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Σ_1^1-FORMULAE ON FINITE STRUCTURES

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Introduction

Let $\langle M, R_1, \ldots, R_k \rangle$ be a finite structure, where $R_1, \ldots, R_k$ are finitary relations. Suppose $A \subseteq M$. We are interested in the following question: What are those properties of A which can be described by a

(1) first-order formula;

(2) second-order formula, if we allow only existential or only universal second-order quantifiers in the prenex normal form, and we restrict the arity of the quantified relations.

For rigorous definitions cf. Notations and Preliminaries.

In Section 1 we prove that the property '$|A|$ is even' cannot be described by a first-order formula, moreover for every formula, among the A's which satisfy it, the number of the odd ones and even ones is almost the same (Theorem 1.2′). Theorem 1.4 states that if a property can be described by a first-order formula, then the set of those A's which satisfy it can be approximated by a set of very special structure, namely the approximating set which is the union of disjoint non-trivial cylinders. (C is a cylinder if

$$\exists\, D, E \subseteq M \quad D \cap E = 0 \wedge C = \{X \subseteq M \mid X \cap (M - E) = D\}.$$

In Section 1 we will work with 0, 1 sequences instead of subsets.) Using this notation we can even say $|E| > |M|^{1-\varepsilon}$ for every cylinder C in the union. This result implies for example that $|A| = \frac{1}{2}|M|$ cannot be described by a first-order formula.

In Section 2 we show that if in a second-order formula we restrict the arity of the quantified relations, then we get a nontrivial hierarchy on finite structures. We will prove (roughly speaking) that the following property of an $n \times n$ 0, 1 matrix, $(n \in \omega)$ can be expressed by universally quantifying the subsets of n (trivially) but cannot be expressed by only existentially quantifying the subsets of n: "*in every line of the matrix the number of* 1*'s is even*" (Theorem 2.1).

Section 3: here we give positive results: we show what a first-order formula can say about the number of the elements of a subset of a finite structure. Although the results of Section 1 imply that the number cannot be defined that way, we will show that $|A|$ can be defined approximately, that is e.g., for every $\varepsilon > 0$ there is a first-order formula such that for every $A \subseteq M$

$$|A| < (\tfrac{1}{2} - \varepsilon)\,|M| \rightarrow A \text{ satisfies the formula,}$$

$$|A| > (\tfrac{1}{2} + \varepsilon)\,|M| \rightarrow A \text{ does not satisfy the formula.}$$

Section 4: we reformulate some results of Sections 1 and 2 in the language of the extensions of nonstandard Peano models. This is the original formulation of our results. It seems that only those questions concerning the hierarchies of the properties of finite structures can be managed, which have a simple reformulation in the theory of nonstandard Peano models. This approach was suggested by Attila Máté (cf. also [1]).

We will prove the following theorems:

Theorem 4.3. *Let N be a countable nonstandard model of Peano Arithmetic $n \in N$, $R \in N$, $k \in \omega$, n nonstandard,*

$$N \models \text{“}R \text{ is a } k\text{-ary relation on } n\text{”}.$$

Then there exists a $P \in N$ with

$$N \models \text{“}P \subseteq n \wedge |P| \text{ is even”}$$

and a model of Peano N'; $n', P', R' \in N'$ with

$$N' \models \text{“}R' \text{ is a } k\text{-ary relation on } n' \quad P' \subseteq n'\text{”}$$

such that the structures defined by $\langle n, P, R \rangle$ and $\langle n', P', R' \rangle$ are isomorphic and

$$N' \models \text{“}|P'| \text{ is odd”}.$$

That is if we fix an initial nonstandard segment of N, then we may continue it to an other model of Peano so that we change the parity of a subset of the initial segment. There are other properties which can be changed in this sense with a different method (see [2]).

A stronger version of this theorem is the following:

Theorem 4.4. *Suppose that N, n, k, R are the same as in Theorem 4.3; $l \in \omega$. Then there exists a $P \in N$ with*

$$N \models \text{“}P = \langle P_i \mid i \in n \rangle \wedge \forall\, i \in n\ |P_i| \text{ is even} \wedge P_i \subseteq n\text{”}$$

such that for all $A_1, \ldots, A_l \in N$, $N \models$ ‘$A_1, \ldots, A_l \subseteq n$’ there exists a model of Peano N' and $n', R', P', A'_1, \ldots, A'_l \in N'$ with

$$N' \models \text{“}R' \text{ is a } k\text{-ary relation on } n' \wedge P' = \langle P'_i \mid i \in n' \rangle \wedge (\forall\, i \in n'\ P'_i \subseteq n) \wedge A'_1, \ldots, A'_l \subseteq n'\text{”}$$

such that the structures $\langle n, R, P, A_1, \ldots, A_l \rangle$ and $\langle n', R', P', A'_1, \ldots, A'_l \rangle$ are isomor-

phic and

$$N' \models \text{“}\exists\, i \in n'\ |P'_i| \text{ is odd”}.$$

Most of our proofs are based on combinatorial lemmas proved in Section 1 (Lemma C.1, C.2 and C.3) and in Section 5.

Notation, preliminaries

$\mathscr{D}(f)$ is the domain, $\mathscr{R}(f)$ is the range of the function f. ${}^XY$ is the set of all functions with $\mathscr{D}(f)=X$, $\mathscr{R}(f)\subseteq Y$. $P(a)$ is the set of all subsets of a. $|A|$ is the number of the elements of the set A. $f \restriction X$ is the restriction of the function f to the set X. We consider a function as a set of ordered pairs, therefore $f \supseteq g$ means that f is the extension of g. The functions f and g are compatable if $\mathscr{D}(f\cap g)=\mathscr{D}(f)\cap\mathscr{D}(g)$. Every natural number n is the set of all natural numbers less than n. If A and B are sets, then $A \,\Delta\, B$ is the symmetric difference of A and B, that is

$$A \,\Delta\, B = (A-B)\cup(B-A).$$

Definition. Let $\mathscr{L}$ be a first-order language with infinitely many k-ary relation and function symbols for all $k\in\omega$. $\mathrm{Sym}(\mathscr{L})$ will be the set of all relation and function symbols of $\mathscr{L}$. We assume that $=$ is a relation symbol of $\mathscr{L}$ and at every interpretation it is interpreted in the natural way.

If $H\subseteq\mathrm{Sym}(\mathscr{L})$, then $\mathscr{L}(H)$ will be the restriction of $\mathscr{L}$ onto H (that is the formulae of $\mathscr{L}(H)$ are those formulae of $\mathscr{L}$ which contain relation and function symbols only from H and the symbol $=$).

Let $\mathscr{L}^{(1)}$ be the second-order extension of $\mathscr{L}$. If $\mathrm{A}_1,\ldots,\mathrm{A}_l\in\mathrm{Sym}(\mathscr{L})$ and $\Phi_1,\ldots,\Phi_i$ are formulae or theories of $\mathscr{L}^{(1)}$, then

$$\mathscr{L}(\mathrm{A}_1,\ldots,\mathrm{A}_l,\Phi_1,\ldots,\Phi_k)=\mathscr{L}(H)$$

where H is the set of relation and function symbols of $\mathscr{L}$ which are among the A_i's or contained in some Φ_i.

A language will be always a language of the type $\mathscr{L}(H)$ or its second-order extension $\mathscr{L}^{(1)}(H)$. A first-order (second-order) formula will always be a formula of $\mathscr{L}$ ($\mathscr{L}^{(1)}$), a relation or function symbol will always be an element of $\mathrm{Sym}(\mathscr{L})$. For a language $\mathscr{M}$ $\mathrm{Sym}(\mathscr{M})$ will denote the set of its relation and function symbols. If $\mathscr{M}$ is a language and $\mathscr{P}\in\mathrm{Sym}(\mathscr{L})$, then let $\mathscr{M}-\mathscr{P}=\mathscr{L}(\mathrm{Sym}(\mathscr{M})-\mathscr{P})$. We will denote a second-order quantor of $\mathscr{L}^{(1)}$ by $\forall^{(1)}_k$ ($\exists^{(1)}_k$) if it quantifies k-ary relations.

We define the $\Sigma^1_{i,k}$ ($\Pi^1_{i,k}$) hierarchies of second-order (prefix) formulae, which may contain free variables of $\mathscr{L}^{(1)}$, by recursion on i for a fixed k.

$\Sigma^1_{0,k}$ ($\Pi^1_{0,k}$) is the set of all first-order formulae. Φ is $\Sigma^1_{i+1,k}$ if there exists a $\Psi\in\Pi^1_{i,k}$ with $(\exists^{(1)}_k X_1\cdots\exists^{(1)}_k X_j\,\Psi)=\Phi$ for some $j\in\omega$. The definition of $\Pi^1_{i+1,k}$ formulae is similar.

If $\mathcal{M}$ is a language, then an interpretation of $\mathcal{M}$ will be always an interpretation of $\mathcal{M}$ on a structure whose underlying set is a natural number.

Thus if $\mathcal{M}$ has only finitely many relation and function symbols, then for all $n \in \omega$, $\mathcal{M}$ has only finitely many interpretations whose underlying set has n elements. The set of all interpretations of $\mathcal{M}$ will be denoted by $\mathrm{Int}(\mathcal{M})$.

If $\pi \in \mathrm{Int}(\mathcal{M})$, then $n(\pi)$ will be underlying set of the corresponding structure and if $\mathcal{P} \in \mathrm{Sym}(\mathcal{M})$, then $\pi(\mathcal{P})$ will be the interpreted of $\mathcal{P}$. Suppose $H_1 \subseteq H_2 \subseteq \mathrm{Sym}(\mathcal{L})$ and $\pi \in \mathrm{Int}(\mathcal{L}(H_2))$. Then $\pi \restriction \mathcal{L}(H_1)$ will be the interpretation τ of $\mathcal{L}(H_1)$ with $n(\tau) = n(\pi)$ and $\tau(\mathcal{P}) = \pi(\mathcal{P})$ for all $\mathcal{P} \in H_1$.

If $\mathcal{P} \in \mathrm{Sym}(\mathcal{L})$ and T is a theory (in $\mathcal{L}$) which does not contain $\mathcal{P}$, then a property of $\mathcal{P}$ over T is a subset of $\{\pi \in \mathrm{Int}(\mathcal{L}(\mathcal{P}_1\ T)) \mid \pi \models T\}$. A property of $\mathcal{P}$ means a property over $T = 0$ that is a subset of $\mathrm{Int}(\mathcal{L}(\mathcal{P}))$. A property R of $\mathcal{P}$ is $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ over T if there exists a $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ formula Φ of $\mathcal{L}(\mathcal{P}, T)$ such that

$$\forall\, \pi \in \mathrm{Int}(\mathcal{L}(\mathcal{P}, T))\ \pi \models T \rightarrow (\pi \models \Phi \leftrightarrow \pi \in R).$$

(R is $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ if it is $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ over $T = 0$). Therefore in that case Φ has no relation or function symbols other than $\mathcal{P}$ and $=$. R is $\Delta^1_{i,j}$ if it is both $\Sigma^1_{i,j}$ and $\Pi^1_{i,j}$. A property of R of $\mathcal{P}$ is weakly $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ if there exists a $\Sigma^1_{i,j}\ (\Pi^1_{i,j})$ formula Φ of $\mathcal{L}$ (that is Φ may have arbitrary relation and function symbols from $\mathcal{L}$), such that for infinitely many interpretations π of $\mathcal{L}(\Phi) - \mathcal{P}$ we have

$$\forall\, \pi' \in \mathrm{Int}(\mathcal{L}(\Phi, \mathcal{P}))\, [\pi' \supseteq \pi \rightarrow (\pi' \models \Phi \leftrightarrow (\pi' \restriction \mathcal{L}(\mathcal{P})) \in R)].$$

We will sometimes denote the interpretated of a $\mathcal{P} \in \mathrm{Sym}(\mathcal{L})$ by $\bar{\mathcal{P}}$ (if the identity of the interpretation is clear from the context), e.g. we will give a property by describing a property of the corresponding $\bar{\mathcal{P}}$'s, that is if $\mathcal{P}$ is a relation symbol, the property "$|\bar{\mathcal{P}}|$ is even" means $\{\pi \in \mathrm{Int}(\mathcal{L}(\mathcal{P})) \mid |\pi(\mathcal{P})| \text{ is even}\}$.

