Ludwig Pittner

Algebraic Foundations of Non-Commutative Differential Geometry and Quantum Groups

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Algebraic Foundations of Non-Commutative Differential Geometry and Quantum Groups

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This volume is dedicated to the memory of our dear friend and respected colleague Ansgar Schnizer, who died of injuries from a traffic accident in Kyoto, Japan, where he had been working at the Research Institute of Mathematical Sciences, on October 16th, 1993.

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Preface

The motivation for this volume first arose during many Thursday afternoons in the Seminar on Mathematical Physics at the Institute of Theoretical Physics of the University of Vienna. This seminar is managed and overwhelmed with physical ideas by Harald Grosse, and is aimed at the construction of models that are stimulating from the mathematical viewpoint and maybe relevant for an approximate understanding of nature.

There is an enormous variety of new mathematical theories, or at least concepts of conjectures, concerned with non-commutative algebraic and differential geometry. This variety and the scale of methods from abstract nonsense on the categorical level to new rules of calculus, which then should be applied to new examples, say models of physical phenomena, are responsible for confusion among beginners. This is also true of the notation and the mathematical rigor. Therefore one might try to provide, as a first step, some consistent algebraic framework, which later on should be expanded by topological, measure-theoretic, and differential structures.

As already mentioned, Harald Grosse, as an outstanding expert on models of statistical physics and quantum field theory, stimulated this work during many seminars and discussions. Manfred Scheunert, who stayed at the Institute of Theoretical Physics of the University of Graz, Austria, from October 1994 until March 1995, provided me with many essential hints and corrections, and gave very precise lectures and seminars on Lie superalgebras and their q-deformation during this half year. Peter Uray, who has now finished his diploma, and Veronika Winkler, who is working on her thesis, both on problems related to quantum groups, inspired me by many questions that were maybe beyond my scope, or even beyond the actual state of research. Numerous problems concerned with the electronic preparation of the manuscript, and in particular of the diagrams, were solved with infinite patience by Wolfgang Schweiger, and partly also by C. B. Lang. Parts of the manuscript were carefully typed in their first version by Elisabeth Murtinger, and many helpful hints and corrections are due to Eveline Neuhold. The major part of the typing and electronic manipulations were done by Sabine Fuchs, whose dutiful accuracy and celestial patience enabled me to finish the volume.

October 1995 Ludwig Pittner

Contents

Introduction

The mathematical theory of non-commutative differential geometry, and similarly the variety of mathematical methods denoted by the term “quantum groups”, were induced both by mathematical unification attempts and the detailed study of physical models. Such models cannot be experimentally falsified in the strict sense of science, but nevertheless should be logically consistent, in order to provide an approach to the understanding of nature. Some main origins of these types are indicated by two diagrammatic schemes.

$$
\begin{tikzcd}
\begin{array}{l}\textit{cohomologies of}\\ \textit{Lie algebras and}\\ \textit{Lie superalgebras}\end{array} \arrow[d] & \begin{array}{l}\textit{graded}\\ \textit{Lie-Cartan pairs}\end{array} \arrow[l] \arrow[d] & \\
\begin{array}{l}\textit{differential forms}\\ \textit{on manifolds and}\\ \textit{supermanifolds}\end{array} \arrow[r, dashed, leftrightarrow] & \begin{array}{l}\textit{matrix}\\ \textit{differential}\\ \textit{geometry}\end{array} \arrow[ddr] & \\
 & \begin{array}{l}\textit{cyclic}\\ \textit{cohomology}\\ \textit{of } \Omega(A)\end{array} \arrow[u] \arrow[dd] & \\
 & & \textit{gauge fields} \\
\begin{array}{l}\textit{K-theory of}\\ C^* \textit{-algebras}\end{array} \arrow[r] \arrow[d] & \begin{array}{l}\textit{Fredholm}\\ \textit{modules}\end{array} \arrow[d] & \\
\begin{array}{l}\textit{K-theory of}\\ \textit{vector bundles}\end{array} \arrow[r] & \begin{array}{l}\textit{elliptic}\\ \textit{pseudo-differential}\\ \textit{operators}\end{array} \arrow[uur] &
\end{tikzcd}
$$

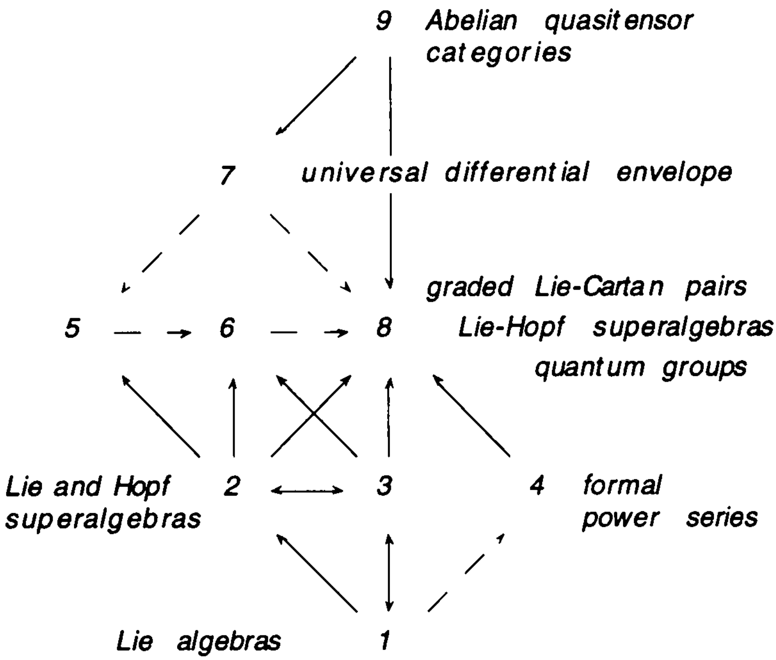
$$
\begin{tikzcd}
& \text{quantum and classical integrable models} & \\
& \text{quantum and classical inverse scattering method} \arrow[u] & \\
\text{braids and knots} \arrow[r] \arrow[dd] & \text{QYBE} \arrow[u] \arrow[r] \arrow[d, leftrightarrow] & \text{CYBE} \arrow[ul] \arrow[d, leftrightarrow] \\
& \text{quantum groups} \arrow[r, "q \longrightarrow 1"'] & \begin{array}{l}\text{affine}\\ \text{Lie algebras}\end{array} \arrow[dd] \\
\begin{array}{l}\text{Hecke algebra}\\ \text{Temperley-Lieb algebra}\\ \text{Birman-Wenzl algebra}\end{array} \arrow[d] & & \\
\text{topological field theory} & & \begin{array}{l}\text{2-dimensional}\\ \text{conformal field theory}\end{array}
\end{tikzcd}
$$

The algebraic structure of non-commutative differential geometry is determined by the notion of graded Lie-Cartan pairs, which can be specialized for instance to the so-called matrix differential geometry, and on an abstract level by the Hochschild and cyclic cohomology of the universal differential envelope $\Omega(A)$ of an associative superalgebra A.

The spirit of quantum groups is sustained by the parameter-dependent quantum Yang-Baxter equation and an according limit with respect to the spectral parameter, thereby encountering representations of the braid group B_3, which is generated by two elements σ_k, $k = 1, 2$, and the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

From the categorial viewpoint, the structures of this new paradise of mathematical physics are characterized by two concepts. The differential calculus is developed from an initial object: the universal differential envelope $\Omega(A)$ of an associative superalgebra A, which provides the multilinear framework for cyclic cohomology. The significance of the parameter-dependent quantum Yang-Baxter equation (QYBE), and consequently of its semiclassical approximation (CYBE), and of the corresponding parameter-independent limits, for the construction of algebras, which in turn serve as ingredients of various physical models, is reflected on the level of abstract nonsense by the definition of braided monoidal (or quasitensor) categories, thereby generalizing the notion of tensor categories.

In view of the high level of abstraction and also enormous complexity of these new methods, this volume may serve as an algebraic guideline. The dependence of chapters $1, \ldots, 9$ is drawn in the following diagram.



Cyclic and Hochschild cohomology within the universal differential envelope of an associative superalgebra can be studied with the basic tools of multilinear algebra. Some quite basic notions of Lie algebras and Lie superalgebras are incorporated into the definition of graded Lie-Cartan pairs and the resulting generalization of the calculus of differential forms. For an understanding of the algebraic essence of quantum groups one needs some knowledge of Hopf superalgebras in general, and in particular of semisimple finite-dimensional complex Lie algebras, and of basic classical Lie superalgebras. Topological prerequisites are necessary in order to treat rigorously those quantum groups, which are defined as unital associative superalgebras with non-polynomial relations, and in order to construct the quantum double of an infinite-dimensional Hopf algebra over a field, which then leads to the universal R-matrix of this quantum double. Real Lie groups are used as groups of bodies of the elements of real Lie-Hopf superalgebras. The categorial view of the last chapter does not reach to the depth of derived functors Ext and Tor.

Although the extent of available literature on the topics in question may be beyond the scope of systematic studies, the author tried to select a reasonable collection of original papers, contributions to proceedings, and monographs, at his own risk of omitting maybe important contributions to these new fields of research.

1. Lie Algebras

This introductory chapter is intended to provide the reader with some definitions and results on Lie algebras, which belong to the basic concepts of algebra, and which in particular are essential with respect to the so-called q-deformation of the universal enveloping algebra of a semisimple finite-dimensional complex Lie algebra. Chap. 8 will be concerned with an extensive presentation of the mathematical framework of these q-deformations, which are called quantum algebras, as some special kind of quantum groups.

The general concepts of linear and multilinear algebra are presented in several books, as for instance those by W. Greub (1978), A. I. Kostrikin and Yu. I. Manin (1989), S. Lang (1984, 1987), D. G. Northcott (1984), and in the volume Algebra I by N. Bourbaki (1989). The theory of universal enveloping algebras of Lie algebras is described extensively in a monograph by J. Dixmier (1977).

The construction of the universal enveloping algebra of an arbitrary Lie algebra is described explicitly, and the Poincaré-Birkhoff-Witt theorem is stated, omitting its proof by means of an appropriate filtration of the universal enveloping algebra. The Cartan classification of semisimple finite-dimensional complex Lie algebras by means of root systems, and their description by means of generators and relations, according to Serre's theorem, is presented almost without any proof, thereby referring to excellent monographs on this subject, for instance by N. Jacobson (1962), J. E. Humphreys (1972), H. Samelson (1990), J.-P. Serre (1987, 1992), J. Tits (1966, 1967), and to the two volumes on Lie groups and Lie algebras by N. Bourbaki (1989, 1968).

The theory of affine Lie algebras, and more generally of Kac-Moody algebras, as important examples of infinite-dimensional Lie algebras, and also of the corresponding q-deformations, can be found for instance in monographs by V. G. Kac (1985), V. G. Kac and A. K. Raina (1987), D. B. Fuks (1986), J. Fuchs (1992), V. Chari and A. N. Pressley (1994), and Wan Zhe-Xian (1991).

1.1 Basic Definitions

Rings are defined to be unital.

(1.1.1) An algebra A over a commutative ring R is defined as an R-bimodule, which is equipped with an R-bilinear mapping: $A \times A \ni \{a, b\} \longrightarrow ab \in A$, such that

(i) $\forall a, b \in A, \forall r, s \in R :$

$$(ar)s = a(rs), \;\; ae_R = a, \;\; (a+b)r = ar + br, \;\; a(r+s) = ar + as;$$

this implies, that

$$\forall a \in A, \forall r \in R : 0_A r = 0_A, \;\; a 0_R = 0_A, \;\; (-a)r = a(-r) = -(ar).$$

$$\forall a \in A, \forall r \in R : ra := ar; \text{ then } \forall a \in A, \forall r, s \in R : r(as) = (ra)s.$$

Here e_R denotes the unit of R.

(ii) $\forall a, b, c \in A, \forall r \in R :$

$$(ar)b = a(br) = (ab)r, \;\; (a+b)c = ac + bc, \;\; a(b+c) = ab + ac;$$

this implies, that $\forall a, b \in A :$

$$0_A a = a 0_A = 0_A, \;\; (-a)b = a(-b) = -(ab), \;\; (-a)(-b) = ab.$$

(1.1.2) An algebra A over R is called associative, if and only if

$$\forall a, b, c \in A : (ab)c = a(bc).$$

An associative algebra over R is called unital, if and only if

$$\exists e_A \in A : \forall a \in A : ae_A = e_A a = a;$$

this unit e_A is then unique. Therefore a unital associative algebra over R is both some ring and bimodule over R.

(1.1.3) An algebra L over R is called Lie algebra over R, if and only if $\forall a, b, c \in L :$

$$aa = 0_L, \;\; a(bc) + b(ca) + c(ab) = 0_L; \;\text{ then } \forall a, b \in L : ab + ba = 0_L.$$

The third order relation is called Jacobi identity.

(1.1.4) Let A, A' be algebras over R. An R-linear map $\phi : A \to A'$ is called homomorphism of algebras, if and only if

$$\forall a, b \in A : \phi(ab) = \phi(a)\phi(b).$$

In the special case of unital associative algebras over R, $\phi \in Hom_R(A, A')$ is called homomorphism of unital associative algebras, if and only if ϕ is a homomorphism of algebras in the above sense, and moreover $\phi(e_A) = e_{A'}$. In the case of Lie algebras over R, a homomorphism of algebras is usually called homomorphism of Lie algebras.

(1.1.5) Let A be an algebra over the commutative ring R. An R-submodule B of A is called subalgebra of A, if and only if $\forall b_1, b_2 \in B : b_1 b_2 \in B$. Every subalgebra of an associative algebra A over R is an associative algebra over R. Similarly every subalgebra of a Lie algebra L over R is again some Lie algebra over R.

(1.1.6) An R-submodule D of an algebra A over R is called an ideal of A, if and only if $\forall a \in A, \forall d \in D : ad \in D, da \in D$; then D is some subalgebra of A.

(1.1.7) The commutator algebra A_L of an associative algebra A over R, with the product defined such that $\forall a, b \in A : [a, b] := ab - ba$, is some Lie algebra over R, the product of A_L being called commutator.

(1.1.8) The R-bimodule $Hom_R(E, E) \equiv End_R(E)$ of endomorphisms of an R-bimodule E is an associative algebra over R with respect to the composition of mappings, with the unit $id\ E$. If a vector space V over a field K is considered, the notation: $(End_K(V))_L \equiv gl(V)$ is used for this so-called general linear algebra.

(1.1.9) An endomorphism $\delta \in End_R(A)$ of an algebra A over R is called derivation of A, if and only if Leibniz' rule holds, i.e.,

$$\forall a, b \in A : \delta(ab) = \delta(a)b + a\delta(b).$$

The set $Der_R(A)$ of derivations of A is an R-submodule of $End_R(A)$. Moreover

$$\forall \delta_1, \delta_2 \in Der_R(A) : \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 =: [\delta_1, \delta_2] \in Der_R(A).$$

Therefore $Der_R(A)$ is some subalgebra of the commutator algebra $(End_R(A))_L$, which is called the derivation algebra of A.

(1.1.10) A homomorphism $\phi : L \longrightarrow (End_R(E))_L$ of Lie algebras over R is called representation of L on the R-bimodule E. Explicitly this means that ϕ is an R-linear map, which moreover fulfills

$$\forall a, b \in L : \phi([a,b]) = \phi(a) \circ \phi(b) - \phi(b) \circ \phi(a).$$

(1.1.11) Let L be a Lie algebra over R, with the product written in terms of $[\ ,\ ]$ as above. The adjoint representation $ad_L \equiv ad : L \longrightarrow (End_R(L))_L$ is defined as the following representation of L on itself:

$$L \ni a \longrightarrow ad\ a : L \ni b \underset{def}{\longrightarrow} [a,b] \in L,\ ad\ a \in Der_R(L).$$

Obviously ad is some homomorphism in the sense of Lie algebras over R.

(1.1.12) The set $ad(L) := \{ad\ a; a \in L\}$ of so-called inner derivations of L is an ideal of the Lie algebra $Der_R(L)$, because one finds, that

$$\forall \delta \in Der_R(L), \forall a, b \in L : [\delta, ad\ a](b) = [\delta(a), b].$$

(1.1.13) The centre of a Lie algebra L over R is the kernel of its adjoint representation; it is an ideal of L.

$$Z(L) := \{c \in L; \forall a \in L : [a,c] = 0_L\} = ker\ ad.$$

The normalizer of an R-linear subspace M of L is defined by

$$N_L(M) := \{a \in L; \forall b \in M : [a,b] \in M\}.$$

The centralizer of a subset S of L is defined by

$$C_L(S) := \{a \in L; \forall b \in S : [a,b] = 0_L\};\ \ C_L(L) = Z(L).$$

Obviously both $N_L(M)$ and $C_L(S)$ are subalgebras of L.

(1.1.14) The so-called derived algebra

$$L^{(1)} := R - lin\ span(\{[a,b]; a,b \in L\}) \equiv [L,L]$$

is an ideal of L. In case of $L^{(1)} = \{0_L\}$, which equivalently means Z $(L) = L$. L is called Abelian.

(1.1.15) A non-Abelian Lie algebra L over R is called simple, if and only if $\{0_L\}$ and L itself are the only ideals of L. If L is simple, then $Z(L) = \{0_L\}$ and $[L,L] = L$.

(1.1.16) Let D be an ideal of an algebra A over a commutative ring R. Then the factor algebra of A modulo D is again an algebra over R.

$$A/D := \{[a]; a \in A\}, [a] := a + D := \{a + d; d \in D\} .$$
$$\forall a, b \in A, \forall r, s \in R : r[a] + s[b] := [ra + sb], \quad [a][b] := [ab];$$
$$0_{A/D} := [0_A] = D.$$

The canonical surjection: $A \ni a \longrightarrow [a] \in A/D$, is some homomorphism of algebras. If A is an associative algebra over R, then A/D is an associative algebra over R too. If A is a Lie algebra over R, then A/D is again some Lie algebra over R.

(1.1.17) Let $\phi : A \longrightarrow A'$ be a homomorphism of algebras. The kernel of ϕ is an ideal of A, and ϕ can be decomposed according to the following diagram, with some isomorphism of algebras $\tilde{\phi}$.

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend right, "\phi"'] & A/ker\ \phi \arrow[r, leftrightarrow, "\tilde{\phi}"'] & Im\ \phi \arrow[r] & A' \end{tikzcd}$$

(1.1.18) Let $\phi : A \longrightarrow A'$ be a homomorphism of algebras, and D' an ideal of the algebra A' over R. Then $\phi^{-1}(D') =: D$ is an ideal of A, and the mapping: $A/D \ni [a] \longrightarrow [\phi(a)] \in A'/D'$ is an injective homomorphism of algebras.

(1.1.19) Let B be a subalgebra of an algebra A over R, and D an ideal of A. Then the sum $B + D$ is some subalgebra of A; the homomorphism of algebras

$$\phi : B \ni b \longrightarrow b + D \in (B + D)/D$$

is surjective, with $ker\ \phi = B \cap D$. Therefore an isomorphism of algebras:

$$\frac{B}{B \cap D} \longleftrightarrow \frac{B + D}{D}$$

is established.

(1.1.20) The derived series of a Lie algebra L over a commutative ring R is defined.

$$L \equiv L^{(0)}, \quad L^{(1)} := R - lin\ span(\{[a, b]; a, b \in L\}) \equiv [L, L],$$
$$\forall n \in \mathbf{N} : L^{(n+1)} := R - lin\ span(\{[a, b]; a, b \in L^{(n)}\}) \equiv [L^{(n)}, L^{(n)}].$$

Due to the Jacoby identity, $\forall k \in \mathbf{N}, n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $L^{(n+k)}$ is an ideal of $L^{(n)}$.
The Lie algebra L over R is called solvable, if and only if $\exists p \in \mathbf{N}$: $L^{(p)} = \{0_L\}$. Of course the factor algebra $L/L^{(1)}$ is Abelian.

(1.1.21) If a Lie algebra L over R is solvable, then every subalgebra B of L is solvable.

(1.1.22) Let $\phi : L \longrightarrow L'$ be a homomorphism in the sense of Lie algebras over R. If L is solvable, then $\phi(L)$ is solvable too, because $\forall n \in \mathbf{N}_0$: $\phi(L^{(n)}) = \phi(L)^{(n)}$.

(1.1.23) Let D be an ideal of the Lie algebra L over R. If both D and L/D are solvable, then L is solvable. Take $p, q \in \mathbf{N}_0$;

$$(L/D)^{(p)} = \{0_{L/D}\} \Longleftrightarrow L^{(p)} \subseteq D;$$

$$(L/D)^{(p)} = \{0_{L/D}\} \text{ and } D^{(q)} = \{0_D\} \Longrightarrow L^{(p+q)} = \{0_L\}.$$

(1.1.24) Let $D_k, k = 1, 2$, be solvable ideals of the Lie algebra L over R. Then their sum $D_1 + D_2$, which obviously is an ideal of L, is solvable because of an isomorphism of Lie algebras:

$$\frac{D_1}{D_1 \cap D_2} \longleftrightarrow \frac{D_1 + D_2}{D_2}.$$

(1.1.25) If $D_k, k = 1, 2$, are ideals of an algebra A over R, with $D_1 \subseteq D_2$, then D_1 is an ideal of D_2, and D_2/D_1 is an ideal of A/D_1. Conversely, let D_1 be an ideal of A; if D' is an ideal of A/D_1, then there is a unique ideal D_2 of A, such that $D_2/D_1 = D'$.

(1.1.26) A Lie algebra L over R is called semisimple, if and only if $\{0_L\}$ is the only solvable ideal of L, which equivalently means that $\{0_L\}$ is the only Abelian ideal of L. Obviously every simple Lie algebra L over R is semisimple.

(1.1.27) If the Lie algebra L over R is semisimple, then there exists an isomorphism of Lie algebras:

$$Der_R(L) \supseteq ad(L) \longleftrightarrow L.$$

(1.1.28) Consider a finite-dimensional Lie algebra L over the field K, $dim\ L \in \mathbf{N}$. The K-linear span C of the union of all the solvable ideals of L is an ideal of L. Moreover C is solvable, as is easily shown using the finite dimension of L. $C \equiv rad\ L$ is called the radical of L. Obviously L is semisimple, if and only if $C = \{0_L\}$; L is solvable, if and only if $L = C$. In any case L/C is semisimple.

(1.1.29) The descending central series of the Lie algebra L over R is defined.

$$L^1 \equiv L,\ \ L^n := R - lin\ span(\{[a,b]; a \in L, b \in L^{n-1}\}) \equiv [L, L^{n-1}],$$

$n = 2, 3, \ldots$. Each L^n is an ideal of L, and $L^n \supseteq L^{n+1}$, for $n \in \mathbf{N}$.

$$L^n = R - lin\ span(\{[a_1, [a_2, [\ldots [a_{n-2}, [a_{n-1}, a_n]] \ldots]; a_1, \ldots, a_n \in L\}),$$

$n = 2, 3, \ldots$. L is called nilpotent, if and only if $\exists n \in \mathbf{N} : L^n = \{0_L\}$. If $L/Z(L)$ is nilpotent, so is L.

(1.1.30)

$$\forall k, l \in \mathbf{N} : R - lin\ span(\{[a,b]; a \in L^k, b \in L^l\}) \subseteq L^{k+l},\ \ L^{(k)} \subseteq L^{2^k}.$$

(1.1.31) Obviously every nilpotent Lie algebra L over R is solvable.

(1.1.32) A nilpotent subalgebra H of a Lie algebra L over a commutative ring R is called Cartan subalgebra of L, if and only if H coincides with its own normalizer $N_L(H)$.

(1.1.33) Let $D_k, k = 1, 2$, be nilpotent ideals of L over R. Their sum $D_1 + D_2$ is then also nilpotent, because one finds, that

$$(D_1 + D_2)^n \subseteq D_1^{n/2} + D_2^{n/2}, n = 2, 4, 6, \ldots;$$

$$(D_1 + D_2)^n \subseteq D_1^{\frac{n+1}{2}} + D_2^{\frac{n+1}{2}}, n = 1, 3, 5 \ldots.$$

(1.1.34) A homomorphism of unital associative algebras $\phi : A \longrightarrow End_R(E)$ is called representation of A; ϕ is called faithful, if and only if it is injective.

(1.1.35) A representation $\phi : L \longrightarrow (End_R(E))_L$ of a Lie algebra L on an R-bimodule E is called faithful, if and only if ϕ is injective, which equivalently means that $ker\ \phi = \{0_L\}$.

(1.1.36) Consider a representation $\phi : L \longrightarrow (End_R(E))_L$ of a Lie algebra L over R on an R-bimodule E. An R-submodule F of E is called invariant with respect to ϕ, if and only if $\forall x \in L : \phi(x)(F) \subseteq F$. ϕ is called reducible, if and only if there is a ϕ-invariant R-submodule $\{0\} \neq F \subset E$. ϕ is called irreducible, if and only if it is not reducible. An invariant R-submodule F of E is called irreducible with respect to ϕ, if and only if the correspondingly restricted representation of L on F is irreducible. ϕ is called completely reducible, if and only if E is the direct sum of irreducible R-submodules.

(1.1.37) Two representations ϕ and ψ of a Lie algebra L over R on R-bimodules E and F, respectively, are called equivalent, if and only if there is some R-linear bijection $\tau : E \longleftrightarrow F$, such that $\forall x \in L : \psi(x) \circ \tau = \tau \circ \phi(x)$.

(1.1.38) Let ϕ and ψ be representations of L on E and F, respectively, as above. Then the mapping: $L \ni x \longrightarrow t(\phi, \psi)(x)$:

$$E \otimes F \ni v \otimes w \underset{def}{\longrightarrow} \phi(x)(v) \otimes w + v \otimes \psi(x)(w) \in E \otimes F,$$

is some representation of L on $E \otimes F$; it is called tensor product of these two representations ϕ, ψ.

(1.1.39) Let E be an R-bimodule, and A an associative R-algebra with unit. E is called an A-left module over R, if and only if

(i) E is an A-left module, with the elements of A used as coefficients;

(ii) $\forall x \in E, a \in A, r \in R : r(ax) = (ra)x = a(rx)$.

(1.1.40) Every algebra-left module over R is induced by some representation ϕ of A on E, and conversely. Let E be an R-bimodule, and A a unital associative algebra over R. Then a homomorphism of unital algebras $\phi : A \longrightarrow End_R(E)$ can be used in order to define

$$\forall x \in E, a \in A : ax := \phi(a)(x).$$

One easily checks all the conditions and finds, that E is an A-left module over the commutative ring R. Here the product of $End_R(E)$ is defined, such that $\forall \phi, \psi \in End_R(E) : \phi\psi := \phi \circ \psi$.

(1.1.41) Let again E be an R-bimodule, and L a Lie algebra over R. E is called L-left module over R, if and only if there is some mapping: $E \times L \ni \{x, l\} \longrightarrow lx \in E$, such that $\forall x, y \in E, k, l \in L, r \in R$:

$$(k + l)x = kx + lx, \quad k(x + y) = kx + ky,$$

$$[k, l]x = k(lx) - l(kx), \quad r(kx) = (rk)x = k(rx).$$

Then

$$\forall x \in E, k \in L : 0_L x = 0_E, \quad k0_E = 0_E, \quad k(-x) = (-k)x = -(kx).$$

(1.1.42) Lie algebra-left modules over R are induced by representations of Lie algebras over R, and conversely. A representation $\phi : L \longrightarrow (End_R(E))_L$ on an R-bimodule E can be used in order to define

$$\forall x \in E, k \in L : kx := \phi(k)(x).$$

Thereby E becomes an L-left module over R.

(1.1.43) Let E be an R-bimodule, and consider the subsequently defined universal enveloping algebra $\nu : L \longrightarrow V_L$ of a Lie algebra L over R. Take a representation ϕ of L on E, and consider the induced L-left module E over R. The induced homomorphism of unital associative algebras $\tilde{\phi}$, such that $\bar{\phi} \circ \nu = \phi$, is used to define the V-left module E over R, such that

$$\forall x \in E, a \in V, k \in L : ax := \bar{\phi}(a)(x).\ kx = \nu(k)x.$$

(1.1.44) Let A be a unital associative algebra over R. The mapping $\rho : A \longrightarrow End_R(A)$, defined by the diagram:

$$A \ni a \longrightarrow \rho(a) : A \ni x \longrightarrow ax \in A,$$

is a homomorphism of unital associative algebras, which is called the regular representation of A. Obviously the representation ρ of A on A is faithful.

(1.1.45) The adjoint representation $ad : L \longrightarrow (End_R(L))_L$ can be used in order to view L as the corresponding L-left module over R, i.e., $\forall k, l \in L : kl := ad\ k(l) = [k, l]$.

(1.1.46) Using the regular representation $\rho : A \longrightarrow End_R(A)$ of a unital associative algebra A, A itself can be viewed as an A-left module over R, i.e., $\forall a, b \in A : \rho(a)(b) = ab$.

1.2 Universal Enveloping Algebra

(1.2.1) Let L be a Lie algebra, and V an associative algebra with the unit e_V, both over a commutative ring R. Consider a homomorphism of Lie algebras $\nu : L \longrightarrow V_L$, into the commutator algebra V_L of V; ν is called universal enveloping algebra of L, if and only if it is a universal object in the following sense: If $\alpha : L \longrightarrow A_L$ is an arbitrary homomorphism of Lie algebras, then there is a unique homomorphism of unital associative algebras

$$\bar{\alpha} : V \longrightarrow A,\ \ \bar{\alpha}(e_V) = e_A,$$

such that the following diagram is commutative.

$$\begin{tikzcd}
L \arrow[r, "\nu"] \arrow[dr, "\alpha"'] & V_L \arrow[r, leftrightarrow, "id"] & V \arrow[d, "\bar{\alpha}"] \\
& A_L \arrow[r, leftrightarrow, "id"'] & A
\end{tikzcd}$$

(1.2.2) As some universal object, this enveloping algebra $V \equiv E(L)$ of L is essentially unique: If V' is another universal enveloping algebra of L, then there exists an isomorphism of unital associative algebras: $V \longleftrightarrow V'$.

(1.2.3) Let $L_k, k = 1, 2, 3$, be Lie algebras over R, with the universal enveloping algebras ν_k: $L_k \longrightarrow V_{kL}$. If $\phi : L_1 \longrightarrow L_2$ is a homomorphism of Lie algebras, then there is a unique homomorphism of unital associative algebras $\phi_* : V_1 \longrightarrow V_2$, such that the upper half of the diagram below is commutative. Obviously $(id\ L_1)_* = id\ V_1$. If ϕ is bijective, then ϕ_* is an isomorphism of algebras, such that $(\phi^{-1})_* = (\phi_*)^{-1}$. Moreover the composition of mappings is compatible with this lifting, due to the diagram below.

$$\begin{tikzcd}
L_1 \arrow[r, "\nu_1"'] \arrow[d, "\phi"] & V_1 \arrow[d, "\phi_*"] \arrow[dd, bend left=60, "\psi_* \circ \phi_* = (\psi \circ \phi)_*"] \\
L_2 \arrow[r, "\nu_2"'] \arrow[d, "\psi"] & V_2 \arrow[d, "\psi_*"] \\
L_3 \arrow[r, "\nu_3"'] & V_3
\end{tikzcd}$$

Therefore some covariant functor is established: from the category of Lie algebras into the category of unital associative algebras, both over R.

(1.2.4) Let $S \subseteq A$ be a subset of an associative algebra A with the unit e_A over R. The algebraic span of S is defined by

$$\begin{aligned} R - alg\ span(S) := R - lin\ span(\{s_1 \cdots s_p; s_1, \ldots, s_p \in S; p \in \mathbf{N}\} \\ \cup \{e_A\}); \end{aligned}$$

obviously it is an associative algebra with the unit e_A over R. In the special case of $R - alg\ span(S) = A$ one says, that A is generated by the subset S.

(1.2.5) Let $\nu : L \longrightarrow V_L$ be a universal enveloping algebra of the Lie algebra L over R.

$$\begin{aligned} V &= R - alg\, span(Im\, \nu) \\ &= R - lin\, span(\{\nu(x_1) \cdots \nu(x_p);\ x_1, \ldots, x_p \in L; p \in \mathbf{N}\} \cup \{e_V\}). \end{aligned}$$

The easy proof of this lemma is independent from the special construction of V_L, due to the next diagram.

$$
\begin{tikzcd}
L \arrow[rr, "\nu"] \arrow[dr, "\alpha"'] \arrow[drrr, bend right=40, "\nu"'] & & V \arrow[dl, "\bar{\alpha}"'] & \\
& W := \mathrm{alg\ span}\,(\mathrm{Im}\ \nu) \arrow[rr] & & V
\end{tikzcd}
$$

(1.2.6) An explicit construction of some universal enveloping algebra $\nu : L \longrightarrow V_L$ is performed according to the diagram below.

$$T \equiv T(L) := \bigoplus_{p \in \mathbf{N}_0} T^p(L),$$

as the direct sum of R-bimodules; here $T^p(L)$ denotes the tensor product of p copies of L.

$$S := \{\{0, -[x, y], x \otimes y - y \otimes x, 0, 0, \ldots\}, x, y \in L\},$$
$$D := \mathbf{Z} - span(\{tst';\ s \in S; t, t' \in T\}) \not\ni e_T := \{e_R, 0, 0, \ldots\}.$$

Obviously D is an ideal of T; moreover, if D' is an ideal of T, and if $D' \supseteq S$, then $D' \supseteq D$. The factor algebra V of T modulo D is used in order to define an appropriate R-linear mapping $\nu : L \longrightarrow V$.

$$
\begin{tikzcd}
L \ni x \arrow[r, "\beta"'] \arrow[rr, bend right=40, "\nu\quad \mathrm{def}"'] & \{0, x, 0, 0, \ldots\} \in T \arrow[r, "\kappa"'] & T/D =: V
\end{tikzcd}
$$

Since the canonical surjection κ vanishes on D, one obtains that

$$\forall x, y \in L : [x, y] \underset{\kappa \circ \beta}{\longrightarrow} \{0, 0, x \otimes y - y \otimes x, 0, \ldots\} + D \in V,$$

which means, that $\forall x, y \in L$:

$$[x, y] \underset{\nu}{\longrightarrow} \{0, x, 0, \ldots\}\{0, y, 0, \ldots\} - \{0, y, 0, \ldots\}\{0, x, 0, \ldots\} + D \in V_L.$$

Therefore a homomorphism of Lie algebras $\nu : L \longrightarrow V_L$ is established; it is a universal object in the above sense. Consider an arbitrary homomorphism of Lie algebras α, as indicated in the next diagram.

$$
\begin{tikzcd}
L \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T(L) \arrow[r, "\kappa"] \arrow[d, "\tilde{\alpha}"] & T(L)/D \arrow[dl, "\bar{\alpha}"] \\
& A &
\end{tikzcd}
$$

Due to the universal property of the tensor algebra $T(L)$, for every R-linear map $\alpha : L \longrightarrow A$, there exists a unique homomorphism of unital associative algebras $\tilde{\alpha} : T(L) \longrightarrow A$, such that $\tilde{\alpha} \circ \beta = \alpha$. Moreover $\tilde{\alpha}(t) = 0_A$ for all $t \in D$, because one finds, that $\forall x, y \in L$:

$$\begin{aligned} \{0, [-x, y], x \otimes y - y \otimes x, 0, \ldots\} \underset{\tilde{\alpha}}{\longrightarrow} &- \alpha([x, y]) + \alpha(x)\alpha(y) - \alpha(y)\alpha(x) \\ &= 0. \end{aligned}$$

Therefore $\tilde{\alpha}$ is factorized by D, which enables one to define the homomorphism of unital associative algebras $\bar{\alpha}$. The uniqueness of $\tilde{\alpha}$ implies uniqueness of $\bar{\alpha}$, such that $\bar{\alpha} \circ \nu = \alpha$. One finds, that the unit e_V is not contained in the image of ν; $\forall x \in L : \nu(x) \neq e_V$.

(1.2.7) Let B be an ideal of L, and I the ideal of B with repect to the universal enveloping algebra V of L.

$$I := \mathbf{Z} - span(\{v\nu(x)v'; x \in B; v, v' \in V\}) \subseteq V.$$

Then the homomorphism of Lie algebras λ, which is defined in the next diagram, is factorized by B, because $Im\ \lambda|_B = \{I\}$.

$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[dr] \arrow[rr, bend right, "\lambda"] & V \arrow[r] & V/I \\ & L/B \arrow[ur, "\lambda'"'] & \end{tikzcd}$$

Using the commutative diagram below one finds, that the homomorphism of Lie algebras $\lambda' : L/B \longrightarrow (V/I)_L$ is some universal enveloping algebra. Moreover, since $\alpha := \phi \circ \kappa$ is factorized by B, because $B \subseteq ker\ \alpha$, one obtains that $I \subseteq ker\ \bar{\alpha}$. Uniqueness of $\bar{\phi}$ then follows from $V/I = R{-}alg\ span(Im\ \lambda')$, because of $V = R - alg\ span(Im\ \nu)$.

$$\begin{tikzcd} L/B \arrow[rrr, bend left, "\lambda'"] \arrow[dr, "\phi"'] & L \arrow[l, "\kappa"'] \arrow[r, "\nu"] \arrow[d, "\alpha"] & V \arrow[r] \arrow[dl, "\bar{\alpha}"] & V/I \arrow[dll, "\bar{\phi}"] \\ & A & & \end{tikzcd}$$

(1.2.8) The diagonal mapping $\delta : L \longrightarrow (V \otimes V)_L$, into the tensor product of algebras $V \otimes V$, induces a unique homomorphism of unital associative algebras $\bar{\delta} =: \Delta : V \longrightarrow V \otimes V$, due to the next diagram.

$$\begin{tikzcd}
L \ni x \arrow[rr, "\nu"'] \arrow[d, "\delta"', "def"] & & V \arrow[d, "\bar{\delta}"] \\
\nu(x) \otimes e_V + e_V \otimes \nu(x) \in V \otimes V & & \arrow[ll, dash] \arrow[ll]
\end{tikzcd}$$

(1.2.9) Let $\phi \in End_R(L)$ be a derivation of L, which means that

$$\forall x, y \in L : \phi([x,y]) = [\phi(x), y] + [x, \phi(y)].$$

Then there is a unique derivation $\phi_* \in End_R(V)$, i.e.,

$$\forall v_1, v_2 \in V : \phi_*(v_1 v_2) = \phi_*(v_1) v_2 + v_1 \phi_*(v_2), \text{ such that } \phi_* \circ \nu = \nu \circ \phi.$$

(1.2.9.1) For an easy proof one constructs an appropriate homomorphism of Lie algebras $\theta : L \longrightarrow W_L$, into the associative algebra

$$\left\{ \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} ; v_{11}, \ldots, v_{22} \in V \right\} =: W, \text{ with the unit } \begin{bmatrix} e_V & 0 \\ 0 & e_V \end{bmatrix} =: e_W,$$

according to the diagram below.

$$\begin{tikzcd}
L \ni x \arrow[rr, "\nu"'] \arrow[d, "\theta"', "def"] & & v \in V \arrow[d, "\bar{\theta}"] \\
\begin{bmatrix} \nu(x) & \nu \circ \phi(x) \\ 0 & \nu(x) \end{bmatrix} = \begin{bmatrix} v & \phi_*(v) \\ 0 & v \end{bmatrix} \in W & & \arrow[ll]
\end{tikzcd}$$

One calculates, that $\phi_* \in End_R(V)$ fulfills Leibniz' rule, and $\phi_*(e_V) = 0$.

(1.2.10) Let V, W be associative algebras with the units e_V, e_W, both over R. An R-linear map $\phi : V \longrightarrow W$ is called an antihomomorphism of unital associative algebras, if and only if

(i) $\forall v_1, v_2 \in V : \phi(v_1 v_2) = \phi(v_2)\phi(v_1) \in W$; (ii) $\phi(e_V)) = e_W$.

Then obviously a homomorphism of Lie algebras $-\phi : V_L \longrightarrow W_L$ is established, due to the next diagram.

$$\begin{tikzcd}
V_L \arrow[r, leftrightarrow, "id"'] \arrow[drrr, "-\phi", "\text{Lie algebra - homomorphism}"'] & V \arrow[r, "\phi"'] & W \arrow[r, leftrightarrow, "id"'] & W_L \arrow[d, leftrightarrow, "-id"] \\
 & & & W_L
\end{tikzcd}$$

(1.2.10.1) Let V be the universal enveloping algebra of a Lie algebra L over R, and $\varepsilon : V \longleftrightarrow W$ an antiisomorphism, i.e., bijective antihomomorphism of unital associative R-algebras. For every unital associative algebra over R, such an antiisomorphism trivially exists. Defining

$$-\varepsilon \circ \nu =: \theta, \quad \sigma := \varepsilon^{-1} \circ \bar{\theta}, \quad \bar{\theta} \circ \nu = \theta,$$

one verifies the following implications.

$$\sigma : V \longleftrightarrow V \Longleftarrow \sigma^2 = id\ V \Longleftarrow \sigma^2 \circ \nu = \nu \Longleftarrow \sigma \circ \nu = -\nu.$$

Hence there is a unique antiisomorphism of unital associative algebras $\sigma = \sigma^{-1} : V \longleftrightarrow V$, such that $\sigma \circ \nu = -\nu$, as is shown in the following diagram.

$$\begin{tikzcd}
L \arrow[r, "\nu"'] \arrow[d, "\nu"] \arrow[dr, "\theta"] & V_L \arrow[r, leftrightarrow, "id"'] & V \arrow[d, leftrightarrow, "\bar{\theta}"] & \\
V_L \arrow[r, leftrightarrow, "-\varepsilon"'] & W_L \arrow[r, leftrightarrow, "id"'] & W \arrow[r, leftrightarrow, "\varepsilon^{-1}"'] & V \arrow[ul, "\sigma"']
\end{tikzcd}$$

1.3 Poincaré - Birkhoff - Witt Theorem

(1.3.1) Consider a Lie algebra L over a field K, with the universal enveloping algebra $E(L) \equiv V := T/D$, with the above notations of $T \equiv T(L), S$ and D.

(1.3.2) Let $\{b_i; i \in I\}$ be an arbitrary K-basis of L, with the totally ordered index set I. Then the cosets of e_K, and of all the standard monomials, provide some vector basis of the universal enveloping algebra V in the following sense. The set $\{b_{k_1} \otimes \cdots \otimes b_{k_p}; k_1, \ldots, k_p \in I\}$ is some vector basis of the p-fold tensor product $T^p(L)$, for $p = 2, 3, \ldots$. The set

$$\Big\{e_V = \{e_K, 0, 0, \ldots\} + D, \{0, b_i, 0, 0, \ldots\} + D,$$
$$\{0, 0, \ldots 0, b_{i_1} \otimes \cdots \otimes b_{i_p}, 0, 0, \ldots\} + D;$$
$$i, i_1 \leq \ldots \leq i_p \in I; p = 2, 3, \ldots\Big\}$$

is some vector basis of V. The proof of this theorem is performed by means of the so-called canonical filtration of V.

(1.3.3) Therefore the universal mapping $\nu : L \longrightarrow V_L$ is injective. Moreover

$$\nu(L) \cap Ke_V = \{0\}, \text{ i.e., } \forall x \in L, \forall k \in K : \nu(x) = ke_V \Longrightarrow k = 0,\ x = 0.$$

(1.3.4) The injective mapping ν, which by its very definition sends: $L \ni b_i \longrightarrow \{0, b_i, 0, 0, \ldots\} + D \in V$ for $i \in I$, is used as an inclusion of L into V, which in turn enables one to simplify the notation, such that

$$\forall x, y \in L : xy \equiv \nu(x)v(y) = \{0, 0, x \otimes y, 0, 0, \ldots\} + D \in V.$$

(1.3.5) Hence the above basis of standard monomials is shortly rewritten as

$$V = K(\{e_V, b_i, b_{i_1} \cdots b_{i_p}; i, i_1 \leq \ldots \leq i_p \in I; p = 2, 3, \ldots\}).$$

(1.3.6) In the finite-dimensional case of $I := \{1, \ldots, n\}$, this standard basis reads:

$$V = K(\{b_1^{q_1} \cdots b_n^{q_n}; q_1, \ldots, q_n \in \mathbf{N}_0\}),$$

denoting $\forall x \in L : x^0 := e_V$.

(1.3.7) With this inclusion of L into V one finds, that $V = K - alg\ span(L)$.

(1.3.8) Especially for representations of L on a vector space E over K, the universal property means, that every representation $\phi : L \longrightarrow (End_K(E))_L$ is uniquely extended to an according representation $\bar{\phi} : V \longrightarrow End_K(E)$.

1.4 Free Lie Algebras

(1.4.1) The free algebra $T(R(S))$ over a set S, with the coefficients from a commutative ring R, is constructed by means of the commutative diagram below.

$$
\begin{tikzcd}
S \arrow[r] \arrow[dr, "\alpha"'] & R(S) =: F \arrow[r, "\beta"'] \arrow[d, "\alpha'"] & T(F) =: T \arrow[dl, "\bar{\alpha}"] \\
 & A &
\end{tikzcd}
$$

Here $R(S) =: F$ denotes the free R-bimodule over S. The tensor algebra $T(F)$ over F is a universal object in the following sense. Let A be a unital associative algebra over R, and $\alpha : S \longrightarrow A$ a mapping; then there exists a unique homomorphism of unital associative algebras $\bar{\alpha} : T(F) \longrightarrow A$, such that $\bar{\alpha} \circ \beta|_S = \alpha'|_S = \alpha$. Obviously $T(R(S)) = R - alg\ span(S)$.

(1.4.2) Especially R is assumed to be a field K. The Lie algebra $L(S) \subseteq T_L$ over K,

$$
\begin{aligned}
L(S) := K - lin\ span(\beta(S) \cup \{[\beta(s_1), \beta(s_2)]; s_1, s_2 \in S\} \\
\cup\ multiple\ commutators),
\end{aligned}
$$

which is generated by the set S as subalgebra $L(S)$ of the commutator algebra T_L of T, is a universal object in the following sense. Let λ be an arbitrary mapping from S into a Lie algebra L over K; then there exists a unique homomorphism of Lie algebras $\tilde{\lambda} : L(S) \longrightarrow L$, such that $\tilde{\lambda} \circ \beta|_S = \lambda$, as indicated in the next diagram.

$$
\begin{tikzcd}
S \ni s \arrow[r, leftrightarrow] \arrow[dr, "\lambda"'] & \beta(s) \in \beta(S) \arrow[r] & L(S) \arrow[dl, "\tilde{\lambda}"] \\
 & L &
\end{tikzcd}
$$

This universal property follows from the diagram below, because of the inclusion $Im(\overline{\nu \circ \lambda})|_{L(S)} \subseteq L$; here $\nu : L \longrightarrow V$ denotes the universal enveloping algebra of L.

$$
\begin{tikzcd}
S \arrow[r, "\beta|_S"] \arrow[d, "\lambda"'] & T \arrow[d, "\overline{\nu \circ \lambda}"] & L(S) \arrow[l] \arrow[d, "\tilde{\lambda}"] \\
L \arrow[r, "\nu"'] & V & L \arrow[l, "\nu"]
\end{tikzcd}
$$

Due to this universal property, $L(S)$ is called the free Lie algebra over the set S, with the coefficients $\in K$.

(1.4.2.1) Moreover $T(K(S))$ is the universal enveloping algebra of $L(S)$, due to the next diagram.

$$\begin{tikzcd}
S \arrow[r, leftrightarrow] \arrow[drr, "\phi"'] & \beta(S) \arrow[r] \arrow[dr, "\psi|_{\beta(S)}"] & L(S) \arrow[r] \arrow[d, "\psi"] & T(K(S)) \arrow[dl, "\bar{\phi}"] \\
 & & A &
\end{tikzcd}$$

For a homomorphism of Lie algebras $\psi : L(S) \longrightarrow A_L$ one then finds, that $\psi = \bar{\phi}|_{L(S)}$.

(1.4.2.2) Assume *char* $K = 0$, and denote by $\delta : T \longrightarrow T \otimes T$ the unique homomorphism of unital associative algebras over K, such that

$$\forall s \in S : \delta(s) = s \otimes e_T + e_T \otimes s,$$

with the unit e_T of T. Assume especially S to be finite, $S := \{s_1, \dots, s_n\}$. Then

$$L(S) = \{x \in T(K(S)); \delta(x) = x \otimes e_T + e_T \otimes x\}.$$

For an easy proof one needs the Poincaré-Birkhoff-Witt theorem.

1.5 Classification of Semisimple Finite-Dimensional Complex Lie Algebras

Let K be an algebraically closed field of characteristic zero. With respect to physical applications choose $K := \mathbf{C}$. The simple finite-dimensional complex Lie algebras are classified by means of the Cartan matrices or Dynkin digrams of their irreducible root systems, which in turn allow for an explicit notation of their commutation relations. Detailed proofs are omitted.

(1.5.1)

$$\forall m \in \mathbf{N} : gl(m, \mathbf{C}) \equiv gl(\mathbf{C}^m) := (End_{\mathbf{C}}\ (\mathbf{C}^m))_L \equiv (Mat(m, \mathbf{C}))_L.$$

The trace: $gl(m, \mathbf{C}) \ni x \longrightarrow tr\ x \in \mathbf{C}$ is $\mathbf{C}$-linear, and $ker\ tr = (gl(m, \mathbf{C}))^{(1)}$.

(1.5.2) The set

$$t(m,\mathbf{C}) \equiv t(\mathbf{C}^m) := \{x \in gl(m,\mathbf{C}); \forall_1^m i < j : x_{ij} = 0\}$$

of lower-triangular matrices is some solvable subalgebra of $gl(m,\mathbf{C})$, because $(t(m,\mathbf{C}))^{(m)} = \{0\}$, for $m = 2,3,\ldots$. The set

$$n(m,\mathbf{C}) \equiv n(\mathbf{C}^m) := \{x \in gl(m,\mathbf{C}); \forall_1^m i \le j : x_{ij} = 0\}$$

of strictly lower-triangular matrices is some nilpotent subalgebra of $gl(m,\mathbf{C})$, because $(n(m,\mathbf{C}))^m = \{0\}$, for $m = 2,3,\ldots$.

$$\forall k \in \mathbf{N} : (t(m,\mathbf{C}))^{k+1} = (t(m,\mathbf{C}))^{(1)} = n(m,\mathbf{C}), \ \textit{for } m = 2,3,\ldots.$$

(1.5.3) The classical Lie algebras $A_m, B_m, C_m, D_m, m \in \mathbf{N}$, are defined by their matrix properties.

(1.5.3.1) $\forall m \in \mathbf{N}$:

$$A_m \equiv sl(m+1,\mathbf{C}) \equiv sl(\mathbf{C}^{m+1}) := \{x \in gl(m+1,\mathbf{C}); tr\ x = 0\}.$$

The dimension of this so-called special linear Lie algebra is
$dim\ A_m = (m+1)^2 - 1$.

(1.5.3.2) The symplectic Lie algebra $C_m \equiv sp(2m,\mathbf{C})$ is defined, $m \in \mathbf{N}$.

$$\forall m \in \mathbf{N} : J_m := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \in Mat(2m,\mathbf{C}),$$

$$C_m := \{x \in gl(2m,\mathbf{C}); J_m x + x^t J_m = 0\},$$

using the transposition of matrices; $dim\ C_m = 2m^2 + m,\ \ tr\ \ C_m = \{0\}$.

(1.5.3.3) The orthogonal Lie algebra $D_m \equiv o(2m,\mathbf{C})$ is defined for $m \in \mathbf{N}$.

$$\forall m \in \mathbf{N} : L_m := \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \in Mat\ (2m,\mathbf{C}),$$

$$D_m := \{x \in gl(2m,\mathbf{C}); L_m x + x^t L_m = 0\}.$$

One finds, that $dim\ D_m = 2m^2 - m,\ \ tr\ D_m = \{0\}$.

(1.5.3.4) The orthogonal Lie algebra $B_m \equiv o(2m+1,\mathbf{C})$ is defined, $m \in \mathbf{N}$.

$$\forall m \in \mathbf{N} : K_m := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{bmatrix} \in Mat(2m+1,\mathbf{C}),$$

$$B_m := \{x \in gl(2m+1,\mathbf{C}); K_m x + x^t K_m = 0\}.$$

Again $tr\ B_m = \{0\}$, and $dim\ B_m = 2m^2 + m$.

(1.5.3.5) Due to the isomorphisms of complex Lie algebras:

$$A_1 \longleftrightarrow B_1 \longleftrightarrow C_1, \ B_2 \longleftrightarrow C_2, \ A_3 \longleftrightarrow D_3, \ D_2 \longleftrightarrow A_1 \oplus A_1,$$

and since $dim\ D_1 = 1$, one may restrict the classification to the families

$$\{A_m; m \in \mathbf{N}\}, \ \{B_m; m \geq 2\}, \ \{C_m; m \geq 3\}, \ \{D_m; m \geq 4\}.$$

(1.5.4) Let L be a finite-dimensional Lie algebra over the field $\mathbf{C}$. The so-called Killing form:

$$L \times L \ni \{x, y\} \underset{def}{\xrightarrow{\kappa}} tr(ad\ x \circ ad\ y) \in \mathbf{C}$$

is defined, the trace being independent of the choice of $\mathbf{C}$-basis for L. This $\mathbf{C}$-bilinear form is symmetric and L-invariant in the following sense.

$$\forall x, y, z \in L : \kappa(x, y) = \kappa(y, x), \ \ \kappa([x, y], z) = \kappa(x, [y, z]).$$

Moreover it is invariant under an automorphism α of L in the sense that

$$\forall x, y \in L : \kappa(\alpha(x), \alpha(y)) = \kappa(x, y).$$

Here one uses that $\forall x \in L : ad\ \alpha(x) = \alpha \circ\ ad\ x \circ \alpha^{-1}$.
If D is an ideal of L, then $\kappa\big|_{D \times D}$ is the Killing form of D.
The kernel of κ is some ideal of L.

(1.5.4.1)

$$\forall \delta \in Der_{\mathbf{C}}(L), \forall x, y \in L : \kappa(\delta(x), y) + \kappa(x, \delta(y)) = 0.$$

(1.5.5) Consider again a finite-dimensional Lie algebra L over $\mathbf{C}$. An element $x \in L$ is called ad-nilpotent, if and only if $\exists n \in \mathbf{N} : (ad\ x)^n = 0$ in the sense of $End_{\mathbf{C}}(L)$.

(1.5.5.1) L is nilpotent, if and only if all the elements of L are ad-nilpotent. (Engel's theorem)

(1.5.5.2) If all the elements of $L \subseteq gl(m, \mathbf{C})$ are nilpotent in the sense of $Mat(m, \mathbf{C})$, then L is nilpotent; moreover in this case one can choose some $t \in Mat(m, \mathbf{C}) : det\ t\ \neq 0, \forall x \in L : txt^{-1} \in n(m, \mathbf{C})$.

(1.5.6) Let L be a solvable subalgebra of $gl(m, \mathbf{C})$. Then $\exists t \in Mat(m, \mathbf{C})$: $det\ t \neq 0, \forall x \in L : txt^{-1} \in t(m, \mathbf{C})$; hence $[L, L]$ is nilpotent. (Lie's theorem)

(1.5.7) Every $A \in Mat(n, \mathbf{C})$ can be transformed to its so-called Jordan canonical form.

$$\exists T \in Mat(n, \mathbf{C}) : det\, T \neq 0,\ TAT^{-1} =: B,$$

such that B reads as follows.

(1.5.7.1) The characteristic polynomial of A is defined by

$$\sigma_A(x) := det(A - xI_n) = \prod_{l=1}^{k} (x - \lambda_l)^{p_l}.$$

(1.5.7.2)

$$B = \begin{bmatrix} B_1 & 0 & \dots & & 0 \\ 0 & & & & \\ & & \ddots & & \\ & & & & 0 \\ 0 & & \dots & 0 & B_k \end{bmatrix},$$

$$\forall_1^k l : B_l = \begin{bmatrix} J_{q_1(l)}(\lambda_l) & 0 & \dots & & 0 \\ 0 & & & & \\ & & \ddots & & \\ & & & & 0 \\ 0 & & \dots & 0 & J_{q_{r_l}(l)}(\lambda_l) \end{bmatrix},$$

$$q_1(l) + \dots + q_{r_l}(l) = p_l,\ 1 \leq q_{r_l}(l) \leq \ldots \leq q_1(l),$$

with the Jordan block notation, such that

$$\forall q \geq 2 : J_q(\lambda) := \lambda I_q + \Sigma_q, \Sigma_q := \begin{bmatrix} 0 & 1 & 0 & & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & & \\ 0 & & & \dots & 0 & 1 \\ 0 & & & \dots & & 0 \end{bmatrix},$$

$$\Sigma_q^q = 0,\ J_1(\lambda) := \lambda.$$

(1.5.7.3) Hence one obtains the unique Jordan-Chevalley decomposition of A.

$$A = A_d + A_n,\ A_d \ \textit{diagonalizable},\ A_n \ \textit{nilpotent},\ [A_d, A_n] = 0.$$

Moreover there are polynomials δ and ν in one variable, both without constant term, such that $A_d = \delta(A), A_n = \nu(A)$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^2 - \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

$$\Sigma_3 = -\frac{1}{\lambda^2}(J_3(\lambda))^3 + \frac{3}{\lambda}(J_3(\lambda))^2 - 2J_3(\lambda).$$

(1.5.7.4) If $A \in Mat(n, \mathbf{C})$ is diagonalizable, then $ad\ A$ is also diagonalizable. For an easy proof choose some basis of $\mathbf{C}^n$, such that

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & & & \\ & & \ddots & \\ & & & 0 \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}; \text{ then}$$

$$Mat(n, \mathbf{C}) \ni [b_{ij}; i, j = 1, \dots n] \underset{ad\ A}{\longrightarrow} [(a_{ii} - a_{jj})b_{ij};\ i, j = 1, \dots, n] \in Mat(n, \mathbf{C}).$$

(1.5.7.5) The Jordan-Chevalley decomposition of $ad\ A$ is just $ad\ A = ad\ A_d + ad\ A_n$.

(1.5.8) For every subalgebra L of $gl(n, \mathbf{C})$ Cartan's criterion holds.

$$\forall x \in [L, L], y \in L : tr(xy) = 0 \Longrightarrow L \text{ solvable.}$$

(1.5.8.1) L solvable $\Longleftrightarrow \forall x, y \in [L, L] : \kappa(x, y) = 0$. (Cartan's first criterion)

(1.5.9) L semisimple $\Longleftrightarrow \kappa$ non-degenerate. (Cartan's second criterion)

(1.5.10) Two ideals $D_k, k = 1, 2$, of a Lie algebra L over the commutative ring R, are called complementary, if and only if $L = D_1 \oplus D_2$; in this case $[D_1, D_2] = \{0\}$.

(1.5.11) Let L be a finite-dimensional complex Lie algebra. $L \neq \{0\}$ is semisimple, if and only if $L = \bigoplus_{k=1}^{n} L_k$, with simple ideals L_k of L; this decomposition is then unique in the following sense: if D is a simple ideal of L, then $\exists k \in \{1, \dots, n\} : D = L_k$.

(1.5.11.1) Let L be semisimple; then $L = [L, L]$, every ideal of L is semisimple, and every homomorphic image of L is semisimple.

(1.5.11.2) If L is semisimple, then $Der_{\mathbf{C}}(L) = ad\ L \longleftrightarrow L$, as an isomorphism of Lie algebras over $\mathbf{C}$.

(1.5.12) Consider a faithful representation $\phi : L \longrightarrow gl(m, \mathbf{C})$ of a finite-dimensional complex Lie algebra L, and assume L to be semisimple. The symmetric $\mathbf{C}$-bilinear form defined by

$$\beta : L \times L \ni \{x, y\} \longrightarrow tr(\phi(x)\phi(y)) \in \mathbf{C}$$

is L-invariant in the sense, that $\forall x, y, z \in L : \beta([x, y], z) = \beta(x, [y, z])$. Hence $ker\ \beta$ is some ideal of L. Moreover $\phi(ker\ \beta)$ is solvable, hence β

is non-degenerate. Take a pair of dual bases of L with respect to β, i.e., $\forall_1^n k,l : \beta(x_k, y_l) = \delta_{kl}$, $n := dim\ L$. The Casimir element is defined with respect to these dual bases,

$$c_\phi := \sum_{k=1}^{n} \phi(x_k)\phi(y_k), \quad tr\ c_\phi = n. \ \forall x \in L : [\phi(x), c_\phi] = 0.$$

If ϕ is irreducible, then $c_\phi = \frac{n}{m} I_m$, due to Schur's lemma.

(1.5.13) If the finite-dimensional complex Lie algebra L is semisimple, then every representation $\phi : L \longrightarrow gl(m, \mathbf{C})$ is completely reducible. (Weyl's theorem)

(1.5.14) The complex Lie algebra $sl(2, \mathbf{C})$ is described by the commutation relations

$$[\sigma^+, \sigma^-] = \sigma^3, \quad [\sigma^3, \sigma^\pm] = \pm 2\sigma^\pm,$$

with the Pauli matrices $\sigma^+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\sigma^- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\sigma^3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ used as the so-called standard basis. The dual basis is $\{\sigma^-, \sigma^+, \frac{1}{2}\sigma^3\}$, and

$$c_\phi = \sigma^+\sigma^- + \sigma^-\sigma^+ + \frac{1}{2}(\sigma^3)^2 = \frac{3}{2} I_2.$$

Here ϕ means just the inclusion of $sl(2, \mathbf{C})$ into $gl(2, \mathbf{C})$.

(1.5.15) Let A be a finite-dimensional complex algebra. Then $Der_{\mathbf{C}}(A) \ni \delta = \delta_d + \delta_n$ implies, that both the diagonalizable part δ_d and the nilpotent part $\delta_n \in Der_{\mathbf{C}}(A)$.

(1.5.15.1) Let the finite-dimensional complex Lie algebra L be semisimple. The isomorphism of complex Lie algebras : $Der_{\mathbf{C}}(L) = ad(L) \longleftrightarrow L$ can be used to define an abstract Jordan decomposition, which is unique.

$$\forall \xi = \xi_d + \xi_n = ad\ x \in ad(L) : \xi_d =: ad\ x_d, \quad \xi_n =: ad\ x_n,$$
$$x_d + x_n = x, \quad [x_d, x_n] = 0.$$

If $L \subseteq gl(m, \mathbf{C})$, then this decomposition coincides with the usual Jordan-Chevalley decomposition of $x \in L$, due to Weyl's theorem.

(1.5.15.1.1) Consider a representation ϕ of L on the finite-dimensional complex vector space V. If $x = x_d + x_n$ is the abstract Jordan decomposition of $x \in L$, then $\phi(x) = \phi(x_d) + \phi(x_n)$ is the Jordan-Chevalley decomposition of $\phi(x) \in End_{\mathbf{C}}(V)$.

(1.5.16) Consider an irreducible representation ϕ of the simple Lie algebra $sl(2,\mathbf{C})$ on the complex vector space V of dimension $m+1, m \in \mathbf{N}$. Since $\phi(\sigma^3)$ is diagonalizable, one may construct some $\mathbf{C}$-basis $\{v_0, v_1, \cdots, v_m\}$ of V, such that

$$\begin{aligned}\forall_0^m k : \phi(\sigma^3)v_k &= (m-2k)v_k,\\ \phi(\sigma^-)v_k &= (k+1)v_{k+1}, \phi(\sigma^+)v_k = (m-k+1)v_{k-1},\\ \phi(\sigma^-)v_m &= \phi(\sigma^+)v_0 = 0,\end{aligned}$$

denoting $v_{-1} := 0$, $v_{m+1} := 0$. Here v_0 is called the maximal vector of highest weigt m, and the eigenvalues of $\phi(\sigma^3)$ are called weights of ϕ. All the irreducible representations of $sl(2,\mathbf{C})$ of dimension $m+1$ are equivalent.

(1.5.17) With $\{\sigma^+, \sigma^3, \sigma^-\}$ used as Cartesian basis of $sl(2,\mathbf{C})$, one obtains the adjoint representation:

$$ad\ \sigma^+ = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, ad\ \sigma^- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}, ad\ \sigma^3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence the non-degenerate Killing form of $sl(2,\mathbf{C})$ is represented by the matrix $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ with the determinant -128.

(1.5.18) Let the finite-dimensional complex Lie algebra L be semisimple. A subalgebra T of L is called toral, if and only if $ad_L\ t$ is diagonalizable for all $t \in T$; then T is Abelian. Let the toral subalgebra T of L be maximal in the sense, that there is no toral subalgebra T' of L such that $T' \supset T$.

(1.5.18.1) Then the subalgebra $ad_L(T)$ of $(End_{\mathbf{C}}(L))_L$ is Abelian, and therefore one can choose an appropriate $\mathbf{C}$-basis of L, such that every element of $ad_L(T)$ is some diagonal complex matrix. Hence one finds the direct sum of complex vector spaces

$$L = \bigoplus_{\alpha \in T^*} L_\alpha, \quad L_\alpha := \{x \in L; \forall t \in T : [t,x] = \alpha(t)x\}.$$

The root system of L is defined as the set of so-called roots,

$$\Phi := \{\alpha \in T^*; \alpha \neq 0, L_\alpha \neq \{0\}\}, \quad card\ \Phi \in \mathbf{N}, \quad T^* := Hom_{\mathbf{C}}(T, \mathbf{C}).$$

(1.5.18.2)

$$\begin{aligned}\forall \alpha, \beta \in T^* : \ &[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta},\\ &\lceil \alpha + \beta \neq 0 \Longrightarrow \forall x \in L_\alpha, y \in L_\beta : \kappa(x,y) = 0 \rceil.\end{aligned}$$

$\forall \alpha \in \Phi, \forall x \in L_\alpha, y \in L_0 : \kappa(x,y) = 0.$

$\forall 0 \neq \alpha \in T^* : x \in L_\alpha \Longrightarrow ad_L(x)$ *nilpotent.*

$L_0 = C_L(T) = T, \ \kappa\big|_{T \times T}$ *non − degenerate,* T *nilpotent.*

(1.5.18.3) The resulting direct sum of T and these root spaces L_α,

$$L = T \oplus \bigoplus_{\alpha \in \Phi} L_\alpha, \ \ \forall \alpha \in \Phi, \forall x \in L_\alpha : x \ ad_L - nilpotent,$$

is called Cartan decomposition of L with respect to Φ.

(1.5.18.4) For every classical Lie algebra L, T can be represented by the corresponding subalgebra of diagonal matrices.

(1.5.18.5) Since the restriction of κ to $T \times T$ is non-degenerate, one establishes the complex-linear bijection:

$$T \ni t \longleftrightarrow \alpha_t \in T^*, \ T \ni s \xrightarrow[def]{\alpha_t} \kappa(s,t) \in \mathbf{C};$$

conversely denote:

$$T^* \ni \alpha \longleftrightarrow t_\alpha \in T, \ \ \forall s \in T : \alpha(s) = \kappa(s, t_\alpha).$$

(1.5.18.6) The root system Φ fulfills the following properties. Let $\alpha \in \Phi$.

$\mathbf{C} - lin\ span(\Phi) = T^*.$

$-\alpha \in \Phi, \ \ [L_\alpha, L_{-\alpha}] \subseteq T.$

$\forall x \in L_\alpha, y \in L_{-\alpha} : [x,y] = \kappa(x,y)t_\alpha.$

$[L_\alpha, L_{-\alpha}] = \mathbf{C}(\{t_\alpha\}), \ \ \alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0.$

$$h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = -h_{-\alpha} \in T.$$

$\exists x_\alpha \in L_\alpha, y_\alpha \in L_{-\alpha} : [x_\alpha, y_\alpha] = h_\alpha, \ \ [h_\alpha, x_\alpha] = 2x_\alpha, \ \ [h_\alpha, y_\alpha] = -2y_\alpha.$

Hence one finds an isomorphism of complex Lie algebras:

$\mathbf{C}(\{x_\alpha, y_\alpha, h_\alpha\}) =: S_\alpha \longleftrightarrow sl(2, \mathbf{C}).$

(1.5.18.7) $\forall \alpha \in \Phi : T = \mathbf{C}(\{h_\alpha\}) \oplus ker\ \alpha,$ because of $T/ker\ \alpha \longleftrightarrow \mathbf{C}.$

(1.5.18.8) Choosing $x_\alpha \in L_\alpha$ for $\alpha \in \Phi$, and calculating the trace of the restriction of $ad_L\ (h_\alpha)$ to an endomorphism of $\mathbf{C}(\{y_\alpha, h_\alpha\} \cup \bigcup_{n \in \mathbf{N}} L_{n\alpha})$,which is zero because of $[x_\alpha, y_\alpha] = h_\alpha$, with the above notation, one finds the following results, denoting $H_\alpha := [L_\alpha, L_{-\alpha}] = \mathbf{C}(\{h_\alpha\})$.

$$\forall \alpha \in \Phi, \forall n = 2, 3, \ldots : n\alpha \notin \Phi, \ \ dim\ L_\alpha = 1, \ \ S_\alpha = L_\alpha \oplus L_{-\alpha} \oplus H_\alpha.$$

(1.5.18.9) Let $\alpha, \beta \in \Phi$, and $\alpha \neq \pm\beta$; then $\forall n \in \mathbf{Z} : \beta + n\alpha \neq 0$.

$$L_{\alpha\beta} := \bigoplus_{n\in\mathbf{Z}} L_{\beta+n\alpha}, \quad [S_\alpha, L_{\alpha\beta}] \subseteq L_{\alpha\beta}.$$

Considering the eigenvalues of this irreducible representation of S_α on $L_{\alpha\beta}$ one finds, that there are non-negative integers p and q, such that one obtains the string

$$\{n \in \mathbf{Z}; \beta + n\alpha \in \Phi\} = \{-q, -q+1, \ldots, p-1, p\}.$$

$$\Gamma_{\alpha\beta} := \beta(h_\alpha) = 2\frac{\kappa(t_\alpha, t_\beta)}{\kappa(t_\alpha, t_\alpha)} = q - p \in \mathbf{Z},$$

the so-called Cartan integers. Especially $\beta - \Gamma_{\alpha\beta}\, \alpha \in \Phi$.

(1.5.18.10) $\forall \alpha \in \Phi : \{c \in \mathbf{C}; c\alpha \in \Phi\} = \{+1, -1\}$.

(1.5.18.11) Using the Cartan decomposition of L one calculates, that

$$\forall s, t \in T : \kappa(s,t) = \sum_{\alpha\in\Phi} \alpha(s)\alpha(t).$$

(1.5.18.12) $\forall \alpha, \beta \in \Phi : \alpha + \beta \in \Phi \Longrightarrow [L_\alpha, L_\beta] = L_{\alpha+\beta}$.

(1.5.18.13) For any two roots $\alpha \neq \pm\beta$ one obtains the following implications.

$$\Gamma_{\alpha\beta} > 0 \Longrightarrow \alpha - \beta \in \Phi, \quad \Gamma_{\alpha\beta} < 0 \Longrightarrow \alpha + \beta \in \Phi.$$

(1.5.19) Let $\mathbf{E}$ be a Euclidean space of dimension $m \in \mathbf{N}$, with the scalar product denoted by $< | >$. A finite subset Φ of $\mathbf{E}$ is called root system in $\mathbf{E}$, if and only if the following conditions are fulfilled.

$$\mathbf{E} = \mathbf{R} - \mathit{lin\ span}(\Phi), \Phi \not\ni 0.$$
$$\forall \alpha \in \Phi : \{c \in \mathbf{R}; c\,\alpha \in \Phi\} = \{+1, -1\}.$$
$$\forall \alpha, \beta \in \Phi : \Gamma_{\alpha\beta} := 2\frac{<\alpha|\beta>}{<\alpha|\alpha>} \in \mathbf{Z}, \quad \beta - \Gamma_{\alpha\beta}\alpha \in \Phi.$$

The rank of a root system Φ in $\mathbf{E}$ is defined as the dimension m of $\mathbf{E}$.

(1.5.19.1) Any vector $0 \neq \tau \in \mathbf{E}$ determines the real-linear reflection

$$\sigma_\tau : \mathbf{E} \ni \xi \longmapsto \xi - 2\frac{<\xi|\tau>}{<\tau|\tau>}\tau \in \mathbf{E}, \quad \sigma_\tau \circ \sigma_\tau = id\ \mathbf{E}.$$

The corresponding reflecting hyperplane is

$$\mathbf{P}_\tau := \{\eta \in \mathbf{E}; <\eta|\tau> = 0\} = \{\eta \in \mathbf{E}; \sigma_\tau(\eta) = \eta\}.$$
$$\forall \alpha, \beta \in \Phi : \sigma_\alpha(\beta) = \beta - \Gamma_{\alpha\beta}\alpha \in \Phi.$$

(1.5.19.2) The set

$$\Phi^* := \{ \frac{2\alpha}{<\alpha|\alpha>}; \alpha \in \Phi\}$$

is also a root system in $\mathbf{E}$, and $(\Phi^*)^* = \Phi$. The root systems Φ and Φ^* are called dual. Φ is called self-dual, if and only if $\Phi = \Phi^*$.

(1.5.20) Consider again a semisimple finite-dimensional complex Lie algebra L. Then $\mathbf{E} := \mathbf{R} - lin\ span(\Phi)$ is some real vector space, and the $\mathbf{R}$-bilinear form defined by the Killing form of L, such that

$$\forall \alpha, \beta \in \mathbf{E} : <\alpha|\beta> := \kappa(t_\alpha, t_\beta) \in \mathbf{R},$$

is positive-definite. Φ is some root system in $\mathbf{E}$.
T^* is the complexification of $\mathbf{E}$.

$$rank\ \Phi := dim\ \mathbf{E} = dim\ T, \quad dim\ L = rank\ \Phi + card\ \Phi.$$

(1.5.21) The Weyl group W of a root system Φ in $\mathbf{E}$ is defined as the subgroup of the group of permutations of Φ, which is generated by the set of restrictions of the reflections $\sigma_\alpha, \alpha \in \Phi$, to idempotent bijections of Φ.

(1.5.21.1) Let ψ be an $\mathbf{R}$-linear bijection of $\mathbf{E}$ onto itself, and assume that $\psi(\Phi) = \Phi$. Then

$$\forall \alpha, \beta \in \Phi : \psi \circ \sigma_\alpha \circ \psi^{-1} = \sigma_{\psi(\alpha)}, \quad \Gamma_{\alpha\beta} = \Gamma_{\psi(\alpha),\psi(\beta)}.$$

(1.5.22) Let Φ be a root system of rank m in $\mathbf{E}$. A basis $\Delta := \{\alpha_1, \ldots, \alpha_m\}$ of the real vector space $\mathbf{E}$ is called root basis of Φ, if and only if

$$\Delta \subseteq \Phi, \ \forall \beta = \sum_{k=1}^{m} b_k \alpha_k \in \Phi : \{b_1, \ldots, b_m\} \subset \mathbf{N}_0 \text{ or}$$
$$\{-b_1, \ldots, -b_m\} \subset \mathbf{N}_0.$$

The elements of Δ are called simple roots.

$$\forall \alpha, \beta \in \Delta : \alpha \neq \beta \Longrightarrow [\Gamma_{\alpha\beta} \leq 0 \text{ and } \alpha - \beta \notin \Phi].$$

(1.5.22.1) For any two bases Δ and Δ' of Φ, $\exists \psi \in W : \psi(\Delta) = \Delta'$.
The Weyl group W is generated by the set of reflections $\sigma_\alpha, \alpha \in \Delta$.

$$\forall \alpha \in \Phi : \exists \psi \in W : \psi(\alpha) \in \Delta. \quad \forall \psi \in W : \psi(\Delta) = \Delta \Longrightarrow \psi = id\ \Phi.$$

(1.5.23) A root system Φ in $\mathbf{E}$ is called irreducible, if and only if it cannot be written as the disjoint union of non-empty subsets $\Phi_k, k = 1, 2$, which are orthogonal in the sense, that $\forall \alpha_1 \in \Phi_1, \alpha_2 \in \Phi_2 : \Gamma_{\alpha_1\alpha_2} = 0$.

(1.5.23.1) Let Δ be a root basis of Φ. Then Φ is irreducible, if and only if Δ cannot be decomposed in the above manner.

(1.5.23.2) Let Φ be irreducible. Then one finds the following consequences.

$$\forall \alpha \in \Phi : \mathbf{R} - lin\ span(\{\psi(\alpha); \psi \in W\}) = \mathbf{E}.$$
$$card\ \{\|\alpha\|; \alpha \in \Phi\} \leq 2.$$
$$\forall \alpha, \beta \in \Phi : \|\alpha\| = \|\beta\| \Longrightarrow \exists \psi \in W : \beta = \psi(\alpha).$$

(1.5.24) Take a root basis $\Delta := \{\alpha_1, \ldots, \alpha_m\}$ of Φ in $\mathbf{E}$. The Cartan matrix Γ of Φ is defined with respect to Δ,

$$\Gamma := [\Gamma_{kl} \equiv \Gamma_{\alpha_k \alpha_l}; k, l = 1, \ldots, m].$$

Every other root basis Δ' of Φ can be written as some family, such that the corresponding Cartan matrix $\Gamma' = \Gamma$. Since Δ is some $\mathbf{R}$-basis of $\mathbf{E}$, $det\ \Gamma \neq 0$.

(1.5.25) Two root systems Φ in $\mathbf{E}$, Φ' in $\mathbf{E}'$, are called isomorphic, if and only if $\exists$ an $\mathbf{R}$-linear bijection $\psi : \mathbf{E} \longleftrightarrow \mathbf{E}'$, such that $\psi(\Phi) = \Phi'$, and $\Gamma_{\alpha\beta} = \Gamma'_{\psi(\alpha),\psi(\beta)}$ for all $\alpha, \beta \in \Phi$.

(1.5.26) Two root systems with the same Cartan matrix are isomorphic, because the natural bijection of root bases can be extended according to the diagram below.

$$\begin{tikzcd}
\Delta \arrow[r] & \Phi \arrow[r] \arrow[ddd, leftrightarrow, "\text{isomorphism}"] & E \arrow[r, "\text{complexification}"] \arrow[ddd, leftrightarrow, "\mathbf{R}\text{-linear}"] & T^* \arrow[ddd, leftrightarrow, "\mathbf{C}\text{-linear}"] \\
\alpha_k \arrow[u, phantom, "\in" rotate=90] \arrow[d, leftrightarrow] & & & \\
\alpha'_k \arrow[d, phantom, "\in" rotate=-90] & & & \\
\Delta' \arrow[r] & \Phi' \arrow[r] & E' \arrow[r, "\text{complexification}"'] & T^*
\end{tikzcd}$$

$$\mathbf{E} = \mathbf{R}(\Delta), \ T^* = \mathbf{C}(\Delta) \underset{\mathbf{C}-\text{linear}}{\longleftrightarrow} T = \mathbf{C}(\{h_k; k = 1, \ldots, m\}).$$

Here one assumes, that $\forall_1^m k, l : \Gamma_{\alpha_k \alpha_l} = \Gamma'_{\alpha_k' \alpha_l'}$.

(1.5.27) Every root system Φ in $\mathbf{E}$ can be decomposed into the disjoint union of irreducible root systems, according to the following procedure. If Φ is reducible, then every basis Δ of Φ is the disjoint union of mutually orthogonal subsets, such that

$$\Delta = \bigcup_{k=1}^{p} \Delta_k, \quad \forall_1^p k \neq l : \Delta_k \perp \Delta_l,$$

with respect to the scalar product of $\mathbf{E}$. $\forall_1^p k : \mathbf{E}_k := \mathbf{R}(\Delta_k), \quad \Phi_k := \Phi \cap \mathbf{E}_k$.

$$\mathbf{E} = \bigoplus_{k=1}^{p} \mathbf{E}_k, \text{ as an } \textit{orthogonal sum}. \ \Phi = \bigcup_{k=1}^{p} \Phi_k, \text{ as } \textit{disjoint union}.$$

Each $\Delta_k, k = 1, \dots, p$, is the basis of an irreducible root system Φ_k in $\mathbf{E}_k$.

(1.5.28) The irreducible root systems can be classified. The only possible examples are those of the classical Lie algebras,

$$\{A_m; m \geq 1\}, \ \{B_m; m \geq 2\}, \ \{C_m; m \geq 3\}, \ \{D_m; m \geq 4\},$$

and those of the so-called exceptional ones, denoted by E_6, E_7, E_8, F_4, G_2. The root systems and Cartan matrices of the classical Lie algebras are listed below.

(1.5.29) In order to construct an appropriate $\mathbf{C}$-basis of each one of the classical Lie algebras L, one needs the $\mathbf{N}_0$-*span* of the positive roots on an appropriate root basis $\Delta := \{\alpha_1, \dots, \alpha_m\}$. $\forall_1^m k : h_k := h_{\alpha_k}$.

$$\Phi = \Phi^+ \cup \Phi^-, \ \Phi^+ := \mathbf{N}_0(\Delta) \cap \Phi, \ \Phi^- := -\Phi^+.$$

$$L = T \oplus \bigoplus_{\alpha \in \Phi} L_\alpha = \mathbf{C}(\{h_k; k = 1, \dots, m\}) \oplus \bigoplus_{\alpha \in \Phi^+} \mathbf{C}(\{x_\alpha, y_\alpha\}).$$

(1.5.29.1) Let $\{\varepsilon_1, \dots, \varepsilon_{m+1}\}$ be the Cartesian basis of $\mathbf{R}^{m+1}, m \in \mathbf{N}$, and denote by $\mathbf{E}$ the orthogonal complement of $\{\varepsilon_1 + \dots + \varepsilon_{m+1}\}$, with respect to the usual scalar product $< | >$ of $\mathbf{R}^{m+1}$. The Cartan decomposition of A_m yields the following root system Φ in $\mathbf{E}$.

$$\Delta := \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}; 1 \leq i \leq m\},$$
$$\Phi^+ = \{\varepsilon_i - \varepsilon_j; 1 \leq i < j \leq m+1\} = \{\alpha_i + \dots + \alpha_j; 1 \leq i \leq j \leq m\};$$
$$\forall_1^m k, l : \Gamma_{kl} := 2\frac{< \alpha_k | \alpha_l >}{< \alpha_k | \alpha_k >} = 2\delta_{kl} - \delta_{k,l+1} - \delta_{k+1,l}.$$
$$\forall 1 \leq k < l \leq m : [\cdots [[X_k^\pm, X_{k+1}^\pm], X_{k+2}^\pm], \dots, X_l^\pm] \in L_{\pm\alpha},$$
$$\alpha := \alpha_k + \dots + \alpha_l \in \Phi^+.$$

The Weyl group W of A_m consists of the permutations of Cartesian basis vectors $\varepsilon_1, \dots, \varepsilon_{m+1}$, because the reflection σ_{α_k} only transposes ε_k and $\varepsilon_{k+1}, k = 1, \dots, m$. Hence W is isomorphic with $\mathbf{P}_{m+1}$.

(1.5.29.2) Let $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be the Cartesian basis of $\mathbf{R}^m, m \geq 2$. The Cartan decomposition of B_m yields the following root system Φ in $\mathbf{R}^m$.

$$\Delta := \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{m-1} := \varepsilon_{m-1} - \varepsilon_m, \alpha_m := \varepsilon_m\},$$

$$\begin{aligned}\Phi^+ &= \{\varepsilon_i; 1 \leq i \leq m\} \cup \{\varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq m\} \\ &= \{\alpha_k + \cdots + \alpha_l; 1 \leq k \leq l \leq m\} \\ &\cup \{\alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_m; 1 \leq k < l \leq m\};\end{aligned}$$

$$\forall_1^m k, l : \Gamma_{kl} = 2\delta_{kl} - (1 - \delta_{km})\delta_{k,l+1} - 2\delta_{km}\delta_{k,l+1} - \delta_{k+1,l}.$$

The root vectors of the roots $\alpha_k + \cdots + \alpha_l, 1 \leq k < l \leq m$, are expressed by the same multiple commutators as above. The remaining roots may be decomposed, such that

$$\forall 1 \leq k < l \leq m : \alpha := \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_m = \beta + \gamma,$$

$$\beta := \alpha_k + \cdots + \alpha_m \in \Phi^+, \quad \gamma := \alpha_l + \cdots + \alpha_m \in \Phi^+, \quad L_\alpha = [L_\beta, L_\gamma],$$

with L_β and L_γ being complex-spanned by one multiple-commutator of the generators $X_k^+, k = 1, \ldots, m$.
The Weyl group of B_m consists of the permutations and sign changes of $\varepsilon_1, \ldots, \varepsilon_m$. For $k = 1, \ldots, m-1$, σ_{α_k} only transposes ε_k and ε_{k+1}, and $\sigma_{\alpha_m}(\varepsilon_m) = -\varepsilon_m$. Therefore W is isomorphic to the semidirect product of the invariant subgroup $\mathbf{Z}_2^m$ of sign changes of m Cartesian basis vectors with the symmetric group $\mathbf{P}_m$.

(1.5.29.3) The Cartan decomposition of $C_m, m \geq 3$, yields the following root system in $\mathbf{R}^m$.

$$\Delta := \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{m-1} := \varepsilon_{m-1} - \varepsilon_m, \alpha_m := 2\varepsilon_m\},$$

$$\begin{aligned}\Phi^+ &= \{2\varepsilon_i; 1 \leq i \leq m\} \cup \{\varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq m\} \\ &= \{\alpha_k + \cdots + \alpha_l; 1 \leq k \leq l \leq m\} \\ &\cup \{2\alpha_k + \cdots + 2\alpha_{m-1} + \alpha_m; 1 \leq k < m\} \\ &\cup \{\alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{m-1} + \alpha_m; 1 \leq k < l < m\};\end{aligned}$$

$$\forall_1^m k, l : \Gamma_{kl} = 2\delta_{kl} - \delta_{k,l+1} - \delta_{k+1,l} - \delta_{lm}\delta_{k+1,l}.$$

The corresponding Weyl group is isomorphic to that of B_m, because the reflections $\sigma_{\alpha_k}, k = 1, \ldots, m$, are acting onto $\varepsilon_1, \ldots, \varepsilon_m$ just as in the case of B_m.

(1.5.29.4) For $D_m, m \geq 4$, one finds the following positive roots in $\mathbf{R}^m$.

$$\Delta := \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \dots, \alpha_{m-1} := \varepsilon_{m-1} - \varepsilon_m, \alpha_m := \varepsilon_{m-1} + \varepsilon_m\},$$

$$\begin{aligned}\Phi^+ &= \{\varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq m\} = \{\alpha_k + \dots + \alpha_l; 1 \leq k \leq l < m\} \cup \{\alpha_m\} \\ &\quad \cup \{\alpha_k + \dots + \alpha_{l-1} + 2\alpha_l + \dots + 2\alpha_{m-2} + \alpha_{m-1} + \alpha_m; \\ &\qquad 1 \leq k < l \leq m-2\} \\ &\quad \cup \{\alpha_k + \dots + \alpha_m; 1 \leq k \leq m-2\} \\ &\quad \cup \{\alpha_k + \dots + \alpha_{m-2} + \alpha_m; 1 \leq k \leq m-2\};\end{aligned}$$

$$\forall_1^{m-1} k, l : \Gamma_{kl} = 2\delta_{kl} - \delta_{k,l+1} - \delta_{k+1,l}, \ \Gamma_{km} = \Gamma_{mk} = -\delta_{k,m-2}, \Gamma_{mm} = 2.$$

The reflections $\sigma_{\alpha_k}, k = 1, \dots, m-1$, again only transpose ε_k and ε_{k+1};

$$\sigma_{\alpha_m}(\varepsilon_{m-1}) = -\varepsilon_m, \ \sigma_{\alpha_m}(\varepsilon_m) = -\varepsilon_{m-1}, \ \sigma_{\alpha_m}(\alpha_l) = \alpha_l, \ 1 \leq l \leq m-2.$$

Therefore W is isomorphic to the semidirect product of the invariant subgroup $\mathbf{Z}_2^{m-1}$ with $\mathbf{P}_m$.

(1.5.30)

For A_m, $\Phi = \{\alpha \in \mathbf{Z}(\{\varepsilon_1, \dots, \varepsilon_{m+1}\}); < \alpha|\varepsilon_1 + \dots + \varepsilon_{m+1} >= 0,$
$< \alpha|\alpha >= 2\}.$

For B_m, $m \geq 2$, $\Phi = \{\alpha \in \mathbf{Z}(\{\varepsilon_1, \dots, \varepsilon_m\}); < \alpha|\alpha > \in \{1, 2\}\}.$

For D_m, $m \geq 4$, $\Phi = \{\alpha \in \mathbf{Z}(\{\varepsilon_1, \dots, \varepsilon_m\}); < \alpha|\alpha >= 2\}.$

The root systems of A_m and $D_m, m \geq 4$, are self-dual.
The root systems of B_m and $C_m, m \geq 3$, are dual.

(1.5.30.1) An appropriate $\mathbf{C}$-basis of each of these classical Lie algebras, and also of each of the exceptional ones, consists of $H_k, k = 1, \dots, m$, of certain multiple-commutators of $X_k^+, k = 1, \dots, m$, and of corresponding multiple-commutators of $X_k^-, k = 1, \dots, m$, according to the positive and negative roots listed above, and using the corresponding Serre relations below.

(1.5.31) Consider again a finite-dimensional complex semisimple Lie algebra L with the root system Φ in T^*, corresponding to a maximal toral subalgebra T of L. Then T is its own normalizer, $N_L(T) = T$.

(1.5.31.1) L is simple, if and only if Φ is irreducible.

(1.5.31.2) Let L be non-simple, with the simple ideals $L_k, k = 1, \dots, p$.

$$L = \bigoplus_{k=1}^{p} L_k, \quad T = \bigoplus_{k=1}^{p} T_k, \quad T_k := T \cap L_k, \quad p \geq 2.$$

For $k = 1, \dots, p$, T_k is some maximal toral subalgebra of L_k, with the corresponding irreducible root system Φ_k.

$$\forall_1^p k : \Phi_k \ni \alpha_k \underset{def}{\longleftrightarrow} \tilde{\alpha}_k \in \tilde{\Phi}_k \subset T^*, \quad \tilde{\alpha}_k\big|_{T_k} := \alpha_k, \quad \forall_1^p l \neq k : \tilde{\alpha}_k\big|_{T_l} := 0.$$

$\Phi = \bigcup_{k=1}^{p} \tilde{\Phi}_k$, as disjoint union of subsets of T^*, which are mutually orthogonal with respect to the Killing form κ.

$$\forall_1^p k \neq l, \forall \alpha \in \tilde{\Phi}_k, \beta \in \tilde{\Phi}_l : \Gamma_{\alpha\beta} = 0.$$

Here one uses, that $\forall_1^p k \neq l : \kappa\big|_{L_k \times L_l} = 0$. Obviously

$$\forall_1^p k : T_k^* \ni \alpha_k \longleftrightarrow \tilde{\alpha}_k \longleftrightarrow t_{\tilde{\alpha}_k} \in T_k,$$

with $\tilde{\alpha}_k$ defined as above. The Weyl group W of Φ is isomorphic to the direct product of the Weyl groups W_k of $\Phi_k, k = 1, \dots, p$.

(1.5.31.3) $L = \mathbf{R} - alg\ span\left(\bigcup_{\alpha \in \Delta}(L_\alpha \cup L_{-\alpha})\right)$, inserting a root basis Δ of Φ.

(1.5.31.4) Let $\Delta = \{\alpha_1, \dots, \alpha_m\}$ be a root basis of Φ. $\forall_1^m k, l$:

$$\Gamma_{kl} \equiv \Gamma_{\alpha_k \alpha_l} = 2\frac{\kappa(t_{\alpha_k}, t_{\alpha_l})}{\kappa(t_{\alpha_k}, t_{\alpha_k})} \in \mathbf{Z}, \quad \sigma_k(\alpha_l) = \alpha_l - \Gamma_{kl}\alpha_k, \quad \sigma_k := \sigma_{\alpha_k}.$$

$$\forall_1^m k \neq l : \Gamma_{kl} \leq 0, \quad \{\alpha_l, \alpha_l + \alpha_k, \dots, \alpha_l - \Gamma_{kl}\alpha_k\} \text{ is the } \textit{string} \text{ through } \alpha_l.$$

(1.5.32) Two maximal toral subalgebras T and T' of L are conjugate under an appropriate automorphism of the complex Lie algebra L.

(1.5.33) Choosing $x_k \in L_{\alpha_k}$ and $y_k \in L_{-\alpha_k}$, such that

$$[x_k, y_k] = h_k := h_{\alpha_k} \in T,$$

the complex Lie algebra L is generated by the set $\{x_k, y_k; k = 1, \dots, m\}$. Moreover these generators fulfill the following relations.

$$\begin{aligned} \forall_1^m k, l : &[h_k, h_l] = 0, \quad [x_k, y_l] = \delta_{kl} h_k, \\ &[h_k, x_l] = \Gamma_{kl} x_l, \quad [h_k, y_l] = -\Gamma_{kl}\ y_l; \\ &(ad\ x_k)^{1-\Gamma_{kl}}(x_l) = (ad\ y_k)^{1-\Gamma_{kl}}(y_l) = 0, \text{ if } k \neq l. \end{aligned}$$

The latter relations for $k \neq l$ are called Serre relations.

(1.5.34) Take a root system Φ of rank m in $\mathbf{E}$, with an according root basis $\Delta = \{\alpha_1, \dots, \alpha_m\}$. The components of the corresponding Cartan matrix are denoted by $\Gamma_{kl} \equiv \Gamma_{\alpha_k \alpha_l}$, for $k, l = 1, \dots, m$. Then the complex Lie algebra

$$L := \frac{\textit{free complex Lie algebra} \text{ over the set } \{x_k, y_k, h_k; k = 1, \dots, m\}}{\textit{ideal} \text{ of the above } \textit{relations}}$$

is finite-dimensional and semisimple. $T := \mathbf{C}(\{h_1, \dots, h_m\})$ is some maximal toral subalgebra of L, with the corresponding root system denoted by $\tilde{\Phi} \subset T^*$. Moreover there is some root basis $\tilde{\Delta} = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_m\}$ of $\tilde{\Phi}$, such that the real-linear bijection: $\mathbf{E} \ni \alpha_k \longleftrightarrow \tilde{\alpha}_k \in \mathbf{R}(\tilde{\Delta})$ yields an isomorphism of the root systems Φ and $\tilde{\Phi}$.

(1.5.34.1) This theorem by J.-P. Serre shows, that every semisimple finite-dimensional complex Lie algebra L is determined uniquely, modulo an isomorphism of complex Lie algebras, by the root system of a maximal toral subalgebra of L.

(1.5.34.2) Using the above relations, an isomorphism of root systems is extended naturally to an isomorphism of complex Lie algebras.

(1.5.34.3) Since the Killing form is invariant under an automorphism of L, the root systems of two maximal toral subalgebras of L are isomorphic.

(1.5.34.4) The universal enveloping algebra $E(L)$ of L is obtained by factorizing the free complex algebra of these $3m$ generators with respect to the above relations.

(1.5.35) Let $\sigma_{k_1} \circ \dots \circ \sigma_{k_p}, k_1, \dots, k_p \in \{1, \dots, m\}$, be the longest word of the Weyl group W of the simple Lie algebra $L := A_m, \dots, D_m$, or of an exceptional one with the corresponding rank m. Then $\Phi^+ = \{\beta_1, \dots, \beta_p\}$ with the positive roots

$$\beta_1 := \alpha_{k_1},\ \beta_2 := \sigma_{k_1}(\alpha_{k_2}),\ \beta_3 := \sigma_{k_1} \circ \sigma_{k_2}(\alpha_{k_3}), \dots,$$
$$\beta_p := \sigma_{k_1} \circ \dots \circ \sigma_{k_{p-1}}(\alpha_{k_p}).$$

(1.5.35.1) In the case of A_2 the three positive roots are

$$\beta_1 := \alpha_1 = \sigma_2 \circ \sigma_1(\alpha_2),\ \beta_2 := \sigma_1(\alpha_2) = \alpha_1 + \alpha_2 = \sigma_2(\alpha_1),$$
$$\beta_3 := \sigma_1 \circ \sigma_2(\alpha_1) = \alpha_2.$$

The longest word of W is $\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2$. W is the free group over the set $\{\sigma_1, \sigma_2\}$, factorized with respect to the relations

$$\sigma_1^2 = \sigma_2^2 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \textit{unit}.$$

(1.5.35.2) In the case of B_2 one finds, that

$$\Gamma = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \quad \sigma_1(\varepsilon_1) = \varepsilon_2, \ \sigma_1(\varepsilon_2) = \varepsilon_1, \ \sigma_2(\varepsilon_1) = \varepsilon_1, \ \sigma_2(\varepsilon_2) = -\varepsilon_2.$$

The longest word of W is $\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 = -id\ \mathbf{R}^2$.

$$\beta_1 := \alpha_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2(\alpha_1), \ \beta_2 := \sigma_1(\alpha_2) = \alpha_1 + \alpha_2 = \sigma_2 \circ \sigma_1(\alpha_2),$$
$$\beta_3 := \sigma_1 \circ \sigma_2(\alpha_1) = \alpha_1 + 2\alpha_2 = \sigma_2(\alpha_1), \ \beta_4 := \sigma_1 \circ \sigma_2 \circ \sigma_1(\alpha_2) = \alpha_2.$$

The Weyl group W is obtained from the free group over the set $\{\sigma_1, \sigma_2\}$, factorizing with respect to the relations $\sigma_1^2 = \sigma_2^2 = (\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2)^2 = \ unit$.

(1.5.35.3) In the cases of B_3 and C_3, with the root basis $\Delta := \{\alpha_1, \alpha_2, \alpha_3\}$,

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \ \alpha_2 := \varepsilon_2 - \varepsilon_3, \ \alpha_3 := \varepsilon_3 \text{ or } 2\varepsilon_3,$$

$$\sigma_1 : \varepsilon_1 \longleftrightarrow \varepsilon_2, \ \sigma_2 : \varepsilon_2 \longleftrightarrow \varepsilon_3, \ \sigma_3 : \varepsilon_3 \longleftrightarrow -\varepsilon_3,$$

the longest word of the Weyl group can be chosen as

$$\sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_2 \circ \sigma_3 \circ \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_2 = -id\ \mathbf{R}^3.$$

(1.5.35.4) In the case of D_4, with the root basis $\Delta := \{\alpha_k; k = 1, \ldots, 4\}$,

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \ \alpha_2 := \varepsilon_2 - \varepsilon_3, \ \alpha_3 := \varepsilon_3 - \varepsilon_4, \ \alpha_4 := \varepsilon_3 + \varepsilon_4,$$

$$\sigma_1 : \varepsilon_1 \longleftrightarrow \varepsilon_2, \ \sigma_2 : \varepsilon_2 \longleftrightarrow \varepsilon_3, \ \sigma_3 : \varepsilon_3 \longleftrightarrow \varepsilon_4, \ \sigma_4 : \varepsilon_3 \longleftrightarrow -\varepsilon_4,$$

one may choose the longest word of W as

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1 \circ \sigma_4 \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_2 \circ \sigma_4 = -id\ \mathbf{R}^4.$$

(1.5.35.5) The set of positive roots of G_2, with $\Gamma_{12} = -3$, is

$$\begin{aligned} \Phi^+ = \{&\alpha_1, \sigma_1(\alpha_2) = 3\alpha_1 + \alpha_2, \sigma_1 \circ \sigma_2(\alpha_1) = 2\alpha_1 + \alpha_2, \\ &\sigma_1 \circ \sigma_2 \circ \sigma_1(\alpha_2) = 3\alpha_1 + 2\alpha_2, \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2(\alpha_1) = \alpha_1 + \alpha_2, \\ &\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1(\alpha_2) = \alpha_2\}. \end{aligned}$$

The longest word of W is

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 = -id\ \mathbf{E}.$$

$$\mathbf{E} := \{\xi \in \mathbf{R}^3; \langle \xi | \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \rangle = 0\}, \ \Delta := \{\alpha_1, \alpha_2\},$$

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \ \alpha_2 := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

$$\sigma_1 : \varepsilon_1 \longleftrightarrow \varepsilon_2, \ \sigma_2 : \begin{cases} \varepsilon_1 \longleftrightarrow -\varepsilon_1 \\ \varepsilon_2 \longleftrightarrow -\varepsilon_3 \end{cases}.$$

W is the free group over the set $\{\sigma_1, \sigma_2\}$, factorized with respect to the relations

$$\sigma_1^2 = \sigma_2^2 = (\sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2)^2 = \ unit.$$

(1.5.35.6) In the general case of rank m, W is obtained from the free group over the set $\{\sigma_k, k = 1, \ldots, m\}$, factorizing with respect to the relations $\sigma_1^2 = \cdots = \sigma_m^2 =$ *unit*, and the following braid-like relations. $\forall_1^m k \neq l$:

$$\sigma_k \circ \sigma_l = \sigma_l \circ \sigma_k, \text{ if } \Gamma_{kl} = 0;$$
$$\sigma_k \circ \sigma_l \circ \sigma_k = \sigma_l \circ \sigma_k \circ \sigma_l, \text{ if } \Gamma_{kl} = \Gamma_{lk} = -1;$$
$$\sigma_k \circ \sigma_l \circ \sigma_k \circ \sigma_l = \sigma_l \circ \sigma_k \circ \sigma_l \circ \sigma_k, \text{ if } \Gamma_{kl} = -2;$$
$$\sigma_k \circ \sigma_l \circ \sigma_k \circ \sigma_l \circ \sigma_k \circ \sigma_l = \sigma_l \circ \sigma_k \circ \sigma_l \circ \sigma_k \circ \sigma_l \circ \sigma_k, \text{ if } \Gamma_{kl} = -3.$$

(1.5.35.7) Obviously the choice of a longest word determines some total ordering of positive roots. The number P of positive roots is listed in the following table.

L	A_m	B_m	C_m	D_m	E_6	E_7	E_8	F_4	G_2
p	$\frac{m(m+1)}{2}$	m^2	m^2	$m^2 - m$	36	63	120	24	6

(1.5.36) In case of $L := A_m$ or D_m, the following automorphisms T_k, $k = 1, \ldots, m$, of L are useful for the construction of an appropriate complex-linear basis of L. T_k is defined such that $\forall_1^m k, l$:

$$h_k \longrightarrow -h_k, \ x_k \longrightarrow -y_k, \ y_k \longrightarrow -x_k;$$
$$h_l \longrightarrow h_l, \ x_l \longrightarrow x_l, \ y_l \longrightarrow y_l \text{ for } \Gamma_{kl} = 0;$$
$$h_l \longrightarrow h_k + h_l, \ x_l \longrightarrow [x_k, x_l], \ y_l \longrightarrow -[y_k, y_l] \text{ for } \Gamma_{kl} = -1.$$

Then the inverse automorphisms $T_k^{-1}, k = 1, \ldots, m$, obviously are acting as:

$$h_l \longrightarrow h_k + h_l, \ x_l \longrightarrow -[x_k, x_l], \ y_l \longrightarrow [y_k, y_l] \text{ for } \Gamma_{kl} = -1.$$

These automorphisms fulfill the following braid relations. $\forall_1^m k, l$:

$$T_k T_l = T_l T_k, \text{ if } \Gamma_{kl} = 0; \ T_k T_l T_k = T_l T_k T_l, \text{ if } \Gamma_{kl} = -1.$$

(1.5.37) In the general case of a simple finite-dimensional complex Lie algebra L, appropriate automorphisms T_k of L, $k = 1, \ldots, m$, are defined such that $\forall_1^m k, l$:

$$T_k^{\pm 1} : h_l \longrightarrow h_l - \Gamma_{kl} h_k, \ x_k \longrightarrow -y_k, \ y_k \longrightarrow -x_k;$$

for $k \neq l$, one conveniently defines T_k such that

$$T_k : x_l \longrightarrow \frac{1}{(-\Gamma_{kl})!} (ad\ x_k)^{-\Gamma_{kl}}(x_l), \ T_k : y_l \longrightarrow \frac{(-1)^{\Gamma_{kl}}}{(-\Gamma_{kl})!} (ad\ y_k)^{-\Gamma_{kl}}(y_l),$$
$$T_k^{-1} : x_l \rightarrow \frac{(-1)^{\Gamma_{kl}}}{(-\Gamma_{kl})!} (ad\ x_k)^{-\Gamma_{kl}}(x_l), \ T_k^{-1} : y_l \rightarrow \frac{1}{(-\Gamma_{kl})!} (ad\ y_k)^{-\Gamma_{kl}}(y_l).$$

Their braid-like relations read $\forall_1^m k \neq l$:

$$T_k \circ T_l = T_l \circ T_k, \text{ if } \Gamma_{kl} = 0;$$
$$T_k \circ T_l \circ T_k = T_l \circ T_k \circ T_l, \text{ if } \Gamma_{kl} = \Gamma_{lk} = -1;$$
$$T_k \circ T_l \circ T_k \circ T_l = T_l \circ T_k \circ T_l \circ T_k, \text{ if } \Gamma_{kl} = -2;$$
$$T_k \circ T_l \circ T_k \circ T_l \circ T_k \circ T_l = T_l \circ T_k \circ T_l \circ T_k \circ T_l \circ T_k, \text{ if } \Gamma_{kl} = -3.$$

The last possibility only occurs in the exceptional case of G_2 with $\Gamma_{12} = -3$.

(1.5.37.1) The one-dimensional positive and negative root spaces are then complex-spannned by the following root vectors.

$$x_{\beta_1} := x_{k_1} \in L_{\beta_1}, \ x_{\beta_2} := T_{k_1}(x_{k_2}) \in L_{\beta_2}, \dots,$$
$$x_{\beta_p} := T_{k_1} \circ \dots \circ T_{k_{p-1}}(x_{k_p}) \in L_{\beta_p},$$
$$y_{\beta_1} := y_{k_1} \in L_{-\beta_1}, \dots, \ y_{\beta_p} := T_{k_1} \circ \dots \circ T_{k_{p-1}}(y_{k_p}) \in L_{-\beta_p}.$$

The simple finite-dimensional complex Lie algebra L is then $\mathbf{C}$-spanned by the following basis.

$$L = \mathbf{C}(\{h_k, x_{\beta_j}, y_{\beta_j}; k = 1, \dots, m; j = 1, \dots, p\}).$$

Of course one could also insert here the inverse automorphisms T_k^{-1}, $k = 1, \dots, m$.

(1.5.38) The braid group $B_{n+1}, n \in \mathbf{N}$, is defined as the free group over the set $\{b_1, \dots, b_n\}$, factorized by the relations

$$\{b_k b_l = b_l b_k \text{ for } |k-l| \geq 2,$$
$$b_k b_l b_k = b_l b_k b_l \text{ for } |k-l| = 1; k, l = 1, \dots, n\}.$$

Factorizing in turn B_{n+1} with respect to the relations

$$\{b_k = b_k^{-1}; k = 1, \dots, n\},$$

one obtains the symmetric group P_{n+1} of permutations of $n+1$ elements.

(1.5.38.1) In the cases of A_m, D_m, E_6, E_7, E_8, the above automorphisms provide an according representation of the braid group B_{m+1}.

(1.5.39) Let the Lie algebra L over a field K be finite-dimensional. L is called reductive, if and only if $rad\ L = Z(L)$; then $L = [L, L] \oplus Z(L)$, and $[L, L]$ is semisimple. If L is semisimple, then $L = [L, L]$ is reductive, because then $rad\ L = Z(L) = \{0\}$.

(1.5.40) Consider a Lie algebra $L \subseteq gl(m, \mathbf{C}), m \geq 2$. If L is acting irreducibly on $\mathbf{C}^m$, then L is reductive, and then $dim\ Z(L) \leq 1$; if moreover $L \subseteq sl(m, \mathbf{C})$, then L is semisimple.

(1.5.41) Let the complex Lie algebra L be reductive, and T a maximal toral subalgebra of $[L, L]$, with the corresponding Cartan decomposition

$$[L, L] = T \oplus \bigoplus_{\alpha \in \Phi} [L, L]_\alpha.$$

Then $H := Z(L) \oplus T$ is some Cartan subalgebra of L, and H is Abelian.

$$\forall \beta \in H^* := Hom_{\mathbf{C}}(H, \mathbf{C}) : L_\beta := \{x \in L; \forall h \in H : [h, x] = \beta(h)x\},$$

$$L_0 = C_L(H) = H,\ \beta\big|_{Z(L)} = 0 \text{ and } \beta\big|_T =: \alpha \neq 0 \Longrightarrow L_\beta = [L, L]_\alpha,$$

$$\beta\big|_{Z(L)} \neq 0 \Longrightarrow L_\beta = \{0\}. \quad L = H \oplus \bigoplus_{\beta \in H^*} L_\beta = H \oplus \bigoplus_{\beta \in \Psi} L_\beta.$$

Here Ψ denotes the root system of L with respect to H, which is defined by

$$\Psi := \{\beta \in H^*; \beta\big|_{Z(L)} = 0, \beta\big|_T \in \Phi\},$$

inserting the root system Φ of $[L, L]$ with respect to T.

(1.5.41.1) L is semisimple, if and only if $Z(L) = \{0\}$; in this case $T = C_L(T) = N_L(T)$ is some Cartan subalgebra of L.

(1.5.42) The generators of the classical Lie algebras can be represented by the following real matrices. The resulting representations, which may be called defining ones, are faithful.

(1.5.42.1) For $A_m, m \geq 1,\ \forall_1^m k, \forall_1^{m+1} i, j$:

$$(x_k)_{ij} = (y_k)_{ji} = \delta_{ik}\delta_{j,k+1},\ (h_k)_{ij} = \delta_{ik}\delta_{jk} - \delta_{i,k+1}\delta_{j,k+1}.$$

(1.5.42.2) For $B_m, m \geq 2,\ \forall_1^{m-1} k, \forall_1^{2m+1} i, j$:

$$(x_k)_{ij} = (y_k)_{ji} = \delta_{i,k+1}\delta_{j,k+2} - \delta_{i,m+k+2}\delta_{j,m+k+1},$$
$$(h_k)_{ij} = \delta_{ij}(\delta_{i,k+1} - \delta_{i,k+2} - \delta_{i,m+k+1} + \delta_{i,m+k+2}),$$
$$(x_m)_{ij} = \sqrt{2}(\delta_{i,m+1}\delta_{j1} - \delta_{i1}\delta_{j,2m+1}),\ (h_m)_{ij} = 2\delta_{ij}(\delta_{i,m+1} - \delta_{i,2m+1}).$$

(1.5.42.3) For $C_m, m \geq 3,\ \forall_1^{m-1} k, \forall_1^{2m} i, j$:

$$(x_k)_{ij} = (y_k)_{ji} = \delta_{ik}\delta_{j,k+1} - \delta_{i,m+k+1}\delta_{j,m+k},$$
$$(h_k)_{ij} = \delta_{ij}(\delta_{ik} - \delta_{i,k+1} - \delta_{i,m+k} + \delta_{i,m+k+1}),$$
$$(x_m)_{ij} = (y_m)_{ji} = \delta_{im}\delta_{j,2m},\ (h_m)_{ij} = \delta_{ij}(\delta_{im} - \delta_{i,2m}).$$

(1.5.42.4) For $D_m, m \geq 4$, $\forall_1^{m-1} k$, the generators are represented by the same matrices as for $C_m . \forall_1^{2m} i, j$:

$$(x_m)_{ij} = (y_m)_{ji} = \delta_{i,m-1}\delta_{j,2m} - \delta_{im}\delta_{j,2m-1},$$
$$(h_m)_{ij} = \delta_{ij}(\delta_{i,m-1} + \delta_{im} - \delta_{i,2m-1} - \delta_{i,2m}).$$

2. Lie Superalgebras

Compared to the structure theory of semisimple finite-dimensional complex Lie algebras, an according classification, and also description by means of generators and relations, of the simple finite-dimensional complex Lie superalgebras is much more complicated. Some of the reasons for this fact, which in particular renders necessary numerous case distinctions, are indicated below. As in the non-graded case, in general the coefficients can be taken from an algebraically closed field of characteristic zero.

The Killing form of a simple finite-dimensional Lie superalgebra is either non-degenerate or equal to zero.

The triangular decomposition of basic classical Lie superalgebras in general does not yield maximal solvable subalgebras.

The finite-dimensional representations of simple Lie superalgebras are not completely reducible, except in the cases of $osp(1, 2n) \equiv B(0, n),\ n \in \mathbf{N}$.

The properties "solvable" and "semisimple" of Lie algebras can be generalized to the $\mathbf{Z}_2$-graded case, but the corresponding theorems, concerning the factorization of a finite-dimensional Lie algebra with respect to the radical, and also the direct sum decomposition of a semisimple finite-dimensional Lie algebra, fail to be generalized to the $\mathbf{Z}_2$-graded case.

The irreducible finite-dimensional representations of a solvable Lie superalgebra need not be one-dimensional.

The pioneer work on the theory of finite-dimensional Lie superalgebras was performed by V.G. Kac (1977, 1978). Excellent systematic studies are also due to M. Scheunert (1979, 1985, 1992), D. A. Leites and V. Serganova (1992), I. Penkov and V. Serganova (1992, 1994), I. Penkov (1994).

The full set of relations in order to describe a basic classical Lie superalgebra $\mathcal{L}$ was established recently by H. Yamane (1991, 1994), who also investigated the q-deformation $U_q(\mathcal{L})$ of the universal enveloping superalgebra of $\mathcal{L}$, the corresponding universal R-matrix, and a Poincaré-Birkhoff-Witt-like linear basis of $U_q(\mathcal{L})$.

As in the non-graded case, for the study of infinite-dimensional Lie superalgebras the reader is referred to the literature, for instance the monograph by Yu. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, M. V. Zaicev (1992).

2.1 Basic Definitions

(2.1.1) Let an algebra A over the commutative ring R be $\mathbf{Z}_2$-graded, with

$$\mathbf{Z_2} := \mathbf{Z}/\{\mathit{even\ integers}\} =: \{\bar{0}, \bar{1}\},$$

$$A = A^{\bar{0}} \oplus A^{\bar{1}}, \ \forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall a \in A^{\bar{p}}, b \in A^{\bar{q}} : ab \in A^{\bar{p}+\bar{q}}.$$

Obviously, $A^{\bar{0}}$ is some subalgebra of A. An element $a \in A$ is called homogenous of degree $\bar{p} \in \mathbf{Z}_2$, if and only if $a \in A^{\bar{p}}$. Then A is called superalgebra over R. Moreover let A be associative. With the super-commutator,

$$\forall a = a^{\bar{0}} + a^{\bar{1}}, b = b^{\bar{0}} + b^{\bar{1}} \in A : [a, b] := \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (a^{\bar{p}} b^{\bar{q}} - (-1)^{pq} b^{\bar{q}} a^{\bar{p}}),$$

the super-commutator algebra A_L of A is constructed; A_L is some Lie superalgebra over R. Furthermore one finds, that

$$\forall a, b, c \in A : [a, bc] = [a, b]c + \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq} b^{\bar{q}} [a^{\bar{p}}, c].$$

An associative superalgebra A over R is called graded-commutative, if and only if $\forall a, b \in A : [a, b] = 0$. If an associative superalgebra A is unital, then its unit $e_A \in A^{\bar{0}}$.

(2.1.1.1) A submodule F of an R-bimodule $E = E^{\bar{0}} \oplus E^{\bar{1}}$ is called $\mathbf{Z}_2$-graded, if and only if $F = (F \cap E^{\bar{0}}) \oplus (F \cap E^{\bar{1}})$.
An ideal D of A is called $\mathbf{Z}_2$-graded, if and only if the submodule D of A is $\mathbf{Z}_2$-graded.
Similarly a subalgebra B of A is called $\mathbf{Z}_2$-graded, if and only if the submodule B of A is $\mathbf{Z}_2$-graded; then B is again a superalgebra over R.

(2.1.1.2) An algebra-homomorphism $\phi : A \longrightarrow A'$, of associative superalgebras A, A' over R, is called homomorphism in the sense of associative superalgebras over R, if and only if it preserves the $\mathbf{Z}_2$-grading in the sense that $\forall \bar{p} \in \mathbf{Z}_2 : Im\ \phi\big|_{A^{\bar{p}}} \subseteq A'^{\bar{p}}$; in this case $\forall a, b \in A : \phi([a, b]) = [\phi(a), \phi(b)]$.

(2.1.1.3) Consider the direct sum of R-bimodules $E^{\bar{0}} \oplus E^{\bar{1}} = E$. The associative algebra $End_R(E) \equiv Hom_R(E, E)$, with the unit $id\ E$, is an associative superalgebra over R, with the direct sum $End_R(E) = End_R^{\bar{0}}(E) \oplus End_R^{\bar{1}}(E)$,

$$End_R^{\bar{q}}(E) := \{\phi \in End_R(E); \forall \bar{p} \in \mathbf{Z}_2 : Im\ \phi\big|_{E^{\bar{p}}} \subseteq E^{\bar{p}+\bar{q}}\}.$$

$$\forall \phi \in End_R(E), \forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall x = x^{\bar{0}} + x^{\bar{1}} \in E :$$
$$End_R^{\bar{q}}(E) \ni \phi^{\bar{q}} : x^{\bar{p}} \underset{def}{\longrightarrow} (\phi(x^{\bar{p}}))^{\bar{p}+\bar{q}}.$$

(2.1.2) An algebra $L = L^{\bar{0}} \oplus L^{\bar{1}}$, with the direct sum of R-bimodules, is called Lie superalgebra, if and only if the R-bilinear mapping:
$L \times L \ni \{a, b\} \longrightarrow [a, b] \in L$ fulfills the following relations.

$$\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall a \in L^{\bar{p}}, b \in L^{\bar{q}} : [a, b] \in L^{\bar{p}+\bar{q}};$$

$$\forall a, b, c \in L : [a, b] + \sum_{\bar{p},\bar{q} \in \mathbf{Z}_2} (-1)^{pq} [b^{\bar{q}}, a^{\bar{p}}] = 0,$$

$$[a, [b, c]] = [[a, b], c] + \sum_{\bar{p},\bar{q} \in \mathbf{Z}_2} (-1)^{pq} [b^{\bar{q}}, [a^{\bar{p}}, c]];$$

$$\forall a \in L^{\bar{0}} : [a, a] = 0.$$

The last condition follows from the second one, if R is a field of characteristic $\neq 2$. Obviously $L^{\bar{0}}$ is some Lie algebra over R.

(2.1.2.1) A subalgebra H of L is called $\mathbf{Z}_2$-graded, if and only if the submodule H of L is $\mathbf{Z}_2$-graded; then H is again an R-Lie superalgebra.
An ideal D of L is called $\mathbf{Z}_2$-graded, if and only if the submodule D of L is $\mathbf{Z}_2$-graded.

(2.1.2.2) An R-linear mapping $\phi : L \longrightarrow L'$, of Lie superalgebras L, L' over R, is called homomorphism in the sense of Lie superalgebras over R, if and only if one finds

$$\text{(i)} \quad \forall \bar{p} \in \mathbf{Z}_2 : Im\ \phi\big|_{L^{\bar{p}}} \subseteq L'^{\bar{p}}; \quad \text{(ii)} \quad \forall a, b \in L : \phi([a, b]) = [\phi(a), \phi(b)].$$

(2.1.3) An R-linear mapping $\delta \in End_R^{\bar{q}}(A)$, of an associative superalgebra A over R into itself, is called derivation of degree $\bar{q} \in \mathbf{Z}_2$, if and only if

$$\forall a, b \in A : \delta(ab) = \delta(a)b + \sum_{\bar{p} \in \mathbf{Z}_2} (-1)^{pq} a^{\bar{p}} \delta(b).$$

The R-bimodule of such derivations of degree $\bar{q} \in \mathbf{Z}_2$ is denoted by $Der_R^{\bar{q}}(A)$. The direct sum $Der_R(A) := Der_R^{\bar{0}}(A) \oplus Der_R^{\bar{1}}(A)$ is some subalgebra of the super-commutator algebra $(End_R(A))_L$, i.e., $\forall \bar{q}, \bar{p} \in \mathbf{Z}_2$,

$$\forall \delta \in Der_R^{\bar{q}}(A), \gamma \in Der_R^{\bar{p}}(A) : [\delta, \gamma] = \delta \circ \gamma - (-1)^{pq} \gamma \circ \delta \in Der_R^{\bar{q}+\bar{p}}(A).$$

The elements of the Lie superalgebra $Der_R(A)$ are called super-derivations of A.

(2.1.4) Let $E = E^{\bar{0}} \oplus E^{\bar{1}}$ be the direct sum of two R-bimodules, and consider a homomorphism $\phi : A \longrightarrow End_R(E)$, in the sense of associative superalgebras over R; then ϕ is called representation of A. Similarly a homomorphism $\phi : L \longrightarrow (End_R(E))_L$, in the sense of Lie superalgebras over R, is called representation of L. Obviously the representation $\phi : A \longrightarrow End_R(E)$ induces an according representation of the super-commutator algebra $\phi : A_L \longrightarrow (End_R(E))_L$. An injective representation is called faithful.

(2.1.5) An R-linear mapping $\delta \in End_R^{\bar{q}}(L)$, of the Lie superalgebra L over R into itself, is called derivation of degree $\bar{q} \in \mathbf{Z}_2$, if and only if

$$\forall a, b \in L : \delta([a,b]) = [\delta(a), b] + \sum_{\bar{p} \in \mathbf{Z}_2} (-1)^{pq} [a^{\bar{p}}, \delta(b)].$$

The R-bimodule of such derivations, of degree $\bar{q} \in \mathbf{Z}_2$, is denoted by $Der_R^{\bar{q}}(L)$. The direct sum $Der_R(L) := Der_R^{\bar{0}}(L) \oplus Der_R^{\bar{1}}(L)$ is some subalgebra of the super-commutator algebra $(End_R(L))_L$, i.e., $\forall \bar{q}, \bar{p} \in \mathbf{Z}_2$,

$$\forall \delta \in Der_R^{\bar{q}}(L), \gamma \in Der_R^{\bar{p}}(L) : [\delta, \gamma] = \delta \circ \gamma - (-1)^{qp} \gamma \circ \delta \in Der_R^{\bar{q}+\bar{p}}(L).$$

The elements of the Lie superalgebra $Der_R(L)$ are called super-derivations of L.

(2.1.6) Let L be a Lie superalgebra over R. The homomorphism $ad_L \equiv ad : L \longrightarrow (End_R(L))_L$, in the sense of Lie superalgebras over R, is called adjoint representation of L:

$$L \ni a \underset{ad}{\longrightarrow} ad\ a : L \ni b \underset{def}{\longrightarrow} [a,b] \in L.$$

The R-bimodule $Im\ ad$ is some $\mathbf{Z}_2$-graded ideal of the Lie superalgebra $Der_R(L)$, because one finds

$$\forall \delta \in Der_R(L), \forall a, b \in L : [\delta, ad\ a](b) = [\delta(a), b].$$

The super-derivations $\delta \in Im\ ad$ are called inner, $\delta \in Der_R(L) \backslash Im\ ad$ are called outer ones.

(2.1.7) If an ideal D of an associative superalgebra A over R is $\mathbf{Z}_2$-graded, then the factor algebra A/D is an associative superalgebra over R, with the $\mathbf{Z}_2$-grading:

$$A/D = (A/D)^{\bar{0}} \oplus (A/D)^{\bar{1}},\ \forall a = a^{\bar{0}} + a^{\bar{1}} \in A :$$

$$a + D \in \begin{cases} (A/D)^{\bar{0}}, & \text{if and only if } a^{\bar{1}} \in D \\ (A/D)^{\bar{1}}, & \text{if and only if } a^{\bar{0}} \in D \end{cases}.$$

(2.1.8) Similarly the factorization of a Lie superalgebra with respect to a $\mathbf{Z}_2$-graded ideal yields again a Lie superalgebra over R.

(2.1.9) Let the Lie superalgebra L over R be non-trivial, i.e.,
$L \neq \{0\} \neq [L, L]$. Here the notation is used, $\forall$ subsets $A, B \subseteq L$:

$$[A, B] := R - lin\ span(\{[a, b]; \in A, b \in B\}).$$

Then L is called simple, if and only if there does not exist any $\mathbf{Z}_2$-graded ideal D of L, such that $\{0\} \neq D \neq L$; in this case L itself and $\{0\}$ are the only ideals of L.

(2.1.9.1) Let L be simple; then

$$[L, L] = L,\ [L^{\bar{0}}, L^{\bar{1}}] = L^{\bar{1}},\ L^{\bar{1}} \neq \{0\} \Longrightarrow [L^{\bar{1}}, L^{\bar{1}}] = L^{\bar{0}}.$$

Here one uses, that both $L^{\bar{0}} \oplus [L^{\bar{0}}, L^{\bar{1}}]$ and $[L^{\bar{1}}, L^{\bar{1}}] \oplus L^{\bar{1}}$ are $\mathbf{Z}_2$-graded ideals of L.

(2.1.10) The image of a representation $\phi : L \longrightarrow (End_R(E))_L$, of the Lie superalgebra L over R, is some subalgebra of $(End_R(E))_L$; if L is simple, then the Lie superalgebra $Im\ \phi$ is simple too.

(2.1.11) Let $\phi : L \longrightarrow L'$ be a homomorphism in the sense of Lie superalgebras over R. Obviously $Im\ \phi$ is some subalgebra of L'. Moreover, if D' is a $\mathbf{Z}_2$-graded ideal of L', then $\phi^{-1}(D')$ is some $\mathbf{Z}_2$-graded ideal of L.

(2.1.12) Consider the adjoint representation $ad : L \longrightarrow (End_R(L))_L$ of the Lie superalgebra L over R. The homomorphism of Lie algebras
$ad' : L^{\bar{0}} \longrightarrow (End_R(L^{\bar{1}}))_L$,

$$ad' : L^{\bar{0}} \ni a \longrightarrow ad'\ a : L^{\bar{1}} \ni b \underset{def}{\longrightarrow} [a, b] \in L^{\bar{1}},$$

is called adjoint representation of $L^{\bar{0}}$ on $L^{\bar{1}}$.

(2.1.13) If the Lie superalgebra L over R is simple, and if $L^{\bar{1}} \neq \{0\}$, then ad' is faithful, $ker\ ad' = \{a \in L^{\bar{0}}; \forall b \in L^{\bar{1}} : [a, b] = 0\} = \{0\}$. Moreover in this case one finds, that $\{a \in L; \forall b \in L^{\bar{1}} : [a, b] = 0\} = \{0\}$, because this set is some $\mathbf{Z}_2$-graded ideal of L.

(2.1.14) A representation ϕ of the Lie superalgebra L over R, on an R-bimodule $E = E^{\bar{0}} \oplus E^{\bar{1}}$, is called reducible, if and only if there is a ϕ-invariant $\mathbf{Z}_2$-graded R-submodule $\{0\} \neq F \subset E$, i.e., such that $\forall a \in L : \phi(a)(F) \subseteq F$. ϕ is called irreducible, if and only if it is not reducible.
An invariant $\mathbf{Z}_2$-graded R-submodule F of E is called irreducible with respect to ϕ, if and only if the correspondingly restricted representation of L on F is irreducible. ϕ is called completely reducible, if and only if E is the direct sum of irreducible $\mathbf{Z}_2$-graded R-submodules.

(2.1.15) Any two representations ϕ and ψ of the Lie superalgebra L , on R-bimodules E and F respectively, are called equivalent, if and only if there is an even R-linear bijection $\rho : E \longleftrightarrow F$, i.e., $\forall \bar{p} \in \mathbf{Z}_2 : \rho(E^{\bar{p}}) = F^{\bar{p}}$, such that $\forall a \in L : \psi(a) \circ \rho = \rho \circ \phi(a)$.

(2.1.16) Let ϕ and ψ be representations of L on E and F, respectively. Then the R-linear map $t(\phi, \psi)$, such that $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2$:

$$L^{\bar{p}} \ni a \longrightarrow t(\phi,\psi)(a) :$$

$$E^{\bar{q}} \otimes F \ni v \otimes w \longrightarrow \phi(a)(v) \otimes w + (-1)^{pq} v \otimes \psi(a)(w) \in E \otimes F,$$

is some representation of L on

$$E \otimes F = \bigoplus_{\bar{p} \in \mathbf{Z}_2} (E \otimes F)^{\bar{p}}, \ (E \otimes F)^{\bar{p}} := \bigoplus_{\bar{q},\bar{r} \in \mathbf{Z}_2, \bar{q}+\bar{r}=\bar{p}} (E^{\bar{q}} \otimes F^{\bar{r}});$$

it is called the tensor product of representations ϕ, ψ.

(2.1.17) Let ϕ be a representation of the Lie superalgebra L on an R-bimodule E.

$$E^* := Hom_R(E, R) = E^{*\bar{0}} \oplus E^{*\bar{1}},$$

$$\forall \bar{p} \in \mathbf{Z}_2 : E^{*\bar{p}} := \{f \in E^*; f\big|_{E^{\overline{p+1}}} = 0\}.$$

Then the representation ϕ^* of L on E^*, which is defined such that

$$\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall a \in L^{\bar{p}}, f \in E^{*\bar{q}} : \phi^*(a)(f) := (-1)^{1+pq} f \circ \phi(a),$$

is called contragredient to ϕ.

(2.1.17.1) Let especially R be a field. Then ϕ^* is faithful, if ϕ is faithful.

(2.1.17.2) Let E be a finite-dimensional vector space over the field $R \equiv K$, and use the K-linear bijection $\sigma_E : E \longleftrightarrow E^{**} := (E^*)^*$, such that

$$\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall v \in E^{\bar{p}}, g \in E^{*\bar{q}} : \sigma_E(v)(g) = (-1)^{pq} g(v).$$

Then $\forall a \in L : \phi^{**}(a) \circ \sigma_E = \sigma_E \circ \phi(a)$, denoting $\phi^{**} := (\phi^*)^*$.

(2.1.18) The graded version of Ado's theorem states, that every finite-dimensional Lie superalgebra over a field of characteristic zero admits a faithful finite-dimensional representation.

(2.1.19) Consider an irreducible representation ϕ of a Lie superalgebra L on a finite-dimensional vector space E over a field K. Then

$$C := \{\gamma \in End_K(E); \forall x \in L : [\phi(x), \gamma] = 0\} = C^{\bar{0}} \oplus C^{\bar{1}}$$

is some $\mathbf{Z}_2$-graded subalgebra of the unital associative superalgebra $End_K(E)$. The non-zero homogeneous elements of C are bijective. If K is algebraically closed, then the $\mathbf{Z}_2$-graded version of Schur's lemma holds, such that

$$C^{\bar{0}} = K(\{id\ E\});\ C^{\bar{1}} \neq 0 \Longrightarrow C^{\bar{1}} = K\{(\gamma_1)\}, \gamma_1^2 = -id\ E.$$

(2.1.20) Consider an R-bilinear form $\lambda : L \times L \longrightarrow R$, on the Lie superalgebra L over R. λ is called L-invariant, if and only if

$$\forall a, b, c \in L : \lambda([a, b], c) = \lambda(a, [b, c]).$$

Furthermore λ is called supersymmetric, if and only if

$$\forall a, b \in L : \lambda(a, b) = \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq} \lambda(b^{\bar{q}}, a^{\bar{p}}).$$

(2.1.20.1) Let L be simple, and assume λ to be L-invariant. Then λ is supersymmetric, and moreover λ is either non-degenerate or zero.

(2.1.21) Let the adjont representation ad' of $L^{\bar{0}}$ on $L^{\bar{1}}$ be faithful and irreducible, i.e., ad' is injective, and there is no R-submodule $\{0\} \neq M \subset L^{\bar{1}}$ such that $ad'(M) \subseteq M$. If moreover $[L^{\bar{1}}, L^{\bar{1}}] = L^{\bar{0}} \neq \{0\}$, then the Lie superalgebra L over R is simple.

2.2 Universal Enveloping Superalgebra

(2.2.1) Let L be a Lie superalgebra, and V a unital associative superalgebra, both over a commutative ring R. A homomorphism of Lie superalgebras $\nu : L \longrightarrow V_L$ is called universal enveloping superalgebra of L, if and only if it is a universal object in the following sense. If A is a unital associative algebra over R, and $\lambda : L \longrightarrow A$ an R-linear map such that

$$\forall x, y \in L : \lambda([x, y]) = \lambda(x)\lambda(y) - \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq} \lambda(y^{\bar{q}})\lambda(x^{\bar{p}}),$$

then there exists a unique homomorphism of unital associative R-algebras $\bar{\lambda} : V \longrightarrow A$, such that $\bar{\lambda} \circ \nu = \lambda$. One then denotes $V \equiv E(L)$.

(2.2.1.1) Especially, if $\lambda : L \longrightarrow A_L$ is a homomorphism in the sense of Lie superalgebras over R, into a unital associative superalgebra A, then $\bar{\lambda}$ is a homomorphism in the sense of unital associative superalgebras over R, i.e., $\bar{\lambda}$ preserves the $\mathbf{Z}_2$-grading.

(2.2.1.2) Homomorphisms of Lie superalgebras ϕ, ψ are lifted to homomorphisms ϕ_*, ψ_* in the sense of unital associative superalgebras over R, with the covariant functorial properties:

$$\psi_* \circ \phi_* = (\psi \circ \phi)_*, \ (id\ L)_* = id\ V.$$

If ϕ is bijective, then ϕ_* is bijective too, and one finds that $(\phi^{-1})_* = (\phi_*)^{-1}$.

(2.2.1.3) From this arrow-theoretic definition one concludes, that

$$V = R - alg\ span(Im\ \nu).$$

(2.2.1.4) If V' is another universal enveloping superalgebra of L, then there is some isomorphism of unital associative superalgebras over R: $V' \longleftrightarrow V$.

(2.2.2) An explicit construction of some universal enveloping superalgebra V of L is performed in the next diagram. As the direct sum of R-bimodules,

$$T \equiv T(L) := \bigoplus_{p \in \mathbf{N}_0} T^p(L).$$

$$S := \{\{0, -[x,y], x \otimes y - \sum_{\bar{p},\bar{q} \in \mathbf{Z}_2} (-1)^{pq} y^{\bar{q}} \otimes x^{\bar{p}}, 0, 0, \ldots\}; x, y \in L\},$$

$$D := sum(\{tst'; s \in S; t, t' \in T\}) \not\ni e_T := \{e_R, 0, 0, \ldots\}.$$

$$\begin{tikzcd}
L \arrow[r] \arrow[rrr, bend right, "\nu"', "def"] & T(L) \arrow[r] & T(L)/D =: V \arrow[r, leftrightarrow, "id"'] & V_L
\end{tikzcd}$$

(2.2.2.1) Especially one finds, that $\forall x \in L : \nu(x) \neq e_V$. Moreover the $\mathbf{Z}_2$-grading of V fulfills the following condition.

$$\begin{aligned}
&\forall_1^n k : x_k \in L^{\overline{p_k}} \\
&\Longrightarrow \nu(x_1) \cdots \nu(x_n) = \{0, \ldots, 0, x_1 \otimes \cdots \otimes x_n, 0, 0, \ldots\} + D \in V^{\bar{p}}, \\
&\bar{p} := \overline{p_1 + \cdots + p_n}.
\end{aligned}$$

(2.2.3) Henceforth take coefficients from a field K of *char* $K \neq 2$.
The $\mathbf{Z}_2$-graded version of the Poincaré - Birkhoff - Witt theorem reads:
Let $\{b_i; i \in I\}$ be any basis of the Lie superalgebra L over K, with an arbitrary totally ordered index set I. Then the cosets with respect to D of all the standard monomials, i.e., the set

$$\begin{aligned}&\{e_V = \{e_K, 0, 0, \ldots\} + D \in V^{\bar{0}}, \nu(b_i) = \{0, b_i, 0, 0, \ldots\} + D,\\ &\nu(b_{i_1}) \cdots \nu(b_{i_p}) = \{0, \ldots, 0, b_{i_1} \otimes \cdots \otimes b_{i_p}, 0, 0, \ldots\} + D;\\ &i, i_1, \ldots, i_p \in I; i_1 \leq \ldots \leq i_p;\\ &\{b_{i_r}, b_{i_{r+1}}\} \subseteq L^{\bar{1}} \Longrightarrow i_r < i_{r+1}, r = 1, \ldots, p-1; p = 2, 3, \ldots\},\end{aligned}$$

is an appropriate basis of the vector space V over the field K.
Especially here one must take care of the fact that

$$\forall x \in L^{\bar{1}} : -[x, x] + (e_K + e_K)(x \otimes x) \in S \subseteq D,$$

with the canonical embedding being suppressed for convenience.

(2.2.4) This theorem implies, that the universal mapping ν is injective, hence taken as an inclusion, such that $V = K - alg\ span(L)$.

(2.2.5) Consider the finite-dimensional case.
If $\{b_1, \ldots, b_m\}$ is a K-basis of $L^{\bar{0}}$, and $\{c_1, \ldots, c_n\}$ is a K-basis of $L^{\bar{1}}$, then

$$\{b_1^{q_1} \cdots b_m^{q_m}, b_1^{q_1} \cdots b_m^{q_m} c_{p_1} \cdots c_{p_r}; 1 \leq p_1 < \ldots < p_r \leq n; q_1, \ldots, q_m \in \mathbf{N}_0\}$$

is some countable K-basis of V, denoting $\forall x \in L^{\bar{0}} : x^0 := e_V$.
Here the notation was simplified, due to the inclusion of L into V.

(2.2.6) The diagonal mapping

$$\delta : L \ni x \underset{def}{\longrightarrow} \nu(x) \otimes e_V + e_V \otimes \nu(x) \in (V \hat{\otimes} V)_L,$$

into the super-commutator algebra of the unital associative superalgebra $V \hat{\otimes} V$, with the skew-symmetric tensor product of two unital associative superalgebras, induces a unique homomorphism $\bar{\delta}$ in the sense of unital associative superalgebras over K, according to the diagram below.

$$\begin{tikzcd} L \ni x \arrow[rr, "\nu"'] \arrow[dr, "\delta"'] & & V \arrow[dl, "\bar{\delta}"] \\ & \delta(x) \in V \hat{\otimes} V & \end{tikzcd}$$

Here $\bar{\delta} =: \Delta$ is called the diagonal homomorphism of the universal enveloping superalgebra V of L. Especially one must convince oneself, that $\forall x, y \in L$:

$$\begin{aligned}&\nu([x,y]) \otimes e_V + e_V \otimes \nu([x,y]) \\ &= [\nu(x) \otimes e_V + e_V \otimes \nu(x), \nu(y) \otimes e_V + e_V \otimes \nu(y)],\end{aligned}$$

inserting the super-commutator of the skew-symmetric tensor product of unital associative superalgebras, which follows from the fact that

$$\nu([x,y]) = \{0, 0, x \otimes y - (-1)^{pq} y \otimes x, 0, 0, \ldots\} + D.$$

(2.2.7) Every representation $\phi : L \longrightarrow (End_K(E))_L$ of the Lie superalgebra L, on the direct sum of vector spaces $E = E^{\bar{0}} \oplus E^{\bar{1}}$ over the field K, induces an according representation $\bar{\phi} : V \longrightarrow End_K(E)$ of the universal enveloping superalgebra V of L, such that $\bar{\phi} \circ \nu = \phi$, with the universal mapping $\nu : L \longrightarrow V$.

(2.2.7.1) The previously defined tensor product of representations ϕ, ψ of L is just the composition $t(\phi, \psi) = T(\phi, \psi) \circ \delta$, inserting the skew-symmetric tensor product of K-linear mappings.

(2.2.8) Let L_1, L_2 be Lie superalgebras over the field K. Their direct sum $L_1 \oplus L_2 =: L$, in the sense of vector spaces over K, is equipped with the following $\mathbf{Z}_2$-grading.

$$\forall \bar{p} \in \mathbf{Z}_2, \forall x_1 \in L_1, x_2 \in L_2 : \{x_1, x_2\} \in L^{\bar{p}} \underset{def}{\Longleftrightarrow} \forall_1^2 k : x_k \in L_k^{\bar{p}}.$$

With the K-bilinear mapping, such that

$$\forall_1^2 k, \forall x_k, y_k \in L_k : [\{x_1, x_2\}, \{y_1, y_2\}] := \{[x_1, y_1], [x_2, y_2]\},$$

L becomes some Lie superalgebra over K. Both the canonical embeddings and projections are homomorphisms of Lie superalgebras over K.
The corresponding universal enveloping superalgebras are constructed as inclusions.

$$\text{For } k = 1, 2, \ \nu_k : L_k \longrightarrow V_k; \ \nu : L_1 \oplus L_2 =: L \longrightarrow V.$$

Then one finds an isomorphism: $V_1 \hat{\otimes} V_2 \longleftrightarrow V$ of unital associative superalgebras over K. The proof of this theorem is presented.

(2.2.8.1) Let $\lambda : L \longrightarrow A$ be any K-linear mapping into a unital associative algebra A, such that

$$\forall x, y \in L : [x, y] \xrightarrow[\lambda]{} \lambda(x)\lambda(y) - \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq} \lambda(y^{\bar{q}})\lambda(x^{\bar{p}}) \in A.$$

Then there is a unique $\bar{\lambda} : V \longrightarrow A$, such that $\bar{\lambda} \circ \nu = \lambda$. With the canonical embeddings $\beta_k : L_k \longrightarrow L$ one defines, for $k = 1, 2, \lambda_k := \lambda \circ \beta_k : L_k \longrightarrow A$; obviously λ_k again fulfills the condition, which serves for the existence of a unique $\overline{\lambda_k} : V_k \longrightarrow A$, such that $\overline{\lambda_k} \circ \nu_k = \lambda_k$.
The homomorphisms of unital associative algebras $\bar{\lambda}, \overline{\lambda_k}, k = 1, 2$, are used for the construction of a universal enveloping superalgebra $(V_1 \hat{\otimes} V_2)_L$ for $L_1 \oplus L_2$.

(2.2.8.2) $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall x_1 \in L_1^{\bar{p}}, x_2 \in L_2^{\bar{q}}$:

$$\lambda([\{x_1, 0\}, \{0, x_2\}]) = \lambda_1(x_1)\lambda_2(x_2) - (-1)^{pq}\lambda_2(x_2)\lambda_1(x_1) = 0.$$

Hence one obtains

$$\forall v_1 \in V_1, v_2 \in V_2 : \overline{\lambda_1}(v_1)\overline{\lambda_2}(v_2) = \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq}\overline{\lambda_2}(v_2^{\bar{q}})\overline{\lambda_1}(v_1^{\bar{p}}) \in A.$$

(2.2.8.3) The universal property of the tensor product of vector spaces over K is used in the next diagram.

$$\begin{tikzcd}
V_1 \times V_2 \ni \{v_1, v_2\} \arrow[rr] \arrow[dr, "\mu"'] & & v_1 \otimes v_2 \in V_1 \hat{\otimes} V_2 \arrow[dl, "\mu'"] \\
& \bar{\lambda}_1(v_1)\,\bar{\lambda}_2(v_2) \in A &
\end{tikzcd}$$

In this special case, μ' is some homomorphism of unital associative algebras.

(2.2.8.4) The K-linear mapping ν defined below fulfills $\mu' \circ \nu = \lambda$.

$$\nu : L := L_1 \oplus L_2 \ni \{x_1, x_2\} \underset{def}{\longrightarrow} x_1 \otimes e_{V_2} + e_{V_1} \otimes x_2 \in (V_1 \hat{\otimes} V_2)_L.$$

Moreover ν is some homomorphism of Lie superalgebras. Because of $V_1 \hat{\otimes} V_2 = K - alg\ span(\ Im\ \nu)$, ν is some universal enveloping superalgebra of L; hence the theorem is proved.

(2.2.9) An R-linear mapping $\phi : A \longrightarrow A'$ of unital associative superalgebras, over a commutative ring R, is called antihomomorphism of unital associative superalgebras, if and only if one finds that

(i) $\forall \bar{p} \in \mathbf{Z}_2 : Im\ \phi\big|_{A^{\bar{p}}} \subseteq A'^{\bar{p}}$; (ii) $\phi(e_A) = e_{A'}$;

(iii) $\forall a, b \in A : \phi(ab) = \sum_{\bar{p}, \bar{q} \in \mathbf{Z}_2} (-1)^{pq}\phi(b^{\bar{q}})\phi(a^{\bar{p}})$.

(2.2.9.1) Let V be a universal enveloping superalgebra of the Lie superalgebra L over the field K, *char* $K \neq 2$. Then there exists a unique antihomomorphism of unital associative superalgebras $\sigma : V \longleftrightarrow V$, such that $\sigma \circ \nu = -\nu, \sigma^2 = id\ V$.

2.3 Free Lie Superalgebras

(2.3.1) The free algebra $T(K(S))$ over a set S, with the coefficients from a field K of char $K \neq 2$, is constructed as tensor algebra over the vector space $K(S)$. Due to any disjoint union of sets $S = S^{\bar{0}} \cup S^{\bar{1}}$, and accordingly the direct sum of vector spaces $K(S) = K(S^{\bar{0}}) \oplus K(S^{\bar{1}})$, the above free algebra is equipped with a natural $\mathbf{Z}_2$-grading. Hence one obtains the unital associative superalgebra $T(K(S)) =: T$, with the corresponding super-commutator algebra T_L.

(2.3.2) The Lie superalgebra of multiple super-commutators of elements of S,

$$\begin{aligned} L(S) := K - lin\ span(S \cup \{[s_1, s_2]; s_1, s_2 \in S\} \\ \cup \{multiple\ super - commutators\}), \end{aligned}$$

which is some subalgebra of T_L, is a universal object in the following sense: Let λ be an even map from S into any Lie superalgebra L over K, i.e., $\forall \bar{z} \in \mathbf{Z}_2, \forall s \in S^{\bar{z}} : \lambda(s) \in L^{\bar{z}}$. Then there exists a unique homomorphism of Lie superalgebras $\tilde{\lambda} : L(S) \longrightarrow L$, such that $\tilde{\lambda}|_S = \lambda$. Here the canonical embedding of $K(S)$ into $T(K(S))$ has been suppressed for convenience. $L(S)$ is called the free Lie superalgebra over the set S, with the coefficients $\in K$.

(2.3.3) Moreover $T(K(S))$ is the universal enveloping superalgebra of $L(S)$. The easy proofs of these statements are similar to those for free Lie algebras over a field.

(2.3.4) Of course the above defined homomorphism of Lie superalgebras $\tilde{\lambda}$ can be lifted to an according homomorphism $\hat{\lambda}$ of unital associative superalgebras over K, with the universal enveloping superalgebra V of L, as is shown in the next diagram.

$$\begin{tikzcd} S \arrow[r] \arrow[drr, "\lambda"'] & K(S) \arrow[r] & L(S) \arrow[r, "envelope"'] \arrow[d, "\tilde{\lambda}"] & T(K(S)) \arrow[d, "\hat{\lambda}"] \\ & & L \arrow[r, "envelope"'] & V \end{tikzcd}$$

(2.3.5) Assume *char* $K = 0$, and let the set S be finite. Denote by $\delta : T \longrightarrow T \otimes T$ the unique homomorphism of unital associative superalgebras over K, such that $\forall s \in S : \delta(s) = s \otimes e_T + e_T \otimes s$, with the unit e_T of T. Then

$$L(S) = \{x \in T(K(S)); \delta(x) = x \otimes e_T + e_T \otimes x\},$$

due to the $\mathbf{Z}_2$-graded version of the Poincaré-Birkhoff-Witt theorem.

2.4 Classification of Classical Lie Superalgebras

Let L be a finite-dimensional complex Lie superalgebra.

(2.4.1) Let L be simple, and

$$L^{\bar{1}} = L_1^{\bar{1}} + L_2^{\bar{1}},\ \{0\} \neq L_k^{\bar{1}} \subset L^{\bar{1}},\ [L^{\bar{0}}, L_k^{\bar{1}}] \subseteq L_k^{\bar{1}},\ k = 1, 2.$$

Then

$$L_1^{\bar{1}} \cap L_2^{\bar{1}} = \{0\},\ [L_1^{\bar{1}}, L_2^{\bar{1}}] = L^{\bar{0}},\ [L_k^{\bar{1}}, L_k^{\bar{1}}] = \{0\},$$

and the adjoint representations of $L^{\bar{0}}$ on $L_k^{\bar{1}}$ are irreducible, for $k = 1, 2$.

(2.4.2) A simple Lie superalgebra L is called classical, if and only if the adjoint representation ad' of $L^{\bar{0}}$ on $L^{\bar{1}} \neq \{0\}$ is completely reducible.

(2.4.3) A simple Lie superalgebra L with non-trivial odd component is classical, if and only if $L^{\bar{0}}$ is reductive.

(2.4.4) A complex-bilinear form $\lambda : L \times L \longrightarrow \mathbf{C}$ is called even, if and only if

$$\forall \bar{p} \in \mathbf{Z}_2, \forall a \in L^{\bar{p}}, b \in L^{\overline{1+p}} : \lambda(a, b) = 0.$$

(2.4.5) Let the complex vector space $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be finite-dimensional. The super-commutator algebra $(End_{\mathbf{C}}(V))_L =: gl(V)$ is equipped with the supertrace, which is defined as the complex-linear mapping

$$str : gl(V) \ni a \longrightarrow str\ a := tr(\gamma \circ a) \in \mathbf{C};$$

here $\gamma = \gamma^{-1} : V \longleftrightarrow V$ denotes the complex-linear bijection:

$$V \ni v^{\bar{0}} + v^{\bar{1}} \longleftrightarrow v^{\bar{0}} - v^{\bar{1}} \in V.\ \forall a, b \in gl(V) : str[a, b] = 0.$$

(2.4.6) The Killing form κ of L is defined as the complex-linear mapping:

$$L \times L \ni \{a, b\} \longrightarrow \kappa(a, b) := str(ad\ a \circ ad\ b) = tr(\gamma \circ ad\ a \circ ad\ b) \in \mathbf{C},$$

with the complex-linear bijection $\gamma : L \ni a^{\bar{0}} + a^{\bar{1}} \longleftrightarrow a^{\bar{0}} - a^{\bar{1}} \in L$.
The Killing form κ of L is supersymmetric, L-invariant, and even.

(2.4.6.1) If D is a graded ideal of L, then $\kappa|_{D\times D}$ is the Killing form of D.

(2.4.7) Let D_1, D_2 be graded ideals of L.

$$D_1 \cap D_2 = \{0\} \Longrightarrow [D_1, D_2] = \{0\} \Longrightarrow \kappa|_{D_1\times D_2} = 0.$$

(2.4.8) The Killing form κ of a simple Lie superalgebra L is either non-degenerate or zero; if κ is non-degenerate, then L is classical. In this case, the adjoint representation ad is injective: $L \longleftrightarrow ad(L) = Der_{\mathbf{C}}(L)$.

(2.4.9) Consider a classical Lie superalgebra L with non-trivial odd component; L is called basic classical, if and only if there is an even non-degenerate L-invariant supersymmetric complex-bilinear form $\lambda : L \times L \longrightarrow \mathbf{C}$.

(2.4.10) Let L be classical, and assume the centre $Z(L^{\bar{0}}) \neq \{0\}$. Then $dim\ Z(L^{\bar{0}}) = 1$, and $L^{\bar{1}} = L_1^{\bar{1}} \oplus L_2^{\bar{1}}$ as the direct sum of two irreducible representations of $L^{\bar{0}}$, $[L^{\bar{0}}, L_k^{\bar{1}}] \subseteq L_k^{\bar{1}} \neq \{0\}$ for $k = 1, 2$. Moreover $\exists$ unique $z \in Z(L^{\bar{0}})$, such that $[z, x] = (-1)^k x$ for $x \in L_k^{\bar{1}}$, $k = 1, 2$. Hence the Killing form κ is non-degenerate, because $\kappa(z, z) = -dim\ L^{\bar{1}}$.

(2.4.11) The Killing form κ of L is non-degenerate, if and only if $L = \bigoplus_{k=1}^{p} L_k$ such that the following conditions hold.

(i) $\forall_1^p k \neq l : [L_k, L_l] = \{0\},\ [L_k, L_k] = L_k$, each L_k being a $\mathbf{Z}_2$-graded ideal of L.

(ii) $\forall_1^p k : L_k$ is either even and simple, or classical with the non-degenerate Killing form $\kappa|_{L_k\times L_k}$.

(2.4.11.1) If κ is non-degenerate, then $L^{\bar{0}}$ is reductive, and then the adjoint representation ad' of $L^{\bar{0}}$ on $L^{\bar{1}}$ is completely reducible.

(2.4.12) Let L be basic classical, with an even non-degenerate L-invariant supersymmetric $\mathbf{C}$-bilinear form $\lambda : L \times L \longrightarrow \mathbf{C}$. Consider the Cartan decomposition of the reductive finite-dimensional complex Lie algebra $L^{\bar{0}} = Z(L^{\bar{0}}) \oplus [L^{\bar{0}}, L^{\bar{0}}]$, with respect to a maximal toral subalgebra T of $[L^{\bar{0}}, L^{\bar{0}}]$.

$$L^{\bar{0}} = Z(L^{\bar{0}}) \oplus T \oplus \bigoplus_{\alpha\in\Phi}[L^{\bar{0}}, L^{\bar{0}}]_\alpha = H \oplus \bigoplus_{\beta\in\Psi^0}(L^{\bar{0}})_\beta,\ \ (L^{\bar{0}})_0 = H,$$

with the Cartan subalgebra $H := Z(L^{\bar{0}}) \oplus T = C_{L^{\bar{0}}}(H)$ of $L^{\bar{0}}$.
Here $\Psi^{\bar{0}}$ denotes the root system of $L^{\bar{0}}$ with respect to H, which corresponds to the root system Φ of $[L^{\bar{0}}, L^{\bar{0}}]$ with respect to T.

$$\Psi^{\bar{0}} := \{\beta \in H^*; \beta|_{Z(L^{\bar{0}})} = 0, \beta|_T \in \Phi\}.$$

The rank of L is defined by $r := dim\ H$.

(2.4.12.1)

$$\begin{aligned}\forall \beta \in H^* : L_\beta &:= \{x \in L; \forall h \in H : [h, x] = \beta(h)x\} \\ &= L_\beta^{\bar{0}} \oplus L_\beta^{\bar{1}},\ L_\beta^{\bar{0}} := L_\beta \cap L^{\bar{0}} = (L^{\bar{0}})_\beta,\ L_\beta^{\bar{1}} := L_\beta \cap L^{\bar{1}};\end{aligned}$$

each L_β is some $\mathbf{Z}_2$-graded complex-linear subspace of L.

$$\Psi^{\bar{0}} = \left\{\beta \in H^*; \beta \neq 0, L_\beta^{\bar{0}} \neq \{0\}\right\},\ \Psi^{\bar{1}} := \left\{\beta \in H^*; L_\beta^{\bar{1}} \neq \{0\}\right\},$$

$$\Psi := \Psi^{\bar{0}} \cup \Psi^{\bar{1}},\ \Psi \cap \{0\} = \emptyset.$$

(2.4.12.2)

$$L = \bigoplus_{\beta \in H^*} L_\beta = H \oplus \left(\bigoplus_{\beta \in \Psi^{\bar{0}}} L_\beta^{\bar{0}}\right) \oplus \left(\bigoplus_{\beta \in \Psi^{\bar{1}}} L_\beta^{\bar{1}}\right),\quad H = L_0^{\bar{0}}.$$

Here one uses that every element of $ad_L\ (H)$ is diagonalizable, because ad' is completely reducible.

(2.4.12.3)

$$\begin{aligned}\forall \beta_1, \beta_2 \in H^* : [L_{\beta_1}, L_{\beta_2}] &\subseteq L_{\beta_1+\beta_2}, \\ \beta_1 + \beta_2 \neq 0 &\Longrightarrow \lambda|_{L_{\beta_1} \times L_{\beta_2}} = 0.\end{aligned}$$

The restriction $\lambda|_{H \times H}$ is non-degenerate. Moreover the restriction $\lambda|_{L_\beta \times L_{-\beta}}$ is non-degenerate, for all $\beta \in \Psi$.

(2.4.12.4) One uses the complex-linear bijection:

$$H \ni h \longleftrightarrow \beta_h \in H^*,\ H \ni f \underset{def}{\overset{\beta_h}{\longleftrightarrow}} \lambda(f, h) \in \mathbf{C};$$

conversely:

$$H^* \ni \beta \longleftrightarrow t_\beta \in H,\ \forall f \in H : \beta(f) = \lambda(f, t_\beta).$$

(2.4.12.5) The root system Ψ of L with respect to H fulfills the following conditions. Take any roots $\alpha, \beta, \beta_1, \beta_2 \in \Psi$, and assume $\beta_1 + \beta_2 \neq 0$.

$$\forall x_1 \in L_{\beta_1}, x_2 \in L_{\beta_2} : \lambda(x_1, x_2) = 0.$$
$$\forall x \in L_\beta, y \in L_{-\beta} : [x, y] = \lambda(x, y) t_\beta.$$
$$\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall \alpha \in \Psi^{\bar{p}}, \beta \in \Psi^{\bar{q}} : \alpha + \beta \in \Psi^{\overline{p+q}} \Longrightarrow [L^{\bar{p}}_\alpha, L^{\bar{q}}_\beta] = L^{\overline{p+q}}_{\alpha+\beta}.$$
$$\beta_1 + \beta_2 \in \Psi \Longleftrightarrow [L_{\beta_1}, L_{\beta_2}] \neq \{0\} \Longrightarrow [L_{\beta_1}, L_{\beta_2}] = L_{\beta_1+\beta_2},$$
if and only if $L \neq A(2,2)$.
$$\beta \in \Psi^{\bar{0}} \Longleftrightarrow -\beta \in \Psi^{\bar{0}},\ \beta \in \Psi^{\bar{1}} \Longleftrightarrow -\beta \in \Psi^{\bar{1}},\ [L_\beta, L_{-\beta}] = H_\beta := \mathbf{C}(\{t_\beta\}).$$
Let $\beta \neq 0$;
$$\{n \in \mathbf{Z}; n\beta \in \Psi\} = \{\pm 1, \pm 2\} \Longleftrightarrow \beta \in \Psi^{\bar{1}} \ et\ \lambda(t_\beta, t_\beta) \neq 0;$$
$$\{n \in \mathbf{Z}; n\beta \in \Psi\} = \{+1, -1\} \Longleftrightarrow \beta \notin \Psi^{\bar{1}} \ vel\ \lambda(t_\beta, t_\beta) = 0.$$
$$\forall \beta \in \Psi^{\bar{0}} : L^{\bar{0}}_\beta = [L^{\bar{0}}, L^{\bar{0}}]_{\beta|_T},\quad dim\ L^{\bar{0}}_\beta = 1.$$
$$\forall \beta \in \Psi^{\bar{1}} : 2\beta \in \Psi^{\bar{0}} \Longleftrightarrow \lambda(t_\beta, t_\beta) \neq 0.$$
$$\forall \beta \in \Psi^{\bar{1}} : dim\ L^{\bar{1}}_\beta = 1, \text{ if and only if } L \neq A(1,1);$$
$$\forall \beta \in \Psi^{\bar{1}} : dim\ L^{\bar{1}}_\beta = 2, \text{ if and only if } L := A(1,1).$$

The elements of $\Psi^{\bar{0}}$ and $\Psi^{\bar{1}}$ are called even and odd roots of L with respect to H.

(2.4.13) The general and special linear Lie superalgebras are defined as the following finite-dimensional complex Lie superalgebras. Let $m, n \in \mathbf{N}_0$, $m + n \geq 1$, be the dimensions of the even and odd subspace of $\mathbf{C}^{m+n}$.

$$gl(m, n, \mathbf{C}) \equiv gl(m, n) := (End_{\mathbf{C}}(\mathbf{C}^{m+n}))_L \equiv (Mat(m, n, \mathbf{C}))_L.$$
$$sl(m, n, \mathbf{C}) \equiv sl(m, n) := \{x \in gl(m, n); str\ x = 0\}.$$

(2.4.13.1)

$$\forall m, n \in \mathbf{N}_0, m \neq n : A(m, n) := sl(m+1,\ n+1),$$
$$dim\ A(m, n) = (m + n + 2)^2 - 1.$$

$$\forall n \in \mathbf{N} : A(n, n) := sl(n+1,\ n+1)/\{\lambda I_{2n+2};\ \lambda \in \mathbf{C}\},$$
$$dim\ A(n, n) = 4(n+1)^2 - 2.$$

Here $sl(m, n)$ is some $\mathbf{Z}_2$-graded ideal of $gl(m, n)$ of codimension 1, for $m + n \geq 1$.

