Flatness

The goal of these notes is to provide a concise introduction to flatness. I've tried to include all the algebra you would need to know for a standard algebraic geometry text, e.g. Hartshorne. Most of the material can be found in Eisenbud; I don't like his presentation very much, which is the motivation behind this rewrite. There is no claim to originality - everything not in Eisenbud can be found in Hartshorne, Ravi Vakil's online notes, or another standard reference.

The general outline is as follows. First, I'll recall the basic properties of flatness. The next section discusses the basic properties of flatness which are most clearly expressed using Tor. Then, I'll analyze how flatness interacts with localization, so that we'll be able to bring our results into the realm of algebraic geometry. Finally, we'll do some algebraic geometry proper by giving the algebraic background for several of the theorems in Hartshorne.

Let A be a commutative ring. The tensor product $M \otimes N$ of two (left) A-modules is finite sums of pairs $m \otimes n$, quotiented out by the usual relations. It satisfies the usual universal property; any A-bilinear map from $M \times N$ factors through it. In particular, if A is a subring of B, then $B \otimes M$ is universal for A-module homomorphisms from M to a B-module N.

There are a number of natural canonical isomorphisms that come up for tensor products. In particular, for M an A module, M' a B-module, and M'' an (A, B)-bimodule,

$$(M \otimes_A M'') \otimes_B M' \cong M \otimes_A (M'' \otimes_B M')$$

In general, the functor $D \otimes -$ is right exact, but it is not left exact (it is the right adjoint to Hom). For example, the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ becomes the 0 map when tensored with $\mathbb{Z}/2$. An A-module D is called flat if $D \otimes -$ is a left exact functor.

Example: If A is a field, every A-module (i.e. A vector space) is flat. This is well-known; it is also a special case of the next criteria. $\square$

Example: Any projective A-module is flat. First, note that any free module is flat: this is obvious for finitely generated free modules. If F is free but not finitely generated, any element of $F \otimes M$ is contained in $F' \otimes M$ for some F' a finitely generated submodule of F. Suppose we have an injection $M \hookrightarrow N$. If an element $\gamma \in F \otimes M$ goes to 0 under the induced map $F \otimes -$ on tensor products, then it also goes to 0 for $F' \otimes -$, which implies that the element is 0. So, all free modules are flat.

Any projective module P is a direct summand of a free module D. Recall that tensor products commute with direct sums. Thus, if $M \hookrightarrow N$ is an injection, then $(P \oplus P') \otimes M = (P \otimes M) \oplus (P' \otimes M)$ injects into $(P \otimes N) \oplus (P' \otimes N)$. Since the map also splits over the direct sum, we have that P is flat. $\square$

Example: If S is a multiplicative set in A, then $S^{-1}A$ is a flat A-module: Recall that the localization $S^{-1}M \cong S^{-1}A \otimes_A M$. Since localization is an exact functor, $S^{-1}A \otimes -$ is exact, meaning that $S^{-1}A$ is flat. $\square$

There are a couple propositions that follow immediately from the canonical isomorphisms mentioned in the first section.

Proposition 0.1 *If M is a flat A-module, and $A \to B$ is a homomorphism of rings, then $M \otimes_A B$ is a flat B-module.*

In geometric terms, this means that flatness commutes with base change. Coming at it the other way around, we have

Proposition 0.2 *If M is a flat B-module and B is a flat A-algebra, then M is also a flat A-algebra.*

In geometric terms, this means that flatness is a transitive property under composition of morphisms.

1 Flatness and Tor

There are several properties of flatness which can best be described using the derived functors Tor. One can prove these by an elementary argument, but this amounts to reconstructing the Tor by hand, so I won't do so. If you're not familiar with derived functors, you should at least read the statements of the theorems.

By definition, any flat A-module M will have $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$ and all N. In fact, by simply rewriting the criteria for flatness, we see:

Proposition 1.1 *The A-module M is flat iff $Tor_1(M, N) = 0$ for every A-module N.*

Using the LES of Tor, this immediately implies:

Proposition 1.2 *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, and M' and M'' are flat, then so is M. If M and M'' are both flat, so is M'.*

Another useful consequence is the following, which again is immediate upon passing to the LES of Tor.

