Quantum Communication Complexity

(a survey)

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Abstract

Can quantum communication be more efficient than its classical counterpart? Holevo's theorem rules out the possibility of communicating more than n bits of classical information by the transmission of n quantum bits—unless the two parties are entangled, in which case twice as many classical bits can be communicated but no more. In apparent contradiction, there are distributed computational tasks for which quantum communication cannot be simulated efficiently by classical means. In extreme cases, the effect of transmitting quantum bits cannot be achieved classically short of transmitting an exponentially larger number of bits.

In a similar vein, can entanglement be used to save on classical communication? It is well known that entanglement on its own is useless for the transmission of information. Yet, there are distributed tasks that cannot be accomplished at all in a classical world when communication is not allowed, but that become possible if the non-communicating parties share prior entanglement. This leads to the question of how expensive it is, in terms of classical communication, to provide an exact simulation of the spooky power of entanglement.

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1 Introduction

It is well known that the use of quantum information allows for tasks that would be provably impossible in a classical world, such as the transmission of unconditionally confidential information between parties that share only a short secret key [2, 4]. Apart from this quantum cryptography, are there advantages to be gained in setting up an infrastructure that would facilitate the transmission of quantum information? In particular, are there advantages to be gained in terms of communication efficiency? A different but related question concerns quantum entanglement: can entangled parties make better use of a classical communication channel than their non-entangled counterparts? Better still: can entangled parties benefit from their entanglement if they are not allowed any form of direct communication?

There are good reasons to believe at first that the answer to all the above questions is negative. In particular, Holevo's theorem [18] states that no more than n bits of expected classical information can be communicated between unentangled parties by the transmission of n quantum bits—henceforth called qubits—regardless of the coding scheme that could be used. If the communicating parties share prior entanglement, twice as much classical information can be transmitted [5], but no more. This applies even if the communication is not restricted to be one-way [14]. It is thus reasonable to expect that no significant savings in communication can be achieved by the transmission of qubits, and possibly no savings at all if the communicating parties do not share prior entanglement. As for the last question, it is well known that entanglement alone cannot be used to signal information—otherwise faster-than-light communication would be possible and causality would be violated—and thus it would seem that entanglement is useless if it is not supplemented by direct forms of communication. Here we survey striking results to the effect that all the intuition in this paragraph is wrong.

After a review of classical communication complexity in Section 2, we consider in Section 3 the situation in which quantum communication is allowed. In Section 4, we revert to classical communication but allow unlimited prior entanglement between the communicating parties. Section 5 investigates in more detail the power of prior entanglement when no direct communication is allowed to take place, which we call spooky communication complexity. In Section 6, we determine how expensive it is to simulate the effect of entanglement in a purely classical world. Finally, we conclude with open problems in Section 7. Although we do not cover the important topic of lower bounds for quantum communication complexity, we encourage the reader to consult [20, 14] for early results and [11] for powerful new techniques.

2 Classical Communication Complexity

Let Alice and Bob be two distant parties who wish to collaborate on a common task that depends on distributed inputs. More precisely, let X, Y and Z be sets and consider a function $f: X \times Y \to Z$. Assume Alice and Bob are given some $x \in X$ and $y \in Y$, respectively, and their goal is to compute z = f(x, y). Sometimes, we add a promise P(x, y) for some Boolean function P, in which case Alice and Bob are required to compute the correct answer f(x, y) only when P(x, y) holds. Whether or not there is a promise, the obvious recipe is for Alice to communicate x to Bob, which allows him to compute z. Once obtained, Bob can then communicate z back to Alice if both parties need to know the answer. If we are concerned with the amount of communication required to achieve this task—paying no attention to the computing effort involved in the process—could there be more efficient solutions for some functions f? For all functions?

The answer is obviously positive for the first question. For example, if $X = Y = \{0,1\}^n$ for some integer n, $Z = \{0,1\}$ and f(x,y) is defined to be 0 if and only if x and y have the same Hamming weight (the same number of 1s), it suffices for Alice to communicate the Hamming weight of x to Bob for him to verify the condition. Thus, about $\lg n$ bits 1 of communication are sufficient for this task, which is much more economical than if Alice had transmitted all n bits of her input to Bob. The answer to the second question, however, is negative: There are functions for which the obvious solution is optimal. For instance, n bits of communication are necessary and sufficient in the worst case for Bob to decide whether or not Alice's input is the same as his. This unsurprising statement is not easy to prove, but a host of techniques have been developed to handle that kind of questions. See [21] for a survey.

