Notes on categorical logic

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1 Introduction

Broadly speaking,[1] logic is the study of the mapping

$$\text{syntax} \longrightarrow \text{semantics}$$

taking syntactical expressions (e.g., the statement "$\forall x\, \exists y\, (x \cdot y = 1)$") to their meanings or "values" (e.g., 1 or "True" in every group). A central question in logic is: to what extent is this mapping invertible? That is, how well does mathematical syntax describe its intended semantics? Results addressing aspects of this question fall under the general heading of *completeness* in logic. Categorical logic is a particularly disciplined way of approaching the general completeness question, which is capable of fully answering it in the best cases: we can prove, for certain well-behaved logics, that the syntax is "completely equivalent" to the semantics.

We will be studying two general classes of logics:

- In **propositional logic(s)**, there is only one type of syntactical expression, namely those denoting statements; such expressions are called **formulas** in logic. We start with **atomic formula symbols** $A, B, C, \dots$, which we may combine using the usual Boolean connectives to form compound formulas like $A \wedge (B \vee \neg C)$. Semantics for propositional logic assigns truth values (0 or 1) to every formula. Propositional logic is the baby version of

- **first-order logic(s)**, in which statements are about elements of mathematical structures (like "$\forall x\, \exists y\, (x \cdot y = 1)$" from above). There are two types of expressions: formulas, and **terms**, which denote elements (like $x, x \cdot y, 1$). We now start with **operation symbols** or **function symbols** (like $\cdot, 1$) which are used to build terms, as well as **relation symbols** (like $=, \leq$) which are used to build atomic statements about terms (like $x \cdot y = 1$) which may then be combined using connectives as well as quantifiers ($\forall, \exists$) to form compound formulas. Semantics for first-order logic assigns sets, operations, and relations to the syntax.

Let us introduce some general logical terminology:

- A **signature** or **language** $\mathcal{L}$ is the set of meaningless symbols from which more complicated expressions are constructed: the atomic formulas in propositional logic; the function and relation symbols in first-order logic.

- A **model** $\mathcal{M}$ is a possible semantics for a given language: an assignment of truth values in propositional logic, or of sets, operations, and relations in first-order logic.

[1] There are some subfields of logic (e.g., structural proof theory) that are arguably all about syntax.

- A **theory** $\mathcal{T}$ is a set of formulas which are assumed to be true. Theories are useful for restricting attention to possible semantics in which the symbols in the language are not completely arbitrary. A model $\mathcal{M}$ is a **model of** $\mathcal{T}$ if it obeys the formulas in $\mathcal{T}$.

Thus, for example, the language of groups is given by $\{\cdot, 1, {}^{-1}\}$, the theory of groups consists of formulas like $\forall x\, \exists y\, (x \cdot y = 1)$, and a model of the theory of groups is a group.

Another important syntactical component of logic is given by a **proof system**, consisting of **inference rules** that allow us to deduce formulas from other formulas. For example, here are two inference rules:

$$\frac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \theta}{\phi \Rightarrow \theta} \qquad\qquad \frac{}{\phi \wedge \psi \Rightarrow \phi}$$

The first rule says that from $\phi \Rightarrow \psi$ and $\psi \Rightarrow \theta$, we may deduce $\phi \Rightarrow \theta$. The second rule says that $\phi \wedge \psi \Rightarrow \phi$ may be deduced from no assumptions. The inference rules are designed to be few, simple, and obviously true under all possible semantics, i.e., in all models.

The key feature of categorical logic (and its predecessor for propositional logic, known as *algebraic logic*) is the factorization

$$\begin{tikzcd}
 & \text{syntax/provable equivalence} =: \text{syntactic algebra} \arrow[dr, dashed] & \\
\text{syntax} \arrow[ur, dashed, two heads] \arrow[rr] & & \text{semantics,}
\end{tikzcd}$$

which is well-defined since provable equivalence implies equivalence in all models. The idea is that quotienting by provable equivalence discards irrelevant syntactical details (e.g., the fact that the formulas $\phi \wedge \psi$ and $\psi \wedge \phi$ are technically distinct, as finite sequences of symbols), while remembering logically relevant aspects of the syntax: all of the logical operations (connectives and quantifiers) also respect provable equivalence, hence descend to algebraic operations on the quotient.

More precisely, given a theory $\mathcal{T}$, let

$$\langle \mathcal{T} \rangle := \{\text{expressions}\}/\mathcal{T}\text{-provable equivalence},$$

equipped with algebraic operations induced by the logical operations on formulas. The resulting algebraic structure goes by a variety of names (depending on the community); we will generally refer to it as the **syntactic algebra** of $\mathcal{T}$. In propositional logic, it is a Boolean algebra (or similar); in first-order logic, it is best regarded as a certain type of category equipped with additional structure (a *pretopos* or similar).

In general, taking a set of expressions built from some atomic symbols (like those of a language $\mathcal{L}$) and then quotienting by an equivalence relation to enforce some laws (like forcing formulas in $\mathcal{T}$ to become true) is the construction of an *algebra presented by generators and relations* (think group presentations). In other words,

$$\langle \mathcal{T} \rangle = \text{algebra presented by generators } \mathcal{L}, \text{ relations } \mathcal{T}$$

(the type of algebra depending on the type of logic). On the other hand, every algebra has some presentation (take all elements as generators, and all relations which hold between them). Thus, the syntactic view of a logic, consisting of languages, theories, and proofs, is equivalent to the study of the corresponding kind of algebras:

$$\text{syntax} \underset{\text{presentation}}{\overset{\text{syntactic algebra}}{\rightleftarrows}} \text{algebra}$$

propositional logic	Boolean algebras, etc.
first-order logic	pretoposes, etc.

To connect back to semantics: by the universal property of presentations, a model of $\mathcal{T}$, i.e., an assignment of "values" to the generators in $\mathcal{L}$ making the relations in $\mathcal{T}$ true, is the same thing as a homomorphism from $\langle \mathcal{T} \rangle$ to the algebra of "values", namely the 2-element Boolean algebra $2 = \{0, 1\}$ in propositional logic, or the category of sets Set (consisting of sets, functions, and relations) in first-order logic. Note that in each case, the algebra of "values" is a particularly simple nontrivial algebra of the corresponding type. In general, the set of homomorphisms from an algebra A to a fixed, simple algebra can be regarded as a "dual space" A^* of A (think dual vector space, dual Banach space, Pontryagin dual group). Thus

$$\{\text{models of } \mathcal{T}\} \cong \langle \mathcal{T} \rangle^* = \text{dual of } \langle \mathcal{T} \rangle.$$

Whenever we are considering duals of some type of structure A, it is natural to ask whether the dual A^* may regarded as some kind of (e.g., topological) "space" from which A may be recovered as the "double dual" A^{**} of all structure-preserving (e.g., continuous) functions on A^*. In our logical context, this means that the syntactic algebra $\langle \mathcal{T} \rangle$ of a theory $\mathcal{T}$ may be recovered from the space of models via

$$\langle \mathcal{T} \rangle \cong \langle \mathcal{T} \rangle^{**} \cong \{\text{models of } \mathcal{T}\}^* = \{\text{functions on } \{\text{models of } \mathcal{T}\}\}.$$

When this is possible, we say that we have a **duality theorem** or **strong (conceptual) completeness theorem** for the logic in question.

- For propositional logic, this is given by the classical Stone duality between Boolean algebras and Stone (i.e., compact Hausdorff zero-dimensional) spaces. In other words, we may recover the syntax of $\mathcal{T}$, in the form of $\langle \mathcal{T} \rangle$, from the topological space of models of $\mathcal{T}$.

- For first-order logic, this is a deep theorem of Makkai from the '80s, which says we may recover $\langle \mathcal{T} \rangle$ from the category of models of $\mathcal{T}$ equipped with ultraproduct operations.

When a duality theorem holds, we thus have three equivalent views of a logic:

$$\text{syntax} \underset{\text{presentation}}{\overset{\text{syntactic algebra}}{\rightleftarrows}} \text{algebra} \xleftrightarrow{\text{duality}} \text{semantics}$$

In this course, we will study this three-way correspondence for a variety of logics, beginning with propositional logic before moving on to first-order logic. Our end goal will be a proof (or proof sketch) of Makkai's duality theorem. Along the way, we will also study some well-behaved subsets of propositional and first-order logic and their corresponding duality theorems.

Part I
Categorical background

2 Categories

A **category** C consists of the following data:

- a set C_0 of **objects** (usually we just write $X \in \mathsf{C}$ to mean $X \in \mathsf{C}_0$);
- for each two objects $X, Y \in \mathsf{C}$, a **hom-set** $\mathrm{Hom}_{\mathsf{C}}(X, Y) = \mathsf{C}(X, Y)$ of **morphisms** $f : X \to Y$ with **domain** X and **codomain** Y;
- for each object $X \in \mathsf{C}$, an **identity morphism** $1_X : X \to X$;
- for any three objects $X, Y, Z \in \mathsf{C}$, a **composition operation**

$$\begin{aligned} \mathsf{C}(Y,Z) \times \mathsf{C}(X,Y) &\longrightarrow \mathsf{C}(X,Z) \\ (g, f) &\longmapsto g \circ f; \end{aligned}$$

- satisfying the **unit** and **associativity** laws:

$$\begin{aligned} 1 \circ f &= f, \\ f \circ 1 &= f, \\ (h \circ g) \circ f &= h \circ (g \circ f) \end{aligned}$$

  whenever the domains and codomains match up.

Example 2.1. The motivating examples of categories consist of classes of mathematical structures (as objects) and homomorphisms between them (as morphisms):[2]

- Set := category of sets and functions
- Grp := category of groups and group homomorphisms
- Vec := category of vector spaces (over a fixed field, say $\mathbb{R}$) and linear maps
- Pos := category of posets and monotone (= order-preserving) maps
- Top := category of topological spaces and continuous maps

Example 2.2. A **directed multigraph** (or **quiver**) G consists of a set of G_0 of **objects** (or **vertices**), together with for each two $X, Y \in \mathsf{G}_0$ a set $\mathsf{G}(X, Y)$ of **morphisms** (or **edges**) from X to Y. Thus, a category can be thought of as a directed multigraph equipped with a unital and associative composition operation.

Conversely, for any directed multigraph G, we may build its **path category** (or **free category**) $\langle \mathsf{G} \rangle$, with the same objects as G but with morphisms $x \to y$ now consisting of finite sequences of edges $x = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} x_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} x_n = y$ in G, composed via concatenation. In other words, we freely add identity and composition to G (akin to free groups or free monoids).

[2] The "sets" of objects of these categories are actually proper classes. We will say more about foundational issues such as these in Section 7.

Exercise 2.3. Describe the universal property of free categories (see Section 10).

Example 2.4. For any group (or monoid) G, we can regard G as a category with one object whose morphisms to itself are the elements of G. Often, we abuse notation and also refer to this one-object category as G. However, since we typically think of the objects of a category as its "elements", it is convenient to have a different notation for the category associated to a group G: we call it $\mathsf{B}G$. Thus $\mathsf{B}G$ has:

- objects: only one, denoted (say) $*$;
- morphisms $g : * \to *$: group elements $g \in G$;
- composition and identity as in G.

Conversely, if a category C has a single object $*$, then $\mathsf{C} = \mathsf{B}(\mathsf{C}(*, *))$.

Thus, one good intuition for categories in general is that they are "many-object monoids". For example, for a structure (e.g., vector space) A, we have the **endomorphism monoid** $\mathrm{End}(A)$ of all homomorphisms $A \to A$; whereas for a whole collection of structures, we have the category of all homomorphisms between them.

For a category C and morphism $f : X \to Y \in \mathsf{C}$, the **inverse** of f, if it exists, is the unique morphism $f^{-1} : Y \to X$ such that

$$\begin{aligned} f^{-1} \circ f &= 1_X, \\ f \circ f^{-1} &= 1_Y. \end{aligned}$$

A morphism with an inverse is also called an **isomorphism**. A category C in which every morphism is an isomorphism is called a **groupoid**. Thus, $\mathsf{B}G$ is a groupoid if G is a group; and every one-object groupoid arises in this way, so that a groupoid may be thought of as a "many-object group".

Example 2.5. For any category C, let $\mathsf{Core}(\mathsf{C})$ be the **groupoid core** of C, with the same objects as C but with only the isomorphisms in C.

- $\mathsf{Core}(\mathsf{Set})$ = sets and bijections
- $\mathsf{Core}(\mathsf{Grp})$ = groups and group isomorphisms

Another very useful "small" type of category (not consisting of a "universe of structures") is

Example 2.6. For any poset $P = (P, \leq)$, we may regard P as a category whose objects are the elements $x \in P$, and with a unique morphism $x \to y$ if $x \leq y$ (and no morphism $x \to y$ otherwise).

- reflexivity of $\leq \implies$ there is an identity $1_x : x \to x$ for each x;
- transitivity of $\leq \implies$ composition.

More generally, $\leq$ can be a **preorder** (i.e., only reflexive and transitive, not requiring that $x \leq y \leq x \implies x = y$).

Exercise 2.7. Which preorders are groupoids?

A good heuristic to keep in mind is that general categorical notions tend to be the "join" or "amalgamation" of their special cases for groups (or monoids) and posets, in that these two special cases often reveal distinct key features of the general notion, and understanding these two special cases often goes a long way towards understanding the general notion. We'll see several examples of this (as well as a few counterexamples).

Example 2.8. For a set X, we have a **discrete category**, also denoted X, whose objects are elements of X and with only identity morphisms. (So we can regard X first as a discrete poset, then regard that poset as a category.)

Example 2.9. For any category C, there is an **opposite category** C^{op}, with the same objects as C, but with $\mathsf{C}^{\mathrm{op}}(X, Y) := \mathsf{C}(Y, X)$ and composition given by the reverse of composition in C.

For a group, this is the same as taking the opposite group (same elements, opposite multiplication). For a poset, this is the same as taking the opposite partial order.

Example 2.10. For two categories C, D, there is a **product category** $\mathsf{C} \times \mathsf{D}$, with

- objects $(\mathsf{C} \times \mathsf{D})_0 := \mathsf{C}_0 \times \mathsf{D}_0$, i.e., pairs (X, Y) of objects $X \in \mathsf{C}$ and $Y \in \mathsf{D}$;
- morphisms $(\mathsf{C} \times \mathsf{D})((X_1, Y_1), (X_2, Y_2)) := \mathsf{C}(X_1, X_2) \times \mathsf{D}(Y_1, Y_2)$;
- identity and composition given coordinatewise as in C, D.

Exercise 2.11. Verify that for groups and posets, the product category coincides with the usual product.

3 Functors

A **functor** $F : \mathsf{C} \to \mathsf{D}$ is a "homomorphism of categories", consisting of:

- a function $F_0 : \mathsf{C}_0 \to \mathsf{D}_0$ (usually we just write $F(X) := F_0(X)$);
- for each $X, Y \in \mathsf{C}$, a function $F = F_{X,Y} : \mathsf{C}(X, Y) \to \mathsf{D}(F(X), F(Y))$;
- preserving identity and composition:

$$\begin{aligned} F(1_X) &= 1_{F(X)}; \\ F(g \circ f) &= F(g) \circ F(f). \end{aligned}$$

In other words, a functor is a homomorphism of directed multigraphs preserving $\circ, 1$.

Example 3.1. Let C be a category of sets equipped with some type of structure (e.g., groups, topological spaces), and all functions preserving said structure. Then we have a **forgetful functor** $\mathsf{C} \to \mathsf{Set}$, taking each structure $X \in \mathsf{C}$ to its underlying set (and each structure-preserving function to itself, regarded now as just a function). Note that this forgetful functor is (by abuse of notation) often hidden: e.g., for a topological space X, we often also denote the underlying set by X.

More generally, if C is a category of sets equipped with some structure, and D is a category of sets equipped with even more structure, then we have a forgetful functor $\mathsf{D} \to \mathsf{C}$. For example, there are two different forgetful functors

$$\mathsf{Ring} \longrightarrow \mathsf{Mon}$$

taking a ring to its underlying additive or multiplicative monoid.

Example 3.2. Of course, we have a category Cat of categories and functors,[3] with a forgetful functor $(-)_0 : \mathsf{Cat} \to \mathsf{Set}$.

Exercise 3.3. Let $\mathsf{Gpd} \subseteq \mathsf{Cat}$ be the subcategory of groupoids. Extend Core to a functor $\mathsf{Cat} \to \mathsf{Gpd}$.

Example 3.4. A **subcategory** of a category C is just what it sounds like: a category D whose objects and morphisms are subsets of those of C, with the same operations as in C. For a subcategory $\mathsf{D} \to \mathsf{C}$, the inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ is a functor. We can regard this as a "forgetful functor" which forgets the property of being in the smaller category D. For example, $\mathsf{Grp} \to \mathsf{Mon}$ is (depending on definitions of group and monoid) such an inclusion functor.

Example 3.5. A functor $F : G \to H$ between groups is just a group homomorphism.

A functor $F : P \to Q$ between posets is an order-preserving map (i.e., $x \le y \implies F(x) \le F(y)$).

Example 3.6. For a group (or monoid) G, a functor $F : G \to \mathsf{Set}$ consists of a set $F(*)$ (where $*$ is the unique object of G, i.e., $\mathsf{B}G$) together with a monoid homomorphism $F_{*,*} : G \to \mathsf{Set}(F(*), F(*))$; in other words, we have a G-set $F(*)$.

Similarly, for any other category C, a functor $F : G \to \mathsf{C}$ is an object in C with an action of G.

Example 3.7. There is a functor

$$F : \mathsf{Set} \longrightarrow \mathsf{Grp}$$

taking a set X to the free group $F(X)$ it generates, and taking a function $f : X \to Y$ to the induced group homomorphism $F(X) \to F(Y)$ whose restriction to generators is just f. Similarly, there is a functor

$$\begin{aligned} \mathbb{R}[-] : \mathsf{Set} &\longrightarrow \mathsf{CRing} := \{\text{commutative rings}\} \\ X &\longmapsto \mathbb{Z}[X] \end{aligned}$$

taking a set to the integral polynomial ring generated by its elements, i.e., the "free ring" it generates. More generally, for any category C of algebraic structures, there is a functor

$$\begin{aligned} \mathsf{Set} &\longrightarrow \mathsf{C} \\ X &\longmapsto \text{free algebra generated by } X. \end{aligned}$$

A **contravariant functor** from C to D is just an ordinary functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. This means that F flips the direction of morphisms:

$$f : X \to Y \in \mathsf{C} \implies F(f) : F(Y) \to F(X) \in \mathsf{D}.$$

Preservation of composition then becomes

$$F(g \circ f) = F(f) \circ F(g).$$

Ordinary functors are sometimes called **covariant** for contrast. To avoid confusion, it is best to understand there to be a single notion of functor (the covariant one), and to explicitly write $^{\mathrm{op}}$ if needed.

[3] Again, we are temporarily ignoring foundational issues.

Example 3.8. There is a **contravariant powerset functor**

$$\begin{aligned} \mathsf{Set}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathcal{P}(X) \\ (f : X \to Y) &\longmapsto (\mathcal{P}(f) := f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)). \end{aligned}$$

Functoriality says that for $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $C \subseteq Z$, we have

$$f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C).$$

Of course, there is also a **covariant powerset functor**

$$\begin{aligned} \mathsf{Set} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathcal{P}(X) \\ (f : X \to Y) &\longmapsto (f : \mathcal{P}(X) \to \mathcal{P}(Y)). \end{aligned}$$

Note that these two functors are the same on objects.

Example 3.9. For any category C, there is a **hom functor**

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}} : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} &\longrightarrow \mathsf{Set} \\ (A, B) &\longmapsto \mathrm{Hom}_{\mathsf{C}}(A, B) = \mathsf{C}(A, B) \\ ((A, B) \xrightarrow{(f,g)} (C, D)) &\longmapsto \begin{pmatrix} \mathsf{C}(A, B) \to \mathsf{C}(C, D) \\ (A \xrightarrow{h} B) \mapsto (C \xrightarrow{f} A \xrightarrow{h} B \xrightarrow{g} D) \end{pmatrix} \end{aligned}$$

(note that $f : A \to C \in \mathsf{C}^{\mathrm{op}}$ means $f : C \to A \in \mathsf{C}$).

By fixing one of the variables of $\mathrm{Hom}_{\mathsf{C}}$, we get for each $A \in \mathsf{C}$ two very important **representable functors** (discussed in detail in Section 8)

$$\begin{aligned} \mathsf{C}(A, -) : \mathsf{C} &\longrightarrow \mathsf{Set} \\ B &\longmapsto \mathsf{C}(A, B) \\ (f : B \to C) &\longmapsto (f \circ (-) : \mathsf{C}(A, B) \to \mathsf{C}(A, C)), \\ \mathsf{C}(-, A) : \mathsf{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ B &\longmapsto \mathsf{C}(B, A) \\ (f : C \to B \in \mathsf{C}) &\longmapsto ((-) \circ f : \mathsf{C}(B, A) \to \mathsf{C}(C, A)). \end{aligned}$$

4 Natural transformations

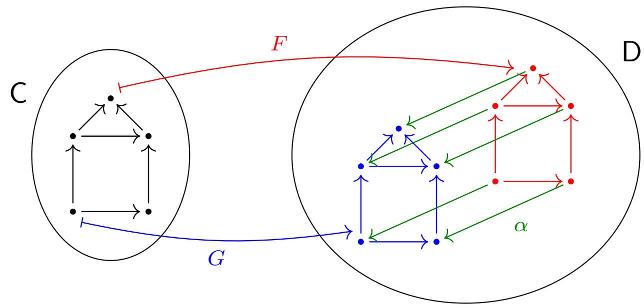
Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A **natural transformation** $\alpha : F \to G$ consists of:

- for each $X \in \mathsf{C}$, a morphism $\alpha_X : F(X) \to G(X) \in \mathsf{D}$, called the **component** of α at X;
- such that for any morphism $f : X \to Y \in \mathsf{C}$, the **naturality square**

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

  commutes.

Here is a picture:



The naturality squares that have to commute are the parallelograms in D consisting of one red, one blue, and two green edges.

Example 4.1. For two monotone maps $F, G : P \to Q$ between posets, there is a unique natural transformation $F \to G$ iff $F \le G$ pointwise. (Since Q is a poset, all diagrams in it commute.)

Example 4.2. Let G be a group. For two functors $A, B : G \to \mathsf{C}$, i.e., two objects (by abuse of notation also denoted) $A, B \in \mathsf{C}$ equipped with G-actions (where $g \in G$ acts via $A(g), B(g)$), a natural transformation $\phi : A \to B$ is a morphism $\phi : A \to B \in \mathsf{C}$ such that

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "A(g)"'] & B \arrow[d, "B(g)"] \\
A \arrow[r, "\phi"'] & B
\end{tikzcd}$$

commutes for every $g \in G$, i.e., a G-equivariant morphism $\phi : A \to B$.

Exercise 4.3. Let G, H be groups, $U, V : G \to H$ be functors (i.e., group homomorphisms). Describe the notion of natural transformation $U \to V$ in terms of conjugacy.

In general, a natural transformation between two functors between categories of structures should be thought of as a “naturally defined” homomorphism between the constructions provided by the two functors, as the following examples illustrate:

Example 4.4. Let $F : \mathsf{Set} \to \mathsf{Mon}$ be the free monoid functor, and let $G : \mathsf{Set} \to \mathsf{Grp} \subseteq \mathsf{Mon}$ be the free group functor. For every set X, there is a monoid embedding

$$\alpha_X : F(X) \hookrightarrow G(X) \in \mathsf{Mon};$$

and as X varies these form the components of a natural transformation, meaning that for each function $f : X \to Y \in \mathsf{Set}$,

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] & G(Y)
\end{tikzcd}$$

commutes. Indeed, for a finite string $x_1 x_2 \cdots x_n \in F(X)$, both compositions take the string to $f(x_1) f(x_2) \cdots f(x_n) \in G(Y)$.

Exercise 4.5. For any set X, we have a diagonal embedding $X \hookrightarrow X^2 \in \mathsf{Set}$. Formalize the naturality of these as X varies (including a description of the functors involved).

For two categories C, D, we have a **functor category** D^{C}, with

- objects: functors $F : \mathsf{C} \to \mathsf{D}$;
- morphisms: natural transformations $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$;
- composition given componentwise by composition in D.

Example 4.6. For a group (or monoid) G and category C, C^G is the category of G-equivariant objects of C (i.e., objects equipped with G-actions, and G-equivariant morphisms between them).

Example 4.7. For a discrete category (i.e., set) I, C^I is (isomorphic to) the product category (Example 2.10).

Example 4.8. Let S be the category $\{0 \xrightarrow{f} 1\}$, with a single non-identity morphism. A functor $F : \mathsf{S} \to \mathsf{C}$ is a morphism $F(f) : F(0) \to F(1) \in \mathsf{C}$. A natural transformation $\alpha : F \to G$ is a commutative square

$$\begin{tikzcd} F(0) \arrow[r, "\alpha_0"] \arrow[d, "F(f)"'] & G(0) \arrow[d, "G(f)"] \\ F(1) \arrow[r, "\alpha_1"'] & G(1). \end{tikzcd}$$

Thus C^{S} is the "category of morphisms in C, and commutative squares between them".

Exercise 4.9. Let $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$ be a natural transformation, each of whose components α_X is an isomorphism. Show that $(\alpha_X^{-1})_{X \in \mathsf{C}}$ is a natural transformation $G \to F$. Thus, α is an isomorphism (in D^{C}) iff each component of α is (in D).

Functor categories are the right notion of "exponential category", as the following shows:

Proposition 4.10. For any categories $\mathsf{C}, \mathsf{D}, \mathsf{E}$, we have an isomorphism

$$\begin{gathered} \mathsf{E}^{\mathsf{C} \times \mathsf{D}} \cong (\mathsf{E}^{\mathsf{D}})^{\mathsf{C}} \\ (\mathsf{C} \times \mathsf{D} \xrightarrow{F} \mathsf{E}) \mapsto \begin{pmatrix} F' : \mathsf{C} \to \mathsf{E}^{\mathsf{D}} \\ \quad X \mapsto F(X, -) \end{pmatrix} \\ \begin{pmatrix} G' : \mathsf{C} \times \mathsf{D} \to \mathsf{E} \\ \quad (X, Y) \mapsto G(X)(Y) \end{pmatrix} \leftarrow\!\shortmid (\mathsf{C} \xrightarrow{G} \mathsf{E}^{\mathsf{D}}). \end{gathered}$$

In particular, functors $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ are in bijection with functors $\mathsf{C} \to \mathsf{E}^{\mathsf{D}}$.

Note that in defining the two mutually inverse functors above, we only specified where objects are sent. Such abbreviations are common when defining functors, due to the large amount of data involved. In general, one gets used to the idea that certain categorical expressions (e.g., $F(X, -)$) are automatically functorial in all of the variables involved (F and X), hence can be used to define a functor simultaneously on objects and morphisms. For example, in the above:

- For fixed $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, for each $X \in \mathsf{C}$, $F(X, -)$ is a functor $\mathsf{D} \to \mathsf{E}$, taking $Y \in \mathsf{D}$ to $F(X, Y)$ and $g : Y \to Y' \in \mathsf{D}$ to $F(X, g) := F(1_X, g)$.
- Allowing X to vary, for a morphism $f : X \to X' \in \mathsf{C}$, $F(f, -)$ is a natural transformation $F(X, -) \to F(X', -)$, whose component at $Y \in \mathsf{D}$ is $F(f, 1_Y) : F(X, Y) \to F(X', Y)$; naturality means commutativity of

$$\begin{tikzcd} F(X,Y) \arrow[r, "F(f{,}1_Y)"] \arrow[d, "F(1_X{,}g)"'] & F(X',Y) \arrow[d, "F(1_{X'}{,}g)"] \\ F(X,Y') \arrow[r, "F(f{,}1_{Y'})"'] & F(X',Y') \end{tikzcd}$$

  for each $g : Y \to Y' \in \mathsf{D}$, which follows from $(f, 1) \circ (1, g) = (1, g) \circ (f, 1) \in \mathsf{C} \times \mathsf{D}$.
- Allowing F to vary, for a natural transformation $\alpha : F \to G : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, we get a natural transformation $\alpha' : F' \to G' : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$, whose component α'_X at each $X \in \mathsf{C}$ is the natural transformation $\alpha_{(X,-)} : F'(X) = F(X, -) \to G(X, -) = G'(X) : \mathsf{D} \to \mathsf{E}$; naturality of each $\alpha'_X = \alpha_{(X,-)}$ follows from naturality of α for morphisms of the form $(1_X, g)$ in $\mathsf{C} \times \mathsf{D}$, while naturality of α' follows from naturality of α for morphisms of the form $(f, 1_Y)$.
- For $G : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$, the definition of G' on objects is clear; for a morphism $(f, g) = (f, 1) \circ (1, g) = (1, g) \circ (1, f) : (X, Y) \to (X', Y') \in \mathsf{C} \times \mathsf{D}$, $G'(f, g) : G(X)(Y) \to G(X')(Y')$ is either of the two composites

$$\begin{tikzcd} G(X)(Y) \arrow[r, "G(f)_Y"] \arrow[d, "G(X)(g)"'] & G(X')(Y) \arrow[d, "G(X')(g)"] \\ G(X)(Y') \arrow[r, "G(f)_{Y'}"'] & G(X')(Y') \end{tikzcd}$$

  which are equal by naturality of $G(f) : G(X) \to G(X') : \mathsf{D} \to \mathsf{E}$.
- Allowing G to vary, for $\alpha : F \to G : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$, we get a natural transformation $\alpha' : F' \to G' : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, with components $\alpha'_{(X,Y)} := (\alpha_X)_Y$.

Proof. Just this once, we will write out the proof in detail, even though there is really nothing going on once we unravel all of the definitions.

To check that the $(-)'' : \mathsf{E}^{\mathsf{C}\times\mathsf{D}} \to (\mathsf{E}^{\mathsf{D}})^{\mathsf{C}} \to \mathsf{E}^{\mathsf{C}\times\mathsf{D}}$ composite is the identity, we must check that it is the identity on objects (i.e., $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$) as well as morphisms (i.e., $\alpha : F \to G : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$). For $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, we have

$$\begin{aligned} F''(X,Y) &= F'(X)(Y) = F(X,Y), \\ F''(f,g) &= F'(f)_{Y'} \circ F'(X)(g) \qquad \text{for } (f,g) : (X,Y) \to (X',Y') \\ &= F(f, 1_{Y'}) \circ F(1_X, g) \\ &= F(f,g). \end{aligned}$$

For $\alpha : F \to G : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, we have

$$\alpha''_{(X,Y)} = (\alpha'_X)_Y = (\alpha_{(X,-)})_Y = \alpha_{(X,Y)}.$$

To check that the other composite is the identity, for $G : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$, we have

$$\begin{aligned}
G''(X)(Y) &= G'(X,Y) = G(X)(Y), \\
G''(X)(g) &= G'(1_X, g) \qquad \text{for } g : Y \to Y' \in \mathsf{D} \\
&= G(X)(g) \circ G(1_X)_Y \\
&= G(X)(g) \qquad (\text{whence } G''(X) = G(X)), \\
G''(f)_Y &= G'(f, 1_Y) \qquad \text{for } f : X \to X' \in \mathsf{C} \\
&= G(X')(1_Y) \circ G(f)_Y \\
&= G(f)_Y,
\end{aligned}$$

while for $\alpha : F \to G : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$, we have

$$(\alpha''_X)_Y = (\alpha'_{(X,-)})_Y = \alpha'_{(X,Y)} = (\alpha_X)_Y. \qquad \square$$

5 2-categories and enriched categories

This section will be rather vague and impressionistic. One could easily teach a full course on 2-categories, or on enriched categories; we will only use them as an informal organizing device, focusing on concrete examples.

We have seen:

- categories: $\mathsf{C}, \mathsf{D}, \mathsf{E}, \ldots$;
- functors between them: $\mathsf{C} \xrightarrow{F} \mathsf{D}$;
- natural transformations between functors: $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \end{tikzcd}$.

These fit into the following kind of structure.

A **2-category** $\mathfrak{C}$ consists of:

- objects $\mathsf{X} \in \mathfrak{C}$;
- for each pair of objects $\mathsf{X}, \mathsf{Y} \in \mathfrak{C}$, a **hom-category** $\mathfrak{C}(\mathsf{X}, \mathsf{Y})$, whose objects $F \in \mathfrak{C}(\mathsf{X}, \mathsf{Y})$ are called **morphisms** $F : \mathsf{X} \to \mathsf{Y}$ in $\mathfrak{C}$, and morphisms are called **2-morphisms** in $\mathfrak{C}$;
- for each $\mathsf{X} \in \mathfrak{C}$, an **identity morphism** $1_{\mathsf{X}} \in \mathfrak{C}$;
- for any three objects, a **composition functor**

$$\circ : \mathfrak{C}(\mathsf{Y}, \mathsf{Z}) \times \mathfrak{C}(\mathsf{X}, \mathsf{Y}) \longrightarrow \mathfrak{C}(\mathsf{X}, \mathsf{Z});$$

- satisfying the unit and associativity laws: e.g., associativity says that the two functors

$$\mathfrak{C}(\mathsf{Y}, \mathsf{Z}) \times \mathfrak{C}(\mathsf{X}, \mathsf{Y}) \times \mathfrak{C}(\mathsf{W}, \mathsf{X}) \rightrightarrows \mathfrak{C}(\mathsf{W}, \mathsf{Z})$$

  are equal.

The quintessential example of a 2-category is the **2-category $\mathfrak{Cat}$ of categories**, whose objects are categories and hom-categories are

$$\mathfrak{Cat}(\mathsf{X}, \mathsf{Y}) := \mathsf{Y}^{\mathsf{X}},$$

i.e., morphisms are functors, and 2-morphisms are natural transformations. We will only ever talk about 2-categories based on $\mathfrak{Cat}$.

Note that the composition (of 1-morphisms) in a 2-category $\mathfrak{C}$, being a functor, also operates on 2-morphisms (i.e., morphisms in hom-categories). In $\mathfrak{Cat}$, these are defined as follows: given

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] & \mathsf{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{C} \arrow[r, "K"] & \mathsf{D}, \end{tikzcd}$$

we may apply K componentwise to α to get

$$K(\alpha) := (K(\alpha_B))_{B \in \mathsf{B}} : K \circ G \to K \circ H : \mathsf{B} \to \mathsf{D},$$

or restrict α to objects in the image of F to get

$$\alpha_F := (\alpha_{F(A)})_{A \in \mathsf{A}} : G \circ F \to H \circ F : \mathsf{A} \to \mathsf{C};$$

these two are, respectively, the following images of 2-morphisms under the composition functor:

$$\begin{aligned} \circ : \mathfrak{Cat}(\mathsf{B}, \mathsf{C}) \times \mathfrak{Cat}(\mathsf{A}, \mathsf{B}) &\longrightarrow \mathfrak{Cat}(\mathsf{A}, \mathsf{C}) \\ (\alpha, 1_F) &\longmapsto \alpha_F, \end{aligned} \qquad \begin{aligned} \circ : \mathfrak{Cat}(\mathsf{C}, \mathsf{D}) \times \mathfrak{Cat}(\mathsf{B}, \mathsf{C}) &\longrightarrow \mathfrak{Cat}(\mathsf{B}, \mathsf{D}) \\ (1_K, \alpha) &\longmapsto K(\alpha). \end{aligned}$$

Sometimes $K(\alpha)$ and α_F are called **whiskerings** of α (due to the above diagram). These are enough to define composition of arbitrary 2-morphisms, via $(\beta, \alpha) = (\beta, 1) \circ (1, \alpha) = (1, \alpha) \circ (\beta, 1)$. That is, we define the "horizontal composition" of two natural transformations

$$\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C} \end{tikzcd}$$

to mean either of the "vertical" (i.e., componentwise) compositions of whiskerings

$$\begin{array}{c} \begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, "H"] & \mathsf{C} \end{tikzcd} \\ \circ \\ \begin{tikzcd} \mathsf{A} \arrow[r, "G"'] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C} \end{tikzcd} \end{array} \qquad \text{or} \qquad \begin{array}{c} \begin{tikzcd} \mathsf{A} \arrow[r, "F"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C} \end{tikzcd} \\ \circ \\ \begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, "K"'] & \mathsf{C}. \end{tikzcd} \end{array}$$

Exercise 5.1. Write out each of these vertical compositions in components, and verify that they are the same.

A 2-category can be thought of as a (1-)category whose hom-sets have been replaced with categories. More generally, for any "sufficiently nice" category V, there is a notion of **V-enriched category** C, with **hom-objects** $\mathsf{C}(X, Y) \in \mathsf{V}$ for each pair of objects $X, Y \in \mathsf{C}$ (composition then becomes a morphism in V). For example:

- category = Set-enriched category
- 2-category = Cat-enriched category (note: Cat, not $\mathfrak{Cat}$)
- preordered set = 2-enriched category (where $2 = \{0 < 1\}$, regarded as a category)

We have seen that every preordered set can be regarded as an ordinary category; from the enriched point of view, this is a forgetful 2-functor from 2-enriched categories to Set-enriched categories induced by the inclusion $2 \hookrightarrow \mathsf{Set}$. We have also seen several examples of categorical notions where it helps to first understand the special case for posets. Actually, it is often a better analogy to regard posets as 2-enriched categories, and to first understand the 2-enriched version of a categorical notion, which may be even simpler than the ordinary notion specialized to posets. We will see examples of this in Section 8 below.

6 Fullness, faithfulness, conservativity, equivalences

When doing field theory (say), one should be interested only in notions which respect field isomorphisms. For a property (i.e., truth value) assigned to fields, that property should be invariant under field isomorphisms; for a (say) group $G(K)$ constructed from each field K, that construction should instead be equivariant with respect to field isomorphisms, meaning a field isomorphism $K \cong L$ should induce, in a functorial manner, a group isomorphism $G(K) \cong G(L)$.

Since we think of a category as an abstraction of a category of structures, this means

- categorical notions should respect isomorphisms *in* a category;
- hence, two functors $F, G : \mathsf{C} \to \mathsf{D}$ being naturally isomorphic should be just as good as them being equal;
- hence, isomorphism *of* categories, i.e., a functor $F : \mathsf{C} \to \mathsf{D}$ with $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F = 1_{\mathsf{C}}$ and $F \circ G = 1_{\mathsf{D}}$, is the "wrong" notion of "sameness" for categories.

The following concrete examples serve to illustrate these points:

Example 6.1. The free group $F(X)$ generated by a set X can be constructed as the set of reduced words (e.g., $xy^{-1}x^{-1}$, not containing, say, xx^{-1} or $x^{-1}x$) over elements of X and their inverses. Alternatively, we can take $F'(X) :=$ all words mod group axioms. The groups $F(X)$ and $F'(X)$ are set-theoretically not *equal*; thus the functors $F, F' : \mathsf{Set} \to \mathsf{Grp}$ are not equal either, but of course it should never matter which construction of free groups we use.

Example 6.2. Let $\mathsf{Cat_{1o}} \subseteq \mathsf{Cat}$ be the subcategory of categories with a single object (and all functors between them). The functor $\mathsf{B} : \mathsf{Mon} \to \mathsf{Cat}$ lands in $\mathsf{Cat_{1o}}$, but is not an isomorphism $\mathsf{B} : \mathsf{Mon} \cong \mathsf{Cat_{1o}}$, since technically the single object of $\mathsf{B}G$ (for a monoid G) must be the symbol $*$, even though every category with a single object is canonically isomorphic to one of the form $\mathsf{B}G$.

Remark 6.3. Of course, when doing (say) concrete computations with a particular field K, it can be very helpful to know that K is (say) a subfield of $\mathbb{C}$ (rather than merely isomorphic to a subfield of $\mathbb{C}$). Likewise, in concrete situations, it can be very helpful to know that a particular functor is in fact an isomorphism, especially because of the subtleties in Remark 11.30 below.

A functor $F : \mathsf{C} \to \mathsf{D}$ is an **equivalence of categories** if there is a functor $G : \mathsf{D} \to \mathsf{C}$ such that

$$G \circ F \cong 1_{\mathsf{C}},$$
$$F \circ G \cong 1_{\mathsf{D}}$$

($\cong$ means naturally isomorphic). Such G is called an **inverse equivalence** (also known as **weak inverse**, **pseudo-inverse**, etc.) of F. Intuitively, equivalences of categories are the least generalization of isomorphisms of categories that also respects isomorphisms *in* categories.

Example 6.4. Consider $\mathsf{B} : \mathsf{Mon} \to \mathsf{Cat}_{1\mathsf{o}}$ as in Example 6.2 above. Let

$$\begin{aligned} F : \mathsf{Cat}_{1\mathsf{o}} &\longrightarrow \mathsf{Mon} \\ \mathsf{C} &\longmapsto \mathsf{C}(X, X) \quad \text{for the unique } X \in \mathsf{C}_0 \end{aligned}$$

(with the obvious action on morphisms, i.e., functors $\mathsf{C} \to \mathsf{D}$). Then $F \circ \mathsf{B} = 1_{\mathsf{Mon}}$ (since $F(\mathsf{B}G) = \mathsf{B}G(*, *) = G$, and similarly for morphisms), while $\mathsf{B} \circ F \cong 1_{\mathsf{Cat}_{1\mathsf{o}}}$, via the natural isomorphism whose component at each category $\mathsf{C} \in \mathsf{Cat}_{1\mathsf{o}}$ with one object X is the functor

$$\begin{aligned} \mathsf{B}(F(\mathsf{C})) &\xrightarrow{\cong} \mathsf{C} \in \mathsf{Cat}_{1\mathsf{o}} \subseteq \mathsf{Cat} \\ \mathsf{B}(F(\mathsf{C})) \ni * &\longmapsto X \in \mathsf{C} \\ \mathsf{C}(X, X) = F(\mathsf{C}) = \mathsf{B}(F(\mathsf{C}))(*, *) \ni g &\longmapsto g \in \mathsf{C}(X, X). \end{aligned}$$

Exercise 6.5. Verify naturality as C varies.

Exercise 6.6. Let

- Met := metric spaces and uniformly continuous maps,
- Met' := complete metric spaces X equipped with a dense subset $D \subseteq X$, with morphisms $(X, D) \to (Y, E)$ given by uniformly continuous maps $f : X \to Y$ such that $f(D) \subseteq E$.

Define (and verify) inverse equivalences $\mathsf{Met} \leftrightarrow \mathsf{Met}'$.

The usual definitions of injectivity and surjectivity for functors, because they refer to equality of objects, are not invariant under equivalences. Instead, we have the following "corrected" notions, capturing aspects of injectivity and surjectivity. A functor $F : \mathsf{C} \to \mathsf{D}$ is:

- **faithful** if it is injective on each hom-set, i.e., each $F : \mathsf{C}(X, Y) \to \mathsf{D}(F(X), F(Y))$ is injective;
- **full** if it is surjective on each hom-set;
- **essentially surjective (on objects)** if each $Y \in \mathsf{D}$ is isomorphic to some $F(X)$.

Proposition 6.7. A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories iff it is full, faithful, and essentially surjective.

Proof. $\Longrightarrow$: Let $G : \mathsf{D} \to \mathsf{C}$ be an inverse equivalence of F, with natural isomorphisms $\alpha : G \circ F \cong 1_{\mathsf{C}}$ and $\beta : F \circ G \cong 1_{\mathsf{D}}$. Essential surjectivity of F is easy: for each $Y \in \mathsf{D}$, we have an isomorphism

$\beta_Y : F(G(Y)) \cong Y$. For faithfulness of F, let $f : X \to Y \in \mathsf{C}$; then by naturality of α, we have a commutative square

$$\begin{tikzcd} G(F(X)) \arrow[r, "\alpha_X", "\cong"'] \arrow[d, "G(F(f))"'] & X \arrow[d, "f"] \\ G(F(Y)) \arrow[r, "\cong", "\alpha_Y"'] & Y, \end{tikzcd}$$

whence f may be recovered from $F(f)$ as $f = \alpha_Y \circ G(F(f)) \circ \alpha_X^{-1}$. Symmetrically, G is faithful. Finally, for fullness of F, let $g : F(X) \to F(Y) \in \mathsf{D}$, and put $f := \alpha_Y \circ G(g) \circ \alpha_X^{-1}$. Then by the above commutative square,

$$\alpha_Y \circ G(F(f)) \circ \alpha_X^{-1} = f = \alpha_Y \circ G(g) \circ \alpha_X^{-1},$$

whence $G(F(f)) = G(g)$ since α_X, α_Y are isomorphisms, whence $F(f) = g$ since G is faithful.

$\Longleftarrow$: Using essential surjectivity of F (and the axiom of choice), choose for each $Y \in \mathsf{D}$ some $X_Y \in \mathsf{C}$ and isomorphism $\beta_Y : F(X_Y) \cong Y$. Define (using fullness and faithfulness of F)

$$\begin{aligned} G : \mathsf{D} &\longrightarrow \mathsf{C} \\ Y &\longmapsto X_Y \\ (Y \xrightarrow{g} Y') &\longmapsto F^{-1}\left(F(X_Y) \xrightarrow[\cong]{\beta_Y} Y \xrightarrow{g} Y' \xrightarrow[\cong]{\beta_{Y'}^{-1}} F(X_{Y'}) \right), \\ \alpha_X := F^{-1}\left(F(X_{F(X)}) \xrightarrow{\beta_{F(X)}} F(X) \right) &: X_{F(X)} = G(F(X)) \to X. \end{aligned}$$

To check that G is a functor, i.e., preserves 1 and $\circ$: from the definition of G, clearly $F \circ G$ preserves 1 and $\circ$; now use faithfulness of F. Naturality of β, i.e., commutativity of

$$\begin{tikzcd} F(G(Y)) \arrow[r, "\beta_Y"] \arrow[d, "F(G(g))"'] & Y \arrow[d, "g"] \\ F(G(Y')) \arrow[r, "\beta_{Y'}"'] & Y' \end{tikzcd}$$

for $g : Y \to Y' \in \mathsf{D}$, is immediate from the definition of $G(g)$. For naturality of α, i.e., commutativity of

$$\begin{tikzcd} G(F(X)) \arrow[r, "\alpha_X"] \arrow[d, "G(F(f))"'] & X \arrow[d, "f"] \\ G(F(X')) \arrow[r, "\alpha_{X'}"'] & X' \end{tikzcd}$$

for $f : X \to X' \in \mathsf{C}$: applying F to this square, we get

$$\begin{tikzcd} F(G(F(X))) \arrow[r, "F(\alpha_X)=\beta_{F(X)}"] \arrow[d, "F(G(F(f)))"'] & F(X) \arrow[d, "F(f)"] \\ F(G(F(X'))) \arrow[r, "F(\alpha_{X'})=\beta_{F(X')}"'] & F(X') \end{tikzcd}$$

which commutes by definition of $G(F(f))$, whence by faithfulness of F, the original square also commutes. □

Example 6.8. Let FDVec be the category of finite-dimensional (say, $\mathbb{R}$-)vector spaces and linear maps, and Mat be the category whose objects are natural numbers $n \in \mathbb{N}$ and morphisms $m \to n$ are $n \times m$ matrices, with composition given by matrix multiplication. There is a functor

$$\begin{aligned} \mathsf{Mat} &\longrightarrow \mathsf{FDVec} \\ n &\longmapsto \mathbb{R}^n \\ (A : m \to n) &\longmapsto A \cdot (-) : \mathbb{R}^m \to \mathbb{R}^n \end{aligned}$$

which is full, faithful, and essentially surjective, hence an equivalence. To construct the inverse equivalence as in Proposition 6.7, choose for each $V \in \mathsf{FDVec}$ a basis, or equivalently, a linear isomorphism $\beta_V : \mathbb{R}^{\dim(V)} \cong V$; the inverse equivalence takes each V to $\mathbb{R}^{\dim(V)}$, and each linear map $T : V \to W$ to its matrix $[T]$ with respect to the chosen bases, which represents the linear map $\beta_W^{-1} \circ T \circ \beta_V$.

$$\begin{tikzcd} \mathbb{R}^{\dim(V)} \arrow[r, "\beta_V", "\cong"'] \arrow[d, "\beta_W^{-1} \circ T \circ \beta_V"'] & V \arrow[d, "T"] \\ \mathbb{R}^{\dim(W)} \arrow[r, "\cong", "\beta_W"'] & W \end{tikzcd}$$

For each n, the linear isomorphism $\alpha_n : \mathbb{R}^{\dim(\mathbb{R}^n)} = \mathbb{R}^n \cong \mathbb{R}^n$ is the matrix of $\beta_{\mathbb{R}^n}$.

Exercise 6.9. Show that the following categories are equivalent:

- $\mathsf{C} :=$ groups with underlying set $\mathbb{Z}$, group homomorphisms,
- $\mathsf{D} :=$ infinite groups with underlying set $\subseteq \mathbb{N}$, group homomorphisms.

Note that every subcategory inclusion $\mathsf{C} \hookrightarrow \mathsf{D}$ is faithful. A subcategory $\mathsf{C} \subseteq \mathsf{D}$ is **full** if the inclusion is full, i.e., C contains all morphisms between its objects in D. (In other words, the underlying directed multigraph of C is an induced subgraph of D.) The **full image** of a functor $F : \mathsf{C} \to \mathsf{D}$ is the full subcategory on the objects in the (actual) image of F; the **essential image** of F is the full subcategory of D on all objects isomorphic to some $F(X)$. Thus, F is essentially surjective iff its essential image is D.

Example 6.10. $\mathsf{AbGrp} \subseteq \mathsf{Grp} \subseteq \mathsf{Mon}$ are full subcategories.

Example 6.11. $\mathsf{Grp} \subseteq \mathsf{SGrp}$ is a full subcategory (where $\mathsf{SGrp} :=$ semigroups).

Example 6.12. $\mathsf{Mon} \subseteq \mathsf{SGrp}$ is a non-full subcategory (e.g., $\infty : (\mathbb{N}, +) \to (\{0, \infty\}, +) \in \mathsf{SGrp}$).

However, every isomorphism in SGrp is in Mon (we say that $\mathsf{Mon} \subseteq \mathsf{SGrp}$ is **full on isomorphisms**).

The above inclusions can also be regarded as forgetful functors (depending on the precise definitions of group, monoid, semigroup).

Example 6.13. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is faithful, but not full or essentially surjective. Similarly if Grp is replaced by Mon, Ring, etc.

Example 6.14. The forgetful functor $(-)_0 : \mathsf{Cat} \to \mathsf{Set}$ is neither full nor faithful (a functor is not determined by what it does on objects), but is essentially surjective (e.g., take discrete categories).

Example 6.15. The forgetful functor $\mathsf{TopGrp} \to \mathsf{Grp}$ is faithful and essentially surjective, but not full (even on isomorphisms).

The above examples illustrate that a forgetful functor $F : \mathsf{D} \to \mathsf{C}$ between two categories of (first-order) structures is:

- faithful, if structures in D are structures in C equipped with additional relations or operations, but not additional elements (thus excluding $\mathsf{Cat} \to \mathsf{Set}$);

- full and faithful, if D only imposes additional properties, which homomorphisms don't care about;

- full on isomorphisms and faithful, if D (possibly) imposes additional structure which homomorphisms must preserve, but which is nonetheless uniquely defined in terms of the C-structure.

  (For logicians: if the additional D-structure is definable (without parameters) using positive existential (but possibly infinitary) formulas in terms of the C-structure, then it is automatically preserved by all homomorphisms. For example, $x \cdot y = y \cdot x = 1$ defines inverses in a monoid, whereas $\forall y\,(x \cdot y = y \cdot x = y)$ defines 1 in a semigroup using a non-positive existential formula.)

In fact, it turns out that abstractly, *every* functor is, up to equivalence of categories, a forgetful functor (so the above descriptions of faithfulness and fullness apply to general functors, at least abstractly). This is illustrated by the following examples. (The general construction requires the Yoneda embedding to represent every category as a concrete category of structures; see Exercise 9.2.)

Example 6.16. Let Met be the category of metric spaces and uniformly continuous functions, $\mathsf{CMet} \subseteq \mathsf{Met}$ be the full subcategory of complete metric spaces. Thus, we can regard the inclusion $\mathsf{CMet} \hookrightarrow \mathsf{Met}$ as forgetting the property of being complete.

We also have a completion functor $\mathsf{Met} \to \mathsf{CMet}$, not usually thought of as "forgetful", which factors as the equivalence $\mathsf{Met} \to \mathsf{Met}'$ from Exercise 6.6, where Met' is the category of complete metric spaces equipped with a dense subset and uniformly continuous maps preserving the dense subset, followed by the forgetful functor $\mathsf{Met}' \to \mathsf{CMet}$ which forgets the dense subset.

Exercise 6.17. Describe the free group functor $F : \mathsf{Set} \to \mathsf{Grp}$ as a forgetful functor, up to equivalence.

The next result shows that (full and) faithful functors are indeed the least generalization of (full) injective functors that also respects equivalences:

Proposition 6.18. Any faithful functor $F : \mathsf{C} \to \mathsf{D}$ factors as an injective functor $G : \mathsf{C} \to \mathsf{E}$ followed by an equivalence $H : \mathsf{E} \to \mathsf{D}$. Moreover, if F is full, then so is G.

Proof. Let

$$\mathsf{E}_0 := \mathsf{C}_0 \sqcup \mathsf{D}_0,$$

$$\mathsf{E}(X, Y) := \begin{cases} \mathsf{D}(F(X), F(Y)) & \text{if } X, Y \in \mathsf{C}, \\ \mathsf{D}(F(X), Y) & \text{if } X \in \mathsf{C} \text{ and } Y \in \mathsf{D}, \\ \mathsf{D}(X, F(Y)) & \text{if } X \in \mathsf{D} \text{ and } Y \in \mathsf{C}, \\ \mathsf{D}(X, Y) & \text{if } X, Y \in \mathsf{D}, \end{cases}$$

with composition and identity as in D. Let

$$\begin{aligned} G : \mathsf{C} &\hookrightarrow \mathsf{E} \\ \mathsf{C}_0 \ni X &\longmapsto X \in \mathsf{C}_0 \subseteq \mathsf{E}_0, \\ (f : X \to Y) &\longmapsto F(f) \in \mathsf{D}(F(X), F(Y)) = \mathsf{E}(X, Y), \end{aligned}$$

$$\begin{aligned} H : \mathsf{E} &\longrightarrow \mathsf{D} \\ X &\longmapsto \begin{cases} F(X) & \text{if } X \in \mathsf{C}_0, \\ X & \text{if } X \in \mathsf{D}_0, \end{cases} \\ (f : X \to Y) &\longmapsto f. \end{aligned}$$

□

Example 6.19. Consider the forgetful functor $F : \mathsf{Grp} \to \mathsf{Set}$. Let E be the category whose objects are either a group or $\varnothing$, and whose morphisms are arbitrary functions. Then $\mathsf{Grp} \subseteq \mathsf{E}$ is a subcategory, while the "forgetful functor" $\mathsf{E} \to \mathsf{Set}$ is an equivalence, being full, faithful, and (essentially) surjective on objects.

(The point is that the objects of a category only have "structure" which can be detected categorically when that structure is required to be preserved by the morphisms. So the category E above is, from an abstract categorical perspective, just another version of the category of sets.)

Exercise 6.20 (set-theoretic)**.** Show that the forgetful functor $\mathsf{E} \to \mathsf{Set}$ above is actually naturally isomorphic to an isomorphism of categories, whence the forgetful functor $F : \mathsf{Grp} \to \mathsf{Set}$ is naturally isomorphic to an injective functor, i.e., Grp is isomorphic to a subcategory of Set.

Remark 6.21. Similarly, every essentially surjective functor factors as a surjection (on objects) onto its full image, followed by the inclusion of the full image which is an equivalence (being full, faithful, and essentially surjective).

Thus (fullness+)faithfulness are the closest equivalence-invariant generalizations of (full) injective functors. Is fullness a notion of "injectivity", or "surjectivity"? It is best thought of as both! Consider the following table:

dimension:	0	1	
	ess. surj. = surjective	surjective	= full
	injective	injective = faithful	

Clearly, a functor $F : \mathsf{C} \to \mathsf{D}$ being faithful is a notion of injectivity on (1-dimensional) morphisms/edges; and being essentially surjective is the same as being surjective on isomorphism classes of (0-dimensional) objects/vertices. The definition of fullness looks like a notion of surjectivity in dimension 1; but this is closely related to *injectivity* in dimension 0, since for example, it implies injectivity on isomorphism classes:

Proposition 6.22. If $F : \mathsf{C} \to \mathsf{D}$ is full and faithful, then the restriction $F : \mathsf{Core}(\mathsf{C}) \to \mathsf{Core}(\mathsf{D})$ is still full (and faithful), i.e., for every isomorphism $g : F(X) \to F(Y)$, there is an isomorphism $f : X \to Y \in \mathsf{C}$ such that $F(f) = g$.

Proof. By fullness, there are $f : X \to Y \in \mathsf{C}$ and $f' : Y \to X \in \mathsf{C}$ such that $F(f) = g$ and $F(f') = g^{-1}$, whence $F(f' \circ f) = F(f') \circ F(f) = g^{-1} \circ g = 1 = F(1)$ and similarly $F(f \circ f') = F(1)$; by faithfulness, this implies $f' \circ f = 1$ and $f \circ f' = 1$. □

Full and faithful functors are the strongest equivalence-invariant notion of categorical "embedding" available. Note that every such functor factors into an equivalence with its full (or essential) image, followed by the inclusion of a full subcategory.

There is another important (equivalence-invariant) notion of "injectivity" for functors, which sits somewhere between faithful and full+faithful: a functor $F : \mathsf{C} \to \mathsf{D}$ is **conservative** (or **reflects isomorphisms**) if for every $f : X \to Y \in \mathsf{C}$, if $F(f)$ is an isomorphism, then f was an isomorphism.

Proposition 6.23. If F is full and faithful, then F is conservative.

Proof. By Proposition 6.22 and faithfulness. □

Fact 6.24. "Usually", conservative implies faithful. (For example, this holds for any free or forgetful functor between "nice" categories of structures. See Remark 23.8 below for the precise statement.)

Example 6.25. The forgetful functor $\mathsf{Grp} \to \mathsf{Mon}$ is conservative (because it's full and faithful).

Example 6.26. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is conservative.

Example 6.27. The forgetful functors $\mathsf{Top} \to \mathsf{Set}$ and $\mathsf{Pos} \to \mathsf{Set}$ are not conservative. However, the forgetful functor $\mathsf{KHaus} \to \mathsf{Set}$, where KHaus = compact Hausdorff spaces, *is* conservative.

Example 6.28. The forgetful functor $\mathsf{TopGrp} \to \mathsf{Top}$ is conservative, but the forgetful functor $\mathsf{TopGrp} \to \mathsf{Grp}$ is not.

As these examples illustrate, conservativity of a forgetful functor to Set tends to mean that the domain category is a category of "algebraic" structures, i.e., there are only operations, no relations. (What are the operations for KHaus?) More generally, a forgetful functor $\mathsf{D} \to \mathsf{C}$ between two categories of structures is conservative if structures in D are "algebraic relative to C".

Exercise 6.29. We might expect notions of "injectivity" to have the following cancellation property: if a composite $\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$ is "injective", then so is F. Show that

- if $G \circ F$ is faithful, then F is faithful;
- if $G \circ F$ is conservative, then F is conservative;
- if $G \circ F$ is full and G is faithful, then F is full;
- if $G \circ F$ is full and faithful, F need not be full.

On the other hand, show that

- if $G \circ F$ is essentially surjective, then G is essentially surjective;

- if $G \circ F$ is full and F is essentially surjective, then G is full.

To summarize, we have the following hierarchy of progressively weaker notions of "injectivity" for functors (including the two degenerate endpoint notions):

$$\text{equivalence} \implies \text{full+faithful} \implies \text{conservative} \overset{\text{(usually)}}{\implies} \text{faithful} \implies \text{arbitrary}$$

7 Some remarks about foundations

So far, we have ignored all questions of sets versus proper classes. Such questions are relevant to category theory, since e.g., a category like Set, consisting of all sets, should be too big to be a set. Moreover, if one wants to study categorical constructions (e.g., functor categories) in their own right, then categories should themselves belong to a well-behaved mathematical universe; and one could continue on to 2-categories, etc. Fortunately, there is a set-theoretic device which allows us to sweep most of the details under the rug, once and for all.

A **(Grothendieck) universe** is a set $\mathcal{U}$ (of sets) such that

- if $A \in B \in \mathcal{U}$, then $A \in \mathcal{U}$;
- if $A \in \mathcal{U}$, then $\mathcal{P}(A) \in \mathcal{U}$;
- if $A \in \mathcal{U}$ and $B_a \in \mathcal{U}$ for each $a \in A$, then $\bigcup_{a \in A} B_a \in \mathcal{U}$;
- $\mathbb{N} \in \mathcal{U}$.

These imply closure under other common set-theoretic operations:

- $\varnothing \in \mathcal{U}$;
- if $A, B \in \mathcal{U}$, then $A \times B, \{A, B\}, (A, B), A^B \in \mathcal{U}$;
- if $A \in \mathcal{U}$ and $B_a \in \mathcal{U}$ for each $a \in A$, then $\prod_{a \in A} B_a \in \mathcal{U}$.

It turns out that all universes are of the form "all sets which are small enough, and whose elements are also small enough, and whose elements of elements are also small enough, etc.". For an infinite cardinal number κ, a set A is **hereditarily of size** $< \kappa$ if

$$\forall A_n \in A_{n-1} \in \cdots \in A_0 = A \, (|A_n| < \kappa).$$

Let $\mathcal{H}_\kappa$ denote the set of all sets of size hereditarily $< \kappa$; this is indeed a set, since it can be constructed from the ground up, starting from the empty set (see Exercise 7.2).

A cardinal number κ is **inaccessible** if

- if $|A| < \kappa$, then $|2^A| < \kappa$ (κ is **strong limit**);
- if $|A| < \kappa$ and $|B_a| < \kappa$ for each $a \in A$, then $|\bigcup_{a \in A} B_a| < \kappa$ (κ is **regular**);
- $|\mathbb{N}| = \aleph_0 < \kappa$ (κ is uncountable).

Fact 7.1. Grothendieck universes are precisely the sets $\mathcal{H}_\kappa$ for inaccessible κ. ($\Longleftarrow$ is trivial to see, and is all we need.)

Officially, we work under the assumption

> Every set belongs to some universe. Equivalently, every cardinal is $<$ some inaccessible.

This is technically a stronger assumption than ZFC alone (by Gödel incompleteness, since universes are models of ZFC); however, modern set theory studies cardinals much larger than inaccessibles, which are widely believed to all be consistent. On the other hand, it is generally believed that any "natural" categorical statement can refined into a (less readable) ZFC statement by inserting cardinality bounds everywhere (although nobody ever bothers to check); so the above assumption is more for convenience than necessity.

From now on, whenever we say "category", we mean a category which is actually a set. This means there is no category Set of *all* sets (nor $\mathsf{Grp}, \mathsf{Top}, \dots$). For an inaccessible cardinal κ, we let

$$\mathsf{Set}_\kappa, \mathsf{Grp}_\kappa, \mathsf{Top}_\kappa, \dots$$

denote the category of sets (groups, topological spaces) belonging to the universe $\mathcal{H}_\kappa$ (and all functions between them, which automatically also belong to $\mathcal{H}_\kappa$). When we write Set (or $\mathsf{Grp}, \dots$), we mean Set_κ for some fixed, but arbitrary, inaccessible κ. Thus, whenever we state a theorem involving Set (say), the theorem has an implicit "$\forall$ inaccessible κ" in front.

We say that a set (or structure) is **(κ-)small** if it belongs to the fixed background universe $\mathcal{H}_\kappa$. For a category C, we say that C is **locally small** if each of its hom-sets $\mathsf{C}(X, Y)$ is small. Note that being small, unlike being locally small, is not invariant under equivalences of categories; we say that C is **essentially small** if it is equivalent to a small category, or equivalently, it is locally small and has a small set of isomorphism classes of objects (since the full subcategory on a set of isomorphism representatives will then have size $< \kappa$, hence be isomorphic to a small category).

Note that Set_κ itself does *not* belong to $\mathcal{H}_\kappa$; we instead have

$$\mathsf{Set}_\kappa \in \mathsf{Cat}_{\kappa'} \in \mathsf{Cat}_{\kappa''} \in \cdots$$

for inaccessibles $\kappa < \kappa' < \kappa'' < \cdots$. Thus, in order to categorically study, say, the collection of all categories of first-order structures, we need at least two inaccessibles $\kappa < \kappa'$; it is easiest just to assume a proper class of inaccessibles.

It will sometimes be necessary to "enlarge universes" by changing our background κ to a larger κ'. When doing so, it is important to know that constructions performed in the smaller universe are still valid in the larger one. For example, note that

$$\mathsf{Set}_\kappa \subseteq \mathsf{Set}_{\kappa'} \subseteq \cdots$$

for $\kappa < \kappa' < \cdots$ are *full* subcategories, and similarly for (say) $\mathsf{Grp}_\kappa \subseteq \mathsf{Grp}_{\kappa'} \subseteq \cdots$ (so the notion of morphism is at least preserved). We will not worry too much about this in general.

Exercise 7.2 (set-theoretic)**.** Let $\mathcal{V}_0 := \varnothing$, $\mathcal{V}_{\alpha+1} := \mathcal{P}(\mathcal{V}_\alpha)$, and $\mathcal{V}_\lambda := \bigcup_{\alpha<\lambda} \mathcal{V}_\alpha$ for a limit ordinal λ be the von Neumann hierarchy of sets. Let κ be an infinite regular cardinal.

- Show that if $A \subseteq \mathcal{V}_\kappa$ and $|A| < \kappa$, then $A \in \mathcal{V}_\kappa$.
- Show that $\mathcal{H}_\kappa \subseteq \mathcal{V}_\kappa$. In particular, $\mathcal{H}_\kappa$ is a set. [Use $\in$-induction.]
- Show that if κ is inaccessible, then $\mathcal{H}_\kappa = \mathcal{V}_\kappa$.

8 Presheaves, the Yoneda lemma

A **presheaf** on a category C is simply a functor $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$; the **category of presheaves** is the functor category

$$\mathsf{PSh}(\mathsf{C}) := \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}.$$

A **copresheaf** on C is a presheaf on C, i.e., a functor $\mathsf{C} \to \mathsf{Set}$. (The placement of the "co" will be explained shortly.)

Even though they are just functors, presheaves play a very special role in category theory. One way to think of a presheaf $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ on C is an assignment of a set of "possible data" $P(X)$ to each object $X \in \mathsf{C}$, such that the data can be "pulled back" along a morphism $f : X \to Y$ via $P(f) : P(Y) \to P(X)$ (or "pushed forward", for a copresheaf).

Example 8.1. Let $\mathcal{P} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ be the contravariant powerset functor. Thus $\mathcal{P}$ assigns to each set X its set of subsets, with a subset "pulled back" along a function by taking preimage.

Example 8.2. Similarly, the presheaf

$$\begin{aligned} \mathcal{O} : \mathsf{Top}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathcal{O}(X) := \{\text{open sets in } X\} \\ (X \xrightarrow{f} Y) &\longmapsto (\mathcal{O}(Y) \xrightarrow{f^{-1}} \mathcal{O}(X)) \end{aligned}$$

assigns to each topological space its set of open sets.

