_m(DM; R) \qquad [\partial M] := \mathfrak{o}_{\partial M, \partial M} \in H_{m-1}(\partial M; R)$$

Since a manifold without boundary is not a manifold, Theorem 8.7 does not give us a fundamental class for M itself, but we remedy that with the following:

Definition 8.17. Let M be a compact smooth m-manifold with boundary, and let $R = \mathbb{Z}/2\mathbb{Z}$ or, if M is oriented, let $R = \mathbb{Z}$. Then the *fundamental class of* M, denoted $[M]$, is the element of $H_m(M, \partial M; R)$ which is given as the image of $[DM] \in H_m(DM; R)$ under the following composition of maps:

$$H_m(DM; R) \longrightarrow H_m(DM, M^+; R) \longrightarrow H_m(M, \partial M; R)$$

where the first map is the usual projection-induced map in the long exact sequence of the pair (DM, M^+), and the second map is the inverse of the excision isomorphism.

If $q \in int(M)$, then we have an excision isomorphism $H_*(int(M), int(M) \setminus \{q\}; R) \cong H_*(M, M \setminus \{q\}; R)$, so we can define $\mathfrak{o}_{M,q}$ to be the image of $\mathfrak{o}_{int(M),q}$ under this isomorphism. To partly justify Definition 8.17, we prove:

Proposition 8.18. *If M is as in Definition 8.17 and $q \in int(M)$ then where $i^q \colon (M, \partial M) \to (M, M \setminus \{q\})$ is the inclusion we have $i^q_*[M] = \mathfrak{o}_{M,q}$.*

(In fact, just like with fundamental classes of compact manifolds without boundary, $[M]$ is the *unique* element of $H_m(M, \partial M; R)$ that has this property for all $q \in int(M)$, but we will neither prove nor use this.)

Proof. We have a composition of excision isomorphisms

$$H_*(int(M), int(M) \setminus \{q\}; R) \longrightarrow H_*(M, M \setminus \{q\}; R) \longrightarrow H_*(DM, DM \setminus \{q\}; R) .$$

As observed in the second paragraph of the proof of Claim 8.11, this composition sends $\mathfrak{o}_{int(M),q}$ to $\mathfrak{o}_{DM,q}$. So the definition of $\mathfrak{o}_{M,q}$ is such that $\mathfrak{o}_{M,q}$ maps to $\mathfrak{o}_{DM,q}$ under the inclusion-induced map $H_*(M, M \setminus \{q\}; R) \to H_*(DM, DM \setminus \{q\}; R)$ (and is the unique element with this property, since the map is an isomorphism).

Now there is a commutative diagram of inclusion-induced maps

$$\begin{tikzcd}
H_*(DM, \varnothing; R) \arrow[r] & H_*(DM, M^+; R) \arrow[r] & H_*(DM, DM \setminus \{q\}; R) \\
 & H_*(M, \partial M; R) \arrow[u, "\cong"] \arrow[r, "i^q_*"] & H_*(M, M \setminus \{q\}; R) \arrow[u, "\cong"]
\end{tikzcd}$$

where the vertical maps are isomorphisms by excision.

Now the composition on the top line sends $[DM]$ to $\mathfrak{o}_{DM,q}$ by the defining property of a fundamental class. Meanwhile $[M] \in H_*(M, \partial M; R)$ is by definition the image of $[DM]$ under the composition of the first map on the top line and the inverse of the first vertical map. So $[M]$ is sent to $\mathfrak{o}_{DM,q}$ by the maps in the commutative square. So the fact that the square commutes implies that $i^q_*[M] \in H_*(M, M \setminus \{q\}; R)$ is sent by the second vertical map to $\mathfrak{o}_{M,q}$. So by the first paragraph of the proof we must have $i^q_*[M] = \mathfrak{o}_{M,q}$. □

The following will have significant consequences for the relationship between characteristic classes and manifolds with boundary.

Theorem 8.19. *In the long exact sequence of the pair $(M, \partial M)$:*

$$H_m(M; R) \longrightarrow H_m(M, \partial M; R) \xrightarrow{\partial} H_{m-1}(\partial M; R) \xrightarrow{j_*} H_{m-1}(M; R)$$

we have

$$\partial[M] = [\partial M]$$

Consequently $j_*[\partial M] = 0$ where $j \colon \partial M \to M$ is the inclusion.

(Actually in the case that $R = \mathbb{Z}$ we will just prove that $\partial[M] = \epsilon_m[\partial M]$ for some undetermined $\epsilon_m \in \{-1, 1\}$ that depends on the dimension m but not otherwise on M; this suffices for most applications. The reader can check on the model case of $M = D^m$ where D^m is the closed m-dimensional unit disk that indeed $\epsilon_m = 1$.)

Proof. The fact that $j_*[M] = 0$ clearly follows from the statement that $\partial[M] = [\partial M]$ using the exactness of the long exact sequence of the pair. So we must show that $\partial[M]$ obeys the property that uniquely characterizes the fundamental class $[\partial M]$, namely that for all $p \in \partial M$ we have $i^p_*(\partial[M]) = \mathfrak{o}_{\partial M,p}$ where $i^p_* \colon H_*(\partial M; R) \to H_*(\partial M, \partial M \setminus \{p\}; R)$ is induced by the inclusion $(\partial M, \varnothing) \to (\partial M, \partial M \setminus \{p\})$.

Let B be a closed coordinate ball around p in the smooth manifold ∂M. Then the collar neighborhood theorem identifies the closure C_0 of a neighborhood of p in M with the product $I \times B$ where $I = [-\frac{1}{2}, 0]$ (and where $C_0 \cap \partial M$ is identified with $\{0\} \times B$). We can slightly "smooth the corners" of the cylinder C_0 to obtain a smooth manifold with boundary $C \subset M$, diffeomorphic to the closed m-ball, such that $\partial C \cap \partial M = B$. Note that, along B, the orientations of ∂C and ∂M are the same. Indeed we may choose an orientation-preserving diffeomorphism $F \colon S^{m-1} \to C$ which maps the north pole $N \in S^{m-1}$ to p and which maps the open northern hemisphere $U_N \subset S^{m-1}$ to B. There is then a commutative diagram

$$\begin{tikzcd} H_{m-1}(U_N, U_N \setminus \{N\}; R) \arrow[r, "F_*"] \arrow[d, "\iota_*"] & H_{m-1}(int(B), int(B) \setminus \{p\}; R) \arrow[d, "j^{BC}_*"] \\ H_{m-1}(S^{m-1}, S^{m-1} \setminus \{N\}; R) \arrow[r, "F_*"] & H_{m-1}(\partial C, \partial C \setminus \{p\}; R) \end{tikzcd}$$