Proposition 1.3 *Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules. If M'' is flat, then for any module N, the sequence tensored with N is also exact.*

In geometric terms, if we have a short exact sequence of sheaves on some scheme Y, and the last sheaf is flat, then the sequence is still exact upon pull-back to some other scheme X.

Yet another consequence is that flatness can be checked on injections of ideals $I \hookrightarrow A$.

Theorem 1.4 *An A-module M is flat iff $M \otimes I \to M \otimes A$ is an injection for every ideal $I \hookrightarrow A$.*

Proof: The necessity of the condition follows from the definition. Conversely, suppose the condition holds. We first show that we can limit ourselves to considering finitely generated submodules, by an argument similar to the one made in the example about free modules. Suppose that we have an injection $N \hookrightarrow N'$. Let $\gamma \in N \otimes M$; since γ only involves finitely many elements in N, there is a finitely generated submodule $\tilde{N}$ such that $\gamma \in \tilde{N} \otimes M$. Then, the fact that γ goes to 0 under the induced map of tensor products can be checked on the finitely generated submodules $\tilde{N}$ and $\tilde{N}'$. So, it suffices to consider the finitely generated case.

We now show that for every injection $N \hookrightarrow N'$ of finitely generated A-modules, we have $M \otimes N \hookrightarrow M \otimes N'$. We can find a sequence

$$N = N_0 \subset N_1 \subset \ldots \subset N_k = N'$$

where each $N_{i+1}/N_i \cong A/I$ for some ideal I. If we can show that $N_i \otimes M \to N_{i+1} \otimes M$ is an injection then induction will finish the proof. However, we have an exact sequence

$$\mathrm{Tor}_1(M, N_{i+1}/N_i) \to M \otimes N_i \to M \otimes N_{i+1}$$

Since $\mathrm{Tor}_1(M, N_{i+1}/N_i) = \mathrm{Tor}_1(M, A/I) = 0$, we have injections everywhere. $\Box$

In fact, it suffices to show that this Tor vanishes for finitely-generated ideals I (by a similar argument). This "finite generation" argument is just a concrete instance of the fact that right adjoints commute with limits; here, our limit is taking the union of finitely-generated submodules.

As an application of this principle, we will characterize flat modules over the simplest kinds of rings, PIDs. Note that a flat module over any domain A must be torsion-free; if a is an element of our ring, then multiplication by a is an injection from $A \to A$, meaning that multiplication by a must also be an injection from $M \to M$. For a PID, this necessary condition is also sufficient.

Proposition 1.5 *Let A be a PID. Then a module M is flat iff it is torsion-free.*

Proof: Note that M is torsion-free iff for every element $a \in A$, multiplication by a is an injection on M. Since every ideal is principal, this amounts to the fact that $\mathrm{Tor}_1(M, A/I) = 0$ for every ideal I. $\Box$

An important special case is the scheme of dual numbers $A = k[t]/(t^2)$, which is useful for deformation theory. This ring has two ideals, (0) and (t). Since we always have that $\mathrm{Tor}_1(M, A) = 0$ (tensoring by A is the identity functor), we only need to check that Tor is 0 for the one quotient A/t.

Proposition 1.6 *The module M is flat over $k[t]/(t^2)$ iff the map $M/tM \to tM$ taking $m + tM \mapsto tm$ is an isomorphism.*

Note that the map is well-defined as $t^2 = 0$. To prove this proposition, we just need to note that the condition is equivalent to the fact that $tm \neq 0$ for any non-zero $m \in M$.

2 Flatness and Localization

If we're going to translate flatness into a property for schemes, then it must behave well under localization. Our first proposition shows that it acts just as we would want it to - flatness is a local property.

Proposition 2.1 *An A-module M is flat iff M_p is a flat A_p-module for every prime $p \subset A$.*

Essentially, this amounts to the fact that an A-module M is 0 iff M_p is 0 for every prime p in A. This implies that we can check if a map has a kernel by passing to localizations. Since flatness amounts to a particular map having a kernel, it can be checked on localizations.

