Geometric Methods in Representation Theory

Kari Vilonen

Geometric Methods in Representation Theory Kari Vilonen

Introduction

The goal of this series of lectures is to survey and provide background for recent joint work with Wilfried Schmid. This work has appeared as a series of papers [SV1,SV2,SV3,SV4]. The type of geometric methods we discuss here were first introduced to representation theory by Beilinson and Bernstein in [BB1], to solve the Kazhdan-Lusztig conjectures. The localization technique of [BB1] can be used to translate questions in representation theory to questions about geometry of complex algebraic varieties. Later, in [K2], Kashiwara initiated a research program, as a series of conjectures, which extends the Beilinson-Bernstein picture. This program was carried out in [KSd] and [MUV] and is explained here in lectures 4 and 5. To explain the joint work with Schmid, we begin by introducing the main technical tool, the characteristic cycle construction, in lecture 6. The primary objective of these lectures, the character formula and the proof of the Barbasch-Vogan conjecture, are explained in lectures 7 and 8, respectively. In the first lecture we present an overview of the lecture series and lectures 2 and 3 provide the necessary background material on sheaf theory and homological algebra.

Acknowledgments

I would like to thank Markus Hunziker for his help in the preparation of the lectures at Park City, for drawing the pictures, and for writing the appendix that appears at the end of these lectures. The notes taken at Park City by Upendra Kulkarni proved to be a valuable help in preparing these notes. The final version of these notes were written when the author was a guest at the Max Planck institute in Bonn. The author was supported by grants from the NSF and NSA, and by a Guggenheim fellowship.

¹Department of Mathematics, Brandeis University, Waltham, MA 02254, USA **E-mail address:** vilonen@math.brandeis.edu

LECTURE 1 Overview

In this first lecture we give, in very rough terms, a basic outline of this lecture series. Let $G_{\mathbb{R}}$ be a semisimple (linear, connected) Lie group. By a representation of $G_{\mathbb{R}}$ we mean a representation on a complete locally convex Hausdorff topological vector space, of finite length. If the group $G_{\mathbb{R}}$ acts on a manifold X and \mathcal{F} is a sheaf on X which is $G_{\mathbb{R}}$ -equivariant, i.e., the group $G_{\mathbb{R}}$ acts on the sections of the sheaf \mathcal{F} , then the cohomology groups $H^k(X,\mathcal{F})$ have a linear $G_{\mathbb{R}}$ -action. Let X denote the flag manifold of the complexification G of $G_{\mathbb{R}}$. Then the above construction gives a functor:

(1.1) {certain
$$G_{\mathbb{R}}$$
-equivariant sheaves on X } \longrightarrow { $G_{\mathbb{R}}$ -representations}.

There is an analogous picture for Harish-Chandra modules. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup of $G_{\mathbb{R}}$ and let K be the complexification of $K_{\mathbb{R}}$. Then there is an equivalence of categories:

$$(1.2) \qquad \{ \text{certain K-equivariant sheaves on X} \} \stackrel{\sim}{-\!\!\!-\!\!\!\!-\!\!\!\!-}} \{ \text{H-C-modules} \} \, ,$$

where the functor is again given by cohomology. In lectures 2 and 3 we make precise the left hand sides of these constructions. We discuss the construction (1.1), due to Kashiwara and Schmid [KSd], in lecture 4. The construction (1.2), due to Beilinson-Bernstein [BB1], which is older, we only discuss briefly in lecture 5. The constructions (1.1) and (1.2) fit together as follows:

$$\{\text{certain } G_{\mathbb{R}}\text{-equivariant sheaves on }X\} \longrightarrow \{G_{\mathbb{R}}\text{-representations}\}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\{\text{certain } K\text{-equivariant sheaves on }X\} \stackrel{\sim}{\longrightarrow} \{\text{H-C-modules}\},$$

where the second vertical arrow associates to a representation its Harish-Chandra module. The first vertical arrow is called the Matsuki correspondence for sheaves which we discuss in some detail in lecture 5. It was shown to be an equivalence in [MUV].

It is clear that equivalences of categories can be used to answer questions of categorical nature. For example, the question of how the standard representations decompose into irreducible representations, i.e., the Kazhdan-Lusztig conjectures,

can be translated by equivalence (1.2) into a question about K-equivariant sheaves and was solved in this way. On the other hand, it is not immediately clear that interesting invariants of representations can be constructed directly from the geometric data. As an example of our techniques we give, in lecture 7, a geometric formula for the character of a representation. Here it is crucial that we use the equivalence (1.1). Finally, in lecture 8, we briefly discuss the solution of the Barbasch-Vogan conjecture, where we use the fact that the constructions (1.1) and (1.2) fit together to form diagram (1.3).

Remark. There does not seem to be any direct way of deciding when a representation is unitary using the construction (1.1). Kostant and Kirillov have suggested that, from the point of view of unitary representations, one should consider the dual Lie algebra $\mathfrak{g}_{\mathbb{R}}^*$ instead of the flag manifold. At this time, it is not clear what sheaves \mathcal{F} one should consider on $\mathfrak{g}_{\mathbb{R}}^*$. For guidance on this matter, one should consult the lectures of Vogan. In his lectures, Vogan points out that in the case of unitary representations and $\mathfrak{g}_{\mathbb{R}}^*$ there probably does not exist as nice a dictionary as (1.1) and (1.2).

To give a more detailed idea of what will be done in lectures 6-8, we first set up some notation and then discuss the case of compact groups.

Notation

We begin by introducing notation which will be used throughout this paper. Consider the flag variety X of the complexification G of $G_{\mathbb{R}}$, and let us view it as the variety of all Borel subalgebras of $\mathfrak{g} = \text{Lie}(G)$. The variety X carries a tautological bundle \mathcal{B} whose fiber over $x \in X$ is the Borel subalgebra \mathfrak{b}_x which fixes x. The bundle \mathcal{B} is G-homogenous. In particular, it is determined by the adjoint action of the Borel group B_x , the stabilizer group of $x \in X$, on the fiber \mathfrak{b}_x . From this we conclude that the G-bundle $\mathcal{B}/[\mathcal{B},\mathcal{B}]$ is trivial. Its fiber \mathfrak{h} is called the universal Cartan algebra; by definition it is canonically isomorphic to $\mathfrak{b}_x/[\mathfrak{b}_x,\mathfrak{b}_x]$ for any $x \in X$. Any concrete Cartan $\mathfrak{t} \subset \mathfrak{g}$ has a set of fixed points on X with the same cardinality as the Weyl group W. A choice of a fixed point of x amounts to a choice of a Borel $\mathfrak{b}_x \supset \mathfrak{t}$ and hence determines a canonical isomorphism $\tau_x : \mathfrak{t} \to \mathfrak{h}$. Via these isomorphisms \mathfrak{h}^* inherits a canonical root system Φ . Furthermore, Φ comes equipped with a canonical system of positive roots Φ^+ such that the roots of $\mathfrak{g}/\mathfrak{b}_x$ are positive. Similarly, we can define the universal Cartan group $H \cong B_x/[B_x,B_x]$ for G whose Lie algebra is \mathfrak{h} . Let $\Lambda \subset \mathfrak{h}^*$ denote the H-integral lattice, i.e., the set of $\lambda \in \mathfrak{h}^*$ which lift to a character of H. The elements in Λ correspond to G-equivariant holomorphic line bundles on X: an element $\mu \in \Lambda$ determines a character e^{μ} of $H \cong B_x/[B_x, B_x]$ which lifts to B_x and hence gives rise to a Gequivariant line bundle \mathbf{L}_{μ} on X. We will build the ρ -shift into our notation from the beginning and write $\mathcal{O}(\mu + \rho)$ for the sheaf of holomorphic sections of \mathbf{L}_{μ} . Here, as usual, $\rho \in \mathfrak{h}^*$ stands for half the sum of positive roots.

The compact case

We will now consider the case when the group $G_{\mathbb{R}}$ is compact. The irreducible representations of $G_{\mathbb{R}}$ are parametrized by $\lambda \in \Lambda + \rho$ such that $\lambda - \rho$ is dominant and the representations themselves are concretely exhibited on the sections of the

G-homogenous line bundle $\mathcal{O}(\lambda)$, i.e., on

(1.4)
$$H^{0}(X, \mathcal{O}(\lambda)) = H^{0}(X, \mathcal{O}(\mathbf{L}_{\lambda-\rho})).$$

This implements constructions (1.1) and (1.2) for compact groups. The element λ determines a coadjoint $G_{\mathbb{R}}$ -orbit $\Omega \subset i\mathfrak{g}_{\mathbb{R}}^*$, and an isomorphism $X \cong \Omega$, as follows. Given $x \in X$, there is a unique Cartan $T_{\mathbb{R}}$ of $G_{\mathbb{R}}$ which fixes x. As was explained above, this gives us a map $\tau_x : \mathfrak{t}_{\mathbb{R}} = \operatorname{Lie}(T_{\mathbb{R}}) \to \mathfrak{h}$, which, in turn, allows us to pull back $\lambda \in \mathfrak{h}^*$ to an element $\lambda_x \in i\mathfrak{t}_{\mathbb{R}}^*$. The direct sum decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus [\mathfrak{t}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}]$ allows us to view λ_x as an element in $i\mathfrak{g}_{\mathbb{R}}^*$. This construction provides a $G_{\mathbb{R}}$ -equivariant isomorphism between X and a $G_{\mathbb{R}}$ -orbit $\Omega \subset i\mathfrak{g}_{\mathbb{R}}^*$. As a coadjoint orbit, Ω has a canonical symplectic form σ_{Ω} . We define the Fourier transform $\hat{\varphi}$ of a tempered function φ , without choosing a square root of -1:

(1.5)
$$\hat{\varphi}(\zeta) = \int_{\mathfrak{g}_{\mathbb{R}}} e^{\zeta(x)} \varphi(x) dx \qquad (\zeta \in i\mathfrak{g}_{\mathbb{R}}^*).$$

Let Θ denote the character of the representation on $\mathrm{H}^0(X,\mathcal{O}(\lambda))$ with highest weight $\lambda - \rho$ and let $\theta = (\det \exp_*)^{1/2} \exp^* \Theta$ denote its character on the Lie algebra. Then, according to Harish-Chandra,

(1.6)
$$\int_{\mathfrak{q}_{\mathbb{R}}} \theta_{\Omega} \varphi \, dx = \frac{1}{(2\pi i)^n n!} \int_{\Omega} \hat{\varphi} \, \sigma_{\Omega}^n \, .$$

In other words,

(1.7) Fourier transform of
$$\theta$$
 = the coadjoint orbit Ω with measure $\frac{\sigma_{\Omega}^{n}}{(2\pi i)^{n} n!}$.

In lecture 7 we generalize this formula for representations of an arbitrary semisimple Lie group $G_{\mathbb{R}}$. A crucial ingredient of this generalization is the characteristic cycle construction of Kashiwara, which we discuss in lecture 6. In the paper [SV2], where the generalization of (1.6) is given, we also obtain an other character formula, resembling the Weyl's character formula, which gives the character of a representation via a Lefschetz type fixed point formula.

Our first goal, in lectures 2-4, is to define the functors

(1.8)
$$\{G_{\mathbb{R}}\text{-equivariant sheaves on }X\} \longrightarrow \{G_{\mathbb{R}}\text{-representations}\}$$

To do so, we first must give a precise meaning to the left hand side. It is the "twisted" $G_{\mathbb{R}}$ -equivariant derived category $D_{G_{\mathbb{R}}}(X)_{\lambda}$ of constructible sheaves on the flag manifold X. Here the twisting parameter $\lambda \in \mathfrak{h}^* =$ the dual space of the universal Cartan. We will explain the construction of $D_{G_{\mathbb{R}}}(X)_{\lambda}$ in three stages. First we introduce the notion of the derived category of constructible sheaves, then we describe how to make this notion $G_{\mathbb{R}}$ -equivariant, and finally, we explain the twisted version. Because the $G_{\mathbb{R}}$ -orbits on the flag manifold are semi-algebraic sets, we will develop the general theory in this context.

LECTURE 2

Derived categories of constructible sheaves

In this lecture we give a brief treatment of constructible sheaves and derived categories. At the end of this lecture series there is an appendix by Markus Hunziker which one may consult for further basic information about derived categories and sheaf cohomology. For a more detailed discussion see, for example, [KSa].

Semi-algebraic sets

Recall that a subset of \mathbb{R}^n is called semi-algebraic if it is the union of finitely many sets of the form

$$S = \{x \in \mathbb{R}^n \mid f_1(x) = \dots = f_r(x) = 0, g_1(x) > 0, \dots, g_s(x) > 0\},\$$

where the f_i and g_j are polynomials in $\mathbb{R}[x_1,\ldots,x_n]$. It follows directly from the definition that the class of semi-algebraic sets is stable under finite intersections, finite unions, and taking the complement. If $S \subset \mathbb{R}^n$ and $S' \subset \mathbb{R}^m$ are semi-algebraic sets then a map $f: S \to S'$ is called semi-algebraic if it is continuous and its graph is a semi-algebraic set in \mathbb{R}^{n+m} . The composition of two semi-algebraic maps is semialgebraic, of course. Furthermore:

For this fact see, for example, $[\mathbf{BM}]$. The definition of a semi-algebraic set generalizes readily to arbitrary algebraic manifolds as follows. If X is a real algebraic manifold then a subset S of X is called semi-algebraic if $S \cap U$ is semi-algebraic for every Zariski open affine subset U of X. The following crucial, non-trivial property of semi-algebraic sets will be important to us:

(2.2) every semi-algebraic set can be triangulated by semi-algebraic simplices.

For a proof of this fact in a far more general context, see [DM, 4.10].

2.3 Remark. In Lecture 8 we have to work in a more general setting of geometric categories arising from the the o-minimal structure $\mathbb{R}_{an,exp}$. The article $[\mathbf{DM}]$ provides an excellent exposition of this theory.

Constructible sheaves

Let X be a semi-algebraic set and let us fix a semi-algebraic triangulation \mathcal{T} of X. A sheaf \mathcal{F} on X is called \mathcal{T} -constructible if its restriction to any (open) simplex σ of \mathcal{T} is a constant sheaf of complex vector spaces of finite rank. We call a sheaf constructible if it is \mathcal{T} -constructible for some \mathcal{T} . The notion of a \mathcal{T} -constructible sheaf is the natural notion of a coefficient system on a simplicial complex: it associates a vector space to each (open) simplex and a linear map from the vector space of one simplex to that of another if the first simplex lies on the boundary of the second.

Another way of thinking about constructible sheaves is based on the notion of a local system. Recall that local systems are locally constant sheaves of finite rank. They constitute a special case of constructible sheaves. Given a locally constant sheaf \mathcal{F} on X and a point $x \in X$ we obtain a representation of $\pi_1(X, x)$ on the stalk \mathcal{F}_x by continuing the sections of \mathcal{F}_x along the loops based at x. This gives (for X connected!) an equivalence of categories:

(2.4) {local systems on
$$X$$
} \leftrightarrow {finite dimensional representations of $\pi_1(X, x)$ }

The category of constructible sheaves on X is the smallest abelian category containing the local systems on all semi-algebraic subsets of X.

Derived categories

Derived categories provide a convenient tool which helps to organize arguments in homological algebra. They provide the appropriate framework for resolutions and for derived functors. The fundamental idea is that one works systematically, from the outset, in the category of complexes. Also, there are functors that exist only in the context of derived categories. An example is provided by the functor $f^!$ introduced later in this section.

Let X be an arbitrary semi-algebraic set. The bounded derived category D(X) of constructible sheaves on X has as its objects bounded complexes of constructible sheaves. Its morphisms are given by chain homotopy classes of maps of chain complexes and, in addition, we formally invert the maps $\varphi: \mathcal{A}^{\bullet} \to \mathcal{B}^{\bullet}$ (the quasi-isomorphisms) which induce isomorphisms $H^{\bullet}(\varphi): H^{\bullet}(\mathcal{B}^{\bullet}) \xrightarrow{\sim} H^{\bullet}(\mathcal{A}^{\bullet})$ on the cohomology sheaves. The notion of exactness looses its meaning in D(X). Exact sequences of chain complexes are called distinguished triangles when viewed in D(X). Objects in D(X) can be shifted: if $\mathcal{A}^{\bullet} \in D(X)$ then $\mathcal{A}^{\bullet}[n]$ denotes the complex such that $\mathcal{A}^{\bullet}[n]$ $\mathcal{A}^{\bullet}[n]$.