where the vertical maps are induced by inclusions; in view of the second paragraph of the proof of Claim 8.11 we have $\iota_* \mathfrak{o}_{U_N,N} = \mathfrak{o}_{S^{m-1},N}$ and $j^{BC}_* \mathfrak{o}_{int(B),p} = \mathfrak{o}_{\partial C,p}$, and since F is orientation-preserving we have $F_* \mathfrak{o}_{U_N,N} = \mathfrak{o}_{int(B),p}$ and $F_* \mathfrak{o}_{S^{m-1},N} = \mathfrak{o}_{\partial C,p}$. Moreover where D^m denotes the m-dimensional closed unit ball (so $\partial D^m = S^{m-1}$), the map $F \colon S^{m-1} \to \partial C$ extends to an orientation-preserving diffeomorphism $\tilde{F} \colon D^m \to C$. If $x \in int(D^m)$ with $F(x) = q \in int(C)$ then $F_* \mathfrak{o}_{D^m,x} = \mathfrak{o}_{C,q}$.

If $q \in int(C)$ one easily sees that the inclusion-induced map $i^{int(C),q}_* \colon H_*(M, M \setminus int(C); R) \to H_*(M, M \setminus \{q\}; R)$ is an isomorphism (for instance by using excision to reduce to the case that $M = \mathbb{R}^m$ and then using arguments similar to those at the start of the proof of Claim 8.13). So since, again for $q \in int(C)$, we have $i^q_*[M] = \mathfrak{o}_{M,q}$ by Proposition 8.18, it follows by splitting $i^q \colon (M, \partial M) \to (M, M \setminus \{q\})$ as a composition

$$(M, \partial M) \xrightarrow{i^{int(C)}} (M, M \setminus int(C)) \xrightarrow{i^{int(C),q}} (M, M \setminus \{q\})$$

that $i^{int(C),p}_*(i^{int(C)}_*[M]) = \mathfrak{o}_{M,q}$, and in particular that $i^{int(C)}_*[M]$ generates $H_m(M, \setminus int(C); R)$ (which is isomorphic to R). Moreover if $x \in D^m$ is such that $\tilde{F}(x) = q$ then in the commutative diagram

$$\begin{tikzcd} H_*(D^m, S^{m-1}; R) \arrow[r, "\tilde{F}_*"] \arrow[d] & H_*(M, M \setminus int(C); R) \arrow[d] \\ H_*(D^m, D^m \setminus \{x\}) \arrow[r, "\tilde{F}_*"] & H_*(M, M \setminus \{q\}; R) \end{tikzcd}$$

we see that all maps are isomorphisms (by excision in the case of the horizontal maps), and that $\tilde{F}_*\mathfrak{o}_{D^m,x} = \mathfrak{o}_{M,q}$. Therefore the class $z \in H_*(D^m, S^{m-1}; R)$ which is sent under the inclusion-induced map $H_*(D^m, S^{m-1}; R) \to H_*(D^m, D^m \backslash \{x\}; R)$ to $\mathfrak{o}_{M,x}$ has the property that $\tilde{F}_* z = i_*^{int(C)}[M] \in H_*(M, M \setminus int(C); R)$. Note that this class z is independent of the manifold M (it also does not depend on the point $x \in int(D^m)$, as can be seen by an argument as in the proof of Claim 8.13).

Now consider the following commutative diagram, where the horizontal maps on the left are all inclusion-induced and those on the right are induced by the composition of an inclusion with $\tilde{F}\colon D^m \cong C$ or $F\colon S^{m-1} \cong \partial C$, and the first row of vertical maps consists of connecting homomorphisms in various long exact sequences of pairs (and where we delete the coefficient ring R from the notation):

$$\begin{tikzcd}
H_m(M, \partial M) \arrow[r, "i_*^{int(C)}"] \arrow[d, "\partial"'] & H_m(M, M \setminus int(C)) \arrow[d, "\partial"'] & H_m(D^m, S^{m-1}) \arrow[l, "\cong"'] \arrow[d, "\partial"'] \\
H_{m-1}(\partial M) \arrow[r] \arrow[d, "i_*^p"'] & H_{m-1}(M \setminus int(C)) \arrow[d] & H_{m-1}(S^{m-1}) \arrow[l] \arrow[d] \\
H_{m-1}(\partial M, \partial M \setminus \{p\}) \arrow[r, "\cong", "\alpha"'] & H_{m-1}(M, M \setminus (int(C) \cup \{p\})) & H_{m-1}(S^{m-1}, S^{m-1} \setminus \{N\}) \arrow[l, "\cong"', "\beta"]
\end{tikzcd}$$

(The indicated isomorphisms are either excision isomorphisms or compositions of excision isomorphisms with the isomorphisms induced by the homeomorphisms $F\colon S^{m-1} \to \partial C$ or $\tilde{F}\colon D^m \to C$.)

Since the fixed class $z \in H_m(D^m, S^{m-1})$ maps to $i_*^{int(C)}[M] \in H_m(M, M \setminus int(C))$, the commutativity of the diagram shows that $z \in H_m(D^m, S^{m-1})$ and $[M] \in H_m(M, \partial M)$ have the same image in the lower center group $H_{m-1}(M, M \setminus (int(C) \cup \{p\}))$. Now both of the vertical maps on the right are isomorphisms if $m > 1$, and if $m = 1$ then their composition is still an isomorphism. Thus in any case the full composition $H_m(D^m, S^{m-1}) \to H_m(M, M \setminus (int(C) \cup \{p\}))$ is an isomorphism. So since z generates $H_m(D^m, S^{m-1}) \cong R$ (and hence the image of z in $H_m(M, M \setminus (int(C) \cup \{p\}))$ generates $H_m(M, M \setminus (int(C) \cup \{p\})) \cong R$), since z and $[M]$ have the same image in $H_m(M, M \setminus (int(C) \cup \{p\}))$, and since α is an isomorphism, this proves that $i_*^p(\partial[M])$ generates $H_{m-1}(\partial M, \partial M \setminus \{p\})$.

In the case that $R = \mathbb{Z}/2\mathbb{Z}$ this completes the proof, since then $\mathfrak{o}_{\partial M,p}$ is the unique generator of $H_{m-1}(\partial M, \partial M \setminus \{p\})$, and so we have shown that $i_*^p(\partial[M])$ is equal to it.

