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ON THE K-THEORY OF LIE GROUPS

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INTRODUCTION

IN THE paper [9] of Atiyah and Hirzebruch where (unitary) K-theory was first defined, the relation of the functor to classical groups was exploited to give particularly simple results (in contrast with cohomology) on certain spaces associated with Lie groups—classifying spaces [9, §4] and some homogeneous spaces [9, §3.6]. It is reasonable to ask whether for Lie groups themselves this functor can be described more simply than the usual cohomology theories,‡ and if so whether the simplicity of the K-theory of Lie groups can throw light on the causes of complications in their cohomology (cf. [14] for some partial explanations of these).

Our concern here is with the first of these problems only (indeed we are forced to deduce the 'simple' K-theory from the 'complicated' cohomology). The result we obtain is the following complete description of $K^*(G)$ (unitary K-theory of the Lie group G) when $\pi_1(G) = H_1(G)$ is torsion-free. In particular this covers the case that G is semi-simple and simply-connected.

THEOREM A. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Then*

(i) *$K^*(G)$ is torsion-free.*

(ii) *$K^*(G)$ can therefore be given the structure of a Hopf algebra over the integers, graded by $\mathbf{Z}_2$.*

(iii) *Regarded as a Hopf algebra $K^*(G)$ is the exterior algebra on the module of primitive elements, which are of degree* 1.

(iv) *A unitary representation $\rho : G \to U(n)$, by composition with the inclusion $U(n) \subset U$, defines a homotopy class $\beta(\rho)$ in $[G, U] = K^1(G)$. The module of primitive elements in $K^1(G)$ is exactly the module generated by all classes $\beta(\rho)$ of this type.*

(v) *In particular, if G is semi-simple of rank l, the l 'basic representations' $\rho_1, \dots, \rho_l$, are defined* (Chapter I, §3) *and the classes $\beta(\rho_1), \dots, \beta(\rho_l)$ form a basis for the above set of primitive elements; we can write*

$$K^*(G) = \Lambda(\beta(\rho_1), \dots, \beta(\rho_l)).$$

The whole object of this paper is the proof of Theorem A, or rather a slight generalization in which torsion in $\pi_1(G)$ is allowed for by enlarging the coefficient ring. The theorem is

† Revised version of thesis submitted for D. Phil. Oxford, 1965.

‡ For surveys of problems in the cohomology of Lie groups see [13, 14].

stated in the text in two parts as Theorems 1.1 and 2.1 of Chapter II. The proof of the second of these theorems, roughly equivalent to parts (ii)–(v) of Theorem A (assuming part (i)), proceeds in relatively straightforward stages which can be paralleled in cohomology.† For Theorem 1.1 of Chapter II on the other hand, which corresponds to part (i) of Theorem A, we give a much more indirect proof, depending on the classification of simple simply-connected compact Lie groups and the calculations of their cohomology mod p for all primes, due to Araki, Borel and others.

Chapter I is a brief exposition of the material we shall need from K-theory and representation theory. Chapter II is devoted to the statement of the main results, and those parts of the proof which have to do with the multiplicative structure; while Chapter III gives the computational proof of II, Theorem 1.1 already referred to.

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I. PRELIMINARIES

§1. *K*-Theory

For the main definitions and theorems of K-theory we refer to the standard sources [6, 8, 9, etc.]. We summarize in this section some of the facts which will be needed.

In what follows $\mathfrak{A}$ (resp. $\hat{\mathfrak{A}}$) is the category of spaces having the homotopy type of a finite CW-complex (resp. CW-complex with finite skeletons); $\mathfrak{A}_0$, $\hat{\mathfrak{A}}_0$ are the corresponding based categories. We shall frequently use the trivial observation that a functor on CW-complexes which is a homotopy invariant admits a unique extension to the corresponding homotopy type category.

An expression $(X, *)$ denotes a pair (space, basepoint). If (X, A) is a pair of spaces, X/A has a natural basepoint. If X, Y are two based topological spaces, $[X, Y]$ denotes the set of based homotopy classes of maps $X \to Y$.

Let U be the stable unitary group, BU the stable complex Grassmannian [15, 17 exp. 5]. Both spaces are in $\hat{\mathfrak{A}}$ and each has a natural choice of basepoint. When (X, A) is a pair in $\mathfrak{A}$, the (unitary) K-theory $K^*(X, A)$ is by definition the $\mathbf{Z}_2$-graded group given by

$$\begin{aligned} K^0(X, A) &= [X/A, \mathbf{Z} \times BU] \\ K^1(X, A) &= [X/A, U], \end{aligned} \tag{1}$$

the group structures arising from (canonical) H-space structures on BU, U. Since the category of complex vector bundles on a space X in $\mathfrak{A}$ is a homotopy invariant, $K^0(X) = K^0(X, \phi)$

† A parallel result is proved by more representation-theoretic methods by ATIYAH, On the K-theory of compact Lie groups, *Topology* **4** (1965), 95–101.

is naturally isomorphic to the Grothendieck group of that category; we shall use the two definitions interchangeably. The way in which the functor K^* is made into a $\mathbf{Z}_2$-graded multiplicative cohomology theory, together with other properties, can be found in [9, §§1, 2]. We shall also use the Künneth formula for K-theory [7].

Coefficient groups. To define K-theory with coefficients in $\mathbf{Z}_q$ ($q \geq 2$) we take a Moore space $M_q = S^1 \cup_q e^2$ (2-cell attached by a map of degree q). Then by definition

(2) $$K^\alpha(X, A; Z_q) = K^\alpha((X, A) \times (M_q, {}^*))$$

for (X, A) a pair in $\mathfrak{A}$ and $\alpha \in \mathbf{Z}_2$. Again we obtain a $\mathbf{Z}_2$-graded cohomology theory on $\mathfrak{A}$ with a multiplication which is always associative and is commutative for $q \not\equiv 2 \bmod 4$. For details see [5]. Reduction mod q, ρ_q, and Bockstein coboundary, β_q, are defined as morphisms of cohomology theories and fit into an exact triangle.

(3) $$\begin{tikzcd} K^*(X, A) \arrow[rr, "q"] & & K^*(X, A) \arrow[dl, "\rho_q"] \\ & K^*(X, A; \mathbf{Z}_q) \arrow[ul, "\beta_q"] & \end{tikzcd}$$

With respect to the multiplicative structures, ρ_q is a homomorphism and β_q a derivation. For the universal coefficient formula see [7].

On a more elementary level let L be any torsion-free abelian group. Then defining $K^*(X, A; L)$ to be simply $K^*(X, A) \otimes L$ again gives a $\mathbf{Z}_2$-graded cohomology theory, since tensoring with L preserves exactness. When L is a ring the multiplicative structure of K-theory carries over to this theory, and an analogous Künneth formula holds. The usual K-theory is of course the case $L = \mathbf{Z}$.

Let G be a $\mathbf{Z}$-graded group; we shall write G_w for the $\mathbf{Z}_2$-graded group defined by

$$G_w^0 = \sum_{k \in \mathbf{Z}} G^{2k}; \; G_w^1 = \sum_{k \in \mathbf{Z}} G^{2k+1}$$

(w means that the grading has been 'weakened'). This is a functor from $\mathbf{Z}$-graded groups (rings, modules, etc.) to $\mathbf{Z}_2$-graded ones. In particular from ordinary cohomology theory we can derive a $\mathbf{Z}_2$-graded cohomology theory $H_w^*(X; L)$. The following result is a convenient restatement of [9, §1.10] in these terms

(1.1) There is a natural isomorphism of $\mathbf{Z}_2$-graded multiplicative cohomology theories

$$ch_{\mathbf{Q}}: K^*(\quad; \mathbf{Q}) \to H_w^*(\quad; \mathbf{Q})$$

The Chern character as usually defined, which we shall write ch, is the composition

$$K^*(X, A) \to K^*(X, A) \otimes \mathbf{Q} = K^*(X, A; \mathbf{Q}) \xrightarrow{ch_{\mathbf{Q}}} H_w^*(X, A; \mathbf{Q})$$

We shall use torsion-free coefficient rings as a substitute for some elementary $\mathscr{C}$-theory [31]. Let Π denote the set of all prime natural numbers: if $\mathbf{p}$ is a subset of Π we shall use $\mathbf{Q}(\mathbf{p})$ to mean the quotient ring of $\mathbf{Z}$ with respect to the multiplicative subset generated by $\mathbf{p}$. The correspondence $\mathbf{p} \to \mathbf{Q}(\mathbf{p})$ is bijective from subsets of Π to rings lying between $\mathbf{Z}$ and $\mathbf{Q}$; $\mathbf{Q}(\phi) = \mathbf{Z}$, $\mathbf{Q}(\Pi) = \mathbf{Q}$. (See [34, §20]).

LEMMA 1.1. *If L is a ring of type* $\mathbf{Q}(\mathbf{p})$ *and* $B \subset A$ *are finitely generated L-modules with A free, then the following conditions are equivalent*

(i) *B is a direct summand in A*

(ii) *For every prime p not in* $\mathbf{p}$, $x \in A$ *and* $px \in B$ *imply* $x \in B$

(iii) *B is the intersection of $A = A \otimes_L L$ with a submodule of $A \otimes_L \mathbf{Q}$; this submodule can be taken to be $B \otimes_L \mathbf{Q}$.*

The proof is left to the reader.

Extension to $\hat{\mathfrak{A}}$. We use the usual inverse limit process [6, §4] to extend the domain of definition of the above K-theories (only a very small extension is needed here). Let X be a CW-complex with finite skeletons, A a finite subcomplex, and write $X^{(i)}$ for the i-skeleton of X. For any coefficient group L of those used above, define

$$\underset{\longleftarrow}{K}^\alpha(X, A; L) = \underset{\longleftarrow}{\lim}\, K^\alpha(X^{(i)} \cup A, A; L) (\alpha \in \mathbf{Z}_2)$$

The resulting $\mathbf{Z}_2$-graded filtered group is a homotopy invariant, so that the definition can be extended to pairs (X, A) with X in $\hat{\mathfrak{A}}$ and A in $\mathfrak{A}$. If X is itself in $\mathfrak{A}$, $K^*(X, A; L)$ and $\underset{\longleftarrow}{K}^*(X, A; L)$ can be identified.

§2. Spectral sequences

Let (X, A) be a pair with X in $\hat{\mathfrak{A}}$, A in $\mathfrak{A}$, and let X be given an increasing filtration: $A = X_0 \subset X_1 \subset \dots, \bigcup_k X_k = X$, where all spaces X_k are in $\mathfrak{A}$. As in [9, §2.1], the filtration defines an $H(p, q)$ system in the sense of Cartan–Eilenberg where

$$H(p, q) = K^*(X_{q-1}, X_{p-1}) \qquad q \geq p. \quad (X_k = A \text{ for } k < 0)$$

(This can of course be done with any coefficient group; to simplify notation we are here considering only the integral case.) Each $H(p, q)$ is $\mathbf{Z}_2$-graded; each E_r in the resulting sequence is $(\mathbf{Z} + \mathbf{Z}_2)$-graded. If $p \in \mathbf{Z}$, $\alpha \in \mathbf{Z}_2$ and $\bar{p}$ denotes p reduced mod 2,

$$E_1^{p\alpha} = K^{\bar{p}+\alpha}(X_p, X_{p-1}).$$

If the filtration is finite, X is itself in $\mathfrak{A}$. Let $K^*(X, A)$ be filtered by defining

$$K^*(X, A)_p = \operatorname{Ker}[K^*(X, A) \to K^*(X_{p-1}, A)]. \tag{1}$$

Standard spectral sequence methods prove that $\{E_r\}$ as defined above converges strongly to the $\mathbf{Z}_2$-graded filtered group $K^*(X, A)$.

Next suppose the filtration infinite, and that X is in $\hat{\mathfrak{A}}$. For each integer $n \geq 0$ the pair (X_n, A) carries a filtration induced from that of (X, A), which is finite and gives rise to a strongly convergent spectral sequence which we shall write $\{E_r(n)\}$; these spectral sequences with the associated filtered groups $K^*(X_n, A)$ form an inverse system in the sense of [6, §3] (except that they include E_1-terms). By definition, $E_1^{p\alpha} \cong E_1^{p\alpha}(n)$ for $n \geq p$, and so inductively $E_r^{p\alpha} \cong E_r^{p\alpha}(n)$ for $n \geq p + r - 1$. Hence certainly

$$\underset{\overleftarrow{n}}{\lim}\, (E_r^{p\alpha}(n)) = E_r^{p\alpha}.$$

Now a filtration can be defined on $\underleftarrow{K}^*(X, A)$ as in (1); certainly if $\{E_r\}$ converges it will be to this filtered group. Sufficient conditions for this are stated in

PROPOSITION 2.1. *Suppose that there exists an integer r_0 such that all differentials d_r are zero in $\{E_r\}$ for $r \geq r_0$. Then $\{E_r\}$ converges strongly to $\underleftarrow{K}^*(X, A)$.*

We begin by proving

LEMMA 2.1. *Under the hypotheses of proposition 2.1, the inverse system $\{K^*(X_p, A)\}$ satisfies (ML)* [6, §3].

Proof. The condition on the differentials implies that for any k.

(A_k) The images of restrictions $K^*(X_{k+p}, X_{k-1}) \to K^*(X_k, X_{k-1})$ coincide for $p \geq r_0 - 1$ (This image is in fact the group $Z^k_{p+1} \subset E^k_1$). We shall prove

(B_k). The images of restrictions $K^*(X_{k+p}, A) \to K^*(X_k, A)$ coincide for $p \geq r_0 - 1$.

This implies the condition (ML). B_0 is trivial and B_1 is the same as A_1. Suppose we have proved $B_1, \ldots, B_k$; consider the commutative diagram

$$\begin{array}{ccccc}
K^*(X_{k+r_0+q}, X_k) & \xrightarrow{j_1} & K^*(X_{k+r_0}, X_k) & \xrightarrow{j_2} & K^*(X_{k+1}, X_k) \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\
K^*(X_{k+r_0+q}, A) & \xrightarrow{i_1} & K^*(X_{k+r_0}, A) & \xrightarrow{i_2} & K^*(X_{k+1}, A) \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\
K^*(X_k, A) & = & K^*(X_k, A) & = & K^*(X_k, A)
\end{array}$$

$(q \geq 0)$

Here all homomorphisms are induced by inclusions; the three columns are exact. By A_{k+1}, $\operatorname{Im} j_2 = \operatorname{Im} j_2 j_1$; by B_k, $\operatorname{Im} g_2 = \operatorname{Im} g_1$. We must show that $\operatorname{Im} i_2 = \operatorname{Im} i_2 i_1$.

Given $u \in K^*(X_{k+r_0}, A)$, pick v in $K^*(X_{k+r_0+q}, A)$ such that $g_1(v) = g_2(u)$. Then $g_2(v - i_1(u)) = 0$ so by exactness we can find w in $K^*(X_{k+r_0}, X_k)$ so that $h_2(w) = u - i_1(v)$. Finally pick x in $K^*(X_{k+r_0+q}, X_k)$ so that $j_2 j_1(x) = j_2(w)$.

Then

$$\begin{aligned}
i_2 i_1(v + h_1(x)) &= i_2 i_1(v) + h_3 j_2 j_1(x) = i_2 i_1(v) + h_3 j_2(w) \\
&= i_2(i_1(v) + h_2(w)) = i_2(u)
\end{aligned}$$

i.e. $i_2(u)$ is in the image of $i_2 i_1$. This proves the lemma.

LEMMA 2.2. *If the inverse system $\{K^*(X_p, A)\}$ satisfies (ML) then so does*

$$\{K^*(X_p, A)_q\} \textit{ for any fixed } q \geq 0.$$

Proof. By definition, $K^*(X_p, A)_q$ is the kernel of the restriction homomorphism $K^*(X_p, A) \to K^*(X_p \cap X_{q-1}, A)$; it is therefore zero if $p < q$, and then the images of

$K^*(X_{p+r}, A)_q \to K^*(X_p, A)_q$ all coincide trivially. Suppose $p \geq q$ and consider the sequence of restrictions

$$K^*(X_{p+s}, A) \xrightarrow{i_1} K^*(X_{p+r}, A) \xrightarrow{i_2} K^*(X_p, A) \xrightarrow{i_3} K^*(X_{q-1}, A)$$

when $0 \leq r \leq s$.

The condition (ML) implies that we can choose r so that $\operatorname{Im} i_2 i_1 = \operatorname{Im} i_1$ for all $s \geq r$. Given

$$x \in K^*(X_{p+r}, A)_q,$$

choose $y \in K^*(X_{p+s}, A)$ so that $i_2 i_1(y) = i_2(x)$. But then $i_3(i_2 i_1(y)) = i_3 i_2(x) = 0$, and y must itself be in $K^*(X_{p+s}, A)_q$. In other words the images of restrictions $K^*(X_{p+s}, A)_q \to K^*(X_p, A)_q$ coincide for all $s \geq r$ and the lemma is proved.