Let us consider complexes of \mathcal{T} -constructible sheaves and let us denote the resulting derived category by $D_{\mathcal{T}}(X)$. The injective \mathcal{T} -constructible sheaves are easy to describe. Any injective \mathcal{T} -constructible sheaf is a direct sums of basic injective \mathcal{T} -constructible sheaf is a constant sheaf on a closure of a simplex in \mathcal{T} . To see that a constant sheaf on closure of simplex σ is injective, it suffices to note that

$$\operatorname{Hom}(\mathbb{C}_{\bar{\sigma}}, \mathcal{F}) = (\mathcal{F}_{\sigma})^*, \quad \text{for any } \mathcal{T}\text{-constructible sheaf } \mathcal{F}.$$

From the discussion above, it follows easily that the category of \mathcal{T} -constructible sheaves has enough injectives. In particular, every $\mathcal{F} \in \mathcal{D}_{\mathcal{T}}(X)$ is isomorphic (in

²One should also multiply the differential of the complex \mathcal{A}^{\bullet} by $(-1)^n$

 $D_{\mathcal{T}}(X)$) to a complex of injectives, its injective resolution. Furthermore, by a standard argument (see, for example, [KSa, Prop. 1.8.7]),

(2.5)
$$D_{\mathcal{T}}(X) \cong \text{homotopy category of injective } \mathcal{T}\text{-complexes}.$$

Except for trivial cases, the category of constructible sheaves on X does not have enough injectives (exercise). However, any $\mathcal{F} \in D(X)$ lies in some $D_{\mathcal{T}}(X)$ and hence has an injective representative in $D_{\mathcal{T}}(X)$. One can develop the theory for various operations on sheaves utilizing this principle. However, it is technically simpler, and perhaps more elegant, to view D(X) as a subcategory of the derived category of all sheaves of \mathbb{C} -vector spaces on X and take the injective representative of $\mathcal{F} \in D(X)$ inside this bigger derived category. In what follows, we will take this point of view. In particular, we view D(X) as a subcategory of the derived category of all sheaves of \mathbb{C} -vector spaces on X consisting of complexes \mathcal{F} such that the cohomology sheaves $H^k(\mathcal{F})$ are constructible and are non-zero for finitely many values of k only.

Operations on sheaves

From now on all of the semi-algebraic sets are assumed to be locally compact. Let $f: X \to Y$ be a map of semi-algebraic sets. We shall define functors $Rf_*, Rf_!: D(X) \to D(Y)$ and $f^*, f^!: D(Y) \to D(X)$.

Direct image. If \mathcal{F} is any sheaf on X then the direct image of \mathcal{F} by f, denoted by $f_*\mathcal{F}$ is the sheaf on Y defined by:

$$V \longmapsto f_* \mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$$

It is not immediately clear that the constructibility of \mathcal{F} implies the constructibility of $f_*(\mathcal{F})$. It follows from that fact that any semi-algebraic map can be Whitney stratified³. For a very general discussion of such matters, see [**DM**]. The functor f_* lifts to a functor $Rf_*: D(X) \to D(Y)$ in the usual way: if \mathcal{J} is an injective resolution of an object $\mathcal{F} \in D(X)$ then $Rf_*(\mathcal{F}) = f_*\mathcal{J}$. Here, again, the fact that $Rf_*(\mathcal{F})$ is constructible is not entirely obvious; recall that the injective resolution \mathcal{J} is not a complex of constructible sheaves. To see that $Rf_*(\mathcal{F}) \in D(X)$, one can argue in the same way as proving the constructibility of $f_*\mathcal{F}$. If $f: X \to \{\text{pt}\}$ then $f_* = \Gamma(X, \)$ is the global section functor on sheaves and hence the complex $Rf_*(\mathbb{C}_X) = R\Gamma(X, \mathbb{C}_X)$ computes the cohomology of X (with coefficients in \mathbb{C}):

(2.7)
$$H^{k}(X, \mathbb{C}_{X}) = R^{k}\Gamma(X, \mathbb{C}_{X}) = R^{k}f_{*}(\mathbb{C}_{X}).$$

The pushforward construction is functorial in the sense that

$$(2.8) R(f \circ g)_* = Rf_* \circ Rg_* \text{when} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Exercise. Assume that X is compact manifold and $f: X \to \mathbb{R}$ is a Morse function. Describe the complex $Rf_*\mathbb{C}_X$.

³Loosely speaking this means that the space Y can be decomposed into strata so that the map f restricted to $f^{-1}(S)$ is locally trivial for any stratum S of Y.

Inverse image. The direct image functor f_* has a left adjoint functor f^* in the category of (constructible) sheaves, i.e., for any sheaf \mathcal{F} on X and any sheaf \mathcal{G} on Y one has

(2.9)
$$\operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F})$$

The sheaf $f^*\mathcal{G}$ is called the inverse image of \mathcal{G} by f. An explicit construction of $f^*\mathcal{G}$ is as follows: $f^*\mathcal{G}$ is the sheaf associated to the presheaf

$$U \longmapsto \lim_{\substack{\longrightarrow \\ V \supset f(U)}} \mathcal{G}(V) .$$

It is clear from this description of $f^*\mathcal{G}$ that if $x \in X$ then

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$$
.

This implies that f^* is exact and hence $Rf^*\mathcal{F} = f^*\mathcal{F}$, for any $\mathcal{F} \in \mathrm{D}(X)$.

Direct image with proper support. For a (constructible) sheaf \mathcal{F} on X we define its direct image with proper support $f_!\mathcal{F}$ as the following subsheaf of $f_*\mathcal{F}$:

$$V \longmapsto f_! \mathcal{F}(V) = \{ s \in \mathcal{F}(f^{-1}(V)) \mid f : \operatorname{supp}(s) \to U \text{ is proper} \}$$

Note that if the map f is proper then $f_*\mathcal{F} = f_!\mathcal{F}$. In the other extreme, if $j: U \to X$ an embedding then

$$(2.10) j! \mathcal{F} = \text{ extension of } \mathcal{F} \text{ by zero }.$$

The functor $f_!$ is closely related to cohomology with compact support. Let

$$\Gamma_c(X, \mathcal{F}) = \text{global sections of } \mathcal{F} \text{ with compact support}$$

If $f: X \to \{\text{pt}\}$ then $f! \mathcal{F} = \Gamma_c(X, \mathcal{F})$ and hence

Base change. Consider a Cartesian square of semi-algebraic sets:

$$X' \xrightarrow{u} X$$

$$v \downarrow \qquad \Box \qquad \downarrow f$$

$$Y' \xrightarrow{g} Y.$$

Recall that a square is Cartesian if it commutes and $X' \simeq X \times_Y Y'$. Then we have a natural isomorphism of functors:

$$(2.12) g^* \circ Rf_! \simeq Rv_! \circ u^*$$

In particular, if $Y' = \{y\}$, then (2.12) implies that

$$(R^k f_! \mathcal{F})_y \simeq R^k \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) \simeq H_c^k(f^{-1}(y), \mathcal{F}).$$

Verdier duality

Unlike f_* , the functor $f_!$ does not have a right adjoint within the category of sheaves. However, the functor $Rf_!$: $D(X) \to D(Y)$ does have a right adjoint functor $f^!$: $D(Y) \to D(X)$:

$$(2.13) R\text{Hom}(Rf_!\mathcal{F},\mathcal{G}) = R\text{Hom}(\mathcal{F},f^!\mathcal{G}).$$

This statement is usually referred to as Verdier duality. The functor $f^!$ can not be obtained by taking the derived functor of a functor on sheaves. If $i: Y \hookrightarrow X$ is locally closed embedding then

$$i^! \mathcal{F} = R\Gamma_Y(\mathcal{F}),$$

where

$$\Gamma_Y(\mathcal{F}) = \text{ sections of } \mathcal{F} \text{ supported on } Y.$$

In particular,

$$\mathrm{H}^k(X, i^! \mathcal{F}) = \mathrm{H}^k(X, R\Gamma_Y(\mathcal{F})) = \mathrm{H}^k_Y(X, \mathcal{F}) = \mathrm{H}^k(X, X - Y; \mathcal{F})$$

the local cohomology of \mathcal{F} along Y.

Dualizing complex and duality functor. Let X be a semi-algebraic set, and $f: X \to \{pt\}$. We define the dualizing complex as

$$\mathbb{D}_X =_{\operatorname{def}} f^! \mathbb{C}_{\{\operatorname{pt}\}}.$$

Applying Verdier duality to the map $f: X \to \{pt\}$, we get

$$R\mathrm{Hom}(Rf_!\mathbb{C}_X,\mathbb{C}_{\{\mathrm{pt}\}})\cong R\mathrm{Hom}(\mathbb{C}_X,\mathbb{D}_X)\cong R\Gamma(X,\mathbb{D}_X).$$

In particular, we see that

$$\mathrm{H}^k(X,\mathbb{D}_X) \cong (\mathrm{H}_c^{-k}(X,\mathbb{C}_X))^* \cong \mathrm{H}_{-k}(X,\mathbb{C}_X).$$

By a slightly more refined computation we see that the dualizing complex can be interpreted as the complex of chains – with closed, not necessarily compact, supports – on X, placed in negative degrees. In particular, there is a canonical isomorphism

(2.15)
$$\operatorname{H}^{k}(\mathbb{D}_{X})_{x} \cong \operatorname{H}_{-k}(X, X - \{x\}; \mathbb{C}), \quad \text{for every } x \in X$$

For $\mathcal{F} \in D(X)$ we define $\mathbb{D}_X \mathcal{F}$, the Verdier dual of \mathcal{F} , by the formula

(2.16)
$$\mathbb{D}_X \mathcal{F} =_{\text{def}} R \mathcal{H}om(\mathcal{F}, \mathbb{D}_X).$$

Here the functor $\mathcal{H}om$ is the sheafification of the functor $U \mapsto \operatorname{Hom}(\mathcal{F}|U,\mathcal{G}|U)$. If $\mathcal{F} \in \mathrm{D}(X)$ then $\mathbb{D}_X \mathcal{F} \in \mathrm{D}(X)$, and

$$\mathcal{F} \xrightarrow{\sim} \mathbb{D}_X \mathbb{D}_X \mathcal{F}$$

is an isomorphism. This latter statement is called biduality. Furthermore, if $f:X\to Y$ is a morphism of semi-algebraic sets, then:

(2.18)
$$Rf_! = \mathbb{D}_X \circ Rf_* \circ \mathbb{D}_Y$$
 and $f^! = \mathbb{D}_Y \circ f^* \circ \mathbb{D}_X$.

Poincaré duality. Let X be an oriented (semi-algebraic) manifold of dimension n, and let $f: X \to \{\text{pt}\}$. Then, by formula (2.15), $\operatorname{H}^k(f^!\mathbb{C}_{\{\text{pt}\}}) \cong \operatorname{H}_{-k}(X, X - \{x\}; \mathbb{C})$. Hence, the specific orientation of X provides a distinguished isomorphism

(2.19)
$$f^! \mathbb{C}_{\{\mathrm{pt}\}} = \mathbb{D}_X \cong \mathbb{C}_X[n].$$

By Verdier duality

(2.20)
$$R\operatorname{Hom}(R\Gamma_c(X,\mathbb{C}_X)[n],\mathbb{C}) \simeq R\Gamma(X,\mathbb{C}_X),$$

and taking the p-th cohomology group gives the isomorphism:

$$(2.21) \qquad (\mathrm{H}_{c}^{n-p}(X,\mathbb{C}_{X}))^{*} \simeq \mathrm{H}^{p}(X,\mathbb{C}_{X}).$$

LECTURE 3

Equivariant derived categories

Let X be a semi-algebraic set with an algebraic $G_{\mathbb{R}}$ -action. Bernstein and Lunts $[\mathbf{BL}]$ defined a category $D_{G_{\mathbb{R}}}(X)$, the $G_{\mathbb{R}}$ -equivariant derived category of constructible sheaves on X, together with a forgetful functor $D_{G_{\mathbb{R}}}(X) \to D(X)$, satisfying the following condition: if $f: X \to Y$ is a $G_{\mathbb{R}}$ -equivariant map between semi-algebraic $G_{\mathbb{R}}$ -spaces, then the functors $Rf_*, Rf_!, f^*, f^!$ lift canonically to functors between the $G_{\mathbb{R}}$ -equivariant derived categories. Here we give a brief account of this theory. For a short summary, see also $[\mathbf{MV}]$.

$G_{\mathbb{R}}$ -equivariant sheaves

As before, we assume that $G_{\mathbb{R}}$ is a connected semisimple Lie group and let us consider a semi-algebraic $G_{\mathbb{R}}$ -space X. Naively, a $G_{\mathbb{R}}$ -equivariant (constructible) sheaf on X is a sheaf \mathcal{F} on X together with isomorphisms of the stalks

(3.1)
$$\varphi_{(q,x)} : \mathcal{F}_{qx} \xrightarrow{\sim} \mathcal{F}_x \quad \text{for all } g \in G_{\mathbb{R}}, x \in X.$$

Of course, one wants the isomorphisms $\varphi_{(g,x)}$ to depend continuously on g and x. To make the notion of a $G_{\mathbb{R}}$ -equivariant map precise consider the maps $a, p : G_{\mathbb{R}} \times X \to X$, a(g,x) = gx, p(g,x) = x. A $G_{\mathbb{R}}$ -equivariant sheaf on X is sheaf \mathcal{F} on X together with an isomorphism of sheaves on $G_{\mathbb{R}} \times X$:

$$(3.2) \varphi : a^* \mathcal{F} \xrightarrow{\sim} p^* \mathcal{F}$$

such that $\varphi|_{\{e\}\times X} = \text{id.}$ (If $G_{\mathbb{R}}$ is not assumed to be connected we need to add a cocycle condition.) Clearly, by restricting to the stalk at (g, x), (3.2) gives (3.1).

Exercise. Let \mathcal{F} be a $G_{\mathbb{R}}$ -equivariant sheaf on X. Construct a canonical linear $G_{\mathbb{R}}$ -action on the cohomology spaces $H^k(X, \mathcal{F})$.

$G_{\mathbb{R}}$ -equivariant local systems

If $G_{\mathbb{R}}$ acts transitively on X then, as is not difficult to see, we have an equivalence of categories

(3.3)
$$\left\{ \begin{array}{l} G_{\mathbb{R}}\text{-equivariant} \\ \text{sheaves on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{finite dimensional representations of} \\ \text{the component group } (G_{\mathbb{R}})_x/(G_{\mathbb{R}})_x^0 \end{array} \right\}$$

Fix $x \in X$. Then we have a fibration $G_{\mathbb{R}} \to X$, $g \mapsto gx$, with fiber the stabilizer $(G_{\mathbb{R}})_x$. Because $G_{\mathbb{R}}$ is connected, the long exact sequence of homotopy groups for a fibration yields a surjection $\pi_1(X,x) \to \pi_0((G_{\mathbb{R}})_x) = (G_{\mathbb{R}})_x/(G_{\mathbb{R}})_x^0$. Thus, a $G_{\mathbb{R}}$ -equivariant sheaf gives rise to a local system on X.

Equivariant derived categories and functors

Let X be a semi-algebraic $G_{\mathbb{R}}$ -space. The equivariant derived category can be characterized by the following properties:

- (1) There exists a forgetful functor $D_{G_{\mathbb{R}}}(X) \to D(X)$ which maps $G_{\mathbb{R}}$ -equivariant sheaves to constructible sheaves.
- (2) For a $G_{\mathbb{R}}$ -equivariant map $f: X \to Y$ the functors $Rf_*, Rf_!.f^*, f^!$ lift to functors between $D_{G_{\mathbb{R}}}(X)$ and $D_{G_{\mathbb{R}}}(Y)$. The same is true for the duality functor \mathbb{D} and the standard properties hold for these lifted functors.
- (3) If $G'_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a normal subgroup and $G'_{\mathbb{R}}$ acts freely on X then

$$\mathrm{D}_{G_{\mathbb{R}}}(X) \simeq \mathrm{D}_{G_{\mathbb{R}}/G'_{\mathbb{R}}}(G'_{\mathbb{R}}\backslash X) \ .$$

In the above, by a "standard property" we mean properties like (2.9), (2.12), (2.13), (2.16), and (2.17). The idea of the construction of $D_{G_{\mathbb{R}}}(X)$ is the same as that of equivariant cohomology: if the action of $G_{\mathbb{R}}$ on X is free then, by property (3), we simply set $D_{G_{\mathbb{R}}}(X) = D(G_{\mathbb{R}} \setminus X)$. If the action of $G_{\mathbb{R}}$ on X is not free, we replace X by $X \times EG_{\mathbb{R}}$ where $EG_{\mathbb{R}}$ is a contractible (infinite dimensional) space on which $G_{\mathbb{R}}$ acts freely and consider the diagonal action of $G_{\mathbb{R}}$ on $X \times EG_{\mathbb{R}}$. Let us organize the various spaces in the following diagram:

$$(3.4) X \xleftarrow{p} X \times EG_{\mathbb{R}} \xrightarrow{q} G_{\mathbb{R}} \setminus (X \times EG_{\mathbb{R}}),$$

where p is the projection to the first factor and q is the map to the quotient. We can now make the following formal definition. An object in $D_{G_{\mathbb{R}}}(X)$ is a triple $(\mathcal{F}, \mathcal{G}, \psi)$, where $\mathcal{F} \in D(X)$, $\mathcal{G} \in D(G_{\mathbb{R}} \setminus (X \times EG_{\mathbb{R}}))$, and ψ is an isomorphism,

$$\psi: p^* \mathcal{F} \xrightarrow{\sim} q^* \mathcal{G} .$$

Here $p: X \times EG_{\mathbb{R}} \to X$ is the projection on the first factor and $q: X \times EG_{\mathbb{R}} \to G_{\mathbb{R}} \setminus (X \times EG_{\mathbb{R}})$ is the quotient map. To make this work in our semi-algebraic context, one approximates $EG_{\mathbb{R}}$ by finite dimensional spaces. For details, see [**BL**] and also [**MV**].