Proof: If M is a flat A-module, then we see that M_p is flat as follows. Suppose we have an injection of A_p-modules $N \hookrightarrow N'$. Note that $M \otimes_A N \cong M_p \otimes_{A_p} N$ canonically. Thus, the injection $M \otimes_A N \hookrightarrow M \otimes_A N'$ immediately yields the flatness of M_p.

Conversely, if M is not flat, there is some injection $N \hookrightarrow N'$ of A-modules such that $M \otimes_A N \to M \otimes_A N'$ has non-trivial kernel. In particular, there is some prime p such that the kernel is not killed by localizing at p. But then, the injection $N_p \hookrightarrow N'_p$ yields a map $M \otimes_A N_p \to M \otimes N'_p$ with non-trivial kernel. Again identifying these as $M_p \otimes_{A_p} N_p$, we find that M_p is not flat. $\square$

In fact, we can verify flatness by checking on all maximal ideals, since these suffice to check whether a module is 0.

Having shown this, our next task is to analyze what it means for M to be flat over a local ring A. It turns out that it is possible to give a very complete description of such modules when A is Noetherian.

Proposition 2.2 *Let A be a local ring with maximal ideal m. Suppose M is a finitely presented A-module (e.g. if A is Noetherian then any finitely generated M will do). Then M is flat iff it is free.*

The proof of course relies on Nakayama's lemma.

Proof: Since M is finitely presented, M/mM is a finite dimensional A/mA vector space. Choose a basis, and lift it to elements $m_1, \ldots, m_n \in M$. These define a natural map $A^n \to M$, and by Nakayama's lemma this map is surjective. We want to show that it is an isomorphism.

Let K denote the kernel of this map - K is finitely generated since M is finitely presented. When we tensor the exact sequence

$$0 \to K \to A^n \to M \to 0$$

with A/mA, the result is still exact, since M is flat (look at the LES of Tor). However, after tensoring we get a sequence of vector spaces. Since $(A/mA)^n$ and M/mM both have dimension n, K/mK must be 0. Using Nakayama's lemma again, we see that $K = 0$. $\square$

In geometric terms, this means that for coherent sheaves on Noetherian schemes, flatness is the same thing as locally free (since local freeness.

In an earlier example, we showed that every projective module is flat. We can now give a partial converse which is particularly relevant for Noetherian rings.

Corollary 2.3 *Let A be a ring and M a finitely-presented A-module. Then M is flat iff M is projective.*

Proof: We know that flatness can be checked by localizing at maximal ideals. By the previous proposition, M is flat iff M_p is a free A_p-module for every maximal ideal p. So it just remains to show that M is projective iff M_p is free for every maximal ideal p. One way is clear - if M is projective, then M_p is also projective, which implies that it is free (this is just the same argument as we used in the previous proposition; indeed, our result there implies this, since we have projective $\Rightarrow$ flat $\Rightarrow$ free). The converse follows from the universal properties of localization and projective modules, along with the fact that $\mathrm{Hom}_{A_p}(M_p, N_p) = \mathrm{Hom}_A(M, N)_p$ for finitely presented M. $\square$

Of course, there may be non-finitely-generated modules M which are flat but not projective. For example, if A is a domain, then the fraction field F is flat but not projective (as the only map from $F \to A$ is the 0 map).

3 Geometric Examples

We first give a few examples to give intuition for what flatness means in the affine case.

Example: Let X be the plane conic corresponding to $A = k[y,t]/(y - t^2)$. We consider the projection $X \to \mathbb{A}^1$ corresponding to the t coordinate. This map should be flat; thinking of the base $\mathbb{A}^1$ as parametrizing the fibers, every fiber is either two separate points or a point of depth two. And of course, A is indeed flat over $k[t]$, being free on the generators $1, y$. The sheaf A is thus locally free of rank 2. $\square$