Example 8.3. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is a copresheaf assigning to each group its elements.

Example 8.4. Let C be any (locally small) category. Recall (Example 3.9) the **hom functor**

$$\mathrm{Hom}_{\mathsf{C}} = \mathsf{C}(-,-) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{Set}.$$

By fixing the second variable (Proposition 4.10), this becomes a functor

$$\begin{aligned} よ = よ_{\mathsf{C}} : \mathsf{C} &\longrightarrow \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \\ X &\longmapsto \mathsf{C}(-, X), \end{aligned}$$

called the **Yoneda embedding**.[4]

For each X, $よX$ is the **representable presheaf** assigning to Y the set of morphisms $g : Y \to X$, pulled back along $h : Z \to Y$ via precomposition. As X varies, for a morphism $f : X \to X'$, the natural transformation $よf : よX \to よX'$ is given componentwise by *post*composition with f.

As always, it helps to understand the special cases of groups/monoids and posets. Let G be a group. Recall (Example 3.6) that a functor $G \to \mathsf{Set}$ is the same thing as a (left) action of G on a set; thus a presheaf $P : G^{\mathrm{op}} \to \mathsf{Set}$ can be thought of as a *right* action of G on a set, namely $P(*)$, where $g \in G$ acts via

$$x \cdot g := P(g)(x).$$

[4] よ is the Japanese character for "yo". We will soon show that よ is indeed an "embedding", i.e., full and faithful.

The representable presheaf よ$*$ is the set G itself, on which G acts via right multiplication.

In general, for any (locally small) category C, we can also think of presheaves on C as "right actions" of C. We define a **right action** of a category C to consist of a family of sets $(P(X))_{X \in \mathsf{C}}$ together with, for each $f : X \to Y \in \mathsf{C}$, an operation

$$\begin{aligned} P(Y) &\longrightarrow P(X) \in \mathsf{Set} \\ a &\longmapsto a \cdot f, \end{aligned}$$

satisfying the usual unit and associativity laws ($a \cdot 1 = 1$ and $(a \cdot g) \cdot f = a \cdot (g \circ f)$). A right action is the same thing as a presheaf $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, with

$$a \cdot f := P(f)(a).$$

For a poset P, we can of course regard P as a category, leading to the usual notion of presheaf $P^{\mathrm{op}} \to \mathsf{Set}$. However, in this case, it is much better to view P as a **2-enriched category**, with "hom-truth-values" instead of hom-sets (Section 5); indeed, hom-sets are the reason why Set-valued functors are so important for general category theory. So for a poset P, by a **2-valued presheaf** on P, we mean monotone map (i.e., functor) $P^{\mathrm{op}} \to 2$. Such a map is the indicator function χ_A of a **lower set** $A \subseteq P$, meaning a set which is downward-closed:

$$x \le y \in A \implies x \in A.$$

Moreover, $A \subseteq B \iff \chi_A \le \chi_B$, i.e., we have an order-isomorphism

$$\{\text{2-valued presheaves on } P\} = \mathsf{Pos}(P^{\mathrm{op}}, 2) \cong \mathrm{Low}(P) := \{\text{lower sets in } P\}.$$

Intuitively, a 2-valued presheaf $A \subseteq P$ assigns to each $x \in P$ only a property (true/false), instead of a set of data. For any $x \in P$, the "representable presheaf" $P(-, x)$ is the indicator function of the set of all $y \le x$, or the **principal ideal below** x, which we denote by

$$\downarrow x = \downarrow_P x := \{y \in P \mid y \le x\}.$$

So $\downarrow : P \to \mathrm{Low}(P)$ is the "2-valued Yoneda embedding".

We now come to (arguably) the single most important result in category theory, the Yoneda lemma, and its key corollary, the (fact that we correctly named the) Yoneda embedding. Before stating the general version, we first consider its special cases for posets and groups, which turn out to be quite familiar results.

Lemma 8.5 (Yoneda lemma for posets)**.** Let P be a poset, $x \in P$, and $A \in \mathrm{Low}(P)$. Then

$$x \in A \iff \downarrow x \subseteq A.$$

Proof. $\implies$: For every $y \in \downarrow x$, we have $y \le x \in A$, whence $y \in A$ since A is lower.

$\impliedby$: We have $x \in \downarrow x \subseteq A$. □

Corollary 8.6 (Yoneda embedding for posets)**.** For every poset P, we have an order-embedding

$$\downarrow : P \longrightarrow \mathrm{Low}(P).$$

Proof. $x \le y \iff x \in \downarrow y \iff \downarrow x \subseteq \downarrow y$ by Lemma 8.5. □

In particular, this implies that every poset embeds into a powerset.

Now consider a group (or monoid) G. The classical representation theorem for groups is Cayley's theorem, which says that every group G embeds into the symmetric group $\mathrm{Sym}(G)$ of bijections $G \cong G$. A stronger form of Cayley's theorem is

Theorem 8.7 (strong Cayley's theorem/Yoneda embedding for groups)**.** For every group G, we have an isomorphism

$$\begin{aligned} G &\cong \mathrm{Aut}(G \text{ as a right } G\text{-set}) \\ g &\mapsto g \cdot (-) \\ f(1) &\mapsfrom f. \end{aligned}$$

Proof. Starting from $g \in G$, we recover g from $g \cdot (-)$ as $g \cdot 1$. Conversely, starting from any automorphism $f : G \cong G$ of G as a right G-set, we have for any $g \in G$

$$\begin{aligned} f(g) &= f(1 \cdot g) \\ &= f(1) \cdot g \qquad \text{since } f \text{ is right-}G\text{-equivariant,} \end{aligned}$$

hence we recover f as left multiplication by $f(1)$. □

The Yoneda lemma in this case comes from observing that the above argument works more generally for right-equivariant maps $G \to X$ for an arbitrary right G-set X:

Lemma 8.8 (Yoneda lemma for groups)**.** For every group G and right G-set X, we have a bijection

$$\begin{aligned} X &\cong \{\text{right-}G\text{-equivariant maps } G \to X\} \\ x &\mapsto x \cdot (-) \\ f(1) &\mapsfrom f. \end{aligned}$$

Proof. Exactly the same as Theorem 8.7. □

Via the equivalence between right G-sets and presheaves on G (and between right-G-equivariant maps and natural transformations), Lemma 8.8 says: for any presheaf $P : G^{\mathrm{op}} \to \mathsf{Set}$ on G, we have a bijection

$$\begin{aligned} P(*) &\cong \mathsf{PSh}(G^{\mathrm{op}})(よ*, P) \\ x &\mapsto P(-)(x) \\ \alpha_*(1) &\mapsfrom \alpha. \end{aligned}$$

We now state the general Yoneda lemma and embedding. The proof is completely analogous to that of Theorem 8.7 (it is a good exercise to try to generalize the proof on your own before reading).

Lemma 8.9 (Yoneda)**.** Let C be a category, $X \in \mathsf{C}$, and $P \in \mathsf{PSh}(\mathsf{C})$. Then we have a bijection

$$\begin{aligned} P(X) &\cong \mathsf{PSh}(\mathsf{C})(よX, P) \\ a &\mapsto \alpha_a := \begin{pmatrix} \mathsf{C}(Y, X) \to P(Y) \\ f \mapsto P(f)(a) \end{pmatrix}_{Y \in \mathsf{C}} : よX \to P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set} \\ \alpha_X(1_X) &\mapsfrom \alpha : よX \to P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}. \end{aligned}$$

Proof. Starting from $a \in P(X)$, we recover a from the natural transformation α_a as $(\alpha_a)_X(1_X) = P(1_X)(a) = 1_{P(X)}(a) = a$. Conversely, starting from any $\alpha : よX \to P$, we have for any $Y \in \mathsf{C}$ and $f \in (よX)(Y) = \mathsf{C}(Y, X)$

$$\begin{aligned}
\alpha_Y(f) &= \alpha_Y(1_X \circ f) \\
&= \alpha_Y((よX)(f)(1_X)) && \text{by definition of } よX \\
&= P(f)(\alpha_X(1_X)) = \alpha_{\alpha_X(1_X),Y}(f)
\end{aligned}$$

by the naturality square

$$\begin{tikzcd}
(よX)(X) = \mathsf{C}(X, X) \arrow[r, "\alpha_X"] \arrow[d, "(よX)(f)"'] & P(X) \arrow[d, "P(f)"] \\
(よX)(Y) = \mathsf{C}(Y, X) \arrow[r, "\alpha_Y"'] & P(Y).
\end{tikzcd}$$

□

Corollary 8.10 (Yoneda embedding)**.** For any category C, the Yoneda embedding $よ : \mathsf{C} \to \mathsf{PSh}(\mathsf{C})$ is full and faithful.

Proof. For $f : X \to Y \in \mathsf{C}$, we have $(よf)_Z = f \circ (-) = (よY)(-)(f)$; in other words, each $よ : \mathsf{C}(X, Y) \to \mathsf{PSh}(\mathsf{C})(よX, よY)$ is the bijection from Lemma 8.9 for $P = よY$. □

Remark 8.11. The Yoneda bijection

$$P(X) \cong \mathsf{PSh}(\mathsf{C})(よX, P)$$

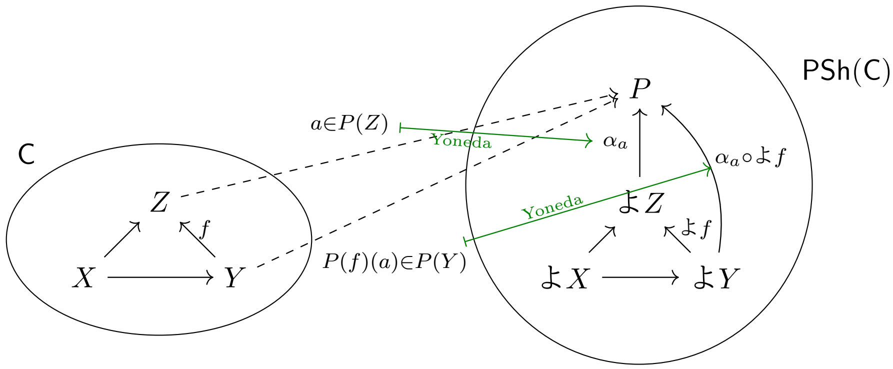
is natural in both variables X, P. For naturality in X, this means that for $f : X \to Y \in \mathsf{C}$, we have a commutative square

$$\begin{tikzcd}
a \arrow[rrr, mapsto, bend left=10] \arrow[ddd, mapsto] \arrow[dr, phantom, "\in" sloped] & & & \alpha_a = (P(-)(a) : \mathsf{C}(Z, Y) \to P(Z))_Z \arrow[ddd, mapsto] \\
& P(Y) \arrow[r, "\cong"] \arrow[d, "P(f)"'] & \mathsf{PSh}(\mathsf{C})(よY, P) \arrow[d, "(-)\circ よf"] \arrow[ur, phantom, "\ni" sloped] & \\
& P(X) \arrow[r, "\cong"] \arrow[dl, phantom, "\in" sloped] & \mathsf{PSh}(\mathsf{C})(よX, P) & \\
P(f)(a) & & & \alpha_a \circ よf = (P(f \circ -)(a) : \mathsf{C}(Z, X) \to P(Z))_Z. \arrow[lll, mapsto, bend left=10] \arrow[ul, phantom, "\ni" sloped]
\end{tikzcd}$$

Exercise 8.12. Verify naturality in P.

The Yoneda lemma and embedding provide another interpretation of presheaves. We can think of a presheaf $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ as a "hypothetical object" (in some category containing C), described solely in terms of "hypothetical morphisms $X \to P$" *from* preexisting objects $X \in \mathsf{C}$; these morphisms are the elements $a \in P(X)$, while $P(f)$ for $f : X \to Y \in \mathsf{C}$ tells us how to compose preexisting morphisms $f : X \to Y$ in C with these hypothetical morphisms $a : X \to P$.[5]

[5]For logicians: this is somewhat similar to "Katětov constructions" of Fraïssé limits, e.g., the construction of the random graph by (repeatedly) adding in new vertices described in terms of their edges to preexisting vertices, or the construction of the Urysohn universal metric space by adding in new points described as distance functions to preexisting points. Indeed, the latter construction can be seen as an "enriched Yoneda embedding", by regarding metric spaces as $[0, \infty)$-enriched categories, a point of view pioneered by Lawvere.



The Yoneda Lemma 8.9 tells us that this way of thinking of presheaves is actually realized in $\mathsf{PSh}(\mathsf{C})$: the hypothetical morphisms $X \to P$ are in bijection with actual morphisms $よ X \to P \in \mathsf{PSh}(\mathsf{C})$. The naturality condition in Remark 8.11 tells us that the rule for composing hypothetical with preexisting morphisms specified by P becomes the actual composition in $\mathsf{PSh}(\mathsf{C})$.

9 Consequences of Yoneda: representing categories and functors; monomorphisms and basic completeness arguments

Since presheaves on C are the same as right actions of C, the Yoneda embedding in particular shows that every (small) category is equivalent to a full subcategory of a category of structures (which are multi-sorted, with one underlying set for each $X \in \mathsf{C}$).

Exercise 9.1. Let $F : \mathsf{D} \to \mathsf{C}$ be an essentially surjective functor between (small) categories. Show that $(-) \circ F : \mathsf{PSh}(\mathsf{C}) \to \mathsf{PSh}(\mathsf{D})$ is conservative.

By taking D to be the discrete category on C_0, conclude that the forgetful functor $\mathsf{PSh}(\mathsf{C}) \to \mathsf{Set}^{\mathsf{C}_0}$ is conservative (as expected, since right actions of C are multi-sorted "algebraic" structures).

Furthermore, show that the disjoint union functor $\bigsqcup : \mathsf{Set}^{\mathsf{C}_0} \to \mathsf{Set}$ is faithful and conservative. Thus, every (small) category admits a faithful and conservative functor

$$\mathsf{C} \xrightarrow{よ} \mathsf{PSh}(\mathsf{C}) \xrightarrow{\text{forget}} \mathsf{Set}^{\mathsf{C}_0} \xrightarrow{\bigsqcup} \mathsf{Set}.$$

Exercise 9.2. In this exercise, we completely ignore size issues.[6]

Let C be a category. We know we can represent C as a category of structures, via $よ : \mathsf{C} \to \mathsf{PSh}(\mathsf{C})$. Sometimes,[7] it is better to represent C via the "double Yoneda embedding"

$$\mathrm{eval}_{\mathsf{C}} : \mathsf{C} \xrightarrow{よ_{\mathsf{C}^{\mathrm{op}}}} \mathsf{PSh}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = (\mathsf{Set}^{\mathsf{C}})^{\mathrm{op}} \xrightarrow{よ_{\mathsf{PSh}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}}} \mathsf{PSh}(\mathsf{PSh}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}) = \mathsf{Set}^{\mathsf{Set}^{\mathsf{C}}}.$$

Note that by Yoneda (twice), $\mathrm{eval}_{\mathsf{C}}$ is full and faithful.

[6] We can assume C, D are small, in which case $よ_{\mathsf{C}^{\mathrm{op}}}, よ_{\mathsf{D}^{\mathrm{op}}}, F^*$ make sense, although $\mathsf{Set}^{\mathsf{C}}, \mathsf{Set}^{\mathsf{D}}$ will no longer be small. We can then replace $\mathsf{Set}^{\mathsf{D}}$ with some small full subcategory $\mathsf{D}' \subseteq \mathsf{Set}^{\mathsf{D}}$ containing the representables, and replace $\mathsf{Set}^{\mathsf{C}}$ with some small full subcategory containing the representables as well as $F^*(P)$ for each $P \in \mathsf{D}'$, so that $F^* : \mathsf{Set}^{\mathsf{D}} \to \mathsf{Set}^{\mathsf{C}}$ restricts to $F^* : \mathsf{D}' \to \mathsf{C}'$.

[7] If we expect C to be a category of structures to which some duality/strong completeness theorem applies, as outlined in Section 1, then $\mathsf{Set}^{\mathsf{C}}$ is its "dual", which should be the "syntactic algebra" of a theory axiomatizing C; thus we should expect to recover C as the "double dual" of "homomorphisms" $\mathsf{Set}^{\mathsf{C}} \to \mathsf{Set}$ (preserving whatever operations on $\mathsf{Set}^{\mathsf{C}}$ correspond to logical operations), which should be a full subcategory of $\mathsf{Set}^{\mathsf{Set}^{\mathsf{C}}}$. See **TODO**.

(a) For $\mathsf{C} = G$ a group, explicitly describe $\mathrm{eval}_G : G \to \mathsf{Set}^{\mathsf{Set}^G}$ (note that eval_G consists of a single functor $\mathrm{eval}_G(*) : \mathsf{Set}^G \to \mathsf{Set}$, together with natural transformations $\mathrm{eval}_G(g) : \mathrm{eval}_G(*) \to \mathrm{eval}_G(*)$ for each $g \in G$; explicitly describe these).

(b) Generalize this description to arbitrary C, thus explaining the name eval_C.

Our goal is to show that just as every abstract category is equivalent to a concrete category of structures, so is every abstract functor equivalent to a "forgetful functor" between two such categories. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Note that precomposition with F naturally induces a functor

$$\begin{aligned} F^* : \mathsf{PSh}(\mathsf{D}) &\longrightarrow \mathsf{PSh}(\mathsf{C}) \\ G &\longmapsto G \circ F, \end{aligned}$$

rather than a functor $\mathsf{PSh}(\mathsf{C}) \to \mathsf{PSh}(\mathsf{D})$.[8] Let

$$\begin{aligned} F^{**} : \mathsf{Set}^{\mathsf{Set}^\mathsf{C}} &\longrightarrow \mathsf{Set}^{\mathsf{Set}^\mathsf{D}} \\ G &\longmapsto G \circ F^*. \end{aligned}$$

F^{**} can be (somewhat perversely) seen as a "forgetful functor" that takes an action of Set^C to its "restriction" along $F^* : \mathsf{Set}^\mathsf{D} \to \mathsf{Set}^\mathsf{C}$ (in analogy with restriction of a group action $H \curvearrowright X$ along a group homomorphism $G \to H$).

(c) Show that the square

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathrm{eval}_\mathsf{C}"] \arrow[d, "F"'] & \mathsf{Set}^{\mathsf{Set}^\mathsf{C}} \arrow[d, "F^{**}"] \\ \mathsf{D} \arrow[r, "\mathrm{eval}_\mathsf{D}"'] & \mathsf{Set}^{\mathsf{Set}^\mathsf{D}} \end{tikzcd}$$

commutes. (In other words, eval is a natural transformation, in fact a "2-natural transformation" between 2-functors $\mathfrak{Cat} \to \mathfrak{Cat}$.)

Thus, F can be regarded as the forgetful functor F^{**} restricted to the full images of $\mathrm{eval}_\mathsf{C}, \mathrm{eval}_\mathsf{D}$ (which are actions of $\mathsf{Set}^\mathsf{C}, \mathsf{Set}^\mathsf{D}$ obeying certain properties, by fullness).

One use of the Yoneda embedding (among many others) is that it immediately reduces many categorical statements to statements in Set.[9] We will give a very simple application of this technique, mostly as an excuse to introduce the following notions.

A morphism $f : X \to Y$ in a category C is **monic**, or a **monomorphism**, if

$$\forall g, h : Z \to X \in \mathsf{C}\,(f \circ g = f \circ h \implies g = h),$$

i.e., each

$$(よ f)_Z = f \circ (-) : (よ X)(Z) = \mathsf{C}(Z, X) \longrightarrow \mathsf{C}(Z, Y) = (よ Y)(W)$$

is injective. This is one standard notion of "injection" in general categories.

[8]There are in fact two canonical functors $\mathsf{PSh}(\mathsf{C}) \to \mathsf{PSh}(\mathsf{D})$ induced by F, called **left and right Kan extensions**, taking a right action of C to the "induced action" and "coinduced action" along F; see **TODO**.

[9]This can be seen as a very basic "completeness theorem", for the unary equational fragment of first-order logic; see **TODO** below.

Example 9.3. $f : X \to Y \in \mathsf{Set}$ is monic iff f is injective (consider $Z = 1$). Similarly for $\mathsf{Pos}, \mathsf{Top}$.

Example 9.4. $f : X \to Y \in \mathsf{Grp}$ is monic iff f is injective (consider $Z = \mathbb{Z}$). Similarly for any category of algebraic structures with free algebras on one generator.

Example 9.5. In the category $\mathsf{DivAbGrp}$ of divisible abelian groups,[10] the quotient map $f : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is monic.

Indeed, let G be another divisible abelian group and $g, h : G \to \mathbb{Q}$ with $g \neq h$. Let $x \in G$ such that $g(x) \neq h(x) \in \mathbb{Q}$, let $n > |g(x) - h(x)|$, and let $y \in G$ with $ny = x$. Then

$$ng(y) = g(x) \neq h(x) = nh(y)$$

whence $g(y) \neq h(y)$, and

$$0 < |g(y) - h(y)| = |g(x) - h(x)|/n < 1,$$

whence $f(g(y)) \neq f(h(y))$.

Example 9.6. All morphisms in a group G, or a poset P, are monic.

Exercise 9.7. Show directly that $\alpha : F \to G \in \mathsf{Set}^{\mathsf{C}}$ is monic iff each component $\alpha_X : F(X) \to G(X) \in \mathsf{Set}$ is injective. [Hint for $\Longrightarrow$: Yoneda.] (This follows from more general principles; see Example 22.16.)

A (generally) much stronger notion of "injection" in a category is that of a **section** $f : X \to Y$, which is a morphism that has a **retraction** $g : Y \to X$ such that $g \circ f = 1_X$.

Example 9.8. $f : X \to Y \in \mathsf{Set}$ is a section iff it is injective, and either $X \neq \varnothing$ or $X = Y = \varnothing$; f is a retraction iff f is surjective (this is one statement of the axiom of choice).

Example 9.9. $2\mathbb{Z} \hookrightarrow \mathbb{Z} \in \mathsf{Grp}$ is a monomorphism but not a section; $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \in \mathsf{Grp}$ is a surjection but not a retraction.

Proposition 9.10. Let C be a category.

(a) If $f : X \to Y \in \mathsf{C}$ is a section, then f is monic.

(b) If $f : X \to Y$ is monic and a retraction, then f is an isomorphism.

(c) If $f : X \to Y$ and $g : Y \to Z$ are monic, then so is $g \circ f$.

(d) If $f : X \to Y$ and $g : Y \to Z$ are such that $g \circ f$ is monic, then f is monic.

Proof 1. (a) Let g be a retraction of f, and let $f \circ k = f \circ l$. Then $k = g \circ f \circ k = g \circ f \circ l = l$.

(b) Let g be a section of f, i.e., $f \circ g = 1_Y$. Then $f \circ g \circ f = f$, whence $g \circ f = 1_X$ since f is monic.

(c) Let $g \circ f \circ k = g \circ f \circ l$. Then $f \circ k = f \circ l$ since g is monic, whence $k = l$ since f is monic.

(d) Let $f \circ k = f \circ l$. Then $g \circ f \circ k = g \circ f \circ l$, whence $k = l$ since $g \circ f$ is monic. ☐

While the above proofs are admittedly very simple, we actually don't have to think at all:

Proof 2. Clearly the above properties hold for injections in Set; and they are both preserved and reflected by よ (i.e., they hold for some morphisms in C iff they hold in $\mathsf{PSh}(\mathsf{C}) = \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ after applying よ), by fullness and faithfulness and the definition of monomorphism. ☐

[10] An abelian group G is **divisible** if for every $x \in G$ and $n \geq 1$, there is (not necessarily unique) $y \in G$ such that $ny = x$.

10 Universal properties

Let C be a (locally small) category. Recall that a presheaf of the form $よX$ for $X \in \mathsf{C}$ is called representable. More generally, we call an arbitrary presheaf $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ **representable** if it is isomorphic to some $よX$. By the Yoneda Lemma 8.9, an isomorphism $よX \cong P$ is of the form

$$\left(\begin{aligned} \mathsf{C}(Y, X) &\xrightarrow{\cong} P(Y) \\ f &\longmapsto P(f)(a) \end{aligned}\right)_{Y \in \mathsf{C}}$$

for some $a \in P(X)$; we call such (X, a) a **representation** of P. In other words, a representation is an object X equipped with "universal" data of the type specified by P, which pulls back to any other data on any other object via a unique map.

Example 10.1. The contravariant powerset presheaf $\mathcal{P} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is representable by $2 = \{0, 1\}$: for each set X, we have a bijection

$$\begin{aligned} \mathsf{Set}(X, 2) &\cong \mathcal{P}(X) \\ f &\mapsto f^{-1}(\{1\}) \end{aligned}$$

which is natural in X. In other words, 2 is a set equipped with a universal subset $\{1\} \subseteq 2$, which pulls back to any other subset via the indicator function of that subset.

Example 10.2. The **Sierpinski space** $\mathbb{S}$ is the topological space with underlying set 2, and with open sets $\varnothing, \mathbb{S}, \{1\}$ (so $\{1\}$ is open but not closed). $\mathbb{S}$ is the topological space equipped with a universal open set $\{1\}$: for any other space X, we have a bijection

$$\begin{aligned} \mathsf{Top}(X, \mathbb{S}) &\cong \mathcal{O}(X) \\ f &\mapsto f^{-1}(\{1\}), \end{aligned}$$

natural in X (so $\mathbb{S}$ represents the open sets presheaf $\mathcal{O}$).

Example 10.3. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is a **representable copresheaf** on Grp (i.e., a representable presheaf on $\mathsf{Grp}^{\mathrm{op}}$), i.e., there is a group with a universal element, namely $\mathbb{Z}$ with $1 \in \mathbb{Z}$: for any other group G, we have a bijection

$$\begin{aligned} \mathsf{Grp}(\mathbb{Z}, G) &\cong G \\ f &\mapsto f(1). \end{aligned}$$

Similarly, there is a universal group with two elements, namely the free group $\mathbb{F}_2 = \langle a, b \rangle$: for any other group G, we have a bijection

$$\begin{aligned} \mathsf{Grp}(\mathbb{F}_2, G) &\cong G^2 \\ f &\mapsto (f(a), f(b)). \end{aligned}$$

In other words, $\mathbb{F}_2$ represents the composite

$$\mathsf{Grp} \xrightarrow{\text{forget}} \mathsf{Set} \xrightarrow{X \mapsto X^2} \mathsf{Set},$$

which is the copresheaf assigning to each group G all the pairs of elements in G.

Exercise 10.4. Describe a presheaf on the category Mon of monoids which is represented by $\mathbb{Z}$.

Example 10.5. For ($\mathbb{R}$-)vector spaces U, V, W, we have the set of bilinear maps $U \times V \to W$, which may be composed with a linear map $W \to X$ to get bilinear maps $U \times V \to X$. Thus we have a copresheaf

$$\begin{aligned} \mathsf{Vec} &\longrightarrow \mathsf{Set} \\ W &\longmapsto \{\text{bilinear maps } U \times V \to W\} \\ (f : W \to X) &\longmapsto f \circ (-), \end{aligned}$$

which is represented by the tensor product $U \otimes V$ equipped with the universal bilinear map $\otimes : U \times V \to U \otimes V$.

Exercise 10.6. The covariant powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is *not* representable. (Thus, the statement "2 is the set with a universal subset $\{1\}$" from Example 10.1 is ambiguous without specifying the "variance" of the universality.)

One of the first things one learns about universal properties is that they characterize objects uniquely up to unique isomorphism. This is a simple consequence of the Yoneda embedding:

Corollary 10.7. Let $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf, and $(X, a), (Y, b)$ be two representations of P. Then there is a unique isomorphism $f : X \cong Y \in \mathsf{C}$ such that $a = P(f)(b)$.

Proof. Let $\alpha : \text{よ}X \cong P$ and $\beta : \text{よ}Y \cong P$ correspond via Yoneda to a, b respectively. Then clearly there is a unique isomorphism $\text{よ}X \cong \text{よ}Y$ making the following diagram commute, which by fullness and faithfulness of よ (Corollary 8.10) is $\text{よ}f$ for a unique isomorphism $f : X \cong Y$.

$$\begin{tikzcd} & P & \\ \text{よ}X \arrow[ur, "\alpha", "\cong"'] \arrow[rr, dashed, "\cong", "\text{よ}f"'] & & \text{よ}Y \arrow[ul, "\beta"', "\cong"] \end{tikzcd}$$

By Remark 8.11, commutativity of this diagram is equivalent to $a = P(f)(b)$. ☐

Example 10.8. For vector spaces V, W and two different versions of the tensor product $V \otimes W$ and $V \otimes' W$, equipped with universal bilinear maps $\otimes : V \times W \to V \otimes W$ and $\otimes' : V \times W \to V \otimes' W$, we get induced linear maps $f : V \otimes W \to V \otimes' W$ and $g : V \otimes' W \to V \otimes W$ such that $f(v \otimes w) = v \otimes' w$ and $g(v \otimes' w) = v \otimes w$, i.e., the following two triangles commute.

$$\begin{tikzcd} & V \otimes W \arrow[d, shift left, "f"] \\ V \times W \arrow[ur, "\otimes"] \arrow[dr, "\otimes'"'] & \\ & V \otimes' W \arrow[uu, shift left, "g"] \end{tikzcd}$$

Then $g(f(v \otimes w)) = g(v \otimes' w) = v \otimes w$, whence $g \circ f = 1_{V \otimes W}$, and similarly $f \circ g = 1_{V \otimes' W}$, so f and g give inverse isomorphisms $V \otimes W \cong V \otimes' W$ exchanging the universal bilinear maps $\otimes, \otimes'$. (This argument is the result of unraveling the Yoneda Lemma 8.9, Corollary 8.10, and Proposition 6.22.)

Note that for posets P, a 2-valued presheaf, i.e., lower set $A \subseteq P$, is representable iff it has a greatest element, i.e., there is a universal element obeying property A (rather than being equipped with universal data).

11 Adjoint functors

Consider again the Example 10.3 of free groups as representable copresheaves. To be more explicit, let $U : \mathsf{Grp} \to \mathsf{Set}$ be the forgetful functor. For each $X \in \mathsf{Set}$, let $F(X)$ be the free group generated by X, and let $\eta_X : X \to F(X) \in \mathsf{Set}$ be the injection of generators. Then $(F(X), \eta_X)$ represents the copresheaf $\mathsf{Set}(X, U(-)) : \mathsf{Grp} \to \mathsf{Set}$: for any other group G, we have a bijection

$$\begin{aligned} \phi_{X,G} : \mathsf{Grp}(F(X), G) &\cong U(G)^X = \mathsf{Set}(X, U(G)) \\ g &\mapsto U(g) \circ \eta_X. \end{aligned}$$

In other words, η_X is the universal map from X to the underlying set of a group: for any other such map $f : X \to U(G)$, there is a unique group homomorphism $g : F(X) \to G$ such that $f = U(g) \circ \eta_X$.

$$\mathsf{Set} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(F(X)) \arrow[d, dashed, "U(g)"] \\ & U(G) \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[d, dashed, "\exists! g"] \\ G \end{tikzcd} \quad \mathsf{Grp}$$

This family of representable presheaves of the form $\mathsf{Set}(X, U(-))$, parametrized (contravariantly) functorially by $X \in \mathsf{Set}$, is an example of an **adjunction**, one of the most important concepts in category theory. Adjunctions have a number of equivalent descriptions; we now give some of the other ones for the free groups example, before giving the general definition.

The free group *functor* $F : \mathsf{Set} \to \mathsf{Grp}$ is defined by letting, for each $f : X \to Y \in \mathsf{Set}$, $F(f) : F(X) \to F(Y) \in \mathsf{Grp}$ be the unique group homomorphism extending f, i.e., such that

$$\mathsf{Set} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & U(F(X)) \arrow[d, dashed, "U(F(f))"] \\ Y \arrow[r, "\eta_Y"] & U(F(Y)) \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[d, dashed, "F(f)"] \\ F(Y) \end{tikzcd} \quad \mathsf{Grp}$$

commutes. In other words, F is defined on morphisms in the unique way which makes η into a natural transformation

$$\eta : 1_{\mathsf{Set}} \longrightarrow U \circ F : \mathsf{Set} \to \mathsf{Set}.$$

In terms of the above bijections $\phi_{X,G} : \mathsf{Grp}(F(X), G) \cong \mathsf{Set}(X, U(G))$, we have the explicit formula

$$F(f) = \phi^{-1}_{X,F(Y)}(\eta_Y \circ f).$$

To say that $(F(X), \eta_X)$ represents $\mathsf{Set}(X, U(-))$ means that $\eta_X \in \mathsf{Set}(X, U(F(X)))$ corresponds, via the Yoneda lemma, to the natural isomorphism

$$\phi_X := (\phi_{X,G})_G : \text{よ}_{\mathsf{Grp}^{\mathrm{op}}}(F(X)) = \mathsf{Grp}(F(X), -) \xrightarrow{\sim} \mathsf{Set}(X, U(-)) \in \mathsf{Set}^{\mathsf{Grp}} = \mathsf{PSh}(\mathsf{Grp}^{\mathrm{op}}).$$

By naturality of the Yoneda lemma (Remark 8.11, in *both* variables), the above naturality square for η is equivalent to the naturality square

$$\begin{tikzcd} \mathsf{Grp}(F(Y), -) = \text{よ}_{\mathsf{Grp}^{\mathrm{op}}}(F(Y)) \arrow[r, "\phi_Y", "\cong"'] \arrow[d, "(-\circ F(f))_{G \in \mathsf{Grp}} = \text{よ}(F(f))"'] & \mathsf{Set}(Y, U(-)) \arrow[d, "(-\circ f)_{G \in \mathsf{Grp}}"] \\ \mathsf{Grp}(F(X), -) = \text{よ}_{\mathsf{Grp}^{\mathrm{op}}}(F(X)) \arrow[r, "\cong", "\phi_X"'] & \mathsf{Set}(X, U(-)) \end{tikzcd}$$

(the lower left composite in this latter square corresponds via Yoneda to $\mathsf{Set}(X, U(F(f)))(\eta_X) = U(F(f)) \circ \eta_X : X \to U(F(Y))$, while the upper right composite corresponds via Yoneda to $\eta_Y \circ f : X \to U(F(Y))$), so that we have a natural isomorphism (in two variables)

$$\begin{aligned} \phi : \mathsf{Grp}(F(?), -) \cong \mathsf{Set}(?, U(-)) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{Grp}} \\ \text{(i.e., } \mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp} \to \mathsf{Set}\text{).} \end{aligned}$$

Conceptually, this ϕ witnesses that F is a lifting up to natural isomorphism of $X \mapsto \mathsf{Set}(X, U(-))$, which lands in the full subcategory $\subseteq \mathsf{Set}^{\mathsf{Grp}}$ of representable copresheaves, across the Yoneda embedding (for $\mathsf{Grp}^{\mathrm{op}}$) which is full and faithful with essential image consisting of the representables:

$$\begin{tikzcd}
\mathsf{Set}^{\mathrm{op}} \arrow[d, dashed, "F"'] \arrow[rrr, bend left=10, "X \mapsto \mathsf{Set}(X, U(-))"] \arrow[dr, phantom, "\overset{\sim}{\Rightarrow}_{\phi}"] & & & \\
\mathsf{Grp}^{\mathrm{op}} \arrow[r, "\text{よ}_{\mathsf{Grp}^{\mathrm{op}}}"'] & \{\text{representables}\} & \subseteq & \mathsf{Set}^{\mathsf{Grp}}
\end{tikzcd}$$

(we can only find a lifting up to natural isomorphism, because $\text{よ}_{\mathsf{Grp}^{\mathrm{op}}}$ is only an equivalence, not an isomorphism). By the formulas in the Yoneda Lemma 8.9, η, ϕ determine each other via

$$\eta_X = \phi_{X,F(X)}(1_{F(X)}), \qquad\qquad \phi_{X,G}(g : F(X) \to G) = U(g) \circ \eta_X.$$

We now give the general definition of an adjunction, in its various equivalent forms. Let C, D be (locally small) categories. In the above example, $\mathsf{C} = \mathsf{Set}$, $\mathsf{D} = \mathsf{Grp}$, and $G = U$ (to avoid confusion with groups G).

(a) A **right adjoint functor** $G : \mathsf{D} \to \mathsf{C}$ is a functor such that for each $X \in \mathsf{C}$, the copresheaf $\mathsf{C}(X, G(-)) \in \mathsf{Set}^{\mathsf{D}}$ is representable, i.e., there is some $F(X) \in \mathsf{D}$ and $\eta_X : X \to G(F(X))$ such that for any other $Y \in \mathsf{D}$, we have a bijection

$$\begin{aligned} \phi_{X,Y} : \mathsf{D}(F(X), Y) &\cong \mathsf{C}(X, G(Y)) \\ g &\mapsto G(g) \circ \eta_X. \end{aligned}$$

In other words, each $X \in \mathsf{C}$ admits a *universal morphism* $\eta_X : X \to G(F(X)) \in \mathsf{C}$ *to an object of the form* $G(Y)$ *for* $Y \in \mathsf{D}$:

$$\begin{tikzcd}
\mathsf{C} & X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & G(F(X)) \arrow[d, dashed, "G(g)"] & & F(X) \arrow[d, dashed, "\exists! g"] & \mathsf{D} \\
& & G(Y) & & Y &
\end{tikzcd}$$

(Note that at this stage, F is only defined on objects, so not yet a functor.)

This definition involves the least amount of data, so is often the easiest to check in concrete situations. However, it masks the perfect symmetry of adjunctions (see (c) below).

Given a right adjoint functor $G : \mathsf{D} \to \mathsf{C}$, as well as chosen representations $(F(X), \eta_X)$ for each $X \in \mathsf{C}$ (which are unique up to unique isomorphism, by Corollary 10.7), the η_X correspond via Yoneda to the natural isomorphisms $\phi_X = (\phi_{X,Y})_{Y \in \mathsf{D}} : \text{よ}_{\mathsf{D}^{\mathrm{op}}}(F(X)) \cong \mathsf{C}(X, G(-))$. We may extend F to a functor $\mathsf{C} \to \mathsf{D}$ in a unique way (by the universal property of the η_X) such that the naturality squares

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & G(F(X)) \arrow[d, "G(F(f))"] \\ X' \arrow[r, "\eta_{X'}"'] & G(F(X')) \end{tikzcd} \qquad \text{or equivalently} \qquad \begin{tikzcd} \text{よ}(F(X')) \arrow[r, "\phi_{X'}", "\cong"'] \arrow[d, "\text{よ}(F(f))"'] & \mathsf{C}(X', G(-)) \arrow[d, "(-\circ f)_{Y \in \mathsf{D}}"] \\ \text{よ}(F(X)) \arrow[r, "\cong", "\phi_X"'] & \mathsf{C}(X, G(-)) \end{tikzcd}$$

commute (these are equivalent by Remark 8.11), i.e., so that

$$\eta : 1_{\mathsf{C}} \to G \circ F : \mathsf{C} \to \mathsf{C} \qquad \text{or equivalently} \qquad \phi : \mathsf{D}(F(?), -) \cong \mathsf{C}(?, G(-)) : \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$$

are natural transformations. The first square says that $F(f)$ is induced via the universal property of η_X by $\eta_{X'} \circ f$, i.e.,

$$\begin{aligned} \phi_{X,F(X')} : \mathsf{D}(F(X), F(X')) &\cong \mathsf{C}(X, G(F(X'))) \\ F(f) &\mapsto \eta_{X'} \circ f. \end{aligned}$$

(b) An **adjunction** between C, D consists of functors $F : \mathsf{C} \to \mathsf{D}$ (the **left adjoint**) and $G : \mathsf{D} \to \mathsf{C}$ (the **right adjoint**) equipped with a natural transformation

$$\eta : 1_{\mathsf{C}} \to G \circ F : \mathsf{C} \to \mathsf{C},$$

called the **unit** of the adjunction, which is componentwise a universal morphism $\eta_X : X \to G(F(X))$ in the sense of (a), i.e., a representation of $\mathsf{C}(X, G(-))$.

(c) Equivalently, an **adjunction** between C, D consists of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with bijections

$$\phi_{X,Y} : \mathsf{D}(F(X), Y) \cong \mathsf{C}(X, G(Y)) : \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set},$$

natural in both variables.

Each η_X corresponds to $\phi_X = (\phi_{X,Y})_Y$ via Yoneda, which means that η, ϕ determine each other via

$$\begin{aligned} \phi_{X,Y} : \overbrace{\mathsf{D}(F(X), Y)}^{\text{よ}_{\mathsf{D}^{\mathrm{op}}}(F(X))(Y)} &\cong \mathsf{C}(X, G(Y)) \\ (F(X) \xrightarrow{g} Y) &\mapsto (X \xrightarrow{\eta_X} G(F(X)) \xrightarrow{G(g)} G(Y)) \\ (F(X) \xrightarrow{1_X} F(X)) &\mapsto (X \xrightarrow{\eta_X} G(F(X))). \end{aligned}$$

We denote an adjunction by

$$F \dashv G$$

(but note that the unit η is also part of the data of the adjunction).

By Corollary 10.7, every right adjoint functor has a unique-up-to-unique-isomorphism left adjoint (more precisely, between any two left adjoints, there is a unique natural isomorphism which also converts between the respective units).

Example 11.1 (free structures)**.** Analogously to the free groups example, for any forgetful functor $U : \mathsf{C} \to \mathsf{Set}$ from a category of structures C, U has a left adjoint (i.e., U *is* a right adjoint) iff each set X admits a universal map $\eta_X : X \to G(F(X))$ into the underlying set of a C-structure $F(X)$; this is the general notion of **free structure generated by** X.

For example, the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ has left adjoint $(-)^{<\mathbb{N}}$, where $X^{<\mathbb{N}}$ for a set X is the free monoid of finite sequences from X, multiplied via concatenation. The unit $\eta_X : X \to X^{<\mathbb{N}}$, given by injection of generators, is the universal map from X to a monoid:

$$\mathsf{Set}\quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(X^{<\mathbb{N}}) \arrow[d, dashed, "U(g)"] \\ & U(M) \end{tikzcd} \qquad \begin{tikzcd} X^{<\mathbb{N}} \arrow[d, dashed, "\exists! g"] \\ M \end{tikzcd} \quad \mathsf{CRing}$$

For forgetful U, one usually doesn't bother to draw the two different categories, instead just drawing

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & X^{<\mathbb{N}} \arrow[d, dashed, "\exists! g"] \\ & M; \end{tikzcd}$$

but note that the morphism g lies in Mon, while the rest of the diagram is in Set.

Example 11.2 (relative free structures)**.** Let $V : \mathsf{Ring} \to \mathsf{Mon}$ be the forgetful functor taking a ring to its *multiplicative* monoid. Its left adjoint is given at each $M \in \mathsf{Mon}$ by the **monoid ring** $\mathbb{Z}[M]$ (when M is a group, called the **group ring**), the free abelian group generated by M with multiplication bilinearly extending that of M. The injection of generators $\eta_M : M \to \mathbb{Z}[M]$ is the universal monoid homomorphism from M into the multiplicative monoid $V(R)$ of a ring R:

$$\begin{tikzcd} M \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathbb{Z}[M] \arrow[d, dashed, "\exists! g"] \\ & R. \end{tikzcd}$$

Adjunctions can be composed: given

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "H"] \arrow[r, phantom, "\perp"] & \mathsf{E}, \arrow[l, shift left, "K"] \end{tikzcd}$$

using definition (c), we have

$$\mathsf{E}(H(F(X)), Z) \cong \mathsf{D}(F(X), K(Z)) \qquad \cong \mathsf{C}(X, G(K(Z))),$$

naturally in X, Z; thus $H \circ F \dashv G \circ K$. The adjunction unit is given via Yoneda by

$$1_{HFX} \mapsto (FX \xrightarrow{\iota_{FX}} KHFX) \mapsto (X \xrightarrow{\eta_X} GFX \xrightarrow{G\iota_{FX}} GKHFX)$$

where η is the unit for $F \dashv G$, and ι is the unit for $H \dashv K$.

Example 11.3. We have

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "(-)^{<\mathbb{N}}"] \arrow[r, phantom, "\perp"] & \mathsf{Mon} \arrow[l, shift left, "U"] \arrow[r, shift left, "\mathbb{Z}[-]"] \arrow[r, phantom, "\perp"] & \mathsf{Ring} \arrow[l, shift left, "V"] \end{tikzcd}$$

where $U \circ V : \mathsf{Ring} \to \mathsf{Set}$ is the forgetful functor, whence $\mathbb{Z}[(-)^{<\mathbb{N}}]$ is its left adjoint, i.e., the free ring on a set X can be constructed in two stages: first by taking all finite strings $X^{<\mathbb{N}}$ (with concatenation), then taking the monoid ring $\mathbb{Z}[X^{<\mathbb{N}}]$, yielding the non-commuting polynomials over X as expected.

Exercise 11.4. What is the group ring construction $\mathbb{Z}[-] : \mathsf{Grp} \to \mathsf{Ring}$ left adjoint to? [Use Example 11.2 and Example 11.7.]

Definition (c) of adjunctions is clearly symmetric: we have

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D} \arrow[l, shift left, "G"] \end{tikzcd} \iff \begin{tikzcd} \mathsf{C}^{\mathrm{op}} \arrow[r, shift left, "F"] \arrow[r, phantom, "\top"] & \mathsf{D} \arrow[l, shift left, "G"] \end{tikzcd},$$

since

$$\mathsf{D}^{\mathrm{op}}(Y, F(X)) = \mathsf{D}(F(X), Y) \cong \mathsf{C}(X, G(Y)) = \mathsf{C}^{\mathrm{op}}(G(Y), X).$$

Thus, the non-symmetric (a) and (b) can be dualized, yielding two more definitions of adjunctions:

(d) A **left adjoint functor** $F : \mathsf{C} \to \mathsf{D}$ is a functor such that for every $Y \in \mathsf{D}$, the presheaf $\mathsf{D}(F(-), Y) \in \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is representable, i.e., every $Y \in \mathsf{D}$ admits a *universal morphism* $\varepsilon_Y : F(G(Y)) \to Y$ *from an object of the form* $F(X)$ *for* $X \in \mathsf{C}$:

$$\mathsf{C} \quad \begin{tikzcd} X \arrow[d, dashed, "\exists! f"'] \\ G(Y) \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[d, dashed, "F(f)"'] \arrow[dr, "g"] & \\ F(G(Y)) \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd} \quad \mathsf{D}$$

Given such $(G(Y), \varepsilon_Y)$ for each Y, there is a unique definition of G on morphisms so that the naturality squares for ε commute.

(e) An **adjunction** between C, D consists of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with a natural transformation

$$\varepsilon : F \circ G \longrightarrow 1_{\mathsf{D}} : \mathsf{D} \to \mathsf{D},$$

called the **counit** of the adjunction, which is componentwise universal in the sense of (d).

ε is related to the bijections $\phi_{X,Y}$ in (c) dually to how η is:

$$\begin{aligned} \phi_{X,Y} : \mathsf{D}(F(X), Y) &\cong \mathsf{C}(X, G(Y)) \\ \varepsilon_Y \circ G(g) &\mapsfrom g \\ \varepsilon_Y &\mapsfrom 1_{G(Y)}. \end{aligned}$$

Exercise 11.5 (adjunctions via unit/counit)**.** Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors and $\eta : 1_{\mathsf{C}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathsf{D}}$ be natural transformations. Show that η, ε are the unit and counit respectively of a (unique, by definition (b) or (e)) adjunction $F \dashv G$ iff

$$\varepsilon_F \circ F(\eta) = 1_F, \qquad\qquad G(\varepsilon) \circ \eta_G = 1_G$$

(recall the **whiskerings** of a natural transformation from Section 5). These two equations are known as the **triangle identities**. Expressed in terms of diagrams in the 2-category $\mathfrak{Cat}$ as in Section 5, they say that the composite natural transformations

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "1"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{C} \arrow[dr, "F"] & \\
& \mathsf{D} \arrow[ur, "G"'] \arrow[rr, "1"'] & \arrow[u, phantom, "\Downarrow\varepsilon"] & \mathsf{D},
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathsf{C} \arrow[rr, "1"] \arrow[dr, "F"] & \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{C} \\
\mathsf{D} \arrow[ur, "G"] \arrow[rr, "1"'] & \arrow[u, phantom, "\Downarrow\varepsilon"] & \mathsf{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

are both the identity (on F, G respectively).

Thus, we have yet another equivalent definition of adjunction:

(f) An **adjunction** between C, D consists of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with natural transformations $\eta : 1_{\mathsf{C}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathsf{D}}$ obeying the triangle identities.

Note that this definition involves purely equational conditions in the categories C, D, unlike the definitions (a)–(e).

Example 11.6. Consider again the free-forgetful adjunction

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Grp}. \arrow[l, shift left, "U"] \end{tikzcd}$$

What is the *counit* $\varepsilon : F \circ U \to 1_{\mathsf{Grp}}$? For each $G \in \mathsf{Grp}$, ε_G is the preimage of $1_{U(G)}$ under

$$\begin{aligned} \phi_{X,G} : \mathsf{Grp}(F(X), G) &\cong \mathsf{Set}(X, U(G)) \\ g &\mapsto U(g) \circ \eta_X \end{aligned}$$

when $X = U(G)$, i.e., $\varepsilon_G : F(U(G)) \to G$ is the unique group homomorphism in

$$\mathsf{Set}\quad \begin{tikzcd} U(G) \arrow[r, "\eta_{U(G)}"] \arrow[dr, "1_{U(G)}"'] & U(F(U(G))) \arrow[d, dashed, "U(\varepsilon_G)"] \\ & U(G) \end{tikzcd} \qquad \begin{tikzcd} F(U(G)) \arrow[d, dashed, "\exists!\varepsilon_G"] \\ G \end{tikzcd} \quad \mathsf{Grp}$$

In other words, ε_G takes a word in the free group $F(U(G))$ over the underlying set of G, and evaluates the word in G.

Example 11.7. The forgetful functor (i.e., inclusion) $J : \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ has a right adjoint, taking a monoid M to the subgroup $I(M) \subseteq M$ of invertible elements (with $\varepsilon_M : I(M) \to M$) the inclusion): for any other monoid homomorphism $g : G \to M$ from a group G, g necessarily lands in $I(M)$.

$$\begin{tikzcd} G \arrow[d, dashed] \arrow[dr, "g"] & \\ I(M) \arrow[r, hook, "\varepsilon_M"'] & M \end{tikzcd}$$

(We are once again abusing notation by conflating the two categories. This abuse isn't as bad as the previous ones, since $\mathsf{Grp} \subseteq \mathsf{Mon}$ is a *full* subcategory, so the morphisms are really the same.)

What is the *unit* η of $J \dashv I$? Each η_G is the forward image of $1_{J(G)}$ under

$$\begin{aligned} \mathsf{Mon}(J(G), M) &\cong \mathsf{Grp}(G, I(M)) \\ g = \varepsilon_M \circ I(g) &\mapsfrom g, \end{aligned}$$

i.e., the dotted arrow in the above triangle when $M = G$ and $g = 1_G$, which is $1_G : G \to I(G) = G$.

Of course, $J : \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ also has a left adjoint $K : \mathsf{Mon} \to \mathsf{Grp}$, taking a monoid M to the free group $K(M)$ it generates while preserving the operations in M (which can be constructed abstractly by taking the free group generated by the underlying set of M, then quotienting by relations that say that $\cdot$ and 1 from M are preserved). So J has both left and right adjoints:

$$\begin{tikzcd} \mathsf{Grp} \arrow[d, hook, "J" description] \\ \mathsf{Mon} \arrow[u, bend left=60, "K"] \arrow[u, bend right=60, "I"'] \end{tikzcd} \qquad K \dashv J \dashv I$$

Note that the unit $M \to K(M)$ (i.e., $J(K(M))$) of $K \dashv J$ is generally *not* injective, in contrast to typical forgetful functors to Set. For example, if $M = ([0, \infty], +)$, then $K(M)$ is the trivial monoid, since for all $x \in [0, \infty]$, we have $x + \infty = \infty$ in $[0, \infty]$, whence $x = 0$ in $K([0, \infty])$.

Example 11.8. Somewhat similarly to the previous example, the inclusion $\mathsf{DivAbGrp} \hookrightarrow \mathsf{AbGrp}$ from the full subcategory of divisible abelian groups has a right adjoint, taking an abelian group A to the subgroup of divisble elements

$$D(A) := \{a \in A \mid \forall n \geq 1 \, \exists b \in A \, (nb = a)\}.$$

Again $\varepsilon_A : D(A) \hookrightarrow A$ is the inclusion. Any other $g : B \to A$ from divisible B lands in $D(A)$.

$$\begin{tikzcd} B \arrow[d, dashed] \arrow[dr, "g"] & \\ D(A) \arrow[r, hook, "\varepsilon_A"'] & A \end{tikzcd}$$

The unit at each $B \in \mathsf{DivAbGrp}$ is the identity $1_B : B \to D(B) = B$.

Exercise 11.9. Show that the inclusion $\mathsf{DivAbGrp} \to \mathsf{AbGrp}$ does *not* have a left adjoint. [Formalize the last sentence of Example 9.4 for faithful right adjoint functors $\mathsf{C} \to \mathsf{Set}$. Then, use Example 9.5 and the fact that the forgetful functor $\mathsf{AbGrp} \to \mathsf{Set}$ *does* have a left adjoint.]

Besides free-forgetful adjunctions, there are some other large families of commonly occurring adjunctions that fit a particular pattern. The following two examples are prototypical of two such families; the second is especially relevant to our purposes.

Example 11.10 (product-hom adjunction)**.** For any sets X, Y, Z, we have a bijection (given by fixing the first variable)

$$\mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, Z^Y),$$

which is easily seen to be natural in all three variables. Fixing Y, we get an adjunction

$$(-) \times Y \dashv (-)^Y : \mathsf{Set} \to \mathsf{Set}.$$

To compute the unit $\eta_X : X \to (X \times Y)^Y$ and counit $\varepsilon_Z : Z^Y \times Y \to Z$:

$$\begin{aligned} \mathsf{Set}(X \times Y, Z) &\cong \mathsf{Set}(X, Z^Y) \\ 1_{X\times Y} &\mapsto \begin{pmatrix} \eta_X : X \to (X \times Y)^Y \\ \quad x \mapsto (y \mapsto (x, y)) \end{pmatrix} \\ \begin{pmatrix} \varepsilon_Z : Z^Y \times Y \to Z \\ \quad (f, y) \mapsto f(y) \end{pmatrix} &\leftarrow\!\shortmid 1_{Z^Y}. \end{aligned}$$

Exercise 11.11. Fix an abelian group B. Work through the details of the **tensor-hom adjunction**

$$\mathsf{AbGrp}(A \otimes B, C) \cong \mathsf{AbGrp}(A, \mathrm{Hom}(B, C)),$$

the "abelianized" version of the product-hom adjunction.

Exercise 11.12 (free and shift actions)**.** Let G be a group, G-Set be the category of G-sets (sets equipped with a (left) G-action, and G-equivariant maps between them), and $U : G\text{-}\mathsf{Set} \to \mathsf{Set}$ be the forgetful functor.

(a) Show that U has a left adjoint, defined on objects by taking each set X to $G \times X$ equipped with a suitable action. Describe the unit and counit.

(b) In group theory, an action $G \curvearrowright X$ is called **free** if only the identity $1 \in G$ has fixed points. Show that free actions (in this sense) are the same as free actions (in the sense of (a)). [However, from the categorical perspective, the free G-set $G \times X$ generated by X is not just the G-set $G \times X$, but $G \times X$ equipped with the unit $\eta_X : X \to G \times X$.]

(c) Show that U has a right adjoint, taking each set X to X^G equipped with a suitable action. Describe the unit and counit.

(d) Does everything work equally well when G is a monoid? (It should, if you did (c) the right way.)

Example 11.13 (hom-hom adjunction)**.** Fixing instead a set Z, we have

$$\mathsf{Set}(X, Z^Y) \cong Z^{X\times Y} \cong \mathsf{Set}(Y, Z^X) = \mathsf{Set}^{\mathrm{op}}(Z^X, Y),$$

naturally in X, Y, yielding an adjunction

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "Z^{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathrm{op}}. \arrow[l, shift left, "Z^{(-)}"] \end{tikzcd}$$

The unit η_X is given by

$$\begin{aligned} \mathsf{Set}^{\mathrm{op}}(Z^X, Y) &\cong Z^{X\times Y} & &\cong \mathsf{Set}(X, Z^Y) \\ 1_{Z^X} &\mapsto \begin{pmatrix} X \times Z^X \to Z \\ \quad (x, f) \mapsto 1_{Z^X}(f)(x) = f(x) \end{pmatrix} & &\mapsto \begin{pmatrix} \eta_X : X \to Z^{Z^X} \\ \quad x \mapsto (f \mapsto f(x)) \end{pmatrix}. \end{aligned}$$

By symmetry, the counit ε is the same as η (but with components regarded as in $\mathsf{Set}^{\mathrm{op}}$, with flipped domain and codomain).

When $Z = 2$, we may identify $2^{(-)}$ with the contravariant powerset functor $\mathcal{P}$ (with precomposition of indicator function $\rightsquigarrow$ preimage). The self-adjunction $\mathcal{P} \dashv \mathcal{P}$ then becomes

$$\mathsf{Set}(X, \mathcal{P}(Y)) \cong \mathsf{P}(X \times Y) \cong \mathsf{Set}(Y, \mathcal{P}(X)),$$

where a subset $A \subseteq X \times Y$ corresponds to the map $X \to \mathcal{P}(Y)$ taking each $x \in X$ to its vertical fiber $A_x = \{y \mid (x, y) \in A\}$, or to the map $Y \to \mathcal{P}(X)$ taking each y to its horizontal fiber.

Adjunctions of the form

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D}^{\mathrm{op}} \arrow[l, shift left, "G"] \end{tikzcd}$$

are called **contravariant** or **dual adjunctions**, and are given by natural bijections

$$\phi_{X,Y} : \mathsf{D}(Y, F(X)) = \mathsf{D}^{\mathrm{op}}(F(X), Y) \cong \mathsf{C}(X, G(Y)),$$

or equivalently by units and/or counits

$$\eta : 1_{\mathsf{C}} \to G \circ F : \mathsf{C} \to \mathsf{C}, \qquad\qquad \varepsilon : 1_{\mathsf{D}} \to F \circ G : \mathsf{D} \to \mathsf{D}$$

satisfying the universality/naturality/triangle identities as in definitions (a,b,d,e,f). Note the completely indistinguishable role of F, G, of C, D, and of η, ε; the adjectives "left" and "right" only make sense once we've chosen where to put the "op". We say simply that F, G are **dual adjoints** of each other. (However, as with contravariant functors, to avoid confusion, it is best to regard there as being only one formal notion of adjunction, the covariant one.)

Exercise 11.14. Let Vec be the category of (say, $\mathbb{R}$-)vector spaces, and $(-)^* = \mathsf{Vec}(-, \mathbb{R})$ be the dual space functor. Show that $(-)^*$ is dual adjoint to itself.

Exercise 11.15. For a poset P, let $\mathrm{Up}(P) = \mathrm{Low}(P^{\mathrm{op}})$ be the poset of **upper sets** in P, i.e., upward-closed subsets, under inclusion. Extend Up to a functor $\mathsf{Pos}^{\mathrm{op}} \to \mathsf{Pos}$ by taking preimage. Show that Up is dual adjoint to itself.

For adjunctions, considering the special case of posets is very helpful (whereas considering the special case of groups is not; see Exercise 11.33). An adjunction between posets P, Q consists of:

(a) a **right adjoint map** $g : Q \to P$, i.e., a monotone map such that for each $x \in P$, the 2-valued copresheaf

$$P(x, g(-)) = \chi_{\{y \in Q \mid x \le g(y)\}}$$

is representable, i.e., $\uparrow f(x) = \{y \in Q \mid x \le g(y)\}$ for some $f(x) \in Q$, i.e., there is a least $f(x) \in Q$ such that $x \le g(f(x))$;

(b) in which case, the map $f : P \to Q$ is automatically monotone, and is called the **left adjoint** of g, and satisfies $1_P \le g \circ f$ (the "unit");

(c) equivalently, two monotone maps $f : P \to Q$ and $g : Q \to P$ such that

$$f(x) \le_Q y \iff x \le_P g(y) \quad \forall x \in P,\, y \in Q;$$

(f) equivalently, two monotone maps $f : P \to Q$ and $g : Q \to P$ such that

$$1_P \leq g \circ f, \qquad\qquad f \circ g \leq 1_Q.$$

The equivalence of these conditions is by Yoneda for posets (Lemma 8.5).

Exercise 11.16. Directly verify the equivalence of these conditions.

Exercise 11.17. Show that for two posets P, Q and arbitrary maps $f : P \to Q$ and $g : Q \to P$ satisfying (c), f, g must be monotone. However, f, g may satisfy (f) without being monotone.

Example 11.18. For any function $f : X \to Y$, we have

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

i.e., for any subsets $A \subseteq X$ and $B \subseteq Y$,

$$f(A) \subseteq B \iff A \subseteq f^{-1}(B).$$

Exercise 11.19. What is the *right* adjoint of $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$?

Example 11.20. In $2 = \{0 < 1\}$, regarded as truth values, we have

$$a \wedge b \leq c \iff a \leq b \to c$$

(where $\wedge$ and $\to$ have their usual logical meanings), i.e.,

$$(-) \wedge b \dashv b \to (-).$$

This can be seen as a "propositional" version of product-hom adjunction (Example 11.10).

Dual adjunctions between posets are also called **Galois connections**, after the original example:

Example 11.21. Let $K \subseteq L$ be a field extension, and let

$$\begin{aligned} P &:= \{\text{intermediate subfields } K \subseteq E \subseteq L\}, \\ Q &:= \{\text{subgroups of } \mathrm{Gal}(L/K)\}, \\ f : P &\longrightarrow Q \\ E &\longmapsto \mathrm{Gal}(L/E), \\ g : Q &\longrightarrow P \\ G &\longmapsto \mathrm{Fix}(G); \end{aligned}$$

then

$$G \subseteq f(E) \iff \forall g \in G,\ x \in E\,(g \text{ fixes } x) \iff E \subseteq g(G),$$

so f, g form a Galois connection (i.e., dual adjunction).

Example 11.22. Let X be a topological space, $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ be the subposet of closed sets, and $i : \mathcal{F}(X) \hookrightarrow \mathcal{P}(X)$ be the inclusion. Then for $A \subseteq X$ and closed $B \subseteq X$,

$$\overline{A} \subseteq_{\mathcal{F}(X)} B \iff A \subseteq_{\mathcal{P}(X)} B = i(B),$$

i.e., $\overline{(-)} \dashv i$.

Similar examples are any subposet $\mathcal{Q} \subseteq \mathcal{P}(X)$ of subsets which are "closed" under some conditions, e.g., $\mathcal{Q} =$ all subgroups of a group X.

Note that in the last example, the right adjoint i was an order-embedding, while the composite $\overline{i(-)} : \mathcal{F}(X) \to \mathcal{F}(X)$ was the identity, i.e., the "counit" $\overline{i(-)} \le 1_{\mathcal{F}(X)}$ was a "natural isomorphism" (equality). These two conditions are equivalent in general:

Proposition 11.23. Let $\begin{tikzcd} P \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & Q \arrow[l, shift left, "g"] \end{tikzcd}$ be an adjunction between posets.

(a) f is an order-embedding iff $1_P = g \circ f$, and dually, g is an order-embedding iff $f \circ g = 1_Q$.

(b) The following are equivalent:

  (i) f is an order-isomorphism;

  (ii) g is an order-isomorphism;

  (iii) f, g are both order-embeddings;

  (iv) $1_P = g \circ f$ and $f \circ g = 1_Q$.

  If these hold, then $f^{-1} = g$.

Proof. (a) For $x, x' \in P$, we have

$$x \le_P x' \implies x \le_P g(f(x')) \iff f(x) \le_Q f(x')$$

where the first $\Rightarrow$ is by the "unit" $x' \le g(f(x'))$; thus f is an order-embedding, i.e., the composite $\implies$ reverses, iff the first one does, i.e., $\downarrow x' = \downarrow g(f(x'))$ for all x', which by Yoneda for posets happens iff $x' = g(f(x'))$ for all x'. Dually, g is an order-embedding iff $f \circ g = 1_Q$.

(b) Suppose f is an order-embedding. Then by (a), $1_P = g \circ f$, i.e., g is a retraction of f, whence f is an order-isomorphism iff g is an order-isomorphism iff g is injective, and in that case $f^{-1} = g$. This yields (i) $\implies$ (iii) $\implies$ (ii). Dually, (ii) $\implies$ (iii) $\implies$ (i). By (a), (iii) $\iff$ (iv). $\square$

For a subposet $Q \subseteq P$, we call Q a **closure system** if the inclusion $Q \hookrightarrow P$ has a left adjoint, i.e., (by definition (a)) for every $x \in P$, there is a least $\overline{x} \in Q$ such that $x \le \overline{x}$, so that

$$\begin{tikzcd} P \arrow[r, shift left, two heads, "\overline{(-)}"] \arrow[r, phantom, "\perp"] & Q. \arrow[l, shift left, hook, "g"] \end{tikzcd}$$

Every adjunction $\begin{tikzcd} P \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & Q \arrow[l, shift left, "g"] \end{tikzcd}$ with g an order-embedding, or equivalently (by Proposition 11.23) $f \circ g = 1_Q$, is isomorphic to a closure system.

Proposition 11.23 has a categorical generalization:

Proposition 11.24. Let $\mathsf{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{D}$ be an adjunction between categories, with unit $\eta : 1_{\mathsf{C}} \to G \circ F$ and counit $\varepsilon : F \circ G \to 1_{\mathsf{D}}$.

(a) F is faithful iff η is componentwise monic.

(b) F is full and faithful iff η is (componentwise) an isomorphism.

(c) The following are equivalent:

(i) F is an equivalence of categories;

(ii) G is an equivalence of categories;

(iii) F, G are both full and faithful;

(iv) η, ε are both natural isomorphisms.

If these hold, then F, G are inverse equivalences.

Proof. For $X, X' \in \mathsf{C}$, $F : \mathsf{C}(X, X') \to \mathsf{D}(FX, FX')$ factors as

$$\mathsf{C}(X, X') \xrightarrow{\eta_{X'} \circ - = (よ\eta_{X'})_X} \mathsf{C}(X, GFX') \xrightarrow[\cong]{\phi^{-1}_{X,FX'}} \mathsf{D}(FX, FX')$$

(by the definition of F on morphisms in definition (b) of adjunction). Thus (a) F is faithful iff $よ\eta_{X'}$ is componentwise injective for all $X' \in \mathsf{C}$, iff $\eta_{X'}$ is monic for all $X' \in \mathsf{C}$; and (b) F is full and faithful iff $よ\eta_{X'}$ is componentwise bijective for all X', iff $\eta_{X'}$ is an isomorphism for all X'.

(c) If F is full and faithful, then η is a natural isomorphism by (b), whence by the triangle identity $\varepsilon_F \circ F(\eta) = 1_F$, each $\varepsilon_{FX} : FGFX \to FX \in \mathsf{D}$ is an isomorphism. If F is also essentially surjective, then for any $Y \in \mathsf{D}$, we have $Y \cong FX$ for some $X \in \mathsf{C}$, whence by naturality of ε, $\varepsilon_Y : FGY \to Y$ is an isomorphism, i.e., G is full and faithful by (b). Thus (i) $\implies$ (iii); and dually, (ii) $\implies$ (iii). If (iii) holds, then η, ε witness that F, G are inverse equivalences, whence (i) and (ii) hold. By (b), (iii) $\iff$ (iv). $\square$

Exercise 11.25. Show that a left adjoint $F : \mathsf{C} \to \mathsf{D}$ is full iff the unit is componentwise a retraction. Combined with (a) above, this gives another proof of (b).