(2.4.13.2) $\forall m, n \in \mathbf{N}_0,\ m+n \geq 1:$

$$\mathbf{C}^{m+n} \times \mathbf{C}^{m+n} \ni \{v, w\} \to < v|w > := \sum_{k=1}^{m+n} v_k w_k = < w|v > \in \mathbf{C}.$$

Let $m \in \mathbf{N},\ n := 2r,\ r \in \mathbf{N}.$

$$W := \begin{bmatrix} I_m & 0 \\ 0 & V_n \end{bmatrix},\ V_n := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix},$$

$$\mathbf{C}^{m+n} \times \mathbf{C}^{m+n} \ni \{v, w\} \xrightarrow[def]{\omega} < v|Ww > \in \mathbf{C}.$$

This complex-bilinear form ω is non-degenerate. Moreover obviously the restrictions of ω onto $\mathbf{C}^m \times \mathbf{C}^m$ and $\mathbf{C}^n \times \mathbf{C}^n$ are symmetric and skew-symmetric, respectively, and

$$\omega|_{\mathbf{C}^m \times \mathbf{C}^n} = \omega|_{\mathbf{C}^n \times \mathbf{C}^m} = 0.$$

The orthosymplectic Lie superalgebra is defined by

$$osp(m, n, \mathbf{C}) \equiv osp(m, n) := \bigoplus_{\bar{p} \in \mathbf{Z}_2} (osp(m, n))^{\bar{p}},$$

$$(osp(m, n))^{\bar{p}} := \Big\{ x \in (gl(m, n))^{\bar{p}};\ \forall \bar{q} \in \mathbf{Z}_2, \forall v \in \mathbf{V}^{\bar{q}}, w \in \mathbf{V}:$$
$$\omega(xv, w) = (-1)^{1+pq} \omega(v, xw) \Big\}.$$

Here $\mathbf{V}^{\bar{0}} := \mathbf{C}^m, \mathbf{V}^{\bar{1}} := \mathbf{C}^n, \mathbf{V} := \mathbf{V}^{\bar{0}} \oplus \mathbf{V}^{\bar{1}} = \mathbf{C}^{m+n}.$

With an appropriate block notation of $osp(m, n) \ni x =: \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the above condition reads $A^t = -A,\ B^t = V_n C,\ D^t V_n = -V_n D$, inserting the transposition of complex matrices. Hence one finds that $osp(m, n) \subset sl(m, n)$.

(2.4.13.3)

$$\forall m \in \mathbf{N}_0,\ n \in \mathbf{N}: B(m, n) := osp(2m+1,\ 2n),$$
$$dim\ B(m, n) = 2(m+n)^2 + m + 3n.$$

$$\forall m \geq 2,\ n \geq 1: D(m, n) := osp(2m,\ 2n),$$
$$dim\ D(m, n) = 2(m+n)^2 - m + n.$$

$$\forall n \geq 2: C(n) := osp(2,\ 2n-2),\ dim\ C(n) = 2n^2 + n - 2.$$

Obviously $\{osp(m, n); m \in \mathbf{N}, n \in 2\mathbf{N}\}$ is the disjoint union of these three sets.

(2.4.13.4)

$$(osp(m,n))^{\bar 0} \longleftrightarrow o(m) \oplus sp(n),$$

$$o(m) \equiv o(\mathbf{C}^m) = \{x \in gl(m,\mathbf{C});\ x^t = -x\},$$
$$sp(n) \equiv sp(\mathbf{C}^n) \equiv C_{n/2},$$

as an isomorphism of complex Lie algebras. $dim(osp(m,n))^{\bar 1} = mn$.

(2.4.13.5)

$$(C(n))^{\bar 0} = \left\{ \left[\begin{array}{cc|cc} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right];\ \beta \in \mathbf{C}; a,\ldots,d \in Mat(n-1,\mathbf{C}); \right.$$
$$\left. a^t = -d, b^t = b, c^t = c \right\},$$

$$(C(n))^{\bar 1} = \left\{ \left[\begin{array}{cc|cc} 0 & 0 & a & b \\ 0 & 0 & c & d \\ \hline -b^t & -d^t & 0 & 0 \\ a^t & c^t & 0 & 0 \end{array}\right];\ a^t,\ldots,d^t \in \mathbf{C}^{n-1} \right\}.$$

(2.4.13.6) $\forall n \geq 2$:

$$Q(n) := \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in sl(n+1,n+1);\ tr\ b = 0 \right\} / \mathbf{C}(\{I_{2n+2}\}),$$

$$P(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in gl(n+1,n+1);\ a^t = -d, b = b^t, c = -c^t, tr\ a = 0 \right\}.$$

Obviously $P(n) \subset sl\ (n+1,n+1)$. $dim\ P(n) = 2(n+1)^2 - 1 = dim\ Q(n) + 1$.

(2.4.14) The following family of 17-dimensional complex Lie superalgebras is constructed. Take the semisimple complex Lie algebra $L^{\bar 0} := A_1 \oplus A_1 \oplus A_1$, with the simple ideals $A_1 \equiv sl(2,\mathbf{C}) = sp(2,\mathbf{C})$ of $L^{\bar 0}$, and consider the representation ϕ of $L^{\bar 0}$ on $\mathbf{V} := \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$:

$$L^{\bar 0} \ni a = a^1 \oplus a^2 \oplus a^3 \longrightarrow \phi(a) \in (End_{\mathbf{C}}(\mathbf{V}))_L,$$

$$\mathbf{V} \ni x^1 \otimes x^2 \otimes x^3$$
$$\underset{\phi(a)}{\longrightarrow} a^1x^1 \otimes x^2 \otimes x^3 + x^1 \otimes a^2x^2 \otimes x^3 + x^1 \otimes x^2 \otimes a^3x^3 \in \mathbf{V}.$$

The complex-bilinear mapping π_1 :

$$\mathbf{C}^2 \times \mathbf{C}^2 \ni \{x,y\} \underset{def}{\longrightarrow} \left[\begin{array}{c|c} -x_1y_2 - x_2y_1 & 2x_1y_1 \\ \hline -2x_2y_2 & x_1y_2 + x_2y_1 \end{array}\right] \in A_1$$

fulfills $\forall x,y,z \in \mathbf{C}^2 : \pi_1(x,y)z = \sigma_1(y,z)x - \sigma_1(z,x)y$, hence

$$\forall x,y \in \mathbf{C}^2, \forall a \in A_1 : \pi_1(ax,y) + \pi_1(x,ay) = [a,\pi_1(x,y)] \in A_1.$$

Here σ_1 denotes the non-degenerate skew-symmetric complex-bilinear form:

$$\mathbf{C}^2 \times \mathbf{C}^2 \ni \{x,y\} \underset{def}{\longrightarrow} < x| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y >= x_1y_2 - x_2y_1 \in \mathbf{C}.$$

This A_1-invariance of π_1 implies an according $L^{\bar{0}}$-invariance of the complex-bilinear mapping $\beta : \mathbf{V} \times \mathbf{V} \longrightarrow L^{\bar{0}}$, which is defined such that $\forall x^1,\ldots,y^3 \in \mathbf{C}^2$:

$$\begin{aligned} \beta(x^1 \otimes \cdots \otimes x^3,\ y^1 \otimes \cdots \otimes y^3) = \lambda_1\pi_1(x^1,y^1)\sigma(x^2,y^2)\sigma(x^3,y^3) \\ + \ \textit{cyclic permutations} \ = \beta(y,x), \end{aligned}$$

with three complex parameters $\lambda_1,\ldots,\lambda_3$.

$$\forall x,y \in \mathbf{V}, \forall a \in L^{\bar{0}} : \beta(\phi(a)x,y) + \beta(x,\phi(a)y) = [a,\beta(x,y)].$$

From now on assume $\lambda_1+\lambda_2+\lambda_3 = 0$; then β fulfills the Jacobi-type condition, that

$$\forall x,y,z \in \mathbf{V} : (\phi \circ \beta(x,y))(z) + \ \textit{cyclic permutations} \ = 0.$$

With the complex-bilinear mappings, such that $\forall a \in L^{\bar{0}}, \forall x,y \in \mathbf{V} =: L^{\bar{1}}$:

$$[x,y] := \beta(x,y) = [y,x], \ \ [a,x] := \phi(a)(x) = -[x,a],$$

the 17-dimensional complex vector space $L := L^{\bar{0}} \oplus L^{\bar{1}}$ becomes some Lie superalgebra, because the condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$ implies the $\mathbf{Z}_2$-graded Jacobi identity. The adjoint representation of $L^{\bar{0}}$ on $L^{\bar{1}}$ is just

$$ad' = \phi = t(A_1, A_1, A_1),$$

as the tensor product of representations. If $\lambda_1 = 0$, then $\{0\} \oplus A_1 \oplus A_1 \oplus L^{\bar{1}}$ is some $\mathbf{Z}_2$-graded ideal of L. Hence L is simple, if and only if $\forall_1^3 k : \lambda_k \neq 0$, and in this case L is classical. Since for any $0 \neq c \in \mathbf{C}$, β and $c\beta$ yield isomorphic complex Lie superalgebras, one obtains the family of classical Lie superalgebras $\{L =: D(2,1;\mu); \mu \in \mathbf{C}\backslash\{0,-1\}\}$, according to the choice $\lambda_1 := 1$, and denoting $\lambda_2 \equiv \mu$. Actually every simple Lie superalgebra over $\mathbf{C}$, with the even component $\longleftrightarrow A_1 \oplus A_1 \oplus A_1$, and $ad' \longleftrightarrow t(A_1, A_1, A_1)$, is isomorphic with some member of the above family. For any choice of the three complex parameters such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the Killing form of L is zero.

(2.4.15) There is exactly one simple complex Lie superalgebra, denoted $G(3)$, such that the even component $\longleftrightarrow G_2 \oplus A_1$, and ad' is the tensor product of the irreducible representations of G_2 and A_1 on $\mathbf{C}^7$ and $\mathbf{C}^2$, respectively.

$$G_2 \subset B_3 := o(7, \mathbf{C}), \ dim\ G_2 = 14.$$

$$G_2 \oplus A_1 \ni g + a \underset{ad'}{\longrightarrow} t(g, a) \in End(\mathbf{C}^7 \otimes \mathbf{C}^2),$$

$$\mathbf{C}^7 \otimes \mathbf{C}^2 \ni x \otimes y \underset{t(g,a)}{\longrightarrow} gx \otimes y + x \otimes ay \in \mathbf{C}^7 \otimes \mathbf{C}^2,$$

suppressing conveniently an explicit notation of these irreducible representations.

$$dim\ \ G(3) = 31, \ G(3) \subset B(8, 7) := osp(17, 14).$$

(2.4.16) Furthermore there is exactly one simple complex Lie superalgebra, denoted $F(4)$, with the even component $\longleftrightarrow B_3 \oplus A_1$, and such that ad' is the tensor product of the irreducible representations of B_3 and A_1 on $\mathbf{C}^8$ and $\mathbf{C}^2$, respectively.

$$dim\ F(4) = 40, \ F(4) \subset D(12, 8) := osp(24, 16).$$

(2.4.17) The Killing forms of $sl(m, n)$ and $osp(m, n)$, for $m+n \geq 1$, are zero for special choices of m and $n \in \mathbf{N}_0$.

$$sl(m, n) \ni \{x, y\} \underset{\kappa}{\longrightarrow} 2(m - n) str(xy) \in \mathbf{C},$$

$$osp(m, n) \ni \{x, y\} \underset{\kappa}{\longrightarrow} (m - n - 2) str(xy) \in \mathbf{C}.$$

(2.4.18) In the cases, which are indicated in the table below as reducible, the adjoint representation ad' of $L^{\bar{0}}$ on $L^{\bar{1}}$ is not irreducible, but completely reducible to two irreducible representations; the latter are contragredient with respect to the corresponding non-degenerate $\mathbf{C}$-bilinear form, except for $P(n)$.

(2.4.19) The following isomorphisms of complex Lie superalgebras may be inserted into the table below.

$$\forall m \neq n \in \mathbf{N}_0 : A(m, n) \longleftrightarrow A(n, m), \ \ A(1, 0) \longleftrightarrow C(2).$$

$$\forall \mu \in \mathbf{C}\backslash\{0, -1\} : D(2, 1; \mu) \longleftrightarrow D(2, 1; -1 - \mu) \longleftrightarrow D(2, 1; \mu^{-1});$$

$$D(2, 1; 1) \longleftrightarrow D(2, 1).$$

(2.4.20) The classical Lie superalgebras with non-zero odd component are classified in the following table. Here κ denotes the Killing form, and r the rank of L. κ is either non-degenerate or equal to zero.

L	$L^{\bar{0}}$	$ad'(L^{\bar{0}})$	κ	r
$A(m,n), m \neq n,$ $m,n \in \mathbf{N}_0$	$A_m \oplus A_n \oplus \mathbf{C}$	reducible	$\neq 0$	$m+n+1$
$A(n,n), n \in \mathbf{N}$	$A_n \oplus A_n$	reducible	0	$2n$
$B(m,n),$ $m,n \in \mathbf{N}$	$B_m \oplus C_n$	irreducible	$\neq 0$	$m+n$
$B(0,n),\ n \in \mathbf{N}$	C_n	irreducible	$\neq 0$	n
$D(m,n),$ $m \geq 2,\ n \geq 1$	$D_m \oplus C_n$	irreducible		$m+n$
$C(n),\ n \geq 2$	$\mathbf{C} \oplus C_{n-1}$	reducible	$\neq 0$	n
$D(2,1;\nu),$ $\nu \in \mathbf{C}\backslash\{0,-1\}$	$A_1 \oplus A_1 \oplus A_1$	irreducible	0	3
$F(4)$	$B_3 \oplus A_1$	irreducible	$\neq 0$	4
$G(3)$	$G_2 \oplus A_1$	irreducible	$\neq 0$	3
$P(n),\ n \geq 2$	A_n	reducible	0	n
$Q(n),\ n \geq 2$	A_n	irreducible	0	n

For $D(m,n), m \geq 2, n \geq 1,\ \kappa = 0$ if and only if $m - n = 1$.

(2.4.20.1) Here the tensor product of representations of $L^{\bar{0}}$, yielding the adjoint representation of $L^{\bar{0}}$ on $L^{\bar{1}}$, is defined in the following natural manner.

$$L^{\bar{0}} = \bigoplus_{l=1}^{k} L_l^{\bar{0}} \ni x = x_1 + \cdots + x_k \underset{ad'}{\longrightarrow} \phi_1(x_1) \otimes \cdots \otimes \phi_k(x_k) \in \bigotimes_{l=1}^{k} \phi_l(L_l^{\bar{0}}),$$

$$L^{\bar{1}} \supseteq \bigotimes_{l=1}^{k} V_l \ni v_1 \otimes \cdots \otimes v_k$$

$$\underset{\phi_1(x_1)\otimes\cdots\otimes\phi_k(x_k)}{\longrightarrow} \sum_{l=1}^{k} v_1 \otimes \cdots \otimes \phi_l(x_l)(v_l) \otimes \cdots \otimes v_k \in L^{\bar{1}},$$

with representations $\phi_l, l = 1, \ldots, k$, of the ideals $L_l^{\bar{0}}$ of $L^{\bar{0}}$, on V_l. Inserting the projections $\pi_l : L^{\bar{0}} \to L_l^{\bar{0}}, l = 1, \ldots, k$, one finds

$$\forall x \in L : \phi_1(x_1) \otimes \cdots \otimes \phi_k(x_k) = t(\phi_1 \circ \pi_1, \ldots, \phi_k \circ \pi_k)(x) \in End_{\mathbf{C}}(L^{\bar{1}}).$$

(2.4.20.2) The classical Lie superalgebra L with non-trivial odd component is basic classical, if and only if $L \notin \{P(n), Q(n); n \geq 2\}$.

(2.4.21) Let L be basic classical, with respect to an even non-degenerate L-invariant supersymmetric complex-bilinear form $\lambda : L \times L \longrightarrow \mathbf{C}$, and take $L \neq A(1,1)$. Note that $L_0 = H$, because $L_0^{\bar{1}} = \{0\}$, which just means that $0 \notin \Psi^{\bar{1}}$.

(2.4.21.1) The properties of the root system were listed previously. Remember that

$$\Psi^{\bar{0}} \cap \Psi^{\bar{1}} = \emptyset.$$

$$\forall 0 \neq \beta \in H^* : \beta \in \Psi := \Psi^{\bar{0}} \cup \ \Psi^{\bar{1}} \Longleftrightarrow L_\beta \neq \{0\} \Longrightarrow dim\ L_\beta = 1.$$

(2.4.21.2) There is some vector basis of L, such that the elements of $ad_L(H)$ are diagonal matrices. Therefore one can find an appropriate triangular decomposition of L, such that $B_+ \cap L^{\bar{0}}$ is some Borel subalgebra of $L^{\bar{0}}$, and

$$L = N_- \oplus H \oplus N_+, \quad H \oplus N_+ = B_+, \quad [H, N_\pm] \subseteq N_\pm,$$

with nilpotent subalgebras $N_\pm$.

$$\Psi_\pm := \{\alpha \in \Psi ; L_\alpha \cap N_\pm \neq \{0\}\} = \Psi_\pm^{\bar{0}} \cup \Psi_\pm^{\bar{1}},$$
$$\forall \bar{p} \in \mathbf{Z}_2 : \Psi_\pm^{\bar{p}} := \{\alpha \in \Psi^{\bar{p}} ; L_\alpha \cap N_\pm \neq \{0\}\}, \quad \Psi_+^{\bar{p}} \cup \Psi_-^{\bar{p}} = \Psi^{\bar{p}},$$
$$\Psi_+ \cap \Psi_- = \emptyset, \quad \Psi_+ \cup \Psi_- = \Psi.$$

The elements of $\Psi_\pm$ are called positive and negative roots, respectively.

(2.4.21.3) A positive root α is called simple, if and only if there do not exist any two positive roots α' and α'' such that $\alpha' + \alpha'' = \alpha$. The set $\Delta := \{\alpha_1, \dots \alpha_s\}$ of simple roots is called root basis of L with respect to B_+. Then H^* is the complex-linear span of Δ. These s simple roots are linearly independent, if and only if $L \notin \{A(n,n); n = 2, 3, \dots\}$, and then $s = r := dim\ H$. For $L := A(n,n), n = 2, 3, \dots, s = r + 1$.

(2.4.21.4) With respect to a root basis $\Delta = \{\alpha_k; k = 1, \dots, s\}$ of L, one can choose elements $x_k \in L_{\alpha_k}$, $y_k \in L_{-\alpha_k}, k = 1, \dots, s$, such that one obtains the following relations.

$$\forall_1^s k, l : [h_k,\ h_l] = 0, \quad [x_k, y_l] = \delta_{kl} h_l,$$
$$[h_k, x_l] = \Gamma_{kl}\ x_l, \quad [h_k, y_l] = -\Gamma_{kl}\ y_l.$$

Here the so-called Cartan matrix

$$\Gamma := [\Gamma_{kl} := \alpha_l(h_k); k, l = 1, \dots, s], \quad h_k := \lambda(x_k, y_k) t_{\alpha_k},$$

can be normalized such that $\Gamma_{kk} \in \{0, 2\}$. Moreover $\forall_1^{s-1} k$, if $\Gamma_{kk} = 0$, then the first non-zero element of $\{\Gamma_{k,k+1}, \dots, \Gamma_{ks}\}$ can be chosen as equal to 1. The components of Γ are integers, if and only if $L \notin \{D(2,1;\mu); \mu \in \mathbf{C}\backslash\mathbf{Z}\}$.

(2.4.21.5) The universal enveloping superalgebra of L is the $\mathbf{C}$ – *alg span* of the set $\{x_k, y_k; k = 1, \ldots, s\}$.

(2.4.21.6) The Lie superalgebra L is uniquely determined by a Cartan matrix, together with the subset $\tau \subseteq \{1, \ldots, s\}$ of indices such that

$$\Delta \cap \Psi_+^{\bar{1}} = \{\alpha_k \in \Delta; k \in \tau\}.$$

Then

$$\forall_1^s k \notin \tau : \{x_k, y_k\} \subset L^{\bar{0}}. \quad \forall_1^s k \in \tau : \{x_k, y_k\} \subset L^{\bar{1}}.$$

One can always choose card $\tau = 1$; this choice is used henceforth.

(2.4.21.7) L is the factor algebra of the free complex Lie superalgebra over the set $\{x_k, y_k, h_k; k = 1, \ldots, s\}$, with respect to the $\mathbf{Z}_2$-graded ideal of the relations above, together with the Serre-like relations just below, and additional relations for $s \geq 3$ and $L \notin \{B(0, n); n \in \mathbf{N}\}$.

$$\forall_1^s k \neq l : (ad\ x_k)^{1-\tilde{\Gamma}_{kl}}(x_l) = (ad\ y_k)^{1-\tilde{\Gamma}_{kl}}(y_l) = 0.$$

$$\forall k \in \tau : \Gamma_{kk} = 0 \Longrightarrow [x_k, x_k] = [y_k, y_k] = 0.$$

Here the modified Cartan matrix $\tilde{\Gamma} := [\tilde{\Gamma}_{kl};\ k, l = 1, \ldots, s]$ is obtained from Γ, substituting -1 for the positive components in the rows with the diagonal component 0. Then obviously

$$\forall_1^s k \neq l : \Gamma_{kk} = 0 \Longrightarrow [x_k, [x_k, x_l]] = [y_k, [y_k, y_l]] = 0.$$

(2.4.21.7.1) For $L := B(0, n), n \in \mathbf{N}$, no additional relations are needed.

(2.4.21.7.2) For $L := A(m, n)$ with $mn \geq 1$, choosing $\tau := \{m + 1\}$, the additional relations can be written as

$$[x_{m+1}, [x_m, [x_{m+1}, x_{m+2}]]] = [y_{m+1}, [y_m, [y_{m+1}, y_{m+2}]]] = 0.$$

(2.4.21.7.3) For the other basic classical Lie superalgebras with $s \geq 3$, occasionally even more complicated relations are needed, as was calculated explicitly by H. Yamane.

(2.4.21.8) Choose *card* $\tau = 1$, and $L \notin \{B(0, n); n \in \mathbf{N}\}$. Then

$$\forall_1^\tau k : \Gamma_{kk} = 0 \Longleftrightarrow k \in \tau.$$

(2.4.22) As an example consider $L := C(2) := \ osp(2, 2)$.

(2.4.22.1) $L^{\bar{0}} \longleftrightarrow \mathbf{C} \oplus C_1, C_1 \equiv sp(2, \mathbf{C})$, as an isomorphism of complex Lie algebras, with the ideals $\mathbf{C}$ and C_1 of $L^{\bar{0}}$, and the centre $\mathbf{C}$ of $L^{\bar{0}}$.

(2.4.22.2) *dim* $L^{\bar 0}$ = *dim* $L^{\bar 1} = 4$.
Choose as **C**-bases of $L^{\bar 0}$ and $L^{\bar 1}$ the families of complex matrices $\{e_1, \ldots, e_4\}$ and $\{e_5, \ldots, e_8\}$, respectively.

$$e_1, \ldots, e_4,$$
$$e_5, \ldots, e_8$$
$$:= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(2.4.22.3) $\forall\ x = \sum_{k=1}^{8} x_k e_k \in L$:

$$ad\ x \ = \begin{bmatrix} 0 & 0 & 0 & 0 & -x_8 & x_7 & x_6 & -x_5 \\ 0 & 0 & -x_4 & x_3 & -x_6 & -x_5 & -x_8 & -x_7 \\ 0 & -2x_3 & 2x_2 & 0 & 0 & -2x_6 & 0 & -2x_8 \\ 0 & 2x_4 & 0 & -2x_2 & 2x_5 & 0 & 2x_7 & 0 \\ -x_7 & x_5 & 0 & x_6 & -x_2 & -x_4 & x_1 & 0 \\ -x_8 & -x_6 & x_5 & 0 & -x_3 & x_2 & 0 & x_1 \\ x_5 & x_7 & 0 & x_8 & -x_1 & 0 & -x_2 & -x_4 \\ x_6 & -x_8 & x_7 & 0 & 0 & -x_1 & -x_3 & x_2 \end{bmatrix}.$$

(2.4.22.4) The Killing form of $C(2)$ is non-degenerate. $\forall x, y \in L$:

$$\kappa(x,y) := str(ad\ x \circ ad\ y) = -2\ str(xy)$$
$$= 4\ x_1y_1 + 4\ x_2y_2 + 2\ x_3y_4 + 2\ x_4y_3$$
$$+ 4\ x_5y_6 - 4\ x_6y_5 + 4\ x_7y_8 - 4\ x_8y_7.$$

(2.4.22.5) $L^{\bar 0}$ is reductive.

$$L^{\bar 0} = Z(L^{\bar 0}) \oplus C_1,\ Z(L^{\bar 0}) = \mathbf{C}(\{e_1\}).$$

$$H := Z(L^{\bar 0}) \oplus T,\ T := \mathbf{C}(\{e_2\}),\ dim\ H = 2.$$

(2.4.22.6) There are two even and four odd roots of $C(2)$.

$$\Psi^{\bar 0} = \{H \ni x_1e_1 + x_2e_2 \to \pm 2x_2 \in \mathbf{C}\},$$

with the root spaces $\mathbf{C}e_3$ and $\mathbf{C}e_4$;

$$\Psi^{\bar 1} = \{H \ni x_1e_1 + x_2e_2 \to \pm ix_1 - x_2, \pm ix_1 + x_2 \in \mathbf{C}\},$$

with the root spaces

$$\mathbf{C}\left(e_5 \pm ie_7 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \pm i & 0 & 0 \end{bmatrix}\right),$$

$$\mathbf{C}\left(e_6 \pm ie_8 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \pm i \\ -1 & \mp i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

(2.4.22.7)

$$\forall x = x_1e_1 + x_2e_2,\ y = y_1e_1 + y_2e_2 \in H : \kappa(x,y) = 4(x_1y_1 + x_2y_2).$$

Hence one obtains the elements of H, which correspond to the above roots:

$$\pm 2x_2 \longleftrightarrow \pm\frac{1}{2}e_2,$$

$$\pm ix_1 - x_2 \longleftrightarrow \frac{1}{4}(\pm ie_1 - e_2),\ \pm ix_1 + x_2 \longleftrightarrow \frac{1}{4}(\pm ie_1 + e_2).$$

(2.4.22.8)

$$N_+ := \mathbf{C}(\{e_4, e_5, e_7\}),\ N_- := \mathbf{C}(\{e_3, e_6, e_8\}),\ L = H \oplus N_+ \oplus N_-.$$

$$[N_+, N_+] = \mathbf{C}e_4,\ [N_-, N_-] = \mathbf{C}e_3,\ [H, N_\pm] = N_\pm.$$

Obviously $N_\pm$ are nilpotent subalgebras of L, and $B := H \oplus N_+$ is some Borel subalgebra of L.

$$\Psi_+ = \{-2x_2,\ \pm ix_1 - x_2\},\ \Psi_- = \{2x_2, \pm ix_1 + x_2\}.$$

$$\Delta = \{\alpha_1, \alpha_2\},\ \alpha_1 := ix_1 - x_2,\ \alpha_2 := -ix_1 - x_2.\ \ \mathbf{C}(\Delta) = H^*.$$

$$L_{\alpha_1} = \mathbf{C}(e_5 + ie_7),\ L_{\alpha_2} = \mathbf{C}(e_5 - ie_7),\ \tau = \{1, 2\}.$$

(2.4.22.9) Hence one obtains the following relations of the odd generators

$$x_1 := \frac{1}{\sqrt{2}}(e_5 + ie_7),\ x_2 := \frac{1}{\sqrt{2}}(e_5 - ie_7),$$

$$y_1 := \frac{1}{\sqrt{2}}(e_6 - ie_8),\ y_2 := \frac{1}{\sqrt{2}}(e_6 + ie_8).$$

$$[x_1, y_1] = ie_1 - e_2 = h_1,\ [x_2, y_2] = -ie_1 - e_2 = h_2;$$

$$[x_1, y_2] = [x_2, y_1] = 0.$$

$$[h_k, x_l] = \Gamma_{kl}x_l,\ [h_k, y_l] = -\Gamma_{kl}y_l,\ \Gamma := [\Gamma_{kl}; k, l = 1, 2] = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

$$[x_1, x_2] = 2e_4,\ [x_1, [x_1, x_2]] = [x_2, [x_2, x_1]] = 0;$$

$$[y_1, y_2] = -2e_3,\ [y_1, [y_1, y_2]] = [y_2, [y_2, y_1]] = 0.$$

$$[x_1, x_1] = [x_2, x_2] = [y_1, y_1] = [y_2, y_2] = 0.$$

(2.4.22.10) Here the following table of super-commutators is used.

$$[[e_k, e_l]; k, l = 1, \dots, 8]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -e_7 & -e_8 & e_5 & e_6 \\ 0 & 0 & 2e_3 & -2e_4 & -e_5 & e_6 & -e_7 & e_8 \\ 0 & -2e_3 & 0 & e_2 & -e_6 & 0 & -e_8 & 0 \\ 0 & 2e_4 & -e_2 & 0 & 0 & -e_5 & 0 & -e_7 \\ e_7 & e_5 & e_6 & 0 & 2e_4 & -e_2 & 0 & -e_1 \\ e_8 & -e_6 & 0 & e_5 & -e_2 & -2e_3 & e_1 & 0 \\ -e_5 & e_7 & e_8 & 0 & 0 & e_1 & 2e_4 & -e_2 \\ -e_6 & -e_8 & 0 & e_7 & -e_1 & 0 & -e_2 & -2e_3 \end{bmatrix}.$$

(2.4.22.11) One may also choose

$$M_+ := \mathbf{C}(\{e_3, e_5 + ie_7, e_6 + ie_8\}),\ M_- := \mathbf{C}(\{e_4, e_5 - ie_7, e_6 - ie_8\});$$

$$L = H \oplus M_+ \oplus M_-,\ [H, M_\pm] = M_\pm,\ [M_\pm, [M_\pm, M_\pm]] = \{0\},$$

$$\Psi_\pm = \{\pm 2x_2, \pm ix_1 + x_2, \pm ix_1 - x_2\},$$

$$\Delta = \{\beta_1, \beta_2\},\ \beta_1 := 2x_2,\ \beta_2 := ix_1 - x_2;$$

$$L_{\beta_1} = \mathbf{C}e_3,\ L_{\beta_2} = \mathbf{C}(e_5 + ie_7),\ \tau = \{2\}.$$

$$\xi_1 := e_3,\ \eta_1 := e_4,\ \xi_2 := \frac{1}{\sqrt{2}}(e_5 + ie_7),\ \eta_2 := \frac{1}{\sqrt{2}}(e_6 - ie_8).$$

$$[\xi_1, \eta_1] = e_2,\ [\xi_2, \eta_2] = ie_1 - e_2.$$

$$[\xi_1, \eta_2] = [\xi_2, \eta_1] = 0,\ [\xi_2, \xi_2] = [\eta_2, \eta_2] = 0.$$

$$[\xi_1, [\xi_1, \xi_2]] = [\xi_2, [\xi_2, \xi_1]] = [\eta_1, [\eta_1, \eta_2]] = [\eta_2, [\eta_2, \eta_1]] = 0.$$

The corresponding Cartan matrix is $\Gamma = \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}$.

There is no automorphism of the Lie superalgebra L, mapping $N_\pm$ onto $M_\pm$.

(2.4.22.12) The adjoint representation of $L^{\bar 0}$ on $L^{\bar 1}$ is not irreducible.

$$L^{\bar 1} = M_+^{\bar 1} \oplus M_-^{\bar 1},\ M_\pm^{\bar 1} := M_\pm \cap L^{\bar 1},\ [L^{\bar 0}, M_\pm^{\bar 1}] = M_\pm^{\bar 1}.$$

Obviously the adjoint representations of $L^{\bar 0}$ on $M_\pm^{\bar 1}$ are contragredient with respect to the non-degenerate Killing form, in the sense that

$$\forall x \in L^{\bar 0},\ y_\pm \in M_\pm^{\bar 1} : \kappa(ad\ x(y_+), y_-) = -\kappa(y_+, ad\ x(y_-));$$

moreover they are irreducible.

(2.4.23) For instance in case of $A(m,n) \neq A(1,1)$, $m,n \in \mathbf{N}$, the root system can be described by the Cartesian unit vectors $\varepsilon_i, i = 1,\ldots,m+n+2$, of $\mathbf{R}^{m+n+2}$.

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_m - \varepsilon_{m+1}, \varepsilon_{m+1} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_n - \delta_{n+1}\},$$

denoting $\forall_1^{n+1} k : \delta_k := \varepsilon_{m+k+1}$;

$$\Psi_{\pm}^{\bar{0}} = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\delta_k - \delta_l); 1 \le i < j \le m+1, 1 \le k < l \le n+1\},$$
$$\Psi_{\pm}^{\bar{1}} = \{\pm(\varepsilon_i - \delta_k); 1 \le i \le m+1, 1 \le k \le n+1\};$$

$$s = m+n+1, \ \tau := \{m+1\}.$$
$$\forall_1^s k,l : \Gamma_{kl} = \left(1 + (-1)^{\delta_{k,m+1}}\right)\delta_{kl} - \delta_{k,l+1} - (-1)^{\delta_{k,m+1}}\delta_{k+1,l},$$
$$\tilde{\Gamma}_{kl} = -\delta_{k,l+1} - \delta_{k+1,l} \text{ for } k \neq l, \ \tilde{\Gamma}_{kk} = \Gamma_{kk}.$$

(2.4.24) In case of rank 1, the defining relations of $B(0,1) := osp(1,2)$ lead immediately to the following defining representation on $\mathbf{C}^3$. $dim\ B(0,1) = 5$.

$$\tau = \{1\}, \ \Gamma = 2, \ \Psi_{+}^{\bar{0}} = \{2\varepsilon_1\}, \ \Psi_{+}^{\bar{1}} = \{\varepsilon_1\} = \Delta.$$
$$[H, X^{\pm}] = \pm 2X^{\pm}, \ X^+X^- + X^-X^+ = H.$$
$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ X^+ = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ X^- = \sqrt{2}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(2.4.24.1) Rewriting and defining

$$H =: 2Q^3, \ X^{\pm} =: 2\sqrt{2}iV^{\pm}, \ Q^{\pm} := \pm 4\left(V^{\pm}\right)^2,$$

one easily finds the super-commutation relations

$$\left[Q^3, Q^{\pm}\right] = \pm 2Q^{\pm}, \ \left[Q^+, Q^-\right] = Q^3,$$

which show that the even part of $B(0,1)$ is just A_1.

(2.4.25) The Cartan matrices of the basic classical Lie superalgebras of rank 2, according to an appropriate choice of τ, are listed below.

L	$A(0,1)$	$B(1,1)$	$B(0,2)$	$C(2)$
τ	$\{1\}$	$\{1\}$	$\{2\}$	$\{2\}$
$dim\ L$	8	12	14	8
Γ	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}$

Furthermore the corresponding properties of the exceptional cases are tabulated below, choosing $\tau := \{1\}$.

For $L := D(2,1;\mu)$, $dim\ L = 17$, $\Gamma = \begin{bmatrix} 0 & 1 & \mu \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

L	$F(4)$	$G(3)$
$dim\ L$	40	31
Γ	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$

(2.4.26) The roots of the basic classical Lie superalgebras $L \neq A(1,1)$ can be written in terms of the Cartesian unit vectors $\varepsilon_i, i = 1, \ldots, p$, of $\mathbf{R}^p$, with an appropriate dimension $p \in \mathbf{N}$. The cases of rank 2 are listed below, with the above choice of τ.

(2.4.26.1) For $A(0,1)$, $\tau := \{1\}$, $p = 3$.

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\},\ \Psi_+^{\bar{0}} = \{\varepsilon_2 - \varepsilon_3\},\ \Psi_+^{\bar{1}} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3\}.$$

(2.4.26.2) For $C(2)$, $\tau := \{2\}$, $p = 3$.

$$\Delta = \{2\varepsilon_3, \varepsilon_1 - \varepsilon_3\},\ \Psi_+^{\bar{0}} = \{2\varepsilon_3\},\ \Psi_+^{\bar{1}} = \{\varepsilon_1 \pm \varepsilon_3\}.$$

(2.4.26.3) For $B(1,1)$, $\tau := \{1\}$, $p = 3$.

$$\Delta = \{\varepsilon_3 - \varepsilon_1, \varepsilon_1\},\ \Psi_+^{\bar{0}} = \{\varepsilon_1, 2\varepsilon_3\},\ \Psi_+^{\bar{1}} = \{\varepsilon_3 \pm \varepsilon_1, \varepsilon_3\}.$$

(2.4.26.4) For $B(0,2)$, $\tau := \{2\}$, $p = 2$.

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2\},\ \Psi_+^{\bar{0}} = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\},\ \Psi_+^{\bar{1}} = \{\varepsilon_1, \varepsilon_2\}.$$

(2.4.27) The following defining representations of basic classical Lie superalgebras of rank 2 are easily calculated from their defining relations. Note that $(X_2^{\pm})^2 \neq 0$ in the case of $B(0,2)$.

L	$X_1^+ = (X_1^-)^t$	X_2^+
$A(0,1)$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
$B(1,1)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$B(0,2)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$	$\sqrt{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$C(2)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 \end{bmatrix}$

For $A(0,1)$ and $B(1,1)$, $X_2^- = (X_2^+)^t$.

L	$B(0,2)$	$C(2)$
X_2^-	$\sqrt{2}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(2.4.28) Consider again the decomposition $L = N_- \oplus H \oplus N_+$ of a classical Lie superalgebra $L \notin \{A(1,1), P(n), Q(n); n \geq 2\}$, defining positive and negative roots, respectively. Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a finite-dimensional complex vector space. An irreducible representation ϕ of L on V is called representation with the highest weight $\mu \in H^*$, if and only if there is some homogeneous vector $v_\mu \in V^{\bar{0}} \cup V^{\bar{1}}$, such that $\phi(L)v_\mu = V$ and the following eigenvalue equations hold.

$$\forall h \in H, x \in N_+ : \phi(h)v_\mu = \mu(h)v_\mu, \quad \phi(x)v_\mu = 0.$$

The tabulated defining representations are of this kind. More generally, every complex finite-dimensional irreducible representation of any basic classical Lie superalgebra $L \notin \{A(n,n); n \in \mathbf{N}\}$ is some highest weight representation.

(2.4.29) Let L be basic classical.

$$L \in \{B(0,n); n \in \mathbf{N}\} \iff \forall \beta \in \Psi^{\bar{1}} : 2\beta \in \Psi^{\bar{0}}.$$

This result is involved into the theorem: Every finite-dimensional representation of L is completely reducible, if and only if $L \in \{B(0,n); n \in \mathbf{N}\}$.

(2.4.30) Let $\Lambda(n) \equiv \Lambda(\mathbf{C}^n)$ be the exterior algebra of $\mathbf{C}^n$, with the basis $\{\theta_1, \ldots, \theta_n\}$ of $\mathbf{C}^n$.

$$\Lambda(n) = \mathbf{C}(\{e_n\} \cup \{\theta_{k_1} \wedge \cdots \wedge \theta_{k_p}; 1 \leq k_1 < \ldots < k_p \leq n, 1 \leq p \leq n\}),$$

$dim\ \Lambda(n) = 2^n, n \in \mathbf{N}$. Here e_n denotes the unit of $\Lambda(n)$.
The Cartan-Lie superalgebra $W(n)$ of super-derivations of $\Lambda(n)$ is defined as some subalgebra of the super-commutator algebra $(End_{\mathbf{C}}(\Lambda(n)))_L$, with respect to the usual $\mathbf{Z}_2$-grading of the graded-commutative complex associative superalgebra $\Lambda(n)$, the generators $\theta_1, \ldots, \theta_n$ being odd.

$$W(n) := Der_{\mathbf{C}}(\Lambda(n)) := \bigoplus_{\bar{p} \in \mathbf{Z}_2} Der_{\mathbf{C}}^{\bar{p}}(\Lambda(n)).$$

(2.4.30.1) There are unique odd derivations $\frac{\partial}{\partial \theta_k} \in Der_{\mathbf{C}}^{\bar{1}}(\Lambda(n))$, for $k = 1, \ldots, n$, such that

$$\forall_1^n k, l : \frac{\partial}{\partial \theta_k}\theta_l = \delta_{kl} e_n, \text{ hence } \frac{\partial}{\partial \theta_k} \circ \frac{\partial}{\partial \theta_l} + \frac{\partial}{\partial \theta_l} \circ \frac{\partial}{\partial \theta_k} = 0.$$

$W(n)$ is the free $\Lambda(n)$-left module over $\{\frac{\partial}{\partial \theta_k}; k = 1, \ldots, n\}$. $dim\ W(n) = n2^n$.
$W(2)$ is isomorphic with the complex Lie superalgebra $C(2)$.

(2.4.30.2) For $n \geq 2$, $W(n)$ is simple.

$$W(1)\Lambda(1) = \Lambda(1), \ [W(1), W(1)]\Lambda(1) = \mathbf{C}e_1.$$

(2.4.31) Including also certain $\mathbf{Z}_2$-graded subalgebras of the above defined Cartan-Lie superalgebras, all the simple finite-dimensional complex Lie superalgebras were classified by V. G. Kac.

3. Coalgebras and $\mathbf{Z}_2$-Graded Hopf Algebras

Hopf algebras play an important role with respect to the investigation of unital associative algebras and their representations for at least two reasons.

For an appropriate representation of unital associative algebras on tensor products of vector spaces, and also on their duals, one needs homomorphisms from unital associative algebras to their two-fold tensor products, and anti-homomorphisms of such algebras onto themselves; these homomorphisms are provided by coproducts and antipodes. Counits serve as one-dimensional representations.

For instance, the cocommutativity of the diagonal mapping on the universal enveloping algebra of a simple finite-dimensional complex Lie algebra is perturbed by an appropriate q-deformation, thereby constructing a quasi-triangular Hopf algebra, which then may be called quantum algebra, as an important type of quantum groups. In order to preserve main features of the representation theory of algebras, quantum groups in general should be constructed preferably as Hopf algebras.

In order to establish the dual $A^* := Hom_K(A, K)$ of a unital associative algebra A over a field K again as such an object, one also needs both coproduct and counit on A. Whereas the dual of an arbitrary coalgebra becomes an associative algebra with unit, the dual of an associative unital algebra can be established as some coalgebra only in the finite-dimensional case, because only in this case the natural injection:

$$A^* \otimes A^* \ni f \otimes g \longrightarrow T(f, g) \longleftrightarrow\in (A \otimes A)^*$$

is surjective. In the infinite-dimensional case one may use either algebraic or topological restrictions of the dual A^*, and an appropriate Hausdorff completion of the tensor product.

Dualizing as bialgebra the q-deformation $U_q(\mathcal{L})$ of the universal enveloping algebra $U(\mathcal{L})$ of a semisimple finite-dimensional complex Lie algebra $\mathcal{L}$, one obtains a so-called matrix quantum semigroup $\mathcal{A}_q$, which in turn may be used in order to transform an according quantum vector space $\mathcal{X}_q$.

The notion of left comodules is useful in particular for the description of such transformations of quantum vector spaces.

In the so-called classical limit of $q \longrightarrow 1$ one ends up with duals of quasitriangular Lie bialgebras. A diagrammatic scheme of the duality of quasitriangular Hopf algebras and matrix quantum semigroups, the latter as quadratic algebras with so-called main commutation relations (MCR), and, in the limit $q \longrightarrow 1$, of quasitriangular Lie bialgebras and Lie algebras with semiclassical MCR, will be presented in the introduction to Chap. 8.

The straightforward generalization of the concept of Hopf algebras to the $\mathbf{Z}_2$-graded case is presented explicitly from the beginning. This superversion will be applied also to the investigation of duality questions, quasitriangularity, and the classical and quantum Yang-Baxter equation (CYBE and QYBE).

For an extensive presentation of the theory of Hopf algebras the reader is referred to monographs by M. E. Sweedler (1969) and E. Abe (1980), a review by J. W. Milnor and J. C. Moore (1965), and to the volume Algebra I by N. Bourbaki (1989) with respect to N_0-graded bialgebras.

The classical Yang-Baxter equation and its geometric interpretation with respect to Poisson-Lie groups is studied for instance in the work of A. Belavin and V. G. Drinfel'd (1983), V. G. Drinfel'd (1983, 1987), M. Jimbo (1985, 1989), Ch. Fronsdal (1993), M. A. Semenov-Tyan-Shanskii (1984, 1992), and in the monograph by V. Chari and A. Pressley (1994).

3.1 Coalgebras over Commutative Rings

(3.1.1.1) Let A be a unital associative algebra over a commutative ring R. One then defines the two structural homomorphisms, or structure mappings, μ and η:

$$\begin{tikzcd} A \times A \ni \{a,b\} \arrow[rr] \arrow[dr, "def"'] & & a \otimes b \in A \otimes A \arrow[dl, "\mu"] \\ & ab \in A & \end{tikzcd}$$

$$R \ni r \xrightarrow[\eta]{} re_A \in A$$

(3.1.1.2) These structure mappings fulfill the commutative diagrams below.

$$\begin{tikzcd} R \otimes A \arrow[r, "T(\eta, id A)"] \arrow[d, phantom, sloped, "\ni"] & A \otimes A \arrow[dd, "\mu"] & A \otimes R \arrow[l, "T(id A, \eta)"'] \arrow[d, phantom, sloped, "\ni"] \\ r \otimes a \arrow[ddr] & & a \otimes r \arrow[ddl] \\ & A \arrow[d, phantom, sloped, "\ni"] & \\ & ra = ar & \end{tikzcd}$$

(3.1.2.1) Conversely suppose an R-bimodule A, and R-linear mappings $\eta : R \longrightarrow A$ and $\mu : A \otimes A \longrightarrow A$, such that the above two diagrams are commutative. Then the R-linear mapping μ induces an algebra A over R, and moreover the commutative diagrams imply, that

$$\forall a \in A : a\eta(e_R) = \eta(e_R)a = a.$$

Hence A is unital, with the unit $e_A := \eta(e_R)$.

(3.1.2.2) This algebra A over R is associative, if and only if the following diagram is commutative.

$$\begin{tikzcd}
A \otimes (A \otimes A) \ni a \otimes (b \otimes c) \arrow[rr, leftrightarrow, "def"'] \arrow[d, "T(id\,A,\,\mu)"] & & (a \otimes b) \otimes c \in (A \otimes A) \otimes A \arrow[d, "T(\mu,\,id\,A)"'] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

(3.1.3) Let $A_k, k = 1, 2$, be algebras over R with the structure maps μ_k and η_k, respectively. An R-linear mapping $\phi : A_1 \longrightarrow A_2$ is a homomorphism of R-algebras, if and only if the diagram (i) is commutative.

(i)
$$\begin{tikzcd}
A_1 \otimes A_1 \arrow[r, "T(\phi,\,\phi)"] \arrow[d, "\mu_1"] & A_2 \otimes A_2 \arrow[d, "\mu_2"] \\
A_1 \arrow[r, "\phi"'] & A_2
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
 & R \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
A_1 \arrow[rr, "\phi"'] & & A_2
\end{tikzcd}$$

(3.1.3.1) Let the algebras $A_k, k = 1, 2$, be associative and unital, with the units $e_{A_k} := \eta_k(e_R)$. Then ϕ is a homomorphism of unital associative algebras, if and only if diagrams (i) and (ii) are commutative.

(3.1.4) Let $A_k, k = 1, \ldots, p$, be algebras over R with the structure maps μ_k and η_k, respectively. The structure maps μ and η of $A := A_1 \otimes \cdots \otimes A_p$ are constructed, using the universal property of tensor products, such that:

$$A \otimes A \ni (a_1 \otimes \cdots \otimes a_p) \otimes (b_1 \otimes \cdots \otimes b_p) \xrightarrow[\mu]{} \bigotimes_{k=1}^{p} (a_k b_k) \in A;$$

$$\eta : R \ni r \underset{def}{\longrightarrow} r\, e_A = r\, e_{A_1} \otimes \cdots \otimes e_{A_p} \in A.$$

With these structure maps μ and η, the algebra A is called the tensor product of algebras $A_k, k = 1, \ldots, p$, over R.

(3.1.4.1) If the algebras $A_k, k = 1, \ldots, p$, are associative, then A is associative too.

(3.1.4.2) Let $\phi_k : A_k \longrightarrow B_k, k = 1, \ldots, p$, be homomorphisms of R-algebras. Then

$$T(\phi_1, \ldots, \phi_p) : \bigotimes_{k=1}^{p} A_k \ni \bigotimes_{k=1}^{p} a_k \longrightarrow \bigotimes_{k=1}^{p} \phi(a_k) \in \bigotimes_{k=1}^{p} B_k$$

is again some homomorphism of R-algebras. If these algebras are associative and unital, and if $\phi_k, k = 1, \ldots, p$, preserve units, then $T(\phi_1, \ldots, \phi_p)$ sends unit to unit too.

(3.1.5.1) Let an algebra A over R, with the structure maps μ and η, be graded with respect to the commutative monoid G, i.e.,

$$A = \bigoplus_{g \in G} A^g, \quad \forall g, h \in G, \forall a \in A^g, b \in A^h : ab \in A^{g+h}.$$

$\forall a \in A : ae = ea = a \Longrightarrow e \in A^{g_0}$. Here g_0 denotes the unit element of G. Then the R-bimodule $A \otimes A$ is G-graded by means of an R-linear bijection:

$$A \otimes A \ni \left(\sum_{g \in G} a^g \right) \otimes \left(\sum_{g \in G} b^g \right) \longleftrightarrow \sum_{g \in G} \sum_{g_1, g_2 \in G; g_1 + g_2 = g} (a^{g_1} \otimes b^{g_2})$$
$$\in \bigoplus_{g \in G} \bigoplus_{g_1, g_2 \in G; g_1 + g_2 = g} (A^{g_1} \otimes A^{g_2}).$$

Obviously the structure mapping μ preserves this grading with respect to G.

(3.1.5.2) Conversely let $A = \bigoplus_{g \in G} A^g$ be an algebra over R with the structure mappings μ and η. The R-bimodule $A \otimes A$ is written as direct sum, as explicated above. If μ preserves this G-grading, then A is some G-graded algebra over R, i.e.,

$$\forall g, h \in G, \forall a \in A^g, b \in A^h : (A \otimes A)^{g+h} \ni a \otimes b \underset{\mu}{\longrightarrow} ab \in A^{g+h}.$$

(3.1.5.3) Let the algebras A_k over R, $k = 1, \dots, p$, with the structure maps μ_k and η_k, be $\mathbf{Z}_2$-graded. Then an according $\mathbf{Z}_2$-grading of the R-bimodule $A := \bigotimes_{k=1}^{p} A_k$ is defined naturally, generalizing the above R-linear bijection. The structure maps $\hat{\mu}$ and $\hat{\eta}$ of the graded (or skew-symmetric) tensor product $\hat{\bigotimes}_{k=1}^{p} A_k$ of these algebras are constructed. Define $\hat{\mu}$, using the universal property of tensor products, such that for homogeneous elements $a_k \in A_k^{\overline{x_k}}, b_k \in A_k^{\overline{y_k}}, \overline{x_k}$ and $\overline{y_k} \in \mathbf{Z}_2$, for $k = 1, \dots, p$:

$$A \otimes A \ni (a_1 \otimes \cdots \otimes a_p) \otimes (b_1 \otimes \cdots \otimes b_p)$$

$$\xrightarrow[\hat{\mu}]{} (-1)^{\sum_{1 \le k < l \le p} x_l y_k} \bigotimes_{k=1}^{p} (a_k b_k) \in A;$$

$$\hat{\eta} : R \ni r \xrightarrow[def]{} r\, e_A = r\, e_{A_1} \otimes \cdots \otimes e_{A_p} \in A.$$

With these structure maps $\hat{\mu}$ and $\hat{\eta}$, A becomes some $\mathbf{Z}_2$-graded R-algebra.

(3.1.5.3.1) The skew-symmetric tensor product of associative superalgebras $A_k, k = 1, \dots, p$, over R is again an associative R-superalgebra. If especially $A_k, k = 1, \dots, p$, are graded-commutative, then their skew-symmetric tensor product is graded-commutative too.

(3.1.5.3.2) Let $\phi_k : A_k \longrightarrow B_k, k = 1, \dots, p$, be homomorphisms of unital associative superalgebras over R. Then

$$T(\phi_k; k = 1, \dots, p) : \hat{\bigotimes}_{k=1}^{p} A_k \longrightarrow \hat{\bigotimes}_{k=1}^{p} B_k,$$

such that for homogeneous $a_k \in A$ and ϕ_k of degrees $\overline{y_k}$ and $\overline{x_k}, k = 1, \dots, p$:

$$a_1 \otimes \cdots \otimes a_p \xrightarrow[T(\phi_1, \dots, \phi_p)]{} (-1)^{\sum_{1 \le k < l \le p} x_l y_k} \phi_1(a_1) \otimes \cdots \otimes \phi_p(a_p),$$

is again some homomorphism of unital associative superalgebras over R.

(3.1.6) Consider an R-bimodule A, and R-linear mappings

$$\Delta : A \longrightarrow A \otimes A, \ \varepsilon : A \longrightarrow R.$$

The family $\{A, \Delta, \varepsilon\}$ is called coalgebra over R, if and only if the corresponding diagrams below are commutative. Here Δ is called comultiplication, and ε counit of A.

$$
\begin{tikzcd}
A \otimes A \arrow[d, "T(\Delta, id A)"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & A \otimes A \arrow[d, "T(id A, \Delta)"] \\
(A \otimes A) \otimes A \arrow[rr, leftrightarrow] & & A \otimes (A \otimes A)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & a \arrow[d, phantom, sloped, "\in"] \arrow[dl, bend right, "def"'] \arrow[dr, bend left, "def"] & \\
e_R \otimes a \arrow[d, phantom, sloped, "\in"] & A \arrow[d, "\Delta"] & a \otimes e_R \arrow[d, phantom, sloped, "\in"] \\
R \otimes A & A \otimes A \arrow[l, "T(\varepsilon, id A)"] \arrow[r, "T(id A, \varepsilon)"'] & A \otimes R
\end{tikzcd}
$$

(3.1.6.1) The commutative ring R itself is some coalgebra over R, with the R-linear maps

$$\Delta_R : R \ni r \underset{def}{\longrightarrow} r(e_R \otimes e_R) \in R \otimes R, \quad e_r := id\ R.$$

(3.1.6.2) Let $\{A_k, \Delta_k, \varepsilon_k\}, k = 1, 2$, be R-coalgebras. An R-linear mapping $\phi : A_1 \longrightarrow A_2$ is called homomorphism of R-coalgebras, if and only if the corresponding diagrams below are commutative.

$$
\begin{tikzcd}
A_1 \arrow[r, "\phi"] \arrow[d, "\Delta_1"] & A_2 \arrow[d, "\Delta_2"] \\
A_1 \otimes A_1 \arrow[r, "T(\phi, \phi)"'] & A_2 \otimes A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\phi"] \arrow[rr, bend right, "\varepsilon_1"'] & A_2 \arrow[r, "\varepsilon_2"] & R
\end{tikzcd}
$$

(3.1.6.2.2) The counit $\varepsilon : A \longrightarrow R$ is some homomorphism of coalgebras, due to the diagram:

$$\Delta_R \circ \varepsilon = T(id\ R, \varepsilon) \circ T(\varepsilon, id\ A) \circ \Delta.$$

(3.1.6.2.3) If $\phi : A_1 \longrightarrow A_2, \psi : A_2 \longrightarrow A_3$ are homomorphisms of coalgebras, then $\psi \circ \phi : A_1 \longrightarrow A_3$ is also some homomorphism of coalgebras, due to the functorial properties of T.

(3.1.6.3) Let $A = \bigotimes_{g \in G} A^g$ be a coalgebra over R, with the costructure maps Δ, ε. Take the usual G-grading of $A \otimes A$, and the trivial one of R, i.e., $R =: R^{g_0}$. Then the coalgebra A over R is called G-graded, if and only if Δ and ε preserve this grading, i.e.,

(i) $\forall g \in G : A^g \ni a \underset{\Delta}{\longrightarrow} \longleftrightarrow \in \bigoplus_{g_1, g_2 \in G; g_1 + g_2 = g} (A^{g_1} \otimes A^{g_2});$

(ii) $\forall g \in G \backslash \{g_0\}, \forall a \in A^g : \varepsilon(a) = 0.$

Of course every coalgebra A over R can be trivially G-graded by $A =: A^{g_0}$. In particular, $\mathbf{Z}_2$-graded coalgebras over R are called super-coalgebras.

(3.1.6.3.1) Let A, B be G-graded R-coalgebras. A homomorphism of coalgebras $\phi : A \longrightarrow B$ is called homomorphism of G-graded coalgebras over R, if and only if it preserves this grading, i.e., $\forall g \in G : Im\ \phi|_{A^g} \subseteq B^g$. The G-grading of R is taken as the trivial one. Hence the counit ε is some homomorphism in the sense of G-graded coalgebras over R, for every G-graded coalgebra $\{A, \Delta, \varepsilon\}$ over R.

(3.1.7) Let $\{A_k, \Delta_k, \varepsilon_k\}, k = 1, \dots, p$, be coalgebras over R. Then the R-bimodule $A := \bigotimes_{k=1}^{p} A_k$ can be established as coalgebra $\{A, \Delta, \varepsilon\}$ over R.

(3.1.7.1) Using an R-linear bijection

$$\tau : \bigotimes_{k=1}^{p} (A_k \otimes A_k) \ni \bigotimes_{k=1}^{p} (a_k \otimes b_k) \longleftrightarrow \left(\bigotimes_{k=1}^{p} a_k \right) \otimes \left(\bigotimes_{k=1}^{p} b_k \right) \in A \otimes A,$$

one defines $\Delta := \tau \circ T(\Delta_1, \dots, \Delta_p)$. An explicit notation, such that

$$\forall_1^p k, \forall a_k \in A_k : \Delta_k(a_k) =: \sum_{l=1}^{L_k} (a'_{kl} \otimes a''_{kl}),$$

yields, inserting an R-linear bijection

$$\zeta : \bigotimes_{k=1}^{p} (A_k \otimes (A_k \otimes A_k)) \ni \bigotimes_{k=1}^{p} (a_k \otimes (b_k \otimes c_k))$$
$$\longleftrightarrow (a_1 \otimes \cdots \otimes a_p) \otimes ((b_1 \otimes \cdots \otimes b_p) \otimes (c_1 \otimes \cdots \otimes c_p)) \in A \otimes (A \otimes A),$$

the commutative diagram:

$$T(id\ A, \Delta) \circ \Delta = \zeta \circ T(T(id\ A_k, \Delta_k) \circ \Delta_k; k = 1, \dots, p),$$

which allows for an easy proof of the coassociative property of Δ.

(3.1.7.2) An R-linear bijection

$$\rho_p : R \otimes \cdots \otimes R \in r_1 \otimes \cdots \otimes r_p \longleftrightarrow r_1 \cdots r_p \in R$$

is used in order to show, that $\forall_1^p k, \forall a_k, b_k \in A_k$:

$$e_R \otimes a_k = e_R \otimes b_k \Longrightarrow e_R \otimes \left(\bigotimes_{k=1}^{p} a_k \right) = e_R \otimes \left(\bigotimes_{k=1}^{p} b_k \right).$$

Here one uses the R-linear bijections:

$$\bigotimes_{k=1}^{p} (R \otimes A_k) \longleftrightarrow \left(\bigotimes^{p} R \right) \otimes A \underset{T(\rho_p, id\ A)}{\longleftrightarrow} R \otimes A.$$

An insertion of the counit diagrams for ε_k and $\Delta_k, k = 1, \ldots, p$, yields the counit diagram for ε and Δ, with the counit $\varepsilon := \rho_p \circ T(\varepsilon_1, \ldots, \varepsilon_p)$.

(3.1.7.3) The coalgebra $\{A, \Delta, \varepsilon\}$ is called tensor product of the coalgebras $\{A_k, \Delta_k, \varepsilon_k\}$, $k = 1, \ldots, p$, over R.

(3.1.7.4) One easily finds, that both ρ_p and ρ_p^{-1} are homomorphisms both of algebras and coalgebras over R. Here $\{R \otimes \cdots \otimes R, \Delta_R^p, \varepsilon_R^p\}$ is defined as the tensor product of R-coalgebras, with the comultiplication

$$\Delta_R^p : \bigotimes^{p} R \ni r_1 \otimes \cdots \otimes r_p \longrightarrow r_1 \cdots r_p \left(\bigotimes^{p} e_R \right) \otimes \left(\bigotimes^{p} e_R \right),$$

and the counit $\varepsilon_R^p := \rho_p$.

(3.1.8) Let $\phi_k : A_k \longrightarrow B_k, k = 1, \ldots, p$, be homomorphisms of coalgebras over R. Then $T(\phi_1, \ldots, \phi_p) : \bigotimes_{k=1}^{p} A_k \longrightarrow \bigotimes_{k=1}^{p} B_k$ is some homomorphism of R-coalgebras too. Here again one must insert an explicit notation of $\Delta_k(a_k), a_k \in A_k$, for the coalgebras $\{A_k, \Delta_k, \varepsilon_k\}, k = 1, \ldots, p$.

(3.1.9) The natural linear bijections of tensor products of R-coalgebras:

$$(A_1 \otimes A_2) \otimes A_3 \longleftrightarrow A_1 \otimes A_2 \otimes A_3 \longleftrightarrow A_1 \otimes (A_2 \otimes A_3),$$

$$A_1 \otimes A_2 \longleftrightarrow A_2 \otimes A_1,$$

are isomorphisms in the sense of coalgebras over R, as is easily shown.

(3.1.10) The map: $A \ni a \longleftrightarrow e_R \otimes a \in R \otimes A$ is some isomorphism of coalgebras over R.

(3.1.11) Let $A_k, k = 1, \ldots, p$, be G-graded coalgebras over R. With the usual G-grading, $A_1 \otimes \cdots \otimes A_p$ is again some G-graded R-coalgebra, as is seen immediately.

(3.1.12) Let $\{A_k, \Delta_k, \varepsilon_k\}, k = 1, \ldots, p$, be $\mathbf{Z}_2$-graded coalgebras over R. Their graded (or twisted) tensor product $\{A, \hat{\Delta}, \hat{\varepsilon}\}$ is constructed.

(3.1.12.1) The counit of A is defined by

$$\hat{\varepsilon} := \rho_p \circ T(\varepsilon_1, \ldots, \varepsilon_p) : A := \bigotimes_{k=1}^{p} A_k \longrightarrow R.$$

(3.1.12.2) With an R-linear bijection, such that for homogenous elements $a_k \in A_k^{\overline{x_k}}, b_k \in A_k^{\overline{y_k}}, \overline{x_k}$ and $\overline{y_k} \in \mathbf{Z}_2$, for $k = 1, \ldots, p$,

$$\begin{aligned} \hat{\tau} : \bigotimes_{k=1}^{p}(A_k \otimes A_k) \ni \bigotimes_{k=1}^{p}(a_k \otimes b_k) \\ \longleftrightarrow (-1)^{\sum_{1 \le k < l \le p} x_l y_k} (a_1 \otimes \cdots \otimes a_p) \otimes (b_1 \otimes \cdots \otimes b_p) \in A \otimes A, \end{aligned}$$

the comultiplication of A is defined by

$$\hat{\Delta} := \hat{\tau} \circ T(\Delta_1, \ldots, \Delta_p).$$

(3.1.12.3) The family $\{A, \hat{\Delta}, \hat{\varepsilon}\}$ is some $\mathbf{Z}_2$-graded coalgebra over R. Especially the proof, that $\hat{\Delta}$ is coassociative, needs some careful notation of signs, and the coassociative property of Δ. The graded (or twisted) tensor product $\{A, \hat{\Delta}, \hat{\varepsilon}\}$ is denoted by $\hat{\bigotimes}_{k=1}^{p} A_k$.

(3.1.12.4) Let $\phi_k : A_k \longrightarrow B_k, k = 1, \ldots, p$, be homomorphisms of $\mathbf{Z}_2$-graded coalgebras over R. Then

$$T(\phi_1, \ldots, \phi_p) : \hat{\bigotimes}_{k=1}^{p} A_k \longrightarrow \hat{\bigotimes}_{k=1}^{p} B_k$$

is again some homomorphism of $\mathbf{Z}_2$-graded coalgebras over R.

(3.1.12.5) Let $A_k, k = 1, 2, 3$, be $\mathbf{Z}_2$-graded coalgebras over R. Then the R-linear bijections, such that:

$$(A_1 \hat{\otimes} A_2) \hat{\otimes} A_3 \ni (a_1 \otimes a_2) \otimes a_3 \longleftrightarrow a_1 \otimes a_2 \otimes a_3 \in A_1 \hat{\otimes} A_2 \hat{\otimes} A_3,$$

$$A_1 \hat{\otimes} A_2 \ni a_1 \otimes a_2 \longleftrightarrow (-1)^{x_1 x_2} a_2 \otimes a_1 \in A_2 \hat{\otimes} A_1,$$

for homogeneous elements $a_k \in A_k^{\overline{x_k}},\ k = 1, 2$, are isomorphisms of $\mathbf{Z}_2$-graded coalgebras over R.

(3.1.13.1) An algebra A over R is called commutative, if and only if its structure map μ fulfills the commutative diagram below.

$$\begin{tikzcd}
A\otimes A \ni a\otimes b \arrow[rr, leftrightarrow, "def"'] \arrow[dr, "\mu"'] & & b\otimes a \in A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

(3.1.13.2) The $\mathbf{Z}_2$-graded algebra A over R is called graded-commutative, if and only if μ satisfies the following commutative diagram, $\forall \bar{x}, \bar{y} \in \mathbf{Z}_2$.

$$\begin{tikzcd}
A^{\bar{x}}\otimes A^{\bar{y}} \ni a\otimes b \arrow[rr, leftrightarrow, "def"'] \arrow[dr, "\mu"'] & & (-1)^{xy}\, b\otimes a \in A^{\bar{y}}\otimes A^{\bar{x}} \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

(3.1.14.1) The coalgebra $\{A, \Delta, \varepsilon\}$ over R is called cocommutative, if and only if the diagram below is commutative.

$$\begin{tikzcd}
A\otimes A \ni a\otimes b \arrow[rr, leftrightarrow, "def"] & & b\otimes a \in A\otimes A \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(3.1.14.2) The $\mathbf{Z}_2$-graded coalgebra $\{A, \Delta, \varepsilon\}$ over R is called graded-cocommutative, if and only if the diagram below is commutative, $\forall \bar{x}, \bar{y} \in \mathbf{Z}_2$.

$$\begin{tikzcd}
A^{\bar{x}}\otimes A^{\bar{y}} \ni a\otimes b \arrow[rr, leftrightarrow, "def"] & & (-1)^{xy}\, b\otimes a \in A^{\bar{y}}\otimes A^{\bar{x}} \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(3.1.15) Consider an R-coalgebra $\{A, \Delta, \varepsilon\}$, and denote $Hom_R(A, R) =: A^*$.

$$\eta^* : R \ni r \underset{def}{\longrightarrow} r\varepsilon \in A^*. \quad \mu_R : R \otimes R \ni r_1 \otimes r_2 \underset{def}{\longrightarrow} r_1 r_2 \in R, \quad \mu_R = \rho_2.$$

The universal property of tensor products is used to construct an R-linear mapping

$$\mu^* : A^* \otimes A^* \ni f \otimes g \longrightarrow \mu_R \circ T(f, g) \circ \Delta \in A^*.$$

With the structure maps μ^* and η^*, A^* becomes an associative algebra over R with the unit ε. Here one uses the following commutative diagrams, $\forall r \in R, \forall f, g, h \in A^*$:

$$\mu_R \circ T(f, r\varepsilon) \circ \Delta = rf = fr = \mu_R \circ T(r\varepsilon, f) \circ \Delta,$$

$$\mu_R \circ T(f, \mu_R \circ T(g, h) \circ \Delta) \circ \Delta = \mu_R \circ T(\mu_R \circ T(f, g) \circ \Delta, h) \circ \Delta.$$

(3.1.15.1) If $\{A, \Delta, \varepsilon\}$ is cocommutative, then A^* is commutative, because the cocommutativity of Δ implies, that

$$\forall f, g \in A^* : \mu_R \circ T(f, g) \circ \Delta = \mu_R \circ T(g, f) \circ \Delta.$$

(3.1.15.2) Let the above coalgebra be $\mathbf{Z}_2$-graded, such that $A = A^{\bar{0}} \oplus A^{\bar{1}}$.

$$\forall \bar{x} \in \mathbf{Z}_2 : A^*_{\bar{x}} := \{f \in A^*; \forall \bar{y} \in \mathbf{Z}_2 \backslash \{\bar{x}\}, \forall a \in A^{\bar{y}} : f(a) = 0\},$$

$$A^*_{\bar{x}} \ni f \longleftrightarrow f\big|_{A^{\bar{x}}} \in Hom_R(A^{\bar{x}}, R).$$

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2 : \bar{x} \neq \bar{y} \Longrightarrow A^*_{\bar{x}} \cap A^*_{\bar{y}} = \{0\}. \quad A^* = A^*_{\bar{0}} \oplus A^*_{\bar{1}}.$$

With the structure maps μ^* and η^*, A^* is some $\mathbf{Z}_2$-graded algebra over R. If $\{A, \Delta, \varepsilon\}$ is graded-cocommutative, then A^* is graded-commutative.

(3.1.15.3) Let the coalgebra A be $\mathbf{N}_0$-graded, such that $A = \bigoplus_{n \in \mathbf{N}_0} A^n$.

$$\forall m \in \mathbf{N}_0 : A^*_m := \{f \in A^*; \forall n \in \mathbf{N}_0 \backslash \{m\}, \forall a \in A^n : f(a) = 0\},$$

$$A^*_m \ni f \longleftrightarrow f\big|_{A^m} \in Hom_R(A^m, R).$$

$$\forall m, n \in \mathbf{N}_0 : m \neq n \Longrightarrow A^*_m \cap A^*_n = \{0\}.$$

$$A^\dagger := \bigoplus_{m \in \mathbf{N}_0} A^*_m \subset A^*. \quad \forall r \in R : \eta^\dagger(r) := \eta^*(r) \in A^\dagger.$$

$$\forall f \otimes g \in A^\dagger \otimes A^\dagger : \mu^\dagger(f \otimes g) = \mu^*(f \otimes g) \in A^\dagger.$$

Then $\{A^\dagger, \mu^\dagger, \eta^\dagger\}$ is some $\mathbf{N}_0$-graded algebra over R. Since $A^\dagger$ is some subalgebra of A^*, $A^\dagger$ is an associative algebra with the unit $\varepsilon \in A^*_0$.
Consider the $\mathbf{Z}_2$-grading, which is naturally induced by the proposed $\mathbf{N}_0$-grading of A. If $\{A, \Delta, \varepsilon\}$ is graded-cocommutative, then $\{A^\dagger, \Delta^\dagger, \varepsilon^\dagger\}$ is graded-commutative.

3.2 Z_2-Graded Bialgebras and Z_2-Graded Hopf Algebras

(3.2.1) Let A be an R-bimodule, and assume that the R-linear mappings

$$\mu : A \otimes A \longrightarrow A, \quad \eta : R \longrightarrow A, \quad \Delta : A \longrightarrow A \otimes A, \quad \varepsilon : A \longrightarrow R$$

satisfy the following conditions, which are not independent of each other.

(i) $\{A, \mu, \eta\}$ is a unital associative algebra over R.

(ii) $\{A, \Delta, \varepsilon\}$ is a coalgebra over R.

(iii) Δ, ε are homomorphisms of unital associative algebras over R.

(iv) μ, η are homomorphisms in the sense of coalgebras over R.

With these conditions the bialgebra $\{A, \mu, \eta, \Delta, \varepsilon\}$ over R is established.

(3.2.2) Let $\{A, \mu, \eta\}$ be a unital associative algebra, and $\{A, \Delta, \varepsilon\}$ a coalgebra, over R. Then the following implications hold.

(3.2.2.1) μ is compatible with Δ, if and only if Δ is a homomorphism of algebras; in this case, the corresponding diagram reads:

$$\Delta \circ \mu = T(\mu, \mu) \circ \tau \circ T(\Delta, \Delta), \text{ with the flip}$$

$$\tau : (A \otimes A) \otimes (A \otimes A) \ni (a \otimes b) \otimes (c \otimes d) \underset{def}{\longleftrightarrow} (a \otimes c) \otimes (b \otimes d) \in (A \otimes A) \otimes (A \otimes A).$$

(3.2.2.2) μ preserves counits, if and only if the counit ε is a homomorphism of algebras, which explicitly means: $\varepsilon \circ \mu = \rho_2 \circ T(\varepsilon, \varepsilon)$.

(3.2.2.3) η is compatible with Δ, if and only if Δ preserves units: $\Delta \circ \eta = T(\eta, \eta) \circ \Delta_R$.

(3.2.2.4) η preserves counits, if and only if ε preserves units: $\varepsilon \circ \eta = id\ R$.

(3.2.2.5) Therefore μ and η are homomorphisms in the sense of coalgebras over R, if and only if Δ and ε are homomorphisms of unital associative algebras over R.

(3.2.3) Let A, B be bialgebras over R. An R-linear mapping $\phi : A \longrightarrow B$ is called homomorphism in the sense of bialgebras over R, if and only if ϕ is a homomorphism both in the sense of unital associative algebras and of coalgebras over R.

(3.2.3.1) The composition $\psi \circ \phi$ of homomorphisms of bialgebras is again of this type. If the homomorphism of bialgebras ϕ is bijective, then ϕ^{-1} is of this type too; in this case ϕ is called isomorphism in the sense of bialgebras over R.

(3.2.4) $\{A, \mu, \eta, \Delta, \varepsilon\}$ is called commutative-cocommutative, if and only if $\{A, \mu, \eta\}$ is commutative and $\{A, \Delta, \varepsilon\}$ is cocommutative.

(3.2.5) Let $\{A_k, \mu_k, \eta_k, \Delta_k, \varepsilon_k\}, k = 1, \ldots, p$, be bialgebras over R. Then $A := \bigotimes_{k=1}^{p} A_k$ is both a unital associative algebra and coalgebra over R, with the corresponding structure maps denoted by $\mu, \eta, \Delta, \varepsilon$. Moreover $\{A, \mu, \eta, \Delta, \varepsilon\}$ is again some bialgebra over R.

(3.2.6) Let $\{A, \mu, \eta\}$ be a unital associative algebra, and $\{A, \Delta, \varepsilon\}$ a coalgebra, over R, both graded with respect to $\mathbf{Z}_2$. Then the R-linear mappings $\mu : A\hat{\otimes}A \longrightarrow A$ and $\eta : R \longrightarrow A$ are homomorphisms of $\mathbf{Z}_2$-graded coalgebras over R, if and only if the R-linear mappings $\Delta : A \longrightarrow A\hat{\otimes}A$ and $\varepsilon : A \longrightarrow R$ are homomorphisms of $\mathbf{Z}_2$-graded unital associative algebras over R. In this case $\{A, \mu, \eta, \Delta, \varepsilon\}$ is called $\mathbf{Z}_2$-graded bialgebra, or super-bialgebra, over R.

(3.2.7) Let A, B be $\mathbf{Z}_2$-graded bialgebras over R. An R-linear mapping $\phi : A \longrightarrow B$ is called homomorphism of $\mathbf{Z}_2$-graded bialgebras over R, if and only if (i) ϕ preserves the $\mathbf{Z}_2$-grading, and (ii) ϕ is a homomorphism both of unital associative algebras and of coalgebras over R. Obviously the composition of such homomorphisms, and the inverse of any bijective homomorphism of this type, are again of this type.

(3.2.8) Let $A_k, k = 1, \ldots, p$, be $\mathbf{Z}_2$-graded bialgebras over R, and denote $\bigotimes_{k=1}^{p} A_k =: A$ as an R-bimodule. Then $\{A, \hat{\mu}, \hat{\eta}\}$ is an associative algebra over R, and $\{A, \hat{\Delta}, \hat{\varepsilon}\}$ is some coalgebra over R, both $\mathbf{Z}_2$-graded, and both denoted by $\hat{\bigotimes}_{k=1}^{p} A_k$. Moreover $\{A, \hat{\mu}, \hat{\eta}, \hat{\Delta}, \hat{\varepsilon}\}$ is some $\mathbf{Z}_2$-graded bialgebra over R, due to the subsequent commutative diagrams.

$$
\begin{tikzcd}
A\hat{\otimes}A \arrow[r, "\hat{\mu}"'] \arrow[rr, bend left, "counit"] \arrow[d, "comult."] & A \arrow[r, "\hat{\varepsilon}"'] \arrow[d, "\hat{\Delta}"'] & R \arrow[r, "\hat{\eta}"'] \arrow[rr, bend left, "id"] \arrow[d, "\Delta_R"] & A \arrow[r, "\hat{\varepsilon}"'] \arrow[d, "\hat{\Delta}"] & R \\
(A\hat{\otimes}A)\otimes(A\hat{\otimes}A) \arrow[r, "T(\hat{\mu}, \hat{\mu})"'] & A\hat{\otimes}A & R\otimes R \arrow[r, "T(\hat{\eta}, \hat{\eta})"'] & A\hat{\otimes}A &
\end{tikzcd}
$$

(3.2.9) The $\mathbf{Z}_2$-graded bialgebra $\{A,\mu,\eta,\Delta,\varepsilon\}$ over R is called graded-commutative-cocommutative, if and only if $\{A,\mu,\eta\}$ is graded-commutative and $\{A,\Delta,\varepsilon\}$ is graded-cocommutative.

(3.2.10) Note that a $\mathbf{Z}_2$-graded bialgebra need not be any bialgebra over R.

(3.2.11) Let $\{C,\Delta_C,\varepsilon_C\}$ be a coalgebra, and $\{A,\mu_A,\eta_A\}$ a unital associative algebra, both over R. Then $H := Hom_R(C,A)$ becomes a unital associative algebra over R with the structure mappings μ_H and η_H defined below.

$$\mu_H : H\otimes H \ni \phi\otimes\psi \longrightarrow \phi{*}\psi := \mu_A\circ T(\phi,\psi)\circ\Delta_C :$$
$$C\ni c\longrightarrow \sum_{l=1}^{L(c)}\phi(c_l')\psi(c_l'')\in A.$$

$$\eta_H : R\ni r\longrightarrow \eta_H(r) : C\ni c \underset{def}{\longrightarrow} r\varepsilon_C(c)e_A\in A,\quad e_H=\eta_A\circ\varepsilon_C.$$

(3.2.11.1) If $\{C,\Delta_C,\varepsilon_C\}$ is cocommutative and $\{A,\mu_A,\eta_A\}$ is commutative, then $\{H,\mu_H,\eta_H\}$ is commutative.

(3.2.11.2) Let $\{A,\mu,\eta,\Delta,\varepsilon\}$ be a $\mathbf{Z}_2$-graded bialgebra over R, and consider $H := Hom_R\ (A,A)$ with the above defined structure mappings η_H and μ_H, the latter called convolution. An element $\sigma\in H^{\bar 0}$ is called antipode of A, if and only if $\sigma{*}id\ A = id\ A{*}\sigma = \eta\circ\varepsilon$, i.e.:

$$\mu_A\circ T(\sigma, id\ A)\circ\Delta = \mu_A\circ T(id\ A,\sigma)\circ\Delta = \eta\circ\varepsilon.$$

(3.2.11.2.1) There exists at most one antipode of A, because $\eta\circ\varepsilon$ is the unit of this convolution.

(3.2.11.2.2) Let σ be the unique antipode of A. Then one concludes:

$$\sigma\circ\eta=\eta.\quad \varepsilon\circ\eta\circ\varepsilon=\varepsilon,\ \varepsilon\circ\sigma=\varepsilon.\quad T(\sigma,\sigma)\circ\Delta=\tau\circ\Delta\circ\sigma.$$

$$\forall a,b\in A:\sigma(ab)=\sum_{\bar x,\bar y\in\mathbf{Z}_2}(-1)^{xy}\sigma(b^{\bar y})\sigma(a^{\bar x}).$$

$$\{A,\mu,\eta\}\ graded-commutative\ \Longrightarrow \sigma\circ\sigma = id\ A.$$
$$\{A,\Delta,\varepsilon\}\ graded-cocommutative\ \Longrightarrow \sigma\circ\sigma = id\ A.$$

Hence σ is an antihomomorphism both of unital associative superalgebras and of $\mathbf{Z}_2$-graded coalgebras over R. For the corresponding proofs one needs an explicit notation of the images of Δ. Here τ denotes the $\mathbf{Z}_2$-graded flip.

(3.2.12) The $\mathbf{Z}_2$-graded bialgebra $\{A, \mu, \eta, \Delta, \varepsilon\}$ is called $\mathbf{Z}_2$-graded Hopf algebra, or Hopf superalgebra, over R, if and only if there exists an antipode σ of A, which is then necessarily unique. A bialgebra A over R is called Hopf algebra, if and only if there exists an antipode of A with respect to the trivial $\mathbf{Z}_2$-grading, such that $A =: A^{\bar{0}}, A^{\bar{1}} := \{0\}$.

(3.2.12.1) The graded (or twisted) tensor product $\hat{\bigotimes}_{k=1}^{p} A_k$ of $\mathbf{Z}_2$-graded Hopf algebras over R, with the unique antipodes σ_k, is again some $\mathbf{Z}_2$-graded Hopf algebra over R with the unique antipode $T(\sigma_k; k = 1, \ldots, p)$. The tensor product of Hopf algebras over R is viewed just as the case of trivial $\mathbf{Z}_2$-grading.

(3.2.13) Let $\{A, \mu, \eta, \Delta, \varepsilon\}$ be a $\mathbf{Z}_2$-graded bialgebra over R. Then $\{A, \mu, \eta, \Delta^{opp}, \varepsilon\}$, $\{A, \mu^{opp}, \eta, \Delta, \varepsilon\}$, and $\{A, \mu^{opp}, \eta, \Delta^{opp}, \varepsilon\}$ are also $\mathbf{Z}_2$-graded R-bialgebras. Here $\mu^{opp} := \mu \circ \tau$, $\Delta^{opp} := \tau \circ \Delta$ denote the opposite structure mappings.

(3.2.13.1) Especially consider a $\mathbf{Z}_2$-graded Hopf algebra $\{A, \mu, \eta, \Delta, \varepsilon\}$ over R with the unique antipode σ. Then also $\{A, \mu^{opp}, \eta, \Delta^{opp}, \varepsilon\}$ is some $\mathbf{Z}_2$-graded R-Hopf algebra with the same antipode σ. Moreover the $\mathbf{Z}_2$-graded bialgebra $\{A, \mu, \eta, \Delta^{opp}, \varepsilon\}$ over R is some $\mathbf{Z}_2$-graded Hopf algebra, if and only if σ is bijective, and then σ^{-1} is the antipode of the latter. The $\mathbf{Z}_2$-graded bialgebra $\{A, \mu^{opp}, \eta, \Delta, \varepsilon\}$ is some $\mathbf{Z}_2$-graded R-Hopf algebra, if and only if σ is bijective, and then again σ^{-1} is the antipode of the latter.

(3.2.14) Let $\{A_k, \mu_k, \eta_k, \Delta_k, \varepsilon_k\}, k = 1, 2$, be Hopf superalgebras over R with the antipodes σ_k. Every homomorphism $\phi : A_1 \longrightarrow A_2$ of super-bialgebras over R is also a homomorphism of Hopf superalgebras over R, in the sense of $\sigma_2 \circ \phi = \phi \circ \sigma_1$. Homomorphisms of Hopf algebras over R are defined with respect to the trivial $\mathbf{Z}_2$-grading. The composition of homomorphisms of Hopf superalgebras over R is again of this type. Especially, if a homomorphism ϕ of Hopf superalgebras over R is bijective, then ϕ^{-1} is also of this type, thereby establishing an isomorphism of Hopf superalgebras over R. Isomorphisms of Hopf algebras over R are defined with respect to the trivial $\mathbf{Z}_2$-grading.

(3.2.15) Let D be a graded linear subspace of the $\mathbf{Z}_2$-graded coalgebra $\{C, \Delta, \varepsilon\}$ over a field K. Then D is called coideal of C, if and only if

$$\Delta(D) \subseteq D \otimes C + C \otimes D, \quad \varepsilon(D) = \{0\}.$$

(3.2.15.1) The factor module $C/D =: C'$ becomes some $\mathbf{Z}_2$-graded coalgebra $\{C', \Delta', \varepsilon'\}$ over K, with the canonical projection defined by

$$\pi : C \ni c \longrightarrow c + D \in C',$$

and according to the following commutative diagrams.

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[rr, bend right, "\varepsilon"'] & C' \arrow[r, "\varepsilon'", "def"'] & K \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\pi"] & C\otimes C \arrow[d, "T(\pi\,,\pi)"] \\ C' \arrow[r, "\Delta'"] & C'\otimes C' \end{tikzcd}$$

Here π itself is some homomorphism of $\mathbf{Z}_2$-graded coalgebras over K.

(3.2.15.2) Let $\phi : C_1 \longrightarrow C_2$ be a homomorphism of $\mathbf{Z}_2$-graded coalgebras over K. Then

$$\Delta_1(ker\ \phi) \subseteq ker\ T(\phi, \phi) = C_1 \otimes ker\ \phi + ker\ \phi \otimes C_1,$$
$$\varepsilon_1(ker\ \phi) = \varepsilon_2 \circ \phi(ker\ \phi) = \{0\}.$$

Hence $ker\ \phi$ is some coideal of C_1. The resulting homomorphism theorem is expressed by the subsequent diagram.

$$\begin{tikzcd} C_1 \arrow[r, "\pi_1"'] \arrow[rrr, bend right, "\phi"'] & C_1\ /ker\ \phi \arrow[r, leftrightarrow, "\text{isomorphism of}"', "Z_2\text{ -graded coalgebras}"' near end] & Im\ \phi \arrow[r] & C_2 \end{tikzcd}$$

Here one uses the $\mathbf{Z}_2$-graded coalgebra $\{Im\ \phi, \Delta_2\big|_{Im\ \phi}, \varepsilon_2\big|_{Im\ \phi}\}$ over K.

(3.2.16) Let D be a graded linear subspace of the $\mathbf{Z}_2$-graded bialgebra $\{A, \mu, \eta, \Delta, \varepsilon\}$ over K, and assume that D is both an ideal of $\{A, \mu, \eta\}$ and coideal of $\{A, \Delta, \varepsilon\}$. Then obviously the factor module $A/D =: A'$ is some $\mathbf{Z}_2$-graded bialgebra over K, and the canonical projection π is a homomorphism of $\mathbf{Z}_2$-graded bialgebras over K.

(3.2.16.1) Let σ be the unique antipode of A. If especially $Im\ \sigma\big|_D \subseteq D$, then one finds the unique antipode σ' of $\{A', \mu', \eta', \Delta', \varepsilon'\}$:
$A' \ni [a] \longrightarrow [\sigma(a)] \in A'$.

(3.2.16.2) Let $\phi : A_1 \longrightarrow A_2$ be a homomorphism of $\mathbf{Z}_2$-graded bialgebras over K. The corresponding homomorphism theorem is shown in the subsequent diagram.

$$\begin{tikzcd} A_1 \arrow[r, "\pi_1"'] \arrow[rrr, bend right, "\phi"'] & A_1\ /ker\ \phi \arrow[r, leftrightarrow, "\text{isomorphism of}"', "\text{super-bialgebras}"' near end] & Im\ \phi \arrow[r] & A_2 \end{tikzcd}$$

3.3 Comodules

(3.3.1) Let M be a bimodule, and C a coalgebra, both over the commutative ring R. Then M is called C-right comodule over R, if and only if there is an R-linear map $\gamma : M \longrightarrow M \otimes C$, such that the following diagrams are commutative.

$$\begin{tikzcd}
M \ni m \arrow[rr, leftrightarrow, "def"'] \arrow[dr, "\gamma"'] & & m \otimes e_R \in M \otimes R \\
& M \otimes C \arrow[ur, "T(id\,M,\, \varepsilon)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes C \arrow[d, "T(id\,M,\, \Delta)"] & M \arrow[l, "\gamma"'] \arrow[r, "\gamma"'] & M \otimes C \arrow[d, "T(\gamma,\, id\,C)"] \\
M \otimes (C \otimes C) \arrow[rr, leftrightarrow] & & (M \otimes C) \otimes C
\end{tikzcd}$$

(3.3.2) C-left comodules over R are defined correspondingly. Especially let the coalgebra C be cocommutative; then every C-right comodule M over R can be viewed as an according C-left comodule over R too, with respect to the following commutative diagrams.

$$\begin{tikzcd}
M \ni m \arrow[r, leftrightarrow, "def"'] \arrow[dr, "\gamma"'] \arrow[drr, bend right, "\delta \quad def"'] & m \otimes e_R \in M \otimes R \arrow[r, leftrightarrow, "flip"'] & R \otimes M \\
& M \otimes C \arrow[r, leftrightarrow, "flip"] & C \otimes M \arrow[u, "T(\varepsilon,\, id\,M)"']
\end{tikzcd}$$

$$\begin{tikzcd}
C\otimes M \arrow[r, leftrightarrow, "flip"] \arrow[d, "T(\Delta, id M)"] & M\otimes C \arrow[d, "T(id M, \Delta)"] & M \arrow[l, "\gamma"'] \arrow[r, "\gamma"] & M\otimes C \arrow[r, leftrightarrow, "flip"] \arrow[d, "T(\gamma, id C)"] & C\otimes M \arrow[d, "T(id C, \delta)"] \\
(C\otimes C)\otimes M \arrow[r, leftrightarrow] & M\otimes(C\otimes C) \arrow[rr, leftrightarrow, "assoc."'] & & (M\otimes C)\otimes C \arrow[r, leftrightarrow] & C\otimes(C\otimes M) \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
(c\otimes d)\otimes m \arrow[r, leftrightarrow] \arrow[rrrr, leftrightarrow, bend right, "associative"'] & m\otimes(c\otimes d) \arrow[rr, leftrightarrow] & & (m\otimes c)\otimes d \arrow[r, leftrightarrow] & d\otimes(c\otimes m)
\end{tikzcd}$$

(3.3.3) Let the $\mathbf{Z}_2$-graded coalgebra C be graded-cocommutative, and assume γ to be even. Then the even R-linear map δ defined in the diagram below establishes an according C-left comodule M over R.

$$\begin{tikzcd}
M \arrow[r, "even"', "\gamma"] \arrow[rr, bend right, "\delta \quad def"'] & M\otimes C \ni m\otimes c \arrow[r, leftrightarrow, "flip"] & (-1)^{\bar{m}\,\bar{c}}\, c\otimes m \in C\otimes M
\end{tikzcd}$$

Here $M = M^{\bar{0}} \oplus M^{\bar{1}}$, and $\bar{m}, \bar{c}$ denote for the moment the $\mathbf{Z}_2$-degrees of m, c, respectively.

(3.3.4) A K-linear subspace N of the C-right comodule M over a field K is called C-right subcomodule of M, if and only if $Im\ \gamma|_N \subseteq N \otimes C$.

(3.3.5) Let $\gamma_k : M_k \longrightarrow M_k \otimes C, k = 1, 2$, be C-right comodules over R. An R-linear map $\phi : M_1 \longrightarrow M_2$ is called homomorphism of C-right comodules over R, if and only if the subsequent diagram is commutative.

$$\begin{tikzcd}
M_1 \arrow[r, "\phi"] \arrow[d, "\gamma_1"] & M_2 \arrow[d, "\gamma_2"] \\
M_1 \otimes C \arrow[r, "T(\phi, id C)"'] & M_2 \otimes C
\end{tikzcd}$$

Especially, if R is a field, then $ker\ \phi$ is some C-right subcomodule of M_1.

3.4 Duality of Finite-Dimensional Z_2-Graded Hopf Algebras

(3.4.1) For a super-coalgebra $\{H, \Delta, \varepsilon\}$ over a commutative ring R, its dual $H^* := Hom_R(H, R)$ becomes an associative superalgebra over R with the unit ε, with respect to the natural $\mathbf{Z}_2$-grading of H^*, due to the structure mappings

$$\mu^* : H^* \otimes H^* \ni f \otimes g \longrightarrow \mu_R \circ T(f, g) \circ \Delta \in H^*, \quad \eta^* : R \ni r \underset{def}{\longrightarrow} r\varepsilon \in H^*.$$

If the super-coalgebra $\{H, \Delta, \varepsilon\}$ is graded-cocommutative, then the associative superalgebra $\{H^*, \mu^*, \eta^*\}$ is graded-commutative.

(3.4.2) Conversely, let $\{H, \mu, \eta\}$ be an associative superalgebra over R with the unit $e_H := \eta(e_R)$.

(3.4.2.1)

$$\varepsilon^* : H^* := Hom_R(H, R) \ni f \underset{def}{\longrightarrow} f(e_H) \in R.$$

(3.4.2.2) Due to the universal property of tensor products one concludes, that

$$\forall f, g \in H^* : \exists \textit{ unique } \phi \in (H \otimes H)^* : \forall a, b \in H : \phi(a \otimes b) = f(a)g(b).$$

(3.4.2.3) The R-linear map ν defined by the following commutative diagram is injective.

$$\begin{tikzcd}
H^* \times H^* \arrow[rr] & & H^* \otimes H^* \arrow[dd, "\nu"] \\
\rotatebox{90}{$\in$} & & \\
\{f,g\} \arrow[r, "def"] & \phi \in (H \otimes H)^* & {} \arrow[l]
\end{tikzcd}$$

(3.4.2.4) Especially let H be finite-dimensional over a field K. Then ν is bijective, and one can define an even comultiplication Δ^* on H^*.

(3.4.2.4.1) The universal property of tensor products implies, that

$$\forall f \in H^* : \exists \textit{ unique } \phi_f \in (H \otimes H)^* : \forall a, b \in H : f(ab) = \phi_f(a \otimes b).$$

(3.4.2.4.2) The K-linear bijection ν from above is used in order to define

$$\Delta^* : H^* \in f \longrightarrow \sum_{l=1}^{L(f)} f_l' \otimes f_l'' \in H^* \otimes H^*,$$

such that $\forall f \in H^* : \nu \circ \Delta^*(f) = \phi_f$, which explicitly means that

$$\forall f \in H^*, \forall a, b \in H : f(ab) = \sum_{l=1}^{L(f)} f_l'(a) f_l''(b).$$

Thereby one establishes the $\mathbf{Z}_2$-graded coalgebra $\{H^*, \Delta^*, \varepsilon^*\}$ over K.
If the associative superalgebra $\{H, \mu, \eta\}$ is graded-commutative, then the super-coalgebra $\{H^*, \Delta^*, \varepsilon^*\}$ is graded-cocommutative.

(3.4.3) Hence the dual $H^* := Hom_K(H, K)$ of a finite-dimensional $\mathbf{Z}_2$-graded bialgebra $\{H, \mu, \eta, \Delta, \varepsilon\}$ over K is again some finite-dimensional $\mathbf{Z}_2$-graded bialgebra $\{H^*, \mu^*, \eta^*, \Delta^*, \varepsilon^*\}$ over the field K.

(3.4.3.1) If especially H is a $\mathbf{Z}_2$-graded Hopf algebra over K with the antipode σ, then H^* is again such an object with the antipode σ^*, which is defined such that

$$\forall f \in H^*, a \in H : \sigma^*(f)(a) := f \circ \sigma(a).$$

(3.4.3.2) With the trivial $\mathbf{Z}_2$-grading of $H =: H^{\bar{0}}, H^{\bar{1}} := \{0\}$, one obtains the duality of finite-dimensional bialgebras over K.

(3.4.3.3) The opposite structure maps of H are denoted by $\mu^{opp} := \mu \circ \tau$, $\Delta^{opp} := \tau \circ \Delta$, and similarly for H^*, with the $\mathbf{Z}_2$-graded flip τ. Obviously the dual of $\{H, \mu^{opp}, \eta, \Delta^{opp}, \varepsilon\}$ is $\{H^*, (\mu^*)^{opp}, \eta^*, (\Delta^*)^{opp}, \varepsilon^*\}$. The latter is equipped with the antipode σ^*, if $\{H, \mu, \eta, \Delta, \varepsilon\}$ allows for the antipode σ.

(3.4.3.4) The antipode σ of $\{H, \mu, \eta, \Delta, \varepsilon\}$ is bijective, because H is finite-dimensional. Therefore both $\{H, \mu, \eta, \Delta^{opp}, \varepsilon\}$ and $\{H, \mu^{opp}, \eta, \Delta, \varepsilon\}$ are $\mathbf{Z}_2$-graded Hopf algebras over K with the antipode σ^{-1}. Then one easily establishes the following dual $\mathbf{Z}_2$-graded Hopf algebras over K:

$$\{H, \mu, \eta, \Delta^{opp}, \varepsilon\} \underset{dual}{\longrightarrow} \{H^*, (\mu^*)^{opp}, \eta^*, \Delta^*, \varepsilon^*\},$$

$$\{H, \mu^{opp}, \eta, \Delta, \varepsilon\} \underset{dual}{\longrightarrow} \{H^*, \mu^*, \eta^*, (\Delta^*)^{opp}, \varepsilon^*\},$$

both duals H^* with the antipode $(\sigma^*)^{-1} = (\sigma^{-1})^*$.
Moreover $\sigma : \{H, \mu, \eta, \Delta, \varepsilon\} \longleftrightarrow \{H, \mu^{opp}, \eta, \Delta^{opp}, \varepsilon\}$ is some isomorphism of $\mathbf{Z}_2$-graded Hopf algebras over K, because of the commutative diagrams:

$$\sigma \circ \mu \circ \tau = \mu \circ T(\sigma, \sigma), \;\; T(\sigma, \sigma) \circ \Delta = \tau \circ \Delta \circ \sigma.$$

3.5 Examples of Z_2-Graded Hopf Algebras

(3.5.1) Consider the alternating algebra $\Lambda(E)$ of an R-bimodule E. The corresponding structure maps explicitly read, such that

$$\forall p, q \in \mathbf{N}, \forall r, s \in R : r \otimes s \xrightarrow[\mu]{} rs,$$

$$\Lambda(E) \otimes \Lambda(E) \ni (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \wedge \cdots \wedge y_q) \xrightarrow[\mu]{} x_1 \wedge \cdots \wedge y_q \in \Lambda(E),$$

$$r \otimes (y_1 \wedge \cdots \wedge y_q) \xrightarrow[\mu]{} r(y_1 \wedge \cdots \wedge y_q),$$

$$(x_1 \wedge \cdots \wedge x_p) \otimes s \xrightarrow[\mu]{} s(x_1 \wedge \cdots \wedge x_p),$$

and η is just the canonical embedding of R into $\Lambda(E)$.

(3.5.1.1) The canonical projection $\varepsilon : \Lambda(E) \longrightarrow R$ is some homomorphism of unital associative R-superalgebras, with respect to the natural $\mathbf{Z}_2$-grading due to the $\mathbf{N}_0$-grading of $\Lambda(E)$.

(3.5.1.2) Consider the R-linear map

$$\delta : E \ni x \longrightarrow x \otimes e_R + e_R \otimes x \in \Lambda(E) \hat{\otimes} \Lambda(E),$$

into the skew-symmetric tensor product of unital associative superalgebras over R. One immediately finds, that $\forall x \in E : \delta(x)\delta(x) = 0$. Hence one establishes a unique homomorphism of unital associative superalgebras Δ, due to the universal property of alternating algebras over R, such that the following diagram becomes commutative.

$$\begin{tikzcd} E \arrow[rr, "\text{embedding}"] \arrow[dr, "\delta"'] & & \Lambda(E) \arrow[dl, "\Delta"] \\ & \Lambda(E) \hat{\otimes} \Lambda(E) & \end{tikzcd}$$

(3.5.1.3) Obviously the unital associative superalgebra $\Lambda(E)\hat{\otimes}\Lambda(E)$ is again graded-commutative, just as $\Lambda(E)$ itself.

(3.5.1.4) The family $\{\Lambda(E), \mu, \eta, \Delta, \varepsilon\}$ is some $\mathbf{Z}_2$-graded Hopf algebra over R, which is graded-commutative-cocommutative, with the unique antipode σ, such that $\forall x \in E : \sigma(x) = -x$.

(3.5.1.5) For every $f \in Hom_R(E,F)$, $\Lambda(f)$ defined by the commutative diagram below is some homomorphism of $\mathbf{Z}_2$-graded Hopf algebras over R.

$$\begin{array}{ccccc}
E & \longrightarrow & T(E) \ni x_1 \otimes \ldots \otimes x_p & \longrightarrow & x_1 \wedge \ldots \wedge x_p \in \Lambda(E) \\
\Big\downarrow f & & \Big\downarrow T(f) & & \Big\downarrow \Lambda(f) \\
F & \longrightarrow & T(F) \ni \bigotimes_{k=1}^{p} f(x_k) & \longrightarrow & \bigwedge_{k=1}^{p} f(x_k) \in \Lambda(F)
\end{array}$$

(3.5.2) An element a of a coalgebra $\{A, \Delta, \varepsilon\}$ over R is called group-like, if and only if $\Delta(a) = a \otimes a$. An element $b \in A$ is called primitive with respect to the group-like element a, if and only if $\Delta(b) = a \otimes b + b \otimes a$.

(3.5.2.1) Let $\{A, \mu, \eta, \Delta, \varepsilon\}$ be a $\mathbf{Z}_2$-graded Hopf algebra over R. Denote the set of group-like elements of $\{A, \Delta, \varepsilon\}$ by G. Then obviously

$$e_A \in G := \{a \in A; \Delta(a) = a \otimes a\} \subseteq A^{\bar{0}}, \ \ \forall a, b \in G : ab \in G, \ \ \varepsilon(a) = e_R.$$

Now let σ be the unique antipode of $\{A, \mu, \eta, \Delta, \varepsilon\}$. Then

$$\forall a \in G : a\sigma(a) = \sigma(a)a = e_A.$$

Hence the set G becomes some group, with the group operation induced by μ. The elements, which are primitive with respect to e_A, are called primitive. The linear subspace of these elements is graded, such that

$$L_A := \{a \in A; \Delta(a) = e_A \otimes a + a \otimes e_A\} = (L_A \cap A^{\bar{0}}) \oplus (L_A \cap A^{\bar{1}}).$$

$$\forall a, b \in L_A : [a, b] := ab - \sum_{\bar{x}, \bar{y} \in \mathbf{Z}_2} (-1)^{xy} b^{\bar{y}} a^{\bar{x}} \in L_A, \ \ \varepsilon(a) = 0.$$

Hence the Lie superalgebra L_A over R, consisting of the primitive elements of A, is established.

(3.5.3) Let $E(L)$ denote the universal enveloping superalgebra of a Lie superalgebra L, over a field K of characteristic *char* $K \neq 2$.

(3.5.3.1) The homomorphism of Lie superalgebras over K:

$$L \oplus L \ni \{x, y\} \longrightarrow x \otimes e + e \otimes y \in E(L) \hat{\otimes} E(L),$$

with the unit e of $E(L)$, fulfills the defining property of universal enveloping superalgebras. Hence there is an isomorphism of unital associative K-superalgebras:

$$E(L \oplus L) \longleftrightarrow E(L) \hat{\otimes} E(L).$$

(3.5.3.2) The universal property of $E(L)$ is used in order to define the homomorphism Δ of unital associative superalgebras over K, according to the following diagram.

$$\begin{tikzcd}
L \ni x \arrow[rr, "\text{homomorphism of Lie-superalgebras}"'] \arrow[dr] & & x \otimes e + e \otimes x \in E(L) \hat{\otimes} E(L) \\
& E(L) \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(3.5.3.3) A homomorphism $\varepsilon : E(L) \longrightarrow K$ of unital associative superalgebras is defined, such that $\forall x \in L : \varepsilon(x) = 0,\ \varepsilon(e) = e_K$, with the unit e_K of K.

(3.5.3.4) Hence one obtains the $\mathbf{Z}_2$-graded Hopf algebra $\{E(L), \mu, \eta, \Delta, \varepsilon\}$ over K, with the unique antipode σ of $E(L)$ defined as the so-called principal graded anti-automorphism of $E(L)$, such that

$$\forall x \in L : \sigma(x) = -x, \quad \sigma \circ \sigma = id\ E(L).$$

Here μ, η denote the structure maps of the unital associative superalgebra $E(L)$.

(3.5.3.5) The $\mathbf{Z}_2$-graded coalgebra $\{E(L), \Delta, \varepsilon\}$ is graded-cocommutative.

(3.5.3.6) The Lie superalgebra of primitive elements of $E(L)$ is L itself. For an appropriate proof one needs the $\mathbf{Z}_2$-graded version of the Poincaré-Birkhoff-Witt theorem. Obviously the only group-like element of $E(L)$ is 0.

(3.5.4) Let G be any group, and denote by $H := K(G)$ the vector space over a field K, which is generated by the basis G.

(3.5.4.1) H becomes a unital associative algebra over K with the structure maps

$$\mu_H : H \otimes H \ni \sum_{k,l \in I_{fin}} c_{kl} g_k \otimes g_l \longrightarrow \sum_{k,l \in I_{fin}} c_{kl} g_k g_l \in H,$$

$$K \ni c \underset{def}{\longrightarrow} c\, e_G \in H,$$

with the unit $e_G =: e_H$ of H. Here I_{fin} denotes an arbitrary finite subset of the index set: $I \longleftrightarrow G$.

(3.5.4.2) With the structure maps defined by the subsequent diagrams one establishes the cocommutative coalgebra $\{H, \Delta_H, \varepsilon_H\}$ over K.

$$
\begin{tikzcd}
G \ni g \arrow[rr, "\text{free over } K"] \arrow[d, "def"'] & & H \arrow[dll, "\Delta_H"] \arrow[drr, "\varepsilon_H"'] & & G \ni g \arrow[ll, "\text{free over } K"'] \arrow[d, "def"] \\
g\otimes g \in H\otimes H & & & & K \ni e_K
\end{tikzcd}
$$

(3.5.4.3) Moreover $\{H, \mu_H, \eta_H, \Delta_H, \varepsilon_H\}$ is some Hopf algebra over K with the unique antipode σ_H, which is constructed by the diagram below, and $\sigma_H \circ \sigma_H = id\ H$.

$$
\begin{tikzcd}
G \ni g \arrow[r, "\text{free over } K"] \arrow[d, "def"'] & H \arrow[d, "\sigma_H"] \\
g^{-1} \in H & g^{-1} \in H \arrow[l, equal]
\end{tikzcd}
$$

3.6 Smash Product of $\mathbf{Z}_2$-Graded Hopf Algebras

(3.6.1.1) Consider two $\mathbf{Z}_2$-graded bialgebras E and H over a commutative ring R, with the trivial grading of $H =: H^{\bar{0}}$, and let $\mu : H \otimes E \longrightarrow E$ be an even R-linear map, which fulfills the following conditions.

(i) $\forall x \in E, \forall f, g \in H : \mu(e_H \otimes x) = x,\ \ \mu(fg \otimes x) = \mu(f \otimes \mu(g \otimes x))$.

Hence E becomes an H-left module over R.

(ii) The pair $\{\mu, H\}$ measures E to E in the sense, that with the notation:

$$H \ni f \xrightarrow[\Delta_H]{} \sum_{l=1}^{L(f)} f_l' \otimes f_l'' \in H \otimes H,\ \ \forall x, y \in E, \forall f \in H :$$

$$\mu(f \otimes xy) = \sum_{l=1}^{L(f)} \mu(f_l' \otimes x)\mu(f_l'' \otimes y),\ \ \mu(f \otimes e_E) = \varepsilon_H(f)e_E.$$

(iii) μ is a homomorphism of $\mathbf{Z}_2$-graded coalgebras over R.

(3.6.1.2) The twisted tensor product of $\mathbf{Z}_2$-graded coalgebras $H\hat{\otimes}E$ is constructed over R, the twist being ineffective due to the trivial $\mathbf{Z}_2$-grading of H.

(3.6.1.3) The so-called smash (or crossed) product of $H \otimes E$ is defined, such that $\forall x, y \in E, \forall f, g \in H$:

$$(f \otimes x)(g \otimes y) = \sum_{l=1}^{L(f)} (f_l'' g) \otimes x\mu(f_l' \otimes y),$$

which especially implies that $(e_H \otimes x)(g \otimes e_E) = g \otimes x$. One thereby obtains some $\mathbf{Z}_2$-graded bialgebra over R, which is denoted by $H \otimes_\mu E$, with the unit $e_H \otimes e_E$.

(3.6.1.4) If both E and H are $\mathbf{Z}_2$-graded Hopf algebras over R, then $H \otimes_\mu E$ is also such an object with the antipode:

$$H \otimes_\mu E \ni f \otimes x \xrightarrow[\sigma_\mu]{} \sum_{l=1}^{L(f)} \sigma_H(f_l') \otimes \mu(\sigma_H(f_l'') \otimes \sigma_E(x)) \in H \otimes_\mu E.$$

$$\forall x \in E, f \in H : \sigma_\mu(f \otimes e_E) = \sigma_H(f) \otimes e_E, \;\; \sigma_\mu(e_H \otimes x) = e_H \otimes \sigma_E(x).$$

(3.6.2) Consider the graded-cocommutative Hopf algebras $H := K(G)$ and $E := E(L)$ over a field K of *char* $K \neq 2$, the former with the trivial grading of $H =: H^{\bar{0}}$, which were defined previously.

(3.6.2.1) The graded tensor product of unital associative superalgebras, and the graded tensor product of $\mathbf{Z}_2$-graded coalgebras, are constructed over K. Denote $F := H\hat{\otimes}E$.

$$\forall h_1, h_2 \in H, \forall a_1, a_2 \in E : (h_1 \otimes a_1)(h_2 \otimes a_2) = (h_1 h_2) \otimes (a_1 a_2).$$

$$\forall g \in G : \varepsilon_F(g \otimes e_E) = e_K, \;\; \Delta_F(g \otimes e_E) = (g \otimes e_E) \otimes (g \otimes e_E),$$

$$\forall x \in L : \varepsilon_F(g \otimes x) = 0,$$

$$\Delta_F(g \otimes x) = (g \otimes x) \otimes (g \otimes e_E) + (g \otimes e_E) \otimes (g \otimes x).$$

Hence one obtains the $\mathbf{Z}_2$-graded Hopf algebra $\{F, \mu_F, \eta_F, \Delta_F, \varepsilon_F\}$ over K. Obviously the $\mathbf{Z}_2$-graded coalgebra $\{F, \Delta_F, \varepsilon_F\}$ is graded-cocommutative. Here $\sigma_F := T(\sigma_H, \sigma_E)$ is the unique antipode of F, and $\sigma_F \circ \sigma_F = id\, F$, with the principal graded anti-automorphism σ_E of E. One finds, that

$$\forall g \in G, x \in L : \sigma_F(g \otimes x) = g^{-1} \otimes (-x), \;\; \sigma_F(g \otimes e_E) = g^{-1} \otimes e_E.$$

(3.6.2.2) The group of group-like elements of F is: $\{g \otimes e_E; g \in G\} \longleftrightarrow G$, indicating an isomorphism in the sense of groups.
The Lie superalgebra of primitive elements of F is: $\{e_H \otimes x; x \in L\} \longleftrightarrow L$, with a natural isomorphism of Lie superalgebras over K.
Here again one uses the $\mathbf{Z}_2$-graded version of the Poincaré-Birkhoff-Witt theorem.

(3.6.3) This graded tensor product of $\mathbf{Z}_2$-graded Hopf algebras over K is generalized with respect to a homomorphism of groups $\pi_L : G \longrightarrow Aut(L)$, into the group of isomorphisms of the Lie superalgebra L over K onto itself. Due to the universal property of $E := E(L)$, for any $g \in G, \pi_L(g)$ is lifted to an isomorphism $\pi(g) : E \longleftrightarrow E$ of unital associative superalgebras over K. Hence π_L is lifted to some homomorphism of groups $\pi : G \longrightarrow Aut(E)$, into the group of isomorphisms of $\mathbf{Z}_2$-graded Hopf algebras over the field K, *char* $K \neq 2$.

(3.6.3.1) The universal property of tensor products allows for the definition of an appropriate structure map

$$\mu_\pi : F_\pi \otimes F_\pi \longrightarrow F_\pi, \; F_\pi := H \otimes E, \; H := K(G)$$

as vector space over K, such that

$$\forall h = \sum_{k \in I_{fin}} c_k g_k, h' \in H, \forall a, a' \in E :$$

$$F_\pi \otimes F_\pi \ni (h \otimes a) \otimes (h' \otimes a')$$
$$\underset{\mu_\pi}{\longrightarrow} (h \otimes a) \circ_\pi (h' \otimes a') := \sum_{k \in I_{fin}} c_k(g_k h') \otimes (a\pi(g_k)(a')) \in F_\pi,$$

thereby constructing an associative superalgebra F_π with the unit $e_H \otimes e_E$, which is denoted by $H \otimes_\pi E$.

(3.6.3.2) More explicitly, the following inclusions are shown in the diagrams below.

$$F_\pi \xleftarrow[\text{subalgebra}]{} \{e_H \otimes a; a \in E(L)\} \xleftrightarrow[\text{isomorphism}]{} E(L)$$

$$F_\pi \xleftarrow[\text{subalgebra}]{} \{h \otimes e_E; h \in K(G)\} \xleftrightarrow[\text{isomorphism}]{} K(G)$$

(3.6.3.3)

$$\forall a \in E, h \in H : (e_H \otimes a) \circ_\pi (h \otimes e_E) = h \otimes a,$$
$$\forall g \in G : (g \otimes e_E) \circ_\pi (e_H \otimes a) \circ_\pi (g^{-1} \otimes e_E) = e_H \otimes \pi(g)(a).$$

Here one uses, that $\forall g \in G : \pi(g)(e_E) = e_E,\ \pi(e_H) = id\ E$.
Obviously, in case of $Im\ \pi := \{id\ E\}$ one recovers the skew-symmetric tensor product $H \hat{\otimes} E$ from above.

(3.6.3.4) The comultiplication Δ_π and counit ε_π on F_π are defined in order to fulfill

$$\begin{aligned}\forall g \in G, x \in L : &\Delta_\pi(g \otimes x) = (g \otimes x) \otimes (g \otimes e_E) + (g \otimes e_E) \otimes (g \otimes x),\\ &\Delta_\pi(g \otimes e_E) = (g \otimes e_E) \otimes (g \otimes e_E),\\ &\varepsilon_\pi(g \otimes e_E) = e_K,\ \varepsilon_\pi(g \otimes x) = 0.\end{aligned}$$

The $\mathbf{Z}_2$-graded coalgebra $\{F_\pi, \Delta_\pi, \varepsilon_\pi\}$ over K is graded-cocommutative.

(3.6.3.5) The $\mathbf{Z}_2$-graded coalgebras $\{F_\pi, \Delta_\pi, \varepsilon_\pi\}$ and $\{F, \Delta_F, \varepsilon_F\}$ coincide. For instance, $\forall g \in G, \forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall x_1 \in L^{\overline{z_1}}, x_2 \in L^{\overline{z_2}}$:

$$\begin{aligned}\Delta_F(g \otimes x_1x_2) &= \Delta_\pi(g \otimes x_1x_2)\\ &= (g \otimes x_1x_2) \otimes (g \in e_E) + (g \otimes e_E) \otimes (g \otimes x_1x_2)\\ &+ (g \otimes x_1) \otimes (g \otimes x_2) + (-1)^{z_1z_2}(g \otimes x_2) \otimes (g \otimes x_1),\\ \varepsilon_F(g \otimes x_1x_2) &= \varepsilon_\pi(g \otimes x_1x_2) = 0.\end{aligned}$$

(3.6.3.6) Hence one obtains the $\mathbf{Z}_2$-graded Hopf algebra $\{F_\pi, \mu_\pi, \eta_\pi, \Delta_\pi, \varepsilon_\pi\}$ over K with the unique antipode σ_π, such that

$$\forall g \in G, a \in E : \sigma_\pi(g \otimes a) = g^{-1} \otimes (\pi(g^{-1}) \circ \sigma_E(a)).$$

Especially

$$\forall g \in G, x \in L : \sigma_\pi(g \otimes e_E) = g^{-1} \otimes e_E,\ \ \sigma_\pi(e_H \otimes x) = e_H \otimes (-x).$$

$\forall g \in G : \pi(g) \circ \sigma_E = \sigma_E \circ \pi(g)$, and therefore $\sigma_\pi \circ \sigma_\pi = id\ F_\pi$.

(3.6.3.7) Again the $\mathbf{Z}_2$-graded version of the Poincaré-Birkhoff-Witt theorem is used.
The group of group-like elements of F_π is $\{g \otimes e_E; g \in G\} \longleftrightarrow G$.
The Lie superalgebra of primitive elements is $\{e_H \otimes x; x \in L\} \longleftrightarrow L$.
These bijections are isomorphisms, the former in the sense of groups, and the latter in the sense of Lie superalgebras over K, respectively.

(3.6.3.8) This $\mathbf{Z}_2$-graded Hopf algebra $K(G) \otimes_\pi E(L)$ is some special case of the above defined smash product $H \otimes_\mu E$, defining μ such that

$$\forall a \in E(L), g \in G : \mu(g \otimes a) = \pi(g)(a).$$

3.7 Graded Star Operations on Z_2-Graded Hopf Algebras

(3.7.1) Let the commutative ring R of coefficients, of an associative superalgebra $\{H, \mu, \eta\}$ with the unit $\eta(e_R) =: e_H$, be equipped with an involutive isomorphism of rings: $R \ni r \longleftrightarrow \bar{r} \in R$, for instance complex conjugation in case $R := \mathbf{C}$. The bijection: $H \ni a \longleftrightarrow a^* \in H$ is called graded star operation on H, if and only if the following conditions are fulfilled.

$$\forall a, b \in H, \forall r, s \in R : (ra + sb)^* = \bar{r}a^* + \bar{s}b^*, \quad a^{**} := (a^*)^* = a;$$
$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2 : a \in H^{\bar{x}}, b \in H^{\bar{y}} \Longrightarrow a^* \in H^{\bar{x}}, \quad (ab)^* = (-1)^{xy} b^* a^*.$$

Obviously $e_H^* = e_H$.

(3.7.1.1) This graded star operation is then lifted to some graded star operation on the skew-symmetric tensor product of associative superalgebras over R, such that $\forall p \in \mathbf{N}, \forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2$:

$$H^{\overline{z_1}} \otimes \cdots \otimes H^{\overline{z_p}} \ni a_1 \otimes \cdots \otimes a_p \underset{*}{\longrightarrow} (-1)^{\sum_{1 \le k < l \le p} z_k z_l} a_p^* \otimes \cdots \otimes a_1^* \in H \hat{\otimes} \cdots \hat{\otimes} H.$$

(3.7.1.2) The structure mapping μ is compatible with this graded star operation and its lifting in the sense, that: $* \circ \mu = \mu \circ *$.

(3.7.2) Consider a $\mathbf{Z}_2$-graded coalgebra $\{H, \Delta, \varepsilon\}$ over the commutative ring R, the latter again being equipped with an involution. An idempotent even antilinear bijection $*$ on H is called graded star operation on H, if and only if the structure mappings Δ and ε are compatible with it in the sense, that:

$$\Delta \circ * = * \circ \Delta, \quad \forall a \in H : \varepsilon(a^*) = \overline{\varepsilon(a)}.$$

Here again the bijection $*$ is lifted to the twisted tensor product $H \hat{\otimes} H$ of coalgebras over R, namely to $\tau \circ T(*, *)$, inserting the $\mathbf{Z}_2$-graded twist τ, and more generally to $H \hat{\otimes} \cdots \hat{\otimes} H$ with finitely many copies of H.

(3.7.2.1) Inserting the natural R-linear bijections:

$$(H \otimes H) \otimes H \ni (a \otimes b) \otimes c \underset{\alpha_{12}}{\longleftrightarrow} a \otimes b \otimes c \underset{\alpha_{23}^{-1}}{\longleftrightarrow} a \otimes (b \otimes c) \in H \otimes (H \otimes H),$$

one finds the commuting diagram:

$$* \circ \alpha_{12} \circ T(\Delta, id\ H) \circ \Delta = \alpha_{23} \circ T(id\ H, \Delta) \circ \Delta \circ *.$$

(3.7.2.2) The twisted tensor product of two $\mathbf{Z}_2$-graded coalgebras over R is thereby equipped with an appropriate graded star operation, such that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in H^{\bar{x}}, b \in H^{\bar{y}} : (a \otimes b)^* = (-1)^{xy} b^* \otimes a^* \in H \hat{\otimes} H.$$

(3.7.3) An idempotent even antilinear bijection $*$ on a $\mathbf{Z}_2$-graded bialgebra H over R is called graded star operation on H, if and only if it is compatible with the four structure mappings in the above described sense.

(3.7.3.1) In the special case of a $\mathbf{Z}_2$-graded Hopf algebra H one demands, that a graded star operation be compatible with the antipode σ in the sense, that: $\sigma \circ * = * \circ \sigma$.

(3.7.3.2) In the special case of $R \ni r \longleftrightarrow \bar{r} := r \in R$, for instance real coefficients, and of $\sigma \circ \sigma = id\ H$, the antipode σ is some graded star operation on H.

3.8 $\mathbf{Z}_2$-Graded Lie Coalgebras and Lie Bialgebras

(3.8.1) Let $L = L^{\bar{0}} \oplus L^{\bar{1}}$ be a vectorspace over a field K of *char* $K \neq 2$, and

$$\Theta : L \ni a \longrightarrow \sum_{l=1}^{L(a)} a'_l \otimes a''_l \in L \otimes L$$

a K-linear map, which is even with respect to the natural grading of tensor products. L is called $\mathbf{Z}_2$-graded Lie coalgebra, or Lie super-coalgebra, over K, if and only if the following conditions hold.

(i) Θ is skew-symmetric in the sense of: $\tau \circ \Theta = -\Theta$.

(ii) The so-called co-Jacobi identity holds:

$$\alpha \circ T(\Theta, id) \circ \Theta - T(id, \Theta) \circ \Theta = T(id, \tau) \circ \alpha \circ T(\Theta, id) \circ \Theta.$$

Here $id \equiv id\ L, \tau$ denotes the $\mathbf{Z}_2$-graded flip on $L \otimes L$, and α merely changes the succession of tensor products: $(L \otimes L) \otimes L \longleftrightarrow L \otimes (L \otimes L)$, both being identified with $L \otimes L \otimes L$ henceforth.

(3.8.2) Consider a Lie superalgebra, which is also a $\mathbf{Z}_2$-graded Lie coalgebra over K. L is called $\mathbf{Z}_2$-graded Lie bialgebra, or Lie super-bialgebra, over K, if and only if the above denoted structure map Θ is compatible with the super-commutator in the sense, that $\forall$ homogeneous $a, b \in L$:

$$[a,b] \xrightarrow[\Theta]{} \sum_{l=1}^{L(a)} (a'_l \otimes [a''_l, b] + (-1)^{\overline{x''_l} y}[a'_l, b] \otimes a''_l)$$
$$+ \sum_{l=1}^{L(b)} ([a, b'_l] \otimes b''_l + (-1)^{x \overline{y'_l}} b'_l \otimes [a, b''_l]),$$

inserting explicitly $\Theta(a)$ and $\Theta(b)$, and denoting by $\bar{x}, \bar{y}, \overline{x''_l}, \overline{y'_l}$ the degrees of a, b, a''_l, b'_l, respectively.

(3.8.2.1) Via the universal enveloping superalgebra $E(L)$ of L, using the super-commutator of $E(L)\hat{\otimes}E(L)$, one may rewrite the above compatibility condition as the following 1-cocycle property:

$$L \ni [a,b] \xrightarrow[\Theta]{} [\Theta(a), \delta(b)] + [\delta(a), \Theta(b)] \in L \otimes L \subset E(L)\hat{\otimes}E(L),$$

inserting the diagonal map, which is defined with the unit e of $E(L)$:

$$L \ni a \xrightarrow[def]{\delta} a \otimes e + e \otimes a \in E(L)\hat{\otimes}E(L).$$

(3.8.3) The dual vector space $L^* := Hom_K(L, K)$ of a $\mathbf{Z}_2$-graded Lie coalgebra L over K, with the natural grading of K-linear forms, becomes some Lie superalgebra over K with the following super-commutator.
For $a \in L$, f and $g \in L^*$, inserting explicitly $\Delta(a)$, and denoting by $\overline{x'_l}$ and $\bar{z}$ the degrees of a'_l and g, respectively,

$$[f,g](a) := \sum_{l=1}^{L(a)} f(a'_l) g(a''_l)(-1)^{z x'_l}.$$

This definition may be conveniently rewritten, such that

$$\forall a \in L, \forall f, g \in L^* : \langle [f,g] | a \rangle = \langle f \otimes g | \Theta(a) \rangle.$$

(3.8.3.1) Here one uses the notation, that

$$\langle f_1 \otimes \cdots \otimes f_n | a_1 \otimes \cdots \otimes a_n \rangle = (-1)^{\sum_{1 \le l < k \le n} z_k x_l} \langle f_1 | a_1 \rangle \cdots \langle f_n | a_n \rangle$$

for homogeneous $f_k \in L^*$ and $a_k \in L$ of degrees z_k and x_k, respectively.

(3.8.3.2) The duality of Jacobi and co-Jacobi identity is shown easily, using that $\forall a \in L, \forall f, g, h \in L^*$:

$$\langle [[f,g], h] \, | a \rangle = \langle f \otimes g \otimes h | T(\Theta, id) \circ \Theta(a) \rangle,$$
$$\langle [f, [g,h]] \, | a \rangle = \langle f \otimes g \otimes h | T(id, \Theta) \circ \Theta(a) \rangle.$$

(3.8.4) Conversely, consider a finite-dimensional Lie superalgebra L over a field K. Then the dual vector space L^* becomes some $\mathbf{Z}_2$-graded Lie coalgebra over K, such that

$$\forall a, b \in L, \forall f \in L^* : \langle \Theta(f) | a \otimes b \rangle = \langle f | [a, b] \rangle .$$

(3.8.5) Hence one establishes the dual vector space L^* of a finite-dimensional $\mathbf{Z}_2$-graded Lie bialgebra over a field K as some object of the same category.

(3.8.6) A K-linear map $\phi : L \longrightarrow L'$ of $\mathbf{Z}_2$-graded Lie coalgebras over K is called homomorphism of such objects, if and only if: $T(\phi, \phi) \circ \Theta = \Theta' \circ \phi$, denoting by Θ and Θ' the involved structure maps. Thereby one establishes an according category.

(3.8.7) Moreover one uses the category of $\mathbf{Z}_2$-graded Lie bialgebras over K, its morphisms being expected to be compatible both with the super-commutators and the co-structure mappings.

(3.8.8) The corresponding non-graded statements are obtained, taking as odd vector subspace of L the trivial one.

(3.8.9) The complex Lie algebra $sl(2, \mathbf{C}) \equiv A_1$ becomes some Lie bialgebra with the structure maps:

$$[h, x^\pm] = \pm 2x^\pm, \;\; [x^+, x^-] = h; \;\; h \underset{\Theta}{\longrightarrow} 0, \;\; x^\pm \underset{\Theta}{\longrightarrow} x^\pm \otimes h - h \otimes x^\pm .$$

The dual vectorspace A_1^*, with the dual basis $\{g, f^\pm\}$ such that

$$\langle g|h \rangle = 1, \;\; \langle g|x^\pm \rangle = 0, \;\; \langle f^\pm|h \rangle = 0, \;\; \langle f^\pm|x^\pm \rangle = 1, \;\; \langle f^\pm|x^\mp \rangle = 0,$$

is some complex Lie bialgebra, the structure maps of which may be denoted as those of A_1 for convenience.

$$[g, f^\pm] = -f^\pm, \;\; [f^+, f^-] = 0;$$

$$g \underset{\Theta}{\longrightarrow} f^+ \otimes f^- - f^- \otimes f^+, \;\; f^\pm \underset{\Theta}{\longrightarrow} \pm 2(g \otimes f^\pm - f^\pm \otimes g).$$

(3.8.10) The complex Lie superalgebra $B(0, 1)$ is also some $\mathbf{Z}_2$-graded Lie bialgebra, with the structure mappings looking like those of A_1, merely inserting the anti-commutator of $x^\pm$. In order to construct the complex $\mathbf{Z}_2$-graded Lie bialgebra $B(0, 1)^*$, one may use the dual bases $\{h, x^\pm, (x^\pm)^2\}$ and $\{g, f^\pm, (f^\pm)^2\}$, h and g being even, $x^\pm$ and $f^\pm$ odd.

$$\langle g|h \rangle = \langle f^\pm|x^\pm \rangle = 1, \;\; \langle (f^\pm)^2|(x^\pm)^2 \rangle = \mp 2, \;\; \langle g|x^\pm \rangle = \cdots = 0.$$

One then calculates easily the following structure mappings.

$$[g, f^\pm] = -f^\pm, \;\; [f^+, f^-] = 0;$$

$$f^{\pm} \xrightarrow[\Theta]{} \pm 2(g \otimes f^{\pm} - f^{\pm} \otimes g) + (f^{\pm})^2 \otimes f^{\mp} - f^{\mp} \otimes (f^{\pm})^2,$$

$$g \xrightarrow[\Theta]{} f^- \otimes f^+ - f^+ \otimes f^- + \frac{1}{2}\left((f^+)^2 \otimes (f^-)^2 - (f^-)^2 \otimes (f^+)^2\right).$$

3.9 Quasitriangular $\mathbf{Z}_2$-Graded Lie Bialgebras

(3.9.1) Consider a Lie superalgebra L over a field K of *char* $K \neq 2$. Assume existence of an even element $r \in L \otimes L$, such that $\forall a \in L$:

(i) $[r + \tau(r), \delta(a)] = 0, \ \delta(a) := a \otimes e + e \otimes a,$

(ii) $[[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], T(\delta, id\ L)(a)] = 0,$

identifying: $(L \otimes L) \otimes L \longleftrightarrow L \otimes (L \otimes L) \longleftrightarrow L \otimes L \otimes L$. Here one usually denotes, with the unit e of the universal enveloping superalgebra $E(L)$,

$$r_{12} := \sum_{l=1}^{I} r_l' \otimes r_l'' \otimes e, \quad r_{13} := \sum_{l=1}^{I} r_l' \otimes e \otimes r_l'', \quad r_{23} := \sum_{l=1}^{I} e \otimes r_l' \otimes r_l'',$$

hereby inserting an explicit notation of r, such that for instance $r_{12} = r \otimes e$, and using the super-commutators of $\bigotimes^n E(L), n = 2, 3$. Then the even K-linear map:

$$L \ni a \xrightarrow[def]{\Theta} [\delta(a), r] \in L \otimes L$$

defines some $\mathbf{Z}_2$-graded Lie bialgebra L over K. This structure map Θ is called coboundary of L.

(3.9.1.1) If r fulfills the so-called classical Yang-Baxter equation (CYBE), such that

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

then L is called quasitriangular with respect to the classical R-matrix r. If moreover $r + \tau(r) = 0$, then L is called triangular.

(3.9.2) The above defined complex Lie bialgebra A_1 is quasitriangular with respect to the classical R-matrix

$$r := \frac{1}{2} h \otimes h + 2x^+ \otimes x^-.$$

(3.9.3) The $\mathbf{Z}_2$-graded complex Lie bialgebra $B(0,1)$ is quasitriangular with respect to the classical R-matrix

$$r := \frac{1}{2} h \otimes h - 2x^+ \otimes x^- - 2(x^+)^2 \otimes (x^-)^2.$$

(3.9.4) Let the finite-dimensional complex Lie superalgebra $L \neq A(1,1)$ be basic classical, or purely even and simple, with the generators $h_k, x_k^+ \equiv x_k$, $x_k^- \equiv y_k, k = 1, \ldots, s$. Then the coboundary $\Theta : L \longrightarrow L \otimes L$, such that

$$\forall_1^s k : h_k \xrightarrow[\Theta]{} 0,\ x_k^{\pm} \xrightarrow[\Theta]{} d_k (x_k^{\pm} \otimes h_k - h_k \otimes x_k^{\pm}),$$

establishes L as some $\mathbf{Z}_2$-graded Lie bialgebra over $\mathbf{C}$.
The integers $d_k, k = 1, \ldots, s$, are chosen such that

(i) $\forall_1^s k, l : d_k \Gamma_{kl} = d_l \Gamma_{lk} =: B_{kl},\ d_k \neq 0,\ \tilde{B}_{kl} := \Gamma_{kl} d_l^{-1} = \tilde{B}_{lk}$.

(ii) In the purely even case of $\tau = \emptyset, \forall_1^s k : d_k > 0$. $\forall k \in \tau : d_k > 0$.

(iii) The greatest common divisor of $\{d_k; k = 1, \ldots, s\}$ is equal to 1.
Here $\tau \subseteq \{1, \ldots, s\}$ denotes the set of indices k such that $x_k^{\pm}$ are odd.

(3.9.5) Consider a simple finite-dimensional complex Lie algebra L of rank m, with the $3m$ generators $h_k, x_k \equiv x_k^+, y_k \equiv x_k^-, k = 1, \ldots, m$. The Lie bialgebra L is quasitriangular with respect to the classical R-matrix

$$r := \sum_{k,l=1}^{m} \left(\tilde{B}^{-1}\right)_{kl} h_k \otimes h_l + 2 \sum_{j=1}^{p} d_{k_j} x^+(\beta_j) \otimes x^-(\beta_j).$$

Here the positive and negative root vectors are defined most conveniently by means of the automorphisms $T_k, k = 1, \ldots, m$, of the complex Lie algebra L, as was explicated in Chap. 1. Along the longest word $\sigma_{k_1} \circ \cdots \circ \sigma_{k_p}$ of the Weyl group, $\beta_1 := \alpha_{k_1}, x^{\pm}(\beta_1) := x_{k_1}^{\pm}$, and $\forall_2^p j$:

$$x^{\pm}(\beta_j) := T_{k_1} \circ \cdots \circ T_{k_{j-1}}(x_{k_j}^{\pm}),\ \ \beta_j := \sigma_{k_1} \circ \cdots \circ \sigma_{k_{j-1}}(\alpha_{k_j}).$$

The braid-like relations of these automorphisms serve for the independence of r from the choice of the longest word.

(3.9.5.1) Inserting the usual defining representation ρ of $A_m, m \in \mathbf{N}$, on $\mathbf{C}^{m+1}$ one obtains:

$$r \xrightarrow[T(\rho,\rho)]{} \frac{m}{m+1} \sum_{k=1}^{m+1} E_{m+1}^{kk} \otimes E_{m+1}^{kk} - \frac{1}{m+1} \sum_{1 \leq k \neq l \leq m+1} E_{m+1}^{kk} \otimes E_{m+1}^{ll} + 2 \sum_{1 \leq k < l \leq m+1} E_{m+1}^{kl} \otimes E_{m+1}^{lk}.$$

(3.9.5.2) For $C_m, m \geq 3$, and $D_m, m \geq 4$, one obtains the defining representations:

$$\begin{aligned} r \longrightarrow & \sum_{1\leq k\leq 2m} \left(E^{kk} - E^{2m+1-k,2m+1-k}\right) \otimes E^{kk} \\ & + 2 \sum_{1\leq k<l\leq 2m} E^{kl} \otimes E^{lk} \pm 2 \sum_{1\leq k\leq m, m+1\leq l\leq 2m} E^{kl} \otimes E^{2m+1-k,2m+1-l} \\ & - 2 \sum_{1\leq k<l\leq m, m+1\leq k<l\leq 2m} E^{kl} \otimes E^{2m+1-k,2m+1-l}, \end{aligned}$$

with the upper sign for C_m, and the lower one for D_m.

(3.9.5.3) For $B_m, m \geq 2$, the defining representation reads:

$$\begin{aligned} r \longrightarrow 2 & \sum_{1\leq k\leq 2m+1, k\neq m+1} \left(E^{kk} - E^{2m+2-k,2m+2-k}\right) \otimes E^{kk} \\ & + 4 \sum_{1\leq k<l\leq 2m+1} E^{kl} \otimes \left(E^{lk} - E^{2m+2-k,2m+2-l}\right). \end{aligned}$$

3.10 Duals of Quasitriangular Z2-Graded Lie Bialgebras

Consider a $\mathbf{Z}_2$-graded Lie bialgebra L over a field K, *char* $K \neq 2$, which is quasitriangular with respect to the classical R-matrix r.

(3.10.1) Let $\rho : L \longrightarrow Mat(d, K)$ be a representation of L on K^d, in the sense of unital associative superalgebras. The $\mathbf{Z}_2$-grading on K^d is arranged such that its Cartesian unit vectors E_d^k are even for $1 \leq k \leq m$, odd for $m+1 \leq k \leq m+n = d$. The classical R-matrix is then represented on the two-fold skew-symmetric tensor product of $Mat(d, K)$:

$$r \xrightarrow[T(\rho,\rho)]{} r_\rho =: \sum_{i,j,k,l=1}^{d} r_{ijkl}\, E_d^{ij} \otimes E_d^{kl} \in Mat(d,K)\hat{\otimes} Mat(d,K),$$

with the basis matrices $(E_d^{ij})_{kl} := \delta_{ik}\, \delta_{jl}, i, \ldots, l = 1, \ldots, d$. Sometimes one conveniently uses an isomorphism of unital associative superalgebras over K:

$$Mat(d,K)\hat{\otimes} Mat(d,K) \longleftrightarrow Mat(d^2, K),$$

which is denoted explicitly in Chap. 8.

(3.10.2) The $\mathbf{Z}_2$-graded Lie bialgebra A_r over the field K is defined by the following super-commutation relations and Lie comultiplication of generators $a_{kl}, k, l = 1, \ldots, d$. Define $\hat{k} := 0$ for $1 \leq k \leq m$, and 1 for $m+1 \leq k \leq d$; the generators a_{kl} are taken as even or odd for $\hat{k} + \hat{l}$ being so. The coboundary Θ of A_r is defined such that:

$$A_r \ni a_{kl} \underset{\Theta}{\longrightarrow} \sum_{j=1}^{d} (-1)^{(\hat{k}+\hat{j})(\hat{j}+\hat{l})} \ a_{kj} \otimes a_{jl} - a_{jl} \otimes a_{kj} \in A_r \otimes A_r.$$

For $i, \ldots, l = 1, \ldots, d$, one demands that

$$(-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l})} a_{ij} a_{kl} - a_{kl} a_{ij} = \sum_{p=1}^{d} \Big(a_{ip} \ r_{pjkl} - r_{ijkp} \ a_{pl}$$
$$+ (-1)^{(\hat{i}+\hat{j})(1+\hat{k}+\hat{l})} a_{kp} \ r_{ijpl} - (-1)^{(\hat{i}+\hat{j}+1)(\hat{k}+\hat{l})} r_{ipkl} \ a_{pj} \Big).$$

(3.10.3) Two $\mathbf{Z}_2$-graded Lie bialgebras L_1 and L_2 over K are called dual with respect to a K-bilinear form: $L_1 \times L_2 \longrightarrow K$, if and only if $\forall a_1, b_1 \in L_1, a_2, b_2 \in L_2$:

$$\langle [a_1, b_1] \big| a_2 \rangle = \langle a_1 \otimes b_1 \big| \Theta_2(a) \rangle, \ \ \langle a_1 \big| [a_2, b_2] \rangle = \langle \Theta_1(a_1) \big| a_2 \otimes b_2 \rangle,$$

defining an according K-bilinear form on tensor products as explained previously.

(3.10.4) The above defined $\mathbf{Z}_2$-graded Lie bialgebras L and A_r are dual with respect to the K-bilinear form:

$$L \times A_r \ni \{x, a_{kl}\} \longrightarrow \langle x \big| a_{kl} \rangle = (\rho(x))_{kl} \in K.$$

(3.10.5) The super-commutation relations of A_r may be rewritten conveniently in terms of the graded tensor product over an associative unital superalgebra A over K, inserting an injective homomorphism $\alpha : A_r \longrightarrow A$ of such objects. This construction is explained in detail in Chap. 8, in order to rewrite the so-called main commutation relations (MCR).

$$Mat(d, A) \ni [b_{kl}; k, l = 1, \ldots, d] \longleftrightarrow \sum_{k,l=1}^{d} b_{kl} \otimes E_d^{kl} \in A \otimes Mat(d, K).$$

$$[b_{ij}; \ i, j = 1, \ldots, d] \otimes_A [c_{kl}; k, l = 1, \ldots, d]$$

$$\longleftrightarrow \ \sum_{i,j,k,l=1}^{d} \ (-1)^{(\hat{i}+\hat{j})\hat{c}_{kl}} \ b_{ij} \ c_{kl} \otimes E_d^{ij} \otimes E_d^{kl},$$

$$(a \otimes E_d^{ij} \otimes E_d^{kl}) \, (b \otimes E_d^{pq} \otimes E_d^{rs}) =$$

$$= (-1)^{(\hat{i}+\hat{j}+\hat{k}+\hat{l})\hat{b}+(\hat{k}+\hat{l})(\hat{p}+\hat{q})}\delta_{jp}\ \delta_{lr}\ ab \otimes E_d^{iq} \otimes E_d^{ks},$$

for $i, \dots, s = 1, \dots, d$. With this product, the K-vector space

$$Mat(d, A)\hat{\otimes}_A Mat(d, A) \longleftrightarrow A \otimes Mat(d, K) \otimes Mat(d, K)$$

becomes an associative superalgebra with the unit $e_A \otimes I_d \otimes I_d$.

(3.10.6) Denoting $I_A := e_A \otimes I_d, r_A := e_A \otimes r$, with the unit e_A of A, inserting the graded flip τ on $Mat(d, K)\hat{\otimes}Mat(d, K), T(id\ A, \tau) =: \tau_A$, such that $\tau_A(r_A) = e_A \otimes \tau(r)$, and abbreviating $[\alpha(a_{kl}); k, l = 1, \dots, d] =: [a]$, one obtains

$$\begin{aligned}&([a] \otimes_A I_A)\,(I_A \otimes_A [a]) - (I_A \otimes_A [a])\,([a] \otimes_A I_A)\\ &= ([a] \otimes_A I_A + I_A \otimes_A [a])\,r_A - r_A\,([a] \otimes_A I_A + I_A \otimes_A [a])\,.\end{aligned}$$

(3.10.6.1) A family of quasitriangular complex $\mathbf{Z}_2$-graded Hopf algebras, inserting for its universal R-matrices appropriate representations on $\mathbf{C}^d \otimes \mathbf{C}^d$:

$$R(q) \longrightarrow I_d \otimes I_d + r \ln q + O\left((\ln q)^2\right), \quad q \in \mathbf{C}\backslash] - \infty, 0],$$

tends on $\mathbf{C}^d$, for $q \longrightarrow 1$, to an according quasitriangular $\mathbf{Z}_2$-graded bialgebra with respect to r, to the first order in $\ln q$. Especially the quantum Yang-Baxter equation tends in this limit to the classical one. Therefore the above super-commutation relations may be called semiclassical main commutation relations. This limit will be discussed in Chap. 8.

(3.10.7) In the case of $sl(2, \mathbf{C})$ one finds the following semiclassical dual, denoting $a_{11} \equiv a, a_{12} \equiv b, a_{21} \equiv c, a_{22} \equiv d$.

$$r \longleftrightarrow \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau \longleftrightarrow \mathit{transposition},$$

$$\begin{aligned}&ab - ba = -b,\ bd - db = -b,\ ac - ca = -c,\ cd - dc = -c,\\ &ad = da,\ bc = cb.\end{aligned}$$

With the additional relation $a = -d$ one recovers the dual A_1^* of A_1.

(3.10.8) In the case of A_m, $m \in \mathbf{N}$, inserting the usual defining representation on $\mathbf{C}^{m+1}$ such that

$$\forall_1^{m+1} i, j, k, l : r_{ijkl} = \delta_{ij}\ \delta_{kl}\ \delta_{ik} - \frac{1}{m+1}\delta_{ij}\delta_{kl} + 2\delta_{il}\ \delta_{jk}\theta(k - l),$$

one obtains the super-commutation relations

$$\begin{aligned}[][a_{ij},\ a_{kl}] &= a_{ij}\ \delta_{kl}(\delta_{jk} - \delta_{ik}) + a_{kl}\delta_{ij}(\delta_{il} - \delta_{ik})\\ &\quad + 2a_{il}\ \delta_{jk}\left(\theta(k - l) - \theta(k - i)\right) + 2a_{kj}\ \delta_{il}\ \left(\theta(j - l) - \theta(k - l)\right),\end{aligned}$$

denoting $\theta(n) := 1$ for $n \in \mathbf{N}$, and 0 for $n \in -\mathbf{N}_0$.

3.11 Deformed Tensor Product of Semiclassical MCR-Type Algebras

(3.11.1) Consider a $\mathbf{Z}_2$-graded Lie bialgebra L over K, which is quasitriangular with respect to the classical R-matrix $r \in L \otimes L$, and let α and β be representations of the Lie superalgebra L on K^p and K^q, respectively. The corresponding dual $\mathbf{Z}_2$-graded Lie bialgebras $A(r;\alpha)$ and $A(r;\beta)$ are then generated by a_{ij} and $b_{kl}, 1 \le i,j \le p$, $1 \le k,l \le q$, with the previously described grading and semiclassical MCR. The tensor product of these two representations is constructed as usual, inserting the diagonal map δ, which is uniquely extended to the comultiplication on the universal enveloping superalgebra $E(L)$:

$$L \ni x \xrightarrow{\delta} x \otimes e + e \otimes x \xrightarrow{T(\alpha,\beta)} \longleftrightarrow T_\delta(\alpha,\beta)(x):$$
$$K^p \otimes K^q \ni u \otimes v \longrightarrow \alpha(x)(u) \otimes v + u \otimes \beta(x)(v) \in K^p \otimes K^q.$$

Denoting $\gamma \longleftrightarrow T_\delta(\alpha,\beta)$, and the generators of the $\mathbf{Z}_2$-graded Lie bialgebra $A(r;\gamma)$ by $c_{rs}, 1 \le r,s \le pq$, one easily finds that

$$\langle x|c_{(i-1)q+k,(j-1)q+l}\rangle = (-1)^{(\hat{i}+\hat{j})\hat{k}}\langle x|a_{ij}\rangle\delta_{kl} + \delta_{ij}\langle x|b_{kl}\rangle.$$

(3.11.2) On the other hand, denoting $r =: \sum_{\lambda=1}^{L} r'_\lambda \otimes r''_\lambda$, one calculates that

$$\begin{aligned}
&\langle \Theta(x)|a_{ij} \otimes b_{kl}\rangle = \langle [\delta(x), r]|a_{ij} \otimes b_{kl}\rangle = -\langle [\delta(x), \tau(r)]|a_{ij} \otimes b_{kl}\rangle \\
&= \sum_{\lambda=1}^{L}\Bigg(\sum_{m=1}^{p}\Big(\langle x|a_{im}\rangle\langle r'_\lambda|a_{mj}\rangle\langle r''_\lambda|b_{kl}\rangle(-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l})} \\
&\qquad\qquad -\langle x|a_{mj}\rangle\langle r'_\lambda|a_{im}\rangle\langle r''_\lambda|b_{kl}\rangle(-1)^{\hat{k}+\hat{l}}\Big) \\
&\quad + \sum_{n=1}^{q}\Big(\langle x|b_{kn}\rangle\langle r'_\lambda|a_{ij}\rangle\langle r''_\lambda|b_{nl}\rangle(-1)^{\hat{i}+\hat{j}} \\
&\qquad\qquad -\langle x|b_{nl}\rangle\langle r'_\lambda|a_{ij}\rangle\langle r''_\lambda|b_{kn}\rangle(-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l})}\Big)\Bigg) \\
&= \sum_{\lambda=1}^{L}\Bigg(\sum_{m=1}^{p}\Big(\langle x|a_{mj}\rangle\langle r'_\lambda|b_{kl}\rangle\langle r''_\lambda|a_{im}\rangle \\
&\qquad\qquad -\langle x|a_{im}\rangle\langle r'_\lambda|b_{kl}\rangle\langle r''_\lambda|a_{mj}\rangle(-1)^{(\hat{i}+\hat{j}+1)(\hat{k}+\hat{l})}\Big) \\
&\quad + \sum_{n=1}^{q}\Big(\langle x|b_{nl}\rangle\langle r'_\lambda|b_{kn}\rangle\langle r''_\lambda|a_{ij}\rangle(-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l}+1)} \\
&\qquad\qquad -\langle x|b_{kn}\rangle\langle r'_\lambda|b_{nl}\rangle\langle r''_\lambda|a_{ij}\rangle\Big)\Bigg).
\end{aligned}$$

(3.11.3) Denoting $r^{(\alpha,\beta)} := T(\alpha,\beta)(r)$, and similarly $r^{(\beta,\alpha)}$, one is thereby heuristically led, remembering that the involved bilinear forms may be degenerate, to the following theorem. A homomorphism of Lie superalgebras over K is established:

$$A(r;\gamma) \ni c_{(i-1)q+k,(j-1)q+l} \longrightarrow (-1)^{(\hat{\imath}+\hat{\jmath})\hat{k}} a_{ij}\delta_{kl} + \delta_{ij} b_{kl} \in A(r;\alpha)\tilde{\otimes}A(r;\beta),$$

into the Lie superalgebra of generators a_{ij} and $b_{kl}, 1 \le i,j \le p,\ 1 \le k,l \le q$, with respect to the following relations: The generators a_{ij} among themselves, and b_{kl} among themselves, fulfill the super-commutation relations of $A(r;\alpha)$ and $A(r;\beta)$, respectively. Moreover they obey the mixed relations

$$\begin{aligned}
&(-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{l})} a_{ij} b_{kl} - b_{kl} a_{ij} \\
&= \sum_{m=1}^{p} \left(r^{(\alpha,\beta)}_{mjkl} a_{im} - r^{(\alpha,\beta)}_{imkl} a_{mj} (-1)^{(\hat{\imath}+\hat{\jmath}+1)(\hat{k}+\hat{l})} \right) \\
&+ \sum_{n=1}^{q} \left(r^{(\alpha,\beta)}_{ijnl} b_{kn} (-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{l}+1)} - r^{(\alpha,\beta)}_{ijkn} b_{nl} \right), \\
&(-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{l})} b_{kl} a_{ij} - a_{ij} b_{kl} \\
&= \sum_{n=1}^{q} \left(r^{(\beta,\alpha)}_{nlij} b_{kn} - r^{(\beta,\alpha)}_{knij} b_{nl} (-1)^{(\hat{k}+\hat{l}+1)(\hat{\imath}+\hat{\jmath})} \right) \\
&+ \sum_{m=1}^{p} \left(r^{(\beta,\alpha)}_{klmj} a_{im} (-1)^{(\hat{k}+\hat{l})(\hat{\imath}+\hat{\jmath}+1)} - r^{(\beta,\alpha)}_{klim} a_{mj} \right).
\end{aligned}$$

(3.11.3.1) Using again faithful copies into an associative unital superalgebra A, with the previously introduced abbreviations I_A, $[a]$ and $[b]$, and denoting $r_A^{(\alpha,\beta)} := e_A \otimes r^{(\alpha,\beta)}$, and similarly $r_A^{(\beta,\alpha)}$, these relations can be short-written as the subsequent mixed semiclassical MCR.

$$\begin{aligned}
&([a] \otimes_A I_A)(I_A \otimes_A [b]) - (I_A \otimes_A [b])([a] \otimes_A I_A) \\
&= ([a] \otimes_A I_A + I_A \otimes_A [b]) r_A^{(\alpha,\beta)}) - r_A^{(\alpha,\beta)}([a] \otimes_A I_A + I_A \otimes_A [b]),
\end{aligned}$$

and similar relations, interchanging $[a]$ with $[b]$, and α with β. The above indicated homomorphism can be abbreviated as:

$$Mat(pq, A) \ni [c] \longrightarrow [a] \otimes_A I_A + I_A \otimes_A [b] \in Mat(p, A)\hat{\otimes}_A Mat(q, A),$$

and may be called deformed tensor product of semiclassical MCR-type algebras.

(3.11.3.2) The proof of this theorem is based on the following almost trivial lemma:

$$T(\gamma,\gamma)(r) \longleftrightarrow r_{13}^{(\alpha,\alpha)} + r_{14}^{(\alpha,\beta)} + r_{23}^{(\beta,\alpha)} + r_{24}^{(\beta,\beta)},$$

where the lower indices denote embeddings into an appropriate four-fold tensor product of matrices over K. One then calculates that $[a] \otimes_A I_A + I_A \otimes_A [b]$ fulfills the semiclassical MCR-type relations with respect to $T(\gamma,\gamma)(r)$.

4. Formal Power Series with Homogeneous Relations

Since the relations and costructure mappings of quantum algebras, which are investigated in Chap. 8, contain exponentials in the generators, and in particular with respect to the universal R-matrix, one is forced to introduce an appropriate topology on the algebra or coalgebra in question. If the ring of coefficients cannot be equipped with a suitable natural topology, one just uses its discrete topology, thereby ending up with formal power series.

The theory of formal power series over an arbitrary set of commuting indeterminates is presented in the volume Algèbre. Chapitres 4 à 7, by N. Bourbaki (1981).

Formal power series in finitely many generators without any relations, and their topological tensor product, can be used in order to prove on an algebraic level the Baker-Campbell-Hausdorff formula, as is shown for instance in the monograph on Lie algebras by N. Jacobson (1962).

Formal power series with homogeneous relations, and also their topological tensor product, are constructed easily, factorizing with respect to the smallest closed ideal of the set of relations. Inhomogeneous relations, for instance of exponential type, are not compatible with the topology of formal power series. Possibilities to escape this difficulty, which is just typical for quantum algebras, are presented in Chap. 8.

4.1 Polynomials in Finitely Many Commuting Indeterminates

(4.1.1) Let $X := \{x_k; k \in I\}$ be a finite set, with the index set $I := \{1, \ldots, p\}$, and R a commutative ring. The commutative unital associative algebra of polynomials in the commuting indeterminates $x_1, \ldots, x_p$ over R is defined.

$$R[X] \equiv R[x_1, \ldots, x_p] := T(E)/C, \; E := R(X),$$
$$C := ideal(\{x_k x_l - x_l x_k; k, l \in I\}),$$

with the unit $e := \{e_R, 0, 0, \ldots\} + C$.
The equivalence classes $x_k + C$, $k \in I$, are conveniently denoted as x_k itself.

(4.1.2)

$$R[X] = R(\{x_1^{n_1} \cdots x_p^{n_p}; n_1, \ldots, n_p \in \mathbf{N}_0\}),$$

with an R-basis of monomials. Here one denotes $\forall k \in I : x_k^0 := e$.

(4.1.3) This algebra of polynomials, which is just the symmetric algebra $S(E)$ over the R-bimodule E, is universal in the following sense. Let $\omega : E \longrightarrow A$ be an arbitrary Weyl map into a unital associative algebra A over R, which means that

$$\forall x, y \in E : \omega(x)\omega(y) = \omega(y)\omega(x).$$

Then $\exists$ unique homomorphism $\omega_* : S(E) \longrightarrow A$ of unital associative R-algebras, such that $\omega_* \circ \kappa \circ \beta = \omega$, according to the diagram below.

$$\begin{tikzcd}
E \arrow[r, "\beta"] \arrow[dr, "\omega"', "\text{Weyl map}"] & T(E) \arrow[r, "\kappa"] \arrow[d, "\text{algebra-homomorphism}"] & S(E) \arrow[dl, "\omega_*", "\text{algebra-homomorphism}"'] \\
 & A &
\end{tikzcd}$$

(4.1.4) Let M be an R-bimodule. The elements of $M[X] := M \otimes R[X]$ are called polynomials in the commuting indeterminates $x_1, \ldots, x_p$, with the coefficients $\in M$.

$$M = R(\{m_k; k \in K\}) \Longrightarrow$$
$$M[X] = R(\{m_k \otimes x_1^{n_1} \cdots x_p^{n_p}; n_1, \ldots, n_p \in \mathbf{N}_0; k \in K\}).$$

(4.1.5) Both $R[X]$ and $M[X]$ are $\mathbf{N}_0$-graded with respect to the grading, such that $\forall n_1, \ldots, n_p \in \mathbf{N}_0$:

$$deg\ x_1^{n_1} \cdots x_p^{n_p} := n_1 + \cdots + n_p, \quad deg\ e = 0, \quad deg\ 0 := -\infty.$$

(4.1.6) Let A be a unital associative algebra over R, and $\{a_k; k \in I\}$ a family of pairwise commuting elements of A. Due to the above universal property, $\exists$ unique homomorphism $\alpha : R[X] \longrightarrow A$ of unital associative algebras over R, such that $\forall k \in I : \alpha(x_k) = a_k$. The family $\{a_k; k \in I\}$ is called generic, if and only if α is injective. One then conveniently denotes

$$\forall \phi \in R[X] : \alpha(\phi) =: \phi(a_1, \ldots, a_p).$$

(4.1.6.1) In particular let A be commutative. Then the elements of $Im\ \alpha$ are called polynomial functions in p variables $\in A$.

(4.1.6.2) An insertion of polynomials into polynomials is defined by the special choice of $A := R[Y]$, $Y := \{y_1, \dots, y_q\}$, considering the image of α.

$$R[X] \ni x_k \underset{def}{\xrightarrow{\alpha}} \sum_{m_1,\dots,m_q \in \mathbf{N}_0} r^{(k)}_{m_1\dots m_q} y_1^{m_1} \cdots y_q^{m_q} \in R[Y].$$

(4.1.7) Let $X := \{x_1, \dots, x_p\}$ and $Y := \{y_1, \dots, y_q\}$ be finite sets, and A a commutative unital associative algebra over R. Take

$$\psi_k := \sum_{m_1,\dots,m_q \in \mathbf{N}_0} r^{(k)}_{m_1\dots m_q} y_1^{m_1} \cdots y_q^{m_q} \in R[Y],\ k = 1, \dots, p,$$

and denote by $\xi : R[X] \longrightarrow R[Y]$ the unique homomorphism of unital associative algebras over R, such that $\forall_1^p k : \xi(x_k) = \psi_k$. Furthermore take $a_1, \dots, a_q \in A$, and denote by $\alpha : R[Y] \longrightarrow A$ the unique homomorphism of unital associative algebras over R, such that $\forall_1^q l : \alpha(y_l) = a_l$. Thereby one obtains the mapping:

$$R[X] \times (\prod^p R[Y]) \times (\prod^q A) \longrightarrow A,$$

which may be called composite polynomial in variables $\in A$.

$$\left.\begin{array}{c} R[X] \ni \phi := \sum_{n_1,\dots,n_p \in \mathbf{N}_0} r_{n_1\dots n_p} x_1^{n_1} \cdots x_p^{n_p} \\ R[Y] \ni \psi_1, \dots, \psi_p \\ A \ni a_1, \dots, a_q \end{array}\right\}$$

$$\longrightarrow \phi(\psi_k(a_l; l = 1, \dots, q); k = 1, \dots, p) = \alpha \circ \xi(\phi)$$

$$= \sum_{n_1,\dots,n_p \in \mathbf{N}_0} r_{n_1\dots n_p} \prod_{k=1}^{n} \Bigg(\sum_{m_1,\dots,m_q \in \mathbf{N}_0} r^{(k)}_{m_1\dots m_q} a_1^{m_1} \cdots a_q^{m_q} \Bigg)^{n_k} \in A.$$

(4.1.8) The partial derivations $\frac{\partial}{\partial x_k} \in Der_R(R[X])$ of such polynomials are defined, such that

$$\forall_1^p k : R[X] \ni x_1^{n_1} \cdots x_p^{n_p} \underset{def}{\xrightarrow{\partial/\partial x_k}} \begin{cases} x_1^{n_1} \cdots x_k^{n_k - 1} \cdots x_p^{n_p},\ n_k \geq 1 \\ 0,\ n_k = 0 \end{cases} \in R[X].$$

(4.1.9) Consider again the insertion of $R[Y]$ into $R[X]$, with the notation from above, and denote $\forall_1^p k : \alpha(x_k) =: \psi_k \in R[Y]$. One easily calculates the following chain rule.

$$\forall \phi \in R[X], \forall_1^q l : \frac{\partial}{\partial y_l} \circ \alpha(\phi) = \sum_{k=1}^{p} \alpha \circ \frac{\partial}{\partial x_k}(\phi) \frac{\partial}{\partial y_l} \psi_k \in R[Y].$$

(4.1.10) Let the field of rational numbers $\mathbf{Q}$ be a subring of R. Then one finds easily Taylor's formula. For two disjoint sets of indeterminates $X := \{x_1, \ldots, x_p\}$ and $Y := \{y_1, \ldots, y_p\}$, consider the following polynomial $\in R[X \cup Y]$ with the coefficients $r : \prod^p \mathbf{N}_0 \longrightarrow R$.

$$\sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1\ldots n_p}(x_1+y_1)^{n_1}\cdots(x_p+y_p)^{n_p}$$

$$= \sum_{k_1,\ldots,k_p \in \mathbf{N}_0} \frac{1}{k_1!\cdots k_p!}\frac{\partial^{k_1+\cdots+k_p}}{\partial x_1^{k_1}\cdots\partial x_p^{k_p}}\Bigg(\sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1\ldots n_p}x_1^{n_1}\cdots x_p^{n_p}\Bigg) y_1^{k_1}\cdots y_p^{k_p}.$$

(4.1.10.1) This formula also holds for the images of polynomials in a commutative unital associative algebra over R.