If $R = \mathbb{Z}$, let y denote the image of z in $H_{m-1}(S^{m-1}, S^{m-1} \setminus \{N\})$, so y has the same image as does $i_*^p(\partial[M])$ in $H_{m-1}(M, M \setminus (int(C) \cup \{p\}))$. Since the composition of the two right vertical maps of the above large commutative diagram is an isomorphism, so that y is a generator, we can write $\mathfrak{o}_{S^{m-1},N} = \epsilon_m y$ for some $\epsilon_m \in \{-1, 1\}$ depending only on m and not on M. (In fact $\epsilon_m = 1$ as the reader can check based on our conventions, but we'll omit the proof.) We have a commutative diagram

$$\begin{tikzcd}
 & H_{m-1}(\partial M, \partial M \setminus \{p\}) \arrow[dr, "\alpha"] & \\
H_{m-1}(U_N, U_N \setminus \{N\}) \arrow[ur] \arrow[dr] & & H_{m-1}(M, M \setminus (int(C) \cup \{p\})) \\
 & H_{m-1}(S^{m-1}, S^{m-1} \setminus \{p\}) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where each map is an isomorphism (given by excision together with a map induced by the orientation-preserving diffeomorphism $F\colon (S^{m-1}, U_N) \to (\partial C, B)$). Since the left maps above are obtained by composing inclusions with orientation-preserving diffeomorphisms, they respect the orientation classes: $\mathfrak{o}_{U_N,N}$ is mapped respectively to $\mathfrak{o}_{\partial M,p}$ and $\mathfrak{o}_{S^{m-1},N}$, and hence since the diagram commutes we have $\alpha(\mathfrak{o}_{\partial M,p}) = \beta(\mathfrak{o}_{S^{m-1},N})$.

Meanwhile we saw earlier that $\alpha(i^p_*(\partial[M])) = \beta(y) = \epsilon_m \beta(\mathfrak{o}_{S^{m-1},N})$. So since α is injective it follows that $i^p_*(\partial[M]) = \epsilon_m \mathfrak{o}_{\partial M,p}$ where again $\epsilon_m \in \{-1,1\}$ depends neither on p nor on M. So by the uniqueness of $[\partial M]$ this proves that $\epsilon_m \partial[M] = [\partial M]$. □

9. Characteristic classes and cobordism

We now bring the discussion back to vector bundles and Stiefel–Whitney classes. From the collar neighborhood theorem one can see that if M is a smooth manifold with boundary then the restriction $TM|_{\partial M}$ splits as a direct sum:

$$TM|_{\partial M} \cong \underline{\mathbb{R}} \oplus T(\partial M)$$

where the trivial line bundle $\underline{\mathbb{R}}$ comes from the tangent bundle to the $(-1,0]$ factor in $(-1,0] \times \partial M$. So where $j\colon \partial M \to M$ is the inclusion (so $TM|_{\partial M} = j^*TM$), the total Stiefel–Whitney classes obey

$$j^*w(TM) = w(j^*TM) = w(\underline{\mathbb{R}}) \cup w(T(\partial M)) = w(T(\partial M))$$

Definition 9.1. Let Y be a compact smooth n-manifold and let $I = (i_1,\dots,i_k) \in \mathbb{N}^k$ be a tuple of natural numbers with $i_1 + \cdots + i_k = n$. Then the *Ith Stiefel–Whitney number of Y* is the element of $\mathbb{Z}/2\mathbb{Z}$ given by

$$w_I(Y) = \langle w_{i_1}(TY) \cup \cdots \cup w_{i_k}(TY), [Y] \rangle$$

Given a smooth manifold Y, one might ask whether there is a smooth manifold with boundary M such that $\partial M = Y$. As stated the answer is obviously yes: let $M = [0,1) \times Y$. But if Y is compact and we require M to also be compact then the story is more complicated.

Theorem 9.2. *Let Y be a compact smooth manifold such that there is a compact smooth manifold with boundary M such that $\partial M = Y$. Then all Stiefel–Whitney numbers $w_I(Y)$ are equal to zero.*

Proof. We have seen that for all i we have $w_i(TY) = j^*w_i(TM)$ where $j\colon Y = \partial M \to M$ is the inclusion. So for any $I = (i_1,\dots,i_k)$ with $i_1 + \cdots + i_k = TY$ we have

$$\begin{aligned} \langle w_I(Y) &= \langle w_{i_1}(TY) \cup \cdots \cup w_{i_k}(TY), [Y] \rangle \\ &= \langle (j^*w_{i_1}(TM)) \cup \cdots \cup (j^*w_{i_k}(TM)), [Y] \rangle \\ &= \langle j^*(w_{i_1}(TM) \cup \cdots \cup w_{i_k}(TM)), [Y] \rangle \\ &= \langle w_{i_1}(TM) \cup \cdots \cup w_{i_k}(TM), j_*[Y] \rangle \\ &= 0 \end{aligned}$$

since by Theorem 8.19 $j_*[Y] = 0$.

□

Example 9.3. For a somewhat silly example, let Y be a compact zero-manifold, *i.e.* a finite discrete collection of points. The only Steifel–Whitney number to speak of is $w_{(0)}(Y)$, and the fundamental class $[Y]$ is, as one can easily see, represented by the chain given by the sum of the various points of Y. So since TY, like any vector bundle, has $w_0(TY) = 1$, the Stiefel–Whitney number $w_{(0)}(Y)$ is equal to the number of points of Y, reduced modulo 2.

Consistently with this, it is easy to see that a compact zero-manifold occurs as the boundary of a compact 1-manifold with boundary if and only if the zero-manifold has an even number of points.

Example 9.4. Let $Y = \mathbb{R}P^n$. By Proposition 7.10 we have $w(TY) = (1+e)^{n+1}$ where $H^*(Y;\mathbb{Z}/2\mathbb{Z}) = \frac{\mathbb{Z}/2\mathbb{Z}[e]}{\langle e^{n+1}\rangle}$. Meanwhile $[Y]$ is the unique nonzero element of $H_n(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$, so $\langle e^n, [Y]\rangle = 1$. So for $I = (i_1, \ldots, i_k)$ we have

$$w_I(Y) = \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \mod 2$$

In particular if n is even, so that $\binom{n+1}{n} = n+1$ is odd, then $w_{(n)}(\mathbb{R}P^n) = 1$, proving that $\mathbb{R}P^n$ does not arise as the boundary of a compact manifold when n is even.

On the other hand if n is odd then for any $I = (i_1, \ldots, i_k)$ with $i_1 + \cdots + i_k = n$, some i_j must be odd. But then since $n+1$ is even $\binom{n+1}{i_j}$ is also even (writing $n+1 = 2r$, one has, modulo 2, $(1+x)^{n+1} = (1+x^2)^r$, which has no odd powers of x in its expansion). So for n odd all Stiefel–Whitney numbers of $\mathbb{R}P^n$ are zero. There are indeed various ways of constructing manifolds with boundary equal to $\mathbb{R}P^n$ for any odd n. For instance, writing $n = 2m-1$, the map $p\colon \mathbb{R}P^{2m-1} \to \mathbb{C}P^{m-1}$ that sends a line in $\mathbb{R}^{2m}$ to the complex line in $\mathbb{C}^m$ which it spans is a fiber bundle with fiber $\mathbb{R}P^1 \cong S^1$, and this bundle can be extended to a D^2-bundle $E \to \mathbb{C}P^{m-1}$ such that $\partial E = \mathbb{R}P^{2m-1}$.