Remark 11.26. "Usually", a left adjoint $F : \mathsf{C} \to \mathsf{D}$ is conservative iff the unit is componentwise an extremal monomorphism (see **TODO**); see [Kelly, (3.35)].

Example 11.27. The fact that $J : \mathsf{Grp} \subseteq \mathsf{Mon}$ is a *full* subcategory means that the unit $1_G : G \to I(J(G)) = \{\text{invertible } g \in G\} = G$ of $J \dashv I$ is an isomorphism (see Example 11.7), and also that the counit $K(J(G)) \to G$ of $K \dashv J$ is an isomorphism, i.e., if we regard a group as a monoid and then freely adjoin inverses, nothing changes.

Example 11.28. $\mathsf{Mon} \subseteq \mathsf{SGrp} = \{\text{semigroups}\}$ is not a full subcategory. Indeed, if we take a monoid M, regard it as a semigroup, and then freely adjoin an identity, the resulting monoid is $M \sqcup \{1\}$, with the counit $M \sqcup \{1\} \to M$ collapsing the new identity onto the old one.

A full subcategory $\mathsf{D} \subseteq \mathsf{C}$ is a **reflective subcategory** if the inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ has a left adjoint (called the **reflector**). Every full and faithful right adjoint functor is equivalent to the inclusion of a reflective subcategory (its full image). The dual notion is a **coreflective subcategory**.

Example 11.29.

- $\mathsf{Grp} \subseteq \mathsf{Mon}$ is reflective as well as coreflective (Example 11.27).
- $\mathsf{DivAbGrp} \subseteq \mathsf{Grp}$ is coreflective but not reflective (Example 11.8 and Exercise 11.9).
- $\mathsf{CMet} := \{\text{complete metric spaces}\} \subseteq \mathsf{Met}$ is reflective (exercise).

Remark 11.30. An **adjoint equivalence** is an adjunction $F \dashv G$ such that both functors are equivalences of categories, as witnessed by the unit $\eta : 1 \cong G \circ F$ and counit $\varepsilon : F \circ G \cong 1$.

It turns out that given any old equivalence of categories $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$, with natural isomorphisms $\eta : 1 \cong G \circ F$ and $\varepsilon : F \circ G \cong 1$ (not necessarily the unit and counit of an adjunction, i.e., not necessarily obeying the triangle identities), there are times when one wants to transport categorical stuff between C, D using F, G, necessarily allowing isomorphisms to enter (since F, G are equivalences, not isomorphisms), but then finds that the resulting isomorphisms are "insufficiently coherent". This happens because e.g., for $X \in \mathsf{C}$, there are two canonical isomorphisms $FGFX \cong FX$, namely $F(\eta_X^{-1})$ and ε_{FX}, which one would want to be equal; this is exactly the first triangle identity. Thus, the "right" notion of equivalence of categories is really that of an adjoint equivalence, which allows everything to be transported seamlessly between C, D.

Exercise 11.31. Let $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$, $\eta : 1 \cong G \circ F$, and $\varepsilon : F \circ G \cong 1$. Show that one triangle identity is satisfied iff the other is. [Hint: by naturality, $\eta_{GF} \circ \eta = GF\eta \circ \eta$.]

However, we have:

Proposition 11.32. Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be inverse equivalences of categories, with natural isomorphisms $\eta : 1 \cong G \circ F$ and $\varepsilon : F \circ G \cong 1$. Then there is another natural isomorphism $\varepsilon' : F \circ G \cong 1$ such that η, ε' satisfy the triangle identities, so that F, G, η, ε' form an adjoint equivalence. (Dually, we can alternatively modify η to η' and keep ε.)

Proof. We will prove the dual version. Since F is an equivalence, an inverse equivalence G for it is constructed in the proof of Proposition 6.7; in that proof, we take β to be the present ε (and G to be the present G), and let α^{-1} from that proof be $\eta' : 1 \cong G \circ F$. By the definition of α, we have $F(\alpha) = \beta_F$, i.e., $\varepsilon_F \circ F(\eta') = 1_F$. By the preceding exercise, this implies $G(\varepsilon) \circ \eta'_G = 1_G$. ☐

Here are the "correct" and "incorrect" ways to specify an equivalence of categories:

- a functor $F : \mathsf{C} \to \mathsf{D}$ for which there exists an inverse equivalence, or equivalently (by Proposition 6.7), which is full, faithful, and essentially surjective;
- ~~functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that there exist natural isomorphisms $\eta : 1_\mathsf{C} \cong G \circ F$ and $\varepsilon : F \circ G \cong 1_\mathsf{D}$~~ (either the unit or counit must be part of the data);
- functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with a natural isomorphism $\eta : 1_\mathsf{C} \cong G \circ F$, such that there exists $\varepsilon : F \circ G \cong 1_\mathsf{D}$ (equivalently, F is an equivalence);
- ~~functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with natural isomorphisms $\eta : 1_\mathsf{C} \cong G \circ F$ and $\varepsilon : F \circ G \cong 1_\mathsf{D}$~~ (the triangle identities may fail);
- functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ together with natural isomorphisms $\eta : 1_\mathsf{C} \cong G \circ F$ and $\varepsilon : F \circ G \cong 1_\mathsf{D}$ satisfying the triangle identities, i.e., an adjoint equivalence.

Exercise 11.33. Let G, H be groups (regarded as one-object categories). Characterize all adjunctions between G, H. Thus, for adjunctions, considering the special case of groups isn't very enlightening.

Part II

Propositional logic

12 Lattices and Boolean algebras

Let P be a poset. The **meet** (aka **infimum**) of a family of elements $a_i \in P$ $(i \in I)$ is their greatest lower bound, i.e., the (unique) greatest element of

$$\bigcap_i {\downarrow} a_i,$$

if it exists. We denote the meet by $\bigwedge_i a_i$. In other words, we have

$$\begin{aligned} b = \bigwedge_i a_i &\iff {\downarrow} b = \bigcap_i {\downarrow} a_i, \\ b \le \bigwedge_i a_i &\iff \forall i\, (b \le a_i). \end{aligned}$$

The first characterization says that "meets are lifted along the Yoneda embedding ${\downarrow} : P \to \mathrm{Low}(P)$". Hence, to prove a universal statement about meets, it is enough to prove it for intersection of sets. For example:

Proposition 12.1. If $\bigwedge_j a_{ij}$ exists for each i, then

$$\bigwedge_{i,j} a_{ij} = \bigwedge_i \bigwedge_j a_{ij},$$

one side existing iff the other does.

Proof. ${\downarrow} \bigwedge_{i,j} a_{ij} = \bigcap_{i,j} {\downarrow} a_{ij} = \bigcap_i \bigcap_j {\downarrow} a_{ij} = \bigcap_i {\downarrow} \bigwedge_j a_{ij} = {\downarrow} \bigwedge_i \bigwedge_j a_{ij}$. □

The second characterization of meets above says

$$b \le_P \bigwedge_i a_i \iff (b)_i \le_{P^I} (a_i)_i.$$

Thus, if all meets indexed by I exist in P, then we have an adjunction

$$\begin{tikzcd} P^I \arrow[d, shift left, "\bigwedge"] \\ P \arrow[u, shift left, "\delta"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

where δ is the diagonal embedding $\delta(b) := (b)_{i \in I}$.

Dually, the **join** (**supremum**) of $a_i \in P$, denoted $\bigvee_i a_i$, is their least upper bound, if it exists:

$$\begin{aligned} b = \bigvee_i a_i &\iff {\uparrow} b = \bigcap_i {\uparrow} a_i, \\ b \ge \bigvee_i a_i &\iff \forall i\, (b \ge a_i). \end{aligned}$$

Example 12.2. $\mathcal{P}(X)$ has arbitrary meets $\bigcap$ and joins $\bigcup$.

Example 12.3. $2 = \{0 < 1\}$ has arbitrary meets and joins. (We can regard $\mathcal{P}(X) \cong 2^X$ as a product of copies of 2, with meets and joins computed pointwise.)

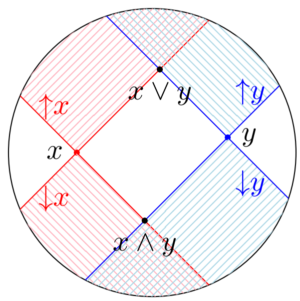
Example 12.4. $\mathbb{R}$ has nonempty bounded meets and joins; $[0,1]$ has arbitrary meets and joins.

Example 12.5. Any linear order has binary meets (min) and joins (max).

Binary and nullary meets and joins are denoted

$$a_1 \wedge a_2 := \bigwedge_{i \in \{1,2\}} a_i, \qquad \top := \bigwedge_{i \in \varnothing} a_i,$$
$$a_1 \vee a_2 := \bigvee_{i \in \{1,2\}} a_i, \qquad \bot := \bigvee_{i \in \varnothing} a_i.$$

So $\top$ = greatest element, $\bot$ = least element.



A **(unital) meet-semilattice**, or just **$\wedge$-lattice**, is a poset with finite meets, equivalently $\top$ and binary meets.[11] Note that in a (not necessarily unital) $\wedge$-lattice, we have

$$(*) \qquad a \le b \iff a = a \wedge b,$$

i.e., the partial order is determined by the $\wedge$ operation. Thus, a **$\wedge$-lattice homomorphism** $f : A \to B$, i.e., map preserving $\top, \wedge$, is automatically monotone. Let

$$\wedge\mathsf{Lat} := \text{category of } \wedge\text{-lattices}.$$

We have faithful forgetful functors

$$\wedge\mathsf{Lat} \hookrightarrow \mathsf{Pos} \longrightarrow \mathsf{Set}.$$

Note that the composite (unlike the latter) is conservative, reflecting the fact that $\wedge$-lattices can be defined purely algebraically in terms of the $\wedge$ operation. In fact, we have:

Exercise 12.6. Show that for any set X, $(*)$ above gives a bijection between partial orders on X with finite (resp., binary) meets, and commutative monoid structures $(X, \wedge, \top)$ (respectively, commutative semigroup structures $(X, \wedge)$) such that $\wedge$ is **idempotent**, i.e., $x \wedge x = x$ for all x.

Remark 12.7. For any monotone map $f : P \to Q \in \mathsf{Pos}$, we have

$$f(\bigwedge_i a_i) \le \bigwedge_i f(a_i)$$

(as long as both meets exist), since $f(\bigwedge_i a_i) \le f(a_j)$ for each j by monotonicity. Thus, when checking preservation of meets (or joins), only one inequality is nontrivial.

[11] For us, lattices are always unital by default; in lattice theory, the opposite convention is often used.

Dually, a **(unital) join-semilattice** or **$\vee$-lattice** is a poset with finite joins. Let

$$\vee\mathsf{Lat} := \text{category of } \vee\text{-lattices}.$$

Remark 12.8. Both $\wedge$-lattices and $\vee$-lattices are equivalent (by Exercise 12.6) to commutative idempotent monoids, which do not have a built-in notion of ordering; composing these two equivalences takes a $\wedge$-lattice A to its opposite $\vee$-lattice A^{op}.

A **(bounded) lattice** is a poset which is both a $\wedge$-lattice and a $\vee$-lattice.

Exercise 12.9. Show that two commutative idempotent monoids $(X, \wedge, \top), (X, \vee, \top)$ on the same underlying set X are meet and join for the same partial order iff the **absorption laws** hold:

$$x \wedge (x \vee y) = x, \qquad\qquad x \vee (x \wedge y) = x.$$

Thus, lattices are (like $\wedge$-lattices and $\vee$-lattices) purely algebraic structures, axiomatized by equational axioms.

A lattice A is **distributive** if for all $a, b, c \in A$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad\qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

i.e., for all $a \in A$, the (monotone) map $a \wedge (-) : A \to A$ preserves binary joins, and $a \vee (-)$ preserves binary meets (note that they automatically preserve $\bot, \top$ respectively). By Remark 12.7, only one inequality in each of these equations is nontrivial.

Proposition 12.10. The two distributive laws above are equivalent.

Proof. By duality ($A := A^{\mathrm{op}}$), it is enough to assume the first and prove the second. We have

$$(a \vee b) \wedge (a \vee c) = (a \wedge a) \vee (a \wedge c) \vee (b \wedge a) \vee (b \wedge c)$$

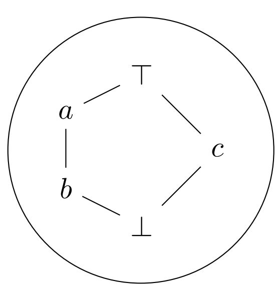
by the first distributive law. Each term on the RHS is clearly $\le a \vee (b \wedge c)$. $\square$

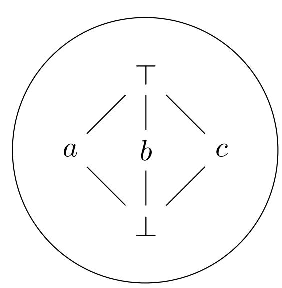
Let

$$\mathsf{DLat} := \text{category of distributive lattices}.$$

Example 12.11. $2 = \{0 < 1\}$ is clearly distributive. Hence, so is any power of it, i.e., any powerset $\mathcal{P}(X)$, as well as any sublattice of $\mathcal{P}(X)$.

Example 12.12. The following lattices are not distributive:

$N_5 :=$ 

$M_3 :=$ 

In fact, these are the minimal (unbounded) counterexamples: every non-distributive (possibly unbounded) lattice contains a copy of N_5 or M_3 as a sublattice (not necessarily preserving $\bot, \top$).

Exercise 12.13. Let A be a (possibly unbounded) lattice. For $a, b \in A$, put

$$[a, b] := \{x \in A \mid a \le x \le b\}.$$

(a) Show that for any $a, b \in A$, we have an adjunction

$$\begin{tikzcd} {[a \wedge b, a]} \arrow[r, shift left, "b \vee (-)"] \arrow[r, phantom, "\perp"] & {[b, a \vee b]}. \arrow[l, shift left, "a \wedge (-)"] \end{tikzcd}$$

A is **modular** if these two maps are inverses of each other, for any $a, b \in A$.

(b) Draw a picture showing why the above definition of modularity is sometimes called the **diamond isomorphism**.

(c) Show that distributive lattices are modular.

(d) Show that for any ($\mathbb{R}$-)vector space V, the lattice of linear subspaces of V is modular, but not necessarily distributive.

(e) Show that M_3 is modular, but N_5 is not.

(f) Show that every non-modular (possibly unbounded) lattice contains an isomorphic copy of N_5.

Showing that modular, non-distributive lattices contain M_3 involves a longer computation; see e.g., [Birkhoff, *Lattice theory*].

Remark 12.14. Of course, every lattice A order-embeds into a powerset, via $\downarrow : A \to \mathrm{Low}(A) \subseteq \mathcal{P}(A)$; however, this embedding only preserves meets, not joins.

However, if A is a linear order, then $\downarrow : A \to \mathrm{Low}(A)$ preserves binary joins; thus linear orders are (possibly unbounded) distributive lattices (which is also easily seen directly).

For a distributive lattice A, two elements $a, b \in A$ are **complements** of each other if

$$a \vee b = \top, \qquad\qquad a \wedge b = \bot.$$

Proposition 12.15. The complement of $a \in A \in \mathsf{DLat}$ is unique, if it exists.

Proof. Let b, c be complements of a. Then

$$\begin{aligned} b &= b \wedge \top \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) \\ &= \bot \vee (b \wedge c) \\ &= b \wedge c \le c, \end{aligned}$$

and similarly, $c \le b$. $\square$

We denote the complement of $a \in A$, if it exists, by $\neg a$. Note that any distributive lattice homomorphism automatically preserves existing complements. A **Boolean algebra** is a distributive lattice with all complements; equivalently, we may regard $\neg$ as an additional unary operation, which is automatically preserved by distributive lattice homomorphisms. So

$$\mathsf{Bool} := \{\text{Boolean algebras}\} \subseteq \mathsf{DLat}$$

is a full subcategory.

Example 12.16. Any powerset $\mathcal{P}(X)$ is a Boolean algebra.

Exercise 12.17. For any commutative ring R, show that the **idempotents** $\{r \in R \mid r^2 = r\}$ form a Boolean algebra, with meets given by multiplication (this uniquely determines the partial order, hence the rest of the Boolean algebra structure).

Exercise 12.18. Show that Bool is isomorphic to the category of **Boolean rings**, the full subcategory of Ring consisting of rings in which every element is idempotent.

Exercise 12.19. Show that in any Boolean algebra, we have

$$a \wedge b \leq c \iff a \leq \neg b \vee c.$$

In other words, $(-) \wedge b \dashv \neg b \vee (-)$.

13 Ideals and filters

A poset P is **(up-)directed** if every finite $F \subseteq P$ has an upper bound, or equivalently, $P \neq \varnothing$ (i.e., $\varnothing$ has an upper bound), and every $a, b \in P$ have an upper bound $c \geq a, b$.

Example 13.1. Any $\vee$-lattice is directed, e.g., $\mathcal{P}_\omega(X) := \{A \subseteq X \mid A \text{ finite}\}$.

Example 13.2. Any poset with $\top$ is directed. (In particular, joins need not exist below $\top$.)

Example 13.3. Any nonempty linear order is directed.

A subset $A \subseteq P$ of a poset is an **ideal** if it is lower and directed.

- If P is a $\vee$-lattice, this is equivalent to being lower and closed under finite joins.
- If P is a $\wedge$-lattice, being lower is equivalent to being closed under $a \wedge (-)$ for all $a \in P$ (since for $a \leq b \in P$ we have $a = a \wedge b$).
  Thus if P is a lattice, an ideal $A \subseteq P$ is equivalently a $\vee$-sublattice closed under all $a \wedge (-)$.

The dual notion (upper down-directed subset) is called a **filter**.

Example 13.4. For any $a \in P$, the principal ideal $\downarrow a = \{b \in P \mid b \leq a\}$ is an ideal.

Example 13.5. $\mathcal{P}_\omega(X) \subseteq \mathcal{P}(X)$ is an ideal. Dually, $\{\text{cofinite subsets}\} \subseteq \mathcal{P}(X)$ is a filter.

Example 13.6. For any $\vee$-lattice homomorphism $f : A \to B$, $f^{-1}(\perp) \subseteq A$ is an ideal.

However, unlike in ring theory, failure of injectivity of $\vee$-lattice homomorphisms is not completely captured by ideals: there can be non-injective $f : A \to B \in \vee\mathsf{Lat}$ with $f^{-1}(\perp) = \{\perp\}$. For example, take any $f : A \to B \in \vee\mathsf{Lat}$, and consider $f_\perp : A_\perp \to B_\perp$, where $(-)_\perp : \vee\mathsf{Lat} \to \vee\mathsf{Lat}$ freely adjoins a new least element.

Let A be a lattice. For any lattice homomorphism $f : A \to 2$, $f^{-1}(0) \subseteq A$ is an ideal, while its complement $f^{-1}(1) \subseteq A$ is a filter. An ideal $I \subseteq A$ is **prime** if $\neg I$ is a filter (equivalently, being upper, closed under $\top, \wedge$); likewise, a filter $F \subseteq A$ is **prime** if $\neg F$ is an ideal. We have bijections

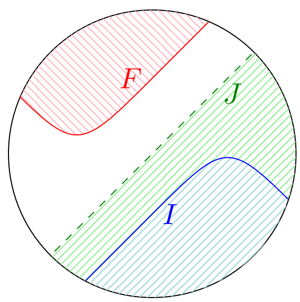
$$\{\text{prime ideals } I \subseteq A\} \xleftrightarrow{\neg} \{\text{prime filters } F \subseteq A\} \underset{(-)^{-1}(1)}{\overset{\chi_{(-)}}{\longleftrightarrow}} \mathsf{Lat}(A, 2),$$

the first order-reversing, the second order-preserving. We call

$$\mathrm{Spec}(A) := \{\text{prime filters } F \subseteq A\} \cong \mathsf{Lat}(A, 2).$$

the **spectrum** of A.

Theorem 13.7 (prime ideal theorem for distributive lattices)**.** Let A be a distributive lattice, $F \subseteq A$ be a filter, and $I \subseteq A$ be an ideal disjoint from F. Then there is a prime ideal $J \subseteq A$ containing I and disjoint from F, i.e., a lattice homomorphism $\mu = \chi_{\neg J} : A \to 2$ such that $\mu(I) = \{0\}$ and $\mu(F) = \{1\}$.



The proof of this fundamental result is essentially the same as that of the familiar result for rings. The following general statement includes both results:

Lemma 13.8. Let A be a **commutative rig** (i.e., set equipped with two commutative monoid operations $+, \cdot$, the latter distributing over the former), $F \subseteq A$ be a multiplicative submonoid (i.e., closed under $1, \cdot$), and $I \subseteq A$ be an ideal (i.e., closed under $0, +$ and $a \cdot (-)$ for all $a \in A$) which is maximal among ideals disjoint from F. Then I is prime (i.e., $\neg I$ is a multiplicative submonoid).

Proof. Since $1 \in F$, $1 \notin I$. Let $a, b \notin I$; we must show that $ab \notin I$. Clearly

$$aA + I = \{ac + e \mid c \in A \ \& \ e \in I\}$$

is an ideal containing a, I, hence intersects F, i.e., there is some $c \in A$ and $e \in I$ with $ac + e \in F$. Similarly, there is some $d \in A$ and $f \in I$ with $bd + f \in F$. Then

$$\begin{aligned} F &\ni (ac + e)(bd + f) \\ &= abcd + \underbrace{acf + bde + ef}_{\in I}, \end{aligned}$$

whence $ab \notin I$ or else $F \cap I \neq \varnothing$. □

Proof of Theorem 13.7. Let $J \supseteq I$ be maximal disjoint from F, using Zorn's lemma. □

Corollary 13.9. For any distributive lattice A and $a \not\leq b \in A$, there is a homomorphism $\mu : A \to 2$ such that $\mu(a) = 1 > 0 = \mu(b)$.

Proof. Apply Theorem 13.7 to $\uparrow a, \downarrow b$. ☐

Corollary 13.10. A distributive lattice A admits a homomorphism $A \to 2$ iff A is nontrivial. ☐

Exercise 13.11. Let A be a distributive lattice.

(a) Let $I \subseteq A$ be an ideal. For $a, b \in A$, put

$$\begin{aligned} a \lesssim_I b \;&:\Longleftrightarrow\; \exists c \in I\,(a \leq b \vee c), \\ a \sim_I B \;&:\Longleftrightarrow\; a \lesssim_I b \lesssim_I a. \end{aligned}$$

Prove that $\sim_I$ is a **congruence relation**, i.e., $\sim_I \subseteq A^2$ is an equivalence relation as well as a sublattice. It follows that the distributive lattice structure on A descends to $A/{\sim_I}$ (this is a general fact about algebraic structures axiomatized by universal equational axioms).

(b) Verify that $[\bot]_{\sim_I} = I$.

(c) Dually, for any filter $F \subseteq A$, there is a congruence relation $\sim^F$ on A such that $[\top]_{\sim^F} = F$.

(d) Deduce Theorem 13.7 from Corollary 13.10.

Corollary 13.12. Every distributive lattice A admits an embedding

$$\begin{aligned} \eta_A : A &\longrightarrow 2^{\mathsf{DLat}(A,2)} &&\cong \mathcal{P}(\mathrm{Spec}(A)) \\ a &\longmapsto (\mu(a))_{\mu \in \mathsf{DLat}(A,2)} &&\mapsto \{F \in \mathrm{Spec}(A) \mid a \in F\} =: [a]. \end{aligned}$$

Hence, a lattice A is distributive iff it embeds into a powerset. ☐

For a Boolean algebra A, prime filters $F \subseteq A$ are also called **ultrafilters**.

Exercise 13.13. Let A be a Boolean algebra, $F \subseteq A$ be a filter. The following are equivalent:

(a) F is an ultrafilter;

(b) F is a maximal filter (where as in ring theory, *maximal* means *maximal proper*);

(c) $\bot \notin F$, and for any $a \in A$, exactly one of $a, \neg a$ is in F;

(d) for any pairwise disjoint $a_1, \ldots, a_n \in A$ (i.e., $a_i \wedge a_j = \bot$ for $i \neq j$), we have

$$\bigvee_i a_i \in F \iff \exists i\,(a_i \in F).$$

In other words, $\chi_F : A \to 2$ is a 2-valued finitely additive measure.

14 Stone duality

The set $2 = \{0, 1\}$ is equipped with the structure of a Boolean algebra, as well as a topological space with the discrete topology, so we can regard it as belonging to both categories:

$$\mathsf{Bool} \ni 2 \in \mathsf{Top}.$$

These two types of structures on 2 are compatible, in the sense that the Boolean operations

$$\wedge : 2^2 \to 2, \qquad \vee : 2^2 \to 2, \qquad \neg : 2 \to 2, \qquad \top : 2^0 \to 2, \qquad \bot : 2^0 \to 2$$

are clearly continuous maps.

Let A be another Boolean algebra. We equip the set

$$\mathsf{Bool}(A, 2) \subseteq 2^A$$

with the subspace topology induced by the product topology on 2^A. Given a Boolean homomorphism $f : A \to B$, the precomposition map $(-) \circ f : \mathsf{Bool}(B, 2) \to \mathsf{Bool}(A, 2)$ is continuous, since for each $a \in A$, the composite with the coordiate projection at a

$$\mathsf{Bool}(B, 2) \xrightarrow{(-)\circ f} \mathsf{Bool}(A, 2) \xrightarrow{(-)(a)} 2$$

is the coordinate projection $(-)(f(b))$ at $f(b)$. Thus we have a functor

$$\mathsf{Bool}(-, 2) : \mathsf{Bool}^{\mathrm{op}} \longrightarrow \mathsf{Top}$$

lifting $よ_{\mathsf{Bool}} 2 = \mathsf{Bool}(-, 2) : \mathsf{Bool}^{\mathrm{op}} \to \mathsf{Set}$ along the forgetful functor $\mathsf{Top} \to \mathsf{Set}$.

We call the space $\mathsf{Bool}(A, 2)$ the **Stone dual space** of A. By the identification of Boolean homomorphisms $\to 2$ with ultrafilters, we have

$$\begin{aligned} \mathsf{Bool}(A, 2) &\cong \mathrm{Spec}(A). \\ \mu &\mapsto \mu^{-1}(1) \\ \chi_F &\mapsfrom F \end{aligned}$$

The corresponding topology on $\mathrm{Spec}(A)$ is generated by the basic clopen sets

$$\mathsf{Bool}(A, 2) \supseteq \{\mu \in \mathsf{Bool}(A, 2) \mid \mu(a) = 1\} \cong [a] := \{F \in \mathrm{Spec}(A) \mid a \in F\} \subseteq \mathrm{Spec}(A)$$

for each $a \in A$. For a homomorphism $f : A \to B \in \mathsf{Bool}$, the continuous map $\mathsf{Bool}(f, 2) : \mathsf{Bool}(B, 2) \to \mathsf{Bool}(A, 2)$ corresponds to

$$\begin{aligned} \mathrm{Spec}(f) : \mathrm{Spec}(B) &\longrightarrow \mathrm{Spec}(A) \\ F &\longmapsto f^{-1}(F). \end{aligned}$$

Exercise 14.1. Show that under the identification of Boolean algebras with Boolean rings (Exercise 12.18), the Stone spectrum is the same as the Zariski spectrum of a commutative ring.

Let X be another topological space. The set

$$\mathsf{Top}(X,2) \subseteq 2^X$$

is a sub-Boolean algebra, by continuity of the Boolean operations. Given a continuous map $g : X \to Y$, the precomposition map $(-) \circ g : \mathsf{Top}(Y,2) \to \mathsf{Top}(X,2)$ is a Boolean homomorphism, since for each $x \in X$, the composite with the coordinate projection at x

$$\mathsf{Top}(Y,2) \xrightarrow{(-)\circ g} \mathsf{Top}(X,2) \xrightarrow{(-)(x)} 2$$

is the coordinate projection $(-)(g(x))$ at $g(x)$. Thus we have a functor

$$\mathsf{Top}(-,2) : \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$$

lifting $よ_{\mathsf{Top}}2 = \mathsf{Top}(-,2) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ along the forgetful functor $\mathsf{Bool} \to \mathsf{Set}$. We call $\mathsf{Top}(X,2)$ the **Stone dual algebra** of the space X. We can identify

$$\begin{aligned} \mathsf{Top}(X,2) &\cong \{\text{clopen sets in } X\} =: \mathcal{KO}(X). \\ k &\mapsto k^{-1}(1) \\ \chi_X &\mapsfrom C \end{aligned}$$

For continuous $f : X \to Y \in \mathsf{Top}$, $\mathcal{KO}(f) : \mathcal{KO}(Y) \to \mathcal{KO}(X)$ is given by preimage.

For $A \in \mathsf{Bool}$ and $X \in \mathsf{Top}$, we have natural bijections (see Example 11.13)

$$\begin{array}{rcl} \mathsf{Set}(A, \mathsf{Set}(X,2)) \cong & \mathsf{Set}(A \times X, 2) & \cong \mathsf{Set}(X, \mathsf{Set}(A,2)) \\ f \mapsto & ((a,x) \mapsto f(a)(x)) & \mapsto (x \mapsto f(-)(x)) \\ (a \mapsto g(-)(a)) \mapsfrom & ((a,x) \mapsto g(x)(a)) & \mapsfrom g. \end{array}$$

For $f \in \mathsf{Set}(A, \mathsf{Set}(X,2))$, we have

$$f \in \mathsf{Set}(A, \mathsf{Top}(X,2)) \iff \forall a \in A\,(f(a)(-) : X \to 2 \text{ is continuous});$$

on the other hand,

$$f \in \mathsf{Bool}(A, \mathsf{Set}(X,2)) \iff \forall x \in X\,(f(-)(x) : A \to 2 \text{ is a Boolean homomorphism}),$$

since the Boolean structure on $\mathsf{Set}(X,2)$ is defined pointwise. Now

$$f \in \mathsf{Bool}(A, \mathsf{Top}(X,2)) \iff f \in \mathsf{Set}(A, \mathsf{Top}(X,2)) \;\&\; f \in \mathsf{Bool}(A, \mathsf{Set}(X,2)),$$

since $\mathsf{Top}(X,2) \subseteq \mathsf{Set}(X,2)$ is a Boolean subalgebra. Thus $f \in \mathsf{Bool}(A, \mathsf{Top}(X,2))$ iff the map $A \times X \to 2$ corresponding to f via the above bijection is a **bihomomorphism**, meaning continuous for each fixed $a \in A$ and a Boolean homomorphism for each fixed $x \in X$. We write

$$(\mathsf{Bool}, \mathsf{Top})(A \times X, 2) := \{f : A \times X \to 2 \mid f \text{ is a bihomomorphism}\}.$$

By analogous reasoning, a map $g \in \mathsf{Set}(X, \mathsf{Set}(A,2))$ is in $\mathsf{Top}(X, \mathsf{Bool}(A,2))$ iff the map $A \times X \to 2$ corresponding to g via the above bijection is a bihomomorphism (now using that the topology on

$\mathsf{Set}(A,2)$ is given by pointwise convergence, while $\mathsf{Bool}(A,2) \subseteq \mathsf{Set}(A,2)$ has the subspace topology). So the above natural bijections restrict to

$$\begin{aligned}\mathsf{Bool}(A, \mathsf{Top}(X,2)) \cong (\mathsf{Bool}, \mathsf{Top})(A \times X, 2) &\cong \mathsf{Top}(X, \mathsf{Bool}(A,2)) \\ &\cong \mathsf{Top}^{\mathrm{op}}(\mathsf{Bool}(A,2), X),\end{aligned}$$

i.e., we have a dual adjunction

$$\begin{tikzcd} \mathsf{Bool} \arrow[r, shift left, "\mathsf{Bool}(-,2)\cong\mathrm{Spec}"] \arrow[r, phantom, "\perp"] & \mathsf{Top}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{Top}(-,2)\cong\mathcal{KO}"] \end{tikzcd},$$

called the **Stone dual adjunction**.

The unit

$$\eta : 1_{\mathsf{Bool}} \longrightarrow \mathsf{Top}(\mathsf{Bool}(-,2),2) \cong \mathcal{KO} \circ \mathrm{Spec}$$

on the Bool side is computed via (see Example 11.13)

$$\begin{aligned}\mathsf{Top}(\mathsf{Bool}(A,2), \mathsf{Bool}(A,2)) &\cong (\mathsf{Bool}, \mathsf{Top})(A \times \mathsf{Bool}(A,2), 2) \\ 1_{\mathsf{Bool}(A,2)} &\mapsto ((a,\mu) \mapsto \mu(a)) \\ &\cong \mathsf{Bool}(A, \mathsf{Top}(\mathsf{Bool}(A,2),2)) \\ &\mapsto \left(\begin{aligned} \eta_A : A &\longrightarrow \mathsf{Top}(\mathsf{Bool}(A,2),2) \cong \mathcal{KO}(\mathrm{Spec}(A)) \\ a &\longmapsto (\mu \mapsto \mu(a)) \qquad \mapsto \{F \in \mathrm{Spec}(A) \mid a \in F\} = [a] \end{aligned} \right).\end{aligned}$$

In other words, each $\eta_A(a) : \mathsf{Bool}(A,2) \to 2$ is the evaluation map at a, or equivalently, the coordinate projection at a.

Analogously, the unit

$$\varepsilon : 1_{\mathsf{Top}} \to \mathsf{Bool}(\mathsf{Top}(-,2),2) \cong \mathrm{Spec} \circ \mathcal{KO}$$

on the Top side (i.e., counit when we write the adjunction as $\mathsf{Bool} \rightleftarrows \mathsf{Top}^{\mathrm{op}}$ as above) is given componentwise by

$$\begin{aligned}\varepsilon_X : X &\longrightarrow \mathsf{Bool}(\mathsf{Top}(X,2),2) \cong \mathrm{Spec}(\mathcal{KO}(X)) \\ x &\longmapsto (k \mapsto k(x)) \qquad \mapsto \{C \in \mathcal{KO}(X) \mid x \in C\},\end{aligned}$$

so that $\varepsilon_X(x) : \mathsf{Top}(X,2) \to 2$ is again the evaluation map or projection at x.

Everything above more-or-less followed from abstract nonsense. The Stone duality *theorem* characterizes when the η_A, ε_X are isomorphisms, and involves concrete (nontrivial) arguments about Boolean algebras and topology.

By definition, the topology on $\mathsf{Bool}(A,2) \cong \mathrm{Spec}(A)$ is induced by the coordinate projection maps $\eta_A(a) : \mathsf{Bool}(A,2) \to 2$ for $a \in A$, i.e., generated by the basic clopen sets from above

$$[a] = \{F \in \mathrm{Spec}(A) \mid a \in F\}$$

and their complements. Since $\eta_A = [-]$ is a Boolean homomorphism, we have $\neg[a] = [\neg a]$; thus it is enough to take the sets $[a]$ as basic open sets.

Lemma 14.2. Each Stone dual space $\mathsf{Bool}(A, 2) \cong \mathrm{Spec}(A)$ is compact.

Proof. It is enough to show that every family of basic closed sets $[a_i] \subseteq \mathrm{Spec}(A)$ with the finite intersection property has nonempty intersection. For each $a_{i_1}, \dots, a_{i_n}$, we have

$$\varnothing \neq [a_{i_1}] \cap \cdots \cap [a_{i_n}] = [a_{i_1} \wedge \cdots \wedge a_{i_n}]$$

whence $a_{i_1} \wedge \cdots \wedge a_{i_n} > \bot$, since $[-]$ is a Boolean homomorphism. Thus the filter generated by these elements, namely

$$\{b \in A \mid \exists i_1, \dots, i_n \, (b \geq a_{i_1} \wedge \cdots \wedge a_{i_n})\},$$

is proper, hence extends by the Prime Ideal Theorem 13.7 (with $I := \{\bot\}$) to an ultrafilter $G \in \mathrm{Spec}(A)$ containing each a_i, i.e., belonging to each $[a_i]$. ☐

Theorem 14.3 (Stone duality – algebraic half)**.** For every Boolean algebra A, the unit $\eta_A : A \to \mathsf{Top}(\mathsf{Bool}(A, 2), 2) \cong \mathcal{KO}(\mathrm{Spec}(A))$ is an isomorphism.

Proof. Injectivity: by the PIT, $\eta_A : A \to \mathsf{Top}(\mathsf{Bool}(A, 2), 2) \subseteq 2^{\mathsf{Bool}(A,2)}$ is injective (Corollary 13.12).

Surjectivity: let $C \subseteq \mathrm{Spec}(A)$ be clopen. Since C is open, $C = \bigcup\{[a] \mid [a] \subseteq U\}$. Since C is closed, hence compact, $C = [a_1] \cup \cdots \cup [a_n] = [a_1 \vee \cdots \vee a_n]$ for some $[a_1], \dots, [a_n] \subseteq C$. ☐

By general properties of adjunctions (Proposition 11.24), this is equivalent to:

Corollary 14.4. $\mathsf{Bool}(-, 2) \cong \mathrm{Spec} : \mathsf{Bool} \to \mathsf{Top}^{\mathrm{op}}$ is full and faithful. ☐

In other words, the spectrum functor $\mathrm{Spec} : \mathsf{Bool}^{\mathrm{op}} \to \mathsf{Top}$ is equivalent to the inclusion of a full (reflective, since it has left adjoint $\mathcal{KO}$; note that we moved the "op") subcategory of Top, namely the essential image of Spec, i.e., all those spaces which are homeomorphic to $\mathrm{Spec}(A)$ for some Boolean algebra A. We now seek to characterize these spaces in a more topological manner.

A topological space X is:

- **zero-dimensional** if it has a basis of clopen sets;
- **totally separated** (or sometimes **ultra-Hausdorff**[12]) if for any $x \neq y \in X$, there is a clopen set containing x but not y;
- **totally disconnected** if every subset $X' \subseteq X$ with at least two points is disconnected, i.e., has a nontrivial clopen (in X') set.

Exercise 14.5. Show the following:

(a) T_0 + zero-dimensional $\implies$ totally separated $\implies$ Hausdorff + totally disconnected;

(b) compact + totally separated $\implies$ zero-dimensional;

(c) compact + Hausdorff + totally disconnected $\implies$ totally separated.
[Let $x \in X$, and let C be the intersection of all clopen neighborhoods of x. If $C \neq \{x\}$, then $C = F \sqcup G$ for nonempty closed sets F, G, say with $x \in F$. By normality, there are disjoint open sets $U \supseteq F$ and $V \supseteq G$. Using the definition of C and compactness, there is a clopen set D disjoint from C such that $U \cup D \cup V = X$. Then $U \setminus D = \neg V \setminus D$ is clopen, contains F, and is disjoint from G, whence $G = \varnothing$.]

[12] "ultra" here is by analogy with ultrametric, and has nothing to do with ultrafilters

A **Stone space** is, equivalently:

- a compact Hausdorff zero-dimensional space;
- a compact T_0 zero-dimensional space;
- a compact totally separated space;
- a compact Hausdorff totally disconnected space.

Let

$$\mathsf{Stone} \subseteq \mathsf{Top}$$

denote the full subcategory of Stone spaces.

Lemma 14.6. Each Stone dual space $\mathsf{Bool}(A, 2) \cong \mathrm{Spec}(A)$ is a Stone space.

Proof. Since the sets $[a]$ form a clopen basis, $\mathrm{Spec}(A)$ is zero-dimensional. To check T_0: for distinct ultrafilters $F, G \in \mathrm{Spec}(A)$, say there is $a \in F \setminus G$; then $F \in [a] \not\ni G$. □

Theorem 14.7 (Stone duality – spatial half)**.** For every Stone space X, the unit $\varepsilon_X : X \to \mathsf{Bool}(\mathsf{Top}(X, 2), 2) \cong \mathrm{Spec}(\mathcal{KO}(X))$ is a homeomorphism.

Proof. Injectivity: for $x \neq y \in X$, since X is totally separated, there is $C \in \mathcal{KO}(X)$ with $x \in C \not\ni y$, whence $\varepsilon_X(x)(\chi_C) = \chi_C(x) = 1 \neq 0 = \chi_C(y) = \varepsilon_X(y)(\chi_C)$.

Density of $\mathrm{im}(\varepsilon_X)$: for any nonempty basic clopen set $[C] \subseteq \mathrm{Spec}(\mathcal{KO}(X))$, where $C \in \mathcal{KO}(X)$, we must have $C \neq \varnothing$ (since $[C] \neq \varnothing$), whence there is $x \in C$, i.e., $\varepsilon_X(x)(\chi_C) = \chi_C(x) = 1$, i.e., $\varepsilon_X(x) \in [C]$.

Since both spaces are compact Hausdorff, injectivity and density of image are enough. □

Again by Proposition 11.24, we now have

Theorem 14.8 (Stone duality)**.** The Stone adjunction

$$\begin{tikzcd} \mathsf{Bool} \arrow[r, shift left, "\mathsf{Bool}(-,2)\cong\mathrm{Spec}"] \arrow[r, phantom, "\perp"] & \mathsf{Stone}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{Top}(-,2)\cong\mathcal{KO}"] \end{tikzcd}$$

is a dual adjoint equivalence of categories. □

Exercise 14.9. Show that the forgetful functor $\mathsf{Stone} \to \mathsf{Set}$ has a left adjoint.

Exercise 14.10. A morphism $f : A \to B \in \mathsf{C}$ in a category C is an **epimorphism** if it is a monomorphism in C^{op}, i.e., if for all $g, h : B \to C$, if $g \circ f = h \circ f$, then $g = h$.

(a) Show that if C admits a faithful ("forgetful") functor $U : \mathsf{C} \to \mathsf{Set}$, then if $U(f)$ is surjective, then f is an epimorphism.

(b) Show that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of torsion-free abelian groups, but not in the category of all abelian groups.

(c) Show that epimorphisms in Stone are precisely the surjective maps.

(d) Show that Bool has the **amalgamation property**: for every $A, B, C \in \mathsf{Bool}$ together with injective homomorphisms $f : A \to B$ and $g : A \to C$, there is a $D \in \mathsf{Bool}$ together with injective homomorphisms $h : B \to D$ and $k : C \to D$ making the following diagram commute.

$$\begin{tikzcd}
& D & \\
B \arrow[ur, hook, dashed, "h"] & & C \arrow[ul, hook', dashed, "k"'] \\
& A \arrow[ul, hook', "f"] \arrow[ur, hook, "g"'] &
\end{tikzcd}$$

15 Propositional logic

A **propositional language** (or **signature**) is a set $\mathcal{L}$, whose elements $P, Q, \ldots \in \mathcal{L}$ are called **atomic formulas**. **Propositional $\mathcal{L}$-formulas** are constructed inductively as follows:

- $P \in \mathcal{L}$ is an $\mathcal{L}$-formula;
- $\top, \bot$ are $\mathcal{L}$-formulas;
- for $\mathcal{L}$-formulas ϕ, ψ, also $\phi \wedge \psi, \phi \vee \psi, \neg\phi$ are $\mathcal{L}$-formulas.

We use the abbreviation

$$\phi \to \psi := \neg\phi \vee \psi.$$

We will denote the set of propositional $\mathcal{L}$-formulas by $\mathcal{L}_{\omega 0}$ (ω for finitary, 0 for propositional).

A **model** $\mathcal{M}$ of $\mathcal{L}$ is an assignment of truth values $\mathcal{M} : \mathcal{L} \to 2$ to the atomic formulas. We extend this assignment to all $\mathcal{L}$-formulas in the obvious way:

$$\mathcal{M}(\top) := 1, \quad \mathcal{M}(\bot) := 0,$$
$$\mathcal{M}(\phi \wedge \psi) := \mathcal{M}(\phi) \wedge \mathcal{M}(\psi), \quad \mathcal{M}(\phi \vee \psi) := \mathcal{M}(\phi) \vee \mathcal{M}(\psi), \quad \mathcal{M}(\neg\phi) := \neg\mathcal{M}(\phi).$$

We also sometimes write

$$\phi^{\mathcal{M}} := \mathcal{M}(\phi),$$

and call it the **interpretation of ϕ in $\mathcal{M}$**. We denote the set of models by

$$\mathrm{Mod}(\mathcal{L}) := \{\text{models of } \mathcal{L}\} \cong 2^{\mathcal{L}}.$$

The above definition only uses the Boolean algebra structure in 2. For a Boolean algebra A, we define a **model $\mathcal{M}$ of $\mathcal{L}$ in A** (or an **A-valued model of $\mathcal{L}$**) to be a map $\mathcal{M} : \mathcal{L} \to A$, which is then extended to $\mathcal{M} : \mathcal{L}_{\omega 0} \to A$ in the same way as above. Put

$$\mathrm{Mod}(\mathcal{L}, A) := \{\text{models of } \mathcal{L} \text{ in } A\} \cong A^{\mathcal{L}}.$$

Example 15.1. For a set X, a $\mathcal{P}(X) \cong 2^X$-valued model of $\mathcal{L}$, i.e., $\mathcal{M} \in (2^X)^{\mathcal{L}} \cong (2^{\mathcal{L}})^X \cong \mathrm{Mod}(\mathcal{L})^X$, can be thought of as an X-indexed family $(\mathcal{M}_x)_{x \in X}$ of (2-valued) models. For a formula ϕ, $\phi^{\mathcal{M}} \in \mathcal{P}(X) \cong 2^X$ is the set of models in which ϕ holds, or the family of truth values $(\phi^{\mathcal{M}_x})_{x \in X}$.

We now define a **proof system**[13] for propositional logic. In general, a proof system consists of a list of **inference rules** of the form

$$\frac{\sigma_1 \qquad \cdots \qquad \sigma_n}{\tau}$$

which intuitively say that "from the hypotheses $\sigma_1, \dots, \sigma_n$, we may deduce τ". There are several styles of proof system in common use;[14] we will use one called a **(Gentzen) sequent calculus**. In a sequent calculus, the statements σ, τ being proved are *implications between formulas*, denoted

$$\sigma := (\phi \Rightarrow \psi).$$

It is best to think of this toplevel implication $\Rightarrow$ as different from the logical connective $\rightarrow$ used in the formulas themselves (even though they have the same intended meaning). We call such an implication σ a **sequent**.[15]

Here are the inference rules. Note that these are actually **rule schema**, i.e., these rules apply for all possible choices for the formulas ϕ, ψ, θ.

$$(\textsc{Id})\ \frac{}{\phi \Rightarrow \phi} \qquad\qquad (\textsc{Cut})\ \frac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \theta}{\phi \Rightarrow \theta}$$

$$(\wedge{\Rightarrow}_1)\ \frac{}{\phi \wedge \psi \Rightarrow \phi} \qquad (\wedge{\Rightarrow}_2)\ \frac{}{\phi \wedge \psi \Rightarrow \psi} \qquad ({\Rightarrow}\wedge)\ \frac{\theta \Rightarrow \phi \qquad \theta \Rightarrow \psi}{\theta \Rightarrow \phi \wedge \psi} \qquad ({\Rightarrow}\top)\ \frac{}{\theta \Rightarrow \top}$$

$$({\Rightarrow}\vee_1)\ \frac{}{\phi \Rightarrow \phi \vee \psi} \qquad ({\Rightarrow}\vee_2)\ \frac{}{\psi \Rightarrow \phi \vee \psi} \qquad (\vee{\Rightarrow})\ \frac{\phi \Rightarrow \theta \qquad \psi \Rightarrow \theta}{\phi \vee \psi \Rightarrow \theta} \qquad (\bot{\Rightarrow})\ \frac{}{\bot \Rightarrow \theta}$$

$$(\textsc{Dist})\ \frac{}{\phi \wedge (\psi \vee \theta) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \theta)} \qquad (\wedge\neg)\ \frac{}{\phi \wedge \neg\phi \Rightarrow \bot} \qquad (\vee\neg)\ \frac{}{\top \Rightarrow \phi \vee \neg\phi}$$

A **proof** (or **derivation**) is a finite rooted tree, drawn with the root at the bottom, in which (in graph-theoretic terminology) edges are labeled by sequents and each internal node is an inference rule. For example, here is a proof:

$$(\textsc{Cut})\ \dfrac{\phi \Rightarrow \psi \vee \theta \qquad (\vee{\Rightarrow})\ \dfrac{({\Rightarrow}\vee_2)\ \dfrac{}{\psi \Rightarrow \theta \vee \psi} \qquad ({\Rightarrow}\vee_1)\ \dfrac{}{\theta \Rightarrow \theta \vee \psi}}{\psi \vee \theta \Rightarrow \theta \vee \psi}}{\phi \Rightarrow \theta \vee \psi}$$

[13] also known as **formal system**, **deductive system**, **deductive apparatus**, etc.

[14] In **Hilbert systems**, the σ, τ are just formulas; Hilbert systems are easy to define, but very hard to work with, and usually yield formal proofs that do not resemble informal mathematical arguments at all. The system we are using is actually a loose generalization of both **natural deduction** systems and **sequent calculi**; both types of proof system involve sequents, and differ (roughly speaking) in where the Cut rule may be used.

[15] One reason for considering sequents is that later (**TODO**), we will consider subsets of full propositional logic which do not admit $\rightarrow$ as a logical connective in formulas; however, sequents may still use one toplevel $\Rightarrow$. Another, related, reason is that we want to regard theories $(\mathcal{L}, \mathcal{T})$ as presentations of Lindenbaum–Tarski algebras $\langle \mathcal{L} \mid \mathcal{T} \rangle$. Relations in an algebra presentation are always of the form $\phi = \psi$ (or $\phi \leq \psi$, in the presence of a $\wedge$-lattice structure) for terms ϕ, ψ; only in certain types of algebras, like groups or Boolean algebras, may we move all of the terms to one side of the equation.

The root edge label is called the **conclusion** ($\phi \Rightarrow \theta \vee \psi$); the non-inference rule leaf edge labels are called the **hypotheses** ($\phi \Rightarrow \psi \vee \theta$). A **propositional $\mathcal{L}$-theory** $\mathcal{T}$ is a set of $\mathcal{L}$-sequents, called the **axioms** of $\mathcal{T}$. If there is a proof of conclusion σ from hypotheses in $\mathcal{T}$, we write

$$\mathcal{T} \vdash \sigma$$

and say $\mathcal{T}$ **proves** σ (so in the above example, $\{\phi \Rightarrow \psi \vee \theta\}$ proves $\phi \Rightarrow \theta \vee \psi$). We also write

$$\begin{aligned} \vdash \sigma \; &:\Longleftrightarrow \; \varnothing \vdash \sigma, \\ \phi \equiv_{\mathcal{T}} \psi \; &:\Longleftrightarrow \; \mathcal{T} \vdash \phi \Leftrightarrow \psi \; :\Longleftrightarrow \; \mathcal{T} \vdash \phi \Rightarrow \psi \;\&\; \mathcal{T} \vdash \psi \Rightarrow \phi \end{aligned}$$

and say ϕ, ψ are **$\mathcal{T}$-(provably)equivalent** in the last case.

When needed, we can regard a formula ϕ as the sequent $\top \Rightarrow \phi$; we silently do so from now on. For example, it makes sense to say $\{\phi, \psi\} \vdash \theta$, by which we really mean $\{\top \Rightarrow \phi, \top \Rightarrow \psi\} \vdash \theta$.

Note that the following notions are *a priori* all distinct:

$$\begin{aligned} &\vdash \phi \Rightarrow \psi, \\ &\vdash \phi \to \psi \quad \text{(i.e., } \vdash \top \Rightarrow \phi \to \psi) \\ &\{\phi\} \vdash \psi \quad \text{(i.e., } \{\top \Rightarrow \phi\} \vdash \top \Rightarrow \psi) \end{aligned}$$

However, they can be shown to be equivalent; see Exercises 15.4 and 15.5 for the trivial cases.

Exercise 15.2. Derive the converse (with $\Rightarrow$ flipped) of the Dist rule from the other rules, without mentioning $\neg$.

Exercise 15.3. An inference rule is **invertible** if whenever its conclusion is provable (from any theory), then so are all of its hypotheses. Show that the ($\Rightarrow\wedge$) and ($\vee\Rightarrow$) rules are invertible.

Exercise 15.4. Derive the rule

$$\frac{\phi \wedge \psi \Rightarrow \theta}{\phi \Rightarrow \psi \to \theta}$$

and its inverse (with conclusion and hypothesis swapped). [See Exercise 12.19.]

Exercise 15.5. Show that if $\mathcal{T} \vdash \phi \wedge \psi \Rightarrow \theta$, then $\mathcal{T} \cup \{\phi\} \vdash \psi \Rightarrow \theta$ (where ϕ really means $\top \Rightarrow \phi$).

For a model $\mathcal{M}$ of $\mathcal{L}$ in a Boolean algebra A, we write

$$\mathcal{M} \models \phi \Rightarrow \psi \; :\Longleftrightarrow \; \phi^{\mathcal{M}} \leq \psi^{\mathcal{M}}$$

and say $\mathcal{M}$ **satisfies** (or **models**) $\phi \Rightarrow \psi$. If $\mathcal{M}$ satisfies every sequent in a theory $\mathcal{T}$, we write

$$\mathcal{M} \models \mathcal{T}$$

and say $\mathcal{M}$ is a **model of** $\mathcal{T}$ (or of $(\mathcal{L}, \mathcal{T})$). Put

$$\begin{aligned} \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A) &:= \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A) \mid \mathcal{M} \models \mathcal{T}\}, \\ \mathrm{Mod}(\mathcal{L}, \mathcal{T}) &:= \mathrm{Mod}(\mathcal{L}, \mathcal{T}, 2). \end{aligned}$$

Proposition 15.6 (soundness)**.** If $\mathcal{T} \vdash \sigma$, then for every Boolean algebra A and model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$, we have $\mathcal{M} \models \sigma$.

Proof. We show by induction on the proof of $\sigma = (\phi \Rightarrow \psi)$ from $\mathcal{T}$ that $\mathcal{M} \models \sigma$, i.e., $\phi^{\mathcal{M}} \le \psi^{\mathcal{M}}$. Basically, this amounts to showing that each inference rule corresponds to an axiom of Boolean algebras. For example:

- If $\mathcal{T} \vdash \phi \Rightarrow \theta$ by the Cut rule applied to $\mathcal{T} \vdash \phi \Rightarrow \psi$ and $\mathcal{T} \vdash \psi \Rightarrow \theta$, then by the IH, we have $\phi^{\mathcal{M}} \le \psi^{\mathcal{M}}$ and $\psi^{\mathcal{M}} \le \theta^{\mathcal{M}}$, whence $\phi^{\mathcal{M}} \le \theta^{\mathcal{M}}$ by transitivity.
- If $\mathcal{T} \vdash \phi \wedge \psi \Rightarrow \psi$ by the ($\wedge{\Rightarrow}_2$) rule, then we have $(\phi \wedge \psi)^{\mathcal{M}} = \phi^{\mathcal{M}} \wedge \psi^{\mathcal{M}} \le \psi^{\mathcal{M}}$.

etc. □

For a Boolean homomorphism $f : A \to B \in \mathsf{Bool}$ and model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A)$, we get an **induced model**

$$\mathrm{Mod}(\mathcal{L}, f)(\mathcal{M}) := f_*(\mathcal{M}) := (f(P^{\mathcal{M}}))_{P \in \mathcal{L}} = f \circ \mathcal{M} \in \mathrm{Mod}(\mathcal{L}, B) = B^{\mathcal{L}}.$$

In other words, for $P \in \mathcal{L}$,

$$P^{f_*(\mathcal{M})} := f(P^{\mathcal{M}}).$$

Lemma 15.7. For every formula $\phi \in \mathcal{L}_{\omega 0}$, we have

$$\phi^{f_*(\mathcal{M})} = f(\phi^{\mathcal{M}}).$$

Proof. By induction on ϕ. For example, if $\phi = \psi \wedge \theta$:

$$\begin{aligned} \phi^{f_*(\mathcal{M})} &= \psi^{f_*(\mathcal{M})} \wedge \theta^{f_*(\mathcal{M})} && \text{by definition of } \phi^{f^*(\mathcal{M})} \\ &= f(\psi^{\mathcal{M}}) \wedge f(\theta^{\mathcal{M}}) && \text{by the induction hypothesis} \\ &= f(\psi^{\mathcal{M}} \wedge \theta^{\mathcal{M}}) && \text{because } f \text{ is a homomorphism} \\ &= f(\phi^{\mathcal{M}}) && \text{by definition of } \phi^{\mathcal{M}}. \end{aligned}$$

□

Corollary 15.8. If $\mathcal{M} \models \phi \Rightarrow \psi$, then $f_*(\mathcal{M}) \models \phi \Rightarrow \psi$.

Proof. $\phi^{f_*(\mathcal{M})} = f(\phi^{\mathcal{M}}) \le f(\psi^{\mathcal{M}}) = \psi^{f_*(\mathcal{M})}$. □

Thus for an $\mathcal{L}$-theory $\mathcal{T}$, we get a map

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}, f) := f_* : \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A) \longrightarrow \mathrm{Mod}(\mathcal{L}, \mathcal{T}, B),$$

yielding a functor

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}, -) : \mathsf{Bool} \longrightarrow \mathsf{Set}$$

which is a copresheaf on Bool assigning to each A the set of models of $\mathcal{T}$ in A.

16 Lindenbaum–Tarski algebras

The **Lindenbaum–Tarski algebra** of a propositional theory $(\mathcal{L}, \mathcal{T})$ is

$$\langle \mathcal{L} \mid \mathcal{T} \rangle := \mathcal{L}_{\omega 0}/\mathcal{T} := \mathcal{L}_{\omega 0}/{\equiv_{\mathcal{T}}}.$$

We partially order $\langle \mathcal{L} \mid \mathcal{T} \rangle$ via

$$[\phi]_{\mathcal{T}} \le [\psi]_{\mathcal{T}} \;:\Longleftrightarrow\; \mathcal{T} \vdash \phi \Rightarrow \psi;$$

this is a partial order by the Id and Cut rules.

Proposition 16.1. $\langle \mathcal{L} \mid \mathcal{T} \rangle$ is a Boolean algebra, with operations induced by the logical connectives.

Proof. Because each axiom of Boolean algebras corresponds to an inference rule. For example, the meet of $[\phi]_{\mathcal{T}}, [\psi]_{\mathcal{T}} \in \langle \mathcal{L} \mid \mathcal{T} \rangle$ is $[\phi \wedge \psi]_{\mathcal{T}}$: we have $[\phi \wedge \psi] \le [\phi], [\psi]$ since $\mathcal{T} \vdash \phi \wedge \psi \Rightarrow \phi$ and $\mathcal{T} \vdash \phi \wedge \psi \Rightarrow \psi$ by $(\wedge{\Rightarrow_1})$ and $(\wedge{\Rightarrow_2})$, and for any other lower bound $[\theta] \le [\phi], [\psi]$, i.e., $\mathcal{T} \vdash \theta \Rightarrow \phi$ and $\mathcal{T} \vdash \theta \Rightarrow \psi$, we have $\mathcal{T} \vdash \theta \Rightarrow \phi \wedge \psi$ by $({\Rightarrow}\wedge)$, i.e., $[\theta] \le [\phi \wedge \psi]$. $\square$

The **universal model** $\mathcal{M}_{\mathcal{T}} = \mathcal{M}_{(\mathcal{L},\mathcal{T})}$ of $(\mathcal{L}, \mathcal{T})$ in $\langle \mathcal{L} \mid \mathcal{T} \rangle$ is given by

$$P^{\mathcal{M}_{\mathcal{T}}} := [P]_{\mathcal{T}} \in \langle \mathcal{L} \mid \mathcal{T} \rangle \qquad \text{for } P \in \mathcal{L}.$$

Lemma 16.2. For each formula $\phi \in \mathcal{L}_{\omega 0}$, we have

$$\phi^{\mathcal{M}_{\mathcal{T}}} = [\phi]_{\mathcal{T}} \in \langle \mathcal{L} \mid \mathcal{T} \rangle.$$

Proof. By induction on ϕ and the definition of Boolean operations on $\langle \mathcal{L} \mid \mathcal{T} \rangle$ from Proposition 16.1. For example, we have $(\phi \wedge \psi)^{\mathcal{M}_{\mathcal{T}}} = \phi^{\mathcal{M}_{\mathcal{T}}} \wedge \psi^{\mathcal{M}_{\mathcal{T}}} = [\phi]_{\mathcal{T}} \wedge [\psi]_{\mathcal{T}} = [\phi \wedge \psi]_{\mathcal{T}}$. $\square$

Corollary 16.3. $\mathcal{M}_{\mathcal{T}}$ is indeed a model of $\mathcal{T}$.

Proof. For an axiom $\phi \Rightarrow \psi \in \mathcal{T}$, we have $\phi^{\mathcal{M}_{\mathcal{T}}} = [\phi]_{\mathcal{T}} \le [\psi]_{\mathcal{T}} = \psi^{\mathcal{M}_{\mathcal{T}}}$ since $\mathcal{T} \vdash \phi \Rightarrow \psi$. $\square$

Corollary 16.4 (trivial completeness theorem for propositional logic)**.** If a sequent $\phi \Rightarrow \psi$ holds in every model of $\mathcal{T}$ in every Boolean algebra A, then $\mathcal{T} \vdash \phi \Rightarrow \psi$.

Proof. $\mathcal{M}_{\mathcal{T}} \models \phi \Rightarrow \psi$ means $[\phi]_{\mathcal{T}} = \phi^{\mathcal{M}} \le \psi^{\mathcal{M}} = [\psi]_{\mathcal{T}}$, i.e., $\mathcal{T} \vdash \phi \Rightarrow \psi$. $\square$

Theorem 16.5 (universality of $\mathcal{M}_{\mathcal{T}}$)**.** For any Boolean algebra A, we have a bijection

$$\begin{aligned} \mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, A) &\cong \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A) \\ f &\mapsto f_*(\mathcal{M}_{\mathcal{T}}) \\ ([\phi]_{\mathcal{T}} \mapsto \phi^{\mathcal{M}}) &\mapsfrom \mathcal{M}. \end{aligned}$$

In other words, $(\langle \mathcal{L} \mid \mathcal{T} \rangle, \mathcal{M}_{\mathcal{T}})$ represents the copresheaf $\mathrm{Mod}(\mathcal{L}, \mathcal{T}, -)$.

Proof. The map $f \mapsto f_*(\mathcal{M}_\mathcal{T})$ takes each f to the composite

$$\mathcal{L} \xrightarrow[P \mapsto [P]_\mathcal{T}]{\mathcal{M}_\mathcal{T}} \langle \mathcal{L} \mid \mathcal{T} \rangle \xrightarrow{f} A.$$

Since the image $\{[P]_\mathcal{T} \mid P \in \mathcal{L}\}$ of $\mathcal{M}_\mathcal{T}$ generates $\langle \mathcal{L} \mid \mathcal{T} \rangle$ as a Boolean algebra, this map is injective; and is surjective iff each model $\mathcal{M} : \mathcal{L} \to A$ of $\mathcal{T}$ extends along $\mathcal{M}_\mathcal{T}$ to a Boolean homomorphism $\langle \mathcal{L} \mid \mathcal{T} \rangle \to A$. Given $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$, the map $[\phi]_\mathcal{T} \mapsto \phi^\mathcal{M}$ is well-defined by soundness (since $\phi \equiv_\mathcal{T} \psi \implies \phi^\mathcal{M} = \psi^\mathcal{M}$), and is a Boolean homomorphism by definition of the Boolean operations on $\langle \mathcal{L} \mid \mathcal{T} \rangle$ from Proposition 16.1 as well as of the the interpretations of formulas in $\mathcal{M}$; and clearly its composite with $\mathcal{M}_\mathcal{T}$ is $\mathcal{M}$. $\square$

In words, a model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$ is a map $\mathcal{M} : \mathcal{L} \to A$ such that for each axiom $\phi \Rightarrow \psi \in \mathcal{T}$, the corresponding inequality $\mathcal{M}(\phi) \le \mathcal{M}(\psi) \in A$ holds; thus, Theorem 16.5 says

"$\langle \mathcal{L} \mid \mathcal{T} \rangle$ is the Boolean algebra presented by generators $\mathcal{L}$ and relations $\mathcal{T}$".

From now on, by abuse of notation, we will usually identify models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$ with Boolean homomorphisms $\langle \mathcal{L} \mid \mathcal{T} \rangle \to A$ via Theorem 16.5.

Conversely, every Boolean algebra A has a canonical presentation: take

$$\begin{aligned} \mathcal{L}_A &:= A, \\ \mathcal{M}_A &:= 1_A \in \mathrm{Mod}(\mathcal{L}_A, A), \\ \mathcal{T}_A &:= \{\phi \Rightarrow \psi \mid \phi, \psi \in (\mathcal{L}_A)_{\omega 0} \ \&\ \mathcal{M}_A \models \phi \Rightarrow \psi\}. \end{aligned}$$

We call $\mathcal{L}_A$ the **internal language** of A, and $\mathcal{T}_A$ the **internal theory** of A. Clearly, $\mathcal{M}_A$ is a model of $\mathcal{T}_A$.

Proposition 16.6. $\mathcal{M}_A : \langle \mathcal{L}_A \mid \mathcal{T}_A \rangle \to A$ is a Boolean algebra isomorphism.

Proof. By definition, $\mathcal{M}_A([\phi]_{\mathcal{T}_A}) = \phi^{\mathcal{M}_A}$; in particular, $\mathcal{M}_A([a]_{\mathcal{T}_A}) = a^{\mathcal{M}_A} = a$ for all $a \in A$. So it suffices to show that every $\mathcal{L}_A$-formula ϕ is $\mathcal{T}_A$-equivalent to an atomic formula. This follows from the definition of $\mathcal{T}_A$, since $\phi^{\mathcal{M}_A} = (\phi^{\mathcal{M}_A})^{\mathcal{M}_A}$, i.e., $\mathcal{M}_A \models \phi \Leftrightarrow \phi^{\mathcal{M}_A}$, whence $\mathcal{T}_A \vdash \phi \Leftrightarrow \phi^{\mathcal{M}_A}$. $\square$

17 Strong completeness and interpretations

We now apply Stone duality to the Lindenbaum–Tarski algebra $\langle \mathcal{L} \mid \mathcal{T} \rangle$ of a propositional theory $(\mathcal{L}, \mathcal{T})$.

Via Theorem 16.5, the Stone dual space topology on $\mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2)$, which is generated by the evaluation maps

$$\begin{aligned} \eta_{\langle \mathcal{L} \mid \mathcal{T} \rangle}([\phi]_\mathcal{T}) : \mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2) &\longrightarrow 2 \\ \mu &\longmapsto \mu([\phi]_\mathcal{T}), \end{aligned}$$

transports to the topology on $\mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$ generated by the maps

$$\begin{aligned} \mathrm{Mod}(\mathcal{L}, \mathcal{T}, 2) &\longrightarrow 2 \\ \mathcal{M} &\longmapsto \phi^\mathcal{M} \end{aligned}$$

for $\phi \in \mathcal{L}_{\omega 0}$. Note that since $\eta_{\langle \mathcal{L} | \mathcal{T} \rangle}$ is a Boolean homomorphism, it is enough to consider the generators $[P]_\mathcal{T} \in \langle \mathcal{L} \mid \mathcal{T} \rangle$ for $P \in \mathcal{L}$, i.e., to consider the maps $\mathcal{M} \mapsto P^\mathcal{M}$, i.e., the topology on $\mathrm{Mod}(\mathcal{L}, \mathcal{T}, 2) \subseteq \mathrm{Mod}(\mathcal{L}, 2) \subseteq 2^\mathcal{L}$ is just the usual subspace topology induced by the product topology.

