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Financial modeling and quantum mathematics

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ABSTRACT

Financial instruments have a random evolution and can be described by a stochastic process. It is shown that another approach for modeling financial instruments – considered as a (classical) random system – is by employing the mathematics that results from the formalism of quantum mechanics. Financial instruments are described by the elements of a linear vector state space and its evolution is determined by a Hamiltonian operator. It is further shown that interest rates can be described by a random function – which is mathematically equivalent to a two dimensional Euclidean quantum field.

1. Financial modeling

Financial instruments are pieces of paper that represent ownership over real economic assets. Assets strongly depend on time, with their value changing randomly due to uncontrollable exogenous and endogenous factors. Financial engineering consists of designing financial instruments and modeling their time evolution. Financial instruments cover a vast range of products; in particular, *financial derivatives*, derived from underlying financial instruments, provide a fertile ground for mathematical modeling. In this paper, the following problems in financial engineering are discussed.

- Options on stocks and foreign exchange.
- Modeling the interest rates yield curve.
- Pricing options on interest rates.

Global derivatives market

Fig. 1(a) gives, for 2011, the breakdown – into the equity, foreign exchange and debt markets – of the global derivatives market's notional amount, which is a staggering $647 trillion. Fig. 1(b) gives the breakdown of the international derivatives markets, with the total number of contracts being 25.21 billion. Fig. 1(c) gives the breakdown of the interest rate derivatives market, which accounted for about 70% of the global derivatives market, with a notional value of $473 trillion [1].

Quantum mathematics and financial instruments

Quantum mechanics and quantum field theory – bedrocks of theoretical physics and of modern technology – provide a vast range of powerful mathematical tools and are examples of the application of quantum mathematics to Physics.

Quantum mathematics follows from the general approach of quantum mechanics and describes an entity using the concept of a *degree of freedom* that carries the superstructure of a linear vector *state space*; the dynamics of the degree of freedom is determined by *operators* acting on this state space; the *expectation values* of random quantities – which are functions of the degree of freedom – can be obtained by a Feynman path integral (functional integration), that entails summing over all possible random evolutions of the degree of freedom [2].

It is worth noting that quantum mathematics is a discipline that extends far beyond the domain of quantum mechanics. A case in point is the theory of second order phase transitions that is based on classical statistical mechanics. Wilson used a

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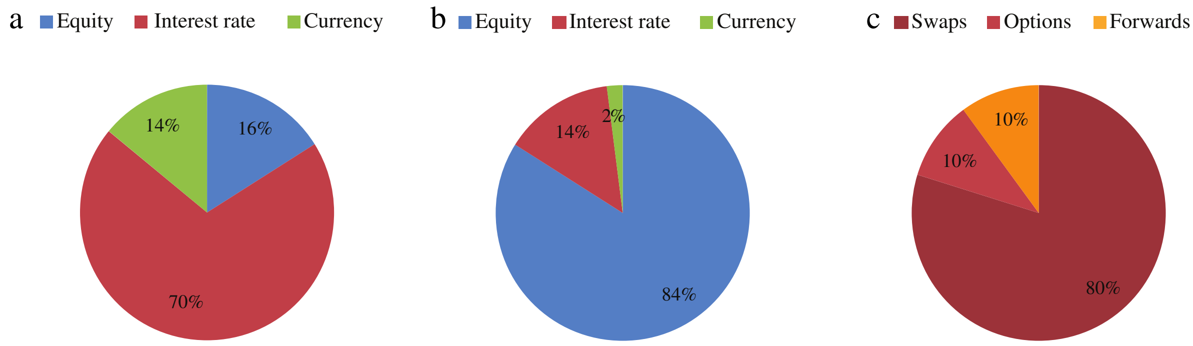

Fig. 1. Financial markets in 2011. (a) Notional amounts of global derivative markets. (b) Volume of global derivative markets. (c) Notional amounts of interest rate derivatives.

model of quantum field theory in Euclidean time to solve the problem of phase transitions and for which he was given the Nobel Prize in 1981 [3]. If one limits quantum mathematics to quantum mechanics, then one may ask questions such as 'is probability conserved in phase transitions?' Such questions are clearly irrelevant since there is no concept of time evolution in phase transitions.

