Hecke algebras of finite groups with applications to spectral graph theory

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These are lecture notes for (parts of) the course "Representation theory" given at Chalmers University in Gothenburg during the fall of 2015. They outline the basic theory of Hecke algebras, with special emphasis on the ones associated to Gelfand pairs, and their role in the spectral theory of finite vertex-transitive graphs and their quotients. Due to laziness and time-constraints, examples covered during the lectures, and related digressions, have not been included in these notes. As with most things, one should approach these notes with scepticism and be ready for numerous misprints and inaccuracies.

Contents

1 Basics

1.1 Linear algebra

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a finite-dimensional complex Hilbert space and define its *unitary group* $U(\mathcal{H})$ by

$$U(\mathcal{H}) = \{A \in \mathrm{GL}(\mathcal{H}) \,:\, \langle Au, Av \rangle_{\mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}, \text{ for all } u, v \in \mathcal{H}\}.$$

Let $\mathcal{H}^*$ denote the dual vector space to $\mathcal{H}$, and note that the map $v \mapsto v^*$ from $\mathcal{H}$ to $\mathcal{H}^*$ defined by

$$v^*(u) = \langle v, u \rangle, \quad \text{for } u \in \mathcal{H} \tag{1.1}$$

is a skew-linear isomorphism (the inner product on $\mathcal{H}$ is assumed to be skew-linear in the first argument). We can equip $\mathcal{H}^*$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ defined by

$$\langle u^*, v^* \rangle_{\mathcal{H}^*} = \langle v, u \rangle_{\mathcal{H}}, \quad \text{for all } u, v \in \mathcal{H}, \tag{1.2}$$

which is readily checked to be skew-linear in the first argument. If $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_{\mathcal{H}_1})$ and $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_2})$ are finite-dimensional Hilbert spaces, we can equip the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ with an inner product defined by

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle u_1, v_1 \rangle_{\mathcal{H}_1} \langle u_2, v_2 \rangle_{\mathcal{H}_2}, \tag{1.3}$$

for $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$. Let $\mathrm{Hom}(\mathcal{H}_2, \mathcal{H}_1)$ denote the linear space of all linear map from $\mathcal{H}_2$ into $\mathcal{H}_1$, and note that the map which takes a tensor $u_1 \otimes u_2^*$ to the linear map of rank 1 between $\mathcal{H}_2$ and $\mathcal{H}_1$ defined by

$$(u_1 \otimes u_2^*)(v_2) = u_2^*(v_2) u_1, \quad \text{for } v_2 \in \mathcal{H}_2, \tag{1.4}$$

can be linearly extended to give a linear isomorphism between $\mathrm{Hom}(\mathcal{H}_2, \mathcal{H}_1)$ and $\mathcal{H}_1 \otimes \mathcal{H}_2^*$. We can also equip the *direct sum* $\mathcal{H}_1 \oplus \mathcal{H}_2$ by an inner product by

$$\langle u_1 \oplus u_2, v_1 \oplus v_2 \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle u_1, v_1 \rangle_{\mathcal{H}_1} + \langle u_2, v_2 \rangle_{\mathcal{H}_2}, \tag{1.5}$$

for $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$.

1.2 Unitary representations

Let G be a finite group. A homomorphism $\pi : G \to U(\mathcal{H})$ is called a *unitary representation* of the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. To avoid cluttering, we shall suppress the dependence of the inner product, and sometimes even the Hilbert space $\mathcal{H}$, which we assume have been chosen once and for all, and we shall refer to the pair $(\mathcal{H}, \pi)$, or simply π, as a *unitary representation* of G.

- A linear subspace $\mathcal{H}_o \subset \mathcal{H}$ is called a *sub-representation* if $\pi(g)\mathcal{H}_o \subset \mathcal{H}_o$ for all $g \in G$. If we denote by π_o the restriction of π to $\mathcal{H}_o$, then $(\mathcal{H}_o, \pi_o)$ is a unitary representation of G. The two sub-representations $\{0\}$ and $\mathcal{H}$ are called *trivial*, and $(\mathcal{H}, \pi)$ is *irreducible* if it has only trivial sub-representations.

- Given a subgroup $H < G$, we define

$$\mathcal{H}^H = \{ v \in \mathcal{H} \, : \, \pi(h)v = v, \ \text{for all } h \in H \}. \tag{1.6}$$

- The *dual unitary representation* $(\mathcal{H}^*, \pi^*)$ is defined by

$$\pi^*(g)v^*(u) = v^*(\pi(g)^{-1}u) = \langle \pi(g)v, u \rangle_{\mathcal{H}}, \quad \text{for } u, v \in \mathcal{H}. \tag{1.7}$$

  One readily checks that $(\mathcal{H}^*, \pi^*)$ is unitary, and

$$\langle u^*, \pi^*(g)v^* \rangle_{\mathcal{H}^*} = \langle \pi(g)v, u \rangle, \quad \text{for all } u, v \in \mathcal{H} \text{ and } g \in G. \tag{1.8}$$

- The *direct sum* and *tensor product* of two unitary representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of G are defined by

$$(\pi_1 \oplus \pi_2)(g)(u_1 \oplus u_2) = \pi_1(g)u_1 \oplus \pi_2(g)u_2, \tag{1.9}$$

  and

$$(\pi_1 \otimes \pi_2)(g)(u_1 \otimes u_2) = \pi_1(g)u_1 \otimes \pi_2(g)u_2, \tag{1.10}$$

  respectively, for $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$. One readily checks that these are unitary representations of G.