Theorem 17.1 (strong completeness theorem for propositional logic)**.**

(a) For any propositional theory $(\mathcal{L}, \mathcal{T})$, we have a Boolean algebra isomorphism

$$\begin{aligned} [\![-]\!] : \langle \mathcal{L} \mid \mathcal{T} \rangle &\cong \mathcal{KO}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \\ [\phi]_\mathcal{T} &\mapsto \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \mid \mathcal{M} \models \phi\} =: [\![\phi]\!] \end{aligned}$$

between $\mathcal{T}$-equivalence classes of formulas and clopen sets of models. Equivalently:

   (i) (completeness) Every sequent $\phi \Rightarrow \psi$ satisfied by every (2-valued) model of $\mathcal{T}$ is provable from $\mathcal{T}$.

   (ii) (definability) Every clopen set of models is $[\![\phi]\!]$ for some formula $\phi \in \mathcal{L}_{\omega 0}$.

(b) Every Stone space is homeomorphic to $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ for some propositional theory $(\mathcal{L}, \mathcal{T})$.

Proof. The map $[\![-]\!]$ is the composite

$$\begin{array}{ccccccc} \langle \mathcal{L} \mid \mathcal{T} \rangle & \xrightarrow[\cong]{\eta_{\langle \mathcal{L} | \mathcal{T} \rangle}} & \mathsf{Top}(\mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2), 2) & \underset{\cong}{\to} & \mathcal{KO}(\mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2)) & \xrightarrow[\cong]{16.5} & \mathcal{KO}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \\ [\phi]_\mathcal{T} & \longmapsto & (\mu \mapsto \mu([\phi]_\mathcal{T})) & \longmapsto & \{\mu \mid \mu([\phi]_\mathcal{T}) = 1\} & \longmapsto & \{\mathcal{M} \mid \mathcal{M} \models \phi\}, \end{array}$$

whence (a) is a restatement of the algebraic half of Stone duality (Theorem 14.3). Part (i) says that if $[\![\phi]\!] \le [\![\psi]\!]$, then $[\phi]_\mathcal{T} \le [\psi]_\mathcal{T}$, i.e., $[\![-]\!]$ is injective. Part (ii) says that $[\![-]\!]$ is surjective.

By Theorem 16.5 and Proposition 16.6, (b) is equivalent to essential surjectivity of $\mathsf{Bool}(-, 2) : \mathsf{Bool}^{\mathrm{op}} \to \mathsf{Stone}$; since (a) is equivalent to full+faithfulness of $\mathsf{Bool}(-, 2)$ (Corollary 14.4), (a)+(b) is equivalent to $\mathsf{Bool}(-, 2)$ being an equivalence. $\square$

We may also directly interpret the functorial statement of Stone duality (Theorem 14.8) in logical terms, using the following general notion.

An **interpretation** $\mathcal{F} : (\mathcal{L}_1, \mathcal{T}_1) \to (\mathcal{L}_2, \mathcal{T}_2)$ between propositional theories is a way of mapping the syntax of $\mathcal{T}_1$ to (not truth values but) the syntax of $\mathcal{T}_2$. By Theorem 16.5, we may define $\mathcal{F}$ as:

- a Boolean homomorphism $\mathcal{F} : \langle \mathcal{L}_1 \mid \mathcal{T}_1 \rangle \to \langle \mathcal{L}_2 \mid \mathcal{T}_2 \rangle$;
- equivalently, a model $\mathcal{F} \in \mathrm{Mod}(\mathcal{L}_1, \mathcal{T}_1, \langle \mathcal{L}_2 \mid \mathcal{T}_2 \rangle)$;
- explicitly, for each $P \in \mathcal{L}_1$, a ($\mathcal{T}_2$-equivalence class of) $\mathcal{L}_2$-formula(s) $\mathcal{F}(P)$, such that, defining inductively $\mathcal{F}(\phi)$ for each $\mathcal{L}_1$-formula ϕ in the obvious way, we have

$$\forall (\phi \Rightarrow \psi) \in \mathcal{T}_1 \, (\mathcal{T}_2 \vdash \mathcal{F}(\phi) \Rightarrow \mathcal{F}(\psi)).$$

Given $\mathcal{F}$, we get a syntactic "recipe" for turning models $\mathcal{M}$ of $\mathcal{T}_2$ into models $\mathcal{F}^*(\mathcal{M})$ of $\mathcal{T}_1$, which via Theorem 16.5 is simply the composite

$$\mathcal{F}^*(\mathcal{M}) := \left(\langle \mathcal{L}_1 \mid \mathcal{T}_1 \rangle \xrightarrow{\mathcal{F}} \langle \mathcal{L}_2 \mid \mathcal{T}_2 \rangle \xrightarrow{\mathcal{M}} 2 \right).$$

Explicitly, $\mathcal{F}^*(\mathcal{M})$ is defined by, for each $\phi \in (\mathcal{L}_1)_{\omega 0}$,

$$\phi^{\mathcal{F}^*(\mathcal{M})} = \mathcal{F}(\phi)^{\mathcal{M}}.$$

Let

$$\mathsf{PropThy} := \text{category of propositional theories } (\mathcal{L}, \mathcal{T}), \text{ interpretations.}$$

This is nothing but a different guise for Bool, in that we have an equivalence

$$\langle - \rangle : \mathsf{PropThy} \xrightarrow{\sim} \mathsf{Bool}$$

taking theories to their Lindenbaum–Tarski algebras and interpretations to the corresponding Boolean homomorphisms (i.e., themselves); essential surjectivity is by the internal theory construction (Proposition 16.6). The composite

$$\mathsf{PropThy} \xrightarrow[\sim]{\langle - \rangle} \mathsf{Bool} \xrightarrow{\mathsf{Bool}(-,2)} \mathsf{Stone}^{\mathrm{op}}$$

is then (naturally isomorphic to)

$$(\mathcal{L}, \mathcal{T}) \longmapsto \langle \mathcal{L} \mid \mathcal{T} \rangle \longmapsto \mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2) \cong \mathrm{Mod}(\mathcal{L}, \mathcal{T}).$$

Stone duality is thus equivalent to:

Theorem 17.2 (duality for propositional logic)**.** We have an equivalence of categories

$$\mathrm{Mod} : \mathsf{PropThy} \xrightarrow{\sim} \mathsf{Stone}^{\mathrm{op}}.$$

In other words:

(a) For any propositional theories $(\mathcal{L}_1, \mathcal{T}_1), (\mathcal{L}_2, \mathcal{T}_2)$, we have a bijection

$$\begin{aligned} \mathsf{PropThy}(\mathcal{T}_1, \mathcal{T}_2) &\cong \mathsf{Stone}(\mathrm{Mod}(\mathcal{T}_2), \mathrm{Mod}(\mathcal{T}_1)) \\ \mathcal{F} &\mapsto \mathcal{F}^* \end{aligned}$$

between interpretations and continuous maps between spaces of models.

(b) Every Stone space is homeomorphic to $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ for some propositional theory $(\mathcal{L}, \mathcal{T})$.

Exercise 17.3. Directly prove the equivalence between Theorem 17.1(a) and Theorem 17.2(a) (without mentioning Boolean algebras or categories, using the explicit definition of interpretation).

18 Commuting structures and dual adjunctions

Recall from Section 14 that Stone duality has an abstract nonsense component, namely that the two compatible structures on the set 2 give rise to a dual adjunction $\mathsf{Bool} \rightleftarrows \mathsf{Top}^{\mathrm{op}}$, as well as a concrete component characterizing when this adjunction is an equivalence. In this section, we discuss the abstract nonsense component for two general classes of (first-order) structures. Before doing so, we must first introduce some basic notions from first-order logic.

A **(single-sorted infinitary) first-order language** $\mathcal{L}$ is a set of symbols, classified into two subsets $\mathcal{L} = \mathcal{L}_{\text{fun}} \sqcup \mathcal{L}_{\text{rel}}$ of **function symbols** $f \in \mathcal{L}_{\text{fun}}$ and **relation symbols** $R \in \mathcal{L}_{\text{rel}}$, each of which has an associated set called its **arity** $\text{ar}(f), \text{ar}(R)$ (which can be thought of as a cardinal number, but may as well be an arbitrary set). An **$\mathcal{L}$-structure** (or **model of $\mathcal{L}$**)

$$\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}})$$

consists of:

- an **underlying set** M;
- for each function symbol $f \in \mathcal{L}_{\text{fun}}$, an **interpretation of f in $\mathcal{M}$** as a function
$$f^{\mathcal{M}} = \mathcal{M}(f) : M^{\text{ar}(f)} \longrightarrow M;$$
- for each relation symbol $R \in \mathcal{L}_{\text{rel}}$, an **interpretation of R in $\mathcal{M}$** as a relation
$$R^{\mathcal{M}} = \mathcal{M}(R) \subseteq M^{\text{ar}(R)}.$$

Example 18.1. The **language of groups** is $\mathcal{L}_{\mathsf{Grp}} := \{\cdot, 1, ^{-1}\}$, all function symbols, with arities $2, 0, 1$ respectively. A $\mathcal{L}_{\mathsf{Grp}}$-structure is a set equipped with an arbitrary binary operation $\cdot$, constant 1, and unary operation $^{-1}$ (note that the notion of $\mathcal{L}$-structure does not take axioms into account).

Example 18.2. The **language of partial orders** is $\mathcal{L}_{\mathsf{Pos}} := \{\leq\}$ where $\leq$ is a binary relation symbol; an $\mathcal{L}_{\mathsf{Pos}}$-structure is a set equipped with an arbitrary binary relation $\leq$.

Similarly, the language of partially ordered groups is $\{\cdot, 1, ^{-1}, \leq\}$, where the first three symbols are function symbols and $\leq$ is a relation symbol.

An **$\mathcal{L}$-homomorphism** $f : \mathcal{M} \to \mathcal{N}$ between $\mathcal{L}$-structures is a function $f : M \to N$ between their underlying sets such that

- for each $g \in \mathcal{L}_{\text{fun}}$, we have
$$f(g^{\mathcal{M}}(\vec{a})) = g^{\mathcal{N}}(f(\vec{a})) \quad \forall \vec{a} \in M^{\text{ar}(g)};$$
- for each $R \in \mathcal{L}_{\text{rel}}$, we have
$$R^{\mathcal{M}}(\vec{a}) \implies R^{\mathcal{N}}(f(\vec{a})) \quad \forall \vec{a} \in M^{\text{ar}(R)}.$$

Let

$$\mathsf{Mod}(\mathcal{L}) := \text{category of } \mathcal{L}\text{-structures, } \mathcal{L}\text{-homomorphisms.}$$

Example 18.3. $\mathsf{Grp} \subseteq \mathsf{Mod}(\mathcal{L}_{\mathsf{Grp}})$ is a full subcategory.

A **substructure** of an $\mathcal{L}$-structure $\mathcal{M}$ is an $\mathcal{L}$-structure $\mathcal{N}$ with $N \subseteq M$, $f^{\mathcal{N}} = f^{\mathcal{M}}|N$ for all $f \in \mathcal{L}_{\text{fun}}$, and $R^{\mathcal{N}} = R^{\mathcal{M}}|N$ for all $R \in \mathcal{L}_{\text{rel}}$. Equivalently, it is enough to specify an arbitrary subset of M closed under each $f^{\mathcal{M}}$.

The **product** $\prod_i \mathcal{M}_i$ of $\mathcal{L}$-structures $\mathcal{M}_i$ for $i \in I$, I some set, is the $\mathcal{L}$-structure with underlying set $\prod_i M_i$ and

$$\begin{aligned} f^{\prod_i \mathcal{M}_i}((\vec{a}^j)_{j \in \text{ar}(f)}) &:= (f^{\mathcal{M}_i}((a_i^j)_j))_{i \in I} && \text{for } f \in \mathcal{L}_{\text{fun}} \text{ and } \vec{a}^j \in \textstyle\prod_i M_i \text{ for each } j \in \text{ar}(f), \\ R^{\prod_i \mathcal{M}_i}((\vec{a}^j)_{j \in \text{ar}(R)}) &:\iff \forall i \in I\, (R^{\mathcal{M}_i}((a_i^j)_j)) && \text{for } R \in \mathcal{L}_{\text{rel}} \text{ and } \vec{a}^j \in \textstyle\prod_i M_i \text{ for each } j \in \text{ar}(R). \end{aligned}$$

Example 18.4. For $\mathcal{L} = \mathcal{L}_{\mathsf{Pos}}$, this says that for $\vec{a}^0, \vec{a}^1 \in \prod_i M_i$,

$$\vec{a}^0 \le_{\prod_i \mathcal{M}_i} \vec{a}^1 \;:\Longleftrightarrow\; \forall i \in I\,(a_i^0 \le_{\mathcal{M}_i} a_i^1).$$

Exercise 18.5. Verify that each projection from a product structure is a homomorphism.

Exercise 18.6. Verify that for an $\mathcal{L}$-substructure $\mathcal{N} \subseteq \prod_i \mathcal{M}_i$ of a product, for another $\mathcal{L}$-structure $\mathcal{K}$, a function $K \to N$ is a homomorphism iff its composite with each projection $\prod_i \mathcal{M}_i \to \mathcal{M}_{i_0}$ is.

Example 18.7. The **language of topological spaces** $\mathcal{L}_{\mathsf{Top}}$ consists of, for each directed poset I, an $(I+1)$-ary (i.e., $(I \sqcup 1)$-ary) relation symbol Lim_I; given a topological space X, we may regard X as a $\mathcal{L}_{\mathsf{Top}}$-structure where

$$\mathrm{Lim}_I^X(\vec{a}, b) \;:\Longleftrightarrow\; \lim_I \vec{a} = b \qquad \text{for } \vec{a} \in X^I,\, b \in X$$

(but as usual, there are many other $\mathcal{L}_{\mathsf{Top}}$-spaces whose Lim_I-relations are not compatible with each other). A substructure of a topological space is a subset equipped with the subspace topology; likewise, the product structure has the product topology.

Note that $\mathcal{L}_{\mathsf{Top}}$ is a *large* language: in order to capture all topological spaces in a fixed universe $\mathcal{U}$ (see Section 7), we need to consider limits indexed by all directed sets in $\mathcal{U}$, whence $\mathcal{L}_{\mathsf{Top}}$ itself does not belong to $\mathcal{U}$; hence, a topological space $X \in \mathcal{U}$ regarded as a $\mathcal{L}_{\mathsf{Top}}$-structure *also does not belong to* $\mathcal{U}$ (only its underlying set does). Size issues like this are unavoidable in general duality theory.[16]

Exercise 18.8. Let $\mathcal{L}$ be an infinitary first-order language. A **universal Horn ($\mathcal{L}$-)axiom** is a first-order axiom of the form

$$\forall \vec{x}\,(\textstyle\bigwedge \Phi(\vec{x}) \Rightarrow \psi(\vec{x}))$$

where $\vec{x}$ is a (possibly infinite) collection of variables, $\psi(\vec{x})$ is an **atomic first-order formula** of the form $R((t_i(\vec{x}))_{i \in \mathrm{ar}(R)})$ where the $t_i(\vec{x})$ are terms (expressions) built from the function symbols in $\mathcal{L}_{\mathrm{fun}}$ and the variables $\vec{x}$ and R is either a relation symbol in $\mathcal{L}_{\mathrm{rel}}$ or the symbol $=$, and $\bigwedge \Phi(\vec{x})$ is a (possibly infinite) conjunction of a set $\Phi(\vec{x})$ of such atomic formulas. For example,

$$\forall x, y, z\,(x + z \le y + z \implies x \le y)$$

is a universal Horn axiom in the language of partially ordered abelian groups.

(a) Show that satisfaction of universal Horn axioms is inherited by products and substructures.

(b) Show that for any $\mathcal{L}$-structure $\mathcal{M}$, relation symbol $R \in \mathcal{L}_{\mathrm{rel}} \cup \{=\}$, and tuple $\vec{a} \in M^{\mathrm{ar}(R)}$, there is a universal Horn axiom which is satisfied by precisely those $\mathcal{L}$-structures $\mathcal{N}$ that do not admit a homomorphism $f : \mathcal{M} \to \mathcal{N}$ with $\neg R^{\mathcal{N}}(f(\vec{a}))$.

(c) Show that for any set $\mathcal{S}$ of $\mathcal{L}$-structures, another $\mathcal{L}$-structure $\mathcal{M}$ satisfies every universal Horn axiom satisfied by all $\mathcal{N} \in \mathcal{S}$ iff $\mathcal{M}$ embeds as a substructure of a product of structures in $\mathcal{S}$.

(d) Let $\mathsf{C} \subseteq \mathsf{Mod}(\mathcal{L})$ be a full subcategory, closed under isomorphisms. Show that C is closed under products and substructures iff it is axiomatized by a set of universal Horn axioms.

[16] **TODO**

(Here $\mathcal{L}$ and the set of axioms may be large (i.e., outside the background universe), but $\mathsf{Mod}(\mathcal{L})$ denotes the category of $\mathcal{L}$-structures with small underlying set, and each $\vec{x}, \Phi(\vec{x})$ should be small.)

Now fix two infinitary first-order languages $\mathcal{L}_1, \mathcal{L}_2$, let $\mathsf{C}_1 \subseteq \mathsf{Mod}(\mathcal{L}_1)$ and $\mathsf{C}_2 \subseteq \mathsf{Mod}(\mathcal{L}_2)$ be full subcategories closed under isomorphisms, products, and substructures,[17] and let $\mathcal{K}_1 \in \mathsf{C}_1$ and $\mathcal{K}_2 \in \mathsf{C}_2$ be structures on the same underlying set K. We say that these two structures on K **commute** if

- for each $f \in (\mathcal{L}_1)_{\mathrm{fun}}$, $f^{\mathcal{K}_1} : K^{\mathrm{ar}(f)} \to K$ is an $\mathcal{L}_2$-homomorphism $\mathcal{K}_2^{\mathrm{ar}(f)} \to \mathcal{K}_2$;
- for each $g \in (\mathcal{L}_2)_{\mathrm{fun}}$, $g^{\mathcal{K}_2} : K^{\mathrm{ar}(g)} \to K$ is an $\mathcal{L}_1$-homomorphism $\mathcal{K}_1^{\mathrm{ar}(f)} \to \mathcal{K}_1$.

We let $\mathcal{K}$ denote the set K equipped with both structures $\mathcal{K}_1, \mathcal{K}_2$, and call it a **dualizing object**. We may then set up a dual adjunction, exactly as in Section 14:

- For all $\mathcal{M} \in \mathsf{C}_1$, $\mathsf{C}_1(\mathcal{M}, \mathcal{K}_1) \subseteq K^M$ is an $\mathcal{L}_2$-substructure of $\mathcal{K}_2^M$, hence in C_2. Indeed, for a function symbol $g \in (\mathcal{L}_2)_{\mathrm{fun}}$, $\mathsf{C}_1(\mathcal{M}, \mathcal{K}_1)$ is closed under $g^{\mathcal{K}_2^M}$: for $\mathcal{L}_1$-homomorphisms $f_i \in \mathsf{C}_1(\mathcal{M}, \mathcal{K}_1)$ for each $i \in \mathrm{ar}(g)$, $g^{\mathcal{K}_2^M}((f_i)_i)$ is the composite $\mathcal{L}_1$-homomorphism

  $$\mathcal{M} \xrightarrow{(f_i)_i} \mathcal{K}_1^{\mathrm{ar}(g)} \xrightarrow{g^{\mathcal{K}_2}} \mathcal{K}_1,$$

  using that $g^{\mathcal{K}_2}$ is an $\mathcal{L}_1$-homomorphism. We write

  $$\mathsf{C}_1(\mathcal{M}, \mathcal{K}) := \mathsf{C}_1(\mathcal{M}, \mathcal{K}_1) \text{ as an } \mathcal{L}_2\text{-structure of } \mathcal{K}_2^M,$$

  to indicate the dependence on $\mathcal{K}_2$ as well as $\mathcal{K}_1$.
- For an $\mathcal{L}_1$-homomorphism $f : \mathcal{M} \to \mathcal{M}' \in \mathsf{C}_1$, the precomposition map

  $$\mathsf{C}_1(f, \mathcal{K}_1) : \mathsf{C}_1(\mathcal{M}', \mathcal{K}_1) \longrightarrow \mathsf{C}_1(\mathcal{M}, \mathcal{K}_1)$$

  is an $\mathcal{L}_2$-homomorphism. Indeed, using Exercise 18.6, its composite with each coordinate projection $(-)(a) : \mathsf{C}_1(\mathcal{M}, \mathcal{K}_1) \to \mathcal{K}_1$ for $a \in M$ is the projection $(-)(f(a)) : \mathsf{C}_1(\mathcal{M}', \mathcal{K}_1) \to \mathcal{K}_1$. We thus get a functor

  $$\mathsf{C}_1(-, \mathcal{K}) : \mathsf{C}_1^{\mathrm{op}} \longrightarrow \mathsf{C}_2$$

  lifting $\text{よ}_{\mathsf{C}_1}\mathcal{K}_1 = \mathsf{C}_1(-, \mathcal{K}_1) : \mathsf{C}_1^{\mathrm{op}} \to \mathsf{Set}$ across the forgetful functor $\mathsf{C}_2 \to \mathsf{Set}$.
- Similarly, we get a functor

  $$\mathsf{C}_2(-, \mathcal{K}) : \mathsf{C}_2^{\mathrm{op}} \longrightarrow \mathsf{C}_1$$

  lifting $\text{よ}_{\mathsf{C}_2}\mathcal{K}_2 = \mathsf{C}_2(-, \mathcal{K}_2) : \mathsf{C}_2^{\mathrm{op}} \to \mathsf{Set}$ across the forgetful functor $\mathsf{C}_1 \to \mathsf{Set}$.
- For $\mathcal{M} \in \mathsf{C}_1$ and $\mathcal{N} \in \mathsf{C}_2$, we have natural bijections

  $$\begin{aligned} \mathsf{Set}(M, \mathsf{Set}(N, K)) &\cong \mathsf{Set}(M \times N, K) \cong \mathsf{Set}(N, \mathsf{Set}(M, K)) \\ f &\mapsto ((a, b) \mapsto f(a)(b)) \mapsto (b \mapsto f(-)(b)). \end{aligned}$$

[17] It is possible to work under slightly more general closure conditions on $\mathsf{C}_1, \mathsf{C}_2$; see **TODO**.

For $f \in \mathsf{Set}(M, \mathsf{Set}(N, K))$, f lands in $\mathsf{C}_2(\mathcal{N}, \mathcal{K}_2)$ iff each $f(a)(-) : N \to K$ is an $\mathcal{L}_2$-homomorphism; in that case, $f \in \mathsf{C}_1(\mathcal{M}, \mathsf{C}_2(\mathcal{N}, \mathcal{K}))$ iff each $f(-)(b)$ is an $\mathcal{L}_1$-homomorphism (by Exercise 18.6). Similar reasoning applies to $g \in \mathsf{Set}(N, \mathsf{Set}(M, K))$. Thus the above natural bijections restrict to

$$\mathsf{C}_1(\mathcal{M}, \mathsf{C}_2(\mathcal{N}, \mathcal{K})) \cong (\mathsf{C}_1, \mathsf{C}_2)(\mathcal{M} \times \mathcal{N}, \mathcal{K}) \cong \mathsf{C}_2(\mathcal{N}, \mathsf{C}_1(\mathcal{M}, \mathcal{K})),$$

where

$$(\mathsf{C}_1, \mathsf{C}_2)(\mathcal{M} \times \mathcal{N}, \mathcal{K}) := \{\textbf{bihomomorphisms } \mathcal{M} \times \mathcal{N} \to \mathcal{K}\},$$

i.e., maps $M \times N \to K$ which are homomorphisms in each variable once the other is fixed.

- We thus get a dual adjunction

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, shift left, "\mathsf{C}_1(-,\mathcal{K})"] \arrow[r, phantom, "\perp"] & \mathsf{C}_2^{\mathrm{op}} . \arrow[l, shift left, "\mathsf{C}_2(-,\mathcal{K})"] \end{tikzcd}$$

The units are easily computed (as in Example 11.13) to be, for $\mathcal{M} \in \mathsf{C}_1$ and $\mathcal{N} \in \mathsf{C}_2$,

$$\begin{aligned} \eta_{\mathcal{M}} : \mathcal{M} &\longrightarrow \mathsf{C}_2(\mathsf{C}_1(\mathcal{M}, \mathcal{K}), \mathcal{K}) \\ a &\longmapsto (f \mapsto f(a)), \end{aligned} \qquad \begin{aligned} \varepsilon_{\mathcal{N}} : \mathcal{N} &\longrightarrow \mathsf{C}_1(\mathsf{C}_2(\mathcal{N}, \mathcal{K}), \mathcal{K}) \\ b &\longmapsto (g \mapsto g(b)). \end{aligned}$$

We say that the dual adjunction is a **duality between** $\mathsf{C}_1, \mathsf{C}_2$ if it is a dual adjoint equivalence of categories, or equivalently (by Proposition 11.24), each $\eta_{\mathcal{M}}$ and $\varepsilon_{\mathcal{N}}$ is an isomorphism.

We now consider what it means to have two commuting structures $\mathcal{K}_1, \mathcal{K}_2$ on $\mathcal{K}$. It is helpful to consider what it means for $f \in (\mathcal{L}_1)_{\mathrm{fun}}$ (say) to be a homomorphism with respect to each individual symbol in $\mathcal{L}_2$:

- For a function symbol $g \in (\mathcal{L}_2)_{\mathrm{fun}}$, f being a g-homomorphism means that for any tuple of tuples $((a_i^j)_{i \in \mathrm{ar}(f)})_{j \in \mathrm{ar}(g)} \in (K^{\mathrm{ar}(f)})^{\mathrm{ar}(g)}$, we have

$$f^{\mathcal{K}_1}((g^{\mathcal{K}_2}((a_i^j)_j))_i) = f^{\mathcal{K}_1}(g^{\mathcal{K}_2^{\mathrm{ar}(f)}}(((a_i^j)_i)_j)) = g^{\mathcal{K}_2}((f^{\mathcal{K}_1}((a_i^j)_i))_j).$$

In other words, for any $\mathrm{ar}(f) \times \mathrm{ar}(g)$ matrix of elements $(a_i^j)_{i,j}$, applying f to each column and then applying g yields the same result as applying g to each row and then applying f:

$$\begin{tikzcd}
 & & \mathrm{ar}(g) & & & \\
 & a_{i_1}^{j_1} & a_{i_1}^{j_2} & a_{i_1}^{j_3} & \cdots \arrow[r, mapsto, "g"] & g^{\mathcal{K}_2}((a_{i_1}^j)_j) \\
\mathrm{ar}(f) & a_{i_2}^{j_1} & a_{i_2}^{j_2} & a_{i_2}^{j_3} & \cdots \arrow[r, mapsto, "g"] & g^{\mathcal{K}_2}((a_{i_2}^j)_j) \\
 & a_{i_3}^{j_1} & a_{i_3}^{j_2} & a_{i_3}^{j_3} & \cdots \arrow[r, mapsto, "g"] & g^{\mathcal{K}_2}((a_{i_3}^j)_j) \\
 & \vdots \arrow[d, mapsto, "f"'] & \vdots \arrow[d, mapsto, "f"'] & \vdots \arrow[d, mapsto, "f"'] & & \vdots \arrow[d, mapsto, "f"'] \\
 & f^{\mathcal{K}_1}((a_i^{j_1})_i) & f^{\mathcal{K}_1}((a_i^{j_2})_i) & f^{\mathcal{K}_1}((a_i^{j_3})_i) & \cdots \arrow[r, mapsto, "g"] & g^{\mathcal{K}_2}((f^{\mathcal{K}_1}((a_i^j)_i))_j) = f^{\mathcal{K}_1}((g^{\mathcal{K}_2}((a_i^j)_j))_i).
\end{tikzcd}$$

In particular, note that *f is a g-homomorphism iff g is an f-homomorphism.*

- For a relation symbol $R \in (\mathcal{L}_2)_{\mathrm{rel}}$, f being an R-homomorphism means that for any tuple of tuples $((a_i^j)_{i \in \mathrm{ar}(f)})_{j \in \mathrm{ar}(R)} \in (K^{\mathrm{ar}(f)})^{\mathrm{ar}(g)}$, we have

$$\forall i\, R^{\mathcal{K}_2}((a_i^j)_j) \iff R^{\mathcal{K}_2^{\mathrm{ar}(f)}}(((a_i^j)_i)_j) \implies R^{\mathcal{K}_2}((f^{\mathcal{K}_1}((a_i^j)_i))_j).$$

  So f *is an R-homomorphism iff R is closed under f.*

Example 18.9. If neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ has function symbols, then $\mathcal{K}_1$ and $\mathcal{K}_2$ vacuously commute. For example, a topology always commutes with itself (note that this says nothing about commutation of limits in the usual sense).

Example 18.10. For $\mathcal{L}_2 = \mathcal{L}_{\mathsf{Pos}}$, $\mathcal{K}_1$ commutes with $\mathcal{K}_2$ iff for each $f \in (\mathcal{L}_1)_{\mathrm{fun}}$, for all $\vec{a}^0, \vec{a}^1 \in \mathrm{ar}(f)$, we have

$$\forall i\, (a_i^0 \le_{\mathcal{K}_2} a_i^1) \implies f^{\mathcal{K}_1}(\vec{a}^0) \le_{\mathcal{K}_2} f^{\mathcal{K}_1}(\vec{a}^1).$$

For example:

- A monoid structure commutes with a partial order iff $1 \le 1$ (always true) and

$$a \le c \ \&\ b \le d \implies ab \le cd \qquad \forall a, b, c, d.$$

- A (semi)lattice structure commutes with its induced partial order.
- A group structure commutes with a partial order iff the monoid structure does, and also

$$a \le b \implies a^{-1} \le b^{-1}.$$

  This is *not* what is usually called a partially ordered group; in fact, it implies that the partial order is $=$ (Exercise).

- Likewise, a Boolean algebra structure commutes with its induced partial order iff the algebra is trivial.

Example 18.11. Similarly, for $\mathcal{L}_2 = \mathcal{L}_{\mathsf{Top}}$ and $\mathcal{K}_2$ given by a topology, $\mathcal{K}_1$ commutes with $\mathcal{K}_2$ iff for each $f \in (\mathcal{L}_1)_{\mathrm{fun}}$, for all directed posets J, for all nets $(a_i^j)_{j \in J}$ and points b_i for each $i \in \mathrm{ar}(f)$, we have

$$\forall i\, (\lim_{j \in J} a_i^j = b_i) \implies \lim_{j \in J} f^{\mathcal{K}_1}((a_i^j)_i) = f((b_i)_i).$$

By definition of the product topology, this means precisely that each $f^{\mathcal{K}_1}$ is continuous.

In particular, if $\mathcal{K}_2$ is the discrete topology on K, then all structures $\mathcal{K}_1$ on K with finitary operations commute with $\mathcal{K}_2$.

Example 18.12. Two monoid structures $(K, \cdot, 1)$ and $(K, *, e)$ on the same set commute with each other iff

$$\begin{aligned} 1 &= e, \\ 1 &= 1 * 1, \\ e \cdot e &= e, \\ (a * b) \cdot (c * d) &= (a \cdot c) * (b \cdot d) \qquad \forall a, b, c, d \in K. \end{aligned}$$

From the last equation, we have

$$1 = 1 \cdot 1 = (1 * e) \cdot (e * 1) = (1 \cdot e) * (e \cdot 1) = e * e = e$$

which implies the other three equations. Again from the last equation, we have

$$a \cdot d = (a * e) \cdot (e * d) = (a \cdot 1) * (1 \cdot d) = a * d,$$

whence the two monoid structures are the same. Finally,

$$b \cdot c = (e * b) \cdot (c * e) = (1 \cdot c) * (b \cdot 1) = c * b,$$

whence the monoid is commutative. Conversely, it is easily seen that any commutative monoid structure commutes with itself. This is called the **Eckmann–Hilton argument**: two commuting monoid structures on the same set are the same thing as a single commutative monoid structure (and similarly for groups).

Example 18.13. As noted in the previous example, two constants commute iff they are equal; and more generally, a constant commutes with an operation iff that operation takes the constant tuple to the constant.

Hence, no structure with two different constants (e.g., a nontrivial ring) commutes with itself.

Example 18.14. By Proposition 12.1, any meets that exist in a poset commute with each other.

A poset P with all meets is called a **complete meet-semilattice** (or **$\bigwedge$-lattice**). These can be regarded as structures in the (large) language $\mathcal{L}_{\bigwedge}$ consisting of an operation $\bigwedge_I$ of arity I for each (small) set I. Thus, every $\bigwedge$-lattice structure commutes with itself (and by the Eckmann–Hilton argument, two commuting $\bigwedge$-lattice structures must in fact be the same).

Exercise 18.15. Show that a poset has all meets iff it has all joins. [Take $\bigvee_i a_i = \bigwedge \bigcap_i \uparrow a_i$.]

Thus, the terms **$\bigwedge$-lattice** and **$\bigvee$-lattice** refer to the same objects, usually called **complete lattices**. However, when we say "$\bigwedge$-lattice", we are only regarding meets as part of the structure. In other words, the categories

$$\begin{aligned} \bigwedge\mathsf{Lat} &:= \text{category of } \bigwedge\text{-lattices, } \bigwedge\text{-preserving maps,} \\ \bigvee\mathsf{Lat} &:= \text{category of } \bigvee\text{-lattices, } \bigwedge\text{-preserving maps} \end{aligned}$$

have the same objects, but different morphisms.

Commutation of meets and joins is more interesting, even in the simplest nontrivial lattice $2 = \{0 < 1\}$. A general rule of thumb is

"a class of meets in 2 commutes with the *opposite* class of joins."

Example 18.16. The empty meet 1 commutes with precisely all nonempty joins (by Example 18.13).

Exercise 18.17. Show that conversely, if an index set I is such that I-ary meets in 2 commute with binary joins, then $I = \varnothing$ or $|I| = 1$. (Of course, the identity function commutes with everything.)

Example 18.18. Finite meets commute with *directed* joins, meaning that for any directed poset J and monotone families[18] $(a_{1j})_{j\in J}, \dots, (a_{nj})_{j\in J}$ in 2, we have

$$(\bigvee_j a_{1j}) \wedge \cdots \wedge (\bigvee_j a_{nj}) = \bigvee_j (a_{1j} \wedge \cdots \wedge a_{nj}).$$

To see the nontrivial inequality $\le$: if the LHS is 1, meaning that there are $j_1, \dots, j_n \in J$ such that $a_{1j_1} = \cdots = a_{nj_n} = 1$, then by directedness there is $j_1, \dots, j_n \le j \in J$, whence by monotonicity $a_{1j} = \cdots = a_{nj} = 1$, which witnesses that the RHS is 1.

Exercise 18.19. Show that conversely, if a poset J is such that monotone J-ary joins in 2 commute with finite meets, then J is directed.

Exercise 18.20.

(a) Let P be a poset with finite meets and directed joins. Show that finite meets commute with directed joins in P iff finite meets *distribute* over directed joins:

$$a \wedge \bigvee_j b_j = \bigvee_j (a \wedge b_j) \qquad \text{for monotone directed } (b_j)_{j\in J}.$$

(b) Show that finite meets commute with directed joins in any complete linear order (e.g., $[0,1]$), hence also in any subposet of a power of $[0,1]$ closed under finite meets and directed joins.

(c) Find an example of a complete lattice in which finite meets do *not* commute with directed joins.

Exercise 18.21. Let κ be a regular cardinal > 1. A poset is **κ-directed** if every subset of size $< \kappa$ has an upper bound. Show that for any poset J, κ-ary (i.e., of size $< \kappa$) meets commute with monotone J-ary joins in 2 iff J is κ-directed.

What does this mean for finite κ? (According to the definition in Section 7, which finite cardinals > 1 are regular?)

19 General dualities

Given a dual adjunction $\mathsf{C}_1 \rightleftarrows \mathsf{C}_2^{\mathrm{op}}$ induced by a dualizing object $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2)$, we now consider the question of when it is a duality, i.e., when the units $\eta_{\mathcal{M}} : \mathcal{M} \to \mathsf{C}_2(\mathsf{C}_1(\mathcal{M}, \mathcal{K}), \mathcal{K})$ and $\varepsilon_{\mathcal{N}} : \mathcal{N} \to \mathsf{C}_1(\mathsf{C}_2(\mathcal{N}, \mathcal{K}), \mathcal{K})$ are isomorphisms for each $\mathcal{M} \in \mathsf{C}_1$ and $\mathcal{N} \in \mathsf{C}_2$.

Example 19.1. For $\mathsf{C}_1 := \mathsf{Bool}$, $\mathsf{C}_2 := \mathsf{Top}$, and $\mathcal{K} := 2$ with the usual Boolean algebra structure and the discrete topology, commutativity of $\mathcal{K}_1, \mathcal{K}_2$ means that each Boolean operation on 2 is continuous (see Example 18.11); Stone duality says that the dualization functor $\mathsf{Bool}(-, 2) : \mathsf{Bool} \to \mathsf{Top}^{\mathrm{op}}$ is full+faithful and has essential image consisting of Stone spaces.

Recall that the key ingredient in the proof of Stone duality (Theorem 14.3) was the PIT for Boolean algebras, i.e., that every Boolean algebra admits enough homomorphisms to 2 to separate points. It is a general (easy) observation that such a condition is necessary to have a duality:

[18]Note that this example does not contradict the (dual of the) preceding example, because it does not quite fit into our framework of first-order structures axiomatized by universal Horn theories: directed joins are a *partial* operation (only defined for monotone families). One could encompass directed joins by working with "Pos-enriched" structures whose operations can have arity given by a poset instead of a set, or alternatively by allowing general partial operations with Horn-definable domain (see again **TODO**).

Lemma 19.2. For every $\mathcal{M} \in \mathsf{C}_1$, the unit $\eta_{\mathcal{M}} : \mathcal{M} \to \mathsf{C}_2(\mathsf{C}_1(\mathcal{M}, \mathcal{K}), \mathcal{K}) \subseteq \mathcal{K}_1^{\mathsf{C}_1(\mathcal{M}, \mathcal{K}_1)} \in \mathsf{C}_1$ at $\mathcal{M}$ is an $\mathcal{L}_1$**-embedding** (i.e., isomorphism with its image substructure) iff for every relation symbol $R \in (\mathcal{L}_1)_{\mathrm{rel}} \cup \{=\}$ and every tuple $\vec{a} \in M^{\mathrm{ar}(R)}$ such that $\neg R^{\mathcal{M}}(\vec{a})$, there is an $\mathcal{L}_1$-homomorphism $f : \mathcal{M} \to \mathcal{K}_1$ such that $\neg R^{\mathcal{K}_1}(f(\vec{a}))$. $\square$

Note that this condition does not depend on $\mathcal{L}_2$ or $\mathcal{K}_2$. If it holds for all $\mathcal{M} \in \mathsf{C}_1$, we say that $\mathcal{K}_1 \in \mathsf{C}_1$ is an **initial cogenerator**. If $\mathcal{L}_1$ has no relation symbols, this just means every $\mathcal{M} \in \mathsf{C}_1$ admits enough homomorphisms to $\mathcal{K}_1$ to separate points (we call such $\mathcal{K}_1$ a **cogenerator**).

Example 19.3. The PIT for distributive lattices (Corollary 13.9) says that $2 \in \mathsf{DLat}$ is a initial cogenerator, when we regard distributive lattices as structures in the language $\{\wedge, \vee, \top, \bot, \le\}$. (However, because $\le$ can be defined via $a \le b \iff a = a \wedge b$, in this case including $\le$ is redundant.)

Thus, given a (universal Horn-axiomatizable) class of structures C_1, the first step to finding a duality for C_1 is to find an initial cogenerator $\mathcal{K}_1 \in \mathsf{C}_1$. This immediately rules out certain C_1:

Exercise 19.4.

(a) Show that for every infinite set X, the alternating group A_X of all bijections $g : X \cong X$ which are the composite of an even (finite) number of transpositions is simple.

(b) Conclude that Grp does not have an (initial) cogenerator.

Given an initial cogenerator $\mathcal{K}_1 \in \mathsf{C}_1$, it remains to find a suitable structure $\mathcal{K}_2 \in \mathsf{C}_2$ commuting with $\mathcal{K}_1$ such that $\mathcal{K}$ induces a duality $\mathsf{C}_1 \simeq \mathsf{C}_2^{\mathrm{op}}$. Modulo some caveats, the rule is

"$\mathcal{K}$ induces a duality iff $\mathcal{K}_1, \mathcal{K}_2$ are initial cogenerators, and each of $\mathcal{K}_1, \mathcal{K}_2$ consists of *all* structure on K commuting with the other structure."

This is illustrated by the following:

Example 19.5. Take $\mathcal{L}_2 := \varnothing$ and $\mathsf{C}_2 := \mathsf{Set}$, i.e., $\mathcal{K}_2$ is just K as a pure set. Then of course, any structure $\mathcal{K}_1$ on K commutes with $\mathcal{K}_2$. For $\mathcal{M} \in \mathsf{C}_1$, the unit map is

$$\begin{aligned} \eta_{\mathcal{M}} : \mathcal{M} &\longrightarrow \mathsf{Set}(\mathsf{C}_1(\mathcal{M}, \mathcal{K}), \mathcal{K}) = \mathcal{K}_1^{\mathsf{C}_1(\mathcal{M}, \mathcal{K}_1)} \\ a &\longmapsto (f \mapsto f(a)), \end{aligned}$$

which will usually not be surjective, i.e., not every function $\mathsf{C}_1(\mathcal{M}, \mathcal{K}_1) \to K$ is given by evaluation at some $a \in \mathcal{M}$. (For example, take also $\mathsf{C}_1 := \mathsf{Set}$ and $K := 2$.)

Remark 19.6. We now prove a rigorous version of part of the above statement: *if* a dualizing object $\mathcal{K}$ induces a dual adjunction such that $\mathsf{C}_1(-, \mathcal{K}) : \mathsf{C}_1 \to \mathsf{C}_2^{\mathrm{op}}$ is full+faithful, then at least the function symbols in $\mathcal{L}_1$ (interpreted in $\mathcal{K}_1$) must generate *all* functions on K commuting with $\mathcal{K}_2$. This will also give a useful general application of duality theorems.

Let $U : \mathsf{C}_1 \to \mathsf{Set}$ be the forgetful functor. Let us assume that U has a left adjoint $F : \mathsf{Set} \to \mathsf{C}_1$, i.e., every set X generates a free structure $F(X) \in \mathsf{C}_1$. Such $F(X)$, if it exists,[19] can always be constructed as follows. First, take

$$\widetilde{F}(X) := \{\text{all } (\mathcal{L}_1)_{\mathrm{fun}}\text{-terms built from } X\},$$

[19] The resulting $F(X)$ will always satisfy the universal property for the free structure generated by X; however, it will be small iff $\mathcal{L}_1$ is. (C_1 consists only of structures whose underlying set is small.) Even if $F(X)$ is not small, its underlying set may still have small cardinality, in which case it may be replaced by an isomorphic copy which is in C_1. For example, this can always be done if $(\mathcal{L}_1)_{\mathrm{fun}}$ is small, so that $\widetilde{F}(X)$ is also small.

i.e., all expressions built from the elements $x \in X$ (which can be thought of as "variables") using the operations in $(\mathcal{L}_1)_{\text{fun}}$. Now let simultaneously $\sim \subseteq \widetilde{F}(X)$ and $\widetilde{R}^{F(X)} \subseteq \widetilde{F}(X)^{\text{ar}(R)}$ for each $R \in (\mathcal{L}_1)_{\text{rel}}$ be smallest such that $\sim$ is an equivalence relation; each $\widetilde{R}^{F(X)}$ is $\sim$-invariant, hence descends to a relation $R^{F(X)}$ on $F(X) := \widetilde{F}(X)/\sim$; and $F(X)$ equipped with these relations is in C_1. Such smallest relations are obtained by closing under the universal Horn axioms for C_1; thus, relations which hold in $F(X)$ are precisely those forced by the axioms.

$$
\begin{tikzcd}
\mathsf{C}_1 \arrow[r, hook, shift left, "\mathsf{C}_1(-{,}\mathcal{K})"] \arrow[d, shift left, "U"] \arrow[d, phantom, "\dashv"] & \mathsf{C}_2^{\text{op}} \arrow[l, two heads, shift left, "\mathsf{C}_2(-{,}\mathcal{K})"] \arrow[l, phantom, "\perp"] \arrow[dl, bend left, "\mathsf{C}_2(-{,}\mathcal{K}_2)"] \arrow[dl, phantom, "\perp"] \\
\mathsf{Set} \arrow[u, shift left, "F"] \arrow[ur, bend left, "\mathcal{K}_2^{(-)}"] &
\end{tikzcd}
$$

Note that the composite of right adjoints $U \circ \mathsf{C}_2(-, \mathcal{K}) : \mathsf{C}_2^{\text{op}} \to \mathsf{Set}$ is the dualization functor $\mathsf{C}_2(-, \mathcal{K}_2) : \mathsf{C}_2^{\text{op}} \to \mathsf{Set}$ from the preceding example, with C_1 replaced by Set. It follows that the composite of left adjoints is the other dualization functor:

$$\mathsf{C}_1(F(-), \mathcal{K}) \cong \mathsf{Set}(-, \mathcal{K}_2) = \mathcal{K}_2^{(-)} : \mathsf{Set} \longrightarrow \mathsf{C}_2^{\text{op}}.$$

Now since $\mathsf{C}_1(-, \mathcal{K})$ is full+faithful, i.e., $\eta : 1_{\mathsf{C}_1} \to \mathsf{C}_2(\mathsf{C}_1(-, \mathcal{K}), \mathcal{K})$ is a natural isomorphism,

$$\begin{aligned} F(X) &\cong \mathsf{C}_2(\mathsf{C}_1(F(X), \mathcal{K}), \mathcal{K}) \\ &\cong \mathsf{C}_2(\mathcal{K}_2^X, \mathcal{K}). \end{aligned}$$

This equation says that the elements of $F(X)$, i.e., $\mathcal{L}_1$-terms over X modulo provable equivalence, i.e., all X-ary $\mathcal{L}_1$-operations generated by the $\mathcal{L}_1$-function symbols, are exactly all X-ary operations on K which are homomorphisms $\mathcal{K}_2^X \to \mathcal{K}_2$.

The above equation also gives an explicit construction of free C_1-structures, quite different from the general construction via equivalence classes of terms. For example, from Stone duality ($\mathsf{C}_1 := \mathsf{Bool}$, $\mathsf{C}_2 := \mathsf{Top}$, $\mathcal{K} := 2$), we get

Corollary 19.7. The free Boolean algebra generated by a set X is $\mathsf{Top}(2^X, 2) \cong \mathcal{KO}(2^X)$. □

Exercise 19.8. Similar reasoning as in the above remark shows that if $\mathsf{C}_1(-, \mathcal{K})$ is full+faithful, then $\mathcal{L}_1$ must "generate" *all* relations on K commuting with $\mathcal{K}_2$ (i.e., which are $\mathcal{L}_2$-substructures); however, there is a subtlety regarding what "generate" means here.

TODO(also sketch proof of sufficiency)

The above rule gives a "universal" way of producing dualities, once we have found an initial cogenerator $\mathcal{K}_1 \in \mathsf{C}_1$: namely, take $\mathcal{K}_2$ to be all structure commuting with $\mathcal{K}_1$. However, this is not so useful in practice, since we are typically interested in dualities between familiar types of structures (e.g., Boolean algebras and topological spaces). It is more useful as a guiding principle: given $\mathcal{K}_1$, we should take $\mathcal{K}_2$ to be *some* (familiar) structure commuting with $\mathcal{K}_1$; if the resulting dual adjunction is not a duality, we should add more structure to $\mathcal{K}_2$, etc.

Exercise 19.9 (Priestley duality)**.** By the PIT, $2 \in \mathsf{DLat}$ is an (initial) cogenerator. To find a duality for DLat, we should take $\mathcal{K}_2$ to be some structure on 2 commuting with the lattice structure. As in Stone duality, the discrete topology does; however, we also need to take some additional structure not commuting with the Boolean complementation $\neg$:

(a) Show that for a distributive lattice A, the double dual $\mathsf{Top}(\mathsf{DLat}(A,2),2)$ with respect to 2 as a distributive lattice and a discrete space is the free Boolean algebra generated by A (i.e., the left adjoint of the full inclusion $\mathsf{Bool} \hookrightarrow \mathsf{DLat}$ at A). (This is analogous to Remark 19.6.)

Note that $\neg : 2 \to 2$ is not monotone, unlike the lattice operations. Let TopPos be the category of **topological posets**, meaning topological spaces which are also equipped with a partial order (not yet requiring any compatibility between the two); a morphism is a monotone continuous map. We thus have a dual adjunction

$$\mathsf{DLat} \underset{\mathsf{TopPos}(-,2)\cong\mathcal{KO}\mathrm{Up}}{\overset{\mathsf{DLat}(-,2)\cong\mathrm{Spec}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{TopPos}$$

where for $X \in \mathsf{TopPos}$,

$$\mathcal{KO}\mathrm{Up}(X) := \{\text{clopen upper } C \subseteq X\}.$$

Since 2 is a cogenerator, the unit

$$\begin{aligned} A &\longrightarrow \mathcal{KO}\mathrm{Up}(\mathrm{Spec}(A)) \\ a &\longmapsto [a] := \{F \in \mathrm{Spec}(A) \mid a \in F\} \end{aligned}$$

is injective for each distributive lattice A.

(b) Verify that the topology on $\mathrm{Spec}(A)$ has a basis of (cl)open sets of the form $[a] \setminus [b]$ for $a, b \in A$.

(c) $\mathrm{Spec}(A) \cong \mathsf{DLat}(A,2) \subseteq 2^A$ is closed, hence compact.

(d) For every $F, G \in \mathrm{Spec}(A)$ such that $F \not\subseteq G$, there is some $a \in A$ such that $F \in [a] \not\ni G$.

(e) For every closed upper $C \subseteq \mathrm{Spec}(A)$ and closed lower $D \subseteq \mathrm{Spec}(A)$, there is some $a \in A$ such that $C \subseteq [a] \subseteq \neg D$.

(f) $[-] : A \to \mathcal{KO}Up(\mathrm{Spec}(A))$ is surjective, hence an isomorphism.

We say that $X \in \mathsf{TopPos}$ is **totally order-separated** if for any $x \not\leq y$, there is a clopen upper set containing x but not y, and a **Priestley space** if it is compact (as a topological space) and totally order-separated.

(g) In a totally order-separated $X \in \mathsf{TopPos}$, the partial order $\leq \subseteq X^2$ is closed, i.e., X is **order-Hausdorff**.

(h) Priestley spaces are Stone spaces (once the partial order is forgotten).

(i) The essential image of $\mathrm{Spec} \cong \mathsf{DLat}(-,2) : \mathsf{DLat} \to \mathsf{TopPos}^{\mathrm{op}}$ is the full subcategory $\mathsf{Pries} \subseteq \mathsf{TopPos}$ of Priestley spaces.

Thus, we have a dual adjoint equivalence

$$\mathsf{DLat} \underset{\mathsf{TopPos}(-,2)\cong\mathcal{KO}\mathrm{Up}}{\overset{\mathsf{DLat}(-,2)\cong\mathrm{Spec}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Pries}^{\mathrm{op}}.$$

(j) We have a commutative diagram of adjunctions (in the obvious sense, i.e., the left, resp., adjoints commute, up to canonical natural isomorphism)

$$\begin{tikzcd}
\mathsf{Bool} \arrow[rr, shift left=2, "\mathrm{Spec}"] \arrow[d, shift left=2, "\mathrm{forget}"] & & \mathsf{Stone}^{\mathrm{op}} \arrow[ll, shift left=2, "\mathcal{KO}"] \arrow[ll, phantom, "\perp\ \sim"] \arrow[d, shift left=2] \arrow[dll, shift left=2, "\mathcal{KO}"] \\
\mathsf{DLat} \arrow[u, shift left=2, "\text{(a)}"] \arrow[u, phantom, "\dashv"] \arrow[rr, shift left=2, "\mathrm{Spec}"] \arrow[urr, shift left=2, "\mathrm{Spec}"] \arrow[urr, phantom, "\perp"] & & \mathsf{Pries}^{\mathrm{op}}. \arrow[ll, shift left=2, "\mathcal{KO}\mathrm{Up}"] \arrow[ll, phantom, "\perp\ \sim"] \arrow[u, shift left=2, "\mathrm{forget}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

(k) What does the *right* adjoint of $\mathsf{Bool} \to \mathsf{DLat}$ correspond to on the spatial side?

Exercise 19.10 (Hofmann–Mislove–Stralka duality)**.** We have an analogous duality for $\wedge$-lattices:

(a) For a $\wedge$-lattice A, we have an order-isomorphism

$$\wedge\mathsf{Lat}(A, 2) \cong \mathrm{Filt}(A) := \{\text{filters in } A\}.$$

(b) $2 \in \wedge\mathsf{Lat}$ is a cogenerator.

(c) The $\wedge$-lattice structure on 2 commutes with itself (Example 18.12) as well as with the discrete topology (Example 18.11).

Let $\mathsf{Top}\wedge\mathsf{Lat}$ denote the category of **topological $\wedge$-lattices**, i.e., $\wedge$-lattices on an underlying topological space such that $\wedge$ is continuous.

(d) For $X \in \mathsf{Top}\wedge\mathsf{Lat}$, we have an order-isomorphism

$$\mathsf{Top}\wedge\mathsf{Lat}(X, 2) \cong \mathcal{KO}\mathrm{Filt}\{X\} := \{\text{clopen filters in } A\}.$$

(e) For $A \in \wedge\mathsf{Lat}$, every closed set $C \subseteq \mathrm{Filt}(A)$ is closed under up-directed joins and down-directed meets.

(f) For $A \in \wedge\mathsf{Lat}$, every closed filter $C \subseteq \mathrm{Filt}(A)$ is $\uparrow F$ for some $F \in \mathrm{Filt}(A)$, namely $F := \bigcap C$.

(g) For $A \in \wedge\mathsf{Lat}$, for a filter $F \in \mathrm{Filt}(A)$, $\uparrow F$ contains the up-directed join $\bigcup_{a \in F} \uparrow a = F$; thus if $\uparrow F$ is open, then $\uparrow a \in \uparrow F$, i.e., $F \subseteq \uparrow a$, i.e., $F = \uparrow a$, for some $a \in F$.

(h) For $A \in \wedge\mathsf{Lat}$, we have an isomorphism

$$\begin{aligned} A &\cong \mathcal{KO}\mathrm{Filt}(\mathrm{Filt}(A)) \\ a &\mapsto [a] := \{F \in \mathrm{Filt}(A) \mid a \in F\}. \end{aligned}$$

A **Stone $\wedge$-lattice** is a topological $\wedge$-lattice whose topology is Stone.

(i) Every Hausdorff topological $\wedge$-lattice is order-Hausdorff.

(j) For a compact (order-)Hausdorff $\wedge$-lattice X, for any closed (open) $C \subseteq X$, the upward closure $\uparrow C$ of C is still closed (open).

(k) For a compact (order-)Hausdorff $\wedge$-lattice X, for any closed (open) $C \subseteq X$,

$$C' := \{x \in X \mid \forall y \in C\,(x \wedge y \in C)\}$$

is still closed (open), and is a (not necessarily unital) filter.

(l) The essential image of $\mathrm{Filt} \cong \wedge\mathsf{Lat}(-,2) : \wedge\mathsf{Lat} \to \mathsf{Top}\wedge\mathsf{Lat}^{\mathrm{op}}$ is the full subcategory $\mathsf{Stone}\wedge\mathsf{Lat}$ of Stone $\wedge$-lattices.

Thus, we have a dual adjoint equivalence

$$\begin{tikzcd}
\wedge\mathsf{Lat} \arrow[r, shift left, "\wedge\mathsf{Lat}(-,2)\cong\mathrm{Filt}"] \arrow[r, phantom, "\perp"] & \mathsf{Stone}\wedge\mathsf{Lat}^{\mathrm{op}}. \arrow[l, shift left, "\mathsf{Top}\wedge\mathsf{Lat}(-,2)\cong\mathcal{KO}\mathrm{Filt}"]
\end{tikzcd}$$

(m) We have a commutative square of adjunctions

$$\begin{tikzcd}
\mathsf{DLat} \arrow[rr, shift left, "\mathrm{Spec}"] \arrow[rr, phantom, "\perp\ \sim"] \arrow[d, hook, shift left, "\mathrm{forget}"] & & \mathsf{Pries}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{KO}\mathrm{Up}"] \arrow[d, shift left] \arrow[dll, shift left, "\mathcal{KO}\mathrm{Up}"] \\
\wedge\mathsf{Lat} \arrow[u, shift left] \arrow[u, phantom, "\dashv"] \arrow[urr, shift left, "\mathrm{Filt}"] \arrow[urr, phantom, "\perp"] \arrow[rr, shift left, "\mathrm{Filt}"] \arrow[rr, phantom, "\perp\ \sim"] & & \mathsf{Stone}\wedge\mathsf{Lat}^{\mathrm{op}}. \arrow[ll, shift left, "\mathcal{KO}\mathrm{Filt}"] \arrow[u, shift left, "\mathrm{forget}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

Exercise 19.11 (Hofmann–Mislove–Stralka duality, order-theoretic version)**.** Again let A be a $\wedge$-lattice. Note that

$$\begin{aligned}
[a] &= \{F \in \mathrm{Filt}(A) \mid a \in F\} \\
&= \{F \in \mathrm{Filt}(A) \mid \uparrow_A a \subseteq F\} \quad \text{by Yoneda (Lemma 8.5)} \\
&= \uparrow_{\mathrm{Filt}(A)} \uparrow_A a,
\end{aligned}$$

i.e., $[-]$ is the "double Yoneda embedding" (see Exercise 9.2)

$$A \xrightarrow{\uparrow_A = \downarrow_{A^{\mathrm{op}}}} \mathrm{Filt}(A)^{\mathrm{op}} = \mathrm{Idl}(A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\uparrow_{\mathrm{Filt}(A)} = \downarrow_{\mathrm{Idl}(A^{\mathrm{op}})^{\mathrm{op}}}} \mathcal{KO}\mathrm{Filt}(\mathrm{Filt}(A)) \subseteq \mathrm{Up}(\mathrm{Filt}(A)) = \mathrm{Low}(\mathrm{Idl}(A^{\mathrm{op}})^{\mathrm{op}}).$$

Furthermore, (f) and (g) in the preceding exercise really show that $[-] : A \to \mathrm{Up}(\mathrm{Filt}(A))$ has image consisting of all upper sets $C \subseteq \mathrm{Filt}(A)$ which are closed under arbitrary meets, whose complements are closed under directed joins, which is isomorphic (via $\chi_{(-)}$) to

$$\textstyle\bigwedge\bigvee\!\!\!\!\uparrow\mathsf{Pos}(\mathrm{Filt}(A), 2) := \{f : \mathrm{Filt}(A) \to 2 \mid f \text{ preserves } \bigwedge, \text{ directed } \bigvee\} \subseteq \mathsf{Pos}(\mathrm{Filt}(A), 2).$$

Since arbitrary meets and directed joins in 2 commute with finite meets (Examples 18.14 and 18.18), we also get a duality

$$\begin{tikzcd}
\wedge\mathsf{Lat} \arrow[r, hook, shift left, "\wedge\mathsf{Lat}(-,2)\cong\mathrm{Filt}"] \arrow[r, phantom, "\perp"] & \bigwedge\bigvee\!\!\!\!\uparrow\mathsf{Pos}^{\mathrm{op}}. \arrow[l, two heads, shift left, "\bigwedge\bigvee\!\!\!\!\uparrow\mathsf{Pos}(-,2)"]
\end{tikzcd}$$

between $\wedge\mathsf{Lat}$ and a full subcategory of the category $\bigwedge\bigvee\!\!\!\!\uparrow\mathsf{Pos}$ of posets with arbitrary meets and directed joins. This full subcategory consists of the **algebraic lattices**, which are often simply defined as posets isomorphic to $\mathrm{Filt}(A)$ for some $\wedge$-lattice A.

Exercise 19.12 (non-unital Hofmann–Mislove–Stralka duality)**.** Prove that there is a duality between non-unital $\wedge$-lattices and Stone $\wedge$-lattices with a least element, or alternatively a full subcategory of the category of posets with arbitrary meets, directed joins, and a least element.

Exercise 19.13 (Lindenbaum, Tarski, Papert, Strauss, Hofmann, Keimel, Mislove, Stralka, ???)**.**

(a) Prove that there is a duality between posets and Stone distributive lattices, or alternatively a full subcategory of the category CLat of **complete lattices** (posets with arbitrary meets, equivalently arbitrary joins, with both meets and joins regarded as being part of the structure).

(b) Prove that this duality restricts to the full subcategories $\mathsf{Set} \subseteq \mathsf{Pos}$ and $\mathsf{StoneBool} \subseteq \mathsf{StoneDLat}$, or alternatively **complete atomic Boolean algebras** $\mathsf{CABool} \subseteq \mathsf{CLat}$ (complete Boolean algebras in which every element is a join of **atoms**, meaning minimal elements above $\bot$).

Remark 19.14. While $[0, 1]$ is an example of a non-Stone compact (order-)Hausdorff distributive lattice, it can be shown that every compact Hausdorff Boolean algebra is Stone, hence atomic. However, there does not appear to be an elementary proof of this: all known proofs use the representation theory of compact groups. (See [Johnstone, *Stone spaces*, VI 4.11].)

The "front face" of Fig. 19.15 depicts the various order-theoretic dualities we have considered so far, together with commutative squares relating them, as in Exercise 19.9(j) and 19.10(m). Note that as the amount of structure on the left half of the diagram decreases (from Bool down to Set), the amount of structure commuting with it on the other side increases. Furthermore, note that the structures on the right half are themselves topological versions of the structures on the left half (in the reverse order), giving rise to the "back face" of the diagram; the dualities on the back are the same as the ones on the front, but "rotated 180°" and with the $^{\mathrm{op}}$ moved to the other side.

Exactly as in Exercise 19.9(a) and Corollary 19.7, each of the free $\dashv$ forget adjunctions on the left half of the diagram (say) becomes a forget $\dashv$ free adjunction on the right half, yielding explicit constructions of each of these free functors, e.g.,

Exercise 19.16. For a poset P, the free distributive lattice generated by P is (isomorphic to) $\mathcal{KO}\mathrm{Up}(\mathrm{Up}(P))$, with unit

$$\begin{aligned} \eta_P : P &\longrightarrow \mathcal{KO}\mathrm{Up}(\mathrm{Up}(P)) \\ x &\longmapsto \{A \in \mathrm{Up}(P) \mid x \in P\} =: [x]. \end{aligned}$$

Exercise 19.17. For a set X, the free Stone space generated by X is

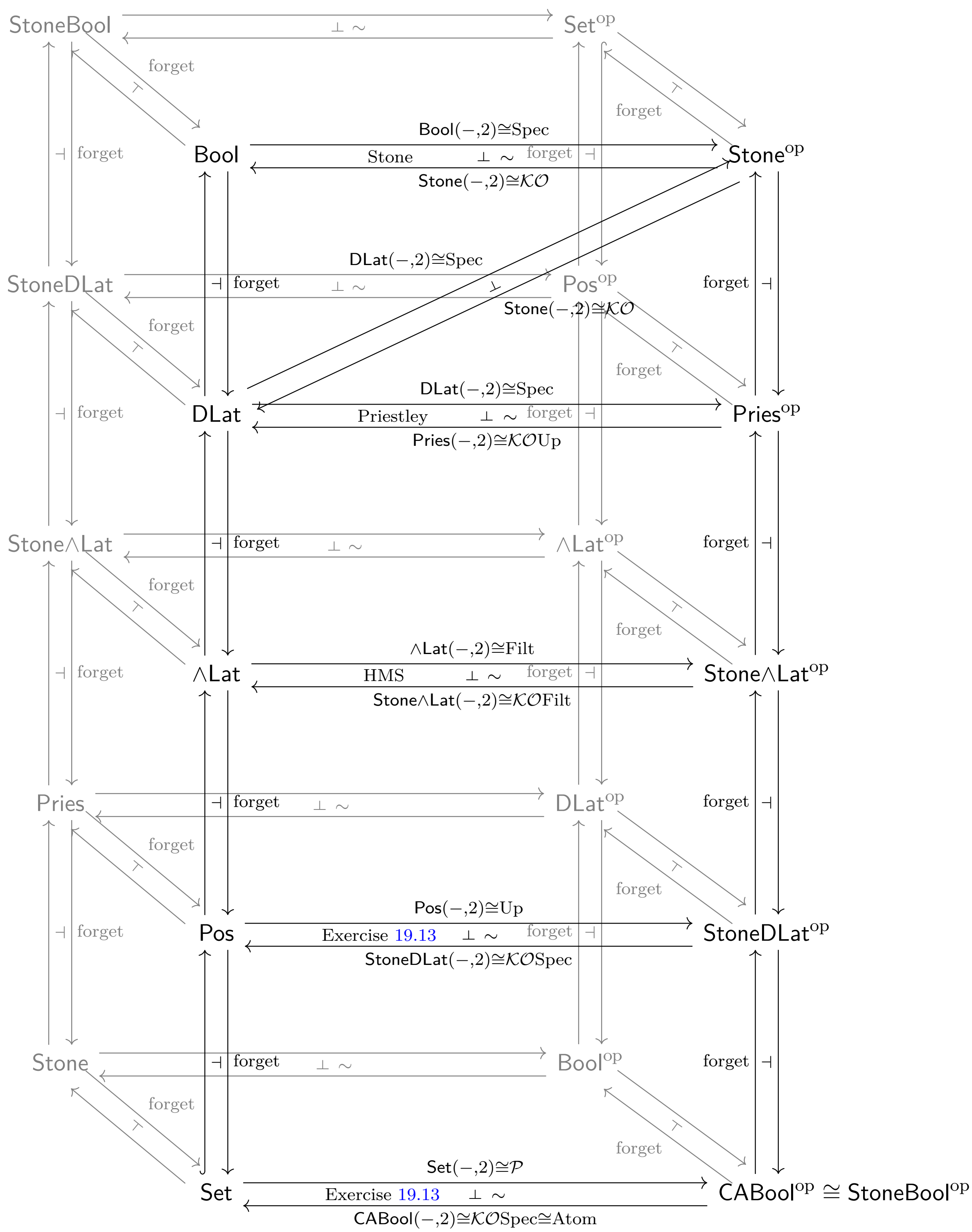
$$\begin{aligned} \eta_X : X &\longrightarrow \mathrm{Spec}(\mathcal{P}(X)) = \{\text{ultrafilters on } X\} =: \beta X \\ x &\longmapsto \{A \in \mathcal{P}(X) \mid x \in A\}, \end{aligned}$$

called the **Stone–Čech compactification** of (the discrete space) X.