Now we turn to Proposition 2.1. First, by the left-exactness of $\varprojlim$, the limit of the exact sequences

$$0 \to K^*(X_p, A)_q \to K^*(X_p, A) \to K^*(X_{q-1}, A)$$

(q fixed, $p \geq q$) is again exact. The natural map

$$\underleftarrow{K}^*(X, A)_q \to \varprojlim K^*(X_p, A)_q$$

is therefore an isomorphism. Next, the vanishing of d_r for $r \geq r_0$ in $\{E_r\}$ implies the same condition holds in $\{E_r(p)\}$ for each p. $E_\infty = E_{r_0}$ is therefore isomorphic with $\varprojlim E_\infty(p) = \varprojlim E_{r_0}(p)$. From these considerations, the limit of the exact sequences

$$0 \to K^*(X_p, A)_{q+1} \to K^*(X_p, A)_q \to E_\infty^q(p) \to 0$$

is a sequence

$$0 \to \underleftarrow{K}^*(X, A)_{q+1} \to \underleftarrow{K}^*(X, A)_q \to E_\infty^q \to 0.$$

But Lemmas 2.1 and 2.2 imply that under the conditions of the theorem the system $\{K^*(X_p, A)_q\}$ (q fixed) satisfies (ML). The sequence (2) is therefore exact; this completes the proof of Proposition 2.1.

We now consider the special case of the filtration by skeletons of a finite CW-complex X. This gives rise to a spectral sequence which we shall write $\{E_r(X; L)\}$ and call the ***standard** spectral sequence* of X; $E_2(X; L) = H^*(X; L)$, $E_\infty(X; L) \sim K^*(X; L)$. (For the case $L = \mathbf{Z}$ see [9, §2.1]). This spectral sequence is strongly convergent and is a homotopy invariant from the E_2 term on; we take advantage of this to consider $\{E_r(X; L)\}$ as defined on the category $\mathfrak{A}$. We shall need to use the fact that ρ_p, being a homomorphism of cohomology theories, induces a homomorphism of spectral sequences

$$\rho_p : \{E_r(X; \mathbf{Z})\} \to \{E_r(X; \mathbf{Z}_p)\}$$

which is the cohomology reduction mod p on the E_2 term.

LEMMA 2.3. *The spectral sequence $\{E_r(X; L)\}$ is multiplicative for $L = \mathbf{Z}$, $\mathbf{Z}_p$; i.e. the E_r's are $(\mathbf{Z} + \mathbf{Z}_2)$-graded rings, the d_r's are derivations, $K^*(X; L)$ is filtered by ideals and $E_\infty(X; L)$ is the corresponding graded ring.*

Proof. By Dold [20 Satz 15.3].

§3. Representation rings

In all that follows 'Lie group' will mean 'compact Lie group over $\mathbf{R}$'. For such a group G the *representation* ring $R(G)$ [9, §4.2] can be characterized as the free abelian group on the isomorphism classes of irreducible representations of G, with multiplication induced by the tensor product of representations. $R(G)$ is a contravariant functor on the category of Lie groups and continuous homomorphisms; the inclusion $1 \to G$ of the trivial group induces an augmentation $\varepsilon : R(G) \to R(1) = \mathbf{Z}$ which assigns to each representation its dimension. The kernel of ε is written $I(G)$. We need to collect a few traditional results on representation rings.

LEMMA 3.1. *If G is commutative, with character group Γ, then $R(G)$ is naturally isomorphic to the group ring* $\mathbf{Z}[\Gamma]$.

Proof. Immediate from the definition.

LEMMA 3.2. *If G, H are Lie groups, then $R(G \times H)$ is naturally isomorphic to $R(G) \otimes R(H)$.*

Proof. Let π_1, π_2 be the projections of $G \times H$ to G, H respectively; the irreducible representations of $G \times H$ are exactly the representations $\pi_1^* \rho \otimes \pi_2^* \sigma$ where ρ, σ run over the irreducible representations of G, H respectively.

Next suppose that G is semi-simple and simply-connected. We recall some facts about the representation theory of G, which is equivalent to that of the Lie algebra $\mathfrak{g}$. Let Λ_+ be the set of dominant weights of $\mathfrak{g}$ [23, VII.3]; Λ_+ is additively the free monoid generated by weights $\omega_1, \ldots, \omega_l$ called 'fundamental weights' where $l = \operatorname{rank} G$. For $\lambda = \Sigma k_i \omega_i \in \Lambda_+$ define the height $h(\lambda)$ to be Σk_i.

Let $\mathrm{Ir}(G)$ be the set of isomorphism classes of irreducible representations of G. If a representation ρ admits a highest weight $\lambda \in \Lambda_+$ we write $w^+(\rho) = \lambda$. In particular any ρ in $\mathrm{Ir}(G)$ admits a highest weight and the correspondence $\rho \to w^+(\rho)$ is bijective from $\mathrm{Ir}(G)$ to Λ_+. The *basic representations* $\rho_1, \ldots, \rho_l$ are those irreducible representations defined by $w^+(\rho_i) = \omega_i$ $(i = 1, \ldots, 1)$.

LEMMA 3.3. *With the above notation $R(G)$ is the polynomial algebra* $\mathbf{Z}[\rho_1, \ldots, \rho_l]$.

Proof. For any representation ρ of G define $h(\rho)$ to be the maximum of $h(\lambda)$ as λ runs over the (non-empty) set of weights of ρ in Λ_+. Then the following facts are immediate consequences of the definitions:

If ρ, ρ' admit the same highest weight and ρ' is irreducible, $\rho = \rho' + \sigma$ where $h(\sigma) < h(\rho)$.

If ρ, ρ' admit highest weights, so does $\rho \otimes \rho'$ and $w^+(\rho \otimes \rho') = w^+(\rho) + w^+(\rho')$.

From these two observations it is a simple algebraic exercise to deduce that any representation of height k has a unique expression as a polynomial of degree k in $\rho_1, \ldots, \rho$. This proves the lemma.

§4. Representations in K-theory

There are two simple ways in which the representations of a Lie group G enter into the K-theory of associated spaces. First, let ξ be a principal G-bundle with base B_ξ in $\mathfrak{A}$.

Then assigning to a representation ρ of G the vector bundle $\rho(\xi)$ over B_ξ defines a homomorphism $\alpha(\xi) : R(G) \to K_t^0(B_\xi)$ which is natural with respect to bundle maps and group homomorphisms. If B_ξ is in $\hat{\mathfrak{A}}$ we can take restrictions to finite subcomplexes, pass to the limit, and derive a continuous homomorphism $\alpha(\xi)\hat{} : R(G)\hat{} \to \underleftarrow{K}^0(B_\xi)$ where the completion of $R(G)$ is with respect to the $I(G)$-adic topology. All these results are stated in full and proved in [9, §4], where it is also shown for G connected that $\alpha(\xi)\hat{}$ is an isomorphism when ξ is a universal G-bundle and $B_\xi = BG$ a classifying space for G.

On the other hand G as a compact differentiable manifold is in $\mathfrak{A}$ so that we can discuss its own K-theory. In particular, we can consider a representation of G simply as a map $\rho : G \to U(n)$ for some n, and then compose with the inclusion $u^n : U(n) \to U$. Since $U(n)$ is connected the homotopy class of ρ, and so of $u^n \circ \rho$, depends only on the equivalence class of ρ as a representation. Write $\beta(\rho)$ for the homotopy class $\{u^n \circ \rho\} \in [G, U] = K^1(G)$. β is a function from equivalence classes of representatons of G to $K^1(G)$.

We now derive some properties of β. The group structure in K^1 is induced from the canonical H-structure (Whitney sum) of U or equivalently from the stable group-structure (matrix multiplication) [8, Lemma 2.4.6]. Applying the first we find immediately that given representations ρ, σ of G with direct sum $\rho \oplus \sigma$

(1) $$\beta(\rho \oplus \sigma) = \beta(\rho) + \beta(\sigma).$$

As an additive function, β therefore has a unique extension to $R(G)$, which we continue to denote by $\beta : R(G) \to K^1(G)$.

Next let the degrees of ρ, σ be m, n respectively and let ι_k be the trivial representation of G of degree k. For any $g \in G$ we have the identity $(\rho \otimes \sigma)(g) = (\rho \otimes \iota_n)(g) \cdot (\iota_m \otimes \sigma)(g)$ (matrix multiplication). Since $\rho \otimes \iota_k$ is the direct sum of k copies of ρ, (1) implies that $\beta(\rho \otimes \iota_k) = k.\beta(\rho)$. From the second description of the group structure in K^1, and the above formulae, we can deduce

(2) $$\beta(\rho \otimes \sigma) = n.\beta(\rho) + m.\beta(\sigma).$$

LEMMA 4.1. $$\beta(\mathbf{Z}) = \beta((I(G))^2) = 0.$$

Proof. First, since $u^k \circ \iota_k : G \to U$ is the constant map, $\beta(\iota_k) = 0$ which implies that β is trivial on $\mathbf{Z} \subset R(G)$.

Next, $I(G)$ is generated as a group by elements of form $\rho - m$ where m is the degree of ρ. Hence $(I(G))^2$ is generated by elements of form $(\rho - m)(\sigma - n)$ where n is the degree of σ. But

$$\beta((\rho - m)(\sigma - n)) = \beta(\rho \otimes \sigma) - n\beta(\rho) - m\beta(\sigma) + \beta(mn) = 0$$

using (2) and the first part of the lemma. Hence β is trivial on the whole of $(I(G))^2$.

Let us define $J(G)$ to be the quotient $I(G)/(I(G))^2$ ('indecomposable elements', of $R(G)$, cf. [28, 3.7]). There are natural projections $R(G) \to I(G) \to J(G)$; let j be their composition.

Lemma 4.1 implies that β factors uniquely through j to give a commutative diagram

(3)
$$\begin{tikzcd} R(G) \arrow[r, "\beta"] \arrow[d, "j"'] & K^1(G) \\ J(G) \arrow[ur, "\bar{\beta}"'] & \end{tikzcd}$$

We digress for a moment on the structure of $J(G)$. Let us define a *standard covering group* to be a group which is the product of a torus and a semi-simple simply-connected group (For the justification of this name, see [15, 2.9]).

LEMMA 4.2. *If G is a standard covering group of rank l, $J(G)$ is a free abelian group of rank l.*

Proof. Write $G = T \times G_0$ with T a torus and G_0 semi-simple and simply-connected. Then $R(G) = R(T) \otimes R(G_0)$ as an augmented algebra (Lemma 3.2), hence $J(G) = J(T) \oplus J(G_0)$. It is therefore sufficient to prove Lemma 4.2 in the two cases $G = T$, $G = G_0$. The second is an immediate consequence of Lemma 3.3; if $\rho_1, \ldots, \rho_l$ are the basic representations of G, $j(\rho_1), \ldots, j(\rho_l)$ form a basis for $J(G)$.

Suppose now G is a torus, with character group Γ. Γ is free abelian of rank l, and $R(G) = \mathbf{Z}[\Gamma]$ by Lemma 3.1. If $\gamma_1, \ldots, \gamma_l$ are a set of generators for Γ it is easy to see that $j(\gamma_1), \ldots, j(\gamma_l)$ form a basis for $J(G)$, completing the proof in this case.

(The smallest connected counterexample to this lemma is the projective unitary group $PU(3)$. $PU(3)$ has rank 2, but $J(PU(3))$ is a free abelian group of rank 3.)

Having introduced α and β we need to relate them. Essentially, our result states that β is (up to sign) the suspension of α. Let $\xi(E_\xi \xrightarrow{\pi} B_\xi)$ be a principal G-bundle as before and b_0 a basepoint of B_ξ; we shall identify $\pi^{-1}(b_0)$ with the standard fibre G as usual. $\alpha(\xi)$ commutes with augmentations and so maps $I(G)$ into $\tilde{K}^0(B_\xi) = K^0(B_\xi, b_0)$, regarded as a subgroup of $K^0(B_\xi)$.

PROPOSITION 4.1. *The diagram*

(4)
$$\begin{tikzcd} K^1(G) \arrow[r, "\delta"] & K^0(E_\xi, G) \\ I(G) \arrow[u, "\beta"] \arrow[r, "\alpha(\xi)"] & K^0(B_\xi, b_0) \arrow[u, "\pi^!"'] \end{tikzcd}$$

is anti-commutative, i.e. $\pi^! \circ \alpha(\xi) + \delta \circ \beta = 0$.

Proof. Given a pair (X, A) we shall write elements of $K^0(X, A)$ as difference elements [10, §3] $\mathrm{d}(\xi_0, \xi_1, \tau)$ where ξ_0, ξ_1 are bundles over X and τ is an isomorphism of their restrictions to A.

Let $\rho : G \to U(n)$ be a representation. $I(G)$ is generated by elements of form $\rho - n$, so it is sufficient to check that $\delta \circ \beta(\rho - n) = \delta \circ \beta(\rho)$ is equal to $-\pi^! \circ \alpha(\xi)(\rho - n)$. Let ε_n denote a trivial bundle of dimension n. $\alpha(\xi)$ sends ρ to $\rho(\xi)$, n to ε_n, and $\alpha(\xi)(\rho - n) \in K^0(B_\xi, b_0)$

can be represented as $d(\rho(\xi), \varepsilon_n; \imath)$ where $\imath$ is any vector space isomorphism of the fibres of $\rho(\xi)$, ε_n over b_0. The total space of $\rho(\xi)$ is $E_\xi \times {}_\rho\mathbf{C}^n$; we shall take $\imath: G \times {}_\rho\mathbf{C}^n \to \mathbf{C}^n$ to be defined by

$$\imath(g \times {}_\rho v) = \rho(g^{-1}) \cdot v \qquad g \in G, v \in \mathbf{C}^n$$

(This can be checked to be a well-defined isomorphism).

Next, it follows that $\pi^!\alpha(\xi)(\rho - n)$ can be represented as $d(\pi^!\rho(\xi), \varepsilon_n; \bar{\imath})$ where $\bar{\imath}$ is $\imath$ lifted to G via π. The total space of $\pi^!\rho(\xi)$ is composed of pairs $(u, u' \times {}_\rho v)$ where $u, u' \in E_\xi$, $v \in \mathbf{C}^n$, and $\pi(u) = \pi(u')$ (so u' is a translate of u by an element of G). The following map φ defines an isomorphism $\varepsilon^n \to \pi^!\rho(\xi)$:

$$\varphi(u, v) = (u, u \times {}_\rho v) \text{ } (u, v \text{ as above}).$$

By the properties of difference elements, $d(\pi^!\rho(\xi), \varepsilon_n; \bar{\imath}) = d(\varepsilon_n, \varepsilon_n; \bar{\imath} \circ (\varphi|G))$. Now

$$\bar{\imath} \circ (\varphi|G)(g, v) = \bar{\imath}(g, g \times {}_\rho v) = (g, \rho(g^{-1}) \cdot v).$$

Hence $\bar{\imath} \circ (\varphi|G)$ is an isomorphism of trivial bundles over G determined by the map $G \to U(n)$ which sends g to $\rho(g^{-1})$. This map composed with u^n clearly determines the element $-\beta(\rho)$ in $K^1(G)$; hence $\pi^! \circ \alpha(\xi)(\rho - n) = d(\varepsilon_n, \varepsilon_n; \bar{\imath} \circ (\varphi|G)) = -\delta \circ \beta(\rho)$ as required.

II. THE MULTIPLICATIVE STRUCTURE OF $K^*(G)$

§1. Introduction

We are now ready to begin discussing the K-theory of Lie groups in detail. At the basis of the discussion is the following result:

THEOREM 1.1. *Let G be a connected, semi-simple and simply connected Lie group; then $K^*(G)$ is torsion-free.*

This theorem will be proved in Chapter III. The proof given there uses the classification theory of semi-simple Lie groups, and detailed knowledge of their cohomology; it is therefore not a 'natural' proof. In this chapter we shall use more straightforward methods to deduce more specific results on the rings $K^*(G)$ from Theorem 1.1.

We begin by making a simple extension of theorem 1.1 to arbitrary connected Lie groups.

PROPOSITION 1.1. *Let $\mathbf{Q(p)}$ be as in* I 1.8 *and suppose $H_1(G; \mathbf{Q(p)})$ is torsion-free; then $K^*(G; \mathbf{Q(p)})$ is torsion-free.*

Proof. To begin with, we can reduce the proof to the case where G is semi-simple. In fact, in general there is a homeomorphism

$$G = G_0 \times T \tag{1}$$

where G_0 is semi-simple and T is a torus (perhaps trivial) [14, 3.1]. Now $K^*(T)$ is torsion-free since $H^*(T; \mathbf{Z})$ is; hence the same is true for $K^*(T; \mathbf{Q(p)})$ and we can apply the Künneth formulae to give

$$K^*(G; \mathbf{Q(p)}) \approx K^*(G_0; \mathbf{Q(p)}) \otimes K^*(T; \mathbf{Q(p)})$$
$$H_1(G; \mathbf{Q(p)}) \approx H_1(G_0; \mathbf{Q(p)}) \oplus H_1(T; \mathbf{Q(p)})$$

Hence $K^*(G; \mathbf{Q}(\mathbf{p}))$ (resp. $H_1(G; \mathbf{Q}(\mathbf{p}))$) has torsion if and only if $K^*(G_0; \mathbf{Q}(\mathbf{p}))$ (resp. $H_1(G_0; \mathbf{Q}(\mathbf{p}))$) has; it is therefore sufficient to prove the proposition for semi-simple groups.