Section 1

In this section $\mathcal{P}$ always will be a unary relation symbol unless it is explicitly stated otherwise. If φ is a first-order (or second-order) sentence and $\pi \in \mathrm{Int}(\mathcal{L}(\varphi) - \mathcal{P})$, then let

$$S^\pi_\varphi = \{P \subseteq n(\pi) \mid \exists\, \pi' \in \mathrm{Int}(\mathcal{L}(\varphi, \mathcal{P}))\ \pi' \supseteq \pi \wedge \pi'(\mathcal{P}) = P \wedge \pi' \models \varphi\},$$

$$\bar{S}^\pi_\varphi = \{f \in {}^{n(\pi)}2 \mid \exists\, P \in S^\pi_\varphi\ \forall\, i \in n(\pi)\ f(i) = 1 \leftrightarrow i \in P\}.$$

We want to show that those subsets of the power set of $n(\pi)$ which are of the form S^π_φ for some first-order φ and π (where $n(\pi)$ is large compared to the length of φ) have some special structure. First consider the case where φ is Σ_0 formula and suppose for the sake of simplicity that φ does not contain function symbols with nonzero arity. Let $c_1, \ldots, c_k$ be the constant symbols contained in φ and let $H = \{\pi(c_1), \ldots, \pi(c_k)\}$. It is clear that in this case $f \in \bar{S}^\pi_\varphi$ depends only on the values of f taken on H. This motivates the following definitions.

Definition. Let A be a finite set. C is a cylinder on A if there exists a function g mapping a subset of A into 2 such that

$$C = \{f \mid f \in {}^A 2, f \supseteq g\}.$$

g will be called the base of the cylinder and will be denoted by $b(C)$.

We will use the following notations:

$$s(C) = \mathscr{D}(b(C)), \qquad \|C\| = |b(C)| = |s(C)|.$$

Note that according to this definition if $C = {}^A 2$, then

$$\|C\| = |b(C)| = |s(C)| = 0.$$

We will use this notion mainly in the case $A \in \omega$.

If X is a set of cylinders, then let

$$\|X\| = \max\{\|C\| \mid C \in X\rangle, \qquad s(X) = \bigcup \{s(C) \mid C \in X\}.$$

Using the notion of cylinders we may say that if φ is the Σ_0 formula described above, then $\bar{S}^\pi_\varphi$ will be the union of some (pairwise disjoint) cylinders C with $\|C\| = k$. If φ is a Σ_1 (or Π_1) formula, then $\bar{S}^\pi_\varphi$ will be the union (or intersection) of sets of the previous type, and alternately taking unions and intersections we get S^π_φ for a Σ_i (Π_i) formula φ as well.

We try to follow this procedure by the hierarchy given below. We will consider also the number of those sets which are necessary for the construction of a fixed set A in this procedure. It will be $w_i(A)$ in the ith level of the hierarchy. The superscript s will be an upper bound for the $w_i(A)$'s in the corresponding set.

Definition. Let $n \in \omega$. For all $i, s \in \omega$ we define a set $U_i^s \subseteq P({}^n 2)$ and a function $w_i : \bigcup_{s \in \omega} U_i^s \to \omega$ by induction on i. If $s \in \omega$, then

$$U_0^s = \{C \mid C \text{ is a cylinder on } n, \|C\| = 1\}.$$

$$w_0(A) = 1 \quad \text{for all } A \in U_0^s.$$

If $A \in {}^n 2$, then let

$$\bar{w}_{i+1}(A) = \min\Big\{\sum_{B \in G} w_i(B) \mid G \subseteq U_i^s \wedge (A = \bigcup G \vee A = \bigcap G)\Big\}.$$

$$U_{i+1}^s = \{A \subseteq {}^n 2 \mid \bar{w}_{i+1}(A) \leqslant s\}$$

and let $w_{i+1} = \bar{w}_{i+1} \restriction U_{i+1}^s$.

Remarks. (1) Since U_0^j is closed under the complementation every U_i^j is also closed under the complementation.

(2) U_i^j obviously depends on n, so if it is necessary we will write $U_i^j(n)$.

Lemma 1.1. (a) *If φ is a first-order sentence, then there are $i, j \in \omega$ such that*

$$\bar{S}^{\pi}_{\varphi} \in U^{n^j}_i(n)$$

for any $\pi \in \mathrm{Int}(\mathcal{L}(\varphi) - \mathcal{P})$ where $n = n(\pi)$.

(b) *Conversely: suppose $i, j \in \omega$, then there exists a first-order sentence φ such that for all $n \in \omega$ and $A \in U^{n^j}_i$ there exists an interpretation π of $\mathcal{L}(\varphi) - \mathcal{P}$ with $n = n(\pi)$ and $A = \bar{S}^{\pi}_{\varphi}$.*

Proof. (a) Suppose that φ is a prefix formula. We prove (a) by induction on the number of quantifiers in φ. For $n(\pi) = 1$ our assertion is trival so we suppose $n(\pi) > 1$.

If φ is quantifier free, then it can be written in the form $\varphi = \mathcal{B}(\mathcal{P}(t_1), \ldots, \mathcal{P}(t_p), \psi_1, \ldots, \psi_q)$ where $\mathcal{B}$ is a Boolean function $t_1, \ldots, t_p$ are terms and $\psi_1, \ldots, \psi_q$ are quantifier free sentences of $\mathcal{L}(\varphi) - \mathcal{P}$. Hence the validity of "$f \in \bar{S}^{\pi}_{\varphi}$" depends only on the values $f(\pi(t_1)), \ldots, f(\pi(t_p))$, that is $\bar{S}^{\pi}_{\varphi} = \bigcup X$ where X is a set of cylinders C with $s(C) = \{\pi(t_1), \ldots, \pi(t_p)\}$. Therefore $n > 1$ implies

$$\bar{S}^{\pi}_{\varphi} \in U^{p \cdot 2^p}_2 \subseteq U^{n^{2p}}_2.$$

(Note that if n is sufficiently large compared to the length of φ, then our proof implies that $\bar{S}^{\pi}_{\varphi} \subseteq U^{n}_2$.) Assume φ contains k quantifiers and $\varphi \doteq \forall x\, \psi(X)$ where ψ is a prefix formula with $k-1$ quantifiers.

Let c be a constant symbol (0-ary function symbol) which does not occur in φ. By the inductive hypothesis we have i, j such that for any interpretation τ of $\mathcal{L}(\psi(c)) - \mathcal{P} = \mathcal{L}(\varphi, c) - \mathcal{P}$, we have $\bar{S}^{\tau}_{\psi(c)} \subseteq U^{n^j}_i(n(\tau))$.

If π is a fixed interpretation of $\mathcal{L}(\varphi) - \mathcal{P}$ and $x \in n(\pi)$, then let $\tau_x \supseteq \pi$ $\tau_x \in \mathrm{Int}(\mathcal{L}(\varphi, c) - \mathcal{P})$ with $\tau_x(c) = x$. Clearly $\bar{S}^{\pi}_{\varphi} = \bigcap_{x \in n(\pi)} \bar{S}^{\tau_x}_{\psi(c)}$ that is $\bar{S}^{\pi}_{\varphi} \in U^{n^{j+1}}_{i+1}$.

For $\exists x\, \psi(x) = \varphi$ we get the same result using

$$\bar{S}^{\pi}_{\varphi} = \bigcup_{x \in n(\pi)} \bar{S}^{\tau_x}_{\psi(c)}.$$

(b) can be proved easily by induction on i. □

(We will not use this part of the lemma.)

Definition. If $A \subseteq {}^{n}2$, then let

$$A^{\mathrm{odd}} = \{f \in A \mid |\{i \in n \mid f(i) = 1\}| \text{ is odd}\},$$
$$A^{\mathrm{even}} = A - A^{\mathrm{odd}}.$$

We want to prove that for a fixed φ if $n(\pi)$ is sufficiently large, then $({}^{n(\pi)}2)^{\mathrm{even}} \neq \bar{S}^{\pi}_{\varphi}$. If $\bar{S}^{\pi}_{\varphi}$ is a cylinder or the union of pairwise disjoint cylinders C with $\|C\| < n(\pi)$ (e.g. if φ is Σ_0), then obviously $|(\bar{S}^{\pi}_{\varphi})^{\mathrm{odd}}| = |(\bar{S}^{\pi}_{\varphi})^{\mathrm{even}}|$ which implies our assertion.

Generally $\bar{S}_\varphi^\pi$ cannot be represented in this way. Consider e.g. the Σ_1 formula $\exists x\, \mathcal{P}(x)$. Since $|\bar{S}_\varphi^\pi| = 2^{n(\pi)} - 1$, $|\bar{S}_\varphi^\pi|$ is odd so $\bar{S}_\varphi^\pi$ is not a union of pairwise disjoint nontrivial cylinders.

Although it is not always possible to represent $\bar{S}_\varphi^\pi$ as a union of disjoint nontrivial cylinders, we will approximate it by such a union. The existence of an approximation will imply

$$\|(\bar{S}_\varphi^\pi)^{\text{odd}}| - |(\bar{S}_\varphi^\pi)^{\text{even}}\| < 2^{n(\pi)-1}$$

that is

$$\bar{S}_\varphi^\pi \neq ({}^{n(\pi)}2)^{\text{even}}.$$

Actually we will prove that the two numbers are very close to each other (Theorem 1.2′).

In the proofs we will use the representation of $\bar{S}_\varphi^\pi$ given in Lemma 1.1 and we will prove the assertions concerning a $B \in U_i^s$ by induction on i.

In the following lemma we consider the simplest nontrivial case. (The next three Lemmas C.1 C.2 C.3 are of combinatorial character, dealing with the unions of sets of cylinders.)

Definition. T is a complete set of cylinders on n if T is a set of cylinders,

$$(C_1 \neq C_2 \wedge C_1, C_2 \in T) \to C_1 \cap C_2 = 0,$$

and $\bigcup T = {}^n2$.

If T is an arbitrary set of cylinders, then let $l(T) = i$ if $\|C\| = i$ for all $C \in T$. If there is no such $i \in \omega$, then let $l(T) = \infty$.