Consistent with the above two examples is the following celebrated result of Thom, which gives a converse to Theorem 9.2.

Theorem 9.5. [T, Théorème IV.10] *If Y is a compact smooth manifold such that all Stiefel–Whitney numbers $w_I(Y)$ are zero then there is a compact smooth manifold with boundary M such that $\partial M = Y$.*

The proof of this (which together with related work is a large part of the reason that Thom won a Fields medal in 1958) is beyond the scope of the course. For a partial exposition see [MS, Chapter 18].

Definition 9.6.
- If M_0 and M_1 are two compact smooth m-dimensional manifolds, a *cobordism* between M_0 and M_1 is a compact smooth $(m+1)$-dimensional manifold with boundary W such that ∂W is diffeomorphic to the disjoint union $M_0 \coprod M_1$.
- Two compact smooth m-manifolds M_0 and M_1 are said to be *cobordant* if there exists a cobordism between them (in which case we write $M_0 \sim M_1$).

It should not be difficult for you to check that $\sim$ is an equivalence relation (the Collar Neighborhood Theorem helps with transitivity). If we denote by $\mathfrak{N}_m$ the set of equivalence classes, then $\mathfrak{N}_m$ forms an *abelian group* under the operation of disjoint union: indeed, associativity and commutativity are obvious, the empty manifold serves as an identity, and since $[0,1] \times M$ can be seen as giving a cobordism between $M \coprod M$ and the empty manifold every element of $\mathfrak{N}_m$ is its own inverse. Better still, $\mathfrak{N}_* = \oplus_m \mathfrak{N}_m$ is a commutative ring under Cartesian product, with the one-point manifold serving as multiplicative identity.

Corollary 9.7. *Two compact m-manifolds M_0 and M_1 are cobordant if and only if all of their Stiefel–Whitney numbers are equal.*

Proof. This follows directly from Theorems 9.2 and 9.5 together with the general identity

$$w_I\left(M_0 \coprod M_1\right) = w_I(M_0) + w_I(M_1) \tag{33}$$

Indeed, since w_I takes values in $\mathbb{Z}/2\mathbb{Z}$, if $w_I(M_0) = w_I(M_1)$ then $w_I(M_0 \coprod M_1) = 0$, so by Theorem 9.5 there is a cobordism W between $M_0 \coprod M_1$ and $\varnothing$, which can be equivalently seen as a cobordism between M_0 and M_1. Conversely if $M_0 \sim M_1$ then Theorem 9.2 shows that $w_I(M_0 \coprod M_1) = 0$ for all I and so $w_I(M_0) = w_I(M_1)$ by (33).

To prove (33), let $M = M_0 \coprod M_1$ and for $j = 0, 1$ let $i_j : M_j \to M$ be the inclusion. Evidently we have

$$i_j^* TM = TM_j.$$

If $k: (M, \varnothing) \to (M, M \backslash M_j)$ is the inclusion we see that $k_*(i_j)_*[M_j] = k_*\big((i_0)_*[M_0] + (i_1)_*[M_1]\big)$, and the defining property of M_j implies that for $p \in M_j$ the class $k_*(i_j)_*[M_j]$ is sent to $\mathfrak{o}_{M,p}$ under the inclusion $(M, M \setminus M_j) \to (M, M \setminus \{p\})$. Since this holds for both $j = 0$ and $j = 1$ and since $k_*(i_j)_*[M_j] = k_*\big((i_0)_*[M_0] + (i_1)_*[M_1]\big)$ it follows that $(i_0)_*[M_0] + (i_1)_*[M_1]$ obeys the defining property of the fundamental class. So

$$(i_0)_*[M_0] + (i_1)_*[M_1] = [M].$$

Thus

$$\begin{aligned} w_I(M) &= \langle w_{i_1}(TM) \cup \cdots \cup w_{i_k}(TM), (i_0)_*[M_0] + (i_1)_*[M_1] \rangle \\ &= \Big\langle i_0^*\big(w_{i_1}(TM) \cup \cdots \cup w_{i_k}(TM)\big), [M_0] \Big\rangle + \Big\langle i_0^*\big(w_{i_1}(TM) \cup \cdots \cup w_{i_k}(TM)\big), [M_1] \Big\rangle \\ &= \langle w_{i_1}(TM_0) \cup \cdots \cup w_{i_k}(TM_0), [M_0] \rangle + \langle w_{i_1}(TM_1) \cup \cdots \cup w_{i_k}(TM_1), [M_1] \rangle \\ &= w_I(M_0) + w_I(M_1) \end{aligned}$$

□

Using this together with an easily-proven result describing the behavior of Stiefel–Whitney numbers under Cartesian products (which we leave to the reader to formulate) and some more sophisticated results about properties of Stiefel–Whitney classes, one can in fact precisely determine the whole unoriented cobordism ring $\mathfrak{N}_*$, see [MS, Problem 16-F]. $\mathfrak{N}_*$ turns out to be freely generated as a $(\mathbb{Z}/2\mathbb{Z})$-algebra by the cobordism classes of certain specific n-manifolds Y_n where n varies through natural numbers such that $n+1$ is not a power of two. Thus any manifold at all is cobordant to a disjoint union of products of these standard manifolds. As a special case of this, one can show that any compact 3-manifold has all of its Stiefel–Whitney numbers equal to zero—hence every compact 3-manifold is the boundary of some compact 4-manifold with boundary.

9.1. **Pontryagin classes and oriented cobordism.** In the case that M is a compact oriented smooth m-manifold, we have a fundamental class $[M] \in H_m(M; \mathbb{Z})$, and so if we can associate some characteristic class $a(TM) \in H^m(M; \mathbb{Z})$ to the tangent bundle TM then we would obtain a characteristic number $\langle a(TM), [M] \rangle \in \mathbb{Z}$ similar to the Stiefel–Whitney numbers but now living in $\mathbb{Z}$ rather than in $\mathbb{Z}/2\mathbb{Z}$.

