Notes on Mirror Symmetry

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Contents

1 Equivariant Cohomology

We begin by introducing the notion of equivariant cohomology of a topological space with a group action and studying some of its properties in Subsections 1.1-1.3. These three subsections contain everything needed to formulate the Localization Theorem of [AB]. Some of the statements in Subsections 1.1-1.3 are used only in the proof of the theorem, while the examples concerning the complex projective space are used later in the proof of the mirror symmetry for a quintic threefold. The equivariant pushforward of a continuous equivariant map is constructed in Subsection 1.4; it is used in a simple setting in the proof of the Localization Theorem and in a more general setting in the proof of the mirror symmetry. Subsection 1.5 contains basic notions from commutative algebra that are used in the proof of the Localization Theorem. The theorem itself is stated and proved in Subsection 1.6; the proof follows Section 3 in [AB].

1.1 Group cohomology

Let G be a Lie group. A classifying space for G is a contractible CW complex[1] EG on which G acts freely, i.e.

$$g\cdot e\neq g \qquad \forall\, e\in EG,\, g\in G-\mathrm{id}.$$

For example, $\mathbb{C}^*$ acts freely on $E\mathbb{C}^*\equiv\mathbb{C}^\infty-0$ by complex multiplication, while S^1 acts freely on $ES^1\equiv S^\infty$ viewed as a subset of $\mathbb{C}^\infty$.

Exercise 1.1. Show that the spaces S^∞ and $\mathbb{C}^\infty-0$ are in fact contractible.
`Hint:` Show that S^∞ retracts onto $S^\infty\cap(0\times\mathbb{C}^\infty)$, i.e. the subspace of elements $v\in S^\infty$ whose first coordinate is zero.

Let $\mathbb{T}$ denote the n-torus, i.e. either $(S^1)^n$ or $(\mathbb{C}^*)^n$. By Exercise 1.1, we can take $E\mathbb{T}$ to be $(S^\infty)^n$ if $\mathbb{T}=(S^1)^n$ and $(\mathbb{C}^\infty-0)^n$ if $\mathbb{T}=(\mathbb{C}^*)^n$. For example, the $\mathbb{T}$-action in the former case is given by

$$\big(e^{\mathrm{i}\theta_1},\ldots,e^{\mathrm{i}\theta_n}\big)\cdot(\underline{z}_1,\ldots,\underline{z}_n)=\big(e^{\mathrm{i}\theta_1}\underline{z}_1,\ldots,e^{\mathrm{i}\theta_n}\underline{z}_n\big).$$

If G is a subgroup of $\mathbb{T}$, G certainly acts freely on $E\mathbb{T}$; thus, we can take $EG=E\mathbb{T}$.

The geometric construction of [M] produces a CW complex EG for any given Lie group G; thus, EG always exists. Standard topological arguments imply that EG is unique up to homotopy commuting with the G-action. Thus, the homotopy type of

$$BG\equiv EG/G$$

is well-defined, i.e. depends only on G. In particular, the group cohomology of G, with $\mathbb{C}$-coefficients,

$$H^*_G\equiv H^*(BG;\mathbb{C})$$

is well-defined as well. We do not need the fact that EG is well-defined up to a G-equivariant homotopy in general. However, we will use the fact that our two constructions of $B\mathbb{T}'$ for a torus $\mathbb{T}'$ produce the same groups $H^*_{\mathbb{T}'}$ whether $\mathbb{T}'$ is viewed as a subgroup of a larger torus or on its own.

Exercise 1.2. Suppose $k\leq n$, $\mathbb{T}=(S^1)^n$, and $\mathbb{T}'=(S^1)^k$. Show that

(a) any injective Lie group homomorphism $f:\mathbb{T}'\longrightarrow\mathbb{T}$ extends to a Lie group isomorphism

$$g\colon \mathbb{T}'\times(S^1)^{n-k}\longrightarrow\mathbb{T};$$

(b) if $g\colon\mathbb{T}\longrightarrow\mathbb{T}$ is a Lie group isomorphism, there exists g-equivariant map $(S^\infty)^n\longrightarrow(S^\infty)^n$;

[1] For our purposes, *CW complex* can be replaced with *topological space* here

(c) if $\mathbb{T}'$ is identified with the subgroup $\mathbb{T}'\times\{\mathrm{id}\}^{n-k}$ of $\mathbb{T}$, $(S^\infty)^k/\mathbb{T}'$ and $(S^\infty)^n/\mathbb{T}'$ are homotopy equivalent;

(d) if $f:\mathbb{T}'\longrightarrow\mathbb{T}$ is any injective Lie group homomorphism and $\mathbb{T}'$ acts on $(S^\infty)^n$ via f, $(S^\infty)^n/\mathbb{T}'$ and $(S^\infty)^k/\mathbb{T}'$ are homotopy equivalent.

If $\mathbb{T}'\subset G$ are closed subgroups of $\mathbb{T}$, the inclusion maps between the three groups induce quotient maps

$$B\mathbb{T}'\longrightarrow BG\longrightarrow B\mathbb{T}.$$

These maps in turn induce ring homomorphisms

$$\rho_{G,\mathbb{T}}\colon H^*_{\mathbb{T}}\longrightarrow H^*_G,\quad \rho_{\mathbb{T}',G}\colon H^*_G\longrightarrow H^*_{\mathbb{T}'},\quad \rho_{\mathbb{T}',\mathbb{T}}\colon H^*_{\mathbb{T}}\longrightarrow H^*_{\mathbb{T}'}\quad \text{s.t.}\quad \rho_{\mathbb{T}',\mathbb{T}}=\rho_{\mathbb{T}',G}\circ\rho_{G,\mathbb{T}}. \tag{1.1}$$

Lemma 1.4 below gives an intrinsic description of the homomorphism $\rho_{\mathbb{T},\mathbb{T}'}$ whenever $\mathbb{T}'$ is a subtorus of $\mathbb{T}$.

Exercise 1.3. Suppose $\mathbb{T}$ is the n-torus and $G\subset\mathbb{T}$ is a closed subgroup; thus, $G\approx\mathbb{T}'\times D$, where $\mathbb{T}'\subset G$ is the identity component of G and D is a finite group. Show that the homomorphism

$$\rho_{\mathbb{T}',G}\colon H^*_G\longrightarrow H^*_{\mathbb{T}'}$$

induced by the inclusion $\mathbb{T}'\longrightarrow G$ is an isomorphism.[2]

Let $\gamma\longrightarrow\mathbb{P}^\infty$ denote the tautological line bundle. If $\mathbb{T}$ is the n-torus,

$$B\mathbb{T}\equiv E\mathbb{T}/\mathbb{T}\approx(\mathbb{P}^\infty)^n$$

is the n-fold product of infinite-dimensional complex projective spaces. Thus,

$$H^*_{\mathbb{T}}\equiv H^*(B\mathbb{T};\mathbb{C})\approx\mathbb{C}[\alpha_1,\ldots,\alpha_n],\quad \deg\alpha_i=2, \tag{1.2}$$

where $\alpha_i=\pi_i^*c_1(\gamma^*)$ and $\pi_i\colon(\mathbb{P}^\infty)^n\longrightarrow\mathbb{P}^\infty$ is the projection onto the ith component.

If $f_c\colon S^1\longrightarrow S^1$ is the homomorphism $f_c(t)=t^c$ for some $c\in\mathbb{Z}$, the map

$$F_c\colon S^\infty\longrightarrow S^\infty,\qquad (z_1,z_2,\ldots)\longrightarrow\begin{cases}(z_1^c,z_2^c,\ldots), & \text{if } c\geq 0,\\ (\bar z_1^{-c},\bar z_2^{-c},\ldots), & \text{if } c\leq 0,\end{cases}$$

is f_c-equivariant. Thus, it descends to a map on the quotients

$$\bar F_c\colon BS^1\approx\mathbb{P}^\infty\longrightarrow BS^1.$$

It is straightforward to check directly from the relevant definitions that

$$\bar F_c^*\gamma=\gamma^{\otimes c}; \tag{1.3}$$

if $c<0$, $\gamma^{\otimes c}\equiv\gamma^{*\otimes(-c)}\approx\bar\gamma^{\otimes c}$. Similarly, if $c_1,\ldots,c_n\in\mathbb{Z}$, the map

$$F_{c_1\ldots c_n}\colon S^\infty\longrightarrow\big(S^\infty\big)^n,\qquad \underline{z}\longrightarrow\big(F_{c_1}(\underline{z}),\ldots,F_{c_n}(\underline{z})\big),$$

[2]This statement depends on our use of cohomology with coefficients in a field of characteristic 0.

is equivariant with respect to the homomorphism

$$f_{c_1\dots c_n}\colon S^1 \longrightarrow (S^1)^n, \qquad t \longrightarrow \big(t^{c_1},\dots,t^{c_n}\big).$$

The map induced by F on the quotients is given by

$$\bar{F}_{c_1\dots c_n}\colon BS^1 \longrightarrow B\mathbb{T}, \qquad [\underline{z}] \longrightarrow \big(\bar{F}_{c_1}([\underline{z}]),\dots,\bar{F}_{c_n}([\underline{z}])\big).$$

Thus, by (1.2) and (1.3),

$$\bar{F}^*_{c_1\dots c_n}\alpha = c_1\bar{F}^*_{10\dots0}\alpha + \dots + c_n\bar{F}^*_{0\dots01}\alpha \quad \forall \alpha \in H^2_{\mathbb{T}}; \tag{1.4}$$

note that by (1.2) and linearity of both sides in α, it is sufficient to check (1.4) for $\alpha=\alpha_i$.

Lemma 1.4. *If $\mathbb{T}=(S^1)^n$ is the n-torus and $\mathfrak{t}_{\mathbb{C}}$ is the complexified Lie algebra of $\mathbb{T}$, there is a natural isomorphism*

$$\Psi_{\mathbb{T}}\colon H^2_{\mathbb{T}} \longrightarrow \mathfrak{t}^*_{\mathbb{C}}.$$

If $\mathbb{T}'\subset\mathbb{T}$ is a closed connected subgroup of $\mathbb{T}$ and $\mathfrak{t}'_{\mathbb{C}}$ is the complexified Lie algebra of $\mathbb{T}'$, then

$$\tilde{\rho}_{\mathbb{T}',\mathbb{T}} \equiv \Psi_{\mathbb{T}'}\circ\rho_{\mathbb{T}',\mathbb{T}}\circ\Psi_{\mathbb{T}}^{-1}\colon \mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathfrak{t}'^*_{\mathbb{C}} \tag{1.5}$$

is the restriction map.[3]

Proof. We identify the Lie algebra of S^1 with $\mathfrak{i}\mathbb{R}$ by the condition that

$$\exp^{S^1}(\mathfrak{i}\theta) = e^{\mathfrak{i}\theta} \in S^1 \subset \mathbb{C}.$$

If $\alpha\in H^2_{\mathbb{T}}$, we define $\Psi_{\mathbb{T}}(\alpha)\in\mathfrak{t}^*_{\mathbb{C}}$ by

$$\bar{F}^*_{c_1\dots c_n}\alpha = \Big(\big\{\Psi_{\mathbb{T}}(\alpha)\big\}\big(df_{c_1\dots c_n}|_{\mathrm{id}}(2\pi\mathfrak{i})\big)\Big)c_1(\gamma^*) \in H^2_{S^1} \qquad \forall\, c_1,\dots,c_n\in\mathbb{Z}. \tag{1.6}$$

By (1.2) with $\mathbb{T}=S^1$,

$$\big\{\Psi_{\mathbb{T}}(\alpha)\big\}\big(df_{c_1\dots c_n}|_{\mathrm{id}}(2\pi\mathfrak{i})\big) \in \mathbb{C}$$

is well-defined by (1.6). By (1.4), $\Psi_{\mathbb{T}}(\alpha)$ is linear on a generating set for $\mathfrak{t}_{\mathbb{C}}$ and thus defines an element of $\mathfrak{t}^*_{\mathbb{C}}$. It is immediate from (1.6) that the map

$$\Psi_{\mathbb{T}}\colon H^2_{\mathbb{T}} \longrightarrow \mathfrak{t}^*_{\mathbb{C}}$$

is linear and satisfies (1.5). Since

$$\big\{\Psi_{\mathbb{T}}(\alpha_i)\big\}\big(df_{\underbrace{0\dots0}_{j-1}10\dots0}|_{\mathrm{id}}(2\pi\mathfrak{i})\big) = \delta_{ij} \qquad \forall\, i,j=1,\dots,n,$$

$\Psi_{\mathbb{T}}$ is injective by (1.2) and thus an isomorphism. □

[3]If $\mathbb{T}=(\mathbb{C}^*)^n$, $\mathfrak{t}_{\mathbb{C}}$ should denote the usual Lie algebra of $\mathbb{T}$.

Corollary 1.5. *If $\mathbb{T}=(S^1)^n$ is the n-torus, there is a natural isomorphism*

$$\Psi_{\mathbb{T}}\colon H^*_{\mathbb{T}} \longrightarrow \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}.$$

If $\mathbb{T}'\subset\mathbb{T}$ is a closed connected subgroup of $\mathbb{T}$, the homomorphism

$$\tilde{\rho}_{\mathbb{T}',\mathbb{T}} \equiv \Psi_{\mathbb{T}'}\circ\rho_{\mathbb{T}',\mathbb{T}}\circ\Psi_{\mathbb{T}}^{-1}\colon \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}} \tag{1.7}$$

is the restriction map.

This corollary follows immediately from the proof of Lemma 1.4 and (1.2). Note that $\mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}$ is the space of polynomials on the vector space $\mathfrak{t}_{\mathbb{C}}$.

A representation ρ of G, i.e. a linear action of G on $\mathbb{C}^k$, induces a vector bundle over BG:

$$V_\rho \equiv EG\times_G\mathbb{C}^k.$$

If ρ is one-dimensional, we will call

$$c_1(V_\rho^*) = -c_1(V_\rho)\in H^*_G$$

the weight of ρ. For example, α_i is the weight of the representation

$$\pi_i\colon\mathbb{T}\longrightarrow\mathbb{C}^*, \qquad \big(e^{\mathrm{i}\theta_1},\ldots,e^{\mathrm{i}\theta_n}\big)\cdot z = e^{\mathrm{i}\theta_i}z. \tag{1.8}$$

More generally, if a representation ρ of G on $\mathbb{C}^k$ splits into one-dimensional representations[4] with weights $\beta_1,\ldots,\beta_k$, we will call $\beta_1,\ldots,\beta_k$ the weights of ρ. In such a case,

$$e(V_\rho^*) = \beta_1\cdot\ldots\cdot\beta_k. \tag{1.9}$$

We will call the representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$ with weights $\alpha_1,\ldots,\alpha_n$ the standard representation of $\mathbb{T}$.

1.2 Equivariant cohomology of topological spaces

If G is a Lie group acting on a topological space M, let

$$B_GM = EG\times_G M.$$

The G-equivariant cohomology of M is defined to be

$$H^*_G(M)\equiv H^*(B_GM;\mathbb{C}).$$

Example 1.6. The G-equivariant cohomology of a point is the group cohomology of G:

$$B_Gpt\equiv EG\times_Gpt = EG/G\times pt = BG \quad\Longrightarrow\quad H^*_G(pt)\equiv H^*(B_Gpt;\mathbb{C}) = H^*(BG;\mathbb{C})\equiv H^*_G.$$

If G acts trivially on M, then $H^*_G(M)$ is simply the tensor product of H^*_G and $H^*(M;\mathbb{C})$:

$$\begin{aligned} & g\cdot x = x\ \forall\, g\in G,\ x\in M \quad\Longrightarrow\quad B_GM\equiv EG\times_GM = EG/G\times M = BG\times M\\ \Longrightarrow\quad & H^*_G(M)\equiv H^*(B_GM;\mathbb{C}) = H^*(BG\times M;\mathbb{C})\\ &\qquad = H^*(BG;\mathbb{C})\otimes H^*(M;\mathbb{C})\equiv H^*_G\otimes H^*(M;\mathbb{C}). \end{aligned} \tag{1.10}$$

In this case, $H^*_G(M)$ and $H^*_G\otimes H^*(M;\mathbb{C})$ are isomorphic as rings.

[4] this is necessarily the case if $G=\mathbb{T}$

If G is a Lie group acting on M, there is a fibration

$$\pi_M\colon B_GM \longrightarrow BG, \qquad [e,x] \longrightarrow [e] \ \ \forall\,(e,x) \in EG\times M, \tag{1.11}$$

with fiber M. It induces an action of H^*_G on $H^*_G(M)$:

$$\alpha\cdot\eta = (\pi_M^*\alpha)\cup\eta \qquad \alpha \in H^*_G \equiv H^*(BG;\mathbb{C}),\ \eta\in H^*_G(M)\equiv H^*(B_GM;\mathbb{C}).$$

Thus, $H^*_G(M)$ is a module over the ring H^*_G. Since the M-fibration π_M is generally not trivial, one cannot generally expect $H^*_G(M)$ and $H^*_G\otimes H^*(M;\mathbb{C})$ to be isomorphic as rings.

Exercise 1.7. Suppose $\mathbb{T}$ is the n-torus, G is a closed subgroup of $\mathbb{T}$, and $\mathbb{T}'$ is the identity component of G. The n-torus $\mathbb{T}$ then acts on the quotient $\mathbb{T}/G$. Show that

$$H^*_{\mathbb{T}}(\mathbb{T}/G) = H^*_G \qquad \text{and} \qquad \rho_{\mathbb{T}',G}\big(\alpha\cdot\eta\big) = \rho_{\mathbb{T}',\mathbb{T}}(\alpha)\cdot\rho_{\mathbb{T}',G}(\eta) \ \ \forall\,\alpha\in H^*_{\mathbb{T}},\ \eta\in H^*_{\mathbb{T}}(\mathbb{T}/G).$$

Thus, the diagram

$$\begin{tikzcd} H^*_{\mathbb{T}} \arrow[r, "\pi^*_{\mathbb{T}/G}"] \arrow[drr, "\rho_{\mathbb{T}',\mathbb{T}}"'] & H^*_{\mathbb{T}}(\mathbb{T}/G) \arrow[r, equal] & H^*_G \arrow[d, "\rho_{\mathbb{T}',G}", "\approx"'] \\ & & H^*_{\mathbb{T}'} \end{tikzcd}$$

commutes.

Exercise 1.8. Let $\mathbb{T}$ be the n-torus as before. The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ is the action on $\mathbb{P}^{n-1}$ induced by the standard representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$:

$$\big(e^{\mathrm{i}\theta_1},\ldots,e^{\mathrm{i}\theta_n}\big)\cdot[z_1,\ldots,z_n] = \big[e^{\mathrm{i}\theta_1}z_1,\ldots,e^{\mathrm{i}\theta_n}z_n\big].$$

(a) Show that $B_{\mathbb{T}}\mathbb{P}^{n-1} = \mathbb{P}V_\rho$.

(b) Use the Thom Isomorphism Theorem to show that there is an isomorphism of graded rings

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \equiv H^*\big(B_{\mathbb{T}}\mathbb{P}^{n-1};\mathbb{C}\big) \approx \mathbb{C}[\alpha_1,\ldots,\alpha_n,x]/\big(x^n+c_1(V_\rho)x^{n-1}+\ldots+c_n(V_\rho)\big),$$

where $x=c_1(\tilde\gamma^*)$ and $\tilde\gamma\longrightarrow\mathbb{P}V_\rho$ is the tautological line bundle.

(c) Conclude that there is an isomorphism of graded rings

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \approx \mathbb{C}[x,\alpha_1,\ldots,\alpha_n]/(x-\alpha_1)\ldots(x-\alpha_n). \tag{1.12}$$

This isomorphism commutes with the isomorphism (1.2).