- If G_1 and G are finite groups and $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are unitary representations of G_1 and G_2 respectively, then their *exterior tensor product* $(\mathcal{H}_1 \otimes \mathcal{H}_2, \pi_1 \boxtimes \pi_2)$ is a unitary representation of the direct product $G_1 \times G_2$ defined by

$$(\pi_1 \boxtimes \pi_2)(g_1, g_2)(u_1 \otimes u_2) = \pi_1(g_1)u_1 \otimes \pi_2(g_2)u_2 \tag{1.11}$$

  for $(g_1, g_2) \in G_1 \times G_2$ and $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$.

- Two unitary representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are *equivalent* if there exists a unitary linear isomorphism $\psi : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\pi_1(g)\psi = \psi\pi_2(g)$ for all $g \in G$. If this is the case, we write $(\mathcal{H}_1, \pi_1) \cong (\mathcal{H}_2, \pi_2)$. More generally, we define the linear spaces

$$\mathrm{Hom}_G(\pi_2, \pi_1) = \{\psi : \mathcal{H}_2 \to \mathcal{H}_1 \,:\, \pi_1(g)\psi = \psi\pi_2(g), \text{ for all } g \in G\} \tag{1.12}$$

  and

$$\mathrm{End}_G(\pi_1) = \mathrm{Hom}_G(\pi_1, \pi_1). \tag{1.13}$$

  The maps in $\mathrm{Hom}_G(\pi_2, \pi_1)$ and $\mathrm{End}_G(\pi)$ are called *intertwiners*.

- Given a unitary representation $(\mathcal{H}, \pi)$, we write $[\pi]$ for the set of all unitary representations which are equivalent to $(\mathcal{H}, \pi)$. The *unitary dual* $\widehat{G}$ is the set

$$\widehat{G} = \{[\pi] \,:\, \pi \text{ is irreducible}\}. \tag{1.14}$$

1.3 Basic results on unitary representations

Let G be a finite group and let $(\mathcal{H}_\lambda, \lambda)$ be a (finite-dimensional) unitary representation of G. Suppose that $\mathcal{H}_o \subset \mathcal{H}_\lambda$ is a sub-representation, and note that

$$\mathcal{H}_o^\perp = \{v \in \mathcal{H}_\lambda \,:\, \langle v, u\rangle = 0, \text{ for all } u \in \mathcal{H}_o\}$$

is again a sub-representation of $\mathcal{H}_\lambda$. Indeed, suppose that $v \in \mathcal{H}_o$, and fix $u \in \mathcal{H}_o$. Then, $\pi(s)u \in \mathcal{H}_o$ for all $s \in G$, and thus

$$\langle \pi(s)v, u\rangle = \langle v, \pi(s)^{-1}u\rangle = 0, \quad \text{for all } s \in G.$$

Since $\mathcal{H}_\lambda$ is assumed to be finite-dimensional, we can successively break any unitary representation (and their orthogonal complements) into sub-representations until we reach irreducible sub-representations. Some of these may be isomorphic and can be bunched together. Such a package of unitary representations, all isomorphic to a given unitary representation $(\mathcal{H}_\pi, \pi)$, can be written of the form $(\mathcal{H}_\pi \otimes \mathcal{C}^m, \pi \otimes I_m)$ for some uniquely determined integer m which we shall refer to as the *multiplicity of* π *in* λ, where I_m denotes the identity representation on $\mathbb{C}^m$. Henceforth we shall denote the multiplicity of π in λ by $\mathrm{mult}(\pi, \lambda)$. We summarize this discussion in the following lemma.

Lemma 1.1. *Let $(\mathcal{H}_\lambda, \lambda)$ be a unitary representation of G. Then there are* irreducible *unitary representations $(\mathcal{H}_j, \pi_j)$ for $j = 1, \ldots, n$ and positive integers $m_1, \ldots, m_n$ such that*

$$\mathcal{H}_\lambda \cong \bigoplus_{j=1}^{n} \mathcal{H}_{\pi_j} \otimes \mathbb{C}^{m_j}.$$

1.4 Schur's Lemma

Suppose that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are two (finite-dimensional) *irreducible* unitary representations of G and let $T \in \mathrm{Hom}_G(\pi_2, \pi_1)$. Since both $\ker T$ and $\mathrm{Im}\, T$ are sub-representations of π_2 and π_1 respectively, we conclude that they must be trivial. In particular, either $T = 0$ or T is an isomorphism between π_2 and π_1. In particular, if $\pi_1 = \pi_2$, then $T - \lambda I$ is an intertwiner for every complex number λ. By standard linear algebra, we know that the $T - \lambda I$ must have a non-trivial kernel from some λ, which forces $T = \lambda I$. We summarize:

Lemma 1.2 (Schur)**.** *An element $T \in \mathrm{Hom}_G(\pi_2, \pi_1)$ is either zero or an isomorphism, and every $T \in \mathrm{End}_G(\pi_1)$ is a complex multiple of the identity.*

Schur's Lemma is perhaps best phrased in terms of tensor products. Recall that any linear map between $\mathcal{H}_2$ and $\mathcal{H}_1$ can be identified with an element in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ via the correspondence which associates to every tensor $u \otimes v^*$ the linear map $T : \mathcal{H}_2 \to \mathcal{H}_1$ defined by

$$Tw = v^*(w)u, \quad \text{for } w \in \mathcal{H}_2.$$

Note that $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ can be equipped with the unitary representation $\pi_1 \boxtimes \pi_2^*$ of $G \times G$, and under the correspondence above, we have

$$\mathrm{Hom}_G(\pi_2, \pi_1) \cong (\mathcal{H}_1 \otimes \mathcal{H}_2^*)^{\Delta G},$$

where $\Delta G = \{(g, g) \,:\, g \in G\}$. Schur's Lemma above can now be formulated as follows. Note that all unitary representations are always assumed to be finite-dimensional.