Lemma C.1. *Let X be a set of cylinders on A, $|A| = n$, $\|X\| \leq k$, $0 \leq l \leq n$. Then there exists a complete set of cylinders T, $T_1 \subseteq T$, $T_2 \subseteq T$ such that $l(T) = l$, and $\bigcup T_1 \subseteq \bigcup X$ and $\bigcup T_2 \subseteq {}^A2 - \bigcup X$ and*

$$2^{-n}\,|{}^A2 - \bigcup T_1 - \bigcup T_2| \leq (1 - 2^{-k})^{[l/k]}.$$

Proof. We will prove the lemma by induction on $[l/k]$ for a fixed k. For $[l/k] = 0$ the assertion is obvious. Suppose $[l/k] > 0$ and let $C_0 \in X$ be fixed. Let $J \subseteq A$ with $|J| = k$ $s(C_0) \supseteq J$.

For all $g \in {}^J2$ let us define a set of cylinders X_g on $A - J$ by

$$X_g = \{E \mid E \text{ is a cylinder on } A - J \text{ and}$$
$$\exists\, C \in X\ ((b(C) \restriction (A - J)) = b(E) \wedge b(C) \text{ and } g \text{ are compatible})\}.$$

Let us apply the inductive hypothesis with $A \to A - J$, $l \to l - k$, $X \to X_g$ for every fixed $g \in {}^J2$. We get a complete system of cylinders T^g on $A - J$ and T_1^g, $T_2^g \subseteq T^g$ with $l(T^g) = l - k$, $\bigcup T_1^g \subseteq \bigcup X_g$, $\bigcup T_2^g \subseteq {}^{A-J}2 - \bigcup X_g$ and $d_g = 2^{-(n-k)}\,|{}^{A-J}2 - \bigcup T_1^g - \bigcup T_2^g| \leq (1 - (1/2^k))^{[l/k-1]}$. (We may suppose that if $\bigcup X_g = {}^{A-J}2$, then $d_g = 0$.)

There exists at least one $g_0 \in {}^n2$ with $d_{g_0} = 0$ (namely any g with $b(c_0) \subseteq g$ can be g_0). Let

$$T = \{D \mid D \text{ is a cylinder on } A \wedge \exists\, g \in {}^J2, E \in T^g\ b(D) = g \cup b(E)\},$$
$$T_i = \{D \in T \mid \exists\, g \in {}^J2, E \in T_i^g\ b(D) = g \cup b(E)\}$$

for $i = 1, 2$. Obviously $D \in T \to \|D\| = k + (l-k) = l$ and $\bigcup T_1 \subseteq \bigcup X$, $\bigcup T_2 \subseteq {}^A2 - \bigcup X$

$$2^{-n}\,|{}^A2 - \bigcup T_1 - \bigcup T_2| \leqslant 2^{-k} \sum_{g \neq g_0} d_g$$
$$\leqslant 2^{-k}(2^k - 1)(1 - 2^{-k})^{[l/k]-1}. \quad \square$$

This lemma shows that the unions of cylinders (with small base) can be approximated by the unions of disjoint cylinders, and the same is true for the complement of the union. If we want to proceed further in the hierarchy taking unions and complements (instead of intersections), we have to consider in the next step a union of cylinders with larger base. If we want to guarantee that

$$2^{-n}\,|{}^A2 - \bigcup T_1 - \bigcup T_2| \leqslant n^{-j}$$

for some constant j (and we need such a good approximation, because we want to take unions where the number of the sets is a power of n), then supposing that $k = \|X\|$ is a constant we get that $l \geqslant c(k, j) \log n$ for some large constant $c(k, j)$.

Hence in the next step we would have to use the lemma with $k \geqslant c \log n$. However, if c is large, then $(1 - 2^{-k})^{[l/k]}$ is very close to 1 for any $l \leqslant n$. So we have to avoid taking unions of cylinders with large base.

We try to reduce the sizes of the cylinders. Suppose that X is a set of cylinder on n and $\|X\| \leqslant n^\varepsilon$. If $Y \subseteq n$, so that $|s(C) \cap Y| \leqslant t$ for all $C \in X$, and $T = \{D \mid$ "D is a cylinder on n" $\wedge\, s(D) = n - Y\}$, then for every fixed $D \in T$ $D \cap \bigcup X$ can be written in the form $D \cap \bigcup X_D$ where X_D is a set of cylinders on n with $|X_D| \leqslant |X|$ and $\|X_D\| \leqslant t$. Indeed let

$$X_D = \{E \mid \text{"}E \text{ is a cylinder on } n\text{"} \wedge \exists\, C \in X\, C \cap D \neq 0 \wedge b(E) = b(C) \restriction Y\}.$$

So 'inside' D, $\bigcup X$ can be represented as the union of cylinders with smaller base. The following lemma guarantees the existence of a 'large' set Y with the required properties.

Lemma C.2. $\forall\, \varepsilon > 0$, $0 < \varepsilon' < \varepsilon$, $j \in \omega\ \exists t$, $N_0 \in \omega\ \forall\, n > N_0$ *if* $H \subseteq P(n)$, $|H| < n^j$, $A \in H \to |A| < n^{\varepsilon'}$, *then there exists an* $Y \subseteq n$ *with* $|Y| \geqslant n^{1-\varepsilon}$ *and* $|A \cap Y| < t$ *for all* $A \in H$.

The number of Y's which are disallowed by one $A \in H$ is at most

$$\frac{n^{\varepsilon'}(n^{\varepsilon'} - 1) \cdots (n^{\varepsilon'} - t + 1)(n - t)(n - t - 1) \cdots (n - n^{1-\varepsilon} + 1)}{t!\,(n^{1-\varepsilon} - t)!}.$$

$\therefore$ Require this number to be less than

$$\frac{n(n-1)\cdots(n-n^{1-\varepsilon}+1)}{(n^{1-\varepsilon})!\,n^{j}}.$$

Given j we can pick t and large n such that this is true.

Proof. This is a well-known fact which can be proved easily by taking a random subset Y of n. $\square$

The situation described before the preceding lemma motivates the following definition.

Suppose that T is a complete set of cylinders on n and $f(D)$ is a set of cylinders for every $D \in T$ then let

$$\bigvee_D (T, f(D)) = \bigcup\{\bigcup\{C \cap D \mid C \in f(D)\} \mid D \in T\}.$$

Lemma C.1 and C.2 implies the following assertion which roughly speaking says that if we have no more than n^j unions of small ($\|C\|$ is constant) cylinders, then there is a complete set of cylinder R with $l(R) \leqslant n - n^{1-\varepsilon}$ so that inside each $D \in R$ the complementers of the unions can be approximated by the unions of small cylinders.

Lemma C.3. *$\forall\, \varepsilon > 0$, k, j, $r \in \omega$ $\exists\, t \in \omega$ such that if n is sufficiently large and $\mathcal{B}$ is a set of systems of cylinders on n with $|\mathcal{B}| = n^j$, $X \in \mathcal{B} \to \|X\| \leqslant k$, then there exists a complete set of cylinders R with*

(a) $l(R) = [n - n^{1-\varepsilon}]$,

(b) *for all $X \in \mathcal{B}$, $E \in R$ there exists a set of cylinders $Z(X, E)$ such that $\|Z(X, E)\| \leqslant t$ and*

$$2^{-n}\left|({}^{n}2 - \bigcup X) \,\Delta\, \bigvee_E (R, Z(X, E))\right| \leqslant \frac{1}{n^r}.$$

Proof. Let us apply Lemma C.1 for each $X \in \mathcal{B}$ with $l = [c \cdot \log n]$ where c is sufficiently large compared to ε, k, j, r but does not depend on n. Let T^X, T_1^X, T_2^X be the complete set of cylinders and its subsets guaranteed by the lemma. Let $H = \{s(C) \mid \exists\, X \in \mathcal{B}\ C \in T^X\}$. Clearly

$$|H| \leqslant n^j\,|T^X| \leqslant n^j\,2^{c \cdot \log n} \leqslant n^{j+c}$$

and $A \in H$ implies $|A| \leqslant c \cdot \log n \leqslant n^{\varepsilon/2}$ if n is sufficiently large. ($|T| \leqslant 2^{l(T)}$ for any complete T.)

Applying Lemma C.2. We get a $Y \subseteq n$ and a $t \in \omega$ (it does not depend on n) with $|Y| \geqslant [n^{1-\varepsilon} + 1]$ and $|A \cap Y| \leqslant t$ for all $A \in H$.

Let

$$R = \{E \mid \text{“}E \text{ is a cylinder on } n\text{”} \wedge s(E) = n - Y\},$$

$$Z_i(X, E) = \{C \mid \text{“}C \text{ is a cylinder on } n\text{”} \wedge s(C) \subseteq Y \wedge \exists\, D \in T_i^X \; b(D) \restriction Y = b(C) \wedge D \cap E \neq 0\}, \quad i = 1, 2,$$

$$Z(X, E) = Z_2(X, E).$$

(a) follows from $|Y| = [n^{1-\varepsilon} + 1]$, $\|Z(X, E)\| \leq t$ is a consequence of $\forall A \in H\, |A \cap Y| \leq t$.

$$\bigvee_E (R, Z(X, E)) = \bigcup T_2^X$$

so according to Lemma C.1

$$2^{-n} \left| ({}^n 2 - \bigcup X) \mathbin{\Delta} \bigvee_E (R, Z(X, E)) \right| \leq (1 - 2^{-k})^{[(c \cdot \log n)/k]} \leq \frac{1}{n^r}$$

if c is sufficiently large compared to r and k. □

We will prove a stronger version of this lemma. In fact we may substitute $1/n^r$ by $2^{-n/(\log n)^v}$ in the last inequality of the lemma where v is sufficiently large compared to ε, k and j but does not depend on n. For our results concerning Σ_1^1 formulae we need this stronger version so in the following proofs we use this form but we will prove it only later (Lemma 5.7). Actually the last section is devoted to this proof. So we suggest the reader to accept Lemma 5.7 without its proof and read the rest of this section (we will indicate what can be proved using the weaker version).

Theorem 1.2. *For all $\varepsilon > 0$, $i, j \in \omega$ if n is large enough, then*

$$A \in U_i^{n^j} \rightarrow \frac{1}{2^n} \big| |A^{\text{even}}| - |A^{\text{odd}}| \big| \leq 2^{-n^{1-\varepsilon}}.$$

Moreover there exists a complete set of cylinders T and a $T' \subseteq T$ such that

$$l(T) \leq n - n^{1-\varepsilon} \quad \textit{and} \quad 2^{-n} \left|\bigcup T' \mathbin{\Delta} A\right| \leq 2^{-n^{1-\varepsilon}}.$$

The proof of this theorem is based on Lemma 5.7. If we use (in the same way) Lemma C.3, then we get that for all $r \in \omega$ the assertion of the theorem holds if we substitute everywhere $2^{-n^{1-\varepsilon}}$ for n^{-r}. The same applies for Theorem 1.4, Theorem 1.2′ as well.

First we prove a more general assertion.