Exercise. Let $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)$ and $a, p : G_{\mathbb{R}} \times X \to X$ the action map and the projection. Show that the above definition of $D_{G_{\mathbb{R}}}(X)$ yields a morphism $\varphi : a^*\mathcal{F} \to p^*\mathcal{F}$ in $D(G_{\mathbb{R}} \times X)$. Show that this gives a linear $G_{\mathbb{R}}$ -action on $H^*(X, \mathcal{F})$; here \mathcal{F} is viewed non-equivariantly, i.e., as an element of D(X).

Twisting

Let us return to the situation of lecture 1 and the notation used there. Now X will stand for the flag manifold on the complexification G of $G_{\mathbb{R}}$. The enhanced flag variety \hat{X} is defined as

(3.6)
$$\hat{X} = G/N$$
, where $N =$ unipotent radical of a Borel B

The group $G \times H$, where $H \cong B/N$ denotes the universal Cartan group, acts transitively on \hat{X} by the formula $(g,h) \cdot g'N = gg'h^{-1}N$. A "sheaf with twist $\lambda \in \mathfrak{h}^*$ on X" is a sheaf \mathcal{F} on \hat{X} such that for any $\hat{x} \in \hat{X}$ the pullback of \mathcal{F} to H under $h \mapsto h \cdot \hat{x}$ is locally constant and has the same monodromy as the function

 $e^{\lambda-\rho}$. We will think of twisted sheaves as objects on X. Note that if $\lambda=\rho$ or, more generally, if λ is an H-integral translate of ρ , then the λ -twisted sheaves are just ordinary sheaves on X. The notions of lecture 2, as well as the notions of the equivariant derived category, extend readily to the twisted case. In particular, we have the notion of the λ -twisted, $G_{\mathbb{R}}$ -equivariant derived category $D_{G_{\mathbb{R}}}(X)_{\lambda}$. If $\mu \in \Lambda$, i.e., if μ is H-integral, then $D_{G_{\mathbb{R}}}(X)_{\lambda} = D_{G_{\mathbb{R}}}(X)_{\lambda+\mu}$. The derived category $D_{G_{\mathbb{R}}}(X)_{\lambda}$ is generated⁴ by standard sheaves. For technical reasons, which will become apparent in the next lecture, we give a classification of standard sheaves in $D_{G_{\mathbb{R}}}(X)_{-\lambda}$. By definition, standard sheaves are associated to pairs (X, \mathcal{L}) , where S is a $G_{\mathbb{R}}$ -orbit on X and \mathcal{L} is an irreducible, $(-\lambda)$ -twisted $G_{\mathbb{R}}$ -equivariant local system on S. Given such a pair (S, \mathcal{L}) we can attach to it two types of standard sheaves:

(3.7)
$$Rj_*\mathcal{L}$$
 and $j_!\mathcal{L}$, where $j:S\hookrightarrow X$ denotes the inclusion.

We can use either type to generate $D_{G_{\mathbb{R}}}(X)_{-\lambda}$. Let us fix $x \in S$ and a Cartan $T_{\mathbb{R}} \subset G_{\mathbb{R}}$ which fixes x. As was shown in lecture 1, this data gives an identification $\tau_x : \mathfrak{t} \xrightarrow{\sim} \mathfrak{h}$, where \mathfrak{t} denotes the complexification of the Lie algebra of $T_{\mathbb{R}}$. Then

(3.8)
$$\left\{ \begin{array}{l} \text{Irreducible, } G_{\mathbb{R}}\text{-equivariant} \\ (-\lambda)\text{-twisted local systems on } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{characters } \chi: T_{\mathbb{R}} \to \mathbb{C}^* \\ \text{with } d\chi = \tau_x^*(\lambda - \rho) \end{array} \right\}.$$

To verify this statement, we apply (3.3) for the action of $G_{\mathbb{R}} \times \mathfrak{h}$ on \hat{S} . Here \hat{S} stands for the inverse image of S in \hat{X} and \mathfrak{h} acts on \hat{X} via the exponential map $\exp: \mathfrak{h} \to H$. Statement (3.3) can be applied because $(-\lambda)$ -twisted, $G_{\mathbb{R}}$ -equivariant local systems on \hat{S} are $G_{\mathbb{R}} \times \mathfrak{h}$ -equivariant local systems on \hat{S} .

⁴In the sense of a triangulated category, i.e., by shifting and forming distinguished triangles.

LECTURE 4

Functors to representations

In this lecture we make precise the ideas of lecture 1 and define the functors from our geometric parameter space $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ to representations. Recall that by a representation of a semisimple Lie group $G_{\mathbb{R}}$ we mean a representation on a complete, locally convex, Hausdorff topological vector space of finite length. We denote the infinitesimal character corresponding to $\lambda \in \mathfrak{h}^*$ by χ_{λ} and use Harish-Chandra's normalization so that the value $\lambda = \rho$ corresponds to the trivial infinitesimal character.

Let $\mathcal{O}(\lambda)$ denote the sheaf of λ -twisted holomorphic functions on X. More precisely, $\mathcal{O}(\lambda)$ is the subsheaf of $\mathcal{O}_{\hat{X}}$ which consists of functions whose restriction to any fiber of the map $\hat{X} \to X$ is a constant multiple of the function $e^{\lambda-\rho}$ when we identify the fibers of $\hat{X} \to X$ with H via the map $h \mapsto h \cdot gN = gh^{-1}N$, as usual. We note that if $\lambda \in \Lambda + \rho$, the ρ -translate of the H-integral lattice, then $\mathcal{O}(\lambda)$ can be viewed as an ordinary sheaf on X and it coincides with the sheaf of sections of the line bundle $\mathbf{L}_{\lambda-\rho}$.

In [KSd], Kashiwara and Schmid define two functors M and m

$$(4.1a) D_{G_{\mathbb{R}}}(X)_{-\lambda} \longrightarrow \left\{ \begin{array}{c} \text{virtual admissible } G_{\mathbb{R}}\text{-representations of} \\ \text{finite length with infinitesimal character } \chi_{\lambda} \end{array} \right\}$$

given by the formulas

(4.1b)
$$M : \mathcal{F} \longmapsto \sum (-1)^k \operatorname{Ext}^k(\mathbb{D}\mathcal{F}, \mathcal{O}(\lambda)) m : \mathcal{F} \longmapsto \sum (-1)^k \operatorname{H}^k(X, \mathcal{F} \otimes \mathcal{O}(\lambda)).$$

A few comments are in order. In [KSd] the target of the functors (4.1) is a derived category of representations. Here we have introduced the simplified version only where the functors land in virtual representations, as we do not need the more refined version in these lectures. To explain the formulas (4.1b), note that the duality functor \mathbb{D} takes $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ to $D_{G_{\mathbb{R}}}(X)_{\lambda}$. The groups $\operatorname{Ext}^{k}(\mathbb{D}\mathcal{F}, \mathcal{O}(\lambda))$ are to be interpreted as Ext-groups in the category of λ -twisted sheaves. As to the groups $H^{k}(X, \mathcal{F} \otimes \mathcal{O}(\lambda))$, note that the sheaf $\mathcal{F} \otimes \mathcal{O}(\lambda)$ has trivial monodromy along the fibers of $\hat{X} \to X$. Therefore $\mathcal{F} \otimes \mathcal{O}(\lambda)$ descends to X and $H^{k}(X, \mathcal{F} \otimes \mathcal{O}(\lambda))$ stands simply for cohomology on X. The topology on the spaces $\operatorname{Ext}^{k}(\mathbb{D}\mathcal{F}, \mathcal{O}(\lambda))$ and $H^{k}(X, \mathcal{F} \otimes \mathcal{O}(\lambda))$ is induced by the usual topology on the sheaf of holomorphic

functions $\mathcal{O}(\lambda)$. Although in general the formulas (4.1) define functors into virtual representations only, there is a subcategory of $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ such that restricted to this subcategory the groups $\operatorname{Ext}^k(\mathbb{D}\mathcal{F},\mathcal{O}(\lambda))$ and $\operatorname{H}^k(X,\mathcal{F}\otimes\mathcal{O}(\lambda))$ are non-zero only in degree zero. Restricted to this subcategory, the functors M and m land in representations, and every representation, up to infinitesimal equivalence, arises in this fashion. We will discuss this subcategory in more detail in lecture 5. Moreover, given \mathcal{F} , the representations $M(\mathcal{F})$ and $m(\mathcal{F})$ are the maximal and minimal globalizations of Schmid [S1] of the same Harish-Chandra module. Strictly speaking, this statement is not made in [KSd]. However, it follows from the results in [KSd] and the statement (4.2) below. As was noted earlier, the categories $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ and $D_{G_{\mathbb{R}}}(X)_{-\lambda-\mu}$ are canonically equivalent if μ lies in the H-integral lattice Λ . Passing from the parameter value λ to $\lambda + \mu$ in the construction (4.1) amounts to coherent continuation on the representation theoretic side.

The functors M and m turn the duality operation $\mathbb D$ into duality of representations:

(4.2) The virtual representation
$$\sum (-1)^p \operatorname{Ext}^p(\mathbb{D}\mathcal{F}, \mathcal{O}_X(\lambda))$$
 is, up to infinitesimal equivalence, the dual of $\sum (-1)^{n+p} \operatorname{Ext}^p(\mathcal{F}, \mathcal{O}_X(-\lambda))$.

In the above statement, n stands for the complex dimension of X. This statement does not appear in [KSd] but a short argument for it can be found in [SV2].

The simplest example of the functors (4.1) is the case when \mathcal{F} is the constant sheaf \mathbb{C}_X . For the constant sheaf to be $(-\lambda)$ -twisted, the parameter λ has to lie in $\Lambda + \rho$. In this case, the functors (4.1) coincide and amount to the Borel-Weil-Bott realization of a finite dimensional representation of $G_{\mathbb{R}}$.

Next we apply the functors M and m to standard sheaves. To that end, let \mathcal{L} be an irreducible $G_{\mathbb{R}}$ -equivariant local system with twist $-\lambda - \rho$ on a $G_{\mathbb{R}}$ -orbit S, let $j: S \hookrightarrow X$ denote the inclusion, and set $\mathcal{F} = Rj_*\mathcal{L}$. Let us first deal with the case of discrete series and hence suppose that $G_{\mathbb{R}}$ has a compact Cartan. Furthermore, we assume that the orbit S is open in X. By (3.8) there is a $G_{\mathbb{R}}$ -equivariant $(-\lambda - \rho)$ -twisted irreducible local system on S precisely when $\lambda \in \Lambda + \rho$, and such a local system is , by necessity, the trivial one. Hence, in our special case, $\mathcal{F} = Rj_*\mathbb{C}_S$. The representation associated to \mathcal{F} by (4.1) is a discrete series representation if λ is regular antidominant, as the following calculation will show. This coincides with the usual parametrization of the discrete series representations as explained in the lectures of Zierau. To do the calculation, we first note that, as $G_{\mathbb{R}}$ -equivariant sheaves,

$$(4.3) \mathbb{D}\mathcal{F} \cong \mathbb{D}Rj_*\mathbb{C}_S \cong j_!\mathbb{D}_S \cong j_!\mathbb{C}_S[2n],$$

where $n = \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} S$. From (4.3) and using (2.13) we conclude:

(4.4)
$$\operatorname{Ext}^{k}(\mathbb{D}\mathcal{F}, \mathcal{O}(\lambda)) = \operatorname{Ext}^{k}(j_{!}\mathbb{C}_{S}[2n], \mathcal{O}(\lambda))$$
$$= \operatorname{Ext}^{k-2n}(j_{!}\mathbb{C}_{S}, \mathcal{O}(\lambda))$$
$$= \operatorname{Ext}^{k-2n}(\mathbb{C}_{S}, \mathcal{O}(\lambda)|S)$$
$$= \operatorname{H}^{k-2n}(S, \mathcal{O}(\lambda));$$

The cohomology groups $H^{k-2n}(S, \mathcal{O}(\lambda))$ are non-zero precisely when $s = k - 2n = \frac{1}{2} \dim K_{\mathbb{R}}/T_{\mathbb{R}}$, and give a discrete series representation of $G_{\mathbb{R}}$. For this fact, see the lectures of Zierau. We thus get

(4.5)
$$M(\mathcal{F}) = (-1)^s \{ \text{discrete series representation attached to } (S, \lambda) \},$$

as virtual representations. Let us now return to the case of a general $G_{\mathbb{R}}$ -orbit S with a $(-\lambda - \rho)$ -twisted, $G_{\mathbb{R}}$ -equivariant local system \mathcal{L} on S, and we set $\mathcal{F} = Rj_*\mathcal{L}$. Then,

$$(4.6) \mathbb{D}\mathcal{F} = \mathbb{D}Rj_*\mathcal{L} = j_!\mathbb{D}\mathcal{L} = j_!\mathcal{L}^* \otimes \operatorname{or}_S[\dim S],$$

where $\mathcal{L}^* = \text{Hom}(\mathcal{L}, \mathbb{C}_S)$ denotes the dual of \mathcal{L} as a local system and or_S denotes the orientation sheaf of S. From (4.6) and using (2.13) we conclude:

$$\operatorname{Ext}^{k}(\mathbb{D}\mathcal{F},\mathcal{O}(\lambda)) = \operatorname{Ext}^{k}(j_{!}(\mathcal{L}^{*} \otimes \operatorname{or}_{S}[\dim S]), \mathcal{O}(\lambda))$$

$$= \operatorname{Ext}^{k-\dim S}(j_{!}(\mathcal{L}^{*} \otimes \operatorname{or}_{S}), \mathcal{O}(\lambda))$$

$$= \operatorname{Ext}^{k-\dim S}(\mathcal{L}^{*} \otimes \operatorname{or}_{S}, j^{!}\mathcal{O}(\lambda))$$

$$= \operatorname{Ext}^{k-\dim S}(\mathbb{C}_{S}, \mathcal{L} \otimes \operatorname{or}_{S} \otimes j^{!}\mathcal{O}(\lambda))$$

$$= \operatorname{Ext}^{k-\dim S}(\mathbb{C}_{S}, j^{!}(\widetilde{\mathcal{L}} \otimes \widetilde{\operatorname{or}_{S}} \otimes \mathcal{O}(\lambda)))$$

$$= \operatorname{H}_{S}^{k-\dim S}(X, \widetilde{\mathcal{L}} \otimes \widetilde{\operatorname{or}_{S}} \otimes \mathcal{O}(\lambda)).$$

Here $\widetilde{\mathcal{L}}$ and $\widetilde{\text{or}_S}$ denote extensions of the sheaves \mathcal{L} and or_S to a small neighborhood of S. For λ antidominant, these groups are non-zero in one degree only $[\mathbf{SW}]$.

Let us turn to the functor m. Attempting to apply m to the standard sheaf $Rj_*\mathcal{F}$ leads to a seemingly very difficult calculation. However it is easy to apply it to the standard sheaf $\mathcal{F} = j_!\mathcal{L}$. This gives

If we take, in (4.8), $\mathcal{F} = \mathbb{D}Rj_*\mathcal{L} = j_!(\mathcal{L}^* \otimes \operatorname{or}_S)[\dim S] \in D_{G_{\mathbb{R}}}(X)_{\lambda}$, then we get

which is, by (4.2), dual to the representation in (4.7). Note that there is a pairing

$$(4.10) (\mathcal{L} \otimes \operatorname{or}_{S} \otimes \mathcal{O}(\lambda)) \otimes (\mathcal{L}^{*} \otimes \operatorname{or}_{S} \otimes \mathcal{O}(-\lambda)|S)) \longrightarrow \mathbf{L}_{-2\rho} = \Omega_{X}.$$

Hence, the duality between (4.7) and (4.9) is an extension of Serre duality.

In (4.1) the categories $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ and $D_{G_{\mathbb{R}}}(X)_{-\mu}$ map to representations of the same infinitesimal character if μ lies in the W-orbit of λ . If $\mu = w \cdot \lambda$ then there is functor

(4.11)
$$I_w: \mathcal{D}_{G_{\mathbb{R}}}(X)_{-\lambda} \longrightarrow \mathcal{D}_{G_{\mathbb{R}}}(X)_{-\mu}, \text{ such that}$$

$$(M \circ I_w)(\mathcal{F}) = M(\mathcal{F}), \text{ for } \mathcal{F} \in \mathcal{D}_{G_{\mathbb{R}}}(X)_{-\lambda}.$$

The functors I_w are called intertwining functors and were first introduced in [**BB2**]. To give a formula for the functors I_w we assume, for simplicity, that $\lambda - \rho$ is integral. Let us set

$$Y_w \ = \ \left\{\, (x,y) \in X \times X \mid y \text{ is in position } w \text{ with respect to } x \,\right\},$$

and denote the projections to the first and the second factor by $p,q:Y_w\to X$, respectively. The functor $I_w:\mathrm{D}(X)\to\mathrm{D}(X)$ is then given by:

$$I_w(\mathcal{F}) = Rq_*p^*(\mathcal{F})[\ell(w)], \qquad \mathcal{F} \in \mathrm{D}(X),$$

where $\ell(w)$ stands for the length of w.