If G is a Lie group acting on M and S is a subspace of M preserved by G, i.e. $g\cdot x\in S$ for all $x\in S$ and $g\in G$, G also acts on S. The inclusion $S\longrightarrow M$ then induces an inclusion $B_GS\longrightarrow B_GM$ and thus a restriction homomorphism

$$H^*_G(M)\equiv H^*(B_GM;\mathbb{C}) \longrightarrow H^*_G(S)\equiv H^*(B_GS;\mathbb{C}), \qquad \eta\longrightarrow \eta|_S.$$

Exercise 1.9. The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ has n fixed points:

$$P_1 = [1,0,\ldots,0], \qquad P_2 = [0,1,0,\ldots,0], \quad \ldots \quad P_n = [0,\ldots,0,1]. \tag{1.13}$$

(a) Show that $\tilde{\gamma}|_{B_{\mathbb{T}}P_i} = V_{\pi_i}$, where π_i is as in (1.8).

(b) Show that the restriction map on the equivariant cohomology induced by the inclusion $P_i \longrightarrow \mathbb{P}^{n-1}$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \approx \mathbb{C}[x,\alpha_1,\ldots,\alpha_n] \Big/ \prod_{k=1}^{k=n}(x-\alpha_k) \longrightarrow H^*_{\mathbb{T}}(P_i) \approx \mathbb{C}[\alpha_1,\ldots,\alpha_n], \qquad x \longrightarrow \alpha_i. \tag{1.14}$$

(c) Conclude that for all $\eta \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$

$$\eta = 0 \qquad \Longleftrightarrow \qquad \eta|_{P_i} = 0 \ \ \forall\, i = 1,2,\ldots,n. \tag{1.15}$$

Thus, an element $\eta \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$ is determined by its restrictions $\eta|_{P_i}$ with $i = 1,2,\ldots,n$.

More generally, if a Lie group G acts on M and M', a G-equivariant continuous map $h\colon M \longrightarrow M'$ induces a continuous map $\tilde{h}\colon B_G M \longrightarrow B_G M'$ and thus a homomorphism

$$\tilde{h}^*\colon H^*_G(M') \equiv H^*(B_G M') \longrightarrow H^*_G(M) \equiv H^*(B_G M).$$

Since the diagram

$$\begin{tikzcd} B_G M \arrow[rr, "\tilde{h}"] \arrow[dr, "\pi_M"'] & & B_G M' \arrow[dl, "\pi_{M'}"] \\ & BG & \end{tikzcd}$$

commutes, $\tilde{h}^*$ commutes with the action of H^*_G, i.e. $\tilde{h}^*$ is a homomorphism of H^*_G-modules (in fact, of H^*_G-algebras).

Exercise 1.10. Suppose the n-torus $\mathbb{T}$ acts on a topological space M, G is a closed subgroup of $\mathbb{T}$, and $\mathbb{T}'$ is the identity component of G. Show that if there exists a $\mathbb{T}$-equivariant map $h\colon M \longrightarrow \mathbb{T}/G$, then

$$\alpha \cdot \eta = \tilde{h}^* \rho^{-1}_{\mathbb{T}',G}\big(\rho_{\mathbb{T}',\mathbb{T}}(\alpha)\big) \cup \eta \qquad \forall\ \alpha \in H^*_{\mathbb{T}},\ \eta \in H^*_{\mathbb{T}}(M),$$

i.e. the action of $H^*_{\mathbb{T}}$ on $H^*_{\mathbb{T}}(M)$ factors through the natural homomorphism $H^*_{\mathbb{T}} \longrightarrow H^*_{\mathbb{T}'}$.

This observation is a key ingredient in the proof of the Localization Theorem. Along with Corollary 1.5, it implies that the action of $\mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}$, i.e. of the space of polynomials on the vector space $\mathfrak{t}_{\mathbb{C}}$, on $H^*_{\mathbb{T}}(M)$ factors through the restriction

$$\mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}}$$

and an action of $\mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}}$ on $H^*_{\mathbb{T}}(M)$. Note that the homomorphism $\rho_{\mathbb{T}',G}$ is invertible; see Exercise 1.3.

1.3 Equivariant vector bundles

If G is a Lie group acting on M and $\pi_V\colon V \longrightarrow M$ is a vector bundle, a lift of the G-action on M to V is an action of G on V such that

$$\pi_V(g\cdot v) = g\cdot \pi_V(v) \qquad \forall\, g\in G,\ v\in V.$$

Some G-actions have natural lifts. For example, if M is a smooth manifold and $\rho\colon G\longrightarrow \mathrm{Diff}(M)$ is an action of G by diffeomorphisms, then ρ lifts to an action on the tangent bundle of M by

$$g\cdot v = d\{\rho(g)\}|_x v \qquad \forall\, g\in G,\ v\in T_xM,\ x\in M.$$

As another example, suppose $\mathbb{T}$ is the n-torus acting in the standard way on $\mathbb{P}^{n-1}$ and ρ is the standard representation of $\mathbb{T}$ on $\mathbb{C}^n$. Since the tautological line bundle $\gamma_{n-1}\longrightarrow\mathbb{P}^{n-1}$ is contained in $\mathbb{P}^{n-1}\times\mathbb{C}^n$ and is preserved by the diagonal $\mathbb{T}$-action, the restriction of the $\mathbb{T}$-action induces an action on γ_{n-1} and thus on $\gamma_{n-1}^{\otimes c}$ for all $c\in\mathbb{Z}$. These $\mathbb{T}$-actions on the line bundles $\gamma_{n-1}^{\otimes c}$ will be called the standard lifts of the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ to $\gamma_{n-1}^{\otimes c}$.

In all cases, given one lift of the G-action on M to V and a one-dimensional representation ρ of G, we can obtain another lift of the G-action on M to V by tensoring V, with its G-action, and $M\times\mathbb{C}_\rho$, with G acting by ρ on $\mathbb{C}$. Once a lift has been chosen, V is called a G-vector bundle on M. In such a case,

$$B_GV \equiv EG\times_G V \longrightarrow B_GM \equiv EG\times_G M$$

is a vector bundle. If $V\longrightarrow M$ is oriented as a vector bundle, so is $B_GV\longrightarrow B_GM$. If this is the case, let

$$\mathbf{e}(V)\equiv e(B_GV)\in H_G^*(M)\equiv H^*(B_GM;\mathbb{C})$$

denote the equivariant euler class of V.

Exercise 1.11. Let $\mathbb{T}$ be the n-torus acting in the standard way on $\mathbb{P}^{n-1}$ and let ρ be the standard representation of $\mathbb{T}$ on $\mathbb{C}^n$. Show that

(a) the equivariant euler class of the line bundle $\gamma_{n-1}^{\otimes c}\longrightarrow\mathbb{P}^{n-1}$, for the standard lift of the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$, is characterized by

$$\mathbf{e}\big(\gamma_{n-1}^{\otimes c}\big)\big|_{P_i} = -c\,\alpha_i \qquad \forall\ i=1,2,\ldots,n; \tag{1.16}$$

(b) there is an exact sequence of $\mathbb{T}$-vector bundles on $\mathbb{P}^{n-1}$

$$0\longrightarrow \gamma_{n-1}^*\otimes\gamma_{n-1}\longrightarrow\gamma_{n-1}^*\otimes\big(\mathbb{P}^{n-1}\times\mathbb{C}^n_\rho\big)\longrightarrow T\mathbb{P}^{n-1}\longrightarrow 0;$$

(c) the equivariant euler class of $T\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}$, for the standard lift of the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$, is characterized by

$$\mathbf{e}\big(T\mathbb{P}^{n-1}\big)\big|_{P_i} = \prod_{k\neq i}(\alpha_i-\alpha_k) \qquad \forall\ i=1,2,\ldots,n. \tag{1.17}$$

1.4 Equivariant pushforward

If $\mathbb{T}$ acts on a compact oriented manifold M, there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H^*_{\mathbb{T}}(M) \equiv H^*(B_{\mathbb{T}}M;\mathbb{C}) \longrightarrow H^*_{\mathbb{T}} \equiv H^*(B\mathbb{T};\mathbb{C}) \tag{1.18}$$

for the fiber bundle (1.11). If $f: M \longrightarrow M'$ is a $\mathbb{T}$-equivariant map between two compact oriented manifolds, we will show that there is a well-defined pushforward homomorphism

$$f_*: H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}(M') \tag{1.19}$$

characterized by the property that

$$\int_{M'} \psi(f_*\eta) = \int_M (f^*\psi)\eta \qquad \forall\, \eta \in H^*_{\mathbb{T}}(M),\ \psi \in H^*_{\mathbb{T}}(M'). \tag{1.20}$$

The homomorphism $\int_M$ of (1.18) corresponds to M' being a point.

Exercise 1.12. Let $f: M \longrightarrow M'$ be a $\mathbb{T}$-equivariant map between two compact oriented manifolds. Deduce from the characterization (1.20) of f_* that

$$f_*\big((f^*\psi)\eta\big) = \psi(f_*\eta) \qquad \forall\, \eta \in H^*_{\mathbb{T}}(M),\ \psi \in H^*_{\mathbb{T}}(M'). \tag{1.21}$$

If $g: M' \longrightarrow M''$ is also a $\mathbb{T}$-equivariant map between two compact oriented manifolds, show that

$$(g \circ f)_* = g_* \circ f_*. \tag{1.22}$$

Hint: An implicit assumption in the statement that f_* is characterized by (1.20) is that the pairing

$$H^*_{\mathbb{T}}(M') \otimes H^*_{\mathbb{T}}(M') \longrightarrow H^*_{\mathbb{T}}, \qquad \eta_1 \otimes \eta_2 \longrightarrow \int_{M'} \eta_1\eta_2\,,$$

is non-degenerate; this is shown below.

If $f: M \longrightarrow M'$ is a map between two compact oriented manifolds of dimensions m and m', respectively, the homomorphism

$$f_*: H^q(M) \longrightarrow H^{q+m'-m}(M'), \qquad \eta \longrightarrow \mathrm{PD}_{M'} f_* \mathrm{PD}_M \eta, \tag{1.23}$$

is characterized by the property

$$\big\langle \psi(f_*\eta), M' \big\rangle = \big\langle (f^*\psi)\eta, M \big\rangle \qquad \forall\, \eta \in H^*(M),\ \psi \in H^*(M').$$

This pushforward homomorphism f_* has a more local description if f is the inclusion map of a submanifold.

Exercise 1.13. Suppose S is a compact oriented (embedded) submanifold of a compact oriented manifold M with normal bundle $\mathcal{N}$. Let $u \in H^*(\mathcal{N}, \mathcal{N}-S)$ be the Thom/orientation class for $\mathcal{N}$. Let

$$\Phi_S: H^{*-\mathrm{rk}\,\mathcal{N}}(S) \longrightarrow H^*(\mathcal{N}, \mathcal{N}-S), \qquad \eta \longrightarrow (\pi^*_{\mathcal{N}}\eta)u,$$

be the Thom isomorphism. If $\iota\colon S\longrightarrow M$ is the inclusion map, show that the diagram

$$\begin{tikzcd}
H^*(\mathcal{N},\mathcal{N}-S) \arrow[r, leftrightarrow, "\approx"] & H^*(M,M-S) \arrow[r] & H^*(M) \\
H^{*-\mathrm{rk}\,\mathcal{N}}(S) \arrow[u, "\Phi_S"] \arrow[urr, "\iota_*"] & &
\end{tikzcd}$$

where the two horizontal arrows are the excision isomorphism and the restriction homomorphism, commutes.

We will construct the homomorphism (1.19) using (1.23) and a sequence of finite-dimensional approximations to $E\mathbb{T}$ and $B_{\mathbb{T}}M$. For each $r\in\mathbb{Z}^+$, let

$$E_r\mathbb{T} = \big(S^{2r+1}\big)^n \subset \big(\mathbb{C}^{r+1}\big)^n, \qquad B_r\mathbb{T} = E_r\mathbb{T}/\mathbb{T}, \quad \text{and} \quad B_rM = E_r\mathbb{T}\times_{\mathbb{T}}M.$$

Let $\pi_{M;r} : B_rM \longrightarrow B_r\mathbb{T}$ be the M-fibration obtained by restricting the fibration (1.11). The $\mathbb{T}$-equivariant inclusion maps $E_r\mathbb{T}\longrightarrow E_{r+1}\mathbb{T}$ induce homomorphisms of the homotopy exact sequences for the $\mathbb{T}$-fibrations $E_r\mathbb{T}\times M\longrightarrow B_rM$:

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_s(\mathbb{T}) \arrow[r] \arrow[d] & \pi_s(E_r\mathbb{T})\times\pi_s(M) \arrow[r] \arrow[d] & \pi_s(B_rM) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \pi_s(\mathbb{T}) \arrow[r] & \pi_s(E_{r+1}\mathbb{T})\times\pi_s(M) \arrow[r] & \pi_s(B_{r+1}M) \arrow[r] & \ldots
\end{tikzcd}$$

Since the first two arrows above are isomorphisms for $s\leq 2r$, the inclusions $B_rM\longrightarrow B_{r+1}M$ induce isomorphisms

$$\pi_s(B_rM) \longrightarrow \pi_s(B_{r+1}M), \quad H_s(B_rM;\mathbb{Z}) \longrightarrow H_s(B_{r+1}M;\mathbb{Z}), \quad H^s(B_{r+1}M;\mathbb{C}) \longrightarrow H^s(B_rM;\mathbb{C})$$

for $s\leq 2r$. Thus, the restriction map

$$\varphi_r\colon H^s(B_{\mathbb{T}}M;\mathbb{C}) \longrightarrow H^s(B_rM;\mathbb{C})$$

is an isomorphism for $s\leq 2r$. If $f: M\longrightarrow M'$ is a $\mathbb{T}$-equivariant map between two compact oriented manifolds of dimensions m and m', respectively, we define

$$\begin{gathered} f_*\colon H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}(M') \qquad \text{by} \qquad \varphi_r(f_*\eta) = f_{r*}(\varphi_r\eta) \\ \text{if} \quad \eta\in H^s(B_{\mathbb{T}}M;\mathbb{C}),\ s\leq 2r, 2r+m-m', \end{gathered} \tag{1.24}$$

where $f_r\colon B_rM\longrightarrow B_rM'$ is the map induced by f.

Exercise 1.14. With notation as above, show that

(a) the normal bundle of B_rM in $B_{r+1}M$ is isomorphic to the $\pi_{M;r}$-pullback of the normal bundle of $B_r\mathbb{T}$ in $B_{r+1}\mathbb{T}$;

(b) the homomorphism f_* is well-defined by (1.24);

(c) the requirement (1.20) specifies f_*.

`Hint:` Exercise 1.13 may be helpful for (b).

Exercise 1.15. Suppose $\mathbb{T}$ acts smoothly on a smooth compact oriented manifold M preserving a compact oriented submanifold S. Let $\mathcal{N}$ be the normal bundle of S in M. Show that

(a) the action of $\mathbb{T}$ on S has a natural lift to $\mathcal{N}$ and $B_{\mathbb{T}}\mathcal{N} \longrightarrow B_{\mathbb{T}}S$ is the normal bundle of $B_{\mathbb{T}}S$ in $B_{\mathbb{T}}M$;

(b) if $\iota\colon S \longrightarrow M$ is the inclusion map, $\mathbf{u} \in H^*(B_{\mathbb{T}}\mathcal{N}, B_{\mathbb{T}}\mathcal{N} - B_{\mathbb{T}}S)$ is the Thom class for the vector bundle $B_{\mathbb{T}}\mathcal{N} \longrightarrow B_{\mathbb{T}}S$, and

$$\Phi_{B_{\mathbb{T}}S}\colon H^{*-\mathrm{rk}\,\mathcal{N}}(B_{\mathbb{T}}S) \longrightarrow H^*(B_{\mathbb{T}}\mathcal{N}, B_{\mathbb{T}}\mathcal{N} - B_{\mathbb{T}}S), \qquad \eta \longrightarrow (\pi^*_{B_{\mathbb{T}}\mathcal{N}}\eta)\mathbf{u},$$

is the Thom isomorphism, then the diagram

$$\begin{tikzcd}
H^*(B_{\mathbb{T}}\mathcal{N}, B_{\mathbb{T}}\mathcal{N} - B_{\mathbb{T}}S) \arrow[r, leftrightarrow, "\approx"] & H^*(B_{\mathbb{T}}M, B_{\mathbb{T}}M - B_{\mathbb{T}}S) \arrow[r] & H^*_{\mathbb{T}}(M) \\
H^*_{\mathbb{T}}(S) \arrow[u, "\Phi_{B_{\mathbb{T}}S}"] \arrow[urr, "\iota_*"] & &
\end{tikzcd}$$

where the two horizontal arrows are the excision isomorphism and the restriction homomorphism, commutes.

1.5 Supports

Let $\mathbb{T}$ be the n-torus. By Corollary 1.5, an element

$$f \in H^*_{\mathbb{T}} = \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}$$

is a polynomial on $\mathfrak{t}_{\mathbb{C}}$. Let

$$V_f = f^{-1}(0) \subset \mathfrak{t}_{\mathbb{C}}.$$

If $\mathcal{K} \subset H^*_{\mathbb{T}}$, let

$$V_{\mathcal{K}} = \bigcap_{f\in\mathcal{K}} V_f \equiv \bigcap_{f\in\mathcal{K}} f^{-1}(0) \subset \mathfrak{t}_{\mathbb{C}}.$$

Exercise 1.16. If $\mathcal{K}_1, \mathcal{K}_2 \subset H^*_{\mathbb{T}}$, define

$$\mathcal{K}_1 \cdot \mathcal{K}_2 = \big\{ f_1 \cdot f_2\colon\ f_1 \in \mathcal{K}_1,\ f_2 \in \mathcal{K}_2 \big\}.$$

If $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subset H^*_{\mathbb{T}}$, show that

$$\mathcal{K}_1 \cdot \mathcal{K}_2 \subset \mathcal{K} \qquad \Longrightarrow \qquad V_{\mathcal{K}} \subset V_{\mathcal{K}_1} \cup V_{\mathcal{K}_2}. \tag{1.25}$$

If W is a module over $H^*_{\mathbb{T}}$, we define the annihilator of W to be

$$\mathcal{K}(W) = \big\{ f \in H^*_{\mathbb{T}} \colon\ f\cdot w = 0 \ \forall\, w \in W \big\} \subset H^*_{\mathbb{T}}.$$

The support of a module W over $H^*_{\mathbb{T}}$ is defined by

$$\mathrm{Supp}(W) = V_{\mathcal{K}(W)} \equiv \bigcap_{f\in\mathcal{K}(W)} f^{-1}(0) \subset \mathfrak{t}_{\mathbb{C}}.$$

If $\mathbb{T}$ acts on a topological space, $H^*_{\mathbb{T}}(M)$ is a module over $H^*_{\mathbb{T}}$ and thus

$$\mathrm{Supp}\big(H^*_{\mathbb{T}}(M)\big) \subset \mathfrak{t}_C.$$

Exercise 1.17. Show that

(a) if $W \neq \{0\}$ is a torsion-free module over $H^*_{\mathbb{T}}$, i.e. $f\cdot w \neq 0$ for all $f \in H^*_{\mathbb{T}} - 0$ and $w \in W - 0$, then $\mathrm{Supp}(W) = \mathfrak{t}_{\mathbb{C}}$;

(b) if $\mathfrak{t}'_{\mathbb{C}}$ is a vector subspace of $\mathfrak{t}_{\mathbb{C}}$ and W is a module over $H^*_{\mathbb{T}}$ such that the action of $H^*_{\mathbb{T}}$ on W is the composition of the restriction homomorphism

$$H^*_{\mathbb{T}} \approx \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}} \tag{1.26}$$

with an action of $\mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}}$ on W, then $\mathrm{Supp}(W) \subset \mathfrak{t}'_{\mathbb{C}}$.

Proposition 1.18. *Suppose $\mathbb{T}$ is the n-torus acting on M, $G \subset \mathbb{T}$ is a closed subgroup, and $\mathbb{T}' \subset G$ is the identity component of G. If there exists a $\mathbb{T}$-equivariant map $h\colon M \longrightarrow \mathbb{T}/G$, then*

$$\mathrm{Supp}\big(H^*_{\mathbb{T}}(M)\big) \subset \mathfrak{t}'_{\mathbb{C}} = \mathrm{Lie}(G)_{\mathbb{C}}.$$

Proof. By Exercise 1.10, the existence of such a map h implies that the action of $H^*_{\mathbb{T}}$ on $H^*_{\mathbb{T}}(M)$ is the composition of the homomorphism (1.26) with an action of $H^*_{\mathbb{T}'} \approx \mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}}$ on $H^*_{\mathbb{T}}(M)$. The conclusion of Proposition 1.18 then follows from part (b) of Exercise 1.17. $\square$

Proposition 1.18 is perhaps the key insight in the proof of the Localization Theorem in Section 3 of [AB]. The second part of the following exercise is also used in the proof.