A more interesting scenario takes place when we do not insist on the correct answer to be obtained with certainty. If we accept a small error probability, we can do significantly better on the above-mentioned equality-testing problem. Let $\varepsilon > 0$ be the tolerated error probability and let p be the smallest prime number larger than n/ε . Let \mathbb{F} be the finite field with p elements. Upon receiving their inputs x and y, Alice forms polynomial $P(z) = x_1 + x_2z + x_3z^2 + \cdots + x_nz^{n-1}$ over \mathbb{F} and Bob forms $Q(z) = y_1 + y_2z + y_3z^2 + \cdots + y_nz^{n-1}$. Then, Alice chooses a random element $w \in \mathbb{F}$. She computes v = P(w) and transmits both w and v to Bob, who computes Q(w) and compares the answer with v. If $Q(w) \neq v$, it has been established that $x \neq y$. Otherwise, Bob can claim with confidence that x = y because two distinct polynomials of degree smaller than n cannot agree on more than n distinct points and therefore the proportion of points in \mathbb{F} on which P(z) and Q(z) agree must be less than $n/\#\mathbb{F} = n/p < \varepsilon$.

 $^{^{1}\,\}mathrm{The}$ symbol "lg" is used to denote the base-two logarithm.

Note that $p \leq 2n/\varepsilon$ since there is always a prime number between any number and its double, and hence each of w and v can be written with no more than $2 + \lg n + \lg \frac{1}{\varepsilon}$ bits. The communication complexity of this protocol is therefore at most twice this many bits, which is much less than n for any fixed error probability ε when n is large enough.

In the classical model of communication complexity, it is often allowed for Alice and Bob to share random variables even though one may argue that this does not make much sense from a mathematical point of view. In this scenario, we assume that Alice draws a random bit string (or integer) according to some specific distribution or sometimes even a random real number—and she tells Bob the outcome of this draw in an *initialization phase*. This communication is not accounted for in the complexity of the protocol because it takes place before Alice and Bob are given their respective inputs. When correctness of the protocol is analysed for a given input, probabilities are taken over the possible choices of that random variable, as if it had been chosen after the inputs were determined ². In this model, the equality-testing problem can be solved with error probability ε with only $m = \lceil \lg 1/\varepsilon \rceil$ bits of communication, regardless of the value of n. In the initialization phase, Alice and Bob share m random bit strings a_1, a_2, \ldots, a_m , each of length n. Once they receive their inputs x and y, Alice computes $b_i = x \cdot a_i$ for each i, where $x \cdot a$ is the inner product ³ between bit strings x and a. Alice transmits b_1, b_2, \ldots, b_m to Bob, who verifies whether or not $b_i = y \cdot a_i$ for each i. If not, it has been established that $x \neq y$. Otherwise, Bob can claim with confidence that x = ybecause the probability of error of this strategy is $2^{-m} \le \varepsilon$ since it is 1/2 independently for each i.

 $^{^2}$ Note that drawing shared random bits once Alice and Bob are separated makes perfect sense in a quantum world if they share entanglement: they simply have to measure corresponding qubits from $|\Phi^+\rangle=\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|11\rangle$ pairs in the computational basis.

³ To compute the inner product between two bit strings of equal length, line them up one under the other and count the number of positions in which they both have a 1. If this number is even, the inner product is 0; otherwise it is 1. Mathematically, it is the exclusive or (sum modulo 2) of the bitwise AND of the two strings.

3 Quantum Communication Complexity

The topic of classical communication complexity was introduced and first studied by Andrew Yao in 1979 [26]. Almost 15 years elapsed before the same pioneer thought of asking how the situation might change if Alice and Bob were allowed to exchange quantum rather than classical bits [27]. It seems at first that no savings in communication are to be expected at all because of Holevo's theorem [18], which states that no more than n bits of expected classical information can be communicated between unentangled parties by the transmission of n qubits. (It was implicit in Yao's original model that Alice and Bob were not allowed to share prior entanglement in the initialization phase.)