LECTURE 5

Matsuki correspondence for sheaves

In this lecture we explain geometric induction, the Beilinson-Bernstein localization, and the Matsuki correspondence for sheaves.

Geometric induction

Let A and B be (linear) Lie groups such that $A \subset B$ and assume that B acts on a semi-algebraic set X. We construct a right adjoint Γ_A^B to the forgetful functor $\operatorname{Forget}_B^A: \operatorname{D}_B(X) \to \operatorname{D}_A(X)$. To this end, let us consider the diagram

$$(5.1) X \xleftarrow{a} B \times X \xrightarrow{q} B/A \times X \xrightarrow{p} X$$

where $a(b,x)=b^{-1}x$, q(b,x)=(bA,x), and p(bA,x)=x. The spaces in the diagram have an action by $B\times A$ in such a way that the maps a,q,p are $B\times A$ -equivariant. This action of $B\times A$ on the spaces in (5.1) is given, reading from left to right, by $(b,a)\cdot x=a\cdot x$, $(b,a)\cdot (b',x)=(bb'a^{-1},b\cdot x)$, $(b,a)\cdot (b'A,x)=(bb'A,b\cdot x)$, $(b,a)\cdot x=b\cdot x$. To give a formula for the functor Γ_A^B , let us pick $\mathcal{F}\in D_A(X)$. As the B-action is trivial on X, we can view $\mathcal{F}\in D_{B\times A}(X)$. Then, by property (3) of the characterization of the equivariant derived category, there is a unique $\tilde{\mathcal{F}}\in D_B(B/A\times X)$ such that $q^*\tilde{\mathcal{F}}=a^*\mathcal{F}$. We then set

(5.2)
$$\Gamma_A^B \mathcal{F} = Rp_* \tilde{\mathcal{F}}$$
, where $\tilde{\mathcal{F}}$ is the unique sheaf such that $q^* \tilde{\mathcal{F}} = a^* \mathcal{F}$.

Intuitively, the operation Γ_A^B amounts to averaging \mathcal{F} over "B/A-orbits".

Parabolic induction. We pause briefly to explain how to phrase parabolic induction in terms of the geometric induction functors. For simplicity, we do it for the trivial infinitesimal character only. Let $P_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a parabolic subgroup with Levi decomposition $P_{\mathbb{R}} = L_{\mathbb{R}} N_{\mathbb{R}}$ with P = LN the corresponding complexified Levi decomposition. We denote by X_L the flag manifold of the group L. Associated to $\mathcal{F} \in \mathcal{D}_{L_{\mathbb{R}}}(X_L)$ we have a (virtual) representation $M(\mathcal{F})$ of $L_{\mathbb{R}}$. In the fibration $X \to G/P$ the fiber over the point eP can be identified with X_L and we denote by $i: X_L \hookrightarrow X$ the inclusion. Then we have the following formula for parabolic induction:

(5.3)
$$\operatorname{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(M(\mathcal{F})) = M(\Gamma_{L_{\mathbb{R}}}^{G_{\mathbb{R}}}i_{*}\mathcal{F}).$$

Beilinson-Bernstein localization

In lecture 4 we explained how to associate a $G_{\mathbb{R}}$ -representation $M(\mathcal{F})$ to an element $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)_{-\lambda}$. We will now explain how to construct the Harish-Chandra module associated to $M(\mathcal{F})$. For this we fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and denote by K the complexification of $K_{\mathbb{R}}$. The answer is provided by the following commutative diagram:

The arrow on the top row associates to a representation its Harish-Chandra module. The arrow α , due to Beilinson and Bernstein [**BB1**], amounts to taking the cohomology of the \mathcal{D} -module that is associated to the element in $D_K(X)_{-\lambda}$ by the Riemann-Hilbert correspondence⁵. One can write the functor α , in analogy with M, as

(5.5)
$$\alpha : \mathcal{F} \longmapsto \sum (-1)^k \operatorname{Ext}^k(\mathbb{D}\mathcal{F}, \mathcal{O}^{\operatorname{alg}}(\lambda)),$$

where $\mathcal{O}^{\text{alg}}(\lambda)$ is the sheaf of twisted algebraic functions on X. It is a subsheaf of $\mathcal{O}(\lambda)$. Note that a similar analogue of the functor m does not make sense, i.e., it does not produce a Harish-Chandra module. Finally, as to the functor Γ ,

(5.6a)
$$\Gamma: D_{G_{\mathbb{R}}}(X)_{-\lambda} \xrightarrow{\sim} D_{K}(X)_{-\lambda}, \qquad \Gamma = \Gamma_{K_{\mathbb{R}}}^{K} \circ \operatorname{Forget}_{K_{\mathbb{R}}}^{G_{\mathbb{R}}}$$

is an equivalence of categories [MUV]. The functor Γ has the following property which justifies calling it the Matsuki correspondence for sheaves:

(5.6b)
$$\Gamma(Rj_!\mathbb{C}_{\mathcal{O}}) = Rj'_*\mathbb{C}_{\mathcal{O}'}[-2\operatorname{codim}_{\mathbb{C}}\mathcal{O}']$$

if the $G_{\mathbb{R}}$ -orbit \mathcal{O} corresponds to the K-orbit \mathcal{O}' under the Matsuki correspondence, which we recall below in (5.7). Here $j:\mathcal{O}\hookrightarrow X$ and $j':\mathcal{O}'\hookrightarrow X$ denote the inclusions of the orbits \mathcal{O} and \mathcal{O}' to the flag manifold X. We will explain in some detail below the geometric idea behind the proof of (5.6).

Matsuki Correspondence for sheaves

The main ingredient of the proof of the Matsuki correspondence for sheaves is a Morse theoretic interpretation and refinement of the original result of Matsuki. Let us recall Matsuki's statement: there is a bijection

$$(5.7) G_{\mathbb{R}} \backslash X \longleftrightarrow K \backslash X$$

between $G_{\mathbb{R}}$ -orbits on X and K-orbits on X such that a $G_{\mathbb{R}}$ -orbit \mathcal{O}' corresponds to a K-orbit \mathcal{O} if and only if $\mathcal{O}' \cap \mathcal{O}$ is non-empty and compact. Furthermore, $G_{\mathbb{R}}$ -equivariant local systems on \mathcal{O}' correspond bijectively to K-equivariant local

⁵For a treatment of \mathcal{D} -modules and the Riemann-Hilbert correspondence, see [**Bo**]

systems on \mathcal{O} . The fundamental idea, due to Uzawa, is that there exists a Bott-Morse function f on X whose stable manifolds, with respect to a particular metric, are the K-orbits and whose unstable manifolds are the $G_{\mathbb{R}}$ -orbits. To construct the metric and the function, we write $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ for the Cartan decomposition, and let $U_{\mathbb{R}}$ denote the compact form corresponding to $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. We choose an H-integral, regular, dominant $\lambda \in \mathfrak{h}^*$, and denote the corresponding highest weight representation of G by V_{λ} . Then the line bundle \mathbf{L}_{λ} gives us an embedding $X \hookrightarrow \mathbb{P}(V_{\lambda})$. We fix a $U_{\mathbb{R}}$ -invariant Hermitian scalar product on V_{λ} . The real part of the scalar product induces a $U_{\mathbb{R}}$ -invariant Riemannian metric, the Fubini-Study metric on the projective space $\mathbb{P}(V_{\lambda})$ and hence on X. This is the metric we use.

To construct the desired function f, we note that the element $\lambda \in \mathfrak{h}^*$ gives us an embedding

(5.8)
$$X \xrightarrow{\sim} \Omega_{\lambda} \subset i\mathfrak{u}_{\mathbb{R}}^*, \qquad \Omega_{\lambda} \text{ a } U_{\mathbb{R}}\text{-orbit}$$

as follows. Given $x \in X$, there is a unique Cartan $T_{\mathbb{R}}$ of $U_{\mathbb{R}}$ which fixes x. As explained in lecture 1, this gives us a map $\tau_x : \mathfrak{t}_{\mathbb{R}} \to \mathfrak{h}$ which lifts to a map $T_{\mathbb{R}} \to H$ and thus $\lambda \in \mathfrak{h}^*$ gives rise to an element $\lambda_x \in i\mathfrak{t}_{\mathbb{R}}^*$. Via the direct sum decomposition $\mathfrak{u}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus [\mathfrak{t}_{\mathbb{R}}, \mathfrak{u}_{\mathbb{R}}]$, we can view $\lambda_x \in i\mathfrak{u}_{\mathbb{R}}^*$ and the association $x \mapsto \lambda_x$ gives a map from X to $i\mathfrak{u}_{\mathbb{R}}^*$. As $i\mathfrak{u}_{\mathbb{R}}^* = i\mathfrak{t}_{\mathbb{R}}^* \oplus \mathfrak{p}_{\mathbb{R}}^*$, we get a map

$$(5.9) m: X \longrightarrow i\mathfrak{u}_{\mathbb{R}}^* \longrightarrow \mathfrak{p}_{\mathbb{R}}^*.$$

The Killing form induces a metric on $\mathfrak{p}_{\mathbb{R}}$ and hence on its dual $\mathfrak{p}_{\mathbb{R}}^*$. We define

$$f: X \longrightarrow \mathbb{R}$$
, by $f(x) = ||m(x)||^2$.

Here is the refined version of the Matsuki correspondence:

The function f is a Bott-Morse function. Its gradient flow ∇f ,

with respect to the metric on X described above, has the

(5.10) K-orbits as stable manifolds and $G_{\mathbb{R}}$ -orbits as unstable manifolds.

The critical set consists of a finite set of $K_{\mathbb{R}}$ -orbits.

The statement (5.10) is illustrated below in figure 1 for the case $G_{\mathbb{R}} = SL(2, \mathbb{R})$.

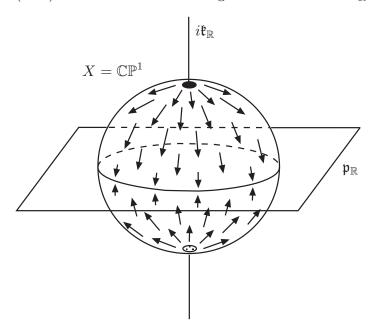


Figure 1

In proving the Matsuki correspondence (5.6) for sheaves it is crucial to know that if a K-orbit \mathcal{O} and a $G_{\mathbb{R}}$ -orbit \mathcal{O}' are related by the Matsuki correspondence, then both \mathcal{O} and \mathcal{O}' can be retracted to $\mathcal{O} \cap \mathcal{O}'$. This is the content of statement (5.10).

Remark. In [N], Ness defined a "moment map" $\mu : \mathbb{P}(V_{\lambda}) \to i\mathfrak{u}_{\mathbb{R}}^*$ for the action of the compact group $U_{\mathbb{R}}$ on $\mathbb{P}(V_{\lambda})$. The map (5.8) is the restriction of this moment map to X via the embedding $X \hookrightarrow \mathbb{P}(V_{\lambda})$. The construction of Ness readily extends to actions of semisimple groups, see, for example, [MUV]. The moment map for the $G_{\mathbb{R}}$ -action on $\mathbb{P}(V_{\lambda})$ composed with the the embedding $X \hookrightarrow \mathbb{P}(V_{\lambda})$ is exactly the map m in (5.9).

Representations and perverse sheaves

In these lectures we emphasize the role of $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ over $D_K(X)_{-\lambda}$. However, for certain things it is preferable to work with the category $D_K(X)_{-\lambda}$ instead, as we will explain below. Let $\lambda \in \mathfrak{h}^*$ be dominant and let $P_K(X)_{-\lambda}$ denote the subcategory of $D_K(X)_{-\lambda}$ of perverse sheaves [**BBD**]. If $\mathcal{F} \in P_K(X)_{-\lambda}$ then, as was shown in [**BB1**] in the language of \mathcal{D} -modules,

$$\operatorname{Ext}^{k}(\mathbb{D}\mathcal{F},\mathcal{O}(\lambda)) = 0, \quad \text{if } k \neq 0.$$

Thus, the functor α of (5.5), restricted to $P_K(X)_{-\lambda}$, gives a functor

(5.11)
$$\alpha : P_K(X)_{-\lambda} \longrightarrow \{\text{Harish-Chandra modules}\}_{\chi_{\lambda}},$$

which is an equivalence of categories if λ is regular. If λ is not regular then there is a kernel which can be described explicitly [**BB1**]. Under (5.11) the irreducible Harish-Chandra modules correspond to intersection homology sheaves of K-equivariant irreducible local systems on K-orbits.

Note that the category $P_K(X)_{-\lambda}$, as well as intersection homology sheaves, are characterized by conditions on objects of $D_K(X)_{-\lambda}$ which are local on X. This does not appear to be possible on the $G_{\mathbb{R}}$ -side. The nice subcategory analogous to $P_K(X)_{-\lambda}$ exists on the $G_{\mathbb{R}}$ -side for formal reasons. By the commutativity of (5.4) we can simply take it to be $\Gamma^{-1}(P_K(X)_{-\lambda})$. The functor $\gamma = \Gamma^{-1}: D_K(X)_{-\lambda} \to D_{G_{\mathbb{R}}}(X)_{-\lambda}$ is given analogously to Γ , by switching the roles of K and $G_{\mathbb{R}}$, and by replacing all the *'s in the construction by !'s. From this description one can see that there can not be a characterization of $\Gamma^{-1}(P_K(X)_{-\lambda})$ as a subcategory of $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ using conditions which are local on X.

Cohomological induction. Finally, let us describe cohomological induction in geometric terms. For simplicity, we do so only in the case of trivial infinitesimal character. Let $P \subset G$ be a parabolic subgroup such that its Levi L is θ -stable and defined over \mathbb{R} , i.e., $L_{\mathbb{R}} \subset G_{\mathbb{R}}$ and $L_{\mathbb{R}} \cap K_{\mathbb{R}} \subset L_{\mathbb{R}}$ is maximal compact. Let us consider the fibration $X \to G/P$. Its fiber over eP can be identified with the flag manifold of X_L of L, and we denote the inclusion of that fiber in X by $i: X_L \to X$. Then

$$\operatorname{Ind}_{(\mathfrak{l},L\cap K)}^{(\mathfrak{g},K)}\alpha(\mathcal{F}) = \alpha(\Gamma_{L\cap K}^{K}i_{*}\mathcal{F}) ,$$

where $\operatorname{Ind}_{(\mathfrak{l},L\cap K)}^{(\mathfrak{g},K)}$ stands for the cohomological induction (in the sense of $[\mathbf{EW}]$) from $(\mathfrak{l},L\cap K)$ -modules to (\mathfrak{g},K) -modules. For this fact see, for example, $[\mathbf{MP}]$ and $[\mathbf{S2}]$.

LECTURE 6

Characteristic cycles

Let X be a real algebraic manifold. For simplicity, we assume that X is oriented. As usual, let D(X) be the bounded derived category of constructible sheaves on X. A simple invariant that one can associate to an object $\mathcal{F} \in D(X)$ is its *Euler characteristic*, $\chi(X, \mathcal{F})$, defined by

(6.1)
$$\chi(X,\mathcal{F}) = \sum_{k} (-1)^k \dim_{\mathbb{C}} H^k(X,\mathcal{F}).$$

The Euler characteristic is additive in distinguished triangles (recall that triangles arise from exact sequences of complexes): if

$$(6.2) \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'[1]$$

is a distinguished triangle then

(6.3)
$$\chi(X,\mathcal{F}) = \chi(X,\mathcal{F}') + \chi(X,\mathcal{F}'').$$

As a particular special case we get $\chi(X, \mathcal{F}[1]) = -\chi(X, \mathcal{F})$. From (6.2) we also conclude that $\chi(X,): D(X) \to \mathbb{Z}$ descends to the K-group K(D(X)). Recall that the K-group is the free abelian group generated by all the $\mathcal{F} \in D(X)$ subject to the relation $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ for all distinguished triangles (6.2). The local Euler characteristic

$$\chi(\mathcal{F}) : X \longrightarrow \mathbb{Z}$$

is defined by

(6.5)
$$\chi(\mathcal{F})_x = \sum_k (-1)^k \dim_{\mathbb{C}} H^k(\mathcal{F})_x.$$

The local Euler characteristic $\chi(\mathcal{F})$ is a constructible function: there is a triangulation \mathcal{T} of X such that for any $\sigma \in \mathcal{T}$ the restriction $\chi(\mathcal{F})|\sigma$ to the open simplex σ is constant.

Remark. It is not very difficult to verify that the homomorphism (both sides are abelian groups)

(6.6)
$$K(D(X)) \longrightarrow \{\text{constructible functions } X \to \mathbb{Z} \}$$

$$\mathcal{F} \longmapsto \chi(\mathcal{F})$$

is an isomorphism.

To give a more geometric description of the category K(D(X)), we recall another point of view to the Euler characteristic. Assume now that X is compact. Then, by a classical theorem of Hopf, the Euler characteristic $\chi(X) = [X].[X]$, the self intersection product of the zero section [X] in $T^*X \simeq TX$. To be able to take the self intersection product, one perturbes one copy of the zero section so that we stay in the same homology class. For example, we can replace one of the copies of X by a generic section of the tangent bundle.