Exercise 1.19. Show that

(a) if $W_1 \longrightarrow W \longrightarrow W_2$ is an exact sequence of $H^*_{\mathbb{T}}$-modules, then

$$\mathrm{Supp}(W) \subset \mathrm{Supp}(W_1) \cup \mathrm{Supp}(W_2).$$

(b) if $\mathbb{T}$ acts on $M = \mathcal{U}_1 \cup \mathcal{U}_2$, preserving the open subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of M (i.e. $g\cdot x \in \mathcal{U}_i$ if $g \in \mathbb{T}$ and $x \in \mathcal{U}_i$), then

$$\mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_1 \cup \mathcal{U}_2)\big) \subset \mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_1)\big) \cup \mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_2)\big). \tag{1.27}$$

1.6 Localization theorem

The Atiyah-Bott Localization Theorem reduces computation of integrals on a compact oriented manifold (evaluation of top cohomology classes against the fundamental class) with a torus action to integrals over the fixed loci. The fixed loci are generally much simpler than the entire space. Each contributes a rational function on $\mathfrak{t}_{\mathbb{C}}$, i.e. in the variables $\alpha_1, \ldots, \alpha_n$ in the notation of (1.2). Once these fractions are added together, the denominators cancel and we end up with a *number*.

Formally, the above fractions are elements of the field of fractions of $H_{\mathbb{T}}^*$, which we denote by $\mathcal{H}_{\mathbb{T}}^*$:

$$\mathcal{H}_{\mathbb{T}}^* \approx \mathbb{C}(\alpha_1, \ldots, \alpha_n).$$

If $\mathbb{T}$ acts on M, let

$$\mathcal{H}_{\mathbb{T}}^*(M) = H_{\mathbb{T}}^*(M) \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^*.$$

Exercise 1.20. Suppose $h\colon W \longrightarrow W'$ is a homomorphism of $H_{\mathbb{T}}^*$-modules such that

$$\operatorname{Supp}(\ker h), \operatorname{Supp}(\operatorname{coker} h) \subsetneq \mathfrak{t}_{\mathbb{C}}.$$

Show that the induced homomorphism

$$W \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^* \longrightarrow W' \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^*$$

is an isomorphism.

Exercise 1.21. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact manifold M. Show that

(a) the manifold M admits a $\mathbb{T}$-invariant Riemannian metric;

(b) if γ is a geodesic with respect to the Levi-Civita connection of such a metric, then so is $g \cdot \gamma$ for every $g \in \mathbb{T}$;

(c) the fixed point set of the $\mathbb{T}$-action,

$$F \equiv \big\{x \in M:\ g \cdot x = x\ \forall\, g \in G\big\},$$

is a disjoint union of finitely many smooth compact submanifolds of M;

(d) if $\mathcal{N}_i$ is the normal bundle to a component F_i of F, the fiber of $\mathcal{N}_i$ over any point of F_i contains no nonzero linear subspace on which $\mathbb{T}$-acts trivially (via the natural lift of the $\mathbb{T}$-action to $\mathcal{N}_i$).

A representation of $\mathbb{T}$ on $\mathbb{R}^m$ splits into trivial representations and two-dimensional representations, with $\mathbb{T}$ acting by rotations. By part (d) of Exercise 1.21, the action of $\mathbb{T}$ on any fiber of $\mathcal{N}_i$ splits into two-dimensional real representations of $\mathbb{T}$. Requiring that each of the rotations determined by the first non-trivially acting component S^1 of $\mathbb{T}$ be counterclockwise determines an orientation on each of the two-dimensional subspaces invariant under the $\mathbb{T}$-action and thus on every fiber of $\mathcal{N}_i$. Since the action of $\mathbb{T}$ on $\mathcal{N}_i$ is continuous, these orientations vary continuously over F_i. Thus, they determine an orientation on $\mathcal{N}_i$.[5] If M is oriented, we then obtain an orientation on F_i by requiring that

$$TM|_{F_i} \approx TF_i \oplus \mathcal{N}_i$$

[5] The choice of orientation on $\mathcal{N}_i$ is well-defined once one fixes an isomorphism $\mathbb{T} \approx (S^1)^n$. A different isomorphism may give rise to the opposite orientation on $\mathcal{N}_i$. However, it would then also change the induced orientation on F_i and thus will not change the F_i-summand in (1.28).

be an isomorphism of oriented vector bundles.

Since the $\mathbb{T}$-equivariant vector bundle $\mathcal{N}_i \longrightarrow F_i$ is oriented, there is a well-defined equivariant euler class

$$\mathbf{e}(\mathcal{N}_i) \in H^*_{\mathbb{T}}(F_i).$$

Since $\mathbb{T}$ acts trivially on F_i, by part (b) of Example 1.6 there is a decomposition

$$\mathbf{e}(\mathcal{N}_i) = \beta \otimes 1 + \sum_{l=1}^{l=N} \mu_l \otimes \eta_l, \qquad \text{where} \qquad \beta, \mu_l \in H^*_{\mathbb{T}},\ \eta_l \in H^{>0}(F_i).$$

If $x \in F_i$ is any point,

$$\beta = \mathbf{e}(\mathcal{N}_i)\big|_x = \mathbf{e}(\mathcal{N}_i|_x) \in H^*_{\mathbb{T}}(x) = H^*_{\mathbb{T}}.$$

Since $\mathcal{N}_i|_x$ splits into non-trivial irreducible representations of $\mathbb{T}$, the euler class of $\mathcal{N}_i|_x$, i.e. the product of the negative weights of these representations, is nonzero; see (1.9). Thus, $\beta \neq 0$, and $\mathbf{e}(\mathcal{N}_i)$ is invertible in $\mathcal{H}^*_{\mathbb{T}}(F_i)$:

$$\begin{aligned}
\mathbf{e}(\mathcal{N}_i)^{-1} &= \big(\beta^{-1} \otimes 1\big)\bigg(1 + \sum_{r=1}^{\infty} (-1)^r \big(\beta^{-r} \otimes 1\big)\bigg(\sum_{l=1}^{l=N} \mu_l \otimes \eta_l\bigg)^r\bigg) \\
&= \big(\beta^{-1} \otimes 1\big)\bigg(1 + \sum_{r=1}^{\dim F_i} (-1)^r \big(\beta^{-r} \otimes 1\big)\bigg(\sum_{l=1}^{l=N} \mu_l \otimes \eta_l\bigg)^r\bigg).
\end{aligned}$$

Theorem 1.22. *If the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact oriented manifold M, the fixed locus F is a disjoint union of smooth compact oriented submanifolds F_i of M. Furthermore, the equivariant euler class of the normal bundle $\mathcal{N}_i$ of F_i in M is well-defined in $H^*_{\mathbb{T}}(F_i)$ and invertible in $\mathcal{H}^*_{\mathbb{T}}(F_i)$. Finally,*

$$\int_M \psi = \sum_{F_i} \int_{F_i} \frac{\psi|_{F_i}}{\mathbf{e}(\mathcal{N}_i)} \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\ \psi \in \mathcal{H}^*_{\mathbb{T}}(M), \tag{1.28}$$

where the sum is taken over all components F_i of F.

This theorem is completely straightforward to use if $\mathbb{T}$ acts with isolated fixed points, i.e. F is a finite union of one-point sets F_i. In such a case, $\mathcal{N}_i = TM|_{F_i}$ and each term on the right-hand side of (1.28) is an element of $\mathcal{H}^*_{\mathbb{T}}$, i.e. a rational function in $\alpha_1, \ldots, \alpha_n$. For example, we can then immediately compute the euler characteristic of M:

$$\chi(M) = \int_M \mathbf{e}(TM) = \sum_{x \in F} \frac{\mathbf{e}(TM)|_x}{\mathbf{e}(T_x M)} = |F|.$$

Corollary 1.23. *If the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact oriented manifold M and there are only finitely many $\mathbb{T}$-fixed points in M, then the euler characteristic of M is the number of $\mathbb{T}$-fixed points. In particular, a smooth compact oriented manifold of negative euler characteristic admits no smooth torus action with only isolated fixed points.*

For example, $(S^1)^n$ acts on $S^{2n-1} \subset \mathbb{C}^n$ by complex multiplication without fixed points and thus $\chi(S^{2n-1}) = 0$. Since the action of $(S^1)^n$ on $S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$ by complex multiplication has two fixed points, $\chi(S^{2n}) = 2$. Since the standard action of $(S^1)^n$ on $\mathbb{P}^{n-1}$ has n fixed points, $\chi(\mathbb{P}^{n-1}) = n$.

Exercise 1.24. Find the euler characteristic of the Grassmannian $G(k,n)$ of k-planes in $\mathbb{C}^n$.

Example 1.25. Suppose the n-torus is acting in the standard way on $\mathbb{P}^{n-1}$. Let $P_i \in \mathbb{P}^{n-1}$ be the ith fixed point as in (1.13). With the notation as in (1.2), let

$$\phi_i = \prod_{k\neq i}(x-\alpha_k) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}), \qquad i=1,2,\ldots,n. \tag{1.29}$$

By (1.28), (1.14), and (1.17),

$$\int_{\mathbb{P}^{n-1}} \psi\phi_i = \int_{P_i} \psi|_{P_i} \equiv \psi|_{P_i} \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\ \psi \in \mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}),\ i=1,2,\ldots,n. \tag{1.30}$$

Thus, ϕ_i is the equivariant Poincare dual of P_i in $\mathbb{P}^{n-1}$.

It does not happen often that a $\mathbb{T}$-action on M has only isolated fixed points. Nevertheless, as will be demonstrated in the next two sections, even if the fixed loci are not isolated points, they are still much simpler than the entire space, and Theorem 1.22 still provides a method for computing seemingly unmanageable integrals. It remains to prove Theorem 1.22.

Exercise 1.26. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth manifold M, $x\in M$, and

$$\mathbb{T}_x \equiv \big\{g\in\mathbb{T}:\ g\cdot x = x\big\} \subset \mathbb{T} \qquad \text{and} \qquad \mathbb{T}x \equiv \big\{g\cdot x:\ g\in\mathbb{T}\big\} \subset M$$

are the stabilizer and the orbit of x, respectively. Show that

(a) the subspace $\mathbb{T}x$ of M is a smooth compact submanifold, which is $\mathbb{T}$-equivariantly diffeomorphic to $\mathbb{T}/\mathbb{T}_x$;

(b) there exist a $\mathbb{T}$-invariant open neighborhood $\mathcal{U}_x$ of $\mathbb{T}x$ in M (i.e. $g\cdot y \in \mathcal{U}_x$ for all $g\in\mathbb{T}$, $y\in\mathcal{U}_x$) and a continuous $\mathbb{T}$-equivariant map $h_x\colon \mathcal{U}_x \longrightarrow \mathbb{T}/\mathbb{T}_x$.

Exercise 1.27. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact manifold M, $F\subset M$ is the $\mathbb{T}$-fixed locus, $\mathcal{U}\subset M$ is a $\mathbb{T}$-invariant open neighborhood of F in M, and $\iota_{\mathcal{U}}: \mathcal{U} \longrightarrow M$ is the inclusion map. Show that

(a) the support of the $H^*_{\mathbb{T}}$-module $H^*_{\mathbb{T}}(M-F)$ is a proper subspace of $\mathfrak{t}_{\mathbb{C}}$;

(b) the restriction homomorphism $\iota^*_{\mathcal{U}}\colon \mathcal{H}^*_{\mathbb{T}}(M) \longrightarrow \mathcal{H}^*_{\mathbb{T}}(\mathcal{U})$ is an isomorphism.

`Hints:` (a) show that $M-F$ is $\mathbb{T}$-equivariantly homotopy equivalent to a finite union of the open sets $\mathcal{U}_x$ as in part (b) of Exercise (1.26) and then use Proposition 1.18 and Exercise 1.19-(b);
(b) use Exercise 1.20.

Example 1.28. The fixed point set F for the action of S^1 on S^2 by rotations around the z-axis consists of the north and south poles. The action of S^1 on the cylinder S^2-F is free and

$$H^*_{\mathbb{T}}(S^2-F) \equiv H^*\big(E\mathbb{T}\times_{\mathbb{T}}(S^2-F);\mathbb{C}\big) \approx H^*\big((S^2-F)/\mathbb{T};\mathbb{C}\big) = H^*\big(I^0;\mathbb{C}\big),$$

where I^0 is the open interval $(-1,1)$. However, the action of $H^{>0}_{\mathbb{T}} \subset H^*_{\mathbb{T}}$ on $H^*_{\mathbb{T}}(S^2-F)$ is trivial and thus

$$\mathcal{H}^*_{\mathbb{T}}(S^2-F) \equiv H^*_{\mathbb{T}}(S^2-F)\otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} = 0,$$

as follows from part (a) of Exercise 1.27.

We now return to the setting of Theorem 1.22. Let $\iota_{F_i}\colon F_i\longrightarrow M$ be the inclusion map. If

$$\pi_M\colon M\longrightarrow pt \qquad \text{and} \qquad \pi_{F_i}\colon F_i\longrightarrow pt$$

are the projections to a point,

$$\pi_{M*}=\int_M\colon \mathcal{H}_{\mathbb{T}}^*(M)\longrightarrow \mathcal{H}_{\mathbb{T}}^*, \qquad \pi_{F_i*}=\int_{F_i}\colon \mathcal{H}_{\mathbb{T}}^*(F_i)\longrightarrow \mathcal{H}_{\mathbb{T}}^*, \qquad \pi_{F_i*}=\pi_{M_*}\circ\iota_{F_i*}; \tag{1.31}$$

the last identity is a special case of (1.22). Denote by $\mathbf{u}_i'\in H_{\mathbb{T}}^*(M)$ the image of the equivariant Thom class of the vector bundle $\mathcal{N}_i\longrightarrow F_i$ under the inverse of the excision isomorphism; see the diagram in Exercise 1.15. In particular,

$$\begin{gathered}\iota_{F_i}^*\mathbf{u}_i'=\mathbf{e}(\mathcal{N}_i)\in H_{\mathbb{T}}^*(F_i), \quad \iota_{F_j}^*\mathbf{u}_i'=0\in H_{\mathbb{T}}^*(F_j) \;\;\text{if}\;\; i\neq j;\\ \iota_{F_i}^*(\iota_{F_i*}\eta)=\eta\cdot\mathbf{e}(\mathcal{N}_i), \quad \iota_{F_j}^*(\iota_{F_i*}\eta)=0 \;\;\text{if}\;\; i\neq j, \qquad \forall\,\eta\in H_{\mathbb{T}}^*(F_i);\end{gathered} \tag{1.32}$$

the last two equalities follow from the first two and Exercise 1.15.

Let $\iota_F\colon F\longrightarrow M$ be the inclusion. By part (b) of Exercise 1.27, the restriction homomorphism

$$\iota_F^*\colon H_{\mathbb{T}}^*(M)\longrightarrow H_{\mathbb{T}}^*(F)=\bigoplus_{F_i}H_{\mathbb{T}}^*(F_i)$$

is an isomorphism. Thus, so is the homomorphism

$$\tilde\iota_F^*\colon H_{\mathbb{T}}^*(M)\longrightarrow \bigoplus_{F_i}H_{\mathbb{T}}^*(F_i), \qquad \psi\longrightarrow \sum_{F_i}\frac{\iota_{F_i}^*\psi}{\mathbf{e}(\mathcal{N}_i)}\,.$$

On the other hand, by (1.32)

$$\tilde\iota_F^*(\iota_{F*}\eta)=\eta \qquad \forall\;\eta\in\bigoplus_{F_i}H_{\mathbb{T}}^*(F_i).$$

Since $\tilde\iota_F^*$ is an isomorphism, it follows that

$$\psi=\iota_{F*}(\tilde\iota_F^*\psi)=\sum_{F_i}\iota_{F_i*}\bigg(\frac{\iota_{F_i}^*\psi}{\mathbf{e}(\mathcal{N}_i)}\bigg) \qquad \forall\;\psi\in H_{\mathbb{T}}^*(M). \tag{1.33}$$

Applying π_{M*} to both sides of (1.33) and using (1.31), we obtain (1.28).

2 Moduli Spaces of Stable Maps

2.1 Stable curves

A (compact complex) curve Σ is called nodal if every singular point of Σ is a simple node. Such a curve can be obtained from a smooth curve $\tilde\Sigma$ by identifying pairs of distinct points. In other words, if Σ is a nodal curve, there exists a smooth curve $\tilde\Sigma$ and a finite set

$$S\equiv\big\{(x_1,x_1'),\ldots,(x_m,x_m')\big\}\subset\tilde\Sigma\times\tilde\Sigma$$

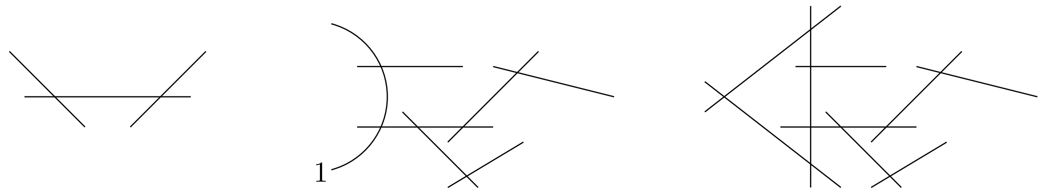

Figure 1: Examples of connected nodal curves of genus 0 and genus 1; the label 1 indicates that the genus of the corresponding irreducible component is 1; all other irreducible components are of genus 0.

such that $x_i \neq x_j$ and $x'_i \neq x'_j$ if $i \neq j$, $x_i \neq x'_j$ for all i and j, and

$$\Sigma = \tilde{\Sigma}/\sim, \qquad \text{where} \quad x_i \sim x'_i \quad \forall\, i = 1, 2, \ldots, m. \tag{2.1}$$

Let $g(\Sigma)$ denote the *arithmetic* genus of Σ. If $\tilde{\Sigma}$ is as above and $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_N$ are the connected components of $\tilde{\Sigma}$, then

$$g(\Sigma) = \sum_{l=1}^{l=N} g(\tilde{\Sigma}_l) - (N-1) + m.$$

Thus, if Σ is connected and of genus 0, Σ is a tree of spheres; see the first diagram in Figure 1. If Σ is connected and of genus 1, Σ consists of a *principal* (irreducible) component (or components) Σ_P, which is either a smooth torus or a loop of $N_P \geq 1$ spheres, and trees of spheres descendant from Σ_P; see the other two diagrams in Figure 1.

A prestable genus g k-marked curve is a tuple $(\Sigma, y_1, \ldots, y_k)$, where Σ is a connected nodal curve of genus g and $y_1, \ldots, y_k \in \Sigma$ are distinct smooth points of Σ. Two such tuples $(\Sigma, y_1, \ldots, y_k)$ and $(\Sigma', y'_1, \ldots, y'_k)$ are equivalent if there exists an invertible morphism[6]

$$\tau\colon \Sigma \longrightarrow \Sigma' \qquad \text{s.t.} \qquad \tau(y_i) = y'_i \;\; \forall\; i = 1, 2, \ldots, k.$$

We denote by $\mathrm{Aut}(\Sigma, y_1, \ldots, y_k)$ the group of automorphisms, i.e. self-equivalences, of $(\Sigma, y_1, \ldots, y_k)$. A prestable genus g k-marked curve $(\Sigma, y_1, \ldots, y_k)$ is called stable if the group $\mathrm{Aut}(\Sigma, y_1, \ldots, y_k)$ is finite. We denote by $\overline{\mathcal{M}}_{g,k}$ the set of equivalence classes of stable genus g k-marked curves; this set is topologized in the next subsection.