Example: Let X be the plane curve corresponding to $A = k[y,t]/(yt - 1)$, and consider the projection onto the t-coordinate as before. This is again a flat family, as the algebra $k[y,t]/(yt-1)$ is isomorphic to the localization $k[t]_{(t)}$, and hence flat. Another way to see this is to note that $k[t]$ is a PID, and A is torsion free. Note that A is not a free $k[t]$-module, and in fact is not even locally free; there is no contradiction to our theorems as A is not finitely generated over $k[t]$. Of course, on the region $t \neq 0$ (where t is invertible) A_t is isomorphic to the structure sheaf $k[t, t^{-1}]$, and hence free of rank 1. However, the fiber above $t = 0$ is empty, and the sheaf is not locally free around this point; the stalk is $A_{(t)} = k[y,t]_{(t)}/(yt - 1)$ which is not a free module.

This example becomes more natural when we projectivize to the complete curve in $\mathbb{P}^2$ given by the homogenous equation $yt - z^2$. Then, the projection to $\mathbb{P}^1$ given by the t coordinate is an isomorphism, so of course this map is flat. $\square$

Example: Here's a non-example. Let X be the (reducible) plane curve corresponding to $A = k[y,t]/(yt - t)$. The fibers are points everywhere except over $t = 0$, where we get an $\mathbb{A}^1$. This bad behavior of the fibers is precisely the thing that flatness is created to avoid. To see that this algebra is not flat over $k[t]$, note that $k[t]$ is a PID but A is not torsion-free, as $t(y-1) = 0$. This corresponds to the fact that multiplication by t from $k[t] \to k[t]$ is injective, but the induced map on tensor products is not. $\square$

4 Algebraic Geometry

Because flatness is a local property, we can define a notion of flatness for schemes. We say that a sheaf $\mathcal{F}$ on X is flat if every stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{x,X}$ module. (This is equivalent to saying that for open affines $U = \mathrm{Spec}(A)$ we have $\mathcal{F}(U)$ is a flat A-module). By Proposition 2.2, this is equivalent to F being a locally free sheaf.

The relative notion is more useful. Suppose we are given a morphism of schemes $f : X \to Y$. A sheaf $\mathcal{F}$ on X is flat at $x \in X$ over Y if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{y,Y}$ module. We say that $\mathcal{F}$ is flat over Y if it is flat at every point of X. Finally, we say that the morphism $f : X \to Y$ is flat if $\mathcal{O}_x$ is flat over Y. Important caution: flatness over Y is a condition on points of X, not points of Y. In particular, flatness of a map $f : X \to Y$ does not imply that $f_*(\mathcal{O}_X)$ is locally free! The correct thing to do is to pull $\mathcal{O}_{\mathcal{Y}}$ back to X.

We have now covered all the basic background that Hartshorne assumes for his chapter on flatness (Hartshorne III.9). In particular, we have proved all of Proposition 9.2. I think it will be useful to continue studying the algebraic results most relevant for chapter 9. In particular, I will use algebra to rederive the proofs in chapter 9 that are done in an geometric way, as I feel that the algebra clarifies what is really going on. I won't discuss the proofs which are already clear, e.g. higher pushforwards commute with flat base change and the constancy of the Hilbert polynomial in flat families. Throughout I will assume that all schemes are Noetherian, so that finitely presented and finitely generated are equivalent conditions.

4.1 Local Criterion for Flatness

Since flatness is a local property, our next step is to analyze algebraically how flatness works with local homomorphisms of local rings, i.e. maps $\phi : (A, m) \to (B, m')$ such that $\phi^{-1}(m') = m$. Suppose we have such a local homomorphism ϕ. Of course, if M is a flat A-module then $M \otimes_A B$ is a flat A module (we showed this in the first section). (Geometrically, this means that the pullback of a locally free sheaf is locally free, which we knew already.) What we want is a criteria for a B-module M to be flat as an A-module. (This is the right way round; this is equivalent to checking the flatness of a sheaf over a map.)

The main theorem of this section is the following.

Theorem 4.1 *Let $\phi : (A, m) \to (B, m')$ be a local homomorphism of local Noetherian rings, and let M be a finitely generated B-module. Then M is a flat A module iff $Tor_1^A(A/mA, M) = 0$.*

We already know that flatness can be checked by the vanishing of Tor for every quotient A/I over all ideals I; the content of this theorem is that, under certain circumstances, it suffices to consider only the maximal ideal m.