4.2 Power Series in Finitely Many Commuting Indeterminates

(4.2.1) Let G be a group, and also a Hausdorff topological space. G is called topological group, if and only if the mappings:

$$G \times G \ni \{g, h\} \longrightarrow gh \in G, \quad G \ni g \longrightarrow g^{-1} \in G$$

are continuous, the former with respect to the product topology. The direct product of a family of topological groups is some topological group, with respect to the product topology.

(4.2.1.1) A subgroup F of a topological group G becomes a topological group be means of the subspace topology; if F is open, then it is also closed in G; the closure of F is some subgroup of G.

(4.2.1.2) Let N be an invariant subgroup of a topological group G. The quotient topology on the factor group G/N is defined in the following manner. A subset V of G/N is called open, if and only if $\pi^{-1}(V)$ is an open subset of G. In this case the canonical projection $\pi : G \ni g \longrightarrow gN \in G/N$ is both open and continuous. Let $\phi : G/N \longrightarrow T$ be a map into a topological space T; if $\phi \circ \pi$ is continuous, then ϕ itself is continuous.

(4.2.1.3) This quotient topology on G/N is Hausdorff, if and only if the subset N of G is closed; in this case G/N is some topological group. If especially N is an open subset of G, then the quotient topology is discrete.

(4.2.1.4) Let $G_k, k = 1, 2$, be topological groups, and consider a homomorphism of groups $\phi : G_1 \longrightarrow G_2$. ϕ is continuous, if and only if it is continuous at the unit e_1 of G_1; in this case its kernel $ker\ \phi$ is some closed subset of G_1, and the three group homomorphisms:

$$G_1 \longrightarrow G_1/ker\ \phi \longrightarrow Im\ \phi \longrightarrow G_2$$

are then continuous.
Two topological groups G_1, G_2 are called isomorphic, if and only if there is an isomorphism of groups: $G_1 \longleftrightarrow G_2$, which is also a homeomorphism.

(4.2.1.5) Assume existence of some countable basis $\{V_n; n \in \mathbf{N}\}$ of neighbourhoods of the unit. Let $\{g_k; k \in \mathbf{N}\}$ be a sequence of elements of G.
It is called convergent to $g \in G$, if and only if

$$\forall n \in \mathbf{N} : \exists m \in \mathbf{N} : \forall k \geq m : g_k g^{-1} \in V_n;$$

in this case one writes $g = \lim_{k \to \infty} g_k$.
It is called Cauchy sequence, if and only if

$$\forall n \in \mathbf{N} : \exists m \in \mathbf{N} : \forall k, l \geq m : g_k g_l^{-1} \in V_n.$$

Obviously every convergent sequence is of Cauchy type.

(4.2.1.6) G is called complete, if and only if every Cauchy sequence converges to an element of G. In this case G/N is complete too, for every closed invariant subgroup N of G.

(4.2.1.7) For every topological Abelian group G, there is a (modulo isomorphism) unique complete topological Abelian group $\tilde{G}$, the so-called completion of G, such that G is isomorphic to some topological subgroup of $\tilde{G}$, and G lies dense in $\tilde{G}, \tilde{G} = \overline{G}$.

(4.2.1.8) Let $G_k, k = 1, 2$, be topological Abelian groups, F_1 a subgroup of G_1, and assume G_2 to be complete. Then every continuous homomorphism of groups: $F_1 \longrightarrow G_2$ can be extended uniquely to a continuous homomorphism of groups: $\overline{F_1} \longrightarrow G_2$.

(4.2.1.9) The topological ring $R[X]$, with the I_1-adic topology defined below, is not complete with respect to the addition of polynomials.

(4.2.2) The commutative unital associative algebra of polynomials $R[X]$ over a commutative ring R, in the indeterminates $x_1, \ldots, x_p \in X$, is equipped with the so-called I_1-adic topology. Here I_1 denotes the ideal of polynomials without constant term,

$$I_1 := \left\{ \sum_{n_1, \ldots, n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} x_1^{n_1} \cdots x_p^{n_p};\ r_{n_1 \ldots n_p} \in R,\ r_{0 \ldots 0} = 0 \right\}.$$

(4.2.2.1)

$$\forall m \in \mathbf{N} : V_m(0) := I_1^m := sum(\{f_1 \cdots f_m; f_1, \ldots, f_m \in I_1\})$$
$$= \left\{ \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} x_1^{n_1} \cdots x_p^{n_p};\ n_1 + \cdots + n_p < m \Longrightarrow r_{n_1 \ldots n_p} = 0 \right\},$$
$$V_0(0) := R[X],\ V_1(0) = I_1,\ V_m(0) \supset V_{m+1}(0),\ \bigcap_{m \in \mathbf{N}_0} V_m(0) = \{0\}.$$

For every $m \in \mathbf{N}_0$, $V_m(0)$ is an ideal of $R[X]$.

(4.2.2.2) There is an obvious isomorphism of unital associative algebras: $R[X]/I_1 \longleftrightarrow R$.

(4.2.2.3) $\forall m \in \mathbf{N}_0, \forall f \in R[X]$: $V_m(f) := \{f+g; g \in V_m(0)\}$, $V_0(f) = R[X]$.

(4.2.2.4) With this countable basis of neighbourhoods $\{V_m(f); m \in \mathbf{N}_0\}$ of $f \in R[X]$, the topology τ_1 is constructed.

$$\forall f \in R[X] : W_f := \{V \subseteq R[X]; \exists m \in \mathbf{N} : V_m(f) \subseteq V\}.$$

The set $\bigcup_{f \in R[X]} W_f$ fulfills the axioms for the neighbourhoods of an according topology. Therefore

$$\tau_1 := \{V \subseteq R[X]; \forall f \in V : V \in W_f\} \ni \emptyset$$

is the unique topology on $R[X]$ with exactly these neighbourhoods. By its very construction, $\forall f \in R[X]$, $\forall m \in \mathbf{N}_0$: $V_m(f) \in \tau_1$.

(4.2.2.5) The ring $R[X]$ is topological, because the following mappings are continuous with respect to the product topology.

$$\prod^2 R[X] \ni \{f,g\} \longrightarrow f+g \in R[X],\ R[X] \ni f \longleftrightarrow -f \in R[X],$$
$$\prod^2 R[X] \ni \{f,g\} \longrightarrow fg \in R[X].$$

(4.2.2.6) Obviously the topology τ_1 is of Hausdorff type.

(4.2.3) Let $\{f_k; k \in \mathbf{N}\}$ be a sequence of polynomials $\in R[X]$. This sequence is called convergent to $f \in R[X]$, if and only if

$$\forall n \in \mathbf{N} : \exists m \in \mathbf{N} : \forall k \geq m : f_k - f \in V_n(0);$$

in this case one writes $f = \lim_{k \to \infty} f_k$.
It is called Cauchy sequence, if and only if

$$\forall n \in \mathbf{N} : \exists m \in \mathbf{N} : \forall k, l \geq m : f_k - f_l \in V_n(0).$$

Of course every convergent sequence is of Cauchy type.

(4.2.4) The commutative unital associative algebra $R[[X]] \equiv R[[x_1, \ldots, x_p]]$ of formal power series in the indeterminates $x_1, \ldots, x_p \in X$ over R is defined.

$$R[[X]] := \left\{ \phi := \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} x_1^{n_1} \cdots x_p^{n_p} : \prod^p \mathbf{N}_0 \ni \{n_1, \ldots, n_p\} \longrightarrow r_{n_1 \ldots n_p} \in R \right\}.$$

An R-linear combination of such power series is defined naturally, and the product of formal power series is defined by an appropriate rearrangement. Let $\phi^{(k)}$, $k = 1, 2$, be such power series, with the coefficients $r^{(k)}_{n_1 \ldots n_p}$, respectively. $\forall s_1, s_2 \in R$:

$$s_1\phi^{(1)} + s_2\phi^{(2)} := \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} (s_1 r^{(1)}_{n_1 \ldots n_p} + s_2 r^{(2)}_{n_1 \ldots n_p}) x_1^{n_1} \cdots x_p^{n_p}.$$

$$\phi^{(1)}\phi^{(2)} := \sum_{m \in \mathbf{N}_0} \sum_{\substack{n_1,\ldots,m_p \in \mathbf{N}_0 \\ n_1 + \cdots + m_p = m}} r^{(1)}_{n_1 \ldots n_p} r^{(2)}_{m_1 \ldots m_p} x_1^{n_1+m_1} \cdots x_p^{n_p+m_p}.$$

(4.2.4.1) $R[X]$ is considered as some subalgebra of $R[[X]]$ in the natural sense.

(4.2.4.2) For an R-bimodule M, the elements of $M[[X]] := M \otimes R[[X]]$ are called formal power series in the indeterminates $x_1, \ldots, x_p$, with the coefficients $\in M$.

(4.2.5) The following family of ideals of $R[[X]]$ is used as some countable basis of neighbourhoods of 0. $\forall m \in \mathbf{N}$:

$$\overline{V}_m(0) := \left\{ \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} x_1^{n_1} \cdots x_p^{n_p}; \; n_1 + \cdots + n_p < m \Longrightarrow r_{n_1 \ldots n_p} = 0 \right\},$$

$$\overline{V}_m(0) \supset \overline{V}_{m+1}(0), \;\; \overline{V}_0(0) := R[[X]].$$

With the resulting Hausdorff topology one establishes the topological ring $R[[X]]$.

(4.2.6) Convergence of sequences, and Cauchy sequences of formal power series, are defined with respect to their addition.

(4.2.7) The topological ring $R[[X]]$ is complete with respect to the addition of formal power series. Obviously $R[X]$ lies dense in $R[[X]]$. Moreover the subspace topology of $R[X]$, which is induced from $R[[X]]$, coincides with the topology τ_1 of polynomials defined previously. Therefore $R[[X]]$ is the unique completion of $R[X]$.
$\forall m \in \mathbf{N}_0 : \overline{V}_m(0) = \overline{V_m(0)}$, the closure of $V_m(0)$ in $R[[X]]$.

(4.2.8) The R-bimodule $R[[X]]$ is the following direct product of bimodules over R.

$$R[[X]] = \prod_{i \in \mathbf{I}} R_i, \ \forall i \in \mathbf{I} := \prod^{p} \mathbf{N}_0 : R_i := R.$$

The product topology of $R[[X]]$, inserting the discrete topology of R, coincides with the topology, which is induced by the basis of zero-neighbourhoods defined above, all of which are ideals.

(4.2.9) Let A be a commutative unital associative algebra over R, equipped with a Hausdorff topology, which is compatible with the addition. Let A be complete with respect to the addition. Moreover assume the existence of some countable basis of zero-neighbourhoods, all of which are ideals of A; then A is some topological ring.

(4.2.9.1) Let $\alpha : R[[X]] \longrightarrow A$ be a homomorphism of unital associative algebras over R, which is continuous. Then $\forall_1^p k : \lim_{n\to\infty}(\alpha(x_k))^n = 0$.

(4.2.9.2) Let $\{a_1, \ldots, a_p\}$ be a family of elements of A, such that $\forall_1^p k : \lim_{n\to\infty} a_k^n = 0$. Then $\exists$ unique homomorphism $\alpha : R[[X]] \longrightarrow A$ of unital associative algebras over R, such that α is continuous, and $\forall_1^p k : \alpha(x_k) = a_k$. One conveniently denotes $\forall \phi \in R[[X]] : \alpha(\phi) =: \phi(a_1, \ldots, a_p)$.

(4.2.9.2.1) There is a unique homomorphism $\dot\alpha : R[X] \longrightarrow A$ of unital associative algebras over R, such that $\forall_1^p k : \dot\alpha(x_k) = a_k$. Since $\dot\alpha$ is uniformly continuous, it is uniquely extended to some continuous map $\alpha : R[[X]] \longrightarrow A$. Moreover α is some homomorphism of unital associative algebras over R, which is uniformly continuous.

(4.2.10) A formal power series $\phi \in R[[X]]$ fulfills the condition $\lim_{n\to\infty} \phi^n = 0$, if and only if its constant term vanishes, such that $\phi \in \overline{V}_1(0)$.

(4.2.11) An insertion of formal power series $\in R[[Y]]$ into a power series $\phi \in R[[X]]$ is constructed. Denote $X := \{x_1, \ldots, x_p\}$, let $\psi_1, \ldots, \psi_p \in R[[Y]]$, and assume that $\forall_1^p k : \lim_{n\to\infty} \psi_k^n = 0$. Then $\exists$ unique continuous homomorphism of unital associative algebras $\alpha : R[[X]] \longrightarrow R[[Y]]$, such that $\forall_1^p k : \alpha(x_k) = \psi_k$.

(4.2.12) One easily calculates, that

$$(e_R - t)^{-1} = \sum_{n \in \mathbf{N}_0} t^n \in R[[t]].$$

(4.2.13) The formal power series $\phi \in R[[X]]$ is invertible, if and only if $\phi = r_0 + \phi_1$, such that $\phi_1 \in \overline{V}_1(0)$, and such that $\exists s_0 \in R : r_0 s_0 = e_R$.

(4.2.13.1) Let $\phi = r_0 + \phi_1$ with the above assumptions. Then $\exists \psi \in \overline{V}_1(0) : \phi = r_0(e_R - \psi)$. Moreover $\exists$ unique continuous homomorphism $\alpha : R[[t]] \longrightarrow R[[X]]$ of unital associative algebras, such that $\alpha(t) = \psi$. Hence

$$\alpha(e_R - t)\alpha((e_R - t)^{-1}) = e_R, \; \alpha(e_R - t) = e_R - \psi = s_0 \phi.$$

(4.2.14) For any $l_1, \ldots, l_p \in \mathbf{N}_0$, the partial derivation:

$$R[[X]] \ni \phi := \sum_{n_1, \ldots, n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} x_1^{n_1} \cdots x_p^{n_p}$$
$$\longrightarrow D^{l_1 \ldots l_p} \phi := \sum_{n_1, \ldots, n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} \prod_{k=1}^{p} \frac{\partial^{l_k}}{\partial x_k^{l_k}} x_k^{n_k} \in R[[X]],$$

$$\frac{\partial^l}{\partial x_k^l} x_k^n := n(n-1) \cdots (n-l+1) x_k^{n-l} \text{ for } l, n \in \mathbf{N} \text{ and } n \geq l,$$
$$\frac{\partial^0}{\partial x_k^0} x_k^n := x_k^n \text{ for } n \in \mathbf{N}_0, \; \frac{\partial^l}{\partial x_k^l} x_k^n := 0 \text{ for } \mathbf{N}_0 \ni n < l,$$

is an R-linear uniformly continuous mapping.

(4.2.15) Let $X := \{x_1, \ldots, x_p\}$ and $Y := \{y_1, \ldots, y_q\}$ be disjoint sets. The mapping:

$$R[[X]][[Y]] \ni \sum_{m_1, \ldots, m_q \in \mathbf{N}_0} \left(\sum_{n_1, \ldots, n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p}^{(m_1 \ldots m_q)} x_1^{n_1} \cdots x_p^{n_p} \right) y_1^{m_1} \cdots y_q^{m_q}$$
$$\longleftrightarrow \sum_{n_1, \ldots, m_q \in \mathbf{N}_0} r_{n_1 \ldots n_p}^{(m_1 \ldots m_q)} x_1^{n_1} \cdots x_p^{n_p} y_1^{m_1} \cdots y_q^{m_q} \in R[[X \cup Y]]$$

is an R-linear bijection, which is uniformly continuous from the left to the right hand side.

(4.2.16) This R-linear bijection is used in order to write down Taylor's formula. Let
$X := \{x_1, \ldots, x_p\}$, $Y := \{y_1, \ldots, y_p\}$, $X \cap Y = \emptyset$, $Z := \{z_1, \ldots, z_p\}$, and

$$\phi := \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} z_1^{n_1} \cdots z_p^{n_p} \in R[[Z]].$$

The image of ϕ with respect to an insertion of formal power series, such that $\forall_1^p k : R[[Z]] \ni z_k \longrightarrow x_k + y_k \in R[[X \cup Y]]$, is denoted naturally:

$$R[[Z]] \ni \phi \longrightarrow \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1 \ldots n_p} (x_1 + y_1)^{n_1} \cdots (x_p + y_p)^{n_p}$$
$$=: \phi(x + y) \in R[[X \cup Y]].$$

With this notation, using the above R-linear bijection, one obtains:

$$R[[X \cup Y]] \ni \phi(x + y)$$
$$\longleftrightarrow \sum_{l_1,\ldots,l_p \in \mathbf{N}_0} \left(\sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{l_1+n_1 \ldots l_p+n_p} \prod_{k=1}^{p} \frac{(l_k + n_k)!}{l_k! n_k!} x_k^{n_k} \right)$$
$$y_1^{l_1} \cdots y_p^{l_p} \in R[[X]][[Y]].$$

(4.2.16.1) If especially the field $\mathbf{Q}$ is some subring of R, then one obtains Taylor's formula, which is written here with an obvious abuse of notation.

$$\phi(x + y) = \sum_{l_1,\ldots,l_p \in \mathbf{N}_0} \frac{1}{l_1! \cdots l_p!} D_x^{l_1 \ldots l_p} \phi(x) y_1^{l_1} \cdots y_p^{l_p}.$$

(4.2.17) Consider the unique continuous homomorphism $\alpha : R[[X]] \longrightarrow A$ of unital associative algebras over R, such that $\forall_1^p k : \alpha(x_k) = a_k$, for $a_k \in A$ such that $\forall_1^p k : \lim_{n \to \infty} a_k^n = 0$.

(4.2.17.1) A is an $R[[X]]$-left module over R, with the module-multiplication: $R[[X]] \times A \ni \{\phi, a\} \longrightarrow \alpha(\phi) a \in A$. An R-linear mapping $D : R[[X]] \longrightarrow A$ is called an R-derivation of these formal power series into A, if and only if

$$\forall \phi, \psi \in R[[X]] : D(\phi\psi) = \alpha(\phi) D(\psi) + D(\phi) \alpha(\psi).$$

(4.2.17.2) For every family $\{b_k \in A; k = 1, \ldots, p\}$, $\exists$ unique continuous R-derivation D of $R[[X]]$ into A, such that $\forall_1^p k : D(x_k) = b_k$. Moreover D is uniformly continuous, and

$$\forall \phi \in R[[X]] : D(\phi) = \sum_{k=1}^{p} \alpha\left(\frac{\partial}{\partial x_k} \phi\right) b_k.$$

(4.2.17.3) Let $\Delta \in Der_R(A)$ be continuous. Then the above theorem immediately implies, that

$$\forall \phi \in R[[X]] : \Delta \circ \alpha(\phi) = \sum_{k=1}^{p} \alpha(\frac{\partial}{\partial x_k}\phi)\Delta(a_k).$$

(4.2.17.4) $\forall_1^p l$, $\frac{\partial}{\partial x_l}$ is the unique continuous R-derivation of $R[[X]]$ into itself, such that

$$\forall_1^p k : x_k \longrightarrow \begin{cases} unit & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}.$$

(4.2.18) Every R-derivation D of $R[[X]]$ into itself is uniformly continuous, because $\forall m \in \mathbf{N}_0 : Im\ D|_{\overline{V}_{m+1}(0)} \subseteq \overline{V}_m(0)$.

(4.2.18.1) Let $d : R[X] \longrightarrow R[[X]]$ be an R-linear map, and assume that

$$\forall f, g \in R[X] : d(fg) = (df)g + f(dg).$$

Then $\exists$ unique R-derivation D of $R[[X]]$ into itself, such that $D\big|_{R[X]} = d$.

(4.2.18.2) The family $\{\frac{\partial}{\partial x_k}; k = 1, \ldots, p\}$ is some basis of the $R[[X]]$-left module $Der_R(R[[X]])$.

$$\forall D \in Der_R(R[[X]]), \forall \phi \in R[[X]] : D(\phi) = \sum_{k=1}^{p} (\frac{\partial}{\partial x_k}\phi)D(x_k).$$

(4.2.19) If R is an integral domain, then $R[X]$ and $R[[X]]$ are integral domains too.

(4.2.20) Let the field $\mathbf{Q}$ be some subring of R, and consider $R[[t]]$.

(4.2.20.1) The exponential function is defined as

$$\exp(t) \equiv e^t := \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad e(t) := e^t - 1, \quad l(t) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}.$$

(4.2.20.2)

$$\exp(t + s) = \exp(t)\exp(s) \in R[[t, s]].$$

$$\frac{\partial}{\partial t} e^t = \frac{\partial}{\partial t} e(t) = e^t = 1 + e(t), \quad \frac{\partial}{\partial t} l(t) = \sum_{n=0}^{\infty} (-t)^n = (1 + t)^{-1}.$$

(4.2.20.3) In the sense of inserting formal power series into formal power series defined above one finds, that

$$\frac{\partial}{\partial t} l \circ e(t) = 1, \quad \frac{\partial}{\partial t} e \circ l(t) = (1 + e \circ l(t))(1+t)^{-1}, \text{ hence}$$

$$l \circ e(t) = e \circ l(t) = t.$$

(4.2.20.4)

$$e(t+s) = e(t) + e(s) + e(t)e(s), \quad l(t) + l(s) = l(t+s+ts).$$

(4.2.21) Let again $\mathbf{Q}$ be some subring of R, and denote $X := \{x_1, \ldots, x_p\}$. The ideal $\overline{V}_1(0)$ of $R[[X]]$ is an open and therefore also closed subset of $R[[X]]$. With the notation $\overline{V}_1(0) =: \mathbf{I}_1$ it is considered as topological subgroup of $R[[X]]$, with respect to the addition.
The subset $\overline{V}_1(1) := \{1 + \phi; \phi \in \overline{V}_1(0)\}$ of $R[[X]]$ is also open and therefore closed; as topological subgroup of $R[[X]]$, with respect to the multiplication, it is denoted by $\mathbf{L}_1$.

(4.2.21.1) Let $\phi, \psi \in \mathbf{I}_1$, and consider the unique continuous homomorphisms of unital associative R-algebras

$$\alpha_1 : R[[t]] \longrightarrow R[[X]], \quad \alpha_2 : R[[t,s]] \to R[[X]], \quad \alpha : R[[t]] \to R[[X]],$$

such that $\alpha_1(t) = \alpha_2(t) = \phi$, $\alpha_2(s) = \psi$, $\alpha(t) = \phi + \psi$. Denote the exponential function of ϕ by

$$\exp(\phi) := \alpha_1(\exp(t)), \quad \exp(\psi) := \alpha_2(\exp(s)), \quad \exp(\phi + \psi) := \alpha(\exp(t)).$$

Then from the partial sums one concludes, that

$$\exp(\phi + \psi) = \exp(\phi)\exp(\psi).$$

Here one uses that $\alpha_2(\exp(t+s)) = \exp(\phi + \psi)$.

(4.2.21.2) $\forall \phi \in \mathbf{I}_1$: $\exp(\phi) - 1 = e(\phi) := \alpha_1(e(t))$, $l(\phi) := \alpha_1(l(t))$.

$$\begin{aligned}\forall m \in \mathbf{N} : \phi \in \overline{V}_m(0) \Longrightarrow \quad & e(\phi) \in \overline{V}_m(0) \\ \Longleftrightarrow \; & \exp(\phi) \in \overline{V}_m(1) := \{1 + \phi; \phi \in \overline{V}_m(0)\}.\end{aligned}$$

Therefore the group homomorphism $\exp : \mathbf{I}_1 \longrightarrow \mathbf{L}_1$ is continuous.

(4.2.21.3)

$$\forall \phi, \psi \in \mathbf{I}_1 : l(\phi) + l(\psi) = l(\phi + \psi + \phi\psi).$$

(4.2.21.4)

$$\forall \phi \in \mathbf{I}_1 : e \circ l(\phi) = l \circ e(\phi) = \phi.$$

Hence one establishes the homeomorphisms $e : \mathbf{I}_1 \longleftrightarrow \mathbf{I}_1$, $e^{-1} = l$.

(4.2.21.5) One therefore obtains the homeomorphic group isomorphism $\exp : \mathbf{I}_1 \longleftrightarrow \mathbf{L}_1$, and denotes the logarithm $\ln := \exp^{-1}$.

(4.2.21.6)

$$\forall \phi, \psi \in \mathbf{L}_1 : \ln(\phi) + \ln(\psi) = \ln(\phi\psi).$$
$$\forall \phi \in \mathbf{I}_1 : \ln(1 + \phi) = l(\phi).$$

(4.2.21.7) Let $\Delta \in Der_R(R[[X]])$ be continuous.

$$\forall \phi \in \mathbf{I}_1 : \Delta \circ \exp(\phi) = \exp(\phi)\Delta(\phi).$$
$$\forall \phi \in \mathbf{L}_1 : \Delta(\phi) = \phi\Delta(\ln \phi).$$

4.3 Power Series in Finitely Many Indeterminates with Homogeneous Relations

(4.3.1) Let $X := \{x_k; k \in I\}$ be a finite set, with the index set $I := \{1, \ldots, p\}$, and denote by

$$F \equiv R\langle X\rangle := T(R(X)) := \bigoplus_{r \in \mathbf{N}_0} T^r(R(X))$$

the free algebra over X, consisting of polynomials in these generators, with the coefficients from a commutative ring R, and with the unit $e \equiv e_F$.

$$R\langle X\rangle = R(\{x_{k_1} \cdots x_{k_n}; k_1, \ldots, k_n \in I; n \in \mathbf{N}\} \cup \{e\}).$$

(4.3.2) The direct product of R-bimodules

$$\overline{F} \equiv R\langle\langle X\rangle\rangle := \prod_{r \in \mathbf{N}_0} T^r(R(X))$$

is an R-bimodule, which can be viewed as the following set of families of mappings.

$$R\langle\langle X\rangle\rangle = \left\{\left\{r_0 \in R, \left\{\prod^n I \ni \{k_1, \ldots, k_n\} \longrightarrow r_{k_1 \ldots k_n} \in R; n \in \mathbf{N}\right\}\right\}\right.$$
$$\left. =: r_0 + \sum_{n \in \mathbf{N}} \sum_{k_1, \ldots, k_n \in I} r_{k_1 \ldots k_n} x_{k_1} \cdots x_{k_n}\right\}.$$

Obviously F is an R-submodule of $\overline{F}$.

(4.3.2.1) With the product of these formal power series in p algebraically independent indeterminates defined below, one obtains an associative algebra $\overline{F}$ over R, with the subalgebra F of $\overline{F}$, and the unit

$$\bar{e} \equiv e_{\overline{F}} = e_F := \left\{ e_R, \left\{ \prod^n I \longrightarrow \{0\} \longrightarrow R; n \in \mathbf{N} \right\} \right\}.$$

$$\overline{F} \times \overline{F} \ni \left\{ r_0^{(i)} + \sum_{n \in \mathbf{N}} \sum_{k_1,\ldots,k_n \in I} r^{(i)}_{k_1\ldots k_n} x_{k_1} \cdots x_{k_n}; i = 1,2 \right\}$$
$$\longrightarrow r_0^{(1)} r_0^{(2)} + \sum_{i,j=1,2; i \neq j} r_0^{(i)} \sum_{n \in \mathbf{N}} \sum_{k_1,\ldots,k_n \in I} r^{(j)}_{k_1\ldots k_n} x_{k_1} \cdots x_{k_n}$$
$$+ \sum_{q \in \mathbf{N}\setminus\{1\}} \sum_{m,n \in \mathbf{N}; m+n=q} \sum_{k_1,\ldots,k_n, l_1,\ldots,l_m \in I} r^{(1)}_{k_1\ldots k_n} r^{(2)}_{l_1\ldots l_m} x_{k_1} \cdots x_{l_m} \in \overline{F}.$$

(4.3.2.2) With the countable basis of zero-neighbourhoods $V_0(0) := F$, $\forall m \in \mathbf{N}$:

$$V_m(0) := \left\{ \sum_{n \in \mathbf{N}} \sum_{k_1,\ldots,k_n \in I} r_{k_1\ldots k_n} x_{k_1} \cdots x_{k_n};\ \forall n < m : r_{k_1\ldots k_n} = 0 \right\}$$
$$\supset V_{m+1}(0),$$

each of them being an ideal of F, and $\bigcap_{m \in \mathbf{N}} V_m(0) = \{0\}$, F becomes some topological ring of Hausdorff type.

(4.3.2.3) By means of an analogous basis of zero-neighbourhoods $\{\overline{V}_m(0); m \in \mathbf{N}_0\}$, which are ideals of $\overline{F}$, one constructs the topological ring $\overline{F}$, which also is of Hausdorff type. $\overline{F}$ is the unique completion of F with respect to the addition. Obviously $\forall m \in \mathbf{N}_0 : \overline{V}_m(0) = \overline{V_m(0)}$, the closure of $V_m(0)$ in $\overline{F}$, is both closed and open in $\overline{F}$. $\overline{V}_0(0) = \overline{F}$.

(4.3.2.4) This topology of $\overline{F}$ is just the product topology of the Cartesian product

$$R \times \Big(\prod_{\mathbf{I}} R\Big), \mathbf{I} := \bigcup_{n \in \mathbf{N}} \Big(\Pi^n I\Big),$$

inserting the discrete topology of R.

(4.3.3) Let S be a subset of $\overline{F}$, and denote by $\overline{J}$ the closure of the ideal J of S in $\overline{F}$. Since the closure of an ideal in a topological ring is again an ideal of this ring, one can define the factor algebra $\overline{F}/\overline{J} =: R\langle\langle X, S\rangle\rangle$. With the usual quotient topology, such that the projection $\overline{\pi} : \overline{F} \longrightarrow \overline{F}/\overline{J}$ is both continuous and open, one obtains some topological group $\overline{F}/\overline{J}$ with respect to the addition; it is of Hausdorff type, because $\overline{J}$ is closed in $\overline{F}$ by definition.

Moreover, since the set of ideals $\{\overline{V}_m^S(0) := \overline{\pi}(\overline{V}_m(0)); m \in \mathbf{N}_0\}$ is some basis of zero-neighbourhoods for this quotient topology, the ring $\overline{F}/\overline{J}$ is topological. Here

$$\forall m \in \mathbf{N}_0 : \ \overline{V}_{m+1}^S(0) \subseteq \overline{V}_m^S(0) \subseteq \overline{\overline{\pi}(V_m(0))},$$

and $\overline{V}_m^S(0)$ is both open and closed in $\overline{F}/\overline{J}$. Obviously $\overline{F}/\overline{J}$ is complete with respect to the addition.

(4.3.3.1) If $\gamma : \overline{F}/\overline{J} \longrightarrow T$ is such a map into a topological space T, that $\gamma \circ \overline{\pi}$ is continuous, then γ itself is continuous.

(4.3.3.2) Relations S are called polynomial, if and only if $S \subseteq F$; in this case the closures in $\overline{F}$, of the ideal J_{fin} of S in F, and of the ideal J of S in $\overline{F}$, coincide.

(4.3.3.3) Polynomial relations S are called homogeneous of degree $r \in \mathbf{N}$, if and only if $S \subseteq T^r(R(X))$. The relations S are called homogeneous, if and only if S is the disjoint union of homogeneous relations of pairwise distinct degrees. In this case the factorization is compatible with the natural $\mathbf{N}_0$-grading of the tensor algebra F, which was used in order to construct the topology of $\overline{F}$. Therefore relations S are assumed to be homogeneous, if one aimes at the factorization with respect to their closed ideal. Otherwise the resulting factor algebra over R might collapse.

(4.3.4) Let the relations S be homogeneous, and assume that they allow for an appropriate rearrangement of monomials, such that

$$F/J_{fin} = R\left(\{x_1^{n_1} \cdots x_p^{n_p} + J_{fin}; n_1, \ldots, n_p \in \mathbf{N}_0\}\right), \ \forall_1^p k : x_k^0 := e.$$

Then the R-linear bijection:

$$R[[X]] \ni \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1\ldots n_p} x_1^{n_1} \cdots x_p^{n_p}$$

$$\longleftrightarrow \sum_{n_1,\ldots,n_p \in \mathbf{N}_0} r_{n_1\ldots n_p} x_1^{n_1} \cdots x_p^{n_p} + \overline{J} \in R\langle\langle X, S\rangle\rangle$$

is some homeomorphism. In the special case of commuting indeterminates, i.e., $S := \{x_k x_l - x_l x_k; k, l \in I\}$, this homeomorphism is an isomorphism of unital associative algebras over R.

(4.3.5) Let an associative algebra A over R with the unit e_A be equipped with a Hausdorff topology, which is compatible with the addition, and let A be complete with respect to the addition. Moreover assume existence of a countable basis $\{I_m; m \in \mathbf{N}\}$ of zero-neighbourhoods in A, all of which are ideals of A; then A is some topological ring.

(4.3.5.1) Consider a continuous homomorphism $\alpha : R\langle\langle X\rangle\rangle \longrightarrow A$ of unital associative R-algebras. Then

$$\forall \overline{m} \in \mathbf{N} : \exists \overline{n} \in \mathbf{N} : \forall n \geq \overline{n}, \forall k_1, \ldots, k_n \in I : \alpha(x_{k_1}) \cdots \alpha(x_{k_n}) \in I_{\overline{m}}.$$

(4.3.5.2) Let $\{a_1, \ldots, a_p\}$ be a family of elements of A, such that

$$\forall \overline{m} \in \mathbf{N} : \exists \overline{n} \in \mathbf{N} : \forall n \geq \overline{n}, \forall k_1, \ldots, k_n \in I : a_{k_1} \cdots a_{k_n} \in I_{\overline{m}}.$$

Then $\exists$ unique continuous homomorphism $\alpha : R\langle\langle X\rangle\rangle \longrightarrow A$ of unital associative R-algebras, such that $\forall k \in I : \alpha(x_k) = a_k$; moreover α is uniformly continuous. One then conveniently denotes $\forall \phi \in R\langle\langle X\rangle\rangle : \alpha(\phi) =: \phi(a_1, \ldots, a_p)$.

(4.3.5.3) For a family $\{\psi_k + \overline{J}; k = 1, \ldots, p\}$ of formal power series $\psi_k \in \overline{V}_1(0) \subset R\langle\langle Y\rangle\rangle$, $Y := \{y_1, \ldots, y_q\}$, $\exists$ unique continuous homomorphism $\alpha : R\langle\langle X\rangle\rangle \longrightarrow R\langle\langle Y, T\rangle\rangle$ of unital associative R-algebras, such that $\forall k \in I : \alpha(x_k) = \psi_k + \overline{J}$; α is uniformly continuous. Here $\overline{J}$ denotes the closure in $R\langle\langle Y\rangle\rangle$ of the ideal J of a homogeneous subset T of $R\langle Y\rangle$.

(4.3.6) A formal power series which homogeneous relations $\phi \in R\langle\langle X, S\rangle\rangle$ is invertible, if and only if $\exists\ \phi_1 \in \overline{V}_1(0)$, and $r_0, s_0 \in R$, such that $r_0 s_0 = e_R$ and $\phi = r_0 + \phi_1$. This lemma is proved just as in the special case of commuting indeterminates.

(4.3.7) Let $\alpha : R\langle\langle X\rangle\rangle \longrightarrow A$ be a continuous homomorphism of unital associative R-algebras, into a topological ring A of the kind introduced above. If α vanishes on homogeneous relations $S \subseteq R\langle X\rangle$, then α is factorized by the closed ideal $\overline{J}$ of S in $\overline{F}$, and then one obtains some continuous homomorphism of unital associative R-algebras: $R\langle\langle X, S\rangle\rangle \longrightarrow A$.

(4.3.8) Let the field of rational numbers $\mathbf{Q}$ be some subring of R. Take $\phi \in \overline{V}_1(0)$, and denote by $\alpha : R[[x]] \longrightarrow R\langle\langle X, S\rangle\rangle$ the unique continuous homomorphism of unital associative R-algebras, such that

$$\alpha(x) = \overline{\phi} := \phi + \overline{J} \in \{\psi + \overline{J}; \psi \in \overline{V}_1(0)\} =: \mathbf{I}_1,$$

with the closed ideal $\overline{J}$ of homogeneous relations S in $R\langle\langle X\rangle\rangle$.

$$e(\overline{\phi}) := \alpha(e(x)),\ \ \exp(\overline{\phi}) := \alpha(\exp(x)) = 1 + e(\overline{\phi}),\ \ l(\overline{\phi}) := \alpha(l(x)).$$
$$\forall \overline{\phi} \in \mathbf{I}_1 : e \circ l(\overline{\phi}) = l \circ e(\overline{\phi}) = \overline{\phi}.$$
$$\forall \overline{\phi}, \overline{\psi} \in \mathbf{I}_1, \forall m \in \mathbf{N}_0 : \ \overline{\phi} - \overline{\psi} \in \{\chi + \overline{J}; \chi \in \overline{V}_m(0)\} =: \mathbf{I}_m$$
$$\Longrightarrow e(\overline{\phi}) - e(\overline{\psi}), l(\overline{\phi} - l(\overline{\psi}) \in \mathbf{I}_m.$$

Hence one establishes the homeomorphism $\exp : \mathbf{I}_1 \longleftrightarrow \mathbf{L}_1 := \{1+\overline{\phi}; \overline{\phi} \in \mathbf{I}_1\}$, and defines the logarithm as inverse of the exponential function,

$$\ln := \exp^{-1}.\ \forall \overline{\phi} \in \mathbf{I}_1 : \ln(1 + \overline{\phi}) = l(\overline{\phi}).$$

(4.3.8.1) Especially let R be the field $\mathbf{K} := \mathbf{R}$ or $\mathbf{C}$. Then the homeomorphism $\exp = \ln^{-1}$ is naturally extended to some homeomorphism:

$$\mathbf{K}\langle\langle X, S\rangle\rangle \longleftrightarrow \mathbf{K}\langle\langle X, S\rangle\rangle \backslash \mathbf{I}_1, \text{ which is again denoted by } \exp = \ln^{-1}.$$

$$\forall c_0, d_0 \in \mathbf{K}, d_0 \neq 0, \forall \phi_1 \in \overline{V}_1(0), \phi := c_0 + \phi_1 \in \mathbf{K}\langle\langle X\rangle\rangle :$$

$$\exp(\phi + \overline{J}) := e^{c_0} \exp(\phi_1 + \overline{J}), \ \ln(d_0 + \phi_1) = \ln d_0 + \ln(1 + d_0^{-1}\phi_1).$$

Here one assumes $\overline{J} \neq \mathbf{K}\langle\langle X\rangle\rangle$, which equivalently means $\overline{e} \notin \overline{J}$, such that $c_0 \in \mathbf{C}$ is uniquely determined by $\phi + \overline{J}$.

4.4 Tensor Product of Formal Power Series with Homogeneous Relations

(4.4.1) Let J_i be an ideal of an algebra A_i over the commutative ring R, for $i = 1, 2$. Then one finds an isomorphism of R-algebras:

$$\frac{A_1}{J_1} \otimes \frac{A_2}{J_2} \ni [a_1] \otimes [a_2] \longleftrightarrow [a_1 \otimes a_2] \in \frac{A_1 \otimes A_2}{(A_1 \otimes J_2) + (J_1 \otimes A_2)}.$$

(4.4.2) Let F_i be the free R-algebra over a set X_i, and J_i the ideal in F_i of relations $S_i \subseteq F_i$, for $i = 1, 2$. Denote by F the free R-algebra over $X_1 \cup X_2$, with the sets X_1 and X_2 being disjoint, and by J the ideal in F of the set of relations

$$S_1 \cup S_2 \cup \{x_1 x_2 - x_2 x_1; x_i \in X_i, \text{ for } i = 1, 2\} \subseteq F,$$

suppressing the natural injection of F_i into F. Then one finds an isomorphism of unital associative R-algebras:

$$\frac{F_1}{J_1} \otimes \frac{F_2}{J_2} \ni [f_1] \otimes [f_2] \longleftrightarrow [f_1 f_2] \in \frac{F}{J}.$$

(4.4.3) Consider topological rings of Hausdorff type $\overline{F_i}/\overline{J_i} =: R\langle\langle X_i, S_i, \rangle\rangle$, $i = 1, 2$, $X_1 := \{x_1, \ldots, x_p\}$, $X_2 := \{y_1, \ldots, y_q\}$, $X_1 \cap X_2 = \emptyset$, with the closure $\overline{J_i}$ of the ideal J_i in $\overline{F_i}$ of homogeneous relations $S_i \subseteq F_i$. The tensor product

$$\frac{\overline{F_1}}{\overline{J_1}} \otimes \frac{\overline{F_2}}{\overline{J_2}} \longleftrightarrow \frac{\overline{F_1} \otimes \overline{F_2}}{J}, \ J := (\overline{F_1} \otimes \overline{J_2}) + (\overline{J_1} \otimes \overline{F_2}),$$

of unital associative R-algebras is equipped with the Hausdorff topology, which is due to the countable basis of zero-neighbourhoods

$$W_m^J(0) := sum(\{\phi_1 \otimes \phi_2 + J; \phi_i \in \overline{V}_{m_i}(0), \text{ for } i = 1, 2;$$
$$m_1, m_2 \in \mathbf{N}_0; m_1 + m_2 = m \in \mathbf{N}_0\}) \supseteq W_{m+1}^J(0),$$

all of which are ideals of $(\overline{F_1} \otimes \overline{F_2})/J = W_0^J(0)$.
In case of no relations, such that $J_i = \{0\}, i = 1, 2$, these zero-neighbourhoods are denoted by $W_m(0)$, and then $\bigcap_{m \in \mathbf{N}_0} W_m(0) = \{0\}$.

(4.4.3.1) The unique completion of this topological ring $\overline{F_1/J_1} \otimes \overline{F_2/J_2}$ is again some topological space of Hausdorff type.

(4.4.3.2) Denote by D the ideal of the homogeneous relations

$$S := S_1 \cup S_2 \cup \{x_k y_l - y_l x_k; k = 1, \dots, p; l = 1, \dots, q\}$$

in $R\langle\langle X_1 \cup X_2 \rangle\rangle$. Then one obtains some homeomorphism of the above defined completion onto the formal power series in the indeterminates $x_1, \dots, y_q$, with relations S:

$$\overline{R\langle\langle X_1, S_1\rangle\rangle \otimes R\langle\langle X_2, S_2\rangle\rangle} \ni (\phi_1 + \overline{J_1}) \otimes (\phi_2 + \overline{J_2})$$
$$\longleftrightarrow \phi_1\phi_2 + \overline{D} \in R\langle\langle X_1 \cup X_2, S\rangle\rangle.$$

This homeomorphism is then used to establish this completion as some unital associative R-algebra, which is some topological ring of Hausdorff type.

(4.4.3.3) Since the polynomials in $x_1, \dots, y_q$ with homogeneous relations S lie dense in $R\langle\langle X_1 \cup X_2, S\rangle\rangle$, the image of $\{\phi + J; \phi \in F_1 \otimes F_2\}$ lies dense in the completion $\overline{F_1/J_1 \otimes F_2/J_2}$.

(4.4.4) Let α_1, α_2, α be the unique continuous homomorphisms: $R[[t]] \longrightarrow R\langle\langle X_1 \cup X_2, S\rangle\rangle$ of unital associative R-algebras, with the notations from above, such that

$$\alpha_1(t) = \phi_1 + \overline{D}, \quad \alpha_2(t) = \phi_2 + \overline{D}, \quad \alpha(t) = \phi_1 + \phi_2 + \overline{D},$$

considering $\phi_1 \in R\langle\langle X_1, S_1\rangle\rangle$, $\phi_2 \in R\langle\langle X_2, S_2\rangle\rangle$ as elements of $R\langle\langle X_1 \cup X_2, S\rangle\rangle$, and assume both ϕ_1 and $\phi_2 \in \overline{V}_1^S(0)$.

$$\exp(t) \xrightarrow[def]{\alpha_1} \exp(\phi_1) + \overline{D} \longleftrightarrow (\exp(\phi_1) + \overline{J_1}) \otimes e_R,$$

$$\exp(t) \xrightarrow[def]{\alpha_2} \exp(\phi_2) + \overline{D} \longleftrightarrow e_R \otimes (\exp(\phi_2) + \overline{J_2}),$$

$$\exp(t) \xrightarrow[def]{\alpha} \exp(\phi_1 + \phi_2) + \overline{D} = \exp(\phi_1)\exp(\phi_2) + \overline{D}$$
$$\longleftrightarrow \exp(\phi_1) \otimes \exp(\phi_2) + J.$$

Hence one finds, that

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!} (\phi_1 \otimes e_R)^n + J \right) = (\exp(\phi_1) + \overline{J_1}) \otimes e_R,$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!} (e_R \otimes \phi_2)^n + J \right) = e_R \otimes (\exp(\phi_2) + \overline{J_2}).$$

(4.4.5) Consider the free Lie algebra $L(X)$ over a set X with the coefficients $\in R \equiv K$, over a field K of *char* $K = 0$.

$$L(X) := K - \mathit{lin\ span}(X \cup \{\mathit{mult.\ commutators} \text{ of } \mathit{elements}\ \in X\}).$$

Then $F := T(K(X))$ is the universal enveloping algebra of $L(X)$.
Denote by $\Delta : F \longrightarrow F \otimes F$ the unique homomorphism of unital associative algebras, such that

$$\forall \phi \in L(X) : \Delta(\phi) = \phi \otimes e_K + e_K \otimes \phi,$$

suppressing the embedding of e_K into F.

(4.4.5.1) The composite mapping: $F \longrightarrow F \otimes F \longrightarrow \overline{\overline{F} \otimes \overline{F}}$, inserting Δ, is uniformly continuous. Therefore $\exists$ unique continuous extension

$$\overline{\Delta} : \overline{F} \longrightarrow \overline{\overline{F} \otimes \overline{F}}.$$

This extension $\overline{\Delta}$ of Δ is uniformly continuous, and $\overline{\Delta}$ is also some homomorphism of unital associative K-algebras. Moreover the closure $\overline{L(X)}$ of $L(X)$ in $\overline{F}$ is some subalgebra of the commutator algebra $\overline{F}_L$, and

$$\overline{L(X)} = \{\phi \in K\langle\langle X\rangle\rangle; \overline{\Delta}(\phi) = e_K \otimes \phi + \phi \otimes e_K\}.$$

(4.4.5.2)

$$\forall \phi, \psi \in \overline{L(X)} : \exp(\phi)\exp(\psi) \xrightarrow[\overline{\Delta}]{} \exp(\phi)\exp(\psi) \otimes \exp(\phi)\exp(\psi),$$

and therefore $\ln(\exp(\phi)\exp(\psi)) \in \overline{L(X)}$.

(4.4.5.3) $\forall \phi, \psi \in \overline{L(X)} \subset \overline{V}_1(0)$:

$$\exp(\phi)\exp(\psi) = \exp(\phi + \psi + \tfrac{1}{2}[\phi, \psi] + \tfrac{1}{12}([[\phi, \psi], \psi] + [\phi, [\phi, \psi]]) + \chi),$$

with an element $\chi \in \overline{L(X)} \cap \overline{V}_4(0)$.

(4.4.5.4) The Jacobi identity shows, that

$$L(X) = \bigoplus_{r \in \mathbf{N}} L^r(X),$$
$$L^r(X) := K - \mathit{lin\ span}(\{[\dots[[x_{k_1}, x_{k_2}], x_{k_3}]\dots, x_{k_r}];$$
$$k_1, \dots, k_r = 1, \dots, p\}) \subset T^r(K(X)).$$

Consider the K-linear map κ:

$$T(K(X)) \ni x_{k_1} \cdots x_{k_r} \xrightarrow[def]{} [\dots[[x_{k_1}, x_{k_2}], x_{k_3}]\dots, x_{k_r}] \in L(X),$$

$$\kappa(e_K) := 0,\ \forall_1^p k : \kappa(x_k) := x_k.$$

One shows by an induction with respect to $r \in \mathbf{N}$, that

$$\forall r \in \mathbf{N} : L^r(X) = \{\phi \in T^r(K(X)); \kappa(\phi) = r\phi\}.$$

Here one uses, that $\kappa\big|_{L(X)}$ is some derivation of $L(X)$.

(4.4.5.5) Consider formal power series $\in K\langle\langle\{s,t\}\rangle\rangle$. Since

$$\ln(\exp(s)\exp(t)) = \sum_{m\in\mathbf{N}} \sum_{\Sigma(m)} \frac{(-1)^{m+1}}{mp_1!q_1!\cdots p_m!q_m!} s^{p_1}t^{q_1}\cdots s^{p_m}t^{q_m} \in \overline{L(X)},$$

this series can be rearranged by means of the unique continuous extension $\overline{\kappa} : K\langle\langle X\rangle\rangle \longrightarrow \overline{L(X)}$ of κ. An insertion of formal power series then yields the Baker-Campbell-Hausdorff formula. $\forall \phi, \psi \in \overline{V}_1(0)$:

$$\ln(\exp(\phi)\exp(\psi)) = \sum_{m\in\mathbf{N}} \sum_{\Sigma(m)} \frac{(-1)^{m+1}}{mp_1!q_1!\cdots p_m!q_m!} \left(\sum_{k=1}^{m}(p_k + q_k)\right)^{-1}$$
$$[\ldots \underbrace{[\phi,\phi]\ldots,\phi]}_{p_1}, \underbrace{\psi]\ldots,\psi]}_{q_1}\ldots, \underbrace{\phi]\ldots,\phi]}_{p_m}, \underbrace{\psi]\ldots,\psi]}_{q_m},$$
$$\Sigma(m) := \{p_1,\ldots,q_m \in \mathbf{N}_0; \forall_1^m k : p_k + q_k > 0\}.$$

(4.4.5.6) $\forall \phi, \psi \in \overline{L(X)} : \exp(\phi)\psi\exp(-\phi) \in \overline{L(X)}$. Hence one finds, that

$$\forall \phi, \psi \in \overline{V}_1(0) : \exp(\phi)\psi\exp(-\phi) = \psi + [\phi,\psi] + \sum_{m=2}^{\infty} \frac{1}{m!} \underbrace{[\phi,\ldots[\phi}_{m},\psi]\ldots].$$

(4.4.5.7) These series of multiple commutators can be factorized with respect to the closure $\overline{J}$ in $\overline{F}$ of an ideal J of homogeneous relations $S \subset F$.

5. $\mathbf{Z}_2$-Graded Lie-Cartan Pairs

Lie-Cartan pairs serve for a suitably unified algebraic description of the covariant exterior derivation of differential forms with respect to a connection on a finite-dimensional real differentiable manifold, of the covariant derivation on real or complex vector bundles, and of cohomologies of Lie algebras.

For instance as a prototype, consider the pair $\{L := T(\mathbf{M}),\ A := C^\infty(\mathbf{M})\}$ of smooth vector fields and scalar fields on an m-dimensional real differentiable manifold $\mathbf{M}$, and the $\mathbf{R}$-bilinear mappings:

$$L \times A \ni \{X, f\} \underset{def}{\longrightarrow} L_X f \in A, \quad A \times L \ni \{f, X\} \underset{def}{\longrightarrow} fX \in L,$$

inserting the Lie derivation L_X of scalar fields with respect to a vector field X, and the module-multiplication of the A-left module L over $\mathbf{R}$.

On the other hand, the E-valued Chevalley cohomologies of a Lie algebra L over a field K provide an easy application of the Lie-Cartan pair $\{L, K\}$ with the K-bilinear mapping: $L \times K \longrightarrow \{0\}$, and $K \times L \longrightarrow L$ defined as the module-multiplication of L; here E denotes a vector space over K.

In case of an atlas consisting of two or more charts, the $C^\infty(\mathbf{M})$-left module $T(\mathbf{M})$ of smooth vector fields on an m-dimensional real differentiable manifold $\mathbf{M}$ is projective, but not free over the commutative unital associative algebra of scalar fields $C^\infty(\mathbf{M})$. Correspondingly one constructs projective-finite right modules over an associative unital algebra, thereby generalizing the concept of real or complex vector bundles.

An appropriate generalization to Lie superalgebras and graded-commutative unital associative superalgebras leads to the notion of $\mathbf{Z}_2$-graded Lie-Cartan pairs. These provide an algebraic scheme for the calculus of real $\mathbf{Z}_2$-graded differential forms, as one step towards a theory of supermanifolds. On the other hand one obtains an elegant approach to the theory of cohomologies of Lie superalgebras.

The concept of Lie-Cartan pairs was developed by D. Kastler and R. Stora (1985), their $\mathbf{Z}_2$-graded generalization is due to A. Jadczyk and D. Kastler (1987). An according differential geometry on Grassmann algebras, including the so-called Berezin integral, is presented in a review by R. Coquereaux, A. Jadczyk, and D. Kastler (1991).

The matrix differential geometry, which was constructed by M. Dubois-Violette, R. Kerner, and J. Madore (1990), can be obtained from an appropriate $Mat_n(\mathbf{C})$-connection of the Lie-Cartan pair $\{Der_{\mathbf{C}}(Mat_n(\mathbf{C})), \mathbf{C}\}$. Generalizing this framework to $Mat_n(\mathbf{C})$-valued smooth functions on an m-dimensional real connected differentiable manifold, the above authors developed models of gauge theory.

An algebraic approach to the notions of covariant derivation, its curvature, and connection was performed by R. Matthes (1991), starting from an associative algebra over a field and the Lie algebra of its derivations. The differential geometry on real or complex fibre bundles is then reformulated within this algebraic framework.

For the classical theory of differentiable manifolds the reader is referred to the corresponding volume by J. Dieudonné (1972), and to the monograph by S. Lang (1972).

Cohomologies of Lie algebras over a field are treated on the abstract level of homological algebra, which is due to H. Cartan and S. Eilenberg (1956), by N. Jacobson (1962). Cohomologies of Lie superalgebras were studied by D. A. Leites (1976).

5.1 Tensor Products over Graded-Commutative Algebras

(5.1.1) Let $E = E^{\bar{0}} \oplus E^{\bar{1}}$ be an A-left module over R, with a unital associative superalgebra A over a commutative ring R. Assume A to be graded-commutative, and moreover, that

$$\forall \bar{z}_1, \bar{z}_2 \in \mathbf{Z}_2, \forall a \in A^{\bar{z}_1}, x \in E^{\bar{z}_2} : ax \in E^{\overline{z_1+z_2}}.$$

Then E is called graded A-left module over R. With the R-bilinear map:

$$E \times A \ni \{x, a\} \underset{def}{\longrightarrow} \sum_{\bar{z}_1, \bar{z}_2 \in \mathbf{Z}_2} (-1)^{z_1 z_2} a^{\bar{z}_1} x^{\bar{z}_2} =: xa \in E,$$

E becomes also an A-right module over R. Then E is called graded A-bimodule over R. One immediately finds, that

$$\forall a, b \in A, \forall x \in E : a(xb) = (ax)b.$$

(5.1.2) Let δ be a homogenous derivation of A, $\delta \in Der_R^{\bar{z}}(A)$, i.e.,

$$\forall \bar{y} \in \mathbf{Z}_2, a \in A^{\bar{y}}, b \in A : \delta(ab) = \delta(a)b + (-1)^{yz} a\delta(b),\ Im\ \delta\big|_{A^{\bar{y}}} \subseteq A^{\overline{y+z}},$$

with the degree $\bar{z}$ of δ being fixed. An endomorphism d of E is then called δ-derivation of E, if and only if $d \in End_R^{\bar{z}}(E)$ fulfills the property that

$$\forall \bar{y} \in \mathbf{Z}_2, a \in A^{\bar{y}}, x \in E : d(ax) = \delta(a)x + (-1)^{yz} ad(y).$$

(5.1.3) One then immediately finds the following lemma. Let d be a δ-derivation, and d' a δ'-derivation of E. Then

$$[d, d'] := d \circ d' - (-1)^{zz'} d' \circ d \in End_R^{\overline{z+z'}}(E)$$

is some $[\delta, \delta']$-derivation of E, with the super-commutator

$$[\delta, \delta'] := \delta \circ \delta' - (-1)^{zz'} \delta' \circ \delta \in Der_R^{\overline{z+z'}}(A),$$

and with the degrees $\bar{z}$ of δ and d, $\overline{z'}$ of δ' and d', respectively.

(5.1.4.1) Let E, F be graded A-bimodules over R. Their tensor product over A is defined as the following factor module over R.

$$E \otimes_A F := (E \otimes F) / N_A^2,$$
$$N_A^2 := sum(\{(xa) \otimes y - x \otimes (ay); x \in E, y \in F, a \in A\}).$$

(5.1.4.2) The diagram below, inserting $a \in A$, allows for the definition of an A-left module $E \otimes F$ over R, such that

$$\forall x \in E, y \in F, a \in A : a(x \otimes y) = (ax) \otimes y.$$

$$\begin{tikzcd}
E \times F \ni \{x,y\} \arrow[rr] \arrow[drr, "R\text{-bilinear}"', "def"] & & x \otimes y \in E \otimes F \arrow[d, "R\text{-linear}"'] \\
& & (ax) \otimes y \in E \otimes F
\end{tikzcd}$$

(5.1.4.3) Obviously this A-left module $E \otimes F$ over R is graded with respect to the natural $\mathbf{Z}_2$-grading of $E \otimes F$. Hence $E \otimes F$ is some graded A-bimodule over R.

(5.1.4.4) Since the above factorization is compatible with this A-left module structure of $E \otimes F$, there is an A-left module $E \otimes_A F$ over R such that, with respect to this factor module,

$$\forall a \in A, x \in E, y \in F : a(x \otimes y + N_A^2) = (ax) \otimes y + N_A^2.$$

(5.1.4.5) Since the above R-submodule is $\mathbf{Z}_2$-graded,

$$N_A^2 = \bigoplus_{\bar{z} \in \mathbf{Z}_2} (N_A^2 \cap (E \otimes F)^{\bar{z}}),$$

one can define, that $\forall \bar{z} \in \mathbf{Z}_2$, $\forall\ t + N_A^2 \in E \otimes_A F$:

$$t + N_A^2 \in (E \otimes_A F)^{\bar{z}} \underset{def}{\Longleftrightarrow} \exists t' \in (E \otimes F)^{\bar{z}} : t' - t \in N_A^2.$$

Thereby one establishes the graded A-bimodule $E \otimes_A F$ over R. For homogeneous elements one finds, that $\forall \overline{z_1}, \overline{z_2}, \bar{y} \in \mathbf{Z}_2, \forall x_1 \in E^{\overline{z_1}}, x_2 \in F^{\overline{z_2}}, a \in A^{\bar{y}}$:

$$\begin{aligned}(-1)^{z_1 y} a\big(x_1 \otimes x_2 + N_A^2\big) &= (x_1 a) \otimes x_2 + N_A^2 \\ &= x_1 \otimes (a x_2) + N_A^2 = (-1)^{y z_2}\big(x_1 \otimes x_2 + N_A^2\big) a.\end{aligned}$$

(5.1.5) More generally, let $E_1, \ldots, E_p$ be graded A-bimodules over R. One then factorizes with respect to the R-submodule

$$\begin{aligned}N_A^p := sum\big(\{&x_1 \otimes \cdots \otimes (a x_i) \otimes \cdots \otimes x_p \\ &- (-1)^{y(z_i + \cdots + z_{j-1})} x_1 \otimes \cdots \otimes (a x_j) \otimes \cdots \otimes x_p; \\ &\forall_1^p k : x_k \in E_k^{\overline{z_k}}, \bar{z}_k \in \mathbf{Z}_2;\ 1 \leq i < j \leq p;\ a \in A^{\bar{y}}, \bar{y} \in \mathbf{Z}_2\}\big),\end{aligned}$$

and obtains the graded A-bimodule over R

$$\bigotimes_{A\ k=1}^{p} E_k := \left(\bigotimes_{k=1}^{p} E_k\right) / N_A^p.$$

(5.1.6) Let E, F be graded A-bimodules over R. With the natural $\mathbf{Z}_2$-grading of the R-bimodule $Hom_R(E, F)$, one defines $\forall \bar{z} \in \mathbf{Z}_2$:

$$\begin{aligned}Hom_A^{\bar{z}}(E, F) := \big\{&\phi \in Hom_R^{\bar{z}}(E, F);\ \forall \bar{y} \in \mathbf{Z}_2, a \in A^{\bar{y}}, x \in E : \\ &\phi(ax) = (-1)^{zy} a \phi(x)\big\},\end{aligned}$$

as an R-submodule of $Hom_R(E, F)$. Then

$$Hom_A(E, F) := \bigoplus_{\bar{z} \in \mathbf{Z}_2} Hom_A^{\bar{z}}(E, F).$$

With the definition, that

$$\forall a \in A, x \in E, \phi \in Hom_A(E, F) : (a\phi)(x) := a\phi(x),$$

one then establishes the graded A-bimodule $Hom_A(E, F)$ over R.

(5.1.7) Let $E_1, \ldots, E_p, F$ be graded A-bimodules over R. Consider the graded A-bimodules $E_1 \otimes_A \cdots \otimes_A E_p$ and $Hom_A\big(E_1 \otimes_A \cdots \otimes_A E_p, F\big)$ over R, in order to establish the universal property of the tensor product over A, according to the subsequent diagram.

$$\begin{tikzcd}
\times_{k=1}^{p} E_k \arrow[r] \arrow[dr, "\lambda_p", "R\text{-multilinear}"'] & \bigotimes_{k=1}^{p} E_k \arrow[r] & \bigotimes_{A\ k=1}^{p} E_k \arrow[dl, "\lambda_p^*"'] \\
& F &
\end{tikzcd}$$

$$L_A(E_1,\dots,E_p;F) := \Big\{\lambda_p : \prod_{k=1}^{p} E_k \longrightarrow F;$$
$$\forall \bar{z}, \overline{z_1},\dots,\overline{z_p} \in \mathbf{Z}_2, \forall x_1 \in E_1^{\overline{z_1}},\dots,x_p \in E_p^{\overline{z_p}}, \forall a \in A^{\bar{z}}, \forall_1^p j :$$
$$\lambda_p(x_1,\dots,ax_j,\dots,x_p) = (-1)^{z(z_j+\cdots+z_p)}\lambda_p(x_1,\dots,x_p)a\Big\}.$$

Here obviously the natural $\mathbf{Z}_2$-grading of λ_p corresponds to that of the uniquely induced map $\lambda_p^* \in Hom_A(E_1 \otimes_A \cdots \otimes_A E_p, F)$.

(5.1.8) Let expecially $E_1 = \cdots = E_p =: E$, and consider the following graded-alternating forms.

$$\Lambda_A^p(E,F) := \Big\{\lambda_p \in L_A^p(E,F) := L_A(\underbrace{E,\dots,E}_{p\ times};F);$$
$$\forall \overline{z_1},\dots,\overline{z_p} \in \mathbf{Z}_2, \forall x_1 \in E^{\overline{z_1}},\dots,x_p \in E^{\overline{z_p}}, \forall P := \begin{bmatrix} 1\dots p \\ j_1\dots j_p \end{bmatrix} \in \mathbf{P}_p :$$
$$\lambda_p(x_{j_1},\dots,x_{j_p}) = \lambda_p(x_1,\dots,x_p)(-1)^{\tau_p+\sum_{\{1\le k<l\le p;j_k>j_l\}} z_{j_k}z_{j_l}}\Big\}.$$

With an appropriate R-submodule, namely

$$N_\Lambda^p := sum\Big(\Big\{x_1 \otimes \cdots \otimes x_p$$
$$- x_{j_1} \otimes \cdots \otimes x_{j_p}(-1)^{\tau_p+\sum_{\{1\le k<l\le p;j_k>j_l\}} z_{j_k}z_{j_l}};$$
$$\forall_1^p k : x_k \in E^{\overline{z_k}}, \overline{z_k} \in \mathbf{Z}_2;\ P := \begin{bmatrix} 1\dots p \\ j_1\dots j_p \end{bmatrix} \in \mathbf{P}_p\Big\}\Big),$$

one establishes the double-factorization with respect to N_A^p and N_Λ^p, i.e., the R-linear bijections:

$$\left(\left(\bigotimes^p E\right)/N_A^p\right)/N_{A\Lambda}^p \longleftrightarrow \left(\bigotimes^p E\right)/(N_A^p + N_\Lambda^p)$$
$$\longleftrightarrow \left(\left(\bigotimes^p E\right)/N_\Lambda^p\right)/N_{\Lambda A}^p,$$

such that

$$\forall t \in \bigotimes^p E :\ (t + NA^p) + N_{A\Lambda}^p \longleftrightarrow t + N_A^p + N_\Lambda^p \longleftrightarrow (t + N_\Lambda^p) + N_{\Lambda A}^p,$$

with the R-submodules

$$N^p_{A\Lambda} := sum\big(\{t + N^p_A; t \in N^p_\Lambda\}\big),\ N^p_{\Lambda A} := sum\big(\{t + N^p_\Lambda; t \in N^p_A\}\big).$$

The graded tensor product over A is then defined as

$$\hat{\bigotimes}^p_A E := \left(\bigotimes^p E\right) / (N^p_A + N^p_\Lambda).$$

The implicit definition, such that $\forall a \in A, \forall x_1, \ldots, x_p \in E$:

$$a\big(x_1 \otimes \cdots \otimes x_p + N^p_A + N^p_\Lambda\big) = (ax_1) \otimes x_2 \otimes \cdots \otimes x_p + N^p_A + N^p_\Lambda,$$

yields the graded A-bimodules $\hat{\bigotimes}^p_A E$, and accordingly, $Hom_A\left(\hat{\bigotimes}^p_A E, F\right)$.

(5.1.8.1) One then establishes the universal property of the graded tensor product over A, according to the next diagram , with $\lambda_p \in \Lambda^p_A(E, F)$, and an according unique A-linear graded-alternating form $\hat{\lambda}_p \in Hom_A\left(\hat{\bigotimes}^p_A E, F\right)$.

$$\begin{tikzcd}
\overset{p}{\times} E \arrow[r] \arrow[rrd, "\lambda_p"'] & \overset{p}{\otimes} E \arrow[r] & \overset{p}{\otimes}_A E \arrow[r] \arrow[d, "\lambda_p^*"] & \overset{\hat{p}}{\otimes}_A E \arrow[ld, "\hat{\lambda}_p"] \\
 & & F &
\end{tikzcd}$$

(5.1.9) Let E be a graded A-bimodule over R. The tensor algebra of E over A is constructed. For $p = 2, 3, \ldots,$

$$T^p_A(E) := \bigotimes^p_A E,\ T^1_A(E) := E,\ T^0_A(E) := A,\ T_A(E) := \bigoplus_{p \in \mathbf{N}_0} T^p_A(E).$$

(5.1.9.1) For convenience, denote $\forall p \geq 2$,

$$\forall x_1, \ldots, x_p \in E : x_1 \otimes \cdots \otimes x_p + N^p_A =: x_1 \otimes_A \cdots \otimes_A x_p \in T^p_A(E).$$

(5.1.9.2) The tensor product over A is associative, modulo an A-linear bijection, as is shown in the following diagrams.

$$\begin{tikzcd}
 & E \otimes_A (E \otimes_A E) \ni \\
E \times E \ni \{x_1, x_2\} \arrow[r, "def"', "\lambda_{x_3}"] \arrow[d] & x_1 \otimes (x_2 \otimes x_3 + N^2_A) + N^2_A \arrow[d, "\lambda^*_{x_3}"] \\
E \otimes E \ni x_1 \otimes x_2 \arrow[r] & x_1 \otimes x_2 + N^2_A \in E \otimes_A E,\ x_3 \in E
\end{tikzcd}$$

$$
\begin{tikzcd}
(E\otimes_A E)\times E \ni \{t+N_A^2, x_3\} \arrow[r, "def"', "\rho"] \arrow[d] & \lambda^*_{x_3}(t+N_A^2) \in E\otimes_A (E\otimes_A E) \\
(E\otimes_A E)\otimes E \ni (t+N_A^2)\otimes x_3 \arrow[r] & (t+N_A^2)\otimes x_3 + N_A^2 \in (E\otimes_A E)\otimes_A E \arrow[u, "\rho_*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(E\otimes_A E)\otimes_A E \ni (x_1\otimes x_2+N_A^2)\otimes x_3+N_A^2 \arrow[d, mapsto] \arrow[d, bend right, "A\text{-linear}"', <->] \\
x_1\otimes(x_2\otimes x_3+N_A^2)+N_A^2 \in E\otimes_A(E\otimes_A E)
\end{tikzcd}
$$

One then similarly contructs $\sigma_* : E\otimes_A (E\otimes_A E) \longrightarrow (E\otimes_A E)\otimes_A E$, and finds that $\sigma_*\circ\rho_* = id$, $\rho_*\circ\sigma_* = id$.

(5.1.9.3) The product of $T_A(E)$ is defined, omitting the canonical embeddings, such that $\forall p,q\in\mathbf{N}, \forall x_1,\ldots,x_p,y_1,\ldots,y_q\in E, \forall a,b\in A$:

$$
\begin{aligned}
&(x_1\otimes_A\cdots\otimes_A x_p)(y_1\otimes_A\cdots\otimes_A y_q)\\
&= x_1\otimes_A\cdots\otimes_A x_p\otimes_A y_1\otimes_A\cdots\otimes_A y_q,
\end{aligned}
$$

ab is defined from A, ay_1 and x_1b are defined from E,

$$a(y_1\otimes_A\cdots\otimes_A y_q) = (ay_1)\otimes_A y_2\otimes_A\cdots\otimes_A y_q,$$

$$(x_1\otimes_A\cdots\otimes_A x_p)b = x_1\otimes_A\cdots\otimes_A x_{p-1}\otimes_A (x_pb).$$

(5.1.9.4) Hence $T_A(E)$ is an associative algebra over R with the unit e_A. Moreover $T_A(E)$ is some graded A-bimodule over R, with the usual $\mathbf{Z}_2$-grading of $T(E)$ used for an appropriate grading of $T_A(E)$. Especially $T_A(E)$ is some graded A-algebra over R, because one finds, that

$$\forall a\in A, \forall t_1,t_2\in T_A(E) : a(t_1t_2) = (at_1)t_2.$$

(5.1.9.5) The graded A-bimodule B over R is called graded A-algebra over R, if and only if B is a unital associative algebra over R, and moreover

$$\forall a\in A, \forall b_1,b_2\in B : a(b_1b_2) = (ab_1)b_2.$$

(5.1.9.5.1) Let B, B' be graded A-algebras over R. An A-linear mapping $\phi = \phi^{\bar 0} + \phi^{\bar 1} \in Hom_A(B, B')$ is called graded algebra-homomorphism over A, if and only if $\phi^{\bar 0}(e_B) = e_{B'}$, and

$$\forall \bar z, \bar y \in \mathbf{Z}_2, \forall b_1 \in B^{\bar z}, b_2 \in B : \phi^{\bar y}(b_1 b_2) = (-1)^{yz_1}\phi^{\bar y}(b_1)\phi^{\bar y}(b_2).$$

In this case, if especially ϕ is even, then it is a homomorphism in the sense of unital associative algebras over R.

(5.1.9.6) The mappings $\phi_k \in Hom_A(E_k, F_k), k = 1, \dots, p$, induce an A-linear map

$$T_A^p(\phi_1, \dots, \phi_p) \in Hom_A\big(E_1 \otimes_A \cdots \otimes_A E_p, F_1 \otimes_A \cdots \otimes_A F_p\big),$$

due to the universal property of tensor products, such that
$\forall x_1 \in E_1, \dots, x_p \in E_p$:

$$\begin{aligned} &T_A^p(\phi_1, \dots, \phi_p)\left(\bigotimes_{A\ k=1}^{p} x_k\right) \\ &= \sum_{\overline{z_1}, \dots, \overline{z_p}, \bar y \in \mathbf{Z}_2} (-1)^{\sum_{1 \le k < l \le p} z_k y_l} \bigotimes_{A\ k=1}^{p} \phi_k^{\bar y}(x_k^{\overline{z_k}}). \end{aligned}$$

(5.1.9.7) Accordingly an A-linear mapping $\phi \in Hom_A(E, F)$ induces $T_A(\phi) \in Hom_A(T_A(E), T_A(F))$. Moreover $T_A(\phi)$ is some graded algebra-homomorphism over A, because one finds, that $\forall \overline{z_1}, \dots, \overline{z_p}, \bar y \in \mathbf{Z}_2$,
$\forall x_1 \in E^{\overline{z_1}}, \dots, x_p \in E^{\overline{z_p}}, \forall y_1, \dots, y_q \in E, \forall \phi \in Hom_A^{\bar y}(E, F)$:

$$\begin{aligned} &T_A(\phi)(x_1 \otimes_A \cdots \otimes_A x_p) T_A(\phi)(y_1 \otimes_A \cdots \otimes_A y_q) \\ &= (-1)^{qy(z_1 + \cdots + z_p)} T_A(\phi)(x_1 \otimes_A \cdots \otimes_A x_p \otimes_A y_1 \otimes_A \cdots \otimes_A y_q). \end{aligned}$$

With the definition, that $\forall a \in T_A^0(E) : T_A(\phi)(a) := a \in T_A^0(F)$, one establishes the covariant functor T_A, according to the diagram below.

$$\begin{tikzcd}
E \arrow[r, "\text{embedding}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\xrightarrow{\text{covariant functor}}"] & T_A(E) \arrow[d, "T_A(\phi)"] \\
F \arrow[r, "\text{embedding}"] & T_A(F)
\end{tikzcd}$$

(5.1.9.8) The universal property of the tensor algebra of E over A is shown in the subsequent diagram.

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "\phi"'] & & T_A(E) \arrow[dl, "\phi_*"] \\ & B & \end{tikzcd}$$

Here E and B are graded A-bimodules over R, and moreover B is some graded A-algebra over R. Then $\phi \in Hom_A(E, B)$ is extended to a unique graded algebra-homomorphism ϕ_* over A, which is explicitly contructed by means of the universal property of tensor products.

(5.1.9.8.1) For $p = 2, 3, 4, \ldots$, $\exists$ *unique* $\phi_*^p \in Hom_A\big(T_A^p(E), B\big)$, such that

$$\phi_*^p\left(\bigotimes_{A}{}^{p}_{k=1} x_k\right) = \sum_{\bar{y}\in\mathbf{Z}_2} (-1)^{\sum_{k=1}^{p-1} z_k(p-k)y} \phi^{\bar{y}}(x_1)\cdots\phi^{\bar{y}}(x_p).$$

(5.1.9.8.2) The direct sum of ϕ_*^p, $\overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2$, $p \in \mathbf{N}_0$, $\phi_*^1 := \phi$, $\forall a \in A : \phi_*^0(a) := ae_B$, yields the desired homomorphism ϕ_*.

(5.1.9.9) The alternating algebra of E over A is constructed, using the graded tensor product over A.

$$\Lambda_A(E) := \bigoplus_{p\in\mathbf{N}_0} \Lambda_A^p(E),\ \Lambda_A^0(E) := A,\ \Lambda_A^1(E) := E,$$

$$\forall p \geq 2 : \Lambda_A^p(E) := \hat{\bigotimes}{}^{p}_{A} E.$$

Then the product of $\Lambda_A(E)$ is defined, such that
$\forall x_1, \ldots, x_p, y_1, \ldots, y_q \in E, \forall a, b \in A$:

$$\begin{aligned}&\big(x_1 \otimes \cdots \otimes x_p + N_A^p + N_\Lambda^p\big)\big(y_1 \otimes \cdots \otimes y_q + N_A^q + N_\Lambda^q\big)\\ &= x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q + N_A^{p+q} + N_\Lambda^{p+q},\end{aligned}$$

$$\big(x_1 \otimes \cdots \otimes x_p + N_A^p + N_\Lambda^p\big)b = x_1 \otimes \cdots \otimes x_{p-1} \otimes (x_p b) + N_A^p + N_\Lambda^p,$$
$$a\big(y_1 \otimes \cdots \otimes y_q + N_A^q + N_\Lambda^q\big) = (ay_1) \otimes y_2 \otimes \cdots \otimes y_q + N_A^q + N_\Lambda^q,$$

with the elements of $\Lambda_A^0(E)$ being inserted in the natural manner. Thereby $\Lambda_A(E)$ becomes some graded A-algebra over R, with the following universal property: For any $\phi \in Hom_A(E, B)$, which is graded-alternating in the sense defined below, there exists a unique graded A-algebra homomorphism $\hat{\phi}$ over R, such that the following diagram is commutative.

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "\phi"'] & T_A(E) \arrow[r] \arrow[d, "\phi_*"] & \Lambda_A(E) \arrow[dl, "\hat{\phi}"] \\ & B & \end{tikzcd}$$

Here $\phi = \phi^{\bar{0}} + \phi^{\bar{1}}$ is assumed to be graded-alternating in the sense, that $\forall \overline{y_1}, \overline{y_2}, \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall x_1 \in E^{\overline{z_1}}, x_2 \in E^{\overline{z_2}}$:

$$\phi^{\overline{y_1}}(x_1)\phi^{\overline{y_2}}(x_2) = (-1)^{1+z_1z_2+z_1y_2+z_2y_1}\phi^{\overline{y_2}}(x_2)\phi^{\overline{y_1}}(x_1).$$

(5.1.9.10) Obviously both $\bigoplus_{p\in\mathbf{N}_0} N_A^p$ and $\bigoplus_{p\in\mathbf{N}_0} N_\Lambda^p$ are ideals of the tensor algebra $T(E)$. One trivially finds an R-linear bijection:

$$\frac{T(E)}{\bigoplus_{p\in\mathbf{N}_0}(N_A^p + N_\Lambda^p)} \ni \big\{\{t^p + d^p; p \in \mathbf{N}_0\}; \forall p \in \mathbf{N}_0 : d^p \in N_A^p + N_\Lambda^p\big\}$$

$$\longleftrightarrow \big\{\{t^p + d^p; d^p \in N_A^p + N_\Lambda^p\}; p \in \mathbf{N}_0\big\} \in R \oplus E \oplus \left(\bigoplus_{p\geq 2} \frac{T^p(E)}{N_A^p + N_\Lambda^p}\right),$$

with the definitions: $N_A^0 + N_A^1 + N_\Lambda^0 + N_\Lambda^1 = \{0\}$.

(5.1.9.11) Obviously the alternating algebra of E over A is spanned by the following monomials.

$$\Lambda_A(E) = A \oplus E \oplus sum(\{x_1 \wedge_A \cdots \wedge_A x_p; x_1, \ldots, x_p \in E; p \geq 2\}),$$

$$\begin{aligned} x_1 \wedge_A \cdots \wedge_A x_p &:= x_1 \otimes \cdots \otimes x_p + N_A^p + N_\Lambda^p \\ &\in \Lambda_A^p(E) := T^p(E)/(N_A^p + N_\Lambda^p). \end{aligned}$$

(5.1.9.12) Every even A-linear mapping $\phi \in Hom_A^{\bar{0}}(E, F)$ induces $\Lambda_A(\phi) \in Hom_A(\Lambda_A(E), \Lambda_A(F))$, due to the subsequent diagram.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\phi"] & T_A(E) \arrow[r] \arrow[d, "T_A(\phi)"] & \Lambda_A(E) \arrow[d, "\Lambda_A(\phi)"] \\ F \arrow[r] & T_A(F) \arrow[r] & \Lambda_A(F) \end{tikzcd}$$

Here again one uses the $\mathbf{Z}_2$-grading of $\Lambda_A(E)$ by homogeneous representatives, because both N_A^p and N_Λ^p are $\mathbf{Z}_2$-graded R-submodules of $T^p(E)$, for $p \geq 2$.

5.2 Projective-Finite Modules

(5.2.1) Consider an A-right module E over R, with the unital associative algebra A over R. $A^n := \bigoplus^n A$, as an R-bimodule. Furthermore with the definitions, that $\forall \{a_1, \ldots, a_n\} \in A^n, \forall b, c \in A$:

$$\{a_1, \ldots, a_n\}b := \{a_k b; k = 1, \ldots, n\}, \; c\{a_1, \ldots, a_n\} := \{ca_k; k = 1, \ldots, n\},$$

A^n becomes an A-A-bimodule over R, i.e., by definition,

(i) A^n is an A-left module over R, and also an A-right module over R;

(ii) $(c\{a_1, \ldots, a_n\})b = c(\{a_1, \ldots, a_n\}b)$.

(5.2.2) The map

$$\gamma \in Hom_A(A^n, E) \subseteq Hom_R(A^n, E)$$

is called generating n-map for E, if and only if γ is surjective; in this case,

$$\lambda \in Hom_A(E, A^n) \subseteq Hom_R(E, A^n)$$

is called lift for γ, if and only if $\gamma \circ \lambda = id\ E$. If λ lifts γ, then $\gamma\big|_{Im\ \lambda}$ is bijective and λ injective.

(5.2.3) The canonical projections and embeddings of A^n are denoted, such that $\forall_1^n k$:

$$A^n \ni \{a_1, \ldots, a_n\} \xrightarrow[\pi^k]{} a_k \xrightarrow[\pi_k]{} \{0, \ldots, 0, a_k, 0, \ldots, 0\} \in A^n.$$

Obviously,

$$\sum_{k=1}^{n} \pi_k \circ \pi^k = id\ A^n, \; \forall_1^n k, l : \pi^k \circ \pi_l = \begin{cases} id\ A, & k = l \\ 0, & k \neq l \end{cases}.$$

(5.2.4) Assume, that there is some lift λ for an appropriate generating n-map γ for E.

$$\forall_1^n k : \varepsilon_k := \gamma \circ \pi_k \in Hom_A(A, E) \subseteq Hom_R(A, E),$$

$$\varepsilon^k := \pi^k \circ \lambda \in Hom_A(E, A) \subseteq Hom_R(E, A).$$

$$\sum_{k=1}^{n} \varepsilon_k \circ \pi^k = \gamma, \; \sum_{k=1}^{n} \pi_k \circ \varepsilon^k = \lambda, \; \sum_{k=1}^{n} \varepsilon_k \circ \varepsilon^k = \gamma \circ \lambda = id\ E.$$

Here the family $\{\varepsilon_k, \varepsilon^k; k = 1, \ldots, n\}$ is called dual n-basis, or coordinatization of E. Especially $\{\pi_k, \pi^k; k = 1, \ldots, n\}$ is some coordinatization of A^n. An A-right module E over R is called projective-finite, if and only if there is some dual n-basis of E, for some $n \in \mathbf{N}$.

(5.2.5) One conveniently uses the natural R-linear bijection:

$$E \ni x \longleftrightarrow \dot{x} : A \ni a \longrightarrow xa \in E, \ \dot{x} \in Hom_A(A, E).$$

Obviously, $\forall \phi \in Hom_A(A, E) : \phi = (\phi(e_A))^\bullet$. Especially one finds:

$$A \ni a \longleftrightarrow \dot{a} : A \ni b \longrightarrow ab \in A, \ \dot{a} \in End_A(A) \subseteq End_R(A).$$

In the latter case, $\forall \alpha \in End_A(A) : \alpha = (\alpha(e_A))^\bullet$.
Here A is considered as an A-right module over R.

(5.2.6)

$$\forall_1^n k, \forall x \in E : x^k := \varepsilon^k \circ \dot{x}, \ \dot{x} = \sum_{k=1}^{n} \varepsilon_k \circ x^k.$$

$$\forall_1^n k, \forall \psi \in Hom_A(E, A) : \psi_k := \psi \circ \varepsilon_k, \ \psi = \sum_{k=1}^{n} \psi_k \circ \varepsilon^k.$$

(5.2.7) Let E_2 be an R_2-right module, and F an R_1-R_2-bimodule, i.e.,

$$\forall y \in F, r_1 \in R_1, r_2 \in R_2 : (r_1 y) r_2 = r_1 (y r_2).$$

In this case, defining

$$\forall \phi \in Hom_{R_2}(E_2, F), r_1 \in R_1, x \in E_2 : (r_1 \phi)(x) := r_1 \phi(x),$$

one obtains an R_1-left module $Hom_{R_2}(E_2, F)$. Here R_1 and R_2 are rings.

(5.2.8) Obviously $Hom_A(E, A)$ is an R-submodule of $Hom_R(E, A)$. Moreover, defining

$$\forall \phi \in Hom_A(E, A), a \in A : a\phi := \dot{a} \circ \phi,$$

which equivalently means, that $\forall x \in E : (a\phi)(x) = a\phi(x)$, $Hom_A(E, A)$ becomes an A-left module over R.

(5.2.9) $\forall \phi \in Hom_A(E, A), x \in E, a \in A :$

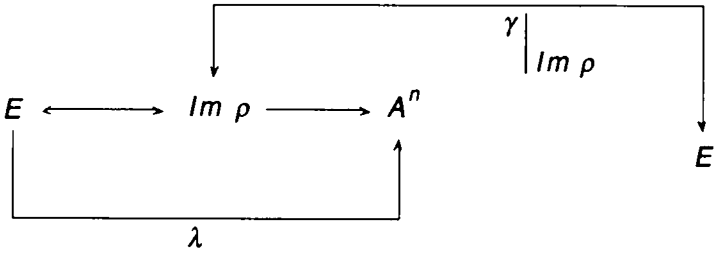
$$(\phi \circ \dot{x})(a) = \phi(x) a \in A, \ \phi \circ \dot{x} = (\phi(x))^\bullet \in End_A(A).$$

(5.2.10) The A-right module E over R is assumed to be projective-finite. Denoting

$$\rho := \lambda \circ \gamma \in End_A(A^n), \ \rho \circ \rho = \rho,$$

one finds the following direct sum decomposition of the A-right module A^n over R, according to the next diagram.

$$Im\ \lambda = Im\ \rho, \ A^n = Im\ \rho \oplus Im(id\ A^n - \rho).$$



(5.2.10.1) A right (left) module over a ring is called projective, if and only if it is some direct summand of a free right (left) module over this ring. Obviously the A-right module E is projective, because it is isomorphic with the direct summand $Im\ \rho$ of A^n. The projective-finite A-right module E over R is also called a finitely generated projective right module over A.

(5.2.11.1) Consider the ring $Mat_n(R)$ of matrices $[r_{ik}; i,k = 1,\dots,n]$ with the components $r_{ik} \in R$. With module-multiplications, such that $\forall [r_{ik}; i,k = 1,\dots,n] \in Mat_n(R), r \in R$:

$$r[r_{ik}; i,k = 1,\dots,n] := [rr_{ik}; i,k = 1,\dots,n]$$
$$= [r_{ik}r; i,k = 1,\dots,n] =: [r_{ik}; i,k = 1,\dots,n]r,$$

$Mat_n(R)$ is an R-bimodule, hence an associative algebra over R, with the unit
$I_n := \begin{bmatrix} e_R & & 0 \\ & \ddots & \\ 0 & & e_R \end{bmatrix}$. Then $Mat_n(R) \otimes A$, as the tensor product of algebras over R, is an associative algebra over R, with the unit $I_n \otimes e_A$.

(5.2.11.2) On the other hand, consider the ring $Mat_n(A)$ of matrices $[a_{ik} : i,k = 1,\dots,n]$ with the components $a_{ik} \in A$, and define module-multiplications, such that $\forall [a_{ik}; i,k = 1,\dots,n] \in Mat_n(A), r \in R$:

$$r[a_{ik}; i,k = 1,\dots,n] := [ra_{ik}; i,k = 1,\dots,n]$$
$$= [a_{ik}r; i,k = 1,\dots,n] =: [a_{ik}; i,k = 1,\dots,n]r,$$

in order to obtain an associative algebra $Mat_n(A)$ over R, with the unit $\begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix}$. Then the universal property of the tensor product allows for an establishment of an isomorphism in the sense of unital associative algebras over R:

$$Mat_n(R) \otimes A \ni [r_{ik}; i, k = 1, \dots, n] \otimes a$$
$$\longleftrightarrow [r_{ik}a; i, k = 1, \dots, n] \in Mat_n(A).$$

(5.2.11.3) Let $\phi \in End_A(A^n) \subseteq End_R(A^n)$, with respect to the A-right module A^n over R, and define

$$\forall_1^n i, k : End_A(A) \ni \phi_k^i := \pi^i \circ \phi \circ \pi_k : A \ni a \longrightarrow \phi_k^i(e_A)a \in A.$$

Here A is considered as an A-right module over R. Then, in this sense,

$$End_A(A) := \{\phi \in End_R(A); \forall a, b \in A : \phi(ab) = \phi(a)b\},$$

with the unit $id\ A$, is an associative algebra over R. Hence one establishes an isomorphism in the sense of unital associative algebras over R:

$$End_A(A^n) \ni \phi \longleftrightarrow [\phi_k^i; i, k = 1, \dots, n] \in Mat_n(End_A(A)),$$

which can be extended to endomorphisms over R:

$$End_R(A^n) \ni \psi \longleftrightarrow [\pi^i \circ \psi \circ \pi_k; i, k = 1, \dots, n] \in Mat_n(End_R(A)).$$

(5.2.11.4) An A-endomorphism ϕ of the projective-finite A-right module E over R can be written as matrix, the components of which are A-endomorphisms of A, according to the diagram below, the bijections of which are isomorphisms in the sense of unital associative algebras over R.

$$\begin{tikzcd}
End_A(E) \ni \phi \arrow[r, leftrightarrow] \arrow[d] & \lambda \circ \phi \circ \gamma \in \{p\psi p\ ;\ \psi \in End_A(A^n)\} \arrow[d] \\
{[\varepsilon^i \circ \phi \circ \varepsilon_k\ ;\ i,k=1,\dots,n]} \in Mat_n(End_A(A)) \arrow[r, leftrightarrow] & End_A(A^n)
\end{tikzcd}$$

To every $\phi \in End_A(E)$ belongs an R-linear map:

$$Hom_A(A, E) \ni \dot{x} = \sum_{k=1}^{n} \varepsilon_k \circ x^k$$
$$\longrightarrow \phi \circ \dot{x} =: \dot{y} = \sum_{l=1}^{n} \varepsilon_l \circ y^l \in Hom_A(A, E),$$

with components such that

$$\forall_1^n l : y^l = \sum_{k=1}^{n} \varepsilon^l \circ \phi \circ \varepsilon_k \circ x^k.$$

5.3 Lie-Cartan Pairs

(5.3.1) Let A be an Abelian unital associative algebra, and L a Lie algebra, both over the commutative ring R. The Lie-Cartan pair $\{L, A\}$ is established by the R-bilinear maps:

$$L \times A \ni \{k, a\} \longrightarrow ka \in A, \;\; A \times L \ni \{a, k\} \longrightarrow ak \in L,$$

which obey the following conditions. $\forall a, b \in A, \forall k, l \in L$:

$$[k,l]a = k(la) - l(ka), \;\; k(ab) = (ka)b + a(kb), \;\; a(kb) = (ak)b,$$
$$(ab)k = a(bk), \;\; e_A k = k, \;\; [k, al] = a[k,l] + (ka)l,$$

which then imply, that

$$0_L a = k 0_A = 0_A, \;\; (-k)a = k(-a) = -(ka),$$
$$0_A k = a 0_L = 0_L, \;\; (-a)k = a(-k) = -(ak),$$
$$k e_A = 0_A, \;\; [ak, l] = a[k,l] - (la)k.$$

These conditions expecially imply, that the following two left modules over R are constructed.

(5.3.1.1) An A-left module L over R is established, i.e., L is an A-left module, with the elements of the ring A used as coefficients, and

$$\forall a \in A, k \in L, r \in R : r(ak) = (ra)k = a(rk).$$

(5.3.1.2) An L-left module A over R is established.

(5.3.1.3) Consider the derivation algebra of A, i.e., the subalgebra $Der_R(A)$ of the commutator algebra $(End_R(A))_L$. Define an R-linear map
$\lambda : L \longrightarrow (End_R(A))_L$, such that

$$\forall k \in L : \;\; A \ni a \underset{\lambda(k)}{\longrightarrow} ka \in A.$$

Then $Im\ \lambda \subseteq Der_R(A)$, and λ is some homomorphism in the sense of Lie algebras over R, i.e.,

$$\forall a, b \in A, \forall k, l \in L : \lambda(k)(ab) = (\lambda(k)(a))b + a(\lambda(k)(b)),$$
$$\lambda([k,l]) = [\lambda(k), \lambda(l)] = \lambda(k) \circ \lambda(l) - \lambda(l)\lambda(k).$$

Here the commutator algebra $(End_R(A))_L$ is defined, such that

$$\forall \phi, \psi \in End_R(A) : [\phi, \psi] := \phi \circ \psi - \psi \circ \phi.$$

(5.3.2) Let $\{L, A\}$ be a Lie-Cartan pair over R, and consider an A-left module E over R, which explicitly means that

(i) E is an R-bimodule;

(ii) E is an A-left module, which may be considered as an A-bimodule, defining $\forall x \in E, a \in A : xa := ax$;

(iii) $\forall x \in E, a \in A, r \in R : r(ax) = (ra)x = a(rx).$

An R-linear mapping $\rho : L \longrightarrow (End_R(E))_L$ is called an E-connection, if and only if

$$\forall k \in L, a \in A, x \in E : \rho(k)(ax) = a(\rho(k)(x)) + (ka)x.$$

An E-connection ρ is called local, if and only if

$$\forall k \in L, a \in A, x \in E : \rho(ak)(x) = a(\rho(k)(x)).$$

(5.3.2.1) The curvature of an E-connection ρ is defined as an R-bilinear mapping

$$\kappa : L \times L \ni \{k, l\} \underset{def}{\longrightarrow} \rho(k) \circ \rho(l) - \rho(l) \circ \rho(k) - \rho([k, l]) \in (End_R(E))_L.$$

An E-connection ρ is called flat, if and only if $\kappa = 0$, which equivalently means that ρ is some representation of L on E, i.e., a homomorphism of Lie algebras over R.

(5.3.2.2) Usually an A-left module E over R is induced by some representation μ of A on E, i.e., a homomorphism of unital associative algebras

$$\mu : A \longrightarrow End_R(E), \ \forall a, b \in A : \mu(ab) = \mu(a) \circ \mu(b).$$

One then defines the bilinear mapping:

$$A \times E \ni \{a, x\} \longrightarrow ax := \mu(a)(x) \in E.$$

(5.3.2.3) Let ρ_1, ρ_2 be E-connections, and $r_1, r_2 \in R$. Then

$$r_1 + r_2 = e_R \Longrightarrow r_1\rho_1 + r_2\rho_2 \text{ is an } E-\text{connection}.$$

(5.3.3) The Lie-Cartan pair $\{L, A\}$ is called

(i) degenerate, if and only if $\forall k \in L, a \in A : ka = 0$; it is called

(ii) injective, if and only if the implication $\lceil \forall a \in A : ka = 0 \Longrightarrow k = 0 \rceil$ holds, which equivalently means, that the homomorphism of Lie algebras λ is injective.

(5.3.3.1) Let L', A' be subalgebras of L, A. The family $\{L', A'\}$ is called subpair of $\{L, A\}$, if and only if $\forall k \in L', a \in A' : ka \in A', ak \in L'$; obviously in this case $\{L', A'\}$ is some Lie-Cartan pair. Especially the subpair $\{L, Re_A\}$ is called the depletion of $\{L, A\}$.

(5.3.3.2) Consider the Lie-Cartan pair $\{Der(A), A\}$, thereby short-writing $Der(A) \equiv Der_R(A)$, with two R-bilinear maps defined such that

$$\forall a, b \in A, \phi \in Der(A) : (a\phi)(b) := a\phi(b), \;\; \phi a := \phi(a).$$

Obviously this Lie-Cartan pair is injective. Moreover every subpair $\{L, A\}$ of it is injective. Here the subpair $\{L, A\}$ of $\{Der(A), A\}$ is established by the assumption, that $\forall a \in A, \phi \in L : a\phi \in L$; L is an appropriate subalgebra of $Der(A)$.

(5.3.3.2.1) Conversely let the Lie-Cartan pair $\{L, A\}$ be injective, i.e., λ injective, as indicated in the following diagram.

$$\begin{tikzcd} L \arrow[r, leftrightarrow, "\text{Lie algebra-isomorphism}"'] \arrow[rrr, bend right, "\lambda"'] & Im\ \lambda \arrow[r] & Der(A) \arrow[r] & (End_R(A))_L \end{tikzcd}$$

The Lie-Cartan pair $\{Im\ \lambda, A\}$, which is established by the products, such that

$$\forall \phi = \lambda(k) \in Im\ \lambda, a \in A : \phi a := ka, \;\; a\phi := \lambda(ak),$$

is some subpair of $\{Der(A), A\}$.

(5.3.3.3) Let the Lie-Cartan pair $\{L, A\}$ be degenerate. Then its defining properties establish an A-left module L, with the commutative ring A providing the coefficients. Moreover an E-connection ρ yields, for $k \in L$, some homomorphism $\rho(k)$ in the sense of A-left modules.

(5.3.4) Important special cases of E-connections, with respect to the Lie-Cartan pair $\{L, A\}$, are established by the choices $E := A$, $E := L$, with the corresponding left module-multiplications provided by the product of A, and the Lie-Cartan bilinear map: $A \times L \longrightarrow L$, respectively.

(5.3.5) In order to generalize the calculus of differential forms, consider an E-connection ρ, with respect to the Lie-Cartan pair $\{L, A\}$. One defines the direct sums of R-bimodules

$$\Lambda^*(L,E) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p(L,E),\ \Lambda^0(L,E) := E,\ \Lambda^1(L,E) := Hom_R(L,E),$$

$$\Lambda^*_A(L,E) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p_A(L,E),\ \Lambda^0_A(L,E) := E,$$

with the A-linear mappings as elements of

$$\begin{aligned}\Lambda^1_A(L,E) : &= \{\lambda_1 \in Hom_R(L,E); \forall k \in L, a \in A : \lambda_1(ak) = a\lambda_1(k)\} \\ &= Hom_A(L,E),\end{aligned}$$

and with the alternating R-multilinear, and especially alternating A-multilinear mappings, $\forall p \geq 2$, as elements of

$$\begin{aligned}\Lambda^p(L,E) := \Bigg\{\lambda_p : \prod^p L \longrightarrow E; \forall k_1, \dots, k_p \in L : \\ \lceil \exists 1 \leq i < j \leq p : k_i = k_j \rceil \Longrightarrow \lambda_p(k_1, \dots, k_p) = 0\Bigg\},\end{aligned}$$

$$\begin{aligned}\Lambda^p_A(L,E) := \{\lambda_p \in \Lambda^p(L,E); \forall k_1, \dots, k_p \in L, \forall 1 \leq j \leq p, \forall a \in A : \\ \lambda_p(k_1, \dots, ak_j, \dots, k_p) = a\lambda_p(k_1, \dots, k_p)\}.\end{aligned}$$

Obviously these alternating mappings are skew-symmetric, i.e.,
$\forall p \geq 2, \forall \lambda_p \in \Lambda^p(L,E), \forall k_1, \dots, k_p \in L, \forall \begin{bmatrix} 1 \dots p \\ j_1 \dots j_p \end{bmatrix} =: P \in \mathbf{P}_p :$

$$\lambda_p(k_1, \dots, k_p) = (-1)^{\tau_p} \lambda_p(k_{j_1}, \dots k_{j_p}),$$

with the sign τ_p of the permutation P of p elements.

(5.3.5.1) The mappings $i(k), \rho(k), \theta_0(k), \theta_\rho(k), \delta_0, \hat{\rho}, \delta_\rho$, which shall be defined now, are R-endomorphisms of $\Lambda^*(L,E)$, $\forall k \in L$, as one proves easily. $\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L,E), x \in E, \forall k, k_0, k_1, \dots, k_p \in L :$

$$\begin{aligned}&(i(k)(\lambda_{p+1}))(k_1, \dots, k_p) := \lambda_{p+1}(k, k_1, \dots, k_p), \\ &i(k)(x) := 0,\ i(k)(\lambda_1) := \lambda_1(k), \\ &(\rho(k)(\lambda_p))(k_1, \dots, k_p) := \rho(k)(\lambda_p(k_1, \dots, k_p)),\end{aligned}$$

as an extension of the E-connection ρ,

$$(\theta_0(k)(\lambda_p))(k_1,\dots,k_p) := -\sum_{j=1}^{p}\lambda_p(k_1,\dots,k_{j-1},[k,k_j],k_{j+1},\dots,k_p),$$
$$\theta_0(k)(x) := 0,\ (\theta_0(k)(\lambda_1))(k_1) = -\lambda_1([k,k_1]),$$
$$\theta_\rho(k) := \theta_0(k)+\rho(k),\ \theta_\rho(k)(x) = \rho(k)(x),$$
$$(\delta_0(\lambda_p))(k_0,k_1,\dots,k_p)$$
$$:= \sum_{0\le i<j\le p}(-1)^{i+j}\lambda_p([k_i,k_j],k_0,\dots,\not{k}_i,\dots,\not{k}_j,\dots,k_p),$$
$$(\delta_0(\lambda_1))(k_0,k_1) = -\lambda_1([k_0,k_1]),\ \delta_0(x) := 0,$$
$$(\hat\rho(\lambda_p))(k_0,k_1,\dots,k_p) := \sum_{j=0}^{p}(-1)^j\rho(k_j)(\lambda_p(k_0,\dots,\not{k}_j,\dots,k_p)),$$
$$(\hat\rho(\lambda_1))(k_0,k_1) = \rho(k_0)(\lambda_1(k_1)) - \rho(k_1)(\lambda_1(k_0)),$$
$$\hat\rho(x)(k) := \rho(k)(x),\ \delta_\rho := \delta_0+\hat\rho,\ \delta_\rho(x) = \hat\rho(x).$$

Here for instance the notation of $\not{k}_j$ means, that this argument is omitted.

(5.3.5.2) Suppressing the canonical embeddings one finds, that $\forall p \in \mathbf{N}, k \in L:$

$$\Lambda^p(L,E)\underset{i(k)}{\longrightarrow}\Lambda^{p-1}(L,E),\ \Lambda^{p-1}(L,E)\underset{\rho(k),\theta_0(k),\theta_\rho(k)}{\longrightarrow}\Lambda^{p-1}(L,E),$$

$$\Lambda^p(L,E)\underset{\delta_0,\hat\rho,\delta_\rho}{\longrightarrow}\Lambda^{p+1}(L,E),\ \delta_0\big|_E = \theta_0(k)\big|_E = i(k)\big|_E = 0.$$

(5.3.5.3) With the R-bilinear mapping: $A\times\Lambda^*(L,E)\longrightarrow\Lambda^*(L,E)$, such that $\forall a\in A, p\in\mathbf{N}, \lambda_p\in\Lambda^p(L,E), x\in E, \forall k_1,\dots,k_p\in L:$

$$(a\lambda_p)(k_1,\dots,k_p) := a\lambda_p(k_1,\dots,k_p),$$

ax being defined in the sense of the A-left module E over R, $\Lambda^*(L,E)$ becomes an A-left module over R.

(5.3.5.4) An endomorphism $\Delta(a)$ of $\Lambda^*(L,E)$, in the sense of A-left modules, is defined.
$\forall p\in\mathbf{N}, \lambda_p\in\Lambda^p(L,E), x\in E, a\in A, \forall k,k_0,k_1,\dots,k_p\in L:$

$$(\Delta(a)(\lambda_p))(k_0,\dots,k_p) := \sum_{j=0}^{p}(-1)^j(k_ja)\lambda_p(k_0,\dots,\not{k}_j,\dots,k_p),$$
$$(\Delta(a)(\lambda_1))(k_0,k_1) = (k_0a)\lambda_1(k_1)-(k_1a)\lambda_1(k_0),\ (\Delta(a)(x))(k) := (ka)x.$$

$$\forall p\in\mathbf{N},\forall a\in A:\Lambda^p(L,E)\underset{A-linear}{\overset{\Delta(a)}{\longrightarrow}}\Lambda^{p+1}(L,E).$$

(5.3.5.5) Obviously $\theta_0(k), \delta_0$, and $\Delta(a)$ are A-endomorphisms of $\Lambda^*(L,E)$. $\forall a, b \in A, \lambda \in \Lambda^*(L,E), k \in L$:

$$\Delta(a)(b\lambda) = b(\Delta(a)(\lambda)),\ \theta_0(k)(a\lambda) = a(\theta_0(k)(\lambda)),\ \delta_0(a\lambda) = a\delta_0(\lambda),$$
$$\rho(k)(a\lambda) = a(\rho(k)(\lambda)) + (ka)\lambda,\ \theta_\rho(k)(a\lambda) = a(\theta_\rho(k)(\lambda)) + (ka)\lambda,$$
$$\hat{\rho}(a\lambda) = a\hat{\rho}(\lambda) + \Delta(a)(\lambda),\ \delta_\rho(a\lambda) = a\delta_\rho(\lambda) + \Delta(a)(\lambda).$$

(5.3.5.6) Furthermore R-endomorphisms $\kappa(k,l), \tilde{\kappa}(k)$, and $\hat{\kappa}$ of $\Lambda^*(L,E)$ are defined.
$\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L,E), x \in E, \forall k, l, k_0, k_1, \ldots, k_{p+2} \in L$:

$$(\kappa(k,l)(\lambda_p))(k_1,\ldots,k_p) := \kappa(k,l)(\lambda_p(k_1,\ldots,k_p)),$$

as an extension of the curvature,

$$(\tilde{\kappa}(k)(\lambda_p))(k_0,\ldots,k_p) := \sum_{j=0}^{p}(-1)^j(\kappa(k,k_j)(\lambda_p))(k_0,\ldots,\not{k}_j,\ldots,k_p),$$
$$(\tilde{\kappa}(k)(\lambda_1))(k_0,k_1) = \kappa(k,k_0)(\lambda_1(k_1)) - \kappa(k,k_1)(\lambda_1(k_0)),$$
$$(\tilde{\kappa}(k)(x))(l) := \kappa(k,l)(x),$$
$$(\hat{\kappa}(\lambda_p))(k_1,\ldots,k_{p+2}) :=$$
$$= \frac{1}{2(p!)} \sum_{\begin{bmatrix} 1 \ldots p+2 \\ j_1 \ldots j_{p+2}\end{bmatrix} =: P \in \mathbf{P}_{p+2}} (-1)^{\tau_P}(\kappa(k_{j_1},k_{j_2})(\lambda_p))(k_{j_3},\ldots,k_{j_{p+2}}),$$
$$(\hat{\kappa}(\lambda_1))(k_1,k_2,k_3)$$
$$= \kappa(k_1,k_2)(\lambda_1(k_3)) + \kappa(k_2,k_3)(\lambda_1(k_1)) + \kappa(k_3,k_1)(\lambda_1(k_2)),$$
$$\hat{\kappa}(x)(k,l) := \kappa(k,l)(x).$$

The definition of $\hat{\kappa}$ is valid for an arbitrary commuting ring R, with $\frac{1}{2(p!)}$ understood to drop out. $\forall p \in \mathbf{N}, k \in L$:

$$\Lambda^{p-1}(L,E) \underset{\kappa(k,l)}{\longrightarrow} \Lambda^{p-1}(L,E),$$
$$\Lambda^p(L,E) \underset{\tilde{\kappa}(k)}{\longrightarrow} \Lambda^{p+1}(L,E),\ \Lambda^p(L,E) \underset{\hat{\kappa}}{\longrightarrow} \Lambda^{p+2}(L,E).$$

(5.3.5.7) $\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L,E), \forall k, l, k_1, \ldots, k_p \in L$:

$$(i(k) \circ i(l)(\lambda_{p+2}))(k_1,\ldots,k_p) = \lambda_{p+2}(l,k,k_1,\ldots,k_p),$$
$$i(k) \circ i(l)(\lambda_2) = \lambda_2(l,k),\ i(k) \circ i(l)\big|_{E \oplus \Lambda^1(L,E)} = 0,$$
$$i(k) \circ i(k) = 0,\ i(k) \circ i(l) + i(l) \circ i(k) = 0,$$
$$\theta_0(k) \circ \theta_0(l) - \theta_0(l) \circ \theta_0(k) = \theta_0([k,l]),$$
$$\rho(k) \circ \rho(l) - \rho(l) \circ \rho(k) = \kappa(k,l) + \rho([k,l]),$$

extending the curvature to $\Lambda^*(L,E)$,

$$\begin{aligned}
&\theta_\rho(k)\circ\theta_\rho(l)-\theta_\rho(l)\circ\theta_\rho(k)=\theta_\rho([k,l])+\kappa(k,l),\\
&\delta_0\circ\delta_0=0,\ \hat{\rho}\circ i(k)+i(k)\circ\hat{\rho}=\rho(k),\\
&\delta_0\circ i(k)+i(k)\circ\delta_0=\theta_0(k),\ \delta_\rho\circ i(k)+i(k)\circ\delta_\rho=\theta_\rho(k),\\
&i(k)\circ\theta_0(l)-\theta_0(l)\circ i(k)=i([k,l]),\ i(k)\circ\rho(l)=\rho(l)\circ i(k),\\
&i(k)\circ\theta_\rho(l)-\theta_\rho(l)\circ i(k)=i([k,l]),\\
&\rho(k)\circ\delta_0=\delta_0\circ\rho(k),\ \theta_0(k)\circ\delta_0=\delta_0\circ\theta_0(k),\\
&\delta_\rho\circ\delta_\rho=\hat{\kappa},\ \theta_\rho(k)\circ\delta_\rho-\delta_\rho\circ\theta_\rho(k)=\tilde{\kappa}(k)=i(k)\circ\hat{\kappa}-\hat{\kappa}\circ i(k).
\end{aligned}$$

(5.3.5.8) If the E-connection ρ is flat, one especially obtains $\forall k,l\in L$:

$$\begin{aligned}
&\kappa(k,l)=0,\ \tilde{\kappa}(k)=0,\ \hat{\kappa}=0,\\
&\rho(k)\circ\rho(l)-\rho(l)\circ\rho(k)=\rho([k,l]),\\
&\theta_\rho(k)\circ\theta_\rho(l)-\theta_\rho(l)\circ\theta_\rho(k)=\theta_\rho([k,l]),\\
&\delta_\rho\circ\delta_\rho=0,\ \theta_\rho(k)\circ\delta_\rho=\delta_\rho\circ\theta_\rho(k)=\delta_\rho\circ i(k)\circ\delta_\rho.
\end{aligned}$$

(5.3.5.9)

$$\forall k\in L: Im\ i(k)\big|_{\Lambda^*_A(L,E)}\cup Im\ \theta_\rho(k)\big|_{\Lambda^*_A(L,E)}\subseteq\Lambda^*_A(L,E).$$

(5.3.5.10) With the definition, that

$$\forall\phi\in End_R(E),a\in A,x\in E:(a\phi)(x):=a\phi(x),$$

$End_R(E)$ obviously becomes an A-left module over R, as E itself.

(5.3.5.11) If the connection ρ is local, i.e., if $\rho: L\longrightarrow(End_R(E))_L$ is A-linear, then one especially finds, that $\forall\lambda\in\Lambda^*_A(L,E):\delta_\rho(\lambda)\in\Lambda^*_A(L,E)$.

(5.3.6.1) Let E be equipped with an R-bilinear map:

$$E\times E\ni\{x,y\}\longrightarrow xy\in E,$$

thereby establishing an algebra E over R. The so-called wedge product is defined as an R-bilinear map: $\Lambda^*(L,E)\times\Lambda^*(L,E)\longrightarrow\Lambda^*(L,E)$, such that

$$\forall p,q\in\mathbf{N}_0:\Lambda^p(L,E)\times\Lambda^q(L,E)\ni\{\lambda_p,\mu_q\}\longrightarrow\lambda_p\wedge\mu_q\in\Lambda^{p+q}(L,E).$$

$\forall x,y\in E,\forall p,q\in\mathbf{N},\forall\lambda_p\in\Lambda^p(L,E),\mu_q\in\Lambda^q(L,E),\forall k_1,\ldots,k_{p+q}\in L$:

$$x \wedge y := xy,$$
$$(x \wedge \lambda_p)(k_1, \ldots, k_p) := x\lambda_p(k_1, \ldots, k_p),$$
$$(\lambda_p \wedge y)(k_1, \ldots, k_p) := \lambda_p(k_1, \ldots, k_p)y,$$
$$(\lambda_p \wedge \mu_q)(k_1, \ldots, k_{p+q})$$
$$:= \frac{1}{p!q!} \sum_{\begin{bmatrix} 1 \ldots p+q \\ j_1 \ldots j_{p+q} \end{bmatrix} =: P \in \mathbf{P}_{p+q}} (-1)^{\tau P} \lambda_p(k_{j_1}, \ldots, k_{j_p}) \mu_q(k_{j_{p+1}}, \ldots, k_{j_{p+q}}),$$
$$(\lambda_1 \wedge \mu_1)(k_1, k_2) = \lambda_1(k_1)\mu_1(k_2) - \lambda_1(k_2)\mu_1(k_1).$$

Hence $\Lambda^*(L, E)$ is some $\mathbf{N}_0$-graded algebra over R.

(5.3.6.1.1) Especially let the product of E be A-bilinear, i.e.,

$$\forall x, y \in E, \forall a, b \in A : (ax)(by) = (ab)(xy).$$

In this case the wedge product is A-bilinear too, i.e.,

$$\forall \lambda, \mu \in \Lambda^*(L, E), \forall a, b \in A : (a\lambda) \wedge (b\mu) = (ab)(\lambda \wedge \mu).$$

In this case $\Lambda^*_A(L, E)$ is some subalgebra of $\Lambda^*(L, E)$.

(5.3.6.2) If E is an associative algebra over R, then $\Lambda^*(L, E)$ is also associative, i.e., $\forall p, q, r \in \mathbf{N}$,
$\forall \lambda_p \in \Lambda^p(L, E), \mu_q \in \Lambda^q(L, E), \nu_r \in \Lambda^r(L, E), \forall k_1, \ldots, k_{p+q+r} \in L :$

$$(\lambda_p \wedge \mu_q \wedge \nu_r)(k_1, \ldots, k_{p+q+r}) = \frac{1}{p!q!r!} \sum_{\begin{bmatrix} 1 \ldots p+q+r \\ j_1 \ldots j_{p+q+r} \end{bmatrix} =: P \in \mathbf{P}_{p+q+r}} (-1)^{\tau P}$$
$$\lambda_p(k_{j_1}, \ldots, k_{j_p}) \mu_q(k_{j_{p+1}}, \ldots, k_{j_{p+q}}) \nu_r(k_{j_{p+q+1}}, \ldots, k_{j_{p+q+r}}).$$

In particular, if E is unital, then of course $\Lambda^*(L, E)$ is unital too.

(5.3.6.3) If the algebra E over R is commutative, then one finds that

$$\forall p, q \in \mathbf{N}_0, \forall \lambda_p \in \Lambda^p(L, E), \mu_q \in \Lambda^q(L, E) : \lambda_p \wedge \mu_q = (-1)^{pq} \mu_q \wedge \lambda_p.$$

(5.3.6.4) If E is a Lie algebra over R, then $\Lambda^*(L, E)$ is some Lie superalgebra over R, with an appropriate $\mathbf{Z}_2$-grading.

$$\forall \bar{p} \in \mathbf{Z}_2 : (\Lambda^*(L, E))^{\bar{p}} := \bigoplus_{p \in \bar{p} \cap \mathbf{N}_0} \Lambda^p(L, E).$$

$\forall p, q, r \in \mathbf{N}_0, \forall \lambda_p \in \Lambda^p(L, E), \mu_q \in \Lambda^q(L, E), \nu_r \in \Lambda^r(L, E) :$

$$\lambda_p \wedge \mu_q + (-1)^{pq} \mu_q \wedge \lambda_p = 0,$$
$$\lambda_p \wedge (\mu_q \wedge \nu_r) = (\lambda_p \wedge \mu_q) \wedge \nu_r + (-1)^{pq} \mu_q \wedge (\lambda_p \wedge \nu_r).$$

(5.3.6.5) The following above defined R-endomorphisms of $\Lambda^*(L,E)$ are super-derivations of the superalgebra $\Lambda^*(L,E)$, with the $\mathbf{Z}_2$-grading as above, such that $\forall p,q \in \mathbf{N}_0, \forall \lambda_p \in \Lambda^p(L,E), \mu_q \in \Lambda^q(L,E), \forall k \in L:$

$$\begin{aligned}
i(k)(\lambda_p \wedge \mu_q) &= (i(k)(\lambda_p)) \wedge \mu_q + (-1)^p \lambda_p \wedge (i(k)(\mu_q)),\\
\theta_0(k)(\lambda_p \wedge \mu_q) &= (\theta_0(k)(\lambda_p)) \wedge \mu_q + \lambda_p \wedge (\theta_0(k)(\mu_q)),\\
\delta_0(\lambda_p \wedge \mu_q) &= (\delta_0(\lambda_p)) \wedge \mu_q + (-1)^p \lambda_p \wedge (\delta_0(\mu_q)).
\end{aligned}$$

(5.3.6.6) Especially assume, that $\forall k \in L, \forall x,y \in E$:

$$\rho(k)(xy) = (\rho(k)(x))y + x(\rho(k)(y)).$$

In this case, $\forall p,q \in \mathbf{N}_0, \forall \lambda_p \in \Lambda^p(L,E), \mu_q \in \Lambda^q(L,E), \forall k \in L:$

$$\begin{aligned}
\rho(k)(\lambda_p \wedge \mu_q) &= (\rho(k)(\lambda_p)) \wedge \mu_q + \lambda_p \wedge (\rho(k)(\mu_q)),\\
\theta_\rho(k)(\lambda_p \wedge \mu_q) &= (\theta_\rho(k)(\lambda_p)) \wedge \mu_q + \lambda_p \wedge (\theta_\rho(k)(\mu_q)).
\end{aligned}$$

Furthermore in this case, $\forall p,q \in \mathbf{N}, \forall \lambda_p \in \Lambda^p(L,E), \mu_q \in \Lambda^q(L,E):$

$$\begin{aligned}
\hat{\rho}(\lambda_p \wedge \mu_q) &= (\hat{\rho}(\lambda_p)) \wedge \mu_q + (-1)^p \lambda_p \wedge (\hat{\rho}(\mu_q)),\\
\delta_\rho(\lambda_p \wedge \mu_q) &= (\delta_\rho(\lambda_p)) \wedge \mu_q + (-1)^p \lambda_p \wedge (\delta_\rho(\mu_q)).
\end{aligned}$$

(5.3.6.7) An R-endomorphism $\hat{\lambda}_1 : \Lambda^{q-1}(L,E) \longrightarrow \Lambda^q(L,E)$ is established for $q \in \mathbf{N}$.
$\forall \lambda_1 \in \Lambda^1(L,E), \forall q \in \mathbf{N}, \forall \mu_q \in \Lambda^q(L,E), \forall k_0, k_1, \ldots, k_q \in L, \forall y \in E:$

$$\begin{aligned}
(\hat{\lambda}_1(\mu_q))(k_0,k_1,\ldots,k_q) &:= (\lambda_1 \wedge \mu_q)(k_0,k_1,\ldots,k_q)\\
&= \sum_{j=0}^{q} (-1)^j \lambda_1(k_j)\mu_q(k_0,\ldots,\cancel{k_j},\ldots,k_q),\\
(\hat{\lambda}_1(y))(k_0) &:= (\lambda_1 \wedge y)(k_0) = \lambda_1(k_0)y.
\end{aligned}$$

5.4 Real Differential Forms

(5.4.1) Consider an m-dimensional real C^∞-manifold $\mathbf{M}$, with the Abelian unital associative algebra $C^\infty(\mathbf{M})$, and the Lie algebra $T(\mathbf{M})$ of C^∞-vector fields, both over $\mathbf{R}$. On an arbitrary chart ϕ of an atlas for $\mathbf{M}$, the smooth vector field X is acting with the components $\xi^k, k = 1,\ldots,m$.

$$X : \mathbf{M} \supseteq \mathit{dom}\,\phi \ni q = \phi^{-1}(x) \longrightarrow \{q,[u]\} \in T(\mathbf{M}),\ [u] = \xi^k(x)\frac{\partial}{\partial x^k},$$

with an equivalence class $[u]$ of smooth curves $u :]-1,+1[\longrightarrow \mathbf{M},\ u(0) = q$.

(5.4.2) The Lie-Cartan pair $\{T(\mathbf{M}), C^\infty(\mathbf{M})\}$ is established with the following two $\mathbf{R}$-bilinear maps. $\forall q \in dom\ \phi$,

$$\forall f \in C^\infty(\mathbf{M}), X \in T(\mathbf{M}) : (fX)(q) := \left\{q, f(q)\xi^k(x)\frac{\partial}{\partial x^k}\right\},$$
$$(Xf)(q) := L_X f(q) = \xi^k(x)\frac{\partial}{\partial x^k} f(x),$$

with the natural basis vectors $\frac{\partial}{\partial x^k}$ used for the partial derivations of the scalar field f on the chart ϕ. Obviously $Im\ e_{C^\infty(\mathbf{M})} := \{1\}$, defining the unit scalar field.

(5.4.3) The real Lie algebra of Lie derivations of smooth scalar fields,

$$L(\mathbf{M}) := \{L_X : C^\infty(\mathbf{M}) \longrightarrow C^\infty(\mathbf{M}); X \in T(\mathbf{M})\},$$

is isomorphic with the real Lie algebra of C^∞-vector fields:

$$T(\mathbf{M}) \ni X \longleftrightarrow L_X \in L(\mathbf{M}),$$

as some subalgebra of the real Lie algebra $Der_{\mathbf{R}}(C^\infty(\mathbf{M})) \equiv Der(C^\infty(\mathbf{M}))$. Moreover the family $\{L(\mathbf{M}), C^\infty(\mathbf{M})\}$ is some subpair of the Lie-Cartan pair $\{Der(C^\infty(\mathbf{M})), C^\infty(\mathbf{M})\}$, defining

$$\forall f, g \in C^\infty(\mathbf{M}), \forall X \in T(\mathbf{M}) : (fL_X)(g) := f(L_X g) = L_{fX} g.$$

Hence in the sense of the above isomorphism of Lie algebras, the pair $\{T(\mathbf{M}), C^\infty(\mathbf{M})\}$ is some subpair of $\{Der(C^\infty(\mathbf{M}), C^\infty(\mathbf{M})\}$.

(5.4.4) Consider the Lie-Cartan pair $\{T(\mathbf{M}), C^\infty(\mathbf{M})\}$ and an E-connection ρ, with $E := C^\infty(\mathbf{M})$, using the product of this algebra for the left module-multiplication. This real-linear map is then acting as

$$\rho : T(\mathbf{M}) \ni X \longrightarrow \phi \in (End_{\mathbf{R}}(C^\infty(\mathbf{M})))_L,$$
$$\forall X \in T(\mathbf{M}), \forall f, g \in C^\infty(\mathbf{M}) : \phi(fg) = f\phi(g) + (L_X f)g.$$

Obviously the $C^\infty(\mathbf{M})$-connection defined by: $T(\mathbf{M}) \ni X \longrightarrow L_X$ is both local and flat.

(5.4.5) Again with respect to the above Lie-Cartan pair, consider the choice $E := T(\mathbf{M})$, using one of the two Lie-Cartan bilinear maps for the definition of the $C^\infty(\mathbf{M})$-left module E over $\mathbf{R}$. Here this real-linear map is acting as

$$\rho : T(\mathbf{M}) \ni X \longrightarrow \phi \in (End_{\mathbf{R}}(T(\mathbf{M})))_L,$$
$$\forall X, Y \in T(\mathbf{M}), \forall f \in C^\infty(\mathbf{M}) : \phi(fY) = f\phi(Y) + (L_X f)Y.$$

Obviously the $T(\mathbf{M})$-connection:

$$T(\mathbf{M}) \ni X \underset{def}{\longrightarrow} L_X : T(\mathbf{M}) \ni Y \longrightarrow [X, Y] \in T(\mathbf{M})$$

is flat, but non-local.

(5.4.6) Defining, with respect to the $C^\infty(\mathbf{M})$-left module of real C^∞-tensor fields of rank $\{s,r\} \in \mathbf{N}_0 \times \mathbf{N}_0$, $E := T^s_r(\mathbf{M})$, the Lie derivation of smooth tensor fields with respect to $X \in T(\mathbf{M})$ provides an E-connection

$$\rho : T(\mathbf{M}) \ni X \underset{def}{\longrightarrow} L_X \in (End_{\mathbf{R}}(E))_L,$$

because one finds, that $\forall r,s \in \mathbf{N}_0$,

$$\forall X \in T(\mathbf{M}), f \in C^\infty(\mathbf{M}), t \in T^s_r(\mathbf{M}) : L_X(ft) = fL_Xt + (L_Xf)t.$$

Obviously, for $r \geq 1$ or $s \geq 1$, this connection is flat, but non-local.

(5.4.7) On every chart ϕ of an atlas for $\mathbf{M}$, the $C^\infty(dom\ \phi)$-left modules of smooth vector fields, more generally of smooth $\{r,s\}$-tensor fields, and especially of differential forms, are free with respect to the bases indicated below.

$$\left\{\frac{\partial}{\partial x^k}; k = 1,\ldots,m\right\} \longrightarrow T(dom\ \phi) \equiv T^1_0(dom\ \phi),$$

$$\left\{\frac{\partial}{\partial x^{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{k_s}} \otimes dx^{l_1} \otimes \cdots \otimes dx^{l_r}; k_1,\ldots,l_r = 1,\ldots,m\right\}$$
$$\longrightarrow T^s_r(dom\ \phi),$$
$$\{dx^{l_1} \wedge \cdots \wedge dx^{l_r}; 1 \leq l_1 < \ldots < l_r \leq m; 1 \leq r \leq m\} \cup \{unit\}$$
$$\longrightarrow E(dom\ \phi),$$

inserting the unit of $C^\infty(dom\ \phi)$.

(5.4.8) Every differential form

$$t_r \in E_r(\mathbf{M}),\ 1 \leq r \leq m,\ \bigoplus_{r=0}^{m} E_r(\mathbf{M}) =: E(\mathbf{M}),$$

can be viewed as an alternating $\mathbf{R}$-multilinear map

$$\lambda_r : \prod^r T(\mathbf{M}) \longrightarrow C^\infty(\mathbf{M}),$$

such that $\forall X_1,\ldots,X_r \in T(\mathbf{M}), \forall q \in \mathbf{M}$:

$$t_r(q)(X_k(q); k = 1,\ldots,r) = (\lambda_r(X_1,\ldots,X_r))(q).$$

One denotes $E_0(\mathbf{M}) \equiv C^\infty(\mathbf{M})$, $E_r(\mathbf{M}) := \{0\}$ for $r \in \mathbf{Z} \setminus \{0,1,\ldots,m\}$.

(5.4.9) Obviously not every element of $\Lambda^*(T(\mathbf{M}), C^\infty(\mathbf{M}))$ is induced by an appropriate differential form $t \in E(\mathbf{M})$. For instance consider the convolution form $\lambda \in \Lambda^1(T(\mathbf{R}^m, C^\infty(\mathbf{R}^m))$, such that:

$$T(\mathbf{R}^m) \ni X =: \xi^l \frac{\partial}{\partial x^l} \underset{\lambda}{\longrightarrow} f : \mathbf{R}^m \ni x \underset{def}{\longrightarrow} \int_{\mathbf{R}^m} \tau_l(x-y)\xi^l(y)dy,$$
$$\forall_1^m l : \tau_l \in C_0^\infty(\mathbf{R}^m),$$

inserting smooth integral kernels with compact support.

(5.4.10) The usual wedge product of differential forms on $\mathbf{M}$ can then be written as wedge product of the corresponding special elements of $\Lambda^*(T(\mathbf{M}), C^\infty(\mathbf{M}))$.

(5.4.11) Obviously $\Lambda^*(T(\mathbf{M}), C^\infty(\mathbf{M}))$ is some unital associative algebra over $\mathbf{R}$. Moreover the above defined injection

$$j_r : E_r(\mathbf{M}) \ni t_r \longrightarrow \lambda_r \in \Lambda^r(T(\mathbf{M}), C^\infty(\mathbf{M})) \text{ for } 0 \le r \le m,$$
$$j_0 := id\ C^\infty(\mathbf{M}),$$

is linear with respect to the coefficients $\in C^\infty(\mathbf{M})$; hence its image is some submodule of the $C^\infty(\mathbf{M})$-left module $\Lambda^*(T(\mathbf{M}), C^\infty(\mathbf{M}))$. Obviously $\forall_0^m r$:

$$Im\ j_r \subseteq \Lambda^r_{C^\infty(\mathbf{M})}(T(\mathbf{M}), C^\infty(\mathbf{M})).$$
$$\forall r \ge m+1 : \Lambda^r_{C^\infty(\mathbf{M})}(T(\mathbf{M}), C^\infty(\mathbf{M})) = \{0\}.$$

(5.4.12) The real endomorphisms d, L_X, and i_X of $E(\mathbf{M})$, are written explicitly on the domain of one chart. Here d denotes the exterior derivation, and L_X the Lie derivation with respect to the vector field $X \in T(\mathbf{M})$, of real differential forms. Let

$$X := \xi^k \frac{\partial}{\partial x^k} \in T(\mathbf{M}),\ f \in C^\infty(\mathbf{M}),$$

$$t_r := \sum_{1 \le l_1 < \ldots < l_r \le m} \tau_{l_1 \ldots l_r} dx^{l_1} \wedge \cdots \wedge dx^{l_r} \in E_r(\mathbf{M}),$$

for $1 \le r \le m$, with $x := \phi(q) \in ran\ \phi$, on one chart ϕ, $dom\ \phi =: \mathbf{M}$.
$\forall 1 \le r \le m$:

$$i_X t_{r+1}(q) = \sum_{1 \le l_1 < \ldots < l_r \le m} \xi^k(x)\tau_{kl_1 \ldots l_r}(x) dx^{l_1} \wedge \cdots \wedge dx^{l_r},$$

$$L_X t_r(q) = \sum_{1 \le l_1 < \ldots < l_r \le m} dx^{l_1} \wedge \cdots \wedge dx^{l_r}$$
$$(\xi^k \nabla_k \tau_{l_1 \ldots l_r} + \tau_{kl_2 \ldots l_r} \nabla_{l_1} \xi^k + \cdots + \tau_{l_1 \ldots l_{r-1} k} \nabla_{l_r} \xi^k)(x).$$

$\forall 1 \le r \le m-1$:

$$dt_r(q) = \sum_{1 \leq l_1 < \ldots < l_{r+1} \leq m} dx^{l_1} \wedge \cdots \wedge dx^{l_{r+1}}$$
$$(\nabla_{l_1} \tau_{l_2 \ldots l_{r+1}} - \nabla_{l_2} \tau_{l_1 l_3 \ldots l_{r+1}} + - \cdots + (-1)^r \nabla_{l_{r+1}} \tau_{l_1 \ldots l_r})(x).$$
$$L_X f(q) = \xi^k(x) \nabla_k f \circ \phi^{-1}(x), \; i_X f := 0, \; i_X t_1(q) = \xi^k(x) \tau_k(x),$$
$$df(q) = dx^l \nabla_l f \circ \phi^{-1}(x), \; dt_m := 0.$$

(5.4.12.1) These real endomorphisms of $E(\mathbf{M}_k), k = 1, 2$, are compatible with the pull back $\psi^* : E(\mathbf{M}_2) \longrightarrow E(\mathbf{M}_1)$ of differential forms, which is induced by a diffeomorphism $\psi : \mathbf{M}_1 \longleftrightarrow \mathbf{M}_2$, as is indicated in the diagram below.

$$\begin{tikzcd}
M_1 \ni q_1 \arrow[rr] \arrow[dd] & & q_2 \in M_2 \arrow[dd] \\
E(M_1) \ni t_1 & & t_2 \in E(M_2) \arrow[ll, "\psi^*"'] \\
t_1(q_1) & & t_2(q_2) \arrow[ll, "T(\psi)"']
\end{tikzcd}$$

$$i_{X_1} \circ \psi^* = \psi^* \circ i_{X_2}, \; L_{X_1} \circ \psi^* = \psi^* \circ L_{X_2}, \; d \circ \psi^* = \psi^* \circ d,$$

with the according push forward of vector fields, such that $X_2 := \psi_*(X_1)$.

(5.4.12.2) Obviously the unital associative algebra $E(\mathbf{M})$ over $\mathbf{R}$ is $\mathbf{Z}_2$-graded by

$$E_{\bar{r}}(\mathbf{M}) := \bigoplus_{r \in \bar{r}} E_r(\mathbf{M}), \; \bar{r} \in \mathbf{Z}_2.$$

Then L_X is some even derivation of $E(\mathbf{M}), i_X$ and d are odd derivations of $E(\mathbf{M})$, for every smooth vector field X on $\mathbf{M}$.

(5.4.12.3)

$$L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}, \; i_X \circ i_X = 0, \; d \circ d = 0,$$
$$L_X = i_X \circ d + d \circ i_X, \; L_X \circ d - d \circ L_X = 0,$$
$$i_X \circ L_Y - L_Y \circ i_X = i_{[X,Y]},$$

for smooth vector fields X, Y. Therefore the real linear span of the set $\{d, L_X, i_X; X \in T(\mathbf{M})\}$ is some subalgebra of the real Lie superalgebra

$$Der_{\mathbf{R}}(E(\mathbf{M})) := \bigoplus_{\bar{z}\in \mathbf{Z}_2} Der_{\mathbf{R}}^{\bar{z}}(E(\mathbf{M}))$$

of super-derivations of $E(\mathbf{M})$.

(5.4.12.4) The de Rham-cohomologies are defined with respect to the restrictions $d\big|_{E_r(\mathbf{M})}$, $0 \le r \le m$.

$$\cdots \longrightarrow \{0\} \xrightarrow[d_{-2}]{} \{0\} \xrightarrow[d_{-1}]{} C^\infty(\mathbf{M}) \xrightarrow[d_0]{} E_1(\mathbf{M}) \xrightarrow[d_1]{} E_2(\mathbf{M}) \longrightarrow$$

$$\cdots \longrightarrow E_m(\mathbf{M}) \xrightarrow[d_m]{} \{0\} \xrightarrow[d_{m+1}]{} \{0\} \longrightarrow \cdots.$$

$$\forall r \in \mathbf{Z} : H_r(\mathbf{M}) := ker\ d_r / Im\ d_{r-1}.$$

The above defined pull back ψ_*, and the Lie derivations L_X with respect to vector fields $X \in T(\mathbf{M})$, are complex-morphisms of degree zero, i.e., compatible with d, and not changing the rank r of a differential form.

(5.4.13) The interior product of the smooth vector field X with the differential form t_{r+1}, $0 \le r \le m-1$, is obtained from the endomorphism $i(X)$ of the real vector space $\Lambda^*(T(\mathbf{M}), C^\infty(\mathbf{M}))$. $\forall q \in dom\ \phi$,

$$\forall 1 \le r \le m-1,\ \forall X := \xi^l \frac{\partial}{\partial x^l}, X_1 := \xi_1^l \frac{\partial}{\partial x^l}, \ldots, X_r := \xi_r^l \frac{\partial}{\partial x^l},$$

$$\forall t_{r+1} := \sum_{1 \le l_1 < \ldots < l_{r+1} \le m} \tau_{l_1 \ldots l_{r+1}} dx^{l_1} \wedge \cdots \wedge dx^{l_{r+1}} \in E_{r+1}(\mathbf{M}),$$

$$\lambda_{r+1} := j_{r+1}(t_{r+1}) :$$

$$\begin{aligned}((i(X)(\lambda_{r+1}))(X_1, \ldots, X_r))(q) &= (i_X t_{r+1})(q)(X_1(q), \ldots, X_r(q)) \\ &= \tau_{l l_1 \ldots l_r}(x) \xi^l(x) \xi_1^{l_1}(x) \cdots \xi_r^{l_r}(x),\end{aligned}$$

$\forall Y := \eta_l dx^l \in E_1(\mathbf{M}), \forall \lambda_1 := j_1(Y)$:

$$(i(X)(\lambda_1))(q) = (\lambda_1(X))(q) = Y(q)(X(q)) = \eta_l(x)\xi^l(x) = i_X Y(q),$$

$\forall f \in C^\infty(\mathbf{M}) : i(X)(f) = i_X f = 0$ by definition.

(5.4.14) For $1 \le r \le m-1$, consider the differential form $t_r \in E_r(\mathbf{M})$, and denote

$$\lambda_r := j_r(t_r),\ \mu_{r+1} := j_{r+1}(dt_r).$$

(5.4.14.1)

$$
\begin{aligned}
&((\delta_0(\lambda_r))(X_0,\ldots,X_r))(q)\\
&= \sum_{0\le i<j\le r}(-1)^{i+j}\sum_{l_1,\ldots,l_r=1}^{m}\tau_{l_1\ldots l_r}(x)\big(\xi_i^k\nabla_k\xi_j^{l_1}-\xi_j^k\nabla_k\xi_i^{l_1}\big)(x)\\
&\qquad\qquad\qquad\qquad \xi_0^{l_2}(x)\cdots\xi_r^{l_r}(x),\\
&(\mu_{r+1}(X_0,\ldots,X_r))(q) = \sum_{1\le l_1<\ldots<l_r\le m}\sum_{k=1}^{m}(\nabla_k\tau_{l_1\ldots l_r}(x))\\
&\qquad\qquad \sum_{\begin{bmatrix} kl_1\ldots l_r\\ jj_1\ldots j_r\end{bmatrix}=:P\in\mathbf{P}_{r+1}}(-1)^{\tau_P}\xi_0^{j}(x)\xi_1^{j_1}(x)\cdots\xi_r^{j_r}(x),\\
&(L_{X_j}(\lambda_r(X_0,\ldots,\cancel{X}_j,\ldots,X_r)))(q)\\
&= \xi_j^k(x)\nabla_k\left(\sum_{l_1,\ldots,l_r=1}^{m}\tau_{l_1\ldots l_r}(x)\xi_0^{l_1}(x)\cdots\xi_r^{l_r}(x)\right),
\end{aligned}
$$

with an obvious notation of components of t_r, and of $X_0,\ldots,X_r\in T(\mathbf{M})$, on *dom* $\phi\ni q$.

(5.4.14.1.1) Especially, for $r=1\le m-1$, one finds, denoting $x:=\phi(q)\in ran\ \phi$, that

$$
\begin{aligned}
&(\mu_2(X_0,X_1))(q) = \sum_{1\le k<l\le m}(\nabla_k\tau_l-\nabla_l\tau_k)(x)\xi_0^k(x)\xi_1^l(x),\\
&((\delta_0(\lambda_1))(X_0,X_1))(q) = -\tau_l(x)\big(\xi_0^k\nabla_k\xi_1^l-\xi_1^k\nabla_k\xi_0^l\big)(x),\\
&\sum_{j=0}^{1}(-1)^j(L_{X_j}(\lambda_1(X_{i_j})))(q) = \big(\xi_0^k\nabla_k(\tau_l\xi_1^l)-\xi_1^k\nabla_k(\tau_l\xi_0^l)\big)(x),
\end{aligned}
$$

with the notation $i_0:=1,\ i_1:=0$.

(5.4.14.2) Hence for $1\le r\le m-1$ the exterior derivation can be written as

$$
\begin{aligned}
\mu_{r+1}(X_0,\ldots,X_r) &= (\delta_0(\lambda_r))(X_0,\ldots,X_r)\\
&\quad+\sum_{j=0}^{r}(-1)^jL_{X_j}(\lambda_r(X_0,\ldots,\cancel{X}_j,\ldots,X_r))\\
&= (\delta_\rho(\lambda_r))(X_0,\ldots,X_r),
\end{aligned}
$$

inserting the $C^\infty(\mathbf{M})$-connection $\rho: T(\mathbf{M})\ni X\longrightarrow L_X$, which is local and flat.

(5.4.14.3) For $r = m$ one obtains, that
$\forall \lambda_m \in \Lambda^m(T(\mathbf{M}), C^\infty(\mathbf{M})), \forall X_0, \ldots, X_m \in T(\mathbf{M})$:

$$\sum_{j=0}^{m}(-1)^j L_{X_j}(\lambda_m(X_0, \ldots, \not{X}_j, \ldots, X_m)) = -(\delta_0(\lambda_m))(X_0, \ldots, X_m).$$

(5.4.14.4) For scalar fields one finds, that $\forall f \in C^\infty(\mathbf{M}), X \in T(\mathbf{M})$:

$$j_1(df) =: \mu_1, \; \mu_1(X) = \delta_\rho(f)(X) = \hat{\rho}(f)(X) = \rho(X)(f) := L_X f.$$

(5.4.15) Similarly the Lie derivation of differential forms on $\mathbf{M}$ can be rewritten, such that $\forall 1 \leq r \leq m, \forall X, X_1, \ldots, X_r \in T(\mathbf{M}), \forall t_r \in E_r(\mathbf{M})$:

$$\begin{aligned}\nu_r(X_1, \ldots, X_r) &= L_X(\lambda_r(X_1, \ldots, X_r)) + (\theta_0(X)(\lambda_r))(X_1, \ldots, X_r) \\ &= (\theta_\rho(X)(\lambda_r))(X_1, \ldots, X_r),\end{aligned}$$

denoting $\lambda_r := j_r(t_r)$, and $\nu_r := j_r(L_X t_r)$.

(5.4.15.1)

$$\forall f \in C^\infty(\mathbf{M}) : L_X f = \rho(X)(f) = \theta_\rho(X)(f).$$

(5.4.16) Therefore the Cartan calculus of differential forms on $\mathbf{M}$ provides an example of the above machinery of Lie-Cartan pairs. Take $X \in T(\mathbf{M})$;

$$\begin{aligned}
&\forall_1^m r : i_X\big|_{E_r(\mathbf{M})} = j_{r-1}^{-1} \circ i(X) \circ j_r, \; i_X\big|_{C^\infty(\mathbf{M})} = i(X)\big|_{C^\infty(\mathbf{M})} = 0; \\
&\forall_0^m r : L_X\big|_{E_r(\mathbf{M})} = j_r^{-1} \circ \theta_\rho(X) \circ j_r; \\
&\forall_0^{m-1} r : d\big|_{E_r(\mathbf{M})} = j_{r+1}^{-1} \circ \delta_\rho \circ j_r, \; d\big|_{E_m(\mathbf{M})} = \delta_\rho \circ j_m = 0,
\end{aligned}$$

inserting the above defined local flat $C^\infty(\mathbf{M})$-connection ρ, and with an obvious meaning of j_r^{-1}.

5.5 Cohomologies of Lie-Cartan Pairs

(5.5.1) The diagrams $j_{r+1} \circ d \circ j_r^{-1}$, $j_r \circ L_X \circ j_r^{-1}$, $0 \leq r \leq m$, $j_{m+1} := 0$, are generalized to an arbitrary Lie-Cartan pair $\{L, A\}$ over a commutative ring R, with respect to an A-connection ρ. Here A is also considered as an A-left module over R, using the product of A for the module-multiplication too.

(5.5.1.1) The R-endomorphisms d, L_k, i_k of $\Lambda^*(L, A)$ are defined appropriately.

$$\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L, A), \forall k, k_0, k_1, \dots, k_p \in L, \forall a \in A : (da)(k_0) := k_0 a,$$

$$(d\lambda_p)(k_0, k_1, \dots, k_p) := (\delta_0(\lambda_p))(k_0, k_1, \dots, k_p) + \sum_{j=0}^{p} (-1)^j k_j \lambda_p(k_0, \dots, \not{k}_j, \dots, k_p),$$

$$(L_k \lambda_p)(k_1, \dots, k_p) := k\lambda_p(k_1, \dots, k_p) + (\theta_0(k)(\lambda_p))(k_1, \dots, k_p),$$

$$L_k a := ka, \; i_k := i(k), \; i_k a = 0, \; i_k \lambda_1 = \lambda_1(k).$$

(5.5.1.2) Now take the flat local A-connection ρ_0 :

$$L \ni k \longrightarrow \rho_0(k) : A \ni a \underset{def}{\longrightarrow} ka \in A.$$

Then obviously $\forall k \in L, a \in A$:

$$d - \delta_{\rho_0}\big|_{\bigoplus_{p \geq 1} \Lambda^p(L,A)} = 0, \; \delta_{\rho_0} := \delta_0 + \hat{\rho}_0, \; \delta_{\rho_0}(a) = 0,$$

$$L_k = \theta_{\rho_0}(k) := \theta_0(k) + \rho_0(k).$$

(5.5.1.3) One then obtains the following relations. $\forall k, l \in L$:

$$i_k \circ i_l + i_l \circ i_k = 0, \; L_k \circ L_l - L_l \circ L_k = L_{[k,l]}, \; d \circ d = 0,$$

$$L_k \circ d - d \circ L_k = 0, \; i_k \circ d + d \circ i_k = L_k, \; i_k \circ L_l - L_l \circ i_k = i_{[k,l]}.$$

Therefore the real linear span of $\{d, L_k, i_k; k \in L\}$ is some subalgebra of the Lie superalgebra

$$Der_R(\Lambda^*(L, A)) := \bigoplus_{\bar{z} \in \mathbf{Z}_2} Der_R^{\bar{z}}(\Lambda^*(L, A))$$

of super-derivations of alternating R-multilinear forms on L, with the values in A.

(5.5.1.4) On an m-dimensional real C^∞-manifold $\mathbf{M}$, choose

$$A := C^\infty(\mathbf{M}), \; L := T(\mathbf{M}),$$

and restrict the above Lie superalgebra to the images of j_r, $0 \leq r \leq m$, and apply j_r^{-1}. Then one obtains the real linear span of $\{d, L_X, i_X; X \in T(\mathbf{M})\}$, in the sense of the foregoing subchapter.

(5.5.1.5) The de Rham-cohomologies are generalized to the cohomologies of d on $\Lambda^*(L, A)$.

(5.5.2) In order to investigate cohomologies of Lie algebras, consider the Lie-Cartan pair $\{L, A\}$ over R, and an E-connection ρ. One finds, that $\forall k \in L, \lambda \in \Lambda^*(L, E), a \in A$:

$$i(k)(a\lambda) = a(i(k)(\lambda)),$$

$$\theta_\rho(k)(a\lambda) = a(\theta_\rho(k)(\lambda)) + (ka)\lambda,\ \delta_\rho(a\lambda) = a\delta_\rho(\lambda) + \Delta(a)(\lambda).$$

(5.5.2.1) Let ρ be flat. The cohomologies $H^p_\rho(L, E)$ are defined with respect to δ_ρ.

$$\delta_\rho \circ \delta_\rho = 0,\ \forall p \in \mathbf{Z} : H^p_\rho(L, E) := ker\ \delta^p_\rho / Im\ \delta^{p-1}_\rho,\ H^0_\rho(L, E) = E/\{0\}.$$

$$\cdots \longrightarrow \{0\} \xrightarrow[\delta_\rho^{-2}]{} \{0\} \xrightarrow[\delta_\rho^{-1}]{} E \xrightarrow[\delta_\rho^{0}]{} \Lambda^1(L, E) \longrightarrow$$

$$\cdots \longrightarrow \Lambda^p(L, E) \xrightarrow[\delta_\rho^{p}]{} \Lambda^{p+1}(L, E) \longrightarrow \cdots.$$

(5.5.2.2) $\forall k, l \in L$:

$$\theta_\rho(k) \circ \theta_\rho(l) - \theta_\rho(l) \circ \theta_\rho(k) = \theta_\rho([k, l]),\ i(k) \circ i(k) = 0,$$

$$\theta_\rho(k) \circ \delta_\rho - \delta_\rho \circ \theta_\rho(k) = 0,\ i(k) \circ \delta_\rho + \delta_\rho \circ i(k) = \theta_\rho(k),$$

$$i(k) \circ \theta_\rho(l) - \theta_\rho(l) \circ i(k) = i([k, l]).$$

Therefore the R-linear span of $\{\delta_\rho, i(k), \theta_\rho(k); k \in L\}$ is some subalgebra of the Lie superalgebra $(End_R(\Lambda^*(L, E)))_L$.

(5.5.2.3) The elements of $\Lambda^*(L, E)$ are called E-cochains. Especially the elements of $ker\ \delta_\rho$ are called E-cocycles, those of $Im\ \delta_\rho$ E-coboundaries. Obviously $\theta_\rho(k)$ turns E-cocycles into E-coboundaries, because

$$\forall \lambda \in \Lambda^*(L, E) : [\delta_\rho(\lambda) = 0 \Longrightarrow \forall k \in L : \theta_\rho(k)(\lambda) = \delta_\rho \circ i(k)(\lambda)].$$

(5.5.2.3.1) Inserting the flat local A-connection ρ_0, choosing $E := A$, one obtains the A-valued Chevalley cohomologies of L with respect to d.

(5.5.2.4) Consider again the choice $E := A$, hence an L-left module E over R. Especially choose $\forall k \in L, a \in A : ka := 0$. Then the Lie-Cartan pair means, that

$$\forall a, b \in A, \forall k, l \in L : (ab)k = a(bk),\ e_A k = k,\ [k, al] = [ak, l] = a[k, l].$$

The R-linear mappings $d, L_k, i_k, k \in L$, are then acting such that

$$\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L,E), \forall k, k_0, k_1, \dots, k_p \in L, \forall a \in A : da = 0,$$
$$d\lambda_p(k_0, k_1, \dots, k_p) = \sum_{0 \le i < j \le p} (-1)^{i+j} \lambda_p([k_i, k_j], k_0, \dots, \not{k}_i, \dots, \not{k}_j, \dots, k_p),$$
$$d\lambda_1(k_0, k_1) = -\lambda_1([k_0, k_1]),$$
$$d\lambda_2(k_0, k_1, k_2) = -\lambda_2([k_0, k_1], k_2) + \lambda_2([k_0, k_2], k_1) - \lambda_2([k_1, k_2], k_0),$$
$$L_k\lambda_p(k_1, \dots, k_p) = -\sum_{j=1}^{p} \lambda_p(k_1, \dots, k_{j-1}, [k, k_j], k_{j+1}, \dots, k_p), \; L_k a = 0,$$
$$L_k\lambda_1(k_1) = -\lambda_1([k, k_1]), \; L_k\lambda_2(k_1, k_2) = -\lambda_2([k, k_1], k_2) - \lambda_2(k_1, [k, k_2]),$$
$$i_k\lambda_{p+1}(k_1, \dots, k_p) = \lambda_{p+1}(k, k_1, \dots, k_p),$$
$$i_k a = 0, \; i_k\lambda_1 = \lambda_1(k), \; i_k\lambda_2(k_1) = \lambda_2(k, k_1).$$

The Lie superalgebra of these super-derivations of A-cochains for L was written down above. Due to the special case of $\rho_0 = 0$ one obtains the cohomologies of L with respect to $\delta_0 = d$.

(5.5.2.5) In the special case of $A := R$, and demanding that

$$\forall a \in A, k \in L : ka := 0,$$

the R-bilinear map: $A \times L \longrightarrow L$ is defined as the module-multiplication of L, and E is an R-bimodule. Therefore the defining conditions on Lie-Cartan pairs are essentially reduced to the zero-map: $L \times A \longrightarrow \{0\} \longrightarrow A$. Obviously every E-connection ρ is local, because ρ is an R-linear map.

$$\forall k \in L, a \in A, x \in E : \rho(k)(ax) = \rho(ak)(x).$$

In the case of $\rho := 0$, one finds the E-valued Chevalley cohomologies $H_0^p(L,E)$, with respect to the R-endomorphism δ_0 of $\Lambda^*(L,E)$, $\delta_0 \circ \delta_0 = 0$.

(5.5.2.5.1) Especially choose $E := A := R$. Then $d = \delta_0$ induces the cohomologies $H_0^p(L,R)$. Using the universal property of the alternating product $\Lambda^p(L)$, with the alternating algebra

$$\Lambda(L) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p(L)$$

of L over R, one obtains an isomorphism of $(\Lambda^p(L))^*$ with $\Lambda^p(L,R)$, $p \in \mathbf{N}_0$, in the sense of R-bimodules, due to the diagram below.

$$\begin{tikzcd} \overset{p}{\times} L \arrow[rr] \arrow[dr, "f"'] & & \Lambda^p(L) \arrow[dl, "\hat{f}"] \\ & R & \end{tikzcd}$$

Here $f \in \Lambda^p(L,R)$ induces uniquely $\hat{f} \in (\Lambda^p(L))^* := Hom_R(\Lambda^p(L),R)$, with the following R-linear bijections. $\forall p \in \mathbf{N}$:

$$\Lambda^p(L,R) \ni f \longleftrightarrow \hat{f} \in (\Lambda^p(L))^*,\ \Lambda^0(L) = \frac{R}{\{0\}} \longleftrightarrow R =: \Lambda^0(L,R),$$

$$(\Lambda^1(L))^* = \left(\frac{L}{\{0\}}\right)^* \longleftrightarrow Hom_R(L,R) =: \Lambda^1(L,R).$$

(5.5.2.5.1.1) In the special case of real coefficients, $R := \mathbf{R}$, and $dim\ L \in \mathbf{N}$, one finds real-linear bijections:

$$\Lambda^p(L,\mathbf{R}) \longleftrightarrow (\Lambda^p(L))^* \longleftrightarrow \Lambda^p(L),\ p \in \mathbf{N}_0.$$

Hence one obtains an isomorphism: $\Lambda(L) \longleftrightarrow \Lambda^*(L,\mathbf{R})$, in the sense of unital associative algebras over $\mathbf{R}$, using carefully an appropriate basis of

$$\Lambda(L) := \bigoplus_{p=0}^{m} \Lambda^p(L)$$

over $\mathbf{R}$. Here one finds, denoting $dim\ L =: m$, that $\forall p > m : \Lambda^p(L) = \{0\}$.

$$\forall\ 0 \le p \le m : dim\ \Lambda^p(L) = \binom{m}{p},\ dim\ \Lambda(L) = 2^m.$$

(5.5.3) The so-called matrix geometry, i.e. non-commutative differential geometry over unital associative algebras of complex matrices, which was constructed by M. Dubois-Violette and collaborators (1990), can be formulated within the framework of Lie-Cartan pairs $\{L,A\}$ over R, and their E-connections, choosing

$$R := \mathbf{C},\ A := \mathbf{C},\ L := Der_{\mathbf{C}}(E),\ L \times \mathbf{C} \underset{def}{\longrightarrow} \{0\} \longrightarrow \mathbf{C},$$

using the module-multiplication of the complex Lie algebra L for the complex-bilinear map: $\mathbf{C} \times L \longrightarrow L$, and inserting the connection
$\rho : L \longrightarrow (End_{\mathbf{C}}(E))_L$ by means of derivations of E, such that

$$\forall k \in L, x \in E : \rho(k)(x) := k(x) \equiv kx.$$

The complex vector space E is chosen as some unital associative algebra, such that derivations on E can be defined. Then $\Lambda^*(L,E)$ is such an object too, moreover carrying an $\mathbf{N}_0$-grading and corresponding $\mathbf{Z}_2$-grading. The connection ρ is flat and local. The endomorphism d of E, which is defined such that $\forall p \in \mathbf{N},\ x \in E,\ k \in L$:

$$d - \delta_\rho\big|_{\Lambda^p(L,E)} := 0,\ (dx)(k) := kx \in E,\ d \circ d = 0,$$

is an odd nilpotent derivation of $\Lambda^*(L,E)$. More explicitly
$\forall p \in \mathbf{N},\ \forall k_0, k_1, \dots, k_p \in L,\ \forall \lambda_p \in \Lambda^p(L,E):$

$$\begin{aligned}
&(d\lambda_p)(k_0,\dots,k_p)\\
&= \sum_{0\le i<j\le p} (-1)^{i+j}\lambda_p\left([k_i,k_j],k_0,\dots,\not{k}_i,\dots,\not{k}_j,\dots,k_p\right)\\
&+\sum_{j=0}^{p}(-1)^j k_j\lambda_p(k_0,\dots,\not{k}_j,\dots,k_p).
\end{aligned}$$

(5.5.3.1) These differential forms over E may be restricted to the smallest $\mathbf{Z}_2$-graded differential subalgebra of $\Lambda^*(L,E)$, which contains E itself as subalgebra, in order to obtain an appropriate differential envelope of E.

(5.5.3.2) The above indicated version of matrix geometry is obtained by the choice of $E := Mat_n(\mathbf{C})$. In this case, every derivation of E is an inner one, i.e., can be expressed by means of the adjoint representation of the commutator algebra $gl(n,\mathbf{C})$ of $Mat_n(\mathbf{C})$. Therefore the complex Lie algebra $Der_{\mathbf{C}}(Mat_n(\mathbf{C}))$ is isomorphic to $sl(n,\mathbf{C})$. Moreover the above indicated differential envelope of $Mat_n(\mathbf{C})$ is $\Lambda^*(Der_{\mathbf{C}}(Mat_n(\mathbf{C})), Mat_n(\mathbf{C}))$ itself.

(5.5.3.3) Furthermore models of gauge theory were constructed, starting from

$$E := C^\infty(\mathbf{M}) \otimes Mat_n(\mathbf{C}),$$

on a finite-dimensional real connected differentiable manifold $\mathbf{M}$.

5.6 $\mathbf{Z}_2$-Graded Lie-Cartan Pairs

(5.6.1) Let L be a Lie superalgebra, and A a unital associative superalgebra, both over a commutative ring R. Assume A to be graded-commutative, i.e., with vanishing super-commutator, such that

$$\forall \bar{p},\bar{q} \in \mathbf{Z}_2, \forall a \in A^{\bar{p}}, b \in A^{\bar{q}} : ab = (-1)^{pq} ba.$$

The family $\{L, A\}$ is called $\mathbf{Z}_2$-graded Lie-Cartan pair, if and only if the following conditions hold.

(i) L is some graded A-left module over R, i.e., L is an A-left module over R, and

$$\forall \bar{p},\bar{q} \in \mathbf{Z}_2, \forall a \in A^{\bar{p}}, k \in L^{\bar{q}} : ak \in L^{\overline{p+q}}.$$

(ii) A homomorphism of Lie superalgebras

$$\lambda : L \longrightarrow Der_R(A) := Der_R^{\bar{0}}(A) \oplus Der_R^{\bar{1}}(A),$$

into the Lie superalgebra of super-derivations of A, is established, i.e., an even R-linear map λ conserving the super-commutator:

$$L \ni k \longrightarrow \lambda(k) : A \ni a \underset{def}{\longrightarrow} ka \in A.$$

(iii) $\forall a, b \in A, \forall k \in L : a(kb) = (ak)b.$

(iv) $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2, \forall a \in A^{\bar{p}}, k \in L^{\bar{q}}, l \in L : [k, al] = (-1)^{pq} a[k, l] + (ka)l.$

(5.6.1.1) These conditions on the pair $\{L, A\}$ explicitly mean an establishment of R-bilinear mappings: $A \times L \longrightarrow L$, $L \times A \longrightarrow A$, which obey the following relations.
$\forall \bar{p}, \bar{q}, \bar{r} \in \mathbf{Z}_2, \forall a \in A^{\bar{p}}, k \in L^{\bar{q}}, l \in L^{\bar{r}}, b \in A :$

$$(ab)k = a(bk),\ e_A k = k,\ ak \in L^{\overline{p+q}},\ ka \in A^{\overline{p+q}},$$

$$k(ab) = (ka)b + (-1)^{pq} a(kb),\ [k, l]a = k(la) - (-1)^{qr} l(ka),$$
$$a(kb) = (ak)b,\ [k, al] = (-1)^{pq} a[k, l] + (ka)l.$$

These relations imply

$$ke_A = 0_A,\ [ak, l] = a[k, l] - (-1)^{(p+q)r} (la)k.$$

(5.6.2) Let $E = E^{\bar{0}} \oplus E^{\bar{1}}$ be a graded A-bimodule over R. An even R-linear map $\rho : L \longrightarrow (End_R(E))_L$ is called E-connection, if and only if $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2$,

$$\forall a \in A^{\bar{p}}, k \in L^{\bar{q}}, x \in E : \rho(k)(ax) = (-1)^{pq} a(\rho(k)(x)) + (ka)x.$$

(5.6.2.1) One then immediately finds, that $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2$,

$$\forall k \in L^{\bar{p}}, x \in E^{\bar{q}}, a \in A : \rho(k)(xa) = (\rho(k)(x))a + (-1)^{pq} x(ka).$$

(5.6.2.2) An E-connection ρ is called local, if and only if

$$\forall k \in L, a \in A, x \in E : \rho(ak)(x) = a(\rho(k)(x)).$$

(5.6.3) The curvature κ of an E-connection ρ is defined as an R-bilinear map, such that $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2$:

$$L^{\bar{p}} \times L^{\bar{q}} \ni \{k, l\}$$
$$\longrightarrow \kappa(k, l) := \rho(k) \circ \rho(l) - (-1)^{pq} \rho(l) \circ \rho(k) - \rho([k, l]) \in (End_R(E))_L^{\overline{p+q}}.$$

In particular ρ is called flat, if and only if its curvature vanishes, i.e., if and only if ρ is a homomorphism of Lie superalgebras over R, which means a representation of L on E.