At the moment the only characteristic classes at our disposal that live in integer-coefficient cohomology are the Chern classes, but these are defined only for complex vector bundles. So if we happen to have the structure of a rank-n complex vector bundle on the tangent bundle TM (in which case M is $2n$-dimensional as a real manifold), then for a tuple of natural numbers $I = (i_1, \ldots, i_k)$ with $i_1 + \cdots + i_k = n$ we can form the Chern number

$$c_I(TM) := \langle c_{i_1}(TM) \cup \cdots \cup c_{i_k}(TM), [M] \rangle \in \mathbb{Z}$$

If M is a complex manifold (*i.e.* if M has an atlas of coordinate charts whose transition functions are holomorphic diffeomorphisms between open subsets of $\mathbb{C}^n$) then TM naturally has the structrure of a complex vector bundle; it is also true that a symplectic manifold has a natural isomorphism class of complex bundle structures on its tangent bundle (namely the ones that are "compatible" with the symplectic form in a standard sense). So for a complex or symplectic manifold we have Chern classes and Chern numbers; these in principle depend on the complex or symplectic structure and not just on the smooth structure (or the oriented smooth structure) of the underlying manifold.

But the typical oriented smooth manifold does not have any natural complex structure on its tangent bundle, and so does not have Chern numbers in the above sense. There is however a simple trick that allows us to move into the world of complex vector bundles: we can tensor the tangent bundle with $\mathbb{C}$.

More specifically, for any vector space V over $\mathbb{R}$ we may form a vector space over $\mathbb{C}$ by taking

$$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = \{v_1 + iv_2 | v_1, v_2 \in V\}$$

Of course scalar multiplication in this complex vector space is given by

$$(a+ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2)$$

(So evidently as a special case we have $(\mathbb{R}^n)^{\mathbb{C}} = \mathbb{C}^n$.) We will find it somewhat more notationally convenient to regard $V^{\mathbb{C}}$ as the real vector space $V \oplus V$ together with the complex scalar multiplication operation given by combining the given real scalar multiplication on V with following rule for multiplication by i:

$$i(v_1, v_2) = (-v_2, v_1)$$

Now if $\pi \colon E \to M$ is a real rank-n vector bundle we obtain a new complex rank-n vector bundle $E^{\mathbb{C}}$ (the "complexification" of E): the underlying topological space of $E^{\mathbb{C}}$ is the Whitney sum $E \oplus E$, and on each fiber $E^{\mathbb{C}}_m = E_m \oplus E_m$ we put the complex vector space structure of the previous paragraph: $i(e_1, e_2) = (-e_2, e_1)$. To each local trivialization

$$\begin{aligned} E|_U &\to U \times \mathbb{R}^n \\ e &\mapsto (\pi(e), \phi(e)) \end{aligned}$$

we associate a local complex-vector bundle trivialization for $E^{\mathbb{C}}$ given by

$$\begin{aligned} E^{\mathbb{C}}|_U &\to U \times \mathbb{C}^n \\ (e_1, e_2) &\mapsto (\pi(e_1), \phi(e_1) + i\phi(e_2)) \end{aligned}$$

These trivializations are complex-linear on each fiber, and so $E^{\mathbb{C}}$ does indeed have the structure of a rank-n complex vector bundle.

The Pontryagin classes of E will (modulo a sign convention) be certain Chern classes of $E^{\mathbb{C}}$—these exist for any vector bundle over a paracompact Hausdorff space. Before making a formal definition we will make some observations about the Chern classes of $E^{\mathbb{C}}$.

To set this up, suppose that $\pi \colon W \to M$ is a rank-n *complex* vector bundle, with local trivializations of the shape

$$w \mapsto (\pi(w), \phi(w)).$$

To W we can associate the *conjugate bundle* $\bar{W}$, defined as follows. As a real vector bundle, $\bar{W}$ is equal to W. However for $w \in \bar{W} = W$, the effect of multiplying w by i is opposite: if $i_W w$ and $i_{\bar{W}} w$ denote the result of multiplying w by i when we consider w as an element of W and $\bar{W}$ respectively, then $i_{\bar{W}} w = -i_W w$. Associated to the above local trivializations $w \mapsto (\pi(w), \phi(w))$ for W we have local complex vector bundle trivializations

$$w \mapsto (\pi(w), \overline{\phi(w)})$$

for $\bar{W}$, where of course $\overline{\phi(w)}$ denotes the complex conjugate of $\phi(w) \in \mathbb{C}^n$

Relatedly, if $F\colon W \to \mathbb{C}^N$ is a complex pre-classifying map for W, then the complex conjugate $\bar{F}\colon \bar{W} \to \mathbb{C}^N$ is a complex pre-classifying map for $\bar{W}$.

Proposition 9.8. *If $\pi\colon W \to M$ is a finite-type rank-n complex vector bundle over a paracompact Hausdorff space then $c_i(\bar{W}) = (-1)^i c_i(W)$ for all $i \in \mathbb{N}$.*

Proof. As topological spaces, the complex projective bundles $\mathbb{P}_{\mathbb{C}}(W)$ and $\mathbb{P}_{\mathbb{C}}(\bar{W})$ are identical, as are their bundle projections $\underline{\pi}\colon \mathbb{P}_{\mathbb{C}}(W) = \mathbb{P}_{\mathbb{C}}(\bar{W}) \to M$. Let $F\colon W \to \mathbb{C}^N$ be a complex pre-classifying map for W, so that $\bar{F}\colon \bar{W} \to \mathbb{C}^N$ is a complex pre-classifying map for $\bar{W}$. Then where $z_W = -\mathbb{P}_{\mathbb{C}}(F)^* e(\gamma^1(\mathbb{C}^N))$ and $z_{\bar{W}} = -\mathbb{P}_{\mathbb{C}}(\bar{F})^* e(\gamma^1(\mathbb{C}^N))$, the Chern classes of W and $\bar{W}$ determined by the relations $c_0(W) = c_0(\bar{W}) = 1$ and

$$\sum_{i=0}^{n} \underline{\pi}^* c_i(W) \cup z_W^{n-i} = 0 \qquad \sum_{i=0}^{n} \underline{\pi}^* c_i(\bar{W}) \cup z_{\bar{W}}^{n-i} = 0$$

Define $b\colon \mathbb{C}P^{N-1} \to \mathbb{C}P^{N-1}$ by $b([x_0 : \cdots : x_{N-1}]) = [\bar{x}_0 : \cdots : \bar{x}_{N-1}]$. Consider the restrictions of b and $e(\gamma^1(\mathbb{C}^N))$ to the standard copy of $S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^{N-1}$ (namely $\mathbb{C}P^1 = \{[x_0 : x_1 : 0 : \cdots : 0]\}$). We see that b restricts to this copy of S^2 as a reflection and hence acts with degree -1. So since the restriction $H^2(\mathbb{C}P^{N-1};\mathbb{Z}) \to H^2(\mathbb{C}P^1;\mathbb{Z})$ is an isomorphism with image generated by the restriction of $e(\gamma^1(\mathbb{C}^N))$, it follows that $b^* e(\gamma^1(\mathbb{C}^N)) = -e(\gamma^1(\mathbb{C}^N))$.