Lemma 1.3. *For all $\varepsilon > 0$, $i, j, s \in \omega$ $\exists\, t \in \omega$ such that if n is large enough and $\mathcal{A} \subseteq U_i^{n^j}(n)$, $|\mathcal{A}| \leq n^s$, then there exists a complete set of cylinders T with $l(T) \leq n - n^{1-\varepsilon}$ and a set of cylinders $Y(A, D)$ for all $A \in \mathcal{A}$, $D \in T$ such that $\|Y(A, D)\| \leq t$ and for all $A \in \mathcal{A}$*

$$2^{-n} \left| A \mathbin{\Delta} \bigvee_D (T, Y(A, D)) \right| \leq 2^{-n^{1-\varepsilon}}. \tag{1.1}$$

Proof. We will use Lemma 5.7 in the proof of this lemma. Using the weaker version, Lemma C.3, we may prove that for all $r \in \omega$ the assertion of the lemma holds if we substitute (1.1) for

$$2^{-n}\left|A \,\Delta \bigvee_D (T, Y(A, D))\right| \leqslant \frac{1}{n^r}$$

(t, T, $Y(A, D)$ depend on r as well).

We will prove the lemma by induction on i.

For $i=0$ the lemma is trivial. Suppose $i>0$ and let $\varepsilon>0$, j, $s \in \omega$, $\mathscr{A} \subseteq U_i^{n^i}$, $|\mathscr{A}| \leqslant n^s$.

Let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ where $\mathscr{A}_1 \cap \mathscr{A}_2 = 0$ and

$$A \in \mathscr{A}_1 \to A = \bigcup G_A, \quad G_A \subseteq U_{i-1}^{n^i}, |G_A| \leqslant n^j,$$

$$A \in \mathscr{A}_2 \to A = \bigcap G_A, \quad G_A \subseteq U_{i-1}^{n^i}, |G_A| \leqslant n^j.$$

Let

$$\mathscr{G} = \Big(\bigcup_{A \in \mathscr{A}_1} G_A\Big) \cup \{{}^n2 - F \mid \exists\, A \in \mathscr{A}_2\; F \in G_A\}.$$

Let us apply the inductive hypothesis with $i \to i-1$, $\varepsilon \to \varepsilon' < \frac{1}{4}\varepsilon$, $\mathscr{A} \to \mathscr{G}$, $s \to s+j$. Then we have a complete set of cylinders T' with $l(T') \leqslant n - n^{1-\varepsilon'}$ and a set of cylinders $Y'(B, D)$ for all $B \in \mathscr{G}$, $D \in T'$ such that $\|Y'(B, D)\| \leqslant t'$ and for all $B \in \mathscr{G}$

$$2^{-n}\left|B \,\Delta \bigvee_D (T', Y'(B, D))\right| \leqslant 2^{-n^{1-\varepsilon'}}. \tag{1.2}$$

We will define T as a refinement of T' (that is $\forall D \in T' \;\; D = \bigcup\{C \mid C \in T \wedge C \subseteq D\}$); and if $D \in T$, $A \in \mathscr{A}_1$, then we will put

$$Y(A, D) = \bigcup_{B \in G_A} Y'(B, D'),$$

where $D' \supset D$, $D' \in T'$. (1.2) implies that for all $A \in \mathscr{A}_1$ (1.1) holds, since $n^j 2^{-n^{1-\varepsilon'}} < 2^{-n^{1-\varepsilon}}$ if n is sufficiently large.

Thus we have to define T and $Y(A, D)$ for all $A \in \mathscr{A}_2$. Let $D \in T'$ fixed. We may suppose that $s(C) \subseteq n - s(D)$ for all $C \in Y'(B, D)$, $B \in \mathscr{G}$.

Let $\quad Y''(A, D) = \bigcup_{F \in G_A} Y'({}^n2 - F, D).$

First note that the complement of the set $\bigvee_D (T', Y''(A, D))$ is a good approximation of the set A. Indeed:

$$\begin{aligned}
&2^{-n}\left|A \,\Delta \Big({}^n2 - \bigvee_D (T', Y''(A, D))\Big)\right| \\
&\quad = 2^{-n}\left|A \,\Delta \Big({}^n2 - \bigcup_{F \in G_A} \bigvee_D (T', Y'({}^n2 - F, D))\Big)\right| \\
&\quad = 2^{-n}\left|({}^n2 - A) \,\Delta \bigcup_{F \in G_A} \bigvee_D (T', Y({}^n2 - F, D))\right| \\
&\quad \leqslant 2^{-n} \sum_{F \in G_A} \left|({}^n2 - F) \,\Delta \bigvee_D (T', Y'({}^n2 - F, D))\right| \leqslant n^j \cdot 2^{-n^{1-\varepsilon'}} \\
&\quad \leqslant 2^{-n^{1-2\varepsilon'}}
\end{aligned} \tag{1.3}$$

if n is sufficiently large. The inequality before the last one is a consequence of (1.2) and $|G_A| \leq n^j$.

Thus we have to approximate the set $2^n - \bigvee_D (T', Y''(A, D))$ with a set of the type $\bigvee_D (T, Y(A, D))$ where T is a refinement of T' and $\|Y(A, D)\| \leq t'$.

For a fixed $D \in T'$

$$D \cap \left(2^n - \bigvee_{\bar{D}} (T', Y''(A, \bar{D}))\right) = D - \bigcup \{C \cap D \mid C \in Y''(A, D)\}. \tag{1.4}$$

If we restrict all of the cylinders in the last expression to the set $n - s(D) = J_D$ we get

$$^{(J_D)}2 - \bigcup Y_{A,D} \quad \text{where} \quad Y_{A,D} = \{C \restriction J_D \mid C \in Y''(A, D)\}$$

where $C \restriction J_D$ is a cylinder on J_D with $b(C) = b(C \restriction J_D)$.

Now we may apply Lemma 5.7 with $\varepsilon \to \varepsilon'$, $k \to t'$, $j \to j$, $\mathcal{B} = \{Y_{A,D} \mid A \in \mathcal{A}_2\}$, $n \to n - \|D\|$ for a fixed D, (here we identify $n - \|D\|$ and $n - s(D) = J_D$). Let R_D be the complete set of cylinders, and $Z_D(X, E)$ the function guaranteed by the lemma. Let $q = n - [n - n^{1-\varepsilon'}] = |J_D|$, then we have for all $A \in \mathcal{A}_2$

$$2^{-q} \left| (^{(J_D)}2 - \bigcup Y_{A,D}) \,\Delta \bigvee_E (R_D, Z(Y_{A,D}, E)) \right| < 2^{-q/(\log q)^{\nu}} \tag{1.5}$$

and $\|Z(Y_{A,D}, E\| \leq t$ for all $A \in \mathcal{A}_2$, $E \in R_D$.

Now we define the complete set of cylinders T on n by

$$K \in T \leftrightarrow (K \text{ is a cylinder on } n \wedge \exists\, D \in T', E \in R_D\; b(T) = b(D) \cup b(E))$$

and put

$$Y(A, K) = \{C \mid C \text{ is a cylinder on } n \wedge \exists\, D \in T', E \in R_D, C' \in Z(Y_{A,D}, E),$$
$$b(K) = b(D) \cup b(E) \wedge b(C) = b(C')\}.$$

(1.4) and (1.5) implies that for all $D \in T'$, $A \in \mathcal{A}_2$

$$2^{-q} \left| \left(D \cap \left(2^n - \bigvee_{\bar{D}} (T', Y''(A, \bar{D}))\right)\right) \Delta \left(D \cap \bigvee_K (T, Y(A, K))\right) \right| \leq 2^{-q/(\log q)^{\nu}}$$

and taking the union of the corresponding sets for all $D \in T'$

$$2^{-n} \left| \left(2^n - \bigvee_D (T', Y''(A, D))\right) \Delta \bigvee_K (T, Y''(A, K)) \right| \leq 2^{-q/(\log q)^{\nu}} \leq 2^{-n^{1-2\varepsilon'}}.$$

This inequality and (1.3) implies the assertion of our lemma for all $A \in \mathcal{A}_2$. $\square$

Now we prove the second assertion of Theorem 1.2.

Let us apply Lemma 1.3 with $\mathcal{A} \to \{A\}$, $\varepsilon \to \varepsilon' < \frac{1}{3}\varepsilon$ let $\bar{T}$ be the complete set of cylinders, $Y(D)$ the function guaranteed by the lemma.

Then we have

$$2^{-n} \left| A \,\Delta \bigvee_D (\bar{T}, Y(D)) \right| \leq 2^{-n^{1-\varepsilon'}} \quad \text{and} \quad \|Y(D)\| \leq t. \tag{1.6}$$

Let $D \in T$ fixed. Then restricting the cylinders of $Y(D)$ to $n - s(D) = J_D$ we get a set of cylinders Y_D on J_D. Now we may apply Lemma C.1 with $A \to J_D$, $X \to Y(D)$, $k \to t$, $l = [\frac{1}{2}q]$ where $q = |J_D| = n - [n - n^{1-\varepsilon'}]$. Let T^D, T_1^D, T_2^D be the sets guaranteed by the lemma. We have $\bigcup T_1^D \subseteq \bigcup Y_D$, $\bigcup T_2^D \subseteq {}^{J_D}2 - \bigcup Y_D$ and

$$2^{-q}\, |{}^{J_D}2 - \bigcup T_1^D - \bigcup T_2^D| \leq \left(1 - \frac{1}{2^t}\right)^{[[\frac{1}{2}q]/t]} \leq c_1(t) e^{-c_2(t)\cdot q}$$
$$\leq 2^{-n^{1-2\varepsilon'}}$$

that is

$$2^{-q}\, |\bigcup Y_D \,\Delta \bigcup T_1^D| \leq 2^{-n^{1-2\varepsilon'}}. \tag{1.7}$$

Let

$$T = \{K \mid K \text{ is a cylinder on } n\ \exists\, D \in \bar{T},\ C \in T^D\ b(K) = b(D) \cup b(C)\},$$

and we define T' similarly with $C \in T_1^D$ substituted for $C \in T^D$. Clearly T is a complete set of cylinders with

$$l(T) = l(\bar{T}) + [\tfrac{1}{2}q] \leq n - n^{1-2\varepsilon'}.$$

(1.6) and (1.7) implies that

$$2^{-n}\, |\bigcup T' \,\Delta\, A| \leq 2^{-n^{1-3\varepsilon'}}. \quad \square$$

The first assertion of the theorem is a consequence of the second one. Indeed for any cylinder C with $\|C\| < n$ $|C^{\text{even}}| = |C^{\text{odd}}|$ and so $|(\bigcup T')^{\text{even}}| = |(\bigcup T')^{\text{odd}}|$ with the T' given in the theorem. $\square$

Theorem 1.4. *For all φ, $\varepsilon > 0$ if n is large enough and π is an interpretation of* $\mathcal{L}(\varphi) - \mathcal{P}$ with $n = n(\pi)$, *then there exists a complete set of cylinders T and a $T' \subseteq T$ such that $l(T) \leq n - n^{1-\varepsilon}$ and*

$$2^{-n}\, |\bar{S}_\varphi^\pi \,\Delta \bigcup T'| \ll 2^{-n^{1-\varepsilon}}.$$

Proof. Lemma 1.1 implies that there exist $i, j \in \omega$ (depending only on φ) such that for all π we have $\bar{S}_\varphi^\pi \in U_i^{n^j}(n(\pi))$. Therefore the first assertion of Theorem 1.2 with $A = \bar{S}_\varphi^\pi$ implies Theorem 1.4 and the second assertion of the same theorem implies the following Theorem 1.2′.