The first hint that quantum communication could be more efficient than classical communication was given in August 1997 by Richard Cleve, Wim van Dam, Michael Nielsen and Alain Tapp [14] in a probabilistic setting ⁴. Alice and Bob are given two-bits vectors x_1x_2 and y_1y_2 , respectively. They must both decide if $x_1y_1 + x_2y_2$ is even or odd, and they are restricted to two bits of communication. Shared random variables are allowed. It is proven in [14] that no classical protocol for this task can give the correct answer with a probability better than $^{7}/_{9}$. Yet, if two quantum bits of communication are allowed, instead of two classical bits, the correct answer can be obtained with an improved probability of $^{4}/_{5}$.

A more convincing case for the superiority of quantum communication came the following year when Harry Buhrman, Richard Cleve and Avi Wigderson [9] proved that quantum communication can be *exponentially* better than classical communication in the error-free model, provided the inputs respect a given promise, and almost quadratically better in the bounded-error promise-free model. Subsequently, Ran Raz proved that an exponential advantage exists to quantum communication even in the bounded-error promise-problem model [23].

The first exponential separation [9] was inspired by the famous Deutsch–Jozsa problem [15], which was used in 1992 to show for the first time that quantum computers could be exponentially faster than classical computers in an oracle-based [6] error-free setting. More specifically, Buhrman, Cleve and Wigderson defined the following scenario, where $\Delta(x,y)$ denotes the Hamming distance between bit strings x and y, which is the number of bit positions on which they differ. Let k be an integer, $n=2^k$, $X=Y=\{0,1\}^n$ and $Z=\{0,1\}$. Function $f:X\times Y\to Z$ is the equality function: f(x,y)=1 if and only if x=y. We have seen already that this communication com-

⁴ For the sake of historical completeness, it is easy to modify a protocol given three months earlier by Buhrman, Cleve and van Dam [8] (original quant-ph version) to achieve a similar goal, and indeed this is done in the final version of that paper, which is to appear in SIAM Journal on Computing.

plexity problem requires n bits of classical communication if errors are not tolerated. Now, we introduce the promise P(x,y), which is defined to be true if and only if $\Delta(x,y) \in \{0,n/2\}$. In other words, P(x,y) holds if and only if either x=y or x and y differ on exactly half their positions. It is proven in [9] that the error-free equality-testing problem requires at least cn classical bits of communication, for some real positive constant c and all sufficiently large n, even when the correct answer is required only when the promise holds. (The hard part of that proof is taken from [16].) Even though quantum communication cannot be significantly more efficient than classical communication for the straight equality-testing problem [11], it is shown in [9] that it can be solved with certainty using as few as k quantum bits of communication whenever the promise holds, which is exponentially better than the cn bits that would be required in a classical scenario. A quantum protocol for this problem is easily derived from the first example of "spooky communication" given in Section 5.

This exponential "superiority" of quantum over classical communication is not entirely convincing because it vanishes as soon as we tolerate an arbitrarily small probability of error. Indeed, we have seen that the equality-testing problem can be solved with a constant number of bits of classical communication, for any fixed error probability, when shared random variables are allowed ⁵. The other problem featured in [9] is more interesting, even though the quantum superiority is merely almost quadratic, because it applies in the more realistic bounded-error model. Consider the following scenario. Alice and Bob are very busy and they would like to find a time when they are simultaneously free for lunch. They each have an engagement calendar, which we think of as an *n*-bit string x (resp y), where $x_i = 1$ (resp. $y_i = 1$) means that Alice (resp. Bob) is free for lunch on day i. Mathematically, they want to find an index i such that $x_i = y_i = 1$ or establish that such an index does not exist. Balasubramanian Kalyanasundaram and Georg Schnitger [19] proved in 1987 that this task requires at least cn classical bits of expected communication in the worst case, for some real positive constant c and all sufficiently large n, even when the answer is only required to be correct with probability at least $\frac{2}{3}$. Intuitively, this means that lunch cannot be scheduled short of exchanging a constant fraction of the appointment calendar. In sharp contrast, it is shown in [9] that this problem can be solved with the exchange of at most $d\sqrt{n} \lg n$ quantum bits for some constant d and all sufficiently large n. This is accomplished by implementing a distributed version of Grover's quantum search algorithm [17] in which we search for a 1 in the bitwise AND of x and y. A $\lceil \lg n \rceil$ -qubit quantum register is shuttled back and forth between Alice and Bob for each of the approximately \sqrt{n} iterations of Grover's algorithm.