Let G be semi-simple, with universal covering group $\tilde{G}$ and projection $\pi : \tilde{G} \to G$. The kernel Γ of π is a finite abelian group; let d be its order. Since $H_1(G) = \Gamma$ has rank zero, $H_1(G; \mathbf{Q}(\mathbf{p}))$ is torsion-free only if it is trivial; and this is the case if and only if $p \in (\mathbf{p})$ for all primes p dividing d. All such primes are then units of $\mathbf{Q}(\mathbf{p})$, so that d is one also.

We now recall some of the properties of the transfer homomorphism in K-theory [6, §2]. Let $f: X \to Y$ be a finite covering, with X, Y in $\mathfrak{A}$. Then the transfer is a homomorphism of abelian groups $f_! : K^*(X) \to K^*(Y)$, preserving filtration and satisfying

$$f_!(x \cdot f^!(y)) = f_!(x) \cdot y \quad (x \in K^*(X), y \in K^*(Y)). \tag{2}$$

Cf. [6, (2.3)]. The extension of $f_!$ under the inclusion $\mathbf{Z} \to \mathbf{Q}(\mathbf{p})$ is a homomorphism of $\mathbf{Q}(\mathbf{p})$-modules $K^*(X; \mathbf{Q}(\mathbf{p})) \to K^*(Y; \mathbf{Q}(\mathbf{p}))$ which we shall also write $f_!$; and this again satisfies (2). Let d be the degree of the covering; then $f_!(1)$ is the class of the direct image of the trivial 1-plane bundle on X, and so has augmentation d. We therefore have as a special case of (2),

$$f_! f^!(y) = f_!(1) \cdot y = (d + y') \cdot y \tag{3}$$

where $y' \in \tilde{K}^*(Y; \mathbf{Q}(\mathbf{p}))$.

Now suppose X is acted on freely by a finite group Γ and $Y = X/\Gamma$ with f as the natural projection. Let us write $\gamma^! : K^*(X; \mathbf{Q}(\mathbf{p})) \to K^*(X; \mathbf{Q}(\mathbf{p}))$ for the homomorphism induced by the action of $\gamma \in \Gamma$ on X.

LEMMA 1.1. *$f^! f_!(x) = \sum_{\gamma \in \Gamma} \gamma^!(x)$ for $x \in K^*(X; \mathbf{Q}(\mathbf{p}))$.*

Before proving Lemma 1.1 let us show how this implies that Proposition 1.1 holds in the semi-simple case. Let G, $\tilde{G}$ be as before; then we can apply Lemma 1.1 with π for f and Γ acting on $\tilde{G}$ by translations. But since $\tilde{G}$ is connected, all translations of $\tilde{G}$ are homotopic to the identity; in particular $\gamma^! =$ identity for each $\gamma \in \Gamma$, so that Lemma 1.1 gives

$$\pi^! \pi_!(x) = d \cdot x \qquad x \in K^*(G; \mathbf{Q}(\mathbf{p})). \tag{4}$$

Now suppose $H_1(G; \mathbf{Q}(\mathbf{p}))$ is torsion free. Then d is a unit in $\mathbf{Q}(\mathbf{p})$; hence d is invertible in $K^*(G; \mathbf{Q}(\mathbf{p}))$, and by the argument of [9, 2.6] so is $d + y'$ for $y' \in \tilde{K}^*(G; \mathbf{Q}(\mathbf{p}))$. Using (3), (4) we see that $\pi_! \pi^!$, $\pi^! \pi_!$ must be isomorphisms; so $\pi^!$, $\pi_!$ are isomorphisms. But $K^*(\tilde{G}; \mathbf{Q}(\mathbf{p}))$ is torsion-free by theorem 1.1; therefore so is $K^*(G; \mathbf{Q}(\mathbf{p}))$.

We now prove Lemma 1.1. For this let D denote the set of integers $\{1, \dots, d\}$ with the discrete topology, and let the elements of Γ be given an enumeration $\gamma_1, \dots, \gamma_d$. Then $X \times D$ is the disjoint union of d copies of X; $K^*(X \times D; \mathbf{Q}(\mathbf{p}))$ is the direct sum of d copies of $K^*(X; \mathbf{Q}(\mathbf{p}))$. Let $u : X \times D \to X$ be the left projection; then u is a (trivial) covering of degree d and it is immediate from the definition of $u_!$ that $u_!(x_1 \oplus \dots \oplus x_d) = x_1 + \dots + x_d$ for $x_1, \dots, x_d \in K^*(X; \mathbf{Q}(\mathbf{p}))$. Now let $t : X \times D \to X$ be given by $t(x, i) = \gamma_i \cdot x (x \in X, i \in D)$. Then by applying the definitions, we see that the covering u is induced from the covering f by the map f, and $t : X \times D \to X$ is the induced map of covering spaces, i.e. we have a commutative diagram

$$\begin{tikzcd} X \times D \arrow[r, "t"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{5}$$

Since the definition of the transfer is natural with respect to maps of coverings, we can deduce from the diagram (5) that $u_! t^! = f^! f_! : K^*(X: \mathbf{Q(p)}) \to K^*(X; \mathbf{Q(p)})$. Now for x_0 in $K^*(X; \mathbf{Q(p)})$, $t^!(x_0) = \gamma_1^! x_0 \oplus \dots \oplus \gamma_d^! x_0$; hence $u_! t^!(x_0) = \Sigma \gamma_i^! x_0 (i = 1, \dots, d)$. This completes the proof.

Note to Proposition 1.1. This is in fact a 'best possible' result, in the sense that the existence of torsion in $H_1(G; \mathbf{Q(p)})$ implies the same for $K^*(G; \mathbf{Q(p)})$. In fact, it can be proved that if G is semi-simple and $\tilde{G}$, Γ are as above, the image of the α-homomorphism $(\Gamma) \to \tilde{K}_0(G; \mathbf{Q(p)})$ is a non-trivial finite summand in $K^*(G; \mathbf{Q(p)})$ if $\Gamma \otimes \mathbf{Q(p)} \neq 0$. Given the reductions made in the proof of Proposition 1.1, this implies the converse.

§2. The Hopf algebra structure

The main theorem

In the remainder of this chapter we shall investigate the structure of $K^*(G; \mathbf{Q(p)})$ when it is a torsion-free $\mathbf{Q(p)}$-module. We have in Proposition 1.1 and the note following it the necessary and sufficient conditions for this to hold.

We begin by introducing the Hopf algebra structure in the K-theory of H-spaces. Let L be one of the rings $\mathbf{Q(p)}$; we shall say a space X is *L-free* if $K^*(X; L)$ is torsion-free. It follows from the Künneth formula that the Cartesian product $\kappa : K^*(X; L) \otimes_L K^*(X; L) \to K^*(X \times X; L)$ is an isomorphism if X is L-free. Suppose now X is an L-free H-space with product map $\mu : X \times X \to X$; then we can define a homomorphism $\kappa^{-1} \circ \mu^! : K^*(X; L) \to K^*(X; L) \otimes_L K^*(X; L)$. The definition of Hopf algebra, like that of algebra, extends immediately from the $\mathbf{Z}$-graded to the $\mathbf{Z}_2$-graded case. The following result is an immediate consequence of the definitions, proved in K-theory exactly as in cohomology [11, §7].

PROPOSITION 2.1.

(i) *Let X be an L-free H-space; then the diagonal homomorphism $\kappa^{-1}\mu^!$ gives the L-algebra $K^*(X; L)$ the structure of a $\mathbf{Z}_2$-graded augmented Hopf algebra with unit over L.*

(ii) *If $f: X_2 \to X_1$ is an H-map* [24] *of the L-free H-spaces X_2, X_1, then $f^! : (X_1; L) \to K^*(X_2; L)$ is a homomorphism of Hopf algebras.*

We are now ready to state the main result of this chapter. In I, §4(3) we defined a homomorphism of abelian groups $\tilde{\beta} : J(G) \to K^1(G)$. Let R be the image of $\tilde{\beta}$; we can regard $R \otimes L$ as embedded in $K^1(G; L)$ as a submodule.

THEOREM 2.1. *If G is an L-free connected Lie group, then*

(a) *$R \otimes L$ is the module of primitive elements in the Hopf algebra $K^*(G; L)$.*

(b) *If the exterior algebra $\Lambda_L(R \otimes L)$ is given the unique Hopf algebra structure in which the elements of $R \otimes L$ are primitive, the inclusion of $R \otimes L$ in $K^1(G; L)$ induces an isomorphism of Hopf algebras.*

$$\Lambda_L(R \otimes L) \approx K^*(G; L) \tag{1}$$

As a particular application of this theorem suppose G is a standard covering group (I, §4) of rank l. Since $H^*(G; \mathbf{Q}) \approx K^*(G; \mathbf{Q})$ is an exterior algebra on a $\mathbf{Q}$-module of rank l [22], l is also the rank of the $\mathbf{Q}$-module $R \otimes \mathbf{Q}$, and of the $\mathbf{Z}$-module R. By I, Proposition 4.1., $J(G)$ is a free $\mathbf{Z}$-module of rank l, so $\tilde{\beta} : J(G) \to R$ which is by definition an epimorphism must be an isomorphism. Since $\pi_1(G)$ is torsion-free, G is $\mathbf{Z}$-free (Proposition 1.1); we therefore have

COROLLARY 2.1. *If G is a standard covering group, $K^*(G)$ can be identified via $\tilde{\beta}$ with $\Lambda_{\mathbf{Z}}(J(G))$.*

COROLLARY 2.2. *If G is semi-simple and simply connected, $K^*(G) = \Lambda_{\mathbf{Z}}(\beta(\rho_1), \dots, \beta(\rho_l))$ where $\rho_1, \dots, \rho_l$ are the basic representations of G.*

In fact, we have seen that $j(\rho_1), \dots, j(\rho_l)$ form a basis for $J(G)$.

The proof of Theorem 2.1 proceeds in two stages. In the rest of §2 we prove a structure theorem for $\mathbf{Z}_2$-graded Hopf algebras (Theorem 2.2) which implies that under the conditions of Theorem 2.1, $K^*(G; L)$ is isomorphic as a Hopf algebra to the exterior algebra on its primitive elements. In §3 we use the fact that $K^*(G; L)$ has this structure to deduce part (a) of Theorem 2.1, whence part (b) follows.

Structure of $\mathbf{Z}_2$-graded Hopf algebras

For the rest of this chapter L will continue to denote a ring of type $\mathbf{Q}(\mathbf{p})$. If F is any free L-module we shall suppose the exterior algebra $\Lambda_L(F)$ is given the unique Hopf algebra structure in which the elements of F are primitive [28, 7.20]; we also $\mathbf{Z}_2$-grade it by giving the elements of F the degree 1. Let A be a $\mathbf{Z}_2$-graded augmented Hopf algebra with unit over L which is a finite-dimensional free L-module. Our structure theorem for Hopf algebras is the following:

THEOREM 2.2. *Let P_0 be the module of primitive elements in $A \otimes \mathbf{Q}$, P the module of primitive elements in A. Then $A \otimes \mathbf{Q} = \Lambda_{\mathbf{Q}}(P_0)$ implies $A = \Lambda_L(P)$ (as Hopf algebras).*

This theorem has the following consequence:

COROLLARY 2.3. *If G is L-free, then $K^*(G; L)$ is isomorphic as a Hopf algebra to the exterior algebra on its primitive elements; and these are homogeneous of degree* 1.

Proof of Corollary 2.3. In Theorem 2.2 set $A = K^*(G; L)$, so that $A \otimes \mathbf{Q} = K^*(G; \mathbf{Q})$ The isomorphism $ch_{\mathbf{Q}}$ (I(1.1)) is a natural transformation of *multiplicative* cohomology theories, and therefore transforms the Hopf algebra structure of $K^*(G; \mathbf{Q})$ into that of $H^*(G; \mathbf{Q})$. Using the theorem of Hopf–Samelson [21, 30] and this isomorphism,

$$H^*(G; \mathbf{Q}) = \Lambda_{\mathbf{Q}}(ch_{\mathbf{Q}}(P_0))$$

as a $\mathbf{Z}$-graded Hopf algebra, i.e.

$$K^*(\mathbf{G}; \mathbf{Q}) = \Lambda_{\mathbf{Q}}(P_0)$$

as a $\mathbf{Z}_2$-graded Hopf algebra. We can now apply Theorem 2.2 to prove the corollary.

Note. The difficulty in applying the standard structural results for Hopf algebras (e.g. [11, Proposition 7.3]) to $K^*(G; L)$ is that the existence of a $\mathbf{Z}$-grading on the Hopf algebra considered enters essentially in the proofs. Theorem 2.2 gets round this difficulty by supposing our $\mathbf{Z}_2$-graded algebra embedded in one which is in fact $\mathbf{Z}$-graded ($A \otimes \mathbf{Q}$). The idea of the proof is to use this $\mathbf{Z}$-grading to define a filtration on A itself, and then determine the structure of A inductively over the filtration subgroups.

Proof of Theorem 2.2. We begin by making some general remarks about Hopf algebras. Let B be an augmented Hopf algebra over L with unit; writing $\bar{B}$. for the kernel of the augmentation ε we have a direct sum decomposition.

$$B = L \oplus \bar{B} \tag{2}$$

Let $\imath : \bar{B} \to B$ be the inclusion, $\pi : B \to \bar{B}$ the projection corresponding to the decomposition (2). If ψ is the diagonal homomorphism of B we recall the *reduced* diagonal $\bar{\psi} : \bar{B} \to \bar{B} \otimes \bar{B}$ is defined to be the composition $\bar{B} \xrightarrow{\imath} B \xrightarrow{\psi} B \otimes B \xrightarrow{\pi \otimes \pi} \bar{B} \otimes \bar{B}$. (All unmarked tensor products are taken over L.) The module of primitive elements $P(B)$ is by definition the kernel of $\bar{\psi}$ [28, 3.7]. Since tensoring with $\mathbf{Q}$ is an exact functor it sends kernels into kernels, in particular

$$\overline{B \otimes \mathbf{Q}} = \bar{B} \otimes \mathbf{Q} \subset B \otimes \mathbf{Q} \tag{3}$$

$$P(B \otimes \mathbf{Q}) = P(B) \otimes \mathbf{Q} \subset \bar{B} \otimes \mathbf{Q}$$

We now turn to the particular Hopf algebra A, and assume that the condition $A \otimes \mathbf{Q} = \Lambda_{\mathbf{Q}}(P_0)$ is satisfied. It is an easy consequence of (I, Lemma 1.1) that if $f: C_1 \to C_2$ is a homomorphism of L-modules with C_2 free, then the kernel of f is a direct summand in C_1; applying this to $\bar{\psi}$ we see that $P = P(A)$ is a direct summand in A. Now (I, Lemma 1.1 (iii)) and (3) give

$$P = (P \otimes \mathbf{Q}) \cap A = P_0 \cap A. \tag{4}$$

The exterior algebra $\Lambda_L(P)$, regarded as embedded in $\Lambda_{\mathbf{Q}}(P_0)$, is the L-subalgebra generated by P. Hence $\Lambda_L(P) \subset A \subset \Lambda_{\mathbf{Q}}(P_0)$; we are also given that $A \otimes \mathbf{Q} = \Lambda_{\mathbf{Q}}(P_0) = \Lambda_L(P) \otimes \mathbf{Q}$.

We can give $\Lambda_L(P)$ and $\Lambda_{\mathbf{Q}}(P_0)$ compatible $\mathbf{Z}$-gradings in a standard way by assigning to the elements of P, P_0 the degree $+1$. These are then $\mathbf{Z}$-graded Hopf algebras; the components of degree $+1$ are exactly P resp. P_0, and the elements of positive degree form the summands $\overline{\Lambda_L(P)}$ resp. $\overline{\Lambda_{\mathbf{Q}}(P_0)}$. These gradings give rise in a natural way to filtrations

$$P = P^1 \subset P^2 \subset \cdots \subset P^n = \overline{\Lambda_L(P)} \tag{5}$$

$$P_0 = P_0^1 \subset P_0^2 \subset \cdots \subset P_0^n = \overline{\Lambda_{\mathbf{Q}}(P_0)}$$

where P^r is the sum of the components of $\overline{\Lambda_L(P)}$ of degree $\leq r$; similarly for P_0^r. Here n, the rank of P, P_0, is also the highest non-zero degree in the exterior algebras. It follows straightforwardly from the definitions that $P^r \otimes \mathbf{Q} = P_0^r$, and that P^r is a direct summand in $\Lambda_L(P)$. Hence by (I, Lemma 1.1), $P_0^r = P_0^r \cap \Lambda_L(P)$. Now if we define an analogous filtration of $\bar{A}$ by $P_A^r = P_0^r \cap \bar{A}$, we have

$$P^r \subset P_A^r \subset P_0^r \tag{6}$$

If we tensor (6) with $\mathbf{Q}$ the inclusions become equalities (since $P_0^r \otimes \mathbf{Q} = P_0^r = P^r \otimes \mathbf{Q}$), and so

$$P^r \otimes \mathbf{Q} = P_A^r \otimes \mathbf{Q}. \tag{7}$$

It is our aim to prove the following relation

$$P^r = P_A^r \qquad (r = 1, \ldots, n) \tag*{(8)$_r$}$$

by induction on r. Since $P^n = \overline{\Lambda_L(P)}$ and $P_A^n = \overline{\Lambda_{\mathbf{Q}}(P_0)} \cap A = \bar{A}$, this will prove the theorem. To start the induction we note that $P^1 = P$, $P_0^1 = P_0$ and so $P_A^1 = P_0 \cap A = P$ by (4). Hence $(8)_1$ is proved. Now assume $(8)_{r-1}$ for $r > 1$. It follows from (6), (7) that P^r is an L-submodule of maximal rank in P_A^r. $(8)_r$ will therefore be proved if we can show that P^r is a direct summand in P_A^r. For this we shall use the criterion (ii) of (I, Lemma 1.1), i.e. we prove that for $p \in \Pi - (\mathbf{p})$ (where $L = \mathbf{Q}(\mathbf{p})$),

$$x \in P_A^r,\ px \in P^r \Rightarrow x \in P^r. \tag{9}$$

We begin by showing that any $x \in P_A^r$ satisfies

$$\bar{\psi}(x) \in P^{r-1} \otimes P^{r-1} \tag{10}$$

In fact it follows, from the definition of the filtration on $\overline{\Lambda_{\mathbf{Q}}(P_0)}$ that $\bar{\psi}(P_0)$ is contained in $P_0^{r-1} \otimes P_0^{r-1}$; this holds for any filtration derived from a grading on a connected Hopf algebra in this way. Hence $\bar{\psi}(x) \in (P_0^{r-1} \otimes P_0^{r-1}) \cap (A \otimes A)$, a module which can also be written $((P^{r-1} \otimes P^{r-1}) \otimes \mathbf{Q}) \cap (A \otimes A)$. Now by the induction hypothesis $P^{r-1} = P_A^{r-1} = P_0^{r-1} \cap A$; by the criterion (iii) of (I, Lemma 1.1) this is a direct summand in A. Hence $P^{r-1} \otimes P^{r-1}$ is a direct summand in $A \otimes A$; we deduce that $((P^{r-1} \otimes P^{r-1}) \otimes \mathbf{Q}) \cap (A \otimes A) = P^{r-1} \otimes P^{r-1}$ which proves (10).