Theorem 1.2′. *If φ is a first-order sentence and $\varepsilon > 0$, then for all but finitely many interpretations π of $\mathcal{L}(\varphi) - \mathcal{P}$ we have*

$$2^{-n}\, \big||(\bar{S}_\varphi^\pi)^{\text{even}}| - |(\bar{S}_\varphi^\pi)^{\text{odd}}|\big| \leq 2^{-n^{1-\varepsilon}}.$$

Corollary 1.5. *There exists a recursive function f, whose recursivity is a theorem of Peano Arithmetic, such that the following statement is a theorem of Peano Arithmetic: if φ is a first-order sentence and $\mathcal{P}$ is a unary relation symbol of $\mathcal{L}$; then for any interpretation π of $\mathcal{L}(\varphi) - \mathcal{P}$ with $n(\pi) > f(\varphi)$ there exist $\pi_1, \pi_2 \in$* Int$(\mathcal{L}(\varphi, \pi))$ *such* *that*

$$\pi_1 \supseteq \pi, \pi_2 \supseteq \pi, \quad (\pi_1 \vDash \varphi) \leftrightarrow (\pi_2 \vDash \varphi)$$

but $|\pi_1(\mathcal{P})|$ *is even and* $|\pi_2(\mathcal{P})|$ *is odd.*

Proof. All of the theorems of this section are theorems of Peano Arithmetic, and all of the functions whose existence are proved are recursive. □

Section 2

Theorem 2.1. *Let $\mathcal{P}$ be a k-ary relation symbol, $k > 1$; then:*

(A) "$|\bar{\mathcal{P}}|$ *is even" is a $\Delta^1_{1,k}$ property of $\mathcal{P}$ which is not weakly $\Sigma^1_{1,k-1}$ or $\Pi^1_{1,k-1}$.*

(B) $\forall x_1, \ldots, x_{k-1}$ $(|\{x_0 \mid \langle x_0, x_1, \ldots, x_{k-1}\rangle \in \bar{\mathcal{P}}\}|$ *is even*$) \equiv S(\bar{\mathcal{P}})$ *is a $\Pi^1_{1,1}$ property over the first-order theory of linear ordering, and it is a $\Pi^1_{1,2}$ property, which is not weakly $\Sigma^1_{1,k-1}$.*

(C) *If $\mathcal{P}$ is a k-ary relation symbol, $k > 1$, then there exists a $\Pi^1_{1,1}$ property of $\mathcal{P}$ which is not weakly $\Sigma^1_{1,k-1}$.*

The essential part of the proof is the proof of the negative part of (A) ("$|\mathcal{P}|$ is even" is not weakly $\Sigma^1_{1,k-1}$ or $\Pi^1_{1,k-1}$). The positive parts of (A) and (B) are trivial and the negative part of (B) is a consequence of the negative part of (A). (We will explain the position of (C) later.)

The idea of the proof is the following.

If "$|\bar{\mathcal{P}}|$ is even" is a $\Sigma^1_{1,k-1}$ property, then it is equivalent with $\exists^1_{k-1}X_1 \cdots \exists^1_{k-1}X_j\, \varphi(X_1, \ldots, X_j, \mathcal{P})$ where φ is first order, so for any fixed interpretations of $\mathcal{P}$ there are $k-1$-ary relations $A_1, \ldots, A_j$ on $n(\pi)$ with $\varphi(A_1, \ldots, A_j, \mathcal{P})$. However, if we fix now the $k-1$-ary relations $A_1, \ldots, A_j$, then according to Theorem 1.2′ there are at most $2^{-(n^k)^{1-\epsilon}} \cdot 2^{n^k}\bar{\mathcal{P}}$ with $\varphi(A_1, \ldots, A_j, \mathcal{P})$. Multiplying it by the number of possible $A_1, \ldots, A_j$'s, we get less than $\frac{1}{2}2^{n^k}$, the number of $|\bar{\mathcal{P}}|$'s with "$|\bar{\mathcal{P}}|$ is even".

Lemma 2.2. *If $\mathcal{A}$ is a unary relation symbol, then "$|\bar{\mathcal{A}}|$ is even" is a $\Delta^1_{1,1}$ property of $\mathcal{A}$ over the first-order theory of linear ordering and it is a $\Delta^1_{1,2}$ property as well.*

Proof. Let LO be the first-order theory of linear ordering, with the relation symbols $\leqslant$, $=$. Let "x covers y" be the abbreviation for

$$x > y \wedge \forall z\ \neg(x > z > y),$$

$$G(x) \quad \text{for } \forall z\, z \leqslant x,$$

$$L(x) \quad \text{for } \forall z\, z \geqslant x.$$

We will define the following second-order formula:

$$\exists^1_1 B\Big[\Big(\forall x\, B(x) \equiv \sum_{x \leqslant y} \mathcal{A}(y) (\operatorname{mod} 2)\Big) \wedge \forall z (G(z) \rightarrow \neg B(z))\Big].$$

Indeed, let

$$\Phi = \exists_1^1 B[\varphi(B) \wedge (\forall z\, G(z) \rightarrow \neg B(z))]$$

where

$$\varphi(B) = \forall\, x, y\, [(x \text{ covers } y) \rightarrow ((B(x) \leftrightarrow B(y)) \leftrightarrow \neg \mathscr{A}(x))]$$
$$\wedge \forall w (L(w) \rightarrow (B(w) \leftrightarrow \mathscr{A}(w)).$$

Clearly for any $\pi \in \mathrm{Int}(\mathscr{L}(\mathscr{A}, \mathrm{LO}))$, $\pi \models \mathrm{LO}$ we have "$\pi(\mathscr{A})$ is even" $\leftrightarrow \pi \models \Phi$, that is $\mathscr{A}$ is even is a $\Sigma_{1,1}^1$ property of $\mathscr{A}$ over LO. Using

$$\phi' = \forall_1^1 B[\varphi(B) \rightarrow (\forall z\, G(z) \rightarrow \neg B(z))]$$

instead of ϕ we get that it is a $\Pi_{1,1}^1$ property over LO.

Let us write Φ in the form $\exists_1^1 B\, \psi(B)$ where ψ is first-order. Put

$$\Psi = \exists_2^1 C\, \exists_1^1 B \text{ ("}C\text{ is a linear ordering of the universe"} \wedge \psi_C'(B)),$$

where ψ_C is obtained by substituting $\leqslant$ for C in Ψ. Clearly if $\pi \in \mathrm{Int}(\mathscr{L}(\mathscr{A}))$, then

$$\text{"}\pi(\mathscr{A}) \text{ is even"} \leftrightarrow \pi \models \Psi,$$

that is "$\tilde{\mathscr{A}}$ is even" is $\Sigma_{1,2}^1$. We get that it is $\Pi_{1,2}^1$ using

$$\Psi' = \forall_2^1 C\, \forall_1^1 B \text{ ("}C\text{ is a linear ordering of the universe"} \rightarrow \psi_C'(B)),$$

where $\phi' = \forall_1^1 B \psi'(B)$.

Proof of (A). Let $\mathscr{D}$ be a binary relation symbol. Obviously there exists a first-order formula $\psi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1})$ of $\mathscr{L}(\mathscr{D})$ such that for all $\pi \in \mathrm{Int}(\mathscr{L}(\mathscr{D}))$ we have $\pi \models \mu$, where μ is a first-order sentence of $\mathscr{L}(\mathscr{D})$ and

$$\pi \models \mu \leftrightarrow \text{"if } \pi(\mathscr{D}) \text{ is a linear ordering of } n(\pi) \text{, then}$$
$$\{\langle\langle x_0, \ldots, x_{k-1}\rangle, \langle y_0, \ldots, y_{k-1}\rangle\rangle \mid \psi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1})\}$$
$$\text{is a linear ordering of } \bigtimes_k n(\pi)\text{"}.$$

Let

$$\bar{\Psi} = \exists_2^1 \mathscr{D}\, \exists_k^1 B \text{ ("}\mathscr{D}\text{ is a linear ordering of the universe"} \wedge \mu \wedge \bar{\psi}_{\mathscr{D}}(B))$$

where $\bar{\psi}_{\mathscr{D}}(B)$ is obtained by performing the following type of substitutions in φ:

$$\forall x \rightarrow \forall\, x_0, \ldots, x_{k-1},$$
$$x \leqslant y \rightarrow \psi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}),$$
$$x = y \rightarrow x_0 = y_0 \wedge \cdots \wedge x_{k-1} = y_{k-1},$$
$$\mathscr{B}(x) \rightarrow \mathscr{B}(x_0, \ldots, x_{k-1}),$$
$$\mathscr{A}(x) \rightarrow \mathscr{A}(x_0, \ldots, x_{k-1}).$$

We have the same situation as in the proof of Lemma 2.2 so

"$\pi(\mathscr{A})$ is even" $\leftrightarrow \pi \models \bar{\Psi}$, that is "$\mathscr{A}$ is even"

is $\Sigma_{1,k}^1$ and defining the corresponding $\Pi_{1,k}^1$ formula $\bar{\Psi}'$ we get that it is $\Pi_{1,k}^1$.

Suppose that "$|\bar{\mathscr{P}}|$ is even" is weakly $\Sigma^1_{1,k-1}$. Then there exists a $\Sigma^1_{1,k-1}$ formula

$$\Psi = \exists^1_{k-1}X_1 \cdots \exists^1_{k-1}X_j\, \psi(X_1, \ldots, X_j)$$

where ψ is a first-order formula such that for infinitely many $\pi \in \mathrm{Int}(\mathscr{L}(\Psi) - \mathscr{P})$, we have

$$\forall \pi' \in \mathrm{Int}(\mathscr{L}(\Psi))\, (\pi' \supseteq \pi \rightarrow (\text{“}|\pi'(\mathscr{P})| \text{ is even”} \leftrightarrow \pi' \vDash \Psi)). \tag{2.1}$$

Let us fix a $\pi \in \mathrm{Int}(\mathscr{L}(\Psi) - \mathscr{P})$ with this property. Later we will suppose that $n(\pi)$ is sufficiently large.