⁵ Even if we do not allow shared random variables, the problem can be solved with 2k + c bits of classical communication, where $k = \lg n$ and c is a constant that depends only on the error probability.

4 Substituting Entanglement for Communication

A slightly different model was introduced by Richard Cleve and Harry Buhrman [13]. Assume Alice and Bob are restricted to communicating classical information. Are there tasks for which they could save on the required amount of communication if they share prior entanglement? Again, it is tempting to think that this is not possible because entanglement cannot be used to increase the capacity of a classical channel: Alice cannot communicate more than n expected bits of classical information to Bob if less than n bits are actually transmitted between them—even if they are allowed two-way communication and unlimited use of entanglement. And again, this intuition is wrong.

In their original paper [13], Cleve and Buhrman were able to show that entanglement can be used to save one bit of classical communication, but only in a three-party scenario. Still, this was the very first example of a distributed task that could be solved more efficiently (in terms of communication) in our quantum world than would be possible in a sad classical world, because it predates [14] by four months. Subsequently, Buhrman, Cleve and van Dam [8] found a two-party distributed problem that can be solved with a probability of success exceeding 85% if prior shared entanglement is available, whereas the probability of success in a classical world could not exceed 75% with the same amount of communication, even if shared random variables are allowed.

The first problem for which communication complexity could be reduced by more than a constant additive amount was also discovered in this shared-entanglement scenario, rather than in Yao's original qubit-transmission scenario described in the previous section: Harry Buhrman, Wim van Dam, Peter Høyer and Alain Tapp [10] gave a k-party distributed task that requires roughly $k \lg k$ bits of communication in a classical world, yet it can be carried out with exactly k bits of classical communication if the parties are allowed to share prior entanglement.

The exponential and almost-quadratic improvements mentioned in the previous section [9, 23] apply just as well in the shared-entanglement scenario. This is obvious since the effect of any protocol that requires the communication of ℓ quantum bits can be achieved by the transmission of 2ℓ classical bits—through quantum teleportation [3]—provided ℓ bits of shared entanglement are available.

There is yet another natural communication complexity scenario, in which the parties are allowed to share prior entanglement *and* to communicate quantum bits. For the sake of brevity, we shall not elaborate on this approach here.

5 Spooky Communication Complexity

An even more intriguing question is to determine if entanglement can be used *instead of* communication. Are there tasks that would be impossible to achieve in a classical world if Alice and Bob were not allowed to communicate, yet those tasks can be performed without *any* form of communication provided the participants share prior entanglement? In the words of Alain Tapp, this would provide a form of *pseudo telepathy* because it would give the *illusion* of communication between Alice and Bob when in fact no such communication takes place. And indeed there would be no communication because entanglement cannot be used to signal information: nothing Alice can do locally on her quantum system can cause a measurable change in Bob's, no matter how they are entangled.

A moment's thought suffices to realize that pseudo telepathy is possible if we are content with probabilistic tasks. Define the EPR task as follows. Once separated, Alice and Bob are given each a real number x and y, respectively, between 0 and π . They are to produce each a single bit: a for Alice and b for Bob. Alice's output must be equally likely to be 0 or 1, and so must Bob's output, but the required correlation is that a=b with probability $\cos^2(x-y)$. It is precisely the essence of Bell's theorem [1] that such correlations cannot be established in a classical world if communication between Alice and Bob is not allowed, even if the inputs are restricted to binary choices $x \in \{0, \pi/6\}$ and $y \in \{0, 5\pi/6\}$. Yet, it is easy for participants who share a $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ state to realize this task. If we think of this state as a pair of entangled polarized photons, it suffices for Alice and Bob to measure their photons at polarization angles x and y, respectively, and the outcomes of the measurements provide the required outputs x and y.