In summary, quantum mathematics can have an interpretation that is very different from quantum mechanics when applied to other disciplines like finance or other classical random systems; the interpretation of the symbols of quantum mathematics in such disciplines have no fixed prescription, but rather have to be arrived at, from first principles.

The future is uncertain and results in the random evolution of financial instruments. The randomness in finance is entirely classical and one may question as to why should quantum mathematics be at all applicable in describing it. Consider a classical macroscopic system that has infinitely many degrees of freedom and which undergoes a classical phase transition. It has been shown that these classical systems are accurately described by quantum field theory. In fact, renormalizable quantum field theories are known to be mathematically equivalent to classical systems that undergo second order phase transitions. With this example in mind, quantum mathematics is applied to the modeling of financial instruments; the models are empirically tested by comparing the models prediction with the market prices of the instrument in question.

Quantum mechanics describes the evolution of classical random systems via the Fokker–Planck Hamiltonian. In the language of physics, the random evolution of a classical system is dissipative and does not conserve energy. This is one of the reasons why the evolution is not unitary. The application of the theory and mathematics of quantum mechanics and quantum field theory to finance is referred to as quantum finance [4,5].

Stochastic calculus currently completely dominates theoretical finance; the quantum formulation of random process in finance is independent of stochastic calculus. In particular, quantum mechanics provides a flexible and powerful framework for modeling options on equities and foreign exchange. Quantum field theory, in turn, provides an efficient and useful framework for modeling interest rates that have imperfect and nontrivial correlations in future time.

2. Equity option $C(S, K, t)$ and quantum mechanics

Options are financial instruments that are derived from underlying quantities such as the stock S of a company [6]. The seller of a call option is obliged to provide the buyer of the option the stock of a company S at some pre-determined price K and at some fixed future time T. The buyer of the option, on the other hand has the right, but not an obligation, to exercise the option at time T.

Payoff function of a call option

The payoff function $g(S)$ is the value of the option $C(S, K, t)$ when it matures, at future time T, and hence is equal to $C(S, K, T)$. The payoff function $g(S)$, given in Fig. 2, has the following properties. If $S < K$ then clearly a call option is valueless since the buyer can buy the security at a lower price from the market. If, however, $S > K$, then the value of the option is $S - K$ and the buyer makes a profit by exercising the call option. Hence, the call option payoff is given by

$$C(S, K, T) \equiv g(S) = [S - K]_+ = \begin{cases} S - K, & S > K \\ 0, & S < K. \end{cases}$$

The fundamental problem in option theory is to find the present value of the option, namely $C(S, K, t)$ $(t < T)$ – shown by dashed line in Fig. 2 – given its value at future time T, namely $C(S, K, T) = g(S)$: namely, a final value problem.

Stochastic formulation

To determine the price of the option $C(S, K, t)$, we need to know how the stock price $S(t)$ evolves in time. In mathematical finance $S(t)$ is assumed to evolve via the following stochastic differential equation

$$\frac{dS}{dt} = \alpha S + \sigma SR \tag{1}$$

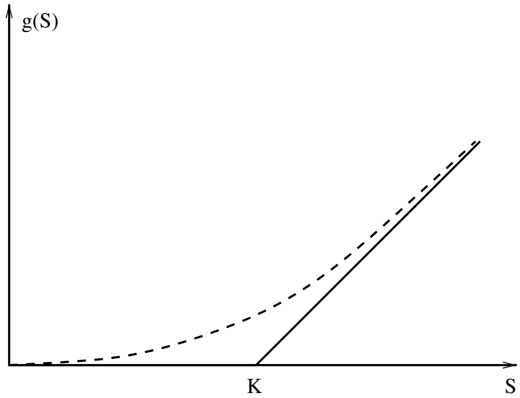

Fig. 2. Payoff function for a call option. The dashed line is the option price $C(S, K, t)$ at some time t before maturity.

where R is white noise. σ is the stock price's volatility and measures the degree of randomness in the evolution of the stock price. The stochastic equation yields the option price based on the requirement that the price is free from opportunities for arbitrage opportunities (options have a single price).