Set $I = \{\pi' \in \mathrm{Int}(\mathscr{L}(\Psi)) \mid \pi' \supseteq \pi\}$. For all $A_1, \ldots, A_j \subseteq \times_{k-1} n(\pi)$ let

$$M(A_1, \ldots, A_j) = \left\{ P \subseteq \underset{k}{\times}\, n(\pi) \mid \exists\, \pi' \in I\, \pi'(\mathscr{P}) = P \wedge \pi' \vDash \psi(A_1, \ldots, A_j) \right\}. \tag{2.2}$$

$P \in M(A_1, \ldots, A_k)$ and (2.1) implies that $|P|$ is even. Now, applying Theorem 2.1′, we want to prove that for any fixed $A_1, \ldots A_j$ the set $M(A_1, \ldots, A_j)$ is small. To make the theorem applicable let $\mathscr{P}'$, $\mathscr{U}$ be unary and $\mathscr{R}_1, \ldots, \mathscr{R}_j$ $k-1$-ary relation symbols and f a k-ary function symbol which does not occur in Ψ. Let ψ' be the formula obtained from ψ by performing the following type of substitutions

$$\forall x(\cdots) \rightarrow \forall x\, \mathscr{U}(x) \rightarrow (\cdots),$$
$$\exists x(\cdots) \rightarrow \exists x\, \mathscr{U}(x) \wedge (\cdots),$$
$$X_i(x_0, \ldots, x_{k-2}) \rightarrow \mathscr{R}_i(x_0, \ldots, x_{k-2}),$$
$$\mathscr{P}(x_0, \ldots, x_{k-1}) \rightarrow \mathscr{P}'(f(x_0, \ldots, x_{k-1})).$$

Let us identify $\times_k n(\pi)$ and $(n(\pi))^k$, let τ be the following interpretation of $\mathscr{L}(\psi') - \mathscr{P}'$: $n(\tau) = (n(\pi))^k$, $\tau(\mathscr{U}) = n(\pi)$, for all relation or function symbols $\mathscr{T}$ which occur both in $\mathscr{L}(\Psi) - \mathscr{P}$ and $\mathscr{L}(\psi') - \mathscr{P}'$ let $\pi(\mathscr{T}) = \tau(\mathscr{T})$, $\tau(\mathscr{R}_i \vDash A_i$ $i = 1, \ldots, j$, $\tau(f)$ is defined by

$$\tau(f)(x_0, \ldots, x_{k-1}) = \langle x_0, \ldots, x_{k-1} \rangle.$$

(2.2) implies that $M(A_1, \ldots, A_k) = S^{\tau}_{\psi'}$. Hence, by Theorem 1.2′ we have that:

$$2^{-n^k} |M(A_1, \ldots, A_j)| \leqslant 2^{-(n^k)^{1-\varepsilon}} = 2^{-n^{k-k\varepsilon}}.$$

Every $P \subseteq \prod_k n(\pi)$ is contained in some $M(A_1, \ldots, A_j)$ if $|P|$ is even, so we have

$$\tfrac{1}{2} = 2^{-n^k} \left| \bigcup \left\{ M(A_1, \ldots, A_j) \mid A_i \subseteq \prod_{k-1} n \right\} \right|$$
$$\leqslant 2^{-n^k} \sum \{ |M(A_1, \ldots, A_j)| \mid \cdots \} \leqslant 2^{jn^{k-1}} 2^{-n^{k-k\varepsilon}} < \tfrac{1}{2}$$

a contradiction. That is "$|\bar{\mathscr{P}}|$ is even" is not weakly $\Sigma^1_{1,k-1}$. The same is true for "$\mathscr{P}$ is odd" that is "$\bar{\mathscr{P}}$ is even" is not $\Pi^1_{1,k-1}$. □

Proof of (B). The first part of the assertion is an immediate consequence of Lemma 2.2. We have to prove that $S(\mathscr{P})$ is not weakly $\Sigma^1_{1,k-1}$. Suppose it is weakly $\Sigma^1_{1,k-1}$ and Ψ is a $\Sigma^1_{1,k-1}$ formula such that for infinitely many $\pi \in \text{Int}(\mathscr{L}(\Psi) - \mathscr{P})$ we have

$$\forall\, \pi' \in \text{Int}(\mathscr{L}(\Psi))\ (\pi' \supseteq \pi \rightarrow (\pi' \vDash \psi \leftrightarrow S(\pi(\mathscr{P})))). \tag{2.3}$$

We claim that our assumption implies that "$|\bar{\mathscr{P}}|$ is even" is weakly $\Sigma^1_{1,k-1}$ in contradiction to (A).

Indeed, using Ψ we can give a $\Sigma^1_{1,k-1}$ formula Ψ_1 of $\mathscr{L}(\Psi, \leqslant)$ (where $\leqslant$ is a binary relation symbol which does not occur in Ψ) with

$$\text{"}|\pi'(\bar{\mathscr{P}})| \text{ is even"} \leftrightarrow \pi' \vDash \Psi_1 \quad \text{for all } \pi' \in \text{Int}(\mathscr{L}(\Psi_1)),\ \pi' \supseteq \pi \tag{2.4}$$

where π satisfies (2.3). Ψ_1 will be equivalent to the following statement "$\leqslant$ is a linear ordering of the universe"$\wedge \exists^1_{k-1} Y$ (if we define the relation Q by

$$Q(x_0, \ldots, x_{k-1}) = \begin{cases} Y(x_1, \ldots, x_{k-1}) & \text{if } x_0 \text{ is the greatest element,} \\ \mathscr{P}(x_0, \ldots, x_{k-1}) & \text{otherwise,} \end{cases}$$

then $S(Q) \wedge |Y| \equiv |\{\langle x_1, \ldots, x_k\rangle |\ \mathscr{P}(\mathbb{1}, x_1, \ldots, x_k) \text{ where } \mathbb{1} \text{ is the greatest element}\}|$ (mod 2)). The relation Q can be given by a first-order formula so our indirect hypothesis implies that $S(Q)$ can be expressed by a $\Sigma^1_{1,k-1}$ formula; (A) implies that the assertion about the parities can be expressed by a $\Sigma^1_{1,k-1}$ formula as well. So we have that Ψ_1 is a $\Sigma^1_{1,k-1}$ formula, and its definition implies that Ψ_1 satisfies (2.4). □

(B) of Theorem 2.1 is somewhat unsatisfactory because $S(\bar{\mathscr{P}})$ is $\Pi^1_{1,1}$ only over a nonempty theory. It is clear that $S(\bar{\mathscr{P}})$ cannot be expressed using only $\bar{\mathscr{P}}$ and $=$. The problem is that to define the parity of a set we need a linear ordering of the universe. So to prove (C) we will divide the universe into two sets and the $\Pi^1_{1,1}$ property will be the following: $\bar{\mathscr{P}}$ restricted to the first set satisfies $S(\cdot)$ and $\bar{\mathscr{P}}$ restricted to the second set codes a linear ordering of the first set. (The division and the linear ordering of the first set will be first-order definable from $\bar{\mathscr{P}}$.)

We will guarantee at the same time that the first set is large, the number of its element will be greater than $\frac{1}{3}$|universe|.

The next difficulty is whether the already proven parts of Theorem 2.1 remain valid if we consider $S(\bar{\mathscr{P}})$ with a $\bar{\mathscr{P}}$ restricted to a large part of the universe.

We will prove that if we have a structure $M = \langle A, R_1, \ldots\rangle$ and $0 \neq W_0 \subseteq A$ $|W| \geqslant \frac{1}{3}|A|$ then there is a structure $M' = \langle W_0, P_1, \ldots\rangle$ (the number and arities of the relations can be larger than in M), so that there is a first-order definable (in M') subset B of $W_0 \times \{\bar{c}_0, \bar{c}_1, \bar{c}_2\}$ where $\bar{c}_0, \bar{c}_1, \bar{c}_2$ are constants in M' with $|B| = |A|$ and first-order definable (in M') relations and functions $R'_1, \ldots$ on B such that the structures $M = \langle A, R_1, \ldots\rangle$ and $\langle B, R'_1, \ldots\rangle$ are isomorphic. (Of course instead of $|W_0| \geqslant \frac{1}{3}|A|$ we may suppose $|W_0| \geqslant c\,|A|$ for any positive constant c.) This isomorphism will guarantee us that we do not have to consider $\bar{\mathscr{P}}$ (and $S(\bar{\mathscr{P}})$)

restricted to a subset (W_0) but, we may take an other structure, with more relations ($M' = \langle W_0, P_1, \ldots \rangle$ where $\bar{\mathscr{P}}$ and $S(\bar{\mathscr{P}})$ is not restricted but the $\Sigma^1_{1,k}$, $\Pi^1_{1,k}$ definability means the same.

The following definitions and lemmas describe this situation. We will use these results in Section 3 as well. Here we do not suppose that $n(\pi)$ is a natural number.

Definition. Suppose that G is a finite subset of $\mathrm{Sym}(\mathscr{L})$.

If $\mathscr{R} \in G$ is a k-ary relation (resp. function) symbol, then let $\mathscr{R}'$ be a new $2k$-ary (resp. $2k+1$-ary) relation (resp. function) symbol. Let c_0, c_1, c_2 be constant symbols and $\mathscr{B}$ be a binary relation symbol different from all of the $\mathscr{R}$, $\mathscr{R}'$-s. Let

$$\bar{G} = \{\mathscr{R}'\}_{\mathscr{R} \in G} \cup \{c_0, c_1, c_2\} \cup \{\mathscr{B}\}.$$

If π is an interpretation of $\mathscr{L}(\bar{G})$ let us define an interpretation $\lambda^{(\pi)}$ of $\mathscr{L}(G)$ as follows

$$n(\lambda) = \{\langle \bar{c}_i, x \rangle \mid x \in n(\pi), \pi \models \mathscr{B}(c_i, x), i = 0, 1, 2\}.$$

If $\mathscr{R}$ is a k-ary relation symbol let

$$\lambda(\mathscr{R})(\langle \bar{c}_{i_0}, x_0 \rangle, \ldots, \langle \bar{c}_{i_{k-1}}, x_{k-1} \rangle) \leftrightarrow \pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1})$$

and if $\mathscr{R}$ is a k-ary function symbol

$$\begin{aligned}&\lambda(\mathscr{R})(\langle \bar{c}_{i_0}, x_0 \rangle, \ldots, \langle \bar{c}_{i_{k-1}}, x_{k-1} \rangle)\\ &\quad = \langle \pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1}, c_0), \pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1}, c_1) \rangle\end{aligned}$$

($\lambda(\mathscr{R})$ is not necessarily defined for all π but we will consider only π's where it is defined). If it is necessary we write $\lambda^{(\pi)}$.

Claim 2.3. *If $\varphi(X)$ is a $\Sigma^1_{1,m}$, $\Pi^1_{1,m}$ resp. first-order formula of $\mathscr{L}(G)$ where X is a second-order p-ary free variable, then there exists a $\bar{\varphi}(X)$ $\Sigma^1_{1,m}$, $\Pi^1_{1,m}$ resp. first-order formula of $\mathscr{L}(G)$ with the p-ary free variable X so that for all $Z \subseteq \times_p n(\pi)$*

$$\lambda \models \varphi(Z_{c_0}) \leftrightarrow \pi \models \bar{\varphi}(Z) \tag{J.1}$$

where $Z_{c_0} = \{\langle c_0, x_0 \rangle, \ldots, \langle c_0, x_{p-1} \rangle \mid \langle x_0, \ldots, x_{p-1} \rangle \in Z\}$.

Proof. For a first-order φ the assertion is true, since every element of $n(\lambda)$ is a pair in $n(\pi)$ and the corresponding set of pairs and the relations of $\lambda(\mathscr{R})$ are first-order definable in λ.

If K is an m-ary relation on $n(\lambda)$, then it can be written in the form

$$\bigcup_{i_0, \ldots, i_{m-1} \in 3} A^{i_0, \ldots, i_{m-1}}_{\bar{c}_{i_0}, \ldots, \bar{c}_{i_{m-1}}} = K$$

where $A^{i_0, \ldots, i_{m-1}}$ is an m-ary relation on $n(\pi)$ and for all m-ary relation A on $n(\pi)$

$$A_{\bar{c}_{i_0}, \ldots, \bar{c}_{i_{m-1}}} = \{\langle \langle \bar{c}_{i_0}, x_0 \rangle, \ldots, \langle \bar{c}_{i_{m-1}}, x_{m-1} \rangle \rangle \mid \langle x_0, \ldots, x_{m-1} \rangle \in A\}.$$

$A^{i_0, \ldots, i_{m-1}}$ will be denoted by $K(i_0, \ldots, i_{m-1})$.