(5.6.4) If ρ_1, ρ_2 are E-connections, then

$$\forall r_1, r_2 \in R : r_1 + r_2 = e_R \Longrightarrow r_1\rho_1 + r_2\rho_2 \text{ is an } E-\text{connection}.$$

(5.6.5) The $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, A\}$ is called degenerate, if and only if $\forall k \in L, a \in A : ka = 0$. In this case every R-linear combination of E-connections is again an E-connection.

(5.6.6) The $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, R\}$, inserting the zero map: $L \times R \longrightarrow \{0\} \longrightarrow R =: R^{\bar{0}}$, and with the module-multiplication of L serving for an R-bilinear map: $R \times L \longrightarrow L$, is called the depletion of $\{L, A\}$; it is isomorphic with the subpair $\{L, Re_A\}$ of $\{L, A\}$.

(5.6.7) Obviously A is some graded A-left module over R. The above defined map λ yields some flat local A-connection ρ_0, according to the diagram below.

$$\begin{tikzcd} L \arrow[r, "\lambda"'] \arrow[rr, bend right, "\rho_0 \qquad def"'] & Der_R(A) \arrow[r] & (End_R(A))_L \end{tikzcd}$$

$$L \ni k \underset{\rho_0}{\longrightarrow} \rho_0(k) : A \ni a \longrightarrow ka \in A.$$

Here

$$\forall k \in L : \rho_0(k) \in Der_R(A) := Der_R^{\bar{0}}(A) \oplus Der_R^{\bar{1}}(A).$$

(5.6.8) In case $E := L$, the adjoint representation of L yields some flat L-connection α.

$$L \ni k \underset{ad}{\longrightarrow} ad\ k : L \ni l \longrightarrow [l, k] \in L,$$

with the homomorphism ad in the sense of Lie superalgebras over R:

$$L \underset{ad}{\longrightarrow} (End_R(L))_L, \ Im\ ad \subseteq Der_R(L).$$

$$L \ni k \longrightarrow \alpha(k) : L \ni l \underset{def}{\longrightarrow} [k, l] \in L.$$

Especially, if $\{L, A\}$ is degenerate, then the L-connection α is local.

(5.6.9) The torsion $T_\rho : L \times L \longrightarrow L$, of an L-connection ρ, is defined as an R-bilinear mapping. $\forall \bar{p}, \bar{q} \in \mathbf{Z}_2$:

$$L^{\bar{p}} \times L^{\bar{q}} \ni \{k, l\} \longrightarrow T_\rho(k, l) := \rho(k)(l) - (-1)^{pq}\rho(l)(k) - [k, l] \in L^{\overline{p+q}},$$

such that obviously $T_\rho(k, l) = (-1)^{pq+1}T_\rho(l, k)$.

(5.6.10) Consider an E-connection ρ, with the $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, A\}$ over R. The R-bimodule of R-multilinear maps $\lambda_p : \prod^p L \longrightarrow E$ is denoted by $L^p(L, E)$, $p \in \mathbf{N}$.

$$L^1(L,E) = Hom_R(L,E),\ L^0(L,E) := E,\ \bigoplus_{p \in \mathbf{N}_0} L^p(L,E) =: L^*(L,E).$$

(5.6.10.1) Henceforth take $R := K \supseteq \mathbf{Q}$, with a field K of characteristic zero. For $p \geq 2$, the vector space $\Lambda^p(L,E)$ of graded-alternating mappings $\lambda_p : \prod^p L \longrightarrow E$ is defined, with the permutations $P \in \mathbf{P}_p$ of p elements.

$$\Lambda^p(L,E) := \Bigg\{ \lambda_p \in L^p(L,E); \forall \overline{z_1}, \dots, \overline{z_p} \in \mathbf{Z}_2, \forall k_1 \in L^{\overline{z_1}}, \dots, k_p \in L^{\overline{z_p}},$$
$$\forall P := \begin{bmatrix} 1 \dots p \\ j_1 \dots j_p \end{bmatrix} \in \mathbf{P}_p :$$
$$\lambda_p(k_{j_1}, \dots, k_{j_p}) = \lambda_p(k_1, \dots, k_p)(-1)^{\tau_p + \sum_{\{1 \leq i < i' \leq p; j_i > j_{i'}\}} z_{j_i} z_{j_{i'}}} \Bigg\},$$

counting just those odd elements of L, which are transposed by P. For instance,

$$\forall \lambda_2 \in \Lambda^2(L,E), \forall k, l \in L^{\bar{1}} : \lambda_2(k,l) = \lambda_2(l,k).$$

For $\lambda_3 \in \Lambda^3(L,E)$ one obtains for example, with the above notation of degrees,

$$\begin{aligned}\lambda_3(k_1,k_2,k_3) &= \lambda_3(k_3,k_1,k_2)(-1)^{z_1 z_3 + z_2 z_3} \\ &= \lambda_3(k_3,k_2,k_1)(-1)^{1 + z_1 z_2 + z_1 z_3 + z_2 z_3}.\end{aligned}$$

One then usually denotes

$$\Lambda^0(L,E) := E,\ \Lambda^1(L,E) := Hom_K(L,E),\ \Lambda^*(L,E) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p(L,E).$$

(5.6.10.2)

$$\begin{aligned}\Lambda^p_A(L,E) := \{ &\lambda_p \in \Lambda^p((L,E); \forall \bar{z}, \overline{z_1}, \dots, \overline{z_p} \in \mathbf{Z}_2, \\ &\forall k_1 \in L^{\overline{z_1}}, \dots, k_p \in L^{\overline{z_p}}, \forall a \in A^{\bar{z}}, \forall 1 \leq j \leq p : \\ &\lambda_p(k_1, \dots, a k_j, \dots, k_p) \\ &= (-1)^{z(z_j + \dots + z_p)} \lambda_p(k_1, \dots k_p) a \\ &= (-1)^{z(y + z_1 + \dots + z_{j-1})} a \lambda_p(k_1, \dots, k_p) \};\end{aligned}$$

here $y \in \bar{y} := \bar{x} - \overline{z_1} - \dots - \overline{z_p} \in \mathbf{Z}_2$, for $\lambda_p(k_1, \dots, k_p) \in E^{\bar{x}}$, $p \in \mathbf{N}$.

$$\Lambda^0_A(L,E) := E,\ \Lambda^*_A(L,E) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p_A(L,E).$$

Especially, for $p = 1$,

$$\Lambda^1_A(L,E) := \{\lambda_1 \in Hom_K(L,E); \forall \bar{z}, \overline{z_1} \in \mathbf{Z}_2, \forall k_1 \in L^{\overline{z_1}}, a \in A^{\bar{z}} : \lambda_1(ak_1) = (-1)^{zz_1}\lambda_1(k_1)a\}.$$

(5.6.10.2.1) Here the $\mathbf{Z}_2$-degree $\bar{y}$ of λ_p is used, such that

$$\Lambda^p(L,E) = \bigoplus_{\bar{y}\in \mathbf{Z}_2} (\Lambda^p(L,E))^{\bar{y}}.$$

(5.6.10.3) In order to understand these permutations, the graded alternator A_p is defined.

$$\forall p \geq 2, \forall \lambda_p \in L^p(L,E), \forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2,$$
$$\forall k_1 \in L^{\overline{z_1}, \ldots, k_p \in L^{\overline{z_p}}}, \forall P := \begin{bmatrix} 1 \ldots p \\ j_1 \ldots j_p \end{bmatrix} \in \mathbf{P}_p :$$
$$(\Sigma(P)\lambda_p)(k_1, \ldots, k_p) := (-1)^{\tau_P + \sum_{\{1\leq i<i'\leq p; j_i > j_{i'}\}} z_{j_i} z_{j_{i'}}} \lambda_p(k_{j_1}, \ldots, k_{j_p}),$$
$$A_p := \frac{1}{p!} \sum_{P\in \mathbf{P}_p} \Sigma(P).$$

Obviously $\Sigma : \mathbf{P}_p \longrightarrow End_K(L^p(L,E))$ is some representation of the symmetric group $\mathbf{P}_p$.

$$A_p^2 = A_p, \text{ because } \forall p \geq 2, \forall P \in \mathbf{P}_p : \Sigma(P)A_p = A_p\Sigma(P) = A_p.$$

One then finds, that $\forall p \geq 2$:

$$\Lambda^p(L,E) = \{\lambda_p \in L^p(L,E); \forall P \in \mathbf{P}_p : \Sigma(P)\lambda_p = \lambda_p\} = Im\ A_p.$$

(5.6.10.4) For $p \in \mathbf{N}$, the vector space $L^p(L,E)$ is $\mathbf{Z}_2$-graded due to the definition, that

$$\forall \bar{y} \in \mathbf{Z}_2 : \lambda_p \in (L^p(L,E))^{\bar{y}} \underset{def}{\Longleftrightarrow}$$
$$\forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2, \forall k_1 \in L^{\overline{z_1}}, \ldots, k_p \in L^{\overline{z_p}} : \lambda_p(k_1, \ldots, k_p) \in E^{\overline{z_1 + \cdots + z_p + y}}.$$
$$L^p(L,E) = \bigoplus_{\bar{y}\in \mathbf{Z}_2} (L^p(L,E))^{\bar{y}}.$$

For $p = 1$, one obtains the usual $\mathbf{Z}_2$-grading of $Hom_K(L,E)$. This intrinsic grading of $L^*(L,E)$ must be distinguished carefully from the total grading defined below.

(5.6.11) The K-endomorphisms $i(k), \rho(k), \theta_0(k), \theta_\rho(k), \delta_0, \hat{\rho}, \delta_\rho$, of $L^*(L,E)$ are defined, inserting the intrinsic grading.
$\forall p \in \mathbf{N}, \forall \bar{y}, \bar{z}, \overline{z_0}, \dots \overline{z_p} \in \mathbf{Z}_2, \forall \lambda_p \in (L^p(L,E))^{\bar{y}}$,
$\forall k \in L^{\bar{z}}, k_0 \in L^{\overline{z_0}}, \dots, k_p \in L^{\overline{z_p}}, \forall x \in E^{\bar{y}}$:

$$(i(k)(\lambda_{p+1}))(k_1,\dots,k_p) := (-1)^{(p+1+y)z}\lambda_{p+1}(k,k_1,\dots,k_p),$$
$$i(k)(x) := 0,\ i(k)(\lambda_1) := (-1)^{(1+y)z}\lambda_1(k),$$
$$(\rho(k)(\lambda_p))(k_1,\dots,k_p) := (-1)^{pz}\rho(k)(\lambda_p(k_1,\dots,k_p)),$$

as an extension of the E-connection ρ,

$$(\theta_0(k)(\lambda_p))(k_1,\dots,k_p) :=(-1)^{1+pz}\sum_{j=1}^{p}(-1)^{z(y+z_1+\cdots+z_{j-1})} \lambda_p(k_1,\dots,k_{j-1},[k,k_j],k_{j+1},\dots,k_p),$$

$$\theta_0(k)(x) := 0,\ (\theta_0(k)(\lambda_1))(k_1) = (-1)^{1+z+zy}\lambda_1([k,k_1]),$$
$$\theta_\rho(k) := \theta_0(k) + \rho(k),$$
$$(\delta_0(\lambda_p))(k_0,k_1,\dots,k_p)$$
$$:= \sum_{0\le i<j\le p}(-1)^{i+j+(z_i+z_j)(z_0+\cdots+z_{i-1})+z_j(z_{i+1}+\cdots+z_{j-1})} \lambda_p([k_i,k_j],k_0,\dots,\cancel{k_i},\dots,\cancel{k_j},\dots,k_p),$$
$$\delta_0(x) := 0,\ (\delta_0(\lambda_1))(k_0,k_1) = -\lambda_1([k_0,k_1]),$$
$$(\delta_0(\lambda_2))(k_0,k_1,k_2) = -\lambda_2([k_0,k_1],k_2)$$
$$+ (-1)^{z_1z_2}\lambda_2([k_0,k_2],k_1) - (-1)^{z_0(z_1+z_2)}\lambda_2([k_1,k_2],k_0),$$

$$(\hat{\rho}(\lambda_p))(k_0,k_1,\dots,k_p) := \sum_{j=0}^{p}(-1)^{j+z_j(y+z_0+\cdots+z_{j-1})} \rho(k_j)(\lambda_p(k_0,\dots,\cancel{k_j},\dots,k_p)),$$

$$\hat{\rho}(x)(k) := (-1)^{yz}\rho(k)(x),$$
$$(\hat{\rho}(\lambda_1))(k_0,k_1) = (-1)^{z_0y}\rho(k_0)(\lambda_1(k_1)) - (-1)^{z_1(y+z_0)}\rho(k_1)(\lambda_1(k_0)),$$
$$\delta_\rho := \delta_0 + \hat{\rho},\ \delta_\rho(x) = \hat{\rho}(x).$$

(5.6.11.1) These endomorphisms show the following properties, for $p \in \mathbf{N}$.

$$L \ni k \underset{linear}{\longrightarrow} i(k) : L^p(L,E) \underset{linear}{\longrightarrow} L^{p-1}(L,E);$$
$$L \ni k \underset{linear}{\longrightarrow} \rho(k);\ \theta_0(k), \theta_\rho(k) : L^{p-1}(L,E) \underset{linear}{\longrightarrow} L^{p-1}(L,E);$$
$$\delta_0, \hat{\rho}, \delta_\rho : L^{p-1}(L,E) \underset{linear}{\longrightarrow} L^p(L,E).$$

(5.6.11.2) Let $\bar{y}, \bar{z} \in \mathbf{Z}_2$; then

$$L^{\bar{z}} \ni k \underset{linear}{\longrightarrow} i(k);\ \rho(k), \theta_0(k), \theta_\rho(k) : (L^*(L,E))^{\bar{y}} \underset{linear}{\longrightarrow} (L^*(L,E))^{\overline{y+z}};$$

$$\delta_0, \hat{\rho}, \delta_\rho : (L^*(L,E))^{\bar{y}} \underset{linear}{\longrightarrow} (L^*(L,E))^{\bar{y}},\ (L^*(L,E))^{\bar{y}} := \bigoplus_{p \in \mathbf{N}_0} (L^p(L,E))^{\bar{y}}.$$

(5.6.11.3) The vector space $\Lambda^*(L,E)$ is invariant under these K-endomorphisms.

(5.6.12) With the convention, that
$\forall a \in A, p \in \mathbf{N}, \lambda_p \in L^p(L,E), x \in E, \forall k_1, \ldots, k_p \in L:$

$$(a\lambda_p)(k_1, \ldots, k_p) := a\lambda_p(k_1, \ldots, k_p),$$

ax being defined in the sense of the graded A-left module E over K, $L^*(L,E)$ becomes some graded A-left module over K. Obviously

$$\forall a \in A, \lambda \in \Lambda_A^*(L,E) : a\lambda \in \Lambda_A^*(L,E).$$

(5.6.12.1) $\forall \bar{x}, \bar{z} \in \mathbf{Z}_2, \forall a \in A^{\bar{x}}, k \in L^{\bar{z}}, \forall \lambda \in L^*(L,E):$

$$\theta_0(k)(a\lambda) = (-1)^{xz} a(\theta_0(k)(\lambda)),\ \delta_0(a\lambda) = a\delta_0(\lambda),$$

$$i(k)(a\lambda) = (-1)^{xz} a(i(k)(\lambda)).$$

(5.6.12.2) Moreover one finds, that $\forall k \in L:$

$$Im\ \theta_\rho(k)|_{\Lambda_A^*(L,E)} \subseteq \Lambda_A^*(L,E),\ Im\ i(k)|_{\Lambda_A^*(L,E)} \subseteq \Lambda_A^*(L,E).$$

If the connection ρ is local, then $\forall k \in L : Im\ \delta_\rho|_{\Lambda_A^*(L,E)} \subseteq \Lambda_A^*(L,E)$.

(5.6.13) An appropriate total grading of $L^*(L,E)$, and accordingly of $End_K(L^*(L,E))$, is defined.

$$\forall p \in \mathbf{N}, \bar{y} \in \mathbf{Z}_2 : \lambda_p \in (L^p(L,E))^{\overline{y+p}}$$

$$\iff \forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2, \forall k_1 \in L^{\overline{z_1}}, \ldots, k_p \in L^{\overline{z_p}} :$$

$$\lambda_p(k_1, \ldots, k_p) \in E^{\overline{z_1 + \cdots + z_p + y}}.$$

The Lie superalgebra $(End_K(\Lambda^*(L,E)))_L$ is understood with respect to this total grading.

(5.6.13.1)

$$\forall \bar{z} \in \mathbf{Z}_2, k \in L^{\bar{z}} : i(k) \in (End_K(L^*(L,E)))^{\overline{z+1}},$$

$$\rho(k), \theta_0(k), \theta_\rho(k) \in (End_K(L^*(L,E)))^{\bar{z}},$$

$$\delta_0, \hat{\rho}, \delta_\rho \in (End_K(L^*(L,E)))^{\bar{1}}.$$

(5.6.13.2) The K-endomorphisms $\kappa(k,l), \hat{\kappa}$ of $L^*(L,E)$ are defined. $\forall p \in \mathbf{N}, \forall \bar{y}, \bar{x}, \bar{z}, \overline{z_1}, \ldots, \overline{z_{p+2}} \in \mathbf{Z}_2, \forall \lambda_p \in (L^p(L,E))^{\bar{y}}, x \in E,$
$\forall k \in L^{\bar{x}}, l \in L^{\bar{z}}, k_1 \in L^{\overline{z_1}}, \ldots, k_{p+2} \in L^{\overline{z_{p+2}}}$:

$$(\kappa(k,l)(\lambda_p))(k_1,\ldots,k_p) := (-1)^{p(x+z)}\kappa(k,l)(\lambda_p(k_1,\ldots,k_p)),$$
$$(\hat{\kappa}(\lambda_p))(k_1,\ldots,k_{p+2})$$
$$:= \sum_{1\le i<j\le p+2} (-1)^{y(z_i+z_j)+i+j+1+(z_i+z_j)(z_1+\cdots+z_{i-1})+z_j(z_{i+1}+\cdots+z_{j-1})}$$
$$\kappa(k_i,k_j)(\lambda_p(k_1,\ldots,\not{k}_i,\ldots,\not{k}_j,\ldots,k_{p+2})),$$
$$(\hat{\kappa}(\lambda_1))(k_1,k_2,k_3) = (-1)^{y(z_1+z_2)}\kappa(k_1,k_2)(\lambda_1(k_3))$$
$$+ (-1)^{y(z_1+z_3)+1+z_3z_2}\kappa(k_1,k_3)(\lambda_1(k_2))$$
$$+ (-1)^{y(z_2+z_3)+(z_2+z_3)z_1}\kappa(k_2,k_3)(\lambda_1(k_1)),$$
$$\hat{\kappa}(x)(k,l) := (-1)^{y(x+z)}\kappa(k,l)(x).$$

One immediately finds, that $\forall p \in \mathbf{N}_0, \forall \bar{x}, \bar{z}, \bar{y} \in \mathbf{Z}_2$:

$$L^{\bar{x}} \times L^{\bar{z}} \ni \{k,l\} \longrightarrow \kappa(k,l) : (L^p(L,E))^{\bar{y}} \longrightarrow (L^p(L,E))^{\overline{y+x+z}},$$
$$(L^p(L,E))^{\bar{y}} \underset{\hat{\kappa}}{\longrightarrow} (L^{p+2}(L,E))^{\bar{y}}.$$

Here the intrinsic grading of $L^*(L,E)$ is inserted. Obviously $\hat{\kappa}$ is even with respect to the total grading too.

(5.6.13.3) Using the total grading for the super-commutator of K-endomorphisms of $\Lambda^*(L,E)$ one finds, assuming henceforth *char* $K \neq 2$, that $\forall k,l \in L$:

$$[i(k),i(l)]\big|_{\Lambda^*(L,E)} = 0, \ [\theta_\rho(k),\theta_\rho(l)] - \theta_\rho([k,l]) - \kappa(k,l)\big|_{\Lambda^*(L,E)} = 0,$$
$$\delta_\rho \circ \delta_\rho - \hat{\kappa}\big|_{\Lambda^*(L,E)} = 0, \ [\delta_\rho, i(k)] - \theta_\rho(k)\big|_{\Lambda^*(L,E)} = 0,$$
$$[i(k),\theta_\rho(l)] - i([k,l])\big|_{\Lambda^*(L,E)} = 0, \ [\delta_\rho,\theta_\rho(k)] + [i(k),\hat{\kappa}]\big|_{\Lambda^*(L,E)} = 0.$$

(5.6.13.3.1) Inserting the connection $\rho := 0$, one obtains the subalgebra $K - lin\ span(\{i(k), \theta_0(k), \delta_0; k \in L\})$ of the Lie superalgebra $(End_K(\Lambda^*(L,E)))_L$, with the total grading of $\Lambda^*(L,E)$.

(5.6.13.3.2)

$$\forall k,l \in L : [\rho(k),\rho(l)] - \rho([k,l]) - \kappa(k,l)\big|_{\Lambda^*(L,E)} = 0,$$
$$[\theta_0(k),\rho(l)]\big|_{\Lambda^*(L,E)} = 0,$$
$$[\delta_0,\hat{\rho}] + \hat{\rho}\circ\hat{\rho} - \hat{\kappa}\big|_{\Lambda^*(L,E)} = 0,$$
$$[\hat{\rho}, i(k)] - \rho(k)\big|_{\Lambda^*(L,E)} = 0, \ [i(k),\rho(l)]\big|_{\Lambda^*(L,E)} = 0.$$

(5.6.14) Especially consider the vector space $\Lambda^*(L,A)$ over a field $K \supseteq \mathbf{Q}$, *char* $K = 0$, taking $E := A$, and using the product of A as module-multiplication too. The intrinsic grading ∂_0 and the total grading ∂ of $\Lambda^*(L,A)$ were defined above, such that

$$\textit{degree}\ \lambda_p(k_1,\ldots,k_p) = \sum_{j=1}^{p} \textit{degree}\ k_j + \partial_0\lambda_p,\ \partial_0\lambda_p + \bar{p} =: \partial\lambda_p,$$

for homogenous elements $\lambda_p \in \Lambda^p(L,A)$, and $k_1,\ldots,k_p \in L,\ p \in \mathbf{N}$.
On $\Lambda^0(L,A) := A$, the gradings ∂_0 and ∂ shall coincide with that of A itself.
On $\Lambda^1(L,A) = Hom_K(L,A), \partial_0$ is just the usual $\mathbf{Z}_2$-grading of homomorphisms.

(5.6.14.1) The A-connection ρ_0 defined below is obviously flat and local.

$$L \ni k \xrightarrow[\rho_0]{} \rho_0(k) : A \ni a \xrightarrow[def]{} ka \in A.$$

Obviously $\forall k \in L : \rho_0(k) \in (End_K(A))_L$.

(5.6.14.2) The K-endomorphisms d, L_k, and i_k of $\Lambda^*(L,A)$ are defined.
$\forall \bar{x}, \bar{z} \in \mathbf{Z}_2, \forall k \in L^{\bar{z}}, a \in A^{\bar{x}}$:

$$d - \delta_{\rho_0}\big|_{\bigoplus_{p\geq 1}\Lambda^p(L,A)} := 0,\ (da)(k) := (-1)^{xz}ka,$$

$$L_k := \theta_{\rho_0}(k)\big|_{\Lambda^*(L,A)},\ i_k := i(k)\big|_{\Lambda^*(L,A)}.$$

One immediately finds the subalgebra $K - lin\ span(\{d, L_k, i_k; k \in L\})$ of the Lie superalgebra $(End_K(\Lambda^*(L,A)))_L$, with respect to the total grading ∂.

$$\begin{aligned}\forall k,l \in L : [i_k, i_l] = 0,\ [L_k, L_l] = L_{[k,l]},\ d \circ d = 0,\\ [d, L_k] = 0,\ [d, i_k] = L_k,\ [i_k, L_l] = i_{[k,l]}.\end{aligned}$$

Remember here the total degrees and intrinsic degrees, for homogenous $k \in L$:

$$\partial_0 d = \bar{0},\ \partial_0 L_k = \partial_0 i_k = \textit{degree}\ k,\ \partial d = \bar{1},\ \partial L_k = \partial i_k + \bar{1} = \textit{degree}\ k.$$

(5.6.14.3) An appropriate wedge product is defined:

$$\prod^{2} \Lambda^*(L,A) \ni \{\lambda,\mu\} \longrightarrow \lambda \wedge \mu \in \Lambda^*(L,A),\ \text{such that}$$

$\forall p,q \in \mathbf{N}, \forall \bar{x}, \bar{y}, \overline{z_1},\ldots,\overline{z_{p+q}} \in \mathbf{Z}_2, \forall a \in A^{\bar{x}}, b \in A^{\bar{y}}$,
$\forall \lambda_p \in (\Lambda^p(L,A))^{\bar{x}}, \mu_q \in (\Lambda^q(L,A))^{\bar{y}}, k_1 \in L^{\overline{z_1}},\ldots,k_{p+q} \in L^{\overline{z_{p+q}}}$:

$$a \wedge b := ab,$$
$$(a \wedge \mu_q)(k_1, \ldots, k_q) := (-1)^{qx} a\mu_q(k_1, \ldots, k_q),$$
$$(\lambda_p \wedge b)(k_1, \ldots, k_p) := (-1)^{y(z_1+\cdots+z_p)} \lambda_p(k_1, \ldots, k_p)b,$$
$$(\lambda_p \wedge \mu_q)(k_1, \ldots, k_{p+q}) := (-1)^{qx} \frac{(p+q)!}{p!q!} (A_{p+q}\nu_{p+q})(k_1, \ldots, k_{p+q}),$$
$$\nu_{p+q}(k_1, \ldots, k_{p+q}) := (-1)^{y(z_1+\cdots+z_p)} \lambda_p(k_1, \ldots, k_p)\mu_q(k_{p+1}, \ldots, k_{p+q}),$$

with the intrinsic degrees $\bar{x}$ of λ_p, $\bar{y}$ of μ_q inserted, i.e.,

$$\textit{degree } \lambda_p(k_1, \ldots, k_p) = \overline{x + z_1 + \cdots + z_p},$$
$$\textit{degree } \mu_q(k_{p+1}, \ldots, k_{p+q}) = \overline{y + z_{p+1} + \cdots + z_{p+q}}.$$

This product explicitly reads, inserting the graded alternator A_{p+q},

$$(\lambda_p \wedge \mu_q)(k_1, \ldots, k_{p+q})$$
$$= \frac{(-1)^{qx}}{p!q!} \sum_{P \in \mathbf{P}_{p+q}} (-1)^{\tau_P + \sigma_P} \lambda_p(k_{j_1}, \ldots, k_{j_p})\mu_q(k_{j_{p+1}}, \ldots, k_{j_{p+q}}),$$
$$P := \begin{bmatrix} 1 \ldots p+q \\ j_1 \ldots j_{p+q} \end{bmatrix}, \ \sigma_P := \sum_{\{1 \le i < i' \le p+q; j_i > j_{i'}\}} z_{j_i} z_{j_{i'}} + y(z_{j_1} + \cdots + z_{j_p}),$$

with the intrinsic degree $\overline{x+y}$ of $\lambda_p \wedge \mu_q$.

(5.6.14.3.1)

$$(\lambda_1 \wedge \mu_1)(k_1, k_2) = (-1)^{x+yz_1} \lambda_1(k_1)\mu_1(k_2) - (-1)^{x+z_1z_2+yz_2} \lambda_1(k_2)\mu_1(k_1).$$

(5.6.14.4) This algebra $\Lambda^*(L, A)$ over K is associative, with the unit e_A. Moreover $\Lambda^*(L, A)$ is graded-commutative with respect to the total grading ∂ defined above. With the above notation this explicitly means, that

$$\partial(\lambda_p \wedge \mu_q) = \partial\lambda_p + \partial\mu_q, \ \lambda_p \wedge \mu_q = (-1)^{(x+p)(y+q)} \mu_q \wedge \lambda_p,$$
$$a \wedge \mu_q = (-1)^{x(y+q)} \mu_q \wedge a, \ a \wedge b = (-1)^{xy} b \wedge a.$$

(5.6.14.5) The algebra $\Lambda^*(L, A)$ over K is some bigraded differential algebra with the $\mathbf{Z}_2$-graded derivation d, with respect to the total grading ∂, and the obvious $\mathbf{N}_0$-grading.
$\forall p, q \in \mathbf{N}_0, \forall \lambda_p \in (\Lambda^p(L, A))^{\partial\lambda_p}, \mu_q \in (\Lambda^q(L, A))^{\partial\mu_q}$:

$$d\lambda_p \in \Lambda^{p+1}(L, A), \ \partial_0(d\lambda_p) = \partial_0\lambda_p, \ \partial(d\lambda_p) = \partial\lambda_p + \bar{1},$$
$$d(\lambda_p \wedge \mu_q) = (d\lambda_p) \wedge \mu_q + (-1)^{\partial\lambda_p} \lambda_p \wedge (d\mu_q), \ d \circ d = 0.$$

Obviously the $\mathbf{Z}_2$-grading ∂ is compatible with the $\mathbf{N}_0$-grading, by its very construction.

(5.6.14.6) $\forall \bar{z} \in \mathbf{Z}_2, k \in L^{\bar{z}}$:

$$i_k \in (Der_K(\Lambda^*(L,A)))^{\overline{z+1}},\ L_k \in (Der_K(\Lambda^*(L,A)))^{\bar{z}},$$

and d is some skew-derivation, with respect to the total grading ∂ of $\Lambda^*(L,A)$.

$$\theta_0(k)\big|_{\Lambda^*(L,A)} \in (Der_K(\Lambda^*(L,A)))^{\bar{z}} \ni \rho_0(k)\big|_{\Lambda^*(L,A)},$$

and $\delta_0\big|_{\Lambda^*(L,A)}$ is some skew-derivation, with the total grading ∂ of $\Lambda^*(L,A)$. Therefore $K - lin\ span(\{d, L_k, i_k; k \in L\})$ is some subalgebra of the Lie superalgebra

$$Der_K(\Lambda^*(L,A)) := \bigoplus_{\bar{z} \in \mathbf{Z}_2} (Der_K(\Lambda^*(L,A)))^{\bar{z}},$$

with respect to the total grading ∂.

(5.6.14.7) Obviously $\Lambda_A^*(L,A)$ is some unital subalgebra of $\Lambda^*(L,A)$ over K. For all $k \in L$, L_k, i_k, and d are reduced by $\Lambda_A^*(L,A)$. Therefore $\Lambda_A^*(L,A)$ is some bigraded differential algebra over K, which is graded-commutative.

(5.6.14.8) These A-valued differential forms are the images of elements of the non-unital universal differential envelope of A, due to its universal property, as is shown in the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\text{canonical embedding}"] \arrow[dr, "\alpha", "\text{canonical embedding}"'] & \Omega(A) \arrow[d, "\alpha_*"] \\
& \Lambda^*(L,A)
\end{tikzcd}$$

Here α_* is some homomorphism in the sense of bigraded differential algebras over K, which is uniquely determined by α. Therefore one finds, that
$\forall n \in \mathbf{N}, \forall a_0, \ldots, a_n \in A, \forall \gamma_0, \gamma_1 \in K$:

$$(\gamma_1 + \gamma_0 a_0) da_1 \cdots da_n \underset{\alpha_*}{\longrightarrow} (\gamma_1 + \gamma_0 a_0) da_1 \cdots da_n,$$

where the universal derivation of $\Omega(A)$ is also denoted by d, for convenience.

(5.6.15) Consider again a graded A-bimodule E over K, and the vector spaces $\Lambda^*(L,E)$ and $\Lambda^*(L,A)$ over $K \supseteq \mathbf{Q}$, $char\ K = 0$.

(5.6.15.1) $\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L, A), \bar{z} \in \mathbf{Z}_2, a \in A^{\bar{z}}, \forall k_1, \dots, k_p \in L:$

$$(a\lambda_p)(k_1, \dots, k_p) := (-1)^{pz} a\lambda_p(k_1, \dots, k_p).$$

Thereby using the wedge product, one obtains some graded A-bimodule $\Lambda^*(L, A)$ over K, with respect to the total grading ∂. One then establishes the K-linear mapping ν, due to the diagram below.

$$\begin{tikzcd}
E^{\bar{z}} \times \Lambda^p(L,A) \ni \{x, \lambda_p\} \arrow[r] \arrow[ddr, "def"'] & x \otimes \lambda_p \arrow[r] & x \otimes_A \lambda_p \in E \otimes_A \Lambda^*(L,A) \arrow[ddl, "\nu"] \\
& \{k_1, \dots, k_p\} \in \overset{p}{\times} L & \\
& \in \Lambda^*(L,E) \arrow[d] & \\
& (-1)^{pz} x \lambda_p(k_1, \dots, k_p) \in E &
\end{tikzcd}$$

Here $\forall x \in E, a \in A : \nu(x \otimes_A a) = xa \in E$.

(5.6.15.2) The next diagram allows for an establishment of the $\Lambda^*(L, A)$-right module $E \otimes_A \Lambda^*(L, A)$ over K, such that

$$\forall x \in E, \forall \lambda, \mu \in \Lambda^*(L, A) : (x \otimes_A \lambda)\mu = x \otimes_A (\lambda \wedge \mu).$$

$$\begin{tikzcd}
E \times \Lambda^*(L,A) \ni \{x, \lambda\} \arrow[r] \arrow[dr] & \arrow[r] & x \otimes_A \lambda \in E \otimes_A \Lambda^*(L,A) \arrow[dl] \\
& x \otimes_A (\lambda \wedge \mu) \in E \otimes_A \Lambda^*(L,A) &
\end{tikzcd}$$

Hence, since $\Lambda^*(L, A)$ is graded-commutative with respect to the total grading ∂, one constructs the graded $\Lambda^*(L, A)$-bimodule $E \otimes_A \Lambda^*(L, A)$ over K, with the total grading of $\Lambda^*(L, A)$.

(5.6.15.3) Let E be projective-finite, with the coordinatization $\{\varepsilon_k, \varepsilon^k; k = 1, \dots, n\}$. Then the $\Lambda^*(L, A)$-right module $E \otimes_A \Lambda^*(L, A)$ over K is also projective-finite.

(5.6.15.3.1) The universal property of the tensor product allows for an isomorphism α in the sense of graded A-bimodules over K, i.e., an even A-linear bijection

$$\alpha \in Hom_A^{\bar{0}}(A \otimes_A \Lambda^*(L,A), \Lambda^*(L,A)):$$
$$A \otimes_A \Lambda^*(L,A) \ni a \otimes_A \lambda \underset{\alpha}{\longleftrightarrow} a \wedge \lambda \in \Lambda^*(L,A).$$

(5.6.15.3.2) One then easily finds an appropriate coordinatization of $E \otimes_A \Lambda^*(L,A)$.

$$\tilde{\varepsilon}_k := T_A^2(\varepsilon_k, id\Lambda^*(L,A)) \circ \alpha^{-1}, \; \tilde{\varepsilon}^k := \alpha \circ T_A^2(\varepsilon^k, id\Lambda^*(L,A)),$$
$$\sum_{k=1}^{n} \tilde{\varepsilon}_k \circ \tilde{\varepsilon}^k = id\; E \otimes_A \Lambda^*(L,A).$$

(5.6.15.3.3) Since every element $f \in E \otimes_A \Lambda^*(L,A)$ is faithfully represented by

$$\dot{f} = \sum_{k=1}^{n} \tilde{\varepsilon}_k \circ f^k, \text{ and } f = (\dot{f}(e_A))^\bullet,$$

one concludes, that

$$\forall f \in E \otimes_A \Lambda^*(L,A): f = \sum_{k=1}^{n} \varepsilon_k(e_A) \otimes_A f^k(e_A).$$

(5.6.15.3.4) Obviously an appropriate coordinatization of $A \otimes_A \Lambda^*(L,A)$ is $\{\alpha^{-1}, \alpha\}$.

(5.6.5.4) Then $\Lambda^*(L,E)$ is established as some graded $\Lambda^*(L,A)$-right module over K.
$\forall \bar{x}, \bar{y}, \overline{z_1}, \dots, \overline{z_{p+q}} \in \mathbf{Z}_2, \forall k_1 \in L^{\overline{z_1}}, \dots, k_{p+q} \in L^{\overline{z_{p+q}}}$,
$\forall x \in E^{\bar{x}}, b \in A^{\bar{y}}, \lambda_p \in (\Lambda^p(L,E))^{\bar{x}}, \mu_q \in (\Lambda^q(L,A))^{\bar{y}}:$

$$(x\mu_q)(k_1,\dots,k_q) := (-1)^{qx} x\mu_q(k_1,\dots,k_q),$$
$$(\lambda_p b)(k_1,\dots,k_p) := (-1)^{y(z_1+\dots+z_p)} \lambda_p(k_1,\dots,k_p) b,$$
$$(\lambda_p\mu_q)(k_1,\dots,k_{p+q}) := \frac{(-1)^{qx}}{p!q!}$$
$$\sum_{\begin{bmatrix} 1 \dots p+q \\ j_1 \dots j_{p+q} \end{bmatrix} =: P \in \mathbf{P}_{p+q}} (-1)^{\tau_P + \sum_{\{1 \le k < l \le p+q : j_k > j_l\}} z_{j_k} z_{j_l} + y(z_{j_1} + \dots + z_{j_p})}$$
$$\lambda_p(k_{j_1}, \dots k_{j_p}) \mu_q(k_{j_{p+1}}, \dots, k_{j_{p+q}}),$$

with the intrinsic degrees $\bar{x}$ of λ_p, $\bar{y}$ of μ_q, and xb defined from E.

(5.6.15.4.1) An appropriate coordinatization of $\Lambda^*(L,E)$ is provided by the dual n-basis $\{\hat{\varepsilon}_k, \hat{\varepsilon}^k; k = 1, \dots, n\}$. For simplicity assume that the dual n-basis of E is homogeneous in the sense, that
$\forall_1^n k : \exists \overline{x_k} \in \mathbf{Z}_2 : \varepsilon_k \in Hom_A^{\overline{x_k}}(A,E),\ \varepsilon^k \in Hom_A^{\overline{x_k}}(E,A),\ \varepsilon_k \circ \varepsilon^k \in End_A^{\bar{0}}(E).$

$$\forall_1^n k, \forall p \in \mathbf{N} : \Lambda^p(L,E) \ni \lambda_p \xrightarrow[\hat{\varepsilon}^k]{def} (-1)^{px_k} \varepsilon^k \circ \lambda_p \in \Lambda^p(L,A),$$
$$\Lambda^p(L,A) \ni \mu_p \xrightarrow[\hat{\varepsilon}_k]{def} (-1)^{px_k} \varepsilon_k \circ \mu_p \in \Lambda^p(L,E),$$
$$\Lambda^0(L,E) \ni x \xrightarrow[\hat{\varepsilon}^k]{def} \varepsilon^k(x) \in \Lambda^0(L,A) \ni a \xrightarrow[\hat{\varepsilon}_k]{def} \varepsilon_k(a) \in \Lambda^0(L,E),$$
$$\hat{\varepsilon}^k \in Hom_{\Lambda^*(L,A)}(\Lambda^*(L,E), \Lambda^*(L,A)),$$
$$\hat{\varepsilon}_k \in Hom_{\Lambda^*(L,A)}(\Lambda^*(L,A), \Lambda^*(L,E)).$$

(5.6.15.4.2) $\forall f \in E \otimes_A \Lambda^*(L,A)$:

$$\dot{f} = \sum_{k=1}^n \tilde{\varepsilon}_k \circ f^k,\ f^k := \tilde{\varepsilon}^k \circ \dot{f} \in End_{\Lambda^*(L,A)}(\Lambda^*(L,A)).$$

$$\forall \lambda \in \Lambda^*(L,E) : \dot{\lambda} = \sum_{k=1}^n \hat{\varepsilon}_k \circ \lambda^k,\ \lambda^k := \hat{\varepsilon}^k \circ \dot{\lambda} \in End_{\Lambda^*(L,A)}(\Lambda^*(L,A)).$$

Here $\Lambda^*(L,A)$ is considered as some $\Lambda^*(L,A)$-right module over K.

(5.6.15.4.3) These coordinatizations are used in order to establish an appropriate isomorphism $\nu : E \otimes_A \Lambda^*(L,A) \longleftrightarrow \Lambda^*(L,E)$, which explicitly is decomposed according to:

$$\Lambda^*(L,E) \ni \lambda \longleftrightarrow \dot{\lambda} = \sum_{k=1}^n \hat{\varepsilon}_k \circ \lambda^k \underset{def}{\longleftrightarrow} \sum_{k=1}^n \tilde{\varepsilon}_k \circ \lambda^k =: \dot{f}$$
$$\longleftrightarrow f \in E \otimes_A \Lambda^*(L,A).$$

Obviously

$$\nu^{-1} \in Hom_{\Lambda^*(L,A)}(\Lambda^*(L,E), \Lambda^*(L,A)).$$

With the degrees $\overline{x_k}$ of ε_k and ε^k, one then finds that $\forall p \in \mathbf{N}$:

$$\Lambda^p(L,E) \ni \lambda_p \xrightarrow[\nu^{-1}]{} \sum_{k=1}^n (-1)^{px_k} \varepsilon_k(e_A) \otimes_A (\varepsilon^k \circ \lambda_p),$$

and especially:

$$E \ni x \xrightarrow[\nu^{-1}]{} \sum_{k=1}^n \varepsilon_k(e_A) \otimes_A \varepsilon^k(x) \in E \otimes_A A,$$

suppressing the involved embeddings.

(5.6.15.5) For the restriction $\nu|_{E\otimes_A \Lambda^*_A(L,A)}$ one immediately finds the following diagram.

$$\begin{tikzcd}
\Lambda^*(L,E) & \Lambda^*_A(L,E) \arrow[l] \arrow[r, leftrightarrow] & E\otimes_A \Lambda^*_A(L,A) \arrow[r] & E\otimes_A \Lambda^*(L,A) \arrow[lll, bend left, "\nu"]
\end{tikzcd}$$

(5.6.15.6) The usual K-endomorphisms of $\Lambda^*(L,E)$ can be calculated from those of $\Lambda^*(L,A)$, due to the diagram below.

$$\begin{tikzcd}
\Lambda^{p+1}(L,E) \arrow[rr, leftrightarrow, "\nu"'] \arrow[d, "i(k)"'] & & E\otimes_A \Lambda^{p+1}(L,A) \arrow[d, "\nu^{-1}\, o\, i(k)\, o\, \nu"] \\
\Lambda^{p}(L,E) \arrow[rr, leftrightarrow, "\nu^{-1}"'] & & E\otimes_A \Lambda^{p}(L,A)
\end{tikzcd}$$

$\forall \bar{x},\bar{z} \in \mathbf{Z}_2, \forall x \in E^{\bar{x}}, k \in L^{\bar{z}}, \lambda \in \Lambda^*(L,A):$

$x \otimes_A \lambda \underset{\nu^{-1}oi(k)o\nu}{\longrightarrow} (-1)^{x(1+z)} x \otimes_A i(k)(\lambda).$

$\forall x \in E, k \in L, a \in A : x \otimes_A a \underset{\nu}{\longrightarrow} xa \underset{i(k)}{\longrightarrow} 0.$

(5.6.15.6.1) $\forall \bar{x},\bar{z} \in \mathbf{Z}_2, \forall x \in E^{\bar{x}}, k \in L^{\bar{z}}, \lambda \in \Lambda^*(L,A), a \in A:$

$x \otimes_A \lambda \underset{\nu^{-1}o\theta_0(k)o\nu}{\longrightarrow} (-1)^{xz} x \otimes_A \theta_0(k)(\lambda),\ x \otimes_A a \underset{\theta_0(k)o\nu}{\longrightarrow} 0,$

$x \otimes_A \lambda \underset{\nu^{-1}o\delta_0 o\nu}{\longrightarrow} (-1)^{x} x \otimes_A \delta_0(\lambda),\ x \otimes_A a \underset{\delta_0 o\nu}{\longrightarrow} 0,$

$x \otimes_A \lambda \underset{\nu^{-1}o\rho(k)o\nu}{\longrightarrow} \rho(k)(x) \otimes_A \lambda + (-1)^{xz} x \otimes_A \rho_0(k)(\lambda),$

$x \otimes_A a \underset{\nu}{\longrightarrow} xa \underset{\rho(k)}{\longrightarrow} (\rho(k)(x))a + (-1)^{xz} x(ka),$

$(\rho(k)(x))a + (-1)^{xz} x(ka) \underset{\nu^{-1}}{\longrightarrow} \rho(k)(x) \otimes_A a + (-1)^{xz} x \otimes_A (ka),$

$x \otimes_A \lambda \underset{\nu^{-1}o\theta_\rho(k)o\nu}{\longrightarrow} \rho(k)(x) \otimes_A \lambda + (-1)^{xz} x \otimes_A \theta_{\rho_0}(k)(\lambda),$

$x \otimes_A a \underset{\nu^{-1}o\theta_\rho(k)o\nu}{\longrightarrow} \rho(k)(x) \otimes_A a + (-1)^{xz} x \otimes_A (ka),$

and moreover, $\forall p \in \mathbf{N}, \lambda_p \in \Lambda^p(L,A):$

$$x \otimes_A \lambda_p \underset{\nu^{-1}\circ\hat{\rho}\circ\nu}{\longrightarrow} (-1)^x x \otimes_A \hat{\rho}_0(\lambda_p) + \nu^{-1}(\rho_x \wedge \lambda_p),$$

$$x \otimes_A \lambda_p \underset{\nu^{-1}\circ\delta_\rho\circ\nu}{\longrightarrow} (-1)^x x \otimes_A \delta_{\rho_0}(\lambda_p) + \nu^{-1}(\rho_x \wedge \lambda_p),$$

$$x \otimes_A a \underset{\hat{\rho}\circ\nu}{\longrightarrow} 0,\ x \otimes_A a \underset{\delta_\rho\circ\nu}{\longrightarrow} 0,$$

with the notation $\rho_x \equiv \hat{\rho}(x)$, which explicitly means:

$$\Lambda^1(L, E) \ni \rho_x : L \ni k \underset{def}{\longrightarrow} (-1)^{xz}\rho(k)(x) \in E.$$

(5.6.15.7) Consider the graded $\Lambda^*(L, A)$-right module $\Lambda^*(L, E)$ over K. For $k \in L$, $i_k, L_k := \theta_{\rho_0}(k), \rho_0(k), \theta_0(k), \delta_0$, and d are super-derivations of $\Lambda^*(L, A)$. Accordingly, the corresponding K-endomorphisms of $\Lambda^*(L, E)$ are module-derivations of $\Lambda^*(L, E)$, in the sense of its total grading.
$\forall \bar{x}, \bar{z} \in \mathbf{Z}_2, \forall p, q \in \mathbf{N}$,
$\forall \lambda_p \in (\Lambda^p(L, E))^{\bar{x}}, \mu_q \in \Lambda^q(L, A), x \in E^{\bar{x}}, b \in A, k \in L^{\bar{z}}$:

$$\begin{aligned}
&i(k)(\lambda_p\mu_q) = i(k)(\lambda_p)\mu_q + (-1)^{(1+z)(x+p)}\lambda_p i(k)(\mu_q),\\
&\theta_0(k)(\lambda_p\mu_q) = \theta_0(k)(\lambda_p)\mu_q + (-1)^{z(x+p)}\lambda_p\theta_0(k)(\mu_q),\\
&\rho(k)(\lambda_p\mu_q) = \rho(k)(\lambda_p)\mu_q + (-1)^{z(x+p)}\lambda_p\rho_0(k)(\mu_q),\\
&\theta_\rho(k)(\lambda_p\mu_q) = \theta_\rho(k)(\lambda_p)\mu_q + (-1)^{z(x+p)}\lambda_p\theta_{\rho_0}(k)(\mu_q),\\
&\delta_0(\lambda_p\mu_q) = \delta_0(\lambda_p)\mu_q + (-1)^{x+p}\lambda_p\delta_0(\mu_q),\\
&\hat{\rho}(\lambda_p\mu_q) = \hat{\rho}(\lambda_p)\mu_q + (-1)^{x+p}\lambda_p\hat{\rho}_0(\mu_q),\\
&\delta_\rho(\lambda_p\mu_q) = \delta_\rho(\lambda_p)\mu_q + (-1)^{x+p}\lambda_p\delta_{\rho_0}(\mu_q),\\
&i(k)(\lambda_p b) = i(k)(\lambda_p)b,\ \theta_0(k)(\lambda_p b) = \theta_0(k)(\lambda_p)b,\\
&\rho(k)(\lambda_p b) = \rho(k)(\lambda_p)b + (-1)^{z(x+p)}\lambda_p(kb),\\
&\theta_\rho(k)(\lambda_p b) = \theta_\rho(k)(\lambda_p)b + (-1)^{z(x+p)}\lambda_p(kb),\\
&\delta_0(\lambda_p b) = \delta_0(\lambda_p)b,\ \hat{\rho}(\lambda_p b) = \hat{\rho}(\lambda_p)b + (-1)^{x+p}\lambda_p(db),\\
&\delta_\rho(\lambda_p b) = \delta_\rho(\lambda_p)b + (-1)^{x+p}\lambda_p(db),
\end{aligned}$$

with the total grading of $\Lambda^*(L, E)$, $\partial\lambda_p = \overline{x+p}$,

$$\begin{aligned}
&i(k)(xb) = 0,\ \theta_0(k)(xb) = 0,\ \delta_0(xb) = 0,\\
&\theta_\rho(k)(xb) = \rho(k)(xb) = \rho(k)(x)b + (-1)^{zx}x(kb),\\
&\rho_{xb} = \rho_x b + (-1)^x x(db),\\
&i(k)(x\mu_q) = (-1)^{(1+z)x}xi(k)(\mu_q),\ \theta_0(k)(x\mu_q) = (-1)^{zx}x\theta_0(k)(\mu_q),\\
&\rho(k)(x\mu_q) = \rho(k)(x)\mu_q + (-1)^{zx}x\rho_0(k)(\mu_q),\\
&\theta_\rho(k)(x\mu_q) = \rho(k)(x)\mu_q + (-1)^{zx}x\theta_{\rho_0}(k)(\mu_q),\\
&\delta_0(x\mu_q) = (-1)^x x\delta_0(\mu_q),\ \hat{\rho}(x\mu_q) = \rho_x\mu_q + (-1)^x x\hat{\rho}_0(\mu_q),\\
&\delta_\rho(x\mu_q) = \rho_x\mu_q + (-1)^x x(d\mu_q).
\end{aligned}$$

These module-derivations are easily derived from the corresponding K-endomorphisms of $E \otimes_A \Lambda^*(L, A)$, using the fact that $\forall x \in E, \forall \mu, \mu' \in \Lambda^*(L, A)$:

$$x(\mu \wedge \mu') = (x\mu)\mu' \underset{\nu^{-1}}{\longrightarrow} (x \otimes_A \mu)\mu' = x \otimes_A (\mu \wedge \mu').$$

5.7 Real Z_2-Graded Differential Forms

(5.7.1) Consider the alternating algebra

$$\Lambda(E) := \bigoplus_{p \in \mathbf{N}_0} \Lambda^p(E)$$

of the vector space E over a field K, *char* $K \neq 2$. For $p \in \mathbf{N}$, any two multi-indices $\{k_1, \ldots, k_p\}, \{l_1, \ldots, l_p\}$, each of them consisting of pairwise different indices $\in I$, card $I := \mathit{dim}\ E$, are called equivalent, if and only if each of them is some permutation of the other one. Selecting one representative from every equivalence class, one obtains the set $\hat{I}_p$. For instance, in case of

$$\mathit{dim}\ E = n \in \mathbf{N},\ \hat{I}_p := \{\{k_1, \ldots, k_p\}; 1 \leq k_1 < \ldots < k_p \leq n\},\ 1 \leq p \leq n.$$

If $\{\theta_k; k \in I\}$ is a basis of E, then $\left\{\theta_{k_1} \wedge \cdots \wedge \theta_{k_p}; \{k_1, \ldots, k_p\} \in \hat{I}_p\right\}$ is some basis of $\Lambda^p(E)$, for $p \in \mathbf{N}$.

(5.7.1.1) $\forall z \in \Lambda(E)$:

$$z = c_0 e + \sum_{\{k_1, \ldots, k_p\} \in \hat{I}_p, p \in \mathbf{N}} c_{k_1 \ldots k_p} \theta_{k_1 \ldots k_p},\ \theta_{k_1 \ldots k_p} := \theta_{k_1} \wedge \cdots \wedge \theta_{k_p},$$

with the unique coefficients $c_0, c_{k_1 \ldots k_p} \in K$, and only finitely many coefficients $\neq 0$. Here $c_0 e$ may be called the body, and $z - c_0 e$ the soul of $z \in \Lambda(E)$. An explicit notation of the unit e of $\Lambda(E)$ is conveniently suppressed.

(5.7.1.2) Obviously,

$$\mathit{dim}\ E = n \Longrightarrow \wedge^{n+1}(z - c_0) = 0.$$

Moreover, in any case,

$$\forall z \in \Lambda(E) : \exists r \in \mathbf{N} : \wedge^r (z - c_0) = 0.$$

(5.7.1.3) Therefore one finds, that

$$\forall z = c_0 + \mathit{soul}\ \in \Lambda(E) : c_0 \neq 0 \Longleftrightarrow \exists\ \mathit{unique}\ y \in \Lambda(E) : z \wedge y = e_K;$$

here e_K denotes the unit of K. In this case

$$c_0 y = e_K + \sum_{k \in \mathbf{N}} \bigwedge^k \left(\frac{z - c_0}{-c_0}\right), \text{ and then obviously } z \wedge y = e_K.$$

(5.7.1.4)

$$dim\ E = n \in \mathbf{N} \Longrightarrow \forall z \in \Lambda(E) : \lceil \forall_1^n k : \theta_k z = 0 \Longrightarrow z = c_{1\ldots n}\theta_{1\ldots n}\rceil.$$

$$dim\ E \notin \mathbf{N} \Longrightarrow \forall z \in \Lambda(E) : \lceil \forall k \in I : \theta_k z = 0 \Longrightarrow z = 0\rceil.$$

(5.7.1.5) The alternating algebra $\Lambda(E)$ is graded-commutative, with the $\mathbf{Z}_2$-grading according to the direct sum

$$\Lambda(E) = \bigoplus_{\bar{z}\in\mathbf{Z}_2} \Lambda^{\bar{z}}(E),\ \Lambda^{\bar{z}}(E) := \bigoplus_{p\in\mathbf{N}_0\cap\bar{z}} \Lambda^p(E),$$

hence being compatible with its $\mathbf{N}_0$-grading.

(5.7.1.6) Let K be equipped with an involution, i.e., an idempotent isomorphism of fields: $K \ni c \longleftrightarrow c^* \in K$, and accordingly $\Lambda(E)$ with the K-antilinear star operation: $\Lambda(E) \ni z \longleftrightarrow z^* \in \Lambda(E)$. $\forall z, y \in \Lambda(E), \forall c, d \in K$:

$$(cz + dy)^* = c^*z^* + d^*y^*,\ (z \wedge y)^* = y^* \wedge z^*,\ (z^*)^* = z.$$

Assume that $\forall k \in I : \theta_k^* = \theta_k$; then

$$\forall\{k_1, \ldots, k_p\} \in \hat{I}_p : \theta^*_{k_1\ldots k_p} = \theta_{k_p} \wedge \cdots \wedge \theta_{k_1} = (-1)^{\frac{p(p-1)}{2}}\theta_{k_1\ldots k_p}.$$

(5.7.1.7) This sign sometimes being rather inconvenient, one might prefer some graded star operation, such that
$\forall\bar{\xi}, \bar{\eta} \in \mathbf{Z}_2, \forall z \in \Lambda^{\bar{\xi}}(E), y \in \Lambda^{\bar{\eta}}(E), \forall c, d \in K$:

$$(cz + dy)^* = c^*z^* + d^*y^*,\ (z \wedge y)^* = (-1)^{\xi\eta}y^* \wedge z^*,\ (z^*)^* = z,$$

and $z^* \in \Lambda^{\bar{\xi}}(E)$. Furthermore, again assume $\forall k \in I : \theta_k^* = \theta_k$; then

$$\forall p \in \mathbf{N}, \{k_1, \ldots, k_p\} \in \hat{I}_p : \theta^*_{k_1\ldots k_p} = \theta_{k_1\ldots k_p},$$

and therefore $\forall z \in \Lambda(E)$:

$$z = z^* \Longleftrightarrow \lceil c_0 = c_0^*,\ \forall p \in \mathbf{N}, \{k_1, \ldots, k_p\} \in \hat{I}_p : c_{k_1\ldots k_p} = c^*_{k_1\ldots k_p}\rceil;$$

here $c_0, c_{k_1\ldots k_p}$ denote the unique components of z.

(5.7.1.8) Appropriate odd derivations of $\Lambda(E)$ are defined.
$\forall k \in I, \{k_1, \ldots, k_p\} \in \hat{I}_p$:

$$\frac{\partial}{\partial\theta_k}\theta_{k_1\ldots k_p} := \begin{cases} 0,\ k \notin \{k_1, \ldots, k_p\} \\ (-1)^{l+1}\theta_{k_1} \wedge \cdots \wedge \theta_{k_{l-1}} \wedge \theta_{k_{l+1}} \wedge \cdots \wedge \theta_{k_p},\ k = k_l \end{cases},$$

$$\frac{\partial}{\partial\theta_k}e := 0,$$

and the unique K-linear extensions, $\frac{\partial}{\partial\theta_k} \in Der_K^{\bar{1}}(\Lambda(E)) \subset End_K(\Lambda(E))$.
$\forall \bar{\xi} \in \mathbf{Z}_2, \forall z \in \Lambda^{\bar{\xi}}(E), y \in \Lambda(E), \forall k \in I$:

$$\frac{\partial}{\partial\theta_k}(z \wedge y) = \frac{\partial z}{\partial\theta_k} \wedge y + (-1)^{\xi} z \wedge \frac{\partial y}{\partial\theta_k}.$$

$$\forall k, l \in I : \frac{\partial}{\partial\theta_k} \circ \frac{\partial}{\partial\theta_l} + \frac{\partial}{\partial\theta_l} \circ \frac{\partial}{\partial\theta_k} = 0,$$

within the Lie superalgebra $Der_K(\Lambda(E))$.

(5.7.2) Consider a real m-dimensional C^∞-manifold $\mathbf{M}$, and take $K := \mathbf{R}$. The map $f : \mathbf{M} \longrightarrow \Lambda(E)$ is called infinitely differentiable, if and only if its components are smooth scalar fields:

$$\mathbf{M} \ni q \xrightarrow[f]{} f_0(q)e + \sum_{\{k_1,\ldots,k_p\} \in \hat{I}_p, p \in \mathbf{N}} f_{k_1\ldots k_p}(q)\theta_{k_1\ldots k_p} \in \Lambda(E),$$

with $f_0 \in C^\infty(\mathbf{M})$, $\forall p \in \mathbf{N}, \{k_1, \ldots, k_p\} \in \hat{I}_p : f_{k_1\ldots k_p} \in C^\infty(\mathbf{M})$.
The unit e of $\Lambda(E)$ shall be suppressed for convenience.

(5.7.2.1) The real vector space $\Lambda_{m,\varepsilon}(\mathbf{M})$ of smooth mappings

$$f : \mathbf{M} \longrightarrow \Lambda(E),\ dim\ E =: \varepsilon,$$

is an associative superalgebra over $\mathbf{R}$, the unit of which shows the image $\{1\}$. Moreover $\Lambda_{m,\varepsilon}(\mathbf{M})$ is graded-commutative.

$$\Lambda_{m,\varepsilon}(\mathbf{M}) = \Lambda^{\bar{0}}_{m,\varepsilon}(\mathbf{M}) \oplus \Lambda^{\bar{1}}_{m,\varepsilon}(\mathbf{M}),$$

$$\forall \bar{z} \in \mathbf{Z}_2 :\ \Lambda^{\bar{z}}_{m,\varepsilon}(\mathbf{M}) := \{f \in \Lambda_{m,\varepsilon}(\mathbf{M}); Im\ f \subseteq \Lambda^{\bar{z}}(E)\}.$$

$\forall \bar{z}, \bar{y} \in \mathbf{Z}_2, \forall f \in \Lambda^{\bar{z}}_{m,\varepsilon}(\mathbf{M}), g \in \Lambda^{\bar{y}}_{m,\varepsilon}(\mathbf{M}), \forall q \in \mathbf{M}$:

$$(f \wedge g)(q) := f(q) \wedge g(q) = (-1)^{zy}(g \wedge f)(q).$$

(5.7.2.2) $\forall f \in \Lambda_{m,\varepsilon}(\mathbf{M}), \forall X \in T(\mathbf{M}), \forall k \in I, \forall q \in \mathbf{M}$:

$$L_X f(q) := L_X f_0(q) + \sum_{\{k\} \in \hat{I}_p} L_X f_{\{k\}}(q)\theta_{\{k\}},$$

$$\frac{\partial}{\partial\theta_k} f(q) := \sum_{\{k\} \in \hat{I}_p} f_{\{k\}}(q) \frac{\partial}{\partial\theta_k}\theta_{\{k\}}.$$

Here obviously L_X and $\frac{\partial}{\partial\theta_k}$ are super-derivations of $\Lambda_{m,\varepsilon}(\mathbf{M})$, i.e.,

$$\forall X \in T(\mathbf{M}), \forall k \in I : L_X \in Der^{\bar{0}}_K(\Lambda_{m,\varepsilon}(\mathbf{M})),\ \frac{\partial}{\partial\theta_k} \in Der^{\bar{1}}_K(\Lambda_{m,\varepsilon}(\mathbf{M})).$$

The corresponding generalization of differential calculus is an easy application of the concept of sheaves over a topological space.

(5.7.3) Let T be a topological space. T is called ringed space, if and only if there are mappings:

$$\textit{topology } \tau \text{ of } T \ni V \longrightarrow \textit{ring } R_V,$$

$$\{V, W \in \tau; V \subseteq W\} \longrightarrow \textit{ring} - \textit{homomorphism } \rho_V^W : R_W \longrightarrow R_V,$$

such that the following properties hold:

(i) $\forall V \in \tau : \rho_V^V = id\ R_V.$

(ii) $\forall U, V, W \in \tau : U \subseteq V \subseteq W \Longrightarrow \rho_U^V \circ \rho_V^W = \rho_U^W.$

(iii) $\left.\begin{array}{l} \forall \alpha \in A : V_\alpha \in \tau,\ \bigcup_{\alpha \in A} V_\alpha =: V \\ f, g \in R_V,\ \forall \alpha \in A : \rho_{V_\alpha}^V(f) = \rho_{V_\alpha}^V(g) \end{array}\right\} \Longrightarrow f = g.$

(iv) $\left.\begin{array}{l} \forall \alpha \in A : V_\alpha \in \tau,\ \bigcup_{\alpha \in A} V_\alpha =: V,\ f_\alpha \in V_\alpha \\ \forall \alpha, \beta \in A : V_\alpha \cap V_\beta \neq \emptyset \Longrightarrow \rho_{V_\alpha \cap V_\beta}^{V_\alpha}(f_\alpha) = \rho_{V_\alpha \cap V_\beta}^{V_\beta}(f_\beta) \end{array}\right\}$

$$\Longrightarrow \exists f \in R_V : \forall \alpha \in A : f_\alpha = \rho_{V_\alpha}^V(f).$$

Here A denotes a set. The family $\{R_V; V \in \tau\}$ is then called sheaf over the topological space T.

(5.7.3.1) Some examples of sheaves over an m-dimensional real C^∞-manifold $\mathbf{M}$ are presented. $T := \mathbf{M}$.

$$\tau \ni V \longrightarrow C^\infty(V),\ \rho_V^W : C^\infty(W) \ni f \longrightarrow f\big|_V \in C^\infty(V).$$

$\tau \ni V \longrightarrow \Lambda_{m,\varepsilon}(V)$, with the restrictions of components.

$\tau \ni V \longrightarrow E(V)$, with the restrictions of smooth differential forms.

(5.7.4) Consider again the above defined manifold $\mathbf{M}$, with an atlas $\{\phi_\alpha; \alpha \in I\}$, $dom\ \phi_\alpha =: V_\alpha$, and take some chart $\phi_\alpha \equiv \phi$, $dom\ \phi =: V$. Let

$$dim\ E =: n \in \mathbf{N},\ \Lambda_n := \Lambda(E),\ \Lambda_{m,n}(V) =: \boldsymbol{\Lambda}_{mn}.$$

(5.7.4.1) Obviously $\boldsymbol{\Lambda}_{mn}$ is some free bimodule over the commutative ring $C^\infty(V)$, with the basis $\{\theta_{k_1 \ldots k_p}; 1 \leq k_1 < \ldots < k_p \leq n\} \cup \{e_{\Lambda_n}\}$.

(5.7.4.2) On the other hand, $\boldsymbol{\Lambda}_{mn}$ is some graded Λ_n-bimodule over $\mathbf{R}$, and moreover $\boldsymbol{\Lambda}_{mn}$ is some graded-commutative associative superalgebra over $\mathbf{R}$.

(5.7.4.3)

$$\Lambda_n \otimes C^\infty(V) \ni z \otimes f_0 \underset{real-linear}{\longleftrightarrow} z f_0 \in \boldsymbol{\Lambda}_{mn}.$$

(5.7.4.4) Obviously the following sequence of real vector spaces is exact:

$$\{0\} \longrightarrow \Lambda_{mn}^{nil} \underset{inclusion}{\longrightarrow} \Lambda_{mn} \underset{projection}{\longrightarrow} C^{\infty}(V) \longrightarrow \{0\}.$$

Here Λ_{mn}^{nil} denotes the ideal of nilpotent elements of $\Lambda_{mn} = C^{\infty}(V) \oplus \Lambda_{mn}^{nil}$. Moreover Λ_{mn}^{nil} is just the intersection of all the ideals J of Λ_{mn} such that $J \supseteq \Lambda_{mn}^{\bar{1}}$.

(5.7.4.5) $\forall_1^n i, k, \forall_1^m j, l$:

$$\frac{\partial}{\partial x^l} \in Der_{\mathbf{R}}^{\bar{0}}(\Lambda_{mn}), \ \frac{\partial}{\partial \theta_k} \in Der_{\mathbf{R}}^{\bar{1}}(\Lambda_{mn}),$$
$$\frac{\partial}{\partial x^l} \circ \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \circ \frac{\partial}{\partial x^l} = 0,$$
$$\frac{\partial}{\partial \theta_k} \circ \frac{\partial}{\partial \theta_i} + \frac{\partial}{\partial \theta_i} \circ \frac{\partial}{\partial \theta_k} = 0, \ \frac{\partial}{\partial x^l} \circ \frac{\partial}{\partial \theta_k} - \frac{\partial}{\partial \theta_k} \circ \frac{\partial}{\partial x^l} = 0.$$

Here $\frac{\partial}{\partial x^l}, l = 1, \ldots, m$, denote the usual Lie derivations with respect to the natural basis on the chart ϕ.

(5.7.4.6) As left modules:

$$\left\{\frac{\partial}{\partial x^l}; l = 1, \ldots, m\right\} \underset{free\ over\ C^{\infty}(V)}{\longrightarrow} Der_{\mathbf{R}}(C^{\infty}(V)),$$

$$\left\{\frac{\partial}{\partial \theta_k}; k = 1, \ldots, n\right\} \underset{free\ over\ \Lambda_n}{\longrightarrow} Der_{\mathbf{R}}(\Lambda_n),$$

with the corresponding module-multiplications defined quite naturally.

(5.7.4.7)

$$Der_{\mathbf{R}}(\Lambda_{mn}) = Der_{\Lambda_n}(\Lambda_{mn}) \oplus Der_{C^{\infty}(V)}(\Lambda_{mn}),$$
$$Der_{\Lambda_n}(\Lambda_{mn}) := \{\delta \in Der_{\mathbf{R}}(\Lambda_{mn}); \delta|_{\Lambda^n} = 0\},$$
$$Der_{C^{\infty}(V)}(\Lambda_{mn}) := \{\delta \in Der_{\mathbf{R}}(\Lambda_{mn}); \delta|_{C^{\infty}(V)} = 0\},$$

with an obvious suppression of the notation of embeddings.

(5.7.4.8) With the module-multiplication defined, such that

$$\forall f, g \in \Lambda_{mn}, \forall \delta \in Der_{\mathbf{R}}(\Lambda_{mn}) : (f\delta)(g) := f\delta(g),$$

one constructs the free left module $Der_{\mathbf{R}}(\Lambda_{mn})$ over Λ_{mn}, with the basis:

$$\left\{\frac{\partial}{\partial x^l}, \frac{\partial}{\partial \theta_k}; k = 1, \ldots, n; l = 1, \ldots, m\right\} \underset{free\ over\ \Lambda_{mn}}{\longrightarrow} Der_{\mathbf{R}}(\Lambda_{mn}).$$

(5.7.4.9) The $\mathbf{Z}_2$-graded Lie-Cartan pair $\{Der_{\mathbf{R}}(\Lambda_{mn}), \Lambda_{mn}\}$ is established with the real-bilinear mappings, such that

$$\forall f, g \in \Lambda_{mn}, \forall \delta \in Der_{\mathbf{R}}(\Lambda_{mn}) : \delta f := \delta(f),\ (f\delta)(g) := f\delta(g).$$

(5.7.4.10) The elements of the graded-commutative associative superalgebra $\Lambda^*(\mathbf{D}_{mn}, \Lambda_{mn})$, $\mathbf{D}_{mn} := Der_{\mathbf{R}}(\Lambda_{mn})$, are called $\mathbf{Z}_2$-graded differential forms. With respect to the generalized exterior derivation d defined previously for $\mathbf{Z}_2$-graded Lie-Cartan pairs, and with the total grading, both $\Lambda^*(\mathbf{D}_{mn}, \Lambda_{mn})$ and $\Lambda^*_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn})$ are bigraded differential algebras over $\mathbf{R}$. Moreover these algebras are graded Λ_{mn}-left modules over $\mathbf{R}$, with the module-multiplication defined such that
$\forall f \in \Lambda_{mn}, \forall p \in \mathbf{N}, \lambda \in \Lambda^p(\mathbf{D}_{mn}, \Lambda_{mn}), \forall \delta_1, \ldots, \delta_p \in \mathbf{D}_{mn}$:

$$(f\lambda)(\delta_1, \ldots, \delta_p) := f\lambda(\delta_1, \ldots, \delta_p).$$

(5.7.4.11) Every element $\lambda \in \Lambda^1_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn})$ is uniquely determined by its values on the basis $\left\{\frac{\partial}{\partial x^l}, \frac{\partial}{\partial \theta_k}; k = 1, \ldots, n; l = 1, \ldots, m\right\}$. $\forall_1^n k, i, \forall_1^m l, j$:

$$dx^l\left(\frac{\partial}{\partial x^j}\right) = \delta_{lj} e_{\Lambda_{mn}},\ d\theta_k\left(\frac{\partial}{\partial \theta_i}\right) = -\delta_{ki} e_{\Lambda_{mn}},$$

$$dx^l\left(\frac{\partial}{\partial \theta_i}\right) = d\theta_k\left(\frac{\partial}{\partial x^j}\right) = 0,$$

$$dx^l \in (\Lambda^1_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn}))^{\bar{1}},\ d\theta_k \in (\Lambda^1_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn}))^{\bar{0}},$$

with respect to the total grading, and therefore

$$dx^l \wedge dx^j = -dx^j \wedge dx^l,\ d\theta_k \wedge d\theta_i = d\theta_i \wedge d\theta_k,\ dx^l \wedge d\theta_k = d\theta_k \wedge dx^l.$$

(5.7.4.12) As some free Λ_{mn}-left module:

$$\{dx^l, d\theta_k; k = 1, \ldots, n; l = 1, \ldots, m\} \longrightarrow \Lambda^1_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn}).$$

(5.7.4.13) With the generalized exterior derivation used above, such that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall f \in \Lambda^{\bar{y}}_{mn}, \delta \in \mathbf{D}^{\bar{x}}_{mn} : df(\delta) := (-1)^{xy}\delta(f),$$

one immediately finds, that

$$df = \sum_{l=1}^{m} \frac{\partial f}{\partial x^l} dx^l + (-1)^{1+y} \sum_{k=1}^{n} \frac{\partial f}{\partial \theta_k} d\theta_k \in \Lambda^1_{\Lambda_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn}).$$

(5.7.4.14) As free $\boldsymbol{\Lambda}_{mn}$-left module, $\Lambda^*_{\boldsymbol{\Lambda}_{mn}}(\mathbf{D}_{mn}, \Lambda_{mn})$ is generated by the basis

$$\begin{aligned}&\{e_{\boldsymbol{\Lambda}_{mn}}, dx^{l_1} \wedge \cdots \wedge dx^{l_s}, d\theta_{k_1} \wedge \cdots \wedge d\theta_{k_r},\\ &dx^{l_1} \wedge \cdots \wedge dx^{l_s} \wedge d\theta_{k_1} \wedge \cdots \wedge d\theta_{k_r};\\ &1 \leq k_1 \leq \ldots \leq k_r \leq n, 1 \leq l_1 < \ldots < l_s \leq m\}.\end{aligned}$$

(5.7.4.15) The so-called Berezin integral may be defined as the monomial of odd derivations

$$\frac{\partial}{\partial\theta_n} \circ \cdots \circ \frac{\partial}{\partial\theta_1} : \boldsymbol{\Lambda}_{mn} \ni f \longrightarrow f_{1\ldots n} \in C^\infty(V) \underset{embedding}{\longrightarrow} \Lambda_{mn}.$$

5.8 Cohomologies of $\mathbf{Z}_2$-Graded Lie-Cartan Pairs

(5.8.1) Consider the $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, A\}$ over a field K, *char* $K \neq 2$. The odd derivation d of $\Lambda^*(L, A)$, with respect to the total grading, explicitly reads,
$\forall p \geq 1, \forall \bar{x}, \bar{y}, \overline{z_0}, \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2, \forall k_0 \in L^{\overline{z_0}}, \ldots, k_p \in L^{\overline{z_p}}$,
$\forall \lambda_p \in (\Lambda^p(L, A))^{\bar{y}}, \forall a \in A^{\bar{x}}$:

$$\begin{aligned}&da(k_0) := (-1)^{xz_0} k_0 a,\\ &(d\lambda_p)(k_0, k_1, \ldots, k_p)\\ &= \sum_{0\leq i<j\leq p} (-1)^{i+j+(z_i+z_j)(z_0+\cdots+z_{i-1})+z_j(z_{i+1}+\cdots+z_{j-1})}\\ &\lambda_p([k_i, k_j], k_0, \ldots, \not{k}_i, \ldots, \not{k}_j, \ldots, k_p)\\ &+\sum_{j=0}^{p}(-1)^{j+z_j(y+z_0+\cdots+z_{j-1})} k_j \lambda_p(k_0, \ldots, \not{k}_j, \ldots, k_p),\\ &(d\lambda_1)(k_0, k_1) = -\lambda_1([k_0, k_1])\\ &\qquad + (-1)^{z_0 y} k_0 \lambda_1(k_1) - (-1)^{z_1(y+z_0)} k_1 \lambda_1(k_0).\end{aligned}$$

The super-derivations i_{k_0} and L_{k_0} of $\Lambda^*(L, A)$ explicitly read, with the same notation of degrees as above,

$$\begin{aligned}&(i_{k_0}(\lambda_{p+1}))(k_1, \ldots, k_p) = (-1)^{(p+1+y)z_0} \lambda_{p+1}(k_0, k_1, \ldots, k_p),\\ &i_{k_0}(\lambda_1) = (-1)^{(1+y)z_0} \lambda_1(k_0), i_{k_0}(a) = 0,\\ &(L_{k_0}(\lambda_p))(k_1, \ldots, k_p)\\ &= (-1)^{1+pz_0} \sum_{j=1}^{p} (-1)^{z_0(y+z_1+\cdots+z_{j-1})} \lambda_p(k_1, \ldots, k_{j-1}, [k_0, k_j], k_{j+1}, \ldots, k_p)\\ &+ (-1)^{pz_0} k_0 \lambda_p(k_1, \ldots, k_p),\end{aligned}$$

$$(L_{k_0}(\lambda_1))(k_1) = (-1)^{1+z_0+z_0 y}\lambda_1([k_0, k_1]) + (-1)^{z_0} k_0 \lambda_1(k_1),$$
$$L_{k_0}(a) = k_0 a.$$

(5.8.2) As indicated previously, $K - lin\ span(\{d, L_k, i_k; k \in L\})$ is some subalgebra of the Lie superalgebra $Der_K(\Lambda^*(L, A))$, with respect to the total grading of $\Lambda^*(L, A)$.

(5.8.3) The elements of $\Lambda^*(L, A)$ are called A-cochains of L, those of $ker\ d$ are called A-cocycles of L, and those of $Im\ d$ are called A-coboundaries of L. Obviously L_k turns A-cocycles into A-coboundaries of L, because $L_k = [d, i_k]$ for $k \in L$.

(5.8.4) The cohomologies of the $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, A\}$ are defined with respect to d, according to the complex of vector spaces over K:

$$\cdots \{0\} \underset{d^{-1}}{\longrightarrow} A \underset{d^0}{\longrightarrow} \Lambda^1(L, A) \longrightarrow \cdots \longrightarrow \Lambda^p(L, A) \underset{d^p}{\longrightarrow} \Lambda^{p+1}(L, A) \longrightarrow \cdots.$$

$$\forall p \in \mathbf{Z} : H^p(L, A) := ker\ d^p / Im\ d^{p-1}.$$

(5.8.4.1) The $\mathbf{Z}_2$-graded Lie-Cartan pair $\{L, A\}$ is degenerate, if and only if $H^0(L, A) = A/\{0\}$.

(5.8.5) More generally, one defines the cohomologies

$$H^p_\rho(L, E) := ker\ \delta^p_\rho / Im\ \delta^{p-1}_\rho,\ p \in \mathbf{Z},$$

with respect to the complex of vector spaces $\Lambda^p(L, E)$ over K, defining

$$\Lambda^0(L, E) := E,\ \forall q \leq -1 : \Lambda^q(L, E) := \{0\},$$

and correspondingly the natural restrictions $\delta^p_\rho : \Lambda^p(L, E) \longrightarrow \Lambda^{p+1}(L, E)$ of δ_ρ. Here the E-connection ρ is assumed to be flat, such that $\delta_\rho \circ \delta_\rho = 0$.

(5.8.5.1) Obviously $\theta_\rho(k)$ turns E-cocycles into E-coboundaries, generalizing straightforward the notation from the case of trivial $\mathbf{Z}_2$-grading.

(5.8.5.2) In the case of $\rho := 0$, $H^0_0(L, E) \longleftrightarrow E$,

$$H^1_0(L, E) \longleftrightarrow \{\lambda_1 \in \Lambda^1(L, E); \forall k_1, k_2 \in L : \lambda_1([k_1, k_2]) = 0\}.$$

(5.8.6) Especially choose $A := K$, and moreover, $\forall a \in A, k \in L : ka := 0$. With the module-multiplication of L inserted for the K-bilinear map: $A \times L \longrightarrow L$, the defining conditions on $\mathbf{Z}_2$-graded Lie-Cartan pairs are reduced to the zero-map: $L \times A \longrightarrow \{0\} \longrightarrow A$. Then every E-connection ρ is local. In case of $\rho := 0$, one finds the E-valued Chevalley cohomologies $H^p_0(L, E)$ with respect to δ_0. In the special case of $E := A := K$, $\delta_0 = d$.

6. Real Lie-Hopf Superalgebras

Several concepts have been developed in order to construct fermionic analogues of finite-dimensional real differentiable manifolds and real Lie groups, aiming at the description of physical systems with both bosonic and fermionic degrees of freedom, by means of additional nilpotent odd coordinates.

The approach by B. DeWitt (1984) and A. Rogers (1986), the crucial point of which concerns the notion of superdifferentiable functions of countably many Grassmann variables, does not fit into a suitable axiomatic scheme of a theory of supermanifolds, as it was proposed on a sheaf-theoretic level by M. J. Rothstein (1986).

The first attempts to construct some consistent framework of supergeometry are due to F. A. Berezin and D. A. Leites (1975), D. A. Leites (1980), and B. Kostant (1977), replacing the sheaf of commutative unital associative algebras of real-valued smooth scalar fields by the sheaf of graded-commutative unital associative algebras of smooth functions on the domains of charts of an atlas, with their values in the exterior algebra over some finite-dimensional real vector space. This sheaf-theoretic approach is performed in the detailed presentations of a theory of supermanifolds by Yu. I. Manin (1988), C. Bartocci, U. Bruzzo, D. Hernández-Ruipérez (1991), and F. Constantinescu, H. F. de Groote (1994).

There is up to now no satisfactory answer to the crucial question:

What is a Lie supergroup?

The concept of real Lie-Hopf superalgebras, which was developed by B. Kostant (1977), may serve for a precise version of the heuristic attempts by V. Rittenberg and M. Scheunert (1978). Lie superalgebras and Lie supergroups of supermatrices are treated extensively by J. F. Cornwell ((1989). The so-called classical Lie supergroups are classified by H. Boseck (1990) by means of his theory of affine Lie supergroups (1987). Another proposal to define a Lie supergroup and its action on a supermanifold can be found in the above-mentioned monograph by C. Bartocci et al. (1991).

For the classical theory of Lie groups and their Lie algebras the reader is referred to the corresponding volumes by N. Bourbaki (1989, 1968), J. Dieudonné (1974), and for instance to the volume edited by A. L. Onishchik (1993). The underlying theory of differentiable manifolds is presented systematically by N. Bourbaki (1983), J. Dieudonné (1972, 1974), and also in several monographs, for instance that by S. Lang (1985), and in three volumes by W. Greub, St. Halperin, R. Vanstone (1972, 1973, 1976). Infinite-dimensional real differentiable manifolds are treated for example in the monograph by R. Abraham, J. E. Marsden, T. Ratiu (1983).

$\mathbf{Z}_2$-graded distributions with finite support on a real Lie group are presented here as an instructive example of real Lie-Hopf superalgebras, and in particular of dualizing topological Hopf superalgebras. The latter aspect is one of the categorial sources of the theory of quantum groups.

6.1 Linear Forms on Graded-Commutative Associative Superalgebras

(6.1.1) Let A be a real graded-commutative associative superalgebra with the unit e_A, and consider the dual vector spaces

$$A^* := Hom_{\mathbf{R}}(A, \mathbf{R}), \quad (A \otimes A)^* := Hom_{\mathbf{R}}(A \otimes A, \mathbf{R}),$$

with the natural $\mathbf{Z}_2$-grading of A^*. Due to the universal property of tensor products, $\forall f, g \in A^* : \exists \phi \in (A \otimes A)^* : \forall a, b \in A :$

$$\phi(a \otimes b) = \sum_{\bar{x}, \bar{y} \in \mathbf{Z}_2} (-1)^{xy} f(a^{\bar{x}}) g^{\bar{y}}(b).$$

One then establishes an injection ν, according to the subsequent diagram.

$$\begin{tikzcd}
A^* \times A^* \arrow[rr] & & A^* \otimes A^* \arrow[d, "\nu"] \\
\{f,g\} \arrow[r, "def"'] & \phi \in (A \otimes A)^* & \arrow[l]
\end{tikzcd}$$

(6.1.2) The definition of a real-linear map: $A^* \ni f \longrightarrow \Delta(f) \in (A \otimes A)^*$ is indicated in the next diagram.

$$\begin{tikzcd}
A \times A \arrow[rr] & & A \otimes A \arrow[d, "\Delta(f)"] \\
\{a,b\} \arrow[r, "def"'] & f(ab) \in \mathbf{R} & \arrow[l]
\end{tikzcd}$$

(6.1.3) Let J be a graded ideal of A. Then $(A \otimes J) + (J \otimes A)$ is some graded ideal of $A \otimes A$, and the following isomorphism of real unital associative superalgebras is established:

$$\frac{A \otimes A}{(A \otimes J) + (J \otimes A)} \ni [a' \otimes a''] \longleftrightarrow [a'] \otimes [a''] \in \frac{A}{J} \otimes \frac{A}{J},$$

inserting the tensor product of real associative algebras, or else the skew-symmetric tensor product of real associative superalgebras.

(6.1.4) One easiliy finds the real-linear bijection:

$$\{f \in A^*; f\big|_J = 0\} =: A_J^* \longleftrightarrow \left(\frac{A}{J}\right)^* := Hom_{\mathbf{R}}\left(\frac{A}{J}, \mathbf{R}\right),$$

due to the next diagram.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & R \\ & A/J \arrow[ur] & \end{tikzcd}$$

(6.1.5) Because of the implications, that

$$\forall f \in A^* : f\big|_J = 0 \Longleftrightarrow \Delta(f)\big|_{(A \otimes J) + (J \otimes A)} = 0,$$

one is led to the real-linear injective map:

$$Im\ \Delta\big|_{A_J^*} \longrightarrow \left(\frac{A}{J} \otimes \frac{A}{J}\right)^* := Hom_{\mathbf{R}}\left(\frac{A}{J} \otimes \frac{A}{J}, \mathbf{R}\right),$$

with the aid of the subsequent diagram, which is commutative $\forall f \in A_J^*$.

$$\begin{tikzcd} A \otimes A \arrow[rr, "\Delta(f)"] \arrow[d] & & R \\ \dfrac{A \otimes A}{(A \otimes J) + (J \otimes A)} \arrow[r, leftrightarrow] & A/J \otimes A/J \arrow[ur] & \end{tikzcd}$$

(6.1.6) Let especially A/J be finite-dimensional, such that one can make use of the real-linear bijection:

$$\left(\frac{A}{J}\right)^* \otimes \left(\frac{A}{J}\right)^* \ni \phi \otimes \psi \underset{def}{\longleftrightarrow} \xi \in \left(\frac{A}{J} \otimes \frac{A}{J}\right)^* :$$

$$\frac{A}{J} \otimes \frac{A}{J} \ni [a] \otimes [b] \underset{\xi}{\longrightarrow} \phi([a])\psi([b]) \in \mathbf{R}.$$

In this case one defines an appropriate comultiplication Δ_J by means of the following real-linear mappings:

$$A_J^* \ni f \longleftrightarrow \tilde{f} \in \left(\frac{A}{J}\right)^*,$$

$$\Delta_J : \left(\frac{A}{J}\right)^* \ni \tilde{f} \underset{def}{\longrightarrow} \sum_{l=1}^{k} \phi_l \otimes \psi_l \in \left(\frac{A}{J}\right)^* \otimes \left(\frac{A}{J}\right)^*,$$

$$\forall a, b \in A : f(ab) = \sum_{l=1}^{k} \phi_l([a])\psi_l([b]).$$

(6.1.7) The comultiplication Δ_J defined above is indeed coassociative. For an easy proof one needs the bijective structure map of $\mathbf{R}$, and the linear bijection:

$$\left(\left(\frac{A}{J}\right)^* \otimes \left(\frac{A}{J}\right)^*\right) \otimes \left(\frac{A}{J}\right)^* \ni (\phi \otimes \psi) \otimes \xi$$
$$\longleftrightarrow T(T(\phi, \psi), \xi) \in Hom_{\mathbf{R}}\left(\left(\frac{A}{J} \otimes \frac{A}{J}\right) \otimes \frac{A}{J}, (\mathbf{R} \otimes \mathbf{R}) \otimes \mathbf{R}\right),$$

due to the finite dimension of A/J.

(6.1.8) With the counit

$$\epsilon_J : \left(\frac{A}{J}\right)^* \ni \phi \underset{def}{\longrightarrow} \phi([e_A]) \in \mathbf{R}$$

one establishes the graded-cocommutative coalgebra $\left(\frac{A}{J}\right)^*$ over $\mathbf{R}$, because of $\Delta_J = \tau \circ \Delta_J$, with respect to the natural $\mathbf{Z}_2$-gradings, and the $\mathbf{Z}_2$-graded flip

$$\tau : \left(\frac{A}{J}\right)^* \otimes \left(\frac{A}{J}\right)^* \ni \phi \otimes \psi \longleftrightarrow (-1)^{xy} \psi \otimes \phi \in \left(\frac{A}{J}\right)^* \otimes \left(\frac{A}{J}\right)^*,$$

for homogeneous elements ϕ, ψ of degrees $\bar{x}, \bar{y}$, respectively.

(6.1.9) Consider a homomorphism of real unital associative superalgebras $\alpha : A_2 \longrightarrow A_1$, and the corresponding real-linear map $\alpha^* : A_1^* \longrightarrow A_2^*$, such that

$$\forall a_2 \in A_2, f_1 \in A_1^* : (\alpha^*(f_1))(a_2) = f_1 \circ \alpha(a_2).$$

Let J_1 be a graded ideal of A_1, the codimension $dim\ A_1/J_1$ of which is finite. Then $J_2 := \alpha^{-1}(J_1)$ is some graded ideal of A_2 with finite codimension, due to an injective homomorphism of real unital associative superalgebras:

$$\frac{A_2}{J_2} \ni [a_2] \longrightarrow [\alpha(a_2)] \in \frac{A_1}{J_1}.$$

$$\forall f_1 \in A_1^* : f_1\big|_{J_1} = 0 \Longrightarrow \alpha^*(f_1)\big|_{J_2} = 0.$$

Hence one obtains some homomorphism α_{co} of $\mathbf{Z}_2$-graded coalgebras over $\mathbf{R}$:

$$\left(\frac{A_1}{J_1}\right)^* \ni \phi_1 \underset{\alpha_{co}}{\longrightarrow} \phi_2 \in \left(\frac{A_2}{J_2}\right)^*,$$

$$\forall a_2 \in A_2 : \phi_2([a_2]) := \phi_1([a_1]),\ \ a_1 := \alpha(a_2).$$

Here one again uses the real-linear bijection:

$$\left(\frac{A_k}{J_k}\right)^* \otimes \left(\frac{A_k}{J_k}\right)^* \longleftrightarrow \left(\frac{A_k}{J_k} \otimes \frac{A_k}{J_k}\right)^*, \text{ for } k = 1, 2.$$

(6.1.9.1) Especially take $\phi_1, \psi_1 \in \left(\frac{A_1}{J_1}\right)^*$, let ϕ_1 be group-like, and ψ_1 primitive with respect to ϕ_1. Then $\alpha_{co}(\phi_1)$ is also group-like, and $\alpha_{co}(\psi_1)$ is primitive with respect to $\alpha_{co}(\phi_1)$.

6.2 Real Lie-Hopf Superalgebras

(6.2.1) Let G be a real Lie group, and $L = L^{\bar{0}} \oplus L^{\bar{1}}$ a real Lie superalgebra. Their smash product is constructed, i.e., the real Hopf superalgebra $\{F_\pi, \mu_\pi, \eta_\pi, \Delta_\pi, \epsilon_\pi\}$, $F_\pi := \mathbf{R}(G) \otimes_\pi E(L)$, with the representation π of G on $E(L)$ indicated in the next diagram.

$$\begin{tikzcd}
G \ni g \arrow[r] \arrow[d, no head] & \pi(g) : E(L) \arrow[r, leftrightarrow] & E(L) \\
{} \arrow[r] & \pi_L(g) : L \arrow[u] \arrow[r, leftrightarrow] & L \arrow[u]
\end{tikzcd}$$

Here an automorphism in the sense of real Lie superalgebras $\pi_L(g)$ is lifted to an automorphism in the sense of real unital associative superalgebras $\pi(g)$, ascending from L to the universal enveloping superalgebra $E(L)$.

(6.2.2) The real Lie algebra of left-invariant vector fields on G is inserted for the even subspace of L, such that $L^{\bar{0}} := L(G)$, $dim\ L^{\bar{0}} =: m \in \mathbf{N}$.

(6.2.3) Let $L^{\bar{1}}$ be finite-dimensional, $dim\ L^{\bar{1}} =: n \in \mathbf{N}, dim\ L = m+n$. The representation π_L is assumed to be a homomorphism of real Lie groups, in the sense of the chart with respect to a homogeneous vector basis for L:

$$\mathbf{A} := Aut_{\mathbf{R}}(L) \longleftrightarrow GL(m+n, \mathbf{R}) \longleftrightarrow \underset{open\ subset}{\longrightarrow} \mathbf{R}^{(m+n)^2}.$$

Here $Aut_{\mathbf{R}}(L)$ denotes the real vector space of real-linear bijections of L onto itself.

(6.2.4) The real Lie algebra $L(\mathbf{A})$ of left-invariant vector fields on $\mathbf{A}$ is identified with the real endomorphisms of L itself, by means of the real-linear bijection β defined in the diagram below.

$$\begin{tikzcd}
L(\mathbf{A}) \arrow[rr, leftrightarrow, "\beta"', "def"'] \arrow[d, leftrightarrow] & & End_R(L) \arrow[d, leftrightarrow] \\
T_{id\ L}(\mathbf{A}) \arrow[rr, leftrightarrow, "bundle\ \ chart"'] & & Mat(m+n, R)
\end{tikzcd}$$

Here $T_{id(L)}(\mathbf{A})$ denotes the tangent space of $\mathbf{A}$ at the unit $id(L)$ of $\mathbf{A}$. The real-linear bijection β obviously does not depend on the choice of vector basis for L.

(6.2.5) The real-linear map α defined in the subsequent diagram is used in order to write down the assumption (ii) below.

$$\begin{tikzcd}
G \ni exp(X) \arrow[r] & \pi_L \circ exp(X) \in Aut_R(L) & \\
L(G) \ni X \arrow[u, "exp"'] \arrow[r, "\alpha", "def"'] \arrow[d, leftrightarrow] & \alpha(X) \in L(\mathbf{A}) \arrow[u, "exp"'] \arrow[r, leftrightarrow, "\beta"] \arrow[d, leftrightarrow, "def"'] & End_R(L) \\
T_{e_G}(G) \ni [u_0] \arrow[r, "def"'] & [\pi_L \circ u_0] \in T_{id\ L}(\mathbf{A}) &
\end{tikzcd}$$

(6.2.6) One then assumes that, on the connected component of the unit e_G of G, π_L is an exponentiation of the adjoint representation of $L^{\bar{0}}$, in the sense of (i) and (ii) below.

(i) $\forall q \in G, X \in L^{\bar{0}} : \pi_L(q)(X) = (Ad\ q)(X) := \rho(q)_*(X)$.

(ii) $\forall X \in L^{\bar{0}}, Y \in L : (\beta \circ \alpha(X))(Y) = [X,Y] = L_X(Y^{\bar{0}}) + [X, Y^{\bar{1}}]$,

inserting the product of L, or equivalently, the super-commutator of $E(L)$. Here $Ad: \ G \longrightarrow Aut_{\mathbf{R}}(L(G))$ denotes the adjoint representation of G, which is induced by the inner automorphism of real Lie groups $\rho(q)$, the latter acting as:

$$G \ni q \longrightarrow \rho(q) : G \ni p \underset{def}{\longrightarrow} qpq^{-1} \in G.$$

Moreover L_X denotes the Lie derivation with respect to $X \in L^{\bar{0}}$.

(6.2.7) A triple $\{G, L, \pi\}$ with the properties assumed above is called real Lie-Hopf superalgebra.

(6.2.8) Let $\{G_k, L_k, \pi_k\}, k = 1,2$, be objects of this type, with the corresponding real Hopf superalgebras, $F_{\pi_k} := \mathbf{R}(G_k) \otimes_{\pi_k} E(L_k), k = 1,2$. A homomorphism of real Hopf superalgebras $\phi : F_{\pi_1} \longrightarrow F_{\pi_2}$ is called homomorphism of real Lie-Hopf superalgebras, if and only if there is a homomorphism $\gamma : G_1 \longrightarrow G_2$ in the sense of Lie groups, such that

$$\forall q_1 \in G_1, X_1 \in L_1^{\bar{0}} : q_1 \otimes X_1 \underset{\phi}{\longrightarrow} \gamma(q_1) \otimes X_2, \ \ q_1 \otimes e_{E_1} \underset{\phi}{\longrightarrow} \gamma(q_1) \otimes e_{E_2},$$

according to the diagram below. Here one denotes $L_k^{\bar{0}} := L(G_k), k = 1,2$.

$$\begin{tikzcd}
L(G_1) \ni X_1 \arrow[r, dash, dashed] \arrow[d, leftrightarrow] & X_2 \in L(G_2) \arrow[d, leftrightarrow] \\
T_{e_{G_1}}(G_1) \ni [u_1] \arrow[r, "def"'] & [\gamma \circ u_1] \in T_{e_{G_2}}(G_2)
\end{tikzcd}$$

One easily verifies that this assumption is compatible with the previous ones.

(6.2.8.1) One then finds, that $\forall n \in \mathbf{N}$, $\forall q_1 \in G_1, \forall X_{11}, \ldots, X_{1n} \in L_1^{\bar{0}}$:

$$q_1 \otimes (X_{11} \cdots X_{1n}) \underset{\phi}{\longrightarrow} \gamma(q_1) \otimes (X_{21} \cdots X_{2n}).$$

(6.2.8.2) Especially ϕ is called an isomorphism in the sense of real Lie-Hopf superalgebras, if and only if ϕ and hence γ are bijective.

6.3 Z_2-Graded Distributions with Finite Support

(6.3.1) Consider the real vector space $A^*_{fin} \equiv A^*_{fin}(G)$ of distributions with finite support on an m-dimensional real Lie group G. The commutative unital associative algebra of smooth real scalar fields is denoted by

$$A := C^\infty(G), \ \ A^* := Hom_{\mathbf{R}}(A, \mathbf{R}).$$

$$A^*_{fin} := \Big\{ \phi \in A^*; \exists p \in \mathbf{N}, \exists q_1, \ldots, q_p \in G,$$

$$\exists \ coefficients \ \Big\{ c^r_{k_1 \ldots k_m} \in \mathbf{R}; k_1, \ldots k_m \in \mathbf{N}_0; r = 1, \ldots, p \Big\} :$$

$$only \ finitely \ many \ coefficients \ \neq 0,$$

$$\forall f \in A : \phi(f) = \sum_{r=1}^{p} \sum_{k_1, \ldots, k_m \in \mathbf{N}_0} c^r_{k_1 \ldots k_m} L^{k_1}_{X_1} \circ \cdots \circ L^{k_m}_{X_m}(f)(q_r) \Big\},$$

inserting a basis $\{X_1, \ldots, X_m\}$ of the real Lie algebra $L(G)$ of left-invariant vector fields on G. The set $supp \ \phi := \{q_1, \ldots, q_p\}$ is called support of ϕ.

(6.3.2) Due to the Poincaré-Birkhoff-Witt theorem, the set

$$\{X_1^{k_1} \ldots X_m^{k_m}; k_1, \ldots, k_m \in \mathbf{N}_0\}$$

is some $\mathbf{R}$-basis of the universal enveloping algebra $E(L(G))$. Correspondingly

$$A^*_m(G) := \mathbf{R}(\{L^{k_1}_{X_1} \circ \cdots \circ L^{k_m}_{X_m}; k_1, \ldots, k_m \in \mathbf{N}_0\})$$

is the universal enveloping algebra of the real Lie algebra of Lie derivations $\mathbf{R}(\{L_{X_k}; k = 1, \ldots, m\})$, as subalgebra of $End_{\mathbf{R}}(C^\infty(G))$.

(6.3.3) Denoting

$$\forall q \in G : A^*_q := \mathbf{R} - lin \ \ span(\{\phi \in A^*_{fin}; supp \ \phi = \{q\}\}),$$

one finds an isomorphism of unital associative real algebras: $A^*_q \longleftrightarrow E(L(G))$.

(6.3.4) More generally one constructs an $\mathbf{R}$-linear bijection

$$\gamma : \mathbf{R}(G) \otimes E(L(G)) \ni q \otimes X_1^{k_1} \cdots X_m^{k_m} \longleftrightarrow \phi \in A^*_{fin},$$

$$\phi : C^\infty(G) \ni f \underset{def}{\longrightarrow} L^{k_1}_{X_1} \circ \cdots \circ L^{k_m}_{X_m}(f)(q) \in \mathbf{R}.$$

(6.3.5) The universal property of tensor products implies existence of an **R**-linear map

$$\Delta : A^* \longrightarrow (A \otimes A)^* := Hom_{\mathbf{R}}(A \otimes A, \mathbf{R}),$$

such that

$$\forall f, g \in A, \forall \phi \in A^* : \Delta(\phi)(f \otimes g) = \phi(fg).$$

(6.3.6) $\forall \phi \in A^*_{fin} : \exists p \in \mathbf{N}, \exists \{\phi'_k, \phi''_k \in A^*_{fin}; k = 1, \dots, p\} :$

$$\forall f, g \in A : \phi(fg) = \sum_{k=1}^{p} \phi'_k(f)\phi''_k(g).$$

Hence one establishes an **R**-linear map

$$\Delta_{fin} : A^*_{fin} \ni \phi \underset{def}{\longrightarrow} \sum_{k=1}^{p} \phi'_k \otimes \phi''_k \in A^*_{fin} \otimes A^*_{fin}.$$

Here one just uses the fact that, that L_X is some derivation on $C^\infty(G)$, for an arbitrary smooth vector field X.

(6.3.7) Moreover defining

$$\epsilon_{fin} : A^*_{fin} \ni \phi \underset{def}{\longrightarrow} \phi(e_A) \in \mathbf{R}, \text{ with the unit } e_A \text{ of } C^\infty(G),$$

one establishes the cocommutative real coalgebra $\{A^*_{fin}, \Delta_{fin}, \epsilon_{fin}\}$.

(6.3.8) The linear bijection γ is an isomorphism of real coalgebras.

(6.3.9) The adjoint representation $Ad : G \longrightarrow Aut_{\mathbf{R}}(L(G))$ is inserted into the real coalgebra $\{F_\pi, \Delta_\pi, \epsilon_\pi\}$. $\forall q \in G, X \in L(G)$:

$$\pi_L(q)(X) := \rho(q)_*(X) \in L(G). \ F_\pi := \mathbf{R}(G) \otimes_\pi E(L(G)).$$

(6.3.10) The structure mappings μ_π, η_π are switched from F_π to A^*_{fin}.

(6.3.10.1) The real-linear form

$$\gamma(e_G \otimes e_{E(L(G))}) = \phi_1 : C^\infty(G) \ni f \underset{def}{\longrightarrow} f(e_G) \in \mathbf{R}$$

is used as unit of the real associative algebra A^*_{fin}.

(6.3.10.2) Let $f \in C^\infty(G)$, and $q, p \in G, k_1, \dots, l_m \in \mathbf{N}_0$; define

$$\phi(f) := L_{X_1}^{k_1} \circ \cdots \circ L_{X_m}^{k_m}(f)(q) \in \mathbf{R}, \ \ \psi(f) := L_{X_1}^{l_1} \circ \cdots \circ L_{X_m}^{l_m}(f)(p) \in \mathbf{R}.$$

Then

$$\phi \circ \psi(f) := L_{X_1}^{k_1} \circ \cdots \circ L_{X_m}^{k_m} \circ L_{X_1'}^{l_1} \circ \cdots \circ L_{X_m'}^{l_m}(f)(qp) \in \mathbf{R},$$

with the transformed invariant vector fields

$$X_k' := (Ad\ \ q)(X_k) \in L(G), \ k = 1, \dots, m.$$

(6.3.11) One obtains the real Hopf algebra $\{A_{fin}^*, \mu_{fin}, \eta_{fin}, \Delta_{fin}, \epsilon_{fin}\}$, with the antipode σ_{fin}, such that $\forall q \in G, X \in L(G)$:

$$\{f(q); f \in C^\infty(G)\} \underset{\sigma_{fin}}{\longrightarrow} \{f(q^{-1}); f \in C^\infty(G)\},$$

$$\{L_X f(e_G); f \in C^\infty(G)\} \underset{\sigma_{fin}}{\longrightarrow} \{-L_X f(e_G); f \in C^\infty(G)\},$$

$$\{L_X f(q); f \in C^\infty(G)\} \underset{\sigma_{fin}}{\longrightarrow} \{-L_{(Ad\ \ q^{-1})(X)}(f)(q^{-1}); f \in C^\infty(G)\}.$$

(6.3.12) Therefore A_{fin}^* is some real Lie-Hopf superalgebra, in the sense of an isomorphism of real Hopf algebras $\gamma : \mathbf{R}(G) \otimes_\pi E(L(G)) \longleftrightarrow A_{fin}^*$, with the trivial $\mathbf{Z}_2$-grading.

(6.3.13) More generally, consider the real graded-commutative associative superalgebra

$$\Lambda_{mn}(G) := \left\{ f : G \underset{smooth}{\longrightarrow} \Lambda_n \right\}, \ dim\ G =: m \in \mathbf{N},$$

of C^∞-mappings from the Lie group G into the alternating algebra

$$\Lambda_n := \Lambda(\mathbf{R}(\{\theta_1, \dots, \theta_n\})), \ dim\ \Lambda_n = 2^n.$$

The above considerations are applied to this case, inserting the universal enveloping superalgebras:

$$\mathbf{R}(\{\theta_1, \dots, \theta_n\}) \underset{universal\ envelope}{\longrightarrow} \Lambda_n,$$

$$L^{\bar{0}} := L(G), \ L^{\bar{1}} := \mathbf{R}\left(\left\{\frac{\partial}{\partial \theta_k}; k = 1, \dots, n\right\}\right), \ E(L^{\bar{1}}) \longleftrightarrow \Lambda_n,$$

$$L := L^{\bar{0}} \oplus L^{\bar{1}} \underset{universal\ envelope}{\longrightarrow} E(L^{\bar{0}}) \hat{\otimes} E(L^{\bar{1}}) \longleftrightarrow E(L^{\bar{0}}) \hat{\otimes} \Lambda_n.$$

(6.3.14) Consider the restriction of $\Lambda^*_{mn}(G) := Hom_{\mathbf{R}}(\Lambda_{mn}(G), \mathbf{R})$, which is defined by the next diagram.

$$\begin{tikzcd}
\Lambda^*_{mn}(G) \arrow[r, leftrightarrow, "real\text{-}linear"] & (C^\infty(G))^* \otimes \Lambda_n \\
\Lambda^{*fin}_{mn}(G) \arrow[u] \arrow[r, leftrightarrow, "real\text{-}linear", "def"'] & A^*_{fin}(G) \otimes \Lambda_n \arrow[u]
\end{tikzcd}$$

Hence one finds an according isomorphism of real $\mathbf{Z}_2$-graded Hopf algebras:

$$\Lambda^{*fin}_{mn}(G) \longleftrightarrow \big(\mathbf{R}(G) \otimes_{\pi_0} E(L(G))\big)\hat{\otimes}\Lambda_n,$$

with the graded-commutative $\mathbf{Z}_2$-graded Hopf algebra Λ_n, and inserting the adjoint representation of G for π_0, such that

$$\forall q \in G : \pi_0(q)\big|_{L(G)} = Ad\ q.$$

The elements of $\Lambda^{*fin}_{mn}(G)$ may be called $\mathbf{Z}_2$-graded distributions with finite support.

(6.3.15) With respect to an appropriate homomorphism π of groups defined in the next diagram, with $X \in L(G)$ and $\partial \in \mathbf{R}\left(\left\{\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}\right\}\right)$, one then establishes an according real Lie-Hopf superalgebra $\Lambda^{*fin}_{mn}(G)$, and an isomorphism of real Hopf superalgebras:

$$\Lambda^{*fin}_{mn}(G) \longleftrightarrow \mathbf{R}(G) \otimes_\pi E(L).$$

$$\begin{tikzcd}
G \ni q \arrow[r] \arrow[dr] & \pi(q) : E(L) \arrow[r, leftrightarrow] & E(L) \\
& \pi_L(q) : L \ni X + \partial \arrow[u] \arrow[r, leftrightarrow] & (Ad\, q)(X) + \partial \in L \arrow[u]
\end{tikzcd}$$

Here an isomorphism of real Lie superalgebras $\pi_L(q)$ is lifted to an isomorphism of real unital associative superalgebras $\pi(q)$, for $q \in G$.

(6.3.16) One therefore obtains the real-linear bijection: $\Lambda^{*fin}_{mn}(G) \longleftrightarrow$

$$\mathbf{R}\Bigg(\Bigg\{L^{k_1}_{X_1} \circ \cdots \circ L^{k_m}_{X_m}\big|_q \circ \left(c + d\frac{\partial}{\partial\theta_{l_1}} \circ \cdots \circ \frac{\partial}{\partial\theta_{l_s}}\right);\ q \in G;\ c, d \in \mathbf{R};$$
$$1 \leq s \leq n,\ 1 \leq l_1 < \ldots < l_s \leq n;\ k_1, \ldots, k_m \in \mathbf{N}_0\Bigg\}\Bigg),$$

the images of which are operating on $\Lambda_{mn}(G)$ as Λ_n-valued real-linear forms with finite support.

(6.3.16.1) Let $\lambda \in \Lambda_n$, $f_\lambda \in C^\infty(G)$, and $f(q) := f_\lambda(q)\lambda$ for $q \in G$. Then the real-linear form ϕ, which corresponds to $L_{X_1}^{k_1} \circ \cdots \circ L_{X_m}^{k_m}\big|_q \circ \partial$ with $\partial \in E(L^{\mathrm{I}})$, $k_1, \ldots, k_m \in \mathbf{N}_0$, is acting onto f according to

$$\phi(f) = L_{X_1}^{k_1} \circ \cdots \circ L_{X_m}^{k_m} f_\lambda(q)(\partial\lambda).$$

Here the above defined isomorphism of real Hopf superalgebras explicitly means:

$$\Lambda_{mn}^{*fin}(G) \ni \phi \longleftrightarrow q \otimes L_{X_1}^{k_1} \circ \cdots \circ L_{X_m}^{k_m} \circ \partial \longleftrightarrow \in \mathbf{R}(G) \otimes_\pi E(L).$$

6.4 Real Lie Supergroups as Real Lie-Hopf Superalgebras

(6.4.1) Let $G_k, k = 1, 2$, be groups, and consider a homomorphism of groups $\phi : G_2 \longrightarrow Aut(G_1)$, into the group of automorphisms of G_1. On the Cartesian product of sets $G_1 \times G_2$ define the map:

$$\begin{aligned}(G_1 \times G_2) \times (G_1 \times G_2) \ni &\{\{x_1, x_2\}, \{y_1, y_2\}\} \\ &\longrightarrow \{x_1(\phi(x_2)(y_1)), x_2 y_2\} \in G_1 \times G_2.\end{aligned}$$

With this product the set $G_1 \times G_2$ becomes some group, the so-called semidirect product, denoted by $G_1 \times_\phi G_2$. In the special case of $Im\ \phi := \{id\ G_1\}$, one obtains the direct product of these two groups. The unit of $G_1 \times_\phi G_2$ is $\{e_1, e_2\}$. Furthermore,

$$\forall x_1 \in G_1, x_2 \in G_2 : \{x_1, x_2\}^{-1} = \{\phi(x_2^{-1})(x_1^{-1}), x_2^{-1}\}.$$

Obviously $\{\{x_1, e_2\}; x_1 \in G_1\}$ is an invariant subgroup, and moreover $\{\{e_1, x_2\}; x_2 \in G_2\}$ is some subgroup of $G_1 \times_\phi G_2$.

$$\forall x_1 \in G_1, x_2 \in G_2 : \{x_1, e_2\}\{e_1, x_2\} = \{x_1, x_2\}.$$

(6.4.1.1) Let $G_k, k = 1, 2$, be topological groups, and assume that the map: $G_1 \times G_2 \ni \{x_1, x_2\} \longrightarrow \phi(x_2)(x_1) \in G_1$ is continuous with respect to the procuct topology of $G_1 \times G_2$. Then $G_1 \times_\phi G_2$ is some topological group, with respect to the product topology of $G_1 \times G_2$.

(6.4.1.2) Especially let $G_k, k = 1, 2$, be real Lie groups. If the map:

$$G_1 \times G_2 \ni \{x_1, x_2\} \longrightarrow \phi(x_2)(x_1) \in G_1$$

is real-analytic on the product manifold $G_1 \times G_2$, then $G_1 \times_\phi G_2$ is some real Lie group.

(6.4.1.2.1) The corresponding left-invariant vector fields are obviously related by the real-linear bijections:

$$L(G_1 \times_\phi G_2) \ni X \longleftrightarrow [u] \longleftrightarrow \{[\pi_1 \circ u], [\pi_2 \circ u]\}$$
$$\longleftrightarrow \{X_1, X_2\} \in \bigoplus_{k=1}^{2} L(G_k),$$

inserting the equivalence classes of curves

$$[\pi_k \circ u] \in T_{e_k}(G_k), k = 1, 2, \quad [u] \in T_{\{e_1, e_2\}}(G_1 \times_\phi G_2).$$

The projections $\pi_k : G_1 \times_\phi G_2 \longrightarrow G_k, k = 1, 2$, are continuous and open.

(6.4.2) For an easy definition of real Lie supergroups one starts from a real Lie-Hopf superalgebra $\{G_0, L, \pi\}$,

$$F_\pi := \mathbf{R}(G_0) \otimes_\pi E(L), \quad L^{\bar{0}} := L(G_0).$$

(6.4.2.1) $E_n := \Lambda_n \hat{\otimes} E(L)$, as some unital associative superalgebra over $\mathbf{R}$. Consider the real vector space

$$M_n := (\Lambda_n^{\bar{0}} \otimes L^{\bar{0}}) \oplus (\Lambda_n^{\bar{1}} \otimes L^{\bar{1}}) \subseteq E_n^{\bar{0}}.$$

As subalgebra of the real Lie superalgebra E_{nL}, M_n is some real Lie algebra; it is called Grassmann envelope of L. Now use the direct sum

$$M_n = (\mathbf{R} \otimes L^{\bar{0}}) \oplus N_n,$$

with an appropriately chosen supplement N_n, such that N_n is some solvable ideal of M_n. With the choice

$$N_n := \left\{ \sum_{1 \le l_1 < \dots < l_r \le n} (\theta_{l_1} \wedge \dots \wedge \theta_{l_r}) \otimes X_{l_1 \dots l_r}; X_{l_1 \dots l_r} \in L^{\bar{r}} \right\}$$

one obtains some isomorphism of real Lie algebras: $M_n/N_n \longleftrightarrow L^{\bar{0}}$.

(6.4.2.2) Let G_1 be a real Lie group, with the real Lie algebra $L(G_1) = N_n$.

(6.4.2.3) A homomorphism of groups $\pi_n : G_0 \longrightarrow Aut(E_n)$ is constructed from π :

$$G_0 \ni g_0 \longrightarrow \pi_n(g_0) : E_n \ni \lambda \otimes X \underset{def}{\longleftrightarrow} \lambda \otimes \pi(g_0)(X) \in E_n,$$

into the group of automorphisms of E_n, the latter in the sense of unital associative superalgebras over $\mathbf{R}$.

(6.4.2.4) For $g_0 \in G_0$ and $\bar{z} \in \mathbf{Z}_2$, $\Lambda_n^{\bar{z}} \otimes L^{\bar{z}}, M_n$ and N_n are invariant with respect to $\pi_n(g_0)$. A homomorphism of groups $\nu : G_0 \longrightarrow Aut(N_n)$ is constructed from π_L:

$$G_0 \ni g_0 \longrightarrow \nu(g_0) : N_n \ni \lambda \otimes X \underset{def}{\longleftrightarrow} \lambda \otimes \pi_L(g_0)(X) \in N_n.$$

Here $Aut(N_n)$ denotes the group of automorphisms of N_n, in the sense of real Lie algebras.

(6.4.2.5) Let $\alpha : G_0 \longrightarrow Aut(G_1)$ be a homomorphism of groups, such that conditions (i) and (ii) are fulfilled. (i) is expressed in the diagram below.

(i)
$$\begin{tikzcd} G_0 \ni g_0 \arrow[r] \arrow[dr] & \alpha(g_0) : G_1 \arrow[r] & G_1 \\ & \nu(g_0) : N_n \arrow[u, "exp"] \arrow[r] \arrow[ur, phantom, "push\ forward"] & N_n \arrow[u, "exp"'] \end{tikzcd}$$

(ii) The map

$$G_0 \times G_1 \ni \{g_0, g_1\} \longrightarrow \alpha(g_0)(g_1) \in G_1$$

is real-analytic. Then one defines the real Lie group $G := G_1 \times_\alpha G_0$, which may be called real Lie supergroup.

(6.4.2.6)

$$dim\ M_n = p + dim\ N_n = 2^{n-1}(p+q),$$

$$p := dim\ G_0 = dim\ L^{\bar{0}} \in \mathbf{N}, \quad q := dim\ L^{\bar{1}} \in \mathbf{N}.$$

(6.4.2.7) The Lie algebra of G is just the subalgebra M_n of E_{nL}. Hence one establishes an isomorphism of real Lie algebras:

$$L(G_1 \times_\alpha G_0) \longleftrightarrow (\Lambda_n^{\bar{0}} \otimes L^{\bar{0}}) \oplus (\Lambda_n^{\bar{1}} \otimes L^{\bar{1}}) \underset{subalgebra}{\longrightarrow} (\Lambda_n \hat{\otimes} E(L^{\bar{0}} \oplus L^{\bar{1}}))_L.$$

This theorem is proved subsequently.

(6.4.2.7.1)

$$L(G) \underset{real-linear}{\longleftrightarrow} L(G_0) \oplus L(G_1) \underset{real-linear}{\longleftrightarrow} M_n.$$

(6.4.2.7.2) The semidirect product with respect to α implies, that $\forall X_0 \in L^{\bar{0}}, X_1 \in N_n, t \in \mathbf{R}$:

$$\begin{aligned}\exp \circ (\nu \circ \exp(X_0))(tX_1) &= \exp(X_0)\exp(tX_1)\exp(-X_0)\\ &= \exp \ \circ \ Ad \ \ \exp(X_0)(tX_1).\end{aligned}$$

(6.4.2.7.3) $\forall X_0 \in L^{\bar{0}}, \lambda \otimes Y \in N_n$:

$$Ad \ \ \exp(X_0) \ \ (\lambda \otimes Y) = \lambda \otimes (\pi_L \circ \exp(X_0))(Y).$$

(6.4.2.7.4) One then finds the diagram below.

$$\begin{tikzcd} G_0 \arrow[r, "\pi_L"'] & Aut(L) \arrow[r] & Aut_R(L) & \\ L^{\bar{0}} \arrow[u, "exp"'] \arrow[rr, "\pi_L' \quad def"'] & & L(Aut_R(L)) \arrow[u, "exp"'] \arrow[r, leftrightarrow, "\beta_L"'] & End_R(L) \end{tikzcd}$$

Here $\forall X_0 \in L^{\bar{0}}$:

$$\beta_L \circ \pi_L'(X_0) : L \ni Y \longrightarrow [X_0, Y],$$

inserting the product in the sense of L, due to the assumptions on π, within the context of real Lie-Hopf superalgebras.

(6.4.2.7.5) Consider the adjoint representation

$$Ad : G \longrightarrow Aut_{\mathbf{R}}(L(G)) =: \mathbf{A},$$

and the exponential map in the subsequent diagram.

$$\begin{tikzcd} G \arrow[r, "\rho"'] & A & \\ L(G) \arrow[u, "exp"'] \arrow[r, "\rho' \quad def"'] & L(A) \arrow[u, "exp"'] \arrow[r, leftrightarrow, "\beta"'] & End_R(L(G)) \end{tikzcd}$$

$\forall X, X' \in L(G) : (\beta \circ Ad'(X))(X') = [X, X']$, with the product of $L(G)$.

(6.4.2.7.6)

$$\forall X_0 \in L^{\bar{0}}, \lambda \otimes Y \in N_n : (\exp \circ Ad'(X_0))(\lambda \otimes Y) = \lambda \otimes (\exp \circ \pi'_L(X_0))(Y),$$

and therefore

$$(\beta \circ Ad'(X_0))(\lambda \otimes Y) = \lambda \otimes (\beta_L \circ \pi'_L(X_0))(Y).$$

(6.4.2.7.7) Hence one finds, that the products of $L(G)$, and of the subalgebra M_n of E_{nL}, coincide.

(6.4.2.8) One thereby obtains the local diffeomorphism:

$$L(G) \underset{inclusion}{\longleftarrow} V_0 \ni X \longleftrightarrow \exp(X) \in U_e \underset{inclusion}{\longrightarrow} G,$$

of an open connected neighbourhood V_0 of $0 \in L(G)$, onto an open connected neighbourhood of the unit $e_G \equiv e \in G$. Using basis vectors X_{0k} of $L^{\bar{0}}$, and X_{1l} of $L^{\bar{1}}$, one then writes

$$V_0 \ni X = \sum_{k=1}^{p} \left(c_0^k + \sum_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{0}} c^k_{l_1 \ldots l_r} \theta_{l_1} \wedge \cdots \wedge \theta_{l_r} \right) \otimes X_{0k}$$
$$+ \sum_{l=1}^{q} \left(\sum_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{1}} d^l_{l_1 \ldots l_r} \theta_{l_1} \wedge \cdots \wedge \theta_{l_r} \right) \otimes X_{1l},$$

with the canonical coordinates of the first kind $c_0^k, c^k_{l_1 \ldots l_r}, d^l_{l_1 \ldots l_r} \in$ real open intervals containing zero.

6.5 Linear Supergroups

(6.5.1) Real or complex supermatrices are defined conveniently as special endomorphisms of real or complex supervectors.

$$\mathbf{W} := \Lambda_n \otimes \mathbf{V}, \ \mathbf{V} := \mathbf{V}^{\bar{0}} \oplus \mathbf{V}^{\bar{1}},$$
$$\mathbf{V}^{\bar{0}} := \mathbf{K}^s, \ \mathbf{V}^{\bar{1}} := \mathbf{K}^t, \ s, t \in \mathbf{N}_0, \ s + t \geq 1,$$
$$\Lambda_n := \Lambda(E), \ E := \mathbf{K}(\{\theta_1, \ldots, \theta_n\}), \ n \in \mathbf{N}, \ \mathbf{K} := \mathbf{R} \text{ or } \mathbf{C}.$$

The elements of this real or complex vector space $\mathbf{W}$ are written as columns, with the $\mathbf{Z}_2$-grading due to those of Λ_n and $\mathbf{V}$, and called supervectors over the field $\mathbf{K}$.

$$\mathbf{W} = \mathbf{W}^{\bar{0}} \oplus \mathbf{W}^{\bar{1}} \ni w = \begin{bmatrix} z_1 \\ \vdots \\ z_{s+t} \end{bmatrix},$$

$$\forall_1^{s+t} l : z_l = c_{l,0} + \sum_{1 \le k_1 < \ldots < k_p \le n} c_{l,k_1 \ldots k_p} \theta_{k_1} \wedge \cdots \wedge \theta_{k_p} \in \Lambda_n.$$

$$\forall \bar{z} \in \mathbf{Z}_2 : w \in \mathbf{W}^{\bar{z}} \iff \forall_1^s k, \forall_{s+1}^{s+t} l : z_k \in \Lambda_n^{\bar{z}}, z_l \in \Lambda_n^{\overline{z+1}}.$$

$$End_{\mathbf{K}}(\mathbf{W}) \underset{\mathbf{K}-linear}{\longleftrightarrow} End_{\mathbf{K}}(\Lambda_n) \hat{\otimes} End_{\mathbf{K}}(\mathbf{V}) \underset{\neq}{\supset} \Lambda_n \hat{\otimes} End_{\mathbf{K}}(\mathbf{V}) =: \mathbf{D},$$

using only Λ_n-multiples of the unit of $End_{\mathbf{K}}(\Lambda_n)$. Then

$$\mathbf{D} = \mathbf{D}^{\bar{0}} \oplus \mathbf{D}^{\bar{1}}, \quad \mathbf{D}^{\bar{z}} \ni M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with the elements $a_{kl} \in A, \ldots, d_{ij} \in D$, such that $\forall_1^s k, l, \forall_1^t i, j$:

$$a_{kl} \in \Lambda_n^{\bar{z}}, \quad b_{kj} \in \Lambda_n^{\overline{1+z}}, \quad c_{il} \in \Lambda_n^{\overline{1+z}}, \quad d_{ij} \in \Lambda_n^{\bar{z}}.$$

Here and henceforth the above $\mathbf{K}$-linear bijection is suppressed. With this $\mathbf{Z}_2$-grading, $\mathbf{D}$ becomes an associative superalgebra over $\mathbf{K}$, with the unit $e_{\mathbf{D}} \equiv I := id\ \mathbf{W}$, the elements of which are called supermatrices over $\mathbf{K}$.

(6.5.2) An appropriate notation of these tensor products is explicated for $s = t = n = 1$.
$\forall a_0, a_1, b_{00}, \ldots, b_{11}, v_0, v_1 \in \mathbf{K}$:

$$(a_0 + \theta a_1) \otimes \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} (a_0 + \theta a_1) v_0 \\ (a_0 - \theta a_1) v_1 \end{bmatrix},$$

$$(a_0 + \theta a_1) \otimes \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} (a_0 + \theta a_1) b_{00} & (a_0 + \theta a_1) b_{01} \\ (a_0 - \theta a_1) b_{10} & (a_0 - \theta a_1) b_{11} \end{bmatrix}.$$

$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall \lambda \in \Lambda_n, \mu \in \Lambda_n^{\bar{x}}, B \in End_{\mathbf{K}}^{\bar{y}}(\mathbf{V}), v \in \mathbf{V}$:

$$(\lambda \otimes B)(\mu \otimes v) = (-1)^{xy} (\lambda \mu) \otimes (Bv),$$

yielding an appropriate matrix multiplication of elements $\in \mathbf{D}, \mathbf{W}$.

(6.5.3) Both $\mathbf{W}$ and $\mathbf{D}$ can be established as graded Λ_n-modules over $\mathbf{K}$. For instance, for $s = t = 1, n = 2$, $\forall v_0, v_1, b_{00}, \ldots, b_{11} \in \mathbf{K}$,

$$\forall \lambda := a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1 \wedge \theta_2 \in \Lambda_2 :$$

$$\lambda \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} := \lambda \otimes \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} (a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1 \wedge \theta_2) v_0 \\ (a_0 - a_1\theta_1 - a_2\theta_2 + a_{12}\theta_1 \wedge \theta_2) v_1 \end{bmatrix},$$

$$\lambda \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} := \lambda \otimes \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$
$$= \begin{bmatrix} (a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_{12}) b_{00} & (a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_{12}) b_{01} \\ (a_0 - a_1\theta_1 - a_2\theta_2 + a_{12}\theta_{12}) b_{10} & (a_0 - a_1\theta_1 - a_2\theta_2 + a_{12}\theta_{12}) b_{11} \end{bmatrix},$$

denoting $\theta_{12} \equiv \theta_1 \wedge \theta_2$.
More generally, with the isomorphism of unital associative $\mathbf{K}$-algebras:

$$\begin{aligned} \Lambda_2 \ni \lambda = a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1 \wedge \theta_2 \\ \longleftrightarrow a_0 - a_1\theta_1 - a_2\theta_2 + a_{12}\theta_1 \wedge \theta_2 =: \bar{\lambda} \in \Lambda_2, \end{aligned}$$

one defines $\forall \lambda \in \Lambda_2, w \in \mathbf{W}, M \in \mathbf{D}$:

$$\lambda w := \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} w, \quad \lambda M := \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} M,$$

$$w\lambda := \sum_{\bar{x},\bar{y} \in \mathbf{Z}_2} (-1)^{xy} \lambda^{\bar{x}} w^{\bar{y}}, \quad M\lambda := \sum_{\bar{x},\bar{y} \in \mathbf{Z}_2} (-1)^{xy} \lambda^{\bar{x}} M^{\bar{y}} = M \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix},$$

with the coefficients λ from the left and right, respectively.

(6.5.4) With an appropriate notation of matrix elements, such that $\forall_1^s k, l$:

$$a_{kl} = a_{kl,0} + \sum_{1 \le k_1 < \ldots < k_p \le n} a_{kl,k_1 \ldots k_p} \theta_{k_1} \wedge \cdots \wedge \theta_{k_p},$$

$$A_0 := [a_{kl,0}; k, l = 1, \ldots, s],$$

etc., one finds the following lemma.

$$\forall M \in \mathbf{D} : [\exists N \in \mathbf{D} : MN = NM = I \iff \det\ (A_0 D_0) \neq 0].$$

In the case of M being invertible, $N =: M^{-1}$ is obviously unique.

(6.5.5) The supertrace $str \in Hom_{\mathbf{K}}(\mathbf{D}, \Lambda_n)$ is defined, such that

$$\forall \bar{z} \in \mathbf{Z}_2, M \in \mathbf{D}^{\bar{z}} : str\ M := tr\ A - (-1)^z tr\ D \in \Lambda_n^{\bar{z}}.$$

$$\forall M = M^{\bar{0}} + M^{\bar{1}} \in \mathbf{D} : str\ M := str\ M^{\bar{0}} + str\ M^{\bar{1}} \in \Lambda_n.$$

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall M \in \mathbf{D}^{\bar{x}}, N \in \mathbf{D}^{\bar{y}} : str(MN) = (-1)^{xy} str(NM).$$

(6.5.6) The superdeterminant is defined, such that

$$\begin{aligned} &\forall M \in \mathbf{D}^{\bar{0}} : det(A_0 D_0) \neq 0 \\ &\qquad \Longrightarrow sdet\ M := \frac{det(A - BD^{-1}C)}{det\ D} = \frac{det\ A}{det(D - CA^{-1}B)} \in \Lambda_n^{\bar{0}}. \end{aligned}$$

$$\forall M, M' \in \mathbf{D}^{\bar{0}} : det(A_0 D_0 A_0' D_0') \neq 0 \Longrightarrow sdet(MM') = (sdet\ M)(sdet\ M').$$

$$\forall M \in \mathbf{D}^{\bar{0}} : \exp(M) := \sum_{k=0}^{\infty} \frac{M^k}{k!} \in \mathbf{D}^{\bar{0}}, \quad \exp(M)\exp(-M) = I,$$

$$sdet \circ \exp(M) = \exp \circ str\ (M).$$

(6.5.7) With the norm, such that

$$\forall z \in \Lambda_n : ||z|| := |z_0| + \sum_{1 \leq k_1 < \ldots < k_p \leq n} |z_{k_1 \ldots k_p}|,$$

Λ_n becomes some Banach algebra, the topology of which is equivalent to the natural one of $\mathbf{K}^{(2^n)}$.

(6.5.8) Consider the group

$$G := GL_n(s,t,\mathbf{K}) := \{M \in \mathbf{D}^{\bar{0}}; det(A_0 D_0) \neq 0\}$$

of even invertible real or complex supermatrices, where $s,t,n \in \mathbf{N}$ are used as above. This so-called general linear supergroup G is established as the real Lie-Hopf superalgebra $\{G_0, L, \pi\}$, such that $G = G_1 \times_\alpha G_0$, with the notation of the preceding chapter. Note here that also in the case of $\mathbf{K} := \mathbf{C}$ the charts of the real-analytic manifold G are real-valued.

(6.5.8.1)

$$G_0 := \left\{ \begin{bmatrix} A_0 & 0 \\ 0 & D_0 \end{bmatrix} \in Mat(s+t,\mathbf{K}); det(A_0 D_0) \neq 0 \right\},$$

$$L^{\bar{0}} := \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} ; X \in Mat(s,\mathbf{K}), Y \in Mat(t,\mathbf{K}) \right\} = L(G_0).$$

(6.5.8.2) $G_1 :=$ real Lie group of real or complex supermatrices $M_1 :=$

$$\begin{bmatrix} I_s + \sum\limits_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{0}} \theta_{l_1 \ldots l_r} A_{l_1 \ldots l_r} & \sum\limits_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{1}} \theta_{l_1 \ldots l_r} B_{l_1 \ldots l_r} \\ \sum\limits_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{1}} \theta_{l_1 \ldots l_r} C_{l_1 \ldots l_r} & I_t + \sum\limits_{1 \leq l_1 < \ldots < l_r \leq n, r \in \bar{0}} \theta_{l_1 \ldots l_r} D_{l_1 \ldots l_r} \end{bmatrix}$$

with real or complex matrices $A_{l_1 \ldots l_r}, \ldots, D_{l_1 \ldots l_r}$ of appropriate dimensions, hereby suppressing an explicit notation of the skew-symmetric tensor product. Then G_1 is an invariant subgroup of G.

$$L^{\bar{1}} := \left\{ \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in Mat(s+t,\mathbf{K}); X_1 = X_4 = 0 \right\}.$$

$$L = L^{\bar{0}} \oplus L^{\bar{1}} \underset{\mathbf{K}-linear}{\longleftrightarrow} Mat(s+t,\mathbf{K}).$$

(6.5.8.3) The group homomorphism $\alpha : G_0 \longrightarrow Aut(G_1)$ used in the preceding chapter explicitly reads, such that

$$\forall M_0 \in G_0 : \alpha(M_0) : G_1 \ni M_1 \underset{def}{\longleftrightarrow} M_0 M_1 M_0^{-1} \in G_1.$$

One then finds the semidirect product $G = G_1 \times_\alpha G_0$ of real Lie groups, such that

$$\forall M = M_1 M_0,\ \ M' = M_1' M_0' \in G : MM' = M_1 \alpha(M_0)(M_1') M_0 M_0' \in G.$$

(6.5.8.4)

$$\forall M_0 \in G_0 : \alpha(M_0) \circ \exp = \exp \circ (\alpha(M_0))_*,$$

with $(\alpha(M_0))_* = \nu(g_0) \in Aut(N_n)$ in the sense of real Lie algebras, using the notation of the preceding chapter. $\forall M_0 \in G_0$:

$$\alpha(M_0) : G_1 \ni \exp(\lambda \hat{\otimes} X) \longleftrightarrow M_0 \exp(\lambda \hat{\otimes} X) M_0^{-1} \in G_1,$$

$$(\alpha(M_0))_* : N_n \ni \lambda \hat{\otimes} X \underset{def}{\longleftrightarrow} M_0(\lambda \hat{\otimes} X) M_0^{-1} = \lambda \hat{\otimes} (M_0 X M_0^{-1}) \in N_n.$$

Here $N_n = L(G_1)$ is the solvable ideal of M_n defined in the preceding chapter.

(6.5.8.5) The corresponding real Lie-Hopf superalgebra $\{G_0, L, \pi\}$ is described by the representation $\pi_L : G_0 \longrightarrow Aut(L)$.

$$\forall M_0 \in G_0 : \pi_L(M_0) : L \ni X \longleftrightarrow M_0 X M_0^{-1} \in L.$$

The derivation of π_L at the unit of G_0 obviously yields the product of the real Lie superalgebra L.

(6.5.8.6) The real Lie algebra

$$L(G) \longleftrightarrow (\Lambda_n^{\bar{0}} \otimes L^{\bar{0}}) \oplus (\Lambda_n^{\bar{1}} \otimes L^{\bar{1}}) = (\mathbf{R} \otimes L^{\bar{0}}) \oplus N_n$$

is some subalgebra of the real Lie superalgebra $(\Lambda_n \hat{\otimes} E(L))_L$.

(6.5.8.7) In case of $\mathbf{K} := \mathbf{R}$, the dimensions of these real Lie algebras obviously are

$$dim\ L^{\bar{0}} = s^2 + t^2,\ \ dim\ L^{\bar{1}} = 2st,$$

$$dim\ L(G) = 2^{n-1}(dim\ L^{\bar{0}} + dim\ L^{\bar{1}}) = dim\ L^{\bar{0}} + dim\ N_n.$$

In case of $\mathbf{K} := \mathbf{C}$, the corresponding real dimensions are twice the above ones, of course.

(6.5.9) Here one always uses, that the skew-symmetric tensor product $\Lambda_n \hat{\otimes} Mat(s+t, \mathbf{K})$ just means an appropriately signed product of $\mathbf{K}$-matrix elements with the elements of Λ_n, in the sense of an isomorphism of unital associative algebras over $\mathbf{K}$:

$$\Lambda_n \hat{\otimes} Mat(s+t, \mathbf{K}) \ni \theta_k \hat{\otimes} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
$$\longleftrightarrow \begin{bmatrix} \theta_k A & \theta_k B \\ -\theta_k C & -\theta_k D \end{bmatrix} \in Mat(s+t, \Lambda_n).$$

(6.5.10) Consider the special case of $s = t, n = 2$. Every element $M \in G = G_1 \times_\alpha G_0$ is uniquely factorized.

$$\forall M = \begin{bmatrix} A_0 + \theta_{12} A_{12} & \theta_1 B_1 + \theta_2 B_2 \\ \theta_1 C_1 + \theta_2 C_2 & D_0 + \theta_{12} D_{12} \end{bmatrix} \in G :$$

$\exists$ *unique* $M_0 \in G_0, M_1 \in G_1$

$$:= \left\{ \begin{bmatrix} I_s + \theta_{12} A'_{12} & \theta_1 B'_1 + \theta_2 B'_2 \\ \theta_1 C'_1 + \theta_2 C'_2 & I_s + \theta_{12} D'_{12} \end{bmatrix} ; A'_{12}, B'_1, \ldots, D'_{12} \in Mat(s, \mathbf{K}) \right\} :$$

$$M = M_1 M_0, \quad M_0 = \begin{bmatrix} A_0 & 0 \\ 0 & D_0 \end{bmatrix}.$$

In case of real supermatrices, obviously $dim\ L(G) = 8s^2, \ \ dim\ N_2 = 6s^2$.

(6.5.11) The special linear supergroup

$$SL_n(s,t,\mathbf{K}) := \{M \in GL_n(s,t,\mathbf{K}); sdet\ M = 1\},$$

$s,t \in \mathbf{N}_0, \ \ s+t \geq 1, \ \ n \in \mathbf{N}$, is generated by the real Lie algebra

$$sl_n(s,t,\mathbf{K}) := \{X \in gl_n(s,t,\mathbf{K}); str\ X = 0\}.$$

Here $gl_n(s,t,\mathbf{K})$ denotes the above described real Lie algebra of $GL_n(s,t,\mathbf{K})$. In the case of $s,t \in \mathbf{N}, s \neq t$, and $\mathbf{K} := \mathbf{C}$, the complexification of the real Lie superalgebra L of the above special linear supergroup is just $A(s-1, t-1)$.

(6.5.12) The super-transpose of $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{D}$ is defined by

$$M^{st} := \begin{bmatrix} A^t & -C^t \\ B^t & D^t \end{bmatrix},$$

inserting the usual transposition of supermatrices over $\mathbf{K}$.

(6.5.12.1) Denoting

$$W := \begin{bmatrix} I_s & 0 \\ 0 & V_t \end{bmatrix}, \quad V_t := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}, \text{ for } s, r \in \mathbf{N}, \ t := 2r,$$

the orthosymplectic linear supergroup is defined for $n \in \mathbf{N}$ by

$$OSP_n(s,t,\mathbf{K}) := \{M \in GL_n(s,t,\mathbf{K}); M^{st}WM = W\}.$$

One then finds that for every element M of this supergroup, $sdet\ M = \pm 1$.

(6.5.12.2) The complexification of the corresponding real Lie superalgebra L is the orthosymplectic Lie superalgebra $osp(s,t,\mathbf{C})$.

(6.5.13) The special unitary supergroup is defined for $n \in \mathbf{N}$, $s,t \in \mathbf{N}_0$, $s + t \geq 1$, by

$$SU_n(s,t,\mathbf{C}) := \{M \in SL_n(s,t,\mathbf{C}); MM^\dagger = I\},$$

inserting the adjoint complex supermatrix $M^\dagger := (M^t)^*$, complex conjugation, and the $\mathbf{C}$-antilinear star operation on Λ_n such that $\theta_k = \theta_k^*$,

$$(\theta_{k_1} \wedge \cdots \wedge \theta_{k_r})^* = \theta_{k_r} \wedge \cdots \wedge \theta_{k_1}, \ \ 1 \leq k_1 < \ldots < k_r \leq n.$$

(6.5.13.1) The complexification of the corresponding real Lie superalgebra L consists of all the matrices $X \in Mat(s+t,\mathbf{C})$, which fulfill the conditions

$$str\ X = 0, \ \ X^\dagger + X = 0.$$

7. Universal Differential Envelope

Theories of fermions are concerned with graded-commutative algebras, containing both even and odd generators, the latter being nilpotent. Hence one is forced to construct supermanifolds and Lie supergroups, in order to develop some kind of differential and integral calculus with respect to both even and odd variables, and moreover to define supersymmetry transformations between bosonic and fermionic objects, for instance creation and annihilation operators, or currents consisting of them, in models of quantum field theory.

From the mathematical viewpoint, such a theory of supersymmetry may be considered as a first step towards a theory of something like "non-commutative spaces". Already on the level of $\mathbf{Z}_2$-grading, non-commutative ingrediences should be incorporated rather in the spaces of functions on the involved differentiable manifolds, than in the definition of superspaces by means of even and odd coordinates.

The structure theorem by I. M. Gel'fand on the duality of locally compact Hausdorff spaces and commutative C algebras, and the structure theorem for an arbitrary C^*-algebra, indicate the possibility to characterize a topological space in terms of continuous mappings defined on the space. The theory of C^*-algebras is presented in several monographs, cf. J. Dixmier (1982), O. Bratteli and D. W. Robinson (1979), M. Takesaki (1979), R. V. Kadison and J. R. Ringrose (1983, 1986). The view of C^*-algebra theory as some kind of non-commutative topology is stressed for instance in the friendly approach to K-theory by N. E. Wegge-Olsen (1993).

Whereas on the levels of measure theory and topology the theory of non-commutative spaces was developed in terms of C^*-algebras and their K-theory, an appropriate non-commutative differential geometry was originally constructed by A. Connes (1985, 1990, 1994). Derivations of associative superalgebras, which are odd and nilpotent, the images of which need not be elements of the algebra itself, are treated systematically, starting from an initial object: the universal differential envelope $\Omega(A)$ of an associative superalgebra A. Multilinear forms on Cartesian products of finitely many copies of A are written as linear forms on $\Omega(A)$, thereby not only reformulating the classical theory of Hochschild cohomologies of associative algebras, but moreover defining Hochschild and cyclic cohomologies of associative superalgebras. Cyclic cohomologies and their relation to the Hochschild complex were introduced by A. Connes in 1981, as an algebraic framework for the Chern character in K-homology.

The global theory of elliptic operators by M. F. Atiyah, for which the reader is referred for instance to the monograph by B. Booss and D. D. Bleecker (1985), was generalized by A. Connes to the concept of Fredholm modules over an associative unital complex superalgebra $\mathcal{A}$. As some easy guideline, consider an elliptic pseudo-differential operator

$$\dot{P} : C^\infty(\mathbf{M}, \mathbf{E}_1) \longrightarrow C^\infty(\mathbf{M}, \mathbf{E}_2),$$

which maps sections of $\mathbf{E}_1$ to sections of $\mathbf{E}_2$, both $\mathbf{E}_1$ and $\mathbf{E}_2$ being complex vector bundles over a finite-dimensional real differentiable manifold $\mathbf{M}$ as base space. Assume that $\dot{P}$ can be extended to some invertible bounded operator

$$P : \mathcal{H}^{\bar{1}} \longrightarrow \mathcal{H}^{\bar{0}}, \; P^{-1} =: Q : \mathcal{H}^{\bar{0}} \longrightarrow \mathcal{H}^{\bar{1}},$$

on complex Hilbert spaces $\mathcal{H}^{\bar{z}}$, $z \in \mathbf{Z}_2$. Then the bounded operator

$$F := \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix} : \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}} =: \mathcal{H} \longleftrightarrow \mathcal{H}$$

is idempotent and odd with respect to the indicated $\mathbf{Z}_2$-grading. Denote by K the Klein operator; then the super-commutators

$$\delta(A) := i[F, A]_g = i(FA - KAKF), \; K := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \; A \in B(\mathcal{H}),$$

with the identity operators of $\mathcal{H}^{\bar{z}}$, $\bar{z} \in \mathbf{Z}_2$, both denoted by I, provide an odd nilpotent derivation δ, thereby establishing the C^*-algebra of bounded linear operators on $\mathcal{H}$ as some $\mathbf{Z}_2$-graded complex differential algebra. Moreover let $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$ be a representation of $\mathcal{A}$ on $\mathcal{H}$. Then the resulting $\mathcal{A}$-right module $\mathcal{H}$ over $\mathbf{C}$, such that $fa := \pi(a)(f)$ for $f \in \mathcal{H}$, $a \in \mathcal{A}$, is called an involutive Fredholm right module over $\mathcal{A}$, if and only if

$$\delta \circ \pi(a) = i(F\pi(a) - K\pi(a)KF) \in B_\infty(\mathcal{H}) \text{ for } a \in \mathcal{A},$$

denoting by $B_\infty(\mathcal{H})$ the C^*-algebra of compact operators on $\mathcal{H}$, which is an ideal of $B(\mathcal{H})$.

The idempotency of F may be replaced by the condition

$$\pi(a)(F^2 - I) \in B_\infty(\mathcal{H}) \text{ for } a \in \mathcal{A},$$

thereby violating the nilpotency of δ, such that the corresponding Fredholm module is no longer involutive. This more general algebraic scheme is achieved for instance from an elliptic pseudo-differential operator $\dot{P}$ with parametrix $\dot{Q}$. The Schatten ideals of $B(\mathcal{H})$ are then used for a detailed investigation and various applications of Fredholm modules. For the study of norm ideals the reader is referred to the monographs by R. Schatten (1970) and B. Simon (1979).

An appropriate generalization to a closable linear operator $\dot{P}$, the closure of which is injective, is provided by the construction of so-called Connes modules.

Excellent monographs on K-theory were written by M. F. Atiyah (1967) and M. Karoubi (1978). The theory of fibre bundles can be learned for instance from D. Husemoller (1966), or M. Nakahara (1990), the latter including applications to anomalies in gauge field theories, and to bosonic string theory. The relevance of differential topology for quantum field theory is also exhibited with respect to instantons and monopoles, bosonic strings, anomalies, conformal and topological quantum field theories by Ch. Nash (1991).

An outstanding application of non-commutative differential geometry is provided by A. Connes' and J. Lott's version (1990) of the standard model of elementary particles, where the Higgs boson appears as some gauge field, which serves for the coupling between two copies of space-time. The latter is proposed as an orientable compact four-dimensional real differentiable manifold with Riemann metric, which is considered as the base space of some complex spinor bundle. More precisely, one constructs some representation π of the unital associative real algebra

$$\mathcal{A} := C^{\infty}(\mathbf{M}, \mathbf{R}) \otimes (\mathbf{C} \oplus \mathbf{H}),$$

inserting the non-commutative field of Hamiltonian quaternions $\mathbf{H}$, on the Hilbert space of leptons and quarks. The odd closed operator F, which is inserted into the super-commutators of an odd derivation δ, such that

$$\delta \circ \pi(a) = i[F, \pi(a)]_g \text{ for } a \in \mathcal{A},$$

is constructed from some self-adjoint Dirac operator D, which is densely defined on the spinor states over $\mathbf{M}$.

Besides presentations of this model by A. Connes himself (1990, 1994), J. C. Várilly and J. M. Gracia-Bondia wrote on extensive review (1993). A related approach to the understanding of the Higgs boson as some gauge field was obtained by R. Coquereaux et al. (1991, 1992).

The Lie algebra homology of matrices was studied by J.-L. Loday and D. Quillen (1984). An extensive presentation of cyclic homology of algebras, cyclic sets and cyclic spaces, the Chern character, homology of Lie algebras of matrices, and the relation to non-commutative differential geometry is due to J.-L. Loday (1992). The relationship between non-commutative geometry and quantum groups is discussed, among other topics, in lectures by Yu. I. Manin (1991, 1992).

In particular the algebraic machinery of non-commutative differential geometry, and also its application to elementary particle physics, was presented quite explicitly by D. Kastler (1988, 1993).

An introduction to non-commutative differential geometry, and an according survey over some of its applications to theoretical physics, is due to R. Coquereaux (1989, 1993).

7.1 Non-Unital Universal Differential Envelope

(7.1.1) Let A be an algebra over the commutative ring R. A is called graded with respect to the commutative monoid G, if and only if $A = \bigoplus_{p \in G} A^p$, as the direct sum of R-bimodules, and $\forall p, q \in G, \forall a \in A^p, b \in A^q : ab \in A^{p+q}$. With the unit p_0 of G, A^{p_0} is some subalgebra of A. An element $a \in A$ is called homogeneous of degree $p \in G$, if and only if $a \in A^p$. For an associative algebra A over R with the unit e_A, obviously $e_A \in G^{p_0}$. Frequently the Abelian monoids $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Z}_2$ are used for grading; one denotes $\mathbf{Z}_2 =: \{\bar{0}, \bar{1}\}$. The gradings of A with respect to $\mathbf{N}_0$ and $\mathbf{Z}_2$ are called compatible, if and only if

$$A = \bigoplus_{\bar{p} \in \mathbf{Z}_2} A^{\bar{p}}, \ \forall \bar{p} \in \mathbf{Z}_2 : A^{\bar{p}} = \bigoplus_{p \in \bar{p} \cap \mathbf{N}_0} A^p.$$

In the case of $\mathbf{Z}_2$, A is called superalgebra over R. The superalgebra A over R is called bigraded, if and only if A is also graded with respect to $\mathbf{N}_0$ in such a manner, that these two gradings are compatible.

(7.1.1.1) The grading automorphism of a superalgebra A over R is defined as an R-linear bijection

$$\theta : A \ni a^{\bar{0}} + a^{\bar{1}} \underset{def}{\longleftrightarrow} a^{\bar{0}} - a^{\bar{1}} \in A, \ \theta^2 = id\ A.$$

(7.1.2) Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be an associative superalgebra over the commutative ring R, which may be unital or non-unital. In any case an appropriate unit is provided by the direct sum of R-bimodules $R \oplus A =: \tilde{A}$. With the R-bilinear mapping, such that

$$\forall \lambda, \mu \in R, \forall a, b \in A : \{\lambda, a\}\{\mu, b\} := \{\lambda\mu, \lambda b + \mu a + ab\} \in \tilde{A},$$

one obtains an associative algebra over R with the unit $\{e_R, 0_A\} =: e_{\tilde{A}}$. With the isomorphism of unital associative algebras over R:

$$\tilde{A}^{\bar{0}} := \{\{\lambda, a\}; \lambda \in R, a \in A^{\bar{0}}\} \longleftrightarrow R \oplus A^{\bar{0}},$$

and an R-linear bijection:

$$\tilde{A}^{\bar{1}} := \{\{0_R, a\};\ a \in A^{\bar{1}}\} \longleftrightarrow A^{\bar{1}},$$

$\tilde{A}$ becomes an associative superalgebra over R. For convenience one usually writes $\forall \lambda \in R, a \in A :$

$$\{\lambda, a\} = \lambda e_{\tilde{A}} + \{0_R, a\} \equiv \lambda + a, \ e_{\tilde{A}} \equiv e_R, \ 0_{\tilde{A}} = \{0_R, 0_A\}.$$

(7.1.3) Let $D = D^{\bar{0}} \oplus D^{\bar{1}}$ be an associative superalgebra over the commutative ring R, and $\delta : D \longrightarrow D$ an R-linear mapping, which fulfills the following three conditions.

$$\text{(i)} \quad \forall \bar{z} \in \mathbf{Z}_2 : \ \mathrm{Im}\ \delta|_{D^{\bar{z}}} \subseteq D^{\overline{z+1}}. \quad \text{(ii)} \quad \delta \circ \delta = 0.$$

$$\text{(iii)} \quad \forall \bar{z} \in \mathbf{Z}_2, \forall a \in D^{\bar{z}}, b \in D : \delta(ab) = \delta(a)b + (-1)^{z} a\delta(b).$$

Here δ is an odd derivation of D. One establishes the closed complex of R-bimodules:

$$D^{\bar{0}} \underset{restriction\ of\ \delta}{\longrightarrow} D^{\bar{1}} \underset{restriction\ of\ \delta}{\longrightarrow} D^{\bar{0}}.$$

Then D is called $\mathbf{Z}_2$-graded differential algebra over R.

(7.1.3.1) Especially let D be $\mathbf{N}_0$-graded, i. e.,

$$D = \bigoplus_{n \in \mathbf{N}_0} D_n, \ \forall n, m \in \mathbf{N}_0, \forall a \in D_n, b \in D_m : ab \in D_{n+m}.$$

Then D is called bigraded differential algebra over R, if and only if these two gradings are compatible, and δ is of $\mathbf{N}_0$-degree 1.

$$\text{(i)} \quad \forall n \in \mathbf{N}_0 : Im\ \delta\big|_{D_n} \subseteq D_{n+1}.$$

$$\text{(ii)} \quad \forall n \in \mathbf{N}_0 : D_n = (D_n \cap D^{\bar{0}}) \oplus (D_n \cap D^{\bar{1}}).$$

Obviously,

$$\text{(ii)} \iff \forall \bar{z} \in \mathbf{Z}_2 : D^{\bar{z}} = \bigoplus_{n \in \mathbf{N}_0} (D^{\bar{z}} \cap D_n).$$

(7.1.4) A homomorphism $\phi : D \rightarrow D'$ of $\mathbf{Z}_2$-graded differential algebras over R is defined as an R-linear mapping, such that the following conditions are fulfilled.

$$\text{(i)} \quad \forall a, b \in D : ab \underset{\phi}{\longrightarrow} \phi(a)\phi(b). \quad \text{(ii)} \quad \forall \bar{z} \in \mathbf{Z}_2 : Im\ \phi\big|_{D^{\bar{z}}} \subseteq D'^{\bar{z}}.$$

$$\text{(iii)} \quad \phi \circ \delta = \delta' \circ \phi.$$

Here ϕ is a homomorphism of associative superalgebras over R, and also a zero-degree complex morphism, with respect to the above indicated complex of R-bimodules.
If and only if both D and D' are unital, one also demands that $\phi(e_D) = e_{D'}$.

(7.1.4.1) An isomorphism ϕ of $\mathbf{Z}_2$-graded differential algebras over R is defined as a bijective homomorphism in this sense; then ϕ^{-1} is also an isomorphism in this sense.

(7.1.4.2) Obviously the composition of such homomorphisms is again some homomorphism in this sense.

(7.1.5) A homomorphism $\phi : D \longrightarrow D'$ of bigraded differential algebras over R is defined as a homomorphism of $\mathbf{Z}_2$-graded differential algebras over R, which also fulfills the condition, that

$$\forall n \in \mathbf{N}_0 : Im\ \phi|_{D_n} \subseteq D_n{}'.$$

Here ϕ is also a complex-morphism of degree 1, with respect to the complex of R-bimodules:

$$D_0 \longrightarrow \cdots \longrightarrow D_n \xrightarrow[\text{restriction of } \delta]{} D_{n+1} \longrightarrow \cdots .$$

(7.1.5.1) An isomorphism ϕ of bigraded differential algebras over R is defined as a bijective homomorphism in this sense; then ϕ^{-1} is an isomorphism in this sense too.

(7.1.5.2) Obviously the composition of such homomorphisms is again some homomorphism in this sense.

(7.1.6) The non-unital universal differential envelope of an associative superalgebra A over R is defined as a homomorphism $\nu : A \longrightarrow \Omega(A)$ of associative superalgebras over R, which fulfills the following universal property: $\Omega(A)$ is some $\mathbf{Z}_2$-graded differential algebra over R, with the odd derivation $d : \Omega(A) \longrightarrow \Omega(A)$. Let D be an arbitrary $\mathbf{Z}_2$-graded differential algebra over R, with the odd derivation $\delta : D \longrightarrow D$, and consider a homomorphism $\phi : A \longrightarrow D$ of associative superalgebras over R, which also means that ϕ is even. Then there is a unique homomorphism $\phi_* : \Omega(A) \longrightarrow D$ of $\mathbf{Z}_2$-graded differential algebras over R, i.e., $\phi_* \circ d = \delta \circ \phi_*$, such that $\phi_* \circ \nu = \phi$. Hence $\phi_* \circ d \circ \nu = \delta \circ \phi$.

(7.1.6.1) An explicit construction of $\Omega(A)$ in the next chapter shows that ν is injective, hence may be viewed as an inclusion, as is indicated in the diagram below.

$$\begin{tikzcd} A \arrow[rr, "\text{inclusion}"] \arrow[dr, "\phi"'] & & \Omega(A) \arrow[dl, "\phi_*"] \\ & D & \end{tikzcd}$$

(7.1.6.2) The usual arrow-theoretic argument shows, that any two such universal objects $\nu : A \longrightarrow \Omega(A)$, $\nu' : A \longrightarrow \Omega'(A)$, are isomorphic in the sense of an isomorphism : $\Omega(A) \longleftrightarrow \Omega'(A)$ of $\mathbf{Z}_2$-graded differential R-algebras.

(7.1.7) Let $\phi : A \longrightarrow B$ be a homomorphism of associative superalgebras over R. Then using abstract nonsense, there is a unique homomorphism $\phi_* : \Omega(A) \longrightarrow \Omega(B)$ in the sense of $\mathbf{Z}_2$-graded differential algebras over R, extending ϕ, according to the diagram below.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi"'] & \Omega(A) \arrow[d, "\phi_*"] \\
B \arrow[r] & \Omega(B)
\end{tikzcd}$$

One easily verifies the covariant functorial properties:

$$(\psi \circ \phi)_* = \psi_* \circ \phi_*, \ (id\ A)_* = id\ \Omega(A).$$

If ϕ is bijective, then ϕ_* ist bijective too, and then $(\phi^{-1})_* = (\phi_*)^{-1}$.