But is pseudo telepathy possible when there is a deterministic criterion to decide if the goal has been achieved and when errors are not tolerated? This brings us to our last form of communication complexity, which we call spooky communication complexity. As usual, let X, Y and Z be sets and consider a function $f: X \times Y \to Z$ such that it is not possible to compute the value of f(x,y) with certainty from knowledge of either x or y alone. It follows that if Alice and Bob are given $x \in X$ and $y \in Y$, respectively, they cannot compute f(x,y) without communication. Can such a function f exist so that Alice and Bob—or at least one of them—could compute f(x,y) nevertheless provided the participants share prior entanglement? Of course not, since this would allow for faster-than-light communication! Thus, we have to define spooky communication complexity in a more subtle way, through a relation rather than a function.

Let X, Y, A and B be sets and consider a relation $R \subseteq X \times Y \times A \times B$. In an *initialization phase*, Alice and Bob are allowed to discuss strategy and share random

variables. They are also allowed to share entanglement. After Alice and Bob are physically separated, they are given $x \in X$ and $y \in Y$, respectively. Without being allowed any forms of communication, their goal is to produce $a \in A$ and $b \in B$, respectively, such that $(x, y, a, b) \in R$. We say that spooky communication, or pseudo telepathy, takes place if this task could not be fulfilled with certainty in a classical world, whereas it can provided Alice and Bob share prior entanglement. The amount of spooky communication complexity is measured in the number of bits of entanglement that are required to succeed in the worst case. The spooky advantage is defined as the function that relates the spooky complexity to the number of classical bits of communication that would be needed in the worst case to succeed in the classical setting.

The first example of spooky communication was provided by Gilles Brassard, Richard Cleve and Alain Tapp [7] as yet another variation on the Deutsch–Jozsa problem [15]. Let k be an integer, $n = 2^k$, $X = Y = \{0, 1\}^n$ and $A = B = \{0, 1\}^k$. The Deutsch–Jozsa relation R is defined as follows, where $\Delta(x, y)$ denotes again the Hamming distance between x and y.

$$(x, y, a, b) \in R \iff \begin{cases} x = y \text{ and } a = b, \text{ or} \\ \Delta(x, y) = n/2 \text{ and } a \neq b, \text{ or} \\ \Delta(x, y) \notin \{0, n/2\} \end{cases}$$

In other words, Alice and Bob are promised that either their inputs are the same, or that they differ on exactly half the bits. They must produce identical outputs if and only if their inputs are the same. But if the promise is not fulfilled, there are no conditions on what Alice and Bob produce. The challenge comes from the fact that the outputs a and b must be exponentially shorter than the inputs x and y.

It is not immediate that the Deutsch–Jozsa relation requires communication to be established in the classical setting. After all, it can be established when n=2 (easily) and n=4 (think about it!). But it is proven in [7] that there exists a positive constant c such that the Deutsch–Jozsa relation cannot be established classically with fewer than cn bits of communication provided n is sufficiently large, based on the similar lower bound from [9] that we had mentioned in Section 3. On the other hand, we show below that the Deutsch–Jozsa relation can be established in the spooky setting with as few as k bits of entanglement. It follows that the spooky advantage of this problem is exponential because n is exponential in k.

To establish the Deutsch-Jozsa relation, Alice creates a 2k-qubit register in state

$$\sum_{z \in \{0,1\}^k} \, 2^{-k/2} \, |z,z\rangle \, ,$$

which is the same as k pairs in state $|\Phi^+\rangle$ up to ordering of the qubits. She keeps the first k qubits of that register and gives the other k qubits to Bob. After Alice and Bob are separated, they receive their respective inputs x and y. To each integer i, $1 \le i \le n$, associate the bit string $z_i \in \{0,1\}^k$ that represents number i-1 in binary. Now, Alice applies to her register the unitary transformation that maps $|z_i\rangle$ to $(-1)^{x_i}|z_i\rangle$ for each i, and Bob does the same to his register, but with y_i instead of x_i . This produces joint state

$$\sum_{i=1}^{n} 2^{-k/2} (-1)^{x_i} (-1)^{y_i} |z_i, z_i\rangle = \sum_{i=1}^{n} 2^{-k/2} (-1)^{x_i \oplus y_i} |z_i, z_i\rangle.$$

Next, Alice applies the Walsh–Hadamard transform, which sends $|0\rangle$ to $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|1\rangle$ to $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, to each of the k qubits of her register, and Bob does the same on his register. Finally, Alice and Bob measure their registers in the computational basis. The resulting classical strings, a and b, are their final output. It is proven in [7] that this process accomplishes the required job.