At first glance it appears that B plays no role in the theorem - however, the crucial fact here is that M is finitely generated over B. Thus, the role of B in the theorem is that it allows us to consider M that are not finitely generated over A itself. If we did not have some kind of finite generation, the theorem would not be true. For example, if $A = B = k[x,y]_{(x,y)}$ and $M = k(x)$ (with y acting as 0), then M is not flat but it does have vanishing first Tor.

Another remark: we already know that a finitely-generated A module N is flat over A iff it is projective. This is prima facie stronger than claiming that $\mathrm{Tor}_1^A(A/mA, N) = 0$. Thus, the content of the theorem in the special case when (B, m') is (A, m) and ϕ is the identity is that the criteria are equivalent.

Proof: Sketch: to show the implication we need, one first notes that for every A-module N of finite length, we have the vanishing of $\mathrm{Tor}_1^A(N, M) = 0$; this follows immediately by induction on the length.

We need to show that multiplication $I \otimes_A M \to M$ has trivial kernel for every ideal $I \subset A$. By Krull's intersection theorem and the Artin-Rees lemma, it suffices to show that the kernel is contained in $(m^t \cap I) \otimes M$ for all t. Using the exact sequence

$$0 \to m^t \cap I \to I \to I/(m^t \cap I) \to 0$$

and tensoring with M, it suffices to show that the kernel maps to 0 in $I/(m^t \cap I) \otimes M = (I + m^t)/m^t \otimes M$. Finally, we compare against the exact sequence

$$0 \to (I + m^t)/m^t \to A/m^t \to A/(I + m^t) \to 0$$

Since the final term has finite A-length, it has vanishing first Tor, proving the needed injectivity. $\square$

This theorem is known as the local criterion for flatness. It is hard to see why since the Tor term is unfamiliar. The following corollary gives a consequence that is more intuitive. It is also sometimes known as the local criterion for flatness.

Corollary 4.2 *Let $\phi : (A, m) \to (B, m')$ be a local homomorphism of local Noetherian rings, and M a finitely generated B-module as above. Suppose that $x \in m$ is a non-zerodivisor on A. Then M is flat over A iff M/xM is flat over A/xA.*

Proof: The forwards implication is always true. For the converse, we first show that for any A/xA module N, we have

$$\mathrm{Tor}_1^A(N, M) = \mathrm{Tor}_1^{A/xA}(N, M/xM)$$

Once we show this, the statement immediately follows from the previous theorem characterizing flatness via the first Tor.

So, let $F. \to M$ be a free resolution of M. Then $A/xA \otimes F.$ is again an exact sequence, since the homology is $\mathrm{Tor}_i^A(A/xA, M)$, and these are all 0 since x is not a zero divisor. Thus, $A/xA \otimes F.$ is a resolution of M/xM, showing that the Tors coincide as claimed. $\square$

Here is a geometric application explaining the name. Suppose that we have a morphism of varieties $X \to Y$ over $\mathbb{A}^1$, such that the maps to $\mathbb{A}^1$ are flat and dominant. For some point p in $\mathbb{A}^1$, choose points p' in Y sitting above p, and p'' in X sitting above p'. If the map of fibers $X_p \to Y_p$ is flat in a neighborhood of p'' (in X_p), the map $X \to Y$ is also flat in a neighborhood of p'' (in X).

Remark: Maybe I'll say a bit more about flatness and regular local rings. Recall that if A is a regular local ring, then there is a regular sequence of elements $\{a_1, \dots, a_r\}$ generating the maximal ideal m, with $r = \dim(A)$. This means that a_i is a non-zerodivisor on $A/(a_1, \dots, a_{i-1})A$.