7.2 Explicit Construction of the Unital Universal Differential Envelope

(7.2.1) Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be an associative superalgebra over the commutative ring R, and $R \oplus A =: \tilde{A}$ the unital associative superalgebra over R, which was defined in the foregoing chapter. With the tensor product of R-bimodules, the unital universal differential envelope $\tilde{\Omega}(A)$ is constructed.

$$\forall n \in \mathbf{N} : \tilde{\Omega}_n(A) \equiv \tilde{\Omega}_n := \tilde{A} \otimes \left(\bigotimes^n A \right), \ \tilde{\Omega}_0(A) \equiv \tilde{\Omega}_0 := \tilde{A},$$

$$\tilde{\Omega}(A) \equiv \tilde{\Omega} := \bigoplus_{n \in \mathbf{N}_0} \tilde{\Omega}_n(A).$$

(7.2.2.1) Due to the universal property of tensor products, $\forall n \in \mathbf{N}$, there is an R-bilinear mapping:

$$\tilde{A} \times \tilde{\Omega}_n \ni \{\tilde{a}, \tilde{\omega}_n\} \longrightarrow \tilde{a} \circ \tilde{\omega}_n \in \tilde{\Omega}_n,$$

such that $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \forall b_1, \ldots, b_n \in A$:

$$\tilde{a} \circ (\tilde{b} \otimes b_1 \otimes \cdots \otimes b_n) = (\tilde{a}\tilde{b}) \otimes b_1 \otimes \cdots \otimes b_n.$$

(7.2.2.2) Again with the universal property of tensor products, $\forall n \in \mathbf{N}$, there is an R-bilinear mapping:

$$\tilde{\Omega}_n \times A \ni \{\tilde{\omega}_n, b\} \longrightarrow \tilde{\omega}_n \circ b \in \tilde{\Omega}_n,$$

such that $\forall \overline{z_1}, \dots, \overline{z_n} \in \mathbf{Z}_2, \forall a_1 \in A^{\overline{z_1}}, \dots, a_n \in A^{\overline{z_n}}, \forall b \in A, \tilde{a} \in \tilde{A}$:

$$\begin{aligned}
&(\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n) \circ b = \tilde{a} \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n b) \\
&+ (-1)^{n+\sum_{l=1}^{n} z_l} (\tilde{a}a_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes b \\
&+\sum_{k=1}^{n-1}(-1)^{n-k+\sum_{l=k+1}^{n} z_l} \tilde{a} \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes (a_k a_{k+1}) \\
&\qquad\qquad\qquad\qquad \otimes a_{k+2} \otimes \cdots \otimes a_n \otimes b.
\end{aligned}$$

(7.2.2.2.1) For $n = 1, 2$, one obtains the following R-bilinear mappings. $\forall \tilde{a} \in \tilde{A}, a_1 \in A^{\overline{z_1}}, a_2 \in A^{\overline{z_2}}, b \in A$:

$$(\tilde{a} \otimes a_1) \circ b = \tilde{a} \otimes (a_1 b) + (-1)^{1+z_1} (\tilde{a}a_1) \otimes b,$$

$$\begin{aligned}
(\tilde{a} \otimes a_1 \otimes a_2) \circ b = {} & \tilde{a} \otimes a_1 \otimes (a_2 b) \\
& + (-1)^{2+z_1+z_2} (\tilde{a}a_1) \otimes a_2 \otimes b \\
& + (-1)^{1+z_2} \tilde{a} \otimes (a_1 a_2) \otimes b.
\end{aligned}$$

(7.2.2.3) An R-bilinear mapping:

$$\tilde{\Omega}_n \times \tilde{A} \ni \{\tilde{\omega}_n, \tilde{b}\} \longrightarrow \tilde{\omega}_n \circ \tilde{b} \in \tilde{\Omega}_n$$

is defined. $\forall n \in \mathbf{N}, \forall \tilde{\omega}_n \in \tilde{\Omega}_n, \forall \tilde{b} := \{\beta, b\} \equiv \beta + b \in \tilde{A}$:

$$\tilde{\omega}_n \circ \tilde{b} := \beta \tilde{\omega}_n + \tilde{\omega}_n \circ b.$$

$\forall n \in \mathbf{N}, \forall \tilde{\omega}_n \in \tilde{\Omega}_n, a \in A^{\bar{z}}, b \in A$:

$$(\tilde{\omega}_n \otimes a) \circ b = \tilde{\omega}_n \otimes (ab) - (-1)^z (\tilde{\omega}_n \circ a) \otimes b.$$

Here for convenience the following natural R-linear bijection is suppressed:

$$\tilde{\Omega}_n \otimes A \ni (\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b \longleftrightarrow \tilde{a} \otimes a_1 \otimes \cdots \otimes a_n \otimes b \in \tilde{\Omega}_{n+1}.$$

(7.2.2.4) There is an R-bilinear mapping:

$$\tilde{\Omega}_n \times \tilde{\Omega}_m \ni \{\tilde{\xi}_n, \tilde{\eta}_m\} \rightarrow \tilde{\xi}_n \circ \tilde{\eta}_m \in \tilde{\Omega}_{n+m},$$

such that $\forall n, m \in \mathbf{N}, \forall \tilde{\omega}_n \in \tilde{\Omega}_n, \tilde{b} \in \tilde{A}, \forall b_1, \dots, b_m \in A$:

$$\tilde{\omega}_n \circ (\tilde{b} \otimes b_1 \otimes \cdots \otimes b_m) = (\tilde{\omega}_n \circ \tilde{b}) \otimes b_1 \otimes \cdots \otimes b_m \in \tilde{\Omega}_{n+m}.$$

(7.2.2.5) Most frequently, for R-bimodules $E_i, F_l, i \in I, l \in L$, an R-linear bijection is used:

$$\left(\bigoplus_{i\in I} E_i\right) \otimes \left(\bigoplus_{l\in L} F_l\right) \ni \{x_i; i \in I\} \otimes \{y_l; l \in L\}$$
$$\longleftrightarrow \{x_i \otimes y_l; i \in I, l \in L\} \in \bigoplus_{i\in I, l\in L} (E_i \otimes F_l).$$

(7.2.2.6) There is an R-bilinear mapping:

$$\tilde{\Omega}(A) \times \tilde{\Omega}(A) \ni \{\tilde{\xi}, \tilde{\eta}\} \longrightarrow \tilde{\xi} \circ \tilde{\eta} \equiv \tilde{\xi}\tilde{\eta} \in \tilde{\Omega}(A),$$

such that for the elements

$$\tilde{\xi} = \{\tilde{\xi}_n; n \in \mathbf{N}_0\} : = \{\tilde{a}_0, \tilde{a}_1 \otimes a_1^1, \dots, \tilde{a}_n \otimes a_n^1 \otimes \cdots \otimes a_n^n, \dots\}$$
$$\equiv \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \otimes a_n^1 \otimes \cdots \otimes a_n^n,$$

$$\tilde{\eta} = \{\tilde{\eta}_n; n \in \mathbf{N}_0\} : = \{\tilde{b}_0, \tilde{b}_1 \otimes b_1^1, \dots, \tilde{b}_n \otimes b_n^1 \otimes \cdots \otimes b_n^n, \dots\}$$
$$\equiv \tilde{b}_0 + \sum_{n=1}^{\infty} \tilde{b}_n \otimes b_n^1 \otimes \cdots \otimes b_n^n,$$

$$\tilde{\xi} \circ \tilde{\eta} = \tilde{a}_0\tilde{b}_0 + \sum_{n\in\mathbf{N}} (\tilde{\xi}_n \circ \tilde{b}_0 + (\tilde{a}_0\tilde{b}_n) \otimes b_n^1 \otimes \cdots \otimes b_n^n) + \sum_{n,m\in\mathbf{N}} \tilde{\xi}_n \circ \tilde{\eta}_m.$$

Here for convenience the canonical embeddings are suppressed. Moreover all the multiple tensor products are taken without any brackets.

(7.2.2.6.1) For instance, one obtains
$\forall \tilde{a} := \alpha + a, \tilde{b} := \beta + b \in \tilde{A}, \forall a_1 \in A^{\overline{z_1}}, a_2 \in A^{\overline{z_2}}, \forall b_1, b_2 \in A$:

$$(\tilde{a} \otimes a_1 \otimes a_2) \circ (\tilde{b} \otimes b_1 \otimes b_2) = \beta\tilde{a} \otimes a_1 \otimes a_2 \otimes b_1 \otimes b_2$$
$$+ \tilde{a} \otimes a_1 \otimes (a_2 b) \otimes b_1 \otimes b_2 + (-1)^{2+z_1+z_2} (\tilde{a}a_1) \otimes a_2 \otimes b \otimes b_1 \otimes b_2$$
$$+ (-1)^{1+z_2} \tilde{a} \otimes (a_1 a_2) \otimes b \otimes b_1 \otimes b_2.$$

(7.2.2.7) $\forall \tilde{\xi}_n \in \tilde{\Omega}_n, \tilde{\eta}_m \in \tilde{\Omega}_m, \forall k \in \mathbf{N}, \forall \omega_k \in \bigotimes^k A$:

$$\tilde{\xi}_n \circ (\tilde{\eta}_m \otimes \omega_k) = (\tilde{\xi}_n \circ \tilde{\eta}_m) \otimes \omega_k \in \tilde{\Omega}_{n+m+k}.$$

(7.2.2.8) This R-bilinear mapping provides an associative product, i.e., one finds, that

$$\forall \tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \in \tilde{\Omega}(A) : (\tilde{\xi} \circ \tilde{n}) \circ \tilde{\zeta} = \tilde{\xi} \circ (\tilde{\eta} \circ \tilde{\zeta}) \in \tilde{\Omega}(A).$$

An easy proof is obtained via an induction with respect to $n, m \in \mathbf{N}$, following the two steps indicated below.
$\forall \tilde{a}, \tilde{b}, \tilde{c} \in \tilde{A}, \forall a_1, \ldots, a_n \in A, \forall b_1, \ldots, b_m \in A, \forall \tilde{\xi} \in \tilde{\Omega}(A)$:

$$\text{(i)} \quad ((\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n) \circ \tilde{b}) \circ \tilde{c} = (\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n) \circ (\tilde{b}\tilde{c}),$$

$$\text{(ii)} \quad (\tilde{\xi} \circ (\tilde{b} \otimes b_1 \otimes \cdots \otimes b_m)) \circ \tilde{c} = \tilde{\xi} \circ ((\tilde{b} \otimes b_1 \otimes \cdots \otimes b_m) \circ \tilde{c}).$$

(7.2.2.9) With the above defined bilinear mapping, the direct sum of R-bimodules

$$\tilde{\Omega}(A) := \tilde{A} \oplus \bigoplus_{n \in \mathbf{N}} \left(\tilde{A} \otimes \left(\bigotimes^{n} A \right) \right)$$

becomes an associative algebra over R, with the unit $\tilde{e} := \{\{e_R, 0_A\}, 0, 0, \ldots\}$. One conveniently writes $\tilde{e} \equiv e_R$.

(7.2.3) The unital associative algebra $\tilde{\Omega}(A)$ is $\mathbf{Z}_2$-graded, according to the definition, that $\forall \bar{z} \in \mathbf{Z}_2, \forall n \in \mathbf{N}$:

$$\tilde{\Omega}_n^{\bar{z}}(A) := \bigoplus_{\overline{z_0}, \overline{z_1}, \ldots, \overline{z_n} \in \mathbf{Z}_2; n + z_0 + z_1 + \cdots + z_n \in \bar{z}} \left(\tilde{A}^{\overline{z_0}} \otimes A^{\overline{z_1}} \otimes \cdots \otimes A^{\overline{z_n}} \right),$$

$$\tilde{\Omega}^{\bar{z}}(A) := \tilde{A}^{\bar{z}} \oplus \left(\bigoplus_{n \in \mathbf{N}} \tilde{\Omega}_n^{\bar{z}}(A) \right).$$

Obviously this $\mathbf{Z}_2$-grading is compatible with the associative product of $\tilde{\Omega}(A)$ in the sense, that

$$\forall \tilde{\xi} \in \tilde{\Omega}(A)^{\bar{x}}, \tilde{\eta} \in \tilde{\Omega}(A)^{\bar{y}} : \tilde{\xi} \circ \tilde{\eta} \in \tilde{\Omega}(A)^{\overline{x+y}}.$$

(7.2.3.1) This $\mathbf{Z}_2$-grading ∂ explicitly reads,
$\forall \overline{z_0}, \overline{z_1}, \ldots, \overline{z_n} \in \mathbf{Z}_2, \forall \tilde{a}_0 \in \tilde{A}^{\overline{z_0}}, a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}$:

$$\partial(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n) = \overline{n + z_0 + \cdots + z_n} \in \mathbf{Z}_2, \;\; \partial \tilde{a}_0 = \overline{z_0}.$$

(7.2.4) The natural $\mathbf{N}_0$-grading ν of $\tilde{\Omega}(A)$, such that

$$\forall \tilde{a}_0 \in \tilde{A}, \forall a_1, \ldots, a_n \in A : \nu(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n) = n \in \mathbf{N}, \;\; \nu \tilde{a}_0 = 0,$$

is also compatible with the above associative product. Moreover, by its very definition, the $\mathbf{Z}_2$-grading ∂ is compatible with ν.

(7.2.5) Due to the universal property of tensor products, there are R-linear mappings

$$d_0 : \tilde{A} \longrightarrow \tilde{\Omega}_1(A),\ d_n : \tilde{\Omega}_n(A) \longrightarrow \tilde{\Omega}_{n+1}(A),\ \text{for } n \in \mathbf{N},$$

such that $\forall \tilde{a} := \{\alpha, a\} \equiv \alpha + a \in \tilde{A}, \forall n \in \mathbf{N}, \forall a_1, \ldots, a_n \in A :$

$$d_0(\tilde{a}) \equiv d_0\tilde{a} = e_{\tilde{A}} \otimes a,\ d_n(\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n) = e_{\tilde{A}} \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

(7.2.5.1) Obviously, $\forall n \in \mathbf{N}_0 : d_{n+1} \circ d_n = 0$. Hence one obtains the following complex of R-bimodules:

$$d := \bigoplus_{n \in \mathbf{N}_0} d_n : \tilde{\Omega}(A) \longrightarrow \tilde{\Omega}(A),\ d \circ d = 0.$$

(7.2.5.2) The $\mathbf{Z}_2$-grading of $\tilde{\Omega}(A)$ fulfills the condition, that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall \tilde{\xi} \in \tilde{\Omega}^{\bar{x}}(A), a \in A^{\bar{y}} : \tilde{\xi} \otimes a \in \tilde{\Omega}^{\overline{x+y+1}}(A).$$

(7.2.5.3) This R-linear mapping d is some odd derivation of $\tilde{\Omega}(A)$, i.e.,

$$\forall \tilde{\xi} \in \tilde{\Omega}^{\bar{z}}(A) : d\tilde{\xi} \in \tilde{\Omega}^{\overline{z+1}}(A),$$
$$\forall \tilde{\eta} \in \tilde{\Omega}(A) :\ \tilde{\xi} \circ \tilde{\eta} \xrightarrow[d]{} (d\tilde{\xi}) \circ \tilde{\eta} + (-1)^{z} \tilde{\xi} \circ (d\tilde{\eta}).$$

An easy proof is provided by an induction with respect to $n \in \mathbf{N}$,

$$\tilde{\xi} := \tilde{a} \otimes a_1 \otimes \cdots \otimes a_n.$$

(7.2.5.4) An induction with respect to $n \in \mathbf{N}$ leads to the convenient expression, that $\forall \tilde{a} \in \tilde{A}, \forall a_1, \ldots, a_n \in A :$

$$\tilde{a} \otimes a_1 \otimes \cdots \otimes a_n = \tilde{a} \circ (d_0 a_1) \circ \cdots \circ (d_0 a_n) \equiv \tilde{a}(da_1) \cdots (da_n),$$

where the embedding of A into $\tilde{A}$ is usually suppressed.
Especially one finds, that $\forall a_1, \ldots, a_n \in A :$

$$e_{\tilde{A}} \otimes a_1 \otimes \cdots \otimes a_n = (d_0 a_1) \circ \cdots \circ (d_0 a_n) \equiv (da_1) \cdots (da_n).$$

(7.2.6) The R-linear submodule

$$\Omega(A) := A \oplus \bigoplus\nolimits_{n \in \mathbf{N}} \Omega_n(A),\ \Omega_n(A) := \tilde{A} \otimes \left(\bigotimes^{n} A \right),$$

such that $\tilde{\Omega}(A) = R \oplus \Omega(A)$, is an ideal of $\tilde{\Omega}(A)$. Obviously the canonical projection of $\tilde{\Omega}(A)$ onto $\Omega(A)$ is commuting both with the canonical projections of $\tilde{\Omega}(A)$ onto $\tilde{\Omega}_n(A)$, and with those onto $\tilde{\Omega}^{\bar{z}}(A)$, for $n \in \mathbf{N}_0, \bar{z} \in \mathbf{Z}_2$. Moreover

$$Im\ d = Im\ d\big|_{\Omega(A)} \subseteq \Omega(A).$$

(7.2.6.1)

$$\forall n \in \mathbf{N} : d\tilde{\Omega}_n(A) \oplus \big(A \circ d\tilde{\Omega}_n(A)\big) = \tilde{\Omega}_{n+1}(A),\ dA \oplus (A \circ dA) = \tilde{A} \otimes A.$$

$$\Omega(A) = d\Omega(A) \oplus (A \circ \Omega(A)).$$

(7.2.7) Both the unital associative algebra $\tilde{\Omega}(A)$, and its ideal $\Omega(A)$, with the two gradings ∂ and ν defined above, are bigraded differential algebras over R with respect to d. $\tilde{\Omega}(A)$ is called the unital universal differential envelope of A, and $\Omega(A)$ is called the non-unital universal differential envelope of A. Note that A need not be unital.

(7.2.8) The $\mathbf{Z}_2$-graded differential algebra $\Omega(A)$ over R fulfills the universal property defined in the foregoing chapter.

(7.2.8.1) Let D be an arbitrary $\mathbf{Z}_2$-graded differential algebra over R, with the odd derivation $\delta : D \longrightarrow D$, and consider a homomorphism $\phi : A \longrightarrow D$ of associative superalgebras over R. Then the universal property of tensor products allows for the definition of an R-linear mapping $\phi_* : \Omega(A) \longrightarrow D$, such that $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in A : \phi_*(a_0) = \phi(a_0)$,

$$a_0(da_1)\cdots(da_n) \underset{\phi_*}{\longrightarrow} \phi(a_0)(\delta \circ \phi(a_1))\cdots(\delta \circ \phi(a_n)),$$

$$(da_1)\cdots(da_n) \underset{\phi_*}{\longrightarrow} (\delta \circ \phi(a_1))\cdots(\delta \circ \phi(a_n)).$$

Here one uses, that $\forall n \in \mathbf{N}$:

$$\Omega_n(A) = \tilde{\Omega}_n(A)$$

$$= R - lin\ span\{a_0 \otimes a_1 \otimes \cdots \otimes a_n, e_{\tilde{A}} \otimes a_1 \otimes \cdots \otimes a_n; a_0, \ldots, a_n \in A\}.$$

(7.2.8.2) One then easily finds, that ϕ_* conserves the $\mathbf{Z}_2$-grading, and that $\phi_* \circ d = \delta \circ \phi_*$, using the fact, that $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in A$:

$$d(a_0(da_1)\cdots(da_n)) = (da_0)(da_1)\cdots(da_n) \in \Omega_{n+1}(A).$$

(7.2.8.3) In order to prove that ϕ_* is a homomorphism of associative superalgebras over R, one proceeds in several steps. $\forall a_0, a_1, \ldots, a_n, b_0, b_1 \in A$:

$$a_0(da_1)\cdots(da_n)(db_1) \underset{\phi_*}{\longrightarrow} \phi_*(a_0(da_1)\cdots(da_n))\phi_*(db_1),$$

$$(da_1)\cdots(da_n)(db_1) \underset{\phi_*}{\longrightarrow} \phi_*((da_1)\cdots(da_n))\phi_*(db_1),$$

$$a_0(da_1)\cdots(da_n)b_0 \underset{\phi_*}{\longrightarrow} \phi_*(a_0(da_1)\cdots(da_n))\phi(b_0),$$

$$(da_1)\cdots(da_n)b_0 \underset{\phi_*}{\longrightarrow} \phi_*((da_1)\cdots(da_n))\phi(b_0),$$

via an induction with respect to $n \in \mathbf{N}$. These results are used for an easy finish of the proof.

(7.2.8.4) Uniqueness of ϕ_* follows from $\phi_* \circ d = \delta \circ \phi_*$.

(7.2.8.5) Especially let D be bigraded, and assume $Im\ \phi \subseteq D_0$, which then implies, that

$$\forall \bar{z} \in \mathbf{Z}_2 : \ Im\ \phi\big|_{A^{\bar{z}}} \subseteq D^{\bar{z}} \cap D_0.$$

Then the induced R-linear mapping ϕ_* is some homomorphism in the sense of bigraded differential algebras over R, i.e.,

$$\forall n \in \mathbf{N}_0 : \ Im\ \phi_*\big|_{\Omega_n A} \subseteq D_n, \ \Omega_0(A) := A.$$

7.3 R-Endomorphisms of $\Omega(A)$

(7.3.1) Consider the non-unital universal differential envelope,

$$\Omega(A) := A \oplus \bigoplus_{n \in \mathbf{N}} \Omega_n(A), \ A =: \Omega_0(A), \ \forall n \in \mathbf{N} :$$

$$\Omega_n(A) = R - lin\ span\{(r + a_0)(da_1) \cdots (da_n); a_0, a_1, \ldots, a_n \in A; r \in R\},$$

of an associative superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$ over the commutative ring R, with the $\mathbf{Z}_2$-grading ∂.

(7.3.1.1) $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n, \ b \in A, \forall \omega, \xi, \eta \in \Omega(A), \forall r \in R$:

$$\partial(a_0(da_1) \cdots (da_n)) = \bar{n} + \partial a_0 + \sum_{k=1}^{n} \partial a_k \text{ for homogeneous } a_0, a_1, \ldots, a_n;$$

$$d(a_0(da_1) \cdots (da_n)) = (da_0)(da_1) \cdots (da_n); \ d \circ d = 0;$$
$$\partial(d\omega) = \bar{1} + \partial\omega \text{ for homogeneous elements } \omega;$$
$$d(\xi\eta) = (d\xi)\eta + (-1)^x \xi(d\eta) \text{ for } \xi \in \Omega^{\bar{x}}(A), \bar{x} \in \mathbf{Z}_2.$$

(7.3.1.2) For $n = 2, 3, \ldots, \forall a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}, \forall a_0, b \in A$:

$$\begin{aligned}
&(r + a_0)(da_1) \cdots (da_n)b = (r + a_0)(da_1) \cdots (da_{n-1})(d(a_n b)) \\
&+ (-1)^{n+z_1+\cdots+z_n}(ra_1 + a_0 a_1)(da_2) \cdots (da_n)(db) \\
&+ \sum_{k=1}^{n-1} (-1)^{n-k+\sum_{l=k+1}^{n} z_l}(r + a_0)(da_1) \cdots (da_{k-1}) \\
&\qquad\qquad (d(a_k a_{k+1}))(da_{k+2}) \cdots (da_n)(db);
\end{aligned}$$

for instance,

$$\begin{aligned}
&(r + a_0)(da_1)b = (r + a_0)(d(a_1 b)) + (-1)^{1+z_1}(ra_1 + a_0 a_1)(db), \\
&a_0(da_1)(da_2)b = a_0(da_1)(d(a_2 b)) \\
&\qquad + (-1)^{2+z_1+z_2} a_0 a_1 (da_2)(db) + (-1)^{1+z_2} a_0 (d(a_1 a_2))(db).
\end{aligned}$$

(7.3.1.3) $\partial(\xi\eta) = \partial\xi + \partial\eta$, for homogeneous elements $\xi, \eta \in \Omega(A)$.

(7.3.2) The R-linear mappings $\beta', \beta, \gamma, \lambda : \Omega(A) \longrightarrow \Omega(A)$, are defined with the tensor products:

$$R \times A \longrightarrow R \otimes A,\ A \times A \longrightarrow A \otimes A,\ \Omega_n(A) \times A \longrightarrow \Omega_{n+1}(A);$$

$\forall n \in \mathbf{N}, \forall r \in R, \forall a_0 \in A^{\overline{z_0}}, a_1 \in A^{\overline{z_1}}, b \in A^{\overline{z}}, \forall \omega_n \in \Omega_n^{\overline{z_n}}(A)$:

$$\begin{aligned}
&r da_1 \xrightarrow[\beta']{} r a_1,\ a_0 da_1 \xrightarrow[\beta']{} (-1)^{z_0} a_0 a_1,\ \omega_n db \xrightarrow[\beta']{} (-1)^{z_n} \omega_n b,\ b \xrightarrow[\beta']{} 0,\\
&r da_1 \xrightarrow[\beta]{} 0,\ a_0 da_1 \xrightarrow[\beta]{} (-1)^{z_0}(a_0 a_1 - (-1)^{z_0 z_1} a_1 a_0),\\
&\omega_n db \xrightarrow[\beta]{} (-1)^{z_n}(\omega_n b - (-1)^{z_n z} b\omega_n),\ b \xrightarrow[\beta]{} 0,\\
&r da_1 \xrightarrow[\gamma]{} 0,\ a_0 da_1 \xrightarrow[\gamma]{} a_0 da_1 - (-1)^{(1+z_1)z_0}(da_1)a_0,\\
&\omega_n db \xrightarrow[\gamma]{} \omega_n db - (-1)^{(1+z)z_n}(db)\omega_n,\ b \xrightarrow[\gamma]{} 0,\\
&r da_1 \xrightarrow[\lambda]{} 0,\ a_0 da_1 \xrightarrow[\lambda]{} (-1)^{(1+z_0)(1+z_1)} a_1 da_0,\\
&\omega_n db \xrightarrow[\lambda]{} (-1)^{(1+z_n)(1+z)} b d\omega_n,\ b \xrightarrow[\lambda]{} b.
\end{aligned}$$

(7.3.2.1) Hence one defines $\alpha := \beta' - \beta,\ \rho := \varepsilon - \gamma,\ \varepsilon := id\ \Omega(A)$, and finds, with the notation from above, that:

$$\begin{aligned}
&r da_1 \xrightarrow[\alpha]{} r a_1,\ a_0 da_1 \xrightarrow[\alpha]{} (-1)^{z_0(1+z_1)} a_1 a_0,\\
&\omega_n db \xrightarrow[\alpha]{} (-1)^{z_n(1+z)} b\omega_n,\ b \xrightarrow[\alpha]{} 0,\\
&r da_1 \xrightarrow[\rho]{} r da_1,\ a_0 da_1 \xrightarrow[\rho]{} (-1)^{z_0(1+z_1)}(da_1)a_0,\\
&\omega_n db \xrightarrow[\rho]{} (-1)^{z_n(1+z)}(db)\omega_n,\ b \xrightarrow[\rho]{} b.
\end{aligned}$$

(7.3.2.2) Furthermore one defines the direct sums

$$\Lambda := \bigoplus_{n \in \mathbf{N}_0} \Lambda_n,\ \sigma := \bigoplus_{n \in \mathbf{N}_0} \sigma_n,\ \sigma_0 := 0,$$

$$\sigma_{n+1} := \sum_{k=1}^{n+1} \rho^{k-n-1}\ \alpha\rho^{n+1-k}\big|_{\Omega_{n+1}(A)},\ \rho^0 := \varepsilon,$$

$$\sigma_1 = \alpha\big|_{\Omega_1(A)},\ \sigma_2 = \rho^{-1}\alpha\rho + \alpha\big|_{\Omega_2(A)},$$

$$\Lambda_n := \sum_{k=0}^{n} \lambda^k\big|_{\Omega_n(A)},\ \lambda^0 := \varepsilon,\ \Lambda_0 := \textit{ embedding} \text{ of } A \text{ into } \Omega(A).$$

(7.3.3) Here one uses, that ρ is an R-linear bijection of $\Omega(A)$ onto itself. $\forall \bar{z}, \bar{y} \in \mathbf{Z}_2, \forall n \in \mathbf{N}, \forall \omega \in \Omega^{\bar{z}}(A), \eta_n \in \Omega_n^{\bar{y}}(A)$:

$$\omega d\eta_n \underset{\rho^{n+1}}{\longrightarrow} (-1)^{z(1+y)}(d\eta_n)\omega.$$

(7.3.3.1)

$$\forall n \in \mathbf{N} : d\Omega_n(A) \oplus (A \circ d\Omega_n(A)) = \Omega_{n+1}(A);$$
$$dA \oplus (A \circ dA) = \Omega_1(A), \ \Omega_0(A) := A.$$
$$d\Omega(A) = dA \oplus \bigoplus_{n\in\mathbf{N}} d\Omega_n(A),$$
$$d\Omega_n(A) = sum\ (\{(da_1)\cdots(da_{n+1}); a_1, \ldots, a_{n+1} \in A\});$$
$$\Omega(A) = A \oplus d\Omega(A) \oplus (A \circ d\Omega(A)),$$
$$A \circ d\Omega(A) = (A \circ dA) \oplus \bigoplus_{n\in\mathbf{N}} (A \circ d\Omega_n(A)).$$

(7.3.3.2) Remember here the R-linear bijections, $\forall n \in \mathbf{N}$:

$$d\Omega_n(A) \longleftrightarrow R \otimes \left(\bigotimes^{n+1} A\right), \ A \circ d\Omega_n(A) \longleftrightarrow \bigotimes^{n+2} A;$$

$$dA \longleftrightarrow R \otimes A, \ A \circ dA \longleftrightarrow A \otimes A.$$

(7.3.3.3) For an easy proof of ρ to be bijective, one needs the following R-linear bijections, which correspond to permutations of tensor products.

$$\rho|_{dA} = embedding \text{ of } dA \text{ into } \Omega(A);$$

$$\forall n \in \mathbf{N} : \rho|_{d\Omega_n(A)} : d\Omega_n(A) \longleftrightarrow d\Omega_n(A) \underset{embedding}{\longrightarrow} \Omega(A).$$

$$\rho|_{d\Omega(A)} : d\Omega(A) \longleftrightarrow d\Omega(A) \underset{embedding}{\longrightarrow} \Omega(A).$$

(7.3.3.4)

$$d\Omega_n(A) \ni (da_1)\cdots(da_{n+1})$$
$$\underset{\rho}{\longrightarrow}(-1)^{(1+z_{n+1})(n+z_1+\cdots+z_n)}(da_{n+1})(da_1)\cdots(da_n),$$

for homogeneous elements $a_k \in A^{\overline{z_k}}, k = 1, \ldots, n+1$, for $n \in \mathbf{N}$.
Hence one obtains, $\forall n \in \mathbf{N}, \forall a_1, \ldots, a_n \in A$:

$$(da_1)\cdots(da_n) \underset{\rho^n}{\longrightarrow} (da_1)\cdots(da_n),$$

$$\rho^{n+1}|_{d\Omega_n(A)} = embedding \text{ into } \Omega(A).$$

(7.3.3.5) One then easily finds the R-linear bijections, $\forall n \in \mathbf{N}_0$:

$$\rho\big|_{\Omega_n(A)} : \Omega_n(A) \longleftrightarrow \Omega_n(A) \underset{embedding}{\longrightarrow} \Omega(A).$$

(7.3.4) In the special case of unital A, one defines an R-linear map $\tau : \Omega(A) \longrightarrow \Omega(A)$. $\forall \bar{z} \in \mathbf{Z}_2, \forall \omega \in \Omega^{\bar{z}}(A)$:

$$\tau(\omega) := (-1)^z \omega(de_A); \ (\varepsilon - \lambda)\tau =: \Gamma_0, \ \Gamma := \Gamma_0 \Lambda.$$

(7.3.5) One easily calculates, that $\forall n \in \mathbf{N}, \forall a_0 \in A^{\overline{z_0}}, \ldots, a_{n+1} \in A^{\overline{z_{n+1}}}$:

$$\begin{aligned}
& a_0(da_1)\cdots(da_{n+1}) \underset{\beta}{\longrightarrow} (-1)^{z_0} a_0 a_1 (da_2)\cdots(da_{n+1}) \\
& - (-1)^{(1+z_{n+1})(z_0+n+z_1+\cdots+z_n)} a_{n+1} a_0 (da_1)\cdots(da_n) \\
& + \sum_{k=1}^{n} (-1)^{k+z_0+\cdots+z_k} a_0(da_1)\cdots(d(a_k a_{k+1}))\cdots(da_{n+1}), \\
& (da_1)\cdots(da_{n+1}) \underset{\beta}{\longrightarrow} a_1(da_2)\cdots(da_{n+1}) \\
& - (-1)^{(1+z_{n+1})(n+z_1+\cdots+z_n)} a_{n+1}(da_1)\cdots(da_n) \\
& + \sum_{k=1}^{n} (-1)^{k+z_1+\cdots+z_k} (da_1)\cdots(d(a_k a_{k+1}))\cdots(da_{n+1}), \\
& \lambda\big|_{d\Omega(A)} = 0, \\
& a_0(da_1)\cdots(da_n) \underset{\lambda}{\longrightarrow} (-1)^{(1+z_n)(n+z_0+\cdots+z_{n-1})} a_n (da_0)\cdots(da_{n-1}).
\end{aligned}$$

(7.3.6) The R-linear mappings $d, \beta', \beta, \alpha, \sigma, \tau, \Gamma_0, \Gamma$ are odd with respect to the $\mathbf{Z}_2$-grading ∂. On the other hand, $\gamma, \rho, \lambda, \Lambda$ are even with respect to ∂.

(7.3.7) One uses the notation, that $\forall n \in \mathbf{N}_0, \forall \bar{z} \in \mathbf{Z}_2$:

$$\Omega_n^{\bar{z}}(A) := \Omega_n(A) \cap \Omega^{\bar{z}}(A).$$

(7.3.8) These R-endomorphisms of $\Omega(A)$ fulfill the following relations.

$$\begin{aligned}
& \beta^2 = \beta'^2 = \alpha\beta + \beta'\alpha = \alpha\beta' + \beta\alpha = 0, \ \beta' d + d\beta' = \varepsilon, \ \alpha d = \lambda, \\
& \alpha d + d\alpha = \rho, \ \gamma + d\alpha = \varepsilon - \lambda, \ \lambda^2 = \lambda\rho, \ \rho = d\alpha + \lambda, \ \beta d + d\beta = \gamma, \\
& \beta\gamma = \gamma\beta, \ \beta\rho = \rho\beta, \ d\gamma = \gamma d, \ d\rho = \rho d, \ \beta\rho^{-1} = \rho^{-1}\beta, \ d\rho^{-1} = \rho^{-1} d, \\
& \beta\lambda - \lambda\beta' + \alpha = 0, \ \beta(\varepsilon - \lambda) = (\varepsilon - \lambda)\beta', \\
& \beta'\lambda - \lambda\beta + \alpha\gamma = 0, \ \beta'(\lambda + \gamma) = (\lambda + \gamma)\beta.
\end{aligned}$$

$$\forall n \in \mathbf{N} : \lambda^{n+1} = \lambda^n \rho = \lambda \rho^n, \ \lambda^n \rho^{-n} = \lambda \rho^{-1}.$$

(7.3.9)

$$\Lambda' := \bigoplus_{n \in \mathbf{N}_0} \lambda^{n+1}\big|_{\Omega_n(A)}, \text{ i.e. :}$$

$$\Omega(A) \ni \{\omega_n; n \in \mathbf{N}_0\} \xrightarrow[def]{\Lambda'} \{\lambda^{n+1}\omega_n; n \in \mathbf{N}_0\} \in \Omega(A).$$

$$\Lambda'\big|_{d\Omega(A)} = 0, \ \Lambda'\big|_{A\circ\Omega(A)} : A \oplus (A \circ d\Omega(A)) = A \circ \Omega(A) \underset{embedding}{\longrightarrow} \Omega(A).$$

(7.3.10) $\forall n \in \mathbf{N}_0, \forall \omega_n \in \Omega_n(A)$:

$$\rho^n \omega_n = (\varepsilon + \rho^{-1}\beta d)\omega_n, \ \rho^{-n}\omega_n = (\varepsilon - \beta d)\omega_n,$$
$$\rho^{n+1}\omega_n = (\varepsilon - d\beta)\omega_n, \ \beta' d\omega_n = \rho^n \lambda \omega_n.$$

Here one uses the equations

$$\varepsilon - \beta d = \rho + d\beta, \ \rho + \beta d = \varepsilon - d\beta,$$

and that

$$Im\ \alpha = Im\ \lambda = A \oplus (A \circ d\Omega(A)).$$

(7.3.11)

$$\pi := \lambda\rho^{-1}, \ \varepsilon - \pi = d\alpha\rho^{-1}, \ \pi(\varepsilon - \pi) = 0.$$
$$\forall n \in \mathbf{N}_0 : \pi\omega_n = \lambda^{n+1}\omega_n = \lambda\rho^n\omega_n.$$
$$\varepsilon - \pi = \Lambda(\varepsilon - \lambda) = (\varepsilon - \lambda)\Lambda, \ Im\ \pi = Im\ \lambda = A \circ \Omega(A),$$
$$\pi\beta\pi = \beta\pi, \ \lambda\pi = \pi\lambda = \lambda.$$

(7.3.12) One finds the following images of projections.

$$Im\ \pi = ker\ (\varepsilon - \pi) = A \oplus (A \circ d\Omega(A)),$$
$$ker\ d = ker\ (\beta' d) = ker\ \pi = Im\ (\varepsilon - \pi) = Im\ (d\beta') = Im\ d = d\Omega(A),$$
$$ker\ \beta' = ker\ (d\beta') = Im\ (\beta' d) = Im\ \beta' \subseteq Im\ \pi.$$

Especially, $Im\ \beta' = Im\ \pi \Longrightarrow A \circ A = \{0\}$, the trivial case.

$$d\big|_{A\circ\Omega(A)} : A \circ \Omega(A) \longleftrightarrow d\Omega(A) \underset{embedding}{\longrightarrow} \Omega(A).$$

$$\Omega(A) = ker\ \pi \oplus Im\ \pi = ker\ (\beta' d) \oplus Im\ (\beta' d) = ker(d\beta') \oplus Im\ (d\beta').$$

(7.3.13) $\forall \omega \in \Omega(A), a \in A^{\bar{x}}, b \in A$:

$$\omega\ da\ db \underset{\beta'\rho^{-1}\beta'\rho}{\longrightarrow} (-1)^{x} \omega ab,$$

$$da\ db \underset{\beta'\rho^{-1}\beta'\rho}{\longrightarrow} (-1)^{x} ab.$$

$\forall n \in \mathbf{N}, \forall l = 0, 1, \ldots, n-1, \forall a_0 \in A^{\overline{z_0}}, \ldots, a_{n+1} \in A^{\overline{z_{n+1}}}$:

$$a_0 da_1 \cdots da_{n+1}$$

$$\underset{\rho^{n-l}}{\longrightarrow} (-1)^{(z_0+\cdots+z_{l+1}+l+1)(z_{l+2}+\cdots+z_{n+1}+n-l)} (da_{l+2}) \cdots (da_{n+1})$$

$$a_0(da_1) \cdots (da_{l+1}),$$

$$a_0 da_1 \cdots da_{n+1} \underset{\rho^{n+1}}{\longrightarrow} (-1)^{z_0(z_1+\cdots+z_{n+1}+n+1)} (da_1) \cdots (da_{n+1}) a_0.$$

Hence one obtains, that $\forall n \geq 3, \forall l = 2, \ldots, n-1, \forall a_l \in A^{\overline{z_l}}$:

$$a_0 da_1 \cdots da_{n+1}$$

$$\underset{\rho^{l-n}\beta'\rho^{-1}\beta'\rho\rho^{n-l}}{\longrightarrow} (-1)^{z_l} a_0 (da_1) \cdots (da_{l-1}) a_l a_{l+1} (da_{l+2}) \cdots (da_{n+1}).$$

$$\forall a_1 \in A^{\overline{z_1}} : a_0\ da_1\ da_2\ da_3 \underset{\rho^{-1}\beta'\rho^{-1}\beta'\rho^2}{\longrightarrow} (-1)^{z_1} a_0\ a_1\ a_2\ da_3.$$

$$a_0\ da_1\ da_2\ da_3 \underset{\beta'\rho^{-1}\beta'\rho^3}{\longrightarrow} 0, \quad a_0\ da_1\ da_2 \underset{\beta'\rho^{-1}\beta'\rho^2}{\longrightarrow} 0.$$

(7.3.13.1)

$$\forall a_1 \in A^{\overline{z_1}} : da_1\ da_2\ da_3 \underset{\rho^{-1}\beta'\rho^{-1}\beta'\rho^2}{\longrightarrow} (-1)^{z_1} a_1\ a_2\ da_3.$$

$\forall n \in \mathbf{N}, \forall l = 0, \ldots, n-1, \forall a_1 \in A^{\overline{z_1}}, \ldots, a_{n+1} \in A^{\overline{z_{n+1}}}$:

$$da_1 \cdots da_{n+1}$$

$$\underset{\rho^{n-l}}{\longrightarrow} (-1)^{(z_1+\cdots+z_{l+1}+l+1)(z_{l+2}+\cdots+z_{n+1}+n-l)} da_{l+2} \cdots da_{n+1}\ da_1 \cdots da_{l+1}.$$

$$\forall n \in \mathbf{N}, \forall a_1, \ldots, a_n \in A : da_1 \cdots da_n \underset{\rho^n}{\longrightarrow} da_1 \cdots da_n.$$

$\forall n \geq 3, \forall l = 2, \ldots, n-1, \forall a_l \in A^{\overline{z_l}}$:

$$da_1 \cdots da_{n+1}$$

$$\underset{\rho^{l-n}\beta'\rho^{-1}\beta'\rho\rho^{n-l}}{\longrightarrow} (-1)^{z_l} (da_1) \cdots (da_{l-1}) a_l a_{l+1} (da_{l+2}) \cdots (da_{n+1}).$$

(7.3.14)

$$\forall n \in \mathbf{N}_0, \forall \omega_n \in \Omega_n(A) : \rho^n \pi \omega_n = \beta' d\rho^n \omega_n, \ \rho^n(\varepsilon - \pi)\omega_n = (\varepsilon - \pi)\omega_n.$$
$$\forall n \in \mathbf{N}_0, \forall \omega_n \in \Omega_n(A) : \beta' \Lambda \omega_n - \Lambda \beta \omega_n = \pi \beta' \rho^n \omega_n.$$

Hence one concludes, that

$$\Lambda \pi = \pi \Lambda = \Lambda + \pi - \varepsilon, \ \beta' \Lambda - \Lambda \beta = \pi \beta'(\varepsilon - \pi) = \alpha \rho^{-1},$$
$$\alpha \rho^{-1} \alpha = 0, \ \beta' \rho^{-1} \beta' \rho = \beta \rho^{-1} \alpha \rho + \alpha \beta.$$

(7.3.14.1)

$$\alpha \rho^{-1} \beta = \alpha \rho^{-1} \beta' = -\beta' \alpha \rho^{-1}, \ \alpha \rho^{-1} \beta \big|_{A \circ \Omega(A)} = 0.$$

$\forall n \geq 2, \forall a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}, a_{n+1} \in A :$

$$\alpha \rho^{-1} \beta(da_1 \cdots da_{n+1}) = (-1)^{z_1 + \cdots + z_n + n} a_1 (da_2) \cdots (da_n) a_{n+1}.$$
$$\forall a_1 \in A^{\overline{z_1}}, a_2 \in A : \alpha \rho^{-1} \beta((da_1)\ (da_2)) = (-1)^{1+z_1} a_1 a_2.$$

(7.3.15)

$$\Sigma := \beta(\sigma - \alpha) + (\sigma + \alpha \rho^{-1})\beta, \ \forall n \in \mathbf{N}, \ \forall \omega_{n+1} \in \Omega_{n+1}(A) :$$
$$(\Sigma - \alpha \rho^{-1} \beta)\omega_{n+1} = \sum_{l=1}^{n} \rho^{l-n} \beta' \rho^{-1} \beta' \rho \rho^{n-l} \omega_{n+1}.$$

Especially one finds, that

$$\Sigma \big|_{A \oplus dA \oplus (A \circ dA)} = \alpha \rho^{-1} \beta \big|_{\Omega_0(A) \oplus \Omega_1(A)} = 0.$$

(7.3.15.1) Therefore, $\forall n \geq 3, \forall a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}, \forall a_0, a_{n+1} \in A :$

$$\begin{aligned}
&\Sigma(a_0\ da_1 \cdots da_{n+1}) \\
&= \sum_{l=2}^{n-1} (-1)^{z_l} a_0 (da_1) \cdots (da_{l-1}) a_l a_{l+1} (da_{l+2}) \cdots (da_{n+1}) \\
&+ (-1)^{z_n} a_0 (da_1) \cdots (da_{n-1}) a_n a_{n+1} + (-1)^{z_1} a_0\ a_1\ a_2\ da_3 \cdots da_{n+1}, \\
&\Sigma(a_0\ da_1\ da_2\ da_3) = (-1)^{z_1} a_0 a_1 a_2\ da_3 + (-1)^{z_2} a_0\ da_1\ a_2\ a_3, \\
&\Sigma(a_0\ da_1\ da_2) = (-1)^{z_1} a_0\ a_1\ a_2.
\end{aligned}$$

(7.3.15.2)

$$\Sigma(da_1\ da_2) = 0,\ \Sigma(da_1\ da_2\ da_3) = (-1)^{z_2} d(a_1 a_2 a_3).$$

$\forall n \geq 3, \forall a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}, a_{n+1} \in A :$

$$\begin{aligned}\Sigma(da_1 \cdots da_{n+1}) &= \sum_{l=2}^{n-1} (-1)^{z_l} (da_1) \cdots (da_{l-1}) a_l a_{l+1} (da_{l+2}) \cdots (da_{n+1}) \\ &+ (-1)^{z_n} (da_1) \cdots (da_{n-1}) a_n a_{n+1} + (-1)^{z_1} a_1\ a_2\ da_3 \cdots da_{n+1} \\ &+ (-1)^{z_1 + \cdots + z_n + n} a_1 (da_2) \cdots (da_n) a_{n+1}.\end{aligned}$$

(7.3.16) One then straightforward calculates, that $\Sigma d = d\Sigma$.

(7.3.17) In the special case of unital A one obtains the relations

$$\begin{aligned}&\tau\pi = \pi\tau = \pi\tau\pi,\ \Gamma_0\pi = \pi\Gamma_0 = \pi\Gamma_0\pi,\ \Gamma\pi = \pi\Gamma = \pi\Gamma\pi, \\ &\tau d = -d\tau = \Gamma_0 d,\ \tau\beta' + \beta'\tau = \varepsilon.\end{aligned}$$

$$\forall \omega \in \Omega(A) : \alpha\tau(\omega) = e_A \omega.$$

Note, that

$$\forall a \in A : e_A\ da - da = \{-e_R, e_A\} \otimes a.$$

In the case of a field R one finds, that $\forall a \in A \setminus \{0_A\} : e_A\ da \neq da$.
The tensor product of R-bimodules may collapse.

$$\begin{aligned}&\tau\beta'\pi = -\beta\tau\pi = -\beta\pi\tau,\ \beta\Gamma_0 + \Gamma_0\beta' = \varepsilon - \lambda, \\ &\Lambda\pi\beta(\varepsilon - \pi) = 0,\ (\beta\Gamma + \Gamma\beta)\pi = 0, \\ &\Gamma^2 = (\varepsilon - \lambda)\tau^2(\varepsilon - \pi) = (\varepsilon - \pi)\tau^2\Lambda(\varepsilon - \pi),\ (\pi\Gamma\pi)^2 = 0.\end{aligned}$$

(7.3.18) All these R-linear mappings are elements of the unital associative superalgebra $End_R(\Omega(A))$, with the unit $id\ \Omega(A) =: \varepsilon$.
Σ is an even R-endomorphism of $\Omega(A)$, with respect to the $\mathbf{Z}_2$-grading ∂.

7.4 Hochschild and Cyclic Cohomology

(7.4.1) Consider the dual bimodule of the non-unital differential envelope $\Omega(A) \equiv \Omega$,

$$\Omega^* \equiv \Omega^*(A) := \ Hom_R(\Omega(A), R),$$

of an associative superalgebra A over R.

(7.4.2) For any set $T \subseteq \Omega(A)$, one denotes

$$\{f \in \Omega^*(A); \forall \omega \in T : f(\omega) = 0\} =: T^0.$$

(7.4.3) The linear form $f \in \Omega^*(A)$ is called closed, if and only if $f \circ d = 0$; f is called cyclic, if and only if $f \circ \lambda = f$.

$$C^* \equiv C^*(A) := \{f \in \Omega^*(A); f \circ d = 0\},$$

$$C^*_\lambda \equiv C^*_\lambda(A) := \{f \in \Omega^*(A); f \circ \lambda = f\} \subseteq C^*(A), \text{ since } \lambda d = \alpha d^2 = 0.$$

(7.4.3.1) Furthermore, $f \in \Omega^*(A)$ is called graded trace, if and only if it vanishes on super-commutators, i.e.,

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall \xi \in \Omega^{\bar{x}}(A), \eta \in \Omega^{\bar{y}}(A) : f(\xi\eta) = (-1)^{xy} \; f(\eta\xi).$$

(7.4.3.2) One easily calculates the following implications, $\forall f \in \Omega^*(A)$:

$$\begin{aligned}
&f \text{ graded trace} \Longleftrightarrow f \circ \beta = f \circ \gamma = 0;\\
&[f \text{ closed, and } f \text{ graded trace}] \Longleftrightarrow f \circ d = f \circ \beta = 0;\\
&f \text{ cyclic} \Longleftrightarrow f \circ d = f \circ \gamma = 0;\\
&[f \circ \beta \text{ cyclic, and } f \text{ closed}] \Longleftrightarrow f \text{ cyclic}.
\end{aligned}$$

Here one uses the convenient notation, that

$$\forall \phi \in End_R \; (\Omega(A)), f \in \Omega^*(A), \omega \in \Omega(A) : \; f \circ \phi(\omega) := f(\phi(\omega));$$

hence $\forall \psi \in End_R \; (\Omega(A)) : (f \circ \phi) \circ \psi = f \circ (\phi\psi)$.

(7.4.3.3) $\forall n \in \mathbf{N}_0$:

$$\begin{aligned}
&\Omega^*_n \equiv \Omega^*_n(A) := \{f \in \Omega^*(A); \forall k \in \mathbf{N}_0 : k \neq n \Longrightarrow f|_{\Omega_k(A)} = 0\},\\
&C^*_n \equiv C^*_n(A) := C^*(A) \cap \Omega^*_n(A),\\
&C^*_{\lambda n} \equiv C^*_{\lambda n}(A) := C^*_\lambda(A) \cap \Omega^*_n(A) \subseteq C^*_n(A).
\end{aligned}$$

(7.4.3.4)

$$\begin{aligned}
&Z^* \equiv Z^*(A) := \{f \in C^*(A); f \circ \beta\pi = 0\},\\
&B^* \equiv B^*(A) := \{f \in \Omega^*(A); \exists \; g \in C^*(A) : g \circ \beta\pi = f\} \subseteq Z^*(A);
\end{aligned}$$

$$\begin{aligned}
\forall n \in \mathbf{N}_0 : \; &Z^*_n \equiv Z^*_n(A) := Z^*(A) \cap \Omega^*_n(A),\\
&B^*_n \equiv B^*_n(A) := B^*(A) \cap \Omega^*_n(A) \subseteq Z^*_n(A);
\end{aligned}$$

here one uses, that $\beta\pi\beta\pi = 0$.
The elements of C^*, Z^*, B^* are called Hochschild cochains, cocycles, and coboundaries of A.

(7.4.3.4.1) The Hochschild cohomologies of A are defined as factor modules over R.

$$H^* \equiv H^*(A) := Z^*(A)/B^*(A);$$
$$\forall n \in \mathbf{N}_0 : H^*_n \equiv H^*_n(A) := Z^*_n(A)/B^*_n(A).$$

(7.4.3.4.2) The Hochschild coboundary operator b is defined as the R-linear surjective mapping:

$$C^*(A) \ni f \xrightarrow[\text{def}]{b} f \circ \beta\pi \in B^*(A).$$

(7.4.3.5) Correspondingly the cyclic cochains, cocycles, coboundaries, and cohomologies of A are defined, and also the Hochschild coboundary operator is restricted to the cyclic one.

$$Z^*_\lambda \equiv Z^*_\lambda(A) = \{f \in C^*_\lambda(A); f \circ \beta = 0\} = \{f \in C^*(A); f \circ \beta = 0\}.$$

$Z^*_\lambda(A)$ is the $R-$submodule of closed, graded traces on $\Omega(A)$.

$$B^*_\lambda \equiv B^*_\lambda(A) := \{f \in \Omega^*(A); \exists g \in C^*_\lambda(A) : g \circ \beta = f\} \subseteq Z^*_\lambda(A),$$

due to the nilpotent endomorphism β.

$$H^*_\lambda \equiv H^*_\lambda(A) := Z^*_\lambda(A)/B^*_\lambda(A).$$

Furthermore one defines $\forall n \in \mathbf{N}_0$:

$$Z^*_{\lambda n} \equiv Z^*_{\lambda n}(A) := Z^*_\lambda(A) \cap \Omega^*_n(A),$$
$$B^*_{\lambda n} \equiv B^*_{\lambda n}(A) := B^*_\lambda(A) \cap \Omega^*_n(A) \subseteq Z^*_{\lambda n}(A),$$
$$H^*_{\lambda n} \equiv H^*_{\lambda n}(A) := Z^*_{\lambda n}(A)/B^*_{\lambda n}(A);$$

$$C^*_\lambda(A) \ni f \xrightarrow[\text{def}]{b_\lambda} f \circ \beta \in B^*_\lambda(A).$$

Obviously the R-linear mapping b_λ is surjective.

$$\forall f \in C^*_\lambda(A) : b_\lambda(f) = b(f).$$

(7.4.3.6)

$$\forall f \in \Omega^*(A) : f \textit{ closed} \iff f \circ \Lambda \textit{ cyclic}.$$

(7.4.3.7) $C^*_{\lambda 0}(A) = C^*_0(A) = \Omega^*_0(A)$, of course. In the cases of $R := \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, one finds that

$$\forall n \in \mathbf{N}_0 : C^*_{\lambda n}(A) = \{g \in \Omega^*_n(A); \exists f \in C^*_n(A) : f \circ \Lambda = g\},$$

because

$$\forall g \in C^*_{\lambda n}(A) : g \circ \Lambda - (n+1)g = 0;$$

hence one obtains

$$C^*_\lambda(A) = \{g \in \Omega^*(A); \exists f \in C^*(A) : f \circ \Lambda = g\}.$$

(7.4.3.8) Consider again the cases of $R := \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$. Then one finds, that $\forall n \in \mathbf{N}_0$,

$$\forall g \in C_n^*(A) : g \circ \Lambda = 0 \Longleftrightarrow \exists f \in C_n^*(A) : f \circ (\varepsilon - \lambda) = g;$$

one uses, that $\forall n \geq 2$:

$$f\big|_{\Omega_n(A)} := \frac{1}{n+1} g \circ \left(n\varepsilon + (n-1)\lambda + \cdots + 2\lambda^{n-2} + \lambda^{n-1}\right)\big|_{\Omega_n(A)};$$
$$f\big|_{\Omega_1(A)} := \frac{1}{2} g\big|_{\Omega_1(A)}.$$

Hence one obtains, that

$$\forall g \in C^*(A) : g \circ \Lambda = 0 \Longleftrightarrow \exists f \in C^*(A) : f \circ (\varepsilon - \lambda) = g.$$

(7.4.3.9)

$$C^*(A) = (d\Omega(A))^\circ, \;\; Z^*(A) = (Im\ d)^\circ \cap (Im\ (\beta\pi))^\circ;$$
$$C_\lambda^*(A) = (Im\ d)^\circ \cap (Im\ \gamma)^\circ,$$
$$Z_\lambda^*(A) = (Im\ d)^\circ \cap (Im\ \gamma)^\circ \cap (Im\ \beta)^\circ = (Im\ d)^\circ \cap (Im\ \beta)^\circ;$$
$$C_\lambda^*(A) = C^*(A) \circ \Lambda, \;\; B^*(A) = C^*(A) \circ \beta\pi, \;\; B_\lambda^*(A) = C_\lambda^*(A) \circ \beta;$$
$$C^*(A) \cap (Im\ \Lambda)^\circ = C^*(A) \circ (\varepsilon - \lambda).$$

(7.4.3.10) Obviously

$$\forall g \in C^*(A) : g \circ \beta'\pi \in C^*(A); \;\; (\beta'\pi)^2 = 0.$$

If the associative superalgebra A is unital, one obtains that

$$\forall g \in C^*(A) : g \circ \beta'\pi = 0 \Longrightarrow \exists f \in C^*(A) : f \circ \beta'\pi = g;$$

choose $g \circ \tau =: f$.

(7.4.4) Let A be unital. In order to relate Hochschild and cyclic cohomology, one defines

$$\forall f \in C^*(A) : Bf := f \circ \Gamma\pi.$$

$$B\left(C_\lambda^*(A)\right) = \{0\}, \;\; B\left(C^*(A)\right) = C_\lambda^*(A), \;\; B^2 = 0, \;\; B \circ b_\lambda = -b_\lambda \circ B;$$
$$B\left(Z^*(A)\right) \subseteq Z_\lambda^*(A), \;\; B\left(B^*(A)\right) \subseteq B_\lambda^*(A).$$

(7.4.5) Let $R := \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$. The so-called periodicity operator

$$S : C^*_\lambda(A) \longrightarrow C^*_\lambda(A)$$

is defined, such that $\forall f \in C^*_\lambda(A), \forall n \in \mathbf{N}_0, \forall a_0, a_1, \ldots, a_{n+2} \in A$:

$$Sf(a_0) = 0,\ Sf(a_0 da_1) = 0,\ Sf \circ d = 0,$$

$$Sf(a_0 da_1 \cdots da_{n+2}) = \frac{1}{n+3} f \circ \Sigma \circ \Lambda\,(a_0 da_1 \cdots da_{n+2})\,;$$

for instance, $\forall z_0, z_1, z_2 \in \mathbf{Z}_2, \forall a_0 \in A^{\overline{z_0}}, \ldots, a_2 \in A^{\overline{z_2}}$:

$$Sf(a_0 da_1 da_2) = \frac{1}{3} f \circ \Sigma \circ (\varepsilon + \lambda + \lambda^2)(a_0 da_1 da_2)$$

$$= \frac{1}{3} f\left((-1)^{z_1} a_0 a_1 a_2 + (-1)^{(1+z_2)(z_0+z_1)+z_0} a_2 a_0 a_1 \right.$$
$$\left. +(-1)^{(1+z_2)(z_0+z_1)+(1+z_1)(z_2+z_0)+z_2} a_1 a_2 a_0\right).$$

(7.4.5.1)

$$\forall f \in Z^*_\lambda(A) : Sf = f \circ \Sigma \in Z^*_\lambda(A) \cap B^*(A);$$

for instance, denoting the $\mathbf{Z}_2$-degrees of $a_k \in A$ by $\overline{z_k}, k = 0, \ldots, 4$,

$$Sf(a_0 da_1 da_2) = (-1)^{z_1} f(a_0 a_1 a_2),$$
$$Sf(a_0 da_1 da_2 da_3) = (-1)^{z_1} f(a_0 a_1 a_2 da_3) + (-1)^{z_2} f(a_0 (da_1) a_2 a_3),$$
$$Sf(a_0\ da_1\ da_2\ da_3\ da_4) = (-1)^{z_1} f(a_0 a_1 a_2 da_3 da_4)$$
$$+ (-1)^{z_2} f(a_0 (da_1) a_2 a_3 da_4 + (-1)^{z_3} f(a_0 (da_1)(da_2) a_3 a_4).$$

(7.4.5.2)

$$\forall n \in \mathbf{N}_0, \forall g \in C^*_{\lambda n}(A) : \frac{n+1}{n+3} S(g \circ \beta) = (Sg) \circ \beta.$$

By means of $S\,(B^*_\lambda(A)) \subseteq B^*_\lambda(A)$ one defines an according R-linear mapping $\tilde{S} : H^*_\lambda(A) \longrightarrow H^*_\lambda(A)$ of equivalence classes. The De Rham cohomology of A is then defined as the R-bimodule

$$H^*_{DR}(A) := \frac{H^*_\lambda(A)}{Im(\tilde{S} - id\ H^*_\lambda(A))}.$$

(7.4.6) For $R := \mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$, the factorized R-linear mappings:

$$H^*_\lambda(A) \xrightarrow{\tilde{\varepsilon}} H^*(A) \xrightarrow{\tilde{B}} H^*_\lambda(A) \xrightarrow{\tilde{S}} H^*_\lambda(A)$$

are defined by means of cyclic and Hochschild cocycles as representatives. Then the following sequence is exact:

$$\xrightarrow{\tilde{\varepsilon}} \circ \xrightarrow{\tilde{B}} \circ \xrightarrow{\tilde{S}} \circ \xrightarrow{\tilde{\varepsilon}}$$

7.5 Graded Tensor Product of Bigraded Differential Algebras

(7.5.1) Let D, D' be $\mathbf{Z}_2$-graded differential algebras over R, with the $\mathbf{Z}_2$-graded nilpotent derivations δ, δ', respectively. The graded (skew-symmetric) tensor product $D\hat{\otimes}D'$ is equipped with an R-linear mapping Δ , according to the commuting diagram below.

$$\begin{tikzcd}
D^{\bar z}\times D' \ni \{a,a'\} \arrow[r] \arrow[d, "def"'] & a\otimes a' \in D^{\bar z}\otimes D' \arrow[dl, "\Delta|_{D^{\bar z}\otimes D'}"] \\
\delta(a)\otimes a' + (-1)^{z}\, a\otimes\delta'(a') \in D\otimes D' &
\end{tikzcd}$$

With this endomorphism Δ, $D\hat{\otimes}D'$ is some $\mathbf{Z}_2$-graded differential algebra over R.

(i) $\forall \bar z \in \mathbf{Z}_2 : Im\ \Delta|_{(D\hat{\otimes}D')^{\bar z}} \subseteq (D\hat{\otimes}D')^{\overline{z+1}}$; (ii) $\Delta\circ\Delta = 0$;

(iii) $\forall \bar z \in \mathbf{Z}_2, \forall s \in (D\hat{\otimes}D')^{\bar z}, t \in D\hat{\otimes}D' : \Delta(st) = \Delta(s)t + (-1)^{z}\, s\Delta(t)$.

(7.5.1.1) In the special case of bigraded differential algebras D and D', $D\hat{\otimes}D'$ is also some bigraded differential algebra over R, with an appropriate $\mathbf{N}_0$-grading due to the direct sum

$$D\hat{\otimes}D' = \bigoplus_{n\in\mathbf{N}_0}\ \bigoplus_{\bar z\in\mathbf{Z}_2} (D\hat{\otimes}D')^{\bar z}_n,$$

$$(D\hat{\otimes}D')^{\bar z}_n \underset{\text{linear}}{\longleftrightarrow} \bigoplus_{k,l\in\mathbf{N}_0;k+l=n}\ \ \bigoplus_{\bar x,\bar y\in\mathbf{Z}_2;\overline{x+y}=\bar z} (D^{\bar x}_k \otimes D'^{\bar y}_l).$$

(7.5.2) Let f, f' be R-linear forms on D, D', respectively; their tensor product is defined as the R-linear form:

$$D\hat{\otimes}D' \ni a\otimes a' \underset{f\otimes f'}{\longrightarrow} f(a)f'(a') \in R,$$

due to the universal property of the tensor product of R-bimodules. Then one easily obtains the following implications.

$$f\circ\delta = 0,\ \ f'\circ\delta' = 0 \Longrightarrow (f\otimes f')\circ\Delta = 0.$$

$$\lceil\forall \bar x,\bar y \in \mathbf{Z}_2, \forall a \in D^{\bar x}, b \in D^{\bar y}, a' \in D'^{\bar x}, b' \in D'^{\bar y} :$$
$$f(ab) = (-1)^{xy} f(ba),\ \ f'(a'b') = (-1)^{xy} f'(b'a')\rceil$$
$$\Longrightarrow \lceil\forall\ \bar x,\bar y \in \mathbf{Z}_2, \forall\ s \in (D\hat{\otimes}D')^{\bar x}, t \in (D\hat{\otimes}D')^{\bar y} :$$
$$(f\otimes f')(st) = (-1)^{xy}(f\otimes f')(ts)\rceil.$$

(7.5.3) $f \in Hom_R\,(D, R) =: D^*$ is called closed, if and only if $f \circ \delta = 0$; f is called graded trace, if and only if it vanishes on super-commutators, i.e.,

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in D^{\bar{x}}, b \in D^{\bar{y}} : f(ab) = (-1)^{xy} f(ba).$$

(7.5.4) Let the differential algebra D over R be bigraded. The R-submodule

$$\bigoplus_{k=n}^{\infty} D_k =: D_{\geq n}$$

is an ideal of D. Hence one obtains the factor algebras $D/D_{\geq n}$, and isomorphisms of associative R-algebras, such that

$$\forall n \in \mathbf{N} : D/D_{\geq n} \ni [d_{<n} + d_{\geq n}] \longleftrightarrow d_{<n} \in D_{<n} := \bigoplus_{k=0}^{n-1} D_k,$$

with an obvious notation.

7.6 Universal Differential Envelope of the Graded Tensor Product of Associative Superalgebras

(7.6.1) Consider the tensor product of direct sums of R-bimodules $E_k = F_k \oplus F'_k$, $k = 1, 2$:

$$E_1 \otimes E_2 \underset{R-linear}{\longleftrightarrow} (F_1 \otimes F_2) \oplus (F_1 \otimes F'_2) \oplus (F'_1 \otimes F_2) \oplus (F'_1 \otimes F'_2).$$

Hence one obtains an isomorphism in the sense of R-bimodules:

$$\begin{aligned} E_1 \otimes E_2 \supset\ & sum\{x_1 \otimes x_2; x_1 \in F_1, x_2 \in F_2\} \\ & \ni x_1 \otimes x_2 \longleftrightarrow x_1 \otimes x_2 \in F_1 \otimes F_2. \end{aligned}$$

(7.6.2) Consider the universal differential envelopes of associative superalgebras A_k over R, $k = 1, 2$, and also that of the skew-symmetric tensor product $A_1 \hat{\otimes} A_2$, and use the universality indicated in the following diagram.

$$\begin{tikzcd} A_1 \hat{\otimes} A_2 \arrow[rr, "embeddings", "v"'] \arrow[drr, "embedding"'] & & \Omega(A_1) \hat{\otimes} \Omega(A_2) \arrow[d, "v_*"] \\ & & \Omega(A_1 \hat{\otimes} A_2) \end{tikzcd}$$

Here ν_* is some homomorphism in the sense of $\mathbf{Z}_2$-graded differential R-algebras.

(7.6.3) Since the tensor product of bimodules is compatible with the direct sum, i.e.:

$$\Omega(A_1)\hat{\otimes}\Omega(A_2) \underset{linear}{\longleftrightarrow} \bigoplus_{n\in\mathbf{N}_0} \bigoplus_{k,l\in\mathbf{N}_0;k+l=n} \bigoplus_{\bar{z}_1,\bar{z}_2\in\mathbf{Z}_2} (\Omega_k^{\overline{z_1}}(A_1)\otimes\Omega_l^{\overline{z_2}}(A_2)),$$

ν_* is some homomorphism in the sense of bigraded differential R-algebras.

(7.6.4) The tensor product of R-linear forms $f_k : E_k \longrightarrow R$, on R-bimodules E_k, $k = 1, \ldots, n$, is defined as the linear form $f_1 \otimes \cdots \otimes f_n$, which is defined using the universal property of tensor products, such that

$$\bigotimes_{k=1}^n f_k : \bigotimes_{k=1}^n E_k \ni \bigotimes_{k=1}^n x_k \longrightarrow f_1(x_1)\cdots f_n(x_n) \in R.$$

(7.6.5) The so-called cup-product of linear forms $f_k \in \Omega^*(A_k), k = 1,2$, is defined as the composite mapping

$$f_1 * f_2 := (f_1 \otimes f_2) \circ \nu_*.$$

(7.6.6) With the nilpotent $\mathbf{Z}_2$-graded derivation Δ on $\Omega(A_1) \hat{\otimes} \Omega(A_2)$, this becomes some bigraded differential algebra over R; here Δ is universally defined, in order to fulfill $\forall\bar{z} \in \mathbf{Z}_2, \forall\omega_1 \in \Omega^{\bar{z}}(A_1), \omega_2 \in \Omega(A_2)$:

$$\Delta(\omega_1 \otimes \omega_2) = (d_1\omega_1) \otimes \omega_2 + (-1)^z\omega_1 \otimes (d_2\omega_2).$$

With an obvious slight abuse of notation one then immediately finds, that $\forall r \in R,\ \forall a_k^l \in A_k,\ l = 1, \ldots, n,\ k = 1,2$:

$$\Omega(A_1\hat{\otimes}A_2) \ni (r + (a_1^0 \otimes a_2^0))d(a_1^1 \otimes a_2^1)\cdots d(a_1^n \otimes a_2^n) \underset{\nu_*}{\longrightarrow}$$

$$(r + (a_1^0 \otimes a_2^0)) \prod_{l=1}^n ((d_1a_1^l) \otimes a_2^l + (-1)^{z_l}a_1^l \otimes (d_2a_2^l)) \in \Omega(A_1)\hat{\otimes}\Omega(A_2),$$

for homogeneous elements $a_1^l \in A_1^{\overline{z_l}}, l = 1, \ldots, n$.

(7.6.7) If $f_k \in \Omega^*(A_k), k = 1,2$, are closed linear forms, then their cup-product is also closed, i.e.,

$$\forall f_1 \in C^*(A_1), f_2 \in C^*(A_2) : f_1 * f_2 \in C^*(A_1\hat{\otimes}A_2).$$

Here one easily calculates, that $\forall f_1 \in \Omega^*(A_1), f_2 \in \Omega^*(A_2)$:

$$f_1 \circ d_1 = 0,\ f_2 \circ d_2 = 0$$

$$\Longrightarrow (f_1 * f_2) \circ d = (f_1 \otimes f_2) \circ \nu_* \circ d = (f_1 \otimes f_2) \circ \Delta \circ \nu_* = 0.$$

(7.6.8) If $f_k \in \Omega^*(A_k), k = 1, 2$, are graded traces, then their cup-product is also of this type, i.e., $\forall f_1 \in \Omega^*(A_1), f_2 \in \Omega^*(A_2)$:

$$f_1 \circ \beta_1 = f_1 \circ \gamma_1 = 0, \; f_2 \circ \beta_2 = f_2 \circ \gamma_2 = 0$$
$$\Longrightarrow (f_1 * f_2) \circ \beta = (f_1 * f_2) \circ \gamma = 0.$$

One straightforward calculates the cup-product acting on super-commutators.

(7.6.9) Since ν_* conserves both the $\mathbf{Z}_2$-grading and the $\mathbf{N}_0$-grading, one finds that

$$\forall n_1, n_2 \in \mathbf{N}_0, \forall f_1 \in \Omega^*_{n_1}(A_1), f_2 \in \Omega^*_{n_2}(A_2) : f_1 * f_2 \in \Omega^*_{n_1+n_2}(A_1 \hat{\otimes} A_2).$$

(7.6.10) $\forall n_1, n_2 \in \mathbf{N}_0$,

$$\forall f_1 \in Z^*_{\lambda n_1}(A_1), f_2 \in Z^*_{\lambda n_2}(A_2) : f_1 * f_2 \in Z^*_{\lambda, n_1+n_2}(A_1 \hat{\otimes} A_2).$$

(7.6.11) An easy calculation yields, that $\forall \omega \in A \oplus dA \oplus A \circ dA$,

$$\forall f_1 \in B^*_\lambda(A_1), f_2 \in Z^*_\lambda(A_2) : f_1 * f_2(\omega) = (g_1 * f_2) \circ \beta(\omega),$$

with the cyclic linear form $g_1 \in C^*_\lambda(A_1)$ defined by the condition $g_1 \circ \beta_1 = f_1$. Here one denotes $A_1 \hat{\otimes} A_2 =: A$.

(7.6.12) More generally one finds, that $\forall n_1, n_2 \in \mathbf{N}_0$,

$$\forall f_1 \in Z^*_{\lambda n_1}(A_1), f_2 \in B^*_{\lambda n_2}(A_2) :$$
$$f_1 * f_2 \in B^*_{\lambda, n_1+n_2}\left(A_1 \hat{\otimes} A_2\right), \; f_2 * f_1 \in B^*_{\lambda, n_1+n_2}\left(A_2 \hat{\otimes} A_1\right).$$

(7.6.12.1) The proof of this theorem by A. Connes is presented in his monograph "Noncommutative Geometry" (Chap. III.1. Cyclic Cohomology, Theorem 12).

(7.6.12.2) This theorem allows for the definition of an according cup-product of cyclic cohomologies, such that $\forall n_1, n_2 \in \mathbf{N}_0$,

$$\forall [f_1] \in H^*_{\lambda n_1}(A_1), [f_2] \in H^*_{\lambda n_2}(A_2) :$$
$$[f_1] * [f_2] = [f_1 * f_2] \in H^*_{\lambda, n_1+n_2}\left(A_1 \hat{\otimes} A_2\right),$$

with an obvious notation of equivalence classes.

(7.6.12.2) Note that the cup-product of cyclic cochains need not be cyclic.

7.7 Universal Differential Envelope of a Commutative Ring

(7.7.1) The commutative ring R is also considered as the free R-bimodule, which is generated by the unit e_R. Moreover R is considered as unital associative superalgebra, with respect to the trivial grading according to $R = R^{\bar{0}} \oplus R^{\bar{1}}$, $R^{\bar{1}} := \{0_R\}$.

(7.7.2) With respect to this trivial $\mathbf{Z}_2$-grading of R, its universal differential envelope is constructed.

$$\begin{aligned} \tilde{\Omega}(R) &= R \oplus \Omega(R) = R \oplus R \oplus d\Omega(R) \oplus R \circ d\Omega(R) \\ &= R - lin\ span\ \{\tilde{e}, e, \underbrace{(de)\cdots(de)}_{n\ times}, e\underbrace{(de)\cdots(de)}_{n\ times}; n \in \mathbf{N}\}, \end{aligned}$$

with the unit $\tilde{e}$ *of* $\tilde{\Omega}(R)$. Here the unit e_R of R is used in the following sense.

$$\begin{aligned} &\tilde{e} := \{e_R, 0, \ldots\},\ e := \{0, e_R, 0, \ldots\}, \\ &de = \{0, 0, \{e_R, 0\} \otimes e_R, 0, \ldots\},\ e\, de = \{0, 0, \{0, e_R\} \otimes e_R, 0, \ldots\}, \\ &(de)^n := \underbrace{(de)\cdots(de)}_{n\ times} = \{\underbrace{0, \ldots, 0}_{(n+1)\ times}, \{e_R, 0\} \otimes \underbrace{e_R \otimes \cdots \otimes e_R}_{n\ times}, 0, \ldots\}, \\ &e(de)^n = \{\underbrace{0, \ldots, 0}_{(n+1)\ times}, \{0, e_R\} \otimes \underbrace{e_R \otimes \cdots \otimes e_R}_{n\ times}, 0, \ldots\}. \end{aligned}$$

(7.7.3) One easily finds the following rules of calculation.

$$\tilde{e}\tilde{e} = \tilde{e},\ \tilde{e}e = e\tilde{e} = e,\ \tilde{e}(de) = (de)\tilde{e} = de,$$

$$e^2 := ee = e,\ e(de) + (de)e = de.$$

(7.7.3.1) $\forall n \in \mathbf{N}$:

$$(de)^n e = \begin{cases} (\tilde{e} - e)(de)^n, & n\ odd \\ e(de)^n, & n\ even \end{cases},\quad e(de)^n e = \begin{cases} 0, & n\ odd \\ e(de)^n, & n\ even \end{cases}.$$

(7.7.3.2)

$$\tilde{\Omega}(R) = R - lin\ span\{(de)^n, e(de)^n; n \in \mathbf{N}_0\},$$

with the notation $\tilde{e} =: (de)^0$.

(7.7.3.3) $\forall n, m \in \mathbf{N}_0$:

$$(de)^n e(de)^m = \begin{cases} (\tilde{e} - e)(de)^{n+m}, & n \text{ odd} \\ e(de)^{n+m}, & n \text{ even} \end{cases},$$

$$e(de)^n e(de)^m = \begin{cases} 0, & n \text{ odd} \\ e(de)^{n+m}, & n \text{ even} \end{cases}.$$

(7.7.4) In the sense of the inner direct sum

$$\tilde{\Omega}(R) = R \oplus \Omega(R), \ \Omega(R) = \bigoplus_{n \in \mathbf{N}_0} \Omega_n(R),$$

one obtains the R-bases:

$$\{\tilde{e}\} \longrightarrow R, \ \{e\} \longrightarrow R, \ \{de\} \longrightarrow dR, \ \{e\, de\} \longrightarrow R \circ dR;$$

$$\forall n \in \mathbf{N} : \{(de)^n\} \longrightarrow d\underbrace{R \circ \cdots \circ d}_{n \text{ times}} R, \{e(de)^n\} \longrightarrow R \circ d\underbrace{R \circ \cdots \circ d}_{n \text{ times}} R.$$

$$\begin{aligned} \tilde{\Omega}(R) &= R \oplus R \oplus dR \oplus (R \circ dR) \oplus \\ &\quad \cdots \oplus (d\underbrace{R \circ \cdots \circ d}_{n \text{ times}} R) \oplus (R \circ dR \circ \cdots \circ dR) \oplus \cdots, \end{aligned}$$

$$\{(de)^n, e(de)^n; n \in \mathbf{N}_0\} \underset{\text{free over } R}{\longrightarrow} \tilde{\Omega}(R).$$

The $\mathbf{N}_0$-grading of $\tilde{\Omega}(R)$ explicitly reads as follows.

$$\tilde{\Omega}(R) = \bigoplus_{n \in \mathbf{N}_0} \tilde{\Omega}_n(R), \ \tilde{\Omega}_0(R) := \tilde{R} := R \oplus R = R(\{\tilde{e}, e\}),$$

$$\forall n \in \mathbf{N}_0 : \tilde{\Omega}_{n+1}(R) = d\Omega_n(R) \oplus R \circ d\Omega_n(R) = \Omega_{n+1}(R).$$

$$\Omega(R) = \bigoplus_{n \in \mathbf{N}_0} \Omega_n(R), \ \Omega_0(R) := R = R(\{e\}).$$

$$\begin{aligned} \forall n \in \mathbf{N}_0 : d\tilde{\Omega}_n(R) &= d\Omega_n(R) = R(\{(de)^{n+1}\}), \\ R \circ d\Omega_n(R) &= R(\{e(de)^{n+1}\}). \end{aligned}$$

(7.7.5) The $\mathbf{Z}_2$-grading of $\tilde{\Omega}(R)$ is non-trivial, i.e.,

$$\forall \bar{z} \in \mathbf{Z}_2 : \tilde{\Omega}^{\bar{z}}(R) = R(\{(de)^n, e(de)^n; n \in \mathbf{N}_0, \bar{n} = \bar{z}\}).$$

(7.7.6) $\forall n \in \mathbf{N}_0 : d\,(e(de)^n) = (de)^{n+1}; \ d\tilde{e} = 0.$

(7.7.7) Obviously the canonical embedding, of an associative algebra A over R into its unital extension $\tilde{A} := R \oplus A$, is an ideal of $\tilde{A}$. The corresponding factor algebra is unital, due to the isomorphism of R-algebras:

$$\tilde{A}/A \ni [\{r, a\}] \longleftrightarrow r \in R.$$

Moreover, since $\tilde{\Omega}(A)$ is the unital extension of $\Omega(A)$, this is an ideal of $\tilde{\Omega}(A)$.

(7.7.8) $\forall n \in \mathbf{N}$:

$$\begin{aligned}
&\beta'(e) = 0,\ \beta'(de) = \beta'(e\ de) = e,\\
&\beta'(e(de)^{2n}) = 0,\ \beta'(e(de)^{2n+1}) = e(de)^{2n},\\
&\beta'((de)^{2n}) = (e - \tilde{e})(de)^{2n-1},\ \beta'((de)^{2n+1}) = e(de)^{2n},\\
&\alpha(e) = 0,\ \alpha(de) = \alpha(e\ de) = e,\\
&\alpha(e(de)^{2n}) = \alpha((de)^{2n}) = -e(de)^{2n-1},\\
&\alpha(e(de)^{2n+1}) = \alpha((de)^{2n+1}) = e(de)^{2n},\\
&\lambda(e) = e,\ \lambda(e\ de) = -e\ de,\ \lambda(e(de)^n) = (-1)^n e(de)^n,\ \lambda \circ d = 0,\\
&\beta(e) = \beta(de) = \beta(e\ de) = 0,\ \beta(e(de)^{2n+1}) = \beta((de)^{2n+1}) = 0,\\
&\beta(e(de)^{2n}) = e(de)^{2n-1},\ \beta((de)^{2n}) = (2e - \tilde{e})(de)^{2n-1},\\
&\rho(e) = e,\ \rho(de) = de,\ \rho(e\ de) = de - e\ de,\\
&\rho(e(de)^n) = (-1)^{n+1}(\tilde{e} - e)(de)^n,\ \rho((de)^n) = (-1)^{n+1}(de)^n,\\
&\gamma(e) = \gamma(de) = 0,\ \gamma(e\ de) = 2e\ de - de,\\
&\gamma((de)^{2n+1}) = 0,\ \gamma((de)^{2n}) = 2(de)^{2n},\\
&\gamma(e(de)^{2n}) = (de)^{2n},\ \gamma(e(de)^{2n+1}) = (2e - \tilde{e})(de)^{2n+1},\\
&\tau(e) = e\ de,\ \tau(de) = -(de)^2,\ \tau(e\ de) = -e(de)^2,\\
&\tau(e(de)^n) = (-1)^n e(de)^{n+1},\ \tau((de)^n) = (-1)^n (de)^{n+1},\\
&\Gamma_0(e) = 2e\ de,\ \Gamma_0(de) = -(de)^2,\ \Gamma_0(e\ de) = 0,\ \Gamma_0(e(de)^{2n+1}) = 0,\\
&\Gamma_0(e(de)^{2n}) = 2e(de)^{2n+1},\ \Gamma_0((de)^n) = (-1)^n(de)^{n+1}.
\end{aligned}$$

(7.7.9) $\forall f \in \Omega^*(R), \forall r_1, r_2 \in R, \forall n \in \mathbf{N}$:

$$f \circ \beta(r_1 e(de)^{2n} + r_2(de)^{2n}) = (r_1 + 2r_2) f(e(de)^{2n-1}) - r_2 f((de)^{2n-1}).$$

Hence for every linear form $f \in \Omega^*(R)$ the following implications hold.

$$f \circ \beta = 0 \Longleftrightarrow f|_{\Omega^1(R)} = 0.$$

(7.7.10)

$$\forall f \in \Omega^*(R), \forall n \in \mathbf{N}_0 : f \circ (\varepsilon - \lambda)(e(de)^n) = \begin{cases} 0, & n \text{ even} \\ 2f(e(de)^n), & n \text{ odd} \end{cases}.$$

Therefore, f is cyclic, if and only if it is some closed graded trace; i.e., $\forall f \in \Omega^*(R)$:

$$f \in C^*_\lambda(R) \Longleftrightarrow f \circ \beta = f \circ d = 0 \Longleftrightarrow f \in Z^*_\lambda(R) \Rightarrow f \circ \gamma = 0.$$

(7.7.10.1)

$$\forall n \in \mathbf{N}_0 : C^*_{\lambda n}(R) = Z^*_{\lambda n}(R) = \begin{cases} \{0\}, & n \text{ odd} \\ C^*_n(R) \underset{R-linear}{\longleftrightarrow} R, & n \text{ even} \end{cases}.$$

(7.7.11) $B^*_\lambda(R) = \{0\}$, and therefore $\forall n \in \mathbf{N}_0$:

$$H^*_{\lambda n}(R) \underset{R-linear}{\longleftrightarrow} Z^*_{\lambda n}(R) \underset{R-linear}{\longleftrightarrow} \begin{cases} \{0\}, & n \text{ odd} \\ R, & n \text{ even} \end{cases}$$

7.8 Universal Differential Envelopes of Finite-Dimensional Clifford Algebras

(7.8.1) Consider a symmetric K-bilinear form $g : E \times E \longrightarrow K$, over a p-dimensional vector space $E, p \in \mathbf{N}$, over the field K of *char* $K \neq 2$. One can choose an orthogonal basis $\{e_1, \ldots, e_p\}$ of E, such that $\forall_1^p k, l$:

$$g(e_k, e_l) = \begin{cases} \delta_k, & k = l \\ 0, & k \neq l \end{cases}, \quad \begin{cases} \delta_k \neq 0, & 1 \leq k \leq m \\ \delta_k = 0, & m+1 \leq k \leq p \end{cases}$$

The form g is non-degenerate, if and only if $m = p$; otherwise $\{e_{m+1}, \ldots, e_p\}$ is some basis of its kernel *ker* g.

(7.8.2) The Clifford algebra C over E with respect to g is constructed by the usual universal diagram, such that

$$\forall x, y \in E : xy + yx = 2g(x, y)e_C.$$

Then C is some associative K-superalgebra with the unit $e_C \equiv e_0$, which is polynomially spanned by E; the $\mathbf{Z}_2$-grading of C is defined such that $E \subseteq C^{\bar{1}}$. The K-dimension of C equals 2^p, due to an isomorphism γ of unital associative superalgebras defined below.

(7.8.2.1) For $1 \leq k \leq p$, let C_k be the Clifford algebra over the one-dimensional vector space $E_1 := K(\{c_1\})$, with the unit $e_{C_k} =: c_2^{(k)}$, with respect to the symmetric K-bilinear form $g_k : E_1 \times E_1 \longrightarrow K$, such that $g_k(c_1, c_1) := \delta_k$.

$$c_1^2 = \delta_k c_2^{(k)},\ c_1 c_2^{(k)} = c_2^{(k)} c_1 = c_1,\ \left(c_2^{(k)}\right)^2 = c_2^{(k)},\ k = 1, \ldots, p.$$

(7.8.2.2) $\forall_1^p k,l : e_k e_l + e_l e_k = \begin{cases} 2\ \delta_k e_C, & k = l \\ 0, & k \neq l \end{cases}.$

(7.8.2.3) With respect to the $\mathbf{Z}_2$-grading of $C_k, k = 1, \ldots, p$, such that $c_1^{(k)} \equiv c_1$ is odd, one establishes an isomorphism γ in the above sense:

$$\gamma : C \supset E \ni e_k \longleftrightarrow$$
$$c_2^{(1)} \otimes \cdots \otimes c_2^{(k-1)} \otimes c_1 \otimes c_2^{(k+1)} \otimes \cdots \otimes c_2^{(p)} \in C_1 \hat{\otimes} \cdots \hat{\otimes} C_p.$$

An appropriate K-linear basis $\{e_{k_1} \cdots e_{k_r}; 1 \le k_1 < \cdots < k_r \le p\} \cup \{e_C\}$ of C is then established, due to the images $\gamma(e_C) =: c_2^{(1)} \otimes \cdots \otimes c_2^{(p)}$, and

$$\gamma(e_{k_1} \cdots e_{k_r}) =: c_{l_1}^{(1)} \otimes \cdots \otimes c_{l_p}^{(p)},\ \forall_1^p k : l_k := \begin{cases} 1, & k \in \{k_1, \ldots, k_r\} \\ 2 & \text{otherwise} \end{cases}.$$

(7.8.3) Within the universal differential envelope $\Omega(C)$, one easily calculates the following realtions, e_k being odd for $1 \le k \le p$, and $e_0 \equiv e_C$ even.

$$(de_0)^n e_0 = \begin{cases} (de_0)^n - e_0(de_0)^n, & n = 1, 3, \ldots \\ e_0(de_0)^n, & n = 2, 4, \ldots \end{cases}. \quad \forall_1^p k :$$

$$(de_k)e_k - e_k de_k = \delta_k de_0,\ (de_k)^2 e_k - e_k(de_k)^2 = \delta_k(de_0 de_k + de_k de_0),$$

$\forall_2^\infty n \in \mathbf{N}$:

$$(de_k)^n e_k = \delta_k \sum_{l=1}^{n} de_k \cdots de_k de_0 de_k \cdots de_k + e_k(de_k)^n \in \Omega_n(C),$$

with the factor de_0 at the position l. Furthermore $\forall_1^p k$:

$$(de_0)e_k + e_0 de_k = de_k,\ (de_0)^2 e_k = e_0 de_0 de_k,$$

$$(de_0)^3 e_k = (de_0)^2 de_k - e_0(de_0)^2 de_k,$$

$$(de_0)^n e_k = \begin{cases} (de_0)^{n-1} de_k - e_0(de_0)^{n-1} de_k, & n = 3, 5, \ldots \\ e_0(de_0)^{n-1} de_k, & n = 2, 4, \ldots \end{cases},$$

$$(de_k)e_0 - e_k de_0 = de_k,\ (de_k)^2 e_0 = e_k de_k de_0 + (de_k)^2 + \delta_k(de_0)^2,$$

$\forall_3^\infty n \in \mathbf{N}$:

$$(de_k)^n e_0 = e_k(de_k)^{n-1} de_0 + (de_k)^n$$
$$+ \delta_k \left((de_k)^{n-2}(de_0)^2 + (de_k)^{n-3} de_0 de_k de_0 + \cdots + de_0(de_k)^{n-2} de_0\right).$$

$$\forall_1^p k \neq l : (de_k)e_l - e_k de_l + (de_l)e_k - e_l de_k = 0.$$

(7.8.4) Consider as an example the first order cyclic and Hochschild cocycles and coboundaries of the Clifford algebra C for $p = 1$, with the unit e_0 and one odd generator e_1, $e_1^2 = \delta_1 e_0$. One easily calculates that

$$\Omega_1(C) \cap \ Im\ \beta = K\left(\{e_0de_0, e_1de_0, e_0de_1\}\right),$$

$$\Omega_1(C) \cap \ Im(\beta\pi) = K\left(\{e_0de_0, e_1de_0, \delta_1 e_0de_1\}\right).$$

Denoting the dual basis vectors of $\{e_kde_l; k,l = 0,1\}$ by ε_{kl}, such that

$$\varepsilon_{kl}(e_rde_s) = \delta_{kr}\delta_{ls},\ k,\dots,s = 0,1,$$

one therefore finds that

$$Z^*_{\lambda 1}(C) = K\left(\{\varepsilon_{11}\}\right),\ Z^*_1(C) = \begin{cases} K\left(\{\varepsilon_{11}, \varepsilon_{01}\}\right), & \delta_1 = 0 \\ K\left(\{\varepsilon_{11}\}\right), & \delta_1 \neq 0 \end{cases}.$$

One then obtains the cohomologies:

$$H^*_{\lambda 1}(C) \longleftrightarrow Z^*_{\lambda 1}(C) \text{ for } \delta_1 = 0,\ dim\ H^*_{\lambda 1}(C) = 0 \text{ for } \delta_1 \neq 0;$$

$$H^*_1(C) \longleftrightarrow Z^*_1(C) \ \text{ for } \delta_1 = 0,\ dim\ \ H^*_1(C) = 0 \text{ for } \delta_1 \neq 0.$$

(7.8.5) The second order cyclic cocycles, again for $p = 1$, for $\delta_1 = 0$ are the elements of

$$Z^*_{\lambda 2}(C) = K\left(\{\varepsilon_{000}, \varepsilon_{100} - \varepsilon_{010} + \varepsilon_{001}, \varepsilon_{111}\}\right),$$

denoting by ε_{ijk} the dual basis vectors of $e_ide_jde_k, i,j,k = 0,1,2$. Since the cyclic coboundaries vanish both on $e_1de_1de_1$ and $e_0de_0de_0$,

$$dim\ H^*_{\lambda 2}(C) = 2.$$

(7.8.6) In the case of $p = 2$, with two odd generators e_1 and e_2, such that

$$e_1^2 = e_2^2 = e_1e_2 + e_2e_1 = 0,$$

the first order cyclic cocycles of this four-dimensional Grassmann algebra are the elements of

$$Z^*_{\lambda 1}(C) = K\left(\{\varepsilon_{11}, \varepsilon_{12} + \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{1(12)} + \varepsilon_{(12)1}, \varepsilon_{2(12)} + \varepsilon_{(12)2}\}\right),$$

denoting by $\varepsilon_{\bar{k}\bar{l}}$ the dual basis vectors with respect to the K-linear basis:

$$\{e_{\bar{k}}de_{\bar{l}}; \bar{k},\bar{l} = 0,1,2,12\} \longrightarrow C \circ dC,\ e_{12} := e_1e_2,$$

such that for instance

$$\varepsilon_{1(12)}\left(e_1d(e_1e_2)\right) = 1,\ \varepsilon_{(12)1}(e_1e_2de_1) = 1.$$

Since β vanishes on the support of $f \in Z^*_{\lambda 1}(C)$, one finds that $B^*_{\lambda 1}(C) = \{0\}$.

(7.8.7) The zero order cyclic and Hochschild cocycles and coboundaries, in the cases of $p = 1, 2$, are calculated easily. For $p = 1$,

$$f \circ \beta(e_1 de_1) = -2f(e_1^2) = -2\delta_1 f(e_0) = 0,$$

$$dim\ Z^*_{\lambda 0}(C) = dim\ Z^*_0(C) = \begin{cases} 1, & \delta_1 \neq 0 \\ 2, & \delta_1 = 0 \end{cases},\ B^*_{\lambda 0}(C) = B^*_0(C) = \{0\}.$$

For $p = 2$, one again finds that $B^*_{\lambda 0}(C) = B^*_0(C) = \{0\}$;

$$Z^*_{\lambda 0}(C) = Z^*_0(C) = Hom_C(C, K) \text{ for } \delta_1 = \delta_2 = 0;$$

$$Z^*_{\lambda 0}(C) = Z^*_0(C) = K(\{\varepsilon_{12}\}) \text{ for } \delta_1\delta_2 \neq 0,$$

with the dual basis vector ε_{12} of $e_1 e_2$.

(7.8.8) Consider again the case of $p = 1$ and $\delta_1 \neq 0$, $e_1^2 = \delta_1 e_0$. With the above notation of dual basis vectors, such that $\varepsilon_k(e_l) = \delta_{kl}, k, l = 0, 1$, one calculates that

$$Z^*_{\lambda 0}(C) = K(\{\varepsilon_1\}),\ B^*_{\lambda 0}(C) = \{0\},\ dim\ H^*_{\lambda 0}(C) = 1;$$

$$Z^*_{\lambda 1}(C) = K(\{\varepsilon_{11}\}) = B^*_{\lambda 1}(C),\ dim\ H^*_{\lambda 1}(C) = 0;$$

$$Z^*_{\lambda 2}(C) = K(\{\varepsilon_{100} - \varepsilon_{010} + \varepsilon_{001}, \varepsilon_{111}\}),$$

$$B^*_{\lambda 2}(C) = K(\{\varepsilon_{100} - \varepsilon_{010} + \varepsilon_{001} + 3\delta_1\varepsilon_{111}\}),\ dim\ H^*_{\lambda 2}(C) = 1;$$

$$S\varepsilon_1 = \varepsilon_{100} - \varepsilon_{010} + \varepsilon_{001} - \delta_1\varepsilon_{111}.$$

(7.8.9) More generally one calculates, that for an arbitrary non-degenerate Clifford algebra C, with $\delta_1 \cdots \delta_p \neq 0$,

$$Z^*_{\lambda 0}(C) = K(\{\varepsilon_{1\ldots p}\}),\ B^*_{\lambda 0}(C) = \{0\},$$

$$H^*_{\lambda, n+2}(C) = \tilde{S}(H^*_{\lambda n}(C)) \text{ for } n \in \mathbf{N}_0,$$

inserting the correspondingly factorized periodicity operator S, and denoting by $\varepsilon_{1\ldots p}$ the dual basis vector of $e_1 \cdots e_p$. For odd $n \in \mathbf{N}$, $H^*_{\lambda n}(C) = \{0\}$.

7.9 Differential Envelopes of Real Scalar Fields

The unital universal differential envelope $\tilde{\Omega}(A)$ of an associative superalgebra A over the commutative ring R can also be constructed using multiple tensor products over the direct sum $\tilde{A} := R \oplus A$.

(7.9.1) Consider the R-linear mapping

$$\tilde{\psi} : \tilde{A} \otimes \tilde{A} \ni \tilde{a} \otimes \tilde{b} \longrightarrow \tilde{a} \otimes \tilde{b} - \tilde{\mu}(\tilde{a} \otimes \tilde{b}) \otimes e_{\tilde{A}} \in \tilde{A} \otimes \tilde{A},$$

with the graded structure mapping

$$\tilde{\mu} : \tilde{A} \otimes \tilde{A} \ni \tilde{a} \otimes \tilde{b} \longrightarrow (-1)^{y}\tilde{a}\tilde{b} \in \tilde{A} \ \text{ for } \ \tilde{b} \in \tilde{A}^{\bar{y}}, \bar{y} \in \mathbf{Z}_2.$$

Due to its properties $\tilde{\psi} \circ \tilde{\psi} = \tilde{\psi}$, $\ ker \ \tilde{\psi} = \tilde{A} \otimes R$, $Im \, \tilde{\psi} = ker \ \tilde{\mu}$, one obtains an R-linear bijection

$$\psi_1 : \tilde{\Omega}_1(A) \ni \tilde{a} \otimes b \longleftrightarrow \tilde{a} \otimes b - \tilde{\mu}(\tilde{a} \otimes b) \otimes e_{\tilde{A}} \in ker \ \tilde{\mu}.$$

Then the R-linear mapping

$$\tilde{\delta}_0 : \tilde{A} \ni \tilde{a} \underset{def}{\longrightarrow} e_{\tilde{A}} \otimes \tilde{a} - (-1)^{x}\tilde{a} \otimes e_{\tilde{A}} = e_{\tilde{A}} \otimes a - (-1)^{x} a \otimes e_{\tilde{A}} \in ker \ \tilde{\mu},$$

for $\tilde{a} =: \alpha + a \in \tilde{A}^{\bar{x}}$, $\alpha \in R$, obviously equals $\tilde{\delta}_0 = \psi_1 \circ d_0$.

(7.9.2) With the left and right module-multiplication, such that $\forall \bar{y}, \bar{z} \in \mathbf{Z}_2, \forall \tilde{a} \in \tilde{A}, \tilde{b} =: \beta + b \in \tilde{A}^{\bar{y}}, \tilde{c} =: \gamma + c \in \tilde{A}^{\bar{z}}$:

$$\psi_1(\tilde{a} \otimes b) \circ \tilde{c} = \psi_1\left((\tilde{a} \otimes b) \circ \tilde{c}\right) = \tilde{a} \otimes (b\tilde{c}) - (-1)^{y}(\tilde{a}b) \otimes \tilde{c},$$

$$\tilde{a} \circ \psi_1(\tilde{b} \otimes c) = \psi_1\big(\tilde{a} \circ (\tilde{b} \otimes c)\big) = (\tilde{a}\tilde{b}) \otimes c - (-1)^{z}(\tilde{a}\tilde{b}c) \otimes e_{\tilde{A}},$$

one establishes an $\tilde{A}$-bimodule $Im \ \psi_1 = ker \ \tilde{\mu}$ over R.

(7.9.3) The n-fold tensor product over $\tilde{A}$ of such objects is constructed as usual, such that for instance

$$\begin{aligned} ker \ \tilde{\mu} \otimes_{\tilde{A}} ker \ \tilde{\mu} \ni \ & (\psi_1(\tilde{a} \otimes b) \circ \tilde{g}) \otimes_{\tilde{A}} \psi_1(\tilde{c} \otimes d) \\ & = \psi_1(\tilde{a} \otimes b) \otimes_{\tilde{A}} (\tilde{g} \circ \psi_1(\tilde{c} \otimes d)) . \end{aligned}$$

(7.9.4) With the above module-multiplications, and the tensor product over $\tilde{A}$, being used as product, the R-direct sum

$$\tilde{H}(A) := \bigoplus_{n \in \mathbf{N}_0} \tilde{H}_n(A), \ \tilde{H}_n(A) := \bigotimes_{\tilde{A}}^{n} ker \ \tilde{\mu} \text{ for } n \geq 1, \ \tilde{H}_0(A) := \tilde{A},$$

becomes an associtative unital superalgebra over R. For instance,

$$\psi_1(\tilde{a} \otimes b) \circ \psi_1(\tilde{c} \otimes d) = \psi_1(\tilde{a} \otimes b) \otimes_{\tilde{A}} \psi_1(\tilde{c} \otimes d) \in \tilde{H}_2(A).$$

(7.9.5) The $\mathbf{Z}_2$-grading of $\tilde{\Omega}(A)$, and also its $\mathbf{N}_0$-grading, are switched to $\tilde{H}(A)$ via an R-linear bijection

$$\psi : \tilde{\Omega}(A) \ni \tilde{a} + \cdots + \tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n + \cdots \longleftrightarrow \tilde{a} +$$
$$\cdots + \psi_1(\tilde{a}_0 \otimes a_1) \otimes_{\tilde{A}} \psi_1(e_{\tilde{A}} \otimes a_2) \otimes_{\tilde{A}} \cdots \otimes_{\tilde{A}} \psi_1(e_{\tilde{A}} \otimes a_n) + \cdots \in \tilde{H}(A).$$

(7.9.6) Moreover ψ is an isomorphism of $\mathbf{Z}_2$-graded differential algebras over R, with respect to the odd derivation $\tilde{\delta} := \psi \circ d \circ \psi^{-1}$. For instance, $\forall \tilde{a}_0 \in \tilde{A}^{\overline{z_0}}, a_1 \in A^{\overline{z_1}}$:

$$\tilde{\delta}(\tilde{a}_0) = \tilde{\delta}_0(\tilde{a}_0) = \tilde{\delta}_0(a_0) = e_{\tilde{A}} \otimes a_0 - (-1)^{z_0} a_0 \otimes e_{\tilde{A}},$$

$$\psi_1(\tilde{a}_0 \otimes a_1) = \tilde{a}_0 \otimes a_1 - (-1)^{z_1} \tilde{a}_0 a_1 \otimes e_{\tilde{A}} = \tilde{a}_0 \circ \tilde{\delta}(a_1)$$
$$\xrightarrow[\tilde{\delta}]{} \psi_1(e_{\tilde{A}} \otimes a_0) \otimes_{\tilde{A}} \psi_1(e_{\tilde{A}} \otimes a_1) = \tilde{\delta}(a_0) \otimes_{\tilde{A}} \tilde{\delta}(a_1).$$

(7.9.7) Now let especially A be unital. Identifying $e_{\tilde{A}}$ with the unit e_A of A, one obtains the following surjective homomorphism $\nu : \tilde{H}(A) \longrightarrow H(A)$ of $\mathbf{Z}_2$-graded differential algebras over R, with respect to the odd derivations $\tilde{\delta}$ and δ, such that $\delta \circ \nu = \nu \circ \tilde{\delta}$.

$$\tilde{A} \ni \tilde{a} =: \alpha + a = \alpha \ e_{\tilde{A}} + a \xrightarrow[def]{\nu} \alpha \ e_A + a \in A,$$

$$ker\ \tilde{\mu} \ni \psi(\tilde{a} \otimes b) \xrightarrow[\nu]{} (\alpha \ e_A + a) \otimes b - (\alpha \ e_A + a) b (-1)^y \otimes e_A \text{ for } b \in A^{\bar{y}},$$

$$\tilde{H}_n(A) \ni \bigotimes_{\tilde{A}\ k=1}^{n} \psi(\tilde{\xi}_k) \xrightarrow[\nu]{} \bigotimes_{A\ k=1}^{n} \nu \circ \psi(\tilde{\xi}_k) \in H_n(A).$$

$$A \ni a \xrightarrow[def]{\delta} e_A \otimes a - (-1)^z a \otimes e_A \ \text{ for } a \in A^{\bar{z}}, \ \delta(e_A) = 0;$$

$$\delta \circ \nu \circ \psi(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n) = \delta(a_0) \otimes_A \delta(a_1) \otimes_A \cdots \otimes_A \delta(a_n),$$

for $\tilde{a}_0 =: \alpha + a_0 \in \tilde{A}$, $a_1, \ldots, a_n \in A$. Note that one uses here, that $ker\ \nu$ is some graded ideal of $\tilde{H}(A)$, and moreover $ker\ \nu \subseteq ker\ (\nu \circ \tilde{\delta})$.
Obviously δ is compatible with the natural $\mathbf{N}_0$-grading of $H(A)$ in the sense, that

$$\delta\left(H_n(A)\right) \subseteq H_{n+1}(A), \ H(A) := \bigoplus_{n \in \mathbf{N}_0} H_n(A), \ H_0(A) := A.$$

(7.9.7.1) As in the case of the odd derivation d of $\tilde{\Omega}(A)$ one finds, that

$$ker(\nu \circ \psi) \subseteq ker(\nu \circ \psi \circ \beta) \cap ker(\nu \circ \psi \circ \beta\pi).$$

Therefore the R-endomorphisms $\beta, \beta\pi$ of $\Omega(A)$ can be pushed onto $H(A)$, and Hochschild cochains, cocycles, and coboundaries of A can be defined with respect to the differential envelope $H(A)$, and conveniently denoted by the same letters as those with respect to $\Omega(A)$.

$$C^*(A) := \left\{f \in H^*(A) \equiv (H(A))^* \,; f \circ \delta = 0\right\};$$

$$Z^*(A) := \{f \in C^*(A); f \circ \beta\pi = 0\}\,,\ H^*_H(A) := Z^*(A)/B^*(A),$$

$$B^*(A) := \{f \in H^*(A); \exists g \in C^*(A) : g \circ \beta\pi = f\} \subseteq Z^*(A);$$

$$\forall f \in H^*(A) : bf := f \circ \beta\pi;\ Im\ b\big|_{C^*(A)} = B^*(A).$$

Since $\beta\pi$ is compatible with ν, the above Hochschild coboundary operator b in the sense of $H(A)$ is acting similarly as that of $\Omega(A)$.

(7.9.7.2) Unfortunately in general λ is not compatible with ν. Therefore one cannot define cyclic linear forms in the sense of $H(A)$, as one does with respect to $\Omega(A)$. For instance, for $a \in A^{\bar{x}}, \bar{x} \in \mathbf{Z}_2$, one finds that $\nu \circ \psi(a \otimes e_A) = 0$, but

$$\nu \circ \psi \circ \lambda(a \otimes e_A) = a \otimes e_A - (-1)^x e_A \otimes a = (-1)^{1+x}\delta(a).$$

(7.9.8) As an example consider the commutative unital associative algebra of real scalar fields,

$$A := C^\infty(\mathbf{R}^m) \ni a : \mathbf{R}^m \longrightarrow \mathbf{R},\ e_A \equiv e.$$

$$\begin{aligned} H_1 &= sum\ (\{a \circ \delta(b) = a \otimes b - ab \otimes e; a, b \in A\}) \\ &\longleftrightarrow sum\ (\{\{a(x)(b(y) - b(x)); x, y \in \mathbf{R}^m\}; a, b \in C^\infty(\mathbf{R}^m)\}) \\ &\subseteq C^\infty(\mathbf{R}^{2m}). \end{aligned}$$

Generally, factorizing with respect to the tensor product over A, one finds that $H_n, n \in \mathbf{N}$, is the real vector space of sums of real smooth mappings of the kind

$$a_0\delta(a_1)\cdots\delta(a_n) \longleftrightarrow \left\{a_0(x_0)\prod_{k=1}^{n}(a_k(x_k) - a_k(x_{k-1}))\,; x_0, \ldots, x_n \in \mathbf{R}^m\right\}$$

with $a_0, \ldots, a_n \in C^\infty(\mathbf{R}^m)$. For instance,

$$\begin{aligned} H_2 \longrightarrow \{\ &\{t(x,y,z) - t(x,y,y) - t(x,x,z) + t(x,x,y); x,y,z \in \mathbf{R}^m\}; \\ &t \in C^\infty(\mathbf{R}^{3m})\}. \end{aligned}$$

(7.9.9) On this purely algebraic level the closed de Rham currents over $\mathbf{R}^m$ can be related to Hochschild cocycles in the following way.

(7.9.9.1) The inclusion of $C^\infty(\mathbf{R}^m)$ into the $\mathbf{Z}_2$-graded real differential algebra $E(\mathbf{R}^m)$ of differential forms, with the exterior derivation denoted by Δ, is extended by universality to some homomorphism: $\Omega(A) \longrightarrow E(\mathbf{R}^m)$ of such objects. This homomorphism vanishes on the kernel of $\nu \circ \psi$, thereby allowing for an according homomorphism of $\mathbf{Z}_2$-graded real differential algebras: $H(A) \longrightarrow E(\mathbf{R}^m)$. This homomorphism is then used in order to construct an injective real-linear map from closed de Rham currents to Hochschild cocycles:

$$\{\phi \in Hom_{\mathbf{R}}(E(\mathbf{R}^m)); \phi \circ \Delta = 0\} =: \mathcal{C} \ni \phi \longrightarrow f \in Z^*(A) \subset H^*(A),$$

such that $\forall a_0, a_1, \ldots, a_n \in A : f(a_0) = \phi(a_0)$,

$$f(a_0\delta(a_1) \otimes_A \cdots \otimes_A \delta(a_n)) = \phi(a_0\Delta(a_1) \wedge \cdots \wedge \Delta(a_n)).$$

(7.9.9.2) Conversely the usual A-linear basis:

$$\{e_A \equiv I, \Delta(x^{k_1}) \wedge \cdots \wedge \Delta(x^{k_p}); 1 \le k_1 < \cdots < k_p \le m\}$$

$$\underset{free\ over\ A}{\longrightarrow} E(\mathbf{R}^m)$$

is used to construct an $\mathbf{R}$-linear map: $Z^*(A) \ni f \longrightarrow \phi \in \mathcal{C}$, such that $\forall a_0, a_1, \ldots, a_n \in A : \phi(a_0) = f(a_0)$,

$$\phi(a_0\Delta(a_1) \wedge \cdots \wedge \Delta(a_n)) = \frac{1}{n!} \sum_{P \in \mathbf{P}_n} (-1)^{\tau_P} f(a_0\delta(a_{j_1}) \otimes_A \cdots \otimes_A \delta(a_{j_n})),$$

inserting the permutations $P := \begin{bmatrix} 1 & \cdots & n \\ j_1 & \cdots & j_n \end{bmatrix} \in \mathbf{P}_n$, and denoting the sign of P by τ_p. Since this real-linear map vanishes on $B^*(A)$, one obtains an according real-linear map:

$$H_H^*(A) \ni [f] \longrightarrow \phi \in \mathcal{C}.$$

(7.9.9.3) One easily verifies the composite mapping: $\mathcal{C} \ni \phi \longrightarrow [f] \longrightarrow \phi$.

(7.9.9.4) Hence one obtains an injective real-linear mapping:

$$\mathcal{C} \ni \phi \longrightarrow [f] \in H_H^*(A).$$

(7.9.9.5) Under certain topological conditions, A. Connes constructed an isomorphism between continuous Hochschild cohomologies and closed de Rham currents.

7.10 Universal Differential Envelope as Factor Algebra

In order to extend conveniently a graded star operation, which is defined on an associative superalgebra A over R, to its unital differential envelope $\tilde{\Omega}(A)$, the latter is reconstructed as an appropriate factor algebra.

(7.10.1)

$$E := A \oplus (R \otimes A) = E^{\bar{0}} \oplus E^{\bar{1}}, \; E^{\bar{z}} := A^{\bar{z}} \oplus \left(R \otimes A^{\overline{z+1}}\right),$$

with an obvious linear bijection.
$T(E) =: \mathbf{T} = \mathbf{T}^{\bar{0}} \oplus \mathbf{T}^{\bar{1}}$, with the usual $\mathbf{Z}_2$-grading of the tensor algebra. The set

$$\mathbf{S} := \{a \otimes b - ab, (e_R \otimes a) \otimes b + (-1)^{z} a \otimes (e_R \otimes b) - e_R \otimes (ab); \\ a \in A^{\bar{z}}, b \in A; \bar{z} \in \mathbf{Z}_2\} \subset \mathbf{T},$$

with the canonical embeddings being conveniently suppressed, generates an ideal $\mathbf{D}$ of $\mathbf{T}$, which is graded with respect to the $\mathbf{Z}_2$-grading of $\mathbf{T}$.

$$\mathbf{D} := sum\{tst'; s \in \mathbf{S}; t, t' \in \mathbf{T}\} = \bigoplus_{\bar{z} \in \mathbf{Z}_2} (\mathbf{D} \cap \mathbf{T}^{\bar{z}}).$$

$\mathbf{F} := \mathbf{T}/\mathbf{D} = \mathbf{F}^{\bar{0}} \oplus \mathbf{F}^{\bar{1}}$, with the $\mathbf{Z}_2$-grading defined according to the following implications.

$$\forall t = t^{\bar{0}} + \mathbf{t}^{\bar{1}} \in \mathbf{T} : [t] \in \mathbf{F}^{\bar{z}} \underset{def}{\Longleftrightarrow} t^{\overline{z+1}} \in \mathbf{D} \Longleftrightarrow \exists t' \in [t] : t' \in \mathbf{T}^{\bar{z}}.$$

(7.10.2) The universal property of tensor products is used to define R-linear mappings $\Delta_p : T^p(E) \longrightarrow T^p(E)$, such that $\forall p \in \mathbf{N}, \forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2$:

$$\Delta_p \left(\bigotimes_{k=1}^{p} (a_k + r_k \otimes b_k) \right)$$
$$= \sum_{k=1}^{p} (-1)^{z_1 + \cdots + z_{k-1}} \bigotimes_{l=1}^{k-1} (a_l + r_l \otimes b_l) \otimes (e_R \otimes a_k) \\ \otimes \bigotimes_{l=k+1}^{p} (a_l + r_l \otimes b_l) \in T^p(E).$$

With the linear mappings

$$\Delta_1 : E \ni a + r \otimes b \underset{def}{\longrightarrow} e_R \otimes a \in R \otimes A, \; \Delta_0 : R \underset{def}{\longrightarrow} \{0\},$$

an R-linear mapping

$$\Delta := \bigoplus_{p=0}^{\infty} \Delta_p$$

is established, with the following additional properties.

$$\forall \bar{z} \in \mathbf{Z}_2 : Im\Delta\big|_{\mathbf{T}^z} \subseteq \mathbf{T}^{\overline{z+1}};\ \Delta \circ \Delta = 0;$$

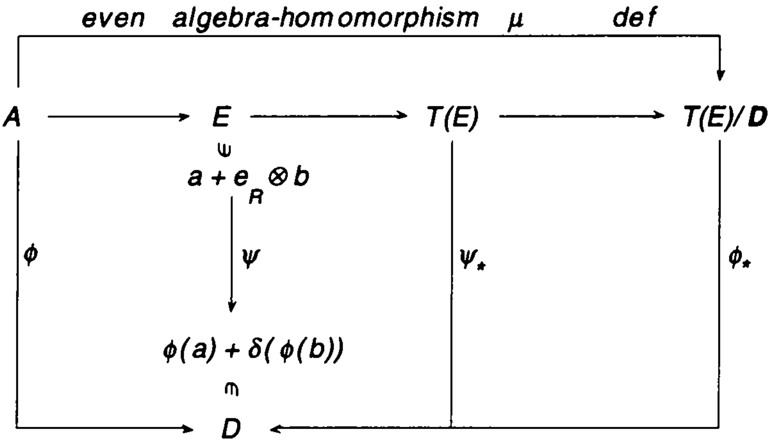
$$\forall \bar{z} \in \mathbf{Z}_2, \forall t_1 \in \mathbf{T}^{\bar{z}}, t_2 \in \mathbf{T} : \Delta(t_1 t_2) = \Delta(t_1)t_2 + (-1)^z t_1 \Delta(t_2).$$

(7.10.3) This $\mathbf{Z}_2$-graded differential algebra $T(E)$, with the nilpotent $\mathbf{Z}_2$-graded derivation Δ, is factorized with respect to $\mathbf{D}$, and Δ is compatible with this factorization.

$$Im\Delta\big|_{\mathbf{D}} \subseteq \mathbf{D};\quad \mathbf{F} \ni [t] \xrightarrow[def]{\Theta} [\Delta(t)] \in \mathbf{F}.$$

Again the latter is some nilpotent $\mathbf{Z}_2$-graded derivation; hence $\mathbf{F}$ is some $\mathbf{Z}_2$-graded differential algebra over R.

(7.10.4) Now take an arbitrary unital $\mathbf{Z}_2$-graded differential algebra D over R, and consider an even algebra-homomorphism $\phi : A \longrightarrow D$, i.e., $\forall \bar{z} \in \mathbf{Z}_2 :$ $Im\phi\big|_{A^z} \subseteq D^{\bar{z}}$. ϕ is extended to $\mathbf{F}$ according to the commuting diagram below.



Here the definition of an even R-linear map ψ is possible, due to the universal property of the tensor product $R \otimes A$. The homomorphism of unital superalgebras ψ_* is factorized by $\mathbf{D}$. Suppressing the embeddings, $\forall p \in \mathbf{N}$,

$$\forall x_1, \ldots, x_p \in E, \forall r \in R : \psi_*(x_1 \otimes \cdots \otimes x_p) = \psi(x_1) \cdots \psi(x_p),$$
$$\psi_*(r) := re_D;$$

$$Im\ \psi_*\big|_{\mathbf{D}} = \{0\};\ \forall t \in T(E) : \phi_*([t]) := \psi_*(t).$$

One therefore obtains some homomorphism of unital superalgebras ϕ_*, which especially fulfills:

$$\phi_* \circ \Theta = \delta \circ \phi_*; \ [\{e_R, 0, \ldots\}] =: e_{\mathbf{F}} \underset{\phi_*}{\longrightarrow} e_D.$$

(7.10.5) For every homomorphism of unital superalgebras $\phi : A \longrightarrow D$, into a unital $\mathbf{Z}_2$-graded differential algebra D over R, there is a unique homomorphism in the sense of unital $\mathbf{Z}_2$-graded differential algebras over R, namely ϕ_*, such that $\phi_* \circ \mu = \phi$.

(7.10.6) Since μ is a homomorphism of unital superalgebras, due to the above factorization, one finds an isomorphism in the sense of unital $\mathbf{Z}_2$-graded differential algebras over R, according to the diagram below.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "\tilde{\beta}"] \arrow[dr, "\mu"'] & \Omega(A) \arrow[r] \arrow[d, "\mu_*"] & \tilde{\Omega}(A) = R \oplus \Omega(A) \arrow[dl, shift left, "\tilde{\mu}_*"] \\
& T(E)/D \arrow[ur, shift left, "\tilde{\beta}_*"] &
\end{tikzcd}$$

Here one defines:

$$(\tilde{\Omega}(A))^{\bar{0}} \ni \{r, 0, 0, \ldots\} \underset{\tilde{\mu}_*}{\longrightarrow} re_{\mathbf{F}} = [\{r, 0, 0, \ldots\}] \in \mathbf{F}^{\bar{0}}.$$

Both $\tilde{\mu}_*$ and $\tilde{\beta}_*$ are homomorphisms in the sense of unital $\mathbf{Z}_2$-graded differential algebras over R. Hence one obtains:

$$\tilde{\mu}_* \circ \tilde{\beta}_* = id\ \mathbf{F},\ \tilde{\beta}_* \circ \tilde{\mu}_* = id\ \tilde{\Omega}(A);\ e_{\tilde{\Omega}(A)} \longleftrightarrow e_{\mathbf{F}}.$$

(7.10.6.1) With all the canonical embeddings written down, i.e., an explicit notation of direct sums, $\forall a, b \in A, \forall r \in R$:

$$\{r, \{a, e_R \otimes b\}, 0, \ldots\} + \mathbf{D} \underset{\tilde{\beta}_*}{\longrightarrow} \{\{r, a\}, \{e_R, 0_A\} \otimes b, 0, \ldots\},$$

with the unital universal differential envelope written as direct sum over R,

$$\tilde{\Omega}(A) := (R \oplus A) \oplus \bigoplus_{n \in \mathbf{N}} \left((R \oplus A) \otimes \bigotimes^{n} A \right).$$

Conveniently one writes $\forall a, b \in A, \forall r \in R$:

$$T(E)/\mathbf{D} \ni [r + a + e_R \otimes b] \underset{\tilde{\beta}_*}{\longrightarrow} r + a + db \in \tilde{\Omega}(A).$$

(7.10.7) There is an isomorphism $\tilde{\beta}_*$ of unital $\mathbf{Z}_2$-graded differential algebras over R:

$$\frac{\textit{tensor algebra} \text{ of } (A \oplus (R \otimes A))}{\textit{ideal} \text{ of } \mathbf{S}} \longleftrightarrow R \oplus \Omega(A).$$

7.11 Extension of Graded Star Operations to the Universal Differential Envelope

(7.11.1) Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be an associative superalgebra over the commutative ring R, and take an involutive isomorphism of rings: $R \ni r \longleftrightarrow \bar{r} \in R$, for instance complex conjugation. The bijection: $A \ni a \longleftrightarrow a^* \in A$ is called graded star operation on A, if and only if the following conditions are fulfilled.

$$\forall a, b \in A, \forall r, s \in R : (ra + sb)^* = \bar{r}a^* + \bar{s}b^*, \; a^{**} := (a^*)^* = a;$$
$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2 : a \in A^{\bar{x}}, b \in A^{\bar{y}} \Longrightarrow a^* \in A^{\bar{x}}, \; (ab)^* = (-1)^{xy} b^* a^*.$$

(7.11.2) Let $T(E)$ be the tensor algebra over the direct sum of R-bimodules $E = E^{\bar{0}} \oplus E^{\bar{1}}$, and consider an even antilinear involution: $E \ni x \longleftrightarrow x^* \in E$, i.e.,

$$\forall x, y \in E, \forall r, s \in R : (rx + sy)^* = \bar{r}x^* + \bar{s}y^*, \; x^{**} = x,$$
$$\forall \bar{z} \in \mathbf{Z}_2 : x \in E^{\bar{z}} \Longrightarrow x^* \in E^{\bar{z}}.$$

(7.11.2.1) $T(E)$ is $\mathbf{Z}_2$-graded, such that

$$\forall p \in \mathbf{N}, \forall_1^p k : x_k \in E^{\overline{z_k}} \Longrightarrow x_1 \otimes \cdots \otimes x_p \in E^{\overline{z_1 + \cdots + z_p}},$$

and the embedding of $T^0(E) := R$ into $T(E)$, $\{\{r, 0, \cdots\}; r \in R\} \subseteq (T(E))^{\bar{0}}$. This $\mathbf{Z}_2$-grading of the unital associative algebra $T(E)$ is defined with the R-linear bijection:

$$T^p(E) := \bigotimes^p E \longleftrightarrow \bigoplus_{\bar{z} \in \mathbf{Z}_2} \bigoplus_{\overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2, \; \overline{z_1 + \cdots + z_p} = \bar{z}} (E^{\overline{z_1}} \otimes \cdots \otimes E^{\overline{z_p}}).$$

(7.11.2.2) There is a unique graded star operation: $T(E) \ni t \longleftrightarrow t^* \in T(E)$, such that

$$\forall r \in R : \{r, 0, \ldots\}^* = \{\bar{r}, 0, \ldots\}, \;\; \forall x \in E : \{0, x, 0, \ldots\}^* = \{0, x^*, 0, \ldots\}.$$

(7.11.2.2.1) The tensor product of R-bimodules $E_1, \ldots, E_p$ can be used to lift a p-antilinear map $f : E_1 \times \cdots \times E_p \longrightarrow F$ to an R-antilinear map $f_* : E_1 \otimes \cdots \otimes E_p \longrightarrow F$, into an R-bimodule F, over the commutative ring R with an involutive isomorphism of rings.

(7.11.2.2.2) There is an R-antilinear map: $T^p(E) \longrightarrow T^p(E)$, for $p \in \mathbf{N}$, such that $\forall \overline{z_1}, \ldots, \overline{z_p} \in \mathbf{Z}_2$:

$$E^{\overline{z_1}} \otimes \cdots \otimes E^{\overline{z_p}} \ni x_1 \otimes \cdots \otimes x_p \\ \longrightarrow (-1)^{\sum_{1 \leq k < l \leq p} z_k z_l} x_p^* \otimes \cdots \otimes x_1^* \in T^p(E).$$

(7.11.2.2.3) Of course, if the bimodules $E^{\bar{0}}, E^{\bar{1}}$ are free over R, one can use their bases for an appropriate definition of this antilinear bijection of $T(E)$ onto itself.

(7.11.2.2.4) The remaining properties of a graded star operation on $T(E)$ are easily shown.

(7.11.3) Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be an associative superalgebra over R, with the graded star operation $*$. There is a unique graded star operation on the universal differential envelope $\Omega(A)$, such that

$$\forall a \in A : \{a, 0, \ldots\}^* = \{a^*, 0, \ldots\}, \ (da)^* = d(a^*) \equiv da^*,$$

which explicitly means that

$$\{0_A, e_{\tilde{A}} \otimes a, 0, \ldots\}^* = \{0_A, e_{\tilde{A}} \otimes a^*, 0, \ldots\}, \ e_{\tilde{A}} := \{e_R, 0_A\}.$$

Here the graded star operation on $\Omega(A)$ is again denoted by $*$. This so-called canonical graded star operation explicitly reads,
$\forall n \geq 2, \forall \overline{z_0}, \ldots, \overline{z_n} \in \mathbf{Z}_2, \forall a_0 \in A^{\overline{z_0}}, a_1 \in A^{\overline{z_0}}, a_1 \in A^{\overline{z_1}}, \ldots, a_n \in A^{\overline{z_n}}$:

$$\begin{aligned} &(a_0 da_1 \cdots da_n)^* \\ &= (-1)^{z_0(z_1 + \cdots + z_n + n) + \sum_{1 \leq k < l \leq n} (z_k + 1)(z_l + 1)} (da_n^*) \cdots (da_1^*) a_0^*, \\ &(da_1 \cdots da_n)^* = (-1)^{\sum_{1 \leq k < l \leq n} (z_k + 1)(z_l + 1)} (da_n^*) \cdots (da_1^*), \\ &(a_0 da_1)^* = (-1)^{z_0(z_1 + 1)} (da_1^*) a_0^*. \end{aligned}$$

(7.11.3.1) Due to the universal property of the tensor product, there is an antilinear map $* : \Omega(A) \longrightarrow \Omega(A)$ with the above properties; obviously it preserves the $\mathbf{Z}_2$-grading of $\Omega(A)$. Moreover uniqueness follows from the very properties of graded star operations.

(7.11.3.2) The idempotency, i.e.: $* \circ * = \varepsilon := id\ \Omega(A)$, follows from the property that

$$\forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall \omega_1 \in \Omega^{\overline{z_1}}(A), \omega_2 \in \Omega^{\overline{z_2}}(A) : (\omega_1 \omega_2)^* = (-1)^{z_1 z_2} \omega_2^* \omega_1^*.$$

For an elegant proof of the latter, in order to avoid straightforward, but tedious calculations, one uses the factor algebra representation of $R \oplus \Omega(A)$.

(7.11.3.2.1) The graded star operation on A is extended, with the universal property of $R \otimes A$, onto:

$$E \ni \{a, r \otimes b\} \longleftrightarrow \{a^*, \bar{r} \otimes b^*\} \in E := A \oplus (R \otimes A).$$

This even antilinear involution on E is then extended to a unique graded star operation on $T(E)$.

(7.11.3.2.2) $Im\ *\big|_{\mathbf{S}} \subseteq \mathbf{S}$, and therefore $Im\ *\big|_{\mathbf{D}} \subseteq \mathbf{D}$. Hence one obtains the graded star operation:

$$T(E)/\mathbf{D} \ni [t] \underset{*}{\longleftrightarrow} [t^*] \in T(E)/\mathbf{D}.$$

(7.11.3.2.3) Hence one defines an even antilinear map:

$$\tilde{\Omega}(A) \ni \{r, \omega\} \underset{*}{\longrightarrow} \{\bar{r}, \omega^*\} \in \tilde{\Omega}(A) = R \oplus \Omega(A).$$

(7.11.3.2.4) This even antilinear map $*$ on $\tilde{\Omega}(A)$, and the graded star operation $*$ on $T(E)/\mathbf{D}$, are compatible with the isomorphism

$$\tilde{\beta}_* : T(E)/\mathbf{D} \longleftrightarrow R \oplus \Omega(A),$$

which was defined in the previous section, due to the following commuting diagram:

$$* \circ \tilde{\beta}_* = \tilde{\beta}_* \circ *,$$

because $\forall a, b \in A,\ \forall r \in R:$

$$\{r, \{a, e_R \otimes b\}, 0, \ldots\} + \mathbf{D} \underset{\tilde{\beta}_*}{\longrightarrow} \{\{r, a\}, \{e_R, 0_A\} \otimes b, 0, \ldots\},$$

with $*$ acting on both sides, such that a, b, r are replaced by $a^*, b^*, \bar{r}$. Extending this diagram to $(r + a_0)da_1 \cdots da_n$, for $a_0, a_1, \ldots, a_n \in A, r \in R$, yields the desired result: $* \circ \tilde{\beta}_* = \tilde{\beta}_* \circ *$, with the notation $*$ used on both algebras.

(7.11.3.2.5) $\forall [t_1] \in (T(E)/\mathbf{D})^{\overline{z_1}}, [t_2] \in (T(E/\mathbf{D})^{\overline{z_2}}$:

$$[t_1][t_2] \xrightarrow{*} (-1)^{z_1 z_2}[t_2^*][t_1^*]$$
$$\xrightarrow{\tilde{\beta}_*} (-1)^{z_1 z_2}\tilde{\omega}_2^*\tilde{\omega}_1^* = * \circ \tilde{\beta}_*([t_1][t_2]) \in (R \oplus \Omega(A))^{\overline{z_1+z_2}}.$$

(7.11.4) In the special case of trivial $\mathbf{Z}_2$-grading of A, $A^{\bar{1}} := \{0\}$, such that $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in A$:

$$a_0 da_1 \cdots da_n \in \Omega^{\bar{n}}(A), \; da_1 \cdots da_n \in \Omega^{\bar{n}}(A),$$

one obtains immediately $\forall r \in R$:

$$((r + a_0)da_1 \cdots da_n)^* = (-1)^{\frac{n(n-1)}{2}}(da_n^*) \cdots (da_1^*)(\bar{r} + a_0^*),$$

$$(a_0 da_1)^* = (da_1^*)a_0^*,$$

with an obvious slightly modified notation, for convenience.
In this case, $E^{\bar{0}} = A, \; E^{\bar{1}} = R \otimes A$.

(7.11.5) $\forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall \omega_1 \in \Omega^{\overline{z_1}}(A), \omega_2 \in \Omega^{\overline{z_2}}(A)$:

$$[\omega_1, \omega_2]_g := \omega_1\omega_2 - (-1)^{z_1 z_2}\omega_2\omega_1 \xrightarrow{*} (-1)^{z_1 z_2}[\omega_2^*, \omega_1^*]_g.$$

$\forall \omega \in \Omega(A) : (d\omega)^* = d(\omega^*) \equiv d\omega^*$; one equivalently writes $* \circ d = d \circ *$.

$\forall \omega \in \Omega(A), b \in A : (\omega db)^* = \rho(\omega^* db^*)$.

$$\rho^{-1}\beta \circ * = - * \circ \beta, \; \rho^{-1}\gamma \circ * = - * \circ \gamma, \; \rho \circ * \circ \rho = *,$$

$$\alpha\rho^{-1} \circ * = * \circ \beta', \; \pi \circ * = * \circ \beta' d.$$

(7.11.6) On $\Omega^*(A) := Hom_R(\Omega(A), R)$, an even antilinear involution is defined, such that

$$\forall f \in \Omega^*(A), \omega \in \Omega(A) : f^*(\omega) := \overline{f(\omega^*)}; \; (f^*)^* = f.$$

Here one uses the $\mathbf{Z}_2$-grading according to the direct sum

$$\Omega^*(A) = \bigoplus_{\bar{z} \in \mathbf{Z}_2} (\Omega^*(A))^{\bar{z}}, \; (\Omega^*(A))^{\bar{z}} := \left\{ f \in \Omega^*(A); Im\, f|_{\Omega^{\overline{z+1}}(A)} = \{0\} \right\}.$$

(7.11.6.1)

$$f^* \circ d = (f \circ d)^*, \; f^* \circ \gamma = -(f \circ \gamma\rho^{-1})^*, \; f^* \circ \beta = -(f \circ \beta\rho^{-1})^*.$$

Hence one obtains the following implications.

f *closed* $\Longleftrightarrow f^*$ *closed*, f *cyclic* $\Longleftrightarrow f^*$ *cyclic*;

f *graded trace* $\Longleftrightarrow f^*$ *graded trace*.

$$* : C^*(A) \longleftrightarrow C^*(A), \; C^*_\lambda(A) \longleftrightarrow C^*_\lambda(A), \; Z^*_\lambda(A) \longleftrightarrow Z^*_\lambda(A).$$

(7.11.6.2)

$\forall f \in C^*_\lambda(A) : f^* \circ \beta = -(f \circ \beta)^*$; hence $* : \; B^*_\lambda(A) \longleftrightarrow B^*_\lambda(A)$.

7.12 Cycles over Associative Superalgebras

(7.12.1) Let D be a $\mathbf{Z}_2$-graded differential algebra with the derivation δ, over a commutative ring R. The linear form $f \in D^* := Hom_R(D,R)$ is called cycle and usually denoted by $\int$, if and only if it is some closed graded trace, which explicitly means that

$$\forall a,b \in D : f \circ \delta(a) \equiv \int \delta(a) = 0, \quad \int [a,b]_g = 0,$$

with the super-commutator

$$[a,b]_g := ab - \sum_{\bar{x},\bar{y} \in \mathbf{Z}_2} (-1)^{xy} b^{\bar{y}} a^{\bar{x}}.$$

(7.12.1.1) Especially let D be bigraded. Then the cycle f is called an n-cycle, if and only if

$$\forall n \neq k \in \mathbf{N}_0 : f\big|_{D_k} = 0.$$

(7.12.2) The tensor product $\int \otimes \int'$ of cycles, on the skew-symmetric tensor product $D\hat{\otimes}D'$ of $\mathbf{Z}_2$-graded differential algebras D and D' over R, is again some cycle. Remember that $\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in D, a' \in D'^{\bar{x}}, b \in D^{\bar{y}}, b' \in D'$:

$$(a \otimes a')(b \otimes b') = (-1)^{xy}(ab) \otimes (a'b'), \quad b \otimes a' \in (D\hat{\otimes}D')^{\overline{x+y}},$$

$$\Delta(b \otimes b') = \delta(b) \otimes b' + (-1)^{y} b \otimes \delta'(b').$$

If both D and D' are bigraded, then

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall k,l \in \mathbf{N}_0, \forall a \in D_k^{\bar{x}}, a' \in D'^{\bar{y}}_l : a \otimes a' \in (D\hat{\otimes}D')^{\overline{x+y}}_{k+l}.$$

$$\forall f \in D^*, f' \in D'^*, a \in D, a' \in D' : f \otimes f'(a \otimes a') = f(a)f'(a').$$

(7.12.3) Let $f \equiv \int$ be a cycle on D, and A an associative superalgebra over R. A homomorphism of associative superalgebras $\phi : A \longrightarrow D$ is uniquely extended to some homomorphism $\phi_* : \Omega(A) \longrightarrow D$ in the sense of $\mathbf{Z}_2$-graded differential algebras over R, i.e.,

$$\forall \bar{z} \in \mathbf{Z}_2 : Im\ \phi_*\big|_{\Omega^{\bar{z}}(A)} \subseteq D^{\bar{z}}, \quad \phi_* \circ d = \delta \circ \phi_*.$$

One then finds, that $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in A, \forall r \in R$:

$$(r + a_0)da_1 \cdots da_n \underset{\phi_*}{\longrightarrow} (r + \phi(a_0))\delta \circ \phi(a_1) \cdots \delta \circ \phi(a_n).$$

(7.12.3.1) The cycle f is pulled back to the linear form $\tau := f \circ \phi_*$, according to the diagram below.

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr] & & D \arrow[dr, "f \equiv \int", "\text{cycle}"'] & \\
& \Omega(A) \arrow[ur, "\phi_*"] \arrow[rr, "\text{character } \tau \quad \text{def}"'] & & R
\end{tikzcd}$$

$$\forall \omega \in \Omega(A) : \tau(\omega) := f \circ \phi_*(\omega) \equiv \int \phi_*(\omega).$$

The family $\{D, \delta, A, \phi, f\} \equiv \{\delta, \phi, \int\}$ is called cycle over A. The pull back $\tau \in \Omega^*(A)$ is called the character of this cycle.

(7.12.3.2) Especially let D be bigraded. The family $\{\delta, \phi, \int\}$ is called $\mathbf{N}_0$-cycle over A, if and only if $Im\ \phi \subseteq D_0$; in this case

$$\forall n \in \mathbf{N}_0 : Im\ \phi_*\big|_{\Omega_n(A)} \subseteq D_n; \text{ here } \Omega_0(A) := A.$$

(7.12.3.3) Obviously the character τ is some closed graded trace, $\tau \in Z^*_\lambda(A)$, which explicitly means that $\tau \circ d = \tau \circ \beta = 0$.

(7.12.3.4) Especially let $\{\delta, \phi, f\}$ be a $\mathbf{N}_0$-cycle over A. If f is a n-cycle for some $n \in \mathbf{N}_0$, then the character τ is some cyclic n-cocycle of A,

$$\tau \in Z^*_{\lambda n}(A) := Z^*_\lambda(A) \cap \Omega^*_n(A).$$

(7.12.4) Conversely every closed graded trace $f \in Z^*_\lambda(A)$ is the character of the cycle $\{d, \phi, f\}$ over A, which is denoted by the diagram below.

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr] & & \Omega(A) \arrow[dr, "f \in Z^*_\lambda(A)"] & \\
& \Omega(A) \arrow[ur, leftrightarrow, "id"] \arrow[rr, "f"'] & & R
\end{tikzcd}$$

(7.12.5) Let

$$\{\delta : D \to D, \phi : A \to D, f : D \to R\},$$

$$\{\delta' : D' \to D', \phi' : A' \to D', f' : D' \to R\}$$

be two cycles over A and A', respectively. The character of the tensor product of these two cycles is the cup-product of their characters, according to the commuting diagram below.

$$
\begin{tikzcd}
 & \Omega(A\hat{\otimes}A') \arrow[dr, "(T(\phi,\phi'))_*"] \arrow[drr, bend left, "\text{character of tensor product}"] \arrow[dd, bend right=80, "\nu_*"'] & & \\
A\hat{\otimes}A' \arrow[ur] \arrow[rr, "T(\phi,\phi')"] \arrow[dr, "\nu"'] & & D\hat{\otimes}D' \arrow[r, "f\otimes f'"'] & R \\
 & \Omega(A)\hat{\otimes}\Omega(A') \arrow[ur, "T(\phi_*,\phi'_*)"'] \arrow[urr, bend right, "\tau\otimes\tau'"'] & &
\end{tikzcd}
$$

(7.12.5.1) The tensor product of these two cycles over A and A', respectively, is defined as the cycle over $A\hat{\otimes}A'$, i.e., the family

$$\{D\hat{\otimes}D', \Delta, A\hat{\otimes}A', T(\phi,\phi'), f\otimes f'\},$$

with the usual derivation Δ on $D\hat{\otimes}D'$. Here the notation $\phi\otimes\phi'$, instead of $T(\phi,\phi')$, would be misleading.

(7.12.5.2) The R-linear map

$$T(\phi,\phi') : A\hat{\otimes}A' \ni a\otimes a' \longrightarrow \phi(a)\otimes\phi'(a') \in D\hat{\otimes}D'$$

is some homomorphism of associative superalgebras. Hence it is extended uniquely to some homomorphism $(T(\phi,\phi'))_*$ in the sense of $\mathbf{Z}_2$-graded differential algebras over R.

(7.12.5.3) On the other hand, since $T(\phi_*,\phi'_*)\circ\nu_*$ is some homomorphism in just this sense, one finds that

$$T(\phi_*,\phi'_*)\circ\nu_* = (T(\phi,\phi'))_*.$$

(7.12.5.4) Hence one obtains, with the characters $\tau := f\circ\phi_*$, $\tau' := f'\circ\phi'_*$, the desired result

$$(f\otimes f')\circ(T(\phi,\phi'))_* = (\tau\otimes\tau')\circ\nu_* = \tau * \tau'.$$

7.13 Fredholm Modules

(7.13.1) Let $\mathcal{A}$ be an associative algebra over the field $\mathbf{C}$ of complex numbers.

(7.13.1.1) $\mathcal{A}$ is called $*$-algebra, if and only if an involution exists, which is defined as an antilinear bijection: $\mathcal{A} \ni A \longleftrightarrow A^* \in \mathcal{A}$, such that $\forall A, B \in \mathcal{A}, \forall \alpha, \beta \in \mathbf{C}$:

$$A^{**} = A, \ (AB)^* = B^* A^*, \ (\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*.$$

In this case, $\forall A \in \mathcal{A} : A = 0 \Longleftrightarrow A^* = 0$. Moreover in this case, a subset $\mathcal{A}_1$ of $\mathcal{A}$ is called self-adjoint, if and only if $\forall A_1 \in \mathcal{A}_1 : A_1^* \in \mathcal{A}_1$.

(7.13.1.2) $\mathcal{A}$ is called normed algebra, if and only if it is some normed space, and the following product inequality holds. $\forall A, B \in \mathcal{A}, \forall \alpha \in \mathbf{C}$:

$$\|A\| \geq 0, \ \|A\| = 0 \Longleftrightarrow A = 0, \ \|\alpha A\| = |\alpha| \ \|A\|,$$

$$\|A + B\| \leq \|A\| + \|B\|, \ \|AB\| \leq \|A \| \ \|B\|.$$

In this case, $\mathcal{A}$ is called Banach algebra, if and only if it is complete with respect to this norm.

(7.13.1.3) Let $\mathcal{A}$ be a Banach algebra with involution. $\mathcal{A}$ is called B^*-algebra, if and only if

$$\forall A \in \mathcal{A} : \|A\| = \|A^*\|.$$

$\mathcal{A}$ is called C^*-algebra, if and only if

$$\forall A \in \mathcal{A} : \|A^* A\| = \|A\|^2;$$

in this case,

$$\forall A \in \mathcal{A} : \|A\| = \|A^*\|, \ \|AA^*\| = \|A\|^2.$$

(7.13.1.4) If the $*$-algebra $\mathcal{A}$ is unital with the unit I, then $I = I^*$ and $\|I\| = 1$; here $\mathcal{A}$ is assumed to be non-zero, which equivalently means $I \neq 0$.

(7.13.1.5) Let $\mathcal{A}$ be a B^*-algebra. Its unital extension $\tilde{\mathcal{A}} := \mathbf{C} \oplus \mathcal{A}$, with the unit $\tilde{I} := \{1, 0_{\mathcal{A}}\}$, is some B^*-algebra, with the involution and norm defined such that

$$\forall \alpha \in \mathbf{C}, A \in \mathcal{A} : \{\alpha, A\}^* := \{\bar{\alpha}, A^*\}, \ \|\{\alpha, A\}\| := |\alpha| + \|A\|.$$

(7.13.1.6) If $\mathcal{A}$ is a C^*-algebra, then its unital extension, with the unit and involution defined above, becomes some C^*-algebra, with the norm such that

$$\forall \alpha \in \mathbf{C}, A \in \mathcal{A} : \|\{\alpha, A\}\| := sup\{\|\alpha B + AB\|; B \in \mathcal{A}, \|B\| = 1\}.$$

(7.13.1.7) Let $\mathcal{I}$ be a left or right ideal of the $*$-algebra $\mathcal{A}$. If $\mathcal{I}$ is self-adjoint, then obviously $\mathcal{I}$ is some two-sided ideal of $\mathcal{A}$.

(7.13.1.8) The B^*-algebra $\mathcal{A}$ is called simple, if and only if the only closed two-sided ideals of $\mathcal{A}$ are $\mathcal{A}$ itself and $\{0\}$.

(7.13.2) Let $\mathcal{H}$ be a Hilbert space with complex coefficients. The set $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is some C^*-algebra, with the adjointness † of operators, and with respect to the operator norm.

(7.13.2.1) Obviously every closed self-adjoint subalgebra of the C^*-algebra $\mathcal{A}$ is some C^*-algebra.

(7.13.2.2) The set $B_\infty(\mathcal{H})$ of compact operators on $\mathcal{H}$ is an ideal of $B(\mathcal{H})$, and $B_\infty(\mathcal{H})$ is some C^*-algebra.

(7.13.3) Let $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$ be a separable Hilbert space with complex coefficients, with the corresponding orthogonal projectors $N^{\bar{0}}, N^{\bar{1}}$, and the so-called Klein operator

$$K := N^{\bar{0}} - N^{\bar{1}} = I - 2N^{\bar{1}}.$$

(7.13.3.1) The C^*-algebra $B(\mathcal{H})$ is decomposed into the direct sum of closed linear subspaces,

$$B(\mathcal{H}) = \bigoplus_{\bar{z} \in \mathbf{Z}_2} B^{\bar{z}}(\mathcal{H}),$$

$$B^{\bar{z}}(\mathcal{H}) := \left\{ A \in B(\mathcal{H}); \forall \bar{x} \in \mathbf{Z}_2 : Im\, A\big|_{\mathcal{H}^{\bar{x}}} \subseteq \mathcal{H}^{\overline{x+z}} \right\}.$$

$$\forall f = f^{\bar{0}} + f^{\bar{1}} \in \mathcal{H}, A = A^{\bar{0}} + A^{\bar{1}} \in B(\mathcal{H}), \forall \bar{z}, \bar{x} \in \mathbf{Z}_2 : f^{\bar{x}} \xrightarrow[A^{\bar{z}}]{} (A(f^{\bar{x}}))^{\overline{x+z}},$$

which equivalently means, that

$$A^{\bar{0}} = N^{\bar{0}} A N^{\bar{0}} + N^{\bar{1}} A N^{\bar{1}},\ A^{\bar{1}} = N^{\bar{1}} A N^{\bar{0}} + N^{\bar{0}} A N^{\bar{1}}.$$

(7.13.3.2) This direct sum is equivalently written in matrix notation.
$\forall A = A^{\bar{0}} + A^{\bar{1}} \in B(\mathcal{H})$:

$$A^{\bar{0}} =: \begin{bmatrix} A^{00} & 0 \\ 0 & A^{11} \end{bmatrix},\ A^{\bar{1}} =: \begin{bmatrix} 0 & A^{01} \\ A^{10} & 0 \end{bmatrix},$$

$$K := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},\ \text{onto}\ \begin{bmatrix} f^{\bar{0}} \\ f^{\bar{1}} \end{bmatrix} \in \mathcal{H}.$$

(7.13.3.3) Obviously the closed subalgebra $B^{\bar{0}}(\mathcal{H})$ is self-adjoint, hence some C^*-algebra.

(7.13.3.4) The unital associative algebra $B(\mathcal{H})$ is $\mathbf{Z}_2$-graded, with the grading automorphism

$$\kappa : B(\mathcal{H}) \ni A = A^{\bar{0}} + A^{\bar{1}} \underset{def}{\longleftrightarrow} A^{\bar{0}} - A^{\bar{1}} = KAK \in B(\mathcal{H}), \ \kappa^2 = id\ B(\mathcal{H}).$$

(7.13.4) Let $\mathcal{A} = \mathcal{A}^{\bar{0}} \oplus \mathcal{A}^{\bar{1}}$ be an associative superalgebra with complex coefficients. Consider some representation π of $\mathcal{A}$ on $\mathcal{H}$, i.e., a homomorphism of complex associative superalgebras:

$$\mathcal{A} \ni a \longrightarrow \pi(a) \in B(\mathcal{H}), \ \forall \bar{z} \in \mathbf{Z}_2 : Im\ \pi\big|_{\mathcal{A}^{\bar{z}}} \subseteq B^{\bar{z}}(\mathcal{H}).$$

(7.13.4.1) If $\mathcal{A}$ is unital with the unit e, one demands $\pi(e) = I$. In this case one obtains an $\mathcal{A}$-right module $\mathcal{H}$ over $\mathbf{C}$, with the definition that

$$\forall f \in \mathcal{H}, a \in \mathcal{A} : fa := \pi(a)(f).$$

(i) $\mathcal{H}$ is some right module over the ring $\mathcal{A}$.

(ii) $\forall f \in \mathcal{H}, a \in \mathcal{A}, \alpha \in \mathbf{C} : \alpha(fa) = (\alpha f)a = f(\alpha a)$.
Here the product of $B(\mathcal{H})$ is used in the sense, that

$$\forall A, B \in B(\mathcal{H}), f \in \mathcal{H} : (A \circ B)(f) := B(A(f)).$$

(7.13.4.2) With the grading automorphism

$$\theta : \mathcal{A} \ni a^{\bar{0}} + a^{\bar{1}} \underset{def}{\longleftrightarrow} a^{\bar{0}} - a^{\bar{1}} \in \mathcal{A}, \ \theta^2 = id\ \mathcal{A},$$

the $\mathbf{Z}_2$-grading of π is written such that

$$\forall a \in \mathcal{A} : \pi \circ \theta(a) = \kappa \circ \pi(a) = K\pi(a)K.$$

(7.13.5) Let F be an odd bounded operator on $\mathcal{H}$,

$$F =: \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}, \ KFK = -F.$$

The $\mathcal{A}$-right module $\mathcal{H}$ over $\mathbf{C}$ is called Fredholm $\mathcal{A}$-right module, if and only if

$$\forall a \in \mathcal{A} : \pi(a)(F^2 - I) \in B_\infty(\mathcal{H}), \ F\pi(a) - K\pi(a)KF \in B_\infty(\mathcal{H}).$$

In this case, $\mathcal{H}$ is called involutive, if and only if one especially finds $F^2 = I$.

(7.13.6) Let $\{\tau_n; n \in \mathbf{N}\}$ be the sequence of repeated singular values of $T \in B_\infty(\mathcal{H})$.

$$]0, +\infty[\cap \textit{spectrum } (T^\dagger T) =: \{\tau_n^2; n \in \mathbf{N}\}, \ \forall n \in \mathbf{N} : 0 < \tau_{n+1} \leq \tau_n;$$

$$0 \leq T^\dagger T = \sum_{n=1}^{\infty} \tau_n^2 < g_n| > g_n,$$

with an arbitrary orthonormal basis of eigenvectors g_n.

(7.13.6.1) For a compact operator $T \in B_\infty(\mathcal{H})$, and $1 \leq p < \infty$, the p-norm is defined by

$$0 \leq \|T\|_p := \left(\sum_{n=1}^{\infty} \tau_n^p \right)^{1/p} \leq \infty.$$

The usual operator norm on $B(\mathcal{H})$ is denoted by $\|T\|_\infty \equiv \|T\|$.

(7.13.6.2)

$$|T| := (T^\dagger T)^{1/2} = \sum_{n=1}^{\infty} \tau_n < g_n| > g_n \geq 0, \ |T| \, |T| = T^\dagger T.$$

$$\forall 1 \leq p < \infty : 0 \leq \|T\|_p = \|T^\dagger\|_p = \| \, |T| \, \|_p = \| \, |T^\dagger| \, \|_p \leq \infty.$$
$$0 \leq \|T\| = \|T^\dagger\| = \| \, |T| \, \| = \| \, |T^\dagger| \, \| < \infty.$$

(7.13.7.1) $\forall \ T_1, T_2 \in B(\mathcal{H}), \forall \ 1 \leq p < \infty :$

$$\|T_1\|_p < \infty \Longrightarrow \ max \ \{\|T_1 T_2\|_p, \|T_2 T_1\|_p\} \leq \|T_1\|_p \|T_2\|.$$

Here T_1 is assumed to be compact, such that both $T_1 T_2$ and $T_2 T_1$ are compact.

(7.13.7.2) $\forall \ T_1, T_2 \in B_\infty(\mathcal{H}), \forall p, q, r \in [1, +\infty[\ :$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Longrightarrow \|T_1 T_2\|_r \leq \|T_1\|_p \|T_2\|_q \leq \infty.$$

(7.13.8) The Schatten ideals $B_p(\mathcal{H}), 1 \leq p < \infty$, are defined as the ideals

$$B_p(\mathcal{H}) := \{T \in B_\infty(\mathcal{H}); \|T\|_p < \infty\}$$

of $B(\mathcal{H})$. With respect to the p-norm, $B_p(\mathcal{H})$ is some Banach space.

$$\forall 1 \leq p < q < \infty : B_p(\mathcal{H}) \subset B_q(\mathcal{H}) \subset B_\infty(\mathcal{H}) \subset B(\mathcal{H}).$$

(7.13.8.1) The trace class $B_1(\mathcal{H})$ is equipped with the trace norm $\|T\|_1$, and the Schmidt class $B_2(\mathcal{H})$ with the Schmidt norm $\|T\|_2$, for $T \in B_\infty(\mathcal{H})$. One then obtains, that

$$\forall\, T \in B_\infty(\mathcal{H}) : 0 \leq \|T\| \leq \|T\|_2 \leq \|T\|_1 \leq \infty.$$

(7.13.9) The trace of $T \in B_1(\mathcal{H})$ is defined as

$$tr\ T := \sum_{n=1}^{\infty} < g_n | T g_n > \in \mathbf{C},$$

which obviously is independent of the choice of an orthonormal basis $\{g_n; n \in \mathbf{N}\}$ for $\mathcal{H}$. One then finds, that

$$\forall p, q \in \,]1, +\infty[\ , \forall\, T_1 \in B_p(\mathcal{H}), T_2 \in B_q(\mathcal{H}) :$$

$$\frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow tr(T_1 T_2) = tr(T_2 T_1).$$

$$\forall\, T \in B_1(\mathcal{H}) : |tr\ T| \leq \|T\|_1;$$

$$\forall\, 1 \leq p < \infty, \forall\, T \in B_p(\mathcal{H}) : \|T\|_p = (tr(|T|^p))^{1/p}.$$

(7.13.10) The Fredholm $\mathcal{A}$-right module $\mathcal{H}$ is called p-summable, for $1 \leq p < \infty$, if and only if one finds that $\forall a \in \mathcal{A}$:

$$\pi(a)(F^2 - I) \in B_p(\mathcal{H}),\ \ F\pi(a) - K\pi(a)KF \in B_p(\mathcal{H}).$$

(7.13.11) The super-commutator algebra $(B(\mathcal{H}))_L$, of the unital associative superalgebra $B(\mathcal{H})$, is defined with respect to the super-commutator.

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall A \in B^{\bar{x}}(\mathcal{H}), B \in B^{\bar{y}}(\mathcal{H}) : [A, B]_g := AB - (-1)^{xy} BA.$$

Obviously π provides an according representation of the super-commutator algebra $\mathcal{A}_L$ on $\mathcal{H}$, i.e., a homomorphism $\pi :\ \mathcal{A}_L \longrightarrow (B(\mathcal{H}))_L$ in the sense of complex Lie superalgebras.

(7.13.12)

$$F^2 = \begin{bmatrix} PQ & 0 \\ 0 & QP \end{bmatrix},\ FF^\dagger = \begin{bmatrix} PP^\dagger & 0 \\ 0 & QQ^\dagger \end{bmatrix},\ \ F^\dagger F = \begin{bmatrix} Q^\dagger Q & 0 \\ 0 & P^\dagger P \end{bmatrix}.$$

$$F^2 = I \Longleftrightarrow [P : \mathcal{H}^{\bar{1}} \longrightarrow \mathcal{H}^{\bar{0}},\ P^{-1} = Q].\quad F = F^\dagger \Longleftrightarrow P = Q^\dagger.$$

F is unitary, if and only if both P and Q are unitary.

(7.13.13) For $A \in B(\mathcal{H})$ and $p \in [1, +\infty]$, assume that

$$FA - KAKF \in B_p(\mathcal{H});$$

then one easily finds, that

$$(F^2 - I)A \in B_p(\mathcal{H}) \Longleftrightarrow A(F^2 - I) \in B_p(\mathcal{H}).$$

(7.13.14) $\forall A \in B(\mathcal{H})$:

$$A(F^2 - I) = \begin{bmatrix} A^{00}(PQ - I) & A^{01}(QP - I) \\ A^{10}(PQ - I) & A^{11}(QP - I) \end{bmatrix},$$

$$[F, A]_g = FA - KAKF = \begin{bmatrix} PA^{10} + A^{01}Q & PA^{11} - A^{00}P \\ QA^{00} - A^{11}Q & QA^{01} + A^{10}P \end{bmatrix}.$$

(7.13.15) Remember the well-known criterion:

$$\forall\, T \in B(\mathcal{H}) : T \ \textit{compact}$$
$$\Longleftrightarrow [\forall g \in \mathcal{H} :< g|f_n > \underset{n\to\infty}{\longrightarrow} 0 \Longrightarrow \|Tf_n\| \underset{n\to\infty}{\longrightarrow} 0].$$

Hence one finds, that $\mathcal{H}$ is some Fredholm $\mathcal{A}$-right module, if and only if the above eight bounded operators, $A^{00}(PQ - I), \ldots, QA^{01} + A^{10}P$, are compact for all $A \in Im\ \pi$.

(7.13.16) The supertrace of $T \in B_1(\mathcal{H})$ is defined as an element of

$$B_1^*(\mathcal{H}) := Hom_{\mathbf{C}}(B_1(\mathcal{H}), \mathbf{C}) \ni str : \ B_1(\mathcal{H}) \ni T \longrightarrow tr(KT) \in \mathbf{C},$$

$$\|KT\|_1 = \|T\|_1.$$

One easily finds, that $\forall\, T \in B_1(\mathcal{H}) \cap B^{\bar{1}}(\mathcal{H}) : str\ \ T = 0$.
With the convenient notation, that

$$\forall\, 1 \leq p < \infty, \forall \bar{z} \in \mathbf{Z}_2 : B_p^{\bar{z}}(\mathcal{H}) := B_p(\mathcal{H}) \cap B^{\bar{z}}(\mathcal{H}),$$

one finds, that $\forall\, p, q \in \,]1, +\infty[\ , \forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall\, T_1 \in B_p^{\bar{x}}(\mathcal{H}), T_2 \in B_q^{\bar{y}}(\mathcal{H})$:

$$\frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow str(T_1T_2) = (-1)^{xy} str(T_2T_1),$$

which just means, that the supertrace vanishes on super-commutators.

(7.13.17)

$$\forall\, T_1 \in B_1(\mathcal{H}), T \in B(\mathcal{H}) : tr([T_1, T]) = 0, \ \ str([T_1, T]_g) = 0.$$

(7.13.18) The $\mathbf{Z}_2$-graded derivation $\delta : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$, which is defined such that $\forall A \in B(\mathcal{H})$:

$$\delta(A) := i[F, A]_g = i(FA - KAKF) = iK[KF, A] \in B(\mathcal{H}),$$

fulfills the subsequent conditions.

$$\forall \bar{z} \in \mathbf{Z}_2 : Im\ \delta\big|_{B^{\bar{z}}(\mathcal{H})} \subseteq B^{\overline{z+1}}(\mathcal{H}).$$

$$\forall \bar{z} \in \mathbf{Z}_2, \forall A \in B^{\bar{z}}(\mathcal{H}), B \in B(\mathcal{H}) : \delta(AB) = \delta(A)B + (-1)^z A\delta(B).$$

$$\forall A \in B(\mathcal{H}) : \delta \circ \delta(A) = \left[A, F^2\right].$$

Hence one obtains some $\mathbf{Z}_2$-graded differential algebra.

$$F^2 = I \Longrightarrow \delta \circ \delta = 0,\ \delta(F) = 2iI.$$

(7.13.19)

$$\forall\, T \in B_1(\mathcal{H}) : str \circ \delta(T) = 0,\ str\ T = \frac{1}{2i} str(F\delta(T)).$$

(7.13.20) One uses the following terminology. Let $D = D^{\bar{0}} \oplus D^{\bar{1}}$ be a $\mathbf{Z}_2$-graded differential algebra over the commutative ring R, with the $\mathbf{Z}_2$-graded derivation δ. The linear form $f \in D^* := Hom_R(D, R)$ is called closed, if and only if $f \circ \delta = 0$; f is called graded trace, if and only if it vanishes on super-commutators, such that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in D^{\bar{x}}, b \in D^{\bar{y}} : f(ab) = (-1)^{xy} f(ba).$$

(7.13.21) As an example, assume $F^2 = I$. In this case $B(\mathcal{H})$ is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$. With the restriction $\delta_1 : B_1(\mathcal{H}) \longrightarrow B_1(\mathcal{H})$ of δ, $B_1(\mathcal{H})$ again is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$. The super-trace $str \in B_{\bar{1}}^*(\mathcal{H})$ is some closed graded trace, because of

$$str \circ \delta\big|_{B_1(\mathcal{H})} = 0.$$

(7.13.22) Let the Fredholm $\mathcal{A}$-right module $\mathcal{H}$ be involutive, $F^2 = I$. The representation π induces some homomorphism $\pi_* : \Omega(\mathcal{A}) \longrightarrow B(\mathcal{H})$ in the sense of $\mathbf{Z}_2$-graded differential algebras over $\mathbf{C}$, according to the diagram below.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "embedding"] \arrow[dr, "\pi"'] & & \Omega(\mathcal{A}) \arrow[dl, "\pi_*"] & & \Omega_{tr}(\mathcal{A}) \arrow[ll, "inclusion"'] \arrow[dl, "surjective"'] \arrow[dr, "t"', "def"] & \\
& B(\mathcal{H}) & & B_1(\mathcal{H}) \arrow[ll, "inclusion"] \arrow[rr, "str"'] & & \mathbf{C}
\end{tikzcd}$$

$$\forall \bar{z} \in \mathbf{Z}_2 : Im\ \pi_*\big|_{\Omega^{\bar{z}}(\mathcal{A})} \subseteq B^{\bar{z}}(\mathcal{H}); \quad \pi_* \circ d = \delta \circ \pi_*.$$

(7.13.22.1) $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in \mathcal{A}$:

$$(1 + a_0)da_1 \cdots da_n \underset{\pi_*}{\longrightarrow} (1 + \pi(a_0))i^n[F, \pi(a_1)]_g \cdots [F, \pi(a_n)]_g.$$

(7.13.22.2)

$$\Omega_{tr}(\mathcal{A}) := \{\omega \in \Omega(\mathcal{A}); \pi_*(\omega) \in B_1(\mathcal{H})\}, \ t := str \circ \pi_*\big|_{\Omega_{tr}(\mathcal{A})}.$$

One easily finds that $\Omega_{tr}(\mathcal{A})$ is an ideal of $\Omega(\mathcal{A})$. Moreover $\Omega_{tr}(\mathcal{A})$ is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$, because of $Im\ d\big|_{\Omega_{tr}(\mathcal{A})} \subseteq \Omega_{tr}(\mathcal{A})$, i.e.,

$$\forall \omega \in \Omega_{tr}(\mathcal{A}) : \pi_*(d\omega) = \delta \circ \pi_*(\omega) = i[F, \pi_*(\omega)]_g \in B_1(\mathcal{H}).$$

The linear form $t \in \Omega^*_{tr}(\mathcal{A}) := Hom_{\mathbf{C}}(\Omega_{tr}(\mathcal{A}), \mathbf{C})$ is an even closed graded trace.

$$\forall \omega \in \Omega(\mathcal{A}) : \pi_*(\omega) \in B_1^{\bar{1}}(\mathcal{H}) \Longrightarrow t(\omega) = 0;$$
$$\forall \omega \in \Omega_{tr}(\mathcal{A}) : t \circ d(\omega) = str \circ \delta \circ \pi_*(\omega) = 0;$$
$$\forall \omega_1, \omega_2 \in \Omega_{tr}(\mathcal{A}) : t([\omega_1, \omega_2]_g) = str([\pi_*(\omega_1), \pi_*(\omega_2)]_g) = 0.$$

Here the super-commutator on $\Omega(\mathcal{A})$ is defined according to its $\mathbf{Z}_2$-grading.