Now we have $\mathbb{P}_{\mathbb{C}}(\bar{F}) = b \circ \mathbb{P}_{\mathbb{C}}(F)$, so it follows from the definitions of z_W and $z_{\bar{W}}$ that $z_{\bar{W}} = -z_W$. So the defining relation for the Chern classes of $\bar{W}$ gives

$$0 = \sum_{i=0}^{n} (-1)^{n-i} c_i(\bar{W}) \cup z_W^{n-i} = (-1)^n \sum_{i=0}^{n} (-1)^i c_i(\bar{W}) \cup z_W^{n-i}$$

But these relations are also satisfied with $c_i(W)$ in place of $(-1)^i c_i(\bar{W})$, so it follows that indeed $c_i(\bar{W}) = (-1)^i c_i(W)$. □

In particular this shows that typically it is not the case that a complex vector bundle is isomorphic to its conjugate bundle (rather, one can use a Hermitian metric on the bundle to show that the conjugate bundle is isomorphic to the *dual* $Hom(\gamma, \underline{\mathbb{C}})$ of the original vector bundle). However those complex vector bundles that arise as complexifications of real bundles are exceptions to this:

Proposition 9.9. *For any real vector bundle $\pi\colon E \to M$ there is an isomorphism of complex vector bundles $E^{\mathbb{C}} \cong \overline{E^{\mathbb{C}}}$.*

Proof. Viewing the underlying real vector bundle of $E^{\mathbb{C}}$ as $E \oplus E$ define $G\colon E^{\mathbb{C}} \to \overline{E^{\mathbb{C}}}$ by $G(e_1, e_2) = (e_1, -e_2)$. This is obviously an isomorphism of real vector bundles, so to see that it is an isomorphism of complex vector bundles we have to see that it intertwines the operations of multiplication by i in the domain and range. We have

$$G(i_{E^{\mathbb{C}}}(e_1, e_2)) = G(-e_2, e_1) = (-e_2, -e_1)$$

and similarly

$$i_{\overline{E^{\mathbb{C}}}} G(e_1, e_2) = i_{\overline{E^{\mathbb{C}}}}(e_1, -e_2) = -i_{E^{\mathbb{C}}}(e_1, -e_2) = -(e_2, e_1),$$

so indeed $G(i_{E^{\mathbb{C}}}(e_1, e_2)) = i_{\overline{E^{\mathbb{C}}}} G(e_1, e_2)$. □

Corollary 9.10. *For any finite-type real vector bundle E over a paracompact Hausdorff space we have $2c_i(E^{\mathbb{C}}) = 0$ for all odd i.*

Proof. For odd i we have $c_i(E^{\mathbb{C}}) = -c_i(\overline{E^{\mathbb{C}}}) = -c_i(E^{\mathbb{C}})$, where the first identity follows from Proposition 9.8 and the second from Proposition 9.9 □

So there is not much interesting information to be obtained from the odd Chern classes of $E^{\mathbb{C}}$ (in principle there might be a little bit, since they can be nontrivial elements of $H^{2i}(M;\mathbb{Z})$ with order two). Accordingly we ignore them and define:

Definition 9.11. If $\pi\colon E \to M$ is any finite-type real vector bundle over a paracompact Hausdorff space M, the *Pontryagin classes* of E, denoted $p_i(E)$, are the classes

$$p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(M;\mathbb{Z})$$

The total Pontryagin class is $p(E) = \sum_i p_i(E)$.

Nothing significant should be read into the factor of $(-1)^i$; it is just a normalization factor which makes some other formulas work better.

It is straightforward to see that an isomorphism of vector bundles induces a complex isomorphism of their complexifications, and that the complexification of the pullback of a vector bundle is isomorphic as a complex vector bundle to the pullback of the complexification, in view of which the naturality of the Chern classes immediately implies that Pontryagin classes enjoy the naturality property $f^*p_i(E) = p_i(f^*E)$. Obviously $p_0(E) = 1$, and if E has real rank n (so that $E^{\mathbb{C}}$ has complex rank n) we have $p_i(E) = 0$ for $i > n/2$. There is also something that serves well enough as a Whitney sum formula, though we have to adjust for the two-torsion information that is neglected by the Pontryagin classes:

Proposition 9.12. *For two vector bundles $E_0, E_1 \to M$ we have $2p(E_0 \oplus E_1) = 2p(E_0) \cup p(E_1)$*

Proof. For any k, using the Whitney sum formula for Chern classes and Corollary 9.10, as well as the obvious isomorphism $(E_0 \oplus E_1)^{\mathbb{C}} \cong E_0^{\mathbb{C}} \oplus E_1^{\mathbb{C}}$,

$$\begin{aligned} p_k(E_0 \oplus E_1) &= (-1)^k c_{2k}(E_0^{\mathbb{C}} \oplus E_1^{\mathbb{C}}) \\ &= (-1)^k \sum_{l+m=2k} c_l(E_0^{\mathbb{C}}) \cup c_m(E_1^{\mathbb{C}}) \\ &= (-1)^k \sum_{i+j=k} c_{2i}(E_0^{\mathbb{C}}) \cup c_{2j}(E_1^{\mathbb{C}}) + (\text{elements of order } 2) \\ &= (-1)^k \sum_{i+j=k} (-1)^{i+j} p_i(E_0) \cup p_j(E_1) + (\text{elements of order } 2) \end{aligned}$$

and so

$$2p_k(E_0 \oplus E_1) = 2 \sum_{i+j=k} p_i(E_0) \cup p_j(E_1).$$

Since this holds for all k the result follows. □

Definition 9.13. Let M be a compact oriented smooth $4n$-dimensional manifold and let $I = (i_1, \dots, i_k)$ be a tuple of numbers with $i_1 + \dots + i_k = n$. Then the *Ith Pontryagin number* of M is the integer

$$p_I(M) = \langle p_{i_1}(TM) \cup \dots \cup p_{i_k}(TM), [M] \rangle$$

For any oriented manifold M let $-M$ denote the same smooth manifold with the opposite orientation. It is not hard to check that $[-M] = -[M]$, so since reversing the orientation of M does not affect the Pontryagin numbers of TM it follows that $p_I(-M) = -p_I(M)$. In particular if there is a diffeomorphism $\phi\colon M \to M$ which reverses the orientation of M then all Pontryagin numbers of M vanish.