Now, suppose we have a map of nonsingular varieties $X \to Y$ (over an algebraically closed field) which is surjective on tangent spaces. This means that for any point $y \in Y$ below $x \in X$, we have $m_y/m_y^2 \to m_x/m_x^2$ is an injection. In particular, a regular sequence $\{a_1, \dots, a_r\}$ in the local ring at y is also regular in the local ring at x. Now, using the above proposition, we see that the local ring $\mathcal{O}_{x,X}$ is flat over $\mathcal{O}_{y,Y}$ iff $\mathcal{O}_{x,X}/(a_1, \dots, a_r)$ is flat over $\mathcal{O}_{y,Y}/(a_1, \dots, a_r)$. However, this latter ring is just a vector space; in particular, every module over it is free, and hence flat. Thus, we see that the morphism $X \to Y$ is flat. This is the crucial step in showing that a morphism which is a surjection on tangent spaces is smooth. $\square$

4.2 Flat Maps are Open

In this section, we will show that a flat map $X \to Y$ is open, that is, it takes open sets to open sets. This is a result of a "going-down" theorem, which is very similar to its more familiar analogue for integral closures.

Proposition 4.3 *Suppose that $\phi : A \to B$ is a map of rings making B into a flat A-module. Suppose that $p \supset p'$ are two primes of A. If q is a prime of B with $\phi^{-1}(q) = p$, then there is a prime q' of B such that $\phi^{-1}(q') = p'$.*

In fact, we can identify q' explicitly - we can pick any prime of B contained in q and minimal over $p'B$.

Proof: Let q' be any prime of B contained in q and minimal over $p'B$. Tensoring preserves flatness; if we tensor by $A/p'A$, we may assume that p' is 0, and so q' is a minimal prime of B. Since B is flat over A, every non-zerodivisor of A is also a non-zerodivisor on B. However, a minimal prime consists of zero-divisors, so q' contains no elements of A, i.e. $\phi^{-1}(q') = p'$. $\square$

This proposition has a natural geometric interpretation, just as the usual going-down theorem does. We claim that any flat map of finite type between Noetherian schemes is open, i.e. it takes open sets to open sets. Suppose given such a map $f : X \to Y$. Since openness is a local property, we may assume X and Y are affine. One first notes that the image of an open set is constructible - see Hartshorne exercises II.3.18 and II.3.19. By these same exercises, a constructible subset is open precisely when it is stable under generization. That is, suppose that $y, y' \in Y$ are points such that y is in the closure of y'. We must show that if y is in the image $f(X)$, so is y'. Translating this to rings, we get exactly the statement of the proposition above.

4.3 Flatness and Dimension of Fibers

In Hartshorne Proposition III.9.5, we learn that for flat maps $X \to Y$ the dimension of the fibers is the difference of the dimensions of X and Y (at least locally). This is a precursor to the constancy of Hilbert polynomials in flat families. Hartshorne's proof there is sort of unsatisfying, so here is an algebraic way of saying exactly the same thing. Hartshorne's proof relies on the Principal Ideal Theorem by cutting down the dimension of Y inductively. Of course, the PIT is the centerpiece of the theorem no matter how you look at it, so I'll recall it here.

Theorem 4.4 *(Principal Ideal Theorem)*

Let R be a Noetherian ring. Let $x_1, \ldots, x_c$ be any elements of R, and suppose p is a minimal prime of R over $(x_1, \ldots, x_c)$. Then dim$(R_p) \leq$

c. Conversely, any prime p with dim$(R_p) = c$ is minimal over some ideal generated by c elements.

Keep in mind that the dimension of the local ring R_p is the codimension of the variety corresponding to p. Thus, a variety defined as the intersection of c hyperplanes has codimension at least c.

We now come to the main proposition of this section.

Proposition 4.5 *Let $\phi : (A, m) \to (B, m')$ be a local homomorphism of Noetherian local rings. Suppose that this map makes B a flat A-module. Then*

$$\mathit{dim}(B) = \mathit{dim}(A) + \mathit{dim}(B/mB)$$

We'll prove this theorem by showing that we have an inequality in each direction. We'll first need a lemma, which will give us one of the inequalities.

Lemma 4.6 *Suppose A is a local ring with maximal ideal m. Then dim(A) is the smallest integer d such that there are d elements $x_1, \ldots, x_d \in m$ such that $m^n \subset (x_1, \ldots, x_d)$ for some sufficiently large n.*

Proof: If $m^n \subset (x_1, \ldots, x_d) \subset m$, then m is a minimal prime over the ideal generated by the x_i. The PIT shows $\dim(A) \leq d$.