(7.13.22.3) $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_n \in \mathcal{A}$:

$$\begin{aligned} a_0 da_1 \cdots da_n \in \Omega_{tr}(\mathcal{A}) \Longrightarrow \tau_n(a_0, a_1, \ldots, a_n) &:= t(a_0 da_1 \cdots da_n) \\ &= i^n tr\big(K\pi(a_0)[F, \pi(a_1)]_g \cdots [F, \pi(a_n)]_g\big); \end{aligned}$$

especially,

$$\partial a_0 + \partial a_1 + \cdots + \partial a_n + \bar{n} = \bar{1} \Longrightarrow \tau_n(a_0, a_1, \ldots, a_n) = 0.$$

These $\mathbf{C} - (n+1)-$linear forms τ_n are called cyclic n-cocycles of $\mathcal{A}$.

(7.13.22.4) The linear form $t := str \circ \pi_*\big|_{\Omega_{tr}(\mathcal{A})}$ is an example of (generalized) cycle.

(7.13.23) Let the involutive Fredholm $\mathcal{A}$-right module $\mathcal{H}$ be p-summable in the sense, that

$$\exists\ 1 \leq p < \infty : \forall A \in Im\ \pi : [F, A]_g = FA - KAKF \in B_p(\mathcal{H}).$$

Then one easily obtains, that $\forall n \in \mathbf{N}$:

$$\begin{aligned} n \geq p \Longrightarrow \Omega_n(\mathcal{A}) &= sum\{a_0 da_1 \cdots da_n, da_1 \cdots da_n; a_0, a_1, \ldots, a_n \in \mathcal{A}\} \\ &\subseteq \Omega_{tr}(\mathcal{A}). \end{aligned}$$

(7.13.23.1) Here one uses, that

$$\forall n \in \mathbf{N}, \forall A_1, \ldots, A_n \in B_p(\mathcal{H}) : n \geq p \geq 1 \Longrightarrow A_1 \cdots A_n \in B_1(\mathcal{H}).$$

(7.13.24) Let the involutive Fredholm $\mathcal{A}$-right module $\mathcal{H}$ be p-summable. Then $\forall n \geq p \geq 1$:

$$\Omega_n(\mathcal{A}) \subseteq \Omega_{tr}(\mathcal{A}), \ t \circ \gamma|_{\Omega_{n+1}(\mathcal{A})} = t \circ \beta|_{\Omega_{n+1}(\mathcal{A})} = 0,$$

$$t \circ \rho|_{\Omega_{n+1}(\mathcal{A})} = t \circ \lambda|_{\Omega_{n+1}(\mathcal{A})} = t|_{\Omega_{n+1}(\mathcal{A})},$$

which explicitly means, with the $\int$ notation of closed graded traces on $\mathbf{Z}_2$-graded differential algebras, that $\forall a_0 \in \mathcal{A}^{\overline{z_0}}, \ldots, a_{n+1} \in \mathcal{A}^{\overline{z_{n+1}}}$:

$$\int a_0 da_1 \cdots da_{n+1} = (-1)^{(1+z_n+1)(n+1+z_0+\cdots+z_n)} \int a_{n+1} da_0 da_1 \cdots da_n.$$

(7.13.24.1) The construction of closed graded traces is extended to $\Omega_{p-1}(\mathcal{A})$, using the definition:

$$\bigoplus_{n \geq p-1 \geq 0} \Omega_n(\mathcal{A}) \ni \omega \xrightarrow[def]{t'} -\frac{i}{2} str(F(\delta \circ \pi_*(\omega))) \in \mathbf{C}.$$

One then finds, that $\forall n \geq p-1 \geq 0, \forall a_0, \ldots, a_n \in \mathcal{A}$:

$$t'(a_0) = \frac{1}{2} str\big(F[F, \pi(a_0)]_g\big) = \frac{1}{2} str(\pi(a_0) - KF\pi(a_0)FK).$$

$$t'|_{dom\ t' \cap (d\Omega(\mathcal{A}) \cup \Omega^{\mathrm{I}}(\mathcal{A}))} = 0; \ t' - t|_{\bigoplus_{n+1 \geq p \geq 1} \mathcal{A} \circ d\Omega_n(\mathcal{A})} = 0.$$

(7.13.24.2) For $p \geq 2$, $\forall \omega_1, \omega_2 \in dom\ t'$:

$$t'\big([\omega_1, \omega_2]_g\big) = str\big([\pi_*(\omega_1), \pi_*(\omega_2)]_g\big) = 0,$$

because in this case $\omega_1 \omega_2 \in \Omega_{tr}(\mathcal{A})$. Here one uses, that $\forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall \omega_1 \in \Omega^{\overline{z_1}}(\mathcal{A}), \omega_2 \in \Omega^{\overline{z_2}}(\mathcal{A})$:

$$F(\delta \circ \pi_*([\omega_1, \omega_2]_g))$$
$$= i\big([\pi_*(\omega_*(\omega_1), \pi_*(\omega_2)]_g - (-1)^{z_1+z_2} F[\pi_*(\omega_1), \pi_*(\omega_2)]_g F\big).$$

7.14 Connes Modules

(7.14.1) Let $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$ be a separable Hilbert space with complex coefficients, with the Klein operator K. Let D be densely defined and closable on the $\mathbf{Z}_2$-graded linear subspace $\mathcal{D} = (\mathcal{D} \cap \mathcal{H}^{\bar{0}}) \oplus (\mathcal{D} \cap \mathcal{H}^{\bar{1}})$, and assume D to be odd, which explicitly means that $\forall \bar{z} \in \mathbf{Z}_2, \forall f \in \mathcal{D} \cap \mathcal{H}^{\bar{z}} : Df \in \mathcal{H}^{\overline{z+1}}$. These two assumptions equivalently mean, that

$$K\mathcal{D} = \mathcal{D} := dom\ D,\ \bar{\mathcal{D}} = \mathcal{H},\ KD = -DK.$$

(7.14.1.2) For a closable linear operator $A : \mathcal{H} \supseteq dom\ A \longrightarrow \mathcal{H}$, one finds easily the following lemma. If $K\ dom\ A = dom\ A$ and $KAK = (-1)^z A$, then $K dom\ \bar{A} = dom\ \bar{A}$ and $K\bar{A}K = (-1)^z \bar{A}$. Therefore especially

$$K dom\ \bar{D} = dom\ \bar{D},\ K\bar{D} = -\bar{D}K.$$

(7.14.1.3) Let $\bar{D}$ be injective; then D^{-1} is closable, $\overline{D^{-1}} = \bar{D}^{-1}$. Moreover let $\bar{D}^{-1}$ be bounded; then $ran\ \bar{D} = \overline{ran\ D}$. Assume $\overline{ran\ D} = \mathcal{H}$; in this case $\bar{D}^{-1} \in B(\mathcal{H})$; due to the above lemma, $\bar{D}^{-1} \in B^{\bar{1}}(\mathcal{H})$. Finally assume, for some $1 \leq p < \infty$, that

$$\bar{D}^{-1} \in B_p^{\bar{1}}(\mathcal{H}) := B_p(\mathcal{H}) \cap B^{\bar{1}}(\mathcal{H}).$$

(7.14.1.4) Let $\mathcal{A} = \mathcal{A}^{\bar{0}} \oplus \mathcal{A}^{\bar{1}}$ be an associative superalgebra with complex coefficients, and consider a representation $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$. If $\mathcal{A}$ is unital with the unit e, one demands $\pi(e) = I := id\ \mathcal{H}$.

(7.14.1.5) Furthermore assume, that

$$\forall A \in Im\ \pi : A\ dom\ \bar{D} \subseteq dom\ \bar{D},\ [A, \bar{D}]_g \textit{ bounded};$$

then $\overline{[A, \bar{D}]_g} \in B(\mathcal{H})$. The family $\{\mathcal{H}, D, \mathcal{A}, \pi\}$ is called p-summable Connes module.

(7.14.1.6) Let $\mathcal{A}$ be equipped with some graded star operation $*$, with respect to complex conjugation, i.e., an even antilinear involution $* : \mathcal{A} \longleftrightarrow \mathcal{A}$, such that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in \mathcal{A}^{\bar{x}}, b \in \mathcal{A}^{\bar{y}} : (ab)^* = (-1)^{xy} b^* a^*.$$

Let the representation π be compatible with $*$ and the adjoint in the sense, that

$$\forall a \in \mathcal{A} : \pi(a^*) = (\pi(a))^\dagger;$$

in this case π is called graded $*$-representation of $\mathcal{A}$ on $\mathcal{H}$. The above Connes module is called self-adjoint, if and only if D is essentially self-adjoint; in this case both $\bar{D}$ and $\bar{D}^{-1}$ are self-adjoint.

(7.14.2.1) Consider the p-summable Connes module $\{\mathcal{H}, D, \mathcal{A}, \pi\}$, for some $1 \le p < \infty$. The separable Hilbert space $\mathcal{H} \oplus \mathcal{H}$ is equipped with an appropriate Klein operator L.

$$\mathcal{H} \oplus \mathcal{H} =: \mathcal{L} = \mathcal{L}^{\bar{0}} \oplus \mathcal{L}^{\bar{1}}, \ \forall \bar{z} \in \mathbf{Z}_2 : \mathcal{L}^{\bar{z}} := \mathcal{H}^{\bar{z}} \oplus \mathcal{H}^{\bar{z}};$$

$$L := \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \ L^2 = id\ \mathcal{L}.$$

An according representation of $\mathcal{A}$ on L is defined:

$$\mathcal{A} \ni a \xrightarrow[def]{\rho} \begin{bmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{bmatrix} \in B(\mathcal{L}).$$

If π is some graded $*$-representation, then ρ is also some graded star representation:

$$\mathcal{A} \ni a^* \xrightarrow[\rho]{} (\rho(a))^\dagger \in B(\mathcal{L}).$$

(7.14.2.2) The following odd operator F is densely defined and closable.

$$F := \begin{bmatrix} 0 & D \\ D^{-1} & 0 \end{bmatrix}, \ dom\ F := ran\ D \oplus dom\ D,$$

$$dom\ \bar{F} = \mathcal{H} \oplus dom\ \bar{D}, \ \bar{F} = \begin{bmatrix} 0 & \bar{D} \\ \bar{D}^{-1} & 0 \end{bmatrix}.$$

One easily finds, that

$$L\ dom\ F = dom\ F, \ LFL = -F, \ L\ dom\ \bar{F} = dom\ \bar{F}, \ L\bar{F}L = -\bar{F}.$$

(7.14.2.3) $\forall a \in \mathcal{A}$:

$$\overline{[\bar{F}, \rho(a)]_g} = \begin{bmatrix} 0 & \overline{[\bar{D}, A]_g} \\ [\bar{D}^{-1}, A]_g & 0 \end{bmatrix} = \begin{bmatrix} 0 & \overline{[\bar{D}, A]_g} \\ \bar{D}^{-1}[A, \bar{D}]_g \bar{D}^{-1} & 0 \end{bmatrix}$$
$$\in B(\mathcal{L}),$$

with $A := \pi(a)$, because of the assumption that

$$\forall A \in Im\ \pi : [\bar{D}, A]_g := \bar{D}A - KAK\bar{D}, \ \overline{[\bar{D}, A]_g} \in B(\mathcal{H}).$$

Hence one obtains, denoting $A := \pi(a), \ B := \pi(b)$, that $\forall a, b \in \mathcal{A}$:

$$\overline{[\bar{F}, \rho(a)]_g [\bar{F}, \rho(b)]_g} =$$
$$= \begin{bmatrix} [\bar{D}, A]_g \bar{D}^{-1} [B, \bar{D}]_g \bar{D}^{-1} & 0 \\ 0 & \bar{D}^{-1}[A, \bar{D}]_g \bar{D}^{-1} \overline{[\bar{D}, B]_g} \end{bmatrix} \in B_{p/2}(\mathcal{L}).$$

(7.14.2.3.1) One easily proves, that $\forall A, B \in B(\mathcal{H})$:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in B_\infty(\mathcal{L}) \Longleftrightarrow A \text{ and } B \in B_\infty(\mathcal{H}).$$

(7.14.2.3.2) Considering carefully the non-repeated singular values with their multiplicities, one finds that $\forall\ 0 < p < \infty$,

$$\forall A, B \in B_\infty(\mathcal{H}) : A \text{ and } B \in B_p(\mathcal{H}) \Longleftrightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in B_p(\mathcal{L}).$$

Here an ideal $B_p(\mathcal{H})$ of $B(\mathcal{H})$ is defined for $0 < p < \infty$,

$$B_p(\mathcal{H}) := \left\{ T \in B_\infty(\mathcal{H}); \sum_{n=1}^{\infty} \tau_n^p < \infty \right\},$$

inserting the repeated singular values τ_n of T.

(7.14.2.4) Let $a_1, \ldots, a_m \in \mathcal{A}$, and denote $\forall_1^m l : \pi(a_l) =: A_l$. One immediately finds that, for even m,

$$\overline{[\bar{F}, \rho(a_1)]_g} \cdots \overline{[\bar{F}, \rho(a_m)]_g} =$$
$$\begin{bmatrix} [\bar{D}, A_1]_g \bar{D}^{-1} [A_2, \bar{D}]_g \bar{D}^{-1} \cdots [\bar{D}, A_{m-1}]_g \bar{D}^{-1} [A_m, \bar{D}]_g \bar{D}^{-1} & 0 \\ 0 & \bar{D}^{-1} [A_1, \bar{D}]_g \bar{D}^{-1} [\bar{D}, A_2]_g \cdots \bar{D}^{-1} [A_{m-1}, \bar{D}]_g \bar{D}^{-1} \overline{[\bar{D}, A_m]_g} \end{bmatrix}.$$
$$m \geq 2n \geq p \geq 2 \Longrightarrow \overline{[\bar{F}, \rho(a_1)]_g} \cdots \overline{[\bar{F}, \rho(a_m)]_g} \in B_1(\mathcal{L}).$$

(7.14.2.5) Obviously,

$$\bar{F}^2 = \begin{bmatrix} I & 0 \\ 0 & I|_{dom\ \bar{D}} \end{bmatrix}, \quad dom\ \bar{F}^2 = \bar{F} dom\ \bar{F} = dom\ \bar{F}, \ \overline{\bar{F}^2} = \overline{F^2} = id\ \mathcal{L}.$$

(7.14.2.6)

$$\forall A, B \in B_1(\mathcal{H}) : \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in B_1(\mathcal{L}), \ tr \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = tr\ A + tr\ B.$$

(7.14.2.7) $\forall n \in \mathbf{N}, \forall a_0 \in \mathcal{A}^{\overline{z_0}}, \ldots, a_{2n} \in \mathcal{A}^{\overline{z_{2n}}}$:

$$2n \geq p \geq 2, \ \overline{z_0} + \cdots + \overline{z_{2n}} = \bar{0}$$
$$\Longrightarrow \overline{[\bar{F}, \rho(a_1)]_g} \cdots \overline{[\bar{F}, \rho(a_{2n})]_g} \in B_1(\mathcal{L}),$$
$$\tau_{2n}(a_0, a_1, \ldots, a_{2n}) := (-1)^n tr \left(L\rho(a_0) \overline{[\bar{F}, \rho(a_1)]_g} \cdots \overline{[\bar{F}, \rho(a_{2n})]_g} \right)$$
$$= (-1)^{\sum_{k=1}^n z_{2k-1}} tr \left(K \bar{D}^{-1} [\bar{D}, A_0]_g \cdots \bar{D}^{-1} [\bar{D}, A_{2n-1}]_g \bar{D}^{-1} \overline{[\bar{D}, A_{2n}]_g} \right),$$

with the notation, that $\forall_0^{2n} l : A_l := \pi(a_l)$. Here one uses, in order to commute under the trace, that $\forall\ 0 < p < q < \infty : B_p(\mathcal{H}) \subset B_q(\mathcal{H}) \subset B_\infty(\mathcal{H})$.

(7.14.3) For simplicity at the moment, let $D = \bar{D} \in B^{\bar{1}}(\mathcal{H})$; then

$$F = \bar{F} \in B^{\bar{1}}(\mathcal{L}), \ F^2 = id\ \mathcal{L}.$$

(7.14.3.1)

$$B(\mathcal{L}) \ni T \xrightarrow[def]{\delta} i[F,T]_g = i(FT - LTLF) \in B(\mathcal{L}).$$

With this nilpotent $\mathbf{Z}_2$-graded derivation δ, $B(\mathcal{L})$ is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$.

$$\forall \bar{z} \in \mathbf{Z}_2, \ \forall S \in B^{\bar{z}}(\mathcal{L}), T \in B(\mathcal{L}) : \delta(ST) = \delta(S)T + (-1)^z S\delta(T);$$

$$\forall \bar{z} \in \mathbf{Z}_2 : Im\ \delta\big|_{B^{\bar{z}}(\mathcal{L})} \subseteq B^{\overline{z+1}}(\mathcal{L}); \quad \delta \circ \delta = 0.$$

(7.14.3.2)

$$\forall T \in B_1(\mathcal{L}) : str\ T := tr(LT) = \frac{1}{2i} str(F\delta(T)), \ str \circ \delta(T) = 0,$$

because the supertrace vanishes on appropriate super-commutators. Obviously

$$str\ \big|_{B_1(\mathcal{L}) \cap B^{\bar{1}}(\mathcal{L})} = 0.$$

(7.14.3.3) The universal property of the non-unital differential envelope is used for the subsequent diagram.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr] \arrow[dr, "\rho"'] & & \Omega(\mathcal{A}) \arrow[dl, "\rho_*"] & & \Omega_{tr}(\mathcal{A}) \arrow[ll, "inclusion"'] \arrow[dl, "surjective"'] \arrow[dr, "t", "def"'] & \\
& B(\mathcal{L}) & & B_1(\mathcal{L}) \arrow[ll, "inclusion"] \arrow[rr, "str"'] & & \mathbf{C}
\end{tikzcd}$$

With the intertwining property: $\rho_* \circ d = \delta \circ \rho_*$, one then obtains that $\forall m \in \mathbf{N}, \forall a_0, a_1, \ldots, a_m \in \mathcal{A}$:

$$(1 + a_0)da_1 \cdots da_m \xrightarrow[\rho_*]{} i^m(1 + \rho(a_0))[F, \rho(a_1)]_g \cdots [F, \rho(a_m)]_g.$$

One immediately verifies, that

$$Im\ d\big|_{\Omega_{tr}(\mathcal{A})} \subseteq \Omega_{tr}(\mathcal{A}) := \rho_*^{-1}(B_1(\mathcal{L})).$$

The $\mathbf{Z}_2$-graded differential algebra $\Omega_{tr}(\mathcal{A})$ is an ideal of $\Omega(\mathcal{A})$.

(7.14.3.4)

$$t := str \circ \rho_*\big|_{\Omega_{tr}(\mathcal{A})} \in \Omega^*_{tr}(\mathcal{A}) := Hom_{\mathbf{C}}(\Omega_{tr}(\mathcal{A}), \mathbf{C}).$$

One immediately finds, that

$$\forall \omega_1 \in \Omega_{tr}(\mathcal{A}), \omega_2 \in \Omega(\mathcal{A}) : t\left([\omega_1, \omega_2]_g\right) = 0.$$

Moreover this $\mathbf{Z}_2$-graded trace t is even and closed.

$$t\big|_{\Omega_{tr}(\mathcal{A})\cap\Omega^{\mathrm{I}}(\mathcal{A})} = 0,\ t \circ d\big|_{\Omega_{tr}(\mathcal{A})} = 0.$$

(7.14.3.5) $\forall n \in \mathbf{N}, \forall a_0 \in \mathcal{A}^{\overline{z_0}}, \ldots, a_{2n} \in \mathcal{A}^{\overline{z_{2n}}}$:

$$2n \geq p \geq 2,\ \overline{z_0} + \cdots + \overline{z_{2n}} = \bar{0} \Longrightarrow a_0 da_1 \cdots da_{2n} \in \Omega_{tr}(\mathcal{A}),$$

$$\begin{aligned} t(a_0 da_1 \cdots da_{2n}) &= (-1)^n tr\left(L\rho(a_0)[F, \rho(a_1)]_g \cdots [F, \rho(a_{2n})]_g\right) \\ &= \tau_{2n}(a_0, a_1, \ldots, a_{2n}); \end{aligned}$$

$$2n \geq p \geq 2,\ \overline{z_0} + \cdots + \overline{z_{2n}} = \bar{1} \Longrightarrow t(a_0 da_1 \cdots da_{2n}) = 0.$$

(7.14.3.5.1) $\forall n \in \mathbf{N}, \forall a_0, \ldots, a_{2n+1} \in \mathcal{A}$:

$$2n \geq p \geq 2 \Longrightarrow a_0 da_1 \cdots da_{2n+1} \in \Omega_{tr}(\mathcal{A}),\ t(a_0 da_1 \cdots da_{2n+1}) = 0.$$

(7.14.3.6)

$$\bigoplus_{m \geq 2n \geq p \geq 2} \Omega_m(\mathcal{A}) \subseteq \Omega_{tr}(\mathcal{A}).\ \forall m \in \mathbf{N}_0 : Im\ \gamma\big|_{\Omega_m(\mathcal{A})} \subseteq \Omega_m(\mathcal{A}),$$

because the restrictions of ρ and λ onto $\mathcal{A} \circ d\Omega_m(\mathcal{A})$ are acting as linear bijections onto this linear subspace, the restriction of ρ onto $d\Omega_m(\mathcal{A})$ is bijective onto the latter, the restrictions of ρ and λ onto $\mathcal{A}$ are bijective onto $\mathcal{A}$, and $\lambda \circ d = 0$.

(7.14.3.7) Therefore one obtains $\forall n \in \mathbf{N}$:

$$n - 1 \geq p/2 \geq 1 \Longrightarrow t \circ \gamma\big|_{\oplus_{m \geq 2n} \Omega_m(\mathcal{A})} = t \circ \beta\big|_{\oplus_{m \geq 2n} \Omega_m(\mathcal{A})} = 0,$$

$$t \circ \lambda\big|_{\oplus_{m \geq 2n} \Omega_m(\mathcal{A})} = t \circ \rho\big|_{\oplus_{m \geq 2n} \Omega_m(\mathcal{A})} = t\big|_{\oplus_{m \geq 2n} \Omega_m(\mathcal{A})}.$$

Explicitly one finds, that for $n - 1 \geq p/2 \geq 1$,
$\forall m \geq 2n, \forall a_0 \in \mathcal{A}^{\overline{z_0}}, \ldots, a_m \in \mathcal{A}^{\overline{z_m}}$:

$$\begin{aligned} \int a_0 da_1 \cdots da_m &= (-1)^{(1+z_m)(m+z_0+\cdots+z_{m-1})} \int a_m da_0 \cdots da_{m-1} \\ &= (-1)^{(1+z_m)(1+m+z_0+\cdots+z_{m-1})} \int da_m a_0 da_1 \cdots da_{m-1}. \end{aligned}$$

(7.14.3.8) Here in the sense of cycles over associative superalgebras, t is written as fermionic integral. $\forall \bar{z} \in \mathbf{Z}_2, \forall \omega \in \Omega_{tr}(\mathcal{A}) \cap \Omega^{\bar{z}}(\mathcal{A})$:

$$\int \omega \equiv t(\omega) = (-1)^z t(\omega) = str \circ \rho_*(\omega).$$

(7.14.4) In order to omit the above assumption on D to be bounded, consider the linear mapping:

$$Im\ \rho \ni \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} =: T$$

$$\xrightarrow[def]{\delta'} \overline{i[\bar{F}, T]_g} = i \begin{bmatrix} 0 & \overline{[\bar{D}, A]_g} \\ \bar{D}^{-1}[A, \bar{D}]_g \bar{D}^{-1} & 0 \end{bmatrix} \in B(\mathcal{L}).$$

Under the assumption, that $Im\ \delta' \subseteq Im\ \rho$, define $\delta : Im\ \rho \longrightarrow Im\ \rho$, such that $\delta' = \nu \circ \delta$, with the inclusion ν of $Im\ \rho$ into $B(\mathcal{L})$.

(7.14.4.1) The associative superalgebra $D(\mathcal{L}) := Im\ \rho$, with the $\mathbf{Z}_2$-grading defined by the direct sum

$$D(\mathcal{L}) = \bigoplus_{\bar{z} \in \mathbf{Z}_2} D^{\bar{z}}(\mathcal{L}), \ D^{\bar{z}}(\mathcal{L}) := D(\mathcal{L}) \cap B^{\bar{z}}(\mathcal{L}),$$

is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$ with the super-derivation δ, because $\forall A \in Im\ \pi, \forall S, T \in Im\ \rho, \forall \bar{z} \in \mathbf{Z}_2$:

$$KAK = (-1)^z A \Longrightarrow K\overline{[A, \bar{D}]_g}K = (-1)^{z+1}\overline{[A, \bar{D}]_g},$$

due to an according lemma on closable linear operators cited above.

$$LSL = (-1)^z S \Longrightarrow \delta(S)T + (-1)^z S\delta(T)\big|_{dom\ \bar{F}} = i[\bar{F}, ST]_g;$$

$$\delta \circ \delta(T)\big|_{dom\ \bar{F}} = 0.$$

Here one uses, that $\forall T \in Im\ \rho : T\ dom\ \bar{F} \subseteq dom\ \bar{F}$.
If $\mathcal{A}$ is unital, then $D(\mathcal{L}) \ni id\ \mathcal{L}$ is unital too.

(7.14.4.2) Consider the following diagram.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[rr] \arrow[dl, "\rho"'] \arrow[dr, "\rho'"', "surjective"] & & \Omega(\mathcal{A}) \arrow[dl, "\rho'_*"'] & & \Omega'_{tr}(\mathcal{A}) \arrow[ll, "inclusion"'] \arrow[dl, "surjective"'] \arrow[dr, "t"', "def"] & \\
B(\mathcal{L}) & & D(\mathcal{L}) \arrow[ll, "inclusion"] & & D_1(\mathcal{L}) \arrow[ll, "inclusion"] \arrow[rr, "str"'] & & \mathbf{C}
\end{tikzcd}$$

$$t := str \circ \rho'_*\big|_{\Omega'_{tr}(\mathcal{A})}, \Omega'_{tr}(\mathcal{A}) := \rho'^{-1}_*(D_1(\mathcal{L})), \ D_1(\mathcal{L}) := D(\mathcal{L}) \cap B_1(\mathcal{L}).$$

(7.14.4.3) $\forall m \in \mathbf{N}, \forall a_0, a_1, \ldots, a_m \in \mathcal{A}$:

$$(1+a_0)da_1 \cdots da_m \xrightarrow[\rho'_*]{} i^m(1+\rho(a_0))\overline{[\bar{F},\rho(a_1)]_g} \cdots \overline{[\bar{F},\rho(a_m)]_g},$$

with the super-derivation:

$$D(\mathcal{L}) \ni T \xrightarrow{\delta} i\overline{[\bar{F},T]_g} \in D(\mathcal{L}),\ \delta \circ \rho'_* = \rho'_* \circ d.$$

(7.14.4.4) Obviously $\Omega'_{tr}(\mathcal{A})$ is an ideal of $\Omega(\mathcal{A})$. Furthermore one finds, that $\forall A \in\ Im\ \pi \cap B_1(\mathcal{H})$:

$$\overline{[\bar{D},A]_g} \in B_1(\mathcal{H}) \Longrightarrow Im\ \delta\big|_{D_1(\mathcal{L})} \subseteq D_1(\mathcal{L}) \Longrightarrow Im\ d\big|_{\Omega'_{tr}(\mathcal{A})} \subseteq \Omega'_{tr}(\mathcal{A}).$$

In this case $\Omega'_{tr}(\mathcal{A})$ is some $\mathbf{Z}_2$-graded differential algebra over $\mathbf{C}$.
Here, as before for $B_1(\mathcal{L})$, one must carefully show that $D(\mathcal{L})$, $D_1(\mathcal{L})$ are $\mathbf{Z}_2$-graded linear subspaces of $B(\mathcal{L})$, i.e.,

$$D_1(\mathcal{L}) = \bigoplus_{\bar{z}\in\mathbf{Z}_2} D_1^{\bar{z}}(\mathcal{L}),\ D_1^{\bar{z}}(\mathcal{L}) := D_1(\mathcal{L}) \cap B^{\bar{z}}(\mathcal{L}),$$

which in turn implies, that

$$\Omega'_{tr}(\mathcal{A}) = \bigoplus_{\bar{z}\in\mathbf{Z}_2} \left(\Omega'_{tr}(\mathcal{A}) \cap \Omega^{\bar{z}}(\mathcal{A})\right).$$

(7.14.4.5) $\forall n \in \mathbf{N}, \forall a_0 \in \mathcal{A}^{\overline{z_0}}, \ldots, a_{2n} \in \mathcal{A}^{\overline{z_{2n}}}$: If $2n \geq p \geq 2$, then

$$a_0 da_1 \cdots da_{2n} \in \Omega'_{tr}(\mathcal{A}),$$

and the following implications hold.

$$\overline{z_0} + \cdots + \overline{z_{2n}} = \bar{0} \Longrightarrow t(a_0 da_1 \cdots da_{2n}) = \tau_{2n}(a_0, \ldots, a_{2n});$$
$$\overline{z_0} + \cdots + \overline{z_{2n}} = \bar{1} \Longrightarrow t(a_0 da_1 \cdots da_{2n}) = 0.$$

(7.14.4.5.1) $\forall n \in \mathbf{N}, \forall a_0, a_1, \ldots, a_{2n+1} \in \mathcal{A}$:

$$2n \geq p \geq 2 \Longrightarrow a_0 da_1 \cdots da_{2n+1} \in \Omega'_{tr}(\mathcal{A}),\ t(a_0 da_1 \cdots da_{2n+1}) = 0.$$

(7.14.4.6) Let d be reduced by $\Omega_{tr}(\mathcal{A})$; then the even graded trace t is closed.

$$\forall \omega_1 \in \Omega'_{tr}(\mathcal{A}), \omega_2 \in \Omega(\mathcal{A}) : t([\omega_1, \omega_2]_g) = 0;$$
$$t\big|_{\Omega'_{tr}(\mathcal{A}) \cap \Omega^{\bar{1}}(\mathcal{A})} = 0;\ t \circ d\big|_{\Omega'_{tr}(\mathcal{A})} = 0.$$

(7.14.4.7)

$$\bigoplus_{m \geq 2n \geq p \geq 2} \Omega_m(\mathcal{A}) \subseteq \Omega'_{tr}(\mathcal{A}).$$

Therefore the restrictions of $t \circ d$, $t \circ \gamma$, $t \circ \beta$, $t \circ \lambda - t$, $t \circ \rho - t$, onto $\bigoplus_{m \geq 2n} \Omega_m(\mathcal{A})$, vanish for $n - 1 \geq p/2 \geq 1$.

8. Quantum Groups

Following the terminology of V. G. Drinfel'd (1987), an enormous variety of algebraic, topological, and also differential structures is subsummed under the notion of "quantum groups", due to the at least four-fold origin of according research.

One main starting point was the study of solutions of the parameter-dependent classical and quantum Yang-Baxter equation (CYBE, QYBE), which led M. Jimbo to the definition of an appropriate q-deformation of the simple complex Lie algebra associated with a symmetrizable generalized Cartan matrix (1985). The two sources of this new construction were the monograph by V. G. Kac on Kac-Moody algebras (second edition, 1985), the first edition of which had already been printed by Birkhäuser, Boston, 1983, and in particular the pioneering papers by P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin on the parameter-dependent CYBE and QYBE (1981, 1982, 1983, 1984), were the universal enveloping algebra of $sl(2, \mathbf{C})$ was originally q-deformed. The solutions of the parameter-dependent CYBE, with their values in a simple finite-dimensional complex Lie algebra, were classified by A. A. Belavin and V. G. Drinfel'd (1983). The geometric significance of the CYBE with respect to Poisson-Lie groups was pointed out by V. G. Drinfel'd (1983).

The study of quadratic algebras, the relations of which can be written in terms of an R-matrix, which solves the parameter-independent QYBE, was motivated from the quantum inverse scattering method (QISM), which had been developed by L. A. Takhtajan and L. D. Faddeev (1979), P. P. Kulish and E. K. Sklyanin (1982), and collaborators. In their fundamental paper of 1989 (1990), N. Yu. Reshetikhin, L. A. Takhtajan, and L. D. Faddeev established quadratic algebras corresponding to the four series of non-exceptional simple finite-dimensional complex Lie algebras, and also the duality between these Hopf algebras and the q-deformations of the universal enveloping algebras of the involved Lie algebras. These quadratic algebras were also used in order to transform quantum spaces. The q-deformed universal enveloping algebras were finally reconstructed as quantum doubles.

The view of quantum groups as super-bialgebras of transformations of quantum superspaces was pointed out by Yu. I. Manin (1989), who in the sequel also investigated related aspects and examples of non-commutative differential geometry (1991, 1992).

From the aspect of model construction, J. Wess and B. Zumino proposed a new differential calculus on the so-called quantum plane (1990), which was generalized to quantum spaces by A. Schirrmacher (1991).

In his fundamental contribution to the Proceedings of the International Congress of Mathematicians at Berkeley (1987), V. G. Drinfel'd used the quantum double construction in order to write down explicitly the universal R-matrix of the quasitriangular Hopf algebra $U_q(sl(2,\mathbf{C}))$, and to indicate its generalization to Kac-Moody algebras.

Starting from the commutative unital associative algebra of continuous functions on a compact topological group, S. L. Woronowicz (1987, 1989) developed his theory of so-called compact matrix pseudogroups, including in particular an appropriate bicovariant differential calculus, as some concept of quantum groups on the level of C^*-algebras. A purely algebraic approach to compact quantum groups is provided by M. S. Dijkhuizen and T. H. Koornwinder (1994).

Quasitriangular Hopf algebras were introduced and exhaustively investigated by V. G. Drinfel'd (1990). Important contributions are also due to D. E. Radford (1992, 1993).

The coassociativity of comultiplication, and accordingly the conditions on the antipode, were weakened by V. G. Drinfel'd (1990) to the definition of quasi-Hopf algebras, in order to provide an algebraic framework for the construction of gauge transformations in models of conformal field theory. Quasi-Hopf algebras, which are related to the topological gauge field theory of R. Dijkgraaf and E. Witten (1990), were constructed and studied by R. Dijkgraaf, V. Pasquier, and P. Roche (1990).

This concept of quasi-Hopf algebras was further weakened by G. Mack and V. Schomerus (1992), omitting the condition $\Delta(e) = e \otimes e$ for the unit e of the Hopf algebra in question, and thereby truncating the tensor product of representations. This truncation then allows for the construction of quantum symmetries of covariant field operators, which obey local braid relations.

Starting from an algebraic scheme of desired relations, for instance those of $U_q(sl(2,\mathbf{C}))$, which were invented and used in order to describe the symmetries of models as for instance the open XXZ quantum spin-$\frac{1}{2}$ chain, one is forced to introduce an appropriate topology for at least three reasons.

(i) Some relations of q-deformed enveloping algebras are non-polynomial. Since these relations are of exponential type, and also the momentum-like relations are not homogeneous, one cannot use the topology of formal power series with relations over $\mathbf{C}$, but one succeeds with the choice of the commutative ring of coefficients $\mathbf{C}[[h]]$, $h = \ln q$.

(ii) This choice in turn brings about complications with respect to the representation theory of quantum enveloping algebras, which should be performed preferably on real or complex vector spaces. Therefore representations of quantum groups are usually studied on the "image level", forgetting about the homomorphisms, which lead down from the level of abstract generators to that of real- or complex- linear operators. Whereas for generic values of the deformation parameter q the representation theory of semisimple finite-dimensional complex Lie algebras can be mimicried, quite new types of representations appear for q being a root of unity. In this case the centres of quantum enveloping algebras become much larger, which indicates an according reduction of such algebras. In particular there are finite-dimensional representations, which are not of highest weight type; these new representations, which are called cyclic (or periodic), are useful for the construction of models, for instance of the chiral Potts model. On the other hand, aiming at representations in the strict sense of homomorphisms, and avoiding all the troubles of bimodules over commutative rings, one might try to equip quantum groups with locally convex topologies, which are nuclear and either of Fréchet type or the strong dual of this type.

(iii) In terms of the involved exponentials as new generators, one obtains polynomial relations of quantum enveloping algebras, but the definition of the universal R-matrix enforces an appropriate completion, anyhow.

Although a consistent $\mathbf{Z}_2$-grading of quantum groups is straightforward on an abstract level, a detailed performance, i.e. application to examples, turns out to be extremely difficult, due to the problems of (i) establishing the full set of relations of basic classical Lie superalgebras, (ii) constructing PBW-like linear bases, and (iii) calculating the universal R-matrix.

Besides of topological problems, the duality of quantum enveloping superalgebras and matrix quantum super-semigroups, in the sense of super-bialgebras, is defined with respect to bilinear forms, which may be degenerate. Despite of this up to now unsolved problem of degeneracy, the duality of super-bialgebras seems to serve as some very useful guideline for the construction and analysis of such objects, in particular with respect to the so-called classical limit of $q \longrightarrow 1$.

$$
\begin{tikzcd}
\text{q-deformed universal enveloping superalgebra} \arrow[r, "q \longrightarrow 1"] \arrow[d, leftrightarrow, "\text{dual as super-bialgebras}"] & \text{Lie superalgebra} \arrow[d, leftrightarrow, "\text{dual as Lie super-bialgebras}"] \\
\text{matrix quantum super-semigroup} \arrow[r, dashed] & \text{Lie superalgebra with semiclassical MCR}
\end{tikzcd}
$$

Here the semi-classical main commutation relations (MCR) are written in terms of some classical R-matrix, which is the coefficient of $\ln q$ in the power series expansion of some finite-dimensional representation of the involved universal R-matrix. Some parameter-dependent version of semi-classical MCR appears within the context of the classical spectral transform method for the non-linear Schrödinger equation, as was pointed out by P. P. Kulish and E. K. Sklyanin (1982).

The problem of q-deformation of the universal enveloping algebras of infinite-dimensional complex Lie algebras, in particular of affine Kac-Moody algebras, in order to obtain a family of quasitriangular Hopf algebras with similar triangular decompositions and PBW-like linear bases as in the undeformed case, is by far unsolved on the level of mathematical rigor.

Linear bases for a certain class of unital associative algebras over a commutative ring may be constructed by means of the so-called diamond lemma, following G. M. Bergman (1978). This method can be used for instance for an easy construction of a $\mathbf{C}$-linear basis of the Manin algebra $Mat_q(2, \mathbf{C})$. More generally R. Berger (1992) established a class of unital associative R-algebras with Poincaré-Birkhoff-Witt-like linear bases, including many examples of quantum groups. Here R denotes the commutative ring of coefficients, which with respect to q-deformed universal enveloping algebras is specialized to complex formal power series in $h = \ln\ q$. The involved $\mathbf{C}[[h]]$-algebras are assumed to be generated by finitely many generators with inhomogeneous quadratic relations, thereby restricting oneself to quantum superspaces and their transformations, and to the upper (or lower) part of the triangular decomposition of $U_q(sl(2, \mathbf{C}))$.

The first very complicated proof of an expected PBW-like $\mathbf{C}[[h]]$-linear basis of the upper (or lower) part of the triangular decomposition of $U_q(sl(n+1, \mathbf{C}))$, $n \in \mathbf{N}$, is due to M. Rosso (1989), who constructed from these bases via the quantum double the universal R-matrix of this quantum group. An appropriate q-deformation of the Weyl group by means of the so-called Lusztig automorphisms was used by A. N. Kirillov and N. Yu. Reshetikhin (1990) to factorize the universal R-matrix of the q-deformation $U_q(\mathcal{L})$ of an arbitrary simple finite-dimensional complex Lie algebra $\mathcal{L}$. The quantum Weyl group and factorization of the universal R-matrix were also investigated by S. Levendorskii, Y. Soibelman, and L. L. Vaksman (1991).

The duality of the q-deformed universal enveloping algebra of $sl(n, K)$, for $n = 2, 3, \ldots$, which is written in terms of polynomial relations, with the corresponding matrix quantum group, in the sense of Hopf algebras over a field K, $char\ K \neq 2$, was investigated by M. Takeuchi (1992), who solved the problem of degeneracy of the involved K-bilinear form in this case, for generic q.

S. M. Koroshkin and V. N. Tolstoy (1991, 1992) calculated the universal R-matrix of $U_q(\mathcal{L})$ for basic classical Lie superalgebras $\mathcal{L}$, although the full set of relations for $\mathcal{L}$ was established by H. Yamane (1991, 1994). The latter author proposed a PBW-like linear basis and constructed the universal R-matrix for $U_q(\mathcal{L})$, inserting an arbitrary basic classical Lie superalgebra $\mathcal{L}$, over the commutative ring $\mathbf{C}[[h]]$ of coefficients, using thereby carefully the h-adic topology. The full set of relations for $sl(m, n, \mathbf{C})$ and its q-deformation, $m, n \in \mathbf{N}_0$, $m + n \geq 1$, was also studied by M. Scheunert (1992, 1993).

Topological coalgebras were studied by M. Takeuchi (1985). The peculiar topological difficulties, which arise from non-polynomial relations, and in particular with respect to the universal R-matrix, were investigated among others by Ph. Bonneau, M. Flato, M. Gerstenhaber, G. Pinczon (1992, 1994). Ph. Bonneau constructed an appropriate topological quantum double (1994). These authors use the theory of Fréchet spaces and their duals, and topological tensor products of such spaces. T. Masuda and Y. Nakagami (1994) considered the duality of quantum groups within the framework of von Neumann algebras. For the general theory of locally convex spaces the reader is referred to F. Treves (1967).

Generic representations of quantum enveloping algebras of simple finite-dimensional complex Lie algebras were studied by M. Rosso (1988), non-generic representations in particular by C. De Concini and V. G. Kac (1990, 1992), and by G. Lusztig (1990, 1993).

The cyclic homology of algebras with quadratic relations, universal enveloping algebras, and group algebras was investigated by B. L. Feigin and B. L. Tsygan (1987). Yu. I. Manin (1987, 1991) investigated the Koszul complexes associated with certain quadratic algebras, the so-called Koszul algebras, the properties of which hold for instance in case of quantum superspace. Koszul resolutions, in the sense of S. B. Priddy (1970), for the q-deformed universal enveloping algebras $U_q(sl(n+1, \mathbf{C}))$, $n \in \mathbf{N}$, and also for the matrix quantum group $SL_q(2, \mathbf{C})$, were calculated by M. Rosso (1990).

Differential forms on matrix quantum semigroups were studied in particular by A. Sudbery (1993) and B. Tsygan (1993). All possibilities of bicovariant differential calculus on the matrix quantum semigroups corresponding to the series A_m, B_m, C_m, D_m, $m \in \mathbf{N}$, were classified by K. Schmüdgen and A. Schüler (1995).

An appropriate q-deformation of the canonical commutation relations (CCR) with one degree of freedom was proposed independently by L. C. Biedenharn (1989) and A. J. Macfarlane (1989). For a deformation parameter q of modulus $|q| = 1$, there are finite-dimensional representations, and also representations by densely defined closed operators in a separable Hilbert space, of q-deformed CCR with finitely many degrees of freedom.

W. Pusz and S. L. Woronowicz (1989) constructed infinite-dimensional representations of q-deformed CCR, and also of q-deformed canonical anti-commutation relations (CAR), both for $0 < q < 1$.

Starting from two distinguished possibilities of covariant differential calculus on quantum spaces, and using an involution for the case of $|q| = 1$, K. Schmüdgen established a convenient class of faithful irreducible representations of q-deformed CCR by self-adjoint operators (1994).

Oscillator and spinor respresentations of the quantum enveloping algebras of the series A_m, B_m, C_m, D_m, $m \in \mathbf{N}$, were constructed by T. Hayashi (1990). Representations of $U_q(sl(m,n,\mathbf{C}))$, $m,n \in \mathbf{N}_0$, $m+n \geq 1$, in terms of creation and annihilation operators of finitely many bosonic and fermionic q-deformed oscillators were used by M. Chaichian, P. P. Kulish, and J. Lukierski in order to study multimodes of systems of such q-oscillators, and for the definition of systems of finitely many q-deformed superoscillators in terms of q-deformed CCR and CAR, and of an according Hamiltonian (1990, 1991).

Algebraic aspects of the QYBE were investigated extensively by L. A. Lambe and D. E. Radford (1993). V. V. Bazhanov (1987) used certain classes of trigonometric solutions of the parameter-dependent QYBE, which are associated with the non-exceptional simple finite-dimensional complex Lie algebras, for the construction of two types of integrable quantum systems: quantum Toda lattices, and integrable models of magnetics.

A guideline from integrable models to quantum groups is drawn for instance in lectures by L. D. Faddeev (1990, 1993). An overview of integrable models and conformal field theory was presented by H. Grosse in several lectures (1990). Many original papers on the CYBE, QYBE, and integrable systems are reprinted in a volume edited by M. Jimbo (1990).

Quantum groups as quantum deformations of Poisson algebraic groups are introduced and studied for instance in lectures by L. A. Takhtajan (1990), and those by C. De Concini and C. Procesi (1993).

Systematic presentations of quantum groups are provided by the monographs by J. Fuchs (1992), J. Fröhlich and T. Kerler (1993), G. Lusztig (1993), St. Shnider and S. Sternberg (1993), V. Chari and A. N. Pressley (1994), A. Joseph (1995), and Ch. Kassel (1995).

8.1 Duality of $\mathbf{Z}_2$-Graded Hopf Algebras

(8.1.1) Let $\{\mathcal{H}_k, \mu_k, \eta_k, \Delta_k, \varepsilon_k\}, k = 1,2$, be $\mathbf{Z}_2$-graded bialgebras over a commutative ring $\mathcal{R}$. These two $\mathbf{Z}_2$-graded bialgebras are called dual, if and only if there is an $\mathcal{R}$-bilinear form $\beta : \mathcal{H}_1 \times \mathcal{H}_2 \ni \{a_1, a_2\} \to \langle a_1 | a_2 \rangle \in \mathcal{R}$, which fulfills the following conditions. $\forall a_1, b_1 \in \mathcal{H}_1, \forall a_2, b_2 \in \mathcal{H}_2$:

$$\langle a_1 | a_2 b_2 \rangle = \langle \Delta_1(a_1) | a_2 \otimes b_2 \rangle, \quad \langle a_1 b_1 | a_2 \rangle = \langle a_1 \otimes b_1 | \Delta_2(a_2) \rangle,$$

$$\langle e_1 | a_2 \rangle = \varepsilon_2(a_2), \quad \langle a_1 | e_2 \rangle = \varepsilon_1(a_1), \quad \text{with the } \textit{units } e_k \text{ of } \mathcal{H}_k,$$

$$\forall \overline{z_1} \neq \overline{z_2} \in \mathbf{Z}_2 : Im \; \beta\big|_{\mathcal{H}_1^{\overline{z_1}} \times \mathcal{H}_2^{\overline{z_2}}} = 0.$$

Here the universal property of tensor products is used for defining an $\mathcal{R}$-bilinear form: $(\mathcal{H}_1 \otimes \mathcal{H}_1) \times (\mathcal{H}_2 \otimes \mathcal{H}_2) \to \mathcal{R}$, again denoted $\langle \; | \; \rangle$, such that $\forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2, \forall a_1 \in \mathcal{H}_1, b_1 \in \mathcal{H}_1^{\overline{z_1}}, a_2 \in \mathcal{H}_2^{\overline{z_2}}, b_2 \in \mathcal{H}_2$:

$$\langle a_1 \otimes b_1 | a_2 \otimes b_2 \rangle = (-1)^{z_1 z_2} \langle a_1 | a_2 \rangle \langle b_1 | b_2 \rangle.$$

(8.1.1.1) Let especially $\mathcal{H}_k, k = 1, 2$, be $\mathbf{Z}_2$-graded Hopf algebras over $\mathcal{R}$, with the unique antipodes σ_k. These two $\mathbf{Z}_2$-graded Hopf algebras are called dual, if and only if they are dual as $\mathbf{Z}_2$-graded bialgebras over $\mathcal{R}$; then

$$\forall a_1 \in \mathcal{H}_1, a_2 \in \mathcal{H}_2 : \langle \sigma_1(a_1) | a_2 \rangle = \langle a_1 | \sigma_2(a_2) \rangle.$$

(8.1.2) Consider a pair $\mathcal{H}_k, k = 1, 2$, of dual $\mathbf{Z}_2$-graded bialgebras over $\mathcal{R}$. Then the left kernel $\mathcal{K}_1$, and the right kernel $\mathcal{K}_2$, of the $\mathcal{R}$-bilinear form β are both $\mathbf{Z}_2$-graded ideals and coideals of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

$$\mathcal{K}_1 := \{a_1 \in \mathcal{H}_1; \forall a_2 \in \mathcal{H}_2 : \langle a_1 | a_2 \rangle = 0\}, \ \ \tilde{\mathcal{H}}_1 := \mathcal{H}_1 / \mathcal{K}_1,$$

$$\mathcal{K}_2 := \{a_2 \in \mathcal{H}_2; \forall a_1 \in \mathcal{H}_1 : \langle a_1 | a_2 \rangle = 0\}, \ \ \tilde{\mathcal{H}}_2 := \mathcal{H}_2 / \mathcal{K}_2.$$

One obtains the $\mathbf{Z}_2$-graded bialgebras $\{\tilde{\mathcal{H}}_k, \tilde{\mu}_k, \tilde{\eta}_k, \tilde{\Delta}_k, \tilde{\varepsilon}_k\}, k = 1, 2$, over $\mathcal{R}$.

(8.1.2.1) The $\mathcal{R}$-bilinear form $\tilde{\beta} : \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_2 \to \mathcal{R}$ defined below is non-degenerate, i.e., its left kernel and right kernel both are the null space.

$$\tilde{\beta} : \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_2 \ni \{\tilde{a}_1 \equiv a_1 + \mathcal{K}_1, \tilde{a}_2 \equiv a_2 + \mathcal{K}_2\} \longrightarrow \langle a_1 | a_2 \rangle =: \langle \tilde{a}_1 | \tilde{a}_2 \rangle \in \mathcal{R}.$$

(8.1.2.2) These two $\mathbf{Z}_2$-graded bialgebras $\tilde{\mathcal{H}}_k$ over $\mathcal{R}$, $k = 1, 2$, are dual with respect to $\tilde{\beta}$.

(8.1.2.3) Let especially $\mathcal{H}_k, k = 1, 2$, be $\mathbf{Z}_2$-graded Hopf algebras over $\mathcal{R}$, with the unique antipodes σ_k. Then the corresponding $\mathbf{Z}_2$-graded Hopf algebras $\tilde{\mathcal{H}}_k$ over $\mathcal{R}$, with the unique antipodes $\tilde{\sigma}_k$, are dual.

8.2 Quasitriangular $\mathbf{Z}_2$-Graded Hopf Algebras

(8.2.1) Let $\{\mathcal{H}, \mu, \eta, \Delta, \varepsilon\}$ be a $\mathbf{Z}_2$-graded Hopf algebra over the commutative ring $\mathcal{R}$, with the antipode σ. It is called quasitriangular, if and only if there is an even element $R \in \mathcal{H} \hat{\otimes} \mathcal{H}$, which fulfills the conditions (i), (ii), and (iii) below.

$$\text{(i)} \quad \exists\ R^{-1} \in \mathcal{H} \hat{\otimes} \mathcal{H} : RR^{-1} = R^{-1}R = e_{\mathcal{H}} \otimes e_{\mathcal{H}},$$

with the unit $e_{\mathcal{H}}$ of $\mathcal{H}$, the inverse obviously being unique.

(ii) Denoting $R =: \sum_{i=1}^{\bar{i}} R'_i \otimes R''_i$, define $R_{kl}, 1 \le k < l \le 3$, as elements of $\mathcal{H} \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}$ by

$$R_{12} := \sum_{i=1}^{\bar{i}} R'_i \otimes R''_i \otimes e_{\mathcal{H}}, \ \ R_{23} := \sum_{i=1}^{\bar{i}} e_{\mathcal{H}} \otimes R'_i \otimes R''_i,$$

$$R_{13} := \sum_{i=1}^{\bar{i}} R_i' \otimes e_{\mathcal{H}} \otimes R_i''.$$

Then demand

$$\mathcal{H}\hat{\otimes}\mathcal{H}\hat{\otimes}\mathcal{H} \ni R_{13}R_{23} \longleftrightarrow T(\Delta, id\,\mathcal{H})(R) \in (\mathcal{H}\hat{\otimes}\mathcal{H})\hat{\otimes}\mathcal{H},$$

$$\mathcal{H}\hat{\otimes}\mathcal{H}\hat{\otimes}\mathcal{H} \ni R_{13}R_{12} \longleftrightarrow T(id\,\mathcal{H}, \Delta)(R) \in \mathcal{H}\hat{\otimes}(\mathcal{H}\hat{\otimes}\mathcal{H}),$$

with the natural $\mathcal{R}$-linear bijections.

(iii) $\forall a \in \mathcal{H} : \mathcal{H}\hat{\otimes}\mathcal{H} \ni \Delta(a) \underset{\tau}{\longleftrightarrow} R\Delta(a)R^{-1} \in \mathcal{H}\hat{\otimes}\mathcal{H}$,

with the $\mathbf{Z}_2$-graded flip τ, such that $\forall \bar{x}, \bar{y} \in \mathbf{Z}_2$:

$$\mathcal{H}^{\bar{x}} \otimes \mathcal{H}^{\bar{y}} \ni a \otimes b \longleftrightarrow (-1)^{xy} b \otimes a \in \mathcal{H}^{\bar{y}} \otimes \mathcal{H}^{\bar{x}}.$$

(8.2.1.1) If $\{\mathcal{H}, \mu, \eta, \Delta, \varepsilon\}$ is quasitriangular with respect to R, then it is also quasitriangular with respect to $\tau(R^{-1})$, and $\{\mathcal{H}, \mu, \eta, \tau \circ \Delta, \varepsilon\}$ is then quasitriangular with respect to $\tau(R)$.

(8.2.1.2) In the special case of $\tau(R) = R^{-1}$, $\mathcal{H}$ is called triangular $\mathbf{Z}_2$-graded Hopf algebra over $\mathcal{R}$.

(8.2.1.3) More generally, one conveniently uses an appropriate embedding, for $1 \leq k < l \leq n$.

$$\bigotimes^n \mathcal{H} \ni R_{kl}$$
$$:= \sum_{i=1}^{\bar{i}} e_{\mathcal{H}} \otimes \cdots \otimes e \otimes R_i' \otimes e_{\mathcal{H}} \otimes \cdots \otimes e_{\mathcal{H}} \otimes R_i'' \otimes e_{\mathcal{H}} \otimes \cdots \otimes e_{\mathcal{H}},$$

inserting R_i' at the position k, R_i'' at the position l.

(8.2.2) Consider the above quasitriangular $\mathbf{Z}_2$-graded Hopf algebra $\mathcal{H}$ over $\mathcal{R}$, with the unique antipode σ. One calculates easily, with the unit $e_{\mathcal{R}}$ of $\mathcal{R}$, that

(i) $T(\varepsilon, id\,\mathcal{H})(R) = e_{\mathcal{R}} \otimes e_{\mathcal{H}}, \quad T(id\,\mathcal{H}, \varepsilon)(R) = e_{\mathcal{H}} \otimes e_{\mathcal{R}}.$

(ii) Furthermore

$$RT(\sigma, id\,\mathcal{H})(R) = R^{-1}T(id\,\mathcal{H}, \sigma)(R^{-1}) = e_{\mathcal{H}} \otimes e_{\mathcal{H}},$$

because of

$$\begin{aligned} RT(\sigma, id\,\mathcal{H})(R) &= T(\mu \circ T(id\,\mathcal{H}, \sigma), id\,\mathcal{H}) \circ T(\Delta, id\,\mathcal{H})(R) \\ &= T(\eta \circ \varepsilon, id\,\mathcal{H})(R), \end{aligned}$$

using $R_{13}R_{23}$, the antipode property, and the above equations (i). Hence obviously

$$T(\sigma,\sigma)(R) = R, \quad T(\sigma,\sigma)(R^{-1}) = R^{-1}.$$

(iii) Moreover one finds the $\mathbf{Z}_2$-graded quantum Yang-Baxter equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

For an easy proof one calculates

$$T(id\,\mathcal{H}, \tau\circ\Delta)(R) = \sum_{i=1}^{\bar{i}} R_i'\otimes(R\Delta(R_i'')R^{-1}),$$

$$T(id\,\mathcal{H}, \tau\circ\Delta)(R)(e_{\mathcal{H}}\otimes R) \longleftrightarrow R_{12}R_{13}R_{23},$$

$$\sum_{i=1}^{\bar{i}} R_i'\otimes(R\Delta(R_i'')) = (e_{\mathcal{H}}\otimes R)T(id\,\mathcal{H},\Delta)(R) \longleftrightarrow R_{23}R_{13}R_{12}.$$

(8.2.3) The square of the antipode σ of a quasitriangular $\mathbf{Z}_2$-graded Hopf algebra $\{\mathcal{H},\mu,\eta,\Delta,\varepsilon\}$ is generated by an even invertible element u, which can be constructed from the so-called universal R-matrix R, following V.G. Drinfel'd.

$$R =: \sum_{i=1}^{\bar{i}} R_i'\otimes R_i'',$$

$$u := \sum_{i=1}^{\bar{i}} (-1)^{\overline{z_i'}\,\overline{z_i''}}\sigma(R_i'')R_i', \quad \sigma(u) = \sigma^{-1}(u) = \sum_{i=1}^{\bar{i}} R_i'\sigma(R_i''),$$

$$u^{-1} = \sum_{i=1}^{\bar{i}} (-1)^{\overline{z_i'}\,\overline{z_i''}} R_i''\sigma^2(R_i'), \quad \sigma(u^{-1}) = \sigma^{-1}(u^{-1}) = \sum_{i=1}^{\bar{i}} \sigma^2(R_i')R_i'',$$

$$\Delta(u) = (\tau(R)R)^{-1}(u\otimes u) = (u\otimes u)(\tau(R)R)^{-1},$$

$$\Delta\circ\sigma(u) = (\tau(R)R)^{-1}(\sigma(u)\otimes\sigma(u)) = (\sigma(u)\otimes\sigma(u))(\tau(R)R)^{-1};$$

$$\forall a\in\mathcal{H} : \sigma^2(a) = uau^{-1}.$$

Here $\overline{z_i'}, \overline{z_i''}$ denote the $\mathbf{Z}_2$-degrees of $R_i', R_i'', i = 1,\dots,\bar{i}$, respectively. Obviously σ^2, hence σ itself, is bijective. Moreover, $u\sigma(u) = \sigma(u)u$ is central.

$$u^{-1}\sigma(u) =: g, \quad \sigma(g) = g^{-1}, \quad \Delta(g) = g\otimes g, \quad \varepsilon(g) = e_{\mathcal{R}}.$$

8.3 Matrices with Non-Commuting Components

(8.3.1) Let A be an associative algebra with the unit e_A, over a commutative ring R. For $n \in \mathbf{N}$, A^n is defined as an A-left module over R, with the basis:

$$R^n := \bigoplus^n R \supset \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =: E_n^i; i = 1, \ldots, n \right\} \underset{free\ over\ A}{\longrightarrow} A^n,$$

and the notation indicated below. $A^1 = A$.

$$A^n \ni \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \sum_{i=1}^n x_i E_n^i.$$

$$\forall \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in A^n, r \in R : r \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} rx_1 \\ \vdots \\ rx_n \end{bmatrix} \in A^n.$$

(8.3.1.1) Obviously, with the direct sum of bimodules over R:

$$A^n \ni \sum_{i=1}^n x_i E_n^i \underset{R-linear}{\longleftrightarrow} \{x_1, \ldots, x_n\} \in \bigoplus_R^n A \underset{R-linear}{\longleftrightarrow} A \otimes R^n.$$

$$\forall x \in A, \forall_1^n i : A^n \ni x E_n^i \longleftrightarrow x \otimes E_n^i \in A \otimes R^n, \;\; E_n^i \longleftrightarrow e_A \otimes E_n^i.$$

(8.3.2) For $n \in \mathbf{N}$, $Mat(n, A)$ is defined as an A-left module over R, with the basis:

$$\{E_n^{ij}; i, j = 1, \ldots, n\} \underset{free\ over\ A}{\longrightarrow} Mat(n, A), \quad Mat(1, A) = A.$$

Here $E_n^{ij} \in Mat(n, R)$ denotes the matrix with all the components zero, except the component e_R in the i-th column and the j-th row, for $i, j = 1, \ldots, n$. One conveniently uses the notation cited below.

$$Mat(n, A) \ni [a_{ij}; i, j = 1, \ldots, n] := \sum_{i,j=1}^n a_{ij} E_n^{ij}.$$

The module multiplication with elements $\in R$ is defined naturally.
$\forall [a_{ij}; i, j = 1, \ldots, n] \in Mat(n, A), r \in R :$

$$r[a_{ij}; i, j = 1, \ldots, n] := [r a_{ij}; i, j = 1, \ldots, n] \in Mat(n, A).$$

(8.3.2.1) With the R-bilinear mapping:

$$\prod^{2} Mat(n,A) \ni \{[a_{ij}; i,j = 1,\dots,n], [b_{ij}; i,j = 1,\dots,n]\}$$
$$\longrightarrow [\sum_{l=1}^{n} a_{il}b_{lj}; i,j = 1,\dots,n] \in Mat(n,A),$$

$Mat(n, A)$ becomes an associative algebra over R, with the unit

$$\sum_{i=1}^{n} E_n^{ii} = \begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix}.$$

(8.3.2.2) One then establishes an isomorphism of unital associative algebras over R:

$$Mat(n,A) \ni [a_{ij}; i,j = 1,\dots,n] \longleftrightarrow \sum_{i,j=1}^{n} a_{ij} \otimes E_n^{ij} \in A \otimes Mat(n,R).$$

(8.3.2.3) The following homomorphism of unital associative algebras over R is injective.

$$Mat(n,A) \ni [a_{ij}; i,j = 1,\dots,n] \longrightarrow \phi \in End_R(A^n),$$

$$\phi : A^n \ni \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \underset{def}{\longrightarrow} \begin{bmatrix} a_{11}x_1 & + \cdots + & a_{1n}x_n \\ \vdots & & \vdots \\ a_{n1}x_1 & + \cdots + & a_{nn}x_n \end{bmatrix} \in A^n;$$

if especially A is commutative, then $\phi \in End_A(A^n)$.

(8.3.3) The elements $E_n^{ij} \otimes E_n^{kl} \in Mat(n,R) \otimes Mat(n,R), i,j,k,l = 1,\dots,n$, with the tensor product of vector spaces over R, are used as basis of an A-left module $Mat(n,A) \otimes_A Mat(n,A)$ over R:

$$\{E_n^{ij} \otimes E_n^{kl}; i,j,k,l = 1,\dots,n\} \underset{free \ \ over \ \ A}{\longrightarrow} Mat(n,A) \otimes_A Mat(n,A).$$

Again the module-multiplication with coefficients $\in R$ is defined naturally.

(8.3.3.1) One conveniently uses the notation indicated below.

$$\bigotimes_A^2 Mat(n,A) \ni [a_{ij}; i,j = 1,\dots,n] \otimes_A [b_{kl}; k,l = 1,\dots,n]$$
$$:= \sum_{i,j,k,l=1}^{n} a_{ij}b_{kl}E_n^{ij} \otimes E_n^{kl}.$$

(8.3.3.2) With the unique R-bilinear mapping:

$$\prod^2 (Mat(n,A) \otimes_A Mat(n,A)) \longrightarrow Mat(n,A) \otimes_A Mat(n,A),$$

such that $\forall_1^n i_1, \ldots, l_2, \forall a_1, a_2 \in A$:

$$\{a_1 E_n^{i_1 j_1} \otimes E_n^{k_1 l_1}, a_2 E_n^{i_2 j_2} \otimes E_n^{k_2 l_2}\} \longrightarrow a_1 a_2 \delta_{j_1 i_2} \delta_{l_1 k_2} E_n^{i_1 j_2} \otimes E_n^{k_1 l_2},$$

$Mat(n,A) \otimes_A Mat(n,A)$ becomes an associative algebra over R, with the unit

$$\sum_{i,j=1}^{n} E_n^{ii} \otimes E_n^{jj} = \begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix} \otimes_A \begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix}.$$

(8.3.3.2.1) Obviously there is an isomorphism of unital associative algebras over R:

$$Mat(n,A) \otimes_A Mat(n,A) \longleftrightarrow A \otimes (Mat(n,R) \otimes Mat(n,R)).$$

(8.3.3.2.2) This unital associative algebra over R must not be confused with the usual tensor product of algebras over R. For instance one finds the following consequence of assuming A to be actually non-commutative; in this case there are elements $\in Mat(n,A) \otimes_A Mat(n,A)$, such that

$$\begin{aligned}
&[a_{ij}; i,j = 1, \ldots, n] \otimes_A [b_{kl}; k,l = 1, \ldots, n] \\
&= \left([a_{ij}]_{i,j=1}^n \otimes_A \begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix} \right) \left(\begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix} \otimes_A [b_{kl}]_{k,l=1}^n \right) \\
&\neq \left(\begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix} \otimes_A [b_{kl}]_{k,l=1}^n \right) \left([a_{ij}]_{i,j=1}^n \otimes_A \begin{bmatrix} e_A & & 0 \\ & \ddots & \\ 0 & & e_A \end{bmatrix} \right).
\end{aligned}$$

(8.3.4) The bijection:

$$\{E_n^{i_1 j_1} \otimes E_n^{i_2 j_2}; i_1, \ldots, j_2 = 1, \ldots, n\} \longleftrightarrow \{E_{n^2}^{ij}; i,j = 1, \ldots, n^2\},$$

such that for instance: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, is usually lifted to an isomorphism of unital associative algebras over R:

$$Mat(n,R) \otimes Mat(n,R) \longleftrightarrow Mat(n^2,R).$$

Moreover, this bijection of bases is also lifted to an isomorphism of A-left modules over R, i.e., an A-linear and R-linear bijection:

$$Mat(n, A) \otimes_A Mat(n, A) \longleftrightarrow Mat(n^2, A),$$

such that for instance:

$$Mat(2, A) \otimes_A Mat(2, A) \ni \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \otimes_A \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$
$$\longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & ab & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in Mat(4, A).$$

This A-linear and R-linear bijection is also an isomorphism of unital associative algebras over R.

(8.3.5) Consider the unital associative superalgebra

$$Mat(m, n, R) := Mat(m + n, R)$$

over R, with the $\mathbf{Z}_2$-grading such that

$$E^{ij}_{m+n} \in \begin{cases} (Mat(m, n, R))^{\bar{0}} \text{ for } 1 \leq i, j \leq m, \text{ or } m + 1 \leq i, j \leq m + n \\ (Mat(m, n, R))^{\bar{1}} \text{ otherwise} \end{cases}.$$

(8.3.5.1) There is an isomorphism of unital associative algebras over R:

$$Mat(m, n, R) \hat{\otimes} Mat(m, n, R) \longleftrightarrow Mat((m + n)^2, R),$$

such that for instance in the case of $m = n = 1$:

$$Mat(1, 1, R) \hat{\otimes} Mat(1, 1, R) \ni \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}$$
$$\longleftrightarrow \begin{bmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 & \beta_1\alpha_2 & \beta_1\beta_2 \\ \alpha_1\gamma_2 & \alpha_1\delta_2 & -\beta_1\gamma_2 & -\beta_1\delta_2 \\ \gamma_1\alpha_2 & \gamma_1\beta_2 & \delta_1\alpha_2 & \delta_1\beta_2 \\ -\gamma_1\gamma_2 & -\gamma_1\delta_2 & \delta_1\gamma_2 & \delta_1\delta_2 \end{bmatrix} \in Mat(4, R),$$

which may be used in order to establish some $\mathbf{Z}_2$-grading of the latter too.

(8.3.5.2) Let A be a unital associative superalgebra over R. The unital associative R-algebra

$$Mat(m + n, A) \longleftrightarrow A \otimes Mat(m, n, R)$$

is $\mathbf{Z}_2$-graded by demanding this isomorphism of unital associative R-algebras to be compatible with the $\mathbf{Z}_2$-grading, and then denoted by $Mat(m, n, A)$. Here the ungraded tensor product of unital associative superalgebras over R is inserted on the right hand side. With this convention, quantum transformations of finite-dimensional quantum superspace will be constructed. With respect to $\mathbf{Z}_2$-graded main commutation relations, one should rather use the graded tensor product of A with matrices over R.

(8.3.5.3) Furthermore the R-bimodule $Mat(m+n, A) \otimes_A Mat(m+n, A)$ is established as some unital associative superalgebra over R, and denoted by $Mat(m,n,A)\hat{\otimes}_A Mat(m,n,A)$, such that the R-linear bijection:

$$Mat(m,n,A)\hat{\otimes}_A Mat(m,n,A) \longleftrightarrow A \otimes (Mat(m,n,R)\hat{\otimes} Mat(m,n,R))$$

is an isomorphism in the sense of unital associative R-superalgebras.
For instance in the case of $m = n = 1, \forall a_1, \dots, d_4 \in A$:

$$\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes_A \begin{bmatrix} 0 & b_2 \\ c_2 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & b_3 \\ c_3 & 0 \end{bmatrix} \otimes_A \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix}\right)$$
$$= -\begin{bmatrix} b_1c_3 & a_1b_3 \\ d_1c_3 & c_1b_3 \end{bmatrix} \otimes_A \begin{bmatrix} b_2c_4 & b_2d_4 \\ c_2a_4 & c_2b_4 \end{bmatrix}.$$

(8.3.5.4) Correspondingly one establishes an isomorphism of unital associative superalgebras over R:

$$Mat(m,n,A)\hat{\otimes}_A Mat(m,n,A) \longleftrightarrow Mat((m+n)^2, A),$$

such that for instance:

$$Mat(1,1,A)\hat{\otimes}_A Mat(1,1,A) \ni \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes_A \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$
$$\longleftrightarrow \begin{bmatrix} a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2 \\ a_1c_2 & a_1d_2 & -b_1c_2 & -b_1d_2 \\ c_1a_2 & c_1b_2 & d_1a_2 & d_1b_2 \\ -c_1c_2 & -c_1d_2 & d_1c_2 & d_1d_2 \end{bmatrix} \in Mat(4, A).$$

(8.3.6) Let B be a unital associative superalgebra over R, which is generated by the homogeneous elements $b_1, \dots, b_n$, and the set S of homogeneous relations.

(8.3.6.1) The tensor algebra over the R-bimodule $R(\{b_1, \dots, b_n\})$ with the basis $\{b_1, \dots, b_n\}$, $F_n := T(R(\{b_1, \dots, b_n\}))$, is used as the free algebra over this basis, with the following universal property.
$\forall$ map $\gamma : \{b_1, \dots, b_n\} \longrightarrow$ any unital associative algebra C over R :
$\exists$ unique homomorphism $\gamma_* : F_n \longrightarrow C$ of unital associative algebras over R, such that the following diagram is commutative.

$$\begin{tikzcd} \{b_1, \dots, b_n\} \arrow[rr, "\text{free over } R \quad \text{embedding}"] \arrow[dr, "\gamma"'] & & F_n \arrow[dl, "\gamma_*"] \\ & C & \end{tikzcd}$$

(8.3.6.2)

$$B := F_n/J,\ J := ideal(S),$$

with the ideal J generated by the set

$$S \subseteq F_n,\ J = sum(\{tst'; s \in S; t, t' \in F_n\}).$$

(8.3.6.3) The tensor algebra F_n is naturally $\mathbf{Z}_2$-graded by the proposed degrees of homogeneous basis vectors. Since all the elements $\in S$ are assumed to be homogeneous, the ideal J is graded. Therefore the factor algebra B is $\mathbf{Z}_2$-graded too.

(8.3.6.4) Let A also be $\mathbf{Z}_2$-graded. An A-point of B is defined as subset $\{a_1, \ldots, a_n\}$ of A, the elements of which are homogeneous and fulfill the relations of the set S in the following sense.
Assume $\forall_1^n k$: degree a_k = degree b_k, and consider the unique homomorphism α of unital associative superalgebras over R: $F_n \longrightarrow A$, such that

$$\forall_1^n k : \alpha(b_k) = a_k.$$

Then the above indicated property means, that $Im\ \alpha\big|_S = \{0\}$.

(8.3.6.5) Such an A-point $\{a_1, \ldots, a_n\}$ of B is called generic, if and only if α is injective.

(8.3.6.6) These definitions are applied to unital associative algebras A, B over R by their trivial $\mathbf{Z}_2$-gradings, taking their odd parts $= \{0\}$.

8.4 Transformations of the Quantum Plane

(8.4.1) Take the vector space $E_2 := K(\{x, y\})$ with the basis $\{x, y\}$ over the field K. The tensor algebra over E_2 is constructed, $F_2 := T(E_2)$, as the free algebra over the set $\{x, y\}$, with the coefficients $\in K$. The corresponding universal property means, that
$\forall$ map $\alpha : \{x, y\} \longrightarrow$ any unital associative algebra A over K :
$\exists$ unique homomorphism $\alpha_* : F_2 \longrightarrow A$ of unital associative algebras over K, such that the following diagram is commutative.

$$\begin{tikzcd} \{x,y\} \arrow[r, "\text{free over } K"] \arrow[dr, "\alpha"'] & E_2 \arrow[r, "\text{embedding}"] & F_2 \arrow[dl, "\alpha_*"] \\ & A & \end{tikzcd}$$

(8.4.1.1) Let $J_q^{2,0} \equiv J_q$ be the ideal of F_2, which is generated by the relation

$$S_q^{2,0} \equiv S_q \subset F_2, \; J_q := ideal(S_q) = sum(\{tst'; s \in S_q; t, t' \in F_2\}),$$

$$S_q := \{xy - q^{-1}yx\}, \; q \in K \setminus \{0\}.$$

The quantum plane is defined as the corresponding factor algebra

$$K_q^{2,0} \equiv K_q := F_2/J_q, \; [xy - q^{-1}yx] = J_q = 0.$$

$$K_q = K(\{x_q^k y_q^l; k, l \in \mathbf{N}_0\}),$$

denoting

$$\forall t \in F_2 : t_q \equiv [t] := t + J_q, \; t_q^0 := e_q := e_{F_2} + J_q = e_{K_q}.$$

(8.4.1.2) Due to the universal property of the tensor algebra, and the above relation, there are homomorphisms

$$\Delta_q : K_q \longrightarrow K_q \otimes K_q, \; \varepsilon_q : K_q \longrightarrow K$$

of unital associative algebras over K, such that:

$$x_q \xrightarrow[\Delta_q]{} x_q \otimes x_q, \; y_q \xrightarrow[\Delta_q]{} y_q \otimes e_q + x_q \otimes y_q, \; e_q \xrightarrow[\Delta_q]{} e_q \otimes e_q,$$

$$x_q \xrightarrow[\varepsilon_q]{} 1_K, \; y_q \xrightarrow[\varepsilon_q]{} 0, \; e_q \xrightarrow[\varepsilon_q]{} 1_K.$$

Hence one obtains the bialgebra $\{K_q, \mu_q, \eta_q, \Delta_q, \varepsilon_q\}$ over K, with the structure maps μ_q, η_q of the factor algebra F_2/J_q, without an antipode.

(8.4.1.3) This relation can also be written with the K-linear bijection

$$\rho_q : E_2 \otimes E_2 \longleftrightarrow E_2 \otimes E_2,$$

which is defined on the basis of $E_2 \otimes E_2$ by the map:

$$x \otimes x \longrightarrow x \otimes x, \; x \otimes y \longrightarrow qx \otimes y, \; y \otimes x \longrightarrow q^{-1}y \otimes x, \; y \otimes y \longrightarrow y \otimes y.$$

Then obviously

$$sum(\{v \otimes w - \rho_q(w \otimes v); v, w \in E_2\}) = K(\{x \otimes y - q^{-1}y \otimes x\}),$$

and therefore

$$ideal(\{v \otimes w - \rho_q(w \otimes v); v, w \in E_2\}) = ideal(S_q) = J_q,$$

with the ideal generated as indicated above.

(8.4.1.4) With the homomorphisms ε_q from above, and Δ'_q of unital associative algebras over K, such that:

$$x_q \xrightarrow[\Delta'_q]{} x_q \otimes x_q, \quad y_q \xrightarrow[\Delta'_q]{} y_q \otimes e_q + e_q \otimes y_q, \quad e_q \xrightarrow[\Delta'_q]{} e_q \otimes e_q,$$

one obtains the cocommutative bialgebra $\{K_q, \mu_q, \eta_q, \Delta'_q, \varepsilon_q\}$ over K, which does not admit an antipode.

(8.4.2) The so-called matrix quantum semigroup of format $\{\bar{0}, \bar{0}\}$ over the field K is defined for $q \in K \setminus \{0\}$.

$$Mat_q(2,0,K) \equiv Mat_q(2,K) := T(E_4)/I_q^{2,0}, \quad E_4 := K(\{a,b,c,d\}),$$

$$I_q^{2,0} \equiv I_q := ideal(T_q^{2,0}),$$

$$T_q^{2,0} \equiv T_q := \{ab - q^{-1}ba, ac - q^{-1}ca, bd - q^{-1}db, cd - q^{-1}dc,$$
$$ad - da + qbc - q^{-1}cb, ad - da + qcb - q^{-1}bc\}.$$

The corresponding equivalence classes are denoted by $a_q \equiv [a] := a + I_q, \ldots$.

(8.4.2.1) For $q^2 \neq -1_K$, $bc - cb \in K - lin\ span(T_q)$; in this case the element $[ad - q^{-1}bc] = [da - qcb]$ is central, i.e., it commutes with all the elements of $Mat_q(2,K)$.

(8.4.2.2) For $q^2 \neq -1_K$,

$$Mat_q(2,K) = K(\{a^k b^l c^m d^n + I_q; k, \ldots, n \in \mathbf{N}_0\}).$$

This statement follows from an application of the so-called diamond lemma to the case of unital associative K-algebras with quadratic relations.

(8.4.2.3) The mappings:

$$\begin{array}{ll}
a \longrightarrow a_q \otimes a_q + b_q \otimes c_q, & a \longrightarrow 1_K, \\
b \longrightarrow a_q \otimes b_q + b_q \otimes d_q, & b \longrightarrow 0, \\
c \longrightarrow c_q \otimes a_q + d_q \otimes c_q, & c \longrightarrow 0, \\
d \longrightarrow c_q \otimes b_q + d_q \otimes d_q, & d \longrightarrow 1_K,
\end{array}$$

induce an appropriate comultiplication Δ_q and counit ε_q, according to the diagram below.

$$\begin{tikzcd}
 & Mat_q(2,0,K)\otimes Mat_q(2,0,K) & \\
\{a,b,c,d\} \arrow[r] \arrow[ur, "\text{def map}"] \arrow[dr, "\text{map def}"'] & T(K(\{a,b,c,d\})) \arrow[r] \arrow[u, "\text{algebra-homomorphism}"'] \arrow[d, "\text{algebra-homomorphism}"] & Mat_q(2,0,K) \arrow[ul, "\Delta_q"'] \arrow[dl, "\varepsilon_q"] \\
 & K &
\end{tikzcd}$$

(8.4.2.4) The elements $[ad - q^{-1}bc]$, $[ad - q^{-1}cb] \in Mat_q(2, K)$ are group-like with respect to Δ_q; in case of $bc = cb$, this element is called quantum determinant.

(8.4.2.5) The bialgebra $\{Mat_q(2, K), \mu_q, \eta_q, \Delta_q, \varepsilon_q\}$ over K does not admit an antipode. Here for convenience the structure maps of $Mat_q(2, 0, K)$ are denoted $\mu_q, \ldots, \varepsilon_q$, although they must not be confused with those of $K_q^{2,0}$.

(8.4.2.6) The tensor algebra $T(E_4)$ is equipped with the flip $\tau : T(E_4) \longleftrightarrow T(E_4)$, such that

$$\forall p \geq 2 : T^p(E_4) \ni x_1 \otimes \cdots \otimes x_p \underset{\tau}{\longleftrightarrow} x_p \otimes \cdots \otimes x_1 \in T^p(E_4),$$

$\tau^2 = id(T(E_4))$, $T^0(E_4)$ and $T^1(E_4)$ being invariant with respect to τ. The map: $\{a, b, c, d\} \longrightarrow Mat_q(2, K)$, which is defined as

$$\{a \longrightarrow d_q, b \longrightarrow -qb_q, c \longrightarrow -q^{-1}c_q, d \longrightarrow a_q\},$$

induces a unique homomorphism $\tilde{\sigma} : T(E_4) \longrightarrow Mat_q(2, K)$ of unital associative algebras over K, which is then combined with the flip τ. The resulting unital algebra-antihomomorphism $\tau \circ \tilde{\sigma}$ vanishes on the ideal I_q. Hence one obtains a unique algebra-antihomomorphism

$$\sigma_q : Mat_q(2, K) \longrightarrow Mat_q(2, K), \text{ such that :}$$

$$a_q \underset{\sigma_q}{\longrightarrow} d_q, \quad b_q \underset{\sigma_q}{\longrightarrow} -qb_q, \quad c_q \underset{\sigma_q}{\longrightarrow} -q^{-1}c_q, \quad d_q \underset{\sigma_q}{\longrightarrow} a_q, \quad e_q \underset{\sigma_q}{\longrightarrow} e_q,$$

with the unit e_q of $Mat_q(2, K)$. Moreover σ_q is surjective, and

$$[ad - q^{-1}bc] \underset{\sigma_q}{\longleftrightarrow} [ad - q^{-1}cb].$$

(8.4.2.7) The so-called special matrix quantum group over K is defined,

$$SL_q(2,K) := T(K(\{a,b,c,d\}))/ideal(T_q \cup \{ad - q^{-1}bc - e, bc - cb\}),$$

with the unit e of the tensor algebra $T(E_4)$. Of course one finds an according homomorphism of unital associative algebras over K:

$$SL_q(2,K) \longleftrightarrow Mat_q(2,K)/ideal(\{ad - q^{-1}bc - e + I_q, bc - cb + I_q\}).$$

Since the structure maps of $Mat_q(2,K)$ are compatible with the relations

$$\{ad - q^{-1}bc - e, bc - cb\},$$

one obtains the Hopf algebra $\{SL_q(2,0,K), \mu_q, \eta_q, \Delta_q, \varepsilon_q\}$ over K with the unique antipode σ_q, the latter being obtained by factorizing the above anti-homomorphism σ_q of $Mat_q(2,K)$ with respect to the set

$$\{ad - q^{-1}bc - e + I_q, bc - cb + I_q\}.$$

Here the structure maps of $SL_q(2,K)$ are denoted just as those of $Mat_q(2,K)$, for convenience.

(8.4.3) Let A be a unital associative algebra over K. Two subsets B, C of A are called commuting, if and only if $\forall b \in B, c \in C : bc = cb$.

(8.4.3.1) An A-point of K_q is defined as an element $\begin{bmatrix} x \\ y \end{bmatrix} \in A^2$, such that $xy - q^{-1}yx = 0$. For such an A-point of K_q, consider the unique homomorphism $\nu : K_q \longrightarrow A$ of unital associative algebras over K, such that $\nu(x_q) = x$, $\nu(y_q) = y$; then $\begin{bmatrix} x \\ y \end{bmatrix}$ is called generic, if and only if ν is injective.

(8.4.3.2) Let $\alpha : Mat_q(2,K) \longrightarrow A$ be the unique homomorphism of unital associative algebras over K, such that

$$\alpha(a_q) = a, \ldots, \alpha(d_q) = d.$$

An A-point of $Mat_q(2,K)$ is defined as an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat(2,A)$, such that $Im\ \alpha\big|_{T_q} = \{0\}$. Obviously, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an A-point of $Mat_q(2,K)$, then $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is also such an A-point. Such an A-point $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called generic, if and only if α is injective.

(8.4.3.3) The quantum determinant of an A-point of $Mat_q(2,K)$ is defined as

$$det_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - q^{-1}bc = da - qcb \in A.$$

Here one assumes $q^2 \neq -1_K$, such that the set $\{ad - q^{-1}bc\}$ commutes with the set $\{a, b, c, d\}$.

(8.4.3.4) Let the A-points $\begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$ of $Mat_q(2,K)$, $k = 1, 2$, commute in the sense of the sets of their components. Their product $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ is then also an A-point of $Mat_q(2,K)$. Furthermore assume $q^2 \neq -1_K$; then

$$det_q \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) = det_q \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} det_q \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

(8.4.3.5) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an A-point of $Mat_q(2,K)$, for $q^2 \neq -1_K$, and assume $\det_q \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be invertible. Then $\left(det_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1}$ also commutes with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the natural sense of components, and the so-called inverse quantum matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} := \left(det_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \in Mat(2, A)$$

is an A-point of $Mat_{q^{-1}}(2,K)$. Especially $\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}^{-1} = \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}$, for the non-generic point $\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}$ of $Mat_q(2,K)$.

(8.4.3.6) Consider the injective homomorphism of unital associative algebras over K: $Mat(2, A) \longrightarrow End_K(A^2)$, which was defined in the foregoing chapter. Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an A-point of K_q, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an A-point of $Mat_q(2,K)$. If these two A-points commute in the sense of the sets of their components, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \in A^2$$

is also an A-point of K_q.

(8.4.3.7) Let an A-point $\begin{bmatrix} x \\ y \end{bmatrix}$ of K_q be generic, and the subset $\{a, b, c, d\}$ of A commute with the set $\{x, y\}$. Then both $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ are A-points of K_q, if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an A-point of $Mat_q(2, K)$.

(8.4.4) Let A be an associative algebra over the commutative ring R, with the unit e_A. Consider an element $T = \sum_{l=1}^{\bar{l}} a'_l \otimes a''_l \in A \otimes A$, with the tensor product of algebras over R. Then the elements $T_{12}, T_{13}, T_{23} \in A \otimes A \otimes A$,

$$T_{12} := \sum_{l=1}^{\bar{l}} a'_l \otimes a''_l \otimes e_A, \quad T_{23} := \sum_{l=1}^{\bar{l}} e_A \otimes a'_l \otimes a''_l,$$

$$T_{13} := \sum_{l=1}^{\bar{l}} a'_l \otimes e_A \otimes a''_l,$$

are uniquely determined by T itself. Such an element $T \in A \otimes A$ is called solution of the quantum Yang-Baxter equation (QYBE), if and only if

$$T_{12}T_{13}T_{23} = T_{23}T_{13}T_{12}.$$

(8.4.5) The relations defining $Mat_q(2, K)$ can be expressed as according main commutation relations (MCR), using an appropriate solution of the QYBE, for $q \in K \setminus \{0\}$.

(8.4.5.1) Consider the following four-dimensional solution of the quantum Yang-Baxter equation.

$$\bigotimes^{2} Mat(2, K) \ni R(q)$$
$$:= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ q - q^{-1} & 0 \end{bmatrix}$$
$$\longleftrightarrow \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} =: R_q \in Mat(4, K), \quad R_q^{-1} = R_{q^{-1}}.$$

Here one also means an isomorphism of unital associative K-algebras:

$$\bigotimes^{2}_{A} Mat(2, A) \ni e_A R(q) = R(q) \longleftrightarrow R_q = e_A R_q \in Mat(4, A).$$

(8.4.5.2)

$$R_{12}(q)R_{13}(q)R_{23}(q) = R_{23}(q)R_{13}(q)R_{12}(q),$$

with the tensor product of unital associative algebras over K. Here, for instance,

$$R_{13}(q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ q - q^{-1} & 0 \end{bmatrix}.$$

(8.4.5.3) Take $a, \dots, d \in A$. The following three statements are then equivalent.

(i) $$R(q)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes_A \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}\right)\left(\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix} \otimes_A \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix} \otimes_A \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes_A \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}\right) R(q).$$

(ii) $$R_q \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$
$$= \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} R_q.$$

(iii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat(2, A)$ is an A-point of $Mat_q(2, K)$, and $bc = cb$.

8.5 Transformations of the Quantum Superplane

(8.5.1) Consider again the free algebra $T(K(\{\xi, \eta\}))$ over the set $\{\xi, \eta\}$, and assume *char* $K \neq 2$ for the field K of coefficients. For $q \in K \setminus \{0\}$, the quantum superplane of format $\{\bar{1}, \bar{1}\}$ is defined as

$$K_q^{0,2} := T(K(\{\xi, \eta\}))/J_q^{0,2}, \quad J_q^{0,2} := ideal(S_q^{0,2}),$$
$$S_q^{0,2} := \{\xi^2, \eta^2, \xi\eta + q^{-1}\eta\xi\}.$$

(8.5.1.1) The quantum superplane of format $\{\bar{0}, \bar{0}\}$ is defined as $K_q^{2,0}$.

(8.5.1.2)

$$J_q^{0,2} = ideal(\{v \otimes w + \rho_{q^{-1}}(w \otimes v); v, w \in K(\{\xi, \eta\})\}),$$

with the above defined K-linear bijection ρ_p, $p \in K \setminus \{0\}$.

(8.5.1.3)

$$K_q^{0,2} = K(\{e_q, \xi_q, \eta_q, \xi_q\eta_q\}),\ \xi_q^2 = \eta_q^2 = \xi_q\eta_q + q^{-1}\eta_q\xi_q = J_q^{0,2} = 0,$$
$$dim K_q^{0,2} = 4,$$

with an obvious notation for the equivalence classes of the ideal $J_q^{0,2}$, and the unit e_q of $K_q^{0,2}$.

(8.5.1.4) Especially for $q = 1$, one finds the graded-commutative and graded-cocommutative Hopf superalgebra $\Lambda(K(\{\xi, \eta\}))$, i.e, the alternating algebra over $K(\{\xi, \eta\})$, with the comultiplication Δ_1, counit ε_1, and the unique antipode σ_1:

$$\xi_1 \xrightarrow[\Delta_1]{} \xi_1 \otimes e_1 + e_1 \otimes \xi_1,\ \eta_1 \xrightarrow[\Delta_1]{} \eta_1 \otimes e_1 + e_1 \otimes \eta_1,$$
$$\xi_1 \xrightarrow[\varepsilon_1]{} 0,\ \eta_1 \xrightarrow[\varepsilon_1]{} 0,\ \xi_1 \xrightarrow[\sigma_1]{} -\xi_1,\ \eta_1 \xrightarrow[\sigma_1]{} -\eta_1.$$

(8.5.1.5) The $\mathbf{Z}_2$-grading of $K(\{\xi, \eta\})$, which is defined by taking the basis vectors ξ and η of odd degree, is uniquely extended to the free algebra over the set $\{\xi, \eta\}$ and the field K. Since the elements of the relation $S_q^{0,2}$ are homogeneous, the ideal $J_q^{0,2}$ is graded. Therefore $K_q^{0,2}$ is some unital associative superalgebra over K, with ξ_q and η_q being of odd degree.

(8.5.1.6) The relations $S_q^{2,0}$ and $S_{q^{-1}}^{0,2}$ are dual in the following sense. Denote $E_2^* := Hom_K(E_2, K)$, $E_2 := K(\{x, y\})$. Then there is some K-linear bijection:

$$(E_2 \otimes E_2)^* \ni \psi \longleftrightarrow \tilde{\psi} := \sum_{l=1}^{\bar{l}} f_l \otimes g_l \in E_2^* \otimes E_2^*,$$

such that

$$\forall v, w \in E_2 : \psi(v \otimes w) = \sum_{l=1}^{\bar{l}} f_l(v) g_l(w) \in K.$$

Inserting the dual basis of E_2^*, such that

$$E_2^* = K(\{\xi, \eta\}),\ \xi(x) = 1_K,\ \xi(y) = 0,\ \eta(x) = 0,\ \eta(y) = 1_K,$$

one easily calculates the following implication. $\forall \psi \in (E_2 \otimes E_2)^*$:

$$\psi(x \otimes y - q^{-1} y \otimes x) = 0$$
$$\Longleftrightarrow \tilde{\psi} \in K - lin\ span(\{\xi \otimes \xi, \eta \otimes \eta, \xi \otimes \eta + q\eta \otimes \xi\}).$$

(8.5.1.6.1) An appropriate dual of the quantum plane $K_q^{2,0}$ is defined by means of the dual basis of E_2^*.

$$K_q^{2,0*} := T(K(\{\xi,\eta\}))/ideal(S_q^{2,0*}),$$

with the dual relations

$$S_q^{2,0*} := \{\xi^2, \eta^2, \xi\eta + q\eta\xi\},$$

and an obvious isomorphism of unital associative superalgebras over K: $K_q^{2,0*} \longleftrightarrow K_{q^{-1}}^{0,2}$. Then the above duality may be rewritten, such that

$$\forall \psi \in (E_2 \otimes E_2)^* : \psi|_{S_q^{2,0}} = 0 \Longleftrightarrow \tilde{\psi} \in K - lin\ span(S_q^{2,0*}).$$

(8.5.2) Let A be a unital associative superalgebra over K. Two subsets B, C of A are called super-commuting, if and only if the super- commutators $[b, c] = 0$ for all $b \in B$, $c \in C$.

(8.5.2.1) An A-point $\begin{bmatrix} x \\ y \end{bmatrix}$ of $K_q^{2,0}$ is now redefined as an element $\in A^2$, such that both x and y are even, and $xy - q^{-1}yx = 0$. Such an A-point $\begin{bmatrix} x \\ y \end{bmatrix}$ is called generic, if and only if the subset $\{x, y\}$ of A is generic in the sense defined previously.

(8.5.2.2) An A-point of $K_q^{0,2}$ is defined as an element $\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in A^2$, such that both ξ and η are odd, and $\xi^2 = \eta^2 = \xi\eta + q^{-1}\eta\xi = 0$. Such an A-point $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ of $K_q^{0,2}$ is called generic, if and only if the subset $\{\xi, \eta\}$ of A is generic.

(8.5.2.3) An A-point of $Mat_q(2, K)$ is now redefined as an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat(2, A)$, such that a, b, c, d are even, and the subset $\{a, b, c, d\}$ of A is an A-point of $Mat_q(2, K)$. Such an A-point $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called generic, if and only if the subset $\{a, \ldots, d\}$ of A is generic.

(8.5.2.4) Let $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ be an A-point of $K_{q^{-1}}^{0,2}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an A-point of $Mat_q(2, K)$. If these two A-points commute in the sense of the sets of their components, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} a\xi + b\eta \\ c\xi + d\eta \end{bmatrix} \in A^2$$

is also an A-point of $K_{q^{-1}}^{0,2}$.

(8.5.2.5) Let the A-points $\begin{bmatrix} x \\ y \end{bmatrix}$ of $K_q^{2,0}$, and $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ of $K_{q^{-1}}^{0,2}$, both be generic, and the subset $\{a, b, c, d\} \subset A^{\bar{0}}$ commute with the set $\{x, y, \xi, \eta\}$. Let both $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ be an A-point of $K_q^{2,0}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ an A-point of $K_{q^{-1}}^{0,2}$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an A-point of $Mat_q(2, K)$.

(8.5.2.6) Let $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ be an A-point of $K_{q^{-1}}^{0,2}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an A-point of $Mat_q(2, K)$, such that $\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ again is an A-point of $K_{q^{-1}}^{0,2}$. Then obviously

$$\xi' \eta' = det_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xi\eta, \text{ for } bc = cb.$$

(8.5.3) The quantum superplane of format $\{\bar{0}, \bar{1}\}$ is defined as

$$K_q^{1,1} := T(K(\{x, y\}))/J_q^{1,1}, \ J_q^{1,1} := ideal(S_q^{1,1}),$$

$$S_q^{1,1} := \{xy - q^{-1}yx, y^2\}, \text{ for } q \in K \setminus \{0\}.$$

(8.5.3.1)

$$K_q^{1,1} = K(\{e_q, y_q, x_q^k, x_q^k y_q; k \in \mathbf{N}\}), \ x_q y_q - q^{-1} y_q x_q = y_q^2 = J_q^{1,1} = 0,$$

with the unit e_q of $K_q^{1,1}$, and $x_q := x + J_q^{1,1}$, $y_q := y + J_q^{1,1}$.

(8.5.3.2) Since the relation $S_q^{1,1}$ is homogeneous with respect to the $\mathbf{Z}_2$-grading, which is due to the basis vectors x being even, and y odd, the ideal $J_q^{1,1}$ is graded. Hence one obtains the unital associative superalgebra $K_q^{1,1}$, with the equivalence classes of $J_q^{1,1}$ being graded such that x_q is even, and y_q odd.

(8.5.3.3) The relations $S_q^{1,1}$, and

$$S_q^{1,1*} := \{\xi^2, \xi\eta - q\eta\xi\} \subset T(K(\{\xi, \eta\})),$$

are dual in the following sense. Denote

$$E_2^* := Hom_K(E_2, K), \ E_2 := K(\{x, y\}),$$

and consider the K-bilinear bijection:

$$(E_2 \otimes E_2)^* \ni \psi \longleftrightarrow \tilde{\psi} := \sum_{l=1}^{\bar{l}} f_l \otimes g_l \in E_2^* \otimes E_2^*,$$

such that

$$\forall v, w \in E_2 : \psi(v \otimes w) = \sum_{l=1}^{I} f_l(v) g_l(w) \in K.$$

With the dual basis of E_2^*, such that

$$E_2^* = K(\{\xi, \eta\}), \ \xi(x) = 1_K, \ \xi(y) = 0, \ \eta(x) = 0, \ \eta(y) = 1_K,$$

one easily calculates, that

$$\forall \psi \in (E_2 \otimes E_2)^* : \psi\big|_{S_q^{1,1}} = 0 \Longleftrightarrow \tilde{\psi} \in K - lin \ \ span(S_q^{1,1*}).$$

(8.5.3.3.1) One therefore defines the dual of the quantum superplane $K_q^{1,1}$,

$$K_q^{1,1*} := T(K(\{\xi, \eta\}))/ideal(S_q^{1,1*}) = K(\{e_q, \xi_q, \eta_q^k, \xi_q \eta_q^k; k \in \mathbf{N}\}),$$

as an associative superalgebra over K with the unit e_q, and with the $\mathbf{Z}_2$-grading due to the basis vectors ξ being odd, and η even, such that ξ_q is odd, and η_q even.
Obviously there is an isomorphism of unital associative superalgebras over K: $K_q^{1,1*} \longleftrightarrow K_q^{1,1}$.

(8.5.4) The corresponding matrix quantum super-semigroup is constructed as

$$Mat_q(1,1,K) := T(K(\{a,b,c,d\}))/I_q^{1,1}, \ I_q^{1,1} := ideal(T_q^{1,1}),$$

$$T_q^{1,1} := \{ab - q^{-1}ba, ac - q^{-1}ca, bd - qdb, cd - qdc, b^2, c^2,$$
$$ad - da + q^{-1}bc + qcb, ad - da - qbc - q^{-1}cb\}.$$

For $q^2 \neq -1_K, \ bc + cb \in K - lin \ \ span(T_q^{1,1})$.

(8.5.4.1) For $q^2 \neq -1_K$,

$$Mat_q(1,1,K) = K(\{a_q^k b_q^l c_q^m d_q^n; k, n \in \mathbf{N}_0; l, m = 0, 1\}),$$

denoting $\forall t \in T(K(\{a,b,c,d\})) : t_q \equiv [t] := t + I_q^{1,1}$.

(8.5.4.2) The vector space $K(\{a, \ldots, d\})$ is equipped with the $\mathbf{Z}_2$-grading, such that the basis vectors a and d are even, b and c are odd. This $\mathbf{Z}_2$-grading is then uniquely extended to the free algebra over the set $\{a, \ldots, d\}$. Since the relations of $T_q^{1,1}$ are homogeneous, one obtains some unital associative K-superalgebra $Mat_q(1,1,K)$.

(8.5.4.3) With the convenient notation $a_{11} \equiv a$, $a_{12} \equiv b$, $a_{21} \equiv c$, $a_{22} \equiv d$, the two mappings:

$$\forall_{\bar{1}}^{2} i,j : a_{ij} \underset{def}{\longrightarrow} \sum_{l=1}^{\bar{l}} [a_{il}] \otimes [a_{lj}] \in Mat_q(1,1,K) \hat{\otimes} Mat_q(1,1,K),$$

$$a_{ij} \underset{def}{\longrightarrow} \delta_{ij} := \begin{cases} 1_K & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases},$$

are uniquely extended to homomorphisms of unital associative superalgebras over K, and then factorized with respect to the ideal $I_q^{1,1}$. Hence one obtains the $\mathbf{Z}_2$-graded bialgebra $Mat_q(1,1,K)$ over K, which does not admit an antipode.

(8.5.5) Let A be a unital associative superalgebra over K. An A-point of $K_q^{1,1}$ is defined as an element $\begin{bmatrix} x \\ y \end{bmatrix} \in A^2$, such that $x \in A^{\bar{0}}$, $y \in A^{\bar{1}}$, and $xy - q^{-1}yx = y^2 = 0$; it is called generic, if and only if the subset $\{x, y\}$ of A is generic.

(8.5.5.1) Correspondingly, an A-point of $Mat_q(1,1,K)$ is defined as an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat(2, A)$, such that a and d are even, b and c are odd, and the subset $\{a, b, c, d\}$ of A is an A-point of $Mat_q(1,1,K)$. Such an A-point $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called generic, if and only if the subset $\{a, b, c, d\}$ of A is generic.

(8.5.5.2) Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an A-point of $K_q^{1,1}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an A-point of $Mat_q(1,1,K)$. If the two subsets $\{x, y\}$ and $\{a, b, c, d\}$ of A are supercommuting, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is again an A-point of $K_q^{1,1}$.

(8.5.5.3) An A-point of $K_q^{1,1*}$ is defined as an element $\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in A^2$, such that $\xi \in A^{\bar{1}}$, $\eta \in A^{\bar{0}}$, and $\xi^2 = \xi\eta - q\eta\xi = 0$; it is called generic, if and only if the subset $\{\xi, \eta\}$ of A is generic.

(8.5.5.4) Let $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ be an A-point of $K_q^{1,1*}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an A-point of $Mat_q(1,1,K)$. If the two subsets $\{\xi, \eta\}$ and $\{a, b, c, d\}$ of A are supercommuting, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ is again an A-point of $K_q^{1,1*}$.

(8.5.5.5) Let the A-points $\begin{bmatrix} x \\ y \end{bmatrix}$ of $K_q^{1,1}$, and $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ of $K_q^{1,1*}$, both be generic, and the subset $\{a,b,c,d\}$ of A super-commuting with $\{x,y,\xi,\eta\}$, a and d being even, b and c odd. If both $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is an A-point of $K_q^{1,1}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ an A-point of $K_q^{1,1*}$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an A-point of $Mat_q(1,1,K)$.

(8.5.6) The relations defining $Mat_q(1,1,K)$ can be obtained from according $\mathbf{Z}_2$-graded main commutation relations, with an appropriate solution of the $\mathbf{Z}_2$-graded QYBE, for $q \in K\backslash\{0\}$.

(8.5.6.1)

$$Mat(1,1,K)\hat{\otimes}Mat(1,1,K) \ni R(q)$$

$$:= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & q^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ q^{-1}-q & 0 \end{bmatrix}$$

$$\longleftrightarrow \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix} =: R_q \in Mat(4,K), \quad R_q^{-1} = R_{q^{-1}}.$$

Here one also means, in the sense of an isomorphism of unital associative superalgebras over R:

$$Mat(1,1,A)\hat{\otimes}_A Mat(1,1,A) \ni e_A R(q) = R(q)$$

$$\longleftrightarrow R_q = e_A R_q \in Mat(4,A).$$

(8.5.6.2) For $q \in K\backslash\{0\}$, $R(q)$ fulfills the $\mathbf{Z}_2$-graded QYBE, which is defined with the $\mathbf{Z}_2$-graded tensor product $Mat(1,1,K)\hat{\otimes}Mat(1,1,K)\hat{\otimes}Mat(1,1,K)$.

(8.5.6.3) Take $a,\ldots,d \in A$, and assume $q^2 \neq -1$. The following three statements are then equivalent.

(i) $$R(q)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes_A \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}\right)\left(\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix} \otimes_A \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix} \otimes_A \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes_A \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix}\right) R(q).$$

(ii) $$R_q \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{bmatrix} \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$
$$= \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{bmatrix} R_q.$$

(iii) $ab = q^{-1}ba, ac = q^{-1}ca, bd = qdb, cd = qdc,$
$b^2 = c^2 = 0, bc = -cb, ad - da = (q - q^{-1})bc.$

In this case, if a and d are even, b and c odd, then $\{a, b, c, d\}$ is an A-point of $Mat_q(1, 1, K)$.

8.6 Transformations of Quantum Superspace

(8.6.1) Choose the format $\{\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1}\}$ of m even and n odd basis vectors. The quantum superspace $K_q^{m,n}$ is constructed over the field K, *char* $K \neq 2$.

$$E_{m,n} := E_{m,n}^{\bar{0}} + E_{m,n}^{\bar{1}},$$
$$E_{m,n}^{\bar{0}} := K(\{x_1, \dots, x_m\}), \; E_{m,n}^{\bar{1}} := K(\{y_1, \dots, y_n\}).$$
$$K_q^{m,n} := T(E_{m,n})/J_q^{m,n}, \; J_q^{m,n} := ideal(S_q^{m,n}),$$
$$S_q^{m,n} := \{x_k x_l - q_{kl}^{-1} x_l x_k, y_i y_j + q_{m+i,m+j}^{-1} y_j y_i, x_k y_j - q_{k,m+j}^{-1} y_j x_k;$$
$$1 \leq k \leq l \leq m, 1 \leq i \leq j \leq n\}.$$

Here the choice of deformation parameters is restricted by the conditions, that

$$\forall_1^m k \leq l, \forall_1^n i \leq j : q_{kk} = q_{m+i,m+i} = 1_K, \; q_{kl} q_{m+i,m+j} q_{k,m+j} \neq 0.$$

For convenience one denotes $\forall_1^m k < l, \forall_1^n i < j, \forall_1^m r, \forall_1^n s :$

$$q_{lk} := q_{kl}^{-1}, \; q_{m+j,m+i} := q_{m+i,m+j}^{-1}, \; q_{m+s,r} := q_{r,m+s}^{-1},$$

and then obtains the following matrix q of deformations parameters.

$$q := [q_{kl}; k, l = 1, \cdots, m+n], \; \forall_1^{m+n} k, l : q_{kl}^{-1} = q_{lk}, q_{kk} = 1_K.$$

Obviously $\forall_1^n j : y_j^2 \in K - lin \;\; span(S_q^{m,n})$.

(8.6.2) The dual quantum superspace is then defined with respect to the above format.

$$E_{m,n}^* := Hom_K(E_{m,n}, K) = E_{m,n}^{*\bar{0}} + E_{m,n}^{*\bar{1}},$$
$$E_{m,n}^{*\bar{1}} := K(\{\xi^1, \dots, \xi^m\}), \; E_{m,n}^{*\bar{0}} := K(\{\eta^1, \dots, \eta^n\}),$$

with the dual basis being inserted, i.e,

$$\forall_1^m k, l, \forall_1^n i, j : \xi^k(x_l) = \delta_{kl}, \; \xi^k(y_j) = 0, \; \eta^i(x_l) = 0, \; \eta^i(y_j) = \delta_{ij}.$$

With this basis one defines the following set of dual relations.

$$S_q^{m,n*} := \{\xi^k\xi^l + q_{kl}\xi^l\xi^k, \eta^i\eta^j - q_{m+i,m+j}\eta^j\eta^i, \xi^k\eta^j - q_{k,m+j}\eta^j\xi^k; \\ 1 \le k \le l \le m, 1 \le i \le j \le n\}.$$

Obviously $\forall_1^m k : (\xi^k)^2 \in K - lin\ \ span(S_q^{m,n*})$.

$$K_q^{m,n*} := T(E_{m,n}^*)/J_q^{m,n*}, \quad J_q^{m,n*} := ideal(S_q^{m,n*}).$$

(8.6.3) The quantum superspace $K_q^{m,n}$ is some unital associative superalgebra over K, with the $\mathbf{Z}_2$-grading due to that of $E_{m,n}$, because the ideal $J_q^{m,n}$ is graded, with the elements of $S_q^{m,n}$ being homogeneous.

(8.6.4) One immediately establishes an isomorphism of unital associative superalgebras over K: $K_q^{m,n*} \longleftrightarrow K_{q^*}^{n,m}$, with the dual matrix q^* of deformation parameters defined such that

$$\forall_1^m k, l, \forall_1^n i, j : \ q_{ij}^* := q_{m+i,m+j}^{-1}, \quad q_{n+k,n+l}^* := q_{kl}^{-1}, \\ q_{i,n+l}^* := q_{l,m+i}, \quad q_{n+k,j}^* := q_{k,m+j}^{-1} = q_{j,n+k}^{*-1}.$$

Obviously $q^{**} = q$.

(8.6.5)

$$\forall \psi \in (E_{m,n} \otimes E_{m,n})^* : \psi\big|_{S_q^{m,n}} = 0 \iff \tilde{\psi} \in K - lin\ \ span(S_q^{m,n*}).$$

Here the K-linear bijection:

$$(E_{m,n} \otimes E_{m,n})^* \ni \psi \longleftrightarrow \tilde{\psi} \in E_{m,n}^* \otimes E_{m,n}^*$$

is inserted. $S_q^{m,n}$ is used as subset of $E_{m,n} \otimes E_{m,n}$, and $S_q^{m,n*}$ as subset of $E_{m,n}^* \otimes E_{m,n}^*$.

(8.6.6) In the case of $m = n = 2$, one finds the following relations, which are written conveniently as equations for the corresponding equivalence classes, with the deformation matrix

$$q := \begin{bmatrix} 1 & q_{12} & r_{11} & r_{12} \\ q_{12}^{-1} & 1 & r_{21} & r_{22} \\ r_{11}^{-1} & r_{21}^{-1} & 1 & p_{12} \\ r_{12}^{-1} & r_{22}^{-1} & p_{12}^{-1} & 1 \end{bmatrix}, \quad q^* = \begin{bmatrix} 1 & p_{12}^{-1} & r_{11} & r_{21} \\ p_{12} & 1 & r_{12} & r_{22} \\ r_{11}^{-1} & r_{12}^{-1} & 1 & q_{12}^{-1} \\ r_{21}^{-1} & r_{22}^{-1} & q_{12} & 1 \end{bmatrix},$$

$$q_{12}p_{12}r_{11}r_{12}r_{21}r_{22} \neq 0.$$
$$x_1x_2 - q_{12}^{-1}x_2x_1 = 0, \quad y_1y_2 + p_{12}^{-1}y_2y_1 = 0,$$
$$\forall_1^2 k, l : x_ky_l - r_{kl}^{-1}y_lx_k = 0, \quad y_l^2 = 0,$$
$$\xi^1\xi^2 + q_{12}\xi^2\xi^1 = 0, \quad \eta^1\eta^2 - p_{12}\eta^2\eta^1 = 0,$$
$$\xi^k\eta^l - r_{kl}\eta^l\xi^k = 0, \quad (\xi^k)^2 = 0,$$
$$\xi^k(x_l) = \eta^k(y_l) = \delta_{kl}, \quad \xi^k(y_l) = \eta^k(x_l) = 0.$$

(8.6.7)

$$\text{For } m=2, n=0,\ q=\begin{bmatrix} 1_K & q_{12} \\ q_{12}^{-1} & 1_K \end{bmatrix},\ q^*=\begin{bmatrix} 1_K & q_{12}^{-1} \\ q_{12} & 1_K \end{bmatrix},$$
$$K_q^{2,0*} \longleftrightarrow K_{q^*}^{0,2}.$$

$$\text{For } m=n=1,\ \ q=q^*=\begin{bmatrix} 1_K & q_{12} \\ q_{12}^{-1} & 1_K \end{bmatrix},\ K_q^{1,1*} \longleftrightarrow K_q^{1,1}.$$

(8.6.8) The corresponding matrix quantum super-semigroup is defined as some unital associative superalgebra over K.

$$Mat_q(m,n,K) := T(K(\{a_{kl}; k,l=1,\ldots,m+n\}))/I_q^{m,n},$$
$$I_q^{m,n} := ideal(T_q^{m,n}).$$

The so-called Manin relations $T_q^{m,n}$ are equipped with the $\mathbf{Z}_2$-grading of basis vectors, such that

$$\forall_1^{m+n} k,l : degree\ a_{kl} := \tilde{k}+\tilde{l},\ \ \hat{k}\in\tilde{k} := \begin{cases} \bar{0} & \text{for } 1\le k\le m \\ \bar{1} & \text{for } m+1\le k\le m+n \end{cases}.$$

This $\mathbf{Z}_2$-grading is naturally extended to the above tensor algebra, and then applied to $Mat_q(m,n,K)$, because the elements of $T_q^{m,n}$ are homogeneous, such that the ideal $I_q^{m,n}$ is graded. The relations of $T_q^{m,n}$ are conveniently written as equations of the equivalence classes $a_{kl}+I_q^{m,n} \equiv [a_{kl}]$.
$\forall_1^{m+n} i,j,k,l$:

$$[a_{ik}^2]=0 \text{ for } \hat{\imath}+\hat{k}\in\bar{1},$$
$$[a_{ik}a_{il}-(-1)^{(\hat{k}+1)(\hat{l}+1)}q_{kl}a_{il}a_{ik}]=0 \text{ for } \hat{\imath}\in\bar{1} \text{ and } k<l,$$
$$[a_{ik}a_{il}-(-1)^{\hat{k}\hat{l}}q_{kl}^{-1}a_{il}a_{ik}]=0 \text{ for } \hat{\imath}\in\bar{0} \text{ and } k<l,$$
$$[a_{ik}a_{jk}-(-1)^{\hat{\imath}\hat{\jmath}}q_{ij}^{-1}a_{jk}a_{ik}]=0 \text{ for } \hat{k}\in\bar{0} \text{ and } i<j,$$
$$[a_{ik}a_{jk}-(-1)^{(\hat{\imath}+1)(\hat{\jmath}+1)}q_{ij}a_{jk}a_{ik}]=0 \text{ for } \hat{k}\in\bar{1} \text{ and } i<j,$$
$$[(-1)^{\hat{k}(\hat{\jmath}+\hat{l})}a_{ik}a_{jl}+(-1)^{\hat{\jmath}\hat{l}}q_{kl}a_{il}a_{jk}]$$
$$=[q_{ij}^{-1}(-1)^{\hat{\imath}\hat{\jmath}}((-1)^{\hat{k}(\hat{\imath}+\hat{l})}a_{jk}a_{il}+(-1)^{\hat{\imath}\hat{l}}q_{kl}a_{jl}a_{ik})] \text{ for } i<j \text{ and } k<l,$$
$$[(-1)^{(\hat{k}+1)(\hat{\jmath}+\hat{l})}a_{ik}a_{jl}+(-1)^{(\hat{\jmath}+1)(\hat{l}+1)}q_{kl}^{-1}a_{il}a_{jk}]$$
$$=[q_{ij}(-1)^{(\hat{\imath}+1)(\hat{\jmath}+1)}((-1)^{(\hat{k}+1)(\hat{\imath}+\hat{l})}a_{jk}a_{il}+(-1)^{(\hat{\imath}+1)(\hat{l}+1)}q_{kl}^{-1}a_{jl}a_{ik})]$$
$$\text{for } i<j \text{ and } k<l.$$

(8.6.9) These quantum matrices are operating onto elements of the corresponding quantum superspace. $\forall_1^{m+n} k$:

$$E_{m,n} \ni x_k \xrightarrow[def]{} \sum_{l=1}^{m+n} (a_{kl} + I_q^{m,n}) \otimes (x_l + J_q^{m,n}) \in Mat_q(m,n,K) \otimes K_q^{m,n}.$$

Here the odd basis vectors $y_1, \ldots, y_n$ are denoted by $x_{m+1}, \ldots, x_{m+n}$. This map is uniquely extended to a homomorphism of unital associative superalgebras over K: $T(E_{m,n}) \longrightarrow Mat_q(m,n,K)\hat{\otimes}K_q^{m,n}$, which vanishes on the set $S_q^{m,n}$ and therefore on its ideal $J_q^{m,n}$. Hence one obtains a unique homomorphism of unital associative superalgebras

$$\alpha_q^{m,n} : K_q^{m,n} \longrightarrow Mat_q(m,n,K)\hat{\otimes}K_q^{m,n},$$

extending the above map of basis vectors over K.

(8.6.10) The relations of the set $S_q^{m,n}$ are rewritten as equations of equivalence classes with respect to $J_q^{m,n}$.

$$\forall_1^{m+n} k,l : [x_k x_l - (-1)^{\hat{k}\hat{l}} q_{kl}^{-1} x_l x_k] = 0,$$

$$\text{hence } [x_{m+1}^2] = \cdots = [x_{m+n}^2] = 0.$$

The equivalence classes of the dual basis vectors fulfill the relations of the set $S_q^{m,n*}$, denoting $\forall_1^n i : \eta^i \equiv \xi^{m+i}$.

$$\forall_1^{m+n} k,l : [\xi^k \xi^l - (-1)^{(\hat{k}+1)(\hat{l}+1)} q_{kl} \xi^l \xi^k] = 0,$$

$$\text{hence } [\xi^1]^2 = \cdots = [\xi^m]^2 = 0;$$

$$\text{here } \xi^k(x_l) = \delta_{kl} := \begin{cases} 1_K & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}.$$

(8.6.11) Correspondingly there is a unique homomorphism $\alpha_q^{m,n*}$ of unital associative superalgebras over K:

$$K_q^{m,n*} \longrightarrow Mat_q(m,n,K)\hat{\otimes}K_q^{m,n*},$$

such that

$$\forall_1^{m+n} k : \alpha_q^{m,n*}([\xi^k]) = \sum_{l=1}^{m+n} [a_{kl}] \otimes [\xi^l].$$

(8.6.12) For an arbitrary format and deformation matrix q, one obtains the diagram below.

$$\begin{tikzcd}
K_q^{m,n\,*} \arrow[rr, leftrightarrow, "\text{superalgebra-isomorphism}"] \arrow[d, "\alpha_q^{m,n\,*}"] & & K_{q^*}^{n,m} \arrow[d, "\alpha_{q^*}^{n,m}"] \\
Mat_q(m,n,K)\,\hat{\otimes}\,K_q^{m,n\,*} \arrow[rr, leftrightarrow] & & Mat_{q^*}(n,m,K)\,\hat{\otimes}\,K_{q^*}^{n,m}
\end{tikzcd}$$

Here an isomorphism of unital associative superalgebras over K is inserted:

$$Mat_q(m,n,K) \longleftrightarrow Mat_{q^*}(n,m,K).$$

(8.6.13) With these K-linear mappings $\alpha_q^{m,n}$ and $\alpha_q^{m,n*}$, both $K_q^{m,n}$ and $K_q^{m,n*}$ become $Mat_q(m,n,K)$-left comodules over K. Moreover these left comodules are $\mathbf{Z}_2$-graded in the sense, that both $\alpha_q^{m,n}$ and $\alpha_q^{m,n*}$ are even with respect to the $\mathbf{Z}_2$-gradings of $Mat_q(m,n,K)$, $K_q^{m,n}$ and $K_q^{m,n*}$.

(8.6.14)

$$Mat_q(0,2,K) := T(K(\{a,b,c,d\})/I_q^{0,2},$$

$$I_q^{0,2} := ideal(T_q^{0,2}),\ T_q^{0,2} = T_{q^{-1}}^{2,0},$$

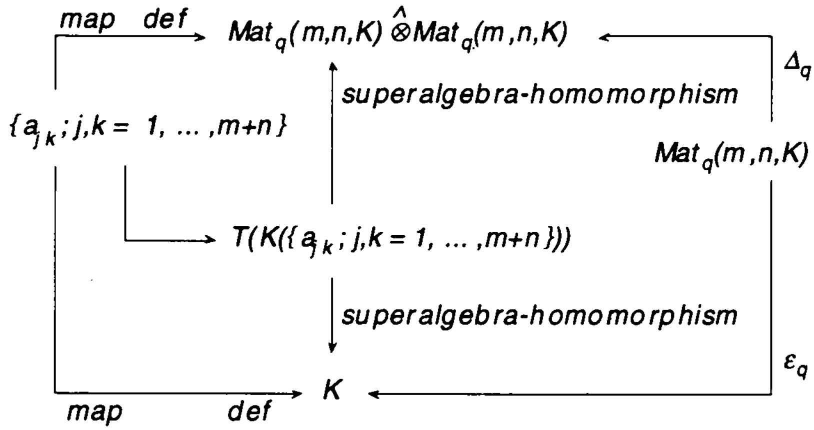
with the notation of the foregoing chapter on the quantum plane.

(8.6.15) Furthermore $Mat_q(m,n,K)$ is established as some $\mathbf{Z}_2$-graded bialgebra over K, with the comultiplication Δ_q and counit ε_q, for $m,n \in \mathbf{N}_0$.

$$\forall_1^{m+n} j,k:\ a_{jk} \underset{def}{\longrightarrow} \delta_{jk} \in K,$$

$$a_{jk} \underset{def}{\longrightarrow} \sum_{l=1}^{m+n} [a_{jl}] \otimes [a_{lk}] \in \bigotimes^2 Mat_q(m,n,K).$$

These two mappings are uniquely extended and then factorized, due to the next diagram. The resulting $\mathbf{Z}_2$-graded bialgebra $Mat_q(m,n,K)$ over K admits an antipode only for special choices of the complex parameters q_{kl}, $k,l = 1,\ldots,m+n$.



(8.6.16) Let A be a unital associative superalgebra over K. An A-point of $K_q^{m,n}$ is defined as an element

$$\sum_{k=1}^{m+n} x_k E_{m+n}^k \in A^{m+n},$$

such that the subset $\{x_1, \ldots, x_{m+n}\}$ of A is an A-point of $K_q^{m,n}$; it is called generic, if and only if this subset of A is generic.

(8.6.16.1) Correspondingly, an A-point of $K_q^{m,n*}$ is defined as an element of A^{m+n}, such that the set of its components is an A-point of $K_q^{m,n*}$. Such an A-point is called generic, if and only if the set of its components is generic.

(8.6.16.2) An A-point of $Mat_q(m,n,K)$ is defined as an element of $Mat(m+n, A)$, such that the set of its components is an A-point of $Mat_q(m,n,K)$; it is called generic, if and only if the set of its components is generic.

(8.6.16.3) Let $[x_k; k = 1, \ldots, m+n]$ be an A-point of the quantum superspace $K_q^{m,n}$, and $[a_{kl}; k,l = 1, \ldots, m+n]$ an A-point of $Mat_q(m,n,K)$. If the corresponding sets of components are super-commuting, then

$$\left[\sum_{l=1}^{m+n} a_{kl} x_l; k = 1, \ldots, m+n\right]$$

again is an A-point of $K_q^{m,n}$.

(8.6.16.4) If $[\xi^k; k = 1, \dots, m+n]$ is an A-point of $K_q^{m,n*}$, furthermore $[a_{kl}; k,l = 1, \dots, m+n]$ an A-point of $Mat_q(m,n,K)$, and if these two A-points are super-commuting, then

$$\left[\sum_{l=1}^{m+n} a_{kl}\xi^l; k = 1, \dots, m+n\right]$$

again is an A-point of $K_q^{m,n*}$.

(8.6.16.5) Let the A-points x of $K_q^{m,n}$, and ξ of $K_q^{m,n*}$, both be generic. Let the set of components of $[a_{kl}; k,l = 1, \dots, m+n]$ be $\mathbf{Z}_2$-graded, such that

$$\forall_1^{m+n} k,l : a_{kl} \in A^{\tilde{k}+\tilde{l}}, \quad \tilde{k} := \begin{cases} \bar{0} & \text{for } 1 \le k \le m \\ \bar{1} & \text{for } m+1 \le k \le m+n \end{cases}$$

Let the two subsets $\{a_{kl}; k,l = 1, \dots, m+n\}$ and $\{x_k, \xi^k; k = 1, \dots, m+n\}$ of A be super-commuting. If both

$$\left[\sum_{l=1}^{m+n} a_{kl}x_l; k = 1, \dots, m+n\right]$$

is an A-point of $K_q^{m,n}$, and

$$\left[\sum_{l=1}^{m+n} a_{kl}\xi^l; k = 1, \dots, m+n\right]$$

an A-point of $K_q^{m,n*}$, then $[a_{kl}; k,l = 1, \dots, m+n]$ is an A-point of $Mat_q(m,n,K)$.

8.7 Topological $\mathbf{Z}_2$-Graded Hopf Algebras

(8.7.1) Let $\mathcal{R}$ be both a ring and a topological space. $\mathcal{R}$ is called topological ring, if and only if the mappings:

$$\mathcal{R} \times \mathcal{R} \ni \{r,s\} \longrightarrow r - s \in \mathcal{R}, \quad \mathcal{R} \times \mathcal{R} \ni \{r,s\} \longrightarrow rs \in \mathcal{R}$$

are continuous with respect to the product topology; then especially $\mathcal{R}$ is an Abelian topological group.

(8.7.2) Let $\mathcal{H}$ be both a topological space, and a left module over a topological ring $\mathcal{R}$. $\mathcal{H}$ is called topological left module over $\mathcal{R}$, if and only if $\mathcal{H}$ is an Abelian topological group, and the map:

$$\mathcal{R} \times \mathcal{H} \ni \{r,a\} \longleftrightarrow ra \in \mathcal{H}$$

is continuous with respect to the product topology.

(8.7.2.1) Every continuous $\mathcal{R}$-linear map from a topological $\mathcal{R}$-left modul into such an object is uniformly continuous.

(8.7.2.2) Topological $\mathcal{R}$-right modules, and topological bimodules, over commutative topological rings, are defined correspondingly.

(8.7.2.3) Two topological $\mathcal{R}$-left modules are called isomorphic, if and only if there exists an $\mathcal{R}$-linear homeomorphism from one of them onto the other.

(8.7.3) Assume $\mathcal{R}$ to be of Hausdorff type, and denote its unique Hausdorff completion by $\tilde{\mathcal{R}}$. Let $\mathcal{H}$ be a topological $\mathcal{R}$-left module of Hausdorff type. Then there is, modulo an $\tilde{\mathcal{R}}$-linear homeomorphism, a unique complete Hausdorff topological $\tilde{\mathcal{R}}$-left module $\tilde{\mathcal{H}}$, called Hausdorff completion of $\mathcal{H}$, such that $\mathcal{H}$ is isomorphic with some dense topological subspace of $\tilde{\mathcal{H}}$. Usually the corresponding injection is viewed as an inclusion, such that $\mathcal{H}$ itself lies dense in $\tilde{\mathcal{H}}, \tilde{\mathcal{H}} = \overline{\mathcal{H}}$.

(8.7.3.1) Let the $\mathcal{R}$-left module $\mathcal{H}$ be free over a basis $\mathcal{B}$. Then the unique Hausdorff completion $\tilde{\mathcal{H}}$ of $\mathcal{H}$ is called topologically free over $\mathcal{B}$, and $\mathcal{B}$ is called topological basis of $\tilde{\mathcal{H}} = \overline{\mathcal{H}}$.

(8.7.4) Let $\mathcal{A}$ be an $\mathcal{R}$-submodule of a topological $\mathcal{R}$-left module $\mathcal{H}$, and $\phi : \mathcal{A} \longrightarrow \mathcal{H}'$ an $\mathcal{R}$-linear continuous map into a complete Hausdorff topological $\mathcal{R}$-left module $\mathcal{H}'$. Then ϕ is uniformly continuous, and there is a unique continuous extension $\overline{\phi} : \overline{\mathcal{A}} \longrightarrow \mathcal{H}'$ of ϕ, and moreover $\overline{\phi}$ is $\mathcal{R}$-linear and uniformly continuous.

(8.7.5) Consider a complete Hausdorff topological bimodule $\mathcal{H}$, over a complete Hausdorff topological commutative ring $\mathcal{R}$. Let the $\mathcal{R}$-tensor product $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ of finitely many copies of $\mathcal{H}$ be equipped with an appropriate Hausdorff topology, such that it becomes a topological bimodule over $\mathcal{R}$. The unique Hausdorff completion of $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ is called topological tensor product and denoted by

$$\mathcal{H}\overline{\otimes} \cdots \overline{\otimes}\mathcal{H} = \overline{\mathcal{H} \otimes \cdots \otimes \mathcal{H}},$$

if and only if the tensor product map: $\mathcal{H} \times \cdots \times \mathcal{H} \longrightarrow \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ is continuous. This property is assumed henceforth.

(8.7.5.1) For instance one may use the strongest topology on $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$, such that the above indicated multilinear tensor product map is continuous. For this purpose a subset of $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ is called open, if and only if it is the image of an open subset of $\mathcal{H} \times \cdots \times \mathcal{H}$.

(8.7.5.2) The Hilbert-tensor product of real or complex Hilbert spaces is not a topological tensor product.

(8.7.5.3) The $\mathcal{R}$-tensor products $\mathcal{H} \otimes (\mathcal{H}\overline{\otimes}\mathcal{H})$ and $(\mathcal{H}\overline{\otimes}\mathcal{H}) \otimes \mathcal{H}$ are also proposed as Hausdorff topological $\mathcal{R}$-bimodules. Their Hausdorff completions are then assumed to be isomorphic in the sense of $\mathcal{R}$-linear homeomorphisms:

$$\begin{aligned}(\mathcal{H}\overline{\otimes}\mathcal{H})\overline{\otimes}\mathcal{H} = \overline{(\mathcal{H}\overline{\otimes}\mathcal{H}) \otimes \mathcal{H}} &\longleftrightarrow \mathcal{H}\overline{\otimes}\mathcal{H}\overline{\otimes}\mathcal{H} \\ &\longleftrightarrow \overline{\mathcal{H} \otimes (\mathcal{H}\overline{\otimes}\mathcal{H})} = \mathcal{H}\overline{\otimes}(\mathcal{H}\overline{\otimes}\mathcal{H}).\end{aligned}$$

More generally such an associative property of topological tensor products is assumed for finitely many copies of $\mathcal{H}$.
Furthermore the flip $\tau : \mathcal{H} \otimes \mathcal{H} \ni a \otimes b \longleftrightarrow b \otimes a \in \mathcal{H} \otimes \mathcal{H}$ is assumed to be continuous, hence extended uniquely to a homeomorphism $\overline{\tau}$ of $\mathcal{H}\overline{\otimes}\mathcal{H}$ onto itself.

(8.7.5.4) Let $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$ be the direct sum of topologically closed $\mathcal{R}$-submodules. In this case one conveniently assumes, that the usual extension of this $\mathbf{Z}_2$-grading to the $\mathcal{R}$-tensor algebra over $\mathcal{H}$ is compatible with the Hausdorff completion of the tensor product in the sense, that

$$\mathcal{H}\overline{\otimes} \cdots \overline{\otimes}\mathcal{H} = \bigoplus_{\bar{z}_1,\ldots,\bar{z}_n \in \mathbf{Z}_2} \overline{(\mathcal{H}^{\bar{z}_1} \otimes \cdots \otimes \mathcal{H}^{\bar{z}_n})},$$

thereby establishing an according $\mathbf{Z}_2$-grading of the topological tensor product.

(8.7.6) Let $\mathcal{H}$ be both an algebra and a topological bimodule, over a commutative topological ring $\mathcal{R}$. $\mathcal{H}$ is called topological $\mathcal{R}$-algebra, if and only if it is also a topological ring.

(8.7.6.1) Let $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$ be the direct sum of topologically closed $\mathcal{R}$-submodules. $\mathcal{H}$ is called topological superalgebra over $\mathcal{R}$, if and only if it is $\mathbf{Z}_2$-graded as an algebra, which explicitly means that

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in \mathcal{H}^{\bar{x}}, b \in \mathcal{H}^{\bar{y}} : ab \in \mathcal{H}^{\overline{x+y}}.$$

(8.7.6.2) Let both $\mathcal{R}$ and $\mathcal{H}$ be complete. The $\mathbf{Z}_2$-graded tensor product $\mathcal{H}\hat{\otimes} \cdots \hat{\otimes}\mathcal{H}$ of topological superalgebras over $\mathcal{R}$ may be equipped with an appropriate uniform Hausdorff topology, such that the tensor product map becomes continuous. Moreover assume that the Hausdorff completion

$$\mathcal{H}\overline{\otimes} \cdots \overline{\otimes}\mathcal{H} = \overline{\mathcal{H}\hat{\otimes} \cdots \hat{\otimes}\mathcal{H}}$$

is compatible with the $\mathbf{Z}_2$-grading, such that it is again some topological $\mathcal{R}$-superalgebra, which is called topological $\mathbf{Z}_2$-graded tensor product of topological $\mathcal{R}$-superalgebras, with the $\mathcal{R}$-subalgebra $\mathcal{H}\hat{\otimes} \cdots \hat{\otimes}\mathcal{H}$. Again one assumes an according associative property of topological tensor products, in the sense of isomorphisms of complete Hausdorff topological $\mathcal{R}$-superalgebras.

(8.7.6.3) In case of a unital associative complete Hausdorff topological $\mathcal{R}$-superalgebra one assumes, that the structure mapping $\mu : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ is continuous, then being uniquely extended to $\overline{\mu} : \mathcal{H}\overline{\otimes}\mathcal{H} \longrightarrow \mathcal{H}$ by continuity. The structure map $\eta : \mathcal{R} \longrightarrow \mathcal{H}$ is obviously continuous. More generally one assumes, that the so-called tensor products of $\mathcal{R}$-linear mappings, for instance $T(\mu, \mu), T(\mu, id\ \mathcal{H}), T(\eta, id\ \mathcal{H})$, are continuous, hence continuously extended, in order to obtain commuting diagrams of uniformly continuous structure mappings of complete Hausdorff topological spaces. The canonical embeddings:

$$\mathcal{H} \ni a \longrightarrow e_{\mathcal{H}} \otimes \cdots \otimes e_{\mathcal{H}} \otimes a \otimes e_{\mathcal{H}} \otimes \cdots \otimes e_{\mathcal{H}} \in \mathcal{H}\overline{\otimes} \cdots \overline{\otimes}\mathcal{H}$$

are continuous, because the tensor product map is so by assumption.

(8.7.7) The tensor product of formal power series with homogeneous relations is equipped with a uniform Hausdorff topology and completed, thereby inserting implicitly the discrete topology of the coefficient ring $\mathcal{R}$. The corresponding tensor product map is continuous. If the indeterminates are labelled with $\mathbf{Z}_2$-degrees, and if the relations are $\mathbf{Z}_2$-homogeneous, such that their closed ideal is some $\mathbf{Z}_2$-graded subspace of the Hausdorff completion of the free $\mathcal{R}$-algebra over these indeterminates, then this $\mathbf{Z}_2$-degree is naturally extended from formal power series with relations to their topological $\mathbf{Z}_2$-graded tensor product, because the involved commutation relations of indeterminates are homogeneous.

(8.7.8) Since the relations of quantum algebras, as for instance $U_q(sl(2, \mathbf{C}))$, are inhomogeneous in the powers of generators, their ideal is blown up to the whole algebra by the closure with respect to powers of generators. Therefore formal power series with inhomogeneous relations must be defined artificially, for instance with respect to powers of the deformation parameter $h = \ln\ q$.

(8.7.8.1) Consider a unital associative algebra $\mathcal{F}$, for instance formal power series in finitely many indeterminates, over the commutative ring $\mathcal{R} := R[[h]]$ of formal power series in h, over a commutative ring R. The ideals

$$V_m(0) := h^m\mathcal{F} := \{h^m a; a \in \mathcal{F}\},\ \ m \in \mathbf{N}_0,\ \ h^0 := e_{\mathcal{R}},$$

constitute a countable basis of neighbourhoods of 0, which induces the so-called h-adic topology of $\mathcal{F}$. The neighbourhoods $\mathcal{U}$ of $b \in \mathcal{F}$ are defined by the condition, that

$$\exists m \in \mathbf{N}_0 : b + V_m(0) \subseteq \mathcal{U}.$$

One thereby establishes the topological $\mathcal{R}$-algebra $\mathcal{F}$. The corresponding tensor product map becomes continuous with respect to the h-adic topology of $\mathcal{F} \otimes \mathcal{F}$. The algebra structure map: $\mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{F}$ is continuous, because it is $\mathcal{R}$-linear. Assume

$$\bigcap_{m \in \mathbf{N}_0} V_m(0) = \{0\};$$

then $\mathcal{F}$ is of Hausdorff type.

(8.7.8.2) Let $\mathcal{J}$ be a closed ideal of $\mathcal{F}$. The factor algebra $\mathcal{H} := \mathcal{F}/\mathcal{J}$ is then equipped with the quotient topology, such that the canonical projection: $\mathcal{F} \ni a \longrightarrow a + \mathcal{J} \in \mathcal{H}$ becomes both continuous and open. Since the ideal is assumed to be closed in $\mathcal{F}$, the quotient topology is again of Hausdorff type. If $\mathcal{F}$ is complete, then $\mathcal{H}$ is complete too.

(8.7.8.3) Of course every $\mathcal{R}$-bimodule $\mathcal{M}$ can be equipped with the h-adic topology. An $\mathcal{R}$-linear map of such bimodules: $\mathcal{M} \longrightarrow \mathcal{M}'$ is continuous. More generally, $\mathcal{R}$-multilinear mappings of such bimodules are continuous with respect to the product topology.

(8.7.9) Let $\mathcal{H}$ be a complete Hausdorff topological $\mathcal{R}$-bimodule. $\mathcal{H}$ is called topological coalgebra over $\mathcal{R}$, if and only if there are $\mathcal{R}$-linear continuous mappings $\Delta : \mathcal{H} \longrightarrow \mathcal{H}\overline{\otimes}\mathcal{H}$ and $\varepsilon : \mathcal{H} \longrightarrow \mathcal{R}$, such that the following diagrams of uniformly continuous $\mathcal{R}$-linear maps are commuting.

(i) $\alpha \circ \overline{T(\Delta, id\ \mathcal{H})} \circ \Delta = \overline{T(id\ \mathcal{H}, \Delta)} \circ \Delta.$

(ii) $\forall a \in \mathcal{H}$:

$$\overline{T(\varepsilon, id\ \mathcal{H})} \circ \Delta(a) = e_{\mathcal{R}} \otimes a,$$
$$\overline{T(id\ \mathcal{H}, \varepsilon)} \circ \Delta(a) = a \otimes e_{\mathcal{R}}.$$

Here the overlined $\mathcal{R}$-linear mappings are defined as the corresponding $\mathcal{R}$-linear unique continuous extensions. The topologies of $\mathcal{R} \otimes \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{R}$ are identified with that of $\mathcal{H}$ itself, by means of the natural $\mathcal{R}$-linear bijections: $\mathcal{R} \otimes \mathcal{H} \longleftrightarrow \mathcal{H} \longleftrightarrow \mathcal{H} \otimes \mathcal{R}$, and $\mathcal{R}$ carries its own topology.

(8.7.9.1) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be topological coalalgebras over $\mathcal{R}$. An $\mathcal{R}$-linear continuous map $\phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called homomorphism of topological coalgebras over $\mathcal{R}$, if and only if the following diagrams of uniformly continuous mappings commute.

$$\overline{T(\phi, \phi)} \circ \Delta_1 = \Delta_2 \circ \phi, \ \varepsilon_2 \circ \phi = \varepsilon_1.$$

(8.7.9.2) In the case of formal power series, $\mathcal{R}$ is equipped with the discrete topology. Then the demanded continuity of ε means, that

$$\exists m \in \mathbf{N} : \varepsilon(\overline{V}_m(0)) = \{0_{\mathcal{R}}\}.$$

Here the zero-neighbourhood $\overline{V}_m(0)$ by definition contains formal power series with monomials of polynomial degree $\geq m$.

(8.7.10) Let the complete Hausdorff topological space $\mathcal{H}$ be both an associative unital topological algebra and topological coalgebra over $\mathcal{R}$, and assume the structure map μ to be continuous. $\mathcal{H}$ is called topological bialgebra over $\mathcal{R}$, if and only if Δ and ε are homomorphisms of unital associative $\mathcal{R}$-algebras, which equivalently means that $\overline{\mu}$ and η are homomorphisms of topological coalgebras over $\mathcal{R}$. Here the structure map: $(\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H}) \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is continuous, because the flip is so, and then extended uniquely to the corresponding Hausdorff completions.

(8.7.10.1) The topological bialgebra $\mathcal{H}$ over $\mathcal{R}$ is called topological Hopf algebra, if and only if there is an $\mathcal{R}$-linear continuous map $\sigma : \mathcal{H} \longrightarrow \mathcal{H}$, such that

$$\overline{\mu} \circ \overline{T(\sigma, id\ \mathcal{H})} \circ \Delta = \overline{\mu} \circ \overline{T(id\ \mathcal{H}, \sigma)} \circ \Delta = \eta \circ \varepsilon.$$

This unique antipode is an antihomomorphism both of algebras and of coalgebras over $\mathcal{R}$.

$$\sigma \circ \overline{\mu} = \overline{\mu} \circ \overline{\tau} \circ \overline{T(\sigma, \sigma)}, \quad \overline{T(\sigma\sigma)} \circ \Delta = \overline{\tau} \circ \Delta \circ \sigma.$$

(8.7.10.2) The topological coalgebra $\mathcal{H}$ over $\mathcal{R}$ is called $\mathbf{Z}_2$-graded, if and only if $\mathcal{H}$ is the direct sum of topologically closed $\mathcal{R}$-submodules, the Hausdorff completion of the $\mathcal{R}$-tensor product of finitely many copies of $\mathcal{H}$ is compatible with this $\mathbf{Z}_2$-grading, and both Δ and ε are even, with respect to the trivial grading of coefficients.

(8.7.10.3) The topological $\mathcal{R}$-bimodule $\mathcal{H}$ is called topological $\mathbf{Z}_2$-graded bialgebra over $\mathcal{R}$, if and only if it is both some topological superalgebra and $\mathbf{Z}_2$-graded topological coalgebra over $\mathcal{R}$, and both Δ and ε are homomorphisms of unital assocative $\mathcal{R}$-superalgebras, with respect to the Hausdorff completion of the $\mathbf{Z}_2$-graded tensor product $\mathcal{H}\hat{\otimes}\mathcal{H}$, which equivalently means that $\overline{\mu}$ and η are homomorphisms of $\mathbf{Z}_2$-graded coalgebras over $\mathcal{R}$. Here the natural coalgebra structure on $\mathcal{R}$ is used, and $\mathcal{R}$ is complete, such that $\mathcal{R}\otimes\mathcal{R}$ is also complete.

$$\mathcal{R} \ni r \xrightarrow[\Delta]{def} r \otimes e_{\mathcal{R}} \in \mathcal{R} \otimes \mathcal{R}, \quad \varepsilon := id\ \mathcal{R}; \quad \mathcal{R} \otimes \mathcal{R} \ni r \otimes s \longmapsto rs \in \mathcal{R}.$$

(8.7.10.4) The $\mathbf{Z}_2$-graded topological bialgebra $\mathcal{H}$ over $\mathcal{R}$ is called $\mathbf{Z}_2$-graded topological Hopf algebra, if and only if there is an even $\mathcal{R}$-linear continuous map $\sigma : \mathcal{H} \longrightarrow \mathcal{H}$ fulfilling the above postulate. This unique antipode σ then also fulfills the two antihomomorphism properties, inserting the unique continuous extension $\overline{\tau}$ of the $\mathbf{Z}_2$-graded flip τ.

(8.7.11) Let $\mathcal{H}$ be a $\mathbf{Z}_2$-graded topological Hopf algebra over $\mathcal{R}$. $\mathcal{H}$ is called quasitriangular, if and only if there is an element $R \in \mathcal{H}\overline{\otimes}\mathcal{H}$, from the Hausdorff completion of $\mathcal{H}\hat{\otimes}\mathcal{H}$, such that the following conditions are fulfilled.

$$\text{(i)} \quad \exists R^{-1} \in \mathcal{H}\overline{\otimes}\mathcal{H} : RR^{-1} = R^{-1}R = e_{\mathcal{H}} \otimes e_{\mathcal{H}},$$

with the unit $e_{\mathcal{H}}$ of $\mathcal{H}$, the inverse then being unique.

$$\begin{aligned}\text{(ii)} \quad & \mathcal{H}\overline{\otimes}\mathcal{H}\overline{\otimes}\mathcal{H} \ni R_{13}R_{23} \longleftrightarrow \overline{T(\Delta, id\,\mathcal{H})}(R) \in (\mathcal{H}\overline{\otimes}\mathcal{H})\overline{\otimes}\mathcal{H}, \\ & \mathcal{H}\overline{\otimes}\mathcal{H}\overline{\otimes}\mathcal{H} \ni R_{13}R_{12} \longleftrightarrow \overline{T(id\,\mathcal{H}, \Delta)}(R) \in \mathcal{H}\overline{\otimes}(\mathcal{H}\overline{\otimes}\mathcal{H}),\end{aligned}$$

with the usual notation. Define $R_{12} \longleftrightarrow R \otimes e_{\mathcal{H}}$, $R_{23} \longleftrightarrow e_{\mathcal{H}} \otimes R$, and, denoting

$$R =: \sum_{k=0}^{\infty} R'_k \otimes R''_k, \; R_{13} := \sum_{k=0}^{\infty} R'_k \otimes e_{\mathcal{H}} \otimes R''_k.$$

$$\text{(iii)} \quad \forall a \in \mathcal{H} : \mathcal{H}\overline{\otimes}\mathcal{H} \ni \Delta(a) \underset{\overline{\tau}}{\longleftrightarrow} R\Delta(a)R^{-1} \in \mathcal{H}\overline{\otimes}\mathcal{H},$$

with the unique continuous extension $\overline{\tau}$ of the $\mathbf{Z}_2$-graded flip τ.

(8.7.11.1) One then finds, that

$$\begin{aligned}&\overline{T(\varepsilon, id\,\mathcal{H})}(R) = e_{\mathcal{R}} \otimes e_{\mathcal{H}}, \; \overline{T(id\,\mathcal{H}, \varepsilon)}(R) = e_{\mathcal{H}} \otimes e_{\mathcal{R}}. \\ &\overline{T(\sigma, id\,\mathcal{H})}(R) = R^{-1}, \; \overline{T(id\,\mathcal{H}, \sigma)}(R^{-1}) = R, \\ &\overline{T(\sigma, \sigma)}(R) = R, \; \overline{T(\sigma, \sigma)}(R^{-1}) = R^{-1}.\end{aligned}$$

Moreover R fulfills the $\mathbf{Z}_2$-graded quantum Yang-Baxter equation.

(8.7.11.2) The square of the antipode σ is generated by an invertible element $u \in \mathcal{H}$, which fulfills similar conditions as in the polynomial case, such that $\forall a \in \mathcal{H}$:

$$\begin{aligned}&\sigma^2(a) = u\, a\, u^{-1}; \\ &u := \sum_{k \in \mathbf{N}_0} (-1)^{z'_k z''_k} \sigma(R''_k) R'_k, \; u^{-1} = \sum_{k \in \mathbf{N}_0} (-1)^{z'_k z''_k} R''_k \sigma^2(R'_k), \\ &\Delta(u) = (\overline{\tau}(R)R)^{-1}(u \otimes u), \\ &\sigma(u) = \sum_{k \in \mathbf{N}_0} R'_k \sigma(R''_k), \; \sigma(u^{-1}) = \sum_{k \in \mathbf{N}_0} \sigma^2(R'_k) R''_k,\end{aligned}$$

with the degrees z'_k, z''_k of R'_k and R''_k, respectively.
Hence σ is bijective. The element $u\sigma(u) = \sigma(u)u$ of $\mathcal{H}$ is central.

8.8 q-Deformation of $sl(2, C)$

The universal enveloping algebra $U(A_1)$ of the Lie algebra $sl(2, \mathbf{C}) \equiv A_1$ over $\mathbf{C}$ is q-deformed to some quasitriangular topological Hopf algebra $U_q(A_1)$, over the ring $\mathcal{R} := \mathbf{C}[[h]]$ of complex formal power series in $h = \ln q$. The according perturbation of cocommutativity is described by the so-called universal R-matrix. An appropriate topological basis of $U_q(A_1)$ over $\mathcal{R}$ is provided by a family of monomials in the generators, the Poincaré series of which coincides with that of the universal enveloping algebra $U(A_1)$.

(8.8.1.1) Using the conventional complex matrix exponential function,

$$\forall M \in Mat(n, \mathbf{C}) : \ [M]_q := \frac{\exp(M \ln q) - \exp(-M \ln q)}{q - q^{-1}} \underset{\mathbf{D} \ni q \longrightarrow 1}{\longrightarrow} M.$$

Here the principal branch of ln is used with the cut $]-\infty, 0]$, and

$$\ln q \in \mathbf{R} \text{ for } q > 0; \ \mathbf{D} := \mathbf{C} \setminus (]-\infty, 0] \cup \{1\}).$$

(8.8.1.2) Consider formal power series ϕ in finitely many indeterminates over $\mathcal{R} := \mathbf{C}[[h]]$, with relations; the latter are introduced by factorizing with respect to their smallest closed ideal, with respect to the h-adic topology.

$$[\phi]_q := \frac{q^\phi - q^{-\phi}}{q - q^{-1}}, \ q := e^h, \ q^\phi := \exp(h\phi).$$

(8.8.1.3) Let $q \in \mathbf{D}, n \in \mathbf{N}$, and assume q as generic, i.e., not any root of 1.

$$[n]_q! := [1]_q \cdots [n]_q \underset{q \longrightarrow 1}{\longrightarrow} n!, \ [1]_q = 1, \ [0]_q! := 1.$$

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]_q!}{[k]_q![n-k]_q!} \frac{q^n \cdots q^{n-k+1}}{q^k \cdots q} \\
&= \frac{(q^{2n} - 1) \cdots (q^{2(n-k+1)} - 1)}{(q^{2k} - 1) \cdots (q^2 - 1)} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q,
\end{aligned}$$

$$\begin{bmatrix} n \\ n \end{bmatrix}_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q := 1,$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \frac{q^{2n} - 1}{q^2 - 1} = 1 + q^2 + q^4 + \cdots + q^{2(n-1)},$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q q^{2k}, \text{ for } k = 1, \ldots, n.$$

For any two elements a, b of a complex associative algebra $\mathcal{A}$ one easily calculates, that $\forall n \in \mathbf{N}$:

$$ba = q^2 ab \Longrightarrow (a+b)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}.$$

The above expressions may also be understood as complex formal power series in h, $e^h =: q$, then using these series as coefficients of $\mathcal{A}$.

(8.8.1.4) Let again $q \in \mathbf{D}$ be generic.

$$\forall n \in \mathbf{N} : \{n\}_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \underset{\mathbf{D} \ni q \longrightarrow 1}{\longrightarrow} n,$$
$$\{n\}_q! := \{1\}_q \cdots \{n\}_q \underset{q \longrightarrow 1}{\longrightarrow} n!, \ \{1\}_q = 1, \ \{0\}_q! := 1.$$

$$\forall M \in Mat(n, \mathbf{C}) : \exp_q(M) := \sum_{n=0}^{\infty} \frac{M^n}{\{n\}_q!} \underset{q \longrightarrow 1}{\longrightarrow} \exp(M).$$

(8.8.1.5) An appropriate q-deformed exponential function is defined, inserting formal power series in finitely many indeterminates over $\mathcal{R}$, with relations.

$$\exp_q(\phi) := \sum_{n=0}^{\infty} \frac{\phi^n}{\{n\}_q!} = e_{\mathcal{R}} + \phi + \frac{\phi^2}{1+q} + \cdots, \ q := e^h.$$

For such objects ϕ, ψ one then verifies the following implication.

$$q\phi\psi = \psi\phi \Longrightarrow \exp_q(\phi + \psi) = \exp_q(\phi)\exp_q(\psi).$$

(8.8.2.1) Consider the unital associative algebra of formal power series in three generators $X^\pm, H$, over $\mathcal{R} := \mathbf{C}[[h]]$. The q-deformation of the universal enveloping algebra of $A_1 \equiv sl(2, \mathbf{C})$ is defined by factorization with respect to the smallest closed ideal $\overline{\mathcal{J}_q}$, in the h-adic topology, of the set of relations

$$\mathcal{S}_q := \{[H, X^\pm] \mp 2X^\pm, \ [X^+, X^-] - [H]_q\},$$
$$U_q(A_1) := \mathcal{R}\langle\langle\{X^\pm, H\}\rangle\rangle / \overline{\mathcal{J}_q}.$$

(8.8.2.2) With the $\mathcal{R}$-linear uniformly continuous structure mappings

$$\Delta : U_q(A_1) \longrightarrow \overline{U_q(A_1) \otimes U_q(A_1)}, \ \varepsilon : U_q(A_1) \longrightarrow \mathcal{R},$$
$$\sigma : U_q(A_1) \longrightarrow U_q(A_1)$$

being defined such that:

$$H \underset{\Delta}{\longrightarrow} H \otimes I + I \otimes H, \ H \underset{\varepsilon}{\longrightarrow} 0, \ H \underset{\sigma}{\longrightarrow} -H,$$
$$X^\pm \underset{\Delta}{\longrightarrow} X^\pm \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes X^\pm, \ X^\pm \underset{\varepsilon}{\longrightarrow} 0, \ X^\pm \underset{\sigma}{\longrightarrow} -q^{\pm 1} X^\pm,$$

with the algebra unit I, $U_q(A_1)$ becomes some topological Hopf algebra over $\mathcal{R}$, with respect to the h-adic topology of formal power series with relations. Here and henceforth the equivalence classes of generators are denoted by the same letters as the generators themselves, for instance shortwriting H instead of $H + \overline{\mathcal{J}_q}$.

(8.8.2.3) Using the Baker-Campbell-Hausdorff formula one easily finds, that $\forall c \in \mathcal{R}$:

$$\exp(cH) \underset{\Delta}{\longrightarrow} \exp(cH) \otimes \exp(cH), \; \exp(cH) \underset{\varepsilon}{\longrightarrow} 1, \; \exp(cH) \underset{\sigma}{\longrightarrow} \exp(-cH).$$

(8.8.2.4) The so-called classical limit of $q \longrightarrow 1$ is described by an isomorphism of unital associative complex algebras:

$$U_q(A_1)/hU_q(A_1) \longleftrightarrow E(A_1),$$

onto the universal enveloping algebra of A_1.

(8.8.3) A unital subalgebra of $Mat(d, \mathbf{C}), d \in \mathbf{N}$, is called finite-dimensional complex representation of the quantum algebra $U_q(A_1)$, if and only if it is polynomially generated by three complex matrices $X^{\pm}, H$, which fulfill the above relations, thereby inserting a complex deformation parameter $q \in \mathbf{D}$.

(8.8.3.1) Under the special condition of generic q, which by definition means that $\forall n \in \mathbf{N} : q^n \neq 1$, all the finite-dimensional complex irreducible representations of $U_q(A_1)$ are obtained by an appropriate deformation of the well-known family of irreducible representations of A_1. Denote by

$$v^j_m, \; m = j, j-1, \ldots, -j, \; \text{for } j = \frac{1}{2}, 1, \frac{3}{2}, \ldots,$$

the Cartesian unit vectors of $\mathbf{C}^{2j+1}$, say with the component 1 of v^j_m at the position $j - m + 1$. One then obtains the following irreducible representation on $Mat(2j+1, \mathbf{C})$.

$$Hv^j_m = 2m\, v^j_m, \; m = j, j-1, \ldots, -j, \; X^{\pm}v_{\pm j} = 0,$$
$$X^{\pm}v^j_m = ([j \mp m]_q [j \pm m + 1]_q)^{1/2} v^j_{m\pm 1} \text{ for } m \neq \pm j.$$

In the limit $\mathbf{D} \ni q \longrightarrow 1$ one obtains the irreducible representations of A_1.

(8.8.3.2) Straightforward calculation yields the following Casimir operator $C \in U_q(A_1)$.

$$D := [\frac{1}{2}(H + I)]_q, \; [D^2, X^+] = [H]_q X^+, \; [D^2, X^-] = -X^-[H]_q,$$

$$C := D^2 + X^-X^+ - \frac{1}{4}I, \; [C, X^{\pm}] = [C, H] = 0.$$

For generic values of $q \in \mathbf{D}$, for $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ and $m = j, j-1, \ldots, -j$,

$$Cv^j_m = c^j_m(q) v^j_m, \; c^j_m(q) := \left([j + \frac{1}{2}]_q\right)^2 - \frac{1}{4} \underset{q \longrightarrow 1}{\longrightarrow} j(j+1).$$

(8.8.3.3) For $q \in \mathbf{D}$, the lowest-dimensional complex representation of $U_q(A_1)$, which may be called fundamental one, is provided by the Pauli matrices themselves, because of $[\sigma^3]_q = \sigma^3$.

$$X^+ := \sigma^+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; X^- := \sigma^- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; H := \sigma^3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(8.8.3.4) For $j = 1$ and $q^4 \neq 1$, the according irreducible representation is provided by

$$H := 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad X^+ := ([2]_q)^{1/2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and the images of $X^\pm$ being mutually transposed complex matrices.

(8.8.3.5) Let $q \in \mathbf{D}$. The above q-deformed irreducible representations, according to some choice of $j \in \left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$, can be constructed, if and only if $\forall n \in \{1, 2, \ldots, 2j\} : q^n \neq \pm 1$.

(8.8.3.6) Without inserting any complex number for the indeterminate h, a finite-dimensional representation of $U_q(A_1)$ is defined as homomorphism of unital associative $\mathcal{R}$-algebras

$$\rho : U_q(A_1) \longrightarrow Mat(d, \mathbf{C}[[h]]).$$

Obviously ρ is continuous with respect to the h-adic topology.

(8.8.4) The topological Hopf algebra $U_q(A_1)$ over $\mathcal{R}$ is quasitriangular with respect to the following so-called universal R-matrix.

$$\begin{aligned} R &:= q^{\frac{1}{2}H \otimes H} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{\frac{n(n-1)}{2}} \left(q^{\frac{1}{2}H} X^+ \otimes q^{-\frac{1}{2}H} X^-\right)^n \\ &\in \overline{U_q(A_1) \otimes U_q(A_1)}, \\ R^{-1} &= T(\sigma, id)(R) \\ &= q^{-\frac{1}{2}H \otimes H} \sum_{n=0}^{\infty} \frac{(1 - q^2)^n}{[n]_q!} q^{-\frac{n(n-1)}{2}} \left(q^{-\frac{1}{2}H} X^+ \otimes q^{+\frac{1}{2}H} X^-\right)^n. \end{aligned}$$

An insertion of this formal power series with relations R into the defining properties of quasitriangularity yields some recursions for the coefficients, which are fulfilled just by the above expression. The equation $T(\sigma, \sigma)(R) = R$ yields some similar expansion of R.

(8.8.4.1) Inserting the Pauli matrices for $j = \frac{1}{2}$ and $q \in \mathbf{D}$, one obtains the following well-known solution of QYBE.

$$R(q) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{bmatrix} \right)$$
$$\left(I_2 \otimes I_2 + (q - q^{-1})\sigma^+ \otimes \sigma^-\right)$$
$$\longleftrightarrow q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \in Mat(4, \mathbf{C}).$$

(8.8.5) The Hopf algebra $SL_q(2, \mathbf{C})$ may also be considered with respect to coefficients $\in \mathcal{R} := \mathbf{C}[[h]]$, $e^h =: q$. Then the topological Hopf algebras $U_q(A_1)$ and $SL_q(2, \mathbf{C})$ are dual with respect to the $\mathcal{R}$-bilinear form $\langle | \rangle$, which is defined by the values

$$\left\langle X^\pm \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle = \sigma^\pm, \quad \left\langle H \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle = \sigma^3,$$

and uniquely extended from polynomials in $X^\pm, H$ to formal power series over $\mathcal{R}$, by continuity with respect to the h-adic topology, the latter being applied also to $SL_q(2, \mathbf{C})$.