Exercise 2.1. Show that

(a) a prestable genus 0 k-marked curve $(\mathbb{P}^1, y_1, \ldots, y_k)$ is stable if and only if $k \geq 3$; if $k \geq 3$, the group $\mathrm{Aut}(\mathbb{P}^1, y_1, \ldots, y_k)$ is trivial;

(b) a prestable genus 0 k-marked curve $(\Sigma, y_1, \ldots, y_k)$ is stable if and only if every (irreducible) component of Σ contains at least 3 special (marked or singular) points of Σ; if it is stable, its group of automorphisms is trivial;

[6]This means that $\tau\colon \Sigma \longrightarrow \Sigma'$ is a homeomorphism induced by a biholomorphic map $\tilde{\tau}\colon \tilde{\Sigma} \longrightarrow \tilde{\Sigma}'$, where $\tilde{\Sigma} \longrightarrow \Sigma$ and $\tilde{\Sigma}' \longrightarrow \Sigma'$ are normalizations as in (2.1).

(c) a smooth prestable genus 1 k-marked curve $(E, y_1, \ldots, y_k)$ is stable if and only if $k \geq 1$;

(d) a prestable genus 1 k-marked curve $(\Sigma, y_1, \ldots, y_k)$ is stable if and only if every genus 0 component of Σ contains at least 3 special (marked or singular) points of Σ and the genus 1 component of Σ (if there is one) contains at least 1 special point.

By Exercise 2.1, the set $\overline{\mathcal{M}}_{g,k}$ is non-empty if $g=0$ and $k \geq 3$ or $g=1$ and $k \geq 1$. Since a smooth curve of genus g has at most finitely many holomorphic automorphisms[7], the space $\overline{\mathcal{M}}_{g,k}$ is non-empty as long as $g, k \geq 0$ and $2g+k \geq 3$.

Example 2.2. If (Σ, y_1, y_2, y_3) is a stable genus 0 3-marked curve, Σ consists of one component, i.e. $\Sigma \approx \mathbb{P}^1$. Since for any two triples, (y_1, y_2, y_3) and (y_1', y_2', y_3'), of distinct points on $\mathbb{P}^1$, there exists a holomorphic automorphism $\tau\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that $\tau(y_i) = y_i'$ for $i=1,2,3$, $\overline{\mathcal{M}}_{0,3}$ consists of a single point: the equivalence class of $\mathbb{P}^1$ with a choice of 3 distinct points.

Example 2.3. If $(\Sigma, y_1, y_2, y_3, y_4)$ is a stable genus 0 4-marked curve, Σ consists of either one component or two components. In the latter case, each of the components carries exactly two of the marked points y_1, y_2, y_3, y_4. Similarly to Example 2.2, each of the three pairings of the points y_1, y_2, y_3, y_4 determines a unique element in $\overline{\mathcal{M}}_{0,4}$. If $\Sigma \approx \mathbb{P}^1$, the only invariant of the distinct points (y_1, y_2, y_3, y_4) on $\mathbb{P}^1$ is the cross ratio:

$$\mathrm{CR}(y_1, y_2, y_3, y_4) \equiv \frac{y_1 - y_3}{y_1 - y_4} : \frac{y_2 - y_3}{y_2 - y_4}\,, \tag{2.2}$$

where we view y_1, y_2, y_3, y_4 as elements of $\mathbb{C} \sqcup \{\infty\}$. In particular, the map

$$\mathrm{CR}\colon \big\{(y_1, y_2, y_3, y_4) \in (\mathbb{P}^1)^4\colon\ y_i \neq y_j \text{ if } i \neq j\big\}/\sim \longrightarrow \mathbb{C} - \{0,1\},$$

is a well-defined bijection. Alternatively, each equivalence class $[\mathbb{P}^1, y_1, y_2, y_3, y_4]$ has a representative of the form $[\mathbb{P}^1, \infty, 0, 1, y]$ with $y \in \mathbb{C} - \{0,1\}$. Thus, $\overline{\mathcal{M}}_{0,4}$ can be naturally identified with $\mathbb{C} - \{0,1\}$ along with 3 points (for the two-component curves), i.e. with $\mathbb{P}^1$.

Example 2.4. If (Σ, y_1) is a stable genus 1 1-marked curve, Σ is either a smooth torus or a sphere with two points identified; y_1 is a smooth point of Σ. Similarly to Example 2.3, the latter determines a unique element in $\overline{\mathcal{M}}_{1,1}$. If Σ is a smooth torus, (Σ, y_1) can be identified with $(\mathbb{C}/\Lambda, 0)$ for some lattice $\Lambda \subset \mathbb{C}$ (i.e. Λ is a discreet subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$). Furthermore, $(\mathbb{C}/\Lambda', 0)$ is equivalent to $(\mathbb{C}/\Lambda, 0)$ if and only if $\Lambda' = \lambda\Lambda$ for some $\lambda \in \mathbb{C}^*$. In particular, the map

$$H \equiv \big\{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\big\} \longrightarrow \mathcal{M}_{1,1}, \qquad z \longrightarrow \big(\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}z), 0\big), \tag{2.3}$$

where $\mathcal{M}_{1,1} \subset \overline{\mathcal{M}}_{1,1}$ is the locus of smooth curves, is surjective. Since the lattices $\mathbb{Z} \oplus \mathbb{Z}z$ and $\mathbb{Z} \oplus \mathbb{Z}z'$, with $z, z' \in H$ are the same if and only if

$$z' = \frac{az+b}{cz+d} \qquad \text{for some} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

the map (2.3) induces a bijection

$$H/\mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathcal{M}_{1,1}, \qquad \text{where} \quad \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm\mathbb{I}\}.$$

[7]If Σ is a smooth curve admitting infinitely many holomorphic automorphisms, the holomorphic bundle $T\Sigma \longrightarrow \Sigma$ admits a holomorphic section. Thus, the degree of $T\Sigma$, i.e. the euler characteristic of Σ, is non-negative.

There is also a bijective map, the j-invariant,

$$\mathrm{j}\colon H/\mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathbb{C};$$

see Section VII-3.3 in [Se]. Thus, $\overline{\mathcal{M}}_{1,1}$ can be naturally identified with $\mathbb{C}$ along with 1 point (for the singular curve), i.e. with $\mathbb{P}^1$.

2.2 Moduli spaces of stable curves

A smooth family of deformations of a prestable curve $(\Sigma, y_1, \ldots, y_k)$ is a $(k+1)$-tuple of smooth morphisms $(\pi, s_1, \ldots, s_k)$[8], where

$$\pi\colon \mathcal{W} \longrightarrow \Delta, \qquad s_i\colon \Delta \longrightarrow \mathcal{W}, \qquad \pi \circ s_i = \mathrm{id}_\Delta,$$

$\mathcal{W}$ is connected, and Δ is an open neighborhood of 0 in $\mathbb{C}^r$ for some $r \in \mathbb{Z}^+$, such that π is surjective,

$$\pi^{-1}(0) = \Sigma, \qquad s_i(0) = y_i \ \ \forall\, i = 1, 2, \ldots, k, \qquad \text{and} \qquad s_i(\Delta) \cap s_j(\Delta) = \emptyset \ \text{ if } \ i \neq j.$$

A sequence of prestable curves $(\Sigma_p, y_{p,1}, \ldots, y_{p,k})$ converges to $(\Sigma, y_1, \ldots, y_k)$ if there exist a deformation $\pi\colon \mathcal{W} \longrightarrow \Delta$ of $(\Sigma, y_1, \ldots, y_k)$ and a sequence $t_p \in \Delta$ converging to 0 such that

$$\pi^{-1}(t_p) = \Sigma_p \quad \text{and} \quad s_i(t_p) = y_{p,i} \ \ \forall\, i = 1, 2, \ldots, k, \qquad \forall\, p = 1, 2, \ldots$$

This defines a topology on $\overline{\mathcal{M}}_{g,k}$.

Exercise 2.5. Let x_1, x_2, x_3, x_4 be the four points in $\mathbb{P}^2$ given by

$$x_1 = [1, 0, 0], \quad x_2 = [0, 1, 0], \quad x_3 = [0, 0, 1], \quad x_4 = [1, 1, 1].$$

(a) Show that any conic (degree 2 curve) in $\mathbb{P}^2$ passing through x_1, x_2, x_3, x_4 is of the form

$$\mathcal{C}_{A,B} = \big\{ [Z_1, Z_2, Z_3] \in \mathbb{P}^2 \colon (A-B) Z_1 Z_2 - A Z_2 Z_3 + B Z_1 Z_3 = 0 \big\}$$

for some $(A, B) \in \mathbb{C}^2 - 0$.

(b) Show that $\mathcal{C}_{A,B}$ is isomorphic to $\mathbb{P}^1$ if $[A, B] \neq [0, 1], [1, 0], [1, 1]$ and in such a case the cross ratio of x_1, x_2, x_3, x_4 on $\mathcal{C}_{A,B}$ is given by

$$\mathrm{CR}_{\mathcal{C}_{A,B}}(x_1, x_2, x_3, x_4) = \frac{B}{A}\,.$$

(c) Conclude that the projection on the first component

$$\pi\colon \mathcal{U} \equiv \big\{ \big([A, B]; [Z_1, Z_2, Z_3]\big) \in \mathbb{P}^1 \times \mathbb{P}^2 \colon (A-B) Z_1 Z_2 - A Z_2 Z_3 + B Z_1 Z_3 = 0 \big\} \longrightarrow \mathbb{P}^1$$

provides a smooth family of deformations for every element of $\overline{\mathcal{M}}_{0,4}$; see Figure 2.

[8] i.e. $\mathcal{W}$ is a complex manifold and $\pi\colon \mathcal{W} \longrightarrow \Delta$ and $s_i\colon \Delta \longrightarrow \mathcal{W}$ are holomorphic maps

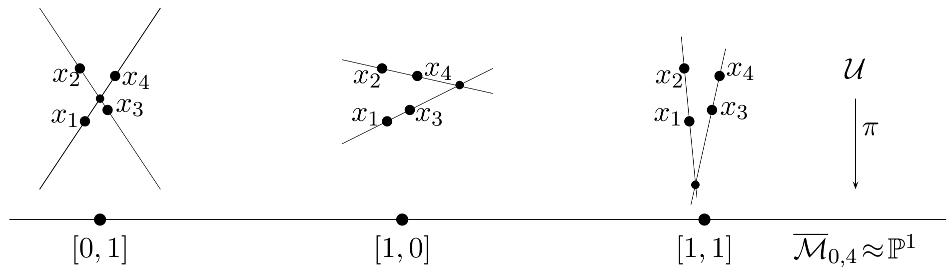

Figure 2: The family $\mathcal{U}\longrightarrow\overline{\mathcal{M}}_{0,4}$, with the 3 special fibers shown; all other fibers are isomorphic to $\mathbb{P}^1$.

Example 2.6. Let $x_1, x_2, \ldots, x_8$ be eight points in general position $\mathbb{P}^2$. Then the space $\mathcal{D}\subset\mathbb{P}^9$ of cubics (degree 3 curves) in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, and the projection on the first component

$$\pi:\mathcal{U}\equiv\big\{\big(\mathcal{C};[Z_1,Z_2,Z_3]\big)\in\mathcal{D}\times\mathbb{P}^2\colon [Z_1,Z_2,Z_3]\in\mathcal{C}\big\}\longrightarrow\mathcal{D}$$

provides a smooth family of deformations for every element of $\overline{\mathcal{M}}_{1,1}$. A general fiber of π is a smooth curve of genus 1. Since the number of rational cubics through 8 general points in $\mathbb{P}^2$ is 12 (see [Z1, Section 2] for example), π has 12 singular fibers. On the other hand, for all $[\Sigma, y_1]\in\overline{\mathcal{M}}_{1,1}$, except for two elements of $\mathcal{M}_{1,1}$, the group of automorphisms of (Σ, y_1) is of order 2 (see Example 2.4 above and [Se, Section VII-1.1]). Thus, the map

$$\mathcal{D}\longrightarrow\overline{\mathcal{M}}_{1,1}, \qquad \mathcal{C}\longrightarrow[\mathcal{C},x_1],$$

is generically 12-to-1.

2.3 Definitions: stable maps

3 Localization on Moduli Spaces of Stable Maps

3.1 Moduli spaces of rational maps to $\mathbb{P}^1$

4 Mirror Symmetry

In this section we state and prove a mirror symmetry formula for the genus 0 Gromov-Witten invariants of a quintic 3-fold, as well as similar formulas for other projective hypersurfaces. The presentation generally follows [MirSym, Chapters 29,30], but we do not treat the Fano cases separately (until the final step) or renormalize the power series involved. Following a suggestion of D. Zagier in a related setting, we also work with power series in $q=e^t$, instead of power series in t and e^t.

4.1 Statement for a quintic

There are a number of related mathematical formulations of mirror symmetry for a quintic 3-fold X_5, i.e. a degree 5 hypersurface in $\mathbb{P}^4$; see [Z2, Appendix B] for a comparison. They originate in the astounding prediction of [CDGP] that relates "counts" of curves in X_5 (later re-interpreted as certain combinations of Gromov-Witten invariants) to the geometry of a one-dimensional mirror family of X_5. Such a relation was completely unexpected mathematically and is still mysterious; it has been proved, but not really explained. The 3-fold X_5 is special from the point of view of string theory because it is Calabi-Yau, i.e.

$$c_1(X) \equiv c_1(TX) = 0 \in H^2(X;\mathbb{Z}) \approx \mathbb{Z}. \tag{4.1}$$

Exercise 4.1. Verify the two statements contained in (4.1).

Mirror symmetry formulas express GW-invariants of X_5 in terms of the hypergeometric series

$$F(w,q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=5d}(5w+r)}{\prod_{r=1}^{r=d}(w+r)^5}\,. \tag{4.2}$$

This is a power series in q with coefficients in rational functions in w that are regular at $w=0$. In particular, F admits a Taylor expansion around $w=0$:

$$F(w,q) = I_0(q)\big(1 + J(q)w + O(w^2)\big), \qquad \text{where} \tag{4.3}$$

$$I_0(q) = F(0,q) = 1 + \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5}\,, \qquad J(q) = \frac{1}{I_0(q)} \sum_{d=1}^{\infty} q^d \bigg(\frac{(5d)!}{(d!)^5} \sum_{r=d+1}^{5d} \frac{5}{r} \bigg). \tag{4.4}$$

Hypergeometric functions, like $F(x,q)$, play a prominent role in complex geometry, including in the study of moduli spaces of complex manifolds (and varieties).

Exercise 4.2. Show that there exists $\tilde{J} \in Q \cdot \mathbb{Q}[[Q]]$ such that

$$q = \big(q\, e^{J(q)}\big) \cdot e^{\tilde{J}(q\, e^{J(q)})} \qquad \text{and} \qquad Q = \big(Q\, e^{\tilde{J}(Q)}\big) \cdot e^{J(Q\, e^{\tilde{J}(Q)})}.$$

In other words, the transformations

$$q \longrightarrow q\, e^{J(q)} \qquad \text{and} \qquad Q \longrightarrow Q\, e^{\tilde{J}(Q)}, \tag{4.5}$$

are mutual inverses.

The maps (4.5) are called *mirror symmetry transformations*.

If $R(w,q) \in \mathbb{Q}(w)[[q]]$ is any power series in q with coefficients in rational functions in w that are regular at $w=0$, denote by

$$[\![R(w,q)]\!]_{w;m} \in \mathbb{Q}[[q]]$$

the coefficient of w^m in the Taylor expansion of $R(w,q)$ at $w=0$:

$$R(w,q) \equiv \sum_{m=0}^{\infty} w^m [\![R(w,q)]\!]_{w^m}.$$

d	n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
11	1017913203569692432490203659468875
12	1512323901934139334751675234074638000
13	2299488568136266648325160104772265542625
14	3565959228158001564810294084668822024070250
15	5624656824668483274179483938371579753751395250
16	9004003639871055462831535610291411200360685606000
17	14602074714589033874568888115959699651605558686799250
18	23954445228532694121482634657728114956109652255216482000
19	39701666985451876233836105884497728824100003703180307231625
20	66408603312404471392397268104340892583652834904833089314920000

Table 1: BPS states n_d for a quintic threefold

Theorem 4.3. *If N_d is the genus 0 degree d GW-invariant of X_5,*

$$\sum_{d=1}^{\infty} N_d Q^d = -\frac{5}{2} \left[\!\!\left[e^{-J(q)w} \frac{F(q,w)}{I_0(q)} \right]\!\!\right]_{w;3} = -\frac{5}{2} \left[\!\!\left[\ln\left(\frac{F(q,w)}{I_0(q)} \right) \right]\!\!\right]_{w;3}, \tag{4.6}$$

where q and Q are related by the mirror transformations (4.5).

By Exercise 4.2, the right-hand side of (4.6) can be written as a power series in Q; thus, (4.6) specifies all N_d. These are rational numbers. Table 1 lists the first 20 numbers n_d defined from N_d by

$$N_d = \sum_{r|d} \frac{n_{d/r}}{r^3} \qquad \forall\ d = 1, 2, \ldots; \tag{4.7}$$

note that these 20 numbers are *integers*!

Exercise 4.4. Show that the identities (4.7) describe the numbers n_d as functions of the numbers N_d, with various d.

Since X_5 is a Calabi-Yau 3-fold, the *expected* dimension of the space of curves in X_5 is zero. Thus, ideally (i.e. if everything is as expected), the space of genus 0 curves consists of isolated elements and every such curve is smooth. The normal bundle to a smooth rational curve in X_5 is holomorphic, of rank 2, and of degree -2. Any such holomorphic bundle splits as

$$\mathcal{O}(-a) \oplus \mathcal{O}(-b) = \gamma^{\otimes a} \oplus \gamma^{\otimes b} \longrightarrow \mathbb{P}^1,$$

with $a+b=2$. Ideally, $a=b$.

Exercise 4.5. Use Gromov's Compactness Theorem to show that if every genus 0 curve in X is smooth and its normal bundle splits as $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, then there are finitely many genus 0 curves in each homology class.

If X_5 is ideal as above, the contribution of every genus 0 degree d curve to the Gromov-Witten invariant N_{rd} is $1/r^3$ by the Aspinwall-Morrison formula. Thus, if n_d is the number of (reduced) genus 0 curves of degree d in an ideal quintic 3-fold X_5, then the number N_d is given by (4.7) for all d. In such a case, the numbers n_d defined from N_d by (4.7) are counts of curves in X_5 and are thus nonnegative integers. However, it is known that even a generic quintic 3-fold (zero set in $\mathbb{P}^4$ of a generic degree 5 homogeneous polynomial in 5 variables) is not ideal. Nevertheless, it is still conjectured that the numbers n_d are integers.

Conjecture 4.6. *The rational numbers n_d defined from the Gromov-Witten invariants N_d of a quintic 3-fold are integers.*

This conjecture is generalized to all compact Calabi-Yau 3-folds X by viewing d and d/r as elements of $H_2(X;\mathbb{Z})$. Using (4.6) and a simple computer program, Conjecture 4.6 can be confirmed to a very high degree.

In a generic quintic 3-fold, rational curves of degree $d\le 4$ are smooth and their normal bundles split as $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Thus, the numbers n_d defined by (4.7) with $d\le 4$ are indeed counts of curves. The first number in Table 1, the number of lines on a general quintic 3-fold, can be confirmed via a straightforward computation on the Grassmannian $\mathbb{G}(2,5)$; see [Ka2, Chapter 7]. A more elaborate Schubert calculus computation [Ka1, Section 3] confirms n_2. A still classical, but even more involved, approach of [ES] verifies n_3.

4.2 Generalization to other hypersurfaces

For the rest of this section we fix positive integers n and a and denote by X a smooth degree a hypersurface in $\mathbb{P}^{n-1}$. If $a>n$ and $d(a-n)>n-5$, the expected dimension of $\overline{\mathfrak{M}}_{0,0}(X,d)$ is negative. Thus, if $a>n$ only finitely many genus 0 GW-invariants of X are nonzero. Our attention will be on the $a\le n$ cases. If $a=n$, X is a Calabi-Yau $(n-2)$-fold; the formula we will obtain for its GW-invariants will look very different from the $a<n$ cases.