(βX also happens to be the free compact Hausdorff space generated by X, although this does not follow abstractly from the above; *a priori*, it could be that the left adjoint of $\mathsf{Stone} \hookrightarrow \mathsf{KHaus} \xrightarrow{\text{forget}} \mathsf{Set}$ is a quotient of the left adjoint of the $\mathsf{KHaus} \xrightarrow{\text{forget}} \mathsf{Set}$.)

Between some of the pairs of categories below, there is also a nice direct construction of the free functor, typically arising from a "normal form" for terms in the general construction of free algebras (see Remark 19.6). In this case, we get an identification between this direct construction, and the construction via duality:

Figure 19.15: Various order-theoretic dualities



Exercise 19.18 (free $\wedge$-lattices)**.**

1. Show that the free $\wedge$-lattice generated by a set X is $\mathcal{P}_\omega(X) := \{\text{finite subsets}\} \subseteq \mathcal{P}(X)$, where a finite subset $A = \{x_1, \dots, x_n\} \subseteq X$ is thought of as a "normal form" for the term $x_1 \wedge \cdots \wedge x_n$.
2. Thus, $\mathcal{P}_\omega(X) \cong \mathcal{KO}\mathrm{Filt}(\mathcal{P}(X))$, where a finite subset $A \subseteq X$ corresponds to $\{B \in \mathcal{P}(X) \mid A \subseteq B\} \in \mathcal{KO}\mathrm{Filt}(\mathcal{P}(X))$.

Exercise 19.19 (Vietoris spaces)**.** For a compact Hausdorff space X, the **Vietoris topology** on the set

$$\mathcal{K}(X) := \{\text{closed subsets of } X\}$$

is generated by the subbasic open sets

$$\Box U := \{K \in \mathcal{K}(X) \mid K \subseteq U\}, \qquad \Diamond U := \{K \in \mathcal{K}(X) \mid K \cap U \neq \varnothing\}$$

for open $U \subseteq X$.

(a) For $F, G \in \mathcal{K}(X)$ such that $F \not\subseteq G$, by regularity, there are disjoint open $U, V \subseteq X$ such that $F \cap U \neq \varnothing$ and $G \subseteq V$, whence $F \in \Diamond U$ and $G \in \Box V$ with $\Diamond U \cap \Box V = \varnothing$. Thus $\mathcal{K}(X)$ is (order-)Hausdorff.

(b) If X is zero-dimensional, then we can take $V = \neg U$, whence $\mathcal{K}(X)$ is totally (order-)separated.

(c) Verify that $\Diamond : \mathcal{O}(X) \to \mathcal{P}(\mathcal{K}(X))$ preserves arbitrary unions, while $\Box : \mathcal{O}(X) \to \mathcal{P}(\mathcal{K}(X))$ preserves directed unions and finite intersections. (Here $\mathcal{O}(X) := \{\text{open sets in } X\}$.)

(d) Thus, $\mathcal{K}(X)$ has a basis of open sets of the form $\Diamond U_1 \cap \cdots \cap \Diamond U_n \cap \Box V$.

(e) **TODO**($\mathcal{K}(X)$ is compact; for now, see [Johnstone, *Stone spaces*, III 4.4])

(f) The binary union operation $\cup : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ is continuous.

(g) For a Stone space X, $\mathcal{K}(X)$ is the free Stone $\wedge$-lattice generated by X, where $K \in \mathcal{K}(X)$ is thought of as the limit of the net of $\bigwedge F$ for finite $F \subseteq K$ (analogously to the previous exercise).

(h) Thus, for a Stone space X, $\mathcal{K}(X) \cong \mathrm{Filt}(\mathcal{KO}(X))$, where $K \in \mathcal{K}(X)$ corresponds to $\{C \in \mathcal{KO}(X) \mid K \subseteq C\}$.

There is an important order-theoretic duality missing from Fig. 19.15, the "midpoint" of the left and right halves (between $\wedge\mathsf{Lat}$ and $\mathsf{Stone}\wedge\mathsf{Lat}$) where the two sides become the same:

Example 19.20 ($\bigwedge$-lattice duality)**. TODO**

Example 19.21. Here are some other well-known dualities induced by dualizing objects:

- $\mathbb{R}$ (or any field K) as an $\mathbb{R}$-vector space, and as a topological $\mathbb{R}$-vector space with the *discrete* topology induces a duality

$$\begin{tikzcd} \mathsf{Vec} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{TopVec}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

  with a full subcategory of TopVec (called the **linearly compact** vector spaces).

- (Pontryagin duality) $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as an abelian group, and as a topological abelian group, induces a duality with compact abelian groups:

$$\mathsf{AbGrp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{KAbGrp}^{\mathrm{op}}.$$

- (Gelfand duality) $[-1, 1]$ (or $[0, 1]$) as a compact Hausdorff space (a cogenerator by Urysohn's lemma) induces a duality with structures consisting of all continuous operations on $[-1, 1]$ (or $[0, 1]$), e.g., truncated $+$, $\cdot$. Such structures are better as known as the (positive) unit balls of **real commutative self-adjoint C^*-algebras** (this characterization of C^*-algebras is due to Duskin, Isbell, Marra–Reggio).

20 Fragments of propositional logic

By a **fragment** of propositional logic, we mean a subclass of all propositional formulas where only certain connectives are allowed. A **theory** in the fragment is then a theory $\mathcal{T}$ whose sequents are all between formulas in the fragment. The following fragments are the main ones of interest to us:[20]

- **Horn** propositional formulas are ones which do not involve negation or disjunction (including $\bot$), hence built from atomic ones using $\wedge, \top$. Thus, Horn sequents are of the form

$$P_1 \wedge \cdots \wedge P_m \Rightarrow Q_1 \wedge \cdots \wedge Q_n$$

  where P_i, Q_j are atomic. Such a sequent is provably equivalent to the set of sequents

$$\begin{gathered} P_1 \wedge \cdots \wedge P_m \Rightarrow Q_1, \\ \vdots \\ P_1 \wedge \cdots \wedge P_m \Rightarrow Q_n \end{gathered}$$

  via the ($\wedge\Rightarrow$) and ($\Rightarrow\wedge$) rules (i.e., each set of sequents $\vdash$ each sequent in the other set; see Section 15). Thus, we may assume without loss that Horn theories consist of sequents of this form (see Exercise 18.8).

- **Coherent** propositional formulas are ones which do not involve negation. Using the Dist rule, every coherent formula may be put into **disjunctive** or **conjunctive normal form**

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} P_{ij}, \qquad\qquad \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} Q_{ij};$$

  a sequent between two such formulas is provably equivalent to a set of sequents of the form

$$P_1 \wedge \cdots \wedge P_m \Rightarrow Q_1 \vee \cdots \vee Q_n,$$

  hence we may assume without loss that coherent theories consist of sequents of this form.

[20] The name "coherent", used in categorical logic, ultimately derives from coherent sheaves in algebraic geometry. Coherent formulas are traditionally known in logic as "positive" formulas. However, because classical logic does not emphasize the role of sequents, traditional names for classes of formulas have the wrong connotation for the corresponding class of theories: for example, a "positive theory" usually means a theory consisting of positive formulas, not implications between them. For this reason, we stick to the categorical logic names for fragments.

(However, the term "Horn formula", which usually refers to what we are calling "Horn sequent", seems to have been co-opted by categorical logicians contrary to traditional usage.)

- The **atomic** fragment is also a (degenerate) example. Atomic theories consist of implications between atomic formulas.

For a propositional language $\mathcal{L}$, we write

$$\mathcal{L} \subseteq \mathcal{L}_{\omega 0}^{\text{Horn}} \subseteq \mathcal{L}_{\omega 0}^{\text{coh}} \subseteq \mathcal{L}_{\omega 0}$$

to denote respectively the sets of atomic, Horn, and coherent $\mathcal{L}$-formulas.

We may interpret formulas ϕ belonging to a fragment in models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A)$ valued in a structure A weaker than a Boolean algebra but still admitting operations corresponding to the allowed connectives:

- coherent ϕ may be interpreted in models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A)$ valued in distributive lattices A;
- Horn ϕ may be interpreted in models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A)$ valued in $\wedge$-lattices A;
- atomic ϕ may be interpreted in models valued in posets.

As before, for $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, A)$ where A is a distributive lattice ($\wedge$-lattice, poset) and a coherent (Horn, atomic) sequent $\phi \Rightarrow \psi$, we write

$$\mathcal{M} \models \phi \Rightarrow \psi \;:\Longleftrightarrow\; \phi^{\mathcal{M}} \leq \psi^{\mathcal{M}}.$$

If $\mathcal{M}$ satisfies every sequent in a coherent (Horn, atomic) theory $\mathcal{T}$, then $\mathcal{M}$ is a **model of** $\mathcal{T}$.

A formal proof (in the proof system defined in Section 15) is **coherent** (**Horn**, **atomic**) if it only involves coherent (Horn, atomic) sequents. Equivalently, the proof only uses rules involving the allowed connectives in the fragment:

- Atomic proofs only use the Id and Cut rules.
- Horn proofs only use the Id, Cut, ($\wedge\Rightarrow_{1,2}$), ($\Rightarrow\wedge$), ($\Rightarrow\top$) rules.
- Coherent proofs only use the Id, Cut, ($\wedge\Rightarrow_{1,2}$), ($\Rightarrow\wedge$), ($\Rightarrow\top$), ($\Rightarrow\vee_{1,2}$), ($\vee\Rightarrow$), ($\bot\Rightarrow$), (Dist) rules.

For a coherent (Horn, atomic) $\mathcal{L}$-theory $\mathcal{T}$ and sequent $\sigma = (\phi \Rightarrow \psi)$, we write

$$\mathcal{T} \vdash_{\text{coh (Horn,at)}} \sigma$$

if there is a coherent (Horn, atomic) proof of σ from $\mathcal{T}$.

Proposition 20.1 (soundness)**.** If $\mathcal{T} \vdash_{\text{coh (Horn,at)}} \sigma$, then for every distributive lattice ($\wedge$-lattice, poset) A and model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$, we have $\mathcal{M} \models \sigma$.

Proof. In the proof of soundness for full propositional logic (Proposition 15.6), to show that each inference rule (e.g., ($\wedge\Rightarrow_2$)) is sound, we only used the corresponding structure on A (e.g., meets). ☐

As before, we may push forward models $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A)$ along distributive lattice ($\wedge$-lattice, poset) homomorphisms $f : A \to B$ to get models $f_*(\mathcal{M}) \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}, B)$, yielding a copresheaf

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}, -) : \mathsf{DLat} \longrightarrow \mathsf{Set} \quad \text{(resp., } \wedge\mathsf{Lat} \to \mathsf{Set},\ \mathsf{Pos} \to \mathsf{Set}\text{)}.$$

The **coherent (Horn, atomic) Lindenbaum–Tarski algebra** of a coherent (Horn, atomic) theory $(\mathcal{L}, \mathcal{T})$ is

$$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh\,(Horn,at)}} := \mathcal{L}_{\omega 0}^{\mathrm{coh\,(Horn,at)}} / {\equiv_{\mathcal{T}}^{\mathrm{coh\,(Horn,at)}}},$$

partially ordered via $[\phi]_{\mathcal{T}} \le [\psi]_{\mathcal{T}} \;:\Longleftrightarrow\; \mathcal{T} \vdash_{\mathrm{coh\,(Horn,at)}} \phi \Rightarrow \psi$ as before.

Proposition 20.2. $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh\,(Horn,at)}}$ is a distributive lattice ($\wedge$-lattice, poset), with operations induced by the logical connectives.

Proof. In the proof that $\langle \mathcal{L} \mid \mathcal{T} \rangle$ is a Boolean algebra (Proposition 16.1), to show each axiom of Boolean algebras, we only used the inference rules involving the corresponding connectives. □

The **universal model** $\mathcal{M}_{\mathcal{T}}$ of $(\mathcal{L}, \mathcal{T})$ in $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh\,(Horn,at)}}$ is given as before by

$$P^{\mathcal{M}_{\mathcal{T}}} := [P]_{\mathcal{T}} \qquad \text{for } P \in \mathcal{L}.$$

As before (Lemma 16.2 and Corollary 16.3), one easily verifies that

$$\phi^{\mathcal{M}_{\mathcal{T}}} = [\phi]_{\mathcal{T}} \qquad \text{for } \phi \in \mathcal{L}_{\omega 0}^{\mathrm{coh\,(Horn,at)}},$$

whence $\mathcal{M}_{\mathcal{T}}$ is a model of $\mathcal{T}$.

Theorem 20.3 (universality of $\mathcal{M}_{\mathcal{T}}$)**.** For any coherent theory $(\mathcal{L}, \mathcal{T})$ and distributive lattice A, we have a bijection

$$\begin{aligned} \mathsf{DLat}(\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}}, A) &\cong \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A) \\ f &\mapsto f_*(\mathcal{M}) \\ ([\phi_{\mathcal{T}}] \mapsto \phi^{\mathcal{M}}) &\leftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

Similarly for any Horn (atomic) theory $(\mathcal{L}, \mathcal{T})$ and $\wedge$-lattice (poset) A.

Proof. Exactly as in Theorem 16.5. □

In other words,

- $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}}$ is the distributive lattice presented by generators $\mathcal{L}$ and relations $\mathcal{T}$;
- $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Horn}}$ is the $\wedge$-lattice presented by generators $\mathcal{L}$ and relations $\mathcal{T}$;
- $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{at}}$ is the poset presented by "generators" $\mathcal{L}$ and relations $\mathcal{T}$.

Conversely, for a distributive lattice ($\wedge$-lattice, poset) A, its **internal theory** $(\mathcal{L}_A, \mathcal{T}_A)$ is given by

$$\begin{aligned} \mathcal{L}_A &:= A, \\ \mathcal{M}_A &:= 1_A \in \mathrm{Mod}(\mathcal{L}_A, A), \\ \mathcal{T}_A &:= \{\phi \Rightarrow \psi \mid \phi, \psi \in (\mathcal{L}_A)_{\omega 0}^{\mathrm{coh\,(Horn,at)}} \;\&\; \mathcal{M}_A \models \phi \Rightarrow \psi\}. \end{aligned}$$

Proposition 20.4. $\mathcal{M}_A : \langle \mathcal{L}_A \mid \mathcal{T}_A \rangle_{\mathrm{coh\,(Horn,at)}} \to A$ is an order-isomorphism.

Proof. By definition, $\mathcal{M}_A([a]_{\mathcal{T}_A}) = a \le b = \mathcal{M}_A([b]_{\mathcal{T}_A}) \iff a \Rightarrow b \in \mathcal{T}_A \implies [a]_{\mathcal{T}_A} \le [b]_{\mathcal{T}_A}$ for all $a, b \in A$, i.e., $\mathcal{M}_A$ is surjective, and an order-embedding when restricted to equivalence classes of atomic formulas; as in Proposition 16.6, these are all of $\langle \mathcal{L}_A \mid \mathcal{T}_A \rangle_{\mathrm{coh}\,(\mathrm{Horn,at})}$. ☐

Theorem 20.5 (strong completeness theorem for coherent propositional logic)**.**

(a) For any coherent propositional theory $(\mathcal{L}, \mathcal{T})$, we have a distributive lattice isomorphism

$$\begin{aligned} [\![-]\!] : \langle \mathcal{L}_A \mid \mathcal{T}_A \rangle_{\mathrm{coh}} &\cong \mathcal{KO}\mathrm{Up}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \\ [\phi]_{\mathcal{T}} &\mapsto \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \mid \mathcal{M} \models \phi\} =: [\![\phi]\!] \end{aligned}$$

between $\mathcal{T}$-equivalence classes of coherent formulas and clopen upper sets of models. Equivalently:

   (i) (completeness) Every coherent sequent $\phi \Rightarrow \psi$ satisfied by every (2-valued) model of $\mathcal{T}$ is *coherently* provable from $\mathcal{T}$.

   (ii) (definability) Every clopen upper set of models is $[\![\phi]\!]$ for some coherent formula $\phi \in \mathcal{L}_{\omega 0}^{\mathrm{coh}}$.

(b) Every Priestley space is order-homeomorphic to $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ for some coherent propositional theory $(\mathcal{L}, \mathcal{T})$.

Theorem 20.6 (strong completeness theorem for Horn propositional logic)**.**

(a) For any Horn propositional theory $(\mathcal{L}, \mathcal{T})$, we have a $\wedge$-lattice isomorphism

$$\begin{aligned} [\![-]\!] : \langle \mathcal{L}_A \mid \mathcal{T}_A \rangle_{\mathrm{Horn}} &\cong \mathcal{KO}\mathrm{Filt}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \\ [\phi]_{\mathcal{T}} &\mapsto \{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \mid \mathcal{M} \models \phi\} =: [\![\phi]\!] \end{aligned}$$

between $\mathcal{T}$-equivalence classes of Horn formulas and clopen filters of models. Equivalently:

   (i) (completeness) Every Horn sequent $\phi \Rightarrow \psi$ satisfied by every (2-valued) model of $\mathcal{T}$ is *Horn* provable from $\mathcal{T}$.

   (ii) (definability) Every clopen filter of models is $[\![\phi]\!]$ for some Horn formula $\phi \in \mathcal{L}_{\omega 0}^{\mathrm{Horn}}$.

(b) Every Stone $\wedge$-lattice is order-homeomorphic to $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ for some Horn propositional theory $(\mathcal{L}, \mathcal{T})$.

Remark 20.7. By **TODO**, closed filters of models $F \in \mathcal{K}\mathrm{Filt}(\mathrm{Mod}(\mathcal{L}, \mathcal{T}))$ are in (order-reversing) bijection with models (namely $\min F$), while clopen filters correspond to compact models. For $\phi \in \mathcal{L}_{\omega 0}^{\mathrm{Horn}}$, the clopen filter $[\![\phi]\!]$ corresponds to the smallest model of ϕ.

Example 20.8. One way in which propositional models arise is as models of a first-order theory on a *fixed* underlying set. For example, consider the (universal Horn) theory of preordered sets. Let X be a set. Then the **theory $\mathcal{T}_{\mathsf{Preord}(X)}$ of preorders on X** consists of atomic formulas

$$(x \le y) \quad \text{for each } x, y \in X,$$

together with Horn axioms

$$\begin{aligned} \top &\Rightarrow (x \le x) \quad \text{for } x \in X, \\ (x \le y) \wedge (y \le z) &\Rightarrow (x \le z) \quad \text{for } x, y, z \in X. \end{aligned}$$

The Stone $\wedge$-lattice of models is a subspace of 2^{X^2}, the space of all binary relations on X, which is closed under arbitrary meets (intersections) and directed joins (unions). A preorder is compact iff it has finitely many off-diagonal pairs, in which case it is $[\![-]\!]$ applied to the Horn formula which is the conjunction of its off-diagonal pairs.

Exercise 20.9.

(a) Give an analogous construction of a Horn propositional theory for any *finitary* (i.e., all relation symbols are finitary, and also the LHS of each axiom is a finite conjunction) universal Horn theory over a relational first-order language that does not mention equality in any axiom.

(b) Explain what goes wrong if an axiom mentions equality.

One can also give an "interpretational" form of the **TODO**

The free/forgetful adjunctions connecting Stone, Priestley, and Hofmann–Mislove–Stralka dualities (Fig. 19.15) also have logical significance. For example, let $(\mathcal{L}, \mathcal{T})$ be a coherent theory. Of course, we can also regard $(\mathcal{L}, \mathcal{T})$ as a full propositional theory. Since the coherent inference rules are a subset of the full proof system, for a coherent sequent $\phi \Rightarrow \psi$, we have

$$\mathcal{T} \vdash_{\mathrm{coh}} \phi \Rightarrow \psi \implies \mathcal{T} \vdash \phi \Rightarrow \psi;$$

thus the inclusion $\mathcal{L}_{\omega 0}^{\mathrm{coh}} \subseteq \mathcal{L}_{\omega 0}$ descends to a canonical map

$$\begin{aligned} \langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}} &\longrightarrow \langle \mathcal{L} \mid \mathcal{T} \rangle \\ [\phi]_{\mathcal{T}} &\longmapsto [\phi]_{\mathcal{T}} \end{aligned}$$

which is clearly a distributive lattice homomorphism into a Boolean algebra.

Lemma 20.10. This map exhibits $\langle \mathcal{L} \mid \mathcal{T} \rangle$ as the free Boolean algebra generated by the distributive lattice $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}}$.

Proof. For any $A \in \mathsf{Bool}$, by Theorems 16.5 and 20.3 we have

$$\mathsf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, A) \cong \mathrm{Mod}(\mathcal{L}, \mathcal{T}, A) \cong \mathsf{DLat}(\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}}, A);$$

the composite of these two bijections is precomposition with the above map. $\square$

Here is an easy consequence. Note that the statement is not at all obvious (although it can be given a purely syntactic proof, via the technique of **cut elimination** for sequent calculus).

Corollary 20.11. Let $\mathcal{T}$ be a coherent theory, ϕ, ψ be coherent formulas. If $\mathcal{T} \vdash \phi \Rightarrow \psi$, then $\mathcal{T} \vdash_{\mathrm{coh}} \phi \Rightarrow \psi$.

Proof. This amounts to the statement that the above canonical map $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}} \to \langle \mathcal{L} \mid \mathcal{T} \rangle$ is an (order-)embedding, i.e., that the unit η_A from a distributive lattice A to its free Boolean algebra is injective. By Stone duality (Fig. 19.15), the free Boolean algebra can be realized as

$$\eta_A = [-] : A \longrightarrow \mathcal{KO}(\mathrm{Spec}(A)),$$

which is injective by the PIT. $\square$

Part III

More categories: limits and colimits

21 Limits

Example 21.1 (products)**.** Everyone knows what the product $X \times Y$ of sets is. To characterize $X \times Y$ categorically, note that for any other set Z, we have a canonical bijection

$$\mathsf{Set}(Z, X \times Y) \cong \mathsf{Set}(Z, X) \times \mathsf{Set}(Z, Y).$$

Indeed, by abuse of notation, this bijection is often treated as the identity: a map $Z \to X \times Y$ is treated as a pair of coordinate maps (f, g).

For objects X, Y in a category C, their **product** is an object $X \times Y \in \mathsf{C}$ together with bijections

$$\mathsf{C}(Z, X \times Y) \cong \mathsf{C}(Z, X) \times \mathsf{C}(Z, Y),$$

natural in Z. By abuse of notation, we often treat this bijection as the identity: for $f : Z \to X \in \mathsf{C}$ and $g : Z \to Y \in \mathsf{C}$, we denote the corresponding morphism $Z \to X \times Y$ simply by (f, g). Naturality means that for any $h : Z' \to Z$, we have a commutative square

$$\begin{tikzcd}
\mathsf{C}(Z', X \times Y) \arrow[r, leftrightarrow] \arrow[d, "(-)\circ h"'] & \mathsf{C}(Z', X) \times \mathsf{C}(Z', Y) \arrow[d, "(f,g)\mapsto(f\circ h, g\circ h)"] \\
\mathsf{C}(Z, X \times Y) \arrow[r, leftrightarrow] & \mathsf{C}(Z, X) \times \mathsf{C}(Z, Y),
\end{tikzcd}$$

i.e., we have $(f, g) \circ h = (f \circ h, g \circ h)$.

$$\begin{tikzcd}
& Z' \arrow[d, "h"] & \\
& Z \arrow[dl, "f"'] \arrow[d, dashed, "(f,g)"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

In other words, a product of X, Y is a representation of the presheaf

$$\mathsf{C}(-, X) \times \mathsf{C}(-, Y) : \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set},$$

thus can be equivalently (by Yoneda, Section 10) described as an object $X \times Y \in \mathsf{C}$ equipped with a universal pair of morphisms

$$\pi_1 : X \times Y \longrightarrow X, \qquad \pi_2 : X \times Y \longrightarrow Y$$

(corresponding via the above bijection to $1_{X \times Y} : X \times Y \to X \times Y$), so that for any other object $Z \in \mathsf{C}$ equipped with a pair of morphisms $f : Z \to X$ and $g : Z \to Y$, there is a unique morphism $(f, g) : Z \to X \times Y$ such that

$$\pi_1 \circ (f, g) = f,$$
$$\pi_2 \circ (f, g) = g.$$

The morphisms π_1, π_2 are called the **projections**. Note that the product consists of the object $X \times Y$ *equipped with the projections* (but we often don't mention them, by abuse of terminology).

By Yoneda (Corollary 10.7), the product $(X \times Y, \pi_1, \pi_2)$, if it exists, is unique up to unique isomorphism: given another product $(X \times' Y, \pi_1', \pi_2')$ of X, Y, there is a unique isomorphism between them compatible with the projections, i.e., making

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, "\cong"] & \\
X & & Y \\
& X \times' Y \arrow[ul, "\pi_1'"] \arrow[ur, "\pi_2'"'] &
\end{tikzcd}$$

commute (namely (π_1', π_2')).

The generalization to arbitrary arity products is straightforward: for a family of objects $(X_i)_{i \in I}$ in C, their **product** is an object $\prod_i X_i$ equipped with natural bijections

$$\mathsf{C}(Z, \prod_i X_i) \cong \prod_i \mathsf{C}(Z, X_i),$$

or equivalently with a universal family of morphisms

$$\pi_i : \prod_j X_j \longrightarrow X_i,$$

called the **projections**, such that for any other object Z equipped with morphisms $f_i : Z \to X_i$, there is a unique morphism

$$(f_i)_i : Z \longrightarrow \prod_i X_i$$

such that

$$\pi_i \circ (f_j)_j = f_i \quad \forall i \in I.$$

When $I = \varnothing$, the product is called the **terminal object**, denoted $\mathbf{1}_{\mathsf{C}}$; there are no projections, and the universal property simply says that any other object Z admits a unique morphism $Z \to \mathbf{1}_{\mathsf{C}}$.

Example 21.2 (sequential inverse limits)**.** Let $\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ be an **inverse sequence** of sets and functions. Their **inverse limit** is

$$\varprojlim_i X_i := \varprojlim (X_i, f_i)_i := \{\vec{x} \in \prod_i X_i \mid \forall i > 0\, (f_i(x_i) = x_{i-1})\}.$$

For another set Z and map into the product $(h_i)_i : Z \to \prod_i X_i$, $(h_i)_i$ lands in $\varprojlim_i X_i$ iff for each i, we have

$$f_i \circ h_i = h_{i-1}.$$

In other words, we have a bijection

$$\mathsf{Set}(Z, \varprojlim_i X_i) \cong \{\vec{h} \in \prod_i \mathsf{Set}(Z, X_i) \mid \forall i > 0\, (f_i \circ h_i = h_{i-1})\} = \varprojlim_i \mathsf{Set}(Z, X_i)$$

(the restriction of $\mathsf{Set}(Z, \prod_i X_i) \cong \prod_i \mathsf{Set}(Z, X_i)$), where the inverse limit on the RHS is of the inverse sequence

$$\cdots \xrightarrow{f_3 \circ (-)} \mathsf{Set}(Z, X_2) \xrightarrow{f_2 \circ (-)} \mathsf{Set}(Z, X_1) \xrightarrow{f_1 \circ (-)} \mathsf{Set}(Z, X_0).$$

For an inverse sequence $\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ in a category C, its **(inverse) limit** is an object $\varprojlim_i X_i$ together with natural bijections

$$\mathsf{C}(Z, \varprojlim_i X_i) \cong \varprojlim_i \mathsf{C}(Z, X_i) \in \mathsf{Set}$$

where the inverse limit on the RHS is of $\cdots \xrightarrow{f_2\circ(-)} \mathsf{C}(Z, X_1) \xrightarrow{f_1\circ(-)} \mathsf{C}(Z, X_0)$. In other words, the inverse limit is a representation of the presheaf

$$\varprojlim_i \mathsf{C}(-, X_i) : \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set},$$

thus equivalently consists of $\varprojlim_i X_i$ together with a universal family of morphisms

$$\pi_i : \varprojlim_j X_j \longrightarrow X_i \quad \text{with} \quad f_i \circ \pi_i = \pi_{i-1},$$

again called the **projections**, such that for any other $Z \in \mathsf{C}$ equipped with morphisms

$$h_i : Z \longrightarrow X_i \quad \text{with} \quad f_i \circ h_i = h_{i-1},$$

there is a unique morphism

$$(h_i)_i : Z \longrightarrow \varprojlim_i X_i$$

such that

$$\pi_i \circ (h_j)_j = h_i \quad \forall i.$$

$$\begin{tikzcd}
Z \arrow[d, dashed, "(h_i)_i"'] \arrow[ddr, "h_2"] \arrow[ddrr, bend left, "h_1"] \arrow[ddrrr, bend left, "h_0"] & & & \\
\varprojlim_i X_i \arrow[dr, "\pi_2"'] \arrow[drr, "\pi_1"] \arrow[drrr, "\pi_0"] & & & \\
\cdots \arrow[r, "f_3"'] & X_2 \arrow[r, "f_2"'] & X_1 \arrow[r, "f_1"'] & X_0
\end{tikzcd}$$

We now give the general definition. Let C be a category, I be another (typically much smaller) category, and $F : \mathsf{I} \to \mathsf{C}$ be a functor; we call F a **diagram of shape I in C**. A **limit** of F is an object

$$\varprojlim F = \varprojlim_{I\in\mathsf{I}} F(I) \in \mathsf{C}$$

equipped with a universal family of morphisms

$$(\pi_I : \varprojlim F \longrightarrow F(I))_{I\in\mathsf{I}} \quad \text{such that} \quad F(f) \circ \pi_I = \pi_J \quad \forall f : I \to J \in \mathsf{I},$$

called the **projections**, such that for any other object $Z \in \mathsf{C}$ equipped with a family

$$(h_I : Z \longrightarrow F(I))_{I\in\mathsf{I}} \quad \text{such that} \quad F(f) \circ h_I = h_J \quad \forall f : I \to J \in \mathsf{I},$$

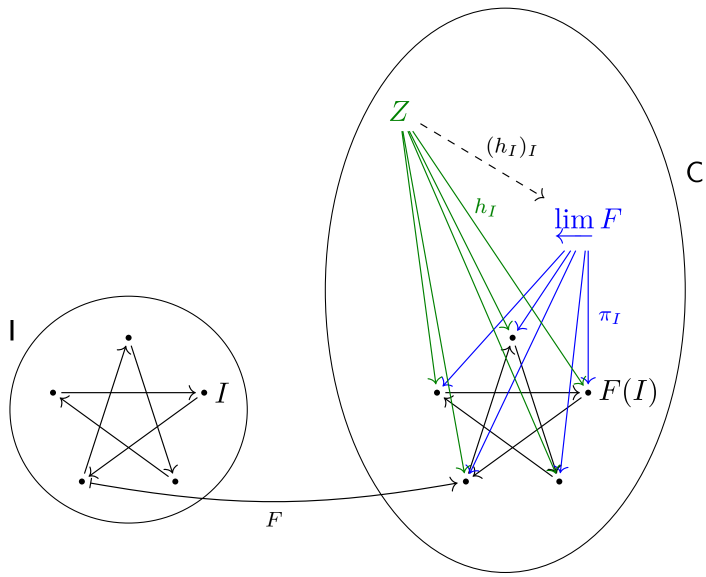
there is a unique morphism

$$(h_I)_I : Z \longrightarrow \varprojlim F$$

such that

$$\pi_I \circ (h_J)_J = h_I \quad \forall I \in \mathsf{I}.$$

The family of morphisms $(h_I : Z \to F(I))_{I \in \mathsf{I}}$ above is called a **cone** from Z to the diagram F. Thus, a limit of F is a *universal cone over* F.



Example 21.3 (sequential inverse limits)**.** For $\mathsf{I} := \mathbb{N}^{\mathrm{op}} = \{\cdots < 2 < 1 < 0\}$ (regarded as a category), an I-shaped diagram $F : \mathsf{I} \to \mathsf{C}$ is the same thing as an inverse sequence $\cdots \xrightarrow{F(2\to 1)} F(1) \xrightarrow{F(1\to 0)} F(0)$; its limit is exactly as defined above.

Example 21.4 (products)**.** For I a set (regarded as a discrete category), an I-shaped diagram $F : \mathsf{I} \to \mathsf{C}$ is the same thing as a family of objects $(F(I))_{I \in \mathsf{I}}$; its limit is the product $\prod_{I \in \mathsf{I}} F(I)$ (note that the commutativity conditions in the definition of cone above are trivial for f an identity morphism).

Proposition 21.5. For a diagram $F : \mathsf{I} \to \mathsf{Set}$ (with I small), its limit can be constructed as

$$\varprojlim F := \{\vec{x} \in \textstyle\prod_{I \in \mathsf{I}} F(I) \mid \forall f : I \to J \in \mathsf{I}\,(F(f)(x_I) = x_J)\}$$

equipped with the usual product projections $\pi_I : \varprojlim F \subseteq \prod_I F(I) \to F(I)$.

Proof. A map $h : Z \to \varprojlim F \subseteq \prod_{I \in \mathsf{I}} F(I)$ is the same thing as a family of maps $(h_I : Z \to F(I))_{I \in \mathsf{I}}$ which lands in $\varprojlim F$, i.e., for all $f : I \to J \in \mathsf{I}$, $F(f) \circ h_I = h_J$, which is the definition of a cone from Z to F; clearly $h = 1_{\varprojlim F}$ corresponds to the family of projections $(\pi_I : \varprojlim F \to F(I))_I$. $\square$

Now for a diagram $F : \mathsf{I} \to \mathsf{C}$, for each $Z \in \mathsf{C}$, we have

$$\begin{aligned} \varprojlim_{I \in \mathsf{I}} \mathsf{C}(Z, F(I)) &= \{\vec{h} \in \textstyle\prod_{I \in \mathsf{I}} \mathsf{C}(Z, F(I)) \mid \forall f : I \to J \in \mathsf{I}\,(F(f) \circ h_I = h_J)\} \\ &= \{\text{cones } Z \to F\}. \end{aligned}$$

Thus by Yoneda,[21]

$$\begin{gathered}
\mathsf{PSh}(\text{よ} Z, \varprojlim_{I\in\mathsf{I}} \text{よ}(F(I))) = \mathsf{PSh}(\text{よ} Z, \varprojlim_{I\in\mathsf{I}} \mathsf{C}(-, F(I))) \cong \varprojlim_{I\in\mathsf{I}} \mathsf{C}(Z, F(I)) = \{\text{cones } Z \to F\} \\
(\alpha : \text{よ} Z \to \varprojlim_{I\in\mathsf{I}} \mathsf{C}(-, F(I))) \mapsto \alpha_Z(1_Z) \\
(\text{よ} h_I)_I = ((h_I \circ (-) : \mathsf{C}(Z', Z) \to \mathsf{C}(Z', F(I)))_I)_{Z'\in\mathsf{C}} \leftarrow\!\shortmid (h_I : Z \to F(I))_I.
\end{gathered}$$

A universal cone $(\pi_I : \varprojlim F \to F(I))_I$ corresponds to a natural isomorphism $\text{よ} Z \cong \varprojlim_{I\in\mathsf{I}} \text{よ}(F(I))$. We have shown:

Corollary 21.6. A limit of $F : \mathsf{I} \to \mathsf{C}$, i.e., universal cone over F, is the same thing as an object $\varprojlim F \in \mathsf{C}$ together with a natural isomorphism

$$\text{よ}(\varprojlim F) = \mathsf{C}(-, \varprojlim F) \cong \varprojlim_{I\in\mathsf{I}} \mathsf{C}(-, F(I)) = \varprojlim_{I\in\mathsf{I}} \text{よ}(F(I)).$$

Moreover, a cone $(h_I : Z \to F(I))_{I\in\mathsf{I}}$ is universal iff $(\text{よ} h_I : \text{よ} Z \to \text{よ}(F(I)))_I$ is (pointwise) universal in $\mathsf{PSh}(\mathsf{C})$. $\square$

Limits should be thought of as categorical analogs of meets in posets: recall from Section 12 that meets are defined via

$$\downarrow \bigwedge_i a_i = \bigcap_i \downarrow a_i,$$

i.e., "meets are lifted from 2 via the Yoneda embedding $\downarrow : P \to \mathrm{Low}(P) \cong 2^{P^{\mathrm{op}}}$". Likewise, Corollary 21.6 says that "limits are lifted from Set via the Yoneda embedding $\text{よ} : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$". As a consequence, many properties of limits can be immediately deduced from the corresponding properties in Set, extending the arguments in Section 9 about monomorphisms (which are in fact an example of limits; see Example 21.16). Here is an example:

Proposition 21.7. Products in any category C are commutative up to isomorphism: we have $X \times Y \cong Y \times X$ (assuming both products exist).

Proof 1. Let p_1, p_2 be the projections from $X \times Y$ and q_1, q_2 be the projections from $Y \times X$. We have morphisms between these two products, given by the universal property:

$$\begin{tikzcd}
& X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, dashed, shift left, "(p_2{,}p_1)"] & \\
X & & Y \\
& Y \times X \arrow[ul, "q_2"] \arrow[ur, "q_1"'] \arrow[uu, dashed, shift left, "(q_2{,}q_1)"] &
\end{tikzcd}$$

We have

$$\begin{aligned}
p_1 \circ (q_2, q_1) \circ (p_2, p_1) = q_2 \circ (p_2, p_1) = p_1, \\
p_2 \circ (q_2, q_1) \circ (p_2, p_1) = q_1 \circ (p_2, p_1) = p_2,
\end{aligned}$$

whence $(q_2, q_1) \circ (p_2, p_1) = 1_{X\times Y}$ since both are morphisms $X \times Y \to X \times Y$ with the same postcomposites with p_1, p_2. Similarly, $(p_2, p_1) \circ (q_2, q_1) = 1_{Y\times X}$. $\square$

[21] For the time being, $\varprojlim_I \text{よ}(F(I))$ is a *pointwise* limit, i.e., we are taking the limits $\varprojlim_I \text{よ}(F(I))(Z) = \varprojlim_I \mathsf{C}(Z, F(I)) \in \mathsf{Set}$ for each $Z \in \mathsf{C}$; in Example 22.15 we will see that pointwise limits are also limits in the functor category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \cong \mathsf{PSh}(\mathsf{C})$.

Proof 2. We have bijections

$$X \times Y \cong Y \times X \in \mathsf{Set},$$

natural in both variables. Thus, for $X, Y, Z \in \mathsf{C}$, we have bijections

$$\mathsf{C}(Z, X \times Y) \cong \mathsf{C}(Z, X) \times \mathsf{C}(Z, Y) \cong \mathsf{C}(Z, Y) \times \mathsf{C}(Z, X) \cong \mathsf{C}(Z, Y \times X),$$

natural in Z (for the middle bijection, naturality uses naturality of the above bijections in Set, with respect to the precomposition maps $\mathsf{C}(f, X), \mathsf{C}(f, Y)$ for $f : Z \to Z' \in \mathsf{C}$), i.e.,

$$\text{よ}(X \times Y) \cong \text{よ}(Y \times X),$$

whence $X \times Y \cong Y \times X$ since よ is full+faithful. □

Here are some more examples of limits:

Example 21.8 (fixed points)**.** Let G be a group, $F : G \to \mathsf{Set}$ be a G-set. Then

$$\varprojlim F = \{x \in F(*) \mid \forall g \in G\,(g \cdot x = F(g)(x) = x)\} = \{\text{fixed points of } F\}.$$

More generally, for a G-action $F : G \to \mathsf{C}$ on an object in C, $\varprojlim F$ (if it exists) is an object equipped with a universal morphism $\varprojlim F \to F(*)$ ("projection") which is fixed by the G-action; we can think of this morphism as the inclusion of the "largest fixed subobject" of $F(*)$.

Exercise 21.9. Show that the "projection" $\varprojlim F \to F(*)$ is monic.

Example 21.10 (equalizers)**.** Let $\mathsf{I} = \{\bullet \rightrightarrows \bullet\}$. An I-shaped diagram in a category C consists of two parallel morphisms

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

The limit of such a diagram is called the **equalizer** of f, g, denoted $\mathrm{eq}(f, g)$.

In Set, by Proposition 21.5,

$$\begin{aligned} \mathrm{eq}(f, g) &= \{(x, y) \in X \times Y \mid f(x) = y = g(x)\} \\ &\cong \{x \in X \mid f(x) = g(x)\} \subseteq X; \end{aligned}$$

we usually use this simpler definition of equalizer. The "projections" are

$$\pi_X : \mathrm{eq}(f, g) \subseteq X, \qquad\qquad \pi_Y := f|\mathrm{eq}(f, g) = g|\mathrm{eq}(f, g) : \mathrm{eq}(f, g) \to Y.$$

Now in an arbitrary category C, we have

$$\begin{aligned} \{\text{cones from } Z \text{ to } X \underset{g}{\overset{f}{\rightrightarrows}} Y\} &\cong \{(h : Z \to X,\ k : Z \to Y) \mid f \circ h = k = g \circ h\} \\ &\cong \{h : Z \to X \mid f \circ h = g \circ h\}; \end{aligned}$$

the equalizer, if it exists, is an object $\mathrm{eq}(f, g) \in \mathsf{C}$ with a universal such morphism $\pi : \mathrm{eq}(f, g) \to X$.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists!"'] \arrow[dr, "h"] & & \\ \mathrm{eq}(f,g) \arrow[r, "\pi"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

Proposition 21.11. The "projection" $\pi : \mathrm{eq}(f, g) \to X$ from an equalizer $\mathrm{eq}(f, g)$ of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is always monic.

Because of this, we usually call the "projection" $\pi : \mathrm{eq}(f, g) \to X$ the "**inclusion**".

Proof 1. By the universal property of limits, if $\pi \circ h = \pi \circ k$ for $h, k : Z \to \mathrm{eq}(f, g)$, then $h = k$. □

Proof 2. $\text{よ}\pi : \text{よ}(\mathrm{eq}(f, g)) \to \text{よ}X$ is (pointwise) the equalizer of $\text{よ}X \underset{\text{よ}g}{\overset{\text{よ}f}{\rightrightarrows}} \text{よ}Y$, hence (pointwise) monic, i.e., π is monic. □

By definition, in Set,

$$\mathrm{eq}(f, g : X \rightrightarrows Y) = \text{solution set of } f(x) = g(x).$$

Given now several such parallel pairs $f_i, g_i : X \rightrightarrows Y_i$,

$$\mathrm{eq}((f_i)_i, (g_i)_i : X \rightrightarrows \textstyle\prod_i Y_i) = \text{solution set of system } (f_i(x) = g_i(x))_i.$$

Applying this to the formula in Proposition 21.5, we get

Proposition 21.12 (construction of limits from products and equalizers)**.** Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram in a category. Then the limit of F may be constructed as

$$\varprojlim F = \mathrm{eq}\left(\textstyle\prod_{I \in \mathsf{I}} F(I) \underset{(\pi_J)_{f:I\to J}}{\overset{(f \circ \pi_I)_{f:I\to J}}{\rightrightarrows}} \textstyle\prod_{f:I\to J \in \mathsf{I}} F(J) \right).$$

That is, if the products and equalizer on the RHS exist, then that equalizer is the limit of F, with projections given by the composites $\varprojlim F \hookrightarrow \prod_J F(J) \xrightarrow{\pi_I} F(I)$.

In particular, if C has all products (of size $< \kappa$, for some infinite regular cardinal κ) and equalizers, then it has all limits (of size $< \kappa$).

Proof. If $\mathsf{C} = \mathsf{Set}$, then the equalizer on the RHS is the construction of $\varprojlim F$ in Proposition 21.5. In general, applying よ to the morphisms $\varprojlim F \hookrightarrow \prod_J F(J) \xrightarrow{\pi_I} F(I)$ yields (pointwise) the limit cone witnessing that $\varprojlim(\text{よ} \circ F) = \text{よ}(\varprojlim F)$ (pointwise) in Set, whence these morphisms form a limit cone in C. □

Example 21.13 (fiber products/pullbacks)**.** Let $\mathsf{I} = \{\bullet \to \bullet \leftarrow \bullet\}$. An I-shaped diagram consists of

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y. \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Its limit is called the **fiber product of** X, Y **over** Z (with respect to f, g), denoted $X \times_Z Y$.

In Set, we have

$$\begin{aligned} X \times_Z Y &= \{(x, y, z) \in X \times Y \times Z \mid f(x) = z = g(y)\} \\ &\cong \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subseteq X \times Y; \end{aligned}$$

again, we usually use this simpler definition. The projections are the restrictions $\pi_1 : X \times_Z \to X$ and $\pi_2 : X \times_Z \to Y$ of the product projections, together with $f \circ \pi_1 = g \circ \pi_2$.

Now in an arbitrary category C, a cone over the above diagram consists of

$$h : W \to X, \qquad k : W \to Y, \qquad \text{such that} \qquad f \circ h = g \circ k : W \to Z;$$

the fiber product, if it exists, is an object $X \times_Z Y$ with a universal such pair of morphisms

$$\pi_1 : X \times_Z Y \to X, \qquad \pi_2 : X \times_Z Y \to Y, \qquad \text{such that} \quad f \circ \pi_1 = g \circ \pi_2 : X \times_Z Y \to Z$$

(called the **projections**). We usually denote the morphism $W \to X \times_Z Y$ induced by another such cone as above by (h, k) instead of $(h, k, f \circ h)$.

$$\begin{tikzcd}
& W \arrow[d, dashed, "{(h,k)}"] \arrow[ddl, bend right, "h"'] \arrow[ddr, bend left, "k"] & \\
& X \times_Z Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

It is often convenient to regard the two sides of a fiber product asymmetrically. We also denote the above situation by

$$\begin{tikzcd}
f^*(Y) := X \times_Z Y \arrow[r] \arrow[d, "f^*(g) := \pi_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

We call $f^*(g) : f^*(Y) \to X$ the **pullback of** $g : Y \to Z$ **along** $f : X \to Z$.

Proposition 21.14. For X, Y, Z, f, g as above, $(\pi_1, \pi_2) : X \times_Z Y \to X \times Y$ is monic (we also call it an "**inclusion**").

Proof 1. To avoid confusion, we denote the fiber product projections by π_1, π_2 as above, and the product projections by p_1, p_2. If $(\pi_1, \pi_2) \circ h = (\pi_1, \pi_2) \circ k : W \to X \times Y$ for some $h, k : W \to X \times_Z Y$, then

$$\pi_1 \circ h = p_1 \circ (\pi_1, \pi_2) \circ h \qquad = p_1 \circ (\pi_1, \pi_2) \circ k = \pi_1 \circ k,$$

and similarly $\pi_2 \circ h = \pi_2 \circ k$, whence $h = k$ by the universal property of $(X \times_Z Y, \pi_1, \pi_2)$. $\square$

Proof 2. In $\mathsf{C} = \mathsf{Set}$, $(\pi_1, \pi_2) : X \times_Z Y \to X \times Y$ is the inclusion; in general, apply よ. $\square$

Fiber products/pullbacks include several familiar constructions in Set:

Example 21.15 (preimages)**.** For $g : Y \subseteq Z \in \mathsf{Set}$ a subset inclusion, we have

$$\begin{aligned} f^*(Y) &= \{(x, y) \in X \times Y \mid f(x) = y \in Y \subseteq Z\} \\ &\cong f^{-1}(Y) \subseteq X, \end{aligned}$$

with $f^*(g) : f^*(Y) \to X$ the inclusion.

It follows by Yoneda that *pullback preserves monomorphisms*: for any monomorphism $g : Y \to Z \in \mathsf{C}$ in an arbitrary category, and any $f : X \to Z \in \mathsf{C}$, if the pullback $f^*(g)$ exists, then it is monic.

Example 21.16 (kernels)**.** For a function $f : X \to Y \in \mathsf{Set}$, we have

$$X \times_Y X = \{(x, x') \in X \mid f(x) = f(x')\} =: \ker(f),$$

the **kernel (pair)** of f, with the usual projections $\pi_1, \pi_2 : \ker(f) \to X$. We likewise define

$$\ker(f) := X \times_Y X$$

for an arbitrary morphism $f : X \to Y$ in an arbitrary category C.

It follows by Yoneda that $f : X \to Y \in \mathsf{C}$ *is monic iff* $\ker(f)$ *is the diagonal* $X \to X \times X$, i.e., iff

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(which is always a cone) is a fiber product.

Exercise 21.17. Directly prove the last sentences of the preceding two examples.

Example 21.18. In Set, we clearly have

$$X \times_{\mathbf{1}} Y = X \times Y.$$

Thus (by Yoneda), *finite products can be constructed from fiber products and terminal object.*

(More generally, one can define **wide pullbacks** to be limits of diagrams consisting of an arbitrary family of morphisms $f_i : X_i \to Y$; these together with terminal object yield all products.)

In Set, we can also construct the equalizer of $f, g : X \rightrightarrows Y$ as the fiber product

$$\begin{tikzcd} \mathrm{eq}(f, g) \cong \{(x, x') \mid (x, f(x)) = (x', g(x'))\} \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "(1_X, g)"] \\ X \arrow[r, "(1_X, f)"'] & X \times Y. \end{tikzcd}$$

Thus, *equalizers can be constructed from fiber products and binary products.*

Combining these two facts and Proposition 21.12, we get that *arbitrary finite limits can be constructed from fiber products and terminal object.*

What kinds of categories have limits? Many familiar ones:

Example 21.19. For any infinitary first-order language $\mathcal{L}$, as in Section 18, the category $\mathsf{Mod}(\mathcal{L})$ of (small) $\mathcal{L}$-structures has all (small) limits. Products are given by product structures; equalizers are given by the equalizer in Set, as in Example 21.10, regarded as a substructure.

Proposition 21.20. Let $\mathsf{C} \subseteq \mathsf{D}$ be a full subcategory, $F : \mathsf{I} \to \mathsf{C}$ be a diagram. Suppose that F has a limit in D, and that the limit object $\varprojlim F$ is in C. Then $\varprojlim F$, with the same projections, is also a limit in C.

Proof. For any other cone $(Z, h_I)_{I \in \mathsf{I}}$ over F in C, the unique induced morphism $(h_I)_I : Z \to \varprojlim F \in \mathsf{D}$ is by fullness also in C. ☐

Corollary 21.21. Any full subcategory $\mathsf{C} \subseteq \mathsf{Mod}(\mathcal{L})$ closed under products and substructures has all (small) limits. ☐

Example 21.22. $\mathsf{Grp}, \mathsf{Bool}, \mathsf{Top}, \mathsf{Pos}, \dots$ have all (small) limits.

Example 21.23. Field does *not* have a terminal object, since fields of different characteristic do not have any homomorphisms between them, or alternatively since all field homomorphisms are injective.

It follows that Field is not universally Horn-axiomatizable (Exercise 18.8), over *any* language.

Exercise 21.24.

(a) Let K be a field, $k \subseteq K$ be its prime subfield (i.e., $\mathbb{F}_p$ if $p = \mathrm{char}(K) > 0$, otherwise $\mathbb{Q}$). Show that $K = k \times K$ in Field.

(b) Let K be non-prime field. Show that $K \times K$ does not exist in Field.

Exercise 21.25. Show that a group G, regarded as a one-object category, has pullbacks, but no binary products, terminal object, or equalizers unless G is trivial.

Unfortunately, due to set-theoretic issues, it is unreasonable to expect interesting categories to have non-small limits:

Example 21.26. In a poset P, since all diagrams commute, limits are just meets:

$$\varprojlim F = \bigwedge_{I \in \mathsf{I}} F(I).$$

In particular, if P is a complete lattice (i.e., has all meets), then it has all limits.

Theorem 21.27 (Freyd)**.** If a category C has products indexed by the set $\mathsf{C}_1 := \bigsqcup_{X,Y \in \mathsf{C}} \mathsf{C}(X, Y)$ of all morphisms in C, then C is a preorder, i.e., there is at most one morphism between any two objects, i.e., C is equivalent to a complete lattice.

Proof. If there were two distinct parallel morphisms $f, g : X \to Y \in \mathsf{C}$, then there would be at least $2^{|\mathsf{C}_1|}$ distinct morphisms $X \to Y^{\mathsf{C}_1}$ by choosing f or g in each coordinate, a contradiction. ☐

Because of this, we call a category C **(small-)complete** if it has all *small* limits, i.e., limits indexed by I in the fixed background universe. By Theorem 21.27, complete categories which are themselves small must be preorders.

22 Limits and functors

Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram, $G : \mathsf{C} \to \mathsf{D}$ be a functor. We say that G

- **preserves** the limit of F, if for any limit cone $(\varprojlim F, (\pi_I : \varprojlim F \to F(I))_I)$ over F, the cone $(G(\varprojlim F), (G(\pi_I) : G(\varprojlim F) \to G(F(I)))_I)$ over $G \circ F$ is a limit cone

  (equivalently, either this holds for *some* limit cone of F, or F does not have a limit);

- **reflects** the limit of F, if for any cone $(Z, (h_I : Z \to F(I))_I)$ over F, if the cone $(G(Z), (G(h_I) : G(Z) \to G(F(I)))_I)$ is a limit cone, then $(Z, (h_I)_I)$ was a limit cone.

Example 22.1. よ $: \mathsf{C} \to \mathsf{PSh}(\mathsf{C})$ preserves and reflects arbitrary limits, by Corollary 21.6.

More generally, we have

Proposition 22.2. Full+faithful functors reflect limits.

Proof. This is essentially Proposition 21.20. Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram, $G : \mathsf{C} \to \mathsf{D}$ be a full+faithful functor, and $(Z, (h_I)_I)$ be a cone over F such that $(G(Z), (G(h_I))_I)$ is a limit cone over $G \circ F$. Then for any cone $(Z', (h'_I)_I)$ over F, we have a cone $(G(Z'), (G(h'_I))_I)$ over $G \circ F$, so there is a unique $g : G(Z') \to G(Z)$, which by full+faithfulness of G is $G(f)$ for a unique $f : Z' \to Z$, such that $G(h_I \circ f) = G(h_I) \circ g = G(h'_I)$ for each $I \in \mathsf{I}$, i.e., $h_I \circ f = h'_I$ for each I, again by full+faithfulness of G. $\square$

Example 22.3. For an infinitary first-order language $\mathcal{L}$ and full subcategory $\mathsf{C} \subseteq \mathsf{Mod}(\mathcal{L})$ closed under products and substructures, the forgetful functor $\mathsf{C} \to \mathsf{Set}$ preserves (small) limits by construction.

Example 22.4. The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ does *not* reflect limits. Take any non-discrete space X, and let Y be $X \times X$ equipped with a finer topology. Then the projections $\pi_1, \pi_2 : Y \to X$ form a non-limit cone which becomes a limit cone in Set.

Proposition 22.5. If $G : \mathsf{C} \to \mathsf{D}$ is a conservative functor, and $F : \mathsf{I} \to \mathsf{C}$ has a limit preserved by G, then G reflects limits of F.

Proof. Let $(Z, (h_I)_I)$ be a cone over F, and $(h_I)_I : Z \to \varprojlim F$ be the induced morphism. Then if $(G(Z), (G(h_I))_I)$ is a limit cone over $G \circ F$, then since $(G(\varprojlim F), (G(\pi_I))_I)$ is also a limit cone, $G((h_I)_I) : G(Z) \to G(\varprojlim F)$ must be an isomorphism, whence since G is conservative, $(h_I)_I$ was an isomorphism, whence $(Z, (h_I)_I)$ was a limit cone. $\square$

Example 22.6. If $\mathcal{L}$ in Example 22.3 is functional (i.e., $\mathcal{L}_{\text{rel}} = \varnothing$), then the forgetful functor $\mathsf{C} \to \mathsf{Set}$ is conservative, hence since it preserves (small) limits, also reflects them.

For example, the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ reflects limits.

Proposition 22.7. Right adjoint functors preserve arbitrary limits.

Proof. Let $H \dashv G : \mathsf{C} \to \mathsf{D}$, and let $F : \mathsf{I} \to \mathsf{C}$ be a diagram. Then for each $Z \in \mathsf{C}$,

$$\begin{aligned}
\{\text{cones } Z \to G \circ F\} &= \{(g_I : Z \to G(F(I)))_{I \in \mathsf{I}} \mid \forall f : I \to J \in \mathsf{I}\,(G(F(f)) \circ g_I = g_J)\} \\
&\cong \{(h_I : H(Z) \to F(I))_{I \in \mathsf{I}} \mid \forall f : I \to J \in \mathsf{I}\,(F(f) \circ h_I = h_J)\} \\
&\qquad\qquad \text{by naturality of the bijections } \mathsf{C}(H(X), Y) \cong \mathsf{D}(X, G(Y)) \text{ in } Y \\
&= \{\text{cones } H(Z) \to F\} \\
&\cong \mathsf{C}(H(Z), \varprojlim F) \\
&\cong \mathsf{D}(Z, G(\varprojlim F));
\end{aligned}$$

and these bijections are themselves natural in Z, using (twice) naturality of $\mathsf{C}(H(X), Y) \cong \mathsf{D}(X, G(Y))$ in X. So $G(\varprojlim F) \in \mathsf{D}$ is a limit of $G \circ F$. To verify that the projections are $G(\pi_I : \varprojlim F \to F(I))$ for each $I \in \mathsf{I}$:

$$\begin{aligned}
1_{G(\varprojlim F)} \in \mathsf{D}(G(\varprojlim F), G(\varprojlim F)) &\mapsto \varepsilon_{\varprojlim F} \in \mathsf{C}(H(G(\varprojlim F)), \varprojlim F) \\
&\mapsto (\pi_I \circ \varepsilon_{\varprojlim F} : H(G(\varprojlim F)) \to F(I))_I \in \{\text{cones } H(Z) \to F\} \\
&\mapsto (G(\pi_I) : G(\varprojlim F) \to G(F(I)))_I \in \{\text{cones } Z \to G \circ F\}
\end{aligned}$$

using again naturality of $\mathsf{C}(H(X), Y) \cong \mathsf{D}(X, G(Y))$ in X in the last step. □

Remark 22.8 (adjoint functor theorem)**.** The converse, that a functor $G : \mathsf{C} \to \mathsf{D}$ which preserves limits must have a left adjoint, is true assuming that C has "enough" limits to begin with. Unfortunately, "enough" involves rather tedious set-theoretic conditions (known as the **solution set condition**).

Such conditions are necessary for functors between interesting categories, due to Theorem 21.27. However, if we are willing to restrict to posets, we may simply require *all* limits (i.e., meets) to exist, yielding

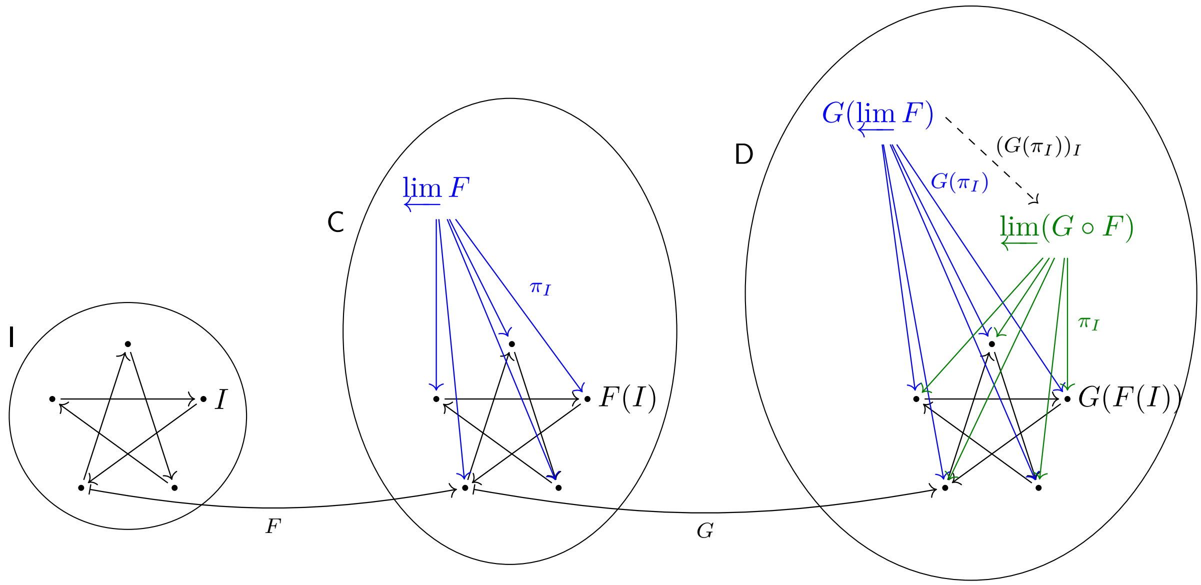
Theorem 22.9 (adjoint functor theorem for posets)**.** A monotone map $g : Q \to P$ between posets, with Q a complete lattice, has a left adjoint iff g preserves all meets.

Proof. g has a left adjoint iff for each $x \in P$, there is a least $y \in Q$ such that $x \le g(y)$; since g preserves meets, we may take the meet of all such y. □

This proof indicates how the general adjoint functor theorem for categories is proved: to construct to the left adjoint of $G : \mathsf{D} \to \mathsf{C}$, i.e., a universal morphism $\eta_X : X \to G(F(X))$ for each $X \in \mathsf{C}$, we would like to set $F(X)$ to be the limit of all $Y \in \mathsf{D}$ equipped with a morphism $X \to G(Y)$, and need for this limit to be preserved by G. In interesting cases, this diagram will be large; but it may have a small "cofinal" subdiagram (see **TODO**).

Suppose a diagram $F : \mathsf{I} \to \mathsf{C}$ and functor $G : \mathsf{C} \to \mathsf{D}$ are such that both F and $G \circ F$ are known to have a limit. In that case, the cone $(G(\pi_I))_I : G(\varprojlim F) \to G \circ F$ induces a **comparison morphism**

$$(G(\pi_I))_I : G(\varprojlim F) \longrightarrow \varprojlim (G \circ F).$$



G preserves $\varprojlim F$ iff this comparison morphism is an isomorphism (easily seen by applying よ). (This is analogous to the observation, Remark 12.7, that when checking that a monotone map preserves meets, only one inequality is nontrivial.)

Example 22.10. Let X be a topological space which is the union of two open sets $U, V \subseteq X$. For each $Y \subseteq X$, let

$$\mathcal{C}(Y) := \{\text{continuous maps } Y \to \mathbb{R}\}.$$

Given $Z \subseteq Y \subseteq X$, we have a restriction map $\mathcal{C}(Y) \to \mathcal{C}(Z)$; we thus get a functor

$$\mathcal{C} : \mathcal{P}(X)^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

We claim that $\mathcal{C}$ preserves the pullback in $\mathcal{P}(X)^{\mathrm{op}}$ (i.e., union)

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
U \arrow[dr] & & V \arrow[dl] \\
& U \cap V. &
\end{tikzcd}$$

The comparison morphism is the first map in the diagram

$$\begin{tikzcd}
& \mathcal{C}(X) \arrow[d, "f \mapsto (f|U, f|V)"] & \\
& \mathcal{C}(U) \times_{\mathcal{C}(U\cap V)} \mathcal{C}(V) \arrow[dl, "(-)|U"'] \arrow[dr, "(-)|V"] & \\
\mathcal{C}(U) \arrow[dr, "(-)|(U\cap V)"'] & & \mathcal{C}(V) \arrow[dl, "(-)|(U\cap V)"] \\
& \mathcal{C}(U \cap V). &
\end{tikzcd}$$

Indeed, for every pair $(g, h) \in \mathcal{C}(U) \times_{\mathcal{C}(U\cap V)} \mathcal{C}(V)$ of continuous functions on U, V which agree on $\mathcal{C}(U \cap V)$, there is a unique $f \in \mathcal{C}(X)$ such that $f|U = g$ and $f|V = h$.

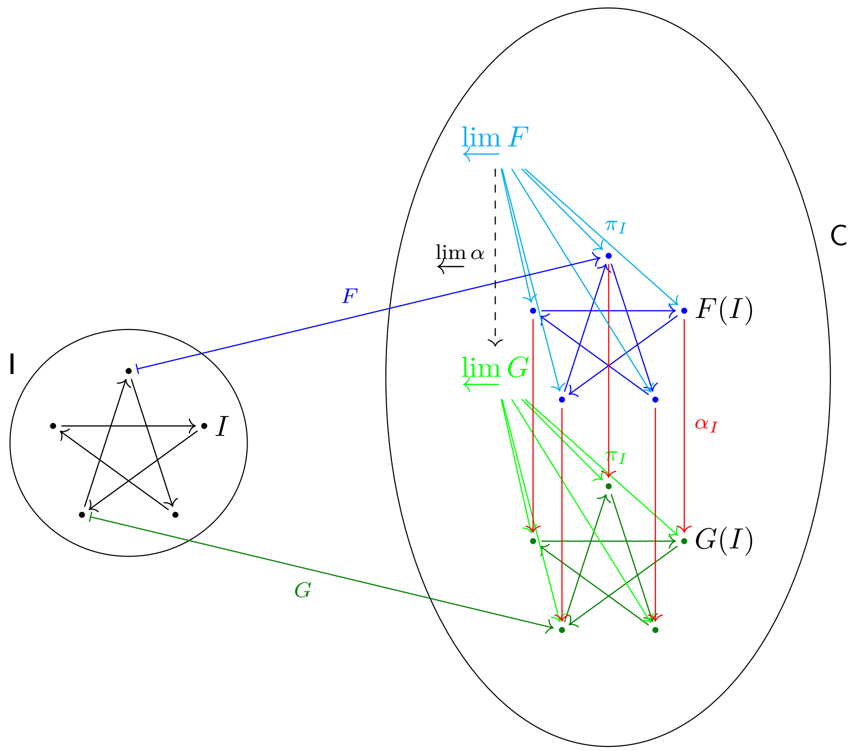
Taking limits is itself a functorial operation: given two diagrams $F, G : \mathsf{I} \to \mathsf{C}$ and a natural transformation $\alpha : F \to G$ between them, if $\varprojlim F, \varprojlim G$ exist, then α induces a morphism

$$\varprojlim \alpha := (\alpha_I \circ \pi_I)_{I \in I} : \varprojlim F \longrightarrow \varprojlim G$$

making the squares

$$\begin{tikzcd}
\varprojlim F \arrow[r, "\pi_I"] \arrow[d, dashed, "\varprojlim \alpha"'] & F(I) \arrow[d, "\alpha_I"] \\
\varprojlim G \arrow[r, "\pi_I"'] & G(I)
\end{tikzcd}$$

commute.