(8.8.6) The structure mappings of $U_q(A_1)$ can be rewritten using only polynomials over $\mathcal{R}$, via an appropriate homomorphism λ of unital associative $\mathcal{R}$-algebras. Let $E_q(A_1)$ be the $\mathcal{R}$-Hopf algebra, which is generated by four elements E, F, K, L, with the following relations and costructure maps.

$$KL = LK = I, \; KE = q^2 EK, \; KF = q^{-2} FK,$$
$$EF - FE = \frac{q^2}{q^2 - 1}(K - L);$$
$$E \xrightarrow[\Delta]{} E \otimes I + L \otimes E, \; F \xrightarrow[\Delta]{} F \otimes K + I \otimes F,$$
$$K \xrightarrow[\Delta]{} K \otimes K, \; L \xrightarrow[\Delta]{} L \otimes L,$$
$$\varepsilon(K) = \varepsilon(L) = 1, \; \varepsilon(E) = \varepsilon(F) = 0,$$
$$K \xrightarrow[\sigma]{} L \xrightarrow[\sigma]{} K, \; E \xrightarrow[\sigma]{} -KE, \; F \xrightarrow[\sigma]{} -FL.$$

The above indicated homomorphism $\lambda : E_q(A_1) \longrightarrow U_q(A_1)$ is established such that:

$$E \longrightarrow X^+ q^{-\frac{1}{2}H}, \; F \longrightarrow X^- q^{\frac{1}{2}H}, \; K \longrightarrow q^H, \; L \longrightarrow q^{-H}.$$

Then obviously

$$\Delta \circ \lambda = T(\lambda, \lambda) \circ \Delta, \; \varepsilon \circ \lambda = \varepsilon, \; \sigma \circ \lambda = \lambda \circ \sigma,$$

just not distinguishing between the structure maps of $U_q(A_1)$ and $E_q(A_1)$ by notation. Note that λ is continuous with respect to the h-adic topology of formal power series with relations. Remembering the classical theory of Lie algebras, the generators of $U_q(A_1)$, or of $E_q(A_1)$, may be called Chevalley generators.

(8.8.6.1) Obviously the above structure mappings can also be used in order to construct a complex Hopf algebra, which may be denoted by $E_q(A_1)$ too, for $q \in \mathbf{D}$. In this case of complex coefficients, complex representations are defined as homomorphisms of unital associative complex algebras. Nevertheless, if one aims at an expression of the universal R-matrix, which works for every finite-dimensional complex representation, one is forced to an appropriate completion procedure, for instance with respect to the h-adic topology.

(8.8.7) An involutive ring isomorphism of $\mathbf{C}[[h]]$ is provided by: $h \longleftrightarrow -h$, and complex conjugation. Inserting this involution of $\mathcal{R}$, the topological Hopf algebras $U_q(A_1)$ and $E_q(A_1)$ can be equipped with appropriate star operations, such that:

$$X^+ \underset{*}{\longleftrightarrow} X^-, \ H \underset{*}{\longleftrightarrow} H; \ K \underset{*}{\longleftrightarrow} L, \ E \underset{*}{\longleftrightarrow} q^{-1}F.$$

These idempotent bijections of $U_q(A_1)$ and $E_q(A_1)$ are compatible with the corresponding structure mappings, such that:

$$* \circ \mu = \mu \circ *, \quad * \circ \eta = \eta \circ *, \quad * \circ \Delta = \Delta \circ *, \quad * \circ \varepsilon = \varepsilon \circ *, \quad * \circ \sigma = \sigma \circ *,$$

with $*$ also denoting the involution of $\mathcal{R}$ for the moment. Obviously this star operation on $U_q(A_1)$ is uniformly continuous with respect to the h-adic topology of formal power series with relations. Note that these star operations are lifted to $\overline{U_q(A_1) \otimes U_q(A_1)}$ and $E_q(A_1) \otimes E_q(A_1)$ as $\tau \circ T(*, *)$ with the flip τ, extending by continuity in the former case, and again denoted by $*$.

(8.8.7.1) For generic q and $|q| = 1$, the irreducible representations of $U_q(A_1)$, or of $E_q(A_1)$, by complex matrices yield according representations of these star operations by transposition of their entries.

(8.8.8) Correspondingly one defines some star operation on the $\mathcal{R}$-Hopf algebra $SL_q(2, \mathbf{C})$, which commutes with all the structure mappings.

$$a^* = a, \ b^* = c, \ c^* = b, \ d^* = d.$$

$$\forall x \in U_q(A_1), y \in SL_q(2, \mathbf{C}) : \langle x^* | y^* \rangle = \langle x | y \rangle.$$

(8.8.9) The topological Hopf algebra $U_q(A_1)$ over $\mathcal{R}$ is quasitriangular with respect to the universal R-matrix

$$\begin{aligned} R = T(\sigma,\sigma)(R) &= \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(n-1)}{2}} (E \otimes F)^n q^{\frac{1}{2}H\otimes H} \\ &= q^{\frac{1}{2}H\otimes H} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(n-1)}{2}} (KE \otimes FL)^n . \end{aligned}$$

$$\begin{aligned} R^{-1} = T(\sigma, id)(R) &= \sum_{n=0}^{\infty} (-1)^n \frac{(1-q^{-2})^n}{[n]_q!} q^{-\frac{n(n-1)}{2}} (KE \otimes FL)^n q^{-\frac{1}{2}H\otimes H} \\ &= T(\sigma,\sigma)(R^{-1}) = q^{-\frac{1}{2}H\otimes H} \sum_{n=0}^{\infty} (-1)^n \frac{(1-q^{-2})^n}{[n]_q!} q^{-\frac{n(n-1)}{2}} (E \otimes F)^n . \end{aligned}$$

(8.8.10) For $q \in \mathbf{D}$, an appropriate $\mathbf{C}$-linear basis of $E_q(A_1)$ is provided by the following Poincaré-Birkhoff-Witt-like family of monomials. Consider the complex Hopf subalgebra $E_q(\mathcal{B}^+)$ of $E_q(A_1)$, which is generated by E, K, L, thereby q-deforming the corresponding Borel subalgebra of A_1.

$$\{E^r K^z; r \in \mathbf{N}_0, z \in \mathbf{Z}\} \underset{free\ over\ \mathbf{C}}{\longrightarrow} E_q(\mathcal{B}^+).$$

The complex subalgebra $E_q(\mathcal{B}^-)$ of $E_q(A_1)$ is $\mathbf{C}$-spanned correspondingly, just replacing E by F. Hence one obtains a basis $\{E^r F^s K^z; r, s \in \mathbf{N}_0; z \in \mathbf{Z}\}$ of the complex vector space $E_q(A_1)$.

(8.8.10.1) Denote by $U_q(\mathcal{B}^+)$ the subalgebra of $U_q(A_1)$, which is generated by X^+ and H. Then every vector $Y \in U_q(\mathcal{B}^+)$ can be written as formal power series with relations,

$$Y = \sum_{r,x\in\mathbf{N}_0} c_{rx} (X^+)^r H^x, \text{ with unique coefficients } c_{rx} \in \mathcal{R}.$$

The vectors of the subalgebra $U_q(\mathcal{B}^-)$ of $U_q(A_1)$, which is generated by X^- and H, are similarly expanded uniquely to powers of X^- and H. Therefore the set $\{(X^+)^r (X^-)^s H^x; r, s, x \in \mathbf{N}_0\}$ is some topological basis of the topologically free $\mathbf{C}[[h]]$-bimodule $U_q(A_1)$, with respect to the h-adic topology.

(8.8.10.2) An inductive proof of an according basis property over $\mathbf{C}[[h]]$ was performed by M. Rosso for the q-deformation of $A_m, m \in \mathbf{N}$, and used in order to construct the universal R-matrix in this more general case. The rather complicated $\mathcal{R}$-linear structure of q-deformation $U_q(\mathcal{L})$ of an arbitrary simple finite-dimensional complex Lie algebra $\mathcal{L}$ is illuminated by means of an appropriate q-deformation of the corresponding Weyl group, yielding the group of so-called Lusztig automorphisms, which are investigated in one of the following chapters.

(8.8.11) The Poincaré series of an associative unital algebra $\mathcal{A}$ of generators $X_i, i \in I$, with relations, over a field, or more generally over a commutative ring $\mathcal{R}$ of coefficients, in the latter case $\mathcal{A}$ being assumed to be free over $\mathcal{R}$, is defined as the sequence of dimensions $d_p, p \in \mathbf{N}_0$, of homogeneous submodules with respect to the sum of powers of generators. Equipping the generators with the degree 1, one establishes a natural $\mathbf{N}_0$-grading of

$$\mathcal{A} = \bigoplus_{p \in \mathbf{N}_0} \mathcal{A}_p, \quad dim\ \mathcal{A}_p =: d_p,$$

with the degree $p_1 + \cdots + p_n$ of monomials $X_{i_1}^{p_1} \cdots X_{i_n}^{p_n}, i_1, \ldots, i_n \in I$.

(8.8.11.1) The universal enveloping algebras of simple finite-dimensional complex Lie algebras are q-deformed in such a manner, that their Poincaré series remain invariant.

(8.8.12) The so-called quantum double construction enables one to construct the universal R-matrix of $U_q(\mathcal{L})$ from an arbitrary topological basis of the q-deformation $U_q(\mathcal{B}^+)$ of the Borel subalgebra $\mathcal{B}^+$ of a simple finite-dimensional complex Lie algebra $\mathcal{L}$. Here $\mathcal{B}^+$ is generated as complex vector space by the basis, which consists of the positive root vectors and toral generators of $\mathcal{L}$.

8.9 q-Deformation of Simple Finite-Dimensional Complex Lie Algebras

(8.9.1) Let $\mathcal{L}$ be a simple finite-dimensional complex Lie algebra with the Cartan matrix $[\Gamma_{kl}; k, l = 1, \ldots, m]$, and denote by $d_k \in \mathbf{N}, k = 1, \ldots, m$, the integers such that

$$\forall_1^m k, l : d_k \Gamma_{kl} = d_l \Gamma_{lk} =: B_{kl},$$

and the greatest common divisor of $\{d_1, \ldots, d_m\}$ is 1, leading to the following table. $[B_{kl}; k, l = 1, \ldots, m]$ is called symmetrized Cartan matrix.

$\mathcal{L}$	d_1	d_2	$\ldots$		d_{m-1}	d_m
$A_m, m \in \mathbf{N}$	1	1	$\ldots$		1	1
$B_m, m \geq 2$	2	2	$\ldots$		2	1
$C_m, m \geq 3$	1	1	$\ldots$		1	2
$D_m, m \geq 4$	1	1	$\ldots$		1	1
E_6	1	1	$\ldots$		1	1
E_7	1	1	$\ldots$		1	1
E_8	1	1	$\ldots$		1	1
F_4	1	2	1	1		
G_2	1	3				

(8.9.1.1) The q-deformation $U_q(\mathcal{L})$ of the universal enveloping algebra of $\mathcal{L}$ is defined as the topological Hopf algebra of formal power series in generators $X_k^{\pm}$ and H_k, $k = 1, \ldots, m$, over $\mathcal{R} := \mathbf{C}[[h]]$, with the following relations and costructure maps. $\forall_1^m k, l$:

$$q_k := q^{d_k} = e^{hd_k},\ q := e^h,$$

$$[H_k, H_l] = 0,\ [H_k, X_l^{\pm}] = \pm\Gamma_{kl}X_l^{\pm},$$

$$[X_k^+, X_l^-] = \delta_{kl}[H_k]_{q_k} = \delta_{kl}\frac{q_k^{H_k} - q_k^{-H_k}}{q_k - q_k^{-1}},$$

$$\sum_{j=0}^{1-\Gamma_{kl}} (-1)^j \frac{[1-\Gamma_{kl}]_{q_k}!}{[j]_{q_k}![1-\Gamma_{kl}-j]_{q_k}!} \left(X_k^{\pm}\right)^j X_l^{\pm} \left(X_k^{\pm}\right)^{1-\Gamma_{kl}-j} = 0 \text{ for } k \neq l;$$

$$H_k \xrightarrow[\Delta]{} H_k \otimes I + I \otimes H_k,\ H_k \xrightarrow[\varepsilon]{} 0,\ H_k \xrightarrow[\sigma]{} -H_k,$$

$$X_k^{\pm} \xrightarrow[\Delta]{} X_k^{\pm} \otimes q_k^{\frac{1}{2}H_k} + q_k^{-\frac{1}{2}H_k} \otimes X_k^{\pm},\ X_k^{\pm} \xrightarrow[\varepsilon]{} 0,\ X_k^{\pm} \xrightarrow[\sigma]{} -q_k^{\pm 1}X_k^{\pm}.$$

Here the relations are introduced by factorizing with respect to their closed ideal in the h-adic topology.

(8.9.1.2) In the special case of $\mathcal{L} := A_m \equiv sl(m+1, \mathbf{C})$, $m \in \mathbf{N}$, the q-deformed Serre relations read, $\forall_1^m k \neq l$:

$$X_l^{\pm}\left(X_k^{\pm}\right)^2 - (q+q^{-1})X_k^{\pm}X_l^{\pm}X_k^{\pm} + \left(X_k^{\pm}\right)^2 X_l^{\pm} = 0 \text{ for } |k-l| = 1,$$
$$X_k^{\pm}X_l^{\pm} - X_l^{\pm}X_k^{\pm} = 0 \text{ for } |k-l| \geq 2.$$

(8.9.1.3) Using the involution of $\mathcal{R}$, which is due to: $h \longleftrightarrow -h$ and complex conjugation, an appropriate star operation on $U_q(\mathcal{L})$ can be defined, such that

$$\forall_1^m k : X_k^+ \underset{*}{\longleftrightarrow} X_k^-,\ H_k \underset{*}{\longleftrightarrow} H_k,$$

which is continuous in the sense of the h-adic topology of formal power series with relations.

(8.9.2) Quantum matrices and quantum coordinates may also be considered with respect to pairwise commuting indeterminates as deformation parameters.
The topological bialgebra $U_q(A_m)$ over $\mathcal{R}$ is in duality with the $\mathcal{R}$-bialgebra $Mat_q(m+1, \mathbf{C}) \equiv Mat_q(m+1, 0, \mathbf{C})$, if and only if the deformation parameters of the latter are chosen such that

$$\forall_1^{m+1} i, j : q_{ii} := e_{\mathcal{R}},\ q_{ij} = q_{ji}^{-1} \neq 0,\ q_{ij} := q \in \mathbf{C}[[h]] \text{ for } i < j.$$

The corresponding $\mathcal{R}$-bilinear form is defined such that $\forall_1^m k, \forall_1^{m+1} i, j$:

$$\langle X_k^+|a_{ij}\rangle = \langle X_k^-|a_{ji}\rangle = \delta_{ik}\delta_{j,k+1},$$
$$\langle H_k|a_{ij}\rangle = \delta_{ik}\delta_{jk} - \delta_{i,k+1}\delta_{j,k+1},$$
$$\langle I|a_{ij}\rangle = \delta_{ij},\ \langle X_k^\pm|e\rangle = \langle H_k|e\rangle = 0,$$

with the corresponding units being denoted by I and e, respectively, and continuously extended by means of the h-adic topology.

(8.9.2.1) The complex bialgebra $Mat_q(m+1, \mathbf{C})$ can be equipped with some star operation, such that $\forall_1^{m+1} i,j : a_{ij}^* = a_{ji}$, under the assumption that $\forall_1^{m+1} i,j : |q_{ij}| = 1$.

(8.9.2.2) With the above choice of deformation parameters aiming at duality, and inserting the indicated involution of $\mathcal{R}$, the star operations on $U_q(A_m)$ and $Mat_q(m+1, \mathbf{C})$ are dual in the sense, that

$$\forall x \in U_q(A_m), a \in Mat_q(m+1, \mathbf{C}) : \langle x^*|a^*\rangle = \langle x|a\rangle.$$

(8.9.3) The above relations can be transformed to polynomials over $\mathcal{R}$, by means of an appropriate homomorphism of unital associative $\mathcal{R}$-algebras $\lambda : E_q(\mathcal{L}) \longrightarrow U_q(\mathcal{L})$. Let $E_q(\mathcal{L})$ be the $\mathcal{R}$-Hopf algebra, which is generated by the set $\{K_k, L_k, E_k, F_k; k = 1, \ldots, m\}$, with the following relations and costructure maps. $\forall_1^m k, l$:

$$[K_k, K_l] = 0,\ K_k L_k = L_k K_k = I,$$
$$K_k E_l = q_k^{\Gamma_{kl}} E_l K_k,\ \ K_k F_l = q_k^{-\Gamma_{kl}} F_l K_k,$$
$$[E_k, F_l] = \delta_{kl} \frac{q_k^2}{q_k^2 - 1}(K_k - L_k),$$
$$(ad^+ E_k)^{1-\Gamma_{kl}}(E_l) = (ad^-\ F_k)^{1-\Gamma_{kl}}(F_l) = 0 \text{ for } k \neq l;$$

$$E_k \xrightarrow[\Delta]{} E_k \otimes I + L_k \otimes E_k,\ \ F_k \xrightarrow[\Delta]{} F_k \otimes K_k + I \otimes F_k,$$
$$\varepsilon(E_k) = \varepsilon(F_k) = 0,$$
$$K_k \xrightarrow[\Delta]{} K_k \otimes K_k,\ \ L_k \xrightarrow[\Delta]{} L_k \otimes L_k,\ \ \varepsilon(K_k) = \varepsilon(L_k) = 1,$$
$$K_k \xleftrightarrow[\sigma]{} L_k,\ \ E_k \xrightarrow[\sigma]{} -K_k E_k,\ \ F_k \xrightarrow[\sigma]{} -F_k L_k.$$

(8.9.3.1) Here the following two q-deformed adjoint representations are used. Denoting:

$$E_q(\mathcal{L}) \ni X \longrightarrow \Delta(X) =: \sum_{l=1}^{L(X)} X_l' \otimes X_l'' \in E_q(\mathcal{L}) \otimes E_q(\mathcal{L}),$$

$$E_q(\mathcal{L}) \ni X \longrightarrow ad^+X : E_q(\mathcal{L}) \ni Y \underset{def}{\longrightarrow} \sum_{l=1}^{L(X)} X_l' Y \sigma(X_l'') \in E_q(\mathcal{L}),$$

$$E_q(\mathcal{L}) \ni X \longrightarrow ad^-X : E_q(\mathcal{L}) \ni Y \underset{def}{\longrightarrow} \sum_{l=1}^{L(X)} X_l'' Y \sigma^{-1}(X_l') \in E_q(\mathcal{L}).$$

Obviously

$$\forall X \in E_q(\mathcal{L}) : ad^-X = \sigma^{-1} \circ ad^+X \circ \sigma.$$

Especially one finds, that $\forall_1^m k, \forall Y \in E_q(\mathcal{L})$:

$$ad^+E_k(Y) := E_kY - L_kYK_kE_k, \; ad^-F_k(Y) := F_kY - K_kYL_kF_k.$$

Moreover $ad^\pm : E_q(\mathcal{L}) \longrightarrow End_{\mathcal{R}}(E_q(\mathcal{L}))$ are homomorphisms of unital associative $\mathcal{R}$-algebras, because

$$\forall X_1, X_2 \in E_q(\mathcal{L}) : ad^\pm(X_1X_2) = ad^\pm X_1 \circ ad^\pm X_2, \; ad^\pm I = id.$$

(8.9.3.2) The above indicated homomorphism λ is defined such that $\forall_1^m k$:

$$K_k \longrightarrow q_k^{H_k}, \; L_k \longrightarrow q_k^{-H_k}, \; E_k \longrightarrow X_k^+ q_k^{-\frac{1}{2}H_k}, \; F_k \longrightarrow X_k^- q_k^{\frac{1}{2}H_k}.$$

Then also

$$\Delta \circ \lambda = T(\lambda, \lambda) \circ \Delta, \; \varepsilon \circ \lambda = \varepsilon, \; \sigma \circ \lambda = \lambda \circ \sigma,$$

with an obvious abuse of notation. Obviously λ is continuous with respect to the h-adic topology of formal power series with relations, because it is $\mathcal{R}$-linear.

(8.9.4) An appropriate q-deformation of the Weyl group leads to Poincare-Birkhoff-Witt-like $\mathcal{R}$-linear bases of the q-deformed positive and negative Borel subalgebra of $E_q(\mathcal{L})$, the latter two being dual as topologically free $\mathcal{R}$-bimodules. The quantum double construction then allows for an explicit computation of the universal R-matrix of the quasitriangular topological Hopf algebra $U_q(\mathcal{L})$ over $\mathcal{R}$.

(8.9.5) Of course the above polynomial relations and costructure maps of $E_q(\mathcal{L})$ can also be understood as constituting a complex Hopf algebra. For this purpose one must assume that $h = \ln q$, with $q \in \mathbf{D} := \mathbf{C}\backslash(]-\infty, 0] \cup \{1\})$,

$$\forall_1^m k, l : q_k := q^{d_k}, \; q_k^j \neq \pm 1 \text{ for } 1 \leq j \leq 1 - \Gamma_{kl} \text{ and } k \neq l.$$

8.10 Finite-Dimensional Quantum Double

The so-called quantum double construction is used in order to construct the universal R-matrix of the q-deformed associative envelopes of finite-dimensional simple complex Lie algebras. The countably infinite dimension of these quantum groups causes topological troubles, which are overcome by techniques rather specific to these examples, namely using some Poincaré-Birkhoff-Witt-like basis, and then constructing the corresponding dual $\mathbf{C}$-linear basis. Therefore the algebraic procedure shall be explained for the case of finite dimension, which occurs for finite-dimensional representations by complex matrices.

(8.10.1) Consider a finite-dimensional Hopf algebra $\{\mathcal{H}, \mu, \eta, \Delta, \varepsilon\}$ over the field K, $dim\ \mathcal{H} =: N \in \mathbf{N}$; then the unique antipode σ is bijective. The dual Hopf algebra with opposite comultiplication $\{\mathcal{H}^*, \mu^*, \eta^*, (\Delta^*)^{opp}, \varepsilon^*\}$ is denoted by $\mathcal{H}^0$, with the antipode $(\sigma^*)^{-1} = (\sigma^{-1})^*$. More precisely, $\mathcal{H}^0$ is the dual K-Hopf algebra of $\{\mathcal{H}, \mu^{opp}, \eta, \Delta, \varepsilon\}$, the latter with the antipode σ^{-1}.

(8.10.2) The coalgebra $D(\mathcal{H}) := \mathcal{H} \otimes \mathcal{H}^0$ over K is established as the usual tensor product of coalgebras over K. Obviously the embeddings:

$$\mathcal{H} \ni a \longrightarrow a \otimes I \in D(\mathcal{H}), \quad \mathcal{H}^0 \ni f \longrightarrow I \otimes f \in D(\mathcal{H})$$

are injective homomorphisms of coalgebras over K. Here the units $e_{\mathcal{H}}$ of $\mathcal{H}$, and ε of $\mathcal{H}^0$, are both denoted by I. The counit of $D(\mathcal{H})$ is the K-linear mapping:

$$D(\mathcal{H}) \ni a \otimes f \longrightarrow \varepsilon(a) f(e_{\mathcal{H}}) \in K.$$

(8.10.3) An appropriate multiplication is provided by the so-called double cross product:

$$\begin{aligned} &D(\mathcal{H}) \otimes D(\mathcal{H}) \ni (a \otimes f) \otimes (b \otimes g) \\ &\longrightarrow \sum_{k=1}^{K(b)} \sum_{l=1}^{L(f)} \left(a b_k^{(2)}\right) \otimes \left(f_l^{(2)} g\right) f_l^{(1)} \circ \sigma^{-1}\left(b_k^{(1)}\right) f_l^{(3)}\left(b_k^{(3)}\right) \in D(\mathcal{H}), \end{aligned}$$

with the notation:

$$\mathcal{H} \ni b \underset{T(\Delta, id)\circ\Delta}{\longrightarrow} \sum_{k=1}^{K(b)} \left(b_k^{(1)} \otimes b_k^{(2)}\right) \otimes b_k^{(3)} \in (\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H},$$

and similarly on $\mathcal{H}^0$, inserting the opposite comultiplication $(\Delta^*)^{opp}$. The unit of this associative K-algebra is $e_{\mathcal{H}} \otimes \varepsilon \equiv I \otimes I$. Obviously

$$\forall a \in \mathcal{H}, f \in \mathcal{H}^0 : (a \otimes I)(I \otimes f) = a \otimes f.$$

(8.10.4) Thereby one establishes the Hopf algebra $D(\mathcal{H})$ over K, with the antipode:

$$D(\mathcal{H}) \ni a \otimes f \longrightarrow \big(I \otimes (\sigma^*)^{-1}(f)\big)\,(\sigma(a) \otimes I) \in D(\mathcal{H}).$$

The above embeddings of $\mathcal{H}$ and $\mathcal{H}^0$ into $D(\mathcal{H})$ are homomorphisms of Hopf algebras over K.

(8.10.5) This so-called quantum double $D(\mathcal{H})$ is some quasitriangular K-Hopf algebra, with respect to the following solution R of the quantum Yang-Baxter equation.
Take any K-basis $\{b_k; k = 1, \ldots, N\}$ of $\mathcal{H}$, and the corresponding dual basis $\{b^k; k = 1, \ldots, N\}$ of $\mathcal{H}^*$, such that $\forall_1^N k, l : b^k(b_l) = \delta_{kl}$. Then the so-called canonical element

$$\sum_{k=1}^{N} b_k \otimes b^k \in D(\mathcal{H})$$

is independent of the choice of basis. Hence also the elements

$$R := \sum_{k=1}^{N} (b_k \otimes I) \otimes (I \otimes b^k) \in D(\mathcal{H}) \otimes D(\mathcal{H}),$$

$$R^{-1} = \sum_{k=1}^{N} (\sigma(b_k) \otimes I) \otimes \big(I \otimes b^k\big) = \sum_{k=1}^{N} (b_k \otimes I) \otimes \big(I \otimes \sigma^*(b^k)\big),$$

$$RR^{-1} = R^{-1}R = (I \otimes I) \otimes (I \otimes I),$$

are independent of the choice of basis. The quasitriangular properties of R are proved applying any element $\in (\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H})$ onto such $\in D(\mathcal{H}) \otimes D(\mathcal{H})$, using the K-linear bijection: $\mathcal{H} \longleftrightarrow \mathcal{H}^{**}$.

8.11 Universal R-Matrix of $U_q(A_m)$

(8.11.1) The Hopf subalgebras of the Hopf algebra $E_q(A_m)$ over the ring $\mathcal{R} := \mathbf{C}[[h]], m \in \mathbf{N}$, which are polynomially spanned by the sets of generators $\{E_k; k = 1, \ldots, m\}$, $\{F_k; k = 1, \ldots, m\}$, $\{E_k, K_k, L_k; k = 1, \ldots, m\}$, and $\{F_k, K_k, L_k; k = 1, \ldots, m\}$, are denoted by $E_q(\mathcal{N}^\pm)$, $E_q(\mathcal{B}^\pm)$. One may also consider $E_q(A_m)$ as complex Hopf algebra, choosing $q \in \mathbf{D}\backslash\{\pm i\}$.

(8.11.2) The q-deformed Serre relations of the q-deformed Borel subalgebra $E_q(\mathcal{B}^+)$ read as follows.

$$\begin{aligned}\forall_1^m k, l: \; & ad^+E_k(E_l) = 0 \text{ for } |k-l| \geq 2, \\ & ad^+E_k \circ ad^+E_k(E_l) = 0 \text{ for } |k-l| = 1,\end{aligned}$$

the latter relations just meaning that

$$E_k^2 E_l + E_l E_k^2 = (q + q^{-1}) E_k E_l E_k \text{ for } |k-l| = 1.$$

The unital associative subalgebra $E_q(\mathcal{B}^-)$ is described by analogous Serre relations of $F_k, k = 1, \ldots, m$.

(8.11.3) The q-deformed adjoint representations are q-deformed derivations in the sense, that $\forall x, y \in E_q(\mathcal{N}^+), \forall_1^m k, \forall c \in \mathbf{C}$:

$$L_k x = q^c x L_k \implies ad^+E_k(xy) = ad^+E_k(x)y + q^c x \; ad^+E_k(y),$$

and similarly for the restriction of ad^-F_k onto $E_q(\mathcal{N}^-)$.

(8.11.4) The positive roots of A_m are strictly ordered according to the longest word ω of the Weyl group.

$$\begin{aligned}\omega := \; & \sigma_1 \circ \cdots \circ \sigma_m \circ \sigma_1 \circ \cdots \circ \sigma_{m-1} \circ \sigma_1 \circ \cdots \circ \sigma_{m-2} \circ \\ & \cdots \circ \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1.\end{aligned}$$

$$\begin{aligned}& \beta_1 := \alpha_1, \; \beta_2 := \sigma_1(\alpha_2) = \alpha_1 + \alpha_2, \ldots, \\ & \beta_m := \sigma_1 \circ \cdots \circ \sigma_{m-1}(\alpha_m) = \alpha_1 + \cdots + \alpha_m, \\ & \beta_{m+1} := \sigma_1 \circ \cdots \circ \sigma_m(\alpha_1) = \alpha_2, \ldots, \\ & \beta_{2m-1} := \sigma_1 \circ \cdots \circ \sigma_m \circ \sigma_1 \circ \cdots \circ \sigma_{m-2}(\alpha_{m-1}) = \alpha_2 + \cdots + \alpha_m, \ldots,\end{aligned}$$

$$\begin{aligned}\beta_{p-1} & := \sigma_1 \circ \cdots \circ \sigma_m \circ \sigma_1 \circ \cdots \circ \sigma_{m-1} \circ \cdots \circ \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_1(\alpha_2) \\ & = \alpha_{m-1} + \alpha_m,\end{aligned}$$

$$\begin{aligned}\beta_p & := \sigma_1 \circ \cdots \circ \sigma_m \circ \sigma_1 \circ \cdots \circ \sigma_{m-1} \circ \cdots \circ \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_1 \circ \sigma_2(\alpha_1) \\ & = \alpha_m,\end{aligned}$$

$p := \frac{1}{2}m(m+1)$. For instance in the case of $m = 3$,

$$\beta_1 = \alpha_1, \; \beta_2 = \alpha_1 + \alpha_2, \; \beta_3 = \alpha_1 + \alpha_2 + \alpha_3,$$

$$\beta_4 = \alpha_2, \; \beta_5 = \alpha_2 + \alpha_3, \; \beta_6 = \alpha_3.$$

(8.11.5) For every positive root $\beta := \alpha_k + \cdots + \alpha_l \in \Phi^+$, define

$$E_\beta := ad^+ E_k \circ \cdots \circ ad^+ E_{l-1}(E_l),\ 1 \le k < l \le m.$$

The q-deformed Serre relations of $E_q(\mathcal{N}^+)$ lead to q-deformed commutation relations of the following kind, calculated quite explicitly by M. Rosso.
$\forall 1 \le k < l \le p$:

$$E_{\beta_k} E_{\beta_l} - \lambda E_{\beta_l} E_{\beta_k} = \begin{cases} 0, \text{ or} \\ \mu E_{\beta_k + \beta_l}, & \beta_k + \beta_l \in \Phi^+, \text{ or} \\ \nu E_{\beta_j} E_{\beta_i}, & k < i < j < l,\ \beta_i + \beta_j = \beta_k + \beta_l\ , \end{cases}$$

with integers i, j, and non-zero coefficients λ, μ, ν, depending on k, l.

(8.11.5.1) For instance in the case of $m = 3$, one obtains the following commutation relations.

$$\begin{aligned}
&ad\ E_1(E_2) = E_{\beta_1} E_{\beta_4} - q E_{\beta_4} E_{\beta_1} = E_{\beta_2},\ ad\ E_1(E_3) = [E_{\beta_1}, E_{\beta_6}] = 0, \\
&ad\ E_1(E_{\alpha_1+\alpha_2}) = E_{\beta_1} E_{\beta_2} - q^{-1} E_{\beta_2} E_{\beta_1} = 0, \\
&ad\ E_1(E_{\alpha_1+\alpha_2+\alpha_3}) = E_{\beta_1} E_{\beta_3} - q^{-1} E_{\beta_3} E_{\beta_1} = 0, \\
&ad\ E_1(E_{\alpha_2+\alpha_3}) = E_{\beta_1} E_{\beta_5} - q E_{\beta_5} E_{\beta_1} = E_{\beta_3}, \\
&ad\ E_2(E_3) = E_{\beta_4} E_{\beta_6} - q E_{\beta_6} E_{\beta_4} = E_{\beta_5}, \\
&ad\ E_2(E_{\alpha_2+\alpha_3}) = E_{\beta_4} E_{\beta_5} - q^{-1} E_{\beta_5} E_{\beta_4} = 0,\ E_{\beta_4} E_{\beta_2} - q E_{\beta_2} E_{\beta_4} = 0, \\
&ad\ E_2(E_{\alpha_1+\alpha_2+\alpha_3}) = [E_{\beta_4}, E_{\beta_3}] = 0,\ E_{\beta_6} E_{\beta_5} - q E_{\beta_5} E_{\beta_6} = 0, \\
&E_{\beta_2} E_{\beta_6} - q E_{\beta_6} E_{\beta_2} = E_{\beta_3}, E_{\beta_6} E_{\beta_3} - q E_{\beta_3} E_{\beta_6} = 0, \\
&E_{\beta_5} E_{\beta_3} - q E_{\beta_3} E_{\beta_5} = 0,\ E_{\beta_3} E_{\beta_2} - q E_{\beta_2} E_{\beta_3} = 0, \\
&[E_{\beta_2}, E_{\beta_5}] = (q^{-1} - q) E_{\beta_4} E_{\beta_3}.
\end{aligned}$$

(8.11.5.2) For any map $\pi : \mathbf{N} \longrightarrow \{1, \ldots, p\}$, and $k \in \mathbf{N}$, the monomial $E_{\beta_{\pi(1)}} \cdots E_{\beta_{\pi(k)}}$ can be rewritten, via an induction on k and $\pi(1)$, as $\mathcal{R}$-linear combination of monomials of the kind $E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1}$, with $r_1, \ldots, r_p \in \mathbf{N}_0$.

(8.11.5.3) Since the above commutation relations are homogeneous with respect to the root lattice $Q := \mathbf{Z}(\{\alpha_1, \ldots, \alpha_m\})$, the $\mathcal{R}$-algebra $E_q(\mathcal{B}^+)$ is Q-graded, such that $E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1}$ is of degree $r_1\beta_1 + \cdots + r_p\beta_p$, the generators $K_1, \ldots, L_m$ being of degree zero by definition.

(8.11.6) By induction one calculates, that $\forall 1 \le k < l \le m$:

$$\begin{aligned}
\Delta(E_{\alpha_k+\cdots+\alpha_l}) &= E_{\alpha_k+\cdots+\alpha_l} \otimes I + L_k \cdots L_l \otimes E_{\alpha_k+\cdots+\alpha_l} \\
&\quad + (1 - q^2) \sum_{j=k}^{l-1} E_{\alpha_k+\cdots+\alpha_j} L_{j+1} \cdots L_l \otimes E_{\alpha_{j+1}+\cdots+\alpha_l},
\end{aligned}$$

$$\varepsilon(E_{\alpha_k+\cdots+\alpha_l}) = 0.$$

(8.11.6.1) Especially for $m = 3$,

$$\begin{aligned}
\Delta(E_{\beta_2}) &= E_{\beta_2} \otimes I + L_1 L_2 \otimes E_{\beta_2} + (1-q^2) E_1 L_2 \otimes E_2, \\
\Delta(E_{\beta_5}) &= E_{\beta_5} \otimes I + L_2 L_3 \otimes E_{\beta_5} + (1-q^2) E_2 L_3 \otimes E_3, \\
\Delta(E_{\beta_3}) &= E_{\beta_3} \otimes I + L_1 L_2 L_3 \otimes E_{\beta_3} \\
&\quad + (1-q^2)(E_1 L_2 L_3 \otimes E_{\beta_5} + E_{\beta_2} L_3 \otimes E_3).
\end{aligned}$$

(8.11.6.2) The comultiplication Δ preserves the $\mathcal{Q}$-degree in the natural sense.

(8.11.6.3) Let any $\mathcal{Q}$-homogeneous $\mathcal{R}$-linear combination of ordered monomials $E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1}$ be zero. An application of Δ allows for an induction on the $\mathcal{Q}$-degree, in order to prove that all the coefficients of monomials of this polynomial are zero. Obviously $E_{\beta_1}, \ldots, E_{\beta_p}$ are $\mathcal{R}$-linearly independent.

(8.11.7) The set $\{E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1}; r_1, \ldots, r_p \in \mathbf{N}_0\}$ is some $\mathcal{R}$-basis of $E_q(\mathcal{N}^+)$. Hence

$$\{E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} K_1^{z_1} \cdots K_m^{z_m}; r_1, \ldots, r_p \in \mathbf{N}_0; z_1, \ldots, z_m \in \mathbf{Z}\}$$

$$\underset{free\ over\ \mathcal{R}}{\longrightarrow} E_q(\mathcal{B}^+).$$

(8.11.8) The following $\mathcal{R}$-linear forms are defined on the complex Hopf algebra $E_q(\mathcal{B}^+)$. $\forall_1^m i, \forall r_1, \ldots, r_p \in \mathbf{N}_0, \forall z_1, \ldots, z_m \in \mathbf{Z}$:

$$\langle k_i^{\pm} | E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} K_1^{z_1} \cdots K_m^{z_m} \rangle$$

$$:= \begin{cases} \exp\big(\pm \ln q \sum\limits_{k=1}^{m} z_k \Gamma_{ik}\big), & \text{if } r_1 + \cdots + r_p = 0 \\ 0 \text{ otherwise} \end{cases},$$

$$\langle f_i | E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} K_1^{z_1} \cdots K_m^{z_m} \rangle := \begin{cases} \phi_i \in \mathbf{C} \setminus \{0\}, & \text{if } r_1\beta_1 + \cdots + r_p\beta_p = \alpha_i \\ 0 \text{ otherwise} \end{cases}.$$

(8.11.8.1) These elements of the unital associative algebra $(E_q(\mathcal{B}^+))^*$ of $\mathcal{R}$-linear forms fulfill at least the following relations.

$$\begin{aligned}
\forall_1^m i, j :\ & k_i^+ k_i^- = k_i^- k_i^+ = I, \text{ hence denote } k_i^+ \equiv k_i,\ k_i^- \equiv k_i^{-1} \equiv l_i; \\
& [k_i, k_j] = [l_i, l_j] = [k_i, l_j] = 0, \\
& f_i k_j = q^{\Gamma_{ij}} k_j f_i,\ f_i l_j = q^{-\Gamma_{ij}} l_j f_i; \\
& [f_i, f_j] = 0, \text{ if } |i-j| \geq 2; \\
& f_i^2 f_j + f_j f_i^2 = (q + q^{-1}) f_i f_j f_i, \text{ if } |i-j| = 1.
\end{aligned}$$

(8.11.8.2) Consider the subalgebra $(E_q(\mathcal{B}^+))^0$ of $(E_q(\mathcal{B}^+))^*$, which is generated by the set $\{k_i^{\pm 1}, f_i; i = 1, \dots, m\}$; it is equipped with comultiplication and counit.

$$\forall_1^m i : k_i^{\pm 1} \xrightarrow[\Delta^0]{} k_i^{\pm 1} \otimes k_i^{\pm 1},\ \varepsilon^0(k_i^{\pm 1}) = 1,$$
$$f_i \xrightarrow[\Delta^0]{} f_i \otimes I + k_i \otimes f_i,\ \varepsilon^0(f_i) = 0.$$

(8.11.8.3) Denote by $(E_q(\mathcal{B}^-))^{opp}$ the Hopf subalgebra of $(E_q(A_m))^{opp}$, which is generated by the set $\{K_l^{\pm 1}, F_l; l = 1, \dots, m\}$, with the opposite comultiplication and inverse antipode. There is some homomorphism ψ both of unital associative algebras and of coalgebras over $\mathcal{R}$, such that $\forall_1^m i : \psi(K_i^{\pm 1}) = k_i^{\pm 1},\ \psi(F_i) = f_i$. In order to prove that this homomorphism $\psi : (E_q(\mathcal{B}^-))^{opp} \longrightarrow (E_q(\mathcal{B}^+))^0$ is bijective, one needs some $\mathcal{R}$-basis of the latter bimodule.

(8.11.8.4)

$$\forall_1^p j : \langle f_{\beta_j} | E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} K_1^{z_1} \cdots K_m^{z_m} \rangle$$
$$:= \begin{cases} \phi(\beta_j) \in \mathcal{R} \setminus \{0\}, & \text{if } r_1\beta_1 + \cdots + r_p\beta_p = \beta_j \\ 0 \text{ otherwise} \end{cases}.$$

Choosing $\forall_1^m k : \phi_k := (1-q^{-2})^{-1}$, and appropriate values of $\phi(\beta_j), 1 \le j \le p$, for instance $\phi(\alpha_1 + \alpha_2) := \phi_1\phi_2(1 - q^2)$, one finds, that

$$\forall 1 \le k < l \le m : f_{\alpha_k + \cdots + \alpha_l} = f_k f_{\alpha_{k+1} + \cdots + \alpha_l} - k_k f_{\alpha_{k+1} + \cdots + \alpha_l} k_k^{-1} f_k.$$

Hence these $\mathcal{R}$-linear forms fulfill the same commutation relations as E_{β_j}, for $1 \le j \le p$. Therefore

$$(E_q(\mathcal{B}^+))^0 = \mathcal{R} - span\big(\{f_{\beta_p}^{r_p} \cdots f_{\beta_1}^{r_1} k_1^{z_1} \cdots k_m^{z_m};$$
$$r_1, \dots, r_p \in \mathbf{N}_0; z_1, \dots, z_m \in \mathbf{Z}\}\big).$$

Moreover this set and the above constructed $\mathcal{R}$-basis of $E_q(\mathcal{B}^+)$ are orthogonal with respect to the $\mathcal{Q}$-degree. $\forall r_1, \dots, s_p \in \mathbf{N}_0, \forall x_1, \dots, y_m \in \mathbf{Z}$:

$$\langle f_{\beta_p}^{r_p} \cdots f_{\beta_1}^{r_1} k_1^{x_1} \cdots k_m^{x_m} | E_{\beta_p}^{s_p} \cdots E_{\beta_1}^{s_1} K_1^{y_1} \cdots K_m^{y_m} \rangle = c\delta_{r_1 s_1} \cdots \delta_{r_p s_p},$$

$c \in \mathcal{R} \setminus \{0\}$. Therefore this set is some $\mathcal{R}$-basis of $(E_q(\mathcal{B}^+))^0$.

(8.11.8.5)

$$\forall 1 \le k < l \le m : F_{\alpha_k + \cdots + \alpha_l} := ad^- F_k \circ \cdots \circ ad^- F_{l-1}(F_l).$$

Since these elements fulfill the same commutation relations as $E_{\beta_j}, 1 \le j \le p$, one obtains an according $\mathcal{R}$-basis of $E_q(\mathcal{B}^-)$.

(8.11.8.6) Therefore

$$\psi : (E_q(\mathcal{B}^-))^{opp} \longleftrightarrow (E_q(\mathcal{B}^+))^0$$

is bijective. This in turn implies, that the generators $k_j, f_j, 1 \leq j \leq m$, fulfill no more relations than the above. One finally establishes an isomorphism of $\mathcal{R}$-Hopf algebras ψ, with respect to the antipode σ^0, such that

$$\forall_1^m j : k_j \underset{\sigma^0}{\longleftrightarrow} k_j^{-1},\ f_j \underset{\sigma^0}{\longrightarrow} -k_j^{-1} f_j.$$

$$\left\{E_q(\mathcal{B}^-),\ (\Delta|_{E_q(\mathcal{B}^-)})^{opp}, \varepsilon|_{E_q(\mathcal{B}^-)}, (\sigma|_{E_q(\mathcal{B}^-)})^{-1}\right\}$$
$$\underset{isomorphism}{\longleftrightarrow} \left\{(E_q(\mathcal{B}^+))^0,\ \textit{above relations}, \Delta^0, \varepsilon^0, \sigma^0\right\},$$

inserting the subalgebra $E_q(\mathcal{B}^-)$ of $E_q(A_m)$.

(8.11.8.7) The Hopf algebras $E_q(\mathcal{B}^+)$ and $(E_q(\mathcal{B}^+))^0$ are dual with respect to the above $\mathcal{R}$-bilinear form, which obviously is non-degenerate.

$$\forall_1^m i, j, \forall z \in \mathbf{Z} : \langle k_i^{\pm} | K_j^z \rangle = q^{\pm z \Gamma_{ij}},\ \langle k_i^{\pm} | E_j \rangle = 0,$$
$$\langle f_i | K_j^{\pm 1} \rangle = 0,\ \langle f_i | E_j \rangle = \frac{\delta_{ij}}{1 - q^{-2}}.$$

(8.11.9) The so-called quantum double of the q-deformed Borel subalgebra $E_q(\mathcal{B}^+)$ is defined as the tensor product of $\mathcal{R}$-coalgebras, and as the so-called double cross product of $\mathcal{R}$-Hopf algebras, inserting the restrictions of Δ from $E_q(A_m)$ onto $E_q(\mathcal{B}^{\pm})$. Note that the comultiplication Δ^0 on $(E_q(\mathcal{B}^+))^0$ must be twisted here.

$$D(E_q(\mathcal{B}^+)) := E_q(\mathcal{B}^+) \otimes E_q(\mathcal{B}^-).$$

(8.11.9.1)

$$\forall_1^m i, j : [K_i \otimes I, I \otimes K_j^{\pm 1}] = [K_i^{-1} \otimes I, I \otimes K_j^{\pm 1}] = 0,$$
$$(K_i \otimes I)(I \otimes F_j) = q^{-\Gamma_{ij}} (I \otimes F_j)(K_i \otimes I),$$
$$(I \otimes K_i)(E_j \otimes I) = q^{\Gamma_{ij}} (E_j \otimes I)(I \otimes K_i),$$
$$[E_i \otimes I, I \otimes F_j] = \delta_{ij} \frac{q^2}{q^2 - 1} (I \otimes K_j - K_j^{-1} \otimes I).$$

The coefficients $\phi_k, k = 1, \ldots, m$, were adjusted with respect to the last relation. Moreover one finds, that $\forall_1^m i, j$:

$$(E_i \otimes I)(K_j^{\pm 1} \otimes I) = E_i K_j^{\pm 1} \otimes I,\ (K_j^{\pm 1} \otimes I)(E_i \otimes I) = K_j^{\pm 1} E_i \otimes I,$$

and similar relations for the generators $F_i, i = 1, \ldots, m$.

(8.11.9.2) Hence there is some surjective homomorphism of Hopf algebras over $\mathcal{R}$: $D(E_q(\mathcal{B}^+)) \longrightarrow E_q(A_m)$, such that $\forall_1^m j$:

$$K_j \otimes I \longrightarrow K_j,\ I \otimes K_j \longrightarrow K_j,\ E_j \otimes I \longrightarrow E_j,\ I \otimes F_j \longrightarrow F_j.$$

(8.11.9.3) $\forall r_1, \ldots, s_p \in \mathbf{N}_0, \forall x_1, \ldots, y_m \in \mathbf{Z} : \exists$ *unique* $c \in \mathcal{R} \setminus \{0\}$:

$$(E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} K_1^{x_1} \cdots K_m^{x_m}) \otimes (F_{\beta_p}^{s_p} \cdots F_{\beta_1}^{s_1} K_1^{y_1} \cdots K_m^{y_m})$$
$$\longrightarrow c E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} F_{\beta_p}^{s_p} \cdots F_{\beta_1}^{s_1} K_1^{x_1+y_1} \cdots K_m^{x_m+y_m}.$$

Hence one finds an $\mathcal{R}$-linear basis of $E_q(A_m)$, with the same Poincaré series as in the non-deformed case of $E(A_m)$.

(8.11.9.4) One then establishes an appropriate $\mathcal{R}$-linear bijection:

$$D(E_q(\mathcal{B}^+)) \longleftrightarrow E_q(A_m) \otimes \frac{\mathcal{R}[K_1, \ldots, L_m]}{\textit{ideal of } \{\forall_1^m j : K_j L_j = I\}},$$

inserting polynomials over $\mathcal{R}$ in $2m$ commuting generators $K_1, \ldots, L_m$.

(8.11.9.5) Obviously these polynomial considerations may also be performed over $\mathbf{C}$, choosing $q \in \mathbf{D} \backslash \{\pm i\}$.

(8.11.10) Consider again the topological Hopf algebra $U_q(A_m)$ over $\mathcal{R}$, which is generated by the set $\{E_k, F_k, H_k; k = 1, \ldots, m\}$ in the sense of formal power series with relations. The subalgebras $U_q(\mathcal{N}^\pm)$ and $U_q(\mathcal{B}^\pm)$ of $U_q(A_m)$ are defined similarly as in the polynomial case, inserting H_k instead of $K_k^{\pm 1}, k = 1, \ldots, m$. Then the q-deformed Borel subalgebras $U_q(\mathcal{B}^\pm)$ are topological Hopf algebras over $\mathcal{R}$. Obviously

$$U_q(\mathcal{N}^+) = \overline{E_q(\mathcal{N}^+)},$$

with respect to the h-adic topology. Therefore

$$\left\{E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} H_1^{x_1} \cdots H_m^{x_m}; r_1, \ldots, x_m \in \mathbf{N}_0\right\}$$

is some topological basis of the q-deformed Borel subalgebra $U_q(\mathcal{B}^+)$ in the sense, that every vector Y of the latter can be written uniquely as

$$Y = \sum_{r_1, \ldots, x_m \in \mathbf{N}_0} c_{r_1 \ldots x_m} E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} H_1^{x_1} \cdots H_m^{x_m},\ c_{r_1 \ldots x_m} \in \mathcal{R}.$$

(8.11.10.1) Define $\mathcal{R}$-linear forms

$$\xi_k, \eta_{\beta_j} \in (U_q(\mathcal{B}^+))^* := Hom_{\mathcal{R}}(U_q(\mathcal{B}^+), \mathcal{R}),$$

vanishing on every basis vector, except

$$\forall_1^m k, \forall_1^p j : \langle \xi_k | H_k \rangle = 1, \ \langle \eta_{\beta_j} | E_{\beta_j} \rangle = 1.$$

$$\zeta_1 := \xi_1 - \frac{1}{2}\xi_2, \ \forall_2^{m-1} k : \zeta_k := \xi_k - \frac{1}{2}(\xi_{k-1} + \xi_{k+1}), \ \zeta_m := \xi_m - \frac{1}{2}\xi_{m-1}.$$

$$\forall_1^m k, l : h_k := \frac{2}{\ln q}\zeta_k, \ \langle h_k | H_l \rangle = \frac{1}{\ln q}\Gamma_{kl}, \ \eta_{\alpha_k}\big|_{E_q(\mathcal{B}^+)} = (1 - q^{-2}) f_k.$$

(8.11.10.2) On the subalgebra of polynomials in E_k and $H_k, k = 1, \ldots, m$,

$$U_q^{fin}(\mathcal{B}^+) := \mathcal{R}\big(\{E_{\beta_p}^{r_p} \cdots E_{\beta_1}^{r_1} H_1^{x_1} \cdots H_m^{x_m}; r_1, \ldots, x_m \in \mathbf{N}_0\}\big),$$

$$Im\ \Delta\big|_{U_q^{fin}(\mathcal{B}^+)} \subset \bigotimes^2 U_q(\mathcal{B}^+).$$

Hence the unital associative algebra $(U_q(\mathcal{B}^+))^0$ of polynomials in the restrictions of $\xi_1, \ldots, \eta_{\beta_p}$ onto $U_q^{fin}(\mathcal{B}^+)$ is defined, quite similarly as for the dual of any coalgebra.

(8.11.10.3)

$$\forall_1^m k, l, \forall x, y \in \mathbf{N}_0 : \langle \xi_k^x | H_l^y \rangle = \delta_{kl}\delta_{xy} x!;$$

$$\forall_1^p i, j, \forall r, s \in \mathbf{N}_0 : \langle \eta_{\beta_i}^r | E_{\beta_j}^s \rangle = \delta_{ij}\delta_{rs} \prod_{k=1}^{r} \frac{1 - q^{-2k}}{1 - q^{-2}};$$

$\forall r_1, \ldots, y_m \in \mathbf{N}_0$:

$$\langle \eta_{\beta_p}^{r_p} \cdots \eta_{\beta_1}^{r_1} \xi_1^{x_1} \cdots \xi_m^{x_m} | E_{\beta_p}^{s_p} \cdots E_{\beta_1}^{s_1} H_1^{y_1} \cdots H_m^{y_m} \rangle$$

$$= \prod_{j=1}^{p} \delta_{r_j s_j} \frac{(1 - q^{-2})(1 - q^{-4}) \cdots (1 - q^{-2r_j})}{(1 - q^{-2})^{r_j}} \prod_{k=1}^{m} \delta_{x_k y_k} x_k!.$$

(8.11.10.4) These forms h_k and $(1 - q^{-2})^{-1}\eta_{\alpha_k}, k = 1, \ldots, m$, fulfill on $U_q^{fin}(\mathcal{B}^+)$ at least the same relations as H_k and F_k. Hence one obtains some surjective homomorphism of unital associative $\mathcal{R}$-algebras

$$\psi : U_q^{fin}(\mathcal{B}^-) \longrightarrow (U_q(\mathcal{B}^+))^0, \text{ such that}$$

$$\forall_1^m k : h_k \longrightarrow H_k, \ \eta_{\alpha_k} \longrightarrow (1 - q^{-2}) F_k,$$

denoting by $U_q^{fin}(\mathcal{B}^-)$ the corresponding restriction to polynomials over $\mathcal{R}$.

(8.11.10.5) Since the monomials $\eta_{\beta_p}^{r_p} \cdots \xi_m^{x_m}, r_1, \dots, x_m \in \mathbf{N}_0$, are $\mathcal{R}$-linearly independent, due to the above orthogonality, ψ is bijective. Then the unique continuous extension of ψ to formal power series with relations yields some isomorphism of topological Hopf algebras

$$\overline{\psi} : (U_q(\mathcal{B}^-))^{opp} \longleftrightarrow \overline{(U_q(\mathcal{B}^+))^0},$$

inserting an appropriate comultiplication, counit, and antipode, on the unique Hausdorff completion of the right hand side.

(8.11.10.6) The resulting $\mathcal{R}$-bilinear form: $U_q^{fin}(\mathcal{B}^-) \times U_q^{fin}(\mathcal{B}^+) \longrightarrow \mathcal{R}$ can be continuously extended to formal power series with relations, with respect to the h-adic topology.

(8.11.10.7) The quantum double $D(U_q(\mathcal{B}^+)) := \overline{U_q(\mathcal{B}^+) \otimes U_q(\mathcal{B}^-)}$ is then constructed by unique continuous extension of structure mappings. One obtains some surjective homomorphism of topological Hopf algebras:

$$D(U_q(\mathcal{B}^+)) \longrightarrow U_q(A_m),$$

such that $\forall_1^m k$:

$$H_k \otimes I \longrightarrow H_k,\ I \otimes H_k \longrightarrow H_k,\ E_k \otimes I \longrightarrow E_k,\ I \otimes F_k \longrightarrow F_k.$$

(8.11.10.8) The completion with respect to the h-adic topology also yields

$$\mathcal{D} := D(U_q(\mathcal{B}^+)) = \overline{U_q^{fin}(\mathcal{B}^+) \otimes U_q^{fin}(\mathcal{B}^-)}.$$

(8.11.11.1) In order to write down explicitly the so-called universal R-matrix of $\mathcal{D}$, remember the following $\mathcal{R}$-bases:

$$\{E_{\beta_p}^{r_p} \cdots H_m^{x_m}; r_1, \dots, x_m \in \mathbf{N}_0\} \underset{free\ over\ \mathcal{R}}{\longrightarrow} U_q^{fin}(\mathcal{B}^+);$$

$$\{\eta_{\beta_p}^{r_p} \cdots \xi_m^{x_m}; r_1, \dots, x_m \in \mathbf{N}_0\} \underset{free\ over\ \mathcal{R}}{\longrightarrow} (U_q(\mathcal{B}^+))^0 \underset{\psi}{\longleftrightarrow} U_q^{fin}(\mathcal{B}^-),$$

$$\{F_{\beta_p}^{r_p} \cdots H_m^{x_m}; r_1, \dots, x_m \in \mathbf{N}_0\} \underset{free\ over\ \mathcal{R}}{\longrightarrow} U_q^{fin}(\mathcal{B}^-).$$

(8.11.11.2)

$$\forall_1^m k : H_k = \frac{1}{\ln q} \sum_{l=1}^{m} \Gamma_{kl} \psi^{-1}(\xi_l).$$

(8.11.11.3)

$$\forall_1^{m-1} k : \eta_{\alpha_k}\eta_{\alpha_{k+1}} - q\eta_{\alpha_{k+1}}\eta_{\alpha_k} = (1-q^2)\eta_{\alpha_k+\alpha_{k+1}}.$$

Therefore

$$\forall 1 \le k < l \le m : (\phi(\alpha_k + \cdots + \alpha_l))^{-1} = \left(1 - \frac{1}{q^2}\right)\left(-\frac{1}{q^2}\right)^{l-k}.$$

$$\forall_1^p j : \psi(F_{\beta_j}) = \phi(\beta_j)\eta_{\beta_j}\big|_{U_q^{fin}(\mathcal{B}^+)}.$$

(8.11.11.4) The universal R-matrix of the quantum double is defined as the following formal power series with relations over $\mathcal{R}$.

$$R_{\mathcal{D}} := \sum_{r_1,\ldots,x_m \in \mathbf{N}_0} \frac{1}{x_1! \cdots x_m!}\Big(\prod_{j=1}^{p} \frac{(1-q^{-2})^{r_j}}{(1-q^{-2})\cdots(1-q^{-r_j})}\Big)$$
$$\big(E_{\beta_p}^{r_p} \cdots H_m^{x_m} \otimes I\big) \otimes \big(I \otimes \psi^{-1}(\eta_{\beta_p}^{r_p}) \cdots \psi^{-1}(\xi_m^{x_m})\big) \in \overline{\mathcal{D} \otimes \mathcal{D}}.$$

The corresponding quasitriangular properties are proved by orthogonality of the inserted topological $\mathcal{R}$-bases.

(8.11.11.5) The homomorphic image of $R_{\mathcal{D}}$ is the universal R-matrix of $U_q(A_m)$.

$$R = \Big(\prod_{j=p}^{1}\sum_{r_j=0}^{\infty} \frac{(1-q^{-2})^{2r_j}(-q^2)^{r_j(1-\lambda_j)}}{(1-q^{-2})\cdots(1-q^{-r_j})} E_{\beta_j}^{r_j} \otimes F_{\beta_j}^{r_j}\Big)$$
$$\exp\Big(\ln q \sum_{k,l=1}^{m} (\Gamma^{-1})_{kl} H_k \otimes H_l\Big),$$

where the product is ordered according to decreasing $j = p, \ldots, 1$. The length of the root $\beta_j := \alpha_k + \cdots + \alpha_l$ is defined by $\lambda_j := l - k + 1,\ 1 \le k \le l \le m$.

(8.11.11.6) Inserting $h = \ln q$ for generic $q \in \mathbf{D}$, one obtains thereby the R-matrix of an arbitrary finite-dimensional complex representation of $U_q(A_m)$.

8.12 Universal R-Matrix of q-Deformed Simple Lie Algebras

The automorphisms $T_k, k = 1, \ldots, m$, of a simple finite-dimensional complex Lie algebra $\mathcal{L}$ of rank m, can be q-deformed to automorphisms of the unital associative algebra $E_q(\mathcal{L})$ over $\mathcal{R}$, which in the sequel are again denoted by T_k. These automorphisms can be used conveniently, being composed along the longest word of the Weyl group, to construct the q-deformed positive and negative root vectors, which in turn are inserted into the ordered monomials of some Poincaré-Birkhoff-Witt-like $\mathcal{R}$-linear basis of $E_q(\mathcal{L})$.

(8.12.1) Let $\mathcal{L}$ be a simple finite-dimensional complex Lie algebra of rank m, and denote

$$q := e^h,\ q_k := q^{d_k} = e^{hd_k}.$$

There are automorphisms T_k of $E_q(\mathcal{L})$, $k = 1, \dots, m$, in the sense of unital associative $\mathcal{R}$-algebras, such that $\forall_1^m k, l$:

$$T_k(K_k) = L_k,\ T_k(E_k) = -F_k L_k,\ T_k(F_k) = -K_k E_k;$$

for $\Gamma_{kl} = 0$,

$$T_k(K_l) = K_l,\ T_k(E_l) = E_l,\ T_k(F_l) = F_l;$$

for $\Gamma_{kl} = -1$,

$$T_k(K_l) = K_k K_l,\ T_k(E_l) = q_k^{-1/2}(E_k E_l - q_k E_l E_k) = q_k^{-1/2} ad^+ E_k(E_l),$$
$$T_k(F_l) = -q_k^{-3/2}(F_k F_l - q_k F_l F_k) = -q_k^{-3/2} ad^- F_k(F_l);$$

for $\Gamma_{kl} = -2$,

$$T_k(K_l) = K_k^2 K_l,$$
$$T_k(E_l) = \frac{1}{q_k[2]_{q_k}}[E_k, E_k E_l - q_k^2 E_l E_k] = \frac{1}{q_k[2]_{q_k}} ad^+ E_k \circ ad^+ E_k(E_l),$$
$$T_k(F_l) = \frac{1}{q_k^3[2]_{q_k}}[F_k, F_k F_l - q_k^2 F_l F_k] = \frac{1}{q_k^3[2]_{q_k}} ad^- F_k \circ ad^- F_k(F_l).$$

(8.12.1.1) The inverse automorphisms of $E_q(\mathcal{L})$ are operating, such that $\forall_1^m k, l$:

$$T_k^{-1}(K_k) = L_k,\ T_k^{-1}(E_k) = -K_k F_k,\ T_k^{-1}(F_k) = -E_k L_k;$$

for $\Gamma_{kl} = -1$,

$$T_k^{-1}(K_l) = K_k K_l,$$
$$T_k^{-1}(E_l) = -q_k^{1/2}(E_k E_l - q_k^{-1} E_l E_k),\ T_k^{-1}(F_l) = q_k^{-1/2}(F_k F_l - q_k^{-1} F_l F_k);$$

for $\Gamma_{kl} = -2$,

$$T_k^{-1}(K_l) = K_k^2 K_l,$$
$$T_k^{-1}(E_l) = \frac{q_k}{[2]_{q_k}}[E_k, E_k E_l - q_k^{-2} E_l E_k],$$
$$T_k^{-1}(F_l) = \frac{1}{q_k[2]_{q_k}}[F_k, F_k F_l - q_k^{-2} F_l F_k].$$

(8.12.1.2) For $\Gamma_{kl} = -3$, which occurs only in the case of $\mathcal{L} := G_2$, for $k = 1, l = 2$, the automorphism T_1 is acting, such that

$$T_1(K_2) = T_1^{-1}(K_2) = K_1^3 K_2,$$

$$T_1(E_2) = \frac{1}{q^{3/2}[3]_q!}[E_1,[E_1,[E_1,E_2]_{q^3}]_q]_{q^{-1}} = \frac{1}{q^{3/2}[3]_q!}(ad^+E_1)^3(E_2),$$

$$T_1(F_2) = \frac{-1}{q^{9/2}[3]_q!}[F_1,[F_1,[F_1,F_2]_{q^3}]_q]_{q^{-1}} = \frac{-1}{q^{9/2}[3]_q!}(ad^-F_1)^3(F_2),$$

$$T_1^{-1}(E_2) = \frac{-q^{3/2}}{[3]_q!}[E_1,[E_1,[E_1,E_2]_{q^{-3}}]_{q^{-1}}]_q,$$

$$T_1^{-1}(F_2) = \frac{1}{q^{3/2}[3]_q!}[F_1,[F_1,[F_1,F_2]_{q^{-3}}]_{q^{-1}}]_q,$$

with the q-deformed commutators $[X,Y]_p := XY - pYX$, X and $Y \in E_q(\mathcal{L})$, for q and $p \in \mathcal{R}$.

(8.12.1.3) These automorphisms, which were introduced by G. Lusztig for the cases of A_m, $m \in \mathbf{N}$, and $D_m, m \geq 4$, fulfill the following braid-like relations. $\forall_1^m k, l$:

$$T_k \circ T_l = T_l \circ T_k \text{ for } \Gamma_{kl} = 0;$$

$$T_k \circ T_l(E_k) = E_l,\ T_k \circ T_l(F_k) = F_l,$$

$$T_k \circ T_l \circ T_k = T_l \circ T_k \circ T_l \text{ for } \Gamma_{kl} = \Gamma_{lk} = -1;$$

for $\Gamma_{kl} = -2$ or $\Gamma_{lk} = -2$,

$$T_k \circ T_l \circ T_k(E_l) = E_l,\ T_k \circ T_l \circ T_k(F_l) = F_l,\ T_k \circ T_l \circ T_k(K_l) = K_l,$$

$$T_k \circ T_l \circ T_k \circ T_l = T_l \circ T_k \circ T_l \circ T_k;$$

for $\Gamma_{kl} = -3$, which means $k = 1, l = 2$, in the case of $\mathcal{L} := G_2$,

$$T_1 \circ T_2 \circ T_1 \circ T_2 \circ T_1 \circ T_2 = T_2 \circ T_1 \circ T_2 \circ T_1 \circ T_2 \circ T_1.$$

(8.12.1.4) In the case of $\mathcal{L} := A_m, m \in \mathbf{N}$, the q-deformed positive root vectors

$$E_\beta := ad^+E_k \circ \cdots \circ ad^+E_{l-1}(E_l),\ \beta := \alpha_k + \cdots + \alpha_l,\ 1 \leq k < l \leq m,$$

can be constructed, inserting the Lusztig automorphisms into the longest word of the Weyl group. For instance in the case of $m = 3$, one finds the six q-deformed positive root vectors with complex coefficients below.

$$E_1 =: E_{\beta_1},\ T_1(E_2) = q^{-1/2}E_{\beta_2},\ T_1 \circ T_2(E_3) = q^{-1}E_{\beta_3},$$

$$T_1 \circ T_2 \circ T_3(E_1) = E_2 =: E_{\beta_4},\ T_1 \circ T_2 \circ T_3 \circ T_1(E_2) = q^{-1/2}E_{\beta_5},$$

$$T_1 \circ T_2 \circ T_3 \circ T_1 \circ T_2(E_1) = E_2 =: E_{\beta_6}.$$

(8.12.2) Let $\sigma_{k_1} \circ \cdots \circ \sigma_{k_p}$, $k_1, \ldots, k_p \in \{1, \ldots, m\}$, be the longest word of the Weyl group of $\mathcal{L}$, and $\beta_j, j = 1, \ldots, p$, the positive roots, which are ordered such that

$$\forall_2^p j : \beta_j := \sigma_{k_1} \circ \cdots \circ \sigma_{k_{j-1}}(\alpha_{k_j}),\ \beta_1 := \alpha_{k_1}.$$

Inserting the q-deformed positive and negative root vectors

$$E(\beta_1) := E_{k_1},\ E(\beta_2) := T_{k_1}(E_{k_2}), \ldots, E(\beta_p) := T_{k_1} \circ \cdots \circ T_{k_{p-1}}(E_{k_p}),$$
$$F(\beta_1) := F_{k_1},\ F(\beta_2) := T_{k_1}(F_{k_2}), \ldots, F(\beta_p) := T_{k_1} \circ \cdots \circ T_{k_{p-1}}(F_{k_p}),$$

one obtains the Poincaré-Birkhoff-Witt-like $\mathcal{R}$-linear basis:

$$\left\{E(\beta_p)^{r_p} \cdots E(\beta_1)^{r_1} K_1^{z_1} \cdots K_m^{z_m}; r_1, \ldots, r_p \in \mathbf{N}_0; z_1, \ldots, z_m \in \mathbf{Z}\right\}$$
$$\underset{free\ over\ \mathcal{R}}{\longrightarrow} E_q(\mathcal{B}^+),$$

of the Hopf subalgebra $E_q(\mathcal{B}^+)$ of $E_q(\mathcal{L})$, which is polynomially spanned by the set of generators $\{E_k, K_k, L_k; k = 1, \ldots, m\}$, and correspondingly some $\mathcal{R}$-linear basis of the q-deformed negative Borel subalgebra $E_q(\mathcal{B}^-)$, just replacing E_k by $F_k, k = 1, \ldots, m$.

(8.12.3) In order to construct the universal R-matrix, one needs the following $\mathcal{R}$-linear bases of $U_q^{fin}(\mathcal{B}^\pm)$, which are defined as the subalgebras of $U_q(\mathcal{L})$ consisting of polynomials in the generators $\{E_k, H_k; k = 1, \ldots, m\}$ and $\{F_k, H_k; k = 1, \ldots, m\}$, respectively:

$$\left\{E(\beta_p)^{r_p} \cdots E(\beta_1)^{r_1} H_1^{x_1} \cdots H_m^{x_m}; r_1, \ldots, x_m \in \mathbf{N}_0\right\} \underset{free\ over\ \mathcal{R}}{\longrightarrow} U_q^{fin}(\mathcal{B}^+),$$

and similarly of $U_q^{fin}(\mathcal{B}^-)$, replacing E_k by F_k, $k = 1, \ldots, m$.

(8.12.4) The universal R-matrix of the quasitriangular complex topological Hopf algebra $U_q(\mathcal{L})$ is calculated explicitly, using the quantum double construction.

(8.12.4.1) The Hopf algebras $E_q(\mathcal{B}^+)$ and $(E_q(\mathcal{B}^-))^{opp}$, the latter being equipped with the restricted opposite comultiplication of $E_q(\mathcal{L})$, are dual with respect to the following $\mathcal{R}$-bilinear form, which is described conveniently in terms of H_k instead of K_k and $L_k, k = 1, \ldots, m$. $\forall_1^m k, l$:

$$B_{kl} := d_k \Gamma_{kl} = B_{lk},\ \tilde{B}_{kl} := \Gamma_{kl} d_l^{-1} = \tilde{B}_{lk},$$

$$\langle \tilde{H}_k | H_l \rangle = \delta_{kl},\ \tilde{H}_k := \ln q \sum_{l=1}^{m} (\tilde{B}^{-1})_{kl} H_l,$$

$$\langle F_k | E_l \rangle = \frac{\delta_{kl}}{1 - q_k^{-2}},\ \langle F_k | H_l \rangle = \langle H_k | E_l \rangle = 0.$$

(8.12.4.2) One then calculates by induction, that $\forall_2^p i, j$:

$$\langle F(\beta_i)|E(\beta_j)\rangle = \frac{\delta_{ij}}{1-q_{k_j}^{-2}},\ \beta_j := \sigma_{k_1}\circ\cdots\circ\sigma_{k_{j-1}}(\alpha_{k_j}),$$

along the longest word of the Weyl group, and generally $\forall r_1,\ldots,y_m \in \mathbf{N}_0$:

$$\langle F(\beta_p)^{r_p}\cdots F(\beta_1)^{r_1}\tilde{H}_1^{x_1}\cdots\tilde{H}_m^{x_m}|E(\beta_p)^{s_p}\cdots E(\beta_1)^{s_1}H_1^{y_1}\cdots H_m^{y_m}\rangle$$
$$= \prod_{k=1}^{m}\delta_{x_k y_k}x_k!\prod_{j=1}^{p}\delta_{r_j s_j}\frac{[r_j]_{q_{k_j}}!}{(1-q_{k_j}^{-2})^{r_j}}q_{k_j}^{-r_j(r_j-1)/2}.$$

(8.12.4.3) The above Poincaré-Birkhoff-Witt-like bases can be used, as was shown explicitly in the case of $A_m, m \in \mathbf{N}$, in order to prove that this $\mathcal{R}$-bilinear form, and similarly the corresponding bilinear form:

$$U_q^{fin}(\mathcal{B}^-)\times U_q^{fin}(\mathcal{B}^+)\longrightarrow \mathcal{R},$$

are non-degenerate. Therefore again the quantum double construction can be used to construct the universal R-matrix of $U_q(\mathcal{L})$, for any simple finite-dimensional complex Lie algebra $\mathcal{L}$.

(8.12.4.4) The surjective homomorphism of topological Hopf algebras over $\mathcal{R}$, which is defined with respect to the h-adic topology on the quantum double:

$$D(U_q(\mathcal{B}^+)) := \overline{U_q(\mathcal{B}^+)\otimes U_q(\mathcal{B}^-)} = \overline{U_q^{fin}(\mathcal{B}^+)\otimes U_q^{fin}(\mathcal{B}^-)} \longrightarrow U_q(\mathcal{L}),$$

such that $\forall_1^m k$:

$$H_k\otimes I\longrightarrow H_k,\ I\otimes H_k\longrightarrow H_k,\ E_k\otimes I\longrightarrow E_k,\ I\otimes F_k\longrightarrow F_k,$$

then yields the universal R-matrix of $U_q(\mathcal{L})$, as image of that of the quantum double.

$$R = \sum_{r_1,\ldots,x_m\in\mathbf{N}_0} E(\beta_p)^{r_p}\cdots H_m^{x_m}\otimes F(\beta_p)^{r_p}\cdots\tilde{H}_m^{x_m}$$
$$\frac{1}{x_1!\cdots x_m!}\prod_{j=1}^{p}\frac{\left(1-q_{k_j}^{-2}\right)^{r_j}}{[r_j]_{q_{k_j}}!}q_{k_j}^{r_j(r_j-1)/2}.$$

Hence one calculates, that

$$R = \Big(\prod_{j=p}^{1}\sum_{r_j=0}^{\infty}\frac{\left(1-q_{k_j}^{-2}\right)^{r_j}}{[r_j]_{q_{k_j}}!}q_{k_j}^{r_j(r_j-1)/2}(E(\beta_j)\otimes F(\beta_j))^{r_j}\Big)$$
$$\exp\Big(\ln q\sum_{k,l=1}^{m}(\tilde{B}^{-1})_{kl}H_k\otimes H_l\Big),$$

the product being ordered according to decreasing $j = p,\ldots,1$.

(8.12.5) One may also use the Chevalley generators, which are defined such that $\forall_1^m k$:

$$\tilde{E}_k := K_k E_k = -\sigma(E_k), \ \tilde{F}_k := F_k L_k = -\sigma(F_k).$$

(8.12.5.1) Along the longest word $\sigma_{k_1} \circ \cdots \circ \sigma_{k_p}$ of the Weyl group, one then defines the q-deformed positive and negative root vectors.

$$\forall_2^p j : \tilde{E}(\beta_j) := T_{k_1}^{-1} \circ \cdots \circ T_{k_{j-1}}^{-1}(\tilde{E}_{k_j}), \ \tilde{F}(\beta_j) := T_{k_1}^{-1} \circ \cdots \circ T_{k_{j-1}}^{-1}(\tilde{F}_{k_j}).$$

The ordered monomials of these q-deformed root vectors, together with $K_k^{\pm 1}$, $k = 1, \ldots, m$, yield again $\mathcal{R}$-linear bases of $E_q(\mathcal{B}^{\pm})$.

(8.12.5.2) The above defined $\mathcal{R}$-bilinear form yields just the same values, inserting the new q-deformed root vectors, ordering their products with increasing $j = 1, \ldots, p$, and placing the toral generators $H_k, k = 1, \ldots, m$, on the left hand side of the involved monomials. Hence one obtains the expression

$$\begin{aligned} T(\sigma,\sigma)(R) = R = &\exp\Big(\ln q \sum_{k,l=1}^{m} (\tilde{B}^{-1})_{kl} H_k \otimes H_l\Big) \\ &\prod_{j=1}^{p} \sum_{r_j=0}^{\infty} \frac{\left(1-q_{k_j}^{-2}\right)^{r_j}}{[r_j]_{q_{k_j}}!} q_{k_j}^{r_j(r_j-1)/2} (\tilde{E}(\beta_j) \otimes \tilde{F}(\beta_j))^{r_j}, \end{aligned}$$

$$\beta_j := \sigma_{k_1} \circ \cdots \circ \sigma_{k_{j-1}}(\alpha_{k_j}).$$

(8.12.5.3) An application of $T(\sigma, \sigma)$ on one of these two expressions of R yields the other one.

(8.12.5.4) The above expressions of R are independent of the choice of the longest word of the Weyl group, due to the braid-like relations of Lusztig automorphisms.

(8.12.6) Denote by R_k an appropriate embedding of the universal R-matrix of the topological Hopf algebra $U_{q_k}(A_1)$, which is polynomially spanned over $\mathcal{R}$ by $X_k^{\pm}$ and H_k, and then completed and factorized in the sense of formal power series with relations, with respect to the h-adic topology. $\forall_1^m k$:

$$\begin{aligned} R_k &:= \sum_{n=0}^{\infty} \frac{\left(1-q_k^{-2}\right)^n}{[n]_{q_k}!} q_k^{n(n-1)/2} (E_k \otimes F_k)^n q_k^{\frac{1}{2} H_k \otimes H_k} \\ &= q_k^{\frac{1}{2} H_k \otimes H_k} \sum_{n=0}^{\infty} \frac{\left(1-q_k^{-2}\right)^n}{[n]_{q_k}!} q_k^{n(n-1)/2} (K_k E_k \otimes F_k L_k)^n =: q_k^{\frac{1}{2} H_k \otimes H_k} \tilde{R}_k. \end{aligned}$$

With these truncated universal R-matrices $\tilde{R}_k, k = 1, \ldots, m$, one obtains the factorization formula

$$R = \exp\Big(\ln q \sum_{k,l=1}^{m} (\tilde{B}^{-1})_{kl} H_k \otimes H_l\Big) \tilde{R},$$

$$\tilde{R} = \tilde{R}_{k_1} \prod_{j=2}^{p} (T_{k_1}^{-1} \otimes T_{k_1}^{-1}) \circ \cdots \circ (T_{k_{j-1}}^{-1} \otimes T_{k_{j-1}}^{-1})(\tilde{R}_{k_j})$$
$$= \left(\prod_{j=1}^{p-1} (T_{k_p} \otimes T_{k_p}) \circ \cdots \circ (T_{k_{j+1}} \otimes T_{k_{j+1}}) (\tilde{R}_{k_j}) \right) \tilde{R}_{k_p}.$$

8.13 Quantum Weyl Group

Let $\mathcal{L}$ be any simple finite-dimensional complex Lie algebra of rank m, and consider the topological Hopf algebra $U_q(\mathcal{L})$ over $\mathcal{R} := \mathbf{C}[[h]]$.

(8.13.1) The Lusztig automorphisms T_k of $U_q(\mathcal{L}), k = 1, \ldots, m$, in the sense of unital associative $\mathcal{R}$-algebras, can be reconstructed from an appropriate extension $\overline{U_q(\mathcal{L})}$ of $U_q(\mathcal{L})$ by means of invertible generators $W_1, \ldots, W_m$, which fulfill the following commutation and braid-like relations.

$$\forall_1^m k, \forall Y \in U_q(\mathcal{L}) : T_k(Y) = W_k Y W_k^{-1}.$$

Here one defines, that

$$\forall_1^m k \neq l : T_k(H_l) := H_l - \Gamma_{kl} H_k, \;\; T_k(H_k) = -H_k.$$

$$\begin{aligned}\forall_1^m k, l : \; & W_k W_l = W_l W_k, \;\; \Gamma_{kl} = 0; \\ & W_k W_l W_k = W_l W_k W_l, \;\; \Gamma_{kl} = -1; \\ & W_k W_l W_k W_l = W_l W_k W_l W_k, \;\; \Gamma_{kl} = -2; \\ & W_k W_l W_k W_l W_k W_l = W_l W_k W_l W_k W_l W_k, \;\; \Gamma_{kl} = -3.\end{aligned}$$

(8.13.2) In the case of A_1, including the new generator W with the relations

$$WEW^{-1} = -FL, \;\; WFW^{-1} = -KE, \;\; WKW^{-1} = L,$$

one obtains some topological Hopf algebra $\overline{U_q(A_1)}$ over $\mathcal{R}$, with the structure mappings:

$$W \underset{\Delta}{\longrightarrow} R^{-1} q^{\frac{1}{2} H \otimes H} (W \otimes W), \;\; \varepsilon(W) = 1,$$

and the antipode being determined by its defining property, just as that of V below. Obviously, denoting

$$q^{-\frac{1}{4}H^2 - H} W =: V, \;\; \Delta(V) = R^{-1}(V \otimes V), \;\; \varepsilon(V) = 1.$$

The coassociative property just means, that R fulfills the QYBE. Moreover the topological Hopf algebra $\overline{U_q(A_1)}$ is quasitriangular with respect to R. Here the corresponding algebraic completion $\overline{U_q(A_1)}$ of $U_q(A_1)$, such that $WH = -HW$, is inserted.

(8.13.3) With the universal R-matrix denoted by $R =: \sum_{k=0}^{\infty} R'_k \otimes R''_k$, the square of the antipode of $U_q(A_1)$ is generated by the following invertible element U.

$$U := \sum_{k=0}^{\infty} \sigma(R''_k)R'_k = q^{-\frac{1}{2}H^2} \sum_{n=0}^{\infty} (-1)^n \frac{(1-q^{-2})^n}{[n]_q!} q^{-\frac{n(n+3)}{2}} F^n E^n L^n,$$

$$U^{-1} = \sum_{k=0}^{\infty} R''_k \sigma^2(R'_k) = q^{\frac{1}{2}H^2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(n+3)}{2}} F^n E^n K^n;$$

$$\forall Y \in U_q(A_1) : \sigma^2(Y) = UYU^{-1}, \; \sigma^2(U) = U.$$

Similarly one calculates, that

$$\sigma(U) = \sum_{k=0}^{\infty} R'_k \sigma(R''_k), \; \sigma(U^{-1}) = \sum_{k=0}^{\infty} \sigma^2(R'_k) R''_k.$$

Applying $T(\sigma, id)$ on $\Delta(V) = R^{-1}(V \otimes V)$ yields, that

$$U = \sigma(V)V, \; \sigma(U) = V\sigma(V).$$

Both V^2 and UL belong to the centre of $\overline{U_q(A_1)}$. Moreover

$$\sigma(UL) = UL, \text{ hence } (UL)^2 = U\sigma(U).$$

(8.13.4) Since the universal R-matrix of $U_q(A_1)$ fulfills the equality

$$(V \otimes V)R(V \otimes V)^{-1} = \tau(R),$$

one finds the following central and group-like element Z of $\overline{U_q(A_1)}$.

$$Z := (UL)^{-1}V^2, \; \Delta Z = Z \otimes Z.$$

(8.13.5) For $q \in \mathbf{D}$, the complex representation of $\overline{U_q(A_1)}$, such that $X^{\pm} = \sigma^{\pm}$, reads:

$$E = \begin{bmatrix} 0 & q^{1/2} \\ 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \\ q^{1/2} & 0 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; W = q^{-1} \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix},$$

$$V = q^{-1/4} \begin{bmatrix} 0 & -q^{-1} \\ 1 & 0 \end{bmatrix}, \; U = q^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & q^{-2} \end{bmatrix}, \; \sigma(U) = q^{-1/2} \begin{bmatrix} q^{-2} & 0 \\ 0 & 1 \end{bmatrix}.$$

More generally for generic q, consider the irreducible complex representations of $U_q(A_1)$ on $\mathbf{C}^{2j+1}$ with highest weights

$$2j,\ j = \frac{1}{2}, 1, \frac{3}{2}, \ldots,\ m = j, j-1, \ldots, -j.$$

Then

$$V_j v^j_m = (-1)^{j-m} q^{m-j(j+1)} v^j_{-m},$$

$$UL = q^{-2j(j+1)} I_{2j+1},\ \ Z = (-1)^{2j} I_{2j+1},$$

hereby denoting by V_j the corresponding representation of V. These representations of Z indicate, that one may identify Z^2 with the unit of $\overline{U_q(A_1)}$. The action of V_j on v^j_j is normalized, such that one obtains the usual tensor product of complex representations of the algebraic extension $\overline{U_q(A_1)}$.

(8.13.6) Again for a simple finite-dimensional complex Lie algebra $\mathcal{L}$ of rank m, the commutation relations of $W_k, k = 1, \ldots, m$, with the generators of $U_q(\mathcal{L})$, are defined by their inner automorphism property. The resulting extension $\overline{U_q(\mathcal{L})}$ becomes some topological Hopf algebra over $\mathcal{R}$, the so-called quantum Weyl group, with the structure maps such that $\forall_1^m k$:

$$W_k \underset{\Delta}{\longrightarrow} R_k^{-1} q_k^{\frac{1}{2} H_k \otimes H_k} (W_k \otimes W_k) = \tilde{R}_k^{-1} (W_k \otimes W_k),\ \ \varepsilon(W_k) = 1.$$

(8.13.6.1) Obviously

$$(W_k \otimes W_k) R_k (W_k^{-1} \otimes W_k^{-1}) q_k^{-\frac{1}{2} H_k \otimes H_k} = q_k^{-\frac{1}{2} H_k \otimes H_k} \tau(R_k),$$

inserting the flip τ of the tensor product. The comultiplication is compatible with the braid-like relations, and also with the inner automorphism property of the Weyl elements $W_k, k = 1, \ldots, m$, which means that $\forall Y \in \overline{U_q(\mathcal{L})}$:

$$\Delta \circ T_k(Y) = \tilde{R}_k^{-1} T(T_k, T_k) \circ \Delta(Y) \tilde{R}_k.$$

(8.13.7) The universal R-matrix of the quasitriangular topological Hopf algebra $U_q(\mathcal{L})$ over $\mathcal{R}$ is factorized such that

$$R = \exp\Big(\ln q \sum_{k,l=1}^{m} (\tilde{B}^{-1})_{kl} H_k \otimes H_l\Big) \tilde{R},\ \ \tilde{R} = (W_{max} \otimes W_{max}) \Delta \left(W_{max}^{-1}\right),$$

with the symmetric matrix $\tilde{B} \in Mat(m, \mathbf{Z})$ such that $\tilde{B}_{kl} := \Gamma_{kl} d_l^{-1}$, and $W_{max} := W_{k_1} \cdots W_{k_p}$, along the longest word $\sigma_{k_1} \circ \cdots \circ \sigma_{k_p}$ of the Weyl group.

8.14 q-Deformation of Oscillator Algebras

(8.14.1) Let $H_q(1)$ be the unital associative $\mathbf{C}[[h]]$-algebra of formal power series in the generators $a, a^\dagger, N$, with the following relations.

$$[N,a] = -a,\ [N,a^\dagger] = a^\dagger,$$
$$[a,a^\dagger] = [N+I]_q - [N]_q = \frac{q^{N+\frac{1}{2}} + q^{-N-\frac{1}{2}}}{q^{1/2}+q^{-1/2}}.$$

(8.14.1.1) The last relation could be gained from the relations

$$aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N},$$

but the latter are not compatible with the following counit. The unital associative $\mathbf{C}[[h]]$-algebra with these stronger relations is denoted by $\tilde{H}_q(1)$.

(8.14.2) With the uniformly continuous structure mappings indicated below, $H_q(1)$ becomes some topological Hopf algebra over $\mathcal{R} := \mathbf{C}[[h]]$, with respect to the h-adic topology.

$$\Delta(a^\ddagger) = e^{-\frac{1}{2}\theta_z}a^\ddagger \otimes q^{\frac{1}{2}(N+\frac{1}{2})} + e^{\frac{1}{2}\theta_z}q^{-\frac{1}{2}(N+\frac{1}{2})} \otimes a^\ddagger,$$
$$\Delta(N) = N\otimes I + I\otimes N + \left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right) I\otimes I,$$
$$\varepsilon(a) = \varepsilon(a^\dagger) = 0,\ \varepsilon(N) = -\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right),$$
$$\sigma(a) = -q^{-1/2}a,\ \sigma(a^\dagger) = -q^{1/2}a^\dagger,\ \sigma(N) = -N - 2\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right),$$

with an angle $\theta_z := \frac{\pi}{2} + 2\pi z,\ e^{i\theta_z} = i,\ e^{2i\theta_z} = -1,\ z\in\mathbf{Z}$, and $a^\ddagger$ denoting both a and $a^\dagger$.

(8.14.3) These structure mappings can be rewritten, via an appropriate homomorphism of unital associative $\mathcal{R}$-algebras, which is also compatible with Δ, ε, and σ, in terms of polynomials over $\mathcal{R}$.

$$K \longrightarrow q^{N+\frac{1}{2}};\ \Delta(K) = -iK\otimes K,\ \varepsilon(K) = i,\ \sigma(K) = -K^{-1};$$
$$K^{\pm 1/2} \longrightarrow q^{\pm\frac{1}{2}(N+\frac{1}{2})};\ \Delta(K^{\pm 1/2}) = e^{\mp\frac{1}{2}\theta_z}K^{\pm 1/2}\otimes K^{\pm 1/2},$$
$$\varepsilon(K^{\pm 1/2}) = e^{\pm\frac{1}{2}\theta_z},\ \sigma(K^{\pm 1/2}) = \pm iK^{\mp 1/2};$$
$$\Delta(a^\ddagger) = a^\ddagger\otimes e^{-\frac{1}{2}\theta_z}K^{1/2} + e^{\frac{1}{2}\theta_z}K^{-1/2}\otimes a^\ddagger,\ \varepsilon(a^\ddagger) = 0,$$

the antipode acting on $a^\ddagger$ as indicated above;

$$[a,a^\dagger] = \frac{K+K^{-1}}{q^{1/2}+q^{-1/2}},\ Ka = q^{-1}aK,\ Ka^\dagger = qa^\dagger K.$$

(8.14.3.1) The $\mathcal{R}$-linear map $T(\Delta, id) \circ \Delta$ is acting such that:

$$a^\dagger \longrightarrow\!\!\!\longleftarrow\!\!\!\longrightarrow$$
$$-ia^\dagger \otimes K^{1/2} \otimes K^{1/2} + K^{-1/2} \otimes a^\dagger \otimes K^{1/2} + iK^{-1/2} \otimes K^{-1/2} \otimes a^\dagger.$$

(8.14.3.2) With respect to these polynomials, one could also insert a complex parameter $q \in \mathbf{D} := \mathbf{C}\backslash(] - \infty, 0] \cup \{1\})$.

(8.14.4) One easily establishes some homomorphism of unital associative $\mathcal{R}$-algebras:

$$U_q(A_1) \longrightarrow \tilde{H}_q(1) \otimes \tilde{H}_q(1),$$

which is called oscillator or boson representation of the former:

$$X^+ \longrightarrow a_1^\dagger \otimes a_2,\ X^- \longrightarrow a_1 \otimes a_2^\dagger,\ H \longrightarrow N_1 \otimes I_2 - I_1 \otimes N_2.$$

Unfortunately this representation is not compatible with the involved comultiplications. Of course, this tensor product over $\mathcal{R}$ describes two non-interacting q-deformed harmonic oscillators.

(8.14.5) The two relations, which are not compatible with the counit, obviously imply

$$aa^\dagger = [N + I]_q,\ a^\dagger a = [N]_q.$$

The Hamilton operator of this q-deformed harmonic oscillator may be defined by

$$H := \frac{1}{2}(aa^\dagger + a^\dagger a) = \frac{1}{2}([N + I]_q + [N]_q).$$

(8.14.6) Consider the non-generic case of $q^p = 1$, for $p \in \{3, 4, \ldots\}$. Then $[p]_q = 0$ implies, that there is some representation of $H_q(1)$ on $\mathbf{C}^p$, with $a^\dagger$ being the transposed of a, such that the complex matrix H is symmetric. $\forall_1^p k, l$:

$$a_{kl} = a^\dagger_{lk} = \delta_{k+1,l}([k]_q)^{1/2},\ N_{kl} = \delta_{kl}(k-1),\ ([N + I]_q)_{pp} = [p]_q = 0.$$

In this case $a^p = (a^\dagger)^p = 0$.

(8.14.6.1) For instance take $p = 3$. $\xi := ([2]_q)^{1/2} = (q + q^{-1})^{1/2}$;

$$q^3 = 1;\ a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{bmatrix},\ a^2 = \begin{bmatrix} 0 & 0 & \xi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ a^3 = 0,\ N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(8.14.7) For any formal power series ϕ over $\mathcal{R}$, the relations of $H_q(1)$ imply that

$$\forall k \in \mathbf{N} : (a^\dagger)^k \phi(N) = \phi(N - kI)(a^\dagger)^k, \ a^k \phi(N) = \phi(N + kI)a^k.$$

(8.14.8) $\forall n \in \mathbf{N}$:

$$a(K^{1/2}a^\dagger)^n = q^{n/2}((K^{1/2}a^\dagger)^n a + (K^{1/2}a^\dagger)^{n-1}K^{1/2}(M_0 + \cdots + M_{n-1})),$$

$$\forall k \in \mathbf{N}_0 : M_k := \frac{q^k K + q^{-k}K^{-1}}{q^{1/2} + q^{-1/2}}, \ M_0 = [a, a^\dagger], \ M_k a^\dagger = a^\dagger M_{k+1}.$$

This recursion will be needed in order to calculate the coefficients of the universal R-matrix.

(8.14.9) The q-deformed harmonic oscillator algebra $H_q(1)$ is quasitriangular, with the following universal R-matrix $R \in H_q(1)\overline{\otimes} H_q(1)$.

(8.14.9.1) Similarly as in the case of $U_q(A_1)$, starting from the ansatz

$$R := \exp(d_1 N \otimes N + d_2(N \otimes I + I \otimes N) + d_3 I \otimes I)$$
$$\sum_{n=0}^{\infty} c_n \left(K^{x/2}a^\dagger \otimes K^{-y/2}a\right)^n,$$

$T(\Delta, id)(R) = R_{13}R_{23}$ determines the parameters d_1, d_2, d_3, and $x = 1$; $T(id, \Delta)(R) = R_{13}R_{12}$ yields also d_1, d_2, d_3, and $y = 1$.

$$d_1 = \ln q, \ d_2 = \frac{1}{2}\ln q - i\theta_z, \ d_3 = d_2\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right).$$

Moreover each of these two equations implies, that

$$\forall k, l \in \mathbf{N}_0 : \frac{c_k c_l}{c_{k+l}} = \begin{bmatrix} k+l \\ k \end{bmatrix}_{q^{-1/2}} = q^{-kl/2}\frac{[k+l]_{q^{1/2}}!}{[k]_{q^{1/2}}![l]_{q^{1/2}}!}, \ c_0 = 1.$$

(8.14.9.2) $R\Delta(K) = \tau \circ \Delta(K)R = \Delta(K)R$ yields no further conditions on the coefficients. $R\Delta(a) = \tau \circ \Delta(a)R$ yields the following recursion formula.

$$c_1 = i(1 + q^{-1})c_0, \ c_2 = -(1 + q^{-1})c_0,$$

$$\forall n \in \mathbf{N} : \frac{c_n}{c_{n-1}} = i(1 + q^{-1})\frac{q^{\frac{n-1}{2}}}{[n]_{q^{1/2}}}.$$

Therefore

$$\forall n \in \mathbf{N} : c_n = i^n(1 + q^{-1})^n \frac{q^{n(n-1)/4}}{[n]_{q^{1/2}}!}, \ c_0 = 1.$$

(8.14.9.3) Hence one obtains

$$R = \exp\left(\ln q\; N \otimes N + \left(\frac{1}{2}\ln q - i\theta_z\right)\Delta(N)\right)$$
$$\sum_{n=0}^{\infty} c_n \left(K^{1/2}a^\dagger \otimes K^{-1/2}a\right)^n .$$

(8.14.9.4)

$$R^{-1} = T(\sigma, id)(R) = \exp\left(-\ln q\; N \otimes N - \left(\frac{1}{2}\ln q - i\theta_z\right)\Delta(N)\right)$$
$$\sum_{n=0}^{\infty}(-i)^n(1+q)^n \frac{q^{-\frac{1}{4}n(n-1)}}{[n]_{q^{1/2}}!}\left(K^{-\frac{1}{2}}a^\dagger \otimes K^{\frac{1}{2}}a\right)^n .$$

(8.14.10) Inserting the involution of $\mathcal{R}$, which is due to: $h \longleftrightarrow -h$ and complex conjugation, the star operation on the topological Hopf algebra $H_q(1)$, such that:

$$N^* = N,\ K \underset{*}{\longleftrightarrow} K^{-1},\ a \underset{*}{\longleftrightarrow} a^\dagger,$$

is compatible with the comultiplication in the sense that:

$$\Delta \circ * = \tau \circ \overline{T(*,*)} \circ \Delta.$$

(8.14.10.1) With respect to the polynomial version, an according star operation can be defined for $q \in \mathbf{D}$ and $|q| = 1$, using complex conjugation.

(8.14.11) The q-deformed harmonic oscillator can be reconstructed by factorizing an appropriate quantum double, from the $\mathcal{R}$-Hopf algebras, which are generated by the sets $\{K^{\pm 1/2}, a\}$ and $\{K^{\pm 1/2}, a^\dagger\}$, with the corresponding restrictions of structure mappings.

(8.14.11.1) An appropriate $\mathcal{R}$-bilinear form is constructed, such that the Hopf algebras $\mathcal{A} := \{K, a^\dagger; \mu, \eta, \Delta, \varepsilon\}$ and $\{K, a; \mu, \eta, \Delta^{opp}, \varepsilon\}$, with the antipodes σ and σ^{-1}, are dual with respect to it; here $\mu, \eta, \Delta, \varepsilon, \sigma$ denote the restrictions of the corresponding structure maps of $H_q(1)$. Denote

$$\langle K^{1/2}|K^{1/2}\rangle =: \kappa,\ \langle a|a^\dagger\rangle =: \alpha;$$
$$\langle K^{\pm 1/2}|K^{\mp 1/2}\rangle = \frac{1}{\kappa}e^{i\theta_z},\ \langle K^{-1/2}|K^{-1/2}\rangle = -\kappa,$$
$$\langle K^{\pm 1/2}|K^{\pm 1}\rangle = \langle K^{\pm 1}|K^{\pm 1/2}\rangle = \kappa^2 e^{\mp i\theta_z/2},$$
$$\langle K^{\pm 1}|K^{\pm 1}\rangle = -\kappa^4,\ \langle K^{\pm 1}|K^{\mp 1}\rangle = \frac{1}{\kappa^4};$$
$$\langle a|Ka^\dagger - qa^\dagger K\rangle = \langle \Delta^{opp}(a)|K \otimes a^\dagger - qa^\dagger \otimes K\rangle = 0 \Longrightarrow \kappa^4 = q.$$

(8.14.11.2) The quantum double is constructed, using this non-degenerate $\mathcal{R}$-bilinear form.

$$D(\mathcal{A}) := \{K, a^\dagger; \mu, \eta, \Delta, \varepsilon\} \otimes \{K, a; \mu, \eta, \Delta, \varepsilon\},$$

with the usual tensor product of $\mathcal{R}$-coalgebras, and the double cross product, inserting the restrictions of Δ from $H_q(1)$ into the latter.

(8.14.11.3) In order to prove that this $\mathcal{R}$-bilinear form is non-degenerate, one needs some $\mathcal{R}$-bilinear bases of the Hopf algebras, which are generated by $\{K^{\pm 1/2}, a\}$ and $\{K^{\pm 1/2}, a^\dagger\}$, respectively. Take for instance:

$$\{a^p K^{z/2}; p \in \mathbf{N}_0, z \in \mathbf{Z}\} \underset{free\ over\ \mathcal{R}}{\longrightarrow} \mathcal{R} - alg\ span\{K^{\pm\frac{1}{2}}, a\},$$

and similarly for $a^\dagger$ and $K^{\pm 1/2}$.

(8.14.11.4) Calculating the double cross product one finds, that

$$\begin{aligned}
&[K \otimes I, I \otimes K] = 0;\\
&(I \otimes K)(a^\dagger \otimes I) = q(a^\dagger \otimes I)(I \otimes K),\\
&(K \otimes I)(a^\dagger \otimes I) = q(a^\dagger \otimes I)(K \otimes I),\\
&(I \otimes K)(I \otimes a) = q^{-1}(I \otimes a)(I \otimes K),\\
&(K \otimes I)(I \otimes a) = q^{-1}(I \otimes a)(K \otimes I);
\end{aligned}$$

$$[I \otimes a, a^\dagger \otimes I] = q^{-1/2}\alpha\kappa K^{1/2} \otimes K^{1/2} - \frac{\alpha}{\kappa} K^{-1/2} \otimes K^{-1/2}.$$

(8.14.11.5) Inserting the parameters

$$\kappa := \pm i q^{1/4}, \ \alpha := \frac{\mp q^{1/4}}{q^{1/2} + q^{-1/2}},$$

one obtains

$$[I \otimes a, a^\dagger \otimes I] = \frac{K^{1/2} \otimes K^{1/2} + K^{-1/2} \otimes K^{-1/2}}{q^{1/2} + q^{-1/2}}.$$

Hence the polynomial version of the q-deformed harmonic oscillator is reconstructed as Hopf algebra over $\mathcal{R}$, factorizing this quantum double with respect to the relation $\{K^{1/2} \otimes I - I \otimes K^{1/2}\}$.

(8.14.12) Concerning systems with finitely many fermionic degrees of freedom, since the generators of different degrees of freedom are assumed to be graded-commuting, it suffices to consider one degree of freedom. Choose the following relations and costructure maps of odd generators $a^{\ddagger} \equiv a$ or $a^{\dagger}$, and an even generator N.

$$\begin{aligned}
&Na - aN = -a, \;\; Na^{\dagger} - a^{\dagger}N = a^{\dagger}, \;\; (a^{\ddagger})^2 = 0,\\
&a^{\dagger}a + aa^{\dagger} = \frac{K + K^{-1}}{q^{1/2} + q^{-1/2}}, \;\; K := q^{N+\frac{1}{2}};\\
&\Delta(a^{\ddagger}) = e^{-\frac{1}{2}\theta_z} a^{\ddagger} \otimes K^{1/2} + e^{\frac{1}{2}\theta_z} K^{-1/2} \otimes a^{\ddagger},\\
&\Delta(N) = N \otimes I + I \otimes N + \left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right) I \otimes I,\\
&\varepsilon(a^{\ddagger}) = 0, \;\; \varepsilon(N) = -\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right),\\
&\sigma(a) = -q^{-\frac{1}{2}}a, \;\; \sigma(a^{\dagger}) = -q^{\frac{1}{2}}a^{\dagger}, \;\; \sigma(N) = -N - 2\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right),
\end{aligned}$$

denoting $\theta_z := \frac{\pi}{2} + 2\pi z, z \in \mathbf{Z}$. Then obviously

$$\Delta(K) = -iK \otimes K, \;\; \varepsilon(K) = i, \;\; \sigma(K) = -K^{-1}.$$

(8.14.12.1) The last one of the above relations could be obtained from

$$aa^{\dagger} + q^{\pm 1}a^{\dagger}a = q^{\pm(N+I)}, \text{ hence } a^{\dagger}a = [N+I]_q, \;\; aa^{\dagger} = -[N]_q,$$

but these relations are not compatible with the above counit. The homomorphic image of $H_q^f(1)$, which is due to these stronger relations, is denoted by $\tilde{H}_q^f(1)$.

(8.14.12.2) With these $\mathcal{R}$-linear structure mappings one establishes some $\mathbf{Z}_2$-graded topological Hopf algebra $H_q^f(1)$, which is quasitriangular with the universal R-matrix

$$\begin{aligned}
R = \; &\exp\left(\ln q N \otimes N + \left(\frac{\ln q}{2} - i\theta_z\right)\Delta(N)\right)\\
&\left(I \otimes I + i\left(1 + q^{-1}\right) K^{\frac{1}{2}} a^{\dagger} \otimes K^{-\frac{1}{2}} a\right).
\end{aligned}$$

As in the bosonic case, R^{-1} is obtained from R by the substitutions:

$$q \longleftrightarrow q^{-1}, \text{ which means}: \; h \longleftrightarrow -h, \text{ and}: \; \theta_z \longleftrightarrow -\theta_z.$$

(8.14.12.3) Again one finds the following graded star operation on $H_q^f(1)$:

$$N^* = N, \;\; K \underset{*}{\longleftrightarrow} K^{-1}, \;\; a \underset{*}{\longleftrightarrow} a^{\dagger}.$$

(8.14.12.4) For $q \in \mathbf{D}$, inserting the complex representation:

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; a^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; N = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

one easily calculates the according representation:

$$R \longleftrightarrow iq^{\frac{1}{4}} \exp\left(-\frac{\theta_z^2}{\ln q}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -iq^{-\frac{1}{2}} & 0 & 0 \\ 0 & -ic_1 & -iq^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \; c_1 := i\left(1 + q^{-1}\right).$$

8.15 Oscillator and Spinor Representations of q-Deformed Universal Enveloping Algebras

(8.15.1) Consider m commuting q-deformed harmonic oscillators, using for instance the tensor product of algebras over $\mathcal{R} := \mathbf{C}[[h]]$, with the natural embeddings:

$$\begin{aligned} \tilde{H}_q(1) \ni X &\longrightarrow I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots I \\ &=: X_k \in \bigotimes^m \tilde{H}_q(1) =: \tilde{H}_q(m) \equiv \tilde{H}_q^b(m), \end{aligned}$$

placing X at the position $k = 1, \ldots, m, \;\; m = 2, 3 \ldots$.
The homomorphism of unital associative $\mathcal{R}$-algebras:

$$U_q(A_m) \longrightarrow \tilde{H}_q^b(m+1),$$

such that

$$\forall_1^m k : X_k^+ \longrightarrow a_k^\dagger a_{k+1}, \; X_k^- \longrightarrow a_k a_{k+1}^\dagger, \; H_k \longrightarrow N_k - N_{k+1},$$

is called oscillator or boson representation of $U_q(A_m), m \in \mathbf{N}$.

(8.15.2) The system $H_q^f(m)$ of m graded-commuting q-deformed fermionic oscillators is defined by the following relations of even generators N_k, and odd generators a_k and $a_k^\dagger$, $k = 1, \ldots, m$. $\forall_1^m k, l$:

$$\begin{aligned} &[N_k, N_l]_- = 0, \\ &[a_k, a_l]_+ = [a_k^\dagger, a_l^\dagger]_+ = [a_k, a_l^\dagger]_+ = 0 \text{ for } k \neq l, \\ &a_k^2 = (a_k^\dagger)^2 = 0, \\ &[N_k, a_l]_- = -\delta_{kl} a_l, \; [N_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger, \\ &[a_k, a_k^\dagger]_+ = \frac{q^{N_k + \frac{1}{2}} + q^{-N_k - \frac{1}{2}}}{q^{1/2} + q^{-1/2}}. \end{aligned}$$

Here the super-commutator is denoted explicitly by

$$[a,b]_{\pm} := ab \pm ba, \text{ for } a,b \in H_q^f(m),$$

the anticommutator occurring if and only if both a and b are odd. The last relation is implied by demanding, that $\forall_1^m k$:

$$a_k a_k^{\dagger} + q^{\pm 1} a_k^{\dagger} a_k = q^{\mp N_k}, \text{ hence } a_k a_k^{\dagger} = [N_k + I]_q,\ a_k^{\dagger} a_k = -[N_k]_q,$$

but these relations are not compatible with the counit below. Nevertheless the latter relations are useful for an establishment of the subsequent spinor representations. Including these stronger relations, the resulting unital associative $\mathcal{R}$-superalgebra is denoted by $\tilde{H}_q^f(m)$.

(8.15.2.1) One establishes $H_q^f(m)$ as some topological $\mathbf{Z}_2$-graded Hopf algebra over $\mathcal{R}$, with the structure mappings such that $\forall_1^m k$:

$$\Delta(a_k^{\dagger}) = e^{-\frac{i}{2}\theta_z} a_k^{\dagger} \otimes K_k^{1/2} + e^{\frac{i}{2}\theta_z} K_k^{-1/2} \otimes a_k^{\dagger},$$

$$\Delta(N_k) = N_k \otimes I + I \otimes N_k + \left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right) I \otimes I,$$

$$\varepsilon(a_k) = \varepsilon(a_k^{\dagger}) = 0,\ \varepsilon(N_k) = -\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right),$$

$$\sigma(a_k) = -q^{-1/2} a_k,\ \sigma(a_k^{\dagger}) = -q^{1/2} a_k^{\dagger},\ \sigma(N_k) = -N_k - 2\left(\frac{1}{2} - \frac{i\theta_z}{\ln q}\right);$$

here one again denotes $K_k^{1/2} := q^{\frac{1}{2}(N_k + \frac{1}{2})}$, such that

$$\Delta(K_k^{1/2}) = e^{-\frac{i}{2}\theta_z} K_k^{1/2} \otimes K_k^{1/2},\ \varepsilon(K_k^{1/2}) = e^{\frac{i}{2}\theta_z},\ \sigma(K_k^{1/2}) = iK_k^{-1/2},$$

with an angle $\theta_z := \frac{\pi}{2} + 2\pi z,\ e^{i\theta_z} = i,\ z \in \mathbf{Z}$;

$$K_k a_k = q^{-1} a_k K_k,\ K_k a_k^{\dagger} = q a_k^{\dagger} K_k,\ a_k a_k^{\dagger} + a_k^{\dagger} a_k = \frac{K_k + K_k^{-1}}{q^{1/2} + q^{-1/2}}.$$

(8.15.2.2) Obviously there is an isomorphism of unital associative $\mathcal{R}$-superalgebras, of $\tilde{H}_q^f(m)$ onto the skew-symmetric tensor product of m copies of $\tilde{H}_q^f(1)$.

(8.15.3) The homomorphism of unital associative $\mathcal{R}$-algebras:

$$U_q(A_m) \longrightarrow \tilde{H}_q^f(m+1),$$

such that $\forall_1^m k$:

$$X_k^+ \longrightarrow a_k^{\dagger} a_{k+1},\ X_k^- \longrightarrow -a_{k+1}^{\dagger} a_k,\ H_k \longrightarrow N_k - N_{k+1},$$

is called spinor or fermion representation of $U_q(A_m), m \in \mathbf{N}$.

(8.15.4) The homomorphism of unital associative $\mathcal{R}$-algebras:

$$U_{q^{1/2}}(B_m) \longrightarrow \tilde{H}_q^f(m),$$

such that

$$\forall_1^{m-1} k : X_k^+ \longrightarrow a_k^\dagger a_{k+1},\ X_k^- \longrightarrow -a_{k+1}^\dagger a_k,\ H_k \longrightarrow N_k - N_{k+1},$$

$$X_m^+ \longrightarrow a_m^\dagger,\ X_m^- \longrightarrow a_m,\ H_m \longrightarrow 2N_m + \left(1 \pm \frac{2\pi i}{\ln q}\right) I,$$

is called spinor or fermion representation of the involved q-deformed universal enveloping algebra, $m \geq 2$. Note that, for instance,

$$[X_m^+, X_m^-] = [H_m]_{q^{1/2}}.$$

(8.15.5) An oscillator or boson representation of $U_q(C_m)$, $m \geq 3$, is defined as the homomorphism of unital associative $\mathcal{R}$-algebras:

$$U_q(C_m) \longrightarrow \tilde{H}_q^b(m),$$

such that

$$\forall_1^{m-1} k : X_k^+ \longrightarrow a_k^\dagger a_{k+1},\ X_k^- \longrightarrow a_k a_{k+1}^\dagger,\ H_k \longrightarrow N_k - N_{k+1},$$

$$X_m^+ \longrightarrow \frac{1}{q+q^{-1}}(a_m^\dagger)^2,\ X_m^- \longrightarrow \frac{-1}{q+q^{-1}} a_m^2,\ H_m \longrightarrow N_m + \frac{1}{2} I.$$

(8.15.6) One also establishes an appropriate spinor or fermion representation for $m \geq 4$:

$$U_q(D_m) \longrightarrow \tilde{H}_q^f(m),$$

such that

$$\forall_1^{m-1} k : X_k^+ \longrightarrow a_k^\dagger a_{k+1},\ X_k^- \longrightarrow -a_{k+1}^\dagger a_k,\ H_k \longrightarrow N_k - N_{k+1},$$

$$X_m^+ \longrightarrow a_{m-1}^\dagger a_m^\dagger,\ X_m^- \longrightarrow a_m a_{m-1},\ H_m \longrightarrow N_{m-1} + N_m + \left(1 \pm \frac{i\pi}{\ln q}\right) I.$$

Here $[X_m^+, X_m^-] = [H_m]_q$.

8.16 Main Commutation Relations and Matrix Quantum Semigroups

(8.16.1) Let $\mathcal{A}_R$ be the unital associative algebra over a field K, which is generated by the set $\{a_{ij}; i,j = 1,\ldots,d\}$, and factorized by the main commutation relations (MCR) with respect to an invertible element

$$R \in Mat(d,K) \otimes Mat(d,K),$$

such that $\forall_1^d i,k,p,q$:

$$\sum_{j,l=1}^{d} (R_{ijkl}a_{jp}a_{lq} - a_{kl}a_{ij}R_{jplq}) = 0, \quad R =: \sum_{i,j,k,l=1}^{d} R_{ijkl}E_d^{ij} \otimes E_d^{kl},$$

with the components of E_d^{ij} being zero, except the entry 1 in the i-th column and j-th row.

(8.16.2) Via an injective homomorphism of unital associative K-algebras $\alpha : \mathcal{A}_R \longrightarrow \mathcal{A}$, the above quadratic relations can be rewritten in terms of the tensor product over $\mathcal{A}$.

$$R_{\mathcal{A}}([a] \otimes_{\mathcal{A}} I_{\mathcal{A}})(I_{\mathcal{A}} \otimes_{\mathcal{A}} [a])$$
$$= (I_{\mathcal{A}} \otimes_{\mathcal{A}} [a])([a] \otimes_{\mathcal{A}} I_{\mathcal{A}})R_{\mathcal{A}} \in Mat(d,\mathcal{A}) \otimes_{\mathcal{A}} Mat(d,\mathcal{A}),$$

$R_{\mathcal{A}} := e_{\mathcal{A}} \otimes R,\ I_{\mathcal{A}} := e_{\mathcal{A}} \otimes I_d$, with the units $e_{\mathcal{A}}$ of $\mathcal{A}$, and I_d of $Mat(d,K)$, and suppressing conveniently the notation of α,

$$[a] := [\alpha(a_{ij}); i,j = 1,\ldots,d].$$

Here

$$Mat(d,\mathcal{A}) := \mathcal{A} \otimes Mat(d,K),$$

which with respect to the R-linear structure means an $\mathcal{A}$-left module over R, which is $\mathcal{A}$-free over the set $\{E_d^{ij}; i,j = 1,\ldots,d\}$.

(8.16.3) With the structure mappings of so-called matrix quantum semigroups, such that

$$\forall_1^d i,j : a_{ij} \underset{\Delta}{\longrightarrow} \sum_{k=1}^{d} a_{ik} \otimes a_{kj}, \quad a_{ij} \underset{\varepsilon}{\longrightarrow} \delta_{ij},$$

$\mathcal{A}_R$ becomes some bialgebra over K.

(8.16.4) Assume that R satisfies the quantum Yang-Baxter equation (QYBE), namely

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},\ R_{13} := \sum_{l=1}^{\bar{l}} R'_l \otimes I_N \otimes R''_l,\ R =: \sum_{l=1}^{\bar{l}} R'_l \otimes R''_l,$$

and with similar meaning of R_{12} and R_{23}.
Using the flip P on $Mat(d,K) \otimes Mat(d,K)$, every solution R of QYBE can be rewritten as an according representation of the braid group B_4 on $K^d \otimes K^d \otimes K^d$.

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23},\ S := PR;\ \forall x, y \in K^d : P(x \otimes y) = y \otimes x,$$

with the corresponding embeddings of S into the threefold tensor product of $Mat(d,K)$. Obviously the components of S are calculated from those of R as

$$S = \sum_{i,j,k,l=1}^{d} S_{ijkl} E_d^{ij} \otimes E_d^{kl},\ S_{ijkl} = R_{kjil},\ P := \sum_{k,l=1}^{d} E_d^{kl} \otimes E_d^{lk}.$$

(8.16.5) Let f be an arbitrary non-constant polynomial in one indeterminate over K. The free K-algebra over the set $\{x_1, \ldots, x_d\}$ may be factorized with respect to the relations below and then denoted by $\mathcal{X}_R(f)$. This so-called quantum K-vector space, associated with R and f, can be established as some $\mathcal{A}_R$-left comodule over K:

$$\mathcal{X}_R(f) \ni x_k \underset{def}{\longrightarrow} \sum_{l=1}^{d} a_{kl} \otimes x_l \in \mathcal{A}_R \otimes \mathcal{X}_R(f);$$

$$\forall_1^d i, k : \sum_{j,l=1}^{d} f(S)_{ijkl} x_j x_l = 0,$$

due to the commutation relation

$$\left[e_{\mathcal{A}} \otimes S, \bigotimes\nolimits_{\mathcal{A}}^{2} [\alpha(a_{ij}); i, j = 1, \ldots, d]\right] = 0.$$

An according $\mathbf{Z}_2$-graded version of the above MCR, and of corresponding relations of so-called quantum super-coordinates, will be presented later on the occasion of duality considerations.

(8.16.5.1) Inserting especially the relations of $Mat_q(2,0,K) \equiv Mat_q(2,K)$, which for $q^2 \neq -1$ can be rewritten as the main commutation relations associated with the R-matrix below:

$$R \longleftrightarrow \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix},\quad S \longleftrightarrow \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{bmatrix},$$

the polynomials $f(t) := t - q$ or $t + q^{-1}$, assuming $char\ K \neq 2$ in the latter case, yield the quantum plane $K_q^{2,0}$ or superplane $K_{1/q}^{0,2}$, with the relations

$$\{x_1x_2 - q^{-1}x_2x_1\} \text{ or } \{x_1^2, x_2^2, x_1x_2 + qx_2x_1\},$$

respectively.

8.17 Fundamental Representation of $U_q(A_2)$

(8.17.1) The system of positive roots, and the longest word σ_{max} of the Weyl group are

$$\Phi^+ = \{\alpha_1, \sigma_1(\alpha_2) = \alpha_1 + \alpha_2,\ \sigma_1 \circ \sigma_2(\alpha_1) = \alpha_2\},$$

$$\sigma_{max} = \sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2.$$

(8.17.2) Let $q \in \mathbf{D}\backslash\{\pm i\}$. Starting from the complex representation of $U_q(A_2)$ on $\mathbf{C}^3$, such that:

$$X_1^+ = (X_1^-)^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ X_2^+ = (X_2^-)^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

one finds the following complex representation of generators of $E_q(A_2)$:

$$E_1 = F_1^t = q^{1/2}X_1^+,\ E_2 = F_2^t = q^{1/2}X_2^+,$$

$$K_1 = \begin{bmatrix} q & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{bmatrix}.$$

The q-deformed positive and negative root vectors are represented by the following complex matrices.

$$\beta_1 := \alpha_1,\ \beta_2 := \alpha_1 + \alpha_2,\ \beta_3 := \alpha_3;\ E(\beta_1) := E_1,\ F(\beta_1) := F_1,$$

$$E(\beta_2) := T_1(E_2) = q^{-1/2}(E_1E_2 - qE_2E_1),\ E(\beta_3) := T_1 \circ T_2(E_1) = E_2,$$
$$F(\beta_2) := T_1(F_2) = -q^{-3/2}(F_1F_2 - qF_2F_1),\ F(\beta_3) := T_1 \circ T_2(F_1) = F_2;$$

$$E(\beta_2) = q^{1/2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (F(\beta_2))^t;\ \forall_1^3 j : (E(\beta_j))^2 = (F(\beta_j))^2 = 0.$$

(8.17.3) Inserting the Cartan matrix Γ, one obtains the corresponding complex representation of the universal R-matrix.

$$\Gamma = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \ \Gamma^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix};$$

$$\exp\left(\ln q \sum_{k,l=1}^{2} (\Gamma^{-1})_{kl} H_k \otimes H_l\right) = q^{\frac{1}{3}(2H_1\otimes H_1 + 2H_2\otimes H_2 + H_1\otimes H_2 + H_2\otimes H_1)}$$

$$\longleftrightarrow q^{2/3}(E_9^{11} + E_9^{55} + E_9^{99}) + q^{-1/3}(E_9^{22} + E_9^{33} + E_9^{44} + E_9^{66} + E_9^{77} + E_9^{88}),$$

$$R(q) \longleftrightarrow q^{-1/3} \left[\begin{array}{ccc|ccc|ccc} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q-q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q-q^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q-q^{-1} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{array}\right].$$

(8.17.4) The two Weyl elements are calculated easily from their commutation relations.

$$W_1 = c\begin{bmatrix} 0 & 1 & 0 \\ -q^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ W_2 = c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -q^{-1} & 0 \end{bmatrix},$$

$$W_{max} = W_1W_2W_1 = W_2W_1W_2 = c^3\begin{bmatrix} 0 & 0 & 1 \\ 0 & -q^{-1} & 0 \\ q^{-2} & 0 & 0 \end{bmatrix},$$

with a non-zero complex number c.

(8.17.5) The element, which generates the square of the antipode, reads

$$U = q^{-2/3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & q^{-2} & 0 \\ 0 & 0 & q^{-4} \end{bmatrix}, \ \sigma(U) = q^{-2/3}\begin{bmatrix} q^{-4} & 0 & 0 \\ 0 & q^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\forall_1^2 k : UE_kU^{-1} = q^2E_k,\ UF_kU^{-1} = q^{-2}F_k,\ UK_kU^{-1} = K_k.$$

Note here that the representation of U can be calculated from the universal R-matrix, namely representing

$$U = \overline{\mu} \circ \overline{T(\sigma, id)} \circ \overline{\tau}(R),\ \sigma(U) = \overline{\mu} \circ \overline{T(id,\sigma)}(R),$$

inserting the flip τ of tensor product, and the algebra structure map μ.

(8.17.6) Inserting $R(q)$ into the MCR yields the complex Hopf algebra $SL_q(3, \mathbf{C})$, which is in duality with the Hopf algebra $E_q(A_2)$ with respect to the complex-bilinear form, which assigns to the elements of $E_q(A_2)$ their representatives. Denoting the above fundamental representation of the unital associative complex algebra $E_q(A_2)$ by

$$\rho : E_q(A_2) \longrightarrow Mat(3, \mathbf{C}),\ \forall Y \in E_q(A_2), \forall_1^3 k, l : \langle Y|a_{kl}\rangle = \rho(Y)_{kl}.$$

8.18 Fundamental Representation of $U_q(B_2)$

(8.18.1) Consider the complex Lie algebra B_2 and its Weyl group.

$$\Gamma = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix},$$

$$\tilde{B} := [\Gamma_{kl}d_l^{-1}; k, l = 1, 2] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},\ \tilde{B}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

$$\Phi^+ = \{\beta_1 := \alpha_1,\ \beta_2 := \sigma_1(\alpha_2) = \alpha_1 + \alpha_2,$$
$$\beta_3 := \sigma_1 \circ \sigma_2(\alpha_1) = \alpha_1 + 2\alpha_2,\ \beta_4 := \sigma_1 \circ \sigma_2 \circ \sigma_1(\alpha_2) = \alpha_2\},$$

along the longest word of the Weyl group,

$$\sigma_{max} = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1.$$

(8.18.2) For $q \in \mathbf{D}$ and $q^4 \neq \pm 1$, consider the following q-deformed complex representation:

$$X_1^+ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ X_2^+ = ([2]_q)^{1/2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$X_1^- = (X_1^+)^t$, $X_2^- = (X_2^+)^t$, with the index t denoting transposition,

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K_1 = q^{2H_1}, \ K_2 = q^{H_2},$$
$$E_1 = X_1^+ q^{-H_1}, \ F_1 = X_1^- q^{H_1}, \ E_2 = X_2^+ q^{-\frac{1}{2}H_2}, \ F_2 = X_2^- q^{\frac{1}{2}H_2} \neq E_2^t.$$

(8.18.3) The positive and negative q-deformed root vectors are

$$E(\beta_1) := E_1 = \begin{bmatrix} 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ F(\beta_1) := F_1 = E_1^t,$$

$$E(\beta_2) := T_1(E_2) = q^{-1}(E_1E_2 - q^2E_2E_1)$$
$$= ([2]_q)^{1/2} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F(\beta_2) := T_1(F_2) = -q^{-3}(F_1F_2 - q^2F_2F_1)$$
$$= \frac{([2]_q)^{1/2}}{q^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$E(\beta_3) := T_1 \circ T_2(E_1) = T_2^{-1}(E_1) = \frac{1}{[2]_q}[E(\beta_2), E_2]$$
$$= \begin{bmatrix} 0 & 0 & 0 & -q & 0 \\ 0 & 0 & 0 & 0 & q^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F(\beta_3) := T_1 \circ T_2(F_1) = T_2^{-1}(F_1) = \frac{-1}{[2]_q}[F(\beta_2), F_2]$$
$$= \frac{1}{q} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -q^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$E(\beta_4) := T_1 \circ T_2 \circ T_1(E_2) = E_2 = ([2]_q)^{1/2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F(\beta_4) := T_1 \circ T_2 \circ T_1(F_2) = F_2 = ([2]_q)^{1/2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(8.18.4)

$$T_2(E_1) = \frac{1}{q[2]_q}[E_2, E_2E_1 - q^2E_1E_2] = q \begin{bmatrix} 0 & 0 & 0 & -q^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_2(F_1) = \frac{1}{q^3[2]_q}[F_2, F_2F_1 - q^2F_1F_2] = q^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 & 0 \end{bmatrix}.$$

(8.18.5)

$$E_1^2 = F_1^2 = E_2^3 = F_2^3 = (E(\beta_2))^3 = (F(\beta_2))^3 = (E(\beta_3))^2 = (F(\beta_3))^2 = 0.$$

(8.18.6) The two Weyl elements, which generate the Lusztig automorphisms, are

$$W_1 = c_1 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -q^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & q^{-2} & 0 \end{bmatrix},$$

$$W_2 = c_2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -q^{-2} & 0 & 0 \\ 0 & -q^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$W_{max} = W_1W_2W_1W_2 = W_2W_1W_2W_1 = c_1^2c_2^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-2} & 0 \\ 0 & 0 & q^{-4} & 0 & 0 \\ 0 & q^{-4} & 0 & 0 & 0 \\ q^{-6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

with non-zero complex numbers c_1, c_2.

(8.18.7) The universal R-matrix of $U_q(B_2)$ is then represented by

$$R(q) \in Mat(5, \mathbf{C}) \otimes Mat(5, \mathbf{C}) \longleftrightarrow Mat(25, \mathbf{C}),$$

which is calculated straightforward. An explicit expression of the fundamental representations of $U_q(\mathcal{L})$, for the series $\mathcal{L} := A_m, m \geq 1, \ldots, D_m, m \geq 4$, and of the corresponding universal R-matrices, is presented later.

(8.18.8) The element, which generates the squared antipode, is calculated from the universal R-matrix.

$$U = \exp\Big(-\ln q \sum_{k,l=1}^{2} H_k(\tilde{B}^{-1})_{kl} H_l\Big)(I + (1 - q^{-4})\sigma(F_1)E_1 + \cdots)$$

$$= \begin{bmatrix} q^{-2} & 0 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 0 & q^{-2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q^{-4} & 0 & 0 & 0 \\ 0 & 0 & q^{-8} & 0 & 0 \\ 0 & 0 & 0 & q^{-8} & 0 \\ 0 & 0 & 0 & 0 & q^{-12} \end{bmatrix}$$

$$= q^{-2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q^{-4} & 0 & 0 & 0 \\ 0 & 0 & q^{-6} & 0 & 0 \\ 0 & 0 & 0 & q^{-8} & 0 \\ 0 & 0 & 0 & 0 & q^{-12} \end{bmatrix}.$$

$$U\sigma(U) = \sigma(U)U = q^{-16} I_5. \ \sigma^2(U) = U.$$

8.19 q-Deformation of Basic Classical Lie Superalgebras

Let the finite-dimensional complex Lie superalgebra $\mathcal{L}$ be basic classical, of rank r, and exclude the case of $A(1,1)$. Denote by s the number of simple roots;

$$s = r, \text{ if and only if } \mathcal{L} \notin \{A(n,n); n = 2, 3, \ldots\};$$

$$\text{for } \mathcal{L} := A(n,n), \ n = 2, 3, \ldots, \ s = r + 1.$$

(8.19.1) The unital associative superalgebra of formal power series with relations $U_q(\mathcal{L})$ is constructed by an appropriate q-deformation of the defining relations of $\mathcal{L}$ itself, over the commutative ring $\mathcal{R} := \mathbf{C}[[h]]$.

(8.19.1.1) The Chevalley generators E_k and F_k are assumed to be even for $k \in \{1, \dots, s\} \setminus \tau$, and odd for $k \in \tau$. One can choose *card* $\tau = 1$, which is proposed henceforth. The Chevalley generators $H_k, k = 1, \dots, s$, and correspondingly the generators K_k and L_k defined below, are assumed to be even. Let $\Gamma := [\Gamma_{kl}; k, l = 1, \dots, s]$ be the Cartan matrix of $\mathcal{L}$, which depends on the choice of the index set $\tau \subsetneq \{1, \dots, s\}$. It can be normalized, such that

$$\Gamma_{kk} = 2 \text{ for } k \in \{1, \dots, s\} \setminus \tau,$$
$$\Gamma_{kk} = 0 \text{ for } k \in \tau \text{ and } \mathcal{L} \notin \{(B(0,n); n \in \mathbf{N}\},$$
$$\Gamma_{kk} = 2 \text{ otherwise.}$$

The so-called symmetrized Cartan matrix is defined by

$$B := [B_{kl} := d_k \Gamma_{kl} = d_l \Gamma_{lk}; k, l = 1, \dots, s],$$

with the integers $d_k \in \mathbf{Z} \setminus \{0\}$ chosen such that the greatest common divisor of $\{d_k; k = 1, \dots, s\}$ is equal to 1, and $\forall k \in \tau : d_k > 0$. For instance in the case of $A(m,n)$, $(m,n) \in \mathbf{N}_0 \times \mathbf{N}_0 \setminus \{\{0,0\}, \{1,1\}\}$, $s = m + n + 1$, choose

$$\tau := \{m+1\}; \text{ then } d_1 = \dots = d_{m+1} = 1,\ d_{m+2} = \dots = d_s = -1.$$

(8.19.1.2) The angular momentum-like relations are deformed, such that

$$q := e^h,\ \forall_1^s k, l : q_k := q^{d_k} = e^{h d_k},$$
$$[H_k, H_l] = 0,\ [H_k, E_l] = \Gamma_{kl} E_l,\ [H_k, F_l] = -\Gamma_{kl} F_l,$$
$$[E_k, F_l] = \delta_{kl} \frac{q_k^2}{q_k^2 - 1} (q_k^{H_k} - q_k^{-H_k}).$$

The Serre-type relations are q-deformed such that $\forall_1^s k \neq l$:

$$(ad^+ E_k)^{1-\tilde{\Gamma}_{kl}}(E_l) = (ad^- F_k)^{1-\tilde{\Gamma}_{kl}}(F_l) = 0,$$
$$E_k^2 = F_k^2 = 0 \text{ for } \Gamma_{kk} = 0.$$

Here the modified Cartan matrix $\tilde{\Gamma}$ is obtained from Γ, substituting -1 for the positive components in the rows with the diagonal component 0.

(8.19.1.3) In case of $\mathcal{L} := A(m,n)$ with $mn \geq 1$, choosing τ as above, the additional fourth order relations are q-deformed such that

$$ad^+ E_{m+1}(ad^+ E_m(ad^+ E_{m+1}(E_{m+2}))) = 0,$$
$$ad^- F_{m+1}(ad^- F_m(ad^- F_{m+1}(F_{m+2}))) = 0.$$

The more complicated additional relations in other cases of $s \geq 3$ are correspondingly q-deformed, in order to obtain a Poincaré-Birkhoff-Witt-like topological basis of $U_q(\mathcal{L})$, which consists of similar monomials as in the undeformed case.

(8.19.1.4) Here one conveniently uses the following q-deformed adjoint representation.

$$\forall_1^s k, \forall \bar{y} \in \mathbf{Z}_2, Y \in U_q^{\bar{y}}(\mathcal{L}) : ad^+ E_k(Y) := E_k Y - (-1)^{yz} L_k Y K_k E_k,$$
$$ad^- F_k(Y) := F_k Y - (-1)^{yz} K_k Y L_k F_k,$$

$z := +1$ for $k \notin \tau$, -1 for $k \in \tau$, and inserting the Chevalley generators defined below.

(8.19.1.5)

$$\forall_1^s k, l : K_k := q_k^{H_k},\ L_k := q_k^{-H_k},\ K_k L_k = I,$$
$$K_k^{\pm 1} E_l = q_k^{\pm \Gamma_{kl}} E_l K_k^{\pm 1},\ K_k^{\pm 1} F_l = q_k^{\mp \Gamma_{kl}} F_l K_k^{\pm 1}.$$

(8.19.1.6) Correspondingly one defines $\forall_1^s k, l$:

$$X_k^+ := E_k q_k^{\frac{1}{2} H_k},\ X_k^- := F_k q_k^{-\frac{1}{2} H_k},\ [X_k^+, X_l^-] = \delta_{kl} \frac{K_k - L_k}{q_k - q_k^{-1}}.$$

(8.19.2) The $\mathbf{Z}_2$-graded topological Hopf algebra $U_q(\mathcal{L})$ is established, in the sense of formal power series with relations over $\mathcal{R}$. $\forall_1^s k$:

$$H_k \underset{\Delta}{\longrightarrow} H_k \otimes I + I \otimes H_k,\ H_k \underset{\varepsilon}{\longrightarrow} 0,\ H_k \underset{\sigma}{\longrightarrow} -H_k,$$
$$E_k \underset{\Delta}{\longrightarrow} E_k \otimes I + L_k \otimes E_k,\ E_k \underset{\varepsilon}{\longrightarrow} 0,\ E_k \underset{\sigma}{\longrightarrow} -K_k E_k,$$
$$F_k \underset{\Delta}{\longrightarrow} F_k \otimes K_k + I \otimes F_k,\ F_k \underset{\varepsilon}{\longrightarrow} 0,\ F_k \underset{\sigma}{\longrightarrow} -F_k L_k.$$

(8.19.2.1) In terms of Chevalley generators $\{E_k, F_k, K_k, L_k; k = 1, \ldots, s\}$, one obtains some $\mathbf{Z}_2$-graded Hopf algebra $E_q(\mathcal{L})$ of polynomials, and an according homomorphism of unital associative $\mathcal{R}$-superalgebras:

$$E_q(\mathcal{L}) \longrightarrow U_q(\mathcal{L}),$$

which is also compatible with Δ, ε, and σ.

(8.19.3) Inserting the involution of $\mathcal{R}$, which is due to: $h \longleftrightarrow -h$ and complex conjugation, one obtains some graded star operation on $U_q(\mathcal{L})$, which is uniformly continuous in the sense of the h-adic topology. In terms of the Chevalley generators, $\forall_1^s k$:

$$X_k^+ \underset{*}{\longleftrightarrow} X_k^-,\ H_k \underset{*}{\longleftrightarrow} H_k.$$

(8.19.3.1) Substituting generic $q \in \mathbf{C}$ for the indeterminate $h = \ln\ q$, one obtains the complex $\mathbf{Z}_2$-graded Hopf algebra $E_q(\mathcal{L})$. For $|q| = 1$, and using complex conjugation, one finds an appropriate graded star operation on $E_q(\mathcal{L})$.

(8.19.4) For instance consider the case of $\mathcal{L} := C(2) \equiv osp(2,2)$. Choose $\tau := \{2\}$, such that

$$\Gamma = \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}, \ d_1 = 2, \ d_2 = 1, \ B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix}, \ \tilde{\Gamma} = \Gamma.$$

The Serre-type relations read

$$E_1^2 E_2 - (q^2 + q^{-2}) E_1 E_2 E_1 + E_2 E_1^2 = 0,$$

$$F_1^2 F_2 - (q^2 + q^{-2}) F_1 F_2 F_1 + F_2 F_1^2 = 0.$$

Due to $\Gamma_{22} = 0$ one finds, that $E_2^2 = F_2^2 = 0$.

(8.19.4.1) For $q \in \mathbf{D}$, $q^4 \neq \pm 1$, one finds the following q-deformed fundamental representation:

$$H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{bmatrix} = F_1^t,$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2q} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2q}} & 0 & 0 & 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 & 0 & 0 & \sqrt{2q} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2q}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the entries of H_1 and H_2 are ± 1, being composed to $\pm\sigma^3$, the q-deformation does not become manifest by the fundamental representation of $X_k^{\pm}$, $k = 1, 2$. Here the usual representation, which yields $H_2 = -\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^3 \end{bmatrix}$, is transformed for convenience.

(8.19.4.2) The roots are written in terms of the Cartesian unit vectors ε_i, $i = 1, 2$, of $\mathbf{R}^2$.

$$\Delta = \{\alpha_1 := 2\varepsilon_2, \alpha_2 := \varepsilon_1 - \varepsilon_2\}, \ \Psi_+^{\bar{0}} = \{\alpha_1\}, \ \Psi_+^{\bar{1}} = \{\alpha_2, \alpha_1 + \alpha_2\}.$$

$$\beta_1 := \alpha_1, \ \beta_2 := \alpha_1 + \alpha_2, \ \beta_3 := \alpha_2.$$

$$E_{\beta_1} := E_1, \ E_{\beta_2} := ad^+ E_1(E_2) = E_1 E_2 - q^2 E_2 E_1, \ E_{\beta_3} := E_2.$$

$$F_{\beta_1} := F_1, \ F_{\beta_2} := ad^- F_1(F_2) = F_1 F_2 - q^2 F_2 F_1, \ F_{\beta_3} := F_2.$$

(8.19.4.3) With respect to the complex Hopf algebra $E_q(\mathcal{L})$ one finds, that

$$q \in \mathbf{D},\ q^4 \neq \pm 1 \Longrightarrow E_{\beta_2}^2 = F_{\beta_2}^2 = 0.$$

(8.19.4.4) The following Poincaré-Birkhoff-Witt-like $\mathcal{R}$-linear bases of the q-deformed subalgebras

$$E_q(\mathcal{B}_+) := \mathcal{R} - alg\ span(\{E_k, K_k^{\pm 1}; k = 1, 2\}),$$
$$E_q(\mathcal{B}_-) := \mathcal{R} - alg\ span(\{F_k, K_k^{\pm 1}; k = 1, 2\}),$$

can be constructed:

$$\{E_{\beta_1}^{x_1} E_{\beta_2}^{x_2} E_{\beta_3}^{x_3} K_1^{z_1} K_2^{z_2}; x_1 \in \mathbf{N}_0; x_2, x_3 \in \{0,1\}; z_1, z_2 \in \mathbf{Z}\}$$
$$\underset{free\ over\ \mathcal{R}}{\longrightarrow} E_q(\mathcal{B}_+),$$

and similarly for $E_q(\mathcal{B}_-)$ with $F_k, k = 1, 2$, instead of E_k. Hence one constructs an according topological basis of $U_q(\mathcal{L})$ over $\mathbf{C}[[h]]$.

(8.19.4.5) Choosing $q \in \mathbf{D}$, $q^4 \neq \pm 1$, one finds according $\mathbf{C}$-linear bases of the q-deformed subalgebras $E_q(\mathcal{B}_\pm)$ of the unital associative complex superalgebra $E_q(\mathcal{L})$.

(8.19.5) Consider the case of $\mathcal{L} := A(m,n), m \neq n$, m and $n \in \mathbf{N}_0$, such that $s = r$, and choose $\tau := \{m+1\}$, such that

$$\Gamma = [(1 + (-1)^{\delta_{k,m+1}})\delta_{kl} - \delta_{k,l+1} - (-1)^{\delta_{k,m+1}}\delta_{k+1,l}; k, l = 1, \ldots, r],$$
$$r = m + n + 1;\ d_1 = \cdots = d_{m+1} = 1,\ d_{m+2} = \cdots = d_r = -1.$$
$$\tilde{\Gamma} = [\Gamma_{kk}\delta_{kl} - \delta_{k,l+1} - \delta_{k+1,l}; k, l = 1, \ldots, r].$$
$$\Delta = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_r := \varepsilon_r - \varepsilon_{r+1}\},$$

inserting the Cartesian unit vectors $\varepsilon_i, i = 1, \ldots, m+n+2$, of $\mathbf{R}^{m+n+2}$. The sets of even and odd positive roots are

$$\Psi_+^{\bar{0}} = \{\alpha_k + \cdots + \alpha_l; 1 \leq k \leq l \leq m, m+2 \leq k \leq l \leq r\},$$
$$\Psi_+^{\bar{1}} = \{\alpha_k + \cdots + \alpha_l; 1 \leq k \leq m+1, m+1 \leq l \leq r\}.$$

(8.19.5.1) According to the set of positive roots

$$\Psi_+ = \Psi_+^{\bar{0}} \cup \Psi_+^{\bar{1}} = \{\alpha_k + \cdots + \alpha_l; 1 \leq k \leq l \leq r\},$$

the following q-deformed root vectors are defined.

$$\forall_1^r k : E_{\alpha_k} := E_k,\ F_{\alpha_k} := F_k.$$
$$\forall 1 \leq k < l \leq r : E_{\alpha_k + \cdots + \alpha_l} := ad^+ E_k \circ \cdots \circ ad^+ E_{l-1}(E_l),$$
$$F_{\alpha_k + \cdots + \alpha_l} := ad^- F_k \circ \cdots \circ ad^- F_{l-1}(F_l).$$

(8.19.5.2) One then finds, that $\forall \beta \in \Psi_+^{\bar{1}} : E_\beta^2 = F_\beta^2 = 0$, if $q \in \mathbf{C}$ is chosen to be generic, or if $q \in \mathbf{C}[[h]]$.

(8.19.5.3) One establishes the following $\mathbf{C}$-linear basis of complex polynomials in E_k, F_k, K_k, L_k, $k = 1, \ldots, r$, for generic q:

$$\{E_{\beta_1}^{x_1} \cdots E_{\beta_p}^{x_p} F_{\beta_1}^{y_1} \cdots F_{\beta_p}^{y_p} K_1^{z_1} \cdots K_r^{z_r}; x_1, \ldots, y_p \in \mathbf{N}_0; z_1, \ldots, z_r \in \mathbf{Z}\}$$

$$\xrightarrow[\mathbf{C}]{} E_q(\mathcal{L}),$$

with x_j and y_j actually $\in \{0, 1\}$ for $\beta_j \in \Psi_+^{\bar{1}}, j = 1, \ldots, p$. Here the set of positive roots $\Psi_+ =: \{\beta_1, \ldots, \beta_p\}$ is strictly ordered such that

$$\forall 1 \le i < j \le p : \beta_i = \alpha_k + \cdots + \alpha_l,\ \beta_j = \alpha_s + \cdots + \alpha_t,$$
$$\text{with } k < s, \text{ or } k = s \text{ and } l < t.$$

(8.19.5.4) Of course one could also use the h-adic topology, in order to construct an according topological basis of $U_q(\mathcal{L})$ over $\mathbf{C}[[h]]$.

(8.19.5.5) In case of $\mathcal{L} := A(0,1)$ with $\tau := \{1\}$,

$$\Gamma = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},\ d_1 = 1,\ d_2 = -1;\ \Delta = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2 - \varepsilon_3\},$$

$$\Psi_+^{\bar{0}} = \{\beta_3 := \alpha_2\},\ \Psi_+^{\bar{1}} = \{\beta_1 := \alpha_1, \beta_2 := \alpha_1 + \alpha_2\}.$$

$$E_{\beta_2} := ad^+ E_1(E_2) = E_1 E_2 - q^{-1} E_2 E_1,$$

$$F_{\beta_2} := ad^- F_1(F_2) = F_1 F_2 - q^{-1} F_2 F_1.$$

The Serre relations read

$$E_2^2 E_1 - (q + q^{-1}) E_2 E_1 E_2 + E_1 E_2^2 = 0,$$

$$F_2^2 F_1 - (q + q^{-1}) F_2 F_1 F_2 + F_1 F_2^2 = 0.$$

Since $E_1^2 = F_1^2 = 0$, also $E_{\beta_2}^2 = F_{\beta_2}^2 = 0$. For $q \in \mathbf{D}$, $q^2 \neq -1$, the fundamental representation is calculated easily:

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$X_1^+ = (X_1^-)^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ X_2^+ = (X_2^-)^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(8.19.6) In case of $B(1,1) := osp(3,2)$ with $\tau := \{1\}$, such that

$$\Gamma = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \ d_1 = 2, \ d_2 = -1, \ \Delta = \{\alpha_1 := \varepsilon_2 - \varepsilon_1, \alpha_2 := \varepsilon_1\},$$

$$\Psi_+^{\bar{0}} = \{\alpha_2, 2(\alpha_1 + \alpha_2)\}, \ \Psi_+^{\bar{1}} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\},$$

inserting above the Cartesian unit vectors $\varepsilon_i, i = 1,2$, of $\mathbf{R}^2$, the subsequent fundamental representation of Chevalley generators of $U_q(B(1,1))$ does not show up any q-deformation, for $q \in \mathbf{D}$ and $q^2 \neq -1$.

$$H_1 = diag[0,0,1,1,0], \ H_2 = diag[1,0,-1,1,-1],$$

$$X_1^+ = (X_1)^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ X_2^+ = (X_2^-)^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(8.19.7) In case of $B(0,2) := osp(1,4)$ with $\tau := \{2\}$, the roots being located in $\mathbf{R}^2$ with Cartesian unit vectors ε_1 and ε_2, such that

$$\Gamma = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \ d_1 = 2, \ d_2 = 1, \ \Delta = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2\},$$

$$\Psi_+^{\bar{0}} = \{\alpha_1, \alpha_1 + 2\alpha_2, 2(\alpha_1 + \alpha_2), 2\alpha_2\}, \ \Psi_+^{\bar{1}} = \{\alpha_1 + \alpha_2, \alpha_2\},$$

one easily finds the following fundamental representation of $U_q(B(0,2))$, for $q \in \mathbf{D}$ and $q^2 \neq -1$. $\xi := ([2]_q)^{1/2}$.

$$H_1 = diag[0,1,-1,-1,1], \ H_2 = diag[0,-2,0,2,0], \ X_1^+ = (X_1^-)^t,$$

$$X_1^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \ X_2^+ = \begin{bmatrix} 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_2^- = \begin{bmatrix} 0 & 0 & 0 & \xi & 0 \\ -\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

8.20 Universal R-Matrix of $U_q\,(B(0,1))$

(8.20.1) The topological $\mathbf{Z}_2$-graded Hopf algebra $U_q\,(B(0,1))$ over the ring $\mathcal{R} := \mathbf{C}[[h]]$ is defined by the following relations and costructure maps.

$$HE - EH = 2E, \;\; HF - FH = -2F, \;\; EF + FE = \frac{q^2}{q^2-1}(q^H - q^{-H}).$$

$$\begin{aligned} &H \xrightarrow[\Delta]{} H \otimes I + I \otimes H, \;\; H \xrightarrow[\varepsilon]{} 0, \;\; H \xrightarrow[\sigma]{} -H, \\ &E \xrightarrow[\Delta]{} E \otimes I + q^{-H} \otimes E, \;\; E \xrightarrow[\varepsilon]{} 0, \;\; E \xrightarrow[\sigma]{} -q^H E, \\ &F \xrightarrow[\Delta]{} F \otimes q^H + I \otimes F, \;\; F \xrightarrow[\varepsilon]{} 0, \;\; F \xrightarrow[\sigma]{} -Fq^{-H}. \end{aligned}$$

Here the generators E and F are odd, whereas H is even. Note especially that E^2 and F^2 are non-zero, due to the Cartan matrix $\Gamma = 2$.

(8.20.2) This q-deformed universal enveloping superalgebra over $\mathcal{R}$ is quasitriangular with respect to the following universal R-matrix.

$$R = \sum_{n=0}^{\infty} c_n (E^n \otimes F^n) q^{\frac{1}{2} H \otimes H}, \;\; c_0 := 1, \;\; c_1 = c_2 := -(1 - q^{-2}),$$

$$\forall n \in \mathbf{N}: \; c_n := (-1)^n (1 - q^{-2})^n q^{\frac{1}{2}n(n-1)} \Big(\prod_{k=1}^{n} \langle k \rangle_q \Big)^{-1},$$

$$\langle k \rangle_q := \frac{(-1)^{k+1} q^k + q^{-k}}{q + q^{-1}}, \;\; \langle 1 \rangle_q = 1.$$

(8.20.3) Let $q \in \mathbf{D}$, $q^2 \neq -1$. Inserting the fundamental representation:

$$H = diag[0, -2, 2],$$

$$E = (q + q^{-1})^{\frac{1}{2}} \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \;\; F = (q + q^{-1})^{\frac{1}{2}} \begin{bmatrix} 0 & 0 & q \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

such that $E^3 = F^3 = 0$, and an isomorphism of unital associative complex superalgebras: $Mat(3, \mathbf{C}) \hat{\otimes} Mat(3, \mathbf{C}) \longleftrightarrow Mat(9, \mathbf{C})$, such that $\forall_1^3 i, j, k, l$:

$$E_3^{ij} \otimes E_3^{kl} \longleftrightarrow (-1)^{(\hat{i} + \hat{j})\hat{k}} E_9^{3(i-1)+k, 3(j-1)+l}, \;\; \hat{1} := 0, \;\; \hat{2} = \hat{3} := 1,$$

one calculates the following representation $R(q)$ of R on $\mathbf{C}^9$:

$$R(q) \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & qr_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 \\ 0 & 0 & qr_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ r_1 & 0 & 0 & 0 & 0 & r_2 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{bmatrix},$$

$r_1 := -q(1-q^{-4})$, $r_2 := (1+q^2)(1-q^{-4})$.

8.21 Duals of Quasitriangular $\mathbf{Z}_2$-Graded Hopf Algebras

Consider a $\mathbf{Z}_2$-graded Hopf algebra $\{\mathcal{H}, \mu, \eta, \Delta, \varepsilon\}$ over a field K, $char\ K = 0$, which is quasitriangular with respect to the universal R-matrix $R \in \mathcal{H}\hat{\otimes}\mathcal{H}$, the latter being even by definition.

(8.21.1) Let $\rho : \mathcal{H} \to Mat\ (d, K)$ be a representation of $\mathcal{H}$ on K^d, in the sense of unital associative superalgebras over K. The $\mathbf{Z}_2$-grading of these K-endomorphisms is arranged such that

$$\forall_1^m k : E_d^k \ even,\ \forall_{m+1}^{m+n} k : E_d^k \ odd,\ \ m+n=d,$$

denoting by E_d^k the Cartesian unit vectors of K^d.

(8.21.2) The corresponding representation of R is denoted by:

$$R \underset{T(\rho,\rho)}{\longrightarrow} R_\rho =: \sum_{i,j,k,l=1}^{d} R_{ijkl} E_d^{ij} \otimes E_d^{kl} \in Mat(d,K)\hat{\otimes}Mat(d,K).$$

(8.21.3) Calculations of examples may be done conveniently using an isomorphism of unital associative superalgebras over K:

$$\begin{aligned} &Mat(d,K)\hat{\otimes}Mat(d,K) \ni E_d^{ij} \otimes E_d^{kl} \\ &\longleftrightarrow (-1)^{(\hat{i}+\hat{j})\hat{k}} E_{d^2}^{(i-1)d+k,(j-1)d+l} \in Mat(d^2,K), \end{aligned}$$

denoting $\hat{k} := 0$ for $1 \le k \le m$, and 1 for $m+1 \le k \le d$.

(8.21.3.1) Denoting the components of $A, B \in Mat(d,K)\hat{\otimes}Mat(d,K)$ similarly as above those of R_ρ, one easily finds that

$$\forall_1^d i,p,k,q : (AB)_{ipkq} = \sum_{j,l=1}^{d} (-1)^{(\hat{j}+\hat{p})(\hat{k}+\hat{l})} A_{ijkl} B_{jplq}.$$

(8.21.4) The matrix quantum super-semigroup $\mathcal{A}_{R_\rho} \equiv \mathcal{A}(R;\rho)$ is defined as the unital associative K-superalgebra, which is generated by even generators a_{kl} for $1 \le k,l \le m$ and $m+1 \le k,l \le m+n$, and odd a_{kl} for the remaining choices of $1 \le k,l \le m+n$, and factorized with respect to the homogeneous main commutation relations (MCR)

$$\begin{aligned} &R_{\mathcal{A}}\left([a] \otimes_{\mathcal{A}} I_{\mathcal{A}}\right)\left(I_{\mathcal{A}} \otimes_{\mathcal{A}} [a]\right) \\ &= \left(I_{\mathcal{A}} \otimes_{\mathcal{A}} [a]\right)\left([a] \otimes_{\mathcal{A}} I_{\mathcal{A}}\right) R_{\mathcal{A}} \in Mat(d,\mathcal{A})\hat{\otimes}_{\mathcal{A}} Mat(d,\mathcal{A}), \end{aligned}$$

suppressing an explicit notation of an injective homomorphism α of unital associative K-superalgebras: $\mathcal{A}(R;\rho) \longrightarrow \mathcal{A}$, i.e. inserting

$$[a] := \sum_{k,l=1}^{d} \alpha(a_{kl}) \otimes E_d^{kl} \in Mat(d,\mathcal{A}) := \mathcal{A}\hat{\otimes}Mat(d,K),$$

and denoting $I_{\mathcal{A}} := e_{\mathcal{A}} \otimes I_d$, $R_{\mathcal{A}} := e_{\mathcal{A}} \otimes R_\rho$. Note that here the graded tensor product $\mathcal{A}\hat{\otimes}Mat(d,K)$ is constructed; one could also use the ungraded one. The latter was used for transformations of finite-dimensional quantum superspace.

(8.21.5) In terms of components these MCR explicitly read, such that $\forall_1^d i,k,p,q$:

$$\sum_{j,l=1}^{d} \left(R_{ijkl}\; a_{jp} a_{lq} (-1)^{(\hat{j}+\hat{p})(\hat{k}+\hat{q})} - (-1)^{(\hat{j}+\hat{p})(\hat{k}+\hat{l})} a_{kl} a_{ij} R_{jplq} \right) = 0.$$

(8.21.6) Obviously $\mathcal{A}(R;\rho)$ becomes some $\mathbf{Z}_2$-graded bialgebra over K, such that

$$\forall_1^d i,j : a_{ij} \xrightarrow[\Delta]{} \sum_{k=1}^{d} a_{ik} \otimes a_{kj}, \quad a_{ij} \xrightarrow[\varepsilon]{} \delta_{ij} \in K.$$

(8.21.7) The representative R_ρ solves QYBE on $\overset{3}{\bigotimes} Mat(d,K)$, due to the quasitriangular properties of R itself.

(8.21.8) Inserting the $\mathbf{Z}_2$-graded flip τ on $\mathcal{H}\hat{\otimes}\mathcal{H}$, one easily finds that $\forall x \in \mathcal{H}, \forall_1^d i, j, k, l$:

$$(T(\rho,\rho) \circ \tau \circ \Delta(x))_{ijkl} = (-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l})} (T(\rho,\rho) \circ \Delta(x))_{klij} .$$

(8.21.9) Writing without loss of generality $\mathcal{H}$ as some free K-algebra factorized by certain homogeneous relations, one then establishes $\mathcal{H}$ and $\mathcal{A}(R;\rho)$ as dual $\mathbf{Z}_2$-graded bialgebras, with respect to the K-bilinear form such that

$$\forall_1^d k, l : \mathcal{H} \times \mathcal{A}(R;\rho) \ni \{x, a_{kl}\} \longrightarrow (\rho(x))_{kl} \in K.$$

This K-bilinear form is compatible with the relations of $\mathcal{H}$, because ρ is some representation of $\mathcal{H}$; it is compatible with the MCR too, due to the above identity involving the representatives of $\Delta(x)$ and $\tau \circ \Delta(x), x \in \mathcal{H}$.

(8.21.10) Consider the matrix quantum semigroup

$$Mat_q(m+1, 0, \mathbf{C}) \equiv Mat_q(m+1, \mathbf{C}), \; m \in \mathbf{N},$$

which is defined by the following Manin relations.

$$\forall_1^{m+1} i < j, k < l : q_{kl} \in \mathbf{C}\backslash\{0\},$$

$$\begin{aligned} a_{ik}a_{il} &= q_{kl}^{-1} a_{il}a_{ik}, \quad a_{ik}a_{jk} = q_{ij}^{-1} a_{jk}a_{ik}, \\ a_{ik}a_{jl} + q_{kl}a_{il}a_{jk} &= q_{ij}^{-1}(a_{jk}a_{il} + q_{kl}a_{jl}a_{ik}), \\ a_{ik}a_{jl} - q_{kl}^{-1}a_{il}a_{jk} &= -q_{ij}(a_{jk}a_{il} - q_{kl}^{-1}a_{jl}a_{ik}). \end{aligned}$$

(8.21.10.1) Especially let $q_{kl} := q$ for $1 \leq k < l \leq m+1$, and assume $q^2 \neq -1$. Then

$$\forall_1^{m+1} i < j, k < l : a_{jk}a_{il} = a_{il}a_{jk}, \;\; a_{ll}a_{kk} - a_{kk}a_{ll} = (q - q^{-1})a_{kl}a_{lk}.$$

Inserting the fundamental representation of the universal R-matrix of $U_q(A_m)$ for $q \in \mathbf{D}$, the resulting MCR are equivalent with the Manin relations. For instance for $m \geq 2$,

$$\begin{aligned} &\langle a_{11}a_{12} - q_{12}^{-1}a_{12}a_{11} | X_1^+ \rangle = 0 \text{ for } q_{12} = q, \\ &\langle a_{11}a_{13} - q_{13}^{-1}a_{13}a_{11} | X_1^+ X_2^+ \rangle = 0 \text{ for } q_{13} = q. \end{aligned}$$

(8.21.10.2) The MCR of the fundamental representation of $U_q(A_m)$ allow for an appropriate definition of the so-called quantum determinant

$$det_q[a_{kl}; k, l = 1, \ldots, m+1] \equiv d := \sum_{P \in P_{m+1}} \left(-q^{-1}\right)^{\lambda_P} a_{1j_1} \cdots a_{m+1, j_{m+1}},$$

denoting the permutations $\begin{bmatrix} 1 & \dots & m+1 \\ j_1 & \dots & j_{m+1} \end{bmatrix} =: P$, and by λ_P the length, i.e. minimal number of transpositions of P. This quantum determinant is both central and group-like.

$$\forall_1^{m+1} k, l : a_{kl} d = d a_{kl}; \quad d \xrightarrow[\Delta]{} d \otimes d, \quad \varepsilon(d) = 1.$$

The factor algebra $SL_q(m+1, \mathbf{C})$, which is obtained from the MCR with the additional relation $d = e :=$ unit, becomes some complex Hopf algebra, because Δ and ε are compatible with the new relation, with the antipode σ such that $\forall_1^{m+1} k, l$:

$$a_{kl} \xrightarrow[\sigma]{} \left(-q^{-1}\right)^{k-l} \sum_{P \in P_m} (-q^{-1})^{\lambda_P} a_{1j_1} \cdots a_{l-1, j_{l-1}} a_{l+1, j_{l+1}} \cdots a_{m+1, j_{m+1}}$$
$$\xrightarrow[\sigma]{} q^{2(l-k)} a_{kl},$$

$$\sum_{j=1}^{m+1} a_{kj} \sigma(a_{jl}) = \sum_{j=1}^{m+1} \sigma(a_{kj}) a_{jl} = \delta_{kl} e,$$

$$P := \begin{bmatrix} 1 \dots k-1, & k+1 \dots m+1 \\ j_1 \cdots j_{l-1} & j_{l+1} \cdots j_{m+1} \end{bmatrix}.$$

This complex Hopf algebra $SL_q(m+1, \mathbf{C})$ may be called special matrix quantum group.