Therefore for any first-order $\psi(Y_0, \ldots, Y_{t-1}, X)$, $Y_0, \ldots, Y_{t-1}$ are m-ary and X is p-ary free variable, there exists a first-order formula of $\mathscr{L}(\bar{G})$ with $t \cdot 3^m + 1$ free variables; $\psi_1(\langle Y_j^{i_0,\ldots,i_{m-1}}\rangle, X)$ so that for all $K_j \subseteq \times_m n(\lambda)$, $j \in t$ and $Z \subseteq n(\pi)$ we have $\lambda \vDash \psi(K_0, \ldots, K_{t-1}, Z_{c_0})$ iff

$$\pi \vDash \psi_1(\langle K_j(i_0, \ldots, i_{m-1})\rangle_{j\in t-1, i_0,\ldots,i_{m-1}\in 3}, Z) \tag{2.5}$$

Now, if φ is e.g. $\Sigma^1_{1,k-1}$ and it is of the form

$$\exists^1_{1,k-1} Y_0 \cdots \exists^1_{1,k-1} Y_{t-1}\, \psi(Y_0, \ldots, Y_{t-1}, X),$$

then (2.5) implies that for all $Z \subseteq n(\pi)$ $\lambda \vDash \varphi(Z_{c_0})$ iff

$$\pi \vDash \langle \exists^1_{1,k-1} Y_j^{i_0,\ldots,i_{m-1}}\rangle_{\substack{j\in t-1\\ i_0,\ldots,i_{m-1}\in 3}} \Bigg[\Bigg(\bigwedge_{\substack{j\in t-1\\ i_0,\ldots,i_{m-1}\in 3}} \forall \langle x_0, \ldots, x_{m-1}\rangle \in Y_j^{i_0,\ldots,i_{m-1}},$$
$$\mathscr{B}(c_{i_0}, x_0) \wedge \cdots \wedge \mathscr{B}(c_{i_{m-1}}, x_{m-1})\Bigg) \wedge \psi_1(\langle Y_j^{i_0,\ldots,i_{m-1}}\rangle, Z)\Bigg]. \quad \square$$

Lemma 2.4. *Let G be a finite subset of* $\mathrm{Sym}(\mathscr{L})$, $\tau \in \mathrm{Int}(\mathscr{L}(G))$, $W_0 \subseteq n(\tau)$, $|W_0| \geq \frac{1}{3}|n(\tau)|$. *Then there exists an interpretation π of $\bar{G}$ so that $n(\pi) = W_0$ and*

$$K_\tau = \langle n(\tau), (\tau(\mathscr{R}))_{\mathscr{R}\in G}\rangle, \qquad K_\lambda = \langle n(\lambda^{(\pi)}), (\lambda^{(\pi)}(\mathscr{R}))_{\mathscr{R}\in G}\rangle$$

are isomorphic, moreover there exists an isomorphism $j: K_\tau \to K_{\lambda^{(\pi)}}$ so that for any $x \in W_0$ $j(x) = \langle \bar{c}_0, x\rangle$.

Proof. Let $n(\pi) = W_0$ and $\pi(c_0)$, $\pi(c_1)$, $\pi(c_2)$ arbitrary but different elements of W_0. To define $\mathscr{B}$ let $W_1, W_2 \subseteq W_0$ so that $|W_0| + |W_1| + |W_2| = n(\tau)$ and put

$$\mathscr{B}(c_i, x) \leftrightarrow x \in W_i \quad \text{for all } x \in W_0.$$

Now we define j and then complete the definition of π. Let $E_1, E_2 \subseteq n(\tau) - W_0$ with $|E_1| = |W_1|$, $|E_2| = |W_2|$ and $W_0 \cup E_1 \cup E_2 = n(\tau)$ (consequently $E_1 \cap E_2 = 0$). Let f_i be a one-to-one map of E_i onto W_i, $i = 1, 2$, and f_0 be the identity mapping on W_0. If $x \in n(\tau)$ put $j(x) = \langle c_0, x\rangle$ if $x \in W_0$ and $j(x) = \langle c_i, x\rangle$ if $x \in E_i$ $i = 1, 2$.

Now we define $\pi(\mathscr{R}')$. If $\mathscr{R}$ is a k-ary relation symbol of $G, i_0, \ldots, i_{k-1} \in 3$, $x_0, \ldots, x_{k-1} \in n(\pi)$, let

$$\pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1}) \leftrightarrow \tau(\mathscr{R})(j^{-i}\langle \bar{c}_{i_0}, x_0\rangle, \ldots, j^{-1}\langle \bar{c}_{i_{k-1}}, x_o\rangle).$$

(for other types of sequences we define $\pi(\mathscr{R}')$ arbitrarily). If $\mathscr{R}$ is a k-ary function symbol, $i_0, \ldots, i_{k-1} \in 3$ $x_0, \ldots, x_{k-1} \in n(\pi)$, then let

$$\pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1}, \bar{c}_1\rangle = f_h^{-1}(y)$$

where $y = \tau(\mathscr{R})(j^{-1}(\langle \bar{c}_{i_0}, x_0\rangle), \ldots, j^{-1}(\langle c_{i_{k-1}}, x_{k-1}\rangle))$ and $h = 0$ if $y \in W_0$, $h = i$ if $y \in E_i$, and let

$$\pi(\mathscr{R}')(\bar{c}_{i_0}, x_0, \ldots, \bar{c}_{i_{k-1}}, x_{k-1}, \bar{c}_0) = \bar{c}_h.$$

(For other types of sequences we define π arbitrarily.)

The definition of π and λ implies that j is an isomorphism. $\square$

Lemma 2.5. *Suppose that $\mathscr{P}$ is a k-ary relation symbol and Φ is a $\Sigma^1_{1,m}$, $\Pi^1_{1,m}$ resp. first-order sentence of $\mathscr{L}$. Then there exists a sentence Ψ of the same type $(\Sigma^1_{1,m},\cdots)$ of $\mathscr{L}$ such that for any*

$$\bar\tau\in\mathrm{Int}(\mathscr{L}(\Phi)-\mathscr{P}),\quad W_0\subseteq n(\bar\tau), |W_0|\geq \tfrac{1}{3}|n(\bar\tau)|,$$

$$D\subseteq\Big(\bigtimes_k n(\bar\tau)\Big)-\bigtimes_k W_0$$

there exists an interpretation π of $\mathscr{L}(\Psi)-\mathscr{P}$ so that $n(\pi)=W_0$ and

$$S^{\pi}_{\Psi}=\Big\{X\cap\bigtimes_k W_0 \mid X\in S^{\bar\tau}_{\Phi}\wedge X\cap\Big[\Big(\bigtimes_k n(\bar\tau)\Big)-\bigtimes_k W_0\Big]=D\Big\}. \tag{2.6}$$

Proof. Let Z be the set defined in (2.6) and let $G=\mathrm{Sym}(\mathscr{L}(\Phi)-\mathscr{P})\cup\mathscr{D}$ where $\mathscr{D}$ is a new k-ary relation symbol. Let τ be an interpretation of $\mathscr{L}(G)$ with $n(\tau)=n(\bar\tau)$, $\tau\supseteq\bar\tau$ and $\tau(\mathscr{D})=D$. Applying Lemma 2.4 we get an interpretation π of $\mathscr{L}(\bar G)$ and an isomorphism j between the structures $\langle n(\tau),\{\tau(\mathscr{R})\}_{\mathscr{R}\in G}\rangle$ and $\langle n(\lambda^{(\pi)}),\{\lambda^{(\pi)}(\mathscr{R})\}_{\mathscr{R}\in G}\rangle$ with $j(x)=\langle c_0,x\rangle$ for all $x\in W_0$. Therefore if we substitute $\bar\tau$ for λ and D for $j(D)$ in (2.6), (where $j(D)=\{\langle j(x_0),\ldots,j(x_{k-1})\rangle\mid\langle x_0,\ldots,x_{k-1}\rangle\in D\}$), then the set defined there will be Z_{c_0}.

Hence the existence of Ψ is a consequence of Claim 2.3.

Lemma 2.6. *If $\mathscr{P}$ is a k-ary $k>1$ relation symbol, then there exist first-order formulae $\varphi_1(x,y)$, $\varphi_2(x)$, ψ of $\mathscr{L}(\mathscr{P})$ such that for any $\lambda\in\mathrm{Int}(\mathscr{L}(\mathscr{P}))$, $\mu\models\psi$ implies that*

$$\text{“}x\leq y\leftrightarrow\varphi_1(x,y) \textit{ is a linear ordering of}$$

$$\{x\in n(\mu)\mid\varphi_2(x)\}=W \quad\textit{and}\quad 2|W|+1\leq n(\lambda)\leq 2|W|+2\text{”}.$$

Moreover if $\pi\in\mathrm{Int}(\mathscr{L}(\mathscr{P},\mathrm{LO}))$, $n(\pi)\geq 2$ and $\pi\models\mathrm{LO}$ and $i=1$ or $i=2$, then there exists a $\tau\in\mathrm{Int}(\mathscr{L}(\mathscr{P}))$ with $n(\tau)=2n(\pi)+i$ such that $\tau\models\psi$,

$$\forall x\in n(\tau)\quad x\in n(\pi)\leftrightarrow\tau\models\varphi_2(x),$$

$$\forall x,y\in n(\pi)\ (\pi\models x\leq y)\leftrightarrow(\tau\models\varphi_1(x,y))$$

and

$$\tau(\mathscr{P})\cap\bigtimes_k n(\pi)=\pi(\mathscr{P}).$$

Proof. First we suppose that $\mathscr{P}$ is binary. ψ will be the following assertion: There exists exactly one element u with $\forall z\,\mathscr{P}(u,z)$ and for this u if $W=\{t\mid\mathscr{P}(t,u)\wedge t\neq u\}$, then $\tau(\mathscr{P})\cap(W\times(n(\tau)-W-\{u\}))$ is a one-to-one map of W onto $V=n(\tau)-W-\{u\}$ if $i=1$, and if $i=2$, then it is a map of W onto a subset of V with $|V|-1$ elements; and $\tau(\mathscr{P})\cap(V\times V)$ is a linear ordering of V.

Obviously ψ is a first-order formula of $\mathscr{L}(\mathscr{P})$. Let $\varphi_2(x)=\mathscr{P}(x,u)\wedge x\neq u$

$$\varphi_1(x,y)=\exists\, p,q\ (p\neq u\wedge q\neq u\wedge\neg\varphi_2(p)\wedge\neg\varphi_2(q)$$
$$\wedge\mathscr{P}(x,p)\wedge\mathscr{P}(x,q)\wedge\mathscr{P}(p,q)).$$

(It means that the one-to-one map in the definition of ψ maps the pair $\langle x, y\rangle$ into the pair $\langle p, q\rangle$ and $p \leq q$ with respect to $\mathscr{P} \cap (V \times V)$.)