Now suppose in addition that $px = x' \in P^r$. The reduction mod p induces a map of Hopf algebras $\rho_p : \Lambda_L(P) \to \Lambda_{\mathbf{Z}_p}(P \otimes \mathbf{Z}_p)$. Now (10) enables us to define $(\rho_p \otimes \rho_p)(\bar{\psi}(x))$; hence $\bar{\psi}(\rho_p(x')) = (\rho_p \otimes \rho_p)(\bar{\psi}(x')) = p(\rho_p \otimes \rho_p)(\bar{\psi}(x)) = 0$, i.e. $\rho_p(x')$ is primitive in $\Lambda_{\mathbf{Z}_p}(P \otimes \mathbf{Z}_p)$. But $P(\Lambda_{\mathbf{Z}_p}(P \otimes \mathbf{Z}_p)) = P \otimes \mathbf{Z}_p$ [28, 7.20], so that $\rho_p(x') \in P \otimes \mathbf{Z}_p$. We can therefore find $x_0 \in P$ and $y \in \Lambda_L(P)$ such that $x' = x_0 + p \cdot y$. Replacing x' by px we have $x_0 = p(x - y)$, so that $p(x - y) \in P$, $x - y \in A$. Since P is a direct summand in A, this implies $x - y \in P$. But $y \in \Lambda_L(P)$, so $x \in \Lambda_L(P) \cap P_A^r = P^r$. This proves (10), and so completes the induction for $(8)_r$.

§3. The primitive elements

To provide a proof for the remaining part (a) of Theorem 2.1. we begin by characterizing the module $R \otimes L$ in a different way. Let $G \xrightarrow{i} EG \xrightarrow{\pi} BG$ be a universal G-bundle. The pair (EG, G) has the homotopy type of the pair (CG, G) and so is in $\mathfrak{A}$; since EG is contractible $\delta : K^1(G; L) \to K^0(EG, G; L)$ is an isomorphism. Define the *universal suspension* σ_G to be the composition $\delta^{-1} \circ \pi^! : \underleftarrow{\tilde{K}}^0(BG; L) \to K^1(G; L)$, by analogy with cohomology.

LEMMA 3.1. *The module $R \otimes L$ is the image of σ_G in $K^1(G; L)$.*

Proof. It is sufficient to consider the case $L = \mathbf{Z}$. In this case we can apply I, Proposition 4.1, modified by taking $\underleftarrow{\tilde{K}}^0(BG)$ instead of $\tilde{K}^0(B_\xi)$. We have that $\delta(R) = \operatorname{Im}(\delta \circ \beta) =$

$\operatorname{Im}(\pi^! \circ \alpha)$. Since $\operatorname{Im} \alpha$ is dense in $\overleftarrow{\tilde{K}^0(BG)}$ [9, §4], $\operatorname{Im}(\pi^! \circ \alpha)$ is dense in $\operatorname{Im} \pi^! \subset K^0(EG, G)$. But as stated above (EG, G) is a pair in $\mathfrak{A}$ and so its filtration topology is the discrete topology. Hence $\operatorname{Im} \sigma_G = \operatorname{Im} \delta^{-1} \circ \pi^! = \operatorname{Im} \delta^{-1} \circ \alpha = \operatorname{Im} \beta = R$ as required.

Theorem 2.1 (a) is therefore equivalent to the statement that the primitive elements in $K^1(G; L)$ are exactly the image of σ_G, or in the traditional language of cohomology the 'universally transgressive elements'. The equivalent result in cohomology has been proved in several different ways—the spectral sequence of the universal bundle [11, §20, etc.], the Eilenberg-Moore spectral sequence [15], and most recently the 'geometric bar construction' of Rothenberg and Steenrod [29]. The first two methods do not carry over directly to K-theory; the third does, and we shall rely on the exposition in [29] with indications of how it should be adapted.

G will continue to be an L-free Lie group; let $\mathscr{E}$ be the Milnor G-resolution. [29, §3]. We have a space $|\mathscr{E}|$, acted on freely by G and filtered by G-subspaces $\mathscr{E}_0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \ldots$; the quotient space is $B = \mathscr{E}/G$ filtered by the quotients $B_i = \mathscr{E}_i/G$. In our previous terminology $|\mathscr{E}| = EG$, $B = BG$. B is in $\hat{\mathfrak{A}}$ and all the other spaces mentioned are in $\mathfrak{A}$. In K-theory with coefficients in L the filtrations of $|\mathscr{E}|$, B give rise to spectral sequences (I, §2) which we write $\{E_r(|\mathscr{E}|; L)\}$, $\{E_r(B; L)\}$ respectively.

Since we have not developed a homology theory dual to K-theory (but see [33]) we must diverge to some extent from the treatment of these spectral sequences in [29], to which our method is dual. Let us write A for $K^*(G; L)$ considered as a coalgebra; for any G-space X in $\mathfrak{A}$ the Künneth formula shows that the structure map of X induces a homomorphism

$$K^*(X; L) \to K^*(X \times G; L) = K^*(X; L) \otimes_L A.$$

This is an A-comodule structure on $K^*(X; L)$ and the correspondence is functorial from G-spaces to A-comodules. A is flat over L, so that A-comodules form an abelian category [28, 2.2]. We can therefore dualize the arguments in [29, §5] to arrive at the following results. Because of this duality, the proof will be omitted, though we shall need to make certain stages explicit later on, referring to the corresponding stages in the homology proof as justification.

THEOREM 3.1. (i) *The initial stage* $\{E_1(|\mathscr{E}|; L), d_1\}$ *of the spectral sequence of* $|\mathscr{E}|$ *is isomorphic to the cobar resolution for* A *as* $\mathbf{Z}_2$*-graded coaugmented* L*-coalgebra* [29, 5.3]

(ii) *There is an isomorphism of complexes*

$$\{E_1(B; L), d_1\} \approx \{E_1(|\mathscr{E}|; L), d_1\} \square_A L$$

(iii) *This induces a homology isomorphism*

$$E_2^{p\alpha}(B; L) \approx \operatorname{Cotor}_A^{p\alpha}(L, L).$$

To do computations we need to know a bit more about the cobar resolution. Let A be any $\mathbf{Z}_2$-graded L-coalgebra with coaugmentation $\eta : L \to A$, supposed free as an L-module. Let the cobar resolution be the complex

$$X : (L \xrightarrow{\eta}) A = X_0 \xrightarrow{\delta_1} X_1 \xrightarrow{\delta_2} X_2 \to \cdots$$

The complex $L \square_A X$ for computing the groups Cotor $^p_A(L, L)$ will be written

$$Y : Y_0 \xrightarrow{\delta_1} Y_1 \xrightarrow{\delta_2} Y_2 \to \cdots$$

with $Y_i = L \square_A X_i$. We can identify X_i with the tensor product $A \otimes \bar{A} \otimes \ldots \otimes \bar{A}$ (i factors are $\bar{A} = \text{Coker } \eta$), and Y_i is then $\bar{A} \otimes \ldots \otimes \bar{A} = L \otimes \bar{A} \otimes \ldots \otimes \bar{A} \subset X_i$.

The following characterization of the cobar resolution can be proved dually to [29, 5.6 Lemma] (in fact forms a necessary part of the proof of Theorem 3.1).

LEMMA 3.2. *The resolution X is characterized by the following conditions*

(i) $X_0 = A$

(ii) X_{n+1} *is isomorphic to* $A \otimes \text{Coker } \delta_n$ *as an A-comodule* (δ_0 *set equal to* η)

(iii) *Under this isomorphism* δ_{n+1} *corresponds to the composition* $X_n \xrightarrow{\Delta_n} A \otimes X_n \to A \otimes \text{Coker } \delta_n$ *where* Δ_n *is the structure homomorphism of the comodule* X_n.

Now suppose that A has the counit $\varepsilon : A \to L$, so $\bar{A}$ is naturally isomorphic to Ker ε. Then we can determine the low degree terms in Cotor.

LEMMA 3.3. *There is a commutative diagram*

$$\begin{array}{ccccc} Y_0 & \xrightarrow{\delta_1} & Y_1 & \xrightarrow{\delta_2} & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ L & \xrightarrow{0} & \bar{A} & \xrightarrow{\bar{\psi}} & \bar{A} \otimes \bar{A} \end{array}$$

where the columns are isomorphisms and $\bar{\psi}$ *is the reduced diagonal* (see §2). *Hence* Cotor ^1_A $(L, L) = H^1(Y)$ *can be identified with the module of primitive elements* $P(A)$.

Proof. First, $X_0 = A$, $X_1 = A \otimes \bar{A}$ and by Lemma 3.2 (iii), δ_1 is the composition

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{1 \otimes \pi} A \otimes \bar{A}$$

(Here π is again the projection $A \to \bar{A}$). In particular $\delta_1(L) = 0$ which establishes the left square in the commutative diagram.

Next, consider the homomorphism $\theta : A \otimes \bar{A} \to \bar{A} \otimes \bar{A}$ defined by $\theta(a \otimes b) = \varepsilon(a)\delta_1(b) - a \otimes b$. θ is equal to -1 on $\bar{A} \otimes \bar{A}$ and so is an epimorphism; it is easy to see that the sequence

$$A \xrightarrow{\delta_1} A \otimes \bar{A} \xrightarrow{\theta} \bar{A} \otimes \bar{A} \to 0$$

is exact. Hence we can use θ to identify Coker δ_1 with $\bar{A} \otimes \bar{A}$. Again by Lemma 3.2 (iii), δ_2 is the composition

$$A \otimes \bar{A} \xrightarrow{\psi \otimes 1} A \otimes A \otimes \bar{A} \xrightarrow{1 \otimes \theta} A \otimes \bar{A} \otimes \bar{A}$$

To identify δ_2 it is sufficient to see how this homomorphism behaves on $L \otimes \bar{A} \subset A \otimes \bar{A}$. Given $a \in \bar{A}$,

$$1 \otimes \delta_2(a) = \delta_2(1 \otimes a) = (1 \otimes \theta)(1 \otimes 1 \otimes a) = 1 \otimes \delta_1(a) - 1 \otimes 1 \otimes a = 1 \otimes \bar{\psi}(a).$$

This proves the commutativity of the second square.

Let us now return to the geometric situation. We have in Lemma 3.3 an identification of Y_1 with $\bar{A} = \tilde{K}^*(G; L)$. We can also identify (B_1, B_0) with $(SG, *)$, and so $E_1^1(B; L)$ with $\tilde{K}^*(SG; L)$. The proof of Theorem 3.1, which in the model of [29, §5] follows closely the analogy between constructions on A-comodules and G-spaces, leads to an identification of $E_1^1(B; L)$ with Y_1 in such a way as to make the diagram

$$\begin{tikzcd}
\tilde{K}^*(G; L) = \bar{A} \arrow[r] \arrow[d, "S"'] & Y_1 \arrow[d] \\
\tilde{K}^*(SG; L) = K^*(B_1, B_0; L) & E_1^1(B; L) \arrow[l, phantom, "="]
\end{tikzcd}$$

of identifications commutative. Lemma 3.3 therefore has the following consequence.

COROLLARY 3.1. *If G is an L-free group, $E_2^1(B; L) \subset E_1^1(B; L) = \tilde{K}^*(SG; L)$ is the suspension of the module P of primitive elements in $\tilde{K}^*(G; L)$.*

We next give a parallel characterization of the 'universally transgressive' elements $\operatorname{Im} \sigma_G$.

PROPOSITION 3.2. *Let G be any Lie group and suppose $\{E_r(B; L)\}$ converges strongly to $K^*(B; L)$. Then $E_\infty^1(B; L) \subset E_1^1(B; L) = \tilde{K}^*(SG; L)$ is the suspension of* $\operatorname{Im} \sigma_G \subset \tilde{K}^*(G; L)$.

Proof. First, the strong convergence implies that $E_\infty^1(B; L)$ is the image of the restriction homomorphism $K^*(B, B_0; L) \to K^*(B_1, B_0; L)$. Call this restriction $i^!$; then the diagram

$$\begin{tikzcd}
\tilde{K}^1(G; L) \arrow[r, "\delta", "\approx"'] \arrow[d, "S"', "\approx"] & K^0(|\mathscr{E}|, G; L) \\
\tilde{K}^0(SG; L) = \tilde{K}^0(B_1; L) \arrow[r, "i^!"] \arrow[r, phantom, "\leftarrow"', shift right=3] & \tilde{K}^0(B; L) \arrow[u, "\pi^!"']
\end{tikzcd}$$

is commutative up to sign (for the proof in homology, which can be imitated in K-theory, see [25, p.10]). Hence $E_\infty^1(B; L) = \operatorname{Im} i^! = \operatorname{Im} (S \circ \delta^{-1} \circ \pi^!) = S(\operatorname{Im} \sigma_G)$.

By putting together the last two results we have

COROLLARY 3.2. *If G is L-free and $\{E_r(B; L)\}$ converges strongly to $K^*(B; L)$, the following are equivalent*

(i) $P = \operatorname{Im} \sigma_G$ *in* $\tilde{K}^*(G; L)$

(ii) $E_2^1 = E_\infty^1$ *in* $\{E_r(B; L)\}$.

It only remains to compute the spectral sequence $\{E_r(B; L)\}$ in the case which interests us, i.e. when G is connected and L-free. Then (Corollary 2.3), $K^*(G; L)$ is the exterior algebra on the module P of primitive elements. This is a self-dual Hopf algebra [18, exp. 9, p.8], in particular, its coalgebra structure is dual to the algebra structure of a plain exterior algebra. Let us call such a coalgebra an 'exterior coalgebra', always supposed to be $\mathbf{Z}_2$-graded in such a way as to give P degree 1.

LEMMA 3.4. *Let A be an exterior coalgebra, with module P of primitive elements. Then* $\Sigma\ \mathrm{Cotor}^{p\alpha}(L, L)$ $(p \in \mathbf{Z}, \alpha \in \mathbf{Z}_2)$ *is isomorphic as a $(\mathbf{Z} + \mathbf{Z}_2)$-graded L-module to the symmetric algebra of* $P = \mathrm{Cotor}^{1,1}_A(L, L)$ (see Lemma 3.3).

Proof. The dual result for algebras is proved in [18]; the lemma can be proved by the dual method.

Note. The sum of the Cotor groups can in fact be given a natural product structure in this case so as to make the isomorphism multiplicative.

PROPOSITION 3.3. *Suppose G is a connected L-free Lie group. Then the spectral sequence $\{E_r(B; L)\}$ collapses, in the sense that $E_2(B; L) = E_\infty(B; L)$.*

Proof. Since $A = K^*(G; L)$ is an exterior coalgebra, Lemma 3.4 (with Theorem 3.1) implies $E_2^{p\alpha}(B; L) = 0$ unless $\alpha = \rho_2(p)$. Since the differential d_r has bidegree $(r, -\rho_2(r-1))$, it must be zero for all $r \geq 2$, and the spectral sequence collapses.

We can now complete the proof of Theorem 2.1(a). In fact, if G is L-free and connected, $\{E_r(B; L)\}$ collapses as has just been shown; collapsing implies strong convergence by I, Proposition 2.1. Hence we can deduce, by Lemma 3.1 and Corollary 3.2 that $P = \mathrm{Im}\ \sigma_G = R \otimes L$.

III. THE ADDITIVE STRUCTURE

§1. General considerations

In this chapter we give the enumerative proof of II, Theorem 1.1 which was referred to there, thus completing the proof of those results in chapter II which depended on it. We begin by reducing the question to explicit computations in a well-defined and countable set of spectral sequences.