6 Classical Simulation of Entanglement

Once we have established that pseudo telepathy is possible, the next natural question is to determine how much classical communication is necessary and sufficient to simulate the effect of k bits of entanglement. It follows from the previous section (and Bell's theorem [1] for small values of k) that at least $c2^k$ bits are required, for some constant c>0 and all $k\geq 1$. But are these many bits of classical communication sufficient to simulate everything that can be accomplished with k bits of entanglement?

In fact, can the effect of entanglement be simulated at all with a finite amount of classical communication? As the simplest possible example, can the EPR task, as defined in Section 5, be simulated by classical communication? In particular, we must have a = b if x = y and $a \neq b$ if $|x - y| = \pi/2$. Surely, it is not possible for Alice to communicate her input x to Bob, for this would require an infinite amount of communication. Yet, it is shown in [7] that four bits of classical communication are sufficient in the worst case for an exact simulation of the EPR task, provided Alice and Bob are allowed to share a continuous real random variable in the initialization phase—admittedly an unreasonable proposition. The essence of the idea is best explained if we further restrict the inputs x and y to be between 0 and 1, rather than between 0 and π . In this case, a single bit of classical communication suffices to simulate the EPR task exactly. This restriction is somewhat bizarre since it translates to requiring the angles to be between 0 and approximately 57.3 degrees. Without this restriction, a rather

painful piecewise construction has to be implemented, as explained in [7], and we need four bits of classical communication to take care of the various possible cases.

In the initialization phase, Alice and Bob share a boolean variable c, which is equally likely to be 0 or 1, and a continuous real variable r chosen uniformly in the interval (0,1). After they are separated, they receive their angles x and y, respectively. Alice outputs a=c, a random bit as required. Then, she tells Bob if a < x with a single bit of classical communication. This allows Bob to determine whether or not r lies in between x and y. In case it does not, Bob outputs the same bit b=c as Alice. Note in particular that if x=y, then we get a=b with certainty, as required. On the other hand, if r does lie between x and y, then Bob outputs b=1-c, a bit complementary to Alice's, with probability $\sin(2|y-r|)$, otherwise he outputs b=c just like Alice.

The probability that a = b is calculated as an integral over the various possibilities for r, as if it had been chosen after x and y are fixed. For simplicity, assume that $x \leq y$. The probability that a = b is 1 if $0 \leq r \leq x$ or $y \leq r \leq 1$ since, in that case, r does not lie between x and y. Otherwise, if x < r < y, the probability that a = b is $1 - \sin(2(y - r))$. Therefore, the global probability that a = b is

$$\int_{r=0}^{x} 1 \, dr + \int_{r=x}^{y} [1 - \sin(2(y-r))] \, dr + \int_{r=y}^{1} 1 \, dr$$
$$= \frac{1}{2} + \frac{1}{2} \cos(2(y-x)) = \cos^{2}(x-y),$$

as required. It is tempting to "improve" on this approach and make it work for all angles between 0 and π , simply with an appropriate change in the probability function $\sin(2|y-r|)$ that determines whether or not Bob will output the same bit as Alice when r lies between x and y. Unfortunately, any such attempt will result in "probabilities" that are either negative or greater than 1!

It is shown in [7] how to simulate an arbitrary von Neumann measurement with only eight classical bits of communication in the worst case, but it is left as an open question to determine whether or not the effect of an arbitrary positive-operator-valued measurement (POVM) can be simulated with a bounded amount of classical communication in the worst case.

A different approach to the classical simulation of entanglement was taken independently by Michael Steiner [25], who showed that the EPR relation can be simulated exactly with significantly fewer *expected* bits of classical communication, provided we accept that there be no upper limit on the required amount of communication in the case of bad luck. Steiner's technique was subsequently refined by Nicolas Cerf, Nicolas Gisin and Serge Massar [12], who showed that as few as 1.19 expected bits of classical communication suffice to simulate exactly an arbitrary von Neumann measurement.