(8.21.10.3) Let again $q \in \mathbf{D}$, $q^2 \neq -1$. The complex Hopf algebra $E_q(A_m)$, consisting of polynomials in the Chevalley generators, is in duality with the complex Hopf algebra $SL_q(m+1, \mathbf{C})$, with respect to the $\mathbf{C}$-bilinear form, such that $\forall_1^m k, \forall_1^{m+1} i, j$:

$$\langle E_k | a_{ij} \rangle = \langle F_k | a_{ji} \rangle = q^{1/2} \delta_{ik} \delta_{j,k+1},$$

$$\langle K_k | a_{ij} \rangle = q \delta_{ik} \delta_{jk} + q^{-1} \delta_{i,k+1} \delta_{j,k+1}.$$

(8.21.10.4) Correspondingly the topological Hopf algebra $U_q(A_m)$ over the ring $\mathcal{R} := \mathbf{C}[[h]]$, consisting of formal power series with relations in $X_k^{\pm}, H_k$, $k = 1, \dots, m$, is in duality with $SL_q(m+1, \mathbf{C})$, with respect to an $\mathcal{R}$-bilinear form such that $\forall_1^m k, \forall_1^{m+1} i, j$:

$$\langle X_k^+ | a_{ij} \rangle = \langle X_k^- | a_{ji} \rangle = \delta_{ik} \delta_{j,k+1}, \quad \langle H_k | a_{ij} \rangle = \delta_{ik} \delta_{jk} - \delta_{i,k+1} \delta_{j,k+1}.$$

Here $SL_q(m+1, \mathbf{C})$ is considered with respect to coefficients $\in \mathcal{R}$.
The representatives of these Chevalley generators of $U_q(A_m)$ are not q-deformed, because $[\sigma^3]_q = \sigma^3$. The universal R-matrix is represented by

$$q^{\frac{1}{m+1}}R(q) = q\sum_{k=1}^{m+1} E_{m+1}^{kk} \otimes E_{m+1}^{kk} + \sum_{1\le k\ne l\le m+1} E_{m+1}^{kk} \otimes E_{m+1}^{ll}$$
$$+ (q-q^{-1}) \sum_{1\le k<l\le m+1} E_{m+1}^{kl} \otimes E_{m+1}^{lk}.$$

(8.21.10.5) The resulting MCR are compatible with the following commutation relations of quantum coordinates, in the sense of an $\mathcal{A}(R;\rho)$-left comodule over $\mathcal{R}$ or $\mathbf{C}$, respectively.

$$\forall_1^{m+1} k < l : x_k x_l = q^{-1} x_l x_k.$$

(8.21.11) Consider the representation ρ of the q-deformed harmonic oscillator $H_q(1)$ on $\mathbf{C}^3$:

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{bmatrix}, \; N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; q := -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \; q^3 = 1,$$

the representatives of a and $a^\dagger$ being mutually transposed, and denoting $\xi := (q+q^{-1})^{1/2}$. The resulting representation of the universal R-matrix R yields some matrix quantum semigroup $\mathcal{A}(R;\rho)$, the MCR of which are compatible with the following commutation relations of quantum coordinates. Denoting by P the flip of $Mat(3,\mathrm{C}) \otimes Mat(3,\mathrm{C})$, $\forall_1^3 i,k$:

$$\sum_{j,l=1}^{3} (PR - R_{1111}I)_{ijkl} \; x_j x_l = \sum_{j,l=1}^{3} R_{kjil} \; x_j x_l - R_{1111} \; x_i x_k = 0.$$

The corresponding factor algebra, which is obtained from the free complex algebra over the set $\{x_1, x_2, x_3\}$ with respect to the relations

$$x_2 x_1 = -iq^{1/2} x_1 x_2, \; x_3 x_1 = -q x_1 x_3, \; x_3 x_2 = iq^{1/2} x_2 x_3,$$
$$q x_2^2 = x_1 x_3, \; 1 + q + q^2 = 0,$$

does not collapse; here one chooses $\xi = i$.

(8.21.12) Inserting the $\mathbf{Z}_2$-graded flip P of $Mat(d,K)\hat{\otimes}Mat(d,K)$, such that $\forall_1^d i,j$:

$$P(E_d^i \otimes E_d^j) = (-1)^{\hat{i}\hat{j}} E_d^j \otimes E_d^i, \; P := \sum_{k,l=1}^{d} (-1)^{\hat{l}} E_d^{kl} \otimes E_d^{lk},$$

the above introduced MCR can be rewritten as the commutation relation

$$\left[e_{\mathcal{A}} \otimes S_\rho, \bigotimes\nolimits_{\mathcal{A}}^{2} [a_{kl}; k,l = 1,\ldots,d] \right] = 0, \; S_\rho := PR_\rho =: S_{ijkl} E_d^{ij} \otimes E_d^{kl}.$$

Then the following quadratic relations of quantum super-coordinates x_k, $k = 1, \ldots, d = m + n$, namely such that for an arbitrary non-constant polynomial f in one indeterminate over K, $\forall_1^d i, k$:

$$\sum_{j,l=1}^{d} f(S_\rho)_{ijkl} x_j x_l \, (-1)^{\hat{j}\hat{k}} = 0, \;\; S_{ijkl} = (-1)^{\hat{i}\hat{j}+\hat{i}\hat{k}+\hat{j}\hat{k}} \, R_{kjil},$$

are compatible with the MCR, in the sense of an $\mathcal{A}(R;\rho)$-left comodule over K, the latter being $\mathbf{Z}_2$-graded in the following natural sense. Consider the generators x_k as even for $1 \leq k \leq m$, odd for $m+1 \leq k \leq d$, such that the above coordinate relations are homogeneous, thereby establishing the unital associative superalgebra $\mathcal{X}(R;\rho)$ over K. Then the structure map: $\mathcal{X}(R;\rho) \to \mathcal{A}(R;\rho) \otimes \mathcal{X}(R;\rho)$ is even. For instance inserting the polynomial $f(t) := t - \lambda$, $\lambda \in K$, one finds the relations, $\forall_1^d i, k$:

$$\sum_{j,l=1}^{d} R_{kjil} \; x_j x_l (-1)^{\hat{i}\hat{j}} = \lambda \; x_i x_k.$$

Of course these MCR and quantum super-coordinate relations may also be understood via an injective homomorphism of unital associative K-superalgebras α into a common pool $\mathcal{A}$, such that

$$\forall_1^d i, k, l : [\alpha(x_i), \alpha(a_{kl})] = 0,$$

using the super-commutator. Then the above relations of quantum super-coordinates and their transformation can be short-written as

$$(e_{\mathcal{A}} \otimes f(S_\rho)) \left([x] \otimes_{\mathcal{A}} [x]\right) = 0, \;\; [x'] = [a][x],$$

the latter explicitly meaning that

$$\forall_1^d k : x'_k := \sum_{l=1}^{d} (-1)^{(\hat{k}+\hat{l})\hat{l}} a_{kl} x_l.$$

Note here that R_ρ, P, and therefore also S_ρ are even elements.

(8.21.13) Some of the above considerations can also be performed over the ring $\mathcal{R} := \mathbf{C}[[h]]$, using complex formal power series in h as matrix elements. As an example consider the q-deformed fermionic oscillator over $\mathcal{R}$. Denoting

$$a_{11} \equiv a, \;\; a_{12} \equiv b, \;\; a_{21} \equiv c, \;\; a_{22} \equiv d, \;\; x_1 \equiv x, \;\; x_2 \equiv y,$$

a and d even, b and c odd, x even, y odd, the MCR read

$$ab = iq^{1/2} ba, \;\; ac = iq^{1/2} ca, \;\; bd = -iq^{1/2} db, \;\; cd = -iq^{1/2} dc,$$

$$bc = -cb, \;\; ad - da = +i \left(q^{1/2} + q^{-1/2}\right) bc.$$

The corresponding coordinate relations, choosing $f(t) := t-\lambda$, are non-trivial only for two values of $\lambda \in \mathcal{R}$; otherwise $x_1^2 = x_2^2 = x_1x_2 = x_2x_1 = 0$. One just inserts the representation of R on $\mathcal{R}^4$, together with the $\mathcal{R}$-linear bijection:

$$P \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \forall_1^2 k,l : E_2^k \otimes E_2^l \longleftrightarrow (-1)^{\hat{k}\hat{l}} E_4^{(k-1)2+l}.$$

$$\lambda := R_{1111} = \exp\left(\frac{1}{4}\ln q + \frac{i\pi}{2} - \frac{\theta_n^2}{\ln q}\right) \Longrightarrow xy = iq^{1/2}yx.$$

$$\lambda := q^{-1}R_{1111} \Longrightarrow x^2 = y^2 = 0, \; xy = iq^{-1/2}yx.$$

8.22 Deformed Tensor Product of Matrix Quantum Super-Semigroups

(8.22.1) Consider a $\mathbf{Z}_2$-graded bialgebra $\{\mathcal{H};\mu,\eta;\Delta,\varepsilon\}$ over K, *char* $K = 0$, and let α,β be representations of the unital associative superalgebra $\mathcal{H}$ on K^p and K^q respectively. The tensor product of these two representations is defined, with an obvious notation of degrees:

$$\mathcal{H} \ni x \xrightarrow{\Delta} \sum_{\lambda=1}^{L(x)} x'_\lambda \otimes x''_\lambda \overset{T(\alpha,\beta)}{\longrightarrow} \longleftrightarrow T_\Delta(\alpha,\beta)(x):$$

$$K^p \otimes K^q \ni u \otimes v \longrightarrow (-1)^{\hat{x}''_\lambda \hat{u}} \sum_{\lambda=1}^{L(x)} \alpha(x'_\lambda)(u) \otimes \beta(x''_\lambda)(v) \in K^p \otimes K^q.$$

In the special cases of $K(\mathcal{G})$ over a group $\mathcal{G}$, such that $\Delta : \mathcal{G} \ni g \longrightarrow g \otimes g$, and of a Lie superalgebra $\mathcal{L}$, such that $\Delta : \mathcal{L} \ni x \longrightarrow x \otimes e + e \otimes x$, with the unit e of $E(\mathcal{L})$ in the latter case, one finds the usual tensor products of representations of such objects.

(8.22.2) Let especially $\mathcal{H}$ be a $\mathbf{Z}_2$-graded Hopf algebra, which is quasitriangular with respect to the universal R-matrix R. One then obtains, according to the proposed representations, two matrix quantum super-semigroups $\mathcal{A}(R;\alpha)$ and $\mathcal{A}(R;\beta)$, with the generators a_{ij} and b_{kl} respectively, for $1 \le i,j \le p$, $1 \le k,l \le q$. Now consider the $\mathbf{Z}_2$-graded bialgebra $\mathcal{A}(R;\gamma)$, $\gamma \longleftrightarrow T_\Delta(\alpha,\beta)$, with the generators c_{rs}, $1 \le r,s \le pq$. One easily calculates that

$$\langle x|c_{(i-1)q+k,(j-1)q+l}\rangle = \sum_{\lambda=1}^{L(x)} \langle x'_\lambda|a_{ij}\rangle\langle x''_\lambda|b_{kl}\rangle(-1)^{(\hat{\imath}+\hat{\jmath})\hat{k}},$$

with the above notation of $\Delta(x)$. Moreover, denoting by $R^{(\alpha,\beta)}_{ijkl}$ the components of $T(\alpha,\beta)(R)$,

$$\begin{aligned}
0 &= T(\alpha,\beta)(R\Delta(x) - \tau\circ\Delta(x)R) \\
&= \sum_{\lambda=1}^{L(x)} \sum_{i,j,m=1}^{p} \sum_{k,l,n=1}^{q} \Big(R^{(\alpha,\beta)}_{ijkl} \langle x'_\lambda \otimes x''_\lambda | a_{jm}\otimes b_{ln}\rangle (-1)^{(\hat{\jmath}+\hat{m})(\hat{k}+\hat{n})} \\
&\qquad - (-1)^{(\hat{\jmath}+\hat{m})(\hat{k}+\hat{l})} \langle x'_\lambda \otimes x''_\lambda | b_{kl}\otimes a_{ij}\rangle R^{(\alpha,\beta)}_{jmln} \Big) E^{im}_p \otimes E^{kn}_q,
\end{aligned}$$

and similarly by an application of $T(\beta,\alpha)$, with corresponding components $R^{(\beta,\alpha)}_{klij}$.

(8.22.3) These identities, although the involved bilinear forms may be degenerate, are hints to the subsequent statement. There is some homomorphism of unital associative K-superalgebras:

$$\mathcal{A}(R;\gamma) \ni c_{(i-1)q+k,(j-1)q+l} \longrightarrow a_{ij}b_{kl}(-1)^{(\hat{\imath}+\hat{\jmath})\hat{l}} \in \mathcal{A}(R;\alpha)\tilde{\otimes}\mathcal{A}(R;\beta),$$

into an appropriately deformed tensor product of unital associative K-superalgebras, both of which were defined by main commutation relations. It is constructed from generators a_{ij} and b_{kl}, $1 \le i,j \le p$, $1 \le k,l \le q$, with respect to the following homogeneous relations. The generators a_{ij} among themselves, and b_{kl} among themselves, fulfill the MCR-type relations of $\mathcal{A}(R;\alpha)$ and $\mathcal{A}(R;\beta)$, respectively. Moreover the mixed relations below are proposed.

$$\sum_{j=1}^{p}\sum_{l=1}^{q} \Big(R^{(\alpha,\beta)}_{ijkl} a_{jm} b_{ln} (-1)^{(\hat{\jmath}+\hat{m})(\hat{k}+\hat{n})} - (-1)^{(\hat{\jmath}+\hat{m})(\hat{k}+\hat{l})} b_{kl} a_{ij} R^{(\alpha,\beta)}_{jmln} \Big) = 0,$$

$$\sum_{j=1}^{p}\sum_{l=1}^{q} \Big(R^{(\beta,\alpha)}_{klij} b_{ln} a_{jm} (-1)^{(\hat{l}+\hat{n})(\hat{\imath}+\hat{m})} - (-1)^{(\hat{l}+\hat{n})(\hat{\imath}+\hat{\jmath})} a_{ij} b_{kl} R^{(\beta,\alpha)}_{lnjm} \Big) = 0.$$

(8.22.3.1) Using copies into an associative unital superalgebra $\mathcal{A}$, denoting

$$R^{(\alpha,\beta)}_{\mathcal{A}} := e_{\mathcal{A}} \otimes R^{(\alpha,\beta)}, \quad R^{(\alpha,\beta)} := T(\alpha,\beta)(R),$$

and similarly for $T(\beta,\alpha)(R)$, these mixed main commutation relations read as follows.

$$\begin{aligned}
R^{(\alpha,\beta)}_{\mathcal{A}}([a]\otimes_{\mathcal{A}} I_{\mathcal{A}})(I_{\mathcal{A}}\otimes_{\mathcal{A}}[b]) &= (I_{\mathcal{A}}\otimes_{\mathcal{A}}[b])([a]\otimes_{\mathcal{A}} I_{\mathcal{A}})R^{(\alpha,\beta)}_{\mathcal{A}}, \\
R^{(\beta,\alpha)}_{\mathcal{A}}([b]\otimes_{\mathcal{A}} I_{\mathcal{A}})(I_{\mathcal{A}}\otimes_{\mathcal{A}}[a]) &= (I_{\mathcal{A}}\otimes_{\mathcal{A}}[a])([b]\otimes_{\mathcal{A}} I_{\mathcal{A}})R^{(\beta,\alpha)}_{\mathcal{A}}.
\end{aligned}$$

The proof of this theorem is performed by means of the following lemma.

$$R^{(\gamma,\gamma)} := T(\gamma,\gamma)(R) \longleftrightarrow T(\alpha,\beta,\alpha,\beta)\circ T(\Delta,\Delta)(R)$$
$$= R_{14}^{(\alpha,\beta)}R_{13}^{(\alpha,\alpha)}R_{24}^{(\beta,\beta)}R_{23}^{(\beta,\alpha)}.$$

Here the lower indices describe embeddings into an appropriate four-fold tensor product of matrices over K. One then easily shows, that the above representation of R on $K^p\otimes K^q\otimes K^p\otimes K^q$ serves for an MCR-type intertwining property of $[a]\otimes_{\mathcal{A}}[b]$. Denoting $R_{\mathcal{A}}^{(\gamma,\gamma)} \longleftrightarrow e_{\mathcal{A}}\otimes R^{(\gamma,\gamma)}$, one calculates that

$$R_{\mathcal{A}}^{(\gamma,\gamma)}([a]\otimes_{\mathcal{A}}[b]\otimes_{\mathcal{A}}I_{\mathcal{A}})(I_{\mathcal{A}}\otimes_{\mathcal{A}}[a]\otimes_{\mathcal{A}}[b])$$
$$= (I_{\mathcal{A}}\otimes_{\mathcal{A}}[a]\otimes_{\mathcal{A}}[b])([a]\otimes_{\mathcal{A}}[b]\otimes_{\mathcal{A}}I_{\mathcal{A}})R_{\mathcal{A}}^{(\gamma,\gamma)}.$$

8.23 Deformed Tensor Product of Quantum Super-Vector Spaces

(8.23.1) Let P,Q,S,T be the graded flips on

$$K^p\otimes K^p,\ \ K^q\otimes K^q,\ \ K^p\otimes K^q,\ \ K^q\otimes K^p,$$

and denote $e_{\mathcal{A}}\otimes P =: P_{\mathcal{A}},\dots$. Here again K denotes a field of *char* $K=0$.

$$S := \sum_{i=1}^{p}\sum_{k=1}^{q}(-1)^{\hat{i}}E_{q,p}^{ki}\otimes E_{p,q}^{ik},\ \ T := \sum_{i=1}^{p}\sum_{k=1}^{q}(-1)^{\hat{k}}E_{p,q}^{ik}\otimes E_{q,p}^{ki},$$
$$TS = id(K^p\otimes K^p),\ \ ST = id(K^q\otimes K^q),$$

involving matrices over K with p rows and q columns, and conversely; for instance, $\left(E_{q,p}^{ki}\right)_{lj} = \delta_{kl}\delta_{ij}$. The mixed MCR-type relations of the foregoing chapter can be rewritten, using again faithful copies into an associative unital superalgebra $\mathcal{A}$, as

$$S_{\mathcal{A}}R_{\mathcal{A}}^{(\alpha,\beta)}([a]\otimes_{\mathcal{A}}[b]) = ([b]\otimes_{\mathcal{A}}[a])S_{\mathcal{A}}R_{\mathcal{A}}^{(\alpha,\beta)},$$
$$T_{\mathcal{A}}R_{\mathcal{A}}^{(\beta,\alpha)}([b]\otimes_{\mathcal{A}}[a]) = ([a]\otimes_{\mathcal{A}}[b])T_{\mathcal{A}}R_{\mathcal{A}}^{(\beta,\alpha)}.$$

Therefore the subsequent relations of x_i and y_k, $1\le i\le p$, $1\le k\le q$, are compatible with their transformation according to

$$[x'] = [a][x],\ \ [y'] = [b][y].$$

For $\mu,\nu\in K$,

$$S_{\mathcal{A}}R_{\mathcal{A}}^{(\alpha,\beta)}([x]\otimes_{\mathcal{A}}[y]) = \mu([y])\otimes_{\mathcal{A}}[x]),$$
$$T_{\mathcal{A}}R_{\mathcal{A}}^{(\beta,\alpha)}([y]\otimes_{\mathcal{A}}[x]) = \nu([x]\otimes_{\mathcal{A}}[y]).$$

(8.23.2) This invariance property induces the following construction. Let $\mathcal{X}(R;\alpha)$ and $\mathcal{X}(R;\beta)$ be the unital associative superalgebras of quantum super-coordinates, which are due to the relations

$$(P_{\mathcal{A}}R_{\mathcal{A}}^{(\alpha,\alpha)} - \xi I_{\mathcal{A}})([x] \otimes_{\mathcal{A}} [x]) = 0, \ \ (Q_{\mathcal{A}}R_{\mathcal{A}}^{(\beta,\beta)} - \eta I_{\mathcal{A}})([y] \otimes_{\mathcal{A}} [y]) = 0,$$

$\xi, \eta \in K$, denoting

$$R_{\mathcal{A}}^{(\alpha,\alpha)} := e_{\mathcal{A}} \otimes R^{(\alpha,\alpha)}, \ \ R^{(\alpha,\alpha)} := T(\alpha,\alpha)(R),$$

and similarly for the representation β of the quasitriangular $\mathbf{Z}_2$-graded Hopf algebra $\mathcal{H}$. Moreover, for $\lambda \in K$, define quantum super-coordinates z_r, $1 \le r \le pq$, with respect to the relations

$$(\mathbf{P}_{\mathcal{A}}R_{\mathcal{A}}^{(\gamma,\gamma)} - \lambda I_{\mathcal{A}})([z] \otimes_{\mathcal{A}} [z]) = 0, \ \ \gamma \longleftrightarrow T_{\Delta}(\alpha,\beta),$$

$\mathbf{P}_{\mathcal{A}} := e_{\mathcal{A}} \otimes \mathbf{P}$, denoting by $\mathbf{P}$ the graded flip on

$$K^{pq} \otimes K^{pq} \longleftrightarrow K^p \otimes K^q \otimes K^p \otimes K^q.$$

$$\mathbf{P} := \sum_{r,s=1}^{pq} E_{pq}^{rs} \otimes E_{pq}^{sr}(-1)^{\hat{s}}$$

$$\longleftrightarrow \sum_{i,j=1}^{p} \sum_{k,l=1}^{q} E_p^{ij} \otimes E_q^{kl} \otimes E_p^{ji} \otimes E_q^{lk}(-1)^{(\hat{\imath}+\hat{\jmath})(\hat{k}+\hat{l})+\hat{\jmath}+\hat{l}} \underset{\tau_{23}}{\longleftrightarrow} P \otimes Q,$$

$\hat{s} = \hat{\jmath} + \hat{l}$ for $s = (j-1)q + l$, where τ_{23} denotes the indicated embedding of the graded flip τ of matrices over K into the involved four-fold tensor product.

(8.23.2.1) On the other hand, a deformed tensor product $\mathcal{X}(R;\alpha)\tilde{\otimes}\mathcal{X}(R;\beta)$ is constructed with respect to the following relations of quantum super-coordinates. The coordinates x_i among themselves, and y_k among themselves, $1 \le i \le p, 1 \le k \le q$, fulfill the relations of $\mathcal{X}(R;\alpha)$ and $\mathcal{X}(R;\beta)$ respectively. Moreover they are assumed to obey the mixed relations above with parameters $\mu, \nu \in K$. Choose $\lambda = \xi\eta\mu\nu$. Then the subsequent theorem holds. There is a homomorphism of unital associative K-superalgebras:

$$\mathcal{X}(R;\gamma) \ni z_{(i-1)q+k} \longrightarrow x_i y_k \in \mathcal{X}(R;\alpha)\tilde{\otimes}\mathcal{X}(R;\beta),$$

which can be short-written as $[z] \longrightarrow [x] \otimes_{\mathcal{A}} [y]$.

(8.23.2.2) The proof of this theorem is due to the following lemma.

$$\begin{aligned}\mathbf{P}R^{(\gamma,\gamma)} &\longleftrightarrow \tau_{23}(P \otimes Q)R_{14}^{(\alpha,\beta)}R_{13}^{(\alpha,\alpha)}R_{24}^{(\beta,\beta)}R_{23}^{(\beta,\alpha)} \\ &= \tau_{23}\left((P \otimes Q)R_{14}^{(\alpha,\beta)}R_{12}^{(\alpha,\alpha)}R_{34}^{(\beta,\beta)}\left(\tau(R)\right)_{23}^{(\alpha,\beta)}\right) \\ &= \left(\tau(R)\right)_{23}^{(\beta,\alpha)} P_{13}R_{13}^{(\alpha,\alpha)}Q_{24}R_{24}^{(\beta,\beta)}R_{23}^{(\beta,\alpha)}.\end{aligned}$$

One then calculates, that

$$(\mathbf{P}_{\mathcal{A}}R_{\mathcal{A}}^{(\gamma,\gamma)} - \lambda I_{\mathcal{A}})([x] \otimes_{\mathcal{A}} [y] \otimes_{\mathcal{A}} [x] \otimes_{\mathcal{A}} [y]) = 0.$$

(8.23.3) As an easy example consider the two-fold tensor product of complex representations of $U_q(A_1)$, $q \in \mathbf{D}$, $q^2 \neq -1$ for odd coordinates, such that:

$$X^{\pm} = \sigma^{\pm}, \ H = \sigma^3, \ R \longleftrightarrow q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

Choosing for instance $\mu = \nu = q^{1/2}$, one obtains the mixed relations

$$x_1 y_1 = y_1 x_1, \ x_2 y_2 = y_2 x_2, \ q x_1 y_2 = y_2 x_1, \ q y_1 x_2 = x_2 y_1, \ x_1 y_2 = y_1 x_2.$$

One thereby obtains the following relations of these new quantum super-coordinates $[z]$, inserting even or odd quantum coordinates x_k and y_k, for $k = 1, 2$.

$$z_1 \longrightarrow x_1 y_1, \ z_2 \longrightarrow x_1 y_2, \ z_3 \longrightarrow x_2 y_1, \ z_4 \longrightarrow x_2 y_2.$$

Case 1. $\xi = \eta = q^{1/2}$, $\lambda = q^2$; $q x_1 x_2 = x_2 x_1$, $q y_1 y_2 = y_2 y_1$;

$$z_1 z_2 = q^{-2} z_2 z_1, \ z_1 z_3 = q^{-2} z_3 z_1, \ z_2 z_4 = q^{-2} z_4 z_2, \ z_3 z_4 = q^{-2} z_4 z_3,$$

$$z_1 z_4 = q^{-4} z_4 z_1, \ z_2 z_3 = z_3 z_2.$$

Case 2. $\xi = \eta = -q^{-3/2}$, $\lambda = q^{-2}$; $x_1 x_2 = -q x_2 x_1$, $y_1 y_2 = -q y_2 y_1$,

$$x_1^2 = x_2^2 = y_1^2 = y_2^2 = 0;$$

$$z_1^2 = z_2^2 = z_3^2 = z_4^2 = 0, \ z_1 z_4 = z_4 z_1, \ z_2 z_3 = z_3 z_2,$$

$$z_1 z_2 = z_2 z_1 = z_1 z_3 = z_3 z_1 = z_2 z_4 = z_4 z_2 = z_3 z_4 = z_4 z_3 = 0.$$

Case 3. $\xi = q^{1/2}$, $\eta = -q^{-3/2}$, $\lambda = -1$;

$$q x_1 x_2 = x_2 x_1, \ y_1 y_2 = -q y_2 y_1, \ y_1^2 = y_2^2 = 0;$$

$$z_1^2 = z_2^2 = z_3^2 = z_4^2 = 0, \ z_1 z_3 = z_3 z_1 = z_2 z_4 = z_4 z_2 = 0,$$

$$z_1 z_2 = -z_2 z_1, \ z_1 z_4 = -q^{-2} z_4 z_1, \ z_2 z_3 = -q^{-2} z_3 z_2, \ z_3 z_4 = -z_4 z_3.$$

8.24 Covariant Differential Calculus on Quantum Superspaces

Let $\mathcal{A}$ be an associative superalgebra over an algebraically closed field K of characteristic 0.

(8.24.1) Let $\Gamma = \Gamma^{\bar{0}} \oplus \Gamma^{\bar{1}}$ be a graded $\mathcal{A}$-$\mathcal{A}$-bimodule over K; the latter assumption especially means, that $\forall \gamma \in \Gamma, \forall a, b \in \mathcal{A} : (a\gamma)b = a(\gamma b)$; moreover the module-multiplications are compatible with the $\mathbf{Z}_2$-grading in the natural sense. Let $d : \mathcal{A} \longrightarrow \Gamma$ be an odd K-linear map, which obeys the $\mathbf{Z}_2$-graded Leibniz rule, i.e.,

$$\forall \bar{x}, \bar{y} \in \mathbf{Z}_2, \forall a \in \mathcal{A}^{\bar{x}}, b \in \mathcal{A}^{\bar{y}} : d(ab) = (da)b + (-1)^x a \; db \in \Gamma^{\overline{x+y+1}}.$$

The pair $\{\Gamma, d\}$ is called first order graded differential calculus over $\mathcal{A}$, if and only if

$$\forall \gamma \in \Gamma : \exists n \in \mathbf{N}, \exists a_1, b_1, \ldots, a_n, b_n \in \mathcal{A} : \gamma = \sum_{k=1}^{n} a_k db_k.$$

(8.24.1.1) For instance let $\mathcal{A}$ be unital, and consider the $\mathbf{Z}_2$-graded differential algebra $H(\mathcal{A})$, which is obtained from the unital universal differential envelope $\tilde{\Omega}(\mathcal{A}) \longleftrightarrow \tilde{H}(\mathcal{A})$ by identifying the unit of $\tilde{\mathcal{A}} := K \oplus \mathcal{A}$ with the unit $e_{\mathcal{A}}$ of $\mathcal{A}$ itself. Denoting the graded structure map of $\mathcal{A}$ by

$$\hat{\mu} : \mathcal{A} \otimes \mathcal{A} \ni a \otimes b \longrightarrow (-1)^y ab \in \mathcal{A} \text{ for } b \in \mathcal{A}^{\bar{y}}, \bar{y} \in \mathbf{Z}_2,$$

one easily constructs the following first order graded differential calculus $\{ker \; \hat{\mu}, \delta\}$ over $\mathcal{A}$:

$$\mathcal{A} \ni a \xrightarrow[def]{\delta} e_{\mathcal{A}} \otimes a - (-1)^x a \otimes e_{\mathcal{A}} \in ker \; \hat{\mu} \text{ for } a \in \mathcal{A}^{\bar{x}}, \bar{x} \in \mathbf{Z}_2.$$

Here $ker \; \hat{\mu}$ becomes a graded $\mathcal{A}$-$\mathcal{A}$-bimodule over K, with respect to the module-multiplications, such that $\forall a, b, c \in \mathcal{A}$:

$$(a \otimes b) \circ c = a \otimes bc, \;\; a \circ (b \otimes c) = ab \otimes c.$$

(8.24.2) Let $\{\Gamma, d\}$ be a first order graded differential calculus over $\mathcal{A}$, and moreover assume $\{\mathcal{A}, \mu, \eta, \Delta, \varepsilon\}$ as $\mathbf{Z}_2$-graded bialgebra over K. This calculus is called left or right covariant, if and only if

$$\Gamma \ni \sum_{k=1}^{n} a_k db_k = 0 \Longrightarrow \sum_{k=1}^{n} \Delta(a_k) \left(T(id \; \mathcal{A}, d) \circ \Delta(b_k)\right) = 0, \text{ or}$$

$$\Longrightarrow \sum_{k=1}^{n} \Delta(a_k) \left(T(d, id \; \mathcal{A}) \circ \Delta(b_k)\right) = 0,$$

respectively. In these two cases one then defines the K-linear mappings:

$$\Gamma \ni \sum_{k=1}^{n} a_k db_k \xrightarrow[\Delta_L]{} \sum_{k=1}^{n} \Delta(a_k)\,(T(id\ \mathcal{A}, d) \circ \Delta(b_k)) \in \mathcal{A} \otimes \Gamma, \text{ or}$$

$$\xrightarrow[\Delta_R]{} \sum_{k=1}^{n} \Delta(a_k)\,(T(d, id\ \mathcal{A}) \circ \Delta(b_k)) \in \Gamma \otimes \mathcal{A}.$$

Here one denotes, inserting the comultiplication images of $a, b \in \mathcal{A}$,

$$\Delta(a)\,(T(id\ \mathcal{A}, d) \circ \Delta(b)) := \sum_{k=1}^{k} \sum_{l=1}^{l} (-1)^{x_k y_l} a_k' b_l' \otimes a_k'' db_l'',$$

denoting by x_k and y_l the degrees of a_k'' and b_l', respectively, and similarly from the right. In these two cases of left or right covariance one thereby finds the commuting diagrams:

$$\Delta_L \circ d = T(id\ \mathcal{A}, d) \circ \Delta, \text{ or } \Delta_R \circ d = T(d, id\ \mathcal{A}) \circ \Delta;$$

$$T(\Delta, id\ \Gamma) \circ \Delta_L = T(id\ \mathcal{A}, \Delta_L) \circ \Delta_L, \text{ or}$$

$$T(id\ \Gamma, \Delta) \circ \Delta_R = T(\Delta_R, id\ \mathcal{A}) \circ \Delta_R.$$

(8.24.2.1) One thereby establishes some $\mathcal{A}$-left or right comodule over K. If both Δ_L and Δ_R are well-defined, then, modulo ordering of tensor products as just above, these comodule-comultiplications moreover obey the commuting diagram:

$$T(\Delta_L, id\ \mathcal{A}) \circ \Delta_R = T(id\ \mathcal{A}, \Delta_R) \circ \Delta_L;$$

in this case the first order graded differential calculus is called bicovariant.

(8.24.2.2) The above defined calculus $\{ker\ \hat{\mu}, \delta\}$ is bicovariant.

(8.24.3) Let $S \in Mat(p, K)\hat{\otimes}Mat(p, K)$ be a solution of both the graded braid relation and the graded Hecke condition, i.e.,

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23},\ S^2 = I_p \otimes I_p + (q - q^{-1})S,\ q \neq 0,\ q^2 \neq -1,$$

with the grading format according to $p = m + n$.
For instance one may insert $S := q^{\frac{1}{p}} PR_\rho$, with the flip P, and the usual defining representation of the universal R-matrix of $U_q(A_{p-1})$, $p \geq 2$, for $q \in \mathbf{D}$, $q^2 \neq -1$.
Then the quantum superspace of even quantum coordinates $x_1, \ldots, x_m$, and odd $x_{m+1}, \ldots, x_{m+n}$, $m + n = p$, is extended to the following graded differential calculus over itself.

(8.24.3.1) Let $\Omega_S(\mathcal{X}_S)$ be the unital associative superalgebra over K, the even generators $x_1, \ldots, x_m, y_{m+1}, \ldots, y_p$, and odd $x_{m+1}, \ldots, x_p, y_1, \ldots, y_m$ of which are subject to the homogeneous relations

$$S_{\mathcal{A}}\left([x] \otimes_{\mathcal{A}} [x]\right) = q\left([x] \otimes_{\mathcal{A}} [x]\right),$$

$$S_{\mathcal{A}}\left([y] \otimes_{\mathcal{A}} [x]\right) = q^{-1}[x] \otimes_{\mathcal{A}} [y],$$

$$S_{\mathcal{A}}\left([y] \otimes_{\mathcal{A}} [y]\right) = -q^{-1}[y] \otimes_{\mathcal{A}} [y],$$

$S_{\mathcal{A}} := e_{\mathcal{A}} \otimes S$, with the convenient notation of skew-symmetric tensor products over $\mathcal{A}$, suppressing an injective homomorphism of unital associative K-superalgebras into the pool of copies $\mathcal{A}$. With the odd derivation d, such that

$$\forall_1^p k : x_k \xrightarrow{d} y_k \xrightarrow{d} 0,$$

one thereby establishes the $\mathbf{Z}_2$-graded differential algebra $\Omega_S(\mathcal{X}_S)$ over K, using the graded Hecke condition on S in order to factorize d.

(8.24.3.2) For the quantum plane one easily calculates the following relations. Let $q \in \mathbf{C}\backslash\{0\}$, $q^2 \neq -1$.

$$S := \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{bmatrix}; \quad q x_1 x_2 = x_2 x_1;$$

$$x_k dx_k = q^2 (dx_k) x_k, \ (dx_k)^2 = 0, \ k = 1, 2; \ dx_1 dx_2 = -q dx_2 dx_1;$$

$$x_1 dx_2 = q(dx_2)x_1, \ x_2 dx_1 + q^{-1} x_1 dx_2 = q(dx_1)x_2 + q^2(dx_2)x_1.$$

(8.24.4) Under the same conditions on S as above, the matrix quantum super-semigroup $\mathcal{A}_S$ of generators $a_{kl}, k, l = 1, \ldots, p$, which obey the graded main commutation relations (MCR) due to the solution PS of the graded QYBE, with the graded flip P, is extended to the following $\mathbf{Z}_2$-graded differential K-algebra $\Omega_S(\mathcal{A}_S)$, again using the graded Hecke condition.

$$S_{\mathcal{A}}\left([a] \otimes_{\mathcal{A}} [a]\right) = \left([a] \otimes_{\mathcal{A}} [a]\right) S_{\mathcal{A}},$$

$$[a] \otimes_{\mathcal{A}} [da] = S_{\mathcal{A}}\left([da] \otimes_{\mathcal{A}} [a]\right) S_{\mathcal{A}},$$

$$[da] \otimes_{\mathcal{A}} [da] = -S_{\mathcal{A}}\left([da] \otimes_{\mathcal{A}} [da]\right) S_{\mathcal{A}}.$$

(8.24.4.1) The comultiplication Δ of the $\mathbf{Z}_2$-graded K-bialgebra $\mathcal{A}_S$, such that $\forall_1^n k, l$:

$$a_{kl} \xrightarrow[\Delta]{} \sum_{j=1}^{n} a_{kj} \otimes a_{jl}, \ a_{kl} \xrightarrow[\varepsilon]{} \delta_{kl} \in K,$$

is then extended uniquely to a homomorphism of $\mathbf{Z}_2$-graded differential algebras: $\tilde{\Omega}(\mathcal{A}_S) \longrightarrow \Omega_S(\mathcal{A}_S)\hat{\otimes}\Omega_S(\mathcal{A}_S)$, from the corresponding unital universal differential envelope into the involved skew-symmetric tensor product of $\mathbf{Z}_2$-graded differential algebras over K, and then factorized by the above relations of $\Omega_S(\mathcal{A}_S)$. Hence one obtains some homomorphism of $\mathbf{Z}_2$-graded differential K-algebras

$$\Delta_* : \Omega_S(\mathcal{A}_S) \longrightarrow \Omega_S(\mathcal{A}_S)\hat{\otimes}\Omega_S(\mathcal{A}_S).$$

(8.24.4.2) Constructing also $\varepsilon_* : \Omega_S(\mathcal{A}_S) \longrightarrow K$ as an appropriately factorized extension of the counit ε, such that $\varepsilon_* \circ d = 0$, one obtains the $\mathbf{Z}_2$-graded bialgebra $\Omega_S(\mathcal{A}_S)$ over K.

(8.24.4.3) The restriction of Δ_* to the $\mathcal{A}_S$-$\mathcal{A}_S$-bimodule $\mathcal{A}_S d\mathcal{A}_S$ over K yields some bicovariant first order graded differential calculus over $\mathcal{A}_S$, such that

$$\forall a, b \in \mathcal{A}_S : \ \Delta_*(adb) = (\Delta_L + \Delta_R)(adb).$$

(8.24.4.4) Let again $q \neq 0$, $q^2 \neq -1$. The relations of the bialgebra $Mat_q(2, K)$ are consistently supplemented by the following ones.

$$\begin{aligned}
a_{11}da_{11} &= q^2(da_{11})a_{11}, \ a_{11}da_{12} = q(da_{12})a_{11},\\
a_{11}da_{21} &= q(da_{21})a_{11}, \ a_{11}da_{22} = (da_{22})a_{11},\\
a_{12}da_{11} &= q(da_{11})a_{12} + (q - q^{-1})a_{11}da_{12}, \ a_{12}da_{12} = q^2(da_{12})a_{12},\\
a_{12}da_{21} &= (da_{21})a_{12} + (q - q^{-1})a_{11}da_{22}, \ a_{12}da_{22} = q(da_{22})a_{12},\\
a_{21}da_{11} &= q(da_{11})a_{21} + (q - q^{-1})a_{11}da_{21}, \ a_{21}da_{21} = q^2(da_{21})a_{21},\\
a_{21}da_{12} &= (da_{12})a_{21} + (q - q^{-1})a_{11}da_{22}, \ a_{21}da_{22} = q(da_{22})a_{21},\\
a_{22}da_{11} &= (da_{11})a_{22} + (q - q^{-1})(a_{12}da_{21} + (da_{12})a_{21}),\\
a_{22}da_{22} &= q^2(da_{22})a_{22},\\
a_{22}da_{12} &= q(da_{12})a_{22} + (q - q^{-1})a_{12}da_{22},\\
a_{22}da_{21} &= q(da_{21})a_{22} + (q - q^{-1})a_{21}da_{22}.
\end{aligned}$$

$$\begin{aligned}
&(da_{11})^2 = (da_{12})^2 = (da_{21})^2 = (da_{22})^2 = 0,\\
&(da_{11})(da_{22}) = -(da_{22})(da_{11}),\\
&(da_{11})(da_{12}) = -q(da_{12})(da_{11}), \ (da_{11})(da_{21}) = -q(da_{21})(da_{11}),\\
&(da_{12})(da_{21}) + (da_{21})(da_{12}) = (q - q^{-1})(da_{11})(da_{22}),\\
&(da_{12})(da_{22}) = -q(da_{22})(da_{12}), \ (da_{21})(da_{22}) = -q(da_{22})(da_{21}).
\end{aligned}$$

For $q = 1$ one obtains the anticommuting differentials of four pairwise commuting even generators.

(8.24.5) The transformation of quantum superspaces by matrix quantum super-semigroups is compatible with the above odd derivations in the sense of the following left covariance. The homomorphism of unital associative K-superalgebras:

$$\mathcal{X}_S \ni x_k \longrightarrow \sum_{l=1}^{p} a_{kl} \otimes x_l \in \mathcal{A}_S \hat{\otimes} \mathcal{X}_S,$$

which is extended to a unique homomorphism of $\mathbf{Z}_2$-graded differential algebras, on the non-unital universal differential envelope:

$$\Omega(\mathcal{X}_S) \longrightarrow \Omega_S(\mathcal{A}_S) \hat{\otimes} \Omega_S(\mathcal{X}_S),$$

can be factorized by the relations of $\Omega_S(\mathcal{X}_S)$, in order to obtain a unique homomorphism of $\mathbf{Z}_2$-graded differential K-algebras, which might be called induced transformation of differential forms on the quantum superspace $\mathcal{X}_S$:

$$\Omega_S(\mathcal{X}_S) \ni dx_k \\ \longrightarrow \sum_{l=1}^{p} \left(da_{kl} \otimes x_l + (-1)^{\hat{k}+\hat{l}} a_{kl} \otimes dx_l \right) \in \Omega_S(\mathcal{A}_S) \hat{\otimes} \Omega_S(\mathcal{X}_S).$$

Here $\hat{k} + \hat{l}$ describes the $\mathbf{Z}_2$-degree of a_{kl}, defining

$$\hat{k} := 0 \text{ for } 1 \le k \le m, \;\; \hat{k} := 1 \text{ for } m+1 \le k \le m+n = p.$$

This left covariance, which is due to compatibility of the relations of $\Omega_S(\mathcal{X}_S)$ and $\Omega_S(\mathcal{A}_S)$, can be short-written, using matrices over a common pool superalgebra $\mathcal{A}$:

$$[a] \otimes [x] \stackrel{d}{\longrightarrow} [da] \otimes [x] + [a] \otimes [dx].$$

8.25 Fundamental Representations of q-Deformed Universal Enveloping Algebras

(8.25.1) With the conventions: $A_m \equiv sl(m+1, \mathbf{C})$,

$$B_m := \{Y \in gl(2m+1, \mathbf{C}); K_m Y + Y^t K_m = 0\},$$
$$C_m := \{Y \in gl(2m, \mathbf{C}); J_m Y + Y^t J_m = 0\},$$
$$D_m := \{Y \in gl(2m, \mathbf{C}); L_m Y + Y^t L_m = 0\},$$

$$L_m := S_{2m},\ K_m := S_{2m+1},\ J_m := \begin{bmatrix} 0 & S_m \\ -S_m & 0 \end{bmatrix},$$

$$S_n := \begin{bmatrix} 000 & \cdots & 001 \\ 000 & \cdots & 010 \\ & \cdots & \\ 010 & \cdots & 000 \\ 100 & \cdots & 000 \end{bmatrix} \quad \text{for } n \in \mathbf{N},$$

inserting above the transposed matrices Y^t of Y, one finds the following defining representations of these complex Lie algebras $\mathcal{L}$.

(8.25.1.1) For $A_m, m \in \mathbf{N}$, $\forall_1^m k, \forall_1^{m+1} i, j$:

$$\left(X_k^+\right)_{ij} = \delta_{ik}\delta_{j,k+1},\ (H_k)_{ij} = \delta_{ik}\delta_{jk} - \delta_{i,k+1}\delta_{j,k+1}.$$

(8.25.1.2) For $B_m, m \geq 2$, $\forall_1^{m-1} k, \forall_1^{2m+1} i, j$:

$$\left(X_k^+\right)_{ij} = \delta_{ik}\delta_{j,k+1} - \delta_{i,2m+1-k}\delta_{j,2m+2-k},$$

$$(H_k)_{ij} = \delta_{ij}\left(\delta_{ik} - \delta_{i,k+1} + \delta_{i,2m+1-k} - \delta_{i,2m+2-k}\right),$$

$$\left(X_m^+\right)_{ij} = \sqrt{2}\left(\delta_{im}\delta_{j,m+1} - \delta_{i,m+1}\delta_{j,m+2}\right),\ (H_m)_{ij} = 2\delta_{ij}\left(\delta_{im} - \delta_{i,m+2}\right).$$

(8.25.1.3) For $C_m, m \geq 3$, $\forall_1^{m-1} k, \forall_1^{2m} i, j$:

$$\left(X_k^+\right)_{ij} = \delta_{ik}\delta_{j,k+1} - \delta_{i,2m-k}\delta_{j,2m+1-k},$$

$$(H_k)_{ij} = \delta_{ij}\left(\delta_{ik} - \delta_{i,k+1} + \delta_{i,2m-k} - \delta_{i,2m+1-k}\right),$$

$$\left(X_m^+\right)_{ij} = \delta_{im}\delta_{j,m+1},\ (H_m)_{ij} = \delta_{ij}\left(\delta_{im} - \delta_{i,m+1}\right).$$

(8.25.1.4) For $D_m, m \geq 4, 1 \leq k \leq m-1$, the representation looks like that of C_m. $\forall_1^{2m} i, j$:

$$\left(X_m^+\right)_{ij} = \delta_{i,m-1}\delta_{j,m+1} - \delta_{im}\delta_{j,m+2},$$

$$(H_m)_{ij} = \delta_{ij}\left(\delta_{i,m-1} + \delta_{im} - \delta_{i,m+1} - \delta_{i,m+2}\right).$$

In all four cases, X_k^+ and X_k^- are represented by mutually transposed real matrices, for $k = 1, \ldots, m$.

(8.25.2) Choose $q \in \mathbf{D}$ such that all the coefficients occurring in the relations are non-zero complex numbers. The fundamental representations of the corresponding q-deformed universal enveloping algebras yield the same complex matrices as above for $q = 1$, except in the following case. For $B_m, m \geq 2$,

$$\forall_1^{2m+1} i, j : \left(X_m^+\right)_{ij} = \left(X_m^-\right)_{ji} = \left([2]_q\right)^{1/2}\left(\delta_{im}\delta_{j,m+1} - \delta_{i,m+1}\delta_{j,m+2}\right),$$

all the other generators being represented by the same real matrices as for $q = 1$.

(8.25.3) For $q \in \mathbf{D}$, let

$$R(q) \in Mat(n, \mathbf{C}) \otimes Mat(n, \mathbf{C}), \ n = m+1, 2m, 2m+1,$$

be the corresponding representation of the universal R-matrix of $U_q(\mathcal{L})$. Then $R(q)$ is some complex matrix solution of QYBE.

(8.25.3.1) For $A_m, m \in \mathbf{N}$,

$$\begin{aligned} q^{\frac{1}{m+1}} R(q) = q \sum_{1 \le k \le m+1} E^{kk} \otimes E^{kk} + \sum_{1 \le k \ne l \le m+1} E^{kk} \otimes E^{ll} \\ + \left(q - q^{-1}\right) \sum_{1 \le k < l \le m+1} E^{kl} \otimes E^{lk}. \end{aligned}$$

(8.25.3.2) For $C_m, m \ge 3$, and $D_m, m \ge 4$,

$$\begin{aligned} R(q) = q \sum_{1 \le k \le 2m} E^{kk} \otimes E^{kk} + \sum_{1 \le k \ne l \le 2m, k+l \ne 2m+1} E^{kk} \otimes E^{ll} \\ + q^{-1} \sum_{1 \le k \le 2m} E^{2m+1-k, 2m+1-k} \otimes E^{kk} \\ + (q - q^{-1}) \Bigg(\sum_{1 \le k < l \le 2m} E^{kl} \otimes E^{lk} \\ \pm \sum_{1 \le k \le m, m+1 \le l \le 2m} q^{k-l\mp 1} E^{kl} \otimes E^{2m+1-k, 2m+1-l} \\ - \sum_{1 \le k < l \le m, m+1 \le k < l \le 2m} q^{k-l} E^{kl} \otimes E^{2m+1-k, 2m+1-l} \Bigg), \end{aligned}$$

with the upper signs for C_m, and lower ones for D_m.

(8.25.3.3) For $B_m, m \ge 2, \ q^2 \ne -1$,

$$\begin{aligned} R(q) = q^2 \sum_{1 \le k \le 2m+1, k \ne m+1} E^{kk} \otimes E^{kk} + E^{m+1, m+1} \otimes E^{m+1, m+1} \\ + \sum_{1 \le k \ne l \le 2m+1, k+l \ne 2m+2} E^{kk} \otimes E^{ll} \\ + q^{-2} \sum_{1 \le k \le 2m+1, k \ne m+1} E^{2m+2-k, 2m+2-k} \otimes E^{kk} \end{aligned}$$

$$+\left(q^2-q^{-2}\right)\Bigg(\sum_{1\le k<l\le 2m+1} E^{kl}\otimes E^{lk}$$
$$-\sum_{1\le k<l\le m,\, m+2\le k<l\le 2m+1} q^{2(k-l)}E^{kl}\otimes E^{2m+2-k,2m+2-l}$$
$$-\sum_{1\le k\le m,\, m+2\le l\le 2m+1} q^{2(k-l+1)}E^{kl}\otimes E^{2m+2-k,2m+2-l}$$
$$-\sum_{1\le k\le m} q^{2(k-m)-1}E^{k,m+1}\otimes E^{2m+2-k,m+1}$$
$$-\sum_{m+2\le l\le 2m+1} q^{2(m-l)+3}E^{m+1,l}\otimes E^{m+1,2m+2-l}\Bigg).$$

Occurrence of coefficients q^2 instead of q in the last expression is due to the choice of $d_k=2$ for $1\le k\le m-1,\ d_m=1$.

(8.25.4) Let $q\in\mathbf{D}$ as above. Denoting by P the flip on $\mathbf{C}^p\otimes\mathbf{C}^p$, $p:=m+1$, $2m, 2m+1$, the complex matrix $\hat{R}(q):=PR(q)$ fulfills the braid relation

$$\hat{R}_{12}(q)\hat{R}_{23}(q)\hat{R}_{12}(q)=\hat{R}_{23}(q)\hat{R}_{12}(q)\hat{R}_{23}(q),$$

with the usual notation of an embedding into the three-fold tensor product. Denote $I_p\otimes I_p=:I$.

(8.25.4.1) For the series $A_m, m\in\mathbf{N}$, the Hecke condition

$$(S(q))^2=I+\left(q-q^{-1}\right)S(q),\ S(q):=q^{\frac{1}{m+1}}PR(q),$$

holds. $S(q)$ is diagonalizable, if and only if $q^2\neq -1$.

(8.25.4.2) For $B_m, m\ge 2,\ S(q):=PR(q)$ fulfills the condition

$$\left(S(q)-q^2I\right)\left(S(q)+q^{-2}I\right)\left(S(q)-q^{-4m}I\right)=0.$$

$S(q)$ is diagonalizable, if and only if $\left(1+q^4\right)\left(1+q^{2-4m}\right)\left(1-q^{-4m-2}\right)\neq 0$.

(8.25.4.3) For $C_m, m\ge 3,\ S(q):=PR(q)$ fulfills the equation

$$\left(S(q)-qI\right)\left(S(q)+q^{-1}I\right)\left(S(q)+q^{-2m-1}I\right)=0.$$

$S(q)$ is diagonalizable, if and only if $\left(1+q^2\right)\left(1+q^{-2m-2}\right)\left(1-q^{-2m}\right)\neq 0$.

(8.25.4.4) For $D_m, m\ge 4,\ S(q):=PR(q)$ fulfills the condition

$$\left(S(q)-qI\right)\left(S(q)+q^{-1}I\right)\left(S(q)-q^{1-2m}I\right)=0.$$

$S(q)$ is diagonalizable, if and only if $\left(1+q^2\right)\left(1+q^{2-2m}\right)\left(1-q^{-2m}\right)\neq 0$.

8.26 Generic Representations of q-Deformed Universal Enveloping Algebras

Let $\mathcal{L}$ be a finite-dimensional complex simple Lie algebra, and assume the deformation parameter $q \in \mathbf{C} \setminus (]-\infty, 0] \cup \{1\})$ to be generic, i.e. not any root of 1. Then quite analogously to the non-deformed case of $q = 1$, the following statements hold, for a representation of the unital associative algebra $E_q(\mathcal{L})$ of polynomials in the generators E_k, F_k, K_k, and $L_k = K_k^{-1}, k = 1, \ldots, m$, on a finite-dimensional complex vector space V. Of course V may be viewed as an $E_q(\mathcal{L})$-left module over $\mathbf{C}$, with the module-multiplication such that $\forall X \in E_q(\mathcal{L}), v \in V : Xv := \rho(X)v$.

(8.26.1) The endomorphisms $\rho(E_k)$ and $\rho(F_k), k = 1, \ldots, m$, are nilpotent. The representatives $\rho(K_k), k = 1, \ldots, m$, can be diagonalized simultaneously.

(8.26.2) A homomorphism of unital associative algebras $\lambda : \mathcal{T}_q \longrightarrow \mathbf{C}$ is called weight of ρ with the weight space V_λ, if and only if

$$V_\lambda := \{v \in V; \forall_1^m k : \rho(K_k)v = \lambda(K_k)v\} \neq \{0\}.$$

Here $\mathcal{T}_q$ denotes the complex subalgebra of the generators K_k and K_k^{-1}, $k = 1, \ldots, m$, of $E_q(\mathcal{L})$. Any two different weight spaces have only the zero vector in common.

(8.26.2.1) If ρ is irreducible, then V is the direct sum of all the weight spaces. Of course, inserting the adjoint representation of a simple Lie algebra $\mathcal{L}$, one finds the root space decomposition of $\mathcal{L}$ itself.

(8.26.3) A non-zero vector $v_{max} \in V$ is called highest weight vector with the highest weight λ_{max}, if and only if λ_{max} is some weight of ρ such that

$$\forall_1^m k : \rho(K_k)v_{max} = \lambda_{max}(K_k)v_{max}, \ \rho(E_k)v_{max} = 0.$$

There is always at least one highest weight vector.

(8.26.4) The $E_q(\mathcal{L})$-left module V over $\mathbf{C}$, or equivalently the representation ρ, is called cyclic, if and only if $V = \{\rho(X)v_{max}; X \in E_q(\mathcal{L})\}$ for some highest weight vector v_{max} with the highest weight λ_{max}. In this case

$$V = \mathbf{C} - lin\ span(\{v_{max}\} \cup \\ \{\rho(F_{j_1} \cdots F_{j_p}); j_1, \ldots, j_p \in \{1, \ldots, m\}, p \in \mathbf{N}\}),$$

and the corresponding weights are easily calculated from the q-deformed commutation relations, such that $\forall_1^m k$:

$$\rho(K_k)\rho(F_{j_1} \cdots F_{j_p})v_{max} = \lambda_{max}(K_k) \prod_{l=1}^{p} q_k^{-\Gamma_{kj_l}} \rho(F_{j_1} \cdots F_{j_p})v_{max}.$$

Every weight of ρ is of this type. Moreover $dim\ V_{\lambda_{max}} = 1$.

(8.26.5) If ρ is irreducible, then there exists only one highest weight, and therefore only one complex ray of highest weight vectors.

(8.26.6) If the irreducible representations ρ and σ of $E_q(\mathcal{L})$, on V and W respectively, are of the same highest weight λ_{max}, then they are equivalent in the usual sense of a **C**-linear bijection $\tau : V \longleftrightarrow W$, such that

$$\forall X \in E_q(\mathcal{L}) : \sigma(X) = \tau \circ \rho(X) \circ \tau^{-1}.$$

(8.26.7) The irreducible representations of $E_q(A_1)$ are classified by their dimension $2j+1$, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, with the highest weights $\lambda_{max}(K) = \pm q^{2j}$, and the corresponding weights $\pm q^{2m}, m = j, j-1, \ldots, -j$.

(8.26.8) The highest weight of any finite-dimensional irreducible representation of $E_q(\mathcal{L})$ is of the type

$$\lambda_{max}(K_k) = s_k q_k^{n_k},\ n_k \in \mathbf{N},\ s_k \in \{+1, -1\},\ k = 1, \ldots, m.$$

(8.26.9) The notion of Verma modules is generalized in order to study representations of $E_q(\mathcal{L})$. For $\lambda := \{\lambda_k \in \mathbf{C}\backslash\{0\}; k = 1, \ldots, m\}$, the Verma module $\mathcal{M}_\lambda$ is defined as the factor algebra

$$\mathcal{M}_\lambda := E_q(\mathcal{L})/ideal\ \ (\{E_k, K_k - \lambda_k I; k = 1, \ldots, m\}).$$

(8.26.9.1) Cyclic highest weight representations are called Verma modules too. Such a representation ρ of $E_q(\mathcal{L})$ on V is reducible, if and only if

$$\exists\ 0 \neq v_{sing} \in V : v_{sing} \neq v_{max},$$
$$\forall_1^m k : \rho(E_k)v_{sing} = 0,\ \rho(K_k)v_{sing} = \mu_k v_{sing},$$

with non-zero complex eigenvalues μ_k; then v_{sing} is called singular vector of ρ. In this case the representation ρ of $E_q(\mathcal{L})$ can be restricted to a cyclic highest weight representation on the linear subspace $\{\rho(X)v_{sing}; X \in E_q(\mathcal{L})\}$ of V, with the highest weight vector v_{sing}, and the highest weight μ_{max} such that

$$\mu_{max}(K_k) = \mu_k,\ k = 1, \ldots, m.$$

(8.26.10) Every finite-dimensional representation of $E_q(\mathcal{L})$ is completely reducible.

(8.26.11) The so-called tensor product of two representations α and β, on finite-dimensional vector spaces V and W over a field K respectively, of an associative unital superalgebra, which moreover is a $\mathbf{Z}_2$-graded bialgebra $\mathcal{H}$ over K, is defined such that:

$$\mathcal{H} \ni x \underset{\Delta}{\longrightarrow} \underset{T(\alpha,\beta)}{\longrightarrow} \longleftrightarrow T_\Delta(\alpha,\beta)(x) \in End_K(V \otimes W).$$

(8.26.12) For instance in the case of $E_q(A_1)$ for positive $q \neq 1$, with the notation $v^j_m \equiv |j,m\rangle$ of normed eigenstates, such that

$$K|j,m\rangle = q^{2m}|j,m\rangle,\ m = j, j-1, \ldots, -j,\ X^{\pm}|j,\pm j\rangle = 0,$$
$$X^{\pm}|j,m\rangle = ([j \mp m]_q [j \pm m + 1]_q)^{1/2}\,|j, m \pm 1\rangle \text{ for } m \neq \pm j,$$

these irreducible representations, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, are combined by means of the following q-deformed Clebsch-Gordan coefficients.

(8.26.12.1) In the lowest case of $j_1 = j_2 = \frac{1}{2}$, one easily calculates that

$$|\frac{1}{2},\frac{1}{2},1,1\rangle = |\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \longleftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$|\frac{1}{2},\frac{1}{2},1,0\rangle = ([2]_q)^{-1/2}\left(q^{-1/2}|\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + q^{1/2}|\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle\right)$$
$$\longleftrightarrow ([2]_q)^{-1/2}\begin{bmatrix} 0 \\ q^{-1/2} \\ q^{1/2} \\ 0 \end{bmatrix},$$

$$|\frac{1}{2},\frac{1}{2},1,-1\rangle = |\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \longleftrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$|\frac{1}{2},\frac{1}{2},0,0\rangle = ([2]_q)^{-1/2}\left(q^{1/2}|\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle - q^{-1/2}|\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle\right)$$
$$\longleftrightarrow ([2]_q)^{-1/2}\begin{bmatrix} 0 \\ q^{1/2} \\ -q^{-1/2} \\ 0 \end{bmatrix},$$

inserting the Pauli spinors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $|\frac{1}{2}, \pm\frac{1}{2}\rangle$. Hence one finds:

$$X^+ \longrightarrow \begin{bmatrix} 0 & q^{-1/2} & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \\ 0 & 0 & 0 & q^{1/2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{transposition}{\longleftrightarrow} \longleftarrow X^-.$$

Obviously these eigenvectors are orthonormal, because q is positive.

(8.26.12.2) One obtains the following q-deformed Van der Waerden formula.

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

$$= \sum_{j=|j_1-j_2|,\ldots,j_1+j_2-1,j_1+j_2} \sum_{m=-j}^{j} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q |j_1, j_2, j, m\rangle,$$

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q$$

$$:= \delta_{m_1+m_2,m} \left(\frac{[j_1+j_2-j]_q![j_2+j-j_1]_q![j+j_1-j_2]_q!}{[j_1+j_2+j+1]_q!} \right)^{1/2}$$

$$q^{\frac{1}{2}(j_1+j_2-j)(j_1+j_2+j+1)+j_1m_2-j_2m_1} \left([2j+1]_q\right)^{1/2}$$

$$\left([j_1+m_1]_q![j_1-m_1]_q![j_2+m_2]_q![j_2-m_2]_q![j+m]_q![j-m]_q!\right)^{1/2}$$

$$\sum_{s\geq 0} \frac{(-1)^s q^{-s(j_1+j_2+j+1)}}{[s]_q![j_1+j_2-j-s]_q!}$$

$$\frac{1}{[j_1-m_1-s]_q![j_2+m_2-s]_q![j-j_2+m_1+s]_q![j-j_1-m_2+s]_q!},$$

denoting $[0]_q! := 1$, and restricting the sum such that

$$j_1 + j_2 - j - s \geq 0,$$

$$j_1 - m_1 - s \geq 0, \; j_2 + m_2 - s \geq 0, \; j - j_2 + m_1 + s \geq 0.$$

(8.26.12.3) Since q is positive, these eigenvectors are orthonormal, and the q-deformed Clebsch-Gordan coefficients are real. Hence one obtains, for $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $m = -j, \ldots, j$, that

$$|j_1, j_2, j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

(8.26.12.4) Due to the transformation of orthonormal bases one enjoys the following relations.

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}_q = \delta_{jj'}\delta_{mm'},$$

$$\sum_{|j_1-j_2|\le j\le j_1+j_2} \sum_{m=-j}^{j} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q = \delta_{m_1 m_1'}\delta_{m_2 m_2'}.$$

(8.26.12.5) Both symmetry relations and recursion formulae for Clebsch-Gordan coefficients can be generalized to these q-deformed coefficients. For instance one finds the following ones.

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_{q^{-1}}.$$

$$\left([j \mp m]_q [j \pm m + 1]_q\right)^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \pm 1 \end{bmatrix}_q$$

$$= q^{m_2}\left([j_1 \pm m_1]_q [j_1 \mp m_1 + 1]_q\right)^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 \mp 1 & m_2 & m \end{bmatrix}_q$$

$$+ q^{-m_1}\left([j_2 \pm m_2]_q [j_2 \mp m_2 + 1]_q\right)^{1/2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 \mp 1 & m \end{bmatrix}_q.$$

8.27 Cyclic Representations of q-Deformed Universal Enveloping Algebras

In the so-called non-generic case of the deformation parameter q being a root of unity, the finite-dimensional representations of q-deformed universal enveloping algebras differ essentially from those of the non-deformed ones. Let $\mathcal{L}$ be a simple finite-dimensional complex Lie algebra with the Cartan matrix $\Gamma_{kl}, 1 \le k, l \le m$, and d_k such that $d_k\Gamma_{kl} = d_l\Gamma_{lk}$, and denote $q_k := q^{d_k}$. Let

$$q := \exp(i\pi n/N), \; n \in \mathbf{N}, \; N \ge 3, \; \forall_1^m k : N > d_k,$$

the latter assumption concerning the exceptional case of G_2 with $\Gamma_{12} = -3$ and $d_2 = 3$. Here the smallest common divisor of n and N is assumed to be equal to 1. Then $q^N = \pm 1$, and therefore $[N]_q = 0$.

(8.27.1) In the non-generic case, the centre of the complex Hopf algebra $E_q(\mathcal{L})$, with the generators $E_k, F_k, K_k^{\pm 1}, k = 1, \ldots, m$, contains certain powers of these generators.

(8.27.1.1) By induction one calculates the following commutation relations. For $1 \leq k, l \leq m$, $\forall n \in \mathbf{N}$:

$$[E_k, F_k^n] = q_k [n]_{q_k} F_k^{n-1} \frac{K_k q_k^{1-n} - K_k^{-1} q_k^{n-1}}{q_k - q_k^{-1}},$$

$$[E_k^n, F_k] = q_k [n]_{q_k} \frac{K_k q_k^{1-n} - K_k^{-1} q_k^{n-1}}{q_k - q_k^{-1}} E_k^{n-1},$$

$$K_k^n E_l = q_k^{n\Gamma_{kl}} E_l K_k^n, \; K_k^n F_l = q_k^{-n\Gamma_{kl}} F_l K_k^n,$$

$$K_k E_l^n = q_k^{n\Gamma_{kl}} E_l^n K_k, \; K_k F_l^n = q_k^{-n\Gamma_{kl}} F_l^n K_k.$$

(8.27.1.2) Moreover the Serre relations imply that $\forall_1^m k, l$:

$$E_k^N E_l = q_k^{-\Gamma_{kl} N} E_l E_k^N, \; F_k^N F_l = q_k^{\Gamma_{kl} N} F_l F_k^N.$$

(8.27.1.3) Hence one finds in the case of even n and odd N, such that $q^N = 1$, that $E_k^N, F_k^N, K_k^{\pm N}$, $k = 1, \ldots, m$, belong to the centre of $E_q(\mathcal{L})$. In case of odd n, such that $q^N = -1$, $E_k^{2N}, F_k^{2N}, K_k^{\pm 2N}, k = 1, \ldots, m$, are contained in this centre.
Moreover in the case of B_m, $m \geq 2$, $q_k^{N\Gamma_{kl}} = 1$ for $1 \leq k, l \leq m$, such that $E_k^N, F_k^N, K_k^{\pm N}$, $k = 1, \ldots, m$, are central for $E_q(B_m), \forall n \in \mathbf{N}$.

(8.27.1.4) Due to the construction of q-deformed positive and negative root vectors by means of Lusztig automorphisms one finds, that the corresponding powers of $E(\beta_j), F(\beta_j)$, $j = 1, \ldots, p$, along the longest word $\sigma_{k_1} \circ \cdots \circ \sigma_{k_p}$ of the Weyl group, are also central for $E_q(\mathcal{L})$.

(8.27.2) The ideal of $E_q(\mathcal{L})$, which is generated as

$$Z_q(\mathcal{L}) := \; ideal \; \left(\{E(\beta_j)^N, F(\beta_j)^N, K_l^{\pm 2N} - I; \; l = 1, \ldots, m; \; j = 1, \ldots, p\}\right),$$

with the unit I of $E_q(\mathcal{L})$, fulfills also the coideal conditions

$$\Delta(Z_q(\mathcal{L})) \subseteq E_q(\mathcal{L}) \otimes Z_q(\mathcal{L}) + Z_q(\mathcal{L}) \otimes E_q(\mathcal{L}), \; \varepsilon(Z_q(\mathcal{L})) = \{0\}.$$

Moreover

$$\sigma(Z_q(\mathcal{L})) \subseteq Z_q(\mathcal{L}),$$

such that one obtains the finite-dimensional complex Hopf algebra

$$E_q^{red}(\mathcal{L}) := E_q(\mathcal{L}) / Z_q(\mathcal{L}),$$

with the $\mathbf{C}$-linear basis

$$\{E(\beta_p)^{r_p} \cdots E(\beta_1)^{r_1} F(\beta_p)^{s_p} \cdots F(\beta_1)^{s_1} K_1^{z_1} \cdots K_m^{z_m};$$
$$r_1, \ldots, s_p \in \mathbf{N}_0; z_1, \ldots, z_m \in \mathbf{Z};$$
$$r_1 < N, \ldots, s_p < N; |z_1| < 2N, \ldots, |z_m| < 2N\}.$$

(8.27.2.1) Of course this ideal is also generated by the set

$$\{E_l^N, F_l^N, K_l^{\pm 2N} - I; l = 1, \ldots, m\}.$$

(8.27.3) Also in case of odd n, the Nth powers of non-toral generators are factorized to zero, in order to construct a universal R-matrix for $U_q^{red}(\mathcal{L})$. On the other hand, the Nth powers of toral generators are not factorized to unit, in order to include for instance boson and fermion representations.

(8.27.3.1) Using the homomorphism of unital associative $\mathbf{C}[[h]]$-algebras: $E_q(\mathcal{L}) \longrightarrow U_q(\mathcal{L})$, which describes the change of generators from $E_k, F_k, K_k^{\pm 1}$, $k = 1, \ldots, m$, to $X_k^{\pm}, H_k$, one can also factorize the topological Hopf algebra $U_q(\mathcal{L})$ with respect to the smallest closed ideal $\bar{Z}_q(\mathcal{L})$ of the above relations, which is obtained as the closure of the image of $Z_q(\mathcal{L})$. Since the latter fulfills the conditions

$$\varepsilon(\bar{Z}_q(\mathcal{L})) = \{0\}, \ \sigma(\bar{Z}_q(\mathcal{L})) \subseteq \bar{Z}_q(\mathcal{L}),$$

$$\Delta\left(\bar{Z}_q(\mathcal{L})\right) \subseteq \overline{U_q(\mathcal{L}) \otimes \bar{Z}_q(\mathcal{L}) + \bar{Z}_q(\mathcal{L}) \otimes U_q(\mathcal{L})},$$

one thereby constructs the topological Hopf algebra

$$U_q^{red}(\mathcal{L}) := U_q(\mathcal{L})/\bar{Z}_q(\mathcal{L}).$$

Here one uses the closure with respect to the unique Hausdorff completion of the tensor product of formal power series with relations, with respect to the h-adic topology.

(8.27.3.2) The reduced quantum algebra $U_q^{red}(\mathcal{L})$ is quasitriangular with respect to some universal R-matrix R^{red}. The latter is obtained formally from the universal R-matrix of $U_q(\mathcal{L})$ itself by omitting the powers $\geq N$ of q-deformed positive and negative root vectors.

(8.27.4) The irreducible finite-dimensional representations of $E_q(A_1)$, which are highest weight representations, are classified modulo equivalence in the following way. Denote by d the dimension, and by $\lambda \in \mathbf{C}$ the highest weight of the representation, the latter not being written explicitly as homomorphism for convenience.

$$K\psi_l = q^{\lambda - 2l}\psi_l,$$

$$X^+\psi_l = \left([l]_q[\lambda + 1 - l]_q\right)^{1/2}\psi_{l-1},$$

$$X^-\psi_l = \left([l+1]_q[\lambda - l]_q\right)^{1/2}\psi_{l+1}, \ l = 0, 1, \ldots, d-1.$$

Here $X^{\pm}$ are inserted as generators of $E_q(A_1)$ instead of E and F, for convenience. Obviously $X^+\psi_0 = 0$; moreover $d \leq N$. One also demands, that

$$X^-\psi_{d-1} = 0.$$

(8.27.4.1) For $d < N$, demanding $[\lambda - d + 1]_q = 0$ means, that

$$\lambda = d - 1 + z\frac{N}{n}, \quad z \in \mathbf{Z}.$$

(8.27.4.2) For $d = N$, demanding $[\lambda - l]_q \neq 0$ for $l = 0, \ldots, N-2$, means that

$$\lambda \notin \left\{ l + z\frac{N}{n}; z \in \mathbf{Z}; l = 0, \ldots, N-2 \right\}.$$

(8.27.4.3) Only for $d < N$ one obtains $K^{2N} = I_d$. Of course one always finds, that $(X^{\pm})^d = 0$. Therefore in case of $d < N$ the above representation can also be used for $E_q^{red}(A_1)$.

(8.27.4.4) The Casimir operator

$$C := \left([\frac{1}{2}(H + I)]_q \right)^2 + X^- X^+ - \frac{1}{4} I$$

takes the eigenvalue $\left([\frac{1}{2}(\lambda+1)]_q\right)^2 - \frac{1}{4}$. Note that this eigenvalue does not suffice to determine one of the above representations. The dimension d depends not only on the highest weight λ, but also on the choice of $z \in \mathbf{Z}$, for $d < N$.

(8.27.4.5) Including the trivial representation with

$$d = 1, \ \lambda = z\frac{N}{n}, \quad z \in \mathbf{Z}, \ X^{\pm} = 0,$$

one finds that every complex number λ can be used as highest weight.

(8.27.5) The non-highest weight representation of $E_q(A_1)$ for $q^4 = 1$, such that:

$$X^+ \longrightarrow \frac{1}{\sqrt{2}}\sigma^1, \ X^- \longrightarrow \frac{-i}{\sqrt{2}}\sigma^2, \ K \longrightarrow q^{\sigma^3}, \ q = \pm i,$$

inserting the Pauli matrices, is cyclic in the sense that

$$\left(X^{\pm}\right)^2 = \pm\frac{1}{2} I_2, \ K^2 = -I_2.$$

(8.27.6) Choose complex parameters λ, α, β, such that $\alpha\beta \neq 0$. The following irreducible N-dimensional representation of $E_q(A_1)$ is cyclic in the sense that $(X^{\pm})^N = c^{\pm} I_N$, with complex numbers $c^{\pm}$ depending on N and the parameters.

$$K\psi_l = q^{\lambda-2l}\psi_l, \ 0 \leq l \leq N-1;$$

$$X^+\psi_l = ([l]_q[\lambda+1-l]_q + \alpha\beta)^{1/2}\psi_{l-1}, \ 1 \leq l \leq N-1;$$
$$X^-\psi_l = ([l+1]_q[\lambda-l]_q + \alpha\beta)^{1/2}\psi_{l+1}, \ 0 \leq l \leq N-2;$$
$$X^+\psi_0 = \beta\psi_{N-1}, \ X^-\psi_{N-1} = \alpha\psi_0.$$

Obviously $K^N = q^{N\lambda} I_N$. Note that here the non-toral generators are also represented by invertible complex matrices.

(8.27.7) The centre of $E_q(A_1)$ consists of all the complex polynomials in C, E^M, F^M, K^M, $M := N$ for even n, $M := 2N$ for odd n.

(8.27.8) Finite-dimensional representations of $E_q(\mathcal{L})$ can be constructed, which are reducible, but indecomposable in the sense, that they cannot be written as direct sum of irreducible representations. Consider for instance the following indecomposable representation of $E_q(A_2)$. Let $\mu + \lambda = -1$.

$$X_1^-\psi_{kl} = \psi_{k+1,l}, \ 0 \leq k \leq N-2, \ 0 \leq l \leq N-1;$$
$$X_1^+\psi_{kl} = [k]_q[\lambda-k+l+1]_q\psi_{k-1,l}, \ 1 \leq k \leq N-1, \ 0 \leq l \leq N-1;$$
$$X_2^-\psi_{kl} = [l+1]_q[\mu-l+k]_q\psi_{k,l+1}, \ 0 \leq k \leq N-1, \ 0 \leq l \leq N-2;$$
$$X_2^+\psi_{kl} = \psi_{k,l-1}, \ 0 \leq k \leq N-1, \ 1 \leq l \leq N-1;$$
$$X_1^-\psi_{N-1,l} = X_1^+\psi_{0l} = X_2^-\psi_{k,N-1} = X_2^+\psi_{k0} = 0, \ 0 \leq k, l \leq N-1;$$
$$K_1\psi_{kl} = q^{\lambda-2k+l}\psi_{kl}, \ K_2\psi_{kl} = q^{\mu-2l+k}\psi_{kl}, \ 0 \leq k, l \leq N-1.$$

Obviously this N-dimensional representation is irreducible for complex numbers μ such that $[\mu - l + k]_q \neq 0$ for $0 \leq k, l \leq N-1$.

(8.27.8.1) Choose for instance $n = 2, N = 3, \lambda = 1$. Then the above representation is reducible, but indecomposable, because there is an invariant subspace spanned by the vectors ψ_{kl}, $0 \leq k, l \leq 2$, $\{k, l\} \neq \{0, 2\}$. For $\lambda = 2$ a similar situation occurs, excluding the indices $\{0, 1\}, \{0, 2\}, \{1, 2\}$.

(8.27.8.2) For $n = 2, N = 5$, and $\lambda = 2$, the corresponding 25-dimensional representation is indecomposable, with the invariant subspace, which is spanned excluding the indices $\{0, 3\}, \{0, 4\}, \{1, 4\}$.

(8.27.8.3) Consider again the case of $n = 2, N = 3, \lambda = 1$. Since $X_k^{\pm}\psi_{20} = 0$ for $k = 1, 2$, one can factorize the above 8-dimensional reducible representation of $E_q(A_2)$, in order to obtain an irreducible 7-dimensional representation on the vector space $\mathbf{C}(\{\psi_{kl}; k, l = 0, 1, 2; |k - l| \leq 1\})$. The above 6-dimensional representation for $\lambda = 2$ is irreducible.

(8.27.9) Consider a representation of $E_q(A_1)$ on $\mathbf{C}^d$, which is spanned by the eigenvectors of K, such that

$$K\psi_l = \kappa_l\psi_l, \; l = 0, \ldots, d-1.$$

The quantum dimension of this representation is defined by

$$d_q := \kappa_0 + \cdots + \kappa_{d-1}.$$

For the irreducible highest weight representations of dimension $d < N$ one finds, that

$$d_q = \sum_{l=0}^{d-1} q^{\lambda - 2l} = [\lambda + 1]_q = [d + z\frac{N}{n}]_q, \; z \in \mathbf{Z}.$$

For $d = N$, the irreducible representations are characterized by $d_q = 0$. Representations with the quantum dimension $d_q = 0$ are called such of type I, whereas those with non-zero quantum dimension are representations of type II.

(8.27.10) Let x, y be elements of an associative algebra over $\mathbf{C}$, with the commutation relation

$$yx = q^2 xy, \; q \neq \pm 1.$$

Then $q^N = \pm 1$ implies, that

$$(x + y)^N = x^N + y^N.$$

Now consider the matrix quantum semigroup $Mat_q(m, 0, \mathbf{C})$, and choose the parameters $q_{kl} := q^2$, $1 \leq k < l \leq m$, and assume $q^4 \neq \pm 1$. Then the Nth powers of generators a_{kl}^N, $1 \leq k < l \leq m$, are central, and moreover

$$a_{kl}^N \xrightarrow[\Delta]{} \sum_{j=1}^{m} a_{kj}^N \otimes a_{jl}^N.$$

(8.27.10.1) With this choice of parameters q_{kl}, $1 \le k < l \le m$, one can define the so-called quantum determinant, which is both central and group-like, by

$$det_q[a_{kl}; k,l = 1,\dots,m] \equiv d := \sum_{P \in P_m} \left(-q^{-2}\right)^{\lambda_P} a_{1j_1} \cdots a_{mj_m};$$

$$\forall_1^m k,l : a_{kl}d = da_{kl};\ \Delta(d) = d \otimes d,\ \varepsilon(d) = 1;$$

here λ_P denotes the length of the permutation $P := \begin{bmatrix} 1 & \dots & m \\ j_1 & \dots & j_m \end{bmatrix}$. One then calculates, that

$$d^N = det[a_{kl}^N; k,l = 1,\dots,m] := \sum_{P \in P_m} (-1)^{\lambda_P} a_{1j_1}^N \cdots a_{mj_m}^N.$$

(8.27.10.2) For $m = 2$ one finds the following N-dimensional representation of $Mat_q(2,0,\mathbf{C})$:

$$a_{11}\psi_k = \alpha(1 - q^{-4k})^{1/2}\psi_{k-1},\ 1 \le k \le N-1,\ a_{11}\psi_0 = 0;$$
$$a_{22}\psi_k = \delta(1 - q^{-4k-4})^{1/2}\psi_{k+1},\ 0 \le k \le N-2,\ a_{22}\psi_{N-1} = 0;$$

$$a_{12}\psi_k = \beta q^{-2k-s}\psi_k,\ a_{21}\psi_k = \gamma q^{-2k-t}\psi_k,\ 0 \le k \le N-1;$$
$$\alpha\delta = -\beta\gamma \ne 0,\ s + t = 2;\ \alpha,\dots,t \in \mathbf{C}.$$

9. Categorial Viewpoint

Investigations of structures, which later on, following V. G. Drinfel'd (1987), were called quantum groups, started from at least four different points of view, as was pointed out in the foregoing chapter, due to the pioneering work by M. Jimbo (1985), S. L. Woronowicz (1987), Yu. I. Manin (1989), N. Yu. Reshetikhin, L. A. Takhtadzhyan, L. D. Faddeev (1990), J. Wess and B. Zumino (1990), and in particular the contribution by V. G. Drinfel'd to the Proceedings of the International Congress of Mathematicians at Berkeley, California, 1987. Therefore the question, what maybe new kinds of categories could be used in order to classify the amazing variety of new mathematical tools, is by far non-trivial.

On the purely algebraic level, forgetting about topological and differential aspects, one is concerned with some generalization of rigid tensor categories. Monoidal categories and their coherence properties were discussed for instance by S. Mac Lane (1971). In particular Tannakian categories as a special kind of tensor categories were investigated extensively by N. R. Saavedra in his thesis (1972), and by P. Deligne, J. S. Milne (1982). Their use for the categorial understanding of quantum groups is described explicitly by A. Joyal and R. Street (1991).

The symmetry constraint, which as some functorial isomorphism defines a tensor category as special kind of a monoidal category, is weakened to a so-called quasisymmetry constraint, thereby establishing quasitensor (braided tensor) categories. The notion of rigidness in terms of the internal $\underline{Hom}$-functor, and the so-called evaluation map, are generalized straightforward from tensor categories to the braided case. Of course the involved categories should be Abelian, and the quasitensor bifunctor is then assumed to be bi-additive.

Finite-dimensional representations of quasitriangular Hopf algebras were identified as the objects of rigid quasitensor categories by S. Majid (1990), who in the sequel used such categories for the study of so-called braided groups and braided matrices (1993).

Abelian tensor categories and their generalization to quasisymmetry constraints are described in the Montréal lectures (1988) and an introductory overview by Yu. I. Manin (1991). An advantage of quasitensor categories is their categorial view of quasitriangular quasi-Hopf algebras; the latter were introduced by V. G. Drinfel'd (1990), aiming, among other applications as for instance invariants of knots, at the construction of gauge transformations in models of conformal field theory. In particular the system of equations by V. G. Knizhnik and A. B. Zamolodchikov (1984), which is satisfied by the correlation functions in the Wess-Zumino-Witten model, provides an example of associativity constraint, which is due to some quasi-coassociative comultiplication.

A rather restrictive notion of quantum category is due to J. Fröhlich and T. Kerler (1993), namely as an Abelian semisimple finite rigid quasitensor category. Topological properties are moreover proposed for the morphisms of so-called C^*-quantum categories. The latter are used as an abstract framework for the study of quantum group symmetries of local quantum field theories in 2 and 3 space-time dimensions.

Topological aspects are also included in the definition of monoidal W^*-categories by M. S. Dijkhuizen in his thesis (1994), in order to classify compact quantum groups, and to establish the Tannaka-Krein duality.

The categorial viewpoint is presented in detail in the monographs by St. Shnider and S. Sternberg (1993), V. Chari and A. Pressley (1994).

9.1 Categories and Functors

(9.1.1) A category Cat is defined by the following conditions.

(i) To every pair of objects A, B from the object class $Ob(Cat)$, a set $Mor(Cat; A, B) \equiv Mor(A, B)$ of morphisms from A to B is assigned, such that $\forall A, B, C, D \in Ob(Cat)$:

$$A \neq C \text{ or } B \neq D \Longrightarrow Mor(A, B) \cap Mor(C, D) = \emptyset.$$

(ii) To every triple of objects $A, B, C \in Ob(Cat)$, an associative product of morphisms is assigned:

$$Mor(B, C) \times Mor(A, B) \ni \{\beta, \alpha\} \longrightarrow \beta\alpha \in Mor(A, C),$$

such that

$$\forall \alpha \in Mor(A, B), \beta \in Mor(B, C), \gamma \in Mor(C, D) : \gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

$$\text{(iii)} \quad \forall A, B \in Ob(Cat) : \exists \varepsilon_A \in Mor(A, A), \varepsilon_B \in Mor(B, B) :$$
$$\forall \alpha \in Mor(A, B) : \alpha\varepsilon_A = \varepsilon_B\alpha = \alpha.$$

This so-called identical morphism ε_A of $A \in Ob(Cat)$ is then unique. Morphisms $\alpha \in Mor(A, B)$ are conveniently denoted by $\alpha : A \longrightarrow B$. The class $Mor(Cat)$ of morphisms of the category Cat is defined as the class of objects α, such that $\exists A, B \in Ob(Cat) : \alpha \in Mor(Cat; A, B)$.

(9.1.2) Let $\alpha \in Mor(A,B)$; α is called monomorphism, if and only if

$$\forall C \in Ob(Cat), \forall \gamma_1, \gamma_2 \in Mor(C,A) : \alpha\gamma_1 = \alpha\gamma_2 \Rightarrow \gamma_1 = \gamma_2;$$

α is called epimorphism, if and only if

$$\forall C \in Ob(Cat), \forall \beta_1, \beta_2 \in Mor(B,C) : \beta_1\alpha = \beta_2\alpha \Longrightarrow \beta_1 = \beta_2;$$

α is called bimorphism, if and only if it is both a monomorphism and an epimorphism. Moreover, define

$$\textit{isomorphism } \alpha \Longleftrightarrow \exists \beta \in Mor(B,A) : \beta\alpha = \varepsilon_A,\ \alpha\beta = \varepsilon_B;$$

then β is unique, hence denoted by α^{-1}. Every isomorphism α is some bimorphism.
A morphism $\alpha : A \longrightarrow A$ is called endomorphism of A. An isomorphism $\alpha : A \longrightarrow A$ is called automorphism of A.
An isomorphism $\alpha \in Mor(A,B)$ is also denoted by $\alpha : A \longleftrightarrow B$.

(9.1.2.1) Two morphisms of a category $\phi_k : A_k \longrightarrow B_k$, $k = 1,2$, are called isomorphic, if and only if there exist isomorphisms $\alpha : A_1 \longrightarrow A_2$, $\beta : B_1 \longrightarrow B_2$, such that $\beta\phi_1 = \phi_2\alpha$.

(9.1.3) The dual category Cat^* with respect to a category Cat is defined, such that

(i) $\forall A, B \in Ob(Cat) = Ob(Cat^*)$:

$$Mor(Cat; A, B) = Mor(Cat^*; B, A);$$

(ii) with $\alpha\beta$ denoting the product of the category Cat,

$$Mor(Cat^*; B, C) \times Mor(Cat^*; A, B) \ni \{\beta, \alpha\} \longrightarrow \alpha\beta \in Mor(Cat^*; A, C).$$

(9.1.4) Let Cat_1, Cat_2 be categories. A covariant (contravariant) functor from Cat_1 into Cat_2 is defined as a pair of maps

$$F_{ob} : Ob(Cat_1) \longrightarrow Ob(Cat_2),\ \ F_{mor} : Mor(Cat_1) \longrightarrow Mor(Cat_2),$$

such that the following conditions are fulfilled.

(i) $A_1 \underset{F_{ob}}{\longrightarrow} A_2,\ B_1 \underset{F_{ob}}{\longrightarrow} B_2$

$$\Longrightarrow Mor(Cat_1; A_1, B_1) \ni \alpha_1 \underset{F_{mor}}{\longrightarrow} \alpha_2 \in Mor(Cat_2; A_2, B_2)$$
$$(Mor(Cat_2; B_2, A_2)).$$

(ii) $A_1 \underset{F_{ob}}{\longrightarrow} A_2 \Longrightarrow \varepsilon_{A_1} \underset{F_{mor}}{\longrightarrow} \varepsilon_{A_2}.$

(iii) If

$$Mor(Cat_1; A_1, B_1) \ni \alpha_1 \underset{F_{mor}}{\longrightarrow} \alpha_2 \in Mor(Cat_2; A_2, B_2),$$
$$Mor(Cat_1; B_1, C_1) \ni \beta_1 \underset{F_{mor}}{\longrightarrow} \beta_2 \in Mor(Cat_2; B_2, C_2),$$

then

$$\beta_1\alpha_1 \underset{F_{mor}}{\longrightarrow} \beta_2\alpha_2 \, (\text{or } \alpha_2\beta_2)\,.$$

Obviously the composition of functors yields again some functor. For instance,

$$contravar.\ functor \circ contravar.\ functor = covariant\ functor.$$

(9.1.5) A subcategory Cat' of Cat is defined by the following conditions.

$$\forall A, B \in Ob(Cat') \subseteq Ob(Cat) : Mor(Cat'; A, B) \subseteq Mor(Cat; A, B);$$
$$\forall A \in Ob(Cat') : \varepsilon_A \in Mor(Cat'; A, A);$$
$$\forall \alpha \in Mor(Cat'; A, B), \beta \in Mor(Cat'; B, C) : \beta\alpha \in Mor(Cat'; A, C);$$

here the product $\beta\alpha$ is defined in the sense of Cat. The product of morphisms in the sense of Cat' is then defined as that of Cat, and $\varepsilon_A \in Mor(Cat'; A, A)$ is also used as the identical morphism of A in the sense of Cat'.
The subcategory Cat' of Cat is called full, if and only if

$$\forall A, B \in Ob(Cat') : Mor(Cat'; A, B) = Mor(Cat; A, B).$$

(9.1.6) Let $A, B \in Ob(Cat)$. The covariant and the contravariant representation functor, $Mor(Cat; A, -)$ and $Mor(Cat; -, B)$, both from Cat into the category Set, the latter with sets as objects and maps as morphisms, are defined:

$$Ob(Cat) \ni X \longrightarrow Mor(Cat; A, X) \text{ or } Mor(Cat; X, B) \in Ob(Set),$$

$$Mor(Cat; X, Y) \ni \xi \longrightarrow \xi_A \text{ or } \xi_B \in Mor(Set),$$

such that in the covariant case:

$$Mor(Cat; A, X) \ni \alpha \underset{\xi_A}{\longrightarrow} \xi\alpha \in Mor(Cat; A, Y),$$

or in the contravariant case:

$$Mor(Cat; X, B) \ni \beta\xi \underset{\xi_B}{\longleftarrow} \beta \in Mor(Cat; Y, B),$$

respectively.

(9.1.7) The product $Cat_1 \times Cat_2$ of two categories $Cat_k, k = 1,2$, is defined in the following way. Let $\alpha_k \in Mor(Cat_k; A_k, B_k), \beta_k \in Mor(Cat_k; B_k, C_k)$, $k = 1,2$. Then

$$Ob(Cat_1 \times Cat_2) \ni \{A_1, A_2\} \underset{\{\alpha_1,\alpha_2\}}{\longrightarrow} \{B_1, B_2\} \in Ob(Cat_1 \times Cat_2)$$

and $\{\beta_1, \beta_2\} \in Mor(Cat_1 \times Cat_2; \{B_1, B_2\}, \{C_1, C_2\})$ are composed according to

$$\{\beta_1, \beta_2\}\{\alpha_1, \alpha_2\} := \{\beta_1\alpha_1, \beta_2\alpha_2\}, \text{ and } \varepsilon_{\{A_1,A_2\}} := \{\varepsilon_{A_1}, \varepsilon_{A_2}\}.$$

The product of finitely many categories is defined correspondingly.

(9.1.8) A functor $F : Cat_1 \times Cat_2 \longrightarrow Cat$, also denoted as bifunctor, is called
(a) co-covariant (b) contra-contravariant (c) co-contravariant (d) contra-covariant, if and only if the following conditions are fulfilled. Let F_{ob}, F_{mor} act as:

$$\{A_1, A_2\} \longrightarrow A, \ \{B_1, B_2\} \longrightarrow B, \ \{A_1, B_2\} \longrightarrow C, \ \{B_1, A_2\} \longrightarrow D,$$

$$\{\alpha_1, \alpha_2\} \longrightarrow \alpha, \ \{\beta_1, \beta_2\} \longrightarrow \beta, \ \{\alpha_1, \beta_2\} \longrightarrow \gamma, \ \{\beta_1, \alpha_2\} \longrightarrow \delta,$$

with the morphisms $\alpha_k, \beta_k, k = 1,2$, denoted above. Then

(a) $\alpha \in Mor(Cat; A, B), \ \{\beta_1\alpha_1, \beta_2\alpha_2\} \longrightarrow \beta\alpha;$
(b) $\alpha \in Mor(Cat; B, A), \ \{\beta_1\alpha_1, \beta_2\alpha_2\} \longrightarrow \alpha\beta;$
(c) $\alpha \in Mor(Cat; C, D), \ \{\beta_1\alpha_1, \beta_2\alpha_2\} \longrightarrow \delta\gamma;$
(d) $\alpha \in Mor(Cat; D, C), \ \{\beta_1\alpha_1, \beta_2\alpha_2\} \longrightarrow \gamma\delta.$

Moreover one demands, that $F_{mor} : \varepsilon_{\{A_1,A_2\}} \longrightarrow \varepsilon_A$.

(9.1.8.1) The product $F_1 \times F_2$ of two functors $F_k : Cat_k \longrightarrow Cat'_k$, $k = 1,2$, is defined as some bifunctor: $Cat_1 \times Cat_2 \longrightarrow Cat'_1 \times Cat'_2$, just assigning pairs to pairs of objects and morphisms. If for instance F_1 is covariant and F_2 contravariant, then $F_1 \times F_2$ is co-contravariant.

(9.1.9) The representation bifunctor

$$Mor(Cat; -, -) \equiv Mor_{Cat} : Cat \times Cat \longrightarrow Set$$

defined below is contra-covariant:

$$Ob(Cat \times Cat) \ni \{A, B\} \longrightarrow Mor(Cat; A, B) \in Ob(Set).$$

Let $\alpha_k \in Mor(Cat; A_k, B_k), k = 1,2$; then to the pair $\{\alpha_1, \alpha_2\}$ the map:

$$Mor(Cat; B_1, A_2) \ni \beta \longrightarrow \alpha_2\beta\alpha_1 \in Mor(Cat; A_1, B_2)$$

is assigned; one finds: $\{\varepsilon_A, \varepsilon_B\} \longrightarrow id(Mor(Cat; A, B))$.

(9.1.10) Consider two functors $F_k, k = 1,2$, from Cat into Cat', which are (i) both covariant, or (ii) both contravariant. A functorial morphism from F_1 to F_2 is a family of morphisms:

$$Ob(Cat) \ni A \underset{\Phi}{\longrightarrow} \phi_A \in Mor(Cat'; A'_1, A'_2), \quad A \underset{F_{k\ ob}}{\longrightarrow} A'_k, k = 1,2,$$

such that the following diagram commutes. Let $\alpha \in Mor(Cat; A, B)$,

$$A \underset{F_{k\ ob}}{\longrightarrow} A'_k, \quad B \underset{F_{k\ ob}}{\longrightarrow} B'_k, \quad \alpha \underset{F_{k\ mor}}{\longrightarrow} \alpha'_k, k = 1,2;$$

then (i) $\alpha'_2\phi_A = \phi_B\alpha'_1$, or (ii) $\alpha'_2\phi_B = \phi_A\alpha'_1$, respectively.

(9.1.10.1) The identical functorial morphism I from F to F is defined such that $I : A \longrightarrow \varepsilon_{A'}$, for $F_{ob} : A \longrightarrow A'$.

(9.1.10.2) Let $F_k, k = 1,2,3$, be functors from Cat into Cat', and consider two functorial morphisms Φ from F_1 to F_2, Ψ from F_2 to F_3. They are composed to a functorial morphism $\Psi \circ \Phi$ from F_1 to F_3, such that:

$$Ob(Cat) \ni A \underset{\Psi\circ\Phi}{\longrightarrow} \psi_A\phi_A \in Mor(Cat'; A'_1, A'_3), \quad A \underset{F_{k\ ob}}{\longrightarrow} A'_k, k = 1,2,3.$$

(9.1.10.3) Denote by $I_k, k = 1,2$, the identical functorial morphisms from F_k to F_k. Then $\Phi \circ I_1 = I_2 \circ \Phi = \Phi$ for every functorial morphism Φ from F_1 to F_2.

(9.1.10.4) A functorial morphism $\Phi : Ob(Cat) \ni A \longrightarrow \phi_A \in Mor(Cat')$ is called functorial isomorphism, if and only if all the images ϕ_A are isomorphisms; in this case the two involved functors F_1 and F_2 are called isomorphic, and one can define the inverse functorial morphism Φ^{-1}, such that $\Phi^{-1} \circ \Phi = I_1$, $\Phi \circ \Phi^{-1} = I_2$, inserting the involved identical functorial morphisms.

(9.1.11) In many categories a product, or a coproduct, can be established.

(9.1.11.1) Let $P \in Ob(Cat)$, and $\pi := \{\pi_k; k \in I\}$ a family of morphisms

$$\pi_k \in Mor(Cat; P, A_k) \equiv Mor(P, A_k).$$

The pair

$$\{P, \pi\} \equiv \prod_{k \in I} A_k$$

is called product of the family $\{A_k; k \in I\}$ in the sense of Cat, if and only if

$$\forall \{\gamma_k \in Mor(C, A_k); k \in I\} : \exists \textit{ unique } \gamma \in Mor(C, P) :$$
$$\forall k \in I : \gamma_k = \pi_k\gamma.$$

The morphisms $\pi_k, k \in I$, are then called projections of P onto A_k.

(9.1.11.2) Let $Q \in Ob(Cat)$, and $\beta := \{\beta_k; k \in I\}$ a family of morphisms

$$\beta_k \in Mor(Cat; A_k, Q) \equiv Mor(A_k, Q).$$

The pair

$$\{Q, \beta\} \equiv \coprod_{k \in I} A_k$$

is called coproduct of the family $\{A_k; k \in I\}$ in the sense of Cat, if and only if

$$\forall \{\gamma_k \in Mor(A_k, C); k \in I\} : \exists \; unique \; \gamma \in Mor(Q, C) : \\ \forall k \in I : \gamma_k = \gamma\beta_k.$$

The morphisms $\beta_k, k \in I$, are then called embeddings of A_k into Q.

(9.1.12) These two constructions are unique in the following sense. Let $\{P, \pi\}$ and $\{P', \pi'\}$ be products, $\{Q, \beta\}$ and $\{Q', \beta'\}$ coproducts of the family of objects $\{A_k; k \in I\}$ of a category Cat. Then there are isomorphisms $\sigma \in Mor(P, P'), \tau \in Mor(Q, Q')$, such that $\forall k \in I : \pi_k = \pi'_k \sigma, \beta'_k = \tau\beta_k$. Both product and coproduct are universal objects, the former universally attracting, the latter universally repelling, in the sense of appropriately chosen categories, due to the subsequent definition.

(9.1.13) An object $P \in Ob(Cat)$ is called universally attracting or terminal, if and only if

$$\forall A \in Ob(Cat) : \exists \; unique \; \alpha \in Mor(Cat; A, P).$$

On the other hand, $Q \in Ob(Cat)$ is called universally repelling or initial, if and only if

$$\forall A \in Ob(Cat) : \exists \; unique \; \alpha \in Mor(Cat; Q, A).$$

(9.1.13.1) If both P and P' are universally attracting, or both repelling, then $\exists$ *unique isomorphism* $\sigma \in Mor(Cat; P, P')$.

(9.1.13.2) For instance, in order to establish the coproduct

$$\{Q, \beta\} \equiv \coprod_{k \in I} A_k$$

as universally repelling object of some category *Coprod*, one defines the latter in the following way. The objects of *Coprod* are pairs

$$\Phi := \{B, \phi\}, \;\; B \in Ob(Cat), \;\; \phi := \{\phi_k; k \in I\}, \;\; \phi_k \in Mor(Cat; A_k, B).$$

Let $\Psi := \{C, \psi\}$ be also an object of *Coprod*; then one defines the set

$$Mor(Coprod; \Phi, \Psi) := \{\mu \in Mor(Cat; B, C); \forall k \in I : \mu\phi_k = \psi_k\}.$$

(9.1.13.3) The product $\{P, \pi\} \equiv \prod_{k \in I} A_k$ is universally attracting in the sense of an appropriately defined category *Prod*. Its objects and morphisms are defined as the following pairs and morphisms of *Cat*.

$$\Phi := \{B, \phi\}, \quad B \in Ob(Cat), \quad \phi := \{\phi_k; k \in I\}, \quad \phi_k \in Mor(Cat; B, A_k);$$

$$Mor(Prod; \Phi, \Psi) := \{\mu \in Mor(Cat; B, C); \forall k \in I : \psi_k \circ \mu = \phi_k\},$$

from Φ to an object $\Psi := \{C, \Psi\}$ of *Prod*.

(9.1.14) Some examples of categories are presented below.

(9.1.14.1) The category *Set* consists of sets as objects, with maps as morphisms.

(9.1.14.2) The category *Top* of topological spaces, with continuous maps as morphisms, is some non-full subcategory of *Set*.

(9.1.14.3) The category *Abel* of Abelian groups is some full subcategory of the category *Group* of groups, with respect to homomorphisms of groups.

(9.1.14.4) The category of left (right) modules over a ring R, with the R-linear mappings as morphisms, is denoted by ${}_R Mod$ (Mod_R).

(9.1.14.5) The category *Ring* consists of rings as objects, with ring homomorphisms as morphisms.

(9.1.14.6) The category of algebras over a commutative ring R, with the algebra homomorphisms as morphisms, is denoted by Alg_R. The categories of associative algebras, and of Lie algebras over R, are full subcategories of Alg_R. The category of unital associative algebras over R is a non-full subcategory of Alg_R, because its morphisms are expected to send unit to unit.

(9.1.14.7) The category of associative (Lie) superalgebras over a commutative ring R, with the corresponding morphisms being understood to be even with respect to the involved $\mathbf{Z}_2$-grading, contains the full subcategory of associative (Lie) algebras over R, the latter just with trivial odd R-submodule.

(9.1.14.8) Similarly one defines the categories of $\mathbf{Z}_2$-graded coalgebras, $\mathbf{Z}_2$-graded bialgebras, and $\mathbf{Z}_2$-graded Hopf algebras over a commutative ring R, with the non-graded objects constituting corresponding full subcategories.

(9.1.15) The Cartesian product of a family of sets $\{A_k; k \in I\}$, with the projections:

$$\prod_{j\in I} A_j \ni \{a_j; j \in I\} \xrightarrow[def]{\pi_k} a_k \in A_k, k \in I,$$

is some product in the categorial sense.

(9.1.16) The direct product G of a family of groups $\{G_k; k \in I\}$ is some product in the categorial sense, with respect to the natural projections.

$$\forall g, h \in G : gh := \{g_k h_k; k \in I\}, \ g^{-1} := \{g_k^{-1}; k \in I\}, \ e := \{e_k; k \in I\},$$

the last definition concerning units.

$$\forall k \in I : G \ni g := \{g_j; j \in I\} \xrightarrow[def]{\pi_k} g_k \in G_k.$$

(9.1.17) There is some product in the sense of the category of rings, with respect to the natural projections. The sum and product of families of ring elements are constructed as the families of sums and products of the involved components. Thereby the Cartesian product of the sets of a family of rings $\{R_k; k \in I\}$, with the units e_k, is established as some ring R with the unit $e := \{e_k; k \in I\}$, such that $\forall r, s \in R$:

$$r + s := \{r_k + s_k; k \in I\}, \ -r := \{-r_k; k \in I\}, \ rs := \{r_k s_k; k \in I\}.$$

$$\forall k \in I : R \ni r := \{r_i; i \in I\} \xrightarrow[def]{\pi_k} r_k \in R_k.$$

(9.1.18) The tensor product of finitely many commutative unital associative algebras over a commutative ring R, for instance of finitely many commutative rings over the integers $\mathbf{Z}$, is some coproduct in the category of such algebras. For instance in the case of two-fold tensor product, the embeddings are defined by:

$$A_1 \ni a_1 \xrightarrow[\beta_1]{} a_1 \otimes e_2 \in A_1 \otimes A_2, \ A_2 \ni a_2 \xrightarrow[\beta_2]{} e_1 \otimes a_2 \in A_1 \otimes A_2,$$

with the units e_k of $A_k, k = 1, 2$.

(9.1.19) Consider a family $\{A_k; k \in I\}$ of left modules over a ring R. The direct product A of the corresponding Abelian groups becomes some R-left module, such that

$$\forall a := \{a_k; k \in I\} \in A, \forall r \in R : ra := \{ra_k; k \in I\}.$$

With respect to the natural projections one thereby obtains some product in the sense of the category ${}_R\mathit{Mod}$. Moreover the direct sum

$$\bigoplus_{k\in I} A_k := \{\{a_k; k \in I\}; card\{k \in I; a_k \neq 0_k\} \in \mathbf{N}_0\}$$

of this family, which is some R-submodule of A, is some coproduct in the sense of ${}_RMod$, with respect to the natural embeddings. $\forall k \in I$:

$$A_k \ni a_k \xrightarrow[def]{\beta_k} \{\delta_{kj} a_k; j \in I\} \in \bigoplus_{j\in I} A_j, \quad \delta_{kj} := \begin{cases} e_R & \text{for } k = j \\ 0_R & \text{for } k \neq j \end{cases} .$$

For any family of R-linear mappings $\{\alpha_k : A_k \longrightarrow B; k \in I\}$, one just uses the diagrams:

$$A_k \ni a_k \xrightarrow{\beta_k} \xrightarrow{\alpha} \alpha_k(a_k),$$

$$\bigoplus_{j\in I} A_j \ni \{a_j; j \in I\} \xrightarrow[def]{\alpha} \sum_{\{j\in I; a_j \neq 0_j\}} \alpha_j(a_j) \in B.$$

(9.1.20) The universal enveloping superalgebra $E(L)$ of a Lie superalgebra L, over a commutative ring R, is some universally repelling object in the sense of the following category. Its objects are the homomorphisms of Lie superalgebras $\lambda : L \longrightarrow A_L$, into the super-commutator algebra A_L of an associative unital superalgebra A over R; its morphisms $\alpha : \lambda_1 \longrightarrow \lambda_2$ are the homomorphisms of unital associative superalgebras such that $\alpha \circ \lambda_1 = \lambda_2$.

(9.1.21) The category of $\mathbf{Z}_2$-graded differential algebras over a commutative ring R, with the corresponding homomorphisms as morphisms, allows for the following universal construction, due to A. Connes.
The injection ν of an associative superalgebra A into its non-unital universal differential envelope $\Omega(A)$, over a commutative ring R of coefficients, is some repelling object in the sense of the following category. Its objects are the homomorphisms of associative superalgebras $\phi : A \longrightarrow D$, into any $\mathbf{Z}_2$-graded differential algebra D over R, with the odd derivation $\delta : D \longrightarrow D$; its morphisms $\mu : \phi_1 \longrightarrow \phi_2$ are the homomorphisms of $\mathbf{Z}_2$-graded differential R-algebras $\mu : D_1 \longrightarrow D_2, \delta_2 \circ \mu = \mu \circ \delta_1$, with an obvious notation of odd derivations, such that $\mu \circ \phi_1 = \phi_2$. Here $\circ$ denotes the composition of homomorphisms of associative superalgebras over R.

9.2 Abelian Categories

(9.2.1) A category $\mathcal{A}$ is called additive, if and only if the axions (Ab 1), ..., (Ab 4) below are fulfilled.

(Ab 1) $\forall A, B \in Ob(\mathcal{A}) : Mor(\mathcal{A}; A, B) \equiv Mor(A, B)$ is an Abelian group, the unit of which is denoted by $0 : A \longrightarrow B$.

(Ab 2) The product of morphisms is $\mathbf{Z}$-bilinear. $\forall \phi, \phi', \psi, \psi' \in Mor(\mathcal{A})$:

$$(\psi + \psi')\phi = \psi\phi + \psi'\phi, \quad \psi(\phi + \phi') = \psi\phi + \psi\phi',$$

$$(-\psi)\phi = \psi(-\phi) = -(\psi\phi); \text{ then } \phi 0 = 0\phi = 0.$$

(Ab 3) $\exists A_0 \in Ob(\mathcal{A}) : \forall A \in Ob(\mathcal{A}) :$

$$card\ Mor(A_0, A) = card\ Mor(A, A_0) = 1.$$

This zero object A_0 is also denoted by 0 sometimes. Then the only element of $Mor\ (A_0, A)$ is monomorphic, and the only element of $Mor(A, A_0)$ is epimorphic, for any object A. These two morphisms are also denoted by 0, for convenience.

(Ab 4) For every finite family of objects, there exist both a product and a coproduct in the categorial sense.
Obviously any two zero objects A_0, A_0' are isomorphic in the sense:

$$A_0 \xrightarrow[\alpha]{} A_0' \xrightarrow[\alpha']{} A_0, \quad \alpha'\alpha = \varepsilon_{A_0}, \quad \alpha\alpha' = \varepsilon_{A_0'}.$$

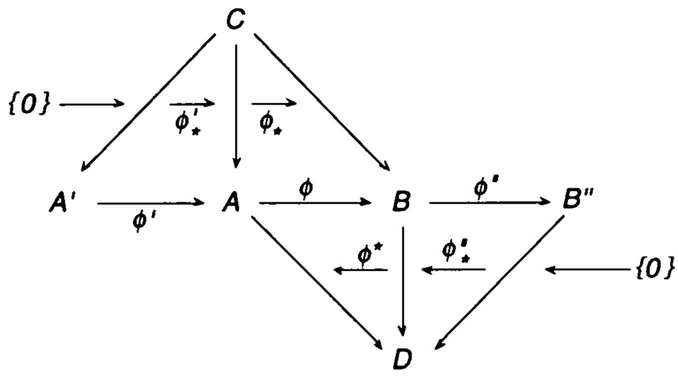
(9.2.2) Let $I := \{1, 2, \ldots, n\} \subset \mathbf{N}$, or $\mathbf{N}$, or $\mathbf{Z}$. A sequence

$$\{\phi_k : M_k \longrightarrow M_{k+1}; k \in I\}$$

of homomorphisms of left (or right) modules over a ring R is called exact, if and only if $\forall k \in I' : Im\ \phi_k = ker\ \phi_{k+1}$; here I' denotes $\{1, \ldots, n-1\}$, or $\mathbf{N}$, or $\mathbf{Z}$, respectively.

(9.2.2.1) Consider a homomorphism $\phi : M \longrightarrow N$ of left (or right) modules over R.
ϕ is injective, if and only if the sequence: $\{0\} \longrightarrow M \ni x \longrightarrow \phi(x) \in N$ is exact.
ϕ is surjective, if and only if the sequence: $M \ni x \longrightarrow \phi(x) \in N \longrightarrow \{0\}$ is exact.

(9.2.3) Consider a morphism $\phi \in Mor(\mathcal{A}; A, B) \equiv Mor(A, B)$ of an additive category $\mathcal{A}$. Take objects A', B'', C, D of this category, and define $\phi'_*, \phi_*, \phi''_*, \phi^*$ by means of the representation functors $Mor(\mathcal{A}; C, -)$ and $Mor(\mathcal{A}; -, D)$, according to the following diagram. Here the mappings of morphisms $\phi'_*, \phi''_*, \phi_*, \phi^*$, which are induced by given ϕ', ϕ'', and ϕ, are homomorphisms of Abelian groups, due to (Ab 2).



Now fix A' and B''. The morphism ϕ' is called kernel of ϕ and denoted by $ker\ \phi$, if and only if

$$\forall C \in Ob(\mathcal{A}) : \{0\} \longrightarrow Mor(C, A') \underset{\phi'_*}{\longrightarrow} Mor(C, A) \underset{\phi_*}{\longrightarrow} Mor(C, B)$$

is an exact sequence. On the other hand, ϕ'' is called cokernel of ϕ and denoted by $coker\ \phi$, if and only if

$$\forall D \in Ob(\mathcal{A}) : \{0\} \longrightarrow Mor(B'', D) \underset{\phi''_*}{\longrightarrow} Mor(B, D) \underset{\phi^*}{\longrightarrow} Mor(A, D)$$

is exact. Hence especially $\phi\phi' = 0$, if ϕ' is a kernel of ϕ; $\phi''\phi = 0$, if ϕ'' is a cokernel of ϕ. Obviously kernels are monomorphic, cokernels are epimorphic. Moreover any two kernels (cokernels) are isomorphic. For an easy proof, choose C and D in the above diagram appropriately.

(9.2.4) For any kernel ϕ' and cokernel ϕ'' of $\phi \in Mor(A, B)$, the following conclusions hold.

$$\phi' = 0 \Longrightarrow \forall C : \phi'_* = 0 \Longrightarrow \forall C : Mor(C, A') = \{0\}, \phi_* \textit{ injective}$$
$$\Longleftrightarrow \phi \textit{ monomorphic.}$$

$$\phi'' = 0 \Longrightarrow \forall D : \phi''_* = 0 \Longrightarrow \forall D : Mor(B'', D) = \{0\}, \phi^* \textit{ injective}$$
$$\Longleftrightarrow \phi \textit{ epimorphic.}$$

Here $A, \ldots, D$ denote objects of an additive category $\mathcal{A}$. Hence, if both $\phi' = 0$ and $\phi'' = 0$, then ϕ is bimorphic. Again denote the zero object of $\mathcal{A}$ by A_0. ϕ is monomorphic, if and only if $A_0 \longrightarrow A$ is some kernel of ϕ. ϕ is epimorphic, if and only if $B \longrightarrow A_0$ is some cokernel of ϕ.

(9.2.5) An additive category $\mathcal{A}$ is called Abelian, if and only if the axioms (Ab 5), ..., (Ab 7) are also valid.

(Ab 5) Every morphism ϕ of $\mathcal{A}$ admits kernels and cokernels.

(Ab 6) Let ϕ' be a kernel, and ϕ'' a cokernel of ϕ. If $\phi' = 0$, then ϕ is some kernel of ϕ''. If $\phi'' = 0$, then ϕ is some cokernel of ϕ'.

(Ab 7) If both a kernel $\phi' = 0$, and a cokernel $\phi'' = 0$, then ϕ is an isomorphism.

(9.2.6) Let the categories $\mathcal{A}, \mathcal{B}$ be additive. A covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called additive, if and only if the map

$$F_{mor} : Mor(\mathcal{A}; A_1, A_2) \longrightarrow Mor(\mathcal{B}; B_1, B_2)$$

is a homomorphism of Abelian groups, for any two objects A_k of $\mathcal{A}$, and $F_{ob} : A_k \longrightarrow B_k, k = 1, 2$.

(9.2.7) In an Abelian category $\mathcal{A}$, the representation bifunctor:

$$Ob(\mathcal{A}) \times Ob(\mathcal{A}) \ni \{A, B\} \longrightarrow Mor(\mathcal{A}; A, B) \equiv Hom(A, B) \in Ob(Abel),$$

into the category of Abelian groups, is denoted by Hom, and the morphisms are called homomorphisms.

(9.2.8) Let $\phi \in Hom(A, B)$ be a homomorphism of an Abelian category, with kernel ϕ' and cokernel ϕ''. Denote by $im\ \phi \equiv \phi_2$ a kernel of ϕ'', and by $coim\ \phi \equiv \phi_1$ a cokernel of ϕ', called image and coimage of ϕ. Then $\phi = \phi_2\phi_1$.

(9.2.8.1) A sequence of homomorphisms $\psi\phi : A \longrightarrow B \longrightarrow C$ is called exact, if and only if any image of ϕ is some kernel of ψ. A finite or infinite sequence of homomorphisms is called exact, if and only if every occurring composition is exact in the above sense.

(9.2.9) An additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Abelian categories is called left (right) exact, if and only if it preserves kernels (cokernels) in the sense, that

$$ker(F_{mor}(\phi)) = F_{mor}(ker\ \phi) \quad (\text{or } coker\ (F_{mor}(\phi)) = F_{mor}(coker\ \phi))$$

for all $\phi \in Hom(\mathcal{A})$. In this case exactness of sequences of the kind:
$0 \longrightarrow A \longrightarrow B \longrightarrow C$ (or $A \longrightarrow B \longrightarrow C \longrightarrow 0$) is preserved. F is called exact, if and only if it is both left and right exact; then it preserves exactness of any sequence.

(9.2.10) The covariant representation functor $Hom(A, -)$ into the morphisms of an Abelian category is left exact.

(9.2.10.1) In order to obtain the dual of an Abelian category, one must interchange kernels and cokernels. Hence one finds that the contravariant representation functor $Hom(-,B)$ is partially exact in the sense, that:

$$coker\ \alpha \underset{Hom(-,B)}{\longrightarrow} ker\,(Hom(-,B)_{mor}(\alpha))$$

for every homomorphism α.

(9.2.10.2) In the category Vec_K of vector spaces A over a field K, the representation functors $Hom(A,-)$ are exact.

(9.2.11) The categories ${}_RMod$ and Mod_R of left and right modules over a ring R are Abelian. In both cases, $Hom_R(A,B)$ is an Abelian group, for any objects A, B. Of course $\{0\}$ is used as zero object. The inclusion: $ker\ \phi \longrightarrow A$ is used as kernel, and the surjection: $B \ni b \longrightarrow b+Im\ \phi \in coker\ \phi$ as cokernel of an R-linear map $\phi : A \longrightarrow B$. Here $ker\ \phi$ denotes the inverse image of $\{0_B\}$, $Im\ \phi$ the image of $\phi \in Hom_R(A,B)$, and $coker\ \phi := B/Im\ \phi$.

(9.2.11.1) In the category of vector spaces over a field K, for any objects A, B,

$$\forall \phi \in Hom_K(A,B) : dim\ A = dim\ ker\ \phi + dim\ Im\ \phi,$$
$$dim\ B = dim\ coker\ \phi + dim\ Im\ \phi.$$

(9.2.12) A sequence $\delta := \{\delta_n : M_n \longrightarrow M_{n+1}; n \in \mathbf{Z}\}$ of homomorphisms of left (or right) modules over a ring R is called complex, if and only if $\forall n \in \mathbf{Z} :\ Im\ \delta_n \subseteq ker\ \delta_{n+1}$, which equivalently means $\delta_{n+1}\delta_n = 0$. Let δ', with modules M'_n over $R, n \in \mathbf{Z}$, be a complex too, and consider a sequence $\phi := \{\phi_n : M_n \longrightarrow M'_n; n \in \mathbf{Z}\}$ of R-bilinear mappings. ϕ is called complex morphism, if and only if it is compatible with δ and δ' in the natural sense that $\forall n \in \mathbf{Z} : \delta'_n\phi_n = \phi_{n+1}\delta_n$. With these morphisms one establishes the category of complexes of left (or right) modules over R, which is Abelian. $\forall n \in \mathbf{Z}$:

$$ker\ \phi_n \underset{inclusion}{\longrightarrow} M_n \underset{\phi_n}{\longrightarrow} M_{n+1} \underset{canonical\ surjection}{\longrightarrow} coker\ \phi_n.$$

For $n \in \mathbf{Z}, \delta_n$ are called the differentials, $ker\ \delta_n$ the n-cycles, $Im\ \delta_{n-1}$ the n-boundaries, and $H_n := ker\ \delta_n/Im\ \delta_{n-1}$ the n-homologies. The complex δ is called finite, if and only if $\exists k \in \mathbf{N} : \forall n \in \mathbf{Z} : |n| > k \Longrightarrow M_n = \{0\}$. The complex δ is called exact, if and only if all the homologies are trivial in the sense that $\forall n \in \mathbf{Z} : H_n = \{0\}$.

9.3 Quasitensor Categories

(9.3.1) A category Mon is called monoidal, if and only if it is equipped with a covariant bifunctor T from $Mon \times Mon$ into Mon, which is associative and contains an identity object $E \in Ob(Mon)$ in the following sense. Denote:

$$Ob(Mon \times Mon) \ni \{A_1, A_2\} \underset{T_{ob}}{\longrightarrow} A_1 \otimes A_2 =: A \in Ob(Mon),$$

$$B := B_1 \otimes B_2,$$

$$\prod_{k=1}^{2} Mor(Mon; A_k, B_k) \ni \{\alpha_1, \alpha_2\}$$

$$\underset{T_{mor}}{\longrightarrow} T(\alpha_1, \alpha_2) \equiv \alpha_1 \otimes \alpha_2 \in Mor(Mon; A, B).$$

Consider the covariant functors $T \circ (T \times I)$ and $T \circ (I \times T)$, denoting by I the identical functor, and suppressing conveniently the order of products of categories, from $Mon \times Mon \times Mon$ into Mon, such that for the objects and morphisms involved:

$$\{A_1, A_2, A_3\} \longrightarrow (A_1 \otimes A_2) \otimes A_3, \ \text{ or } \ A_1 \otimes (A_2 \otimes A_3),$$

$$\{\alpha_1, \alpha_2, \alpha_3\} \longrightarrow (\alpha_1 \otimes \alpha_2) \otimes \alpha_3, \ \text{ or } \ \alpha_1 \otimes (\alpha_2 \otimes \alpha_3);$$

then assume existence of a functorial isomorphism from $T \circ (T \times I)$ to $T \circ (I \times T)$:

$$(A_1 \otimes A_2) \otimes A_3 \longleftrightarrow A_1 \otimes (A_2 \otimes A_3),$$

which may be called an associativity constraint. Moreover define two covariant functors from Mon into itself, such that for objects A and morphisms α, inserting the identical morphism ε_E of E:

$$E \otimes A \longleftarrow A \longrightarrow A \otimes E, \ \ \varepsilon_E \otimes \alpha \longleftarrow \alpha \longrightarrow \alpha \otimes \varepsilon_E;$$

then assume that there are functorial isomorphisms between these two functors and the identity functor of Mon, which explicitly means that for objects A, B and morphisms $\alpha : A \longrightarrow B$ of Mon:

$$E \otimes A \longleftrightarrow A : \lambda_A \longleftarrow A \longrightarrow \mu_A : A \longleftrightarrow A \otimes E,$$

$$(\varepsilon_E \otimes \alpha)\lambda_A = \lambda_B \alpha, \ \ (\alpha \otimes \varepsilon_E)\mu_A = \mu_B \alpha.$$

Let $\lambda_E = \mu_E : E \longleftrightarrow E \otimes E$.

(9.3.1.1) Of course such products of four or more objects can then be written, modulo isomorphisms, with an arbitrary location of brackets; for instance:

$$(A \otimes (B \otimes C)) \otimes D \longleftrightarrow A \otimes (B \otimes (C \otimes D))$$

for any objects $A, \ldots, D$ of Mon. Moreover one demands that the so-called pentagon diagram below is commutative, inserting only the isomorphisms which are due to the associativity constraint.

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (A \otimes B) \otimes (C \otimes D) \arrow[r, leftrightarrow] & A \otimes (B \otimes (C \otimes D)) \arrow[d, leftrightarrow] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, leftrightarrow] & & A \otimes ((B \otimes C) \otimes D)
\end{tikzcd}$$

(9.3.1.2) Finally one postulates that, inserting these functorial isomorphisms, the following triangular diagram commutes:

$$(A \otimes B \longleftrightarrow A \otimes (E \otimes B) \longleftrightarrow (A \otimes E) \otimes B \longleftrightarrow A \otimes B) = \varepsilon_{A \otimes B}$$

for any objects A, B of Mon. By means of the pentagon diagram one can prove that then also:

$$(A \otimes B \longleftrightarrow E \otimes (A \otimes B) \longleftrightarrow (E \otimes A) \otimes B \longleftrightarrow A \otimes B) = \varepsilon_{A \otimes B},$$
$$(A \otimes B \longleftrightarrow A \otimes (B \otimes E) \longleftrightarrow (A \otimes B) \otimes E \longleftrightarrow A \otimes B) = \varepsilon_{A \otimes B},$$

inserting as above the isomorphisms $\lambda_A, \ldots, \mu_{A \otimes B}$, and the identical morphisms; for instance $\lambda_A \otimes \varepsilon_B : A \otimes B \longleftrightarrow (E \otimes A) \otimes B$.

(9.3.1.3) Such a category Mon is called strict monoidal, if and only if the involved functorial morphisms are identical. In this case the above pentagon and triangular diagrams are consequences, of course.

(9.3.1.4) One can prove, that in a monoidal category Mon the so-called coherence property holds, in the sense that all the diagrams of associativity isomorphisms, and isomorphisms of the kind λ_A, μ_A, and ε_A, $A \in Ob(Mon)$, are commuting.

(9.3.1.5) If E' is another identity object of Mon, then the isomorphism $\nu := \mu_{E'}^{-1} \lambda'_E : E \longleftrightarrow E'$ fulfills the diagram: $\lambda'_{E'} \nu = (\nu \otimes \nu) \lambda_E$.

(9.3.2) Let a category *Ten* be monoidal, with the above notations. It is called tensor category, if and only if a symmetry constraint Σ holds, in the sense of a functorial isomorphism of the following two covariant functors:

$$\{A, B\} \longrightarrow A \otimes B \text{ or } B \otimes A, \quad \{\alpha, \beta\} \longrightarrow \alpha \otimes \beta \text{ or } \beta \otimes \alpha;$$

$$\Sigma : Ob(Ten \times Ten) \ni \{A, B\} \longrightarrow \sigma_{A,B} \in Mor(Ten; A \otimes B, B \otimes A).$$

These isomorphisms are assumed to fulfill the following symmetry condition and hexagon diagram:

$$\sigma_{B,A}\sigma_{A,B} = \varepsilon_{A \otimes B}, \quad \sigma_{A \otimes B, C} = (\sigma_{A,C} \otimes \varepsilon_B)(\varepsilon_A \otimes \sigma_{B,C}),$$

for any objects A, B, C of *Ten*. The second condition, which by the symmetry constraint implies that also

$$\sigma_{A, B \otimes C} = (\varepsilon_B \otimes \sigma_{A,C})(\sigma_{A,B} \otimes \varepsilon_C)$$

for any objects, meaning that associativity and symmetry constraints are compatible, is written down here conveniently suppressing an explicit notation of associativity isomorphisms. For instance the second one of these two hexagon diagrams is drawn below.

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (A \otimes B) \otimes C \arrow[r, leftrightarrow] & (B \otimes A) \otimes C \arrow[d, leftrightarrow] \\
(B \otimes C) \otimes A \arrow[r, leftrightarrow] & B \otimes (C \otimes A) \arrow[r, leftrightarrow] & B \otimes (A \otimes C)
\end{tikzcd}$$

Moreover one demands, that $\sigma_{A,E} = \lambda_A \mu_A^{-1}$.

(9.3.3) The category of bimodules over a commutative ring R is of tensor type, with respect to the tensor product over R, and inserting the natural isomorphisms into the associativity and symmetry constraints. The identity object E is R itself.

(9.3.4) A monoidal category *Braid* is called braided, or of quasitensor type, if and only if it is equipped with a commutativity constraint of quasisymmetric type

$$\Sigma : \{A, B\} \longrightarrow \sigma_{A,B} : A \otimes B \longleftrightarrow B \otimes A \text{ for } A, B \in Ob(Braid),$$

which fulfills both hexagon conditions, and $\sigma_{A,E} = \sigma_{E,A}^{-1} = \lambda_A \mu_A^{-1}$. Note that here the two hexagon diagrams do not imply each other to commute.

(9.3.4.1) The functorial property of Σ means, that for any two morphisms $\alpha_k : A_k \longrightarrow B_k, k = 1, 2,$

$$(\alpha_2 \otimes \alpha_1)\sigma_{A_1,A_2} = \sigma_{B_1,B_2}(\alpha_1 \otimes \alpha_2).$$

Hence one obtains, using the hexagon diagrams, the braid relations

$$(\sigma_{B,C} \otimes \varepsilon_A)(\varepsilon_B \otimes \sigma_{A,C})(\sigma_{A,B} \otimes \varepsilon_C)$$
$$= (\varepsilon_C \otimes \sigma_{A,B})(\sigma_{A,C} \otimes \varepsilon_B)(\varepsilon_A \otimes \sigma_{B,C})$$

for objects A, B, C, suppressing again for convenience an explicit notation of associativity isomorphisms.

(9.3.5) As an outstanding example consider a quasitriangular $\mathbf{Z}_2$-graded Hopf algebra $\mathcal{H}$ over a field K, with the universal R-matrix $R \in \mathcal{H}\hat{\otimes}\mathcal{H}$. The quasitensor category *Rep* of representations of $\mathcal{H}$, in the sense of an associative unital superalgebra, on finite-dimensional vector spaces over K is established, with the graded $\mathcal{H}$-left modules over K as objects.

(9.3.5.1) Let ρ_A, ρ_B be representations of $\mathcal{H}$ on $A = A^{\bar{0}} \oplus A^{\bar{1}}, B = B^{\bar{0}} \oplus B^{\bar{1}}$, respectively.

$$\forall x \in \mathcal{H}, a \in A, b \in B : xa := \rho_A(x)a, \quad xb := \rho_B(x)b.$$

An even K-linear map $\phi : A \longrightarrow B$ is considered as morphism in the sense of *Rep*, if and only if it is $\mathcal{H}$-linear too, such that

$$\forall x \in \mathcal{H} : \phi\rho_A(x) = \rho_B(x)\phi.$$

(9.3.5.2) An appropriate covariant bifunctor from *Rep* $\times$ *Rep* into *Rep* is constructed by means of the tensor product of representations of $\mathcal{H}$, inserting the comultiplication Δ, and denoted by T_Δ:

$$\{A, B\} \xrightarrow[T_\Delta\ ob]{} A \otimes B, \ T_\Delta(\rho_A, \rho_B) : \mathcal{H} \xrightarrow[\Delta]{} \xrightarrow[T(\rho_A,\rho_B)]{} \longleftrightarrow End_K(A \otimes B);$$

$$\{\phi, \psi\} \xrightarrow[T_\Delta\ mor]{} T(\phi, \psi) \equiv \phi \otimes \psi,$$

inserting just the tensor product of K-linear mappings.

(9.3.5.3) The natural isomorphism of three-fold tensor products over K:

$$(A \otimes B) \otimes C \ni (a \otimes b) \otimes c \longleftrightarrow a \otimes (b \otimes c) \in A \otimes (B \otimes C)$$

can be used for the associativity constraint, because it is $\mathcal{H}$-linear, due to the associativity of the comultiplication Δ. The pentagon diagram obviously commutes.

(9.3.5.4) The field K is used as identity object, with the counit ε representing $\mathcal{H}$ on K.

$$\forall a \in A \in Ob(Rep) : \lambda_A(a) := 1_K \otimes a, \;\; \mu_A(a) := a \otimes 1_K.$$

$$\lambda_K = \mu_K : K \ni k \longrightarrow 1_K \otimes k = k \otimes 1_K = k(1_K \otimes 1_K) \in K \otimes K.$$

(9.3.5.5) The universal R-matrix is inserted into the quasisymmetry constraint Σ_R.

$$\sigma_{A,B} := P_{A,B} \circ R_{A,B}, \;\; R_{A,B} := T(\rho_A, \rho_B)(R).$$

Here $P_{A,B}$ denotes the $\mathbf{Z}_2$-graded flip on $A \otimes B$. The K-linear bijection $\sigma_{A,B}$ is also $\mathcal{H}$-linear, due to the intertwining property of R, i.e.,

$$\forall x \in \mathcal{H} : R\Delta(x) = \tau \circ \Delta(x)R.$$

The two hexagon diagrams are commutative, because R is quasitriangular:

$$R_{13}R_{23} \longleftrightarrow T(\Delta, id\,\mathcal{H})(R), \;\; R_{13}R_{12} \longleftrightarrow T(id\,\mathcal{H}, \Delta)(R).$$

Moreover

$$\sigma_{A,K} = \sigma_{K,A}^{-1} = \lambda_A \mu_A^{-1} : A \otimes K \ni a \otimes k \longleftrightarrow k \otimes a \in K \otimes A,$$

because

$$T(id\,\mathcal{H}, \varepsilon)(R) = e_{\mathcal{H}} \otimes 1_K, \;\; T(\varepsilon, id\,\mathcal{H})(R) = 1_K \otimes e_{\mathcal{H}}.$$

(9.3.5.6) Obviously

$$\sigma_{B,A}\sigma_{A,B} = T(\rho_A, \rho_B)(\tau(R)R),$$

denoting as above by τ the $\mathbf{Z}_2$-graded flip on $\mathcal{H}\hat{\otimes}\mathcal{H}$. Therefore the category *Rep* is of tensor type, if $\mathcal{H}$ is triangular; in this case

$$\tau(R) = R^{-1} \Longrightarrow \sigma_{B,A}\sigma_{A,B} = id(A \otimes B).$$

(9.3.6) An associative superalgebra $\mathcal{H}$ with unit $e_{\mathcal{H}}$, over a field K, is called $\mathbf{Z}_2$-graded quasibialgebra, if and only if it is equipped with homomorphisms of unital superalgebras $\Delta : \mathcal{H} \longrightarrow \mathcal{H}\hat{\otimes}\mathcal{H}$, $\varepsilon : \mathcal{H} \longrightarrow K$, and an even invertible element $\Phi \in \mathcal{H}\hat{\otimes}\mathcal{H}\hat{\otimes}\mathcal{H}$, such that, suppressing conveniently the natural isomorphisms of associative tensor products over K, the following conditions hold. $\forall x \in \mathcal{H}$:

$$T(\Delta, id\,\mathcal{H}) \circ \Delta(x) = \Phi^{-1}T(id\,\mathcal{H}, \Delta) \circ \Delta(x)\Phi,$$

$$T(id\,\mathcal{H}, id\,\mathcal{H}, \Delta)(\Phi)T(\Delta, id\,\mathcal{H}, id\,\mathcal{H})(\Phi)$$
$$= (e_{\mathcal{H}} \otimes \Phi)T(id\,\mathcal{H}, \Delta, id\,\mathcal{H})(\Phi)(\Phi \otimes e_{\mathcal{H}}),$$

$$T(\varepsilon, id\,\mathcal{H}) \circ \Delta(x) = 1_K \otimes x, \;\; T(id\,\mathcal{H}, \varepsilon) \circ \Delta(x) = x \otimes 1_K,$$

$$T(id\,\mathcal{H}, \varepsilon, id\,\mathcal{H})(\Phi) = e_{\mathcal{H}} \otimes 1_K \otimes e_{\mathcal{H}}.$$

(9.3.6.1) Homomorphisms $\phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ of $\mathbf{Z}_2$-graded quasibialgebras are defined as homomorphisms of unital superalgebras, which are compatible with the costructure mappings in the sense that

$$T(\phi,\phi) \circ \Delta_1 = \Delta_2 \circ \phi, \ \ \varepsilon_2 \circ \phi = \varepsilon_1, \text{ and } T(\phi,\phi,\phi)(\Phi_1) = \Phi_2$$

with respect to the invertible elements in question.

(9.3.6.2) Consider representations, in the sense of unital associative superalgebras, of $\mathcal{H}$ on finite-dimensional vector spaces over K. The resulting graded $\mathcal{H}$-left modules over K are the objects of a monoidal category, with even $\mathcal{H}$-linear mappings as morphisms. Denote by ρ_A, ρ_B, ρ_C such representations of $\mathcal{H}$ on vector spaces A, B, C, respectively. The tensor product $T_\Delta(\rho_A, \rho_B) := T(\rho_A, \rho_B) \circ \Delta$ of representations is used in order to establish the functorial isomorphism:

$$(A \otimes B) \otimes C \ni (a \otimes b) \otimes c \longleftrightarrow \tilde{a} \otimes (\tilde{b} \otimes \tilde{c}) \in A \otimes (B \otimes C),$$

$$T(\rho_A, \rho_B, \rho_C)(\Phi) : a \otimes b \otimes c \longleftrightarrow \tilde{a} \otimes \tilde{b} \otimes \tilde{c},$$

as an $\mathcal{H}$-linear bijection, which fulfills the pentagon diagram. The field K is used as identity object, with ε as representation of $\mathcal{H}$ on K. The functorial isomorphisms $\lambda_A : A \longleftrightarrow K \otimes A$, $\mu_A : A \longleftrightarrow A \otimes K$ are defined such that: $A \ni a \longleftrightarrow 1_K \otimes a$ or $a \otimes 1_K$, respectively.

(9.3.6.2.1) By means of the pentagon diagram one finds, that then also

$$T(id\ K, \rho_A, \rho_B) \circ T(\varepsilon, id\ \mathcal{H}, id\ \mathcal{H})(\Phi) = 1_K \otimes id\ A \otimes id\ B,$$

$$T(\rho_A, \rho_B, id\ K) \circ T(id\ \mathcal{H}, id\ \mathcal{H}, \varepsilon)(\Phi) = id\ A \otimes id\ B \otimes 1_K.$$

(9.3.6.3) Conveniently one also postulates, that

$$T(\varepsilon, id\ \mathcal{H}, id\ \mathcal{H})(\Phi) = 1_K \otimes e_\mathcal{H} \otimes e_\mathcal{H},$$

$$T(id\ \mathcal{H}, id\ \mathcal{H}, \varepsilon)(\Phi) = e_\mathcal{H} \otimes e_\mathcal{H} \otimes 1_K.$$

(9.3.6.4) In order to construct a quasitensor category, furthermore assume existence of an even invertible element $R \in \mathcal{H}\hat{\otimes}\mathcal{H}$, which fulfills the following conditions.

$$T(\Delta, id\ \mathcal{H})(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi,$$

$$T(id\ \mathcal{H}, \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}.$$

This quasitriangular property implies, that

$$T(\varepsilon, id\ \mathcal{H})(R) = 1_K \otimes e_\mathcal{H}, \ \ T(id\ \mathcal{H}, \varepsilon)(R) = e_\mathcal{H} \otimes 1_K,$$

and it implies also the quasi-QYBE

$$R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi = \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}.$$

Here for any permutation $\begin{bmatrix} 1 & 2 & 3 \\ j_1 & j_2 & j_3 \end{bmatrix}$, $\Phi_{j_1 j_2 j_3}$ is obtained from $\Phi_{123} := \Phi$ by means of the $\mathbf{Z}_2$-graded flips, which correspond to the inverse permutation. For instance,

$$\Phi_{231} := T(\tau, id\, \mathcal{H}) \circ T(id\, \mathcal{H}, \tau)(\Phi),$$
$$\Phi_{321} := T(\tau, id\, \mathcal{H}) \circ T(id\, \mathcal{H}, \tau) \circ T(\tau, id\, \mathcal{H})(\Phi).$$

Moreover one demands the usual intertwining property, that

$$\forall x \in \mathcal{H} : \tau \circ \Delta(x) = R\Delta(x)R^{-1}.$$

(9.3.6.5) One thereby obtains the quasitensor category *Rep* of graded $\mathcal{H}$-left modules over K. The functorial isomorphisms

$$\sigma_{A,B} := P_{A,B} \circ T(\rho_A, \rho_B)(R),$$

for any objects A, B carrying representations ρ_A, ρ_B of $\mathcal{H}$, describe the quasisymmetry constraint.

9.4 Rigid Quasitensor Categories

(9.4.1) Consider a quasitensor category *Braid*, and assume that the quasitensor functor is compatible with the representation functor

$$Mor(Braid; -, -) \equiv Mor(-, -)$$

in the following sense. For any objects B, C of *Braid*, define the contravariant functor $F_{B,C}$ by composition of a rather trivial covariant functor and the representation functor $Mor(Braid; -, C) \equiv Mor(-, C)$:

$$Ob(Braid) \ni A \longrightarrow A \otimes B \longrightarrow Mor(A \otimes B, C) \in Ob(Set),$$

$$Mor(Braid) \ni \alpha \longrightarrow \alpha \otimes \varepsilon_B \underset{def}{\longrightarrow} \phi_\alpha \in Mor(Set),$$

which just means that $\phi_\alpha(f_2) = f_2(\alpha \otimes \varepsilon_B)$ for all $\alpha \in Mor(A_1, A_2)$ and $f_2 \in Mor(A_2 \otimes B, C)$. Now assume that $F_{B,C}$ is functorially isomorphic with the representation functor $Mor(-, H)$ for some object $H \equiv \underline{Hom}(B, C)$ of *Braid*:

$$Ob(Braid) \ni A \longrightarrow \phi_A : Mor(A \otimes B, C) \longleftrightarrow Mor(A, H),$$

$$\phi_{A_1} \circ \phi_\alpha = \tilde{\phi}_\alpha \circ \phi_{A_2}, \ \ \tilde{\phi}_\alpha := (Mor(-, H))_{mor}(\alpha), \ \ \tilde{\phi}_\alpha(\tilde{f}_2) = \tilde{f}_2\alpha,$$

for $\tilde{f}_2 \in Mor(A_2, H)$ and the above α.

(9.4.1.1) Consider for instance the category Mod_R of bimodules over a commutative ring R. Define R-linear bijections

$$\phi_A : Hom_R(A \otimes B, C) \ni f \longleftrightarrow \tilde{f} \in Hom_R(A, H), H := Hom_R(B, C)$$

for any objects A, B, C, such that $\forall a \in A, b \in B : f(a \otimes b) = (\tilde{f}(a))(b)$. Then $\forall \alpha \in Hom_R(A_1, A_2), f_2 \in Hom_R(A_2 \otimes B, C), a_1 \in A_1, b \in B$:

$$f_1(a_1 \otimes b) = f_2(a_2 \otimes b), \ \ a_2 := \alpha(a_1), \ \ f_1 := \phi_\alpha(f_2).$$

(9.4.2) Let $f \in Mor(A \otimes B, C)$. Inserting

$$\phi_A(f) =: \alpha \in Mor(A, H), \ \ \nu_H^* := \phi_H^{-1}(\varepsilon_H) \in Mor(H \otimes B, C),$$

one finds that $f = \phi_\alpha(\nu_H^*) = \nu_H^*(\alpha \otimes \varepsilon_B)$.

(9.4.3) Inserting the identity object E of the category *Braid*, one defines the dual object

$$B^* := \underline{Hom}(B, E), \ \ \phi_A : Mor(A \otimes B, E) \longleftrightarrow Mor(A, B^*),$$

assuming hereby, that the so-called internal $\underline{Hom}$-functor can be applied to the pair B, E. The so-called evaluation map is conveniently denoted by

$$\nu_{B^*}^* =: \nu_B : B^* \otimes B \longrightarrow E.$$

Assume existence of a contravariant functor:

$$Ob(Braid) \ni B \longrightarrow B^* \in Ob(Braid),$$

$$Mor(B_1, B_2) \ni f \longrightarrow f^* \in Mor(B_2^*, B_1^*),$$

such that

$$\nu_{B_1}(f^* \otimes \varepsilon_{B_1}) = \nu_{B_2}(\varepsilon_{B_2^*} \otimes f).$$

(9.4.3.1) For instance in the category of bimodules over a commutative ring R, with the notation $B^* := Hom_R(B, R)$ for a bimodule B, one finds that:

$$B^* \otimes B \ni f \otimes b \underset{\nu_B}{\longrightarrow} f(b) \in R.$$

$$\forall f \in Hom_R(B_1, B_2), f_2 \in B_2^* : f^*(f_2) = f_2 \circ f \in B_1^*.$$

(9.4.4) For an object B, the morphism $\sigma_B : B \longrightarrow B^{**} := \underline{Hom}(B^*, E)$ is defined as the image, via the functorial isomorphism ϕ_B, of the composite mapping:

$$B \otimes B^* \underset{symmetry}{\longrightarrow} B^* \otimes B \underset{\nu_B}{\longrightarrow} E.$$

The object B is called reflexive, if and only if σ_B is an isomorphism of *Braid*.

(9.4.4.1) For instance for bimodules over a commutative ring R:

$$B \otimes B^* \ni b \otimes f \longleftrightarrow f \otimes b \underset{\nu_B}{\longrightarrow} f(b) \in R,$$

$$B \ni b \underset{\sigma_B}{\longrightarrow} \sigma_B(b) : B^* \ni f \longrightarrow f(b) \in R.$$

(9.4.5) Denoting $H_k := \underline{Hom}(B_k, C_k)$ for $k = 1, 2$, and $B := B_1 \otimes B_2$, $C := C_1 \otimes C_2$, $H := H_1 \otimes H_2$, consider the morphism: $H \longrightarrow \underline{Hom}(B, C)$, which is assigned via the functorial isomorphism ϕ_H to the composite morphism:

$$(H_1 \otimes H_2) \otimes (B_1 \otimes B_2) \underset{symmetry}{\longrightarrow} (H_1 \otimes B_1) \otimes (H_2 \otimes B_2) \underset{\nu^*_{H_1} \otimes \nu^*_{H_2}}{\longrightarrow} C_1 \otimes C_2.$$

Assume that for any objects $B_k, C_k, k = 1, 2$, this morphism of *Braid* is an isomorphism:

$$\underline{Hom}(B_1, C_1) \otimes \underline{Hom}(B_2, C_2) \longleftrightarrow \underline{Hom}(B_1 \otimes B_2, C_1 \otimes C_2).$$

(9.4.5.1) For instance in the category of vector spaces over a field K:

$$Hom_K(B_1, C_1) \otimes Hom_K(B_2, C_2) \ni f \otimes g \longrightarrow$$
$$T(f, g) \in Hom_K(B_1 \otimes B_2, C_1 \otimes C_2)$$

is an isomorphism, if the involved vector spaces are finite-dimensional. In general, for bimodules over a commutative ring R, the indicated R-linear map is injective.

(9.4.6) A quasitensor category is called rigid, if and only if the above assumptions hold.

(i) The internal $\underline{Hom}$-functor can be applied to every pair of objects.

(ii) Every morphism f can be dualized: $f \longrightarrow f^*$, such that the above diagram commutes.

(iii) Every object is reflexive.

(iv) The internal $\underline{Hom}$-functor is compatible with the quasitensor functor in the sense of the above demanded isomorphism.

(9.4.6.1) The tensor category of finite-dimensional vector spaces over a field K is rigid.

(9.4.7) Consider again a quasitriangular Hopf superalgebra $\mathcal{H}$ over a field K. The quasitensor category *Rep* of representations of $\mathcal{H}$ on finite-dimensional vector spaces $A = A^{\bar{0}} \oplus A^{\bar{1}}$ over K is rigid.

(9.4.7.1) For any pair of such vector spaces, $H := Hom_K(B,C)$ is used as internal $\underline{Hom}$-object. Denoting the representations of $\mathcal{H}$ on B and C by ρ_B and ρ_C, the corresponding representation ρ_H of $\mathcal{H}$ on $H \ni f$ is defined by

$$\rho_H(x)f := \sum_{k=1}^{\bar{k}(x)} (-1)^{zz_k} \rho_C(x'_k) f(\rho_B \circ \sigma)(x''_k), \quad \mathcal{H} \ni x \xrightarrow[\Delta]{} \sum_{k=1}^{\bar{k}(x)} x'_k \otimes x''_k,$$

for homogeneous f, x''_k of degrees $\bar{z}, \overline{z_k}$, respectively.

(9.4.7.2) For an object A, the functorial isomorphism ϕ_A is defined, such that for an even $\mathcal{H}$-linear map $f : A \otimes B \longrightarrow C$,

$$\forall a \in A, b \in B : f(a \otimes b) = (\phi_A(f)(a))\,(b),$$

thereby exhausting all the even $\mathcal{H}$-linear maps: $A \longrightarrow H$ by means of the defining properties of the antipode σ. Therefore

$$\nu_H^* : H \otimes B \ni h \otimes b \longrightarrow h(b) \in C$$

is even and $\mathcal{H}$-linear too.

(9.4.7.3) Inserting the representation ε of $\mathcal{H}$ on K, the dual vector space $B^* := Hom_K(B,K)$ consequently carries the so-called contragredient representation ρ_{B^*} of $\mathcal{H}$, such that

$$\rho_{B^*}(x)f = (-1)^{z_1 z_2} f(\rho_B \circ \sigma(x)) \in B^*$$

for homogeneous $x \in \mathcal{H}, f \in B^*$ of degrees $\overline{z_1}, \overline{z_2}$, respectively. For an even $\mathcal{H}$-linear map f: $B_1 \longrightarrow B_2$, the dual map $f^* : B_2^* \ni f_2 \longrightarrow f_2 \circ f \in B_1^*$ is then again a morphism of the category *Rep*.

(9.4.7.4) The evaluation map ν_B is $\mathcal{H}$-linear and even:

$$B^* \otimes B \ni f \otimes b \xrightarrow[\nu_B]{} f(b) \in K.$$

(9.4.7.5) Denote $B := B_1 \otimes B_2, C := C_1 \otimes C_2$; the K-linear bijection:

$$Hom_K(B_1, C_1) \otimes Hom_K(B_2, C_2) \ni f_1 \otimes f_2$$
$$\longleftrightarrow \tilde{T}(f_1, f_2) \in Hom_K(B, C) :$$

$$B \ni b_1 \otimes b_2$$
$$\longrightarrow \sum_{i=1}^{\bar{i}} (-1)^{(z_i+z_1)(z_i+z_2)} f_1 \circ \rho_{B_1}(R''_i)(b_1) \otimes (\rho_{H_2}(R'_i)(f_2))(b_2) \in C$$

for homogeneous b_1, f_2, R'_i of degrees z_1, z_2, z_i, respectively, is $\mathcal{H}$-linear and even. Here $H_2 := Hom_K(B_2, C_2)$ carries the representation ρ_{H_2} of $\mathcal{H}$, and as usual $R =: \sum_{i=1}^{\bar{i}} R'_i \otimes R''_i$.

(9.4.7.6) The K-linear bijection $\sigma_B : B \longleftrightarrow B^{**}$ is even and $\mathcal{H}$-linear. $\forall \overline{z_1}, \overline{z_2} \in \mathbf{Z}_2$:

$$B^{\overline{z_1}} \ni b \longleftrightarrow \sigma_B(b) : \left(B^{\overline{z_2}}\right)^* \ni f \longrightarrow (-1)^{z_1 z_2} f \circ \rho_B(u)(b) \in K,$$

inserting the even element $u \in \mathcal{H}$, which fulfills $\forall x \in \mathcal{H} : uxu^{-1} = \sigma^2(x)$.

(9.4.7.7) Choose a homogeneous basis $\{b_k; k = 1, \ldots, d\}$ of the K-vector space B, and denote the dual basis of B^* by $\{b^l; l = 1, \ldots, d\}$, such that $\forall_1^d k, l : b^l(b_k) = \delta_{kl}$. The K-linear map

$$\pi_B : K \ni 1_K \longrightarrow \sum_{k=1}^{d} b_k \otimes b^k \in B \otimes B^*,$$

which is obviously independent of the choice of the homogeneous basis, is also $\mathcal{H}$-linear, due to the defining properties of the antipode. The composite mappings:

$$B \underset{\lambda_B}{\longrightarrow} K \otimes B \underset{\pi_B \otimes \varepsilon_B}{\longrightarrow} (B \otimes B^*) \otimes B \longleftrightarrow B \otimes (B^* \otimes B) \underset{\varepsilon_B \otimes \nu_B}{\longrightarrow} B \otimes K \underset{\mu_B^{-1}}{\longrightarrow} B$$

and

$$B^* \underset{\mu_{B^*}}{\longrightarrow} B^* \otimes K \underset{\varepsilon_{B^*} \otimes \pi_B}{\longrightarrow} B^* \otimes (B \otimes B^*)$$
$$\longleftrightarrow (B^* \otimes B) \otimes B^* \underset{\nu_B \otimes \varepsilon_{B^*}}{\longrightarrow} K \otimes B^* \underset{\lambda_{B^*}^{-1}}{\longrightarrow} B^*,$$

inserting the $\mathcal{H}$-linear maps

$$\mu_B^{-1} : B \otimes K \ni b \otimes 1_K \longleftrightarrow b \in B, \;\; \lambda_{B^*}^{-1} : K \otimes B^* \ni 1_K \otimes f \longleftrightarrow f \in B^*,$$

are equal to $\varepsilon_B := id\ B$ and $\varepsilon_{B^*} := id\ B^*$, respectively.

(9.4.8) Consider a $\mathbf{Z}_2$-graded quasibialgebra $\mathcal{H}$ over the field K, with the above notations. It is called $\mathbf{Z}_2$-graded quasi-Hopf algebra, or quasi-Hopf superalgebra, if and only if there are an even K-linear bijection $\sigma : \mathcal{H} \longleftrightarrow \mathcal{H}$, and even elements $\alpha, \beta \in \mathcal{H}^{\overline{0}}$, such that the following conditions hold. The so-called quasiantipode σ is assumed to be an anti-automorphism of unital superalgebras, i.e., for homogeneous elements $x_1, x_2 \in \mathcal{H}$ of degrees $\overline{z_1}, \overline{z_2}$, respectively,

$$\sigma(x_1 x_2) = (-1)^{z_1 z_2} \sigma(x_2) \sigma(x_1), \text{ and } \sigma(e_{\mathcal{H}}) = e_{\mathcal{H}}.$$

Moreover, denoting

$$\Phi =: \sum_{m=1}^{\bar{m}} \phi_m^{(1)} \otimes \phi_m^{(2)} \otimes \phi_m^{(3)}, \;\; \Phi^{-1} =: \Psi =: \sum_{n=1}^{\bar{n}} \psi_n^{(1)} \otimes \psi_n^{(2)} \otimes \psi_n^{(3)},$$

and $\Delta(x) =: \sum_{k=1}^{\bar{k}(x)} x'_k \otimes x''_k$, one demands that $\forall x \in \mathcal{H}$:

$$\sum_{k=1}^{\bar{k}(x)} \sigma(x'_k)\alpha x''_k = \varepsilon(x)\alpha, \quad \sum_{k=1}^{\bar{k}(x)} x'_k \beta\sigma(x''_k) = \varepsilon(x)\beta,$$

$$\sum_{m=1}^{\bar{m}} \phi_m^{(1)}\beta\sigma(\phi_m^{(2)})\alpha\phi_m^{(3)} = \sum_{n=1}^{\bar{n}} \sigma(\psi_n^{(1)})\alpha\psi_n^{(2)}\beta\sigma(\psi_n^{(3)}) = e_{\mathcal{H}}.$$

The last two conditions imply each other. Moreover these conditions imply that $\varepsilon(\alpha\beta) \neq 0$, and $\varepsilon \circ \sigma = \varepsilon$.

(9.4.8.1) Starting from the above properties, one obtains an according quasi-Hopf superalgebra by means of the following replacements:

$$\Delta \longleftrightarrow \tau \circ \Delta, \quad \Phi \longleftrightarrow \left(\Phi^{-1}\right)_{321}, \quad \sigma \longleftrightarrow \sigma^{-1},$$

$$\alpha \longleftrightarrow \sigma^{-1}(\alpha), \quad \beta \longleftrightarrow \sigma^{-1}(\beta).$$

(9.4.8.2) Under the above conditions, there is an invertible even element $F \in \mathcal{H}\hat{\otimes}\mathcal{H}$, such that $\forall x \in \mathcal{H}$:

$$F\Delta \circ \sigma(x)F^{-1} = T(\sigma, \sigma) \circ \tau \circ \Delta(x).$$

(9.4.8.3) Take an even invertible element $\gamma \in \mathcal{H}$; then σ, α, β may be replaced by

$$\sigma'(x) := \gamma\sigma(x)\gamma^{-1} \text{ for } x \in \mathcal{H}, \quad \alpha' := \gamma\alpha, \quad \beta' := \beta\gamma^{-1}.$$

Therefore one may choose $\beta' := e_{\mathcal{H}}$, if β is invertible.

(9.4.9) Consider again the quasitensor category *Rep* of representations of a quasi-Hopf superalgebra $\mathcal{H}$ on finite-dimensional vector spaces over the field K. The representation of $\mathcal{H}$ on $Hom_K(B, C)$, and especially the contragredient representation on $B^* := Hom_K(B, K)$, are defined just as in the case of $\Phi :=$ *unit*, now inserting the quasiantipode σ.

(9.4.9.1) The evaluation map ν_B defined below is even and $\mathcal{H}$-linear:

$$Hom_K(B, K) \otimes B \ni f \otimes b \xrightarrow[\nu_B]{} f \circ \rho_B(\alpha)(b) \in K,$$

inserting a representation ρ_B of $\mathcal{H}$ on a finite-dimensional vector space B over K.

(9.4.9.2) Using a homogeneous basis of $B = B^{\bar{0}} \oplus B^{\bar{1}}$ and the corresponding dual basis of B^*, and inserting the $\mathcal{H}$-linear map

$$\pi_B : K \ni 1_K \longrightarrow \sum_{k=1}^{d} \rho_B(\beta)(b_k) \otimes b^k \in B \otimes B^*,$$

one again calculates the following composite mappings:

$$(\varepsilon_B \otimes \nu_B) \circ T(\rho_B, \rho_{B^*}, \rho_B)(\Phi) \circ (\pi_B \otimes \varepsilon_B) \circ \lambda_B = \mu_B,$$

$$(\nu_B \otimes \varepsilon_{B^*}) \circ T(\rho_{B^*}, \rho_B, \rho_{B^*})(\Phi) \circ (\varepsilon_{B^*} \otimes \pi_B) \circ \mu_{B^*} = \lambda_{B^*},$$

with some marginal abuse of notation. Due to these diagrams one might call the quasitensor category *Rep* rigid.

(9.4.10) The product of finitely many additive categories becomes an additive category, by means of the direct product of Abelian groups of morphisms.

$$\begin{aligned} Mor(Cat_1 \times Cat_2; \{A_1, A_2\}, \{B_1, B_2\}) \ni \{\alpha_1, \alpha_2\} &+ \{\beta_1, \beta_2\} \\ &:= \{\alpha_1 + \beta_1, \alpha_2 + \beta_2\}. \end{aligned}$$

(9.4.11) Consider an additive category, which is also monoidal. One then conveniently demands the covariant bifunctor to be biadditive, which explicitly means that:

$$\{\alpha_1 + \beta_1, \alpha_2\} \underset{T_{mor}}{\longrightarrow} (\alpha_1 + \beta_1) \otimes \alpha_2 = \alpha_1 \otimes \alpha_2 + \beta_1 \otimes \alpha_2,$$

$$\{\alpha_1, \alpha_2 + \beta_2\} \underset{T_{mor}}{\longrightarrow} \alpha_1 \otimes (\alpha_2 + \beta_2) = \alpha_1 \otimes \alpha_2 + \alpha_1 \otimes \beta_2,$$

$$(-\alpha_1) \otimes \alpha_2 = \alpha_1 \otimes (-\alpha_2) = -(\alpha_1 \otimes \alpha_2), \text{ hence } 0 \otimes \alpha_2 = \alpha_1 \otimes 0 = 0,$$

for corresponding morphisms $\alpha_k, \beta_k, k = 1, 2$, of this category.

(9.4.12) An Abelian category, which is also a rigid quasitensor category, thereby assuming the tensor bifunctor to be biadditive, may be called a quantum category.

(9.4.12.1) The quantum category of representations of a quasitriangular Hopf superalgebra $\mathcal{H}$ on finite-dimensional vector spaces over a field K was established above.

(9.4.12.2) This categorial view may be extended to quasitriangular quasi-Hopf superalgebras, weakening the meaning of rigidity as indicated above.

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Notation

Some general notations are collected at first. Then symbols are ordered rather with respect to their first occurrence, and the according page numbers are indicated.

$\mathbf{N}$	natural numbers
$\mathbf{N}_0$	$\mathbf{N} \cup \{0\}$
$\mathbf{Z}$	integer numbers
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{C}$	complex numbers
$\mathbf{K}$	$\mathbf{R}$ or $\mathbf{C}$
$\mathbf{Z}_2$	$:= \mathbf{Z}/\{even\ integers\} = \{\bar{0}, \bar{1}\}$
$:=$	is defined as
$\equiv$	equivalent
$\in$	is element of
$\notin$	is not element of
$\subset$	is subset of, not equal to
$\subseteq$	is subset of, may be equal to
$\cup$	union of sets
$\cap$	intersection of sets
$A \setminus B$	complement of set B in set A
$\emptyset$	empty set
$\forall$	for all
$\exists$	exist
$\Longrightarrow$	implies
$\Longleftrightarrow$	implies each other
$\longrightarrow$	mapping, limit
$\longleftrightarrow$	bijection
$\times$	Cartesian product of sets
$\circ$	composition of mappings
$\oplus$	direct sum
$\otimes$	tensor product
$\hat{\otimes}$	graded tensor product
$\wedge$	wedge product

$[\ ,\ ]$	(super-)commutator, product of Lie (super-)algebra
$\Vert\ \Vert$	norm
$\langle\ \vert\ \rangle$	scalar product, bilinear form
$card$	cardinal number
def	is defined as
dim	dimension
dom	domain of linear operator
ran	range of linear operator
Im	image of mapping
id	identical map
tr	trace
$\mathbf{P}_n$	permutation group of n elements
R	commutative ring
K	field
$Hom_R(E, F)$	R-linear maps from E to F
$End_R(E) \equiv Hom_R(E, E)$	endomorphisms of E
$Aut_R(E)$	automorphisms of E $\equiv$ R-linear bijections of E onto E
$E^* \equiv Hom_R(E, R)$	dual R-bimodule

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str	supertrace	55
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$\Lambda(E)$	alternating algebra of E	93
$\otimes_\mu$	smash product of ($\mathbf{Z}_2$-graded) bialgebras	97
$*$	graded star operation	100
Θ	coboundary of ($\mathbf{Z}_2$-graded) Lie bialgebra	104
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Mon	monoidal category	419
T	tensor bifunctor	419
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