To generalize the theorem of Hopf to arbitrary $\mathcal{F} \in D(X)$, Kashiwara, in [K1], introduced the notion of a *characteristic cycle* $CC(\mathcal{F})$. The characteristic cycle $CC(\mathcal{F})$ is a semi-algebraic Lagrangian cycle (not necessarily with compact support) on the cotangent bundle T^*X . The definition of $CC(\mathcal{F})$ is Morse-theoretic. Heuristically, $CC(\mathcal{F})$ encodes the infinitesimal change of the local Euler characteristic $\chi(\mathcal{F})_x$ to various co-directions in X. We have the following generalization of Hopf's theorem [K1]:

(6.7)
$$\chi(X, \mathcal{F}) = CC(\mathcal{F}).[X].$$

As the definition of CC is quite technical, we will omit it, and refer to [K1] and [KS, chapter IX] for details. Below we will give an axiomatic characterization of CC following [SV1].

Semi-algebraic chains and cycles. Let M be a real algebraic manifold (In our situation, $M = T^*X$). A semi-algebraic p-chain, C, on M is a finite integer linear combination

(6.8)
$$C = \sum n_{\alpha}[S_{\alpha}].$$

Here the S_{α} are oriented p-dimensional semi-algebraic submanifolds of M and the symbols $[S_{\alpha}]$ are subject to the following relations:

- (i) $[S_1 \cup S_2] = [S_1] + [S_2]$ if S_1, S_2 are disjoint;
- (ii) $[S^{-}] = -[S]$, where S^{-} is the manifold S with the opposite orientation;
- (iii) [S] = [S'] if $S' \subset S$ is an open and dense subset of S with the orientation induced from S.

From the relations above we conclude that we can write any chain C in (6.8) in such a way that the S_{α} are disjoint. Once we have done so, the support of C is defined as $|C| = \overline{\bigcup_{\alpha} S_{\alpha}}$, the closure of the union of the S_{α} . As semialgebraic sets can be triangulated, we can define the boundary operator ∂ from p-chains to (p-1)-chains in the usual way. If C is a p-chain and $\partial C = 0$ then we call C a p-cycle.

Lagrangian cycles. Let $M = T^*X$, where X is a real algebraic manifold of dimension n. The manifold T^*X has a canonical symplectic structure. We call a

semi-algebraic subset Z of T^*X Lagrangian if Z has an open dense subset U consisting of smooth points such that U is a Lagrangian submanifold of T^*X . We call a cycle on T^*X Lagrangian if its support is. The group of positive reals \mathbb{R}^+ acts by scaling on T^*X . We denote by $\mathcal{L}^+(X)$ the group of semi-algebraic, \mathbb{R}^+ -invariant Lagrangian cycles on T^*X . Each $C \in \mathcal{L}^+(X)$ is an n-cycle on T^*X and, as is not very hard to show,

(6.9)
$$|C| \subset \bigcup T_{S_i}^* X$$
, $S_1, \dots S_k \subset X$ submanifolds.

Characteristic Cycles

We will now give the axiomatic description of the characteristic cycle construction. For that we assume, for simplicity, that the real algebraic manifold X is orientable, and we fix an orientation of X. The characteristic cycle construction CC is a map

(6.10)
$$CC: D(X) \longrightarrow \mathcal{L}^+(X)$$

satisfying the following properties:

(a) The definition is local, *i.e.*, the following diagram commutes:

$$D(X) \xrightarrow{CC} \mathcal{L}^{+}(X)$$

$$j^{*} \downarrow \qquad \qquad \downarrow j^{*}$$

$$D(U) \xrightarrow{CC} \mathcal{L}^{+}(U);$$

here U is an open subset of X and the j^* on the right denotes the restriction⁶ of cycles from T^*X to T^*U .

- (b) $CC(\mathbb{C}_X) = [X]$; the symbol [X] makes sense because we have fixed an orientation of X.
- (c) CC is additive in exact sequences, i.e., if $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'[1]$ is a distinguished triangle then

$$CC(\mathcal{F}) = CC(\mathcal{F}') + CC(\mathcal{F}'').$$

(d) If $j: U \to X$ is an open embedding, and g is any semi-algebraic defining equation of ∂U – which is at least C^1 and which we assume to be positive on U – then the following diagram commutes:

$$D(U) \xrightarrow{CC} \mathcal{L}^{+}(U)$$

$$Rj_{*} \downarrow \qquad \qquad \downarrow j_{*}$$

$$D(X) \xrightarrow{CC} \mathcal{L}^{+}(X) ;$$

here $j_*: \mathcal{L}^+(U) \to \mathcal{L}^+(X)$ is the following limit operation

$$j_*(C) = \lim_{s \to 0^+} (C + s d \log g), \quad C \in \mathcal{L}^+(U).$$

⁶The restriction can be performed because our cycles do not necessarily have compact support.

In property (d) the notation $C + s d \log g$ stands for the cycle which is obtained by applying the automorphism $(x, \xi) \mapsto (x, \xi + s \frac{dg_x}{g(x)})$ of T^*U to C. The limit operation can be interpreted as

$$\lim_{s \to 0^+} (C + s d \log g) \ = \ -\partial \tilde{C} \,,$$

where we view

$$\tilde{C} = \{C + s d \log g \mid s > 0\}$$

as an (n+1)-chain in $\mathbb{R} \times T^*X$. The property (4) is proved as a theorem in [SV1], where one can also find a more detailed discussion of the notion of limit and precise conventions for orientations.

Example 1. Let $j:(0,1) \hookrightarrow \mathbb{R}$ be the inclusion map of the open interval. As a defining equation of $\partial(0,1)$ we choose g(x)=x(1-x). We will apply (4) with $U=(0,1), X=\mathbb{R}$, and $\mathcal{F}=\mathbb{C}_{(0,1)}$. Because, by (2), $\mathrm{CC}(\mathcal{F})=[(0,1)]$, we get

$$CC(Rj_*\mathbb{C}_{(0,1)}) = \lim_{s \to 0^+} \left[\left\{ s \frac{dg_x}{g(x)} \mid 0 < x < 1 \right\} \right]$$
$$= \lim_{s \to 0^+} \left[\left\{ s \frac{dx}{x} - s \frac{dx}{1 - x} \mid 0 < x < 1 \right\} \right].$$

The result of this calculation is illustrated below in figure 2.

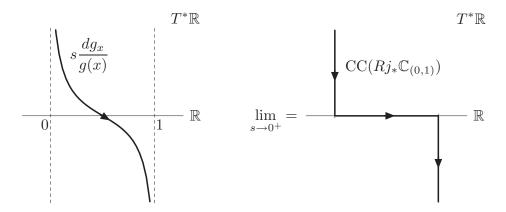


Figure 2

Proceeding in the same manner as in the previous example, we see that

$$CC(\mathbb{C}_Y) = [T_Y^*X], \qquad Y \text{ a closed submanifold of } X,$$

for a particular orientation (see the orientation conventions of [SV1]) of T_Y^*X .

Further properties of CC

Kashiwara's theorem (6.7) extends to the relative situation. Let $f: X \to Y$ be a proper real algebraic map between (oriented) real algebraic manifolds. Then

(6.11)
$$CC(Rf_*\mathcal{F}) = f_*(CC(\mathcal{F})).$$

To define the map

$$(6.12) f_*: \mathcal{L}^+(X) \longrightarrow \mathcal{L}^+(Y)$$

on the right hand side of the equation, let us consider the commutative diagram

(6.13)
$$T^*X \xleftarrow{df} X \times_Y T^*Y \xrightarrow{\tau} T^*Y$$

$$\downarrow \pi_X \qquad \qquad \downarrow \pi_Y$$

$$X \xleftarrow{=} X \xrightarrow{f} Y$$

The assumption that f is proper implies that τ is proper. The map (6.12) is defined by the formula

(6.14)
$$f_*(C) = \tau_*((df)^*C).$$

Here $(df)^*$ is a Gysin map. If the cycle C happens to be transverse to the map df then $(df)^*(C) = (df)^{-1}(C)$. If this is not the case, then we embed C into a family of cycles whose generic members C_s are transverse to df. In this way $(df)^*(C)$ is expressed as the limit $(df)^*(C) = \lim_{s\to 0^+} (df)^{-1}(C_s)$. For a proof of (6.11), see [**KSa**] and for its interpretation in the present language see [**SV1**].

Finally, let us describe the effect of the operation f^* on characteristic cycles for $f: X \to Y$ a submersion. By property (a) it suffices to do so when $X = Z \times Y$ and $f: Z \times Y \to Y$ is the projection. Then

(6.15)
$$\operatorname{CC}(f^*\mathcal{G}) = \operatorname{CC}(\mathbb{C}_Z \boxtimes \mathcal{G}) = [Z] \times \operatorname{CC}(\mathcal{G}).$$

This formula follows from the properties (a-d). On the contrary, there does not appear to be an easy formal way to deduce (6.11) from properties (a-d). Finally, from property (c) we conclude that

(6.16)
$$CC(\mathcal{F}) = \sum (-1)^k CC(H^k(\mathcal{F}))$$

for $\mathcal{F} \in \mathrm{D}(X)$.

Example. Let us deduce (6.7) from (6.11). To this end, let us assume that X is a compact (oriented!) real algebraic manifold, $Y = \{pt\}$, and $f : X \to \{pt\}$. applying (6.16) to $Rf_*\mathcal{F}$, gives $CC(Rf_*\mathcal{F}) = \chi(\mathcal{F})$. By formula (6.11) we get:

$$\chi(\mathcal{F}) = \mathrm{CC}(Rf_*\mathcal{F}) = \tau_*(df)^*(\mathrm{CC}(\mathcal{F})) = \mathrm{CC}(\mathcal{F}).[X];$$

with an appropriate interpretation of the signs in the intersection product.

LECTURE 7 The character formula

In this lecture we give an integral formula for the Lie algebra character of the representation $M(\mathcal{F})$ associated to a $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)_{-\lambda}$ in terms of the characteristic cycle $CC(\mathcal{F})$. We begin by briefly recalling the notion of the character on the Lie algebra.

The Lie algebra character

Let π be a representation, in our previous sense, of the semisimple Lie group $G_{\mathbb{R}}$ with infinitesimal character χ_{λ} . To π , following Harish-Chandra, we can associate its character Θ_{π} . The character Θ_{π} is an invariant eigendistribution on $G_{\mathbb{R}}$, i.e., it is conjugation invariant and the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ acts on Θ_{π} via the character χ_{λ} . Via the exponential map we can, at least in the neighborhood of the origin, pull back the distribution Θ_{π} to the Lie algebra $\mathfrak{g}_{\mathbb{R}}$. We define the Lie algebra character by the formula

(7.1)
$$\theta_{\pi} = \sqrt{\det(\exp_*)} \exp^* \Theta_{\pi}.$$

We have inserted the factor $\sqrt{\det(\exp_*)}$ so that θ_{π} is an invariant eigendistribution on $\mathfrak{g}_{\mathbb{R}}$, i.e., conjugation invariant and the constant coefficient differential operators $S(\mathfrak{g})^G \cong \mathcal{Z}(\mathfrak{g})$ act on θ_{π} via the character χ_{λ} . Harish-Chandra's regularity theorem implies that θ_{π} can be extended from the neighborhood of the origin uniquely to all of $\mathfrak{g}_{\mathbb{R}}$ and that θ_{π} is a locally L^1 -function which is real analytic on the set of regular semisimple elements in $\mathfrak{g}_{\mathbb{R}}$.

Rossmann's formula

Let us now assume that the group $G_{\mathbb{R}}$ has a compact Cartan. As was explained in lecture 4, the discrete series representations are parametrized by regular, antidominant $\lambda \in \Lambda + \rho$ and a choice of an open $G_{\mathbb{R}}$ -orbit on X. To this data we attach a discrete series representation $\pi = \pi(S, \lambda)$. Recall (4.5): as virtual representations, $\pi(S, \lambda) = (-1)^s M(Rj_*(\mathbb{C}_S))$, where $j: S \hookrightarrow X$ denotes the inclusion. Associated to the data (S, λ) we construct a coadjoint $G_{\mathbb{R}}$ -orbit $\Omega_{\lambda}(S)$ in $i\mathfrak{g}_{\mathbb{R}}^*$ exactly the same way as in lecture 1 where we had assumed that $G_{\mathbb{R}}$ is compact. Given $x \in S$, there is a unique compact Cartan $T_{\mathbb{R}}$ that fixes x. This gives us

a map $\tau_x: \mathfrak{t}_{\mathbb{R}} \to \mathfrak{h}$ which, in turn, allows us to view λ as an element $\lambda_x \in i\mathfrak{t}_{\mathbb{R}}^*$. Finally, the direct sum decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus [\mathfrak{t}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}]$, allows us to view $\lambda_x \in i\mathfrak{g}_{\mathbb{R}}^*$. The association $x \mapsto \lambda_x$ gives a $G_{\mathbb{R}}$ -equivariant identification of S with a $G_{\mathbb{R}}$ -orbit $\Omega_{\lambda}(S) \subset i\mathfrak{g}_{\mathbb{R}}^*$. The orbit $\Omega_{\lambda}(S)$ has a canonical symplectic form σ_{λ} . Let θ_{π} denote the Lie algebra character of the discrete series representation $\pi(S,\lambda)$. Rossmann, in $[\mathbf{R}\mathbf{1}]$, proves the following result:

(7.2)
$$\int_{\mathfrak{g}_{\mathbb{R}}} \theta_{\pi} \varphi \, dx = \frac{1}{(2\pi i)^{n} n!} \int_{\Omega_{\lambda}(S)} \hat{\varphi} \, \sigma_{\lambda}^{n},$$

in complete analogy with (1.6), where φ is a tempered test function and the Fourier transform is performed without the i as in (1.5). As in lecture 1, this result can be phrased as

(7.3)
$$\hat{\theta}_{\pi} = \text{the coadjoint orbit } \Omega_{\lambda}(S) \text{ with measure } \frac{\sigma_{\lambda}}{(2\pi i)^{n} n!}.$$

From this one can obtain an analogous formula for all tempered representations.

Formula (7.2,3), as stated, can not be generalized for non-tempered representations: the Fourier transform $\hat{\theta}_{\pi}$ no longer makes sense when θ_{π} is not tempered. However, in [**R2**] Rossmann proposed a way to generalize the integral formula (7.2). More specifically, he showed how to write down the invariant eigendistributions on $\mathfrak{g}_{\mathbb{R}}$ as integrals resembling (7.2). We begin by constructing the twisted moment map

(7.4)
$$\mu_{\lambda} : T^*X \longrightarrow \mathfrak{g}^*.$$

We consider the compact form $U_{\mathbb{R}}$ corresponding to the Lie algebra $\mathfrak{u}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. Using the same construction as in (5.8), we obtain a $U_{\mathbb{R}}$ -equivariant real algebraic map

$$(7.5) m_{\lambda} : X \longrightarrow i\mathfrak{u}_{\mathbb{R}}^* \subset \mathfrak{g}^*.$$

Recall the moment map

which is given, on the level of fibers $T_x^*X \cong (\mathfrak{g}/\mathfrak{b}_x)^*$, by the canonical inclusion $(\mathfrak{g}/\mathfrak{b}_x)^* \to \mathfrak{g}^*$. The moment map μ is G-equivariant and complex algebraic. Its image is the nilpotent cone, once we make the identification $\mathfrak{g} \cong \mathfrak{g}^*$ by means of the Killing form. The twisted moment map μ_{λ} is given by the following formula:

where $\pi: T^*X \to X$ denotes the projection. The twisted moment map, for regular λ , provides an isomorphism

where $\Omega_{\lambda} \subset \mathfrak{g}^*$ is a G-orbit. Let us denote the canonical symplectic form on Ω_{λ} by σ_{λ} and let us write

(7.9)
$$T_{G_{\mathbb{R}}}^* X = \bigcup_{S \text{ a } G_{\mathbb{R}}\text{-orbit}} T_S^* X.$$

Then we see that

$$(7.10) \quad \mathrm{H}^{inf}_{2n}(T^*_{G_{\mathbb{R}}}X,\mathbb{C}) \ = \ \left\{ \mathrm{Lagrangian} \ \mathbb{C}\text{-cycles on} \ T^*X \ \mathrm{supported} \ \mathrm{on} \ T^*_{G_{\mathbb{R}}}X \right\},$$

where $n = \dim_{\mathbb{C}} X$, and the symbol H^{inf}_{2k} stands for homology with closed, i.e., possibly non-compact, supports. We define the Fourier transform $\hat{\varphi}$ of a test function φ in $C_c^{\infty}(\mathfrak{g}_{\mathbb{R}})$ without choosing a square root of -1, as a holomorphic function on \mathfrak{g}^* :

(7.11)
$$\hat{\varphi}(\zeta) = \int_{\mathfrak{g}_{\mathbb{R}}} e^{\zeta(x)} \varphi(x) dx \qquad (\zeta \in \mathfrak{g}^*).$$

Let us assume that λ is regular. Rossmann [R2] shows that

(7.12)
$$H_{2n}^{inf}(T_{G_{\mathbb{R}}}^*X, \mathbb{C}) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{invariant eigendistributions on } \mathfrak{g}_{\mathbb{R}} \\ \text{with infinitesimal character } \chi_{\lambda} \end{array} \right\} \\
C \mapsto \left\{ \varphi \mapsto \frac{1}{(2\pi i)^n n!} \int_{\mu_{\lambda}(C)} \hat{\varphi} \ \sigma_{\lambda}^n \right\}.$$

Here $\varphi \in C_c^{\infty}(\mathfrak{g}_{\mathbb{R}})$ is a test function. The integral $\int_{\mu_{\lambda}(C)} \hat{\varphi} \ \sigma_{\lambda}^n$ converges because $\hat{\varphi}$ decays rapidly in the imaginary directions and the cycle $\mu_{\lambda}(C)$ has bounded real parts: $\mu_{\lambda}(C)$ differs from $\mu(C)$ by a compact set and $\mu(T_{G_{\mathbb{R}}}^*X) \subset i\mathfrak{g}_{\mathbb{R}}^*$. Note that the form $\hat{\varphi}\sigma_{\lambda}^n$ is holomorphic, of top degree in Ω_{λ} . Hence, the cycle $\mu_{\lambda}(C)$ can be replaced by a homologous cycle without changing the value of the integral $\int_{\mu_{\lambda}(C)} \hat{\varphi} \ \sigma_{\lambda}^n$, provided that the chain giving rise to the homology has bounded real parts.