Similarly to (4.2), (4.3), and (4.4), let

$$F(w,q)=\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{r=1}^{r=ad}(aw+r)}{\prod_{r=1}^{r=d}(w+r)^n}\in\mathbb{Q}(w)\big[\big[q\big]\big] \tag{4.8}$$

For each $d\in\mathbb{Z}^+$, let

$$\pi\colon\mathcal{L}\equiv\mathcal{O}(a)=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^{n-1}\qquad\text{and}\qquad \begin{aligned}&\tilde\pi\colon\mathcal{V}_d\equiv\overline{\mathfrak{M}}_{0,k}(\mathcal{L},d)\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d),\\ &\tilde\pi\big([\xi\colon\Sigma\longrightarrow\mathcal{L}]\big)=\big[\pi\circ\xi\colon\Sigma\longrightarrow\mathbb{P}^{n-1}\big].\end{aligned}$$

Exercise 4.7. Show that $\mathcal{V}_d \longrightarrow \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ is a vector orbi-bundle of rank $da+1$ and the vector bundle homomorphism

$$\widetilde{\mathrm{ev}}_1 : \mathcal{V}_d \longrightarrow \mathrm{ev}_1^*\mathcal{L}, \qquad \big[\xi : \Sigma \longrightarrow \mathcal{L}\big] \longrightarrow \xi(x_1(\Sigma)), \tag{4.9}$$

where $x_1(\Sigma)\in\Sigma$ is the first marked point, is surjective.

It follows that $\mathcal{V}_d' \equiv \ker \widetilde{\mathrm{ev}}_1$ is a vector orbi-bundle and there is a short exact sequence of vector orbi-bundles

$$0 \longrightarrow \mathcal{V}_d' \longrightarrow \mathcal{V}_d \longrightarrow \mathrm{ev}_1^*\mathcal{L} \longrightarrow 0 \tag{4.10}$$

over $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ for $k\ge 1$. With ψ_1 and ev_1 denoting the first chern class of the cotangent line bundle at the first marked point and the evaluation map on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$, respectively, let

$$Z(H,Q) = 1 + \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\bigg(\frac{e(\mathcal{V}_d')}{1-\psi_1}\bigg) \in H^*(\mathbb{P}^{n-1})\big[\big[Q\big]\big]. \tag{4.11}$$

Theorem 4.8. *If n and $a\le n$ are positive integers, X is a smooth degree a hypersurface in $\mathbb{P}^{n-1}$, and Y and Z are given by (4.8) and (4.11), then*

$$\begin{aligned} Z(H,Q) &= \begin{cases} F(H,Q), & \textit{if } a < n-1; \\ e^{-a!Q}F(H,Q), & \textit{if } a = n-1; \end{cases} \\ Z(H,Q) &= e^{-J(q)H}F(H,q)/I_0(q) \quad \textit{if} \quad a=n, \end{aligned} \tag{4.12}$$

with Q and q related by the mirror transformation (4.5) in the last case.

Exercise 4.9. Using (1.21), show that

$$Z(H,Q) = 1 + \sum_{d=1}^{\infty} Q^d \bigg(\sum_{m=0}^{n-1} H^m \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_d')\psi_1^{(n-a)d+m-1}\mathrm{ev}_1^* H^{n-1-m}\bigg),$$

where $H=c_1(\gamma^*)\in H^2(\mathbb{P}^{n-1})$ is the Poincare dual to the hyperplane class and $\psi_1^p\equiv 0$ if $p<0$.

Exercise 4.9 implies that $Z(H,Q)$ encodes many Gromov-Witten invariants of X, since

$$\int_{[\overline{\mathfrak{M}}_{0,k}(X,d)]^{vir}} \eta|_{\overline{\mathfrak{M}}_{0,k}(X,d)} = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_d)\eta \qquad \forall\, \eta \in H^*\big(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)\big) \tag{4.13}$$

by the Hyperplane Theorem. Furthermore, if

$$f_1 : \overline{\mathfrak{M}}_{0,1}(X,d) \longrightarrow \overline{\mathfrak{M}}_{0,0}(X,d) \qquad \text{and} \qquad f_2 : \overline{\mathfrak{M}}_{0,2}(X,d) \longrightarrow \overline{\mathfrak{M}}_{0,1}(X,d)$$

are the forgetful morphisms dropping the (first) marked point in the first case and the second marked point in the second case (and contracting any resulting unstable components), then

$$\begin{aligned} \int_{[\overline{\mathfrak{M}}_{0,1}(X,d)]^{vir}} \psi_1^m f_1^*\eta_0 &= \begin{cases} 0, & \text{if } m=0; \\ -2\int_{[\overline{\mathfrak{M}}_{0,0}(X,d)]^{vir}} \eta_0, & \text{if } m=1; \end{cases} \quad \forall\, \eta_0 \in H^*\big(\overline{\mathfrak{M}}_{0,0}(X,d)\big); \\ \int_{[\overline{\mathfrak{M}}_{0,2}(X,d)]^{vir}} \psi_1^m f_2^*\eta_1 &= \begin{cases} 0, & \text{if } m=0; \\ \int_{[\overline{\mathfrak{M}}_{0,1}(X,d)]^{vir}} \psi_1^{m-1}\eta_1, & \text{if } m\ge 1; \end{cases} \quad \forall\, \eta_1 \in H^*\big(\overline{\mathfrak{M}}_{0,1}(X,d)\big). \end{aligned} \tag{4.14}$$

Exercise 4.10. Verify (4.14).
`Remark:` For our purposes it is sufficient to verify (4.14) only for the classes

$$\eta_k = \eta'_k\big|_{\overline{\mathfrak{M}}_{0,k}(X,d)}, \qquad \text{with} \quad \eta'_k \in H^*\big(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)\big).$$

For such classes, (4.14) follows from the analogous identities for $\mathbb{P}^{n-1}$ (in place of X) and (4.13).

Exercise 4.11. Deduce Theorem 4.3 from Theorem 4.8.

4.3 Equivariant version

Theorem 4.8 will be proved by applying the Atiyah-Bott Localization Theorem; see Theorem 1.22. In fact, we will prove a stronger, equivariant, version of Theorem 4.8.

With $\mathbb{T}$ denoting the n-torus, let

$$\mathcal{H}^*_{\mathbb{T}} \approx \mathbb{C}(\alpha_1,\ldots,\alpha_n) \equiv \mathbb{C}_\alpha \tag{4.15}$$

be the field of fractions of the ring $H^*_{\mathbb{T}}$ as in Subsection 1.6; $\alpha_1,\ldots,\alpha_k$ are the generators of $H^*_{\mathbb{T}}$ as in (1.2). With respect to the standard action of $\mathbb{T}$,

$$\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \equiv H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \approx \mathbb{C}_\alpha[x]/(x-\alpha_1)\ldots(x-\alpha_n); \tag{4.16}$$

see Exercise 1.8. Via the standard lift to the line bundle $\gamma \longrightarrow \mathbb{P}^{n-1}$, the $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ lifts to a $\mathbb{T}$-action on the line bundle

$$\mathcal{L} \equiv \mathcal{O}(a) \equiv \gamma^{*\otimes a} \longrightarrow \mathbb{P}^{n-1}.$$

Via composition of maps, the $\mathbb{T}$-actions on $\mathbb{P}^{n-1}$ and $\mathcal{L}$ induce actions on $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ and

$$\mathcal{V}_d \equiv \overline{\mathfrak{M}}_{0,2}(\mathcal{L},d) \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d).$$

Since the evaluation map $\widetilde{\mathrm{ev}}_1$ in (4.9) is $\mathbb{T}$-equivariant, the vector orbi-bundle

$$\mathcal{V}'_d \subset \mathcal{V}_d \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$$

is also $\mathbb{T}$-equivariant and thus has a well-defined equivariant euler class

$$\mathbf{e}(\mathcal{V}'_d) \in \mathcal{H}^*_{\mathbb{T}}\big(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\big).$$

Since the forgetful map

$$\overline{\mathfrak{M}}_{0,3}(\mathbb{P}^{n-1},d) \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$$

is $\mathbb{T}$-equivariant, the cotangent line bundle at the first marked point also has an equivariant euler class

$$\psi_1 \in \mathcal{H}^*_{\mathbb{T}}\big(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\big).$$

Since the morphism $\mathrm{ev}_1\colon \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d) \longrightarrow \mathbb{P}^{n-1}$ is $\mathbb{T}$-equivariant, the power series

$$\mathcal{Z}(x,\hbar,Q) = 1 + \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\bigg(\frac{\mathbf{e}(\mathcal{V}'_d)}{\hbar-\psi_1}\bigg) \in \mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big[\big[Q\big]\big] \tag{4.17}$$

is well-defined. Let

$$\mathcal{Y}(x,\hbar,q) = \sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=ad}(ax+r\hbar)}{\prod_{r=1}^{r=d}\prod_{k=1}^{k=n}(x-\alpha_k+r\hbar)}\,. \tag{4.18}$$

This is a power series in q with coefficients in $\mathbb{C}_\alpha[x][[\hbar^{-1}]]$ if $a \le n$. We will also denote by $\mathcal{Y}(x,\hbar,q)$ the image of $\mathcal{Y}(x,\hbar,q)$ in

$$\big\{\mathbb{C}_\alpha[x]/(x-\alpha_1)\dots(x-\alpha_n)\big\}(\hbar)\big[[q]\big] = \big\{\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big\}(\hbar)\big[[q]\big].$$

Let

$$\sigma_1 = \sum_{k=1}^{k=n}\alpha_k\,, \qquad C_1(q) = \bigg(\sum_{d=1}^{\infty} q^d \frac{(ad)!}{(d!)^n}\sum_{r=1}^{r=d}\frac{1}{r}\bigg)\bigg/ I_0(q) \quad \text{if} \quad a=n\,.$$

Theorem 4.12. *If n and $a \le n$ are positive integers, X is a smooth degree a hypersurface in $\mathbb{P}^{n-1}$, and $\mathcal{Z}$ and $\mathcal{Y}$ are given by (4.17) and (4.18), then*

$$\begin{aligned} \mathcal{Z}(x,\hbar,Q) &= \begin{cases} \mathcal{Y}(x,\hbar,Q), & \text{if } a < n-1;\\ e^{-a!Q/\hbar}\mathcal{Y}(x,\hbar,Q), & \text{if } a = n-1; \end{cases} \\ \mathcal{Z}(x,\hbar,Q) &= e^{-C_1(q)\sigma_1/\hbar}e^{-J(q)x/\hbar}\mathcal{Y}(x,\hbar,q)\big/I_0(q) \quad \text{if} \quad a=n, \end{aligned} \tag{4.19}$$

with Q and q related by the mirror transformation (4.5) in the last case.

Theorem 4.8 follows immediately from Theorem 4.12 by setting $\hbar = 1$ and $\alpha_i = 0$.

By (1.15), the power series

$$\mathcal{Z}(x,\hbar,Q), \mathcal{Y}(x,\hbar,Q) \in \big\{\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big\}\big[[\hbar^{-1},Q]\big]$$

are determined by their values at n points:

$$\mathcal{Z}(x=\alpha_i,\hbar,Q), \mathcal{Y}(x=\alpha_i,\hbar,Q) \in \mathbb{C}_\alpha\big[[\hbar^{-1},Q]\big] \qquad i=1,2,\dots,n.$$

Exercise 4.13. Show that for all $i = 1,2,\dots,n$

$$\mathcal{Z}(\alpha_i,\hbar,Q) = 1 + \sum_{d=1}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} \frac{\mathbf{e}(\mathcal{V}'_d)}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\,,$$

where ϕ_i is the equivariant Poincare dual of the fixed point $P_i \in \mathbb{P}^{n-1}$ given by (1.29).

The proof of (4.19) consists of showing that

Step 1: $\mathcal{Y}(x,\hbar,Q)$ and $\mathcal{Z}(x,\hbar,Q)$ satisfy the recursion (4.31) on the Q-degree;

Step 2: $\mathcal{Y}(x,\hbar,Q)$ and $\mathcal{Z}(x,\hbar,Q)$ satisfy the self-polynomiality condition (SPC) of Lemma 4.17;

Step 3: the two sides of (4.19), viewed as a powers series in $\hbar^{-1}$, agree mod $\hbar^{-2}$;

Step 4: if $\mathcal{F}(x,\hbar,Q)$ satisfies the recursion and the SPC, so do certain transforms of $\mathcal{F}(x,\hbar,Q)$;

Step 5: if $\mathcal{F}(x,\hbar,Q)$ satisfies the recursion and the SPC and $\mathcal{F}(\alpha_i,\hbar,0)\in\mathbb{C}_\alpha^*$ for every $i=1,2,\ldots,n$, then $\mathcal{F}$ is determined by its "mod $\hbar^{-2}$ part".

For the purposes of these statements, in particular in Steps 3 and 5,

$$\mathcal{F}(x,\hbar,Q),\mathcal{Y}(x,\hbar,Q),\mathcal{Z}(x,\hbar,Q)\in\mathbb{C}_\alpha[x]\big[\big[\hbar^{-1},Q\big]\big]\Big/\prod_{k=1}^{k=n}(x-\alpha_k).$$

For example, the statement in Step 5 means

$$\begin{aligned}\mathcal{F}(\alpha_i,\hbar,Q)=\bar{\mathcal{F}}(\alpha_i,\hbar,Q)\quad(\text{mod }\hbar^{-2})\quad\forall\, i=1,2,\ldots,n\\ \Longrightarrow\qquad \mathcal{F}(\alpha_i,\hbar,Q)=\bar{\mathcal{F}}(\alpha_i,\hbar,Q)\quad\forall\, i=1,2,\ldots,n.\end{aligned}$$

None of the many steps involved in the proof of Theorem 4.12 is very difficult. The proof that $\mathcal{Y}(x,\hbar,Q)$ satisfies the recursion (4.31) and the SPC is a straightforward algebraic computation involving the Residue Theorem on S^2 stated in Exercise 4.15 below; see Lemma 4.22. Proofs of Steps 4 and 5 also consist of fairly simple algebraic computations; see Lemma 4.21 and Proposition 4.20, respectively. The Atiyah-Bott Localization Theorem is used to show that $\mathcal{Z}(x,\hbar,Q)$ satisfies the recursion (4.31) and the SPC. However, overall the proof of Theorem 4.3 involves several key insights, originating in [Gi] and clarified in [Pa] (and later in more detail in [MirSym, Part IV]):

- integrating $e(\mathcal{V}_d)$ over $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$, with $k>0$, rather than over $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^{n-1},d)$ as one might expect. This makes it possible to pushforward cohomology classes to $\mathbb{P}^{n-1}$, thus re-formulating the mirror symmetry statement of Theorem 4.3 as a direct relation with the power series (4.2), and to set up a degree recursion for Gromov-Witten invariants;
- augmenting the integrand by the denominator $\hbar-\psi_1$ to beautifully capture the essential nature of the recursion;
- discovering the uniqueness property of Step 5;
- constructing a morphism from $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ to a space with a larger torus action in order to verify that $\mathcal{Z}(x,\hbar,Q)$ satisfies the SPC; see Section 4.7.

Exercise 4.14. Show that both sides of (4.19) equal 1 modulo $\hbar^{-2}$, in the sense described above.
`Hint:` Use (4.14) and Exercise 4.13.

If $f=f(\hbar)$ is a meromorphic function on $\mathbb{C}$, we denote by $\underset{\hbar=\hbar_0}{\mathfrak{R}}\, f$ the residue of f and $f\mathrm{d}\hbar$ at $\hbar=0$:

$$\underset{\hbar=\hbar_0}{\mathfrak{R}}\, f=\underset{\hbar=\hbar_0}{\mathfrak{R}}\, f\mathrm{d}\hbar\equiv\frac{1}{2\pi\mathfrak{i}}\oint f(\hbar)\mathrm{d}\hbar,$$

where the integral is taken over a simple closed positively oriented loop around a enclosing no other pole of f. The value of the residue of a 1-form is independent of the choice of a local holomorphic coordinate. The following statement will be used a number of times in the proof of Theorem 4.12. It holds for all compact Riemann surfaces.

Exercise 4.15. If η is a meromorphic 1-form on S^2, show that

$$\sum_{\hbar_0\in S^2}\underset{\hbar=\hbar_0}{\mathfrak{R}}\,\eta=0.$$

4.4 On rigidity of certain polynomial conditions

This subsection describes the extent of rigidity of double power series that satisfy a certain recursion and a polynomiality condition.

Denote by $\mathbb{Z}^{\ge 0}$ the set of nonnegative integers and by $[n]$, whenever $n \in \mathbb{Z}^{\ge 0}$, the set of positive numbers not exceeding n:

$$\mathbb{Z}^{\ge 0} = \big\{0,1,2,\ldots,\big\}, \qquad [n] = \big\{1,2,\ldots,n\big\}.$$

Whenever f is a function of w (and possibly of other variables) which is holomorphic at $w\!=\!0$ (for a dense subspace of the other variables) and $s\!\in\!\mathbb{Z}^{\ge 0}$, let $[\![f]\!]_{w;s}$ denote the coefficient of w^s in the power series expansion of f around $w\!=\!0$; this is a function of the other variables if there are any. For any ring R, let

$$R[\![\hbar]\!] \equiv R[[\hbar^{-1}]] + R[\hbar]$$

denote the R-algebra of Laurent series in $\hbar^{-1}$. If

$$\mathcal{F}(\hbar,Q) = \sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{\infty}\mathcal{F}_d^{(r)}\hbar^{-r}\bigg)Q^d \in R[\![\hbar]\!]\big[\big[Q\big]\big]$$

for some $\mathcal{F}_d^{(r)}\!\in\! R$, we define

$$\mathcal{F}(\hbar,Q) \cong \sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{p-1}\mathcal{F}_d^{(r)}\hbar^{-r}\bigg)Q^d \pmod{\hbar^{-p}},$$

i.e. we drop $\hbar^{-p}$ and higher powers of $\hbar^{-1}$, instead of higher powers of $\hbar$.

For any element

$$\mathcal{F}\!\equiv\!\mathcal{F}(x,\hbar,Q) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big],$$

we define

$$\Phi_{\mathcal{F}}\!\equiv\!\Phi_{\mathcal{F}}(\hbar,z,Q) \in \mathbb{C}_\alpha[\![\hbar]\!]\big[\big[z,Q\big]\big] \qquad \text{by}$$

$$\Phi_{\mathcal{F}}(\hbar,z,Q) = \sum_{i=1}^{i=n}\frac{a\alpha_i\, e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i\!-\!\alpha_k)}\mathcal{F}\big(\alpha_i,\hbar,Qe^{\hbar z}\big)\mathcal{F}(\alpha_i,-\hbar,Q). \tag{4.20}$$

Lemma 4.16. *For every $\mathcal{F}\!\in\! H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ such that*

$$\mathcal{F}(x\!=\!\alpha_i,\hbar,Q) \in \mathbb{C}_\alpha(\hbar)\big[\big[Q\big]\big] \subset \mathbb{C}_\alpha[\![\hbar]\!]\big[\big[Q\big]\big] \qquad \forall\, i \in [n],$$

there exists a unique collection

$$\big(E_{\mathcal{F};d}\!\equiv\! E_{\mathcal{F};d}(\hbar,\Omega)\big)_{d\in\mathbb{Z}^{\ge 0}} \subset \mathbb{C}_\alpha(\hbar)[\Omega]$$

such that the Ω-degree of $E_{\mathcal{F};d}$ is at most $(d+1)n-1$ for every $d \in \mathbb{Z}^{\geq 0}$ and

$$\Phi_{\mathcal{F}}(\hbar, z, Q) = \sum_{d=0}^{\infty} Q^d \left(\frac{1}{2\pi \mathfrak{i}} \oint e^{\Omega z} \frac{E_{\mathcal{F};d}(\hbar, \Omega)}{\prod\limits_{r=0}^{r=d} \prod\limits_{k=1}^{k=n} (\Omega - \alpha_k - r\hbar)} \mathrm{d}\Omega \right), \tag{4.21}$$

where each path integral is taken over a simple closed loop in $\mathbb{C}$ enclosing all points $\Omega = \alpha_k + r\hbar$ with $k = 1, \ldots, n$ and $r = 0, 1, \ldots, d$. The equality holds for a dense collection of complex parameters $\hbar$.