For the converse, we simply use the converse direction of the PIT to say that there is some set of elements $x_1, \ldots, x_{\dim(A)}$ such that m is minimal over the ideal they generate. But then $A/(x_1, \ldots, x_{\dim(A)})$ has only one prime ideal, the reduction of m. In particular, m must be nilpotent in this quotient ring, giving us the theorem. □

This lemma gives us a way of identifying what we mean by a system of parameters: any $(x_1, \ldots, x_d)$ satisfying the second condition should be considered a system of parameters, since they control the local behavior of the scheme.

The outline of the proof is as follows. One inequality follows easily from the previous lemma. Consider the ideal with generators given by a system of parameters for A and a system of paramters of B/mB. We show that this ideal contains a power of the maximal ideal of B. In particular, this will show that $\dim(B) \leq \dim(A) + \dim(B/mB)$ with no conditions on the flatness.

For the reverse inequality, we will use the "going down" theorem of the previous section to identify $\dim(A)$ with $\dim(B_q)$, where q is a minimal prime over mB. Geometrically, this implies that the codimension of the subvariety corresponding to A is the no more than the codimension of its preimage.

Proof: (of Proposition)

First we show that the left side is no larger than the right. Let $x_1, \ldots, x_d$ be a local system of parameters for A as above, and $y_1, \ldots, y_e$ be a local system of parameters for B/mB. This means that for sufficiently large t and s

$$\begin{aligned} m'^t &\subset mB + (y_1, \ldots, y_e) \\ m'^{ts} &\subset (mB + (y_1, \ldots, y_e))^s \\ &\subset m^s B + (y_1, \ldots, y_e) \\ &\subset (x_1, \ldots, x_d)B + (y_1, \ldots, y_e) \end{aligned}$$

So by the Principal Ideal Theorem, we have the claimed inequality.

For the reverse inequality, let q be a prime of B minimal over mB. So, $\dim(B/q) = \dim(B/mB)$. Because

$$\dim(B) \geq \dim(B/q) + \dim(B_q)$$

it suffices to show that $\dim(B_q) \geq \dim(A)$. This follows immediately from the going-down theorem of the previous section. □

Remark: There are very interesting connections between flatness and regularity. Suppose A is a Cohen-Macaulay local ring (so for example any regular local ring). Then any B as in the proposition is flat iff $\dim(B) = \dim(A) + \dim(B/mB)$, that is, the consequence of the theorem is a necessary condition. □

4.4 Generic Flatness

Earlier, we showed a kind of "openness" of flatness. We'll prove a stronger statement here. The statement is known as Grothendieck's generic freeness.

Theorem 4.7 *Let A be a Noetherian domain, B a finitely-generated A algebra, and M a finitely-generated B module. There exists an $a \in A$ such that $M[a^{-1}]$ is a free $A[a^{-1}]$ module.*

Geometrically, this implies that any finite type morphism between (integral Noetherian) varieties will be flat on an open subset.
Proof: Some other time. $\square$

Here is a first geometric corollary:

Corollary 4.8 *Let $f : X \to Y$ be a dominant morphism between varieties over an algebraically closed field. There is an open subset U of Y such that the fibers of f above U have the expected dimension $dim(X) - dim(Y)$.*

Proof: In terms of algebra, we need to show the following. Suppose given an injection of Noetherian domains $A \to B$ making B a finitely-generated A-algebra. Then, there is some $a \in A$ such that for every prime $p \subset A$ not containing a, there is a prime $q \subset B$ above p (so $q \cap A = p$). Furthermore, for these p we must have

$$\dim(B_q) = \dim(A_p) + \dim(B_q/pB_q)$$

Thus, the locus defined by a represents the complement of the open set U in the theorem.

Of course, this follows from generic freeness and the analysis of dimension of fibers of flat maps. $\square$

If you follow this line of thought in more detail, you eventually reach a proof of upper semicontinuity of fiber dimension.