The character formula

Fix $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)_{-\lambda}$ and let $\theta(\mathcal{F})$ denote the Lie algebra character of the representation $M(\mathcal{F})$. We continue to assume that λ is regular. Then, as is shown in $[\mathbf{SV2}]$:

Theorem. The character $\theta(\mathcal{F})$ is given by taking $C = CC(\mathcal{F})$ in (7.12), i.e.,

(7.13)
$$\int_{\mathfrak{g}_{\mathbb{R}}} \theta(\mathcal{F}) \varphi dx = \frac{1}{(2\pi i)^n n!} \int_{\mu_{\lambda}(CC(\mathcal{F}))} \hat{\varphi} \sigma_{\lambda}^n,$$

for $\varphi \in C_c^{\infty}(\mathfrak{g}_{\mathbb{R}})$.

Remark. The fact that $CC(\mathcal{F})$ is supported on $T_{G_{\mathbb{R}}}^*X$ follows from the $G_{\mathbb{R}}$ -equivariance of \mathcal{F} . Note that $CC(\mathcal{F}) \in H_{2n}^{inf}(T_{G_{\mathbb{R}}}^*X, \mathbb{Z})$, i.e., the characters are given by integral cycles in (7.12).

To extend the validity of the theorem to arbitrary λ , we use the following result:

Lemma. $\mu_{\lambda}^* \sigma_{\lambda} = -\sigma + \pi^* \tau_{\lambda}$, where σ denotes the canonical symplectic form on T^*X and τ_{λ} is a 2-form on X defined by $\tau_{\lambda}(u_x, v_x) = \lambda[u, v]$, $x \in X$, and u_x, v_x are tangent vectors given by $u, v \in \mathfrak{u}_{\mathbb{R}}$.

The following formula is valid for all λ :

(7.14)
$$\int_{\mathfrak{g}_{\mathbb{R}}} \theta(\mathcal{F}) \varphi dx = \frac{1}{(2\pi i)^n n!} \int_{CC(\mathcal{F})} \mu_{\lambda}^* \hat{\varphi} (-\sigma + \pi^* \tau_{\lambda})^n .$$

As an example, let us consider the case $G_{\mathbb{R}} = SL(2,\mathbb{R})$. Then $X = \mathbb{CP}^1$, and there are three $G_{\mathbb{R}}$ -orbits, the upper and lower hemispheres D_+ , D_- , and the equator $\mathbb{RP}^1 \cong S^1$. Let us consider a discrete series representation associated to D_- and a negative $\lambda \in \mathbb{Z} \cong \Lambda$. We take $\mathcal{F} = Rj_*\mathbb{C}_{D_-} = \mathbb{C}_{\bar{D}_-}$. To calculate $CC(Rj_*\mathbb{C}_{D_-})$, we view D_- as a unit disc in \mathbb{C} and, by the defining property (d) of the characteristic cycle in lecture 6, we should choose a defining equation f for the boundary of D_- . Once the equation is chosen, we get:

(7.15)
$$\operatorname{CC}(Rj_*\mathbb{C}_{D_-}) = \lim_{s \to 0^+} \left(\operatorname{CC}(\mathbb{C}_{D_-}) + s \frac{df}{f} \right) \\ = \lim_{s \to 0^+} \left\{ s \frac{df_x}{f(x)} \mid x \in D_+ \right\}.$$

By making the simplest choice $f(z) = 1 - |z|^2$ for the equation of the boundary, we see that $CC(Rj_*\mathbb{C}_{D_-})$ is a cylinder with base D_- . There is a more "sophisticated" choice for the equation of the boundary:

(7.16)
$$f(z) = \left(\frac{1+|z|^2}{1-|z|^2}\right)^{4\lambda}.$$

For this boundary equation we get

(7.17)
$$\mu_{\lambda} \left\{ \frac{df_x}{f(x)} \mid x \in D_{-} \right\} = \Omega_{\lambda}(D_{-}),$$

where, we recall, $\Omega_{\lambda}(D_{-})$ denotes the coadjoint $G_{\mathbb{R}}$ -orbit in $i\mathfrak{g}_{\mathbb{R}}^{*}$ determined by λ and D_{-} . Letting the parameter s vary between 0 and 1 establishes a homology between $\Omega_{\lambda}(D_{-})$ and $\mu_{\lambda}(\mathrm{CC}(Rj_{*}\mathbb{C}_{D_{+}}))$. Thus, our character formula (7.13) agrees with Rossmann's formula (7.3) for discrete series representations of $SL(2,\mathbb{R})$.

The above argument can be generalized to discrete series representations for any group $G_{\mathbb{R}}$. To this end, let us assume that $G_{\mathbb{R}}$ has a compact Cartan. Pick an open $G_{\mathbb{R}}$ -orbit $S \subset X$ and choose $\lambda \in \rho + \Lambda$ antidominant. Recall that the discrete series representation attached to (S,λ) is given by $(-1)^s M(Rj_*\mathbb{C}_S)$, where $j: S \hookrightarrow X$ denotes the inclusion. In analogy with (7.16), we make the following choice for the defining equation of ∂S :

(7.18)
$$f = \frac{G_{\mathbb{R}}\text{-invariant metric on } \mathbf{L}_{\lambda}}{U_{\mathbb{R}}\text{-invariant metric on } \mathbf{L}_{\lambda}};$$

here \mathbf{L}_{λ} denotes the line bundle on X associated to λ . Then, just as in the case $G_{\mathbb{R}} = SL(2,\mathbb{R})$,

(7.19)
$$\mu_{\lambda} \left\{ \frac{df_x}{f(x)} \mid x \in S \right\} = \Omega_{\lambda}(S).$$

Therefore, for discrete series, the character formula (7.13) agrees with Rossmann's formula. This is a crucial step in proving the formulas (7.13) and (7.14). For details, see [SV2].

LECTURE 8

Microlocalization of Matsuki = Sekiguchi

In this final lecture we explain the relationship between the Matsuki and the Sekiguchi correspondences. This relationship is provided by geometry. It is a crucial step in the proof of the Barbasch-Vogan conjecture [SV3] which we will briefly explain at the end. We will continue to use the notation of the previous lectures. Recall that the Matsuki correspondence provides a bijection between the $G_{\mathbb{R}^-}$ and K-orbits on X:

$$(8.1) G_{\mathbb{R}} \backslash X \longleftrightarrow K \backslash X ,$$

where a $G_{\mathbb{R}}$ -orbit \mathcal{O}' corresponds to K-orbit \mathcal{O} if and only if $\mathcal{O} \cap \mathcal{O}'$ is non-empty and compact (in which case $\mathcal{O} \cap \mathcal{O}'$ constitutes a $K_{\mathbb{R}}$ -orbit). Let \mathcal{N} be the nilpotent cone in \mathfrak{g} . The Kostant-Sekiguchi correspondence provides a bijection between nilpotent $G_{\mathbb{R}}$ - and K-orbits:

$$(8.2) G_{\mathbb{R}} \backslash \mathcal{N} \cap i\mathfrak{g}_{\mathbb{R}} \longleftrightarrow K \backslash \mathcal{N} \cap \mathfrak{p}$$

Here a $G_{\mathbb{R}}$ -orbit \mathcal{O}' corresponds to a K-orbit \mathcal{O} if and only if there exists a Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ which is defined over \mathbb{R} and which commutes with the Cartan involution, such that

$$(8.3) j\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \in \mathcal{O}' , j\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathcal{O} .$$

Let us recall, from lecture 5, the Matsuki correspondence for sheaves:

(8.4)
$$\gamma: D_K(X)_{-\lambda} \xrightarrow{\sim} D_{G_{\mathbb{R}}}(X)_{-\lambda}.$$

It satisfies the following property:

(8.5)
$$\gamma(Rj_*\mathbb{C}_{\mathcal{O}}) = Rj'_!\mathbb{C}_{\mathcal{O}'}[2\operatorname{codim}_{\mathbb{C}}\mathcal{O}];$$

here $j: \mathcal{O} \hookrightarrow X$ and $j': \mathcal{O}' \hookrightarrow X$ denote the inclusions of a K-orbit \mathcal{O} and a $G_{\mathbb{R}}$ -orbit \mathcal{O}' which are related under the Matsuki correspondence. Also, recall that $\gamma = \Gamma^{-1}$ is defined exactly in the same way as Γ except that the roles of K and

 $G_{\mathbb{R}}$ are switched and all the *'s are replaced by !'s. Next, we consider the following commutative diagram:

(8.6)
$$D_{K}(X)_{-\lambda} \xrightarrow{\gamma} D_{G_{\mathbb{R}}}(X)_{-\lambda}$$

$$\downarrow^{CC} \qquad \downarrow^{CC}$$

$$H_{2n}^{inf}(T_{K}^{*}X, \mathbb{Z}) \xrightarrow{\Phi=CC(\gamma)} H_{2n}^{inf}(T_{G_{\mathbb{R}}}^{*}X, \mathbb{Z}),$$

where the map $\Phi = \mathrm{CC}(\gamma)$ is the effect of the functor γ on characteristic cycles. From the properties (a-d) of the characteristic cycles one can conclude that apriori that $\Phi = \mathrm{CC}(\gamma)$ exists. We will now give an explicit formula for Φ . To this end, let us define, for s > 0, an automorphism $F_s : T^*X \to T^*X$ by the following formula:

(8.7)
$$F_s(x,\xi) = (\exp(s^{-1}(\text{Re }\xi)) x, \operatorname{Ad}(\exp(s^{-1}(\text{Re }\xi)) \xi),$$

where $(x,\xi) \in T^*X$, $\xi \in T_x^*X \simeq \mathfrak{n}_x$, and $\text{Re}: \mathfrak{g} \to \mathfrak{g}_{\mathbb{R}}$ associates to an element in \mathfrak{g} its real part in $\mathfrak{g}_{\mathbb{R}}$. The automorphisms F_s preserve the real(!) symplectic form on T^*X . The map Φ is given as a limit of these symplectomorphisms:

(8.8)
$$\Phi(C) = \lim_{s \to 0^+} (F_s)_*(C) .$$

Remark. It is not obvious that the limit exists. To give meaning to it, one constructs the chain $\tilde{C} = \{(s,\zeta) \in \mathbb{R} \times \mathfrak{g} \mid s > 0, \zeta \in (F_s)_*(C)\}$ and sets $\lim_{s\to 0^+}(F_s)_*(C) = -\partial \tilde{C}$. For this to make sense, the support $|\tilde{C}|$ should be triangulable. This is by no means obvious. It follows from the fact that the set $|\tilde{C}|$ belongs to the analytic geometric category \mathcal{C} coming from the o-minimal structure $\mathbb{R}_{an,\text{exp}}$. For a beautiful exposition of analytic geometric structures see [DM]. The analytic geometric categories satisfy (essentially) the same good properties as semi-algebraic (and subanalytic) sets, and we could have worked in this more general context in lectures 2 and 6. The reason that one needs to pass to the analytic geometric category \mathcal{C} in Lie theory is that the exponential function $\exp: \mathbb{R} \to \mathbb{R}$ is not real analytic at infinity. However, it is an allowable function in the category \mathcal{C} .

We will briefly explain the idea behind the verification of the formula (8.8), for the details, see [SV3]. To begin with, let us recall the definition of γ . Using the notation in the diagram:

$$(8.9) X \stackrel{a}{\leftarrow} G_{\mathbb{R}} \times X \stackrel{q}{\rightarrow} G_{\mathbb{R}} / K_{\mathbb{R}} \times X \stackrel{p}{\rightarrow} X$$

the functor γ is given by $\gamma(\mathcal{F}) = Rp_!\tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}$ is the object such that $q^!\tilde{\mathcal{F}} = a^!\mathcal{F}$. As the maps a and q are submersions, it is easy to express $\mathrm{CC}(\tilde{\mathcal{F}})$ in terms of $\mathrm{CC}(\mathcal{F})$. The difficulty lies in computing $\mathrm{CC}(Rp_!\tilde{\mathcal{F}})$ in terms of $\mathrm{CC}(\tilde{\mathcal{F}})$, in particular, because the map p is not proper. We will compactify the map p by embedding the symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$ inside a compact real algebraic manifold and taking its closure. At this point it does not matter which compactification of the symmetric space we choose. This gives us the following diagram:

(8.10)
$$G_{\mathbb{R}} \times X \xrightarrow{q} G_{\mathbb{R}}/K_{\mathbb{R}} \times X \xrightarrow{j} \overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X$$

$$\downarrow^{p} \qquad \qquad \downarrow_{\bar{p}}$$

$$X \qquad X = X.$$

From this diagram we conclude that

$$(8.11) Rp_! \tilde{\mathcal{F}} = R\bar{p}_* j_! \tilde{\mathcal{F}},$$

using the fact that $\bar{p}_! = \bar{p}_*$ since \bar{p} is proper. To get a formula for $CC(Rp_!\tilde{\mathcal{F}}) = CC(R\bar{p}_*j_!\tilde{\mathcal{F}})$, we begin by applying (6.11) to $R\bar{p}_*$. The top row of (6.13) becomes, in this case,

$$(8.12) T^*(\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X) \xleftarrow{d\bar{p}^*} G_{\mathbb{R}}/K_{\mathbb{R}} \times T^*X \xrightarrow{\tau} T^*X,$$

and $d\bar{p}^*$ is the canonical embedding. Then, by (6.11),

(8.13)
$$\operatorname{CC}(R\bar{p}_*j_!\tilde{\mathcal{F}}) = \bar{p}_*(\operatorname{CC}(j_!\tilde{\mathcal{F}})) = \tau_*([\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X].\operatorname{CC}(j_!\tilde{\mathcal{F}})),$$

where, $[\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X]$. $CC(j_!\tilde{\mathcal{F}})$ is the intersection product of cycles. To compute $CC(j_!\tilde{\mathcal{F}})$, we choose a defining equation f for the boundary of $G_{\mathbb{R}}/K_{\mathbb{R}}$ and use a variant of the defining property (d) of the characteristic cycle map. Denoting $C = CC(\tilde{\mathcal{F}})$, we get:

(8.14)
$$CC(j_!\tilde{\mathcal{F}}) = j_!C = \lim_{s \to 0^+} (C - s d \log f).$$

Combining (8.13) and (8.14) gives:

(8.15)
$$\operatorname{CC}(Rp_!\tilde{\mathcal{F}}) = \tau_*([\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X].(\lim_{s \to 0^+} (C - s d \log f))).$$

We now rewrite this formula as

(8.16)
$$\operatorname{CC}(Rp_!\tilde{\mathcal{F}}) = \lim_{s \to 0^+} \tau_*([\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X].(C - s d \log f)).$$

If f is chosen appropriately then the intersection $(\overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \times X) \cap (C - s d \log f)$ is transverse and the intersection product in (8.16) can be replaced by an ordinary intersection⁷. This observation is the crux of the computation.

To evaluate the right hand side of (8.16), we choose as $\overline{G_{\mathbb{R}}/K_{\mathbb{R}}}$ the one-point compactification of $G_{\mathbb{R}}/K_{\mathbb{R}} \simeq \mathfrak{p}_{\mathbb{R}}$:

$$(8.17) \overline{G_{\mathbb{R}}/K_{\mathbb{R}}} \simeq \overline{\mathfrak{p}_{\mathbb{R}}} = \mathfrak{p}_{\mathbb{R}} \cup \{\infty\}$$

Furthermore, we choose the defining equation $f: \overline{\mathfrak{p}_{\mathbb{R}}} \to \mathbb{R}$ as follows:

(8.18)
$$f(\zeta) = \begin{cases} e^{-\frac{1}{2}B(\zeta,\zeta)} & \text{if } \zeta \in \mathfrak{p}_{\mathbb{R}} \\ 0 & \text{if } \zeta = \infty. \end{cases}$$

Here B denotes the Killing form. Then $d \log f = -\operatorname{id}$ on $\mathfrak{p}_{\mathbb{R}}$ and a relatively easy computation gives the formula for Φ .