We recall that II, Theorem 1.1 states

(A) *For G semi-simple and simply-connected $K^*(G)$ is torsion free.*

It is in fact sufficient to prove (A) for simple groups. For suppose this done; then by the decomposition theorem for semi-simple and simply-connected groups, we can write $G = G_1 \times \ldots \times G_k$ with $G_1, \ldots, G_k$ simple and simply-connected. Now $K^*(G_i)$ is free for $i = 1, \ldots, k$ by hypothesis; hence by the Künneth formula, $K^*(G) = K^*(G_1) \otimes \ldots \otimes K^*(G_k)$ is also free.

The simple and simply-connected groups can be classified in a well-known way, and a great deal is known about their cohomology. The next reduction we can make is a consequence of the following lemma. (Here as throughout this chapter p denotes a prime, not restricted to be odd unless this is specified).

LEMMA 1.1. *If $X \in \mathfrak{A}$, $K^*(X)$ has torsion of order p only if $H^*(X; \mathbf{Z})$ does.*

Proof. Let $\mathbf{Q}_p$ denote $\mathbf{Z}$ localized at the prime p ($\mathbf{Q}(\Pi - \{p\})$ in the notation of I §1). We recall two elementary results on torsion in abelian groups.

(i) If A is a finitely generated abelian group, A has p-torsion if and only if $A \otimes \mathbf{Q}_p$ does.

(ii) Under the conditions of (i), $A \otimes \mathbf{Q}_p$ has no p-torsion if and only if the natural homomorphism $A \otimes \mathbf{Q}_p \to A \otimes \mathbf{Q}$ is injective.

Now for $X \in \mathfrak{A}$, $H^*(X; \mathbf{Z})$ is finitely generated, and so is $K^*(X)$. If $H^*(X; \mathbf{Z})$ has no p-torsion, by (i), (ii) the homomorphism of spectral sequences

$$\{E_r(X; \mathbf{Q}_p)\} \to \{E_r(X; \mathbf{Q})\}$$

induced by the inclusion $\mathbf{Q}_p \subset \mathbf{Q}$, is injective on the E_2-term:

$$E_2(X; \mathbf{Q}_p) = H^*(X; \mathbf{Q}_p) \to H^*(X; \mathbf{Q}) = E_2(X; \mathbf{Q})$$

But $\{E_r(X; \mathbf{Q})\}$ collapses in the sense that $E_2 = E_\infty$ [9, §2.4]; hence so does $\{E_r(X; \mathbf{Q}_p)\}$. Now the graded group associated to $K^*(X; \mathbf{Q}_p)$ is $E_\infty(X; \mathbf{Q}_p) = E_2(X; \mathbf{Q}_p)$ which has no p-torsion; so $K^*(X; \mathbf{Q}_p)$ has none. The Lemma now follows using (i).

Let us consider pairs (G, p) such that G is a simple simply-connected group and p a prime; let $\mathcal{O}$ be the set of those pairs (G, p) such that $H^*(G; \mathbf{Z})$ has p-torsion. Then if $(G, p) \notin \mathcal{O}$ we can use Lemma 1.1 to deduce that $K^*(G)$ has no p-torsion. To prove (A), therefore it is sufficient to prove

(B) If $(G, p) \in \mathcal{O}$ then $K^*(G)$ has no p-torsion.

The pairs $(G, p) \in \mathcal{O}$ have been listed by Borel [14, 2.5]; they are given by

$$p = 2,\ G = \mathbf{Spin}\,(n)(n \geq 7),\ \mathbf{G}_2,\ \mathbf{F}_4,\ \mathbf{E}_6,\ \mathbf{E}_7,\ \mathbf{E}_8$$

$$p = 3,\ G = \mathbf{F}_4,\ \mathbf{E}_6,\ \mathbf{E}_7,\ \mathbf{E}_8 \tag{1}$$

$$p = 5,\ G = \mathbf{E}_8.$$

LEMMA 1.2. *For $X \in \mathfrak{A}$ $\dim_p K^*(X; \mathbf{Z}_p) \geq \dim_{\mathbf{Q}} K^*(X; \mathbf{Q})$, and equality holds if and only if $K^*(X)$ has no p-torsion.*

Proof. Let $P \subset K^*(X)$ be the subgroup of elements of order a power of p; T the subgroup of elements of finite order prime to p; and F a maximal free subgroup of $K^*(X)$. The standard decomposition for finitely generated abelian groups implies $K^*(X) = F \oplus P \oplus T$ Hence the universal coefficient theorem implies

$$K^*(X; \mathbf{Z}_p) = (F \otimes \mathbf{Z}_p) \oplus (P \otimes \mathbf{Z}_p) \oplus (\mathrm{Tor}\,(P, \mathbf{Z}_p))$$

all other summands vanishing; while $K^*(X; \mathbf{Q}) = F \otimes \mathbf{Q}$. Now $\dim_{\mathbf{Q}} F \otimes \mathbf{Q} = \dim_p F \otimes \mathbf{Z}_p$ (both being equal to the rank of the free $\mathbf{Z}$-module F); while $P \otimes \mathbf{Z}_p$ and $\mathrm{Tor}\,(P, \mathbf{Z}_p)$ are zero if and only if $P = 0$, i.e. $K^*(X)$ has no p-torsion. The lemma follows.

Let G be a simple group of rank l; then $\dim_{\mathbf{Q}}(K^*(G; \mathbf{Q})) = \dim_{\mathbf{Q}}(H^*(G; \mathbf{Q}))$ which is equal to 2^l. Hence (B) is equivalent to

(C) If $(G, p) \in \mathcal{O}$ and G has rank l, then $\dim_p(K^*(G; \mathbf{Z}_p)) = 2^l$.

We shall calculate $\dim_p K^*(G; \mathbf{Z}_p)$ by means of the standard spectral sequence $\{E_r(G; \mathbf{Z}_p)\}$ using the more or less complete descriptions of $H^*(G; \mathbf{Z}_p)$ available for $(G, p) \in \mathcal{O}$ [2, 3, 4, 12, 14]. The following inequalities are applications of general spectral sequence theory, and Lemma 1.2.

$$\dim_p(E_r(G; \mathbf{Z}_p)) \geq \dim_p(E_\infty(G; \mathbf{Z}_p)) = \dim_p(K^*(G; \mathbf{Z}_p)) \geq 2^l.\ (2 \leq r < \infty). \tag{2}$$

In §§2, 3 we prove the following proposition on the spectral sequences $\{E_r(G; \mathbf{Z}_p)\}$; taken together with the inequalities (2) it implies (C) and so II, Theorem 1.1.

PROPOSITION 1.1. (i) *If* (G, p) *is a pair in* $\mathcal{O}$ *other than* $(\mathbf{E}_7, 2)$, $(\mathbf{E}_8, 2)$ *and* G *has rank* l, *then* $\dim_p(E_{2p}(G; \mathbf{Z}_p)) = 2^l$.

(ii) *If* $G = \mathbf{E}_l$ $(l = 7$ *or* $8)$, *then* $\dim_2(E_6(G; \mathbf{Z}_2)) = 2^l$.

The method of proving Proposition 1.1 will be as follows: first we prove that the differential d_{2p-1} in $\{E_r(X; \mathbf{Z}_p)\}$ is the first non-vanishing one and determine it as a stable cohomology operation. In §2 we apply this to finding $\dim_p(E_{2p}(G; \mathbf{Z}_p))$ in case (i) and so prove the first proposition. In §3 we consider the cases $(\mathbf{E}_7, 2)$, $(\mathbf{E}_8, 2)$ in detail, first determining the E_4-term in the spectral sequence and then making an *ad hoc* calculation of the differential d_5 in the one case where it is needed. This will be sufficient to prove part (ii).

The differentials

We prove the following result about $\{E_r(X; \mathbf{Z}_p)\}$.

PROPOSITION 1.2. *In the spectral sequence* $\{E_r(X; \mathbf{Z}_p)\}$

(i) $d_r = 0$ *for* $2 \leq r \leq 2p-2$, *so that for* $2 \leq r \leq 2p-1$ $E_r^q(X; \mathbf{Z}_p)$ *can be identified with* $H^q(X; \mathbf{Z}_p)$.

(ii) *Using the above identification,* d_{2p-1} *is equal (up to multiplication by a non-zero element of* $\mathbf{Z}_p$*) to Milnor's stable cohomology operation* $Q_1 : H^q(X; \mathbf{Z}_p) \to H^{q+2p-1}(X; \mathbf{Z}_p)$.

Proof. This result is closely related to Proposition 7.2 of [10]; it is slightly simpler because of what we know about the Steenrod algebra $\mathscr{A}_p$. In fact, suppose we have proved that $d_i = 0$ for $1 < i < r$ where $r \leq 2p-1$; then by standard arguments $d_r : H^q(X; \mathbf{Z}_p) \to H^{q+r}(X; \mathbf{Z}_p)$ is a stable primary cohomology operation, i.e. an element of $\mathscr{A}_p^r$. But $\mathscr{A}_p^r = 0$ for $2 \leq r \leq 2p-2$, and all even differentials vanish by considering the secondary $(\mathbf{Z}_2-)$ grading (cf. [9, 2.1]). This proves part (i).

Now consider $d_{2p-1} \in \mathscr{A}_p^{2p-1}$. The operations $\mathscr{P}^1\delta$ and $\delta\mathscr{P}^1$ form a $\mathbf{Z}_p$-basis for $\mathscr{A}_p^{2p-1}$ (this is still true for $p = 2$ setting $\mathscr{P}^1 = Sq^2$, $\delta = Sq^1$). Write $d_{2p-1} = a.\mathscr{P}^1\delta + b.\delta\mathscr{P}^1$; using the Adem relations we find

$$d_{2p-1}d_{2p-1} = (a+b)^2\delta\mathscr{P}^2\delta$$

Hence for d_{2p-1} to be a differential we must have $a + b = 0$, so that d_{2p-1} is indeed a multiple of Q_1. It remains to show that $d_{2p-1} \neq 0$.

Consider the space $X = S^{2n}P_p(\mathbf{C}) \cup e^{2n+2p+1}$ of [10, §7] where the cell is attached by a map of degree Mp, M prime to p. We have

$$H^{2i}(X) = \mathbf{Z} \ (n < i < n+p),\ H^{2n+2p+1}(X) = \mathbf{Z}_p$$
$$H^i(X) = 0 \text{ otherwise}$$

so

$$H^{2i}(X; \mathbf{Z}_p) = \mathbf{Z}_p \ (n < i \leq n+p),\ H^{2n+2p+1}(X; \mathbf{Z}_p) = \mathbf{Z}_p$$
$$\tilde{H}^i(X; \mathbf{Z}_p) = 0 \text{ otherwise.}$$

From [10, §7], if $\sigma^{2n}(x)$ generates $H^{2n+2}(X; \mathbf{Z})$, y generates $H^{2p+2\ +1}(X; \mathbf{Z})$, we have in $\{E_r(X; \mathbf{Z})\}$ that

$$d_r(\sigma^{2n}(x)) = 0 \text{ for } 2 \leq r < 2p-1,\ d_{2p-1}(\sigma^{2n}(x)) = \frac{My}{(p-1)!}$$

Hence, using the morphism of spectral sequences induced by ρ_p, in $\{E_r(X; \mathbf{Z}_p)\}$

$$d_r(\rho_p\sigma^{2n}(x)) = 0 \text{ for } 2 \leq r < 2p-1,\ d_{2p-1}(\rho_p\sigma^{2n}(x)) = \frac{M\rho_p(y)}{(p-1)!}$$

Since $\rho_p(y)$ is a generator of $H^{2p+2r+1}(X; \mathbf{Z}_p)$ and so nonzero, and M is prime to p, d_{2p-1} is indeed nonzero.

§2. Proof of Proposition 1.1 (i)

The case $p \neq 2$

The proof of the first part of Proposition 1.1 is now a matter of straightforward computation. In fact for any $X \in \mathfrak{A}$ and prime p we have by Proposition 1.2 an isomorphism of complexes $(E_{2p-1}(X; \mathbf{Z}_p), d_{2p-1}) \approx (H^*(X; \mathbf{Z}_p), Q_1)$. Let $(G, p) \in \mathcal{O}$, and suppose (G, p) is not $(\mathbf{E}_7, 2)$ or $(\mathbf{E}_8, 2)$. Then Proposition 1.1 (i) is equivalent to

(1) $$\dim_p H(H^*(G; \mathbf{Z}_p), Q_1) = 2^l \text{ where } l = \text{rank } G.$$

The rings $H^*(G; \mathbf{Z}_p)$ have been calculated in all cases, with the action of $\mathscr{A}_p$ to a certain extent [2, 3; 12, IV, V; 14, §2]. We shall begin by considering the five cases where p is odd, as all these have a simple structure, as follows.

(2.1). For $(G, p) \in \mathcal{O}$, p odd, the cohomology rings $H^*(G; \mathbf{Z}_p)$ are the following. (x_i denotes a generator of dimension i).

(a) $H^*(\mathbf{F}_4; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$
[12, Théorème 19.2 (c)].

(b) $H^*(\mathbf{E}_6; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})$
[14, §2.3 Théorème 1].

(c) $H^*(\mathbf{E}_7; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})$
[2, Theorem 8, p. 426].

(d) $H^*(\mathbf{E}_8; \mathbf{Z}_3) = \mathbf{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$
[2, Theorem 9, p. 433].

In cases (a)–(d) we have the relations

$\mathscr{P}_3^1 x_3 = x_7$, $\delta_3 x_7 = x_8$ while in case (d) alone

$\mathscr{P}_3^1 x_{15} = -x_{19}$, $\delta_3 x_{19} = x_{20}$

(e) $H^*(\mathbf{E}_8; \mathbf{Z}_5) = \mathbf{Z}_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{35}, x_{39}, x_{47})$ and $\mathscr{P}_5^1 x_3 = x_{11}$, $\delta_5 x_3 = x_{12}$.
[14, 2.3, Théorème 2].

The only part of (2.1) which is not proved explicitly in the sources referred to is the relation $\mathscr{P}_3^1 x_{15} = -x_{19}$ in $H^*(\mathbf{E}_8; \mathbf{Z}_3)$. $H^*(\mathbf{E}_8; \mathbf{Z}_3)$ is determined by Araki [2] by means of the Postnikov system. Using Araki's notation X_k for the complex occupying the k^{th} place in the Postnikov system of $\mathbf{E}_8(\pi_i(X_k) \approx \pi_i(\mathbf{E}_8),\ i \le k;\ \pi_i(X_k) = 0,\ i > k)$, the fibering $X_{17} \to X_{15}$ induces isomorphisms in cohomology mod 3 and by Araki's formula

$$H^*(X_{15}; \mathbf{Z}_3) = \Lambda_3(u_3, \mathscr{P}^1 u_3, \mathscr{P}^3\mathscr{P}^1 u_3, u_{15}, \mathscr{P}^1 u_{15}) \otimes \mathbf{Z}_3[\delta_3\mathscr{P}^1 u_3, \delta_3\mathscr{P}^3\mathscr{P}^1 u_3, \delta_3\mathscr{P}^1 u_{15}]$$

in dimensions ≤ 21. Under the homomorphism induced by the canonical fibering $\mathbf{E}_8 \to X_{17}$ the elements $u_3, u_{15}, \mathscr{P}^3\mathscr{P}^1 u_3$ in $H^*(X_{17}; \mathbf{Z}_3)$ map to x_3, x_{15}, x_{19} in $H^*(\mathbf{E}_8; \mathbf{Z}_3)$, so that the image of the mod 3 k-invariant $k_3^{19}(\mathbf{E}_8) = \mathscr{P}^1 u_{15} + \mathscr{P}^3\mathscr{P}^1 u_3$ [2, (38)] is $\mathscr{P}^1 x_{15} + x_{19}$. Since a mod p k-invariant of X has trivial image in $H^*(X; \mathbf{Z}_p)$ [2, (26')], $\mathscr{P}^1 x_{15} = -x_{19}$ as stated.

Finally, for dimensional reasons $\delta_p(x_3) = 0$ in all cases (a)–(e) above. In case (d), x_{15} is the image of a fundamental class in the cohomology of $K(\mathbf{Z}, 15)$, itself the mod 3 reduction of an integral class, so that $\delta_3(x_{15})$ is also zero.

It is now simple to evaluate $Q_1 = \mathscr{P}^1\delta - \delta\mathscr{P}^1$ on the elements x_3, x_{15}; for $\mathscr{P}^1\delta$ is trivial on them by the previous remark, so that we have the following limited information.

$$\begin{aligned} &Q_1 x_3 = -\delta_3\mathscr{P}_3^1 x_3 = -x_8 \text{ in cases (a)–(d)} \\ (2)\qquad &Q_1 x_{15} = -\delta_3\mathscr{P}_3 x_{15} = x_{20} \text{ in case (d)} \\ &Q_1 x_3 = -\delta_5\mathscr{P}_5^1 x_3 = -x_{12} \text{ in case (e)} \end{aligned}$$

We can now give a unified description of the Q_1-complexes $H^*(G; \mathbf{Z}_p)$.