Proof. It can be assumed that

$$\alpha_k + r\hbar \neq \alpha_{k'} + r'\hbar \qquad \forall\ k, k' \in [n],\ r, r' \in \mathbb{Z}^{\geq 0},\ (r,k) \neq (r', k').$$

Note that for every $i = 1, \ldots, n$ and $d' = 0, 1, \ldots, d$,

$$\begin{aligned}
&\prod_{r=0}^{r=d'-1} (\alpha_i + d'\hbar - \alpha_i - r\hbar) \prod_{r=d'+1}^{r=d} (\alpha_i + d'\hbar - \alpha_i - r\hbar) \prod_{r=0}^{r=d} \prod_{k \neq i} (\alpha_i + d'\hbar - \alpha_k - r\hbar) \\
&\quad = d'!\hbar^{d'}(d-d')!(-\hbar)^{d-d'} \left(\prod_{r=1}^{r=d'} \prod_{k\neq i} (\alpha_i - \alpha_k + r\hbar) \right) \left(\prod_{k\neq i} (\alpha_i - \alpha_k) \right) \left(\prod_{r=1}^{r=d-d'} \prod_{k\neq i} (\alpha_i - \alpha_k - r\hbar) \right) \\
&\quad = \left(\prod_{k\neq i} (\alpha_i - \alpha_k) \right) \Delta_{d'}(\alpha_i, \hbar)\, \Delta_{d-d'}(\alpha_i, -\hbar),
\end{aligned}$$

where

$$\Delta_d(x, \hbar) \equiv \prod_{r=1}^{r=d} \prod_{k=1}^{k=n} (x - \alpha_k + r\hbar) \qquad \forall\ d \in \mathbb{Z}^{\geq 0}. \tag{4.22}$$

By Cauchy's Formula,

$$\begin{aligned}
&\frac{1}{2\pi\mathfrak{i}} \oint e^{\Omega z} \frac{E_{\mathcal{F};d}(\hbar, \Omega)}{\prod_{r=0}^{r=d} \prod_{k=1}^{k=n} (\Omega - \alpha_k - r\hbar)} \mathrm{d}\Omega \\
&\qquad = \sum_{d'=0}^{d'=d} \sum_{i=1}^{i=n} e^{(\alpha_i + d'\hbar)z} \frac{E_{\mathcal{F};d}(\hbar, \alpha_i + d'\hbar)}{\big(\prod\limits_{k\neq i} (\alpha_i - \alpha_k) \big) \Delta_{d'}(\alpha_i, \hbar)\, \Delta_{d-d'}(\alpha_i, -\hbar)} \\
&\qquad = \sum_{d'=0}^{d'=d} \sum_{i=1}^{i=n} \left(\frac{e^{\alpha_i z}}{\prod\limits_{k\neq i} (\alpha_i - \alpha_k)} \right) \left(\frac{(e^{\hbar z})^{d'}}{\Delta_{d'}(\alpha_i, \hbar) \Delta_{d-d'}(\alpha_i, -\hbar)} \right) E_{\mathcal{F};d}(\hbar, \alpha_i + d'\hbar).
\end{aligned} \tag{4.23}$$

On the other hand, by the assumptions on $\mathcal{F}$,

$$\mathcal{F}(x, \hbar, Q) = \sum_{d=0}^{\infty} \frac{N_{\mathcal{F};d}(x, \hbar)}{\Delta_d(x, \hbar)} Q^d \tag{4.24}$$

for a unique $N_{\mathcal{F};d} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]$ such that $N_{\mathcal{F};d}(x=\alpha_i,\hbar) \in \mathbb{C}_\alpha(\hbar)$ for every $i \in [n]$. By (4.20) and (4.24),

$$\begin{aligned}\Phi_{\mathcal{F}}(\hbar,z,Q) &= \sum_{d=0}^{\infty}\sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n}\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\bigg(\frac{N_{\mathcal{F};d'}(\alpha_i,\hbar)}{\Delta_{d'}(\alpha_i,\hbar)}\bigg)(Qe^{\hbar z})^{d'}\bigg(\frac{N_{\mathcal{F};d-d'}(\alpha_i,-\hbar)}{\Delta_{d-d'}(\alpha_i,-\hbar)}\bigg)Q^{d-d'}\\ &= \sum_{d=0}^{\infty}Q^d\Bigg(\sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n}\frac{e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\bigg(\frac{(e^{\hbar z})^{d'}}{\Delta_{d'}(\alpha_i,\hbar)\Delta_{d-d'}(\alpha_i,-\hbar)}\bigg)\\ &\qquad\qquad\times a\alpha_i\,N_{\mathcal{F};d'}(\alpha_i,\hbar)N_{\mathcal{F};d-d'}(\alpha_i,-\hbar)\Bigg).\end{aligned} \tag{4.25}$$

By (4.23) and (4.25), (4.21) is satisfied if and only if

$$E_{\mathcal{F};d}(\hbar,\alpha_i+d'\hbar) = a\alpha_i\,N_{\mathcal{F};d'}(\alpha_i,\hbar)\cdot N_{\mathcal{F};d-d'}(\alpha_i,-\hbar) \quad \forall\, i\in[n],\ d'=0,\ldots,d. \tag{4.26}$$

For a dense collection of complex parameters $\hbar$, there exists a unique polynomial

$$E_{\mathcal{F};d}(\hbar,\Omega) \in \mathbb{C}_\alpha(\hbar)[\Omega]$$

of Ω-degree at most $(d+1)n-1$ that satisfies (4.26). $\square$

Lemma 4.17. *If $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ and $(E_{\mathcal{F};d})_{d\in\mathbb{Z}^{\ge 0}} \subset \mathbb{C}_\alpha(\hbar)[\Omega]$ are as in Lemma 4.16, then*

$$\Phi_{\mathcal{F}} \in \mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big] \qquad\Longleftrightarrow\qquad E_{\mathcal{F};d} \in \mathbb{C}_\alpha[\hbar,\Omega] \ \ \forall\, d\in\mathbb{Z}^{\ge 0}. \tag{4.27}$$

Proof. By Exercise 4.15,

$$\frac{1}{2\pi\mathfrak{i}}\oint\frac{\Omega^s\mathrm{d}\Omega}{\prod\limits_{r=0}^{r=d}\prod\limits_{k=1}^{k=n}(\Omega-\alpha_k-r\hbar)} = \begin{cases}0, & \text{if } s<(d+1)n-1;\\ 1, & \text{if } s=(d+1)n-1;\\ R^d_{s-(d+1)n+1}(\hbar), & \text{if } s>(d+1)n-1,\end{cases} \tag{4.28}$$

where $R^d_s\in\mathbb{C}_\alpha[\hbar]$ is given by

$$R^d_s(\hbar) = \bigg[\!\!\bigg[\prod_{r=0}^{r=d}\prod_{k=1}^{k=n}(1-(\alpha_k+r\hbar)w)^{-1}\bigg]\!\!\bigg]_{w;s} \qquad \forall\ s\in\mathbb{Z}^{\ge 0}.$$

The path integral in (4.28) is again taken over a simple closed loop enclosing all points $\Omega=\alpha_k+r\hbar$ with $r\le d$. Write

$$\Phi_{\mathcal{F}}(\hbar,z,Q) = \sum_{d=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}F_{d,m}(\hbar)z^mQ^d\,, \qquad E_{\mathcal{F};d}(\hbar,\Omega) = \sum_{s=0}^{(d+1)n-1}f_{d,s}(\hbar)\Omega^s. \tag{4.29}$$

By (4.21), (4.28), and (4.29),

$$\begin{aligned} F_{d,m}(\hbar) &= \sum_{s=0}^{(d+1)n-1} \frac{1}{2\pi\mathfrak{i}} \oint \frac{f_{d,s}(\hbar)\,\Omega^{m+s}\mathrm{d}\Omega}{\prod\limits_{r=0}^{r=d}\prod\limits_{k=1}^{k=n}(\Omega-\alpha_k-r\hbar)} \\ &= \sum_{s=\max(0,(d+1)n-1-m)}^{(d+1)n-1} R^d_{m+s-(d+1)n+1}(\hbar)\, f_{d,s}(\hbar). \end{aligned} \tag{4.30}$$

Since $R^d_s\in\mathbb{C}_\alpha[\hbar]$, it follows that $F_{d,m}\in\mathbb{C}_\alpha[\hbar]$ if $f_{d,s}\in\mathbb{C}_\alpha[\hbar]$ for all s. Conversely, since $R^d_0(\hbar)=1$,

$$F_{d,0},\ldots,F_{d,(d+1)n-1}\in\mathbb{C}_\alpha[\hbar] \qquad\Longrightarrow\qquad f_{d,(d+1)n-1},\ldots,f_{d,0}\in\mathbb{C}_\alpha[\hbar].$$

These observations imply Lemma 4.17. □

Definition 4.18. *Let $C\equiv(C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{C}_\alpha$. A Laurent series $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ is C-recursive if*

$$\mathcal{F}\big(\alpha_i,(\alpha_i-\alpha_j)/d,Q\big)\in\mathbb{C}_\alpha[[Q]]$$

is well-defined for all $d\in\mathbb{Z}^+$ and $i,j\in[n]$ and

$$\mathcal{F}(\alpha_i,\hbar,Q)=\sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{r=N_d}\mathcal{F}^r_{i;d}\hbar^{-r}\bigg)Q^d+\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\,\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big) \tag{4.31}$$

for every $i\in[n]$ and for some $N_d\in\mathbb{Z}$ and $\mathcal{F}^r_{i;d}\in\mathbb{C}_\alpha$.

Exercise 4.19. Suppose $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ satisfies (4.31). Show that

(a) if $\mathcal{F}(\alpha_i,\hbar,0)\in\mathbb{C}_\alpha[\hbar,\hbar^{-1}]$ for all $i\in[n]$, then $\mathcal{F}(\alpha_i,\hbar,Q)\in\mathbb{C}_\alpha(\hbar)\big[\big[Q\big]\big]$ for all $i\in[n]$;

(b) $\mathcal{F}$ is determined by $\mathcal{F}(x,\hbar,0)$ and the collections $\{C_i^j(d)\}$ and $\{\mathcal{F}^r_{i;d}\}$.

Proposition 4.20. *Let $C\equiv(C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{C}_\alpha$. If Laurent series $\mathcal{F},\bar{\mathcal{F}}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ are C-recursive, $\Phi_{\mathcal{F}},\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, and*

$$\mathcal{F}(x=\alpha_i,\hbar,Q=0)=\bar{\mathcal{F}}(x=\alpha_i,\hbar,Q=0)\in\mathbb{C}^*_\alpha\subset\mathbb{C}_\alpha[\![\hbar]\!]\qquad\forall\,i\in[n],$$

then $\mathcal{F}\cong\bar{\mathcal{F}}\pmod{\hbar^{-2}}$ if and only $\mathcal{F}=\bar{\mathcal{F}}$.

Proof. Let $\mathcal{F}^r_{i;d},\bar{\mathcal{F}}^r_{i;d}\in\mathbb{C}_\alpha$ be the coefficients in (4.31) corresponding to $\mathcal{F}$ and $\bar{\mathcal{F}}$, respectively,

$$\delta\mathcal{F}^r_{i;d}=\mathcal{F}^r_{i;d}-\bar{\mathcal{F}}^r_{i;d},\quad\text{and}\quad\delta\mathcal{F}(x,\hbar,Q)=\mathcal{F}(x,\hbar,Q)-\bar{\mathcal{F}}(x,\hbar,Q).$$

We show by induction on d that $\delta\mathcal{F}^r_{i;d}=0$ for all i and r. For each $i\in[n]$, let

$$f_i=\mathcal{F}(\alpha_i,\hbar,Q=0)=\bar{\mathcal{F}}(\alpha_i,\hbar,Q=0)\in\mathbb{C}^*_\alpha.$$

Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ are C-recursive and

$$\mathcal{F}(x,\alpha_i,0), \bar{\mathcal{F}}(x,\alpha_i,0) \in \mathbb{C}_\alpha(\hbar) \qquad \forall\, i\in[n],$$

$\mathcal{F}$ and $\bar{\mathcal{F}}$ satisfy the assumptions of Lemmas 4.16 and 4.17. Let

$$N_{\mathcal{F};d}, N_{\bar{\mathcal{F}};d} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]$$

be as in the proof of Lemma 4.16 and

$$\delta N_d = N_{\mathcal{F};d} - N_{\bar{\mathcal{F}};d}.$$

Since $\mathcal{F}(\alpha_i,\hbar,0)=\bar{\mathcal{F}}(\alpha_i,\hbar,0)$, $\mathcal{F}^r_{i;0}=\bar{\mathcal{F}}^r_{i;0}$ for all i and r. Suppose $d>0$ and we have shown that

$$\mathcal{F}^r_{i;d'} = \bar{\mathcal{F}}^r_{i;d'} \qquad \forall\, d'=0,1,\ldots,d-1,\; i\in[n],\; r. \tag{4.32}$$

Then, by (4.31),

$$\mathcal{F}(\alpha_i,\hbar,Q) \equiv \bar{\mathcal{F}}(\alpha_i,\hbar,Q) \pmod{Q^d} \;\; \forall\, i\in[n], \quad N_{\mathcal{F};d'} = N_{\bar{\mathcal{F}};d'} \;\; \forall\, d'=0,1,\ldots,d-1. \tag{4.33}$$

Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ agree modulo $\hbar^{-2}$, by (4.31) and the first equation in (4.33)

$$\delta\mathcal{F}(\alpha_i,\hbar,Q) \equiv Q^d \sum_{r=2}^{r=N} \delta\mathcal{F}^r_{i;d}\hbar^{-r} \pmod{Q^{d+1}} \qquad \forall\, i\in[n] \tag{4.34}$$

for some $N\in\mathbb{Z}^+$ and $\delta\mathcal{F}^r_{i;d}\in\mathbb{C}_\alpha$. Then,

$$N_{\mathcal{F};0}(\alpha_i,\hbar) = N_{\bar{\mathcal{F}};0}(\alpha_i,\hbar) = f_i\,, \quad \delta N_{d'}(\alpha_i,\hbar) = \begin{cases} 0, & \text{if } d'<d; \\ \Delta_d(\alpha_i,\hbar) \sum\limits_{r=2}^{r=N} \delta\mathcal{F}^r_{i;d}\hbar^{-r}, & \text{if } d'=d, \end{cases} \tag{4.35}$$

by (4.24), (4.33), and (4.34). Let $\delta E_d = E_{\mathcal{F};d} - E_{\bar{\mathcal{F}};d}$. Since

$$\delta E_d(\hbar,\alpha_i+d'\hbar) = 0 \;\; \forall\, d'=1,\ldots,d-1,\; i\in[n]$$

by (4.26) and (4.35) and $\delta E_d\in\mathbb{C}_\alpha[\hbar,\Omega]$ by Lemma 4.17,

$$\delta E_d(\hbar,\Omega) = \bigg(\prod_{d'=1}^{d-1}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-d'\hbar\big)\bigg)\cdot R_d(\hbar,\Omega)$$

for some $R_d\in\mathbb{C}_\alpha[\hbar,\Omega]$. Thus,

$$\delta E_d(\hbar,\alpha_i+d\hbar) = \bigg(\prod_{d'=1}^{d-1}\prod_{k=1}^{k=n}\big(\alpha_i+d\hbar-\alpha_k-d'\hbar\big)\bigg)\cdot R_d(\hbar,\alpha_i+d\hbar) = \hbar^{d-1}\tilde{R}_d(\hbar) \tag{4.36}$$

for some $\tilde{R}_d\in\mathbb{C}_\alpha[\hbar]$. On the other hand, by (4.26) and (4.35)

$$\begin{aligned} \delta E_d(\hbar,\alpha_i+d\hbar) = a\alpha_i\,\delta N_d(\alpha_i,\hbar)\cdot f_i &= a\alpha_i f_i\cdot\bigg(d!\hbar^d\prod_{r=1}^{r=d}\prod_{k\neq i}(\alpha_i-\alpha_k+r\hbar)\bigg)\sum_{r=2}^{r=N}\delta\mathcal{F}^r_{i;d}\hbar^{-r} \\ &= a\alpha_i f_i\cdot\bigg(d!\prod_{r=1}^{r=d}\prod_{k\neq i}(\alpha_i-\alpha_k+r\hbar)\bigg)\sum_{r=2}^{r=N}\delta\mathcal{F}^r_{i;d}\hbar^{d-r}. \end{aligned} \tag{4.37}$$

By (4.36) and (4.37),

$$\delta\mathcal{F}^r_{i;d}=0 \quad \forall\, r=2,\ldots,N,\ i\in[n].$$

Along with (4.34), this implies that (4.32) holds with d replaced by $d+1$. $\square$

4.5 Other algebraic observations

In this subsection we show that the recursion and polynomiality conditions of Subsection 4.4 are preserved under certain transformations and are satisfied by the function $\mathcal{Y}$ defined in (4.18). For $i,j\in[n]$ with $i\neq j$ and $d\in\mathbb{Z}^+$, let

$$\mathfrak{C}_i^j(d)=\frac{\prod_{r=1}^{ad}(a\alpha_i+r(\alpha_j-\alpha_i)/d)}{d\prod\limits_{\substack{r=1\\(r,k)\neq(d,j)}}^{r=d}\prod\limits_{k=1}^{k=n}(\alpha_i-\alpha_k+r(\alpha_j-\alpha_i)/d)}\in\mathbb{C}_\alpha. \tag{4.38}$$

Lemma 4.21. *Suppose $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][[Q]]$ is C-recursive and $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$. Then,*

(i) if $f\in\mathbb{C}_\alpha[[Q]]$, then $f\mathcal{Z}$ is C-recursive and $\Phi_{f\mathcal{Z}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$;

(ii) if $f\in Q\cdot\mathbb{C}_\alpha[[Q]]$, then $\bar{\mathcal{F}}\equiv e^{f/\hbar}\mathcal{F}$ is C-recursive and $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$;

(iii) if $g\in Q\cdot\mathbb{C}[[Q]]$ and

$$\bar{\mathcal{F}}(x,\hbar,Q)=e^{g(Q)x/\hbar}\mathcal{F}\big(x,\hbar,Qe^{g(Q)}\big),$$

then $\bar{\mathcal{F}}$ is C-recursive and $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[Q,z]]$.

Proof. (i) Since $\mathcal{F}$ is C-recursive and the multiplication by f preserves the structure of each of the terms in (4.31), $f\mathcal{F}$ is also C-recursive. Since

$$\Phi_{f\mathcal{F}}(\hbar,z,Q)=f\big(Qe^{\hbar z}\big)f(Q)\Phi_{\mathcal{F}}(\hbar,z,Q)$$

and $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, $\Phi_{f\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$.

(ii) Since the coefficient of Q^0 in f is 0, the multiplication by $e^{f(Q)/\hbar}$ preserves the structure of the first term on the right-hand side of (4.31). The (d,j)-summand in the last term becomes

$$\begin{aligned}e^{f(Q)/\hbar}\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big)&=\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\bar{\mathcal{F}}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big)\\&\quad+\Big(e^{f(Q)/\hbar}-e^{f(Q)/((\alpha_j-\alpha_i)/d)}\Big)\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big).\end{aligned}$$

Since $\mathcal{F}$ is C-recursive and

$$\frac{e^{f(Q)/\hbar}-e^{f(Q)/((\alpha_j-\alpha_i)/d)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\in\mathbb{C}_\alpha[\hbar,\hbar^{-1}]\big[[Q]\big],$$

it follows that $\bar{\mathcal{F}}$ is C-recursive as well. On the other hand,

$$\Phi_{\bar{\mathcal{F}}}(\hbar,z,Q)=e^{(f(Qe^{\hbar z})-f(Q))/\hbar}\Phi_{\mathcal{F}}(\hbar,z,Q). \tag{4.39}$$

Since $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ and

$$\big(f(Qe^{\hbar z})-f(Q)\big)/\hbar\in\mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big],$$

(4.39) implies that $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ as well.