Lemma 1.3. *Suppose that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_1)$ are irreducible unitary representations of G. Then,*

$$\dim(\mathcal{H}_1 \otimes \mathcal{H}_2^*)^{\Delta G} = \begin{cases} 1 & \text{if } \mathcal{H}_1 \cong \mathcal{H}_2 \\ 0 & \text{otherwise.} \end{cases}$$

If $(\mathcal{H}_1, \pi_1) = (\mathcal{H}_2, \pi_2)$, then the one-dimensional space $(\mathcal{H}_1 \otimes \mathcal{H}_1^*)^{\Delta G}$ is spanned by the element

$$w = \sum_{i=1}^{n} e_i \otimes e_i^*,$$

where $e_1, \ldots, e_n$ is any choice of an ON-basis of $\mathcal{H}_1$.

An (almost) immediate consequence of this lemma is the following result, which will be crucially used in the next section.

Lemma 1.4. *Suppose that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are inequivalent,* irreducible *unitary representations of G. Then,*

$$\frac{1}{|G|} \sum_{s \in G} \overline{\langle u_1, \pi_1(s) v_1 \rangle_1} \langle u_2, \pi_2(s) v_2 \rangle_2 = 0,$$

for all $u_1, u_2 \in \mathcal{H}_1$ and $v_1, v_2 \in \mathcal{H}_2$. If $(\mathcal{H}, \pi)$ is an irreducible unitary representation of G, then

$$\frac{1}{|G|} \sum_{s \in G} \overline{\langle u_1, \pi(s) v_1 \rangle} \langle u_2, \pi(s) v_2 \rangle = \frac{\langle u_1, u_2 \rangle \langle v_1, v_2 \rangle}{d_\pi},$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$, where $d_\pi = \dim \mathcal{H}$.

Proof. Note that

$$\overline{\langle u_1, \pi_1(s)v_1\rangle} = \langle \pi_1(s)v_1, u_1\rangle = (\pi_1^*(s)v^*)(u) = \langle u_1^*, \pi_1^*(s)v_1^*\rangle$$

and

$$\overline{\langle u_1, \pi_1(s)v_1\rangle}\langle u_2, \pi_2(s)v_2\rangle = \langle u_1^* \otimes u_2, (\pi_1^* \otimes \pi_2)(s)(v_1^* \otimes v_2)\rangle$$

for all $u_1, v_1 \in \mathcal{H}_1$ and $u_2, v_2 \in \mathcal{H}_2$. Hence, if we set

$$z = \frac{1}{|G|}\sum_{s\in G}(\pi_1^* \otimes \pi_2)(s)(v_1^* \otimes v_2),$$

then $z \in (\mathcal{H}_1^* \otimes \mathcal{H}_2)^{\Delta G}$. By Lemma 1.3, we conclude that $z = 0$ if $\pi_1 \not\cong \pi_2$, and if $\pi_1 = \pi_2 = \pi$, then $z = \lambda w$ for some λ, where

$$w = \sum_{i=1}^{d_\pi} e_i^* \otimes e_i,$$

and $\{e_i\}$ is an ON-basis for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Note that we can write

$$u_1^* \otimes u_2 = aw + r_u \quad \text{and} \quad v_1^* \otimes v_2 = bw + r_v,$$

for some r_u, r_v which are orthogonal to w, where (since $\|w\|^2 = d_\pi$)

$$a = \frac{\langle u_1^* \otimes u_2, w\rangle}{d_\pi} = \frac{\langle u_1, u_2\rangle}{d_\pi} \quad \text{and} \quad b = \frac{\langle v_1^* \otimes v_2, w\rangle}{d_\pi} = \frac{\langle v_1, v_2\rangle}{d_\pi}.$$

We conclude that $\lambda = ab$, which finishes the proof. □

1.5 Intertwiners

Let $(\mathcal{H}_\lambda, \lambda)$ be a unitary representation of G and recall that Lemma 1.2 asserts that one can find *irreducible*, and mutually inequivalent, unitary representations $(\mathcal{H}_1, \pi), \ldots, (\mathcal{H}_n, \pi_n)$ and positive integers $m_1, \ldots, m_n$ such that

$$\mathcal{H}_\lambda \cong \bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathbb{C}^{m_j}.$$

Suppose that $T \in \mathrm{End}_G(\lambda)$. Since π_j's are mutually inequivalent, Lemma 1.2 shows that T can be written as direct sum of operators $T_j \in \mathrm{End}_G(\mathcal{H}_j \otimes \mathbb{C}^{m_j})$. Fix an index j and an ON-basis $e_1, \ldots, e_{m_j}$ of $\mathbb{C}^{m_j}$. To avoid cluttering, we write $S = T_j$, and we note that we can, by the general properties of tensor products, find linear maps $S_{kl} : \mathcal{H}_j \to \mathcal{H}_j$ such that

$$S(u \otimes e_k) = \sum_{l=1}^{m_j} S_{kl}(u) \otimes e_l, \quad \text{for } u \in \mathcal{H}_j.$$

The condition that $T_j \in \mathrm{End}_G(\mathcal{H}_j \otimes \mathbb{C}^{m_j})$ now clearly translates to $S_{kl} \in \mathrm{End}_G(\mathcal{H}_j)$ for every k and l. By Lemma 1.2, we conclude that there are complex numbers λ_{kl} such that $S_{kl} = \lambda_{kl} I$, and thus

$$S(u \otimes v) = \sum_{k=1}^{m_j} v_k S(u \otimes e_k) = \sum_{k=1}^{m_j} (v_k \lambda_{kl} u) \otimes e_k = \sum_{k=1}^{m_j}\sum_{l=1}^{m_j} (v_k \lambda_{kl} u) \otimes e_k = u \otimes \Lambda_j v,$$

where the matrix representation of Λ_j relative to the basis $\{e_k\}$ is given by (λ_{lk}). In other words, every $T \in \mathrm{End}_G(\lambda)$ is composed by arbitrary endomorphisms of the multiplicity spaces $\mathbb{C}^{m_j}$ as j ranges from 1 to n. Conversely, every map of this composition defines an element in $\mathrm{End}_G(\lambda)$. We summarize:

Lemma 1.5. *Suppose that*

$$\mathcal{H}_\lambda \cong \bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathbb{C}^{m_j},$$

where $(\mathcal{H}_j, \pi_j)$ *are mutually inequivalent, irreducible unitary representations of* G*. Then,*

$$\mathrm{End}_G(\lambda) = \bigoplus_{j=1}^{n} \mathrm{End}(\mathbb{C}^{m_j}).$$

We say that $(\mathcal{H}_\lambda, \lambda)$ is *multiplicity-free* if all of the m_j's in the decomposition above are equal to one. Using the fact that $\mathrm{End}(\mathbb{C}^m)$ is a commutative $\mathbb{C}$-algebra precisely when $m = 1$, we have the following immediate corollary.

Corollary 1.6. $(\mathcal{H}_\lambda, \lambda)$ *is multiplicity-free if and only if* $\mathrm{End}_G(\lambda)$ *is commutative.*

2 Hecke algebras and homogeneous spaces

2.1 Hecke algebras

Let G be a finite group and $K < G$ a subgroup. Let $\ell^1(G, K)$ denote the complex linear space of complex-valued bi-K-invariant functions on G. We can equip $\ell^1(G, K)$ with a *convolution product* defined by

$$(\rho_1 * \rho_2)(t) = \frac{1}{|K|} \sum_{s \in G} \rho_1(s)\rho_2(s^{-1}t), \quad \text{for } \rho_1, \rho_2 \in \ell^1(G, K),$$

which turns $\ell^1(G, K)$ into a $\mathbb{C}$-algebra, and an *inversion* defined by

$$\check{\rho}(s) = \rho(s^{-1}), \quad \text{for } \rho \in \ell^1(G, K).$$

We shall refer to $\ell^1(G, K)$ with these structures as the *Hecke algebra* of (G, K), and we note that the indicator function δ_K of K is the identity in this $\mathbb{C}$-algebra, i.e.

$$\delta_K * \rho = \rho * \delta_K = \rho, \quad \text{for all } \rho \in \ell^1(G, K). \tag{2.1}$$

Furthermore, we note that

$$\widecheck{\rho_1 * \rho_2} = \check{\rho}_2 * \check{\rho}_1, \quad \text{for all } \rho_1, \rho_2 \in \ell^1(G, K). \tag{2.2}$$

Let $X = G/K$ and let $\ell^2(X)$ denote the complex Hilbert space consisting of all complex-valued functions on X, equipped with the inner product

$$\langle f_1, f_2 \rangle_X = \sum_{xK \in G/K} \overline{f_1(xK)} f_2(xK), \quad \text{for } f_1, f_2 \in \ell^2(X).$$

The *regular representation* $(\ell^2(X), \lambda_X)$ is given by

$$\lambda_X(g)f(xK) = f(g^{-1}xK), \quad \text{for } xK \in G/K.$$

One readily checks that this is indeed a unitary representation of G. Given $\rho \in \ell^1(G, K)$, we can define an element $T_\rho \in \mathrm{End}_G(\lambda_X)$ by

$$(T_\rho f)(xK) = \frac{1}{|K|} \sum_{s \in G} \rho(s) f(xsK), \quad \text{for } f \in \ell^2(X).$$

Lemma 2.1. *The map $\rho \mapsto T_\rho$ is an isomorphism of $\mathbb{C}$-algebras.*

Proof. One readily checks that the map $\rho \mapsto T_\rho$ is linear, injective and satisfies $T_{\rho_1 * \rho_2} = T_{\rho_1} T_{\rho_2}$ for all $\rho_1, \rho_2 \in \ell^1(G, K)$. To prove surjectivity, suppose that $M \in \mathrm{End}_G(\lambda_X)$, and note that we can find a complex-valued function m on $X \times X$, which is invariant under the diagonal G-action, such that

$$Mf(xK) = \sum_{yK \in G/K} m(xK, yK) f(yK), \quad \text{for } f \in \ell^2(X).$$

If we define $|K|\rho(s) = m(K, sK)$, then $M = T_\rho$, which finishes the proof. $\square$

2.2 Spherical unitary representations

A unitary representation $(\mathcal{H}, \pi)$ of G is called K-spherical (or simply *spherical* if K has been fixed) if $\mathcal{H}^K \neq \{0\}$. We note that every $\rho \in \ell^1(G, K)$ defines a linear map $\pi(\rho) : \mathcal{H}^K \to \mathcal{H}^K$ defined by

$$\pi(\rho)v = \frac{1}{|K|} \sum_{s \in G} \rho(s)\pi(s)v, \quad \text{for } v \in \mathcal{H}^K. \tag{2.3}$$

We also note that if $(\mathcal{H}, \pi)$ is K-spherical, then so is the dual representation $(\mathcal{H}^*, \pi^*)$. In fact, if $u \in \mathcal{H}$ is invariant under $\pi(K)$, then u^* defined in (1.1), is invariant under $\pi^*(K)$. Furthermore, being K-spherical is a property inherited under isomorphisms, and we define the *K-spherical unitary dual $\widehat{G}^K$* by

$$\widehat{G}^K = \{[\pi] \in \widehat{G} \,:\, \pi \text{ is } K\text{-spherical}\}, \tag{2.4}$$

where $\widehat{G}$ is defined as in (1.14).