LEMMA 2.1. *For (G, p) as in (2.1), we can write $H^*(G; \mathbf{Z}_p) = A \otimes B$ where*

(i) *A is a Q_1-subalgebra; $H(A)$ is an exterior algebra*

(ii) *B is an exterior algebra*

(iii) *The dimension of $H(A) \otimes B$ over $\mathbf{Z}_p$ is 2^l.*

Proof. Since Q_1 is a derivation, a subalgebra A is a Q_1-subalgebra if its generators form a closed set under Q_1. We list the algebras A for the various cases of (2.1):

$$\begin{aligned} &\text{In (a)–(c)}, \quad A = \Lambda(x_3) \otimes \mathbf{Z}_3[x_8]/(x_8^3) \\ (2)\qquad &\text{In (d)}, \quad A = \Lambda(x_3, x_{15}) \otimes \mathbf{Z}_3[x_8, x_{20}]/[x_8^3, x_{20}^3] \\ &\text{In (e)}, \quad A = \Lambda(x_3) \otimes \mathbf{Z}_5[x_{12}]/(x_{12}^5) \end{aligned}$$

In each case we take B to be the subalgebra generated by all x_i not listed as generators of A; this is an exterior algebra whose rank is $l - 1$ except in case (d) when it is $l - 2$.

The calculation of $H(A)$ from (3) using (2) (and the fact that Q_1 is a derivation) is now easy. We have

$$\begin{aligned} &\text{In cases (a)–(c)}, \quad H(A) = \Lambda(\{x_3 x_8^2\}) \\ (4)\qquad &\text{In case (d)}, \quad H(A) = \Lambda(\{x_3 x_8^2\}, \{x_{15} x_{20}\}) \\ &\text{In case (e)}, \quad H(A) = \Lambda(\{x_3 x_{12}^4\}) \end{aligned}$$

where $\{x\}$ denotes the homology class of x.

It now follows that in all cases $H(A) \otimes B$ is an exterior algebra of rank l, and so of dimension 2^l; this completes the proof of Lemma 2.1.

As a result of Lemma 2.1 we can apply a 'twisted tensor product' argument [18, exposé 3, p. 6]. By this, there is a spectral sequence with $E_0 = A \otimes B$, $E_1 = H(A) \otimes B$

and E_∞ the graded module corresponding to some filtration of $H(A \otimes B)$. We can use standard results on dimension in spectral sequences to deduce

$$\text{(5)} \qquad \dim_p \; H(A \otimes B) = \dim_p E_\infty \leq \dim_p E_1 = \dim_p (H(A) \otimes B).$$

Since $\dim_p (H(A) \otimes B) = 2^l$ by Lemma 2.1 (iii) we can combine (5) and §1(2) to deduce $\dim_p H(A \otimes B) = 2^l$, which is (1).

Structure of $H^*(G; \mathbf{Z}_2)$

The case $p = 2$ of Proposition 1.1 (i) will be proved in the rest of this section. Here we have an infinite number of possible groups G to deal with; it happens, however, that their cohomology mod 2 as a Q_1-algebra can be described by a general scheme (Condition 2.1) and it is possible to compute the homology of any complex with such a description.

Let A be a graded $\mathbf{Z}_2$-algebra with a differential d of degree $+3$ which is a derivation. We say that A satisfies Condition 2.1_r if it has a simple system of generators $y_1, \dots, y_n$ [11, Définition 6.4] such that

(a) $\mathrm{d}(y_i) \neq 0$ for exactly $n - r$ generators y_i
(b) If $\mathrm{d}(y_i) \neq 0$ then $\mathrm{d}(y_i) = y_j$ for some j
(c) If $y_i^2 \neq 0$ then $y_i^2 = y_j$ for some j
(d) If y_j has even degree then $y_j = \mathrm{d}(y_i)$ for some i
(e) There are no elements y_i in degree <3 and at most one y_i in degree 3.

The following two results provide a proof of (1), and so of Proposition 1.1 (i) for $p = 2$.

PROPOSITION 2.1. *Let* $(A, \mathrm{d}) = (H^*(G; \mathbf{Z}_2), Q_1)$ *where* $(G, 2) \in \mathcal{O}$, $G \neq \mathbf{E}_7, \mathbf{E}_8$. *Then* A *satisfies Condition* 2.1_r *where* $r = l = $ *rank* G.

PROPOSITION 2.2. *If* A *satisfies Condition* 2.1_r *then* $\dim_2 H(A) = 2^r$.

Proof of Proposition 2.1. We consider first the exceptional groups $\mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6$. Writing $\Delta(x_{i_1}, \dots, x_{i_k})$ for a $\mathbf{Z}_2$-algebra with a simple system of generators $x_{i_1}, \dots, x_{i_k}$ ($\deg x_{i_q} = i_q$) we have

$$H^*(\mathbf{G}_2; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6)$$

$$\text{(2.2)} \qquad H^*(\mathbf{F}_4; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_{15}, x_{23})$$

$$H^*(\mathbf{E}_6; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_9, x_{15}, x_{17}, x_{23})$$

In each case $Sq^2x_3 = x_5$, $Sq^1x_5 = x_6$, so $x_3^2 = Sq^3x_3 = x_6$; while Sq^1, Sq^2 vanish on all other generators and $x_i^2 = 0$ if $i \neq 3$.

(For $\mathbf{G}_2$, $\mathbf{F}_4$, see [12, Théorèmes 17.3 (c), 19.2 (c)], for $\mathbf{E}_6$ see [3, Theorem 2.7].)

By (2.2), since $Q_1 = Sq^1Sq^2 + Sq^2Sq^1$, it is easy to check that in each of the three cases $H^*(G; \mathbf{Z}_2)$ satisfies Condition 2.1_r with $n = l + 1$, $r = l$.

The remaining case is that of the spinor groups. We reproduce the description of their cohomology from [12, Théorème 12.1 (c)].

$$\text{(2.3)} \qquad H^*(\mathbf{Spin}\,(n); \mathbf{Z}_2) = \Delta(\{x_i \mid i \in S\}, y)$$

where S is the set of all integers $<n$ which are not powers of 2 and $\deg y = 2^{s(n)} - 1$; $s(n)$ is the integer such that $2^{s(n)-1} < n \leq 2^{s(n)}$.

There are $n - s(n)$ generators, and they satisfy

$$\text{(6)} \qquad \begin{aligned} Sq^i x_j &= \binom{j}{i} x_{i+j} \quad \text{if} \quad i+j \in S \\ &= 0 \text{ otherwise} \\ Sq^i y &= 0 \text{ for all } i > 0 \end{aligned}$$

We are considering only the cases $n \geq 7$ (see §1 (1)); hence the possibility $s(n) - 1 = 3$ is excluded and part (e) of Condition 2.1 is satisfied. Parts (b) and (c) follow easily from (6); to prove (a) and (d) we must find exactly when the relation $Q_1 x_j = x_{j+3}$ ($j, j+3 \in S$), holds. It follows from (6) that these are the only possibilities for non-trivial values of Q_1 on generators.

LEMMA 2.2. *If $j, j+3 \in S$, $Q_1 x_j = x_{j+3}$ if and only if j is odd.*

Proof. First suppose j is even. Then, applying (6), $Sq^1 x_j = Sq^1 Sq^2 x_j = 0$ so that $Q_1 x_j = 0$. This proves the 'only if' part of the lemma. We suppose now j is odd. Then $j+2$ is odd and $j+2 \in S$, since $j+3 \in S$ by hypothesis and $j+2$ cannot be a power of 2. Hence by (6),

$$\begin{aligned} Sq^2 x_j &= \tfrac{1}{2} j(j-1) x_{j+2} \\ &= x_{j+2} \text{ if } j \equiv 3 \pmod 4 \\ &= 0 \text{ if } j \equiv 1 \pmod 4 \end{aligned}$$

Since $Sq^1 x_{j+2} = x_{j+3}$ by (6), we have

$$\text{(7)} \qquad Sq^1\, Sq^2 x_j = x_{j+3} \text{ if and only if } j \equiv 3 \pmod 4.$$

It is possible that $j+1 \notin S$ because it is a power of 2. In this case we must have $j \equiv 3 \pmod 4$ ($j+1 = 2$ is impossible). Suppose on the other hand $j+1 \in S$, then applying (6), $Sq^2 Sq^1 x_j = Sq^2 x_{j+1} = \tfrac{1}{2} j(j+1)\, x_{j+3}$. This is x_{j+3} if $j \equiv 1 \pmod 4$ and zero if $j \equiv 3 \pmod 4$; hence whether or not $j+1 \in S$ we have

$$\text{(8)} \qquad Sq^2 Sq^1 x_j = x_{j+3} \text{ if and only if } j \equiv 1 \pmod 4.$$

Combining the relations (7) and (8) the lemma follows.

Condition 2.1 (d) is an immediate consequence of Lemma 2.2; in fact, inspecting the set S it is easy to see that an even integer $i \in S$ only if $i - 3 \in S$; and then x_i must be equal to $Q_1 x_{i-3}$.

We need only therefore prove that part (a) of Condition 2.1_l holds. If $G = \mathbf{Spin}\,(n)$, l is equal to $[n/2]$ so that the relation to be proved is

$$\text{(9)} \qquad N = n - S(n) - [n/2]$$

where N is the number of odd $j \in S$ such that $j+3 \in S$. Using (d), this is the same as the number of even $i \in S$, i.e. the number of even integers $<n$ which are not powers of 2. Now

there are $[(n-1)/2]$ even integers $<n$; and of these $s(n)-1$ are powers of 2. Hence $N=[(n-1)/2]-s(n)+1=[(n+1)/2]-s(n)$; using the identity $[n/2]+[(n+1)/2]=n$ this is equivalent to (9).

Proof of Proposition 2.2. We shall use induction on the integer $n-r$. First suppose $n-r=0$; then Condition 2.1 (a) implies that A is a trivial complex, so that $\dim_2 H(A)=\dim_2 A=2^n$ (since $A=\Delta(y_1,\ldots,y_n)$), which is 2^r as $n=r$. Hence the result is proved in this case; now suppose it proved when $n-r=m$ and let A satisfy condition 2.1_r with $r=n-m-1$. We number the generators of A so that

$$\text{(10)}\qquad \begin{aligned} dy_{2i-1} &= y_{2i} \qquad (i=1,\ldots,m+1)\\ dy_j &= 0 \quad \text{if} \quad j>2m+2 \end{aligned}$$

and y_{2m+1} has minimal degree among the y_i with $i\le 2m+2$.

We note for future reference that, as a consequence of Condition 2.1 (c) and (d),

$$\text{(11)}\qquad y_i^2\neq 0 \text{ implies } y_i^2=y_{2j} \text{ for some } j\le m+1.$$

We also note that if $i\le 2m$, $\deg y_i\ge \deg y_{2m+1}\ge 3$ and $\deg y_i>3$ by (10) and Condition 2.1 (e); so that $\deg y_i^2=2\deg y_i>\deg y_{2m+1}+3=\deg y_{2m+2}$ and $y_i^2=y_{2m+2}$ is impossible.

Now let $B\subset A$ be the subalgebra generated by $y_1,\ldots,y_{2m}$.

LEMMA 2.3. *$y_1,\ldots,y_{2m}$ form a simple system of generators for B.*

Proof. It is sufficient to show that any monomial $f\equiv \Pi\, y_i^{k_i}$ $(i=1,\ldots,2m)$ can be reduced to one with all $k_i\le 1$; or equivalently, if $k_i>1$ for some i, that f can be reduced to a monomial with smaller total degree $k=k_1+\ldots+k_{2m}$. Suppose $k_i>1$, then $f=f'\cdot y_i^2$ with f' a monomial of total degree $k-2$. But by (11) and the remarks which follow it, $y_i^2=y_{2j}$ with $j\le m$ so that $f=f'y_{2j}$ which has total degree $k-1$. This proves the lemma.

Since d is a derivation and maps generators of B into generators, B is a d-subalgebra of A. Hence using Lemma 2.3, for any $a(\in B)$ of form

$$\text{(12)}\qquad \Pi\, y_i^{k_i}(i=1,\ldots,2m;\ k_i\in\{0,1\} \text{ all } i).$$

da is a sum of elements of the same form. Now let $C\subset A$ be the submodule whose basis consists of all monomials $a.b$, with a of form (12) and b of form

$$\Pi\, y_j^{k_j}(j=2m+3,\ldots,n;\ k_j\in\{0,1\} \text{ all } j).$$

Any such b must be a d-cycle, since $y_{2m+3},\ldots,y_n$ are, so that $d(a.b)=(da).b$ is a sum of elements of the same form as $a.b$ itself. C is therefore a d-submodule of A.

LEMMA 2.4. $\dim_2 H(C)=2^{n-m-2}$.

Proof. We want to apply the induction hypothesis to C; but the possibility that $y_i^2=y_{2m+2}$ for some $i>2m+2$ means that C may not be an algebra. To avoid this difficulty we define a differential algebra $\bar{C}$ as follows.

(2.4) $\bar{C}$ has a simple system of generators $\bar{y}_1,\ldots,\bar{y}_{2m},\bar{y}_{2m+3},\ldots,\bar{y}_n$ such that

(i) $d\bar{y}_{2i-1}=\bar{y}_{2i}$ for $1\le i\le m$; $d\bar{y}_j=0$ otherwise.

(ii) $\bar{y}_i^2 = \bar{y}_j$ if $y_i^2 = y_j$ in A and $j \neq 2m + 2$
$\bar{y}_i^2 = 0$ if $y_i^2 = y_{2m+2}$ in A.

It is elementary that the requirements for the multiplicative structure define a unique $\mathbf{Z}_2$-algebra $\bar{C}$. The canonical basis for $\bar{C}$ in terms of monomials is exactly parallel to the basis $\{a.b\}$ we have given for C. Suppose that for a as in (12) we define $\bar{a} = \Pi\, \bar{y}_i^{k_i} (i = 1, \ldots, 2m)$ and similarly for b, $\bar{b}$. Then the homomorphism $g : C \to \bar{C}$ defined by $g(a.b) = \bar{a}.\bar{b}$ is an isomorphism of $\mathbf{Z}_2$-modules.

The requirement that d should be a derivation gives only one possible extension of the differential defined by (i) to this canonical basis of $\bar{C}$. To check that this is indeed a derivation it is sufficient to apply d to the relations (ii) which define $\bar{C}$. We find the requirement $\mathrm{d}(\bar{y}_j) = 0$ if $\bar{y}_j = y_i^2$; but this is a consequence of (i) since j must be equal to $2k$ with $1 \leq k \leq m$ by (11) and (ii). Hence $(\bar{C}, d)$ is a differential $\mathbf{Z}_2$-algebra as stated. The elements $\bar{y}_1, \ldots, \bar{y}_{2m}$ generate a d-subalgebra of $\bar{C}$ isomorphic to B in the obvious way; hence when a is a basis element (12) we have $\mathrm{d}(\bar{a}) = \overline{(\mathrm{d}a)}$. Hence for a standard basis element $a.b$ of C, $\mathrm{d}g(a.b) = \mathrm{d}(\bar{a}.\bar{b}) = \mathrm{d}(\bar{a}).b = (\overline{\mathrm{d}a}).\bar{b} = g((\mathrm{d}a).b) = g\mathrm{d}(a.b)$, that is, g is an isomorphism of complexes and $H(C) \approx H(\bar{C})$ as a $\mathbf{Z}_2$-module. But it is easy to check that $\bar{C}$ satisfies Condition $2.1_{n'-m}$ with $n' = n - 2$ generators for its simple system; the induction hypothesis applies and $\dim_2 H(\bar{C}) = 2^{n'-m} = 2^{n-m-2}$. This proves Lemma 2.4.

To express the homology of A in terms of that of C we use the direct sum decomposition.

$$A = C \oplus C\,.\,y_{2m+1} \oplus C\,.\,y_{2m+2} \oplus C\,.\,y_{2m+1}y_{2m+2} \tag{13}$$

(obtained by using the canonical basis for A in terms of monomials). The differential maps these components as follows:

$$\begin{aligned} &\mathrm{d}C \subset C;\ \mathrm{d}(C\,.\,y_{2m+1}) \subset C\,.\,y_{2m+1} \oplus C\,.\,y_{2m+2}; \\ &\mathrm{d}(C\,.\,y_{2m+2}) \subset C\,.\,y_{2m+2}; \\ &\mathrm{d}(C\,.\,y_{2m+1}y_{2m+2}) \subset C \oplus C\,.\,y_{2m+1}y_{2m+2}. \end{aligned} \tag{14}$$

Proof of (14). We have already proved that $\mathrm{d}C \subset C$. Suppose $x \in C$; then $\mathrm{d}(x.y_{2m+1}) = (\mathrm{d}x)y_{2m+1} + x.y_{2m+2}$, $\mathrm{d}(x.y_{2m+2}) = (\mathrm{d}x)y_{2m+2}$ (since $\mathrm{d}y_{2m+1} = y_{2m+2}$). This proves the second and third inclusions in (14). For the fourth we have $\mathrm{d}(x.y_{2m+1}\ y_{2m+2}) = (\mathrm{d}x).y_{2m+1}\ y_{2m+2} + x.y_{2m+2}^2$; since $y_{2m+2}^2 = y_{2j}$ with $j \leq m$ (using (11)), $x.y_{2m+2}^2 \in C$ which proves the last inclusion.

It follows from *(14)* that A splits as a direct sum of d-submodules $C \oplus C.y_{2m+1}y_{2m+2}$, $C.y_{2m+1} \oplus C.y_{2m+2}^2$. To complete the induction we need only prove $\dim_2 H(A) = 2^{n-m-1} = 2 \dim_2 H(C)$; this will follow from the next lemma.