If C has all limits of shape I, we thus get a functor

$$\varprojlim = \varprojlim{}^{\mathsf{I}}_{\mathsf{C}} : \mathsf{C}^{\mathsf{I}} \longrightarrow \mathsf{C}.$$

Example 22.11. In Set,

$$\varprojlim \alpha : \varprojlim F \subseteq \prod_{I \in \mathsf{I}} F(I) \longrightarrow \prod_{I \in \mathsf{I}} G(I) \supseteq \varprojlim G$$
$$(x_I)_I \longmapsto (\alpha_I(x_I))_I.$$

The functoriality of $\varprojlim$ can be understood conceptually as follows. Note that a cone $(h_I)_I$ from Z to a diagram $F : \mathsf{I} \to \mathsf{C}$ is the same thing as a natural transformation

$$(h_I)_I : \Delta(Z) \longrightarrow F : \mathsf{I} \to \mathsf{C}$$

where

$$\begin{aligned} \Delta = \Delta^I_{\mathsf{C}} : \mathsf{C} &\longrightarrow \mathsf{C}^{\mathsf{I}} \\ Z &\longmapsto (X \mapsto Z) \end{aligned}$$

is the **diagonal functor**. Thus, the definition of $\varprojlim F$ becomes

$$\mathsf{C}(-, \varprojlim F) \cong \{\text{cones } (-) \to F\} = \mathsf{C}^{\mathsf{I}}(\Delta(-), F),$$

i.e., *a limit of F is the same thing as a value of the right adjoint of Δ at F.* Now recall (Section 11) that the value of the left/right adjoint (of a given functor) on morphisms is fixed once the value on objects is chosen; for the right adjoint of Δ, this yields the above definition of $\varprojlim \alpha$. Thus, C has all limits of shape I iff the right adjoint $\varprojlim$ of Δ can be everywhere defined, yielding

$$\begin{tikzcd} \mathsf{C}^{\mathsf{I}} \arrow[d, shift left, "\varprojlim"] \\ \mathsf{C} \arrow[u, shift left, "\Delta"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

(again in analogy with meets, Section 12).

Natural transformations between diagrams can arise as follows. Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram, $G, H : \mathsf{C} \to \mathsf{D}$ be functors, and $\alpha : G \to H$ be a natural transformation. Then we get a natural transformation

$$\alpha_F : G \circ F \longrightarrow H \circ F : \mathsf{I} \to \mathsf{D}$$

(the whiskering of α across F, see Section 5). Suppose, furthermore, that $\varprojlim F$ exists and is preserved by G, H, so that $(G(\pi_I))_{I \in \mathsf{I}}, (H(\pi_I))_{I \in \mathsf{I}}$ form limit cones witnessing $G(\varprojlim F) = \varprojlim(G \circ F)$ and $H(\varprojlim F) = \varprojlim(H \circ F)$.

$$\begin{tikzcd} G(\varprojlim F) \arrow[r, "G(\pi_I)"] \arrow[d, dashed, "\varprojlim \alpha_F = \alpha_{\varprojlim F}"'] & G(F(I)) \arrow[d, "\alpha_{F(I)}"] \\ H(\varprojlim F) \arrow[r, "H(\pi_I)"'] & H(F(I)) \end{tikzcd}$$

Then $\alpha_F : G \circ F \to H \circ F$ induces a unique morphism $\varprojlim \alpha_F : G(\varprojlim F) \to H(\varprojlim F)$ making the above squares commute. But by the naturality squares for α with respect to the morphisms $\pi_I : \varprojlim F \to F(I)$, $\varprojlim \alpha_F := \alpha_{\varprojlim F}$ works. We have shown:

Proposition 22.12. Natural transformations between limit-preserving functors automatically preserve limits: for a diagram $F : \mathsf{I} \to \mathsf{C}$ with a limit preserved by functors $G, H : \mathsf{C} \to \mathsf{D}$, for any natural transformation $\alpha : G \to H$, we have

$$\alpha_{\varprojlim F} = \varprojlim \alpha_F : G(\varprojlim F) \cong \varprojlim(G \circ F) \longrightarrow \varprojlim(H \circ F) \cong H(\varprojlim F). \qquad \square$$

Recall that a category C is **complete** if it has all small limits. We call a functor $F : \mathsf{C} \to \mathsf{D}$ between complete categories **continuous** if it preserves small limits. Let[22]

$\mathfrak{Lim}$:= 2-category of complete categories, continuous functors, and natural transformations $\subseteq \mathfrak{Cat}$.

[22]Because of Theorem 21.27, we must take $\mathfrak{Lim}$ to consist of "medium" categories, living in the next universe above that in which the small shape categories I live.

This is a **locally full** sub-2-category of $\mathfrak{Cat}$, i.e., each hom-category

$$\mathfrak{Lim}(\mathsf{C},\mathsf{D}) = \{\text{continuous functors } F : \mathsf{C} \to \mathsf{D}\} \subseteq \mathsf{D}^{\mathsf{C}}$$

is a full subcategory. The preceding proposition tells us that it is indeed correct to consider arbitrary natural transformations between continuous functors.

The functoriality of limits can be turned around to give **limits of functors**. Let

$$F : \mathsf{I} \longrightarrow \mathsf{C}^{\mathsf{J}}$$

be a diagram in a functor category. We may equivalently (Proposition 4.10) view F as functors

$$\begin{aligned} \widetilde{F} : \mathsf{I} \times \mathsf{J} &\longrightarrow \mathsf{C} \\ (I, J) &\longmapsto F(I)(J), \end{aligned} \qquad\qquad \begin{aligned} F' : \mathsf{J} &\longrightarrow \mathsf{C}^{\mathsf{I}} \\ J &\longmapsto F(-)(J). \end{aligned}$$

Suppose that for each $J \in \mathsf{J}$, the diagram $F'(J) = F(-)(J) : \mathsf{I} \to \mathsf{C}$ has a limit $\varprojlim^{\mathsf{I}}_{\mathsf{C}} F'(J) = \varprojlim_{I \in \mathsf{I}} F(I)(J) \in \mathsf{C}$. Then for $g : J \to J' \in \mathsf{J}$, we have

$$\varprojlim^{\mathsf{I}}_{\mathsf{C}} F'(g) = \varprojlim_{I \in \mathsf{I}} F(I)(g) : \varprojlim_{I \in \mathsf{I}} F(I)(J) \longrightarrow \varprojlim_{I \in \mathsf{I}} F(I)(J')$$

defined as above, yielding a functor

$$\varprojlim^{\mathsf{I}}_{\mathsf{C}} \circ F' = \varprojlim_{I \in \mathsf{I}} F(I)(-) : \mathsf{J} \longrightarrow \mathsf{C},$$

namely the composite $\mathsf{J} \xrightarrow{F'} \mathsf{C}^{\mathsf{I}} \xrightarrow{\varprojlim^{\mathsf{I}}_{\mathsf{C}}} \mathsf{C}$ (where the right adjoint $\varprojlim^{\mathsf{I}}_{\mathsf{C}}$ only needs to be defined on the essential image of F'); we call this the **pointwise limit** of F. For any $G : \mathsf{J} \to \mathsf{C}$, we have

$$\begin{aligned} \mathsf{C}^{\mathsf{J}}(G, \varprojlim^{\mathsf{I}}_{\mathsf{C}} \circ F') &\cong (\mathsf{C}^{\mathsf{I}})^{\mathsf{J}}(\Delta^{\mathsf{I}}_{\mathsf{C}} \circ G, F') \quad \text{by } \Delta^{\mathsf{I}}_{\mathsf{C}} \dashv \varprojlim^{\mathsf{I}}_{\mathsf{C}} : \mathsf{C}^{\mathsf{I}} \to \mathsf{C} \\ &\cong (\mathsf{C}^{\mathsf{J}})^{\mathsf{I}}(\Delta^{\mathsf{I}}_{\mathsf{C}^{\mathsf{J}}}(G), F) \\ &= \{\text{cones } G \to F\}, \end{aligned}$$

which proves

Proposition 22.13 (pointwise limits)**.** Let $F : \mathsf{I} \to \mathsf{C}^{\mathsf{J}}$ be a diagram in a functor category, and suppose that for each $J \in \mathsf{J}$, the limit $\varprojlim_{I \in \mathsf{I}} F(I)(J)$ exists. Then the pointwise limit $\varprojlim_{I \in \mathsf{I}} F(I)(-)$ is the limit of F in C^{J}. □

Corollary 22.14. If C is complete (or has products, pullbacks, ...), then so is C^{J}, for any J. □

Example 22.15. (Small) limits in any presheaf category $\mathsf{PSh}(\mathsf{C}) = \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ are computed pointwise. Thus when reducing limits in arbitrary C to limits in Set via よ (Corollary 21.6), we mean either pointwise limits or limits in $\mathsf{PSh}(\mathsf{C})$.

Example 22.16. If C has pullbacks, then monomorphisms in C^{I} are precisely the pointwise monic (in C) natural transformations. For example, monomorphisms in $\mathsf{PSh}(\mathsf{C})$ are precisely the pointwise injective natural transformations (cf. Exercise 9.7).

Exercise 22.17. Verify that for any I, C, the forgetful functor $\mathsf{C}^{\mathsf{I}} \to \mathsf{C}^{\mathsf{I}_0}$ (forgetting the morphisms in a diagram) preserves and reflects arbitrary limits.

Remark 22.18. It is possible for C to lack certain limits but C^{J} to still have them "by accident". For a trivial example, take $\mathsf{C} := \mathsf{J} := \varnothing$.

Now given $F : \mathsf{I} \to \mathsf{C}^{\mathsf{J}}$ as above, equivalently $\widetilde{F} : \mathsf{I} \times \mathsf{J} \to \mathsf{C}$, such that F has a pointwise limit $\varprojlim^{\mathsf{I}} F = \varprojlim_{I \in \mathsf{I}} \widetilde{F}(I, -) : \mathsf{J} \to \mathsf{C}$, we may ask whether this J-shaped diagram itself has a limit. Comparing this limit with $\varprojlim \widetilde{F}$ yields a "Fubini theorem for limits":

Proposition 22.19. Let $F : \mathsf{I} \times \mathsf{J} \to \mathsf{C}$ be a doubly-indexed diagram such that for each $J \in \mathsf{J}$, the limit $\varprojlim_{I \in \mathsf{I}} F(I, J)$ exists. Then

$$\varprojlim_{J \in \mathsf{J}} \varprojlim_{I \in \mathsf{I}} F(I, J) \cong \varprojlim_{(I,J) \in \mathsf{I} \times \mathsf{J}} F(I, J),$$

either side existing if the other does. Given the LHS, the projections from the RHS are given by

$$\pi_{(I,J)} := \pi_I \circ \pi_J : \varprojlim_{J'} \varprojlim_{I'} F(I', J') \longrightarrow F(I, J);$$

given the RHS, the projections from the LHS are given by

$$\pi_J := (\pi_{(I,J)})_I : \varprojlim_{(I,J')} F(I, J') \longrightarrow \varprojlim_I F(I, J).$$

Proof. In Set, we have

$$\begin{aligned}
\varprojlim_J \varprojlim_I F(I, J) &= \{((x_{I,J})_I)_J \in \textstyle\prod_J \varprojlim_I F(I, J) \mid \forall g : J \to J' \in \mathsf{J}\,((\varprojlim_I F(I, g))((x_{I,J})_I) = (x_{I,J'})_I)\} \\
&= \{((x_{I,J})_I)_J \in \textstyle\prod_J \varprojlim_I F(I, J) \mid \forall g : J \to J' \in \mathsf{J}\,((F(I, g)(x_{I,J}))_I = (x_{I,J'})_I)\} \\
&= \{((x_{I,J})_I)_J \in \textstyle\prod_J \prod_I F(I, J) \mid \forall g : J \to J' \in \mathsf{J}\,((F(I, g)(x_{I,J}))_I = (x_{I,J'})_I) \ \& \\
&\qquad\qquad \forall J \in \mathsf{J}\, \forall f : I \to I' \in \mathsf{I}\,(F(f, J)(x_{I,J}) = x_{I',J})\} \\
&\cong \{(x_{I,J})_{I,J} \in \textstyle\prod_{I,J} F(I, J) \mid \forall (f, g) : (I, J) \to (I', J') \in \mathsf{I} \times \mathsf{J}\,(F(f, g)(x_{I,J}) = x_{I',J'})\} \\
&= \varprojlim_{(I,J)} F(I, J)
\end{aligned}$$

where the second-last step uses that $(f, g) : (I, J) \to (I', J') \in \mathsf{I} \times \mathsf{J}$ may be written as $(f, 1_{J'}) \circ (1_I, g)$; the projections are clearly as claimed. The general case follows from Yoneda. $\square$

Composing two of these isomorphisms, we get

Corollary 22.20 (limits commute)**.** Let $F : \mathsf{I} \times \mathsf{J} \to \mathsf{C}$ be a doubly-indexed diagram. Then

$$((\pi_I \circ \pi_J)_J)_I : \varprojlim_J \varprojlim_I F(I, J) \cong \varprojlim_I \varprojlim_J F(I, J),$$

assuming both sides exist. $\square$

Note that for $F : \mathsf{I} \to \mathsf{C}^{\mathsf{J}}$ corresponding to $\widetilde{F} : \mathsf{I} \times \mathsf{J} \to \mathsf{C}$,

$$((\pi_I \circ \pi_J)_J)_I = (\varprojlim_J (\pi_I : \varprojlim_{I'} \widetilde{F}(I', J) \to \widetilde{F}(I, J)))_I : \varprojlim_J \varprojlim_I \widetilde{F}(I, J) \to \varprojlim_I \varprojlim_J \widetilde{F}(I, J)$$

is the comparison morphism for the functor $\varprojlim^{\mathsf{J}}_{\mathsf{C}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}$ to preserve the pointwise limit $\varprojlim_I \widetilde{F}(I, -) = \varprojlim F \in \mathsf{C}^{\mathsf{J}}$. Thus, the above may be restated as

Corollary 22.21 (limits preserve limits)**.** For any J, C, the limit operation $\varprojlim^{\mathsf{J}}_{\mathsf{C}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}$ preserves pointwise limits in C^{J}. $\square$

Commutation of limits is analogous to commutation of meets in posets (Proposition 12.1), and as in Section 18, can be thought of as saying that "$\varprojlim$ is a $\varprojlim$-homomorphism", which as in Hofmann–Mislove–Stralka duality (Exercise 19.10) will be a key ingredient in the duality theorems for first-order logic that we will consider.

Example 22.22. For $\mathsf{I} = \mathsf{J} = \mathbb{N}^{\mathrm{op}}$ (Example 21.3), the above says that the limit of a diagram $F : \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \to \mathsf{C}$, i.e., a "double inverse sequence", may be computed as either the inverse limit of the inverse limits of rows, or the inverse limit of inverse limits of columns:

$$
\begin{tikzcd}
\varprojlim_j \varprojlim_i F(i,j) \cong \varprojlim_i \varprojlim_j F(i,j) \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] & \cdots \arrow[r, dashed] & \varprojlim_j F(2,j) \arrow[r, dashed] \arrow[d, "\pi_j"'] & \varprojlim_j F(1,j) \arrow[r, dashed] \arrow[d, "\pi_j"'] & \varprojlim_j F(0,j) \arrow[d, "\pi_j"'] \arrow[dd, bend left, "\pi_2"] \arrow[ddd, bend left, "\pi_1"] \arrow[dddd, bend left=40, "\pi_0"] \\
\vdots \arrow[d, dashed] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\varprojlim_i F(i,2) \arrow[r, "\pi_i"] \arrow[d, dashed] & \cdots \arrow[r] & F(2,2) \arrow[r] \arrow[d] & F(1,2) \arrow[r] \arrow[d] & F(0,2) \arrow[d] \\
\varprojlim_i F(i,1) \arrow[r, "\pi_i"] \arrow[d, dashed] & \cdots \arrow[r] & F(2,1) \arrow[r] \arrow[d] & F(1,1) \arrow[r] \arrow[d] & F(0,1) \arrow[d] \\
\varprojlim_i F(i,0) \arrow[r, "\pi_i"] \arrow[rr, bend right=10, "\pi_2"'] \arrow[rrr, bend right=15, "\pi_1"'] \arrow[rrrr, bend right=20, "\pi_0"'] & \cdots \arrow[r] & F(2,0) \arrow[r] & F(1,0) \arrow[r] & F(0,0)
\end{tikzcd}
$$

23 Lex categories, subobjects and relations

The adjective "lex" (historically short for "left exact") is used to describe categorical notions related to finite limits. A **lex category** (or **finitely complete category**) is one which has all finite limits. In such categories, we may work with many familiar notions lifted from Set via the Yoneda embedding, e.g., relations and graphs of functions.

In an arbitrary category C, for an object $X \in \mathsf{C}$, we preorder the monomorphisms into X via

$$(f : A \to X) \subseteq (g : B \to X) \;:\Longleftrightarrow\; \exists h : A \to B\,(f = g \circ h).$$

$$
\begin{tikzcd}
A \arrow[r, dashed, "h"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\
& X
\end{tikzcd}
$$

Since g is monic, such h is unique if it exists; since f is monic, h must also be monic (Proposition 9.10(d)). The induced equivalence relation identifies f, g iff there is $h : A \to B$ as above which is an isomorphism (by uniqueness). A **subobject** of X is an equivalence class of monomorphisms into X. We let

$$\mathrm{Sub}(X) := \mathrm{Sub}_{\mathsf{C}}(X) := \{\text{subobjects of } X\},$$

partially ordered by $\subseteq$ (descended to the quotient).

It is common to abuse terminology and notation regarding subobjects:

- We usually denote a subobject of X by $A \subseteq X$, thought of as the domain of some monomorphism $A \to X$ in the equivalence class.
- The monomorphism $A \to X$ is left unnamed, or referred to as the **inclusion** $A \hookrightarrow X$.
- Likewise, for two subobjects $A, B \subseteq X$ such that $A \subseteq B$, the unique monomorphism $h : A \to B$ commuting with the inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$ is called the **inclusion** $A \hookrightarrow B$.
- We often treat a single monomorphism $f : A \to X$ as a subobject (its equivalence class).

Each subobject poset $\mathrm{Sub}(X)$ always has a greatest element

$$\top_{\mathsf{Sub}(X)} = X \quad (\text{i.e., } 1_X : X \to X).$$

Now suppose that C has pullbacks. Recall (Example 21.15) that pullback preserves monomorphisms. Thus, each morphism $f : X \to Y \in \mathsf{C}$ induces a map

$$f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$$

given by pullback:

$$\begin{tikzcd} f^*(A) \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

It follows from Yoneda, or an easy direct calculation, that f^* is order-preserving, and that $f \mapsto f^*$ is functorial. Moreover, each $\mathrm{Sub}(X)$ also has binary meets or **intersections**, given by pullbacks:

$$\begin{tikzcd} A \cap B := A \times_X B \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & B \arrow[d, hook] \\ A \arrow[r, hook] & X \end{tikzcd}$$

Again by Yoneda or direct calculation, each $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ preserves meets. We thus get a functor

$$\mathrm{Sub} = \mathrm{Sub}_{\mathsf{C}} : \mathsf{C}^{\mathrm{op}} \longrightarrow \wedge\mathsf{Lat}.$$

Example 23.1. $\mathrm{Sub}_{\mathsf{Set}}$ is (naturally isomorphic to) the contravariant powerset functor $\mathcal{P} : \mathsf{Set}^{\mathrm{op}} \to \wedge\mathsf{Lat}$.

Likewise, $\mathrm{Sub}_{\mathsf{Grp}}$ (say) takes a group to its $\wedge$-lattice of subgroups (up to natural isomorphism).

Example 23.2. In Pos, a monomorphism is an arbitrary injective monotone map, i.e., up to isomorphism, the inclusion of a subset equipped with possibly a finer partial order. Thus $\mathrm{Sub}_{\mathsf{Pos}}(P) \cong \{(Q, \le_Q) \mid Q \subseteq P \;\&\; (\le_Q) \subseteq (\le_P)\}$.

Likewise, for a topological space X, $\mathrm{Sub}_{\mathsf{Top}}(X)$ consists of subsets equipped with possibly a finer topology.

Remark 23.3. In order to make sense of $\mathrm{Sub}_{\mathsf{C}}$, C need not be small, but each of its subobject posets must have small cardinality; C is called **well-powered** in this case.

Now let C be a lex category. The **graph** of a morphism $f : X \to Y \in \mathsf{C}$ is

$$\mathrm{graph}(f) := (1_X, f) : X \to X \times Y.$$

Since $(1_X, f)$ has a retraction $\pi_1 : X \times Y \to X$, it is monic, hence can be regarded as a subobject of $X \times Y$. Of course, we call such subobjects **binary relations**.

Proposition 23.4.

(a) For $f, g : X \to Y$, if $\mathrm{graph}(f) \subseteq \mathrm{graph}(g)$, then $f = g$.

(b) $R \subseteq X \times Y$ is $\mathrm{graph}(f)$ for some (unique) f iff the composite

$$R \hookrightarrow X \times Y \xrightarrow{\pi_1} X$$

is an isomorphism.

(c) $R \subseteq X \times Y$ is $\mathrm{graph}(f)$ for an isomorphism f iff the composite

$$R \hookrightarrow X \times Y \xrightarrow{\pi_2} Y$$

is also an isomorphism.

(d) $\mathrm{graph}(1_X)$ is the diagonal $\Delta_X := (1_X, 1_X) : X \hookrightarrow X \times X$.

(e) For $f : X \to Y$ and $g : Y \to Z$, $\mathrm{graph}(g \circ f) \hookrightarrow X \times Z$ is the composite $(\mathrm{graph}(f) \times Z) \cap (X \times \mathrm{graph}(g)) \to X \times Z$ in the following diagram, where each of the squares is a pullback:

$$\begin{tikzcd}[column sep=tiny]
 & & & (\mathrm{graph}(f) \times Z) \cap (X \times \mathrm{graph}(g)) \arrow[dl, hook] \arrow[dr, hook] & & & \\
 & & \mathrm{graph}(f) \times Z \arrow[dl] \arrow[dr, hook] & & X \times \mathrm{graph}(g) \arrow[dl, hook] \arrow[dr] & & \\
 & \mathrm{graph}(f) \arrow[dr, hook] & & X \times Y \times Z \arrow[dl, "\pi_{12}=(\pi_1,\pi_2)"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}=(\pi_2,\pi_3)"] & & \mathrm{graph}(g) \arrow[dl, hook] & \\
 & & X \times Y & X \times Z & Y \times Z & &
\end{tikzcd}$$

(In particular, said composite is monic.)

Proof. By Yoneda and the corresponding familiar facts in Set. □

A **lex functor** $F : \mathsf{C} \to \mathsf{D}$ between lex categories is one which preserves finite limits. Let

$$\mathfrak{Lex} := \text{2-category of lex categories, lex functors, natural transformations.}$$

(As in Proposition 22.12, natural transformations between lex functors are automatically "lex".)

Since monomorphisms are a limit notion (Example 21.16), lex functors preserve monomorphisms. Thus, a lex functor $F : \mathsf{C} \to \mathsf{D}$ induces, for each $X \in \mathsf{C}$, a map

$$F = F_{\mathrm{Sub}(X)} : \mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(F(X)),$$

which is a $\wedge$-lattice homomorphism since F also preserves pullbacks.

Proposition 23.5. A lex functor $F : \mathsf{C} \to \mathsf{D}$ is conservative iff each $F_{\mathrm{Sub}(X)}$ is injective.

Proof. $\Longrightarrow$: For any $A \subseteq B \subseteq X$, if $F(A) = F(B) \subseteq F(X)$, then by conservativity, $A = B \subseteq X$. Replacing A with $A \cap B$, we see that if $F(A) \cap F(B) = F(A \cap B) = F(B) \subseteq F(X)$, i.e., $F(A) \supseteq F(B)$, then $A \cap B = B \subseteq X$, i.e., $A \supseteq B$.

$\Longleftarrow$: Let $f : X \to Y \in \mathsf{C}$ such that $F(f)$ is an isomorphism. The kernel $\ker(f) = X \times_Y X$ can be regarded as a subobject of $X \times X$ as in Proposition 21.14. Since F is lex, we have $F(\ker(f)) = \ker(F(f)) \subseteq F(X) \times F(X) \cong F(X \times X)$. Since $F(f)$ is an isomorphism, $\ker(F(f))$ is the diagonal $\Delta_{F(X)} = (1_{F(X)}, 1_{F(X)}) : F(X) \hookrightarrow F(X) \times F(X)$, which is $F(\Delta_X)$ again since F is lex. Thus since $F_{\mathrm{Sub}(X \times X)}$ is injective, $\ker(f) \subseteq X \times X$ is the diagonal, i.e., f is monic. Now regarding f as a subobject $X \subseteq Y$, since $F(f)$ is an isomorphism, $F(X) = F(Y)$ as subobjects of $F(Y)$, whence $X = Y$ as subobjects of Y, i.e., f is an isomorphism. □

Corollary 23.6. Lex conservative functors are faithful.

Proof. By Proposition 23.5 and Proposition 23.4(a). □

Exercise 23.7. Show that in fact, if $F : \mathsf{C} \to \mathsf{D}$ is conservative, C has equalizers, and F preserves equalizers, then F is faithful.

Remark 23.8. In particular, for right adjoint functors whose domain has finite limits (or just equalizers), conservativity implies faithfulness. This includes many forgetful functors.

Since both conservativity and faithfulness are invariant under flipping morphisms around, conservativity implies faithfulness also for *left* adjoint functors whose domains have *coequalizers* (Example 24.5), e.g., any free functor from Set.

A lex functor $F : \mathsf{C} \to \mathsf{D}$ is called **full on subobjects** if each $F_{\mathrm{Sub}(X)}$ is surjective (whereas conservativity = "faithful on subobjects").

Proposition 23.9. Lex conservative full-on-subobjects functors are full.

Proof. Let $F : \mathsf{C} \to \mathsf{D}$ be such a functor. For a morphism $g : F(X) \to F(Y)$, by fullness on subobjects, there is a binary relation $R \subseteq X \times Y$ such that $F(R) = \mathrm{graph}(g) \subseteq F(X) \times F(Y) \cong F(X \times Y)$. By Proposition 23.4(b), the composite $F(R) \hookrightarrow F(X) \times F(Y) \xrightarrow{\pi_1} F(X)$ is an isomorphism; thus since F is conservative (and lex), the composite $R \hookrightarrow X \times Y \xrightarrow{\pi_1} X$ is an isomorphism, i.e., $R = \mathrm{graph}(f)$ for some $f : X \to Y$, again by Proposition 23.4(b). Again since F is lex, $\mathrm{graph}(F(f)) = F(\mathrm{graph}(f)) = F(R) = \mathrm{graph}(g)$, whence $F(f) = g$ by Proposition 23.4(a). □

Corollary 23.10. A lex, conservative, full-on-subobjects, essentially surjective functor is an equivalence. $\square$

Thus for lex functors, we have an alternative to the full/faithful hierarchy (Section 6), based on subobjects instead of morphisms. The two hierarchies are interleaved as follows:

$$
\begin{tikzcd}
\text{subobjects:} & & \text{cons.+full-on-subs} \arrow[ddr, Rightarrow] & & \text{conservative} \arrow[ddr, Rightarrow] & & \\
& \text{equiv.} \arrow[ur, Rightarrow] & & & & & \text{any} \\
\text{morphisms:} & & & \text{full+faithful} \arrow[uur, Rightarrow] & & \text{faithful} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

24 Colimits

The **colimit** of a diagram $F : \mathsf{I} \to \mathsf{C}$ is the limit of $F : \mathsf{I}^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$, consisting of an object

$$\varinjlim F = \varinjlim_{I \in \mathsf{I}} F(I) \in \mathsf{C}$$

together with bijections

$$\mathsf{C}(\varinjlim F, Z) \cong \varprojlim_{I \in \mathsf{I}^{\mathrm{op}}} \mathsf{C}(F(I), Z),$$

natural in Z. An element of the right-hand side is called a **cocone** from F to Z, consisting of a family of morphisms

$$(h_I : F(I) \longrightarrow Z)_{I \in \mathsf{I}} \quad \text{such that} \quad h_I = h_J \circ F(f) \quad \forall f : I \to J \in \mathsf{I}.$$

Thus, a colimit is equivalently an object $\varinjlim F \in \mathsf{C}$ together with a universal cocone

$$(\iota_I : F(I) \longrightarrow \varinjlim F)_{I \in \mathsf{I}} \quad \text{such that} \quad \iota_I = \iota_J \circ F(f) \quad \forall f : I \to J \in \mathsf{I},$$

such that for any other cocone $(Z, (h_I)_I)$, there is a unique morphism

$$[h_I]_I : \varinjlim F \longrightarrow Z$$

such that

$$[h_I]_I \circ \iota_I = h_I \quad \forall I \in \mathsf{I}.$$

The ι_I are called the **injections** (but as with limit “projections”, they are not necessarily “injective” in any sense; in particular, they need not be monomorphisms).

TODO(picture)

Remark 24.1. Note that colimits in C reduce, via $\text{よ}_{\mathsf{C}^{\mathrm{op}}}$, to *limits* in Set. In particular, general properties of colimits cannot be deduced from properties of colimits in Set “by Yoneda”. Especially, compatibility conditions *between limits and colimits* do not follow from familiar properties in Set (see Example 24.3, **TODO** below). Indeed, categorical logic is in a sense all about determining “natural” compatibility conditions between limits and colimits, and then proving that they imply all others that hold in a familiar category like Set (analogous to the PIT for distributive lattices).

Example 24.2 (coproducts)**.** When I is discrete, $\varinjlim F$ is called the **coproduct** $\coprod_{I \in \mathsf{I}} F(I)$. When $\mathsf{I} = 2$, the coproduct $X \sqcup Y$ is an object equipped with a universal pair of morphisms

$$\iota_1 : X \longrightarrow X \sqcup Y, \qquad \iota_2 : Y \longrightarrow X \sqcup Y,$$

so that for any other Z with a pair of morphisms $f : X \to Z$ and $g : Y \to Z$, there is a unique $[f, g] : X \sqcup Y \to Z$ making

$$\begin{tikzcd} X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commute. When $\mathsf{I} = \varnothing$, the coproduct is an **initial object**, denoted $\varnothing_{\mathsf{C}}$, with a unique morphism $\varnothing_{\mathsf{C}} \to Z$ to any other object Z.

In Set, coproduct is disjoint union, usually constructed as

$$\textstyle\coprod_{i \in I} X_i := \{(i, x) \mid i \in I \ \& \ x \in X_i\}$$

with injections

$$\begin{aligned} \iota_i : X_i &\longrightarrow \textstyle\coprod_i X_i \\ x &\longmapsto (i, x). \end{aligned}$$

Maps $f_i : X_i \to Z$ induce

$$\begin{aligned} [f_i]_i : \textstyle\coprod_i X_i &\longrightarrow Z \\ (i, x) &\longmapsto f_i(x). \end{aligned}$$

(In practice, we often assume the X_i are disjoint and just take the union; this causes problems when e.g., considering $X \sqcup X$.)

Example 24.3. In CRing, the initial object is $\mathbb{Z}$, and coproduct is given by tensor product. Note that the unique morphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is not monic, nor is the coproduct injection $\iota_1 : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} \cong \{0\}$. Thus, coproducts need not interact with limits (e.g., monomorphisms) the same way as in Set.

Example 24.4 (pushouts and epimorphisms)**. TODO**

Example 24.5 (coequalizers)**.** The colimit of an $\mathsf{I} = \{\bullet \rightrightarrows \bullet\}$-shaped diagram, i.e., a parallel pair

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y,$$

is called the **coequalizer** $\operatorname{coeq}(f, g)$, and is equipped with a universal morphism $\iota : Y \to \operatorname{coeq}(f, g)$ such that $\iota \circ f = \iota \circ g$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\iota"] \arrow[dr, dashed, "h"'] & \operatorname{coeq}(f,g) \arrow[d, dashed, "{\exists! [h]}"] \\ & & Z \end{tikzcd}$$

In Set, $\operatorname{coeq}(f, g)$ can be constructed as the quotient

$$\operatorname{coeq}(f, g) := Y/{\sim}$$

by the equivalence relation $\sim$ generated by

$$f(x) \sim g(x) \quad \forall x \in X.$$

The "injection" $\iota : Y \to \operatorname{coeq}(f, g)$ is the quotient projection. Given $h : Y \to Z$, the induced $[h] : \operatorname{coeq}(f, g) \to Z$ descends from h:

$$\begin{aligned} [h] : \operatorname{coeq}(f, g) &\longrightarrow Z \\ [y]_\sim &\longmapsto h(y), \end{aligned}$$

which is well-defined since $h \circ f = h \circ g$ means that $\ker(h)$ contains the generating pairs of $\sim$.

Applying Proposition 21.12 to $\mathsf{Set}^{\mathrm{op}}$, we get

Corollary 24.6. The colimit of any small diagram $F : \mathsf{I} \to \mathsf{Set}$ may be constructed as

$$\varinjlim F = \left(\bigsqcup_{I \in \mathsf{I}} F(I)\right)/{\sim}$$

where $\sim$ is the equivalence relation generated by

$$\iota_I(x) \sim \iota_J(F(f)(x)) \quad \forall f : I \to J \in \mathsf{I}$$

where $\iota_I : F(I) \to \bigsqcup_J F(J)$ are the coproduct injections; the colimit injections $F(I) \to \varinjlim F$ are the composites of ι_I with the quotient map. $\square$

Example 24.7 (directed colimits)**.** Let I be a directed poset, e.g., $\mathbb{N}$. Label the unique morphism $I \to J \in \mathsf{I}$ for $I \leq J$ as (I, J), so that $F(I, J)$ denotes the corresponding morphism in the diagram. Then the above equivalence relation $\sim$ on $\bigsqcup_{I \in \mathsf{I}} F(I)$ is given by

$$\iota_I(x) \sim \iota_J(y) \iff \exists K \geq I, J\, (F(I, K)(x) = F(J, K)(y)).$$

Indeed, if such K exists, then we have $\iota_I(x) \sim F(I, K)(x) = F(J, K)(y) \sim \iota_J(y)$; and the RHS defines an equivalence relation, since if $\iota_I(x) \sim \iota_J(y) \sim \iota_K(z)$ as witnessed by $I \leq L \geq J \leq K \geq M$, then there is $N \geq L, M$ by directedness, which witnesses $\iota_I(x) \sim \iota_K(z)$.

$$\begin{tikzcd}[column sep=small]
 & & & F(I,N)(x) = F(L,N)(F(J,L)(y)) = F(M,N)(F(J,M)(y)) = F(K,N)(z) \in F(N) & & & \\
 & F(I,L)(x) = F(J,L)(y) \in F(L) \arrow[urr, mapsto, "F(L,N)"] & & & & F(J,M)(y) = F(K,M)(z) \in F(M) \arrow[ull, mapsto, "F(M,N)"'] & \\
x \in F(I) \arrow[ur, mapsto, "F(I,L)"] & & & y \in F(J) \arrow[ull, mapsto, "F(J,L)"'] \arrow[urr, mapsto, "F(J,M)"] & & & z \in F(K) \arrow[ul, mapsto, "F(K,M)"']
\end{tikzcd}$$

In this case $\varinjlim F$ is traditionally called the **direct limit** of F, though the name **directed colimit** fits better with the general categorical terminology.

TODO(filtered colimits, final functors, commutativity with finite limits)

Part IV
First-order logic

The following table depicts the roles of categories, structures, etc. in first-order logic as compared to propositional logic:

	propositional logic	first-order logic
values	$\boxed{0, 1}$ −1-dim'l	sets, functions, relations
		0-dim'l
$\{\text{values}\}$	$2 = \{0, 1\}$	Set
$\mathrm{Mod}(\mathcal{L}, \mathcal{T})$	Stone space $\subseteq 2^{\mathcal{L}}$	ultracategory $\subseteq \mathsf{Set}^{\langle \mathcal{L} \rangle}$
$\langle \mathcal{L} \mid \mathcal{T} \rangle$	Boolean algebra $\subseteq 2^{\mathrm{Mod}(\mathcal{L}, \mathcal{T})}$	pretopos $\subseteq \mathsf{Set}^{\mathsf{Mod}(\mathcal{L}, \mathcal{T})}$
		1-dim'l
$\{\text{theories}\}$	$\mathsf{Bool} \simeq \mathsf{Stone}^{\mathrm{op}}$	$\mathfrak{Pretop} \hookrightarrow \mathfrak{UltCat}^{\mathrm{op}}$
		2-dim'l
$\{\text{logics}\}$	$\subseteq \mathfrak{Cat}$	$\boxed{\subseteq \text{2-cat?}}$ 3-dim'l

The "values" of each logic are "that which the syntax denotes". In propositional logic, the syntax consists of propositional formulas, which denote truth values. The collection of all values is then the set $2 = \{0, 1\}$. A propositional theory $(\mathcal{L}, \mathcal{T})$ can be represented as either the space $\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \subseteq 2^{\mathcal{L}}$ of all models of $\mathcal{T}$, a Stone space, or as the Lindenbaum–Tarski Boolean algebra $\langle \mathcal{L} \mid \mathcal{T} \rangle$ which can be thought of as continuous assignments of truth values to models (by Stone duality); both of these are sets equipped with structure induced by 2. So we can regard the category Bool of Boolean algebras, or its dual, the category Stone of Stone spaces, as the category of propositional theories (and interpretations, as in Section 17). Comparing different fragments of propositional logic (Section 20), each can be represented as a corresponding category of algebras. So we can regard the 2-category of categories (of "sufficiently nice" structures) as the "2-category of propositional logics".

In first-order logic, the syntax instead denotes sets, functions, and relations, which are collected together in the (lex) category Set. Thus for a first-order theory $(\mathcal{L}, \mathcal{T})$, its models form a subcategory $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ of a power of Set (equipped with an abstract notion of "ultraproduct", turning it into an "ultracategory"; see **TODO**), while the first-order analog of the Lindenbaum–Tarski algebra $\langle \mathcal{L} \mid \mathcal{T} \rangle$ will likewise be a subalgebra of a power of Set (a "pretopos", **TODO**) which can be thought of as "continuous" assignments of sets to models (by Makkai duality, **TODO**). The collection of all such categories equipped with additional structure then forms a 2-category, which can be regarded as the 2-categories of first-order theories. We can then regard the "3-category" (which we will not discuss at all) of all such 2-categories of categorical structures as the "3-category of first-order logics".

Each row of the table collects together all objects of the same type as in the previous row, and itself forms an object of one higher "dimension";[23] e.g., sets are 0-dimensional discrete collections of points, and are collected together in the category Set. In first-order logic, everything is of one *higher* dimension than in propositional logic; in other words, the same type of object (e.g., a category) represents a *lower*-level concept than in propositional logic (a single first-order theory, versus the collection of all propositional theories).

25 First-order languages and structures

A **(multi-sorted finitary) first-order language** $\mathcal{L}$ is a set of symbols, classified into three subsets $\mathcal{L} = \mathcal{L}_{\text{sort}} \sqcup \mathcal{L}_{\text{fun}} \sqcup \mathcal{L}_{\text{rel}}$ consisting of:

- **sort symbols** $P \in \mathcal{L}_{\text{sort}}$;
- **function sybols** $f \in \mathcal{L}_{\text{fun}}$, each of which has an associated finite tuple of sorts called its **arity** $\mathrm{ar}(f) \in \mathcal{L}_{\text{sort}}^{<\mathbb{N}} = \bigsqcup_{n\in\mathbb{N}} \mathcal{L}_{\text{sort}}^n$, as well as an associated **value sort** $\mathrm{val}(f) \in \mathcal{L}_{\text{sort}}$; we write
$$f : (P_1, \dots, P_n) \longrightarrow Q$$
to mean $\mathrm{ar}(f) = (P_1, \dots, P_n)$ and $\mathrm{val}(f) = Q$, and let
$$\mathcal{L}_{\text{fun}}(\vec{P}; Q) := \{f \in \mathcal{L}_{\text{fun}} \mid f : \vec{P} \to Q\};$$
- **relation symbols** $R \in \mathcal{L}_{\text{rel}}$, each of which has an associated finite tuple of sorts called its **arity** $\mathrm{ar}(f) \in \mathcal{L}_{\text{sort}}^{<\mathbb{N}}$; we write
$$\mathcal{L}_{\text{rel}}(\vec{P}) := \{R \in \mathcal{L}_{\text{rel}} \mid \mathrm{ar}(R) = \vec{P}\}.$$

Intuitively, sorts are names for different underlying sets of structures.

Example 25.1. Any single-sorted finitary first-order language $\mathcal{L}$, as defined in Section 18 (where *finitary* means all the arities are finite, not necessarily that $\mathcal{L}$ is itself finite), can be regarded as having $\mathcal{L}_{\text{sort}} := \{*\}$, where $*$ is thought of as naming the single underlying set of $\mathcal{L}$-structures. In this case, we usually treat arities as natural numbers $n \in \mathbb{N}$ instead of tuples $(*, \dots, *) \in \{*\}^n$.

Example 25.2. The definition of categories given in Section 2 does not directly correspond to a multi-sorted language, since the hom-sets $\mathsf{C}(X, Y)$ of a category C are parametrized by elements X, Y of another sort (the one naming objects).[24] However, a **category** C can alternatively be defined as:

- sets C_0 and $\mathsf{C}_1 = \bigsqcup_{X,Y\in\mathsf{C}_0} \mathsf{C}(X, Y)$ of **objects** and **morphisms**;

[23] Technically, "truncatedness" would be the more correct homotopy-theoretic terminology. For example, the 2-sphere is 2-dimensional but not 2-truncated; regarded as an ∞-groupoid, it would not be equivalent to a 2-groupoid.

[24] Makkai's **first-order logic with dependent sorts (FOLDS)** is a version of first-order logic that provides sorts parametrized by other sorts, and is a small fragment of higher-order **dependent type theory**.

- **source** and **target** maps $\sigma, \tau : \mathsf{C}_1 \to \mathsf{C}_0$;
- an **identity** map $1_{(-)} : \mathsf{C}_0 \to \mathsf{C}_1$;
- a **composition** map $\circ : \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 = \{(f, g) \mid \sigma(f) = \tau(g)\} \to \mathsf{C}_1$;
- satisfying certain axioms: $\sigma(f \circ g) = \sigma(g)$, $1_X \circ f = f$, etc.

This definition can be formalized via the **language of categories $\mathcal{L}_{\mathsf{Cat}}$**, consisting of:

- two sorts $\mathrm{Ob}, \mathrm{Mor}$, naming $\mathsf{C}_0, \mathsf{C}_1$ in a category C;
- two function symbols $\sigma, \tau : \mathrm{Mor} \to \mathrm{Ob}$;
- a function symbol $1_{(-)} : \mathrm{Ob} \to \mathrm{Mor}$;
- a *relation* symbol $\mathrm{graph}(\circ)$ of arity $(\mathrm{Mor}, \mathrm{Mor}, \mathrm{Mor})$, naming the graph of $\circ$, which we are forced to use since $\circ$ is not a *total* operation.

Example 25.3. A 0-*sorted* language $\mathcal{L}$, with $\mathcal{L}_{\mathrm{sort}} = \varnothing$, necessarily has $\mathcal{L}_{\mathrm{fun}} = \varnothing$ as well, since there are no possible value sorts; thus it only consists of relation symbols, all of which must have arity $\varnothing$ (the empty sequence of sorts), i.e., $\mathcal{L}$ is the same thing as a *propositional* language, as defined in Section 15.

Let $\mathcal{L}$ be a language, C be a lex category; recall (Section 23) that C has well-behaved notions of *subobjects* and *relations*. An **$\mathcal{L}$-structure (model of $\mathcal{L}$) $\mathcal{M}$ in** C consists of:

- for each sort symbol $P \in \mathcal{L}_{\mathrm{sort}}$, an **interpretation of P in $\mathcal{M}$** as an object
$$P^{\mathcal{M}} = \mathcal{M}(P) \in \mathsf{C}$$
(usually denoted M when $\mathcal{L}$ is one-sorted);
- for each function symbol $f : (P_1, \dots, P_n) \to Q \in \mathcal{L}_{\mathrm{fun}}$, an **interpretation of f in $\mathcal{M}$** as a morphism
$$f^{\mathcal{M}} = \mathcal{M}(f) : P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}} \longrightarrow Q^{\mathcal{M}};$$
- for each relation symbol $R \in \mathcal{L}_{\mathrm{rel}}(P_1, \dots, P_n)$, an **interpretation of R in $\mathcal{M}$** as a subobject
$$R^{\mathcal{M}} \subseteq P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}}.$$

Example 25.4. An $\mathcal{L}_{\mathsf{Grp}} = \{\cdot, 1, {}^{-1}\}$-structure $\mathcal{M}$ in C is an object $M \in \mathsf{C}$ equipped with morphisms
$$\begin{aligned} \cdot^{\mathcal{M}} &: M \times M \longrightarrow M, \\ 1^{\mathcal{M}} &: \mathbf{1}_{\mathsf{C}} \longrightarrow M, \\ {}^{-1\mathcal{M}} &: M \longrightarrow M. \end{aligned}$$

(As in Example 18.1, the group axioms are not yet taken into account.)

Example 25.5. An $\mathcal{L}_{\mathsf{Pos}} = \{\leq\}$-structure $\mathcal{M}$ in C is an object $M \in \mathsf{C}$ equipped with a binary relation $\leq^{\mathcal{M}} \subseteq M^2$.

For example, an $\mathcal{L}_{\mathsf{Pos}}$-structure $\mathcal{M}$ in Pos is a poset M equipped with a subset $\leq^{\mathcal{M}} \subseteq M^2$ which is itself equipped with a finer partial order $\leq_{\leq^{\mathcal{M}}}$ than the product partial order on M^2 (recall the characterization of subobjects in Pos from Example 23.2).

Example 25.6. An $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{M}$ in C is two objects $M_0 = \mathrm{Ob}^{\mathcal{M}}$, $M_1 = \mathrm{Mor}^{\mathcal{M}} \in \mathsf{C}$ equipped with morphisms $\sigma^{\mathcal{M}}, \tau^{\mathcal{M}} : M_1 \to M_0$, $\iota : M_0 \to M_1$, and a binary relation $\mathrm{graph}(\circ)^{\mathcal{M}} \subseteq M_1^3$ (not yet required to be any kind of function graph).

Example 25.7. For propositional $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$ in C consists of, for each $P \in \mathcal{L} = \mathcal{L}_{\mathrm{rel}}$, a **subterminal object** $P^{\mathcal{M}} \subseteq \mathbf{1}_{\mathsf{C}}$.

When $\mathsf{C} = \mathsf{Set}$, these can be identified with truth values $0, 1 \subseteq 1$, thus recovering the definition of **model of $\mathcal{L}$** from Section 15.

When C is a $\wedge$-lattice regarded as a lex category, the terminal object is $\top \in \mathsf{C}$, and subobjects are arbitrary elements of C (all morphisms being monic), thus recovering the definition of **model of $\mathcal{L}$ in a $\wedge$-lattice** from Section 20. In fact, for a general lex category C, $\mathcal{L}$-structures in C are the same thing as models of $\mathcal{L}$ in the $\wedge$-lattice $\mathrm{Sub}(\mathbf{1}_{\mathsf{C}})$.

An **$\mathcal{L}$-homomorphism** $f : \mathcal{M} \to \mathcal{N}$ between $\mathcal{L}$-structures in C consists of:

- for each $P \in \mathcal{L}_{\mathrm{sort}}$, a morphism $f_P : P^{\mathcal{M}} \to P^{\mathcal{N}}$; such that
- for each $g : (P_1, \dots, P_n) \to Q \in \mathcal{L}_{\mathrm{fun}}$, we have

$$f_Q \circ g^{\mathcal{M}} = g^{\mathcal{N}} \circ (f_{P_1} \times \cdots \times f_{P_n}) : P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}} \to Q^{\mathcal{N}},$$

  i.e., commutativity of

$$\begin{tikzcd}
P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}} \arrow[r, "f_{P_1} \times \cdots \times f_{P_n}"] \arrow[d, "g^{\mathcal{M}}"'] & P_1^{\mathcal{N}} \times \cdots \times P_n^{\mathcal{N}} \arrow[d, "g^{\mathcal{N}}"] \\
Q^{\mathcal{M}} \arrow[r, "f_Q"'] & Q^{\mathcal{N}};
\end{tikzcd}$$

- for each $R \subseteq (P_1, \dots, P_n) \in \mathcal{L}_{\mathrm{rel}}$, we have

$$R^{\mathcal{M}} \subseteq (f_{P_1} \times \cdots \times f_{P_n})^*(R^{\mathcal{N}}) \subseteq P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}},$$

  i.e., (unique) existence of the shorter dashed morphism in the commutative diagram

$$\begin{tikzcd}
R^{\mathcal{M}} \arrow[dr, dashed] \arrow[drr, dashed, bend left] \arrow[ddr, hook] & & \\
& (f_{P_1} \times \cdots \times f_{P_n})^*(R^{\mathcal{N}}) \arrow[r] \arrow[d, hook] & R^{\mathcal{N}} \arrow[d, hook] \\
& P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}} \arrow[r, "f_{P_1} \times \cdots \times f_{P_n}"'] & P_1^{\mathcal{N}} \times \cdots \times P_n^{\mathcal{N}},
\end{tikzcd}$$

  or equivalently (by the universal property of the pullback) of the longer dashed morphism, i.e.,

$$\text{“}f_{P_1} \times \cdots \times f_{P_n} : P_1^{\mathcal{M}} \times \cdots \times P_n^{\mathcal{M}} \to P_1^{\mathcal{N}} \times \cdots \times P_n^{\mathcal{N}} \text{ restricts to } R^{\mathcal{M}} \to R^{\mathcal{N}}\text{”}.$$

Let

$$\begin{aligned} \mathsf{Mod}(\mathcal{L}, \mathsf{C}) &:= \text{category of } \mathcal{L}\text{-structures in } \mathsf{C},\ \mathcal{L}\text{-homomorphisms}, \\ \mathsf{Mod}(\mathcal{L}) &:= \mathsf{Mod}(\mathcal{L}, \mathsf{Set}), \end{aligned}$$

which agrees with the definition in Section 18 for single-sorted $\mathcal{L}$ (modulo the natural isomorphism $\mathrm{Sub}_{\mathsf{Set}} \cong \mathcal{P}$).

Example 25.8. TopGrp can be regarded as a full subcategory of $\mathsf{Mod}(\mathcal{L}_{\mathsf{Grp}}, \mathsf{Top})$; morphisms are continuous by virtue of being in Top, and group homomorphisms by virtue of being $\mathcal{L}_{\mathsf{Grp}}$-homomorphisms.

26 First-order logic

Let $\mathcal{L}$ be a (multi-sorted finitary) first-order language. Fix an infinite set Var of **variables** $x, y, \ldots$. A **variable context over** $\mathcal{L}$ is a finite set of variables $X \subseteq \mathrm{Var}$ together with a family of sorts $\vec{P} = (P_x)_{x \in X} \in \mathcal{L}_{\mathrm{sort}}^X$. When $\mathcal{L}$ is one-sorted, we identify variable contexts with finite $X \subseteq \mathrm{Var}$.

We define the **$\mathcal{L}$-terms** t over each variable context $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^X$, as well as their **value sorts** $\mathrm{val}(t) \in \mathcal{L}_{\mathrm{sort}}$, inductively as follows:

- $x \in X$ is a term over context $\vec{P}$ with $\mathrm{val}(x) := P_x$;
- for a function symbol $f : (Q_1, \ldots, Q_n) \to R \in \mathcal{L}_{\mathrm{fun}}$, as well as terms $t_1, \ldots, t_n$ over variable context $\vec{P}$ with value sorts $Q_1, \ldots, Q_n$ respectively, $f(t_1, \ldots, t_n)$ is a term over context $\vec{P}$ with value sort R. When $\mathrm{ar}(f) = \varnothing$, i.e., f is a **constant symbol**, we write f instead of $f()$.

We write

$$\mathcal{L}_{\omega\omega}(\vec{P}; Q) := \{\text{terms over context } \vec{P} \text{ with value sort } Q\}.$$

Example 26.1. $\cdot(x, {}^{-1}(y))$, usually written $x \cdot y^{-1}$, is an $\mathcal{L}_{\mathsf{Grp}}$-term over the context $\{x, y\}$ (or any bigger context).

We now define the **$\mathcal{L}$-formulas** ϕ over each variable context $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^X$ inductively as follows:

- for two terms s, t over $\vec{P}$ of value sort Q,
$$(s = t)$$
is a formula over $\vec{P}$;
- for a relation symbol $R \in \mathcal{L}_{\mathrm{rel}}(Q_1, \ldots, Q_n)$ and terms $t_1, \ldots, t_n$ over $\vec{P}$ of value sorts $Q_1, \ldots, Q_n$,
$$R(t_1, \ldots, t_n)$$
(or just R if $\mathrm{ar}(R) = \varnothing$) is a formula over $\vec{P}$;

- $\top, \bot$ are formulas over $\vec{P}$;
- for formulas ϕ, ψ over $\vec{P}$, also $\phi \wedge \psi, \phi \vee \psi, \neg\phi$ are formulas over $\vec{P}$;
- for a formula ϕ over a context of the form $(\vec{P} \setminus \{y\}) \sqcup (y \mapsto Q)$ where $\vec{P} \setminus \{y\}$ is $\vec{P}$ with $y \mapsto P_y$ deleted if $y \in X$,

  $$(\exists y \in Q)\phi$$

  (or $(\exists y)\phi$ if $\mathcal{L}$ is one-sorted) is a formula over $\vec{P}$.

We use the abbreviations

$$\begin{aligned}
\phi \to \psi &:= \neg\phi \vee \psi, \\
(\vec{s} = \vec{t}) &:= (s_1 = t_1) \wedge \cdots \wedge (s_n = t_n), \\
(\exists \vec{y} \in \vec{Q})\phi &:= (\exists y_1 \in Q_1) \cdots (\exists y_n \in Q_n)\phi, \\
(\forall y \in Q)\phi &:= \neg(\exists y \in Q)(\neg\phi).
\end{aligned}$$

We write

$$\mathcal{L}_{\omega\omega}(\vec{P}) := \{\text{formulas over context } \vec{P}\}.$$

Example 26.2. Consider $\mathcal{L}$ with two sorts A, B and a single function symbol $f : A \to B$. Here is an $\mathcal{L}$-formula over the context $(x \mapsto A,\, y \mapsto B) \in \mathcal{L}_{\text{sort}}^{\{x,y\}}$ (or any larger context):

$$\phi := (f(x) = y) \vee (\forall z \in B)\, \underbrace{(\exists y \in A)\, \overbrace{((f(x) = z) \wedge (f(y) = z))}^{\text{over } (x\mapsto A,\, y\mapsto A,\, z\mapsto B)}}_{\text{over } (x\mapsto A,\, y\mapsto B,\, z\mapsto B)}.$$

We will use the following notation for substitution into formulas:

$$[x \mapsto y, z \mapsto y \cdot y, \ldots]\phi := \phi \text{ with } x \text{ replaced by } y,\ z \text{ replaced by the term } y \cdot y,\ \ldots.$$

Here are the annoying[25] technical details:

- An occurrence of a variable x in a formula ϕ is **bound** if it occurs underneath an $\exists x$ (or $\forall x$), otherwise **free**. In Example 26.2, each occurrence of x is free; the first occurrence of y is free, while the second is bound (by $\exists y \in A$); and both occurrences of z are free.
- A **substitution** σ between two variable contexts $\vec{P} \in \mathcal{L}_{\text{sort}}^X$ and $\vec{Q} \in \mathcal{L}_{\text{sort}}^Y$ is a function

  $$\sigma \in \textstyle\prod_{x \in X} \mathcal{L}_{\omega\omega}(\vec{Q}; P_x).$$

  If each $\sigma(x)$ is a variable $\in Y$ (with $Q_{\sigma(x)} = P_x$), we call σ a **variable substitution**.

[25]Careful formal treatments of variable substitution are usually given in computer science contexts. See e.g., [Barendregt, *The Lambda Calculus: Its Syntax and Semantics*]. In particular, the elegant solution of **de Bruijn indices** avoids quotienting by α-equivalence, and is more convenient for computer representations of syntax, at the cost of being much less readable for humans.

- Given a substitution σ from $\vec{P} \in \mathcal{L}_{\text{sort}}^X$ to $\vec{Q} \in \mathcal{L}_{\text{sort}}^Y$, we define

  $$\begin{aligned} &\mathcal{L}_{\omega\omega}(\vec{P}) \longrightarrow \mathcal{L}_{\omega\omega}(\vec{Q}) \\ &\qquad \phi \longmapsto [\sigma]\phi := \phi \text{ with all free occurrences of } x \in X \textit{ simultaneously} \text{ replaced by } \sigma(x). \end{aligned}$$

  For example,

  $$\begin{aligned} \mathcal{L}_{\omega\omega}(x \mapsto A,\, y \mapsto A) \ni (x = y) &\xmapsto{[x \mapsto x^2,\, y \mapsto x]} (x^2 = x) \in \mathcal{L}_{\omega\omega}(x \mapsto A,\, y \mapsto B), \\ (x < y) \vee (\exists y)(x < y) &\xmapsto{[x \mapsto x,\, y \mapsto z]} (x < z) \vee (\exists y)(x < y), \\ (x < z) \vee (\exists y)(x < y) &\xmapsto{[x \mapsto y,\, z \mapsto y]} (y < z) \vee (\exists y)(y < y). \end{aligned}$$

  We automatically extend σ via the identity in the notation $[\sigma]\phi$ if needed, so that e.g., in the second example, we needn't write $x \mapsto x$.

- In the last example, the meaning of the $\exists$ subformula has been changed; we say that the substitution has **captured** the bound variable y. This happens if some $x \in X$ occurs free in ϕ underneath some $\exists y$ such that y occurs in $\sigma(x)$. We say the substitution is **safe** if this does not happen.

- We say that two formulas $(\exists x \in P)\phi, (\exists y \in P)[x \mapsto y]\phi$, where the latter substitution is safe, are **α-equivalent**; such formulas trivially have the same meaning. More generally, we say that two formulas are **α-equivalent** if they are related by a finite sequence of such substitutions between subformulas.

- *We henceforth identify α-equivalent formulas. Every formula may be replaced from an α-equivalent one in which all occurrences of the same variable are either all free or refer to the same quantifier. Thus, up to α-equivalence, all variable substitutions may be assumed safe.*

- Substitution $[\sigma]t$ into terms t is defined analogously (but without needing to worry about variable capture, since terms do not bind variables). For $\sigma : \vec{P} \to \vec{Q}$ and $t \in \mathcal{L}_{\omega\omega}(\vec{P}; R)$, we have $[\sigma]t \in \mathcal{L}_{\omega\omega}(\vec{Q}; R)$.

We next define a proof system for first-order logic. As in propositional logic (Section 15), the system will be a sequent calculus. Let $\mathcal{L}$ be a first-order language. A **sequent** between $\mathcal{L}$-formulas is an expression of the form

$$\phi \Rightarrow_{\vec{P}} \psi$$

where ϕ, ψ are two formulas in the same context $\vec{P} \in \mathcal{L}_{\text{sort}}^X$, which is thought of as the *universally quantified implication* $(\forall \vec{x} \in \vec{P})(\phi \to \psi)$, but with this toplevel $\forall$ and $\to$ playing a different role (for similar reasons as in propositional logic).

We include in the proof system all the inference rules from propositional logic (Section 15), *for formulas in the same context*, e.g.,

$$(\textsc{Id})\ \frac{}{\phi \Rightarrow_{\vec{P}} \phi} \qquad (\textsc{Cut})\ \frac{\phi \Rightarrow_{\vec{P}} \psi \qquad \psi \Rightarrow_{\vec{P}} \theta}{\phi \Rightarrow_{\vec{P}} \theta}$$

$$({\Rightarrow}{\vee_1})\ \frac{}{\phi \Rightarrow_{\vec{P}} \phi \vee \psi} \qquad ({\Rightarrow}{\vee_2})\ \frac{}{\psi \Rightarrow_{\vec{P}} \phi \vee \psi} \qquad ({\vee}{\Rightarrow})\ \frac{\phi \Rightarrow_{\vec{P}} \theta \qquad \psi \Rightarrow_{\vec{P}} \theta}{\phi \vee \psi \Rightarrow_{\vec{P}} \theta} \qquad \cdots.$$

In addition, we include the following rules:

$$\text{(Sub)}\ \frac{\phi \Rightarrow_{\vec{P}} \psi}{[\sigma]\phi \Rightarrow_{\vec{Q}} [\sigma]\psi} \quad \text{for any substitution } \sigma : \vec{P} \to \vec{Q},$$

$$\left.\begin{array}{r} \text{(Refl)}\ \dfrac{}{\top \Rightarrow_{\vec{P}} (t = t)} \\[2em] \text{(Leib)}\ \dfrac{}{(s = t) \wedge [y \mapsto s]\phi \Rightarrow_{\vec{P}} [y \mapsto t]\phi} \end{array}\right\} \text{for } s, t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q),\ \phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q)),$$

$$(\Rightarrow\exists)\ \frac{}{[y \mapsto t]\phi \Rightarrow_{\vec{P}} (\exists y \in Q)\phi} \qquad (\exists\Rightarrow)\ \frac{\phi \Rightarrow_{\vec{P} \sqcup (y \mapsto Q)} \psi}{(\exists y \in Q)\phi \Rightarrow_{\vec{P}} \psi}$$

$$\text{(Frob)}\ \frac{}{\phi \wedge (\exists y \in Q)\psi \Rightarrow_{\vec{P}} (\exists y \in Q)(\phi \wedge \psi)} \quad \text{for } y \notin \vec{P}.$$

Some notes:

- The substitution rule (Sub) includes as a special case the so-called **weakening** rule when σ is an inclusion:

  $$\text{(Sub)}\ \frac{\phi \Rightarrow_{\vec{P}} \psi}{\phi \Rightarrow_{\vec{Q}} \psi} \qquad \text{for } \vec{P} \subseteq \vec{Q}.$$

- On the other hand, we do *not* allow unused variables to be removed from the context, since the intended meaning is that we are universally quantifying over those variables, and their sorts may be empty.

  We may however remove unused variables as long as there is a term of that sort in the remaining context (e.g., a constant symbol, or another variable of that sort), using (Sub).

- The reflexivity rule (Refl) implies n-ary analogs allowing us to deduce $\vec{t} = \vec{t}$, e.g.,

  $$(\Rightarrow\wedge)\ \dfrac{\text{(Refl)}\ \dfrac{}{\top \Rightarrow_{\vec{P}} (s = s)} \qquad \text{(Refl)}\ \dfrac{}{\top \Rightarrow_{\vec{P}} (t = t)}}{\top \Rightarrow_{\vec{P}} (s = s) \wedge (t = t)}$$

  Likewise, the Leibniz rule (Leib) implies a generalization which allows us to replace several terms at once:

  $$\frac{}{(\vec{s} = \vec{t}) \wedge [\vec{x} \mapsto \vec{s}]\phi \Rightarrow_{\vec{P}} [\vec{x} \mapsto \vec{t}]\phi}$$

  where $\vec{x}$ should consist of distinct variables not in $\vec{P}$. The proof is by using (Cut) to reduce to repeated applications of (Leib) (exercise).

- The ($\Rightarrow\exists$), ($\exists\Rightarrow$), and Frobenius (Frob) rules should be thought of as analogs for $\exists$ of the ($\Rightarrow\vee_i$), ($\vee\Rightarrow$), and distributivity (Dist) rules for $\vee$. Note that the side condition $y \notin \vec{P}$ can always be assumed, up to replacing the $\exists y$ with an α-equivalent formula.

Example 26.3. Here is a proof of the converse of the (Frob) rule (cf. Exercise 15.2):

$$
\text{(}\exists\Rightarrow\text{)}\,\dfrac{\text{(}\Rightarrow\wedge\text{)}\,\dfrac{\text{(}\wedge\Rightarrow_1\text{)}\,\dfrac{}{\phi\wedge\psi\Rightarrow\phi}\qquad \text{(Cut)}\,\dfrac{\text{(}\wedge\Rightarrow_2\text{)}\,\dfrac{}{\phi\wedge\psi\Rightarrow\psi}\qquad \text{(}\Rightarrow\exists\text{)}\,\dfrac{}{\psi=[y\mapsto y]\psi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}(\exists y\in Q)\psi}}{\phi\wedge\psi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}(\exists y\in Q)\psi}}{\phi\wedge\psi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}\phi\wedge(\exists y\in Q)\psi}}{(\exists y\in Q)(\phi\wedge\psi)\Rightarrow_{\vec{P}}\phi\wedge(\exists y\in Q)\psi}
$$

Example 26.4. Here is a proof of the inverse of the ($\exists\Rightarrow$) rule (cf. Exercise 15.3):

$$
\text{(Cut)}\,\dfrac{\text{(}\Rightarrow\exists\text{)}\,\dfrac{}{[y\mapsto y]\phi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}(\exists y\in Q)\phi}\qquad \text{(Sub)}\,\dfrac{(\exists y\in Q)\phi\Rightarrow_{\vec{P}}\psi}{(\exists y\in Q)\phi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}\psi}}{\phi\Rightarrow_{\vec{P}\sqcup(y\mapsto Q)}\psi}
$$

Exercise 26.5. Show that conversely, the inverse of the ($\exists\Rightarrow$) rule can be used to derive the ($\Rightarrow\exists$) rule.

Exercise 26.6. Derive the duals of the ($\Rightarrow\exists$) and ($\exists\Rightarrow$) rules for $\forall$, using the definition $\forall := \neg\exists\neg$.

A **first-order $\mathcal{L}$-theory** $\mathcal{T}$ is a set of $\mathcal{L}$-sequents, called the **axioms** of $\mathcal{T}$. If there is a proof of a sequent $\phi \Rightarrow_{\vec{P}} \psi$ from hypotheses in $\mathcal{T}$, we write

$$\mathcal{T} \vdash \phi \Rightarrow_{\vec{P}} \psi$$

and say $\mathcal{T}$ **proves** $\phi \Rightarrow_{\vec{P}} \psi$. As in propositional logic, we use the obvious abbreviations like

$$\vdash \phi \;:\Longleftrightarrow\; \varnothing \vdash \top \Rightarrow \phi,$$

etc.

The relative complexity of first-order logic compared to propositional logic means that it will be convenient for us to consider fragments of first-order logic right away. By a **fragment** of first-order logic, we mean a subclass of all first-order formulas where only certain connectives and quantifiers are allowed. We will consider the following fragments, listed in increasing order:

- **Horn** first-order formulas are built from atomic formulas using $\wedge, \top$. Thus, Horn sequents are of the form

  $$\phi_1 \wedge \cdots \wedge \phi_m \Rightarrow_{\vec{P}} \psi_1 \wedge \cdots \wedge \psi_n$$

  where ϕ_i, ψ_j are atomic. Such a sequent is provably equivalent to the set of sequents

  $$\phi_1 \wedge \cdots \wedge \phi_m \Rightarrow_{\vec{P}} \psi_j.$$

  Thus, we may assume without loss that Horn theories consist of sequents of this form. Since $\vec{P}$ may be an arbitrary context, these are really the (finitary) *universal Horn theories* in the sense of Exercise 18.8.

  Despite its simplicity, the Horn fragment is actually not that well-behaved for categorical purposes.

- The **lex** (or **Cartesian**, **finite limit**, **essentially algebraic**) fragment is a slight generalization of the Horn fragment, which is much better behaved categorically. Informally, its formulas are built using $\wedge, \top$ and "provably unique $\exists$". The formal definition is somewhat involved, and will be postponed until Section 31.