2.3 Decomposing $\ell^2(X)$

Given a unitary spherical representation $(\mathcal{H}_\pi, \pi)$ of G with $\mathcal{H}_\pi^K \neq \{0\}$, we can define a non-trivial linear map $\iota_\pi : \mathcal{H}_\pi \otimes (\mathcal{H}^*)_\pi^K \to \ell^2(G/K)$ by

$$\iota_\pi(u \otimes v^*) = \sqrt{\frac{d_\pi}{|X|}}\,(u \otimes v^*)_\pi, \tag{2.5}$$

where $d_\pi = \dim \mathcal{H}_\pi$ and

$$(u \otimes v^*)_\pi(xK) = (\pi^*(x)v^*)(u) = \langle \pi(x)v, u\rangle_\pi, \quad \text{for } u \in \mathcal{H}_\pi \text{ and } v \in \mathcal{H}_\pi^K.$$

We note that

$$(\pi(s)u \otimes v^*)_\pi(xK) = \langle \pi(s^{-1}x)v, u\rangle_\pi = \lambda_X(s)(u \otimes v^*)_\pi(xK) \tag{2.6}$$

for all $s \in G$, and

$$(u \otimes \pi^*(\rho)v^*)_\pi(xK) = \frac{1}{|K|} \sum_{s \in G} \rho(s)\langle \pi(xs)v, u\rangle_\pi = T_\rho(u \otimes v^*)(xK), \tag{2.7}$$

for all $\rho \in \ell^1(G, K)$. In other words,

$$\iota_\pi \circ (\pi(s) \otimes \pi^*(\rho)) = (\lambda_X(s)T_\rho) \circ \iota_\pi, \quad \text{for all } s \in G \text{ and } \rho \in \ell^1(G, K).$$

The following theorem is the main result of this section.

Theorem 2.2. *For every* irreducible *K-spherical unitary representation* $(\mathcal{H}_\pi, \pi)$,

- *the map* $\iota_\pi : \mathcal{H}_\pi \otimes \mathcal{H}_{\pi^*}^K \to \ell^2(X)$ *is unitary.*
- *the image* $\operatorname{Im} \iota_\pi \subset \ell^2(G/K)$ *depends only on* $[\pi] \in \widehat{G}^K$.

Furthermore, if $(\mathcal{H}_{\pi_1}, \pi_1)$ *and* $(\mathcal{H}_{\pi_2}, \pi_2)$ *are inequivalent K-spherical unitary representations of G, then* $\operatorname{Im} \iota_{\pi_1} \perp \operatorname{Im} \iota_{\pi_2}$, *and*

$$\ell^2(X) = \bigoplus_{[\pi] \in \widehat{G}^K} \operatorname{Im} \iota_\pi.$$

In particular, $\operatorname{mult}(\pi, \lambda_X) = \dim \mathcal{H}_\pi^K$ *for every* $\pi \in \widehat{G}^K$.

Corollary 2.3. $(\ell^2(X), \lambda_X)$ *is multiplicity-free if and only if for every irreducible K-spherical unitary representation* $(\mathcal{H}_\pi, \pi)$, *we have* $\dim \mathcal{H}_\pi^K = 1$.

2.4 Proof of Theorem 2.2

The facts that ι_π is unitary, its image only depends on $[\pi] \in \widehat{G}^K$ and the direct sum

$$\bigoplus_{[\pi] \in \widehat{G}^K} \operatorname{Im} \iota_\pi \subset \ell^2(X) \tag{2.8}$$

is orthogonal follow immediately from Lemma 1.4. It remains to prove that the direct sum in (2.8) equals $\ell^2(X)$, or equivalently, if $f \in \ell^2(X)$ is any element such that

$$\langle f, (u \otimes v^*)_\pi \rangle_X = 0, \quad \text{for all } [\pi] \in \widehat{G}^K \text{ and } u \in \mathcal{H}_\pi \text{ and } v \in \mathcal{H}_\pi^K,$$

then $f = 0$. Writing this out, we see that we need to show that

$$\sum_{xK \in G/K} f(xK)\pi(x)v = 0, \text{ for all } \pi \in \widehat{G}^K \text{ and } v \in \mathcal{H}_\pi^K \implies f = 0.$$

Upon multiplying the left hand side with $\pi(s)$ and re-arranging the sum, we see that this is equivalent to the implication:

$$\sum_{xK \in G/K} f(sxK)\pi(x)v = 0, \text{ for all } s \in G \text{ and } \pi \in \widehat{G}^K \text{ and } v \in \mathcal{H}_\pi^K \implies f = 0.$$

If f is not identically zero, there exists $s \in G$ such that $f(sK) \neq 0$, and thus

$$F(xK) = \frac{1}{|K|} \sum_{k \in K} f(skxK), \quad \text{for } xK \in G/K$$

is not identically zero (since $F(K) = f(sK)$), and K-invariant. Furthermore, we have

$$\sum_{xK \in G/K} F(xK)\pi(x)v = 0, \quad \text{for all } \pi \in \widehat{G}^K \text{ and } v \in \mathcal{H}_\pi^K.$$

Let $\mathcal{H}$ denote the sub-representation of λ_X which is spanned by $v(xK) = \overline{F(x^{-1}K)}$ (note that v is well-defined since F is K-invariant) and write it as $v = \sum_j v_j$, where $v_j \in \mathcal{H}_{\pi_j}^K$ and each $\mathcal{H}_{\pi_j}$ is an irreducible representation of G. By assumption,

$$\sum_{xK \in G/K} F(xK)\pi_j(x)v_j = 0, \quad \text{for every } j,$$

and thus, upon summing over j and evaluating at K, we have

$$0 = \Big(\sum_{xK \in G/K} F(xK)\lambda_X(x)v \Big)(K) = \frac{1}{|K|} \sum_{xK \in G/K} |F(xK)|^2,$$

which forces $F = 0$. This contradiction implies that $f = 0$.