Quantum mechanical formulation

Assume that the option price $C(S, K, t)$ is an element of a state space analogous to the wave function of quantum mechanics. However, unlike the wave function, the state function $C(S, K, t)$ is directly observable. Since the stock price is never negative $S = e^x > 0$. In the Dirac state space notation, the option price and payoff are

$$C(S, K, t) = \langle x|C, t\rangle; \qquad g(S) = \langle x|g\rangle = \langle x|C, T\rangle.$$

Note that, unlike the case of quantum mechanics, the state space is larger than a normed Hilbert space—since the norm of the instrument $S = e^x$ is infinite. The option price satisfies a Schrodinger-like equation, with Hamiltonian H evolving the option price

$$H|C, t\rangle = \frac{\partial}{\partial t}|C, t\rangle; \qquad |C, t\rangle = e^{tH}|C, t\rangle.$$

Since $|C, T\rangle = |g\rangle$, for remaining time $\tau = T - t$, the option price is

$$|C, t\rangle = e^{-(T-t)H}|g\rangle = e^{-\tau H}|g\rangle \Rightarrow \langle x|C, t\rangle = \langle x|e^{-\tau H}|g\rangle. \tag{2}$$

Unlike the case of quantum mechanics, the time evolution of the financial instrument does not conserve probability due to the decaying nature of $e^{-\tau H}$; this poses no problem in financial modeling since there is no need to make a probabilistic interpretation of the price of a financial instrument.

Option pricing Hamiltonians

The famous Black–Scholes pricing formula for equity options follows from the stochastic differential equation given in Eq. (1) and imposing a martingale condition on the option price. It can be shown that the Black–Scholes result is given by the following linear Hamiltonian

$$H_{\mathrm{BS}} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r \neq H_{\mathrm{BS}}^{\dagger}. \tag{3}$$

If one assumes stochastic volatility, namely that $\sigma^2 = e^y$ is random, option pricing is then driven by the nonlinear Merton–Garman Hamiltonian

$$\begin{aligned} H_{\mathrm{MG}} = & -\frac{e^y}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{e^y}{2} - r\right)\frac{\partial}{\partial x} - \left(\lambda e^{-y} + \mu - \frac{\xi^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial}{\partial y} \\ & - \rho\xi e^{y(\alpha-1/2)}\frac{\partial^2}{\partial x \partial y} - \frac{\xi^2 e^{2y(\alpha-1)}}{2}\frac{\partial^2}{\partial y^2} + r \neq H_{\mathrm{MG}}^{\dagger}. \end{aligned} \tag{4}$$

Martingale and Hamiltonians

A fundamental theorem of mathematical finance is that option price is free from arbitrage opportunities if the underlying random process evolving the security is a martingale. For security S, the Hamiltonian formulation of martingale requires that

$$H_{\mathrm{BS}}|S\rangle = 0.$$

For example, for the Black–Scholes Hamiltonian, the explicit expression for the martingale condition is given by

$$H_{\mathrm{BS}}S(x) = \langle x|H_{\mathrm{BS}}|S\rangle = \left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r\right]e^x$$
$$= \left[-\frac{\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right) + r\right]e^x = 0.$$

In an efficient market, the condition that risk free financial instruments undergo a martingale evolution results, due to fundamental theorem of finance, in arbitrage free pricing. The martingale condition in turn requires a drift term that makes the Hamiltonian non-Hermitian.

In summary, all Hamiltonians in finance are non-Hermitian to ensure that option prices are free from arbitrage opportunities.

Option pricing kernel

The completeness equation (resolution of identity) of the state space is given by

$$\mathcal{I} = \int_{-\infty}^{\infty} dx|x\rangle\langle x|$$

and yields, from Eq. (2) the following

$$C(S, K, t) = \langle x|C, t\rangle = \int_{-\infty}^{\infty} dx'\langle x|e^{-(T-t)H}|x'\rangle\langle x'|g\rangle.$$

The pricing kernel is given by

$$p(x, x'; T, t) = \langle x|e^{-(T-t)H}|x'\rangle.$$

The conditional probability that the degree of freedom has value a x at time T, given that it had a value of x' at earlier time t, is the following

$$\frac{p(x, x'; T, t)}{\int dxp(x, x'; T, t)}.$$

In contrast to classical random systems, in quantum mechanics the conditional probability is proportional to $|\langle x|e^{-itH/\hbar}|x'\rangle|^2$.