LEMMA 2.5. (i) $H(C \oplus C.y_{2m+1}\ y_{2m+2}) \approx H(C) \oplus H(C)$

(ii) $H(C.y_{2m+1} \oplus C.y_{2m+2}) = 0$.

Proof. (i) First we can write $C \oplus C.y_{2m+1}\ y_{2m+2}$ as $C \oplus C.u$ with $\mathrm{d}u = 0$. In fact $\mathrm{d}(y_{2m+1}\ y_{2m+2}) = y_{2m+2}^2$; if $y_{2m+2} = 0$ we can choose $u = y_{2m+1}\ y_{2m+2}$ and this is immediate. If $y_{2m+2}^2 \neq 0$, $y_{2m+2}^2 = y_{2j} = \mathrm{d}y_{2j-1}$ with $j \leq m$ and we can choose $u = y_{2m+1}\,y_{2m+2} + y_{2j-1}$, $y_{2j-1} \in B$. Since for $x, x' \in C$

$$x + x'.y_{2m+1}\ y_{2m+2} = (x + x'.y_{2j-1}) + x'(y_{2m+1}\,y_{2m+2} + y_{2j-1})$$

and $x + x'.y_{2j-1} \in C$ (using Lemma 2.3), we find that

$$C \oplus C.y_{2m+1}\,y_{2m+2} = C \oplus C.(y_{2m+1}\,y_{2m+2} + y_{2j-1}) = C \oplus C.u$$

and $\mathrm{d}u = 0$ as required.

Now the condition $\mathrm{d}u = 0$ implies that $C \oplus C.u$ is a direct sum of d-modules, and that $x \to x.u$ is an isomorphism of the d-modules C, $C.u$. Hence $H(C \oplus C.u) \approx H(C) \oplus H(C)$, which proves part (i).

(ii) Suppose $x, y \in C$, and $\mathrm{d}(x.y_{2m+1} + y.y_{2m+2}) = 0$. Then $(\mathrm{d}x)y_{2m+1} + (x + \mathrm{d}y)y_{2m+2} = 0$ and $\mathrm{d}x$, $x + \mathrm{d}y \in C$. Hence $\mathrm{d}x = x + \mathrm{d}y = 0$, so $x = \mathrm{d}y$. We can now write

$$x.y_{2m+1} + y.y_{2m+2} = \mathrm{d}(y.y_{2m+1})$$

so that $C.y_{2m+1} \oplus C.y_{2m+2}$ is acyclic as stated.

§3. Proof of Proposition 1.1(ii)

The E_4-term

We consider the spectral sequences $\{E_r(\mathbf{E}_l; \mathbf{Z}_2)\}$, $l = 7, 8$. The complex $(E_3(\mathbf{E}_l; \mathbf{Z}_2), \mathrm{d}_3) = (H^*(\mathbf{E}_l; \mathbf{Z}_2), Q_1)$ can be described from the known information on $H^*(\mathbf{E}_l, \mathbf{Z}_2)$, which we summarize.

(3.1) (i) $H^*(\mathbf{E}_7; \mathbf{Z}_2) = \mathbf{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27})$
$H^*(\mathbf{E}_8; \mathbf{Z}_2) = \mathbf{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29})$

(ii) The restriction $H^*(\mathbf{E}_8; \mathbf{Z}_2) \to H^*(\mathbf{E}_7; \mathbf{Z}_2)$ sends x_i to x_i when they are both defined, to zero otherwise.

(iii) In both groups, $Sq^2x_3 = x_5$, $Sq^4x_5 = x_9$, $Sq^8x_9 = x_{17}$, $Sq^2x_{15} = x_{17}$, $Sq^8x_{15} = x_{23}$, $Sq^4x_{23} = x_{27}$ and in $\mathbf{E}_8$, $Sq^2x_{27} = x_{29}$.

These results are all in the papers of Araki and Shikata [3, 4] except for the relation $Sq^2x_{15} = x_{17}$, which is due to Thomas [32, Theorem 1] (and is essential for the evaluation of Q_1). For $\mathbf{E}_7$ see [3, Theorem 3.13]; for $\mathbf{E}_8$ see [4, Theorems 1, 2]. We now need to deduce some further relations.

(1) In both $\mathbf{E}_7$, $\mathbf{E}_8$ $Sq^1x_{2i-1} = x_i^2 (i = 3, 5, 9)$

In $\mathbf{E}_8$, $Sq^1x_{29} = x_{15}^2$.

In fact, the first three cases are simply applications of the formula $Sq^1Sq^{i-1}x_i = Sq^ix_i = x_i^2$ (for i odd). Since by (3.1)(iii) $x_{29} = Sq^2Sq^4Sq^8x_{15}$ we can deduce the last formula by using the relation $Sq^1Sq^2Sq^4Sq^8 = Sq^{15}$ which can be obtained from the Adem relations. To proceed further we need the following result.

LEMMA 3.1. *In the description* (3.1) *we can suppose that* $Sq^1x_{15} = 0$.

First, by (3.1) (ii) it is sufficient to prove this in $\mathbf{E}_8$. The conditions required of x_{15} are that $x_{15}^4 = 0$ and $Sq^2x_{15} = x_{17}$. Apart from this, altering x_{15} by a decomposable element will not alter the structure (3.1). Suppose then $Sq^1x_{15} \neq 0$. Then Sq^1x_{15} must be equal to the only non-zero element in $H^{16}(\mathbf{E}_8; \mathbf{Z}_2)$, i.e. $x_3^2x_5^2$, which is also equal to $Sq^1x_5^3$, using (1) and the Cartan formula. By similar means, we find that $Sq^2x_3^5 = Sq^2x_5^3 = x_3^4x_5$, and $Sq^1x_3^5 = 0$. Hence the element

$$x'_{15} = x_{15} + x_5^3 + x_3^5$$

satisfies $Sq^1x'_{15} = 0$, $Sq^2x'_{15} = x_{17}$ and $(x'_{15})^4 = 0$. This proves the lemma.

From now on we shall suppose that $Sq^1x_{15} = 0$ in both $\mathbf{E}_7$ and $\mathbf{E}_8$. We can now proceed to determine Q_1 on the generators $\{x_i\}$. The relations

(2) $$Q_1x_3 = x_3^2,\ Q_1x_5 = 0, \text{ in } \mathbf{E}_7, \mathbf{E}_8$$
$$Q_1x_9 = x_3^4 \text{ in } \mathbf{E}_8,\ Q_1x_9 = 0 \text{ in } \mathbf{E}_7$$

follow easily from (3.1)(iii), (1) (using the Adem relations to prove Sq^2 trivial on x_5, x_9).

Next, by Lemma 3.1 and (1) we have

(3) $$Q_1x_{15} = Sq^1Sq^2x_{15} = x_9^2 \text{ in } \mathbf{E}_7, \mathbf{E}_8.$$

For x_{17} we have $Sq^2Sq^1x_{17} = Sq^2x_9^2 = (Sq^1x_9)^2 = x_5^4$; while $Sq^2x_{17} = Sq^2Sq^2x_{15} = Sq^3Sq^1x_{15} = 0$, again by Lemma 3.1. Hence

(4) $$Q_1x_{17} = x_5^4 \text{ in } \mathbf{E}_8,\ Q_1x_{17} = 0 \text{ in } \mathbf{E}_8$$

x_{23} is the only generator to give serious difficulty. We first find $Sq^1x_{23} = Sq^1Sq^8x_{15} = Sq^9x_{15}$. By the Adem relations, we can deduce the formula

$$Sq^4Sq^2Sq^3 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1;$$

applying this to x_{15} and using Lemma 3.1 and (1) gives

$$Sq^9x_{15} = Sq^4Sq^2x_9^2 + Sq^7x_{17}$$

$Sq^7x_{17} = Sq^7Sq^8x_9 = Sq^{15}x_9 = 0$; while repeated application of the Cartan formula gives

$$Sq^2x_9^2 = (Sq^1x_9)^2 = x_5^4,\ Sq^4x_5^4 = (Sq^1x_5)^4 = x_3^8.$$

Hence finally $Sq^1x_{23} = x_3^8$, so $Sq^2Sq^1x_{23} = (Sq^1x_3^4)^2 = 0$ in both $\mathbf{E}_7$ and $\mathbf{E}_8$.

Next we shall prove $Sq^2x_{23} = 0$ in $\mathbf{E}_8$ (and so in $\mathbf{E}_7$ also by (3.1)(ii)). A basis for $H^{25}(\mathbf{E}_8; \mathbf{Z}_2)$ is given by $x_{17}x_5x_3$, $x_{15}x_5^2$, $x_9x_5^2x_3^2$, x_5^5, $x_5^2x_3^5$; $y = Sq^2x_{23}$ must be a linear combination of these elements satisfying $Sq^2y = Sq^3Sq^1x_{23} = 0$ (by what has been proved). Evaluating Sq^2 on the elements of the above basis we see that the kernel of $Sq^2: H^{25}(\mathbf{E}_8; \mathbf{Z}_2) \to H^{27}(\mathbf{E}_8; \mathbf{Z}_2)$ is generated by x_5^5, so that if $y \neq 0$ we must have $y = x_5^5$. To exclude this possibility we must use the diagonal homomorphism ψ of $H^*(\mathbf{E}_8; \mathbf{Z}_2)$. x_3 is primitive in this Hopf algebra for dimensional reasons; hence its iterated squares x_5, x_9, x_{17} are also primitive, and

(4a) $$Sq^2(\psi x_{15}) = \psi x_{17} = x_{17} \otimes 1 + 1 \otimes x_{17}$$

The possible cross terms in ψx_{15} are linear combinations of $x_3 \otimes x_3^4$, $x_3 \otimes x_9x_3$, $x_3^2 \otimes x_3^3$, $x_3 \otimes x_9$, $x_5 \otimes x_5^2$ and their transposes $x_3^4 \otimes x_3$, etc. If we now apply Sq^2 we find that the further restriction (4a) eliminates all possibilities but linear combinations of $x_3 \otimes x_3^4 + x_5 \otimes x_5^2$, $x_3^2 \otimes x_9$ and their transposes. Now

$$Sq^2x_{23} = Sq^2Sq^8x_{15} = (Sq^{10} + Sq^9Sq^1)x_{15} = Sq^{10}x_{15}$$

(using Lemma 3.1 again). But by applying the Cartan formula and known relations again we find that Sq^{10} vanishes on the given possible cross terms in ψx_{15}; hence $Sq^{10}x_{15} = y$ must be primitive. This contradicts $y = x_5^5$, showing that Sq^2x_{23} vanishes as stated. Combining all results on x_{23}, therefore, we have

(5) $$Q_1x_{23} = 0 \text{ in } \mathbf{E}_7, \mathbf{E}_8.$$

Next to find $Sq^1x_{27} = Sq^5x_{23}$ we use the formula $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$ (an Adem relation); this gives $Sq^5x_{23} = Sq^4x_3^8 = (Sq^1x_3^2)^4 = 0$, where we have inserted the information previously obtained on squares of x_{23}. On the other hand $Sq^1Sq^2x_{27} = Sq^1x_{29} = x_{15}^2$ by (3.1)(iii) and (1), so

(6) $$Q_1x_{27} = x_{15}^2 \text{ in } \mathbf{E}_8;\ Q_1x_{27} = 0 \text{ in } \mathbf{E}_7.$$

The last generator to consider is x_{29} (in $\mathbf{E}_8$ only). The above results imply $Sq^2x_{29} = Sq^2Sq^2x_{27} = Sq^3Sq^1x_{27} = 0$; while $Sq^2Sq^1x_{29} = Sq^2x_{15}^2 = (Sq^1x_{15})^2 = 0$. Hence

(7) $$Q_1x_{29} = 0 \text{ in } \mathbf{E}_8.$$

The relations (2)–(7) provide the necessary description of the action of Q_1 in the cohomology of both groups. We can now use this to compute the groups $H(H^*(\mathbf{E}_l; \mathbf{Z}_2), Q_1) = E_4(\mathbf{E}_l; \mathbf{Z}_2)$.

LEMMA 3.2. *Let $\{x\}$ denote the class in $E_4(\mathbf{E}_7; \mathbf{Z}_2)$ of $x \in H^*(\mathbf{E}_7; \mathbf{Z}_2)$. Then*

$$E_4(\mathbf{E}_7; \mathbf{Z}_2) = \mathbf{Z}_2[\{x_5\}]/(\{x_5^4\}) \otimes \Lambda(\{x_3^3\}, \{x_9\}, \{x_{17}\}, \{x_{23}\}, \{x_{27}\}, \{x_9^2x_{15}\}).$$

Proof. We split $H^*(\mathbf{E}_7; \mathbf{Z}_2)$ as a tensor product $A \otimes B \otimes C$ where

$$A = \mathbf{Z}_2[x_3]/(x_3^4),$$
$$B = \mathbf{Z}_2[x_9]/(x_9^4) \otimes \Lambda(x_{15}),\ C = \mathbf{Z}_2[x_5]/(x_5^4) \otimes \Lambda(x_{17}, x_{23}, x_{27})$$

By relations (2)–(6), A, B, C are Q_1-subalgebras, with Q_1 acting trivially on C. Hence by the Künneth formula $H(A \otimes B \otimes C) = H(A) \otimes H(B) \otimes C$. With $Q_1x_3 = x_3^2$, $Q_1x_{15} = x_9^2$, $Q_1x_9 = 0$ (by (2), (3)), it is easy to deduce $H(A) = \Lambda(\{x_3^3\})$, $H(B) = \Lambda(\{x_9\}, \{x_9^2x_{15}\})$, proving the lemma.

For $\mathbf{E}_8$ we shall shorten the calculation to give a less complete result which will still be sufficient. It is

LEMMA 3.3. *With the notations as in Lemma* 3.2,

$$E_4(\mathbf{E}_8; \mathbf{Z}_2) = \mathbf{Z}_2[\{x_5\}]/(\{x_5^4\}) \otimes T$$

where T is a $\mathbf{Z}_2$-algebra of dimension 2^7.

Proof. We split the cohomology as a tensor product $A \otimes B \otimes C$ with

$$A = \mathbf{Z}_2[x_3, x_9, x_{15}]/(x_3^{16}, x_9^4, x_{15}^4) \otimes \Lambda(x_{27}),$$
$$B = \mathbf{Z}_2[x_5]/(x_5^8) \otimes \Lambda(x_{17}),\ C = \Lambda(x_{23}, x_{29}).$$

Again (2)–(7) imply that A, B, C are Q_1-subalgebras, with Q_1 acting trivially on C, so that $H(A \otimes B \otimes C) = H(A) \otimes H(B) \otimes C$. Next, it follows immediately that $H(B) =$

$\mathbf{Z}_2[\{x_5\}]/(\{x_5^4\}) \otimes \Lambda(\{x_{17}x_5^4\})$; the factor $\Lambda(\{x_{17}x_5^4\}) \otimes C$ in $H(A \otimes B \otimes C)$ has dimension 2^3. To prove the lemma, therefore, it is enough to show that $\dim_2 H(A) = 2^4$.

Let A' be the subalgebra of A generated by x_9, x_{15}, x_{27}, and let A_i be the submodule $A' \cdot x_3^i$ for $0 \leq i < 16$, $(A_0 = A')$. If we filter A by the subcomplexes $A(i) = \Sigma A_j (j \geq i)$ we have the usual isomorphism of E_0 with $\sum_{i=0}^{15} A_i$ in the resulting spectral sequence; and the differential is given on A_0 by

$$\text{(8)} \qquad \mathrm{d}_0 x_9 = 0, \ \mathrm{d}_0 x_{15} = x_9^2, \ \mathrm{d}_0 x_{27} = x_{15}^2$$

(since $Q_1 x_9 = x_4^3$ which is in A_4). It is now easy to deduce that the E_1-term in the spectral sequence is the sum $\Sigma H(A_0).x_3^i$ for $i = 0, \ldots, 15$ since the isomorphisms $A_0 \to A_i$ given by multiplying by x_3^i preserve the differential of (8). By a straightforward calculation using (8)

$$H(A_0) = \Lambda(\{x_9\}, \{x_{15}x_9^2\}, \{x_{27}x_{15}^2\}).$$

Now $Q_1 x_9 \in A_4$, $Q_1(x_{15}x_9^2) = Q_1(x_{27}x_{15}^2) = 0$ so that all elements in $H(A_0)$ are d_1-cycles in the spectral sequence, while $\mathrm{d}_1(x_3^k) = kx_3^{k+1}$ for $k = 1, \ldots, 15$. Hence for $\xi \in H(A_0)$ $\mathrm{d}_1(\xi . x_3^k) = k\xi . x_3^{k+1}$, so that $E_2 = H(A_0) + H(A)_0 . \{x_3^{15}\}$.

Since finally $Q_1(x_9 + x_3^3) = 0$, all further differentials vanish on $H(A_0)$ and the spectral sequence collapses. Hence

$$\dim_2 H(A) = \dim_2 E_\infty = \dim_2 E_2 = 2 \dim_2 H(A_0) = 2^4$$

as required.