3 Finite vertex-transitive digraphs and their quotients

3.1 Vertex-transitivity

A finite *digraph* $\mathcal{G} = (X, E)$ consists of a finite set X and a subset $E \subset X \times X$. We refer to the elements in X as *vertices of* $\mathcal{G}$ and to the elements in E as *arcs* of $\mathcal{G}$. A *digraph map* between two finite digraphs (X, E) and (Y, F) is a map $\phi : X \to Y$ such that

$$(x, y) \in E \implies (\phi(x), \phi(y)) \in F.$$

If ϕ is bijective and its inverse ϕ^{-1} is also a digraph map, then ϕ is called an *isomorphism*, and an isomorphism of (X, E) with itself is called an *automorphism* of (X, E). We denote by $\mathrm{Aut}(\mathcal{G})$ the (possibly trivial) finite group of all automorphisms of $\mathcal{G} = (X, E)$.

We say that a subgroup $G < \mathrm{Aut}(\mathcal{G})$ is *vertex-transitive* if for every $x, y \in X$, there exists $g \in G$ such that $gx = y$. If G is vertex-transitive and $x_o \in X$ is any point, then we can write $X = G/K$, where K denotes the stabilizer of x_o, and if we set

$$S = \{KsK \,:\, (K, sK) \in E\} \subset K\backslash G/K,$$

then

$$E = \{(xK, yK) \,:\, Kx^{-1}yK \in S\} \subset G/K \times G/K.$$

Conversely, if G is a finite group, $K < G$ a subgroup and $S \subset K\backslash G/K$, then we can define the *Schreier digraph* (X, E_S) *associated to* (G, K, S) by

$$X = G/K \quad \text{and} \quad E_S = \{(xK, yK) \,:\, Kx^{-1}yK \in S\}.$$

We say that a finite digraph is *vertex transitive* if its automorphism group is vertex-transitive, and we conclude from the discussion above that every such digraph is isomorphic to a Schreier graph.

3.2 Laplace operators on digraphs

Suppose that $\mathcal{G} = (X, E)$ is a finite digraph, and let $\ell^2(X)$ denote the Hilbert space of all complex-valued functions on X equipped with the inner product

$$\langle f_1, f_2 \rangle_X = \sum_{x \in X} \overline{f_1(x)} f_2(x), \quad \text{for } f_1, f_2 \in \ell^2(X).$$

Given a non-negative function m on E, we define the *Laplace operator* Δ_m on $\ell^2(X)$ by

$$(\Delta_m f)(x) = \sum_{y \in X} m(x, y) f(y), \quad \text{for } f \in \ell^2(X). \tag{3.1}$$

We note that if E is a symmetric set, i.e. $(y, x) \in E$ whenever $(x, y) \in E$, and $m(x, y) = m(y, x)$ for all $(x, y) \in E$, then Δ_m is a self-adjoint operator on $\ell^2(X)$. In the case when (X, E) is the Schreier graph associated to a triple (G, K, S), any Laplace operator on (X, E) can be identified with an element in the Hecke algebra $\ell^1(G, K)$ as the following lemma shows.

Lemma 3.1. *Let G be a finite group, $K < G$ a subgroup and $S \subset K\backslash G/K$. Let (X, E) denote the Schreier graph associated to (G, K, S). For any non-negative $\rho \in \ell^1(G, K)$ whose support is contained in S, we have $\Delta_{m_\rho} = T_\rho$, where $|K|\, m_\rho(xK, yK) = \rho(x^{-1}y)$.*

3.3 Quotients of digraphs

Let G be a finite group and let $K, \Gamma < G$ be subgroups. Define

$$X = G/K \quad \text{and} \quad Y = \Gamma\backslash G/K.$$

Given a subset $S \subset K\backslash G/K$, we can equip X and Y with arc sets

$$E = \{(xK, yK) \,:\, Kx^{-1}yK \in S\}$$

and

$$F = \{(\Gamma xK, \Gamma yK) \,:\, Kx^{-1}\gamma yK \in S, \text{ for some } \gamma \in \Gamma\}$$

respectively. We shall refer to (Y, F) as the *quotient graph* of (X, E). Sometimes, when the dependence on G, K, Γ and S ought to be stressed, we shall also denote this digraph by $\mathcal{G}(G, K, \Gamma, S)$.

Note that we can identify the Hilbert spaces $\ell^2(X)^\Gamma$ with $\ell^2(Y)$ via the unitary map

$$(Uf)(\Gamma xK) = |\Gamma \cdot xK|^{-\frac{1}{2}} f(xK), \quad \text{for } f \in \ell^2(X)^\Gamma,$$

where $|\Gamma \cdot xK|$ denotes the cardinality of the Γ-orbit of xK in X. Suppose that $\rho \in \ell^1(G, K)$ is non-negative and supported on S, and define $\tilde{\Delta}_\rho = \Delta_{m_\rho}$ with m_ρ as in Lemma 3.1. Since $T_\rho \in \mathrm{End}_G(\lambda_X)$, the Laplace operator $\tilde{\Delta}_\rho$ restricts to an operator on $\ell^2(X)^\Gamma$. This allows us to define the Laplace operator Δ_ρ on (Y, F) by $\Delta_\rho := U\tilde{\Delta}_\rho U^{-1}$, where U is the unitary equivalence between $\ell^2(X)^\Gamma$ and $\ell^2(Y)$ above.

Lemma 3.2. *We have $\Delta_\rho = \Delta_{m_\rho^\Gamma}$, where*

$$m_\rho^\Gamma(\Gamma xK, \Gamma yK) = \Big(\frac{|\Gamma \cdot yK|}{|\Gamma \cdot xK|}\Big)^{\frac{1}{2}} \sum_{zK \in \Gamma \cdot yK} \rho(x^{-1}z),$$

for $\Gamma xK, \Gamma yK \in Y$.