- **Regular**[26] formulas are built using $\wedge, \top, \exists$. Using the (Frob) rule and its converse (Example 26.3), we may move every $\wedge$ in a regular formula inside of every $\exists$; thus up to provable equivalence, all regular formulas are of the form

  $$(\exists \vec{y} \in \vec{Q})(\phi_1 \wedge \cdots \wedge \phi_n)$$

  with the ϕ_i atomic. Regular sequents between two such formulas are, using the ($\exists\Rightarrow$) rule and its inverse (Example 26.4), equivalent to ones of the form

  $$\phi_1 \wedge \cdots \wedge \phi_m \Rightarrow_{\vec{P}} (\exists \vec{y} \in \vec{Q})(\psi_1 \wedge \cdots \wedge \psi_n).$$

- **Coherent**[27] formulas are built using $\wedge, \top, \vee, \bot, \exists$. It is easily seen that we have provable equivalences $(\exists y \in Q)(\phi \vee \psi) \Leftrightarrow (\exists y \in Q)\phi \vee (\exists y \in Q)\psi$ and $(\exists y \in Q)\bot \Leftrightarrow \bot$ (Exercise), so that we may move $\vee$ *outside* of every $\exists$; combined with disjunctive normal form, this means that coherent formulas are up to provable equivalence all of the form

  $$\bigvee_{i=1}^{m} (\exists \vec{y} \in \vec{Q}) \bigwedge_{j=1}^{n_i} \phi_{ij}$$

  with the ϕ_{ij} atomic. Coherent sequents are equivalent to sets of sequents of the form

  $$\bigwedge_i \phi_i \Rightarrow_{\vec{P}} \bigvee_j (\exists \vec{y} \in \vec{Q}) \bigwedge_k \psi_{jk}.$$

Remark 26.7. We do *not* have the classical **prenex normal form** for formulas, with all quantifiers moved out front. This is because $\vee$ cannot be moved inside of $\exists$: to go from $\phi \vee (\exists x \in P)\top$ to $(\exists x \in P)(\phi \vee \top)$, we need to assume P is nonempty.

For a first-order language $\mathcal{L}$ and variable context $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^X$, we write

$$\mathcal{L}_{\omega\omega}^{\mathrm{Horn}}(\vec{P}) \subseteq \mathcal{L}_{\omega\omega}^{\mathrm{reg}}(\vec{P}) \subseteq \mathcal{L}_{\omega\omega}^{\mathrm{coh}}(\vec{P}) \subseteq \mathcal{L}_{\omega\omega}(\vec{P})$$

to denote respectively the sets of Horn, regular, and coherent formulas in context $\vec{P}$. For a theory $\mathcal{T}$ and sequent $\phi \Rightarrow_{\vec{P}} \psi$ in one of these fragments, we write

$$\mathcal{T} \vdash_{\mathrm{Horn\,(reg,coh)}} \phi \Rightarrow_{\vec{P}} \psi$$

if there is a proof using only sequents in the fragment.

[26] classically known as "positive-primitive"; for the same reasons as before, we will avoid using classical names for formulas in fragments other than Horn

[27] classically known as "positive-existential"

27 Categorical semantics

Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$ be an $\mathcal{L}$-structure in a lex category C. We will define interpretations of the syntax of first-order logic in $\mathcal{M}$, extending the interpretations of the symbols in $\mathcal{L}$.

First, we define inductively for each term $t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$, where $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^X$ is a context and $Q \in \mathcal{L}_{\mathrm{sort}}$ is a sort, an **interpretation of** t **in** $\mathcal{M}$ as a morphism

$$t^{\mathcal{M}} = \mathcal{M}(t) : \prod \vec{P}^{\mathcal{M}} := \prod_{x \in X} P_x^{\mathcal{M}} \longrightarrow Q^{\mathcal{M}},$$

as follows:

- For $x \in X$, $x^{\mathcal{M}} := \pi_x : \prod \vec{P}^{\mathcal{M}} \to P_x^{\mathcal{M}}$.
- For $f : (Q_1, \dots, Q_n) \to R \in \mathcal{L}_{\mathrm{fun}}$ and terms $t_i \in \mathcal{L}_{\omega\omega}(\vec{P}, Q_i)$, $f(t_1, \dots, t_n)^{\mathcal{M}}$ is the composite

$$\prod \vec{P}^{\mathcal{M}} \xrightarrow{(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})} Q_1^{\mathcal{M}} \times \cdots \times Q_n^{\mathcal{M}} \xrightarrow{f^{\mathcal{M}}} R^{\mathcal{M}}.$$

Example 27.1. For a topological group G, regarded as an $\mathcal{L}_{\mathsf{Grp}}$-structure in Top, the term $x \cdot y^{-1}$ is interpreted as the continuous map

$$\begin{aligned} G \times G &\xrightarrow{(\pi_1, \pi_2(-)^{-1})} G \times G \xrightarrow{\cdot} G \\ (g, h) &\longmapsto (g, h^{-1}) \longmapsto g \cdot h^{-1}. \end{aligned}$$

Lemma 27.2 (substitution lemma for terms)**.** For a substitution σ between contexts $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^X$ and $\vec{Q} \in \mathcal{L}_{\mathrm{sort}}^Y$, and a term $t \in \mathcal{L}_{\omega\omega}(\vec{P}; R)$, we have

$$([\sigma]t)^{\mathcal{M}} = t^{\mathcal{M}} \circ \sigma^{\mathcal{M}} : \prod \vec{Q}^{\mathcal{M}} \longrightarrow R^{\mathcal{M}}$$

where

$$\sigma^{\mathcal{M}} := (\sigma_x^{\mathcal{M}})_{x \in X} : \prod \vec{Q}^{\mathcal{M}} \longrightarrow \prod \vec{P}^{\mathcal{M}}$$

(note the contravariance in σ).

$$\begin{tikzcd} \prod \vec{Q}^{\mathcal{M}} \arrow[rr, "([\sigma]t)^{\mathcal{M}}"] \arrow[dr, "\sigma^{\mathcal{M}} := (\sigma_x^{\mathcal{M}})_{x \in X}"'] & & R^{\mathcal{M}} \\ & \prod \vec{P}^{\mathcal{M}} \arrow[ur, "t^{\mathcal{M}}"'] & \end{tikzcd}$$

Proof. By induction on t. If $t = x \in X$, then

$$([\sigma]t)^{\mathcal{M}} = \sigma_x^{\mathcal{M}} = \pi_x \circ \sigma^{\mathcal{M}} = t^{\mathcal{M}} \circ \sigma^{\mathcal{M}}.$$

If $t = f(t_1, \dots, t_n)$ where $t_i \in \mathcal{L}_{\omega\omega}(\vec{P}; R_i)$ and $f : (R_1, \dots, R_n) \to S \in \mathcal{L}_{\mathrm{fun}}$, then

$$\begin{aligned} ([\sigma]t)^{\mathcal{M}} &= f([\sigma]t_1, \dots, [\sigma]t_n)^{\mathcal{M}} \\ &= f^{\mathcal{M}} \circ (([\sigma]t_1)^{\mathcal{M}}, \dots, ([\sigma]t_n)^{\mathcal{M}}) \\ &= f^{\mathcal{M}} \circ (t_1^{\mathcal{M}} \circ \sigma^{\mathcal{M}}, \dots, t_n^{\mathcal{M}} \circ \sigma^{\mathcal{N}}) && \text{by IH} \\ &= f^{\mathcal{M}} \circ (t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}}) \circ \sigma^{\mathcal{M}} && \text{by naturality of } \mathsf{C}(-, \textstyle\prod_i R_i^{\mathcal{M}}) \cong \prod_i \mathsf{C}(-, R_i^{\mathcal{M}}) \\ &= t^{\mathcal{M}} \circ \sigma^{\mathcal{M}}. \end{aligned}$$

□

We now define inductively for certain $\mathcal{L}$-formulas ϕ in contexts $\vec{P} \in \mathcal{L}_{\text{sort}}^X$ an **interpretation of ϕ in $\mathcal{M}$** as a subobject

$$\phi^{\mathcal{M}} = \mathcal{M}(\phi) \subseteq \prod_{x \in X} P_x^{\mathcal{M}}.$$

Such ϕ will be called **interpretable in $\mathcal{M}$**. As we give the definition, we will simultaneously verify the following:

Lemma 27.3 (substitution lemma for formulas)**.** For a substitution σ between contexts $\vec{P} \in \mathcal{L}_{\text{sort}}^X$ and $\vec{Q} \in \mathcal{L}_{\text{sort}}^Y$, and a formula $\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$ interpretable in $\mathcal{M}$, also $[\sigma]\phi$ is interpretable in $\mathcal{M}$, with

$$([\sigma]\phi)^{\mathcal{M}} = (\sigma^{\mathcal{M}})^*(\phi^{\mathcal{M}}),$$

i.e., we have a pullback square

$$\begin{tikzcd} ([\sigma]\phi)^{\mathcal{M}} \arrow[r, dashed] \arrow[d, hook] & \phi^{\mathcal{M}} \arrow[d, hook] \\ \prod \vec{Q}^{\mathcal{M}} \arrow[r, "\sigma^{\mathcal{M}}"'] & \prod \vec{P}^{\mathcal{M}}. \end{tikzcd}$$

(In Set, this says $([\sigma]\phi)^{\mathcal{M}} = (\sigma^{\mathcal{M}})^{-1}(\phi^{\mathcal{M}}) = \{\vec{a} \mid \sigma^{\mathcal{M}}(\vec{a}) \in \phi^{\mathcal{M}}\}$, i.e., $([\sigma]\phi)^{\mathcal{M}}(\vec{a}) \iff \phi^{\mathcal{M}}(\sigma^{\mathcal{M}}(\vec{a}))$.)

Proof. This will be proved by induction on ϕ, as we define $\phi^{\mathcal{M}}$. □

We say that $\mathcal{M}$ **satisfies** a sequent $\phi \Rightarrow_{\vec{P}} \psi$ if

$$\mathcal{M} \models \phi \Rightarrow_{\vec{P}} \psi \;:\iff\; \phi, \psi \text{ interpretable in } \mathcal{M} \;\&\; \phi^{\mathcal{M}} \subseteq \psi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}.$$

If $\mathcal{M}$ satisfies every sequent in a theory $\mathcal{T}$, we write

$$\mathcal{M} \models \mathcal{T}$$

and say $\mathcal{M}$ is a **model of $\mathcal{T}$** (or of $(\mathcal{L}, \mathcal{T})$).

Proposition 27.4 (soundness)**.** If $\mathcal{M} \models \mathcal{T}$, and $\mathcal{T} \vdash \phi \Rightarrow_{\vec{P}} \psi$ using only sequents interpretable in $\mathcal{M}$, then $\mathcal{M} \models \phi \Rightarrow_{\vec{P}} \psi$.

Proof. This will be proved by induction on the proof of $\phi \Rightarrow_{\vec{P}} \psi$ from $\mathcal{T}$.

- The base case, when $\phi \Rightarrow_{\vec{P}} \psi \in \mathcal{T}$, is by the assumption $\mathcal{M} \models \mathcal{T}$.
- If the last step of the proof is Id or Cut, then use reflexivity and transitivity of $\subseteq$ in $\mathrm{Sub}_{\mathsf{C}}(\prod \vec{P}^{\mathcal{M}})$ (together with the IH for Cut).
- If $\mathcal{T} \vdash [\sigma]\phi \Rightarrow_{\vec{Q}} [\sigma]\psi$ by the Sub rule applied to $\mathcal{T} \vdash \phi \Rightarrow_{\vec{P}} \psi$, for a substitution $\sigma : \vec{P} \to \vec{Q}$:

$$\begin{aligned} ([\sigma]\phi)^{\mathcal{M}} &= (\sigma^{\mathcal{M}})^*(\phi^{\mathcal{M}}) && \text{by substitution lemma} \\ &\subseteq (\sigma^{\mathcal{M}})^*(\psi^{\mathcal{M}}) && \text{by IH and monotonicity of pullback} \\ &= ([\sigma]\psi)^{\mathcal{M}} && \text{by substitution lemma.} \end{aligned}$$

The rest of the proof will be completed as we define $\phi^{\mathcal{M}}$. □

Horn formulas are always interpretable, using the lex structure in C:

- For an atomic formula $R(t_1, \dots, t_n)$ where $R \in \mathcal{L}_{\mathrm{rel}}(\vec{Q})$ and $t_i \in \mathcal{L}_{\omega\omega}(\vec{P}; Q_i)$,

$$R(t_1, \dots, t_n)^{\mathcal{M}} := (t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})^*(R^{\mathcal{M}}) \subseteq \prod \vec{P}^{\mathcal{M}},$$

i.e., we have a pullback

$$\begin{tikzcd}
R(t_1, \dots, t_n)^{\mathcal{M}} \arrow[r] \arrow[d, hook] & R^{\mathcal{M}} \arrow[d, hook] \\
\prod \vec{P}^{\mathcal{M}} \arrow[r, "(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})"'] & Q_1^{\mathcal{M}} \times \cdots \times Q_n^{\mathcal{M}}.
\end{tikzcd}$$

Proof of substitution lemma. For $\sigma : \vec{P} \to \vec{P}'$, we have

$$\begin{aligned}
([\sigma]R(t_1, \dots, t_n))^{\mathcal{M}} &= R([\sigma]t_1, \dots, [\sigma]t_n)^{\mathcal{M}} \\
&= (([\sigma]t_1)^{\mathcal{M}}, \dots, ([\sigma]t_n)^{\mathcal{M}})^*(R^{\mathcal{M}}) \\
&= (t_1^{\mathcal{M}} \circ \sigma^{\mathcal{M}}, \dots, t_n^{\mathcal{M}} \circ \sigma^{\mathcal{M}})^*(R^{\mathcal{M}}) && \text{by substitution lemma for terms} \\
&= (\sigma^{\mathcal{M}})^*((t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})^*(R^{\mathcal{M}})) && \text{by functoriality of pullback} \\
&= (\sigma^{\mathcal{M}})^*(R(t_1, \dots, t_n)^{\mathcal{M}}).
\end{aligned}$$

□

- For an atomic formula $s = t$ where $s, t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$,

$$(s = t)^{\mathcal{M}} := \mathrm{eq}\left(\prod \vec{P}^{\mathcal{M}} \overset{s^{\mathcal{M}}}{\underset{t^{\mathcal{M}}}{\rightrightarrows}} Q^{\mathcal{M}}\right).$$

Proof of substitution lemma. In any lex category, given a diagram

$$\begin{tikzcd}
 & A \arrow[d, "h"] & \\
\mathrm{eq}(f, g) \arrow[r, hook] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C,
\end{tikzcd}$$

we have $h^*(\mathrm{eq}(f, g)) = \mathrm{eq}(f \circ h, g \circ h) \subseteq A$, by Yoneda and the computation in Set:

$$\begin{aligned}
h^*(\mathrm{eq}(f, g)) &= \{a \in A \mid h(a) \in \mathrm{eq}(f, g)\} \\
&= \{a \in A \mid f(h(a)) = g(h(a))\} = \mathrm{eq}(f \circ h, g \circ h).
\end{aligned}$$

Now given a substitution $\sigma : \vec{P} \to \vec{P}'$, take $h := \sigma^{\mathcal{M}} : \vec{P}'^{\mathcal{M}} \to \vec{P}^{\mathcal{M}}$, $f := s^{\mathcal{M}}$, and $g := t^{\mathcal{M}}$. □

- For conjunctions,

$$\begin{aligned}
\top^{\mathcal{M}} &:= \top_{\prod \vec{P}^{\mathcal{M}}} = \prod \vec{P}^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}, \\
(\phi \wedge \psi)^{\mathcal{M}} &:= \phi^{\mathcal{M}} \cap \psi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}.
\end{aligned}$$

Proof of substitution lemma. For $\sigma : \vec{P} \to \vec{P}'$, use that $(\sigma^{\mathcal{M}})^* : \mathrm{Sub}(\prod \vec{P}'^{\mathcal{M}}) \to \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ preserves finite meets (by Yoneda):

$$\begin{aligned}
([\sigma]\top)^{\mathcal{M}} &= \top^{\mathcal{M}} = \top_{\prod \vec{P}'^{\mathcal{M}}} = (\sigma^{\mathcal{M}})^*(\top_{\prod \vec{P}^{\mathcal{M}}}), \\
([\sigma](\phi \wedge \psi))^{\mathcal{M}} &= ([\sigma]\phi \wedge [\sigma]\psi)^{\mathcal{M}} \\
&= ([\sigma]\phi)^{\mathcal{M}} \cap ([\sigma]\psi)^{\mathcal{M}} \\
&= (\sigma^{\mathcal{M}})^*(\phi^{\mathcal{M}}) \cap (\sigma^{\mathcal{M}})^*(\psi^{\mathcal{M}}) \quad \text{by IH} \\
&= (\sigma^{\mathcal{M}})^*(\phi^{\mathcal{M}} \cap \psi^{\mathcal{M}}) \\
&= (\sigma^{\mathcal{M}})^*((\phi \wedge \psi)^{\mathcal{M}}).
\end{aligned}$$

□

Proof of soundness, for proofs ending in a rule involving $\wedge, \top, =$.

- If $\mathcal{T} \vdash \phi \wedge \psi \Rightarrow_{\vec{P}} \phi$ by the $(\wedge{\Rightarrow}_1)$ rule: we have $(\phi \wedge \psi)^{\mathcal{M}} = \phi^{\mathcal{M}} \cap \psi^{\mathcal{M}} \subseteq \phi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$. The $(\wedge{\Rightarrow}_2)$ rule is similar.
- If $\mathcal{T} \vdash \theta \Rightarrow_{\vec{P}} \phi \wedge \psi$ by the $({\Rightarrow}\wedge)$ rule applied to $\mathcal{T} \vdash \theta \Rightarrow_{\vec{P}} \phi$ and $\mathcal{T} \vdash \theta \Rightarrow_{\vec{P}} \psi$: by the IH, $\theta^{\mathcal{M}} \subseteq \phi^{\mathcal{M}} \cap \psi^{\mathcal{M}} = (\phi \wedge \psi)^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$.
- If $\mathcal{T} \vdash \top \Rightarrow_{\vec{P}} (t = t)$ by the Refl rule: we have $(t = t)^{\mathcal{M}} = \mathrm{eq}(t^{\mathcal{M}}, t^{\mathcal{M}}) = \prod \vec{P}^{\mathcal{M}} = \top^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$ (since $\mathrm{eq}(f, f) =$ domain of f in any category, by Yoneda).
- If $\mathcal{T} \vdash (s = t) \wedge [y \mapsto s]\phi \Rightarrow_{\vec{P}} [y \mapsto t]\phi$ by the Leib rule, where $s, t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$ and $\phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$: the LHS is interpreted as the limit (using the substitution lemma)

$$\begin{tikzcd}
((s = t) \wedge [y \mapsto s]\phi \Rightarrow_{\vec{P}} [y \mapsto t]\phi)^{\mathcal{M}} \arrow[r, hook, dashed] \arrow[d, hook, dashed] & ([y \mapsto s]\phi)^{\mathcal{M}} \arrow[r, dashed] \arrow[d, hook, dashed] & \phi^{\mathcal{M}} \arrow[d, hook] \\
(s = t)^{\mathcal{M}} = \mathrm{eq}(s^{\mathcal{M}}, t^{\mathcal{M}}) \arrow[r, hook, dashed] & \prod \vec{P}^{\mathcal{M}} \arrow[r, "(1_{\prod \vec{P}^{\mathcal{M}}}, s^{\mathcal{M}})"] \arrow[dr, shift left, "s^{\mathcal{M}}"] \arrow[dr, shift right, "t^{\mathcal{M}}"'] & \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \\
& & Q
\end{tikzcd}$$

which in Set is easily seen to be $\{\vec{a} \in \prod \vec{P}^{\mathcal{M}} \mid (s^{\mathcal{M}}(\vec{a}) = t^{\mathcal{M}}(\vec{a}))\ \&\ \phi^{\mathcal{M}}(s^{\mathcal{M}}(\vec{a}))\}$. Similarly, in Set, we have $([y \mapsto t]\phi)^{\mathcal{M}} = \{\vec{a} \in \prod \vec{P}^{\mathcal{M}} \mid \phi^{\mathcal{M}}(t^{\mathcal{M}}(\vec{a}))\}$, which clearly contains the former set. For general lex C, apply Yoneda. □

28 Unions and complements

To interpret $\bot, \vee, \neg, \exists$, we need to impose additional structure on C beyond being merely a lex category:

- To interpret $\phi \vee \psi \in \mathcal{L}_{\omega\omega}(\vec{P})$: soundness of the $({\Rightarrow}\vee_i)$ and $(\vee{\Rightarrow})$ rules requires that we put

$$(\phi \vee \psi)^{\mathcal{M}} := \phi^{\mathcal{M}} \vee \psi^{\mathcal{M}} \in \mathrm{Sub}(\textstyle\prod \vec{P}^{\mathcal{M}}).$$

However, there is no reason for this join to be preserved by the pullback maps $(\sigma^{\mathcal{M}})^* = ((\sigma(x)^{\mathcal{M}})_x)^* : \mathrm{Sub}(\prod \vec{P}'^{\mathcal{M}}) \to \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ for a substitution $\sigma : \vec{P} \to \vec{P}'$, which is what we would need to prove the substitution lemma, analogous to the proof for $\wedge$ above.[28] Note that the morphism $\sigma^{\mathcal{M}} : \prod \vec{P}'^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}}$ may be any morphism in C with codomain $\prod \vec{P}^{\mathcal{M}}$, depending on $\mathcal{M}, \sigma$ (since for any $(f_x)_{x \in X} : A \to \prod \vec{P}^{\mathcal{M}} \in \mathsf{C}$, we may take the f_x to be unary functions in $\mathcal{M}$, then take $\sigma := (f_x)_x$). Thus, in order to always be able to interpret $\phi \vee \psi$, we should require the join $\phi^{\mathcal{M}} \vee \psi^{\mathcal{M}} \in \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ to be preserved under arbitrary pullbacks.

Example 28.1. In Grp, the join of $\mathbb{Z} \times \{0\}, \{0\} \times \mathbb{Z} \subseteq \mathbb{Z}^2$ is all of $\mathbb{Z}^2$, which is not preserved under pullback along the diagonal $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$.

We call a join of subobjects $A, B \subseteq X \in \mathsf{C}$ in a lex category C **pullback-stable** if it is preserved by $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ for any $f : Y \to X \in \mathsf{C}$. In that case, we also call the join the **union** of A, B, denoted

$$A \cup B := A \vee B.$$

We now define $\phi \vee \psi$ to be interpretable in $\mathcal{M}$ if ϕ, ψ both are, and the union

$$(\phi \vee \psi)^{\mathcal{M}} := \phi^{\mathcal{M}} \cup \psi^{\mathcal{M}}$$

exists. The substitution lemma now follows as for conjunction above.

Proof of soundness, for proofs ending in a rule involving $\vee$. The $(\Rightarrow\vee_i)$ and $(\vee\Rightarrow)$ rules follow from the definition $(\phi \vee \psi)^{\mathcal{M}} = \phi^{\mathcal{M}} \vee \psi^{\mathcal{M}}$. We now verify the (Dist) rule: for any $\phi, \psi, \theta \in \mathcal{L}_{\omega\omega}(\vec{P})$, we have

$$\begin{aligned} (\phi \wedge (\psi \vee \theta))^{\mathcal{M}} &= \\ \phi^{\mathcal{M}} \cap (\psi^{\mathcal{M}} \cup \theta^{\mathcal{M}}) &= \\ i^*(\psi^{\mathcal{M}} \cup \theta^{\mathcal{M}}) &= i^*(\psi^{\mathcal{M}}) \cup i^*(\theta^{\mathcal{M}}) \quad \text{by pullback-stability} \\ &= (\phi^{\mathcal{M}} \cap \psi^{\mathcal{M}}) \cup (\phi^{\mathcal{M}} \cap \theta^{\mathcal{M}}) \\ &= ((\phi \wedge \psi) \vee (\phi \wedge \theta))^{\mathcal{M}} \end{aligned}$$

where $i : \phi^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}}$ is the inclusion. (More precisely, the above shows that the two sides are equal as subobjects of $\phi^{\mathcal{M}}$, which implies that they are equal as subobjects of $\prod \vec{P}^{\mathcal{M}}$.) $\square$

- To interpret $\perp \in \mathcal{L}_{\omega\omega}(\vec{P})$, we likewise require $\mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ to have a pullback-stable least element.

  Proposition 28.2. Let C be a lex category, $X \in \mathsf{C}$. A pullback-stable least subobject $\varnothing_X$ of X is the same thing as an initial object $\varnothing_{\mathsf{C}}$ of C which is **strict**, meaning that every morphism $Y \to \varnothing_{\mathsf{C}}$ is an isomorphism. In particular, if some $X \in \mathsf{C}$ has a pullback-stable least subobject $\varnothing_X$, then $\varnothing_X$ is also the pullback-stable least subobject of every other $Y \in \mathsf{C}$.

[28] Note that we cannot simply appeal to Yoneda and the fact that f^{-1} in Set preserves unions, since joins of subobjects are not expressible solely in terms of limits. In fact, we will see in Section 35 that unions are a type of *colimit*.

Proof. First, we show that a strict initial object $\varnothing_{\mathsf{C}} \in \mathsf{C}$ is a pullback-stable least subobject of every $X \in \mathsf{C}$. By initiality, there is a unique morphism $i : \varnothing_{\mathsf{C}} \to X$. By strictness, the projections $\ker(i) \to \varnothing_{\mathsf{C}}$ are isomorphisms, i.e., i is monic. So $\varnothing_{\mathsf{C}}$ is a subobject of X, which is least by initiality. For any $f : Y \to X$, by strictness, $f^*(\varnothing_{\mathsf{C}}) \cong \varnothing_{\mathsf{C}}$ is the least subobject of Y.

Now we show that a pullback-stable least subobject $\varnothing_X \subseteq X$ is a strict initial object. For any $Y \in \mathsf{C}$, by pullback-stability, we have $\varnothing_X \times Y = \pi_1^*(\varnothing_X) = \varnothing_{X\times Y} \subseteq X \times Y$.

$$\begin{tikzcd}
\varnothing_X \times Y \arrow[r, hook] \arrow[d, "\cong"'] & X \times Y \arrow[d, "\pi_1"] \\
\varnothing_X \arrow[r, hook] & X
\end{tikzcd}$$

It follows that the projection $\varnothing_X \times Y \to \varnothing_X$ is monic, since its kernel is $\varnothing_X \times Y \times Y = \varnothing_{X\times Y\times Y}$ which is contained in the diagonal $\varnothing_X \times Y \hookrightarrow (\varnothing_X \times Y) \times_{\varnothing_X} (\varnothing_X \times Y) \cong \varnothing_X \times Y \times Y$. Since $\varnothing_X \hookrightarrow X$ is the least subobject, it follows that the projection $\varnothing_X \times Y \to \varnothing_X$ is in fact an isomorphism. We thus get a morphism $\varnothing_X \cong \varnothing_X \times Y \hookrightarrow X \times Y \xrightarrow{\pi_2} Y$. Any two morphisms $f, g : \varnothing_X \to Y$ must be equal, since their graphs are subobjects of $\varnothing_X \times Y = \varnothing_{X\times Y} \subseteq X \times Y$. Thus $\varnothing_X$ is an initial object. Finally, for any $f : Y \to \varnothing_X$, f factors as the composite $Y \cong \mathrm{graph}(f) \hookrightarrow Y \times \varnothing_X$ followed by the projection $Y \times \varnothing_X \to \varnothing_X$ which is an isomorphism by the above, whence f is monic, whence f is an isomorphism since $\varnothing_X \subseteq X$ is least. □

If C has a strict initial object $\varnothing_{\mathsf{C}}$, then $\bot$ is interpretable (in all contexts $\vec{P}$), with

$$\bot^{\mathcal{M}} := \varnothing_{\mathsf{C}} \subseteq \prod \vec{P}^{\mathcal{M}}.$$

The proof of the inductive case of the substitution lemma and soundness of the ($\bot\Rightarrow$) rule is straightforward.

- To interpret $\neg\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$, we require $\phi, \bot$ to be interpretable and $\phi^{\mathcal{M}} \in \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ to have a **pullback-stable complement**, meaning $\neg\phi^{\mathcal{M}} \in \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}})$ satisfying

$$\phi^{\mathcal{M}} \cup \neg\phi^{\mathcal{M}} = \prod \vec{P}^{\mathcal{M}}, \qquad\qquad \phi^{\mathcal{M}} \cap \neg\phi^{\mathcal{M}} = \varnothing_{\mathsf{C}}$$

(in particular, the join $\phi^{\mathcal{M}} \vee \neg\phi^{\mathcal{M}}$ exists and is pullback-stable). By the proof of Proposition 12.15, such $\neg\phi^{\mathcal{M}}$ is unique if it exists. We then put

$$(\neg\phi)^{\mathcal{M}} := \neg\phi^{\mathcal{M}}.$$

For a substitution $\sigma : \vec{P} \to \vec{P}'$, the pullback map $(\sigma^{\mathcal{M}})^* : \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}}) \to \mathrm{Sub}(\prod \vec{P}'^{\mathcal{M}})$ preserves finite meets and pullback-stable joins, hence pullback-stable complements; this yields the substitution lemma in this case. Soundness of the ($\wedge\neg$) and ($\wedge\neg$) rules follows easily from the definition of complements.

Thus in a lex category with (pullback-stable) unions, a strict initial object, and pullback-stable complements of subobjects, we may interpret arbitrary quantifier-free formulas.

29 Images

Finally, to interpret $(\exists y \in Q)\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$, where $\phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$ (we may assume $y \notin \vec{P}$, up to α-equivalence): soundness of the $(\exists\Rightarrow)$ rule and its inverse (which is equivalent to $(\Rightarrow\exists)$ by Example 26.4 and Exercise 26.5) says

$$((\exists y \in Q)\phi)^{\mathcal{M}} \subseteq \psi^{\mathcal{M}} \in \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}}) \iff \phi^{\mathcal{M}} \subseteq \psi^{\mathcal{M}} = \pi_1^*(\psi^{\mathcal{M}}) \in \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}})$$

where $\psi^{\mathcal{M}} = \pi_1^*(\psi^{\mathcal{M}})$ is by the substitution lemma for the inclusion $\vec{P} \hookrightarrow \vec{P} \sqcup (y \mapsto Q)$ whose interpretation is $\pi_1 : \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}}$, i.e.,

$$\begin{aligned} ((\exists y \in Q)\phi)^{\mathcal{M}} &= \text{left adjoint of } \pi_1^* : \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}}) \to \mathrm{Sub}(\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}}) \text{ at } \phi^{\mathcal{M}} \\ &= \min\{A \subseteq \prod \vec{P}^{\mathcal{M}} \mid \phi^{\mathcal{M}} \subseteq \pi_1^*(A) \subseteq \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}}\} \\ &=: \pi_1(\phi^{\mathcal{M}}) = \textbf{essential image of } \phi^{\mathcal{M}} \textbf{ under } \pi_1. \end{aligned}$$

So $\pi_1(\phi^{\mathcal{M}}) \subseteq \prod \vec{P}^{\mathcal{M}}$ is the smallest subobject admitting the shorter dashed morphism in the commutative diagram

$$\begin{tikzcd} \phi^{\mathcal{M}} \arrow[dr, dashed] \arrow[drr, dashed, bend left] \arrow[ddr, hook] & & \\ & \pi_1^*(\pi_1(\phi^{\mathcal{M}})) \arrow[r] \arrow[d, hook] & \pi_1(\phi^{\mathcal{M}}) =: ((\exists y \in Q)\phi)^{\mathcal{M}} \arrow[d, hook] \\ & \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \arrow[r, "\pi_1"'] & \prod \vec{P}^{\mathcal{M}}, \end{tikzcd}$$

which by the universal property of $\pi_1^*(\pi_1(\phi^{\mathcal{M}}))$ is equivalent to existence of the longer dashed morphism, i.e.,

$$\pi_1(\phi^{\mathcal{M}}) = \min\{A \subseteq \prod \vec{P}^{\mathcal{M}} \mid \phi^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \xrightarrow{\pi_1} \prod \vec{P}^{\mathcal{M}} \text{ factors through } A\}.$$

In general, for a morphism $f : X \to Y$ in a lex category C, the **essential**[29] **image** of f is

$$\mathrm{im}(f) := \min\{B \subseteq Y \mid f \text{ factors through } B\},$$

if it exists. For $A \subseteq X$, the **essential image of** A **under** f is

$$\begin{aligned} f(A) &:= \mathrm{im}(A \hookrightarrow X \xrightarrow{f} Y) \\ &= \min\{B \subseteq Y \mid A \subseteq f^*(B) \subseteq X\} \\ &= \text{left adjoint of } f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X) \text{ at } A. \end{aligned}$$

$$\begin{tikzcd} & \mathrm{im}(f) \arrow[dr, hook, "g"] & \\ X \arrow[ur, dashed, two heads, "h"] \arrow[rr, "f"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, dashed] \arrow[drr, dashed, bend left] \arrow[ddr, hook] & & \\ & f^*(B) \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

[29]This terminology is nonstandard, and is motivated by the distinction between "joins" and "unions" of subobjects: essential images do not behave like images of functions in general, unless they are pullback-stable.

We say that $f : X \to Y \in \mathsf{C}$ is an **extremal epimorphism** if $\mathrm{im}(f) = Y$, i.e.,

$$f = g \circ h,\ g \text{ monic} \implies g \text{ invertible}.$$

Clearly, if f has an essential image, then in the factorization $f = g \circ h$ through $\mathrm{im}(f)$ as above, h is an extremal epimorphism. Conversely, if $\mathrm{im}(f) \in \mathsf{C}$ is any object through which f factors as $g \circ h$ as above, with h extremal epic and g monic, then since h is extremal epic, $g : \mathrm{im}(f) \hookrightarrow Y$ is a mini*mal* subobject of Y through which f factors; but since $\mathrm{Sub}(Y)$ is a $\wedge$-lattice, this implies mini*mum*. Thus

$$\mathrm{im}(f) = \text{any object} \in \mathsf{C} \text{ s.t. } f \text{ factors as } X \xrightarrow{h}\!\!\!\!\rightarrow \mathrm{im}(f) \overset{g}{\hookrightarrow} Y \text{ with } h \text{ extremal epic, } g \text{ monic.}$$

Proposition 29.1. Extremal epimorphisms in lex categories are epimorphisms.

Proof. If $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ with f extremal epic and $g \circ f = h \circ f$, then f factors through $\mathrm{eq}(g, h) \hookrightarrow Y$, which is hence an isomorphism, i.e., $g = h$. □

Example 29.2. In Set, every surjection (= epimorphism) is extremal.

Example 29.3. In $\mathsf{TFAbGrp} := \{\text{torsion-free abelian groups}\}$, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic (Exercise 14.10(b)) but not extremal epic, since it factors through the proper subobject $\mathbb{Z} \subseteq \mathbb{Q}$.

Exercise 29.4. In Top, epimorphisms are continuous surjections, while extremal epimorphisms are continuous surjections whose codomain has the quotient topology.

Exercise 29.5. Show that in a lex category C:

(a) For any commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "e"'] & C \arrow[d, hook, "m"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "d"] & D \end{tikzcd}$$

with e extremal epic and m monic, there is a unique d making both triangles commute. For morphisms e, m satisfying this condition for all f, g, we say that e is **left orthogonal** to m. A morphism left orthogonal to all monomorphisms is called a **strong epimorphism**.

(b) Strong epimorphisms are (exactly) extremal epimorphisms. (In non-lex categories, strong epimorphism is a strictly stronger notion.)

(c) For any morphism m, the class of morphisms left orthogonal to it is closed under composition.

(d) Thus, extremal epimorphisms are closed under composition.

(e) Retractions (Section 9) are extremal epimorphisms.

(f) Monic + extremal epic $\implies$ isomorphism.

(g) If $f : X \to Y$ and $g : Y \to Z$ with $g \circ f$ extremal epic, then g is extremal epic.

Exercise 29.6. An **extremal monomorphism** is the dual of an extremal epimorphism, i.e., f such that whenever $f = g \circ h$ with h epic, h is an isomorphism.

(a) In Set, extremal monomorphism = monomorphism = injection.

(b) In Top, extremal monomorphism = continuous embedding.

(c) In $\mathsf{Haus} := \{\text{Hausdorff spaces}\}$, extremal monomorphism = continuous embedding with closed image.

(d) In Grp, extremal monomorphism = injective homomorphism.[30]

(e) In $\mathsf{TFAbGrp}$, extremal monomorphism = injective homomorphism whose image is closed under division by all $n \in \mathbb{N}^+$.

As in the case of $\vee$, we must impose additional conditions on weak images to ensure substitution and soundness for the interpretation of $\exists$. Let $(\exists y \in Q)\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$ be as above. The substitution lemma says that for $\sigma : \vec{P} \to \vec{P}'$ (assuming also $y \notin \vec{P}'$, up to α-equivalence), we have

$$\begin{aligned}
([\sigma](\exists y \in Q)\phi)^{\mathcal{M}} &= \\
((\exists y \in Q)[\sigma \times 1_{\{y\}}]\phi)^{\mathcal{M}} &= \\
\pi_1(([\sigma \times 1_{\{y\}}]\phi)^{\mathcal{M}}) &\overset{\text{IH}}{=} \\
\pi_1((\sigma^{\mathcal{M}} \times 1_{Q^{\mathcal{M}}})^*(\phi^{\mathcal{M}})) &\overset{?}{=} (\sigma^{\mathcal{M}})^*(\pi_1(\phi^{\mathcal{M}})) \\
&= (\sigma^{\mathcal{M}})^*(((\exists y \in Q)\phi)^{\mathcal{M}}),
\end{aligned}$$

i.e., in the diagram

$$\begin{tikzcd}
([\sigma \times 1_{\{y\}}]\phi)^{\mathcal{M}} \arrow[rr, dashed, two heads] \arrow[dr, hook] \arrow[dd] & & (\sigma^{\mathcal{M}})^*(((\exists y \in Q)\phi)^{\mathcal{M}}) \arrow[dr, dashed, hook] \arrow[dd] & \\
& \prod \vec{P}'^{\mathcal{M}} \times Q^{\mathcal{M}} \arrow[rr, "\pi_1 = (\sigma^{\mathcal{M}})^*(\pi_1)"] \arrow[dd, "\sigma^{\mathcal{M}} \times 1_{Q^{\mathcal{M}}} = \pi_1^*(\sigma^{\mathcal{M}})"] & & \prod \vec{P}'^{\mathcal{M}} \arrow[dd, "\sigma^{\mathcal{M}}"] \\
\phi^{\mathcal{M}} \arrow[rr, dashed, two heads] \arrow[dr, hook] & & ((\exists y \in Q)\phi)^{\mathcal{M}} \arrow[dr, dashed, hook] & \\
& \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \arrow[rr, "\pi_1"'] & & \prod \vec{P}^{\mathcal{M}}
\end{tikzcd}$$

where the left, right, front, and back squares are pullbacks and the lower dashed composite is an essential image factorization, also the top dashed composite is an essential image factorization; in other words, the essential image $((\exists y \in Q)\phi)^{\mathcal{M}} = \pi_1(\phi^{\mathcal{M}})$ is "preserved" under the pullback $(\sigma^{\mathcal{M}})^*$. As before, $\sigma^{\mathcal{M}}$ may be any morphism with codomain $\prod \vec{P}^{\mathcal{M}}$, depending on $\mathcal{M}, \sigma$.

We call an essential image of $f : X \to Y \in \mathsf{C}$ in a lex category C **pullback-stable** if for any $g : Z \to Y$, in the diagram of pullback squares

$$\begin{tikzcd}
g^*(X) \arrow[r, two heads, "g^*(h)"'] \arrow[rr, bend left, "g^*(f)"] \arrow[d] & g^*(\mathrm{im}(f)) \arrow[r, hook] \arrow[d] & Z \arrow[d, "g"] \\
X \arrow[r, two heads, "h"] \arrow[rr, bend right, "f"'] & \mathrm{im}(f) \arrow[r, hook] & Y
\end{tikzcd}$$

[30]Thanks to Anton Bernshteyn for suggesting this problem.

the upper row yields the essential image factorization of $g^*(f)$. Since pullback automatically preserves monomorphisms, this is equivalent to just requiring $g^*(h)$ to be an extremal epimorphism, for any $g : Z \to Y$; since we may take g to be a composite $Z \xrightarrow{g'} \mathrm{im}(f) \hookrightarrow Y$, whose pullback along $\mathrm{im}(f) \hookrightarrow Y$ is g' (because $\mathrm{im}(f) \hookrightarrow Y$ is monic), this is equivalent to requiring the extremal epimorphism $h : X \twoheadrightarrow \mathrm{im}(f)$ to be preserved under pullback. We call a pullback-stable essential image an **image**. When f above is the composite $A \hookrightarrow X \xrightarrow{f} Y$ with a subobject inclusion $A \subseteq X$, pullback-stability amounts to

$$g^*(f(A)) = g^*(f)(f^*(g)^*(A)) \subseteq Z$$

for any $g : Z \to Y$.

We now define $(\exists y \in Q)\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$ to be interpretable in $\mathcal{M}$ if ϕ is, and the image

$$((\exists y \in Q)\phi)^{\mathcal{M}} = \pi_1(\phi^{\mathcal{M}}) \quad \text{where } \pi_1 : \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}}$$

exists. The substitution lemma follows from the above discussion.

Proof of soundness, for proofs ending in a rule involving $\exists$. The $(\exists\Rightarrow)$ and $(\Rightarrow\exists)$ rules follow from the definition of $((\exists y \in Q)\phi)^{\mathcal{M}}$ and Exercise 26.5. We now verify the (Frob) rule: let $\phi, (\exists y \in Q)\psi \in \mathcal{L}_{\omega\omega}(\vec{P})$ with $\psi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$, $y \notin \vec{P}$. We have

$$\begin{aligned}
(\phi \wedge (\exists y \in Q)\psi)^{\mathcal{M}} &= \phi^{\mathcal{M}} \cap (\pi_1 : \textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}})(\psi^{\mathcal{M}}) \\
&= i^*((\pi_1 : \textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}})(\psi^{\mathcal{M}})) \\
&= (\pi_1 : \phi^{\mathcal{M}} \times Q^{\mathcal{M}} \to \phi^{\mathcal{M}})((\pi_1^*(i) : \phi^{\mathcal{M}} \times Q^{\mathcal{M}} \hookrightarrow \textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}})^*(\psi^{\mathcal{M}})) \\
&\qquad\qquad \text{by pullback-stability} \\
&= (\pi_1 : \phi^{\mathcal{M}} \times Q^{\mathcal{M}} \to \phi^{\mathcal{M}})((\phi^{\mathcal{M}} \hookrightarrow \textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}})^*(\psi^{\mathcal{M}})) \quad \text{by subst.} \\
&= (\pi_1 : \phi^{\mathcal{M}} \times Q^{\mathcal{M}} \to \phi^{\mathcal{M}})((\phi \wedge \psi)^{\mathcal{M}}) \subseteq \phi^{\mathcal{M}} \subseteq \textstyle\prod \vec{P}^{\mathcal{M}}
\end{aligned}$$

where $i : \phi^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}}$ is the inclusion. To finish, we need to know that the image under $\pi_1 : \phi^{\mathcal{M}} \times Q^{\mathcal{M}} \to \phi^{\mathcal{M}}$ is the same as that under $\pi_1 : \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}}$. This is a general fact:

Lemma 29.7. Let C be a lex category, $f : X \to Y \in \mathsf{C}$, $B \subseteq Y$, and $A \subseteq f^*(B) \subseteq X$. Then letting $f|f^*(B) : f^*(B) \to B$ be the pullback of f along $B \hookrightarrow Y$, we have

$$f(A) = (f|f^*(B))(A) \subseteq B \subseteq Y,$$

either side existing if the other does, and either side being pullback-stable if the other is.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, dashed, two heads] & f^*(B) \arrow[r, hook] \arrow[d, "f|f^*(B)"'] & X \arrow[d, "f"] \\
f(A) = (f|f^*(B))(A) \arrow[r, hook, dashed] & B \arrow[r, hook] & Y
\end{tikzcd}$$

Proof. If $(f|f^*(B))(A)$ exists, then the factorization $A \twoheadrightarrow (f|f^*(B))(A) \hookrightarrow B \hookrightarrow Y$ must be the essential image of $A \hookrightarrow f^*(B) \hookrightarrow X \xrightarrow{f} Y$, i.e., $f(A)$. Conversely, if $f(A)$ exists, then since $A \hookrightarrow X \xrightarrow{f} Y$ factors through $B \hookrightarrow Y$, we have $f(A) \subseteq B$, yielding the bottom dashed morphism in the above diagram, with the left square commuting since the outer rectangle commutes and $B \hookrightarrow Y$ is monic; thus $A \twoheadrightarrow f(A) \hookrightarrow B$ must be the essential image factorization of $A \hookrightarrow f^*(B) \xrightarrow{f|f^*(B)} B$. So either essential image $f(A) = (f|f^*(B))(A)$ exists if the other does; and either is pullback-stable iff the extremal epimorphism $A \twoheadrightarrow f(A)$ is preserved under pullback. $\square$

Take $f := \pi_1 : X := \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to Y := \prod \vec{P}^{\mathcal{M}}$, $B := \phi^{\mathcal{M}}$, and $A := (\phi \wedge \psi)^{\mathcal{M}}$. $\square$

30 Regular and coherent categories

A lex category C is called:

- **regular**, if every morphism in it has an image, or equivalently, every subobject $A \subseteq X$ has an image under every morphism $f : X \to Y$;
- **coherent**, if it is regular, and has a strict initial object as well as binary unions of subobjects (equivalently, finite unions of subobjects);
- **Boolean coherent**, if it is coherent and each subobject has a complement.

These are the categories in which it is always possible to interpret the regular, coherent, and full fragments of first-order logic, respectively.

A lex functor $F : \mathsf{C} \to \mathsf{D}$ is called:

- **regular** if C, D are regular and F preserves images (of either morphisms or subobjects), or equivalently, extremal epimorphisms;
- **coherent** if C, D are coherent, F is regular, and F preserves the initial object as well as binary unions of subobjects (hence F also preserves pullback-stable complements that exist).

Let

$$\mathfrak{BoolCoh} \subseteq \mathfrak{Coh} \subseteq \mathfrak{Reg} \subseteq \mathfrak{Lex} \subseteq \mathfrak{Cat}$$

denote the sub-2-categories of (Boolean) coherent/regular categories, coherent/regular functors, and all natural transformations. The first inclusion is full (an isomorphism on each hom-category), while the others are only **locally full** (full on each hom-category).

31 The lex fragment

While the Horn fragment of first-order logic can be interpreted in any lex category, it does not fully capture everything that can be expressed in a lex manner. This is remedied by the **lex** (also known as **Cartesian**, **finite limit**, **essentially algebraic**) fragment. Roughly speaking, a **lex formula** is built using $\wedge, \top$, and "provably unique $\exists$". Note that this is *not* the same as the "exists unique" quantifier $\exists!$: we are only allowed to write down $\exists$, *provided we already know it is unique.* This is illustrated by the following example:

Example 31.1. Recall the language of categories $\mathcal{L}_{\mathsf{Cat}} = \{\mathrm{Ob}, \mathrm{Mor}, \sigma, \tau, 1_{(-)}, \mathrm{graph}(\circ)\}$ from Example 25.2. The **theory of categories** $\mathcal{T}_{\mathsf{Cat}}$ consists of the Horn axioms

$$\begin{aligned}
\top &\Rightarrow_{(X \mapsto \mathrm{Ob})} (\sigma(1_X) = X) \wedge (\tau(1_X) = X), \\
\mathrm{graph}(\circ)(f, g, h) &\Rightarrow_{(f,g,h \mapsto \mathrm{Mor})} (\sigma(f) = \tau(g)) \wedge (\tau(f) = \tau(h)) \wedge (\sigma(g) = \sigma(h)), \\
\top &\Rightarrow_{(f \mapsto \mathrm{Mor})} \mathrm{graph}(\circ)(f, 1_{\sigma(f)}, f) \wedge \mathrm{graph}(\circ)(1_{\tau(f)}, f, f), \\
\mathrm{graph}(\circ)(f, g, k) \wedge \mathrm{graph}(\circ)(g, h, l) \wedge \mathrm{graph}(\circ)(k, h, m) &\Rightarrow_{(f,g,h,k,l,m \mapsto \mathrm{Mor})} \mathrm{graph}(\circ)(f, l, m),
\end{aligned}$$

together with the non-Horn axiom asserting that graph($\circ$) is indeed the graph of a function $\mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \to \mathrm{Mor}$:

$$(\sigma(f) = \tau(g)) \Rightarrow_{(f,g \mapsto \mathrm{Mor})} (\exists! h \in \mathrm{Mor})\, \mathrm{graph}(\circ)(f, g, h).$$

As noted above, this is not yet a lex sequent; however, whenever $\exists!$ appears on the RHS of a sequent, we may split it into two sequents, the first asserting uniqueness which is Horn, the second asserting existence which is unique given the first:

$$\begin{aligned}
\mathrm{graph}(\circ)(f, g, h) \wedge \mathrm{graph}(\circ)(f, g, h') &\Rightarrow_{(f,g,h,h' \mapsto \mathrm{Mor})} (h = h'), \\
(\sigma(f) = \tau(g)) &\Rightarrow_{(f,g \mapsto \mathrm{Mor})} (\exists h \in \mathrm{Mor})\, \mathrm{graph}(\circ)(f, g, h).
\end{aligned}$$

The interpretations of these two sequents in a $\mathcal{L}_{\mathsf{Cat}}$-structure $\mathcal{M}$ in a lex category C will together say that the composite $\mathrm{graph}(\circ)^{\mathcal{M}} \hookrightarrow \mathrm{Mor}^{\mathcal{M}} \times \mathrm{Mor}^{\mathcal{M}} \times \mathrm{Mor}^{\mathcal{M}} \xrightarrow{\pi_{12}} \mathrm{Mor}^{\mathcal{M}} \times \mathrm{Mor}^{\mathcal{M}}$ is an isomorphism with $\mathrm{Mor}^{\mathcal{M}} \times_{\mathrm{Ob}^{\mathcal{M}}} \mathrm{Mor}^{\mathcal{M}} \subseteq \mathrm{Mor}^{\mathcal{M}} \times \mathrm{Mor}^{\mathcal{M}}$, whence $\mathrm{graph}(\circ)^{\mathcal{M}}$ is the graph of a unique morphism $\mathrm{Mor}^{\mathcal{M}} \times_{\mathrm{Ob}^{\mathcal{M}}} \mathrm{Mor}^{\mathcal{M}} \to \mathrm{Mor}^{\mathcal{M}}$ by Proposition 23.4(b) (see Example 31.10).

Exercise 31.2 (categories are not Horn-axiomatizable)**.** Let $(\mathcal{L}, \mathcal{T})$ be a (possibly multi-sorted or infinitary, but small) Horn theory, $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ be the category of models of $\mathcal{T}$ in Set and homomorphisms between them (see Section 32).

(a) Show that extremal epimorphisms in $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ are precisely the surjective homomorphisms.

Let $U : \mathsf{Mod}(\mathcal{L}, \mathcal{T}) \to \mathsf{Set}^{\mathcal{L}_{\mathrm{sort}}}$ be the forgetful functor. As described in Remark 19.6, U has a left adjoint $F : \mathsf{Set}^{\mathcal{L}_{\mathrm{sort}}} \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$.

(b) Show that for any $f : F(\vec{X}) \to \mathcal{N} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T})$, where $\vec{X} = (X_P)_{P \in \mathcal{L}_{\mathrm{sort}}} \in \mathsf{Set}^{\mathcal{L}_{\mathrm{sort}}}$, and any extremal epimorphism $g : \mathcal{M} \to \mathcal{N} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T})$, there is a (not necessarily unique) lift h such that $f = g \circ h$.

$$\begin{tikzcd}
& \mathcal{M} \arrow[d, two heads, "g"] \\
F(\vec{X}) \arrow[ur, dashed, "h"] \arrow[r, "f"'] & \mathcal{N}
\end{tikzcd}$$

This property of $F(\vec{X})$ is called **(extremal) projectivity**. In Set, projectivity of a set X is the axiom of choice for families of sets indexed by X.

(c) Conclude that $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$ has **enough projectives**: every $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ admits an extremal epimorphism from a projective object.

(d) Show that in Cat, a functor $F : \mathsf{C} \to \mathsf{D}$ is extremal epic iff the image of F generates D as a subcategory.

(e) Show that in Cat, any (small) category C with a morphism $f : X \to Y$ between two distinct objects $X \neq Y$ is not projective. [Consider an extremal epimorphism $\mathbb{I}_2 \sqcup \mathbb{I}_2 \to \mathbb{I}_3$, where $\mathbb{I}_n :=$ indiscrete category with n objects.]

(f) Conclude that Cat does not have enough projectives, thus is not equivalent to $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$, for *any* Horn theory $(\mathcal{L}, \mathcal{T})$ (note that $\mathcal{L}$ need not be $\mathcal{L}_{\mathsf{Cat}}$).

The formal definition of the lex fragment is somewhat involved, since (as in the above example) which existentials are provably unique depends on a "background" theory. For any first-order theory $(\mathcal{L}, \mathcal{T})$, we define the following concepts simultaneously inductively:

- The **$\mathcal{T}$-lex** formulas are built from atomic formulas using $\wedge, \top$ and $\mathcal{T}$-lex-provably unique $\exists$, where an existential formula $(\exists y \in Q)\phi$ with $\phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$, $\vec{P} \in \mathcal{L}_{\text{sort}}^X$, $y \notin X$ is **$\mathcal{T}$-lex-provably unique** if
$$\mathcal{T} \vdash_{\text{lex}} \phi \wedge [y \mapsto z]\phi \Rightarrow_{\vec{P} \sqcup (y \mapsto Q) \sqcup (z \mapsto Q)} (y = z);$$
we henceforth refer to this sequent via **"$(\exists y \in Q)\phi$ is unique"**. We write
$$\mathcal{L}_{\omega\omega}^{\text{Horn}}(\vec{P}) \subseteq \mathcal{L}_{\omega\omega}^{\mathcal{T}\text{-lex}} := \{\mathcal{T}\text{-lex formulas}\} \subseteq \mathcal{L}_{\omega\omega}^{\text{reg}}(\vec{P}).$$
- A **$\mathcal{T}$-lex sequent** is one between two $\mathcal{T}$-lex formulas.
- A $\mathcal{T}$-lex sequent $\phi \Rightarrow_{\vec{P}} \psi$ is **$\mathcal{T}$-lex-provable**, denoted
$$\mathcal{T} \vdash_{\text{lex}} \phi \Rightarrow_{\vec{P}} \psi,$$
if there is a proof from hypotheses in $\mathcal{T}$ using only intermediate sequents which are $\mathcal{T}$-lex. (We do not yet demand that the sequents in $\mathcal{T}$ themselves be $\mathcal{T}$-lex.)

We now define a theory $\mathcal{T}$ to be a **lex theory** if there is a well-founded relation $\prec$ on it such that every sequent in $\mathcal{T}$ is lex with respect to its set of $\prec$-predecessors. In other words, we may build up $\mathcal{T}$ by starting with some Horn sequents, then adding some lex sequents relative to those, then adding some lex sequents relative to everything before, and so on.

Example 31.3. The existence axiom from Example 31.1 is lex relative to the uniqueness axiom.

More generally, suppose we have three $\mathcal{T}$-lex formulas $\phi \in \mathcal{L}_{\omega\omega}^{\mathcal{T}\text{-lex}}(\vec{P})$, $\psi \in \mathcal{L}_{\omega\omega}^{\mathcal{T}\text{-lex}}(\vec{Q})$, and $\theta \in \mathcal{L}_{\omega\omega}^{\mathcal{T}\text{-lex}}(\vec{P} \sqcup \vec{Q})$, where $\vec{P} \in \mathcal{L}_{\text{sort}}^X$ and $\vec{Q} \in \mathcal{L}_{\text{sort}}^Y$. Then the three sequents
$$\theta \Rightarrow_{\vec{P} \sqcup \vec{Q}} \phi \wedge \psi,$$
$$\theta \wedge [Y \to Y']\theta \Rightarrow_{\vec{P} \sqcup \vec{Q} \sqcup \vec{Q}'} (Y = Y'),$$
$$\phi \Rightarrow_{\vec{P}} (\exists \vec{Q})\theta,$$
where $\vec{Q}' \in \mathcal{L}_{\text{sort}}^{Y'}$ is a disjoint copy of $\vec{Q}$, and $(Y = Y')$ and $\exists \vec{Q}$ have the obvious meanings, together express that **"θ is the graph of some $\phi \to \psi$"**. The first two sequents are $\mathcal{T}$-lex, while the third is $(\mathcal{T} \cup \{\text{second}\})$-lex, hence $\mathcal{T}$-lex if the second sequent is $\mathcal{T}$-lex-provable.

Lemma 31.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure in a lex category C, and $\phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$ be interpretable in $\mathcal{M}$. Then $\mathcal{M} \models$ "$(\exists y \in Q)\phi$ is unique" iff the composite

$$\phi^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \xrightarrow{\pi_1} \prod \vec{P}^{\mathcal{M}}$$

is monic.

Proof. Using the substitution lemma for ϕ, we have $(\phi \wedge [y \mapsto z]\phi)^{\mathcal{M}} = \pi_{12}^*(\phi^{\mathcal{M}}) \cap \pi_{13}^*(\phi^{\mathcal{M}})$ where $\pi_{1i} : \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}}$ is the projection to the first and ith factors. On the other hand, $(y = z)^{\mathcal{M}} = \mathrm{eq}(\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \times Q^{\mathcal{M}} \underset{\pi_3}{\overset{\pi_2}{\rightrightarrows}} Q^{\mathcal{M}})$. So the claim amounts to:

> In any lex category C, for $X, Y \in \mathsf{C}$ and $A \subseteq X \times Y$, we have
>
> $$\pi_{12}^*(A) \cap \pi_{13}^*(A) \subseteq \mathrm{eq}(\pi_2, \pi_3) \subseteq X \times Y \times Y$$
>
> iff the composite $A \hookrightarrow X \times Y \xrightarrow{\pi_1} X$ is monic.

In Set, this says

$$\{(x, y, z) \in X \times Y \times Y \mid (x, y), (x, z) \in A\} \subseteq \{(x, y, z) \in X \times Y \times Y \mid y = z\}$$

iff the composite $A \hookrightarrow X \times Y \xrightarrow{\pi_1} X$ is injective, which is clearly true. Now apply Yoneda. $\square$

Lemma 31.5. Let C be a lex category, $f : X \hookrightarrow Y \in \mathsf{C}$ be a monomorphism. Then the (pullback-stable) image of f is f itself regarded as a subobject of Y.

Proof. f factors through another subobject $A \subseteq Y$ iff $X \subseteq A \subseteq Y$; clearly the smallest such A is X. For pullback-stability: given $g : Z \to Y$, $g^*(f) : g^*(X) \hookrightarrow Z$ is still monic, hence is its own essential image, which is exactly what pullback-stability means. $\square$

Corollary 31.6. Let C be a lex category, $f : X \to Y \in \mathsf{C}$, and $A \subseteq X$ such that the composite $A \hookrightarrow X \xrightarrow{f} Y$ is monic. Then the (pullback-stable) image $f(A) \subseteq Y$ is given by the composite $A \hookrightarrow X \xrightarrow{f} Y$. $\square$

Corollary 31.7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure in a lex category C, and $\phi \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup (y \mapsto Q))$ be interpretable in $\mathcal{M}$. If $\mathcal{M} \models$ "$(\exists y \in Q)\phi$ is unique", then $(\exists y \in Q)\phi$ is interpretable in $\mathcal{M}$, with

$$((\exists y \in Q)\phi)^{\mathcal{M}} = \pi_1(\phi^{\mathcal{M}}) = (\phi^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \xrightarrow{\pi_1} \prod \vec{P}^{\mathcal{M}}) \subseteq \prod \vec{P}^{\mathcal{M}}. \qquad \square$$

Proposition 31.8 (lex soundness)**.** Let $\mathcal{M}$ be a model of a (not necessarily lex) theory $(\mathcal{L}, \mathcal{T})$ in a lex category C.

(a) $\mathcal{T}$-lex formulas are interpretable in $\mathcal{M}$.

(b) If $\mathcal{T} \vdash_{\mathrm{lex}} \phi \Rightarrow_{\vec{P}} \psi$, then $\mathcal{M} \models \phi \Rightarrow_{\vec{P}} \psi$.

Proof. We prove both parts simultaneously by induction on the definition of $\mathcal{T}$-lex formula and $\mathcal{T}$-lex provability. That is, we show:

(a) If a formula ϕ is built using $\wedge, \top$ and $\exists$ which is unique in $\mathcal{M}$, then it is interpretable in $\mathcal{M}$. This is by the preceding corollary.

(b) If $\mathcal{T} \vdash \phi \Rightarrow_{\vec{P}} \psi$ using only intermediate sequents which are interpretable in $\mathcal{M}$, then $\mathcal{M} \models \phi \Rightarrow_{\vec{P}} \psi$. This is by general soundness (Proposition 20.1).

Thus, the families of formulas interpretable in $\mathcal{M}$, as well as sequents satisfied by $\mathcal{M}$, also satisfy the definitions of $\mathcal{T}$-lex formula and $\mathcal{T}$-lex provability, hence contain the smallest such families. $\square$

Corollary 31.9. Let $\mathcal{M}$ be an $\mathcal{L}$-structure in a lex category C, and let $\mathcal{T}$ be a lex theory. Then either $\mathcal{M} \models \mathcal{T}$, or there is a sequent in $\mathcal{T}$ which is interpretable but not satisfied in $\mathcal{M}$.

Proof. If there is no sequent in $\mathcal{T}$ which is interpretable but not satisfied in $\mathcal{M}$, then by induction on the well-founded relation $\prec$ witnessing that $\mathcal{T}$ is lex, using lex soundness, we have $\mathcal{M} \models \mathcal{T}$. $\square$

Example 31.10. Let $\mathcal{M}$ be an $\mathcal{L}$-structure in a lex category C, and let $\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$, $\psi \in \mathcal{L}_{\omega\omega}(\vec{Q})$, and $\theta \in \mathcal{L}_{\omega\omega}(\vec{P} \sqcup \vec{Q})$ be such that

$$\mathcal{M} \models \text{“}\theta \text{ is the graph of some } \phi \to \psi\text{”}$$

according to the three sequents in Example 31.3. The first of these says

$$\theta^{\mathcal{M}} \subseteq \pi_1^*(\phi^{\mathcal{M}}) \cap \pi_2^*(\psi^{\mathcal{M}}) = \phi^{\mathcal{M}} \times \psi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}} \times \prod \vec{Q}^{\mathcal{M}}$$

where $\pi_1, \pi_2 : \prod \vec{P}^{\mathcal{M}} \times \prod \vec{Q}^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}}, \prod \vec{Q}^{\mathcal{M}}$ are the projections. The second sequent says that “$(\exists \vec{Q})\theta$ is unique”, i.e., the composite

$$\theta^{\mathcal{M}} \hookrightarrow \prod \vec{P}^{\mathcal{M}} \times \prod \vec{Q}^{\mathcal{M}} \xrightarrow{\pi_1} \prod \vec{P}^{\mathcal{M}}$$

is monic, while the third sequent says that this composite contains $\phi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$; since $\theta^{\mathcal{M}} \subseteq \pi_1^*(\phi^{\mathcal{M}})$, the above composite is in fact an isomorphism with $\phi^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$. In other words, by Proposition 23.4(b), we have shown

$$(\mathcal{M} \models \text{“}\theta \text{ is the graph of some } \phi \to \psi\text{”}) \iff (\theta^{\mathcal{M}} \text{ is the graph of some } \phi^{\mathcal{M}} \to \psi^{\mathcal{M}}).$$

For example, the last two sequents in $\mathcal{T}_{\mathsf{Cat}}$ from Example 31.1 say precisely that $\mathrm{graph}(\circ)^{\mathcal{M}} \subseteq (\mathrm{Mor}^{\mathcal{M}})^3$ is the graph of a morphism from $(\sigma(f) = \tau(g))^{\mathcal{M}} = \mathrm{eq}(\mathrm{Mor}^{\mathcal{M}} \times \mathrm{Mor}^{\mathcal{M}} \underset{\tau^{\mathcal{M}} \circ \pi_2}{\overset{\sigma^{\mathcal{M}} \circ \pi_1}{\rightrightarrows}} \mathrm{Ob}^{\mathcal{M}}) \cong \mathrm{Mor}^{\mathcal{M}} \times_{\mathrm{Ob}^{\mathcal{M}}} \mathrm{Mor}^{\mathcal{M}}$ to $\mathrm{Mor}^{\mathcal{M}}$.

32 The 2-copresheaf of models

Let $\mathcal{L}$ be a first-order language and C be a lex category. Recall (Section 25) that $\mathsf{Mod}(\mathcal{L}, \mathsf{C})$ denotes the category of $\mathcal{L}$-structures in C and $\mathcal{L}$-homomorphisms between them. For a first-order $\mathcal{L}$-theory $\mathcal{T}$, we define the full subcategories

$$\begin{aligned} \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C}) &:= \{\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C}) \mid \mathcal{M} \models \mathcal{T}\} \subseteq \mathsf{Mod}(\mathcal{L}, \mathsf{C}), \\ \mathsf{Mod}(\mathcal{L}, \mathcal{T}) &:= \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{Set}) \subseteq \mathsf{Mod}(\mathcal{L}). \end{aligned}$$

These categories are best-behaved when $\mathcal{T}$ belongs to some fragment (e.g., regular), and C is a category of the corresponding kind (e.g., a regular category).

Lemma 32.1. Let $f : \mathcal{M} \to \mathcal{N} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$ be a homomorphism.

(a) For each term $t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$, we have

$$f_Q \circ t^{\mathcal{M}} = t^{\mathcal{N}} \circ (\textstyle\prod_x f_{P_x}) : \prod \vec{P}^{\mathcal{M}} \to Q^{\mathcal{N}},$$

i.e., commutativity of

$$\begin{tikzcd}
\prod \vec{P}^{\mathcal{M}} \arrow[r, "\prod_x f_{P_x}"] \arrow[d, "t^{\mathcal{M}}"'] & \prod \vec{P}^{\mathcal{N}} \arrow[d, "t^{\mathcal{N}}"] \\
Q^{\mathcal{M}} \arrow[r, "f_Q"'] & Q^{\mathcal{N}}.
\end{tikzcd}$$

(b) For each *coherent* formula $\phi \in \mathcal{L}^{\mathrm{coh}}_{\omega\omega}(\vec{P})$ interpretable in both $\mathcal{M}, \mathcal{N}$, we have

$$\phi^{\mathcal{M}} \subseteq (\textstyle\prod_x f_{P_x})^*(\phi^{\mathcal{N}}) \subseteq \prod \vec{P}^{\mathcal{M}},$$

i.e., (unique) existence of either dashed morphism in the commutative diagram

$$\begin{tikzcd}
\phi^{\mathcal{M}} \arrow[dr, dashed] \arrow[drr, dashed, bend left] \arrow[ddr, hook] & & \\
& (\prod_x f_{P_x})^*(R^{\mathcal{N}}) \arrow[r] \arrow[d, hook] & \phi^{\mathcal{N}} \arrow[d, hook] \\
& \prod \vec{P}^{\mathcal{M}} \arrow[r, "\prod_x f_{P_x}"'] & \prod \vec{P}^{\mathcal{M}}.
\end{tikzcd}$$

Proof. (a) By induction on t.