The pricing kernel can be written as an N-fold product of $e^{-\epsilon H}$, where $\epsilon = \tau/N$, in the following manner

$$p(x, x'; \tau) = \lim_{N\to\infty}\langle x|\left[e^{-\epsilon H}\right]^N|x'\rangle$$
$$= \lim_{N\to\infty}\langle x|e^{-\epsilon H}\cdots e^{-\epsilon H}|x'\rangle. \quad (5)$$

The limit of $N \to \infty$ is required for obtaining the pricing kernel for continuous (remaining time) τ.

Option Lagrangian and action

Inserting the completeness equation $(N-1)$ times in Eq. (5) for the pricing kernel yields

$$p(x, x'; \tau) = \left(\prod_{i=1}^{N-1}\int dx_i\right)\prod_{i=1}^{N}\langle x_i|e^{-\epsilon H}|x_{i-1}\rangle \quad (6)$$

with boundary conditions

$$x_N = x, \quad x_0 = x'.$$

The *Lagrangian* $\mathcal{L}$ for the system is defined by the Dirac–Feynman formula

$$\langle x_i|e^{-\epsilon H}|x_{i-1}\rangle \equiv \mathcal{N}_i(\epsilon)e^{\epsilon\mathcal{L}(x_i;x_{i-1};\epsilon)}$$

and the action S is given by

$$S = \epsilon\sum_{i=1}^{N}\mathcal{L}(x_i; x_{i-1}; \epsilon) \to \int_0^{\tau} dt\,\mathcal{L}(x, dx/dt).$$

The pricing kernel is given by taking the $N \to \infty$ in Eq. (6) and yields the *Feynman path integral*

$$p(x, x'; \tau) \equiv \int DXe^S\bigg|_{x_N=x, x_0=x'}.$$

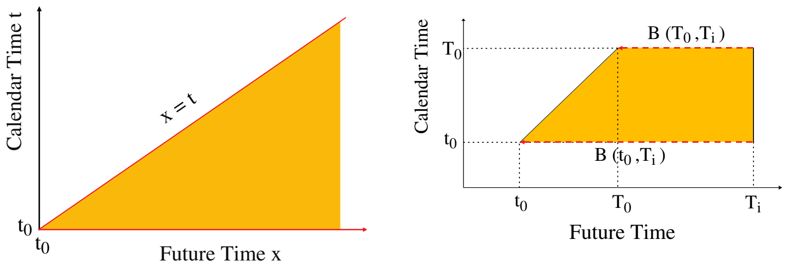

Fig. 3. (a) Shaded area is the domain of the forward interest rates. (b) Discounted zero coupon bond and martingale.

Option Lagrangians

The Black–Scholes Lagrangian, for $\delta x = x_i - x_{i-1}$ and using Eq. (3), is given by

$$H_{BS} \Rightarrow \mathcal{L}_{BS} = -\frac{1}{2\sigma^2}\left(\frac{\delta x}{\epsilon} + r - \frac{\sigma^2}{2}\right)^2 - r + O(\epsilon).$$

The Merton–Garman Lagrangian, for $\delta y = y_i - y_{i-1}$ and $\alpha = 1$ and using Eq. (4), is

$$H_{MG} \Rightarrow \mathcal{L}_{MG} = -\frac{1}{2\xi^2}\left(\frac{\delta y}{\epsilon} + \mu - \frac{1}{2}\xi^2\right)^2 - r - \frac{e^{-y}}{2(1-\rho^2)}\left[\frac{\delta x}{\epsilon} + r - \frac{1}{2}e^y - \frac{\rho}{\xi}e^{y/2}\left(\frac{\delta y}{\epsilon} + \mu - \frac{1}{2}\xi^2\right)\right]^2 + O(\epsilon).$$

The Merton–Garman option price has been exactly evaluated in [4] using path integration.

3. Forward interest rates

The forward interest rate $f(t,x)$ is the interest rate – fixed at time $t < x$ – for an instantaneous (overnight) loan taken at future time x. At every instant $t, f(t,x)$ – as a function of x – is an entire curve. Since $x > t, f(t,x)$ is defined on a two dimensional semi-infinite plane $t \geq t_0, x \geq t$, shown as the shaded domain in Fig. 3(a). The initial condition $f(t_0,x)$ is fixed by market data. Each point t, x in the shaded domain corresponds to one forward interest rate $f(t,x)$.