(iii) Since the coefficient of Q^0 in g is 0, the operation of replacing Q with $Qe^{g(Q)}$ followed by multiplication by $e^{\alpha_i g(Q)/\hbar}$ preserves the structure of the first term on the right-hand side of (4.31). The (d,j)-summand in the last term becomes

$$\begin{aligned}e^{\alpha_i g(Q)/\hbar}\frac{C_i^j(d)Q^d e^{dg(Q)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Qe^{g(Q)}\big) &= \frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\bar{\mathcal{F}}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big)\\ &+\Big(e^{(\alpha_i/\hbar+d)g(Q)}-e^{(\alpha_j/((\alpha_j-\alpha_i)/d))g(Q)}\Big)\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Qe^{g(Q)}\big).\end{aligned}$$

Since $\mathcal{F}$ is C-recursive and

$$\frac{e^{(\alpha_i/\hbar+d)g(Q)}-e^{(\alpha_j/((\alpha_j-\alpha_i)/d))g(Q)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}}=\frac{d}{(\alpha_i+d\hbar)-\alpha_j}e^{g(Q)\frac{dz}{z-\alpha_i}}\Big|_{z=\alpha_j}^{z=\alpha_i+d\hbar}\in\mathbb{C}_\alpha[\hbar,\hbar^{-1}]\big[\big[Q\big]\big],$$

it follows that $\bar{\mathcal{F}}$ is C-recursive as well. On the other hand,

$$\Phi_{\bar{\mathcal{F}}}(\hbar,z,Q)=\Phi_{\mathcal{F}}\big(\hbar,\tilde{z},Qe^{g(Q)}\big),\qquad\text{where}\quad \tilde{z}=z+\frac{g(Qe^{\hbar z})-g(Q)}{\hbar}\in\mathbb{C}[\hbar]\big[\big[z,Q\big]\big].\tag{4.40}$$

Since $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, (4.40) implies that $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ as well. $\square$

Lemma 4.22. *The function $\mathcal{Y}$ defined by (4.18) satisfies the C-recursivity condition of Definition 4.18 with the collection of coefficients given by (4.38) and $\Phi_{\mathcal{Y}}\in\mathbb{C}_\alpha[\hbar][[z,q]]$.*

Proof. (1) In this argument, we view $\mathcal{Y}$ as an element of $\mathbb{C}_\alpha(x,\hbar)[[q]]$. By (4.18) and (4.38),

$$\frac{C_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{Y}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big)=\underset{z=\frac{\alpha_j-\alpha_i}{d}}{\mathfrak{R}}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}.$$

Thus, by the Residue Theorem on S^2,

$$\begin{aligned}\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{Y}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big) &= -\underset{z=\hbar,0,\infty}{\mathfrak{R}}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}\\ &=\mathcal{Y}(\alpha_i,\hbar,q)-\underset{z=0,\infty}{\mathfrak{R}}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}.\end{aligned}\tag{4.41}$$

On the other hand,

$$\underset{z=\infty}{\mathfrak{R}}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}=\begin{cases}1, & \text{if } a<n;\\ \sum\limits_{d=0}^{\infty}q^d\frac{(nd)!}{(d!)^n}, & \text{if } a=n;\end{cases}$$

$$\underset{z=0}{\mathfrak{R}}\Big\{\frac{1}{\hbar-z}[\![\mathcal{Y}_d(\alpha_i,z,q)]\!]_{q;d}\Big\}=\left[\!\!\left[\frac{1}{\hbar-z}\frac{\prod\limits_{r=1}^{r=ad}(a\alpha_i+rz)}{d!\prod\limits_{r=1}^{r=d}\prod\limits_{k\neq i}(\alpha_i-\alpha_k+rz)}\right]\!\!\right]_{z;d-1}\in\mathbb{Q}_\alpha[\hbar^{-1}].$$

Thus, (4.41) implies that $\mathcal{Y}$ satisfies the recursion (4.31).

(2) In this argument, we view $\mathcal{Y}$ as an element of $\mathbb{C}_\alpha[x][[\hbar^{-1},q]]$; in particular,

$$\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,qe^{\hbar z}\big)\mathcal{Y}(x,-\hbar,q)$$

viewed as a function of x has residues only at $x=\alpha_i$ with $i\in[n]$ and $x=\infty$. By (4.18),

$$\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Y}\big(\hbar,\alpha_i,qe^{\hbar z}\big)\mathcal{Y}(-\hbar,\alpha_i,q)=\mathfrak{R}_{x=\alpha_i}\left\{\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(\hbar,x,qe^{\hbar z}\big)\mathcal{Y}(-\hbar,x,q)\right\}.$$

Thus, by Exercise 4.15,

$$\begin{aligned}\Phi_{\mathcal{Y}}(\hbar,z,q)&=-\mathfrak{R}_{x=\infty}\left\{\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,qe^{\hbar z}\big)\mathcal{Y}(x,-\hbar,q)\right\}\\
&=\sum_{d_1,d_2=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^{n-2+p+(n-a)(d_1+d_2)}}{(n-2+p+(n-a)(d_1+d_2))!}q^{d_1+d_2}e^{d_1\hbar z}\\
&\qquad\left[\!\!\left[\frac{1}{\prod\limits_{k\neq i}(1-\alpha_k w)}\frac{\prod\limits_{r=1}^{r=ad_1}(a+r\hbar w)}{\prod\limits_{r=1}^{r=d_1}\prod\limits_{k=1}^{k=n}(1-(\alpha_k-r\hbar)w)}\cdot\frac{\prod\limits_{r=1}^{r=ad_2}(a-r\hbar w)}{\prod\limits_{r=1}^{r=d_2}\prod\limits_{k=1}^{k=n}(1-(\alpha_k+r\hbar)w)}\right]\!\!\right]_{w;p}.\end{aligned}$$

The (d_1,d_2,p)-th summand above is $q^{d_1+d_2}$ times an element of $\mathbb{C}_\alpha[\hbar]\big[\big[z\big]\big]$. Thus, $\Phi_{\mathcal{Y}}\in\mathbb{C}_\alpha[\hbar][[z,q]]$. $\square$

4.6 Proof of the recursion property for GW-invariants

As described in detail in [MirSym, Section 27.3], the fixed loci of the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$ are indexed by decorated graphs that have no loops. A graph consists of a set Ver of vertices and a collection Edg of edges, i.e. of two-element subsets of Ver. A loop in a graph (Ver, Edg) is a subset of Edg of the form

$$\big\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_N,v_1\}\big\},\qquad v_1,\ldots,v_N\in\mathrm{Ver},\ N\geq 1.$$

Neither of the three graphs in Figure 3 has a loop. A decorated graph with two marked points is a tuple

$$\Gamma=\big(\mathrm{Ver},\mathrm{Edg};\mu,\mathfrak{d},\eta\big),\tag{4.42}$$

where (Ver, Edg) is a graph and

$$\mu\colon\mathrm{Ver}\longrightarrow[n],\qquad \mathfrak{d}\colon\mathrm{Edg}\longrightarrow\mathbb{Z}^+,\quad\text{and}\quad \eta\colon\{1,2\}\longrightarrow\mathrm{Ver}$$

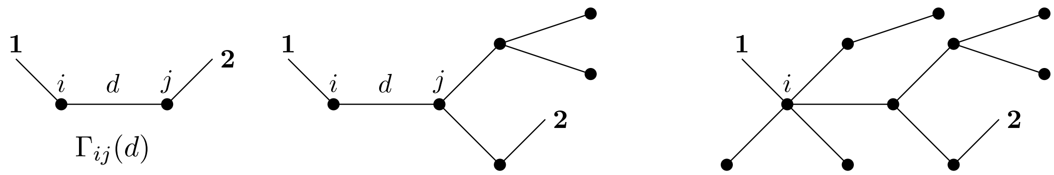

Figure 3: Two graphs of type $A_i(j;d)\subset A_i$ and a graph of type B_i

are maps such that

$$\mu(v_1) \neq \mu(v_2) \qquad \text{if} \quad \{v_1, v_2\} \in \text{Edg}. \tag{4.43}$$

In Figure 3, the values of the map μ on some of the vertices are indicated by letters next to those vertices. Similarly, the value of the map $\mathfrak{d}$ on one of the edges is indicated by a letter next to the edge. The two elements of the set $\{1,2\}$ are shown in bold face. They are linked by line segments to their images under η. By (4.43), no two consecutive vertex labels are the same and thus $j\neq i$.

The fixed locus $\mathcal{Z}_\Gamma$ of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$ corresponding to a decorated graph Γ consists of the stable maps f from a genus-zero nodal curve $\mathcal{C}_f$ with 2 marked points into $\mathbb{P}^{n-1}$ that satisfy the following conditions. The components of $\mathcal{C}_f$ on which the map f is not constant are rational and correspond to the edges of Γ. Furthermore, if $e=\{v_1,v_2\}$ is an edge, the restriction of f to the component $\mathcal{C}_{f,e}$ corresponding to e is a degree-$\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}^1_{\mu(v_1),\mu(v_2)} \subset \mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. The map $f|_{\mathcal{C}_{f,e}}$ is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, $f|_{\mathcal{C}_{f,e}}$ is unique up to isomorphism. The remaining, contracted, components of $\mathcal{C}_f$ are indexed by the vertices $v\in\text{Ver}$ such that

$$\text{val}(v) \equiv \big|\{v'\in\text{Ver}\colon \{v,v'\}\in\text{Edg}\}\big| + \big|\{i\in\{1,2\}\colon \eta(i)=v\}\big| \geq 3.$$

The map f takes such a component $\mathcal{C}_{f,v}$ to the fixed point $\mu(v)$. Thus,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \prod_{v\in\text{Ver}} \overline{\mathcal{M}}_{0,\text{val}(v)}, \tag{4.44}$$

where $\overline{\mathcal{M}}_{0,l}$ denotes the moduli space of stable genus-zero curves with l marked points. For the purposes of this definition, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are one-point spaces. For example, in the case of the last diagram in Figure 3,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \overline{\mathcal{M}}_{0,5}\times\overline{\mathcal{M}}_{0,3}^2\times\overline{\mathcal{M}}_{0,2}^2\times\overline{\mathcal{M}}_{0,1}^5 \approx \overline{\mathcal{M}}_{0,5}$$

is a fixed locus[9] in $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$ for some $d\geq 9$.

[9]after dividing by an appropriate automorphism group $\mathbb{A}_\Gamma$ as in [MirSym, Section 27.3]; in what follows $\int_{\mathcal{Z}_\Gamma}$ will denote integration over the orbifold $\mathcal{Z}_\Gamma/\mathbb{A}_\Gamma$

We will show that the function $\mathcal{Z}(x,\hbar,Q)$ defined in (4.17) is C-recursive in the sense of Definition 4.18 with the collection of coefficients given by (4.38) by determining the contribution to

$$\mathcal{Z}(\alpha_i,\hbar,Q)=1+\sum_{d=1}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{e(\mathcal{V}_d')}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\,, \tag{4.45}$$

from the $\mathbb{T}$-fixed loci of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$, with $d\geq 1$. Suppose Γ is a decorated graph as in (4.42) that contributes to (4.45), in the sense of the localization formula (1.28). By (1.29) and (1.14),

$$\mathrm{ev}_1^*\phi_i\big|_{\mathcal{Z}_\Gamma}=\prod_{k\neq i}\big(\alpha_{\mu(\eta(1))}-\alpha_k\big)=\delta_{i,\mu(\eta(1))}\prod_{k\neq i}(\alpha_i-\alpha_k),$$

where $\delta_{i,\mu(\eta(1))}$ is the Kronecker delta function. Thus, by (1.28), Γ does not contribute to $\mathcal{Z}(\hbar,\alpha_i,Q)$ unless $\mu(\eta(1))=i$, i.e. the marked point 1 of the map is taken to the point $P_i\in\mathbb{P}^{n-1}$. There are two types of graphs that do (or may) contribute to (4.45); they will be called A_i and B_i-types. In a graph of the A_i-type, the marked point 1 is attached to a vertex $v_0\in\mathrm{Ver}$ of valence two which is labeled i. In a graph of the B_i-type, the marked point 1 is attached to a vertex v_0 of valence at least 3, which is still labeled i. Examples of the two types are depicted in Figure 3.

Suppose Γ is a graph of type B_i and

$$\mathcal{Z}_\Gamma\subset\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d),$$

so that Γ contributes to the coefficient of Q^d in (4.45). In this case, the restriction of ψ_1 to $\mathcal{Z}_\Gamma$ is the pull-back of a ψ-class from the component $\overline{\mathcal{M}}_{0,\mathrm{val}(v_0)}$ in the decomposition (4.44). Since the $\mathbb{T}$-action on the corresponding tautological line bundle is trivial,

$$\psi_1^k\big|_{\mathcal{Z}_\Gamma}=0\qquad\forall\ k\geq d>\mathrm{val}(v_0)-3.$$

Thus, Γ contributes a polynomial in $\hbar^{-1}$, of degree at most d, to the coefficient of Q^d in (4.45). Therefore, the contributions of the loci of type B_i to (4.45) are accounted for by the middle term in (4.31).

A graph Γ as in (4.42) of type A_i has a unique vertex v joined to v_0. Denote by $A_i(j;d_0)$ the set of all graphs Γ of type A_i such that $\mu(v)=j$ and $\mathfrak{d}(\{v_0,v\})=d_0$, i.e. the unique vertex v of Γ joined to v_0 is mapped to $P_j\in\mathbb{P}^{n-1}$ and the edge $\{v_0,v\}$ corresponds to the d_0-fold cover of $\mathbb{P}^1_{ij}$ branched only over P_i and P_j. By (4.43),

$$A_i=\bigcup_{d_0=1}^{\infty}\bigcup_{j\neq i}A_i(j;d_0). \tag{4.46}$$

Suppose $\Gamma\in A_i(j;d_0)$ and v is the unique vertex joined to v_0 by an edge. We break Γ at v into two graphs:

(i) Γ_0 consisting of the vertices v_0 and v, the edge $\{v_0,v\}$, and marked points 1 and 2 attached to v_0 and v, respectively;

(ii) Γ_c consisting of all vertices and edges of Γ, other than the vertex v_0 and the edge $\{v_0,v\}$, with a new marked point attached to v;

,

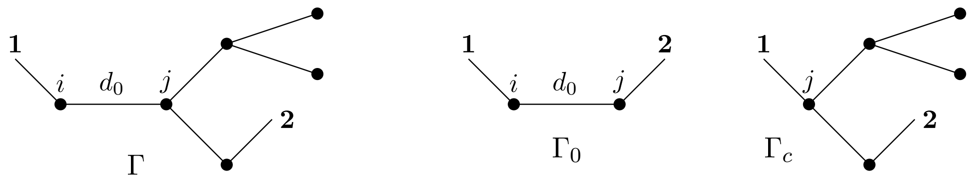

Figure 4: A graph of type $A_i^*(j;d_0)$ and its two subgraphs

see Figure 4. Let d_c denote the degree of Γ_c, i.e. the sum of all edge labels. By (4.44),

$$\mathcal{Z}_\Gamma \approx \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_c}. \tag{4.47}$$

Denote by π_0 and π_c the two component projection maps.

By [MirSym, Section 27.4],

$$\begin{aligned} \mathcal{V}'_{d_0+d_c}\big|_{\mathcal{Z}_\Gamma} &= \pi_0^*\mathcal{V}'_{d_0} \oplus \pi_c^*\mathcal{V}'_{d_c},\\ \frac{\mathcal{N}\mathcal{Z}_\Gamma}{T_{P_i}\mathbb{P}^{n-1}} &= \pi_0^*\bigg(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_0}}{T_{P_i}\mathbb{P}^{n-1}}\bigg) \oplus \pi_c^*\bigg(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_c}}{T_{P_j}\mathbb{P}^{n-1}}\bigg) \oplus \pi_0^*L_2 \otimes \pi_c^*L_1, \end{aligned} \tag{4.48}$$

where $L_2 \longrightarrow \mathcal{Z}_{\Gamma_0}$ and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_c}$ are the tautological tangent line bundles. Thus, by (1.17),

$$\begin{aligned} \frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma} &= \pi_0^*\bigg(\frac{\mathbf{e}(\mathcal{V}'_{d_0})}{\hbar-\psi_1}\bigg) \cdot \pi_c^*\Big(\mathbf{e}(\mathcal{V}'_{d_c})\Big),\\ \frac{\mathrm{ev}_1^*\phi_i|_{\mathcal{Z}_\Gamma}}{\mathbf{e}(N\mathcal{Z}_\Gamma)} &= \pi_0^*\bigg(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_0})}\bigg) \cdot \pi_c^*\bigg(\frac{\mathrm{ev}_1^*\phi_j}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_c})}\bigg) \cdot \frac{1}{\pi_0^*c_1(L_2) - \pi_c^*\psi_1}. \end{aligned} \tag{4.49}$$

By [MirSym, Sections 27.1, 27.2],

$$\begin{aligned} \mathbf{e}(\mathcal{V}'_{d_0})\big|_{\mathcal{Z}_{\Gamma_0}} &= \prod_{r=1}^{ad_0}\Big(a\alpha_i + r\frac{\alpha_j-\alpha_i}{d_0}\Big), \qquad \psi_1\big|_{\mathcal{Z}_{\Gamma_0}} = c_1(L_2) = \frac{\alpha_j-\alpha_i}{d_0},\\ \mathbf{e}(N\mathcal{Z}_{\Gamma_0}) &= (-1)^{d_0}\prod_{r=1}^{r=d_0}\Big(r\frac{\alpha_j-\alpha_i}{d_0}\Big)^2 \prod_{r=0}^{r=d_0}\prod_{k\neq i,j}\Big(\alpha_i-\alpha_k+r\frac{\alpha_j-\alpha_i}{d_0}\Big). \end{aligned} \tag{4.50}$$

Thus, using (1.17) and taking into the account the automorphism group $\mathbb{A}_{\Gamma_0}=\mathbb{Z}_{d_0}$ of $\mathcal{Z}_{\Gamma_0}$, we obtain

$$\int_{\mathcal{Z}_{\Gamma_0}} \frac{\mathbf{e}(\mathcal{V}'_{d_0})\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(N\mathcal{Z}_{\Gamma_0})} = \frac{\mathfrak{C}_i^j(d_0)}{\hbar - \frac{\alpha_j-\alpha_i}{d_0}}. \tag{4.51}$$

By (4.47), (4.50), and (4.51), the contribution of Γ to (4.45) is

$$\begin{aligned} &Q^{d_0+d_c}\int_{\mathcal{Z}_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma}\frac{1}{\mathbf{e}(N\mathcal{Z}_\Gamma)}\\ &\qquad = \frac{\mathfrak{C}_i^j(d_0)Q^{d_0}}{\hbar - \frac{\alpha_j-\alpha_i}{d_0}} \cdot \Bigg(\bigg\{Q^{d_c}\int_{\mathcal{Z}_{\Gamma_c}} \frac{\mathbf{e}(\mathcal{V}'_{d_c})\mathrm{ev}_1^*\phi_j}{\hbar-\psi_1}\frac{1}{\mathbf{e}(N\mathcal{Z}_{\Gamma_c})}\bigg\}\Big|_{\hbar=\frac{\alpha_j-\alpha_i}{d_0}}\Bigg). \end{aligned} \tag{4.52}$$

We next sum (4.52) over $\Gamma \in A_i(j;d_0)$. This is the same as summing the expression in the curly brackets over all 2-pointed graphs with the marked point 1 attached to a vertex v labeled j, i.e. all graphs of types A_j and B_j. By the localization formula (1.28), the sum of the terms in the curly brackets over all such graphs Γ_c is $\mathcal{Z}(Q,\alpha_j,\hbar)$. Thus,

$$\sum_{\Gamma\in A_i(j;d_0)} Q^{d_0+d_c}\int_{\mathcal{Z}_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma}\frac{1}{\mathbf{e}(N\mathcal{Z}_\Gamma)} = \frac{\mathfrak{C}_i^j(d_0)Q^{d_0}}{\hbar-\frac{\alpha_j-\alpha_i}{d_0}}\cdot\mathcal{Z}\big(\alpha_j,(\alpha_j-\alpha_i)/d_0,Q\big). \tag{4.53}$$

We conclude that $\mathcal{Z}(x,\hbar,Q)$ is $\mathfrak{C}$-recursive in the sense of Definition 4.18:

- the middle term in (4.31) consists of the contributions from the graphs of type B_i;
- the (d_0,j)-summand in (4.31) consists of the contributions from the graphs of type $A_i(j;d_0)$.