Here is another situation in which the Pontryagin numbers vanish:

Theorem 9.14. *Suppose that N is a compact smooth oriented $(4n+1)$-manifold, with $\partial N = M$. Then $p_I(M) = 0$ for all I.*

Proof. The proof is of course similar to that of the corresponding statement for Stiefel–Whitney numbers. The assumption implies that where $j\colon M \to N$ is the inclusion we have an isomorphism of oriented bundles $j^*TN \cong \underline{\mathbb{R}} \oplus TM$. So since $p(\underline{\mathbb{R}}) = 1$, it follows from Proposition 9.12 that, for all i, $2p_i(TM) = 2j^*p_i(TN)$. So for $I = (i_1, \dots, i_k)$,

$$\begin{aligned}
2^k p_I(TM) &= \langle (2p_{i_1}(TM)) \cup \cdots \cup (2p_{i_k}(TM)), [M] \rangle \\
&= 2^k \langle j^*(p_{i_1}(TN) \cup \cdots \cup p_{i_k}(TN)), [M] \rangle \\
&= 2^k \langle p_{i_1}(TN) \cup \cdots \cup p_{i_k}(TN), j_*[M] \rangle
\end{aligned}$$

But $j_*[M] = 0$ by Theorem 8.19, so $2^k p_I(M) = 0$, *i.e.* (since we are working in $\mathbb{Z}$, in which 2^k is not a zero-divisor) $p_I(M) = 0$. □

Two oriented compact smooth manifolds M_0 and M_1 are said to be oriented-cobordant if there is an oriented compact smooth manifold W with boundary such that ∂W is diffeomorphic as an oriented manifold to $(-M_0) \coprod M_1$. This is an equivalence relation, and just as in the unoriented case the equivalence classes of oriented n-manifolds form an abelian group Ω_n under disjoint union. Theorem 9.14 and the same argument as in the proof of Corollary 9.7 show that two oriented-cobordant manifolds necessarily have the same Pontryagin numbers. Thom proved a near-converse to Theorem 9.14 ([T, Corollaire IV.16]), to the effect that if M_0 and M_1 have the same Pontryagin numbers then there is a positive integer k such that the disjoint union of k copies of M_0 is cobordant to the disjoint union of k copies of M_1.

To compute Pontryagin classes $p_i(E)$ for some particular bundles E we will first see how they can be expressed in terms of the Chern classes of E in the special case that E is a already a complex vector bundle. First we observe:

Proposition 9.15. *If E is a complex vector bundle then $E^{\mathbb{C}}$ is isomorphic as a complex vector bundle to $E \oplus \bar{E}$.*

Proof. Identifying $E^{\mathbb{C}}$ as a real vector bundle with $E \oplus E$ with the complex structure given as usual by $i(e_1, e_2) = (-e_2, e_1)$, we may define $G\colon E^{\mathbb{C}} \to E \oplus \bar{E}$ by $G(e_1, e_2) = (e_1 + ie_2, e_1 - ie_2)$ (where i denotes multiplication by i in e, which of course is opposite to multiplication by i in $\bar{E}$). This is obviously an isomorphism of real vector bundles, and is easily seen to be complex linear on each fiber. □

Now let us compute the Pontryagin classes of the tangent bundle of the complex manifold $\mathbb{C}P^n$. Arguing in the same way as in the case of $\mathbb{R}P^n$ one checks that there is an isomorphism of complex vector bundles $T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong Hom(\gamma, \underline{\mathbb{C}})^{n+1}$ where $\gamma = \gamma^1(\mathbb{C}^{n+1})$ is the tautological bundle, with $c(\gamma) = 1 + e$ where $H^*(\mathbb{C}P^n; \mathbb{Z}) = \frac{\mathbb{Z}[e]}{\langle e^{n+1} \rangle}$. Now a Hermitian metric on γ yields an isomorphism $Hom(\gamma, \underline{\mathbb{C}}) \cong \bar{\gamma}$. So $T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong \bar{\gamma}^{\oplus(n+1)}$, whence $c(T\mathbb{C}P^n) = (1-e)^{n+1}$. Likewise $\overline{T\mathbb{C}P^n} \oplus \underline{\mathbb{C}} \cong \gamma^{\oplus(n+1)}$, so $c(\overline{T\mathbb{C}P^n}) = (1+e)^{n+1}$. Thus by Proposition 9.15,

$$c\left((T\mathbb{C}P^n)^{\mathbb{C}}\right) = c(T\mathbb{C}P^n) \cup c(\overline{T\mathbb{C}P^n}) = (1-e^2)^{n+1}.$$

So

$$p_i(T\mathbb{C}P^n) = (-1)^i c_{2i}\left((T\mathbb{C}P^n)^{\mathbb{C}}\right) = \binom{n+1}{i} e^{2i}.$$

Equivalently, the total Pontryagin class of $T\mathbb{C}P^n$ is $p(T\mathbb{C}P^n) = (1+e^2)^{n+1}$.

Corollary 9.16. *For each nonnegative integer k, $\mathbb{C}P^{2k}$ is not the boundary of any compact oriented smooth manifold.*

Proof. It follows from what we have just done that the top Pontryagin class $p_k(T\mathbb{C}P^{2k})$ is equal to $(2k+1)e^{2k}$. Since e^{2k} generates $H^{4k}(\mathbb{C}P^{2k};\mathbb{Z})$, it evaluates as ± 1 on $[\mathbb{C}P^{2k}]$. Thus $p_{(k)}(\mathbb{C}P^{2k}) = \pm(2k+1) \neq 0$ and the conclusion follows from Theorem 9.14. □

Aside from applications to cobordism, Pontryagin classes have applications to other topological questions. Here is one:

Theorem 9.17. *For all positive integers k the tangent bundle TS^{4k} is not isomorphic to any complex vector bundle. In particular, S^{4k} is not diffeomorphic to a complex manifold.*

Remark 9.18. Of course S^2 is diffeomorphic to the complex manifold $\mathbb{C}P^1$. It turns out that the only values of m for which TS^m is isomorphic to a complex vector bundle are $m = 2$ and 6. It is still an open question whether S^6 is diffeomorphic to a complex manifold.

Proof. Write $E = TS^{4k}$. The Pontryagin numbers of S^{4k} vanish because it is the boundary of the $(4k+1)$-ball (or alternatively because it admits an orientation-reversing diffeomorphism). In particular $p_{(k)}(S^{4k}) = 0$, and hence $p_k(E) = 0$, *i.e.* $c_{2k}(E^{\mathbb{C}}) = 0$. Of course $c_j(E^{\mathbb{C}}) = 0$ for all $j > 0$ other than $j = 2k$ also, since $H^{2j}(S^{4k};\mathbb{Z}) = \{0\}$ for such j. Thus $E^{\mathbb{C}}$ has total Chern class $c(E^{\mathbb{C}}) = 1$.