(Zero) coupon bond and martingale

A zero coupon bond pays a fixed sum, say \$1, at some future time T_i. Its price, at time T_0, is $B(T_0,T_i)$ is given by discounting \$1 and yields

$$B(T_0,T_i) = e^{-\int_{T_0}^{T_i} dx f(T_0,x)}.$$

Zero coupon bonds obey the martingale condition given by

$$B(t_0,T_i) = E\big[e^{-\int_{t_0}^{T_0} dt r(t)} B(T_0,T_i)\big]; \quad r(t) = f(t,t)$$

and shown in Fig. 3(b). $E[\ldots]$ is the expectation taken over the future random forward interest rates, which are shown by the shaded portion in the Fig. 3(b). A coupon bond pays a series of 'coupons' c_i at future times $T_i = T_0 + i\ell$; its price at calendar time T_0 is given by the following weighted sum of zero coupon bonds $\sum_{i=1}^{N} c_i B(T_0,T_i)$.

European bond option

The payoff function of a coupon bond, with strike price K and maturity time T_0, is given by [6]

$$\left[\sum_{i=1}^{N} c_i B(T_0,T_i) - K\right]_+.$$

The price of the call option, at (earlier) time $t_0 < T_0$, is given by the expectation value of its discounted payoff function [7]

$$\mathcal{C}(t_0;T_0,K) = E\left(e^{-\int_{t_0}^{T_0} dt r(t)}\left[\sum_{i=1}^{N} c_i B(T_0,T_i) - K\right]_+\right) \equiv E\big(\mathcal{F}\big).$$

4. Quantum field theory of interest rates

The (empirical) market behavior of $f(t,x)$, given in Fig. 4, shows that the forward interest rates $f(t,x)$ is a *random function* of both calendar time t and future time x. Hence, $f(t,x)$ is taken to be an independent random variable for each t and each x.

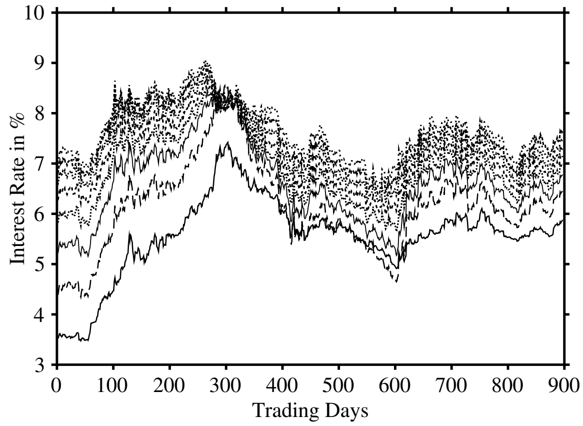

Fig. 4. Daily market forward interest rates, for future time from 1 to 8 years, namely $f(t, t+1), \ldots, f(t, t+8)$.

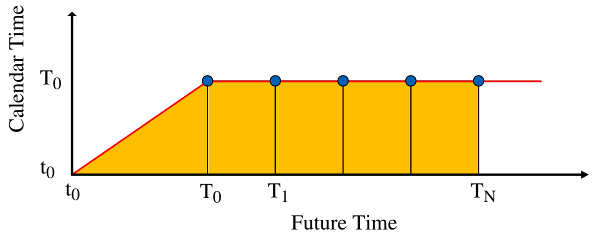

Fig. 5. Coupon bond option payoff function.

A parsimonious quantum finance model for the forward interest rates is

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)A(t,x). \tag{7}$$

$A(t, x)$ is an independent random variable for each t and each x and will be shown to be mathematically equivalent to a two dimensional quantum field.

Forward interest rates—a quantum field

How can we evaluate the option $\mathcal{C}(t_0; T_0, K) = E(\mathcal{F}) = E(\mathcal{F}[A])$? The expectation value $E(\mathcal{F}[A])$ is evaluated using functional integration—by summing $\mathcal{F}[A]$ over all possible functions $A(t, x)$ of t, x. More precisely, one performs infinitely many integrations, one integration $\int dA(t, x)$ for each t and each x. In effect, both $A(t, x)$ and $f(t, x)$ are classical random functions, which are mathematically equivalent to a two dimensional Euclidean quantum field.