4.7 Proof of the polynomiality property for GW-invariants

In this subsection we use Lemma 4.23, which is proved in the next subsection, to show that the function $\mathcal{Z}(x,\hbar,Q)$ defined in (4.17) satisfies the polynomiality property of Lemma 4.17. The argument presented here is a more detailed version of Section 30.2 of [MirSym] that describes what is perhaps the most unexpected idea in [Gi].

We will denote the weight of the standard action of the one-torus $\mathbb{T}^1$ on $\mathbb{C}$ by $\hbar$. Let $\widetilde{\mathbb{T}}=\mathbb{T}^1\times\mathbb{T}$. By (1.2),

$$H^*_{\mathbb{T}^1}\approx\mathbb{C}[\hbar],\quad H^*_{\widetilde{\mathbb{T}}}\approx\mathbb{C}[\hbar,\alpha_1,\ldots,\alpha_n]\qquad\Longrightarrow\qquad \mathcal{H}^*_{\widetilde{\mathbb{T}}}\approx\mathbb{C}_\alpha(\hbar).$$

Throughout this subsection, $V=\mathbb{C}\oplus\mathbb{C}$ will denote the representation of $\mathbb{T}^1$ with the weights 0 and $-\hbar$. The induced action on $\mathbb{P}V$ has two fixed points:

$$q_1\equiv[1,0],\qquad q_2\equiv[0,1].$$

Let $\gamma_1\longrightarrow\mathbb{P}V$ be the tautological line bundle. Then,

$$\mathbf{e}(\gamma_1^*)\big|_{q_1}=0,\quad \mathbf{e}(\gamma_1^*)\big|_{q_2}=-\hbar,\quad \mathbf{e}(T_{q_1}\mathbb{P}V)=\hbar,\quad \mathbf{e}(T_{q_2}\mathbb{P}V)=-\hbar. \tag{4.54}$$

For each $d\in\mathbb{Z}^{\geq0}$, the action of $\widetilde{\mathbb{T}}$ on $\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ induces an action on

$$\overline{\mathfrak{X}}_d\equiv\mathbb{P}\big(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*\big).$$

It has $(d+1)n$ fixed points:

$$P_i(r)\equiv\big[\tilde{P}_i\otimes u^{d-r}v^r\big],\qquad i\in[n],\ r\in\{0\}\cup[d],$$

if (u,v) are the standard coordinates on V and $\tilde{P}_i\in\mathbb{C}^n$ is the i-th coordinate vector (so that $[\tilde{P}_i]=P_i\in\mathbb{P}^{n-1}$). Let

$$\Omega\equiv\mathbf{e}(\gamma^*)\in H^*_{\widetilde{\mathbb{T}}}\big(\overline{\mathfrak{X}}_d\big)$$

denote the equivariant hyperplane class.

For all $i\in[n]$ and $r \in \{0\}\cup[d]$,

$$\Omega|_{P_i(r)} = \alpha_i+r\hbar, \qquad \mathbf{e}(T_{P_i(r)}\overline{\mathfrak{X}}_d) = \bigg\{ \prod_{\substack{s=0\\(s,k)\neq(r,i)}}^{s=d}\prod_{k=1}^{k=n} (\Omega-\alpha_k-s\hbar)\bigg\}\bigg|_{\Omega=\alpha_i+r\hbar}. \tag{4.55}$$

[10]

Since

$$B_{\widetilde{\mathbb{T}}}\overline{\mathfrak{X}}_d = \mathbb{P}\big(B_{\widetilde{\mathbb{T}}}(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*)\big) \longrightarrow B\widetilde{\mathbb{T}} \qquad \text{and}$$
$$c\big(B_{\widetilde{\mathbb{T}}}(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*)\big) = \prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(1-(\alpha_k+s\hbar)\big) \in H^*\big(B\widetilde{\mathbb{T}}\big),$$

[11]

the $\widetilde{\mathbb{T}}$-equivariant cohomology of $\overline{\mathfrak{X}}_d$ is given by

$$\begin{aligned}
H^*_{\widetilde{\mathbb{T}}}\big(\overline{\mathfrak{X}}_d\big) \equiv H^*\big(B_{\widetilde{\mathbb{T}}}\overline{\mathfrak{X}}_d\big) &= H^*\big(B\widetilde{\mathbb{T}}\big)[\Omega]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-(\alpha_k+s\hbar)\big)\\
&\approx \mathbb{Q}\big[\Omega,\hbar,\alpha_1,\ldots,\alpha\big]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-s\hbar\big)\\
&\subset \mathbb{Q}_\alpha[\hbar,\Omega]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-s\hbar\big).
\end{aligned} \tag{4.56}$$

There is a natural $\widetilde{\mathbb{T}}$-equivariant morphism

$$\Theta\colon \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big) \longrightarrow \overline{\mathfrak{X}}_d.$$

A general element b of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$ determines a map

$$(f,g)\colon \mathbb{P}^1 \longrightarrow (\mathbb{P}V,\mathbb{P}^{n-1}),$$

up to an automorphism of the domain $\mathbb{P}^1$. Thus, the map

$$g\circ f^{-1}\colon \mathbb{P}V \longrightarrow \mathbb{P}^{n-1}$$

is well-defined and determines an element $\Theta(b)\in\overline{\mathfrak{X}}_d$. The map Θ extends continuously over the boundary of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$; see Exercise 4.24.[12] We denote the restriction of Θ to the smooth substack

$$\mathfrak{X}_d \equiv \big\{b\in\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)\colon \mathrm{ev}_1(b)\in q_1\times\mathbb{P}^{n-1},\ \mathrm{ev}_2(b)\in q_2\times\mathbb{P}^{n-1}\big\} \tag{4.57}$$

[10]The weight (i.e. negative first chern class) of the $\widetilde{\mathbb{T}}$-action on the line $P_i(r)\subset\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ is $\alpha_i+r\hbar$. The tangent bundle of $\overline{\mathfrak{X}}_d$ at $P_i(r)$ is the direct sum of the lines $P_i(r)^*\otimes P_k(s)$ with $(k,s)\neq(i,r)$.

[11]The vector space $\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ is the direct sum of the one-dimensional representations $P_k(s)$ of $\widetilde{\mathbb{T}}$.

[12]For a complete algebraic proof, see [LLY, Lemma 2.6].

of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$ by θ_d, or simply by θ whenever there is no ambiguity.

Let

$$\pi\colon \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)\longrightarrow \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}^{n-1},d\big)$$

be the natural projection map. In light of (4.56), the following lemma implies that

$$\Phi_{\mathcal{Z}}(\hbar,z,Q)\in\mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big].$$

Lemma 4.23. *With $\mathcal{Z}(\hbar,x,Q)$ as in (4.17) and Φ as in (4.20),*

$$\Phi_{\mathcal{Z}}(\hbar,z,Q)=\sum_{d=0}^{\infty}Q^d\int_{\mathfrak{X}_d}e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_d). \tag{4.58}$$

Exercise 4.24. Suppose $[f,g_s]$ is a sequence of elements in $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ so that the domain of each map is $\mathbb{P}^1$ and the sequence converges to some element

$$[\Sigma,\tilde{f},\tilde{g}]\in\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)).$$

There is a unique irreducible component $\Sigma_0\approx\mathbb{P}^1$ of Σ so that $\tilde{f}_0\equiv\tilde{f}|_{\Sigma_0}$ is not constant and can thus be chosen to be f; let $\tilde{g}_0=\tilde{g}|_{\Sigma_0}$. The maps

$$g_s\circ f^{-1},\ \tilde{g}_0\circ f^{-1}\colon\mathbb{P}V\longrightarrow\mathbb{P}^{n-1}$$

correspond to n-tuples of homogeneous polynomials

$$[\mathbf{R}_s]=\big[R_{s;1},\ldots,R_{s;n}\big]\in\overline{\mathfrak{X}}_d\,,\qquad [\tilde{\mathbf{S}}]=\big[\tilde{S}_1,\ldots,\tilde{S}_n\big]\in\overline{\mathfrak{X}}_{\tilde{d}_0},$$

with no common factors. Let

$$[\mathbf{R}]\equiv\big[(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_1,\ldots,(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_n\big]\in\overline{\mathfrak{X}}_d\,,$$

be a limit point of $\{[\mathbf{R}_s]\}$, i.e. the limit of a subsequence of $\{[\mathbf{R}_s]\}$, where $d_l\in\mathbb{Z}^+$,

$$[u_1,v_1],\ldots,[u_m,v_m]\in\mathbb{P}V$$

are distinct points, and the tuple of homogeneous polynomials

$$[\mathbf{S}]\equiv[S_1,\ldots,S_n]\in\overline{\mathfrak{X}}_{d_0}$$

has no common factor. In particular, $d_0+\ldots+d_m=d$. Show that

(a) $\tilde{d}_0=d_0$, $[\tilde{\mathbf{S}}]=[\mathbf{S}]$, Σ consists of Σ_0 along with connected rational curves $\Sigma_1,\ldots,\Sigma_m$ attached to Σ_0 at $f^{-1}([u_1,v_1]),\ldots,f^{-1}([u_m,v_m])$, and the degree of $\tilde{g}|_{\Sigma_l}$ is d_l;

(b) show that the map Θ extends continuously over $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ as claimed above, and the extension is $\tilde{\mathbb{T}}$-equivariant.

`Remark:` The first part is the hard one, as it requires a hands-on understanding of the topology of $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ either from the algebro-geometric or symplectic point of view. First try the case when $m=1$, $d_1=1$, and $v_1=0$.

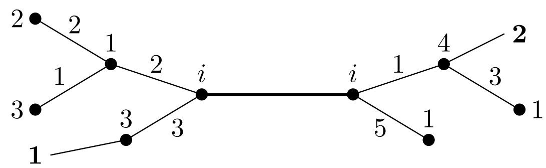

Figure 5: A graph representing a fixed locus in $\mathfrak{X}_d$; $i\!\neq\!1,3,4$

4.8 Proof of Lemma 4.23

In this subsection we use the localization formula (1.28) to prove Lemma 4.23. We show that each fixed locus of the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}_d$ contributing to the right-hand side of (4.58) corresponds to a pair (Γ_1,Γ_2) of a graphs, with Γ_1 and Γ_2 contributing to $\mathcal{Z}(\alpha_i,\hbar,Qe^{\hbar z})$ and $\mathcal{Z}(\alpha_i,-\hbar,Q)$, respectively, for some $i\!\in\![n]$.

Similarly to Subsection 4.6, the fixed loci of the $\widetilde{\mathbb{T}}$-action on $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\!\times\!\mathbb{P}^{n-1},(d',d)\big)$ correspond to decorated graphs Γ with 2 marked points and no loops. Each edge should be labeled by a pair of integers, indicating the degrees of the corresponding maps in $\mathbb{P}V$ and in $\mathbb{P}^{n-1}$. Each vertex should be labeled either $(1,j)$ or $(2,j)$ for some $j\!\in\![n]$, indicating the fixed point, (q_1,P_j) or (q_2,P_j), to which the vertex is mapped. No two consecutive vertex labels are the same, but if two consecutive vertex labels differ in the k-th component, with $k\!=\!1,2$, the k-th component of the label for edge connecting them must be nonzero.

The situation for the $\widetilde{\mathbb{T}}$-action on

$$\mathfrak{X}_d\subset\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\!\times\!\mathbb{P}^{n-1},(1,d)\big)$$

is simpler, however. There is a unique edge of positive $\mathbb{P}V$-degree. We draw it as a thick horizontal line. The first component of all other edge labels must be 0; so we drop it. The first components of the vertex labels change only when the thick edge is crossed. Thus, we drop the first components of the vertex labels as well, with the convention that these components are 1 on the left side of the thick edge and 2 on the right. In particular, the vertices to the left of the thick edge (including the left endpoint) lie in $q_1\!\times\!\mathbb{P}^{n-1}$ and the vertices to its right lie in $q_2\!\times\!\mathbb{P}^{n-1}$. Thus, by (4.57), the marked point 1 is attached to a vertex to the left of the thick edge and the marked point 2 is attached to a vertex to the right. Finally, both vertices of the thick edge have the same (remaining, second) label $i\!\in\![n]$. Let $\mathcal{A}_i$ denote the set of graphs as above so that the two endpoints of the thick edge are labeled i; see Figure 5.

If $\Gamma\!\in\!\mathcal{A}_i$, we break it into three sub-graphs:

(i) Γ_1 consisting of all vertices and edges of Γ to the left of the thick edge, including its left vertex v_1, and a new marked point attached to v_1; we label the new marked point 1, while re-labeling the old marked point 1 by 2;

(ii) Γ_0 consisting of the thick edge, its two vertices v_1 and v_2, and new marked points 1 and 2 attached to v_1 and v_2, respectively;

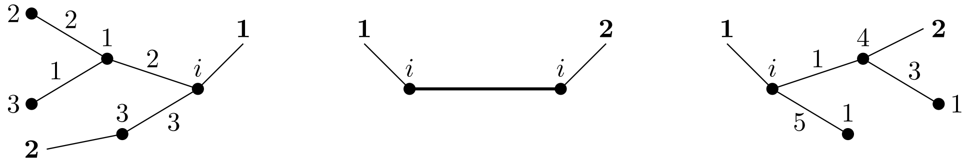

Figure 6: The three sub-graphs of the graph in Figure 5

(iii) Γ_2 consisting of all vertices and edges of Γ to the right of the thick edge, including its right vertex v_2, and a new marked point 1 attached to v_2;

see Figure 6. The fixed locus in $\mathfrak{X}_d$ corresponding to Γ is then

$$\mathcal{Z}_\Gamma \approx \mathcal{Z}_{\Gamma_1} \times \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_2}. \tag{4.59}$$

The middle term is a single point. Let π_1, π_0, and π_2 denote the three component projection maps. Denote by d_1 and d_2 the degrees of Γ_1 and Γ_2, i.e.

$$\mathcal{Z}_{\Gamma_1} \subset \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d_1), \qquad \mathcal{Z}_{\Gamma_2} \subset \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d_2).$$

The exceptional case for the first statement is $d_1 = 0$, in which case the corresponding moduli space does not exist.

Suppose $\Gamma \in \mathcal{A}_i$, d_1 and d_2 are as above, and $d_1 > 0$. Similarly to (4.48),

$$\begin{aligned} \pi^*\mathcal{V}_{d_1+d_2}\big|_{\mathcal{Z}_\Gamma} &= \mathcal{L}_{P_i} \oplus \pi_1^*\mathcal{V}'_{d_1} \oplus \pi_2^*\mathcal{V}''_{d_2}, \\ \frac{\mathcal{N}\mathcal{Z}_\Gamma}{T_{P_i}\mathbb{P}^{n-1}} &= \pi_1^*\left(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_1}}{T_{P_i}\mathbb{P}^{n-1}}\right) \oplus \pi_2^*\left(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_2}}{T_{P_i}\mathbb{P}^{n-1}}\right) \oplus \pi_1^*L_1 \otimes \pi_0^*L_1 \oplus \pi_0^*L_2 \otimes \pi_2^*L_1, \end{aligned} \tag{4.60}$$

where $\mathcal{N}\mathcal{Z}_\Gamma \longrightarrow \mathcal{Z}_\Gamma$ is the normal bundle of $\mathcal{Z}_\Gamma$ in $\mathfrak{X}_d$ and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_1}$, $L_1, L_2 \longrightarrow \mathcal{Z}_{\Gamma_0}$, and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_2}$ are the tautological tangent line bundles. We note that

$$L_1 = T_{q_1}\mathbb{P}^1 \quad \text{and} \quad L_2 = T_{q_2}\mathbb{P}^1 \qquad \text{on} \quad \mathcal{Z}_{\Gamma_0}.$$

Thus, by (4.60), (1.17), and (4.54),

$$\begin{aligned} \pi^*\big(\mathbf{e}(\mathcal{V}_{d_1+d_2})\big)\Big|_{\mathcal{Z}_\Gamma} &= a\alpha_i \cdot \pi_1^*\mathbf{e}(\mathcal{V}'_{d_1}) \cdot \pi_2^*\big(\mathbf{e}(\mathcal{V}'_{d_2})\big), \\ \frac{\prod_{k\neq i}(\alpha_i - \alpha_k)}{\mathbf{e}(N\mathcal{Z}_\Gamma)} &= \pi_1^*\left(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\right) \cdot \pi_2^*\left(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\right) \cdot \frac{1}{\hbar - \pi_1^*\psi_1} \cdot \frac{1}{(-\hbar) - \pi_2^*\psi_1}. \end{aligned} \tag{4.61}$$

The map θ takes the locus $\mathcal{Z}_\Gamma$ to a fixed point $P_k(r) \in \overline{\mathfrak{X}}_d$. It is immediate that $k = i$. By Exercise 4.24, $r = d_1$. Thus, by the first identity in (4.55),

$$\theta^*\Omega\big|_{\mathcal{Z}_\Gamma} = \alpha_i + d_1\hbar.$$

Combining (4.59) and (4.61) with this observation, we obtain

$$\begin{aligned}\int_{\mathcal{Z}_\Gamma}\frac{e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_{d_1+d_2})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}&=\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\\&\times\left\{e^{d_1\hbar z}\int_{\mathcal{Z}_{\Gamma_1}}\frac{\mathbf{e}(\mathcal{V}'_{d_1})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\right\}\left\{\int_{\mathcal{Z}_{\Gamma_2}}\frac{\mathbf{e}(\mathcal{V}'_{d_2})}{(-\hbar)-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\right\}.\end{aligned}\tag{4.62}$$

We note that this identity remains valid for $d_1=0$ if we set the term in the first curly brackets to 1 for $d_1=0$.

We now sum up (4.62), multiplied by $Q^{d_1+d_2}$, over all $\Gamma\in\mathcal{A}_i$. This is the same as summing over all pairs (Γ_1,Γ_2) of graphs such that

(1) Γ_1 is a 2-pointed graph of a degree $d_1\geq 0$ such that the marked point 1 is attached to the vertex labeled i;

(2) Γ_2 is an 2-pointed graph of a degree $d_2\geq 0$ such that the marked point 1 is attached to the vertex labeled i.

By the localization formula (1.28),

$$\begin{aligned}\sum_{\Gamma_1}Q^{d_1}\left\{e^{d_1\hbar z}\int_{\mathcal{Z}_{\Gamma_1}}\frac{\mathbf{e}(\mathcal{V}'_{d_1})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\right\}&=1+\sum_{d=1}^{\infty}(Qe^{\hbar z})^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}'_d)}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\\&=\mathcal{Z}\big(\alpha_i,\hbar,Qe^{\hbar z}\big);\\\sum_{\Gamma_2}Q^{d_2}\left\{\int_{\mathcal{Z}_{\Gamma_2}}\frac{\mathbf{e}(\mathcal{V}'_{d_2})\mathrm{ev}_1^*\phi_i}{(-\hbar)-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\right\}&=\sum_{d=0}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}'_d)}{(-\hbar)-\psi_1}\mathrm{ev}_1^*\phi_i\\&=\mathcal{Z}\big(\alpha_i,-\hbar,Q\big).\end{aligned}\tag{4.63}$$

Finally, by (1.28), (4.62), and (4.63),

$$\sum_{d=0}^{\infty}Q^d\int_{\mathfrak{X}_d}e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_d)=\sum_{i=1}^{i=n}\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Z}\big(\alpha_i,\hbar,Qe^{\hbar z}\big)\mathcal{Z}\big(\alpha_i,-\hbar,Q\big)=\Phi_{\mathcal{Z}}(\hbar,z,Q),$$

as claimed in (4.58).

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