If E were (isomorphic to) a complex vector bundle then we would have $E^{\mathbb{C}} \cong E \oplus \bar{E}$ by Proposition 9.15 and hence

$$\begin{aligned} 1 &= c(E \oplus \bar{E}) = c(E) \cup c(\bar{E}) = (1 + c_{2k}(E)) \cup (1 + c_{2k}(\bar{E})) \\ &= (1 + c_{2k}(E)) \cup (1 + c_{2k}(E)) = 1 + 2c_{2k}(E) \end{aligned}$$

where we have used Proposition 9.8 as well as the fact that $H^j(S^{4k};\mathbb{Z}) = \{0\}$ for $j \notin \{0, 4k\}$. Thus we would have $c_{2k}(E) = 0$. But $c_{2k}(E)$ is the top Chern class of E, so by Theorem 6.25 it is equal to the Euler class of $E = TS^{4k}$. So if TS^{4k} were isomorphic to a complex vector bundle then its Euler class would be zero. The proof of the Theorem will thus be complete if we show that TS^{4k} has nonzero Euler class.

It is a general fact that for a compact oriented smooth manifold M the quantity $\langle e(TM), [M]\rangle$ is equal to the Euler characteristic of M, from which the conclusion follows since S^{4k} has Euler characteristic 2. But let us prove that $e(TS^{4k}) \neq 0$ (and more generally that $e(TS^n) \neq 0$ for n even) independently of this general result.

Again write $E = TS^n$, so we can identify E with $\{(\vec{x}, \vec{v}) \in S^n \times \mathbb{R}^{n+1} | \vec{x} \cdot \vec{v} = 0\}$. By stereographic projection of the second factor onto the sphere, we can identify E with $\{(\vec{x}, \vec{y}) \in S^n \times S^n | \vec{y} \neq -\vec{x}\}$. Under this identification the complement of the zero-section is given by

$$E^0 = \{(\vec{x}, \vec{y}) | \vec{x} \neq \vec{y} \neq -\vec{x}\}.$$

In other words, where

$$\Delta = \{\vec{x}, \vec{x})\} \subset S^n \times S^n \qquad Z = \{(\vec{x}, -\vec{x})\} \subset S^n \times S^n,$$

we have

$$(E, E^0) \cong (S^n \times S^n \setminus Z, S^n \times S^n \setminus (\Delta \cup Z))$$

We have a commutative diagram

$$\begin{tikzcd}
 & H^n(E,E^0;\mathbb{Z}) \arrow[r, "\pi^*"] & H^n(E;\mathbb{Z}) & \\
 & H^n(S^n\times S^n, S^n\times S^n\setminus\Delta;\mathbb{Z}) \arrow[u, "j^*"] \arrow[r, "i^*"] & H^n(S^n\times S^n;\mathbb{Z}) \arrow[u, "k^*"] \arrow[r] & H^n(S^n\times S^n\setminus\Delta;\mathbb{Z})
\end{tikzcd}$$

where j^* is an isomorphism by excision and the bottom row is exact. The desired Euler class $e(E)$ is the image of the Thom class $\tau \in H^n(E,E^0;\mathbb{Z}) \cong \mathbb{Z}$ (which is a generator of that group) under the composition of π^* with the isomorphism $H^*(E;\mathbb{Z}) \to H^*(S^n;\mathbb{Z})$ given by restriction to the zero-section. In particular $e(E) \neq 0$ if and only if $k^* \circ i^* \neq 0$, which by the exactness of the bottom row holds if and only if the restriction of k^* to the kernel of the inclusion-induced map $H^*(S^n \times S^n;\mathbb{Z}) \to H^*(S^n \times S^n \setminus \Delta;\mathbb{Z})$ is nonzero.

Now $k\colon E \to S^n \times S^n$ embeds E as $S^n \times S^n \setminus Z$. Moreover $S^n \times S^n \setminus Z$ deformation retracts to Δ, while $S^n \times S^n \setminus \Delta$ deformation retracts to Z. So after composing with the cohomology isomorphisms induced by these deformation retractions, we see that $e(E) \neq 0$ if and only if there is a class $\alpha \in H^*(S^n \times S^n;\mathbb{Z})$ such that $\alpha|_Z = 0$ but $\alpha|_\Delta \neq 0$. There is indeed such a class when n is even: Let $\pi_1, \pi_2\colon S^n \times S^n \to S^n$ be the projections onto the first and second factors and let ω be a generator for $H^n(S^n;\mathbb{Z})$. Now put $\alpha = \pi_1^*\omega + \pi_2^*\omega$. Where $f : S^n \to Z$ is the embedding $\vec{x} \mapsto (\vec{x}, -\vec{x})$ and $A\colon S^n \to S^n$ is the antipodal map, we see that $\pi_1 \circ f = 1_{S^n}$ while $\pi_2 \circ f = A$, so that $f^*\alpha = \omega + A^*\omega = 0$ when n is even since in this case A has degree -1. So since f is a diffeomorphism to Z this proves that $\alpha|_Z = 0$. Meanwhile where $g\colon S^n \to \Delta$ is the embedding $\vec{x} \mapsto (\vec{x}, \vec{x})$ we see that $g^*\alpha = 2\omega \neq 0$, so that $\alpha|_\Delta \neq 0$, completing the proof. □

References

[BT] R. Bott and L. Tu. *Differential forms in algebraic topology*. Graduate Texts in Mathematics, **82**. Springer-Verlag, New York, 1982.

[FP] R. Fritsch and R. Piccinini. *Cellular structures in topology* Cambridge University Press, 1990.

[H] A. Hatcher. *Algebraic topology*

[Hi] M. Hirsch. *Differential topology* Graduate Texts in Mathematics, **61**. Springer-Verlag, New York, 1976.

[La] S. Lang. *Algebra*, revised 3rd edition. Graduate Texts in Mathematics, **211**. Springer-Verlag, New York, 2002.

[Lee] J. Lee. *Introduction to smooth manifolds*. Graduate Texts in Mathematics, **218**. Springer-Verlag, New York, 2003.

[MS] J. Milnor and J. Stasheff. *Characteristic Classes*. Annals of Mathematics Studies, Princeton University Press, 1974.

[T] R. Thom. *Quelques propriétés globales des variétés différentiables*. Comment. Math. Helv. **28**, (1954), 17–86.

[U1] M. Usher. *Math 8210, Fall 2011 Lecture Notes*, part 1.

[U2] M. Usher. *Math 8210, Fall 2011 Lecture Notes*, part 2.