⁷We are ignoring all issues of orientation.

Remark. The function f is not real analytic at infinity. This forces us outside the semi-algebraic and subanalytic categories and into the analytic geometric category \mathcal{C} coming from the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, as was explained in the previous remark.

We will now relate the map Φ to the Kostant-Sekiguchi correspondence. To do so, we identify $\mathfrak g$ with $\mathfrak g^*$ via the Killing form and let $\mathcal N\subset\mathfrak g$ denote the nilpotent cone. Then the moment map $\mu:T^*X\to\mathcal N$ and furthermore

(8.19)
$$\mu^{-1}(i\mathfrak{g}_{\mathbb{R}}) = T_{G_{\mathbb{R}}}^* X \quad \text{and} \quad \mu^{-1}(\mathfrak{p}) = T_K^* X.$$

Let us fix a G-orbit $\mathcal{O} \subset \mathcal{N}$, write $i\mathfrak{g}_{\mathbb{R}} \cap \mathcal{O}$ as a union of $G_{\mathbb{R}}$ -orbits:

$$(8.20a) i\mathfrak{g}_{\mathbb{R}} \cap \mathcal{O} = \mathcal{O}'_1 \cup \cdots \cup \mathcal{O}'_k,$$

and, similarly, $\mathfrak{p} \cap \mathcal{O}$ as a union of K-orbits:

$$(8.20b) p \cap \mathcal{O} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k.$$

We enumerate the orbits so that \mathcal{O}_i and \mathcal{O}'_i correspond to each other under Sekiguchi. Let us consider the complex Lagrangian cycle $[\mu^{-1}(\mathcal{O}_i)]$ on T^*X . It is, by definition, supported on T_K^*X . Clearly the symplectomorphisms F_s of (8.7) map $\underline{\mu^{-1}(\mathcal{O})} \to \mu^{-1}(\mathcal{O})$ and hence Φ , as the limit of the F_s , maps $\mu^{-1}(\mathcal{O})$ to its closure $\overline{\mu^{-1}(\mathcal{O})}$. Thus, we can write

(8.21)
$$\Phi([\mu^{-1}(\mathcal{O}_i)]) = \sum n_j [\mu^{-1} \mathcal{O}'_j] + \text{lower order terms},$$

where by lower order terms we mean chains which lie over $\partial \bar{\mathcal{O}} = \bar{\mathcal{O}} - \mathcal{O}$. Here the $n_j \in \mathbb{Z}$ and we note that the $[\mu^{-1}\mathcal{O}'_j]$ are chains, not necessarily cycles, as they can have boundary in $\mu^{-1}(\partial \bar{\mathcal{O}})$.

Theorem. The map Φ induces the Sekiguchi correspondence on the nilpotent orbits, i.e.,

(8.22)
$$\Phi([\mu^{-1}\mathcal{O}_i]) = [\mu^{-1}\mathcal{O}_i'] + l.o.t.$$

We do not know of a simple argument for (8.22). The proof is contained in [SV3,SV4]. As a corollary of this result we see that the Sekiguchi correspondence is given by

$$(8.23) \quad c \mapsto \lim_{s \to 0^+} \{ \operatorname{Ad}(\exp(s^{-1}\operatorname{Re}\zeta)\zeta \mid \zeta \in c \} = \lim_{s \to 0^+} \{ s \operatorname{Ad}(\exp(\operatorname{Re}\zeta)\zeta \mid \zeta \in c \} ,$$

where c stands for one of the K-orbits in (8.20b).

Finally, we explain, very briefly, how the above theorem enters the proof of the Barbasch-Vogan conjecture. For details, see [SV3]. Putting together (5.4), (8.6), and (8.22) we get the commutative diagram:

$$\{\text{HC-modules}\}_{\chi_{\lambda}} \qquad \longrightarrow \qquad \{G_{\mathbb{R}}\text{-representations}\}_{\chi_{\lambda}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$D_{K}(X)_{-\lambda} \qquad \longrightarrow \qquad D_{G_{\mathbb{R}}}(X)_{-\lambda}$$

$$\downarrow \text{CC}$$

$$H_{2n}^{inf}(T_{K}^{*}X, \mathbb{Z}) \qquad \stackrel{\Phi}{\longrightarrow} \qquad H_{2n}^{inf}(T_{G_{\mathbb{R}}}^{*}X, \mathbb{Z})$$

$$\downarrow \text{gr}(\mu)_{\lambda} \qquad \qquad \downarrow \text{gr}(\mu)_{\lambda}$$

 $\mathrm{nilpotent}\ \mathrm{orbits}\ \mathrm{in}\ \mathcal{N}\cap\mathfrak{p}\ \xrightarrow{\ \ \mathrm{Sekiguchi}\ \ }\mathrm{nilpotent}\ \mathrm{orbits}\ \mathrm{in}\ \mathcal{N}\cap\mathfrak{ig}_{\mathbb{R}}$

A few remarks are in order. First, we have turned around the arrows in (5.4). As the functors M and α are not invertible, one interprets the top square as a consistent choice of representatives in $D_K(X)_{-\lambda}$ and $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ for representations. Furthermore, both the vertical arrows $\operatorname{gr}(\mu)_{\lambda}$ stand for the operation of taking the leading term of the result of integration of a cycle along the fiber of μ against the form e^{λ} .

Let us consider an irreducible representation (π, V) whose associated Harish-Chandra module we denote by M. By a result of Chang $[\mathbf{C}]$, the left vertical column amounts to the associated cycle construction:

(8.25)
$$\operatorname{Ass}(M) = \sum a_j[\mathcal{O}_j], \quad \text{with } a_j \in \mathbb{Z}_{\geq 0}.$$

For the associated cycle construction, see the lectures of Vogan. From the character formula (7.14) we can conclude that the right hand column amounts to associating to V the Fourier transform of the leading term of the asymptotic expansion of the Lie algebra character of the representation (π, V) . This gives the wave front cycle

(8.26)
$$WF(V) = \sum b_j [\mathcal{O}'_j], \quad \text{with } b_j \in \mathbb{C}.$$

The invariant Ass(M) is purely algebraic whereas the invariant WF(V), introduced in $[\mathbf{BV}]$, is analytic. The commutative diagram (8.24) implies that these two invariants coincide under the Sekiguchi correspondence, i.e.,

$$(8.27) a_j = b_j \text{if} \mathcal{O}_j \leftrightarrow \mathcal{O}'_j \text{under Sekiguchi}.$$

This statement is usually referred to as the Barbasch-Vogan conjecture. Note that it implies, in particular, that the b_j are non-negative integers.

APPENDIX

Homological algebra by Markus Hunziker

Let $\mathfrak A$ be an abelian category. Typical examples of abelian categories are the category of (left) modules over an arbitrary ring R with unit, the category of sheaves of $\mathbb C$ -vector spaces on a topological space X, and the category of $\mathbb C$ -constructible sheaves on a semi-algebraic set X.

The category of complexes $C(\mathfrak{A})$

Definition. Recall that a (cochain) complex of objects in \mathfrak{A} is a sequence of objects A^i , $i \in \mathbb{Z}$, together with morphisms $d^i: A^i \to A^{i+1}$,

$$A^{\cdot} = (\cdots \longrightarrow A^{i} \xrightarrow{d^{i}} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \cdots), \text{ such that } d^{i+1} \circ d^{i} = 0$$

for all i. The morphisms $d^i:A^i\to A^{i+1}$ are called the differentials of the complex. A morphism of complexes $f:A^i\to B^i$ is a sequence of morphisms $f^i:A^i\to B^i$ which commute with the differentials in the sense that $d^i_B\circ f^i=f^{i+1}\circ d^i_A$ for all i. Thus we obtain a category $C(\mathfrak{A})$, which is abelian. We identify \mathfrak{A} with the full subcategory of $C(\mathfrak{A})$ consisting of complexes A^i such that $A^i=0$ for $i\neq 0$. Later we will also need the full subcategory $C^+(\mathfrak{A})$ of bounded below complexes.

Shift functors. For every integer k, we define a functor $[k]: C(\mathfrak{A}) \to C(\mathfrak{A})$ as follows. If A is a complex then A [k] is the complex given by

$$A^i[k] \ = \ A^{k+i} \ , \qquad d^i_{A[k]} \ = \ (-1)^k \, d^{k+i}_A \ .$$

If $f: A^{\cdot} \to B^{\cdot}$ is a morphism of complexes then $f[k]: A^{\cdot}[k] \to B^{\cdot}[k]$ is given by $f^{i}[k] = f^{k+i}$. The functor [k] is called the *shift functor* of degree k.

Cohomology and quasi-isomorphisms. The *i-th cohomology object* of a complex A^{\cdot} is the object $H^{i}(A^{\cdot}) = \ker d^{i}/\operatorname{im} d^{i-1}$ which is well-defined since \mathfrak{A} is an abelian category. A morphism of complexes $f:A^{\cdot}\to B^{\cdot}$ induces morphisms $H^{i}(f):H^{i}(A^{\cdot})\to H^{i}(B^{\cdot})$ between cohomology objects. If all the $H^{i}(f):H^{i}(A^{\cdot})\to H^{i}(B^{\cdot})$ are isomorphisms then we say that f is a *quasi-isomorphism* and we write $f:A^{\cdot}\xrightarrow{q^{is}}B^{\cdot}$.

Example. Let A be an object of \mathfrak{A} and let $0 \to A \to E^0 \to E^1 \to \cdots$ be a resolution of A. Then we have a quasi-isomorphism $A \xrightarrow{qis} E$.

The homotopy category $K(\mathfrak{A})$

Homotopy. Two morphisms $f, g: A^{\cdot} \to B^{\cdot}$ in $C(\mathfrak{A})$ are called *homotopic* if there is a sequence of morphisms $k^{i}: A^{i} \to B^{i-1}$ in \mathfrak{A} such that

$$f^i - g^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i$$
.

If f and g are homotopic then they induce the same morphism $H^i(A^{\cdot}) \to H^i(B^{\cdot})$ on the cohomology objects for all i.

Definition. The category $K(\mathfrak{A})$ has as objects complexes of objects in \mathfrak{A} and as morphisms homotopy equivalence classes of morphisms in $C(\mathfrak{A})$. Similarly, we obtain a category $K^+(\mathfrak{A})$ from the category $C^+(\mathfrak{A})$ of bounded below complexes.

Triangles and long exact sequences

The notion of a short exact sequence is not well-defined in $K(\mathfrak{A})$, which is *not* an abelian category. The substitutes for short exact sequences are so-called distinguished triangles. They generate canonically long exact cohomology sequences.

Definition. A triangle in $K(\mathfrak{A})$ is a diagram $A^{\cdot} \xrightarrow{u} B^{\cdot} \xrightarrow{v} C^{\cdot} \xrightarrow{w} A^{\cdot}[1]$. Often a triangle is written in the mnemonic form

$$\begin{array}{ccc} A^{\cdot} & \xrightarrow{u} & B^{\cdot} \\ & & \swarrow^{+1} & \swarrow_{v} & , \\ & & C^{\cdot} & \end{array}$$

whence the name. A morphism of triangles is given by a commutative diagram:

Mapping cones and distinguished triangles. Let $u: A^{\cdot} \longrightarrow B^{\cdot}$ be a morphism in $C(\mathfrak{A})$. The *mapping cone* of u is the complex $C(u)^{\cdot}$ which is defined by

$$C(u)^i = A^{i+1} \oplus B^i$$
 , $d^i_{C(u)} = \begin{bmatrix} -d^{i+1}_A & 0 \\ u^{i+1} & d^i_B \end{bmatrix}$.

There is a canonical exact sequence $0 \to B^{\cdot} \xrightarrow{v} C(u)^{\cdot} \xrightarrow{w} A^{\cdot}[1] \to 0$ in $C(\mathfrak{A})$, given by $v: b \mapsto (0, b)$ and $w: (a, b) \mapsto a$. The triangle

$$A^{\cdot} \stackrel{u}{\longrightarrow} B^{\cdot} \stackrel{v}{\longrightarrow} C(u)^{\cdot} \stackrel{w}{\longrightarrow} A^{\cdot}[1]$$

is called the *standard triangle* associated to the mapping cone C(u). A distinguished triangle in K(X) is a triangle which is isomorphic to a standard one.

Remark. If two morphisms $u, u' : A \to B$ in $C(\mathfrak{A})$ are homotopic then the mapping cones C(u) and C(u') are isomorphic in $K(\mathfrak{A})$ and also their associated standard triangles are isomorphic. This isomorphism is not unique in general.

Long exact sequences. If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A \cdot [1]$ is a triangle in K(\mathfrak{A}), then the morphisms u, v, and w induce canonically a sequence

$$\cdots \rightarrow H^i(A^{\cdot}) \rightarrow H^i(B^{\cdot}) \rightarrow H^i(C^{\cdot}) \rightarrow H^{i+1}(A^{\cdot}) \rightarrow \cdots .$$

If the triangle is distinguished this sequence is exact.

Remark. Let $0 \to A^{\cdot} \xrightarrow{u} B^{\cdot} \xrightarrow{v} C^{\cdot} \to 0$ be a short exact sequence in $C(\mathfrak{A})$. There is a canonical map $h: C(u)^{\cdot} \to C^{\cdot}$ given by $h^i: A^{i+1} \oplus B^i \to C^i$, $(a,b) \mapsto v^i(b)$. One can show that h is a quasi-isomorphism.

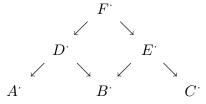
The derived category $D(\mathfrak{A})$

The derived category $D(\mathfrak{A})$ is obtained from $K(\mathfrak{A})$ by "localization" at the multiplicative set of quasi-isomorphisms. It comes together with a natural functor $Q: K(\mathfrak{A}) \to D(\mathfrak{A})$ which sends quasi-isomorphisms to isomorphisms. Similarly, a derived category $D^+(\mathfrak{A})$ is obtained from $K^+(\mathfrak{A})$.

Definition. The objects of $D(\mathfrak{A})$ are again just complexes of objects in \mathfrak{A} . If A and B are two objects in $D(\mathfrak{A})$ then a morphism from A to B is defined as an equivalence classes of diagrams in $K(\mathfrak{A})$ of the form $A \overset{qis}{\longleftarrow} C \to B$. Here the diagram $A \overset{qis}{\longleftarrow} C \to B$ is equivalent to the diagram $A \overset{qis}{\longleftarrow} \tilde{C} \to B$ if there exists a commutative diagram in $K(\mathfrak{A})$ of the form

$$\begin{array}{c} C^{\cdot} \\ \swarrow \uparrow \searrow \\ A^{\cdot} \xleftarrow{qis} D^{\cdot} \longrightarrow B^{\cdot} \\ \searrow \downarrow \nearrow \\ \tilde{C}^{\cdot} \end{array}$$

One can view a diagram $A^{\cdot} \stackrel{s}{\leftarrow} C^{\cdot} \stackrel{u}{\rightarrow} B^{\cdot}$ as a fraction u/s. The composition of morphisms in $D(\mathfrak{A})$ is defined as follows. Let $A^{\cdot} \stackrel{qis}{\leftarrow} D^{\cdot} \rightarrow B^{\cdot}$ and $B^{\cdot} \stackrel{qis}{\leftarrow} E^{\cdot} \rightarrow C^{\cdot}$ be two diagrams in $K(\mathfrak{A})$ representing two morphisms in $D(\mathfrak{A})$. One can show that there always exists a diagram $D^{\cdot} \stackrel{qis}{\leftarrow} F^{\cdot} \rightarrow E^{\cdot}$ such that the following diagram commutes in $K(\mathfrak{A})$:



The diagram $A^{\cdot} \stackrel{q^{is}}{\longleftarrow} F^{\cdot} \to C^{\cdot}$ then defines the composition of the given morphisms in D(\mathfrak{A}). Every morphism $A^{\cdot} \to B^{\cdot}$ in K(\mathfrak{A}) induces a morphism in D(\mathfrak{A}) via the diagram $A \stackrel{=}{\longleftarrow} A \to B$. This defines the functor $Q: K(\mathfrak{A}) \to D(\mathfrak{A})$.

Remark. A morphism $u: A^{\cdot} \to B^{\cdot}$ in $K(\mathfrak{A})$ becomes the zero-map in $D(\mathfrak{A})$ iff there is a quasi-isomorphism $s: B^{\cdot} \to A^{\cdot}$ such that $s \circ u = 0$ in $K(\mathfrak{A})$.

Hyperext and homological dimension. Let A and B be two complexes considered as objects in the derived category $D(\mathfrak{A})$. Then we define the k-th hyperext as the abelian group

$$\operatorname{Ext}^{k}(A^{\cdot}, B^{\cdot}) = \operatorname{Hom}_{D(\mathfrak{A})}(A^{\cdot}, B^{\cdot}[k]).$$

If A and B are objects in \mathfrak{A} , which we may consider as objects in $D(\mathfrak{A})$ concentrated in degree 0, then $\operatorname{Ext}^k(A,B)$ coincides with the usual Ext. This is a result due to Yoneda.