(b) By induction on ϕ, using that each operation in C used to interpret coherent connectives and quantifiers is monotone. For example, if $\phi = \psi \vee \theta$:

$$\begin{aligned}
\phi^{\mathcal{M}} &= \psi^{\mathcal{M}} \cup \theta^{\mathcal{M}} \\
&\subseteq (\textstyle\prod_x f_{P_x})^*(\psi^{\mathcal{N}}) \cup (\prod_x f_{P_x})^*(\theta^{\mathcal{N}}) && \text{by IH and monotonicity of } \cup \\
&= (\textstyle\prod_x f_{P_x})^*(\psi^{\mathcal{N}} \cup \theta^{\mathcal{N}}) && \text{by pullback-stability of } \cup \\
&= (\textstyle\prod_x f_{P_x})^*(\phi^{\mathcal{N}}).
\end{aligned}$$

If $\phi = (\exists y \in Q)\psi$:

$$\begin{aligned}
\phi^{\mathcal{M}} &= \pi_1^{\mathcal{M}}(\psi^{\mathcal{M}}) && \text{where } \pi_1^{\mathcal{M}} : \textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \to \prod \vec{P}^{\mathcal{M}} \\
&\subseteq \pi_1^{\mathcal{M}}((\textstyle\prod_x f_{P_x} \times f_Q)^*(\psi^{\mathcal{N}})) && \text{by IH and monotonicity of image} \\
&= \pi_1^{\mathcal{M}}((\textstyle\prod_x f_{P_x} \times 1_{Q^{\mathcal{M}}})^*((1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q)^*(\psi^{\mathcal{N}}))) \\
&= (\textstyle\prod_x f_{P_x})^*(\pi_1^{\mathcal{N}}((1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q)((1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q)^*(\psi^{\mathcal{N}})))) && \text{by pullback-stability of image } (*) \\
&\subseteq (\textstyle\prod_x f_{P_x})^*(\pi_1^{\mathcal{N}}(\psi^{\mathcal{N}})) && \text{by } 1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q \dashv (1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q)^* \\
&= (\textstyle\prod_x f_{P_x})^*(\phi^{\mathcal{N}})
\end{aligned}$$

where $(*)$ refers to the pullback square

$$\begin{tikzcd}
\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \arrow[rr, "\pi_1^{\mathcal{M}}"] \arrow[d, "\prod_x f_{P_x} \times 1_{Q^{\mathcal{M}}}"'] & & \prod \vec{P}^{\mathcal{M}} \arrow[d, "\prod_x f_{P_x}"] \\
\prod \vec{P}^{\mathcal{N}} \times Q^{\mathcal{M}} \arrow[r, "1_{\prod \vec{P}^{\mathcal{N}}} \times f_Q"'] & \prod \vec{P}^{\mathcal{N}} \times Q^{\mathcal{N}} \arrow[r, "\pi_1^{\mathcal{N}}"'] & \prod \vec{P}^{\mathcal{N}}.
\end{tikzcd}$$

□

For a lex functor $F : \mathsf{C} \to \mathsf{D}$ and $\mathcal{L}$-structure $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$, we get an **induced $\mathcal{L}$-structure**

$$\mathsf{Mod}(\mathcal{L}, F)(\mathcal{M}) := F_*(\mathcal{M}) := F \circ \mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathsf{D})$$

with

$$\begin{aligned}
P^{F_*(\mathcal{M})} &:= F(P^{\mathcal{M}}) && \text{for } P \in \mathcal{L}_{\text{sort}}, \\
f^{F_*(\mathcal{M})} &:= F(f^{\mathcal{M}}) : F(\textstyle\prod \vec{P}^{\mathcal{M}}) \cong \prod F(\vec{P}^{\mathcal{M}}) = \prod \vec{P}^{F_*(\mathcal{M})} \to Q^{F_*(\mathcal{M})} = F(Q^{\mathcal{M}}) && \text{for } f \in \mathcal{L}_{\text{fun}}(\vec{P}; Q), \\
R^{F_*(\mathcal{M})} &:= F(R^{\mathcal{M}}) \subseteq F(\textstyle\prod \vec{P}^{\mathcal{M}}) \cong \prod F(\vec{P}^{\mathcal{M}}) = \prod \vec{P}^{F_*(\mathcal{M})} && \text{for } R \in \mathcal{L}_{\text{rel}}(\vec{P}).
\end{aligned}$$

For an $\mathcal{L}$-homomorphism $f : \mathcal{M} \to \mathcal{N} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$, we get an $\mathcal{L}$-homomorphism

$$\mathsf{Mod}(\mathcal{L}, F)(f) := F_*(f) := (F(f_P))_{P \in \mathcal{L}} : F_*(\mathcal{M}) \longrightarrow F_*(\mathcal{N}) \in \mathsf{Mod}(\mathcal{L}, \mathsf{D});$$

this is a homomorphism by applying F to the commutative squares witnessing that f is a homomorphism:

$$\begin{tikzcd}
F(\prod \vec{P}^{\mathcal{M}}) \cong \prod \vec{P}^{F_*(\mathcal{M})} \arrow[r, "F(\prod_i f_{P_i}) \cong \prod_i F_*(f)_{P_i}"] \arrow[d, "F(g^{\mathcal{M}}) = g^{F_*(\mathcal{M})}"'] & \prod \vec{P}^{F_*(\mathcal{N})} \cong F(\prod \vec{P}^{\mathcal{N}}) \arrow[d, "g^{F_*(\mathcal{N})} = F(g^{\mathcal{N}})"] \\
F(Q^{\mathcal{M}}) = Q^{F_*(\mathcal{M})} \arrow[r, "F(f_Q) = F_*(f)_Q"'] & Q^{F_*(\mathcal{N})} = F(Q^{\mathcal{N}})
\end{tikzcd} \qquad \text{for } g \in \mathcal{L}_{\text{fun}}(\vec{P}; Q),$$

$$\begin{tikzcd}
F(R^{\mathcal{M}}) = R^{F_*(\mathcal{M})} \arrow[r, dashed] \arrow[d, hook] & R^{F_*(\mathcal{N})} = F(R^{\mathcal{N}}) \arrow[d, hook] \\
F(\prod \vec{P}^{\mathcal{M}}) \cong \prod \vec{P}^{F_*(\mathcal{M})} \arrow[r, "F(\prod_i f_{P_i}) \cong \prod_i F_*(f)_{P_i}"'] & \prod \vec{P}^{F_*(\mathcal{N})} \cong F(\prod \vec{P}^{\mathcal{N}})
\end{tikzcd} \qquad \text{for } R \in \mathcal{L}_{\text{rel}}(\vec{P}).$$

Thus we get a functor

$$\mathsf{Mod}(\mathcal{L}, F) := F_* : \mathsf{Mod}(\mathcal{L}, \mathsf{C}) \longrightarrow \mathsf{Mod}(\mathcal{L}, \mathsf{D}).$$

Lemma 32.2. (a) For every term $t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$, we have

$$t^{F_*(\mathcal{M})} = F(t^{\mathcal{M}}) : F(\textstyle\prod \vec{P}^{\mathcal{M}}) \cong \prod F(\vec{P}^{\mathcal{M}}) = \prod \vec{P}^{F_*(\mathcal{M})} \to Q^{F_*(\mathcal{M})} = F(Q^{\mathcal{M}}).$$

(b) For every formula $\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$ such that

- ϕ is interpretable in $\mathcal{M}$, and F is coherent; or
- ϕ is regular, and F is regular; or

- ϕ is $\mathcal{T}$-lex, for any $\mathcal{L}$-theory $\mathcal{T}$ such that $\mathcal{M} \models \mathcal{T}$,

we have

$$\phi^{F_*(\mathcal{M})} = F(\phi^{\mathcal{M}}) \subseteq F(\prod \vec{P}^{\mathcal{M}}) \cong \prod F(\vec{P}^{\mathcal{M}}) = \prod \vec{P}^{F_*(\mathcal{M})}.$$

Proof. (a) By induction on t.

(b) By induction on ϕ. In the lex case, by soundness, every existential occurring in ϕ is unique in $\mathcal{M}$, hence by Corollary 31.7 interpreted as a composition in $\mathcal{M}$, hence preserved by F. $\square$

Corollary 32.3. If F is coherent (regular, lex), and $\mathcal{M} \models \mathcal{T}$ for a (regular, lex) theory $\mathcal{T}$, then $F_*(\mathcal{M}) \models \mathcal{T}$.

Proof. In the lex case, use induction on the well-founded relation $\prec$ witnessing that $\mathcal{T}$ is lex. $\square$

Thus for a coherent (coherent, regular, lex) functor $F : \mathsf{C} \to \mathsf{D}$ between Boolean coherent (coherent, regular, lex) categories C, D, and a (coherent, regular, lex) theory $(\mathcal{L}, \mathcal{T})$, the functor $\mathsf{Mod}(\mathcal{L}, F) = F_* : \mathsf{Mod}(\mathcal{L}, \mathsf{C}) \to \mathsf{Mod}(\mathcal{L}, \mathsf{D})$ restricts to a functor between the full subcategories

$$\mathsf{Mod}(\mathcal{L}, \mathcal{T}, F) = F_* : \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C}) \longrightarrow \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{D}),$$

yielding a functor $\mathsf{Mod}(\mathcal{L}, \mathcal{T}, -)$. We can turn it into a 2-functor, as follows.

For a natural transformation $\alpha : F \to G : \mathsf{C} \to \mathsf{D}$ between lex functors, for an $\mathcal{L}$-structure $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$, we have an $\mathcal{L}$-homomorphism

$$\begin{aligned}\mathsf{Mod}(\mathcal{L}, \alpha)(\mathcal{M}) := \alpha_{\mathcal{M}} := (\alpha_{P^{\mathcal{M}}} : F(P^{\mathcal{M}}) = P^{F_*(\mathcal{M})} \to P^{G_*(\mathcal{M})} = G(P^{\mathcal{M}}))_{P \in \mathcal{L}_{\mathrm{sort}}} \\ : F_*(\mathcal{M}) \longrightarrow G_*(\mathcal{M}) \in \mathsf{Mod}(\mathcal{L}, \mathsf{D}),\end{aligned}$$

by the naturality squares

$$\begin{tikzcd}
F(\prod \vec{P}^{\mathcal{M}}) \cong \prod \vec{P}^{F_*(\mathcal{M})} \arrow[r, "\alpha_{\prod \vec{P}^{\mathcal{M}}} \cong \prod_i \alpha_{P_i^{\mathcal{M}}}"] \arrow[d, "F(f^{\mathcal{M}}) = f^{F_*(\mathcal{M})}"'] & \prod \vec{P}^{G_*(\mathcal{M})} \cong G(\prod \vec{P}^{\mathcal{M}}) \arrow[d, "f^{G_*(\mathcal{M})} = G(f^{\mathcal{M}})"] \\
F(Q^{\mathcal{M}}) = Q^{F_*(\mathcal{M})} \arrow[r, "\alpha_{Q^{\mathcal{M}}}"'] & Q^{G_*(\mathcal{M})} = G(Q^{\mathcal{M}})
\end{tikzcd} \qquad \text{for } f \in \mathcal{L}_{\mathrm{fun}}(\vec{P}; Q),$$

$$\begin{tikzcd}
F(R^{\mathcal{M}}) = R^{F_*(\mathcal{M})} \arrow[r, dashed, "\alpha_{R^{\mathcal{M}}}"] \arrow[d, hook] & R^{G_*(\mathcal{M})} = G(R^{\mathcal{M}}) \arrow[d, hook] \\
F(\prod \vec{P}^{\mathcal{M}}) \cong \prod \vec{P}^{F_*(\mathcal{M})} \arrow[r, "\alpha_{\prod \vec{P}^{\mathcal{M}}} \cong \prod_i \alpha_{P_i^{\mathcal{M}}}"'] & \prod \vec{P}^{G_*(\mathcal{M})} \cong G(\prod \vec{P}^{\mathcal{M}})
\end{tikzcd} \qquad \text{for } R \in \mathcal{L}_{\mathrm{rel}}(\vec{P})$$

(using that natural transformations between product-preserving functors automatically preserve products, Proposition 22.12). These homomorphisms are natural in $\mathcal{M}$, by naturality of α: for a homomorphism $f : \mathcal{M} \to \mathcal{N} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$, for each $P \in \mathcal{L}_{\mathrm{sort}}$, we have the naturality square

$$\begin{tikzcd}
F(P^{\mathcal{M}}) = P^{F_*(\mathcal{M})} \arrow[r, "\alpha_{P^{\mathcal{M}}}"] \arrow[d, "F(f_P) = F_*(f)_P"'] & P^{G_*(\mathcal{M})} = G(P^{\mathcal{M}}) \arrow[d, "G_*(f)_P = G(f_P)"] \\
F(P^{\mathcal{N}}) = P^{F_*(\mathcal{N})} \arrow[r, "\alpha_{P^{\mathcal{N}}}"'] & P^{G_*(\mathcal{N})} = G(P^{\mathcal{N}})
\end{tikzcd}$$

which is the Pth coordinate of the naturality square for $\mathsf{Mod}(\mathcal{L},\alpha)$ at f. Thus we get

$$\mathsf{Mod}(\mathcal{L},\alpha) : \mathsf{Mod}(\mathcal{L},F) = F_* \longrightarrow G_* = \mathsf{Mod}(\mathcal{L},G) : \mathsf{Mod}(\mathcal{L},\mathsf{C}) \to \mathsf{Mod}(\mathcal{L},\mathsf{D}).$$

For a (coherent, regular, lex) theory $\mathcal{T}$ and coherent (coherent, regular, lex) $F, G : \mathsf{C} \to \mathsf{D}$, $\mathsf{Mod}(\mathcal{L},\alpha)$ restricts to

$$\mathsf{Mod}(\mathcal{L},\mathcal{T},\alpha) : \mathsf{Mod}(\mathcal{L},\mathcal{T},F) \longrightarrow \mathsf{Mod}(\mathcal{L},\mathcal{T},G) : \mathsf{Mod}(\mathcal{L},\mathcal{T},\mathsf{C}) \to \mathsf{Mod}(\mathcal{L},\mathcal{T},\mathsf{D}).$$

Putting everything together, we get a 2-functor

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{L},\mathcal{T},-) : (\mathfrak{Bool})\mathfrak{Coh}, \mathfrak{Reg}, \mathfrak{Lex} \arrow[r] & \mathfrak{Cat} \\
\mathsf{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[r, mapsto] & \mathsf{Mod}(\mathcal{L},\mathcal{T},\mathsf{C}) \arrow[d, "\mathsf{Mod}(\mathcal{L},\mathcal{T},F)"', shift right=4ex] \arrow[d, "\mathsf{Mod}(\mathcal{L},\mathcal{T},G)", shift left=4ex] \arrow[d, phantom, "\overset{\mathsf{Mod}(\mathcal{L},\mathcal{T},\alpha)}{\Rightarrow}"] \\
\mathsf{D} \arrow[r, mapsto] & \mathsf{Mod}(\mathcal{L},\mathcal{T},\mathsf{D}),
\end{tikzcd}$$

a "2-copresheaf" assigning to each (Boolean) coherent (regular, lex) category C all "possible data" of models of $\mathcal{T}$ in C.

33 Syntactic categories

Let $(\mathcal{L},\mathcal{T})$ be a (coherent, regular, lex) first-order theory. The **Boolean coherent (coherent, regular, lex) syntactic category**

$$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool\,(coh,reg,lex)}}$$

of $\mathcal{T}$ has

- objects: pairs $(\vec{P},\alpha)$, where $\vec{P} \in \mathcal{L}^X_{\mathrm{sort}}$ is a context and $\alpha \in \mathcal{L}^{(\mathrm{coh,reg},\mathcal{T}\text{-lex})}_{\omega\omega}(\vec{P})$ is a formula in context $\vec{P}$;
- morphisms $(\vec{P},\alpha) \to (\vec{Q},\beta)$: equivalence classes $[\phi]$ of formulas $\phi \in \mathcal{L}^{(\mathrm{coh,reg},\mathcal{T}\text{-lex})}_{\omega\omega}(\vec{P} \sqcup \vec{Q})$ such that

  $$\mathcal{T} \vdash_{(\mathrm{coh,reg,lex})} \text{“}\phi \text{ is the graph of some } \alpha \to \beta\text{”}$$

  according to the three sequents from Example 31.3, with two such formulas ϕ, ψ considered equivalent if they are $\mathcal{T}$-provably equivalent in the corresponding fragment:

  $$\phi \equiv^{(\mathrm{coh,reg,lex})}_{\mathcal{T}} \psi \ :\Longleftrightarrow\ \mathcal{T} \vdash_{(\mathrm{coh,reg,lex})} \phi \Leftrightarrow_{\vec{P}} \psi.$$

Proposition 33.1. $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool\,(coh,reg,lex)}}$ is a lex category, with:

(a) identity $1_{(\vec{P},\alpha)} := [\alpha \wedge \bigwedge_{X \ni x \mapsto x' \in X'} (x = x')]$ where $\vec{P'} \in \mathcal{L}^{X'}_{\mathrm{sort}}$ is a disjoint copy of $\vec{P} \in \mathcal{L}^X_{\mathrm{sort}}$;

(b) composition $((\vec{P},\alpha) \xrightarrow{[\phi]} (\vec{Q},\beta) \xrightarrow{[\psi]} (\vec{R},\gamma)) := [(\exists \vec{Q})(\phi \wedge \psi)]$, for disjoint $\vec{P},\vec{Q},\vec{R}$ (otherwise replace them with disjoint copies and make the obvious variable substitutions);

(c) finite products $\prod_i(\vec{P}_i, \alpha_i) := (\bigsqcup_i \vec{P}_i, \bigwedge_i \alpha_i)$, with projections $\pi_i := [\bigwedge_{X_i \ni x \mapsto x' \in X'_i}(x = x')]$ where $\vec{P}'_i \in \mathcal{L}_{\mathrm{sort}}^{X'_i}$ is a disjoint copy of $\vec{P}_i \in \mathcal{L}_{\mathrm{sort}}^{X_i}$, and with universal property witnessed by

$$([\phi_i] : (\vec{Q}, \beta) \to (\vec{P}_i, \alpha_i))_i := [\bigwedge_i \phi_i] : (\vec{Q}, \beta) \to \prod_i(\vec{P}_i, \alpha_i);$$

(d) equalizers $\mathrm{eq}((\vec{P}, \alpha) \underset{[\psi]}{\overset{[\phi]}{\rightrightarrows}} (\vec{Q}, \beta)) := (\vec{P}, (\exists \vec{Q})(\phi \wedge \psi)) \xhookrightarrow{1_{(\vec{P}, (\exists \vec{Q})(\phi \wedge \psi))}} (\vec{P}, \alpha)$;

(e) monomorphisms $[\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)$ with $\mathcal{T} \vdash_{(\mathrm{coh,reg,lex})} \phi \wedge [X \to X']\phi \Rightarrow_{\vec{P} \sqcup \vec{P}' \sqcup \vec{Q}} (X = X')$, where $\vec{P}' \in \mathcal{L}_{\mathrm{sort}}^{X'}$ is a disjoint copy of $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^{X}$;

(f) subobject $\wedge$-lattices

$$\begin{aligned}
\mathrm{Sub}(\vec{P}, \alpha) &\cong \{\phi \in \mathcal{L}_{\omega\omega}(\vec{P}) \mid \mathcal{T} \vdash_{(\mathrm{coh,reg,lex})} \phi \Rightarrow_{\vec{P}} \alpha\}/{\equiv_{\mathcal{T}}^{(\mathrm{coh,reg,lex})}} \\
([\phi] : (\vec{Q}, \beta) \hookrightarrow (\vec{P}, \alpha)) &\mapsto [(\exists \vec{Q})\phi] \\
((\vec{P} \sqcup (y \mapsto Q), \phi) \hookrightarrow (\vec{P} \sqcup (y \mapsto Q), \alpha) \xrightarrow{\pi_1} (\vec{P}, \alpha)) &\mapsto [(\exists y \in Q)\phi] \\
(1_{(\vec{P}, \phi)} : (\vec{P}, \phi) \hookrightarrow (\vec{P}, \alpha)) &\leftarrow\!\shortmid [\phi] \\
(\vec{P}, \phi) \wedge (\vec{P}, \psi) &\leftarrow\!\shortmid [\phi \wedge \psi] \\
\top &\leftarrow\!\shortmid [\alpha],
\end{aligned}$$

with the partial order on the RHS given by $\mathcal{T}$-provable implication;

(g) $\mathrm{graph}([\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)) = (\vec{P} \sqcup \vec{Q}, \phi) \subseteq (\vec{P} \sqcup \vec{Q}, \alpha \wedge \beta) = (\vec{P}, \alpha) \times (\vec{Q}, \beta)$;

(h) isomorphisms $[\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)$ with $\mathcal{T} \vdash_{(\mathrm{coh,reg,lex})}$ "ϕ is the graph of some $\beta \to \alpha$";

(i) for each term $t \in \mathcal{L}_{\omega\omega}^{(\mathrm{coh,reg},\mathcal{T}\text{-lex})}(\vec{P}; Q)$, a particular morphism

$$[t = y] : (\vec{P}, \top) \cong \prod_x((x \mapsto P_x), \top) \to ((y \mapsto Q), \top);$$

(j) pullbacks of subobjects along morphisms of the form $([\sigma(y) = x])_y : (\vec{P}, \top) \cong \prod_x((x \mapsto P_x), \top) \to \prod_y((y \mapsto Q_y), \top) \cong (\vec{Q}, \top)$, for a substitution $\sigma : \vec{Q} \to \vec{P}$, given by

$$([\sigma(y) = x])_y^*(\vec{Q}, \phi) := (\vec{P}, [\sigma]\phi);$$

(k) general fiber products

$$\begin{tikzcd}
(\vec{P}, \alpha) \times_{(\vec{R}, \gamma)} (\vec{Q}, \beta) := (\vec{P} \sqcup \vec{Q}, (\exists \vec{R})(\phi \wedge \psi)) \subseteq (\vec{P}, \alpha) \times (\vec{Q}, \beta) \arrow[r] \arrow[d] & (\vec{Q}, \beta) \arrow[d, "{[\psi]}"] \\
(\vec{P}, \alpha) \arrow[r, "{[\phi]}"'] & (\vec{R}, \gamma).
\end{tikzcd}$$

$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool}\,(\mathrm{coh,reg})}$ is furthermore regular, with:

(l) images of morphisms $\mathrm{im}([\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)) := (\vec{Q}, (\exists \vec{P})\phi)$;

(m) images of subobjects under the particular product projections $\pi_1 : (\vec{P} \sqcup (y \mapsto Q), \top) \cong (\vec{P}, \top) \times ((y \mapsto Q), \top) \to (\vec{P}, \top)$ given by

$$\pi_1(\vec{P} \sqcup (y \mapsto Q), \phi) := (\vec{P}, (\exists y \in Q)\phi);$$

(n) extremal epimorphisms $[\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)$ with $\mathcal{T} \vdash_{(\mathrm{coh,reg})} \beta \Rightarrow_{\vec{Q}} (\exists \vec{P})\phi$.

$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool\,(coh)}}$ is furthermore coherent, with:

(o) strict initial object $\varnothing := (\varnothing, \bot)$, with $[\bot] : \varnothing \to (\vec{P}, \alpha)$ for any other $(\vec{P}, \alpha)$;

(p) joins of subobjects $(\vec{P}, \phi) \vee (\vec{P}, \psi) := (\vec{P}, \phi \vee \psi) \subseteq (\vec{P}, \alpha)$.

$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool}}$ is furthermore Boolean, with:

(q) complements of subobjects $\neg(\vec{P}, \phi) := (\vec{P}, \alpha \wedge \neg\phi) \subseteq (\vec{P}, \alpha)$.

Proof. By explicit constructions of proofs witnessing that the various properties hold.

(a) We must check that $\mathcal{T} \vdash$ "$\phi := \alpha \wedge \bigwedge_{x \mapsto x'}(x = x')$ is the graph of some $\alpha \to [X \to X']\alpha$". We have

$$\mathcal{T} \vdash \phi \Rightarrow_{\vec{P} \sqcup \vec{P}'} \alpha \wedge [X \to X']\alpha$$

by the ($\wedge\!\Rightarrow_1$), Leibniz, and ($\Rightarrow\!\wedge$) rules. We have

$$\mathcal{T} \vdash \phi \wedge [X' \to X'']\phi \Rightarrow_{\vec{P} \sqcup \vec{P}' \sqcup \vec{P}''} (X' = X'')$$

since $\phi \wedge [X' \to X'']\phi$ contains the conjunct $(X = X') \wedge (X = X'')$ (with the obvious abuse of notation), which by Leibniz implies $(X' = X'')$. And we have

$$\mathcal{T} \vdash \alpha \Rightarrow_{\vec{P}} (\exists \vec{P}')\phi$$

by the ($\Rightarrow\!\exists$) rule with witness $X' := X$.

(b) We must first check that $\mathcal{T} \vdash$ "$(\exists \vec{Q})(\phi \wedge \psi)$ is the graph of some $\alpha \to \gamma$". For example, to prove the third sequent

$$\alpha \Rightarrow_{\vec{P}} (\exists \vec{R})(\exists \vec{Q})(\phi \wedge \psi),$$

by (Cut) it is enough to prove

$$\begin{aligned}
\alpha &\Rightarrow_{\vec{P}} (\exists \vec{Q})\phi, \\
(\exists \vec{Q})\phi &\Rightarrow_{\vec{P}} (\exists \vec{Q})(\phi \wedge (\exists \vec{R})\psi), \\
(\exists \vec{Q})(\phi \wedge (\exists \vec{R})\psi) &\Rightarrow_{\vec{P}} (\exists \vec{R})(\exists \vec{Q})(\phi \wedge \psi).
\end{aligned}$$

The first is $\mathcal{T}$-provable since $[\phi]$ is a morphism $(\vec{P}, \alpha) \to (\vec{Q}, \beta)$. For the second sequent:

$$
{\scriptstyle(\Rightarrow\exists)}\,\dfrac{{\scriptstyle(\exists\Rightarrow)}\,\dfrac{{\scriptstyle(\Rightarrow\wedge)}\,\dfrac{{\scriptstyle(\mathrm{ID})}\,\dfrac{}{\phi \Rightarrow_{\vec{P}\sqcup\vec{Q}} \phi} \qquad {\scriptstyle(\mathrm{CUT})}\,\dfrac{\dfrac{(\text{b/c } [\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta))}{\phi \Rightarrow_{\vec{P}\sqcup\vec{Q}} \beta} \qquad {\scriptstyle(\mathrm{SUBST})}\,\dfrac{\dfrac{(\text{b/c } [\psi] : (\vec{Q}, \beta) \to (\vec{R}, \gamma))}{\beta \Rightarrow_{\vec{Q}} (\exists\vec{R})\psi}}{\beta \Rightarrow_{\vec{P}\sqcup\vec{Q}} (\exists\vec{R})\psi}}{\phi \Rightarrow (\exists\vec{R})\psi}}{\phi \Rightarrow_{\vec{P}\sqcup\vec{Q}} \phi \wedge (\exists\vec{R})\psi}}{\phi \Rightarrow_{\vec{P}\sqcup\vec{Q}} (\exists\vec{Q})(\phi \wedge (\exists\vec{R})\psi)}}{(\exists\vec{Q})\phi \Rightarrow_{\vec{P}} (\exists\vec{Q})(\phi \wedge (\exists\vec{R})\psi)}
$$

The third sequent above is a general tautology, provable using $(\exists\Rightarrow)$, $(\Rightarrow\exists)$, and (Frob).

We must then check that composition is associative and unital. For example, associativity of $(\vec{P}, \alpha) \xrightarrow{[\phi]} (\vec{Q}, \beta) \xrightarrow{[\psi]} (\vec{R}, \gamma) \xrightarrow{[\theta]} (\vec{S}, \delta)$ amounts to

$$\mathcal{T} \vdash (\exists\vec{R})((\exists\vec{Q})(\phi \wedge \psi) \wedge \theta) \Leftrightarrow_{\vec{P}\sqcup\vec{S}} (\exists\vec{Q})(\phi \wedge (\exists\vec{R})(\psi \wedge \theta));$$

this is straightforward using the $(\exists\Rightarrow)$, $(\Rightarrow\exists)$, and (Frob) rules.

etc. (**TODO**: do a few more cases) $\square$

Remark 33.2. The reason that Proposition 33.1 is not as trivial as Proposition 16.1 is that the syntax of first-order logic does not correspond as perfectly to coherent/lex/... categories as the syntax of propositional logic does to Boolean algebras/$\wedge$-lattices/.... From the point of view of first-order logic, it would be more natural to capture the syntax via an algebraic structure $\langle \mathcal{L} \mid \mathcal{T} \rangle$ consisting of "objects, n-ary morphisms, and relations" rather than merely "objects and morphisms"; indeed, the slight difficulty in Proposition 33.1 (and Theorem 33.7 below) is because we are constantly converting between the "morphisms" and "relations" views of a lex category (Section 23).

Such a structure consisting of "objects, morphisms, and relations" is known as a **hyperdoctrine**, invented by Lawvere (see [Jacobs]). One can prove that hyperdoctrines (of a suitable kind) are equivalent to lex categories (with suitable additional structure). Composing this equivalence with the "syntactic hyperdoctrine" of a theory $\mathcal{T}$ then yields the syntactic category of $\mathcal{T}$.

Example 33.3. Let $(\mathcal{L}, \mathcal{T})$ be a propositional theory, regarded as a 0-sorted first-order theory as in Example 25.3. Every $\mathcal{L}$-formula is over the empty context, so the objects of $\langle \mathcal{L} \mid \mathcal{T} \rangle$ are $(\varnothing, \phi)$ for a propositional $\mathcal{L}$-formula ϕ; these are all subobjects of the terminal object $(\varnothing, \top)$, with connectives corresponding to the Boolean operations in $\mathrm{Sub}(\varnothing, \top)$. In other words, the syntactic category $\langle \mathcal{L} \mid \mathcal{T} \rangle$ is a preorder equivalent to the propositional Lindenbaum–Tarski Boolean algebra of $\mathcal{T}$.

Similarly, if $\mathcal{T}$ belongs to the coherent (Horn) fragment of propositional logic, then the syntactic category $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{coh}\,(\mathrm{lex})}$ is equivalent to the Lindenbaum–Tarski distributive lattice ($\wedge$-lattice) of $\mathcal{T}$.

Remark 33.4. In the definition of $\langle \mathcal{L} \mid \mathcal{T} \rangle$, we could have also taken $\mathcal{T}$-provable equivalence classes for the objects. Doing so would have yielded an equivalent category, since if $(\vec{P}, \alpha), (\vec{P}, \beta)$ are $\mathcal{T}$-provably equivalent, then they are isomorphic in $\langle \mathcal{L} \mid \mathcal{T} \rangle$ as we defined it above (via the identity $1_{(\vec{P}, \alpha)}$ which is also a morphism $(\vec{P}, \alpha) \to (\vec{P}, \beta)$).

The **universal model** $\mathcal{M}_\mathcal{T} = \mathcal{M}_{(\mathcal{L},\mathcal{T})}$ of a first-order (coherent, regular, lex) theory $(\mathcal{L}, \mathcal{T})$ in $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool\,(coh,reg,lex)}}$ is given by

$$\begin{aligned}
P^{\mathcal{M}_\mathcal{T}} &:= ((x_0 \mapsto P), \top) && \text{for } P \in \mathcal{L}_{\mathrm{sort}}, \\
f^{\mathcal{M}_\mathcal{T}} &:= [f(\vec{x}) = y] : ((x_i \mapsto P_i)_i, \top) \cong \textstyle\prod_i P_i^{\mathcal{M}_\mathcal{T}} \to Q^{\mathcal{M}_\mathcal{T}} \cong ((y \mapsto Q), \top) && \text{for } f \in \mathcal{L}_{\mathrm{fun}}(\vec{P}; Q), \\
R^{\mathcal{M}_\mathcal{T}} &:= ((x_i \mapsto P_i)_i, R(\vec{x})) \subseteq ((x_i \mapsto P_i), \top) \cong \textstyle\prod_i P_i^{\mathcal{M}_\mathcal{T}} && \text{for } R \in \mathcal{L}_{\mathrm{rel}}(\vec{P}),
\end{aligned}$$

where $x, x_1, x_2, \ldots \in \mathrm{Var}$ are arbitrary fixed variables.

Lemma 33.5. (a) For each term $t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$, we have

$$t^{\mathcal{M}_\mathcal{T}} = [t = y] : (\vec{P}, \top) \cong \prod \vec{P}^{\mathcal{M}_\mathcal{T}} \to Q^{\mathcal{M}}_{\mathcal{T}} \cong ((y \mapsto Q), \top).$$

(b) For each formula $\phi \in \mathcal{L}_{\omega\omega}^{(\mathrm{coh,reg},\mathcal{T}\text{-lex})}(\vec{P})$, we have

$$\phi^{\mathcal{M}_\mathcal{T}} = (\vec{P}, \phi) \subseteq (\vec{P}, \top) \cong \prod \vec{P}^{\mathcal{M}_\mathcal{T}};$$

and $\mathcal{M}_\mathcal{T} \models \mathcal{T}$.

Proof. (a) By induction on t.

(b) By induction on ϕ, using the explicit constructions of limits, unions, and images given by Proposition 33.1. For example, if $\phi = (s = t)$ for terms $s, t \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$:

$$\begin{aligned}
(s = t)^{\mathcal{M}_\mathcal{T}} &= \mathrm{eq}((\vec{P}, \top) \cong \prod \vec{P}^{\mathcal{M}_\mathcal{T}} \underset{t^{\mathcal{M}_\mathcal{T}}}{\overset{s^{\mathcal{M}_\mathcal{T}}}{\rightrightarrows}} Q^{\mathcal{M}_\mathcal{T}} \cong ((y \mapsto Q), \top)) \\
&= \mathrm{eq}((\vec{P}, \top) \underset{[t=y]}{\overset{[s=y]}{\rightrightarrows}} ((y \mapsto Q), \top)) && \text{by (a)} \\
&= (\vec{P}, (\exists y \in Q)((s = y) \wedge (t = y))) \subseteq (\vec{P}, \top) && \text{by Proposition 33.1(d)} \\
&= (\vec{P}, (s = t))
\end{aligned}$$

since $\vdash (\exists y \in Q)((s = y) \wedge (t = y)) \Leftrightarrow_{\vec{P}} (s = t)$ by an easy proof. In the regular/coherent/full first-order case, for each $\phi \Rightarrow_{\vec{P}} \psi \in \mathcal{T}$, we get

$$\phi^{\mathcal{M}_\mathcal{T}} = (\vec{P}, \phi) \subseteq (\vec{P}, \psi) = \psi^{\mathcal{M}_\mathcal{T}} \subseteq (\vec{P}, \top)$$

by Proposition 33.1(f), since $\mathcal{T} \vdash \phi \Rightarrow_{\vec{P}} \psi$.

In the lex case, we first prove as above that for each subtheory $\mathcal{T}' \subseteq \mathcal{T}$, if $\mathcal{M}_\mathcal{T} \models \mathcal{T}'$, then for each $\mathcal{T}'$-lex ϕ, we have $\phi^{\mathcal{M}_\mathcal{T}} = (\vec{P}, \phi)$. In the case $\phi = (\exists y \in Q)\psi$, since ϕ is $\mathcal{T}'$-lex and $\mathcal{M}_\mathcal{T} \models \mathcal{T}'$, by soundness, the existential is unique in $\mathcal{M}_\mathcal{T}$, so

$$\begin{aligned}
\phi^{\mathcal{M}_\mathcal{T}} &= (\psi^{\mathcal{M}_\mathcal{T}} \hookrightarrow (\vec{P} \sqcup (y \mapsto Q), \top) \cong (\vec{P}, \top) \times ((y \mapsto Q), \top) \xrightarrow{\pi_1} (\vec{P}, \top)) \subseteq (\vec{P}, \top) \\
&= ((\vec{P} \sqcup (y \mapsto Q), \psi^{\mathcal{M}_\mathcal{T}}) \hookrightarrow (\vec{P} \sqcup (y \mapsto Q), \top) \xrightarrow{\pi_1} (\vec{P}, \top)) \subseteq (\vec{P}, \top) && \text{by IH} \\
&= (\vec{P}, (\exists y \in Q)\psi) && \text{by Proposition 33.1(f)} \\
&= (\vec{P}, \phi).
\end{aligned}$$

It follows as above that for each $\mathcal{T}' \vdash_{\mathrm{lex}} \phi \Rightarrow_{\vec{P}} \psi \in \mathcal{T}$, we have $\mathcal{M}_\mathcal{T} \models \phi \Rightarrow_{\vec{P}} \psi$. Now by induction on the well-founded relation $\prec$ witnessing that $\mathcal{T}$ is lex, we get $\mathcal{M}_\mathcal{T} \models \mathcal{T}$, hence we may take $\mathcal{T}' = \mathcal{T}$. $\square$

Corollary 33.6 (trivial completeness theorem for first-order logic)**.** If a (coherent, regular, $\mathcal{T}$-lex) sequent $\phi \Rightarrow_{\vec{P}} \psi$ holds in every model of $\mathcal{T}$ in a Boolean coherent (coherent, regular, lex) category C, then $\mathcal{T} \vdash_{(\text{coh,reg,lex})} \phi \Rightarrow_{\vec{P}} \psi$.

Proof. $\mathcal{M}_\mathcal{T} \models \phi \Rightarrow_{\vec{P}} \psi$ means $(\vec{P}, \phi) = \phi^{\mathcal{M}_\mathcal{T}} \subseteq \psi^{\mathcal{M}_\mathcal{T}} = (\vec{P}, \psi) \subseteq (\vec{P}, \top) \in \langle \mathcal{L} \mid \mathcal{T} \rangle_{\text{Bool (coh,reg,lex)}}$, i.e., $\mathcal{T} \vdash_{(\text{coh,reg,lex})} \phi \Rightarrow_{\vec{P}} \psi$ by Proposition 33.1(f). $\square$

Theorem 33.7 (universality of $\mathcal{M}_\mathcal{T}$)**.** Let $\mathcal{T}$ be a coherent (regular, lex) theory. For any coherent (regular, lex) category C, we have an equivalence of categories

$$\begin{aligned}\mathsf{Mod}(\mathcal{L}, \mathcal{T}, -)(\mathcal{M}_\mathcal{T}) : \underset{(\mathfrak{Reg},\mathfrak{Lex})}{\mathfrak{Coh}}(\langle \mathcal{L} \mid \mathcal{T} \rangle_{\text{coh (reg,lex)}}, \mathsf{C}) &\simeq \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C}) \\ F &\mapsto F_*(\mathcal{M}_\mathcal{T}) \\ (\alpha : F \to G) &\mapsto \alpha_{\mathcal{M}_\mathcal{T}}.\end{aligned}$$

Proof. Faithfulness: for a natural transformation $\alpha : F \to G : \langle \mathcal{L} \mid \mathcal{T} \rangle \to \mathsf{C}$, we have

$$\begin{aligned}\alpha_{\mathcal{M}_\mathcal{T}} &= (\alpha_{P^{\mathcal{M}_\mathcal{T}}} : F(P^{\mathcal{M}_\mathcal{T}}) = P^{F_*(\mathcal{M}_\mathcal{T})} \to P^{G_*(\mathcal{M}_\mathcal{T})} = G(P^{\mathcal{M}_\mathcal{T}}))_{P \in \mathcal{L}_{\text{sort}}} \\ &= (\alpha_{((x \mapsto P), \top)} : F((x \mapsto P), \top) \to G((x \mapsto P), \top))_{P \in \mathcal{L}_{\text{sort}}};\end{aligned}$$

thus the homomorphism $\alpha_{\mathcal{M}_\mathcal{T}}$ determines each component $\alpha_{((x \mapsto P), \top)}$. Since α is between lex functors, thus preserves finite products (Proposition 22.12), it follows that each component $\alpha_{(\vec{P}, \top)} \cong \prod_x \alpha_{((x \mapsto P_x), \top)}$ is determined. Now for an arbitrary $(\vec{P}, \beta) \in \langle \mathcal{L} \mid \mathcal{T} \rangle$, by the naturality square

$$\begin{tikzcd} F(\vec{P}, \beta) \arrow[r, "\alpha_{(\vec{P},\beta)}"] \arrow[d, hook] & G(\vec{P}, \beta) \arrow[d, hook] \\ F(\vec{P}, \top) \arrow[r, "\alpha_{(\vec{P},\top)}"'] & G(\vec{P}, \top) \end{tikzcd}$$

in which the right edge is monic, the component $\alpha_{(\vec{P}, \beta)}$ is determined.

Fullness: let $f : F_*(\mathcal{M}_\mathcal{T}) \to G_*(\mathcal{M}_\mathcal{T})$ be a homomorphism; we must show that $f = \alpha_{\mathcal{M}_\mathcal{T}}$ for a natural transformation $\alpha : F \to G$. By the above, we must define $\alpha_{(\vec{P}, \beta)}$ to be the unique morphism making

$$\begin{tikzcd} \beta^{F_*(\mathcal{M}_\mathcal{T})} = F(\vec{P}, \beta) \arrow[r, dashed, "\alpha_{(\vec{P},\beta)}"] \arrow[d, hook] & G(\vec{P}, \beta) = \beta^{G_*(\mathcal{M}_\mathcal{T})} \arrow[d, hook] \\ \prod \vec{P}^{F_*(\mathcal{M}_\mathcal{T})} \cong F(\vec{P}, \top) \arrow[r, "\prod_x f_{P_x}"'] & G(\vec{P}, \top) \cong \prod \vec{P}^{G_*(\mathcal{M}_\mathcal{T})} \end{tikzcd}$$

commute; we have $\beta^{F_*(\mathcal{M}_\mathcal{T})} = F(\beta^{\mathcal{M}_\mathcal{T}}) = F(\vec{P}, \beta) \subseteq F(\vec{P}, \top)$ by Lemmas 32.2 and 33.5, and similarly $\beta^{G_*(\mathcal{M}_\mathcal{T})} = G(\vec{P}, \beta) \subseteq G(\vec{P}, \top)$, so such an $\alpha_{(\vec{P}, \beta)}$ exists by Lemma 32.1 since f is a homomorphism (and we are working with at most coherent formulas). We must show that α is a natural transformation; clearly then $\alpha_{\mathcal{M}_\mathcal{T}} = f$. From the definition, α is clearly natural with respect to the inclusions $(\vec{P}, \beta) \hookrightarrow (\vec{P}, \top) \in \langle \mathcal{L} \mid \mathcal{T} \rangle$ as well as the projections $(\vec{P} \sqcup \vec{Q}, \top) \to (\vec{P}, \top), (\vec{Q}, \top)$.

Now for an arbitrary $[\phi] : (\vec{P}, \beta) \to (\vec{Q}, \gamma) \in \langle \mathcal{L} \mid \mathcal{T} \rangle$, we may factor $[\phi]$ through its graph $(\vec{P} \sqcup \vec{Q}, \phi) \subseteq (\vec{P} \sqcup \vec{Q}, \beta \wedge \gamma)$ (Proposition 33.1(g)), and consider the commutative diagram

$$\begin{tikzcd}
(\vec{P}, \beta) \arrow[d, hook] \arrow[rr, bend left, "{[\phi]}"] & (\vec{P} \sqcup \vec{Q}, \phi) \arrow[l, "\cong"'] \arrow[r, "\pi_2|(\vec{P}\sqcup\vec{Q},\phi)"'] \arrow[d, hook] & (\vec{Q}, \gamma) \arrow[d, hook] \\
(\vec{P}, \top) & (\vec{P} \sqcup \vec{Q}, \top) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & (\vec{Q}, \gamma);
\end{tikzcd}$$

α is natural with respect to every morphism below the top row, hence is also natural with respect to the top row (because the left and right edges are monic), hence with respect to $[\phi]$.

Essential surjectivity: given $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C})$, define

$$\begin{aligned}
F : \langle \mathcal{L} \mid \mathcal{T} \rangle &\longrightarrow \mathsf{C} \\
(\vec{P}, \alpha) &\longmapsto \alpha^{\mathcal{M}} \\
([\phi] : (\vec{P}, \alpha) \to (\vec{Q}, \beta)) &\longmapsto \text{unique } \alpha^{\mathcal{M}} \to \beta^{\mathcal{M}} \text{ with graph } \phi^{\mathcal{M}} \subseteq \alpha^{\mathcal{M}} \times \beta^{\mathcal{M}};
\end{aligned}$$

such a unique $\alpha^{\mathcal{M}} \to \beta^{\mathcal{M}}$ exists by Example 31.10. Using the explicit constructions in Proposition 33.1, it is straightforward to check that F is a coherent (regular, lex) functor. For example, to check that F preserves the composition of $(\vec{P}, \alpha) \xrightarrow{[\phi]} (\vec{Q}, \beta) \xrightarrow{[\psi]} (\vec{R}, \gamma)$: we have

$$\begin{aligned}
\operatorname{graph}(F([\psi] \circ [\phi])) &= \operatorname{graph}(F([(\exists \vec{Q})(\phi \wedge \psi)])) \\
&= ((\exists \vec{Q})(\phi \wedge \psi))^{\mathcal{M}} \\
&= \Big(\pi_{12}^*(\phi^{\mathcal{M}} \subseteq \textstyle\prod \vec{P}^{\mathcal{M}} \times \prod \vec{Q}^{\mathcal{M}}) \cap \pi_{23}^*(\psi^{\mathcal{M}} \subseteq \prod \vec{Q}^{\mathcal{M}} \times \prod \vec{R}^{\mathcal{M}}) \\
&\qquad \hookrightarrow \textstyle\prod \vec{P}^{\mathcal{M}} \times \prod \vec{Q}^{\mathcal{M}} \times \prod \vec{R}^{\mathcal{M}} \xrightarrow{\pi_{13}} \prod \vec{P}^{\mathcal{M}} \times \prod \vec{R}^{\mathcal{M}}\Big) \\
&\qquad\qquad \text{(using the substitution lemma)} \\
&= \Big(\pi_{12}^*(\operatorname{graph}(F([\phi]))) \cap \pi_{23}^*(\operatorname{graph}(F([\psi]))) \\
&\qquad \hookrightarrow \alpha^{\mathcal{M}} \times \beta^{\mathcal{M}} \times \gamma^{\mathcal{M}} \xrightarrow{\pi_{13}} \alpha^{\mathcal{M}} \times \gamma^{\mathcal{M}}\Big) \\
&= \operatorname{graph}(F([\psi]) \circ F([\phi])) \qquad \text{by Proposition 23.4(e).}
\end{aligned}$$

We have

$$\begin{aligned}
F(P^{\mathcal{M}_\mathcal{T}}) &= F((x \mapsto P), \top) = P^{\mathcal{M}} && \text{for } P \in \mathcal{L}_{\mathrm{sort}}, \\
F(R^{\mathcal{M}_\mathcal{T}}) &= F((\vec{x} \mapsto \vec{P}), R(\vec{x})) = R^{\mathcal{M}} \subseteq \textstyle\prod \vec{P}^{\mathcal{M}} && \text{for } R \in \mathcal{L}_{\mathrm{rel}}(\vec{P}), \\
\operatorname{graph}(F(f^{\mathcal{M}_\mathcal{T}})) &= \operatorname{graph}(F([f(\vec{x}) = y])) = (f(\vec{x}) = y)^{\mathcal{M}} \\
&= \operatorname{eq}(\textstyle\prod \vec{P}^{\mathcal{M}} \times Q^{\mathcal{M}} \underset{\pi_2}{\overset{f^{\mathcal{M}} \circ \pi_1}{\rightrightarrows}} Q^{\mathcal{M}}) = \operatorname{graph}(f^{\mathcal{M}}) && \text{for } f \in \mathcal{L}_{\mathrm{fun}}(\vec{P}; Q),
\end{aligned}$$

so $F_*(\mathcal{M}_\mathcal{T}) = \mathcal{M}$. $\square$

In words, a model $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C})$ consists of objects $P^{\mathcal{M}}$ for each $P \in \mathcal{L}_{\mathrm{sort}}$, together with morphisms $f^{\mathcal{M}} : \prod \vec{P}^{\mathcal{M}} \to Q^{\mathcal{M}}$ for each $f \in \mathcal{L}_{\mathrm{fun}}(\vec{P}; Q)$ and subobjects $R^{\mathcal{M}} \subseteq \prod \vec{P}^{\mathcal{M}}$ for each $R \in \mathcal{L}_{\mathrm{rel}}(\vec{P})$, obeying the inequalities in $\mathcal{T}$; thus, Theorem 33.7 says

"$\langle \mathcal{L} \mid \mathcal{T} \rangle_{\text{coh (reg,lex)}}$ is the coherent (regular, lex) category presented by generating objects $\mathcal{L}_{\text{sort}}$, morphisms $\mathcal{L}_{\text{fun}}$, and subobjects $\mathcal{L}_{\text{rel}}$, and relations between subobjects $\mathcal{T}$".

The use of Lemma 32.1 in the proof of fullness means that Theorem 33.7 fails for $\langle \mathcal{L} \mid \mathcal{T} \rangle_{\text{Bool}}$. To remedy this, we must modify the definition of homomorphism to explicitly require preservation of all first-order formulas. For a Boolean coherent category C, an **$\mathcal{L}$-elementary embedding** $f : \mathcal{M} \to \mathcal{N}$ between $\mathcal{L}$-structures in C is a family $f = (f_P : P^{\mathcal{M}} \to P^{\mathcal{N}})_{P \in \mathcal{L}_{\text{sort}}}$ of morphisms such that for each $\phi \in \mathcal{L}_{\omega\omega}(\vec{P})$, we have

$$\phi^{\mathcal{M}} \subseteq (\textstyle\prod_x f_{P_x})^*(\phi^{\mathcal{N}}) \subseteq \prod \vec{P}^{\mathcal{M}}.$$

Replacing ϕ with $\neg\phi$ shows that we in fact have

$$\phi^{\mathcal{M}} = (\textstyle\prod_x f_{P_x})^*(\phi^{\mathcal{N}}) \subseteq \prod \vec{P}^{\mathcal{M}}.$$

Taking $\phi := (x = y) \in \mathcal{L}_{\omega\omega}(x, y \mapsto P)$ for $P \in \mathcal{L}_{\text{sort}}$ shows that

$$\Delta_{P^{\mathcal{M}}} = (x = y)^{\mathcal{M}} = (f_P \times f_P)^*((x = y)^{\mathcal{N}}) = (f_P \times f_P)^*(\Delta_{P^{\mathcal{N}}}) = \ker(f_P) \subseteq P^{\mathcal{M}} \times P^{\mathcal{M}},$$

i.e., $f_P : P^{\mathcal{M}} \to P^{\mathcal{N}}$ is monic. Taking $\phi := (g(\vec{x}) = y) \in \mathcal{L}_{\omega\omega}(\vec{x} \mapsto \vec{P},\, y \mapsto Q)$ for $g \in \mathcal{L}_{\omega\omega}(\vec{P}; Q)$ shows that

$$\operatorname{graph}(g^{\mathcal{M}}) = (g(\vec{x}) = y)^{\mathcal{M}} \subseteq (\textstyle\prod_i f_{P_i} \times f_Q)^*((g(\vec{x}) = y)^{\mathcal{N}}) = (\prod_i f_{P_i} \times f_Q)^*(\operatorname{graph}(g^{\mathcal{N}})),$$

i.e., (by Yoneda)

$$f_Q \circ g^{\mathcal{M}} = g^{\mathcal{N}} \circ \textstyle\prod_i f_{P_i}.$$

Thus, elementary embeddings are sortwise monic homomorphisms. Let

$$\begin{aligned} \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathsf{C}) &:= \{\mathcal{L}\text{-structures, elementary embeddings}\} \subseteq \mathsf{Mod}(\mathcal{L}, \mathsf{C}) \quad \text{(non-full)}, \\ \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}, \mathsf{C}) &:= \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathsf{C}) \cap \mathsf{Mod}(\mathcal{L}, \mathcal{T}, \mathsf{C}), \\ \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}) &:= \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}, \mathsf{Set}). \end{aligned}$$

Let $F : \mathsf{C} \to \mathsf{D} \in \mathfrak{BoolCoh}$ be a coherent functor between Boolean coherent categories. For an elementary embedding $f : \mathcal{M} \to \mathcal{N} \in \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathsf{C})$, $F_*(f) : F_*(\mathcal{M}) \to F_*(\mathcal{N}) \in \mathsf{Mod}(\mathcal{L}, \mathsf{D})$ is an elementary embedding, for exactly the same reason that it was a homomorphism in Section 32. Thus $\mathsf{Mod}(\mathcal{L}, F) = F_* : \mathsf{Mod}(\mathcal{L}, \mathsf{C}) \to \mathsf{Mod}(\mathcal{L}, \mathsf{D})$ restricts to

$$\mathsf{Mod}_{\text{elem}}(\mathcal{L}, F) : \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathsf{C}) \longrightarrow \mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathsf{D}).$$

For a natural transformation $\alpha : F \to G : \mathsf{C} \to \mathsf{D} \in \mathfrak{BoolCoh}$, for each $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathsf{C})$, $\alpha_{\mathcal{M}} : F_*(\mathcal{M}) \to G_*(\mathcal{M})$ is an elementary embedding, for exactly the same reason that it was a homomorphism in Section 32. Thus the 2-copresheaf $\mathsf{Mod}(\mathcal{L}, \mathcal{T}, -) : \mathfrak{BoolCoh} \to \mathfrak{Cat}$ restricts pointwise to a 2-copresheaf

$$\mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}, -) : \mathfrak{BoolCoh} \longrightarrow \mathfrak{Cat}.$$

Theorem 33.8 (universality of $\mathcal{M}_\mathcal{T}$ for first-order theories)**.** Let $(\mathcal{L}, \mathcal{T})$ be a first-order theory. For any Boolean coherent category C, we have an equivalence of categories

$$\mathsf{Mod}_{\mathrm{elem}}(\mathcal{L}, \mathcal{T}, -)(\mathcal{M}_\mathcal{T}) : \mathfrak{BoolCoh}(\langle \mathcal{L} \mid \mathcal{T} \rangle_{\mathrm{Bool}}, \mathsf{D}) \simeq \mathsf{Mod}_{\mathrm{elem}}(\mathcal{L}, \mathcal{T}, \mathsf{C}).$$

Proof. Exactly as in Theorem 33.7, except that in the proof of fullness, $\alpha_{(\vec{P},\beta)}$ exists because f is an elementary embedding. $\square$

Conversely, every Boolean coherent (coherent, regular, lex) category C has a presentation. The **internal language** $\mathcal{L}_\mathsf{C}$ of C is given by

$$\begin{aligned}
(\mathcal{L}_\mathsf{C})_{\mathrm{sort}} &:= \mathsf{C}_0, \\
(\mathcal{L}_\mathsf{C})_{\mathrm{fun}}(\vec{P}; Q) &:= \mathsf{C}(\textstyle\prod \vec{P}, Q), \\
(\mathcal{L}_\mathsf{C})_{\mathrm{rel}}(\vec{P}) &:= \mathrm{Sub}_\mathsf{C}(\textstyle\prod \vec{P}).
\end{aligned}$$

The **canonical model** $\mathcal{M}_\mathsf{C}$ of $\mathcal{L}_\mathsf{C}$ interprets each symbol as itself:

$$\begin{aligned}
P^{\mathcal{M}_\mathsf{C}} &:= P && \text{for } P \in (\mathcal{L}_\mathsf{C})_{\mathrm{sort}}, \\
f^{\mathcal{M}_\mathsf{C}} &:= f : \textstyle\prod \vec{P} \to Q && \text{for } f \in (\mathcal{L}_\mathsf{C})_{\mathrm{fun}}(\vec{P}; Q), \\
R^{\mathcal{M}_\mathsf{C}} &:= R \subseteq \textstyle\prod \vec{P} && \text{for } R \in (\mathcal{L}_\mathsf{C})_{\mathrm{rel}}(\vec{P}).
\end{aligned}$$

The **internal theory** of C is

$$\mathcal{T}_\mathsf{C} := \{\phi \Rightarrow_{\vec{P}} \psi \mid \mathcal{M}_\mathsf{C} \models \phi \Rightarrow_{\vec{P}} \psi\}.$$

Let

$$\mathcal{T}_\mathsf{C}^{\mathrm{Horn}} \subseteq \mathcal{T}_\mathsf{C}^{\mathrm{reg}} \subseteq \mathcal{T}_\mathsf{C}^{\mathrm{coh}} \subseteq \mathcal{T}_\mathsf{C}$$

consist of all sequents in the respective fragment. Let also

$$\mathcal{T}_\mathsf{C}^{\mathrm{lex}} := \mathcal{T}_\mathsf{C}^{\mathrm{Horn}} \cup \{\phi \Rightarrow_{\vec{P}} \psi \in \mathcal{T}_\mathsf{C} \mid \phi, \psi \text{ are } \mathcal{T}_\mathsf{C}^{\mathrm{Horn}}\text{-lex}\};$$

this is clearly a lex theory (we could also add the $\mathcal{T}_\mathsf{C}^{\mathrm{lex}}$-lex sequents, and so on, but they will already be implied by $\mathcal{T}_\mathsf{C}^{\mathrm{lex}}$).

Proposition 33.9. For any Boolean coherent (coherent, regular, lex) category C, $\mathcal{M}_\mathsf{C} : \langle \mathcal{L}_\mathsf{C} \mid \mathcal{T}_\mathsf{C}^{\mathrm{Bool\,(coh,reg,lex)}} \rangle_{\mathrm{Bool\,(coh,reg,lex)}} \to \mathsf{C}$ is an equivalence of categories.

Proof. Since $\mathcal{M}_\mathsf{C}$ is a lex functor, it is enough to show that it is conservative, full-on-subobjects, and essentially surjective (Corollary 23.10).

First, we show that for every $\phi \in (\mathcal{L}_\mathsf{C})_{\omega\omega}^{(\mathrm{coh,reg},\mathcal{T}_\mathsf{C}^{\mathrm{lex}}\text{-lex})}(\vec{P})$, where $\vec{P} \in \mathcal{L}_{\mathrm{sort}}^{\{x_1,\dots,x_n\}}$, we have $\mathcal{T}_\mathsf{C}^{(\mathrm{coh,reg,lex})} \vdash_{\mathrm{Bool\,(coh,reg,lex)}} \phi \Leftrightarrow_{\vec{P}} \phi^{\mathcal{M}_\mathsf{C}}(x_1, \dots, x_n)$. In the non-lex cases, this is immediate from $\mathcal{M}_\mathsf{C} \models \phi \Leftrightarrow_{\vec{P}} \phi^{\mathcal{M}_\mathsf{C}}(x_1, \dots, x_n)$ (whence this sequent is in $\mathcal{T}_\mathsf{C}$).

Lemma 33.10. If every $\exists$ occurring in $\phi \in (\mathcal{L}_\mathsf{C})_{\omega\omega}^{\mathrm{reg}}(\vec{P})$ is unique in $\mathcal{M}_\mathsf{C}$, then ϕ is $\mathcal{T}_\mathsf{C}^{\mathrm{Horn}}$-lex, hence $\mathcal{T}_\mathsf{C}^{\mathrm{Horn}} \vdash_{\mathrm{lex}} \phi \Leftrightarrow_{\vec{P}} \phi^{\mathcal{M}_\mathsf{C}}(\vec{x})$.

In particular, this holds if ϕ is $\mathcal{T}$-lex for *any* theory $\mathcal{T}$ (e.g., $\mathcal{T} = \mathcal{T}_\mathsf{C}^{\mathrm{lex}}$) such that $\mathcal{M}_\mathsf{C} \models \mathcal{T}$.

Proof. By induction on ϕ. In the interesting case $\phi = (\exists y \in Q)\psi$, by the IH, we have $\mathcal{T}_{\mathsf{C}}^{\text{Horn}} \vdash_{\text{lex}} \psi \Leftrightarrow_{\vec{P} \sqcup (y \mapsto Q)} \psi^{\mathcal{M}_{\mathsf{C}}}(\vec{x}, y)$. Since $(\exists y \in Q)\psi$ is unique in $\mathcal{M}_{\mathsf{C}}$, i.e., $\psi^{\mathcal{M}_{\mathsf{C}}} \hookrightarrow \prod \vec{P} \times Q \xrightarrow{\pi_1} \prod \vec{P}$ is monic, we have $\mathcal{M}_{\mathsf{C}} \models$ "$(\exists y \in Q)\psi^{\mathcal{M}_{\mathsf{C}}}(\vec{x}, y)$ is unique", whence these sequents are in $\mathcal{T}_{\mathsf{C}}^{\text{Horn}}$, whence $\mathcal{T}_{\mathsf{C}}^{\text{Horn}} \vdash$ "$(\exists y \in Q)\psi$ is unique", i.e., ϕ is $\mathcal{T}_{\mathsf{C}}^{\text{Horn}}$-lex. $\square$

Now for any $(\vec{P}, \alpha) \in \langle \mathcal{L}_{\mathsf{C}} \mid \mathcal{T}_{\mathsf{C}}^{(\cdots)} \rangle_{(\cdots)}$, we have

$$\begin{aligned}
\operatorname{Sub}(\vec{P}, \alpha) &\cong \{\phi \mid \mathcal{T}_{\mathsf{C}}^{(\cdots)} \vdash_{(\cdots)} \phi \Rightarrow_{\vec{P}} \alpha\}/\equiv_{\mathcal{T}}^{(\cdots)} && \text{by Proposition 33.1(f)} \\
&\cong \{A(\vec{x}) \mid A \subseteq \textstyle\prod \vec{P} \ \&\ \mathcal{T}_{\mathsf{C}}^{(\cdots)} \vdash_{(\cdots)} A(\vec{x}) \Rightarrow_{\vec{P}} \alpha^{\mathcal{M}}(\vec{x})\} && \text{by above} \\
&\overset{\mathcal{M}_{\mathsf{C}}}{\cong} \operatorname{Sub}_{\mathsf{C}}(\alpha^{\mathcal{M}_{\mathsf{C}}}) = \operatorname{Sub}_{\mathsf{C}}(\mathcal{M}_{\mathsf{C}}(\vec{P}, \alpha)),
\end{aligned}$$

using that for $A, B \subseteq \prod \vec{P}$, we have

$$\begin{aligned}
&\mathcal{T}_{\mathsf{C}}^{(\cdots)} \vdash A(\vec{x}) \Rightarrow_{\vec{P}} B(\vec{x}) \\
&\implies \mathcal{M}_{\mathsf{C}} \models A(\vec{x}) \Rightarrow_{\vec{P}} B(\vec{x}) \\
&\iff A(\vec{x})^{\mathcal{M}_{\mathsf{C}}} = A \subseteq B = B(\vec{x})^{\mathcal{M}_{\mathsf{C}}} \subseteq \textstyle\prod \vec{P} \\
&\implies (A(\vec{x}) \Rightarrow_{\vec{P}}) \in \mathcal{T}_{\mathsf{C}}^{\text{Horn}}.
\end{aligned}$$

Thus $\mathcal{M}_{\mathsf{C}}$ is conservative and full-on-subobjects. It is also essentially surjective, because every $P \in \mathsf{C}_0$ is $\mathcal{M}_{\mathsf{C}}((x \mapsto P), \top)$. $\square$

Corollary 33.11 (Morleyization)**.** For any first-order theory $(\mathcal{L}, \mathcal{T})$, there is a coherent theory $(\mathcal{L}', \mathcal{T}')$ such that for any Boolean coherent category C, we have

$$\mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}, \mathsf{C}) \simeq \mathsf{Mod}(\mathcal{L}', \mathcal{T}', \mathsf{C})$$

("2-naturally" in C).

Proof. Take $(\mathcal{L}', \mathcal{T}') := (\mathcal{L}_{\langle \mathcal{L} | \mathcal{T} \rangle_{\text{Bool}}}, \mathcal{T}^{\text{coh}}_{\langle \mathcal{L} | \mathcal{T} \rangle_{\text{Bool}}})$. Then

$$\begin{aligned}
\mathsf{Mod}_{\text{elem}}(\mathcal{L}, \mathcal{T}, \mathsf{C}) &\simeq \mathfrak{Coh}(\langle \mathcal{L} \mid \mathcal{T} \rangle_{\text{Bool}}, \mathsf{C}) && \text{by Theorem 33.8} \\
&\simeq \mathfrak{Coh}(\langle \mathcal{L}' \mid \mathcal{T}' \rangle_{\text{coh}}, \mathsf{C}) && \text{by Proposition 33.9} \\
&\simeq \mathsf{Mod}(\mathcal{L}', \mathcal{T}', \mathsf{C}) && \text{by Theorem 33.7.} \quad \square
\end{aligned}$$

Remark 33.12. In many cases, there is a much more efficient presentation of C than $(\mathcal{L}_{\mathsf{C}}, \mathcal{T}_{\mathsf{C}})$. Suppose given some $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{C}}$ such that

- every $X \in \mathsf{C}$ admits a monomorphism to some finite product $\prod \vec{P}$ with $\vec{P} \in \mathcal{L}_{\text{sort}}$;
- **TODO**

In particular, the Morleyization **TODO**

The internal language of C provides a systematic way of defining categorical analogs of familiar set-based notions, generalizing the analogies for monomorphisms, pullbacks, images, etc.: we just interpret the usual definition in $\mathcal{M}_{\mathsf{C}}$. From now on, we will refer to the interpretation of first-order logic in $\mathcal{M}_{\mathsf{C}}$ as being in C itself:

$$\begin{gathered}
\phi^{\mathsf{C}} := \phi^{\mathcal{M}_{\mathsf{C}}}, \\
\mathsf{C} \models \phi \Rightarrow_{\vec{P}} \psi \ :\iff\ \mathcal{M}_{\mathsf{C}} \models \phi \Rightarrow_{\vec{P}} \psi.
\end{gathered}$$

Proposition 33.13. Let C be a lex category.

(a) $f, g : X \to Y \in \mathsf{C}$ are equal iff $\mathsf{C} \models \top \Rightarrow_{(x \mapsto X)} (f(x) = g(x))$.

(b) $f : X \to Y$ is monic iff $\mathsf{C} \models (f(x) = f(x')) \Rightarrow_{(x, x' \mapsto X)} (x = x')$.

(c) $f : X \to Y$ is a pullback-stable extremal epimorphism iff $\mathsf{C} \models \top \Rightarrow_{(y \mapsto Y)} (\exists x \in X)(f(x) = y)$.

(d) $f : X \to Y$ is an isomorphism iff C satisfies both preceding sequents.

(e) Intersections are given by $A \cap B = (A(x) \wedge B(x))^{\mathsf{C}} \subseteq X$.

(f) Unions are given by $A \cup B = (A(x) \vee B(x))^{\mathsf{C}} \subseteq X$.

(g) $R \subseteq X \times Y$ is a graph of some $X \to Y$ iff $\mathsf{C} \models$ "$R(x, y)$ is the graph of some $X(x) \to Y(y)$".

(h) Equalizers: $\mathrm{eq}(X \overset{f}{\underset{g}{\rightrightarrows}} Y) = (f(x) = g(x))^{\mathsf{C}} \subseteq X$.

(i) Pullbacks: $(f : X \to Y)^*(B \subseteq Y) = B(f(x))^{\mathsf{C}} \subseteq X$.

(j) Images: $(f : X \to Y)(A \subseteq X) = ((\exists x \in X)(A(x) \wedge (f(x) = y)))^{\mathsf{C}} \subseteq Y$.

Proof. We have already used some of these; e.g., (h) is by Example 31.10. The rest are similarly straightforward computations. □

34 Gabriel–Ulmer duality

35 Exact and pretopos completions

36 Barr–Makkai duality

37 Ultracategories and Makkai duality