Feynman path integral

To carry out a path integral over quantum field $A(t, x)$ we need the probability distribution function for $A(t, x)$ given by e^S/Z, where $S[A]$ is the action.

The Feynman path integral for forward interest rates is defined by

$$E(\mathcal{F}[A]) = \frac{1}{Z}\int DAe^{S[A]}\mathcal{F}[A]$$

$$\int DA = \prod_{(t,x)\in\mathcal{D}} \int_{-\infty}^{+\infty} dA(t,x); \qquad Z = \int DAe^{S[A]}.$$

The domain $\mathcal{D}$ of t, x over which the path integration is carried out depends on the financial instrument. In general, the initial value of the forward interest rates $f(t_0, x)$ is one of the boundaries of the domain $\mathcal{D}$, which is a finite subspace of the t, x plane.

Path integral domain for coupon bond option

The coupon bond option payoff $\left[\sum_{i=1}^{N} c_i B(T_0, T_i) - K\right]_+$ is shown in the Fig. 5 as a line at calendar time T_0, with large dots indicating the future times for coupon payments. The domain $\mathcal{D}$ of the path integration for evaluating the price of the coupon bond option is generated by the forward interest rates in the shaded domain: there is one integration variable $A(t, x)$ for each point in the shaded domain.

Since $r(t) = f(t, t)$, the discounting factor $e^{-\int_{t_0}^{T_0} dt r(t)}$ is on the $x = t$ boundary and is indicated by the edge in Fig. 5.

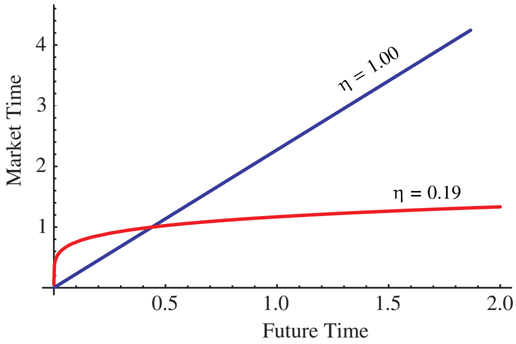

Fig. 6. Market future time.

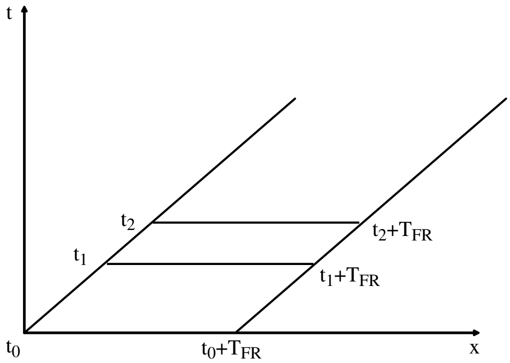

Fig. 7. Time dependent state space for interest rates.

Market future time and stiff action

A careful analysis of the market data shows that, instead of remaining future time $x - t$, interest rates are described by remaining market future time, given by $z = (x - t)^\eta$, where $\eta < 1$ is an index, shown in Fig. 6, that needs to be empirically determined. The forward interest rates' 'stiff' action $S[A]$ is given by

$$S[A] = -\frac{1}{2}\int_{t_0}^{+\infty} dt \int_0^{+\infty} dz \left[A^2(t,z) + \frac{1}{\mu^2}\left(\frac{\partial A}{\partial z}\right)^2 (t,z) + \frac{1}{\lambda^4}\left(\frac{\partial^2 A}{\partial z^2}\right)^2 (t,z)\right]. \tag{8}$$

The rigidity μ and stiffness λ parameters both smoothen the shape of the forward interest rates in the future time direction and are necessary for describing the market behavior of interest rates. The propagator is given by

$$E[A(t,z)A(t',z')] = \frac{1}{Z}\int DAe^{S[A]}A(t,z)A(t',z') = \delta(t-t')D(z,z').$$

For notational simplicity, the index is henceforth set to one, that is, $\eta = 1$.