Even an arbitrary POVM can be simulated by this technique, at the expected cost of 6.38 bits of classical communication.

Building on [25, 12], Serge Massar, Dave Bacon, Nicolas Cerf and Richard Cleve [22] discovered that the exact classical simulation of quantum entanglement can be achieved without any need for shared random variables, provided we are satisfied with an *expected* bounded amount of classical communication. In particular, they show how to simulate the effect of an arbitrary POVM on one bit of entanglement with less than 20 bits of expected classical communication. Conversely, they also show that the exact simulation of quantum entanglement with a worst-case bounded amount of classical communication (as in the scenario of [7] described earlier in this section) is *not* possible without an infinite amount of shared randomness.

Finally, the question asked at the beginning of this section is almost resolved in [7]. It is still unknown if there exists a constant c such that $c2^k$ bits of classical communication are sufficient to simulate exactly the effect of k bits of entanglement for all values of k. However, it is shown in [7] that as few as $(3k+6)2^k$ expected bits of classical communication suffice to simulate the outcome of any POVM that Alice and Bob could perform on their respective shares of k bits of entanglement. This simulation protocol does not require Alice and Bob to share random variables in the initialization phase. No similar results are known for worst-case bounded communication even if we allow the sharing of continuous random variables.

7 Conclusions and Open Problems

We have seen a variety of scenarios according to which quantum mechanics allows for a significant improvement in the efficiency of communication, compared to what would be possible in a classical world. This is surprising because the transmission of n quantum bits cannot serve to communicate more than n classical bits of information, and because quantum entanglement on its own cannot be used to communicate at all. Perhaps the most interesting aspect of quantum communication complexity is that the advantage provided by quantum mechanics has been established rigourously. This is in sharp contrast with the field of quantum computing, in which it is merely believed that quantum mechanics allows for an exponential speedup in some computational tasks, such as the factorization of large numbers [24]. Indeed, it has not yet been ruled out that there might exist an efficient factorization algorithm for the classical computer.

Several interesting questions are still open. The exponential advantage of quantum communication over classical communication has been established only in the case of promise problems, in both the error-free [9] and bounded-error [23] scenarios. Could

there be a (total) function $f: X \times Y \to Z$, where $X = Y = \{0, 1\}^n$, for which the distributed computation of f would be exponentially more efficient with quantum communication compared to classical communication?

We have seen at the end of Section 4 that the amount of classical communication required for the accomplishment of a distributed task in the presence of unlimited entanglement cannot be more than twice the amount of quantum communication that would suffice for the same task, because quantum teleportation can be used to transmit quantum bits through a classical channel. How about the other direction? Could there be a task that can be accomplished with a small amount of classical communication in the presence of unlimited entanglement, but that would require a much larger amount of quantum communication if prior entanglement were not available?

We have seen in Section 5 that the Deutsch–Jozsa relation can be established classically without communication when n=2 or n=4, but not when n is arbitrarily large. But how large must "large" be? In particular, can it be established for n=8? It is interesting to note that the Deutsch–Jozsa relation becomes easier and easier to fake when n becomes larger. Indeed, if Alice and Bob share k random variables $t_1, t_2, \ldots, t_k \in \{0,1\}^n$ in the initialization phase, and if they output $a_i = x \cdot t_i$, and $b_i = y \cdot t_i$, respectively, their probability of being caught with a = b if in fact $\Delta(x,y) = n/2$ goes down as 2^{-k} . A nice open question is to determine the task on n input and k output bits that can be handled with certainty given sufficient entanglement and no communication, but for which the probability of success would be as small as possible in a classical world.

Finally, several open questions are given in Section 6 concerning the classical simulation of quantum entanglement. Is it possible to achieve the EPR task with fewer than four bits of classical communication in the worst case? Is it possible to simulate an arbitrary POVM with a worst-case bounded amount of classical communication? How much classical communication is sufficient in the worst case to simulate the effect of k bits of entanglement? In the expected case? We have seen how classical communication can be used to simulate entanglement for tasks that did not involve classical communication in the quantum setting. How about the classical simulation of tasks that use not only quantum entanglement but also classical (or perhaps quantum) communication?

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