We say that \mathfrak{A} has homological dimension $\leq n$ if $\operatorname{Ext}^k(A,B)=0$ for all k>n and for any objects A and B in \mathfrak{A} . If \mathfrak{A} has homological dimension ≤ 1 then one can show that for any complex A which is bounded from above and below we have an isomorphism in the derived category

$$A^{\cdot} \simeq \bigoplus_{k} H^{k}(A^{\cdot})[-k]$$
.

This holds for example if \mathfrak{A} is the abelian category of vector spaces over a field.

Derived functors

Let $F: \mathfrak{A} \to \mathfrak{B}$ be an additive morphism between abelian categories. The functor extends to a functor $K^+(\mathfrak{A}) \to K^+(\mathfrak{B})$. However, this functor does not send quasi-isomorphisms to quasi-isomorphisms in general. If we assume that the functor F is left exact then under suitable hypotheses (for example if \mathfrak{A} has enough injectives) there exists a derived functor $RF: D^+(\mathfrak{A}) \to D^+(\mathfrak{B})$ which is close to F in the sense that if A is an object in \mathfrak{A} we have a natural isomorphism $F(A) = H^0(RF(A))$.

Injective resolutions. An *injective resolution* is a quasi-isomorphism $A \xrightarrow{qis} I$ such that I^i is an injective object of $\mathfrak A$ for all i. If $\mathfrak A$ has enough injectives then injective resolutions always exist. In K(X) injective resolutions are also unique as follows. Suppose $f: A \to B$ is a morphism in $K(\mathfrak A)$. Let $u: A \xrightarrow{qis} E$ be any quasi-isomorphism and let $v: B \xrightarrow{qis} I$ be an injective resolution. Then there exists a unique morphism $g: E \to I$ such that $v \circ f = g \circ u$.

Theorem. Assume $\mathfrak A$ has enough injectives. Let $\mathfrak J$ be the full category of $\mathfrak A$ of injective objects. Then the natural functor $Q: K^+(\mathfrak A) \to D^+(\mathfrak A)$ induces an equivalence of categories

$$\mathrm{K}^+\!(\mathfrak{J}) \ \simeq \ \mathrm{D}^+(\mathfrak{A}) \ .$$

For a proof of this theorem see, for example, [KSa, Prop. 1.7.10]

Definition. Let $F: \mathfrak{A} \to \mathfrak{B}$ be a left exact, additive functor between abelian categories and assume that \mathfrak{A} has enough injectives. Then the *right derived functor* of F is the functor $RF: D^+(\mathfrak{A}) \to D^+(\mathfrak{B})$ given by

$$RF(A^{\cdot}) = F(I^{\cdot}),$$

where I is any injective resolution of A. Note that RF is well-defined by the remarks above. The *i-th derived functor* of F is the functor $R^iF: D^+(\mathfrak{A}) \to \mathfrak{B}$ given by $R^iF(A) = H^i(F(I))$.

Remark. More generally, one can define define the derived functor RF for a functor $F: K^+(\mathfrak{A}) \to K^+(\mathfrak{B})$ of triangulated categories, i.e., a functor which commutes with the shift functor [1], and transforms distinguished triangles into distinguished triangles.

Example. Fix an object A of $K^+(\mathfrak{A})$. Then the functor $Hom(A, -) : K^+(\mathfrak{A}) \to K^+(\mathfrak{A})$ is a functor of triangulated categories and we may compute the derived functor RHom(A, -) using injective resolutions as above.

Theorem (Yoneda). Assume that \mathfrak{A} has enough injectives. Then

$$\operatorname{Ext}^{k}(A^{\cdot}, B^{\cdot}) = R^{k} \operatorname{Hom}(A^{\cdot}, B^{\cdot}) .$$

In particular, for two objects A and B of \mathfrak{A} , $\operatorname{Ext}^k(A,B)$ is the usual Ext.

F-injective resolutions. To compute the derived functor of F it not necessary to consider injective resolutions. A full additive subcategory \mathfrak{J} of \mathfrak{A} is called F-injective if the following conditions are satisfied:

- (i) Every object of $\mathfrak A$ is isomorphic to a subobject of an object of $\mathfrak J$.
- (ii) If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in \mathfrak{A} , and if A' and A are objects of \mathfrak{J} , then A'' is also an object of \mathfrak{J} .
- (iii) If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in \mathfrak{A} , and if A', A, A'' are objects in \mathfrak{J} , then the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact.

If \mathfrak{J} is any F-injective subcategory of \mathfrak{A} then we may compute $RF: D^+(\mathfrak{A}) \to D^+(\mathfrak{B})$ as above by replacing injective resolutions with F-injective resolutions. This is very useful to compute derived functors in practice by choosing a convenient F-injective category.

Theorem. Let $F: \mathfrak{A} \to \mathfrak{A}'$ and $F': \mathfrak{A}' \to \mathfrak{A}''$ be two left exact additive functors between abelian categories. Assume that there exists an F-injective subcategory \mathfrak{J} of \mathfrak{A} , and a F'-injective subcategory \mathfrak{J}' of \mathfrak{A}' such that F maps objects of \mathfrak{J} to objects of \mathfrak{J}' . Then \mathfrak{J} is $(F' \circ F)$ -injective and we have a natural isomorphism:

$$R(F' \circ F) = RF' \circ RF$$
.

For the proof of this result see, for example, [KSa, Prop. 1.8.7]

Cohomology of sheaves

Let X be a topological space. Let \mathfrak{C} be the category of \mathbb{C} -vector spaces and let $\mathfrak{C}(X)$ be the category of sheaves of \mathbb{C} -vector spaces on X.

Definition. Let $\Gamma(X, -)$ be the global section functor from $\mathfrak{C}(X)$ to \mathfrak{C} . This functor is left exact. We denote by $H^i(X, -)$ the *i*-th derived functor $R^i\Gamma(X, -)$. For a given sheaf \mathcal{F} on X, the vector space

$$H^{i}(X, \mathcal{F}) = R^{i}\Gamma(X, \mathcal{F})$$

is called the *i-th cohomology space* of X with coefficients in the sheaf \mathcal{F} .

Injective, flabby, and c-soft resolutions. Recall that to compute $R\Gamma(X, \mathcal{F})$, we replace \mathcal{F} by a complex of injective sheaves \mathcal{I} quasi-isomorphic to \mathcal{F} and then apply the functor $\Gamma(X, -)$ to \mathcal{I} . It is actually enough to choose the \mathcal{I}^i in a subcategory which is injective with respect to the functor $\Gamma(X, -)$. Examples of such categories are the category of flabby sheaves, and in the case when X is locally compact, the category of c-soft sheaves.

Example. Let X be a real \mathcal{C}^{∞} -manifold of dimension n and let \mathcal{E}^p be the sheaf of smooth p-forms on X. The sheaves \mathcal{E}^p are c-soft. By the Poincaré Lemma, the de $Rham\ complex$

$$0 \to \mathbb{C}_X \to \mathcal{E}^0 \xrightarrow{d} \cdots \to \mathcal{E}^n \to 0$$

is exact. Thus the constant sheaf \mathbb{C}_X is quasi-isomorphic to the complex \mathcal{E} , and for any p the cohomology space $\mathrm{H}^p(X,\mathbb{C}_X)$ is the space of globally closed p-forms modulo the space of globally exact p-forms.

Axiomatic sheaf cohomology. Let $\mathfrak{J}(X)$ be any full additive subcategory of $\mathfrak{C}(X)$ which is injective with respect to $\Gamma(X,-)$. Then the functors $\operatorname{H}^i:\mathfrak{C}(X)\to\mathfrak{C}$ satisfy the following properties:

- (i) There is a natural isomorphism $\Gamma(X, -) = H^0(X, -)$.
- (ii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves then there is a long exact sequence

$$\cdots \to \mathrm{H}^i(X,\mathcal{F}') \to \mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^i(X,\mathcal{F}'') \xrightarrow{\partial^i} \mathrm{H}^{i+1}(X,\mathcal{F}') \to \cdots$$

and the connecting homomorphisms ∂^i behave functorially.

(iii) If \mathcal{J} is any sheaf of the category $\mathfrak{J}(X)$, then $H^i(X,\mathcal{J})=0$ for $i\neq 0$.

The functors H^i are uniquely determined by these axioms.

Local cohomology

Sections with supports. Let Z be a locally closed subset of X. We choose an open subset V of X containing Z as a closed subset, and then define

$$\Gamma_Z(X, \mathcal{F}) = \{ s \in \mathcal{F}(V) : s|_{V-Z} = 0 \}.$$

One checks that $\Gamma_Z(X, \mathcal{F})$ is independent of the choice of the open subset V. We call $\Gamma_Z(X, \mathcal{F})$ the sections of \mathcal{F} with support in Z. If Z = Y is closed then $\Gamma_Y(X, \mathcal{F})$ is just the global sections with support in Y. If Z = V is open then $\Gamma_V(X, \mathcal{F}) = \mathcal{F}(V)$.

Let X, Z, and V be as above. Then if U is an open subset of X, the natural restriction homomorphism $\mathcal{F}(V) \to \mathcal{F}(V \cap U)$ induces a homomorphism $\Gamma_Z(X, \mathcal{F}) \to \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$. The presheaf $U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$ is a sheaf. This sheaf is denoted by $\Gamma_Z(\mathcal{F})$, and is called the *sheaf of sections of* \mathcal{F} *with support in* Z.

The functors $\Gamma_Z(X,-):\mathfrak{C}(X)\to\mathfrak{C}$ and $\Gamma_Z:\mathfrak{C}(X)\to\mathfrak{C}(X)$ are left exact. Moreover, we have $\Gamma_Z(X,-)=\Gamma(X,-)\circ\Gamma_Z(-)$.

Remark. There is a different interpretation of the functors above as follows. Let \mathbb{C}_Z be the constant sheaf on Z, which we also may interpret as a sheaf on X by extending it by zero outside Z. Then we have natural isomorphisms of functors $\Gamma_Z(X,-) = \operatorname{Hom}(\mathbb{C}_Z,-)$, and $\Gamma_Z(-) = \operatorname{Hom}(\mathbb{C}_Z,-)$.

Definition. The *i*-th right derived functors of $\Gamma_Z(X, -)$ and Γ_Z are by denoted by $\mathrm{H}^i_Z(X, -)$ and $\mathrm{H}^i_Z(-)$, respectively. For a given sheaf \mathcal{F} , the vector space $\mathrm{H}^i_Z(X, \mathcal{F})$ (resp., the sheaf $\mathrm{H}^i_Z(\mathcal{F})$) is called the *i*-th cohomology space (resp., cohomology sheaf) of X with coefficients in \mathcal{F} and supports in Z. Note that if Z = X then $\mathrm{H}^i_Z(X, \mathcal{F}) = \mathrm{H}^i(X, \mathcal{F})$ is the usual sheaf cohomology. The natural properties of the functors $\mathrm{H}^i(X, -)$ generalize to properties of the functors $\mathrm{H}^i_Z(X, -)$.

Remark. For any sheaf \mathcal{F} we have a canonical isomorphism

$$H_Z^i(X, \mathcal{F}) = \operatorname{Ext}^i(\mathbb{C}_Z, \mathcal{F}),$$

and similarly, $H_Z^i(\mathcal{F}) = \mathcal{E}xt^i(\mathbb{C}_Z, \mathcal{F})$, where $\mathcal{E}xt$ is the derived functor obtained from $\mathcal{H}om$.

Relative cohomology. We often also write

$$H_Z^i(X, \mathcal{F}) = H^i(X, X - Z; \mathcal{F})$$
,

and think of the cohomology spaces $H_Y^i(X, \mathcal{F})$ as relative cohomology of the pair (X, X - Z) with coefficients in \mathcal{F} .

Long exact sequences. Let Z and X be as above, and let Y be closed in Z. Then for any sheaf \mathcal{F} , there is an exact sequence $0 \to \Gamma_Y(\mathcal{F}) \to \Gamma_Z(\mathcal{F}) \to \Gamma_{Z-Y}(\mathcal{F})$. Moreover, if \mathcal{F} is flabby, then this sequence extends to a short exact sequence. Hence we get long exact sequences

$$\cdots \to \mathrm{H}^i_Y(\mathcal{F}) \to \mathrm{H}^i_Z(\mathcal{F}) \to \mathrm{H}^i_{Z-Y}(\mathcal{F}) \to \mathrm{H}^{i+1}_Y(\mathcal{F}) \to \cdots$$

and

$$\cdots \to \mathrm{H}^i_Y(X,\mathcal{F}) \to \mathrm{H}^i_Z(X,\mathcal{F}) \to \mathrm{H}^i_{Z-Y}(X,\mathcal{F}) \to \mathrm{H}^{i+1}_Y(X,\mathcal{F}) \to \cdots.$$

Excision. Let Z be a locally closed subset of X, and let V be an open subset of X containing Z. Then for any sheaf \mathcal{F} , there exists a natural isomorphism

$$\mathrm{H}^i_Z(X,\mathcal{F}) = \mathrm{H}^i_Z(V,\mathcal{F}|_V)$$
.

BIBLIOGRAPHY

- [BV] D.Barbasch and D.Vogan, *The local structure of characters*, Jour. Func. Anal. **37** (1980), 27–55.
- [BB1] A. Beilinson and J. Bernstein, *Localisations de g-modules*, C. R. Acad. Sci. Paris **292** (1981), 15–18.
- [BB2] A. Beilinson and J. Bernstein, A generalization of Casselman's submodule theorem, in: Representation Theory of Reductive Groups, Progress in Mathematics, Vol 40, Birkhäuser, Boston, 1983, pp. 35–52.
- [BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faiseaux pervers*, Astérisque 100 (1983).
- [BL] J. Bernstein and V. Lunts, Equivariant sheaves and functors, Lecture Notes in Mathematics 1578, Springer, 1994.
- [BM] E. Bierstone and P.D. Milman, Semi-analytic and subanalytic sets, Publications Math. IHES 67 (1988), 5–42.
- [Bo] A. Borel et al., Algebraic D-Modules, Perspectives in Mathematics, Academic Press, 1987.
- [C] J.-T. Chang, Asymptotics and characteristic cycles for representations of complex groups, Compositio Math. (1993), 265 283.
- [DM] L. van den Dries and C. Miller, Geometric categories and o-minimal structures, Duke Math. Jour. 84 (1996), 497 540.
- [EW] T. J. Enright and N. R. Wallach, Notes on homological algebra and representations of Lie algebras, Duke Math. Jour. 47 (1980), 1–15.
- [K1] M. Kashiwara, Index theorem for constructible sheaves, Astérisque 130 (1985), 193 209.
- [K2] M. Kashiwara, Open problems in group representation theory, Proceedings of Taniguchi Symposium held in 1986, RIMS preprint 569, Kyoto University, 1987.
- [KSa] M. Kashiwara and P. Schapira, Sheaves on manifolds, Springer, 1990.
- [KSd] M.Kashiwara and W.Schmid, Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups, Lie Theory and Geometry, in Honor of Bertram Kostant, Progress in Mathematics, Birkhäuser, 1994, pp. 457–488.
- [MP] D. Miličić and P. Pandžić, Equivariant derived categories, Zuckerman functors and localization, Geometry and representation theory of real and p-adic groups (Córdoba, 1995), Progr. Math., 158, Birkhäuser, 1998, pp. 209–242.

- [MUV] I. Mirković, T. Uzawa, and K. Vilonen, *Matsuki correspondence for sheaves*, Inventiones Math. **109** (1992), 231–245.
- [MV] I. Mirković and K. Vilonen, *Characteristic varieties of character sheaves*, Inventiones Math. **93** (1988), 405–418.
- [N] L. Ness, A stratification of the null cone via the moment map, Am. Jour. Math. **106** (1984), 1281 1325.
- [R1] W. Rossmann, Kirillov's character formula for reductive Lie groups, Inventiones Math. 48 (1978), 207–220.
- [R2] W. Rossmann, *Characters as contour integrals*, Lecture Notes in Mathematics 1077, Springer, 1984, pp. 375–388.
- [S1] W. Schmid, Boundary value problems for group invariant differential equations, Astérisque, Numero Hors Serie (1985), 311–321.
- [S2] W. Schmid, Construction and classification of irreducible Harish-Chandra modules, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., 101, Birkhäuser, 1991., pp. 235–275.
- [SV1] W. Schmid and K. Vilonen, *Characteristic cycles of constructible sheaves*, Inventiones Math. **124** (1996), 451–502.
- [SV2] W. Schmid and K. Vilonen, Two geometric character formulas for reductive Lie groups, Jour. AMS 11 (1998), 799 876.
- [SV3] W. Schmid and K. Vilonen, Characteristic cycles and wave front cycles of representations of reductive Lie groups, to appear in the Annals of Math.
- [SV4] W. Schmid and K. Vilonen, On the geometry of nilpotent orbits, to appear in the Atiyah volume of Asian Journal of Mathematics.
- [SW] W. Schmid and J. Wolf, Geometric quantization and derived functor modules for semisimple Lie groups, Jour. Funct. Anal. 90 (1990), 48–112.