Remark 3.3. The exact form of Δ_ρ described in Lemma 3.2 will not be important for our discussions. Note however that if Γ acts freely on X, i.e. $\Gamma \cap x^{-1}Kx = \{e\}$ for every $x \in G/K$, so that $|\Gamma \cdot xK| = |\Gamma|$ for every $xK \in G/K$, then $\Delta_{\chi_S} = \Delta_{\chi_F}$, that is to say, Δ_{χ_S} coincides with the adjacency operator on the graph (Y, F).

We recall from Theorem 2.2 that $(\ell^2(X), \lambda_X)$ admits an orthogonal decomposition of the form

$$\ell^2(X) \cong \bigoplus_{[\pi] \in \widehat{G}^K} \mathcal{H}_\pi \otimes (\mathcal{H}_\pi^*)^K.$$

In particular, we have

$$\ell^2(Y) \cong \ell^2(X)^\Gamma \cong \bigoplus_{[\pi] \in \widehat{G}^K} \mathcal{H}_\pi^\Gamma \otimes (\mathcal{H}_\pi^*)^K,$$

Furthermore, from the proof of Theorem 2.2, we see that upon choosing ON-bases $\{u_{\pi,1}, \ldots, u_{\pi,m_\pi}\}$ and $\{v_{\pi,1}, \ldots, v_{\pi,n_\pi}\}$ for $\mathcal{H}_\pi^\Gamma$ and $\mathcal{H}_\pi^K$ respectively, the functions

$$\tilde{f}_{\pi,ij}(xK) = \sqrt{\frac{d_\pi}{|G/K|}}\, \langle \pi(x) v_{\pi,j}, u_{\pi,i} \rangle_\pi,$$

and

$$f_{\pi,ij}(\Gamma x K) = \sqrt{\frac{d_\pi}{|\Gamma \cdot xK||G/K|}}\, \langle \pi(x) v_{\pi,j}, u_{\pi,i} \rangle_\pi$$

form ON-bases for $\ell^2(X)^\Gamma$ and $\ell^2(Y)$ respectively. Fix $\rho \in \ell^1(G,K)$ and note that by (2.7), we have

$$\widetilde{\Delta}_\rho \tilde{f}_{\pi,ij}(xK) = \sqrt{\frac{d_\pi}{|G/K|}}\, \langle \pi(x)\pi(\rho) v_{\pi,j}, u_{\pi,i} \rangle_\pi,$$

where $\pi(\rho)$ is defined as in (2.3). In particular, if $\check{\rho} = \rho$ so that $\pi(\rho)$ is a self-adjoint operator on $\mathcal{H}_\pi^K$, and if $v_{\pi,1}, \ldots, v_{\pi,n_\pi}$ are chosen to be orthonormal eigenvectors of $\pi(\rho)$ with corresponding (real) eigenvalues $\lambda_{\pi,1}, \ldots, \lambda_{\pi,n_\pi}$, then

$$\widetilde{\Delta}_\rho \tilde{f}_{\pi,ij} = \lambda_{\pi,j} \tilde{f}_{\pi,ij}, \quad \text{for all } i, j. \tag{3.2}$$

In other words, the spectral decomposition of Δ_ρ on $\ell^2(Y)$ is completely determined by

- The eigenvalues and eigenvectors of the action of $\pi(\rho) \curvearrowright \mathcal{H}_\pi^K$ for every $[\pi] \in \widehat{G}^K$.
- A choice of ON-basis for $\mathcal{H}_\pi^\Gamma$.

In the next section we shall confine our attention to an important special case when the action $\pi(\rho) \curvearrowright \mathcal{H}_\pi^K$ for $\rho \in \ell^1(G,K)$ is particularly easy to describe.

4 Gelfand pairs and their spherical functions

Let G be a finite group and $K < G$ a subgroup. We say that (G,K) is a *Gelfand pair* if its Hecke algebra $(\ell^1(G,K), *)$ is commutative. Upon combining Corollary 1.6 and Lemma 2.1 and Corollary 2.3 we get the following alternative characterization of such pairs.

Theorem 4.1. *The following are equivalent:*

- (G,K) *is Gelfand*
- $(\ell^2(X), \lambda_X)$ *is multiplicity-free.*
- $\dim \mathcal{H}_\pi^K = 1$ *for every K-spherical irreducible unitary representation $(\mathcal{H}_\pi, \pi)$.*

Suppose that (G,K) is a Gelfand pair, and let $(\mathcal{H}_\pi, \pi)$ be a K-spherical unitary representation of G. By the previous theorem, there exists a unique (up to scaling by a complex number of unit length) element $u_\pi \in \mathcal{H}_\pi^K$ with $\|u_\pi\|_\pi = 1$. We conclude that there exists a $\mathbb{C}$-algebra homomorphism $\omega_\pi : \ell^1(G,K) \to \mathbb{C}$ characterized by

$$\pi(\rho) u_\pi = \omega_\pi(\rho) u_\pi, \quad \text{for every } \rho \in \ell^1(G,K). \tag{4.1}$$

Suppose that $\Gamma < G$ is a subgroup, and let $\{f_{\pi,ij}\} \subset \ell^2(Y)$ be as in the previous subsection, where $i = 1, \ldots, \dim \mathcal{H}_\pi^\Gamma$ and $n_\pi = 1$, so that $j = 1$. We note that by the relations 2.7 and 4.1, we have

$$\widetilde{\Delta}_\rho \tilde{f}_{\pi,i1} = \overline{\omega_\pi(\rho)}\, \tilde{f}_{\pi,i1}, \quad \text{for } i = 1, \ldots, \dim \mathcal{H}_\pi^\Gamma,$$

i.e. the spectrum of Δ_ρ on (Y, F) equals $\{\overline{\omega_\pi(\rho)}, \ldots, \overline{\omega_\pi(\rho)}\}$ with multiplicities $\dim \mathcal{H}_\pi^\Gamma$.