Interest rates state space and Hamiltonian

For each time slice, forward interest rates $f(t,x)$ are defined only for future time $x > t$. Let T_{FR} be the maximum future time for financial instruments—which for simplicity is later taken to be ∞. The state space is *time dependent* and defined for the trapezoidal domain $t < x < t + T_{FR}$ given in Fig. 7; at time t_1, the state space consists of all possible functions $f(t_1, x)$, with $x \in [t_1, t_1 + T_{FR}]$.

The interest rates stiff action given in Eq. (8), yields, writing $f(x) \equiv f(t,x)$, the following time dependent non-Hermitian Hamiltonian

$$\mathcal{H}(t) = -\frac{1}{2}\int_t^\infty dxdx'\mathcal{M}(x,x';t)\frac{\delta^2}{\delta f(x)\delta f(x')} - \int_t^\infty dx\alpha(t,x)\frac{\delta}{\delta f(x)}$$

where $\delta/\delta f(x)$ is a functional derivative. In most models, $\mathcal{M}(x,x';t)$ is taken to be *independent* of $f(x)$; in particular, for the model of forward interest rates defined in Eq. (7), we have the following

$$\mathcal{M}(x,x';t) = \sigma(t,x)D(x,x';t)\sigma(t,x').$$

The drift $\alpha(t,x)$ is given by the Hamiltonian formulation of the martingale condition, namely

$$\mathcal{H}(t)[B(t,T)] = 0$$

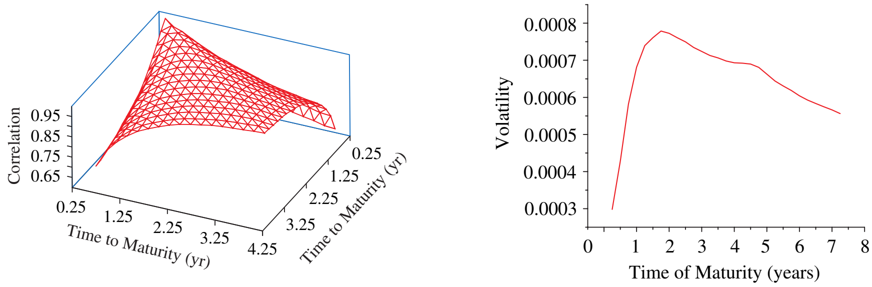

Fig. 8. (a) Correlator $E[\delta f(t,x)\delta f(t,x')]_c$ normalized so that it is ≤ 1 for Libor interest rates from 26 May 1999 to 17 May 2004. (b) Empirical volatility for Libor.

Table 1
Empirical values of the parameters given in Eq. (8).

	λ	μ	η	RMS error for the entire fit (%)
Libor	2.273/year	1.406/year	0.19	0.82
Euribor	2.831/year	1.721/year	0.19	0.69

and, applied to Eq. (7), yields the drift to be the following

$$\alpha(t,x) = \alpha(z) = \sigma(z)\int_0^z dz' D(z,z')\sigma(z').$$

5. Empirical analysis of interest rates model

Recall $A(t,x) = \left[\partial f(t,x)/\partial t - \alpha(t,x)\right]/\sigma(t,x)$. The parameters η, μ, λ and function $\sigma(t,x)$ are all fixed to fit market data. The quantum finance model is calibrated and tested by equating the market and model correlators. Let $\partial f(t,x)/\partial t \equiv \dot{f}(t,x)$; define the connected correlator as

$$E[\dot{f}(t,x)\dot{f}(t',x')]_c = \sigma(z)\sigma(z')E[A(t,z)A(t',z')]$$
$$z = (x-t)^\eta; \qquad z' = (x'-t)^\eta.$$

The left hand side is evaluated from the market and the right hand fixes the parameters μ, λ, η of the quantum finance model.

Empirical interest rates correlator

To empirically evaluate the correlator, calendar time is discretized into $t = n\epsilon$, with $\epsilon = 1$ day and $\epsilon\dot{f}(t,x) \simeq \delta f(t,x) \equiv f(t+\epsilon,x) - f(t,x)$.

$$E[\delta f(t,x)\delta f(t,x')]_c \equiv E[\delta f(t,x)\delta f(t,x')] - E[\delta f(t,x)]E[\delta f(t,x')].$$

Since $\partial f/\partial t \simeq \delta f/\epsilon = \alpha + \sigma A$, the quantum finance model yields

$$\begin{aligned} E[\delta f(t,x)\delta f(t,x')]_c &= \epsilon^2\sigma(z)\sigma(z')E[A(t,z)A(t,z')] \\ &= \epsilon\sigma(z)\sigma(z')D(z,z'). \end{aligned}$$

The (observed) market values of $f(t,x)$ are taken to be the random outcomes of sampling $f(t,x)$. The expectation value $E[\ldots]$ is empirically taken to be equal to a moving average of the market values over the past 180 days and is given for Libor in Fig. 8(a).