The differential d_5

From the preceding results we have a description of $E_4(\mathbf{E}_l; \mathbf{Z}_2) = E_5(\mathbf{E}_l; \mathbf{Z}_2)$ $(l = 7, 8)$. We shall now give a sufficient description of d_5 to prove Proposition 1.1.(ii). The following result will be sufficient.

PROPOSITION 3.1. *In* $E_5(\mathbf{E}_l; \mathbf{Z}_2)$ $(l = 7, 8)$ *the relation* $\mathrm{d}_5(\{x_5\}) = \{x_5^2\}$ *holds.*

To deduce Proposition 1.1 (ii) from this we begin by noting (I, Lemma 2.3) that d_5 is derivation. Hence $E_5(\mathbf{E}_l; \mathbf{Z}_2)$ is a DGA module over the DGA algebra $A = \mathbf{Z}_2[\{x_5\}]/(\{x_5\}^4)$ with $\mathrm{d}_5(\{x_5\}) = \{x_5\}^2$. By Lemmas 3.2, 3.3 the module structure is given by $E_5(\mathbf{E}_l; \mathbf{Z}_2) = A \otimes B_l$ with B_l a subspace of dimension 2^{l-1} over $\mathbf{Z}_2$. Hence (see the remarks following Lemma 2.1), there is a spectral sequence with $E_\infty \sim H(A \otimes B_l)$ and $E_1 = H(A) \otimes B_l$. Now $H(A) = \Lambda_2(\{x_5^3\})$, so $\dim_2 (H(A) \otimes B_l) = 2^l$, $\dim_2 (E_6(\mathbf{E}_l; \mathbf{Z}_2)) = \dim_2 H(A \otimes B_l) \leq 2^l$. This, with inequality (2) of §1, implies Proposition 1.1 (ii).

Proof of Proposition 3.1

We shall consider the case $l = 8$ only; the proof for $l = 7$ again follows from (3.1) (ii). Let $\xi \in H^3(\mathbf{E}_8; \mathbf{Z})$ be a generator, which we regard as a homotopy class of maps $\mathbf{E}_8 \to K(\mathbf{Z}, 3)$. Since $\pi_i(\mathbf{E}_8) = 0$ for $3 < i < 15$, by Theorem V (c) of [17], ξ induces an isomorphism $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_2) \to H^*(\mathbf{E}_8; \mathbf{Z}_2)$ in dimensions < 15. If we turn to the homomorphism of spectral sequences $\{E_r(K(\mathbf{Z}, 3); \mathbf{Z}_2)\} \to \{E_r(\mathbf{E}_8; \mathbf{Z}_2)\}$ induced by ξ, we can deduce that this is an isomorphism for $r = 2, 3$ in dimensions < 15 and for $r = 4, 5$ in dimensions < 12. Let

$\bar{\imath}_3$ be the generator of $H^3(K(\mathbf{Z}, 3); \mathbf{Z}_2)$. ξ^* sends $\bar{\imath}_3$ to x_3 and so $Sq^2\bar{\imath}_3$ to $Sq^2x_3 = x_5$; Proposition 3.1 is therefore equivalent to

$$\text{(9)} \qquad \mathrm{d}_5(\{Sq^2\bar{\imath}_3\}) = \{(Sq^2\bar{\imath}_3)^2\}$$

in $E_5(K(\mathbf{Z}, 3); \mathbf{Z}_2)$.

To verify (9) we shall construct a finite dimensional approximation to $K(\mathbf{Z}, 3)$ simple enough for d_5 to be computed in it. Let P_n denote n-dimensional complex projective space ($n = 1, 2, \ldots, \infty$). P_∞ is a $K(\mathbf{Z}, 2)$ and can be given an associative commutative H-structure $\mu : P_\infty \times P_\infty \to P_\infty$ which maps $P_k \times P_l$ into P_{k+l}.† In the Milnor resolution [29, §3] of P_∞ the classifying space B is a $K(\mathbf{Z}, 3)$.

For our purposes it is convenient to return to Milnor's original formulation [26] in terms of joins: we can make the identification (up to homotopy)

$$\text{(10)} \qquad \mathscr{E}_1 = P_\infty \circ P_\infty,\ B_1 = S(P_\infty);$$

$p_1 : \mathscr{E}_1 \to B_1$ is given by $p_1(tx \oplus (1-t)y) = (\mu(x, y), t)$ where $x, y \in P_\infty$, $t \in I$.

$$B_2 = B_1 \cup_{p_1} C\mathscr{E}_1.$$

Now by the cellularity of μ, p_1 maps $P_2 \circ P_2$ into $S(P_4) \subset B_1$. Let this map be called p and write Y for the subspace $S(P_4) \cup_p C(P_2 \circ P_2)$; $Y \subset B_2 \subset B = K(\mathbf{Z}, 3)$.

LEMMA 3.4. *The inclusion $Y \to K(\mathbf{Z}, 3)$ induces isomorphisms in cohomology in dimensions ≤ 7.*

Proof. First, since P_∞ is 1-connected, $\mathscr{E}_n$ is $(3n+1)$-connected by [26, Lemma 2.3]. The pair (B_n, B_{n-1}) has the connectivity of $(C\mathscr{E}_{n-1}, \mathscr{E}_{n-1})$ i.e. $3n-1$. This is ≥ 8 if $n \geq 3$; hence (B, B_2) is 8-connected.

Now we determine the 8-skeleton of B_2. With the usual cell-structure on P_∞, we have

$$S(P_\infty)^{(8)} = S(P_3);\ C(P_\infty \circ P_\infty)^{(8)} = \bigcup_{i+j=3} C(P_i \circ P_j)$$

and

$$B_2^{(8)} = S(P_\infty)^{(8)} \cup_{p_1} C(P_\infty \circ P_\infty)^{(8)}.$$

Now P_0 is simply a point, so $P_0 \circ P_0$, $P_0 \circ P_3$, $P_3 \circ P_0$ are all cones. Hence $P_0 \circ P_3 \cup P_3 \circ P_0$ is a contractible closed subspace of $P_\infty \circ P_\infty$. It follows that $S(P_3) \cup C(P_1 \circ P_2 \cup P_2 \circ P_1)$ is a homotopy 8-skeleton of B_2; since this is contained in Y the inclusions $Y \to B_2 \to B$ induce cohomology isomorphisms in dimensions ≤ 7.

Lemma 3.4 implies that $H^3(Y; \mathbf{Z}_2)$, $H^5(Y; \mathbf{Z}_2)$ have each only one non-zero element; we write these y_3, y_5 respectively. They are the restrictions of $\bar{\imath}_3$, $Sq^2\bar{\imath}_3$ in the cohomology of $K(\mathbf{Z}, 3)$. Since $\mathrm{d}_3(Sq^2\bar{\imath}_3) = 0$ in $E_3(K(\mathbf{Z}, 3); \mathbf{Z}_2)$, $\mathrm{d}_3(y_5) = 0$ in $E_3(Y; \mathbf{Z}_2)$ and y_5 determines a class $\{y_5\}$ in $E_5(Y; \mathbf{Z}_2)$. We shall prove

LEMMA 3.5. $\mathrm{d}_5(\{y_5\}) \neq 0$.

† The definition is $\mu((x_0, \ldots, x_m), (y_0, \ldots, y_n)) = (w_0, \ldots, w_{m+n})$ where $(\Sigma x_i t^i)(\Sigma y_j t^j) = \Sigma w_k t^k$.

This clearly implies $d_5(\{Sq^2\bar{\imath}_3\}) \neq 0$; since $\{(Sq^2\bar{\imath}_3)^2\}$ is the only non-zero element in $E_5^{10}(K(\mathbf{Z}, 3); \mathbf{Z}_2) = E_5^{10}(\mathbf{E}_8; \mathbf{Z}_2)$ (see the proof of Lemma 3.3), (9) follows.

Proof of Lemma 3.5. We shall compute $H^*(Y; \mathbf{Z}_2)$. Since Y is an adjunction space we have an exact sequence

(11) $\quad \ldots \to \tilde{H}^i(SP_4; \mathbf{Z}_2) \xrightarrow{p^*} \tilde{H}^i(P_2 \circ P_2; \mathbf{Z}_2) \to \tilde{H}^{i+1}(Y; \mathbf{Z}_2) \to \tilde{H}^{i+1}(SP_4; \mathbf{Z}_2) \to \ldots \; (i \in \mathbf{Z})$

The induced homomorphism p^* together with the cohomology of the join are most easily identified using Mayer–Vietoris sequences as in [26, Lemma 3.1]. Let C^+ (C^-) be the set of points (x, t) in $S(P_4)$ satisfying $t \geq 1/3$ $(t \leq 2/3)$; let $D^\pm$ in $P_2 \circ P_2$ be the inverse image $p^{-1}(C^\pm)$. Then we have a map of proper triads $p: (P_2 \circ P_2, D^+, D^-) \to (S(P_4), C^+, C^-)$. Moreover, $p|D^+ \cap D^- : D^+ \cap D^- \to C^+ \cap C^-$ is homotopy equivalent to $\mu : P_2 \times P_2 \to P_4$; D^+, D^- are homotopy equivalent to P_2 and are contractible in $P_2 \circ P_2$; C^+, C^- are contractible. As in [26] we find that the two Mayer-Vietoris sequences give rise to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}^i(P_2; \mathbf{Z}_2) \oplus \tilde{H}^i(P_2; \mathbf{Z}_2) \arrow[r, "\lambda"] & \tilde{H}^i(P_2 \times P_2; \mathbf{Z}_2) \arrow[r, "\Delta", leftrightarrow] & \tilde{H}^{i+1}(P_2 \circ P_2; \mathbf{Z}_2) \arrow[r] & 0 \\ & 0 \arrow[r] & \tilde{H}^i(P_4; \mathbf{Z}_2) \arrow[u, "\mu^*"] \arrow[r, "S", "\approx"'] & \tilde{H}^{i+1}(S(P_4); \mathbf{Z}_2) \arrow[u, "p^*"] \arrow[r] & 0 \end{tikzcd} \tag{12}$$

where the upper row is a split exact sequence, and λ maps $x \oplus y$ to $x \otimes 1 + 1 \otimes y \in \tilde{H}^i(P_2 \times P_2; \mathbf{Z}_2)$ (x, y in $\tilde{H}^i(P_2; \mathbf{Z}_2)$).

Next, using the Künneth formula, we can deduce from (12) that there is a commutative diagram

$$\begin{tikzcd} \tilde{H}^{i+1}(S(P_4); \mathbf{Z}_2) \arrow[rr, "p^*"] & & (\tilde{H}^{i+1}(P_2 \circ P_2; \mathbf{Z}_2) \\ \tilde{H}^i(P_4; \mathbf{Z}_2) \arrow[u, "S"] \arrow[r, "\mu^*"] & \tilde{H}^i(P_2 \times P_2; \mathbf{Z}_2) \arrow[r, "\pi \otimes \pi"] & (\tilde{H}^*(P_2; \mathbf{Z}_2) \otimes \tilde{H}^*(P_2; \mathbf{Z}_2))^i \arrow[u, "\Delta"'] \end{tikzcd} \tag{13}$$

where S, Δ are isomorphisms and π is the projection (as in II §2). We have therefore finally identified p^* in terms of μ^*, which itself stands in an obvious relation to the Hopf algebra structure of $H^*(P_\infty; \mathbf{Z}_2)$.

Now exactly the same argument may be applied in K-theory, using the K-theory mod 2 product structure and Künneth formula [5, 6.2]. The commutativity of this product has not so far entered into our arguments. We therefore have a similar commutative diagram

$$\begin{tikzcd} \tilde{K}^{\alpha+1}(S(P_4); \mathbf{Z}_2) \arrow[rr, "p!"] & & \tilde{K}^{\alpha+1}(P_2 \circ P_2; \mathbf{Z}_2) \\ \tilde{K}^\alpha(P_4; \mathbf{Z}_2) \arrow[u, "S"'] \arrow[r, "\mu!"] & \tilde{K}^\alpha(P_2 \times P_2; \mathbf{Z}_2) \arrow[r, "\pi \otimes \pi"] & (\tilde{K}^*(P_2; \mathbf{Z}_2) \otimes \tilde{K}^*(P_2; \mathbf{Z}_2))^\alpha \arrow[u, "\Delta"'] \end{tikzcd} \tag{13'}$$

with S, Δ isomorphisms $(\alpha \in \mathbf{Z}_2)$.

It remains to compute μ^* and $\mu^!$ in detail. Let u_k be the generator of $H^2(P_k; \mathbf{Z})$, $\bar{u}_k$ its reduction mod 2. Then

$$H^*(P_k; \mathbf{Z}) = \mathbf{Z}[u_k]/(u_k^{k+1}),$$
$$H^*(P_k; \mathbf{Z}_2) = \mathbf{Z}_2[\bar{u}_k]/(\bar{u}_k^{k+1}).$$

Since μ is the restriction of the product on P_∞ we must have

$$\mu^*(u_4) = u_2 \otimes 1 + 1 \otimes u_2.$$

Multiplying out and reducing mod 2 we have the following description of the composite $(\pi \otimes \pi) \circ \mu^*$:

$$(14) \qquad \begin{aligned} &(\pi \otimes \pi) \circ \mu^*(u_4^i) = 0 \quad (i = 1, 2, 4) \\ &(\pi \otimes \pi) \circ \mu^*(u_4^3) = u_2^2 \otimes u_2 + u_2 \otimes u_2^2 \end{aligned}$$

In K-theory, let $\eta_k \in K^0(P_k)$ be the class of the Hopf bundle, characterized by

$$ch(\eta_k) = \exp(u_k) = \sum_{r=}^{k} \frac{u_k^r}{r!}$$

(We are considering u_k, inaccurately, as a rational class.) Since $H^*(P_k; \mathbf{Z})$ is torsion-free, so is $K^*(P_k)$ and to find $\mu^!(\eta_4)$ it is enough to find $ch(\mu^!\eta_4) = \mu^*(ch\, \eta_4)$. But

$$\mu^*(ch\, \eta_4) = \mu^*(\exp(u_4)) = \exp(\mu^*(u_4))$$

since μ^* is a ring homomorphism

$$\begin{aligned} &= \exp(1 \otimes u_2 + u_2 \otimes 1) = \exp(1 \otimes u_2) \cdot \exp(u_2 \otimes 1) \\ &= (1 \otimes \exp u_2)(\exp u_2 \otimes 1) = \exp u_2 \otimes \exp u_2 = ch(\eta_2 \otimes \eta_2). \end{aligned}$$

Hence

$$(15) \qquad u^!(\eta_4) = \eta_2 \otimes \eta_2.$$

Now let $\tilde{\eta}_k = \eta_k - 1$; then

$$K^*(P_k) = \mathbf{Z}[\tilde{\eta}_k]/(\tilde{\eta}_k^{k+1}) \quad [8, \text{Corollary } 2.5.4].$$

Write $\bar{\eta}_k$ for $\rho_2(\tilde{\eta}_k) \in \tilde{K}^0(P_k; \mathbf{Z})$. Since ρ_2 is multiplicative the universal coefficient theorem gives

$$K^*(P_k;\mathbf{Z}_2) = \mathbf{Z}_2[\bar{\eta}_k]/(\bar{\eta}_k^{k+1})$$

Applying (15), multiplying out and reducing mod 2 we have

$$(16) \qquad \begin{aligned} &(\pi \otimes \pi) \circ \mu^!(\tilde{\eta}_4) = \bar{\eta}_2 \otimes \bar{\eta}_2 \\ &(\pi \otimes \pi) \circ \mu^!(\tilde{\eta}_4^2) = \bar{\eta}_2^2 \otimes \bar{\eta}_2^2 \\ &(\pi \otimes \pi) \circ \mu^!(\tilde{\eta}_4^3) = \bar{\eta}_2^2 \otimes \bar{\eta}_2 + \bar{\eta}_2 \otimes \bar{\eta}_2^2 \\ &(\pi \otimes \pi) \circ \mu^!(\tilde{\eta}_4^4) = 0. \end{aligned}$$

The formulae (14), (16), the commutative diagrams (13), (13)′ and the adjunction space exact sequences give the following descriptions:

LEMMA 3.6. $\tilde{H}^i(Y; \mathbf{Z}_2) = \mathbf{Z}_2 \quad i = 3, 5, 6, 8, 9, 10$

$= 0 \quad$ *otherwise*

$$\tilde{K}^0(Y; \mathbf{Z}_2) = \tilde{K}^1(Y; \mathbf{Z}_2) = \mathbf{Z}_2.$$

Now let us compute $\{E_r(Y; \mathbf{Z}_2)\}$. Writing y_i for the generator of $H^i(Y; \mathbf{Z}_2)$ we know already by Lemma 3.4 that $d_3(y_3) = y_6$, $d_3(y_5) = 0$, and so $d_3(y_6) = 0$.

For dimensional reasons d_3 must vanish on the remaining generators. Now suppose $d_5(y_5) = 0$; then again for dimensional reasons d_5 and all subsequent differentials must vanish and $E_\infty(Y; \mathbf{Z}_2) = E_4(Y; \mathbf{Z}_2)$ is a $\mathbf{Z}_2$-space of dimension 4 generated by $\{y_5\}$, $\{y_8\}$, $\{y_9\}$, $\{y_{10}\}$. This contradicts the result of Lemma 3.6 on $\tilde{K}^*(Y; \mathbf{Z}_2)$; hence we must have $d_5(y_5) \neq 0$ which proves Lemma 3.5.

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