In particular, if $\Gamma = K$, we see that $\ell^2(K\backslash G/K)$ is spanned by exactly $|\widehat{G}^K|$ many orthogonal functions, which proves the following lemma.

Lemma 4.2. *For every Gelfand pair (G, K), we have $|\widehat{G}^K| = |K\backslash G/K|$.*

4.1 Weakly symmetric pairs and Gelfand-Selberg's Lemma

Let G be a finite group and $K < G$ a subgroup. We say that (G, K) is *weakly symmetric* if there exists an order two automorphism σ of G such that $s^{-1} \in K\sigma(s)K$ for every $s \in G$. If σ can be chosen to be the identity, then (G, K) is called *symmetric*.

Lemma 4.3 (Gelfand-Selberg)**.** *Every weakly symmetric pair is Gelfand.*

Proof. Given $\rho \in \ell^1(G, K)$, we define $\rho^\sigma(s) = \rho(\sigma(s))$. Since (G, K) is weakly symmetric, we have $\rho^\sigma = \check{\rho}$, and thus by (2.2)

$$\rho_1^\sigma * \rho_2^\sigma = (\rho_1 * \rho_2)^\sigma = \widecheck{\rho_1 * \rho_2} = \check{\rho}_2 * \check{\rho}_1 = \rho_2^\sigma * \rho_1^\sigma,$$

for all $\rho_1, \rho_2 \in \ell^1(G, K)$. We conclude that $(\ell^1(G, K), *)$ is commutative. $\square$

5 Isospectral graphs

We say that two digraphs (Y_1, F_1) and (Y_2, F_2) are *isospectral* if the adjacency operators

$$(\Delta_i f)(y) = \sum_{(y,z)\in F_i} f(z), \quad \text{for } i = 1, 2$$

have the same eigenvalues with the same multiplicities. In the case when

$$Y_1 = \Gamma_1\backslash G/K \quad \text{and} \quad Y_2 = \Gamma_2\backslash G/K$$

for some finite group G and subgroups K, Γ_1, Γ_2 such that

$$\Gamma_i \cap xKx^{-1} = \{e\}, \quad \text{for all } x \in G/K,$$

e.g. if K is trivial, and

$$F_i = \{(\Gamma_i xK, \Gamma_i yK) \,:\, Kx^{-1}\gamma_i yK \in S, \ \text{for some } \gamma_i \in \Gamma_i\}, \quad \text{for } i = 1, 2,$$

for some $S \subset K\backslash G/K$, then by Lemma 3.2, we can identify Δ_i with the action of T_{χ_S} on $\ell^2(G/K)^{\Gamma_i}$. In particular, by the discussion at the end of subsection 3.3, we see that the eigenvalues of Δ_i are completely determined by the action of $\pi(\chi_S)$ on $\mathcal{H}_\pi$ as π ranges over $\widehat{G}^K$, and their multiplicities are completely determined by $\dim \mathcal{H}_\pi^\Gamma$ (which could be zero). In particular, under the assumptions above, (Y_1, F_1) and (Y_2, F_2) are isospectral if and only if

$$\dim \mathcal{H}_\pi^{\Gamma_1} = \dim \mathcal{H}_\pi^{\Gamma_2}, \quad \text{for all } [\pi] \in \widehat{G}^K. \tag{5.1}$$

5.1 Sunada's observation

We shall now discuss a simple observation of Sunada. Let G be a finite group, and given $y \in G$, we denote by y^G the set of all conjugates of y. We say that two subsets $A, B \subset G$ are *almost conjugate* if

$$|A \cap y^G| = |B \cap y^G|, \quad \text{for all } y \in G.$$

In the notation from the previous subsection (with $K = \{e\}$), Sunada proved:

Theorem 5.1 (Sunada)**.** *Let G be a finite group and suppose that $\Gamma_1, \Gamma_2 < G$ are almost-conjugate subgroups. Then, for every symmetric subset $S \subset G$, the graphs (Y_1, F_1) and (Y_2, F_2) are isospectral.*

Proof. It suffices to prove (5.1). Fix an irreducible unitary representation $(\mathcal{H}_{pi}, \pi)$ of G and an ON-basis $\{e_i\}$ for $\mathcal{H}_\pi$. Define

$$P_i = \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} \pi(\gamma_i).$$

Since P_i is an orthogonal projection on $\mathcal{H}_\pi^{\Gamma_i}$, we note that

$$\dim \mathcal{H}_\pi^{\Gamma_i} = \operatorname{tr} P_i = \frac{1}{|\Gamma_i|} \sum_{j=1}^{d_\pi} \sum_{\gamma \in \Gamma_i} \langle e_j, \pi(\gamma) e_j \rangle.$$

Since the trace map is independent of the choice of ON-basis, the function

$$\psi(g) = \sum_{j=1}^{n} \langle e_j, \pi(\gamma) e_j \rangle, \quad \text{for } g \in G$$

is conjugation invariant, and thus

$$\sum_{\gamma \in \Gamma_i} \psi(\gamma) = \sum_{c \in \mathcal{C}} |\Gamma_i \cap c| \psi(c),$$

where $\mathcal{C}$ denotes the set of conjugacy classes in G. By assumption, $|\Gamma_1 \cap c| = |\Gamma_2 \cap c|$ for every $c \in \mathcal{C}$ and thus $|\Gamma_1| = |\Gamma_2|$ and

$$\sum_{\gamma \in \Gamma_1} \psi(\gamma) = \sum_{\gamma \in \Gamma_2} \psi(\gamma),$$

which finishes the proof. $\square$