One can choose $\epsilon D(z,z) = 1$ due to a symmetry of the quantum finance model. The volatility function $\sigma(t,x) = \sigma(z)$ is then given by empirical volatility

$$\sigma^2(z) \equiv E[\left(\delta f(t,x)\right)^2]_c : \text{Market Volatility}$$

and is given in Fig. 8(b).

Table 1 gives the best fit for the parameters for the quantum finance model for Libor and Euribor interest rates. The quantum finance model, for both Libor and Euribor, fits the market to better than 99%, with only three parameters accounting for the entire market correlator. Market future time is $(x-t)^{0.19}$; for $x - t = 2$ years Libor market time is $[\lambda(x-t)]^\eta = 1.33$, in contrast to 4.54 for $\eta = 1$.

An interest rate swaption is an option for entering an interest rate swap and is a special case of a coupon bond option. The daily 2 by 10 Libor swaption model and market prices are plotted in Fig. 9; the model's predicted price has an overall accuracy of over 96%.

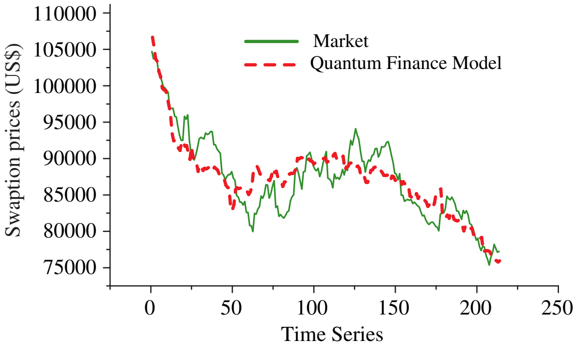

Fig. 9. Daily market and model prices are plotted for a 2 by 10 Libor swaption price, from 6 April 2004 to 28 January 2005. Total root mean square error (RMS) is 3.31%.

6. Summary

Quantum finance provides a comprehensive framework that can, in principle, price and hedge any financial instrument. Quantum finance extends the formalism of quantum mathematics to a new domain; in particular, financial instruments are represented by state vectors that are not normalizable and hence are elements of a (time dependent) state space that is larger than a Hilbert space.

The evolution for financial instruments is not unitary, since, like other classical random systems, the time evolution of financial instruments is dissipative and does not conserve probability. Furthermore, Hamiltonians in finance are not Hermitian due to the requirement of no arbitrage, which is realized by the martingale evolution of risk neutral financial instruments. The martingale condition requires a drift term that results in the Hamiltonian being non-Hermitian.

Forward interest rates' empirically observed imperfect and nontrivial correlations are accurately modeled by quantum field theory. The quantum finance model of interest rates is analytically and computationally tractable and price as well as hedge long duration vanilla, exotic and hybrid interest rate options. The usefulness of the quantum finance models for practitioners of finance is still an open question.

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References

[1] R. Devai, G. Naacke, World federation of exchanges data derivatives market survey 2011, May 2012.
[2] B.E. Baaquie, The Theoretical Foundations of Quantum Mechanics, Springer, USA, 2013.
[3] K.G. Wilson, The renormalization group and critical phenomena, Rev. Modern Phys. 55 (3) (1983) 583–600.
[4] B.E. Baaquie, Quantum Finance, Cambridge University Press, UK, 2004.
[5] B.E. Baaquie, Interest Rates and Coupon Bonds in Quantum Finance, Cambridge University Press, UK, 2009.
[6] J.C. Hull, Options, Futures, and Other Derivatives, Prentice Hall, New Jersey, 2000.
[7] R. Jarrow, S. Turnbull, Derivative Securities, South-Western College Publishing, USA, 2000.