The first assertion of the lemma is a consequence of the definitions of ψ, φ_1 and φ_2. To prove the second one let $n(\tau) = 2n(\pi) + i$. If we choose a $u \in n(\pi)$ and fix the values of $\tau(\mathscr{P})$ with $W = n(\pi)$ everywhere except on $V \times V$ and $W \times W$ in accordance to ψ, then we may choose a linear ordering of V for $\tau(\mathscr{P}) \cap (V \times V)$ so that the linear ordering on W induced by ψ be identical to $\leq_\pi$. Now we may choose $\tau(\mathscr{P})$ arbitrarily on $W \times W$ (in ψ there is no restriction on $\tau(\mathscr{P}) \restriction (W \times W)$ so the requirement

$$\pi(\mathscr{P}) = \tau(\mathscr{P}) \cap (n(\pi) \times n(\pi))$$

can be met. (Note that the uniqueness of u with the property $\forall z\, \mathscr{P}(u, z)$ is guaranteed by $\neg\mathscr{P}(t, u)$ if $t \in V$ and $\neg\mathscr{P}(t, x) \vee \neg\mathscr{P}(t, y)$ if $t \in W$, $x, y \in V$ $x \neq y$). For k-ary $\mathscr{P}$ we define u by $\forall x_1, \ldots, x_{k-1}$ $\mathscr{P}(u, x_1, \ldots, x_{k-1})$ then we use the relation $\mathscr{P}(x, y, u, \ldots, u)$ instead of $\mathscr{P}(x, y)$ in the definition of ψ, φ_1 and φ_2 and otherwise the proof remains unchanged. $\square$

Proof of (C). The property will be the following

$$S_1(\bar{\mathscr{P}}) = \text{“}\bar{\mathscr{P}} \text{ satisfies } \psi\text{”} \wedge S\Big(\bar{\mathscr{P}} \restriction \bigtimes_k \{x \mid \varphi_2(x)\}\Big).$$

First we prove that $S_1(\bar{\mathscr{P}})$ is $\Pi_{1,1}^1$. Let $\pi \in \operatorname{Int}(\mathscr{L}(G))$. ($\pi(\mathscr{P})$ satisfies $\psi) \leftrightarrow \pi \models \psi$ is first-order. If $\pi \models \psi$, then by Lemma 2.6 the first-order formula $\varphi_1(x, y)$ defines a linear ordering on $\{x \mid \varphi_2(x)\}$, so the first assertion of (B) implies that $S_1(\bar{\mathscr{P}})$ is also $\Pi_{1,1}^1$.

Suppose now that Φ is a $\Sigma_{1,k-1}^1$ formula and $\bar{\tau}$ is an interpretation of $\mathscr{L}(\Phi) - \mathscr{P}$ so that for all $\tau' \supseteq \bar{\tau}\, \tau' \models \Phi \leftrightarrow S_1(\mathscr{P})$ and $n(\bar{\tau}) \geq 6$. *Let* $\tau \in \operatorname{Int}(\mathscr{L}(\Phi, \mathscr{P}))$ with $\tau \supseteq \bar{\tau}$ and $\tau \models \psi$ (for $n(\bar{\tau}) \geq 6$ the second assertion of Lemma 2.6 implies the existence of a τ with this property).

Now we apply Lemma 2.5 with $W_0 = \{x \in n(\tau) \mid \tau \models \varphi_2(x)\}$ and $D = \tau(\mathscr{P}) - \bigtimes_k W_0$, Φ, $\bar{\tau}$ are the same as above. Then we have an interpretation π of $\mathscr{L}(\Psi) - \mathscr{P}$ with $n(\pi) = W_0$ and (2.6) and the definition of $S_1(\bar{\mathscr{P}})$ implies that for all $\pi' \supseteq \pi (S(\pi'(\mathscr{P})) \wedge \mu \models \psi) \leftrightarrow \pi' \models \Psi$ where $\mu \supseteq \bar{\tau}$ and $\mu(\mathscr{P}) = \pi'(\mathscr{P}) \cup D$. The definition of ψ implies that ψ does not depend on the values of $\mu(\mathscr{P})$ on $\bigtimes_k W_0$ so $\tau \models \psi$ implies that $\mu \models \psi$ that is $S(\pi'(\mathscr{P})) \leftrightarrow \pi' \models \Psi$. We have proved that for all $\bar{\tau}$, there is a π with this property and with $n(\pi') \geq \frac{1}{3} n(\bar{\tau})$, hence if $S_1(\mathscr{P})$ is $\Sigma_{1,k-1}^1$ so is $S(\mathscr{P})$ in contradiction to (B). $\square$

Corollary 2.7. *There exists a recursive function r whose recursivity is a theorem of Peano Arithmetic, so that the following assertion is also a theorem of Peano Arithmetic: For every $\Sigma_{1,1}^1$ sentence Ψ if $n > r(\Psi)$ and π is an interpretation of $\mathscr{L}(\Psi) - \mathscr{P}$ with $n(\pi) = n$, then*

$$\neg(\forall\, \pi' \in \operatorname{Int}(\mathscr{L}(\Psi, \mathscr{P}))\ \pi' \supseteq \pi \rightarrow (S(\pi'(\mathscr{P})) \leftrightarrow \pi' \models \Psi)).$$

Proof. All of the theorems of this section are theorems of Peano Arithmetic and all of the functions whose existence are proved are recursive. □

Section 3

In this section we always will assume that $\mathcal{L}$ has a relation $\leqslant$ and if π is an interpretation, then $\pi(\leqslant)$ is defined and it is the usual ordering relation on the natural number $n(\pi)$. We will also assume that $\mathcal{P}$ is a unary relation symbol.

Definition. Let $n \in \omega$ and $A, B \subseteq P(n)$. We say that the first-order sentence φ separates A and B if there exists an interpretation $\pi \in \operatorname{Int}(\mathcal{L}(\varphi) - \mathcal{P})$, $n(\pi) = n$ such that for all $\pi' \in \operatorname{Int}(\mathcal{L}(\varphi, \mathcal{P}))$, $\pi' \supseteq \pi$ we have $\pi'(\mathcal{P}) \in A \rightarrow \pi' \models \varphi$ and $\pi'(\mathcal{P}) \in B \rightarrow \pi' \models \neg\varphi$. We will say that A, B are j-separable if there exists a first-order sentence φ of length less than j which separates A, B.

Theorem 3.1. *$\forall\, i \in \omega\ \exists\, j \in \omega$ such that for all $n \in \omega$ we have: $\forall\, a, b \in n$ if $a > (1 + 1/(\log n)^i)b$, then the sets*

$$\{X \subseteq n \mid |X| \geqslant a\}, \qquad \{Y \subseteq n \mid |Y| \leqslant b\}$$

are j-separable.

The results of Section 1 show that one cannot define the number of the elements of a subset of a structure by a first-order formula, however, according to Theorem 3.1 we can say at least approximatively what this number is.

First as an illustration we give a trivial example of a formula which separates

$$A = \{X \subseteq n \mid |X| > \tfrac{1}{2}n\}, \qquad B = \{Y \subseteq n \mid |Y| < \sqrt{n}\}.$$

φ is defined as follows:

$$\varphi \equiv \forall x\ \exists\, y, z \quad \mathcal{P}(y) \wedge \mathcal{P}(z) \wedge x = y + z.$$

If $+$ is interpreted as the addition mod n, then $|\bar{\mathcal{P}}| > \frac{1}{2}n$ implies $|\bar{\mathcal{P}} + \bar{\mathcal{P}}| = n$ since

$$\forall\, x \in n\,[(\exists\, y \in \bar{\mathcal{P}}\ \ x - y \in \bar{\mathcal{P}}) \rightarrow x \in \bar{\mathcal{P}} + \bar{\mathcal{P}}] \quad \text{and} \quad \bigcup_{y} \{y, x - y\} = n \supseteq \bar{\mathcal{P}}.$$

If $|\bar{\mathcal{P}}| < \sqrt{n}$, then clearly $|\bar{\mathcal{P}} + \bar{\mathcal{P}}| < n$ that is $\bar{\mathcal{P}} + \bar{\mathcal{P}} \neq n$.

The following definition has a probabilistic motivation. Suppose we do not know any formula (and the corresponding interpretation) which separates A and B, but we have a formula φ containing a relation symbol $\mathcal{R}$ so that if we take an interpretation of $\mathcal{L}(\varphi, \mathcal{P}, \mathcal{R})$ where $\bar{\mathcal{R}}$ is a random subset of n (or $\times_k n$), then for all $\bar{\mathcal{P}} \in A$ the probability of $\varphi(\bar{\mathcal{P}}, \bar{\mathcal{R}})$ will be greater than if $\bar{\mathcal{P}} \in B$. In that case φ shows some difference between A and B. (If A and B is defined by the cardinality of its elements, then it is easy to give φ's with this property (Lemma 3.4).) If the two probabilities are 1 and 0, then picking an arbitrary $\bar{\mathcal{R}}$ we get that A and B are separated by φ. Even if the probabilities are very close to 1 and 0, then we may choose an $\bar{\mathscr{R}}$ with the same property (Lemma 3.3). Therefore our task will be the following, starting with a formula φ and the corresponding probabilities p, q we will try to construct (in a sequence) new φ's where the probabilities will be closer to 1 and 0. We will use essentially two types of construction for giving the next φ in the sequence:

(1) replacing φ by $\neg\varphi$ (then we must change the role of A and B and we get $1-q$, $1-p$ from the pair p, q).

(2) We take a sequence of random and independent $\bar{\mathscr{R}}$'s of length $l \leq n^s$ (where s is a constant) and the new formula says that all of the $\varphi(\bar{\mathscr{P}}, \bar{\mathscr{R}})$'s are true. Since the new probabilities will be p^l, q^l we may use this step and (1) together to widen the gap between p, q until we get numbers very close to 1 and 0.

Definition. Let p, q be real numbers $1 \geq p, q \geq 0$, $i, n \in \omega$, $A, B \subseteq P(n)$. A, B are $\langle i, p, q\rangle$ separable or $S(i, p, q, A, B)$ if there exists a first-order sentence φ of length less than i, and a relation symbol $\mathscr{R} \neq \mathscr{P}$ with arity k, $k < i$, such that there exists an interpretation π of $\mathscr{L}(\varphi) - \mathscr{P} - \mathscr{R}$ with $n(\pi) = n$, and for all $\pi' \in \mathrm{Int}[\mathscr{L}(\varphi, \mathscr{P}) - \mathscr{R}]$ we have

$$\pi'(\mathscr{P}) \in A \to 2^{-n^k} |\{\tau \in \mathrm{Int}(\varphi, \mathscr{P}, \mathscr{R}) \mid \tau \supseteq \pi' \wedge \tau \vDash \varphi\}| \geq p \tag{3.1}$$

and

$$\pi'(\mathscr{P}) \in B \to 2^{-n^k} |\{\tau \in \mathrm{Int}(\varphi, \mathscr{P}, \mathscr{R}) \mid \tau \supseteq \pi' \wedge \tau \vDash \varphi\}| \leq q. \tag{3.2}$$

Remark. Our definition implies that if $S(i, 1, 0, A, B)$ then A, B are i-separable. Indeed let $\pi'' \in \mathrm{Int}(\mathscr{L}(\varphi, \mathscr{R}) - \mathscr{P})$ arbitrary. Then (3.1) and (3.2) implies that for all $\pi_0 \supseteq \pi''$, $\pi_0 \in \mathrm{Int}(\mathscr{L}(\varphi, \mathscr{R}, \mathscr{P}))$ we have $\pi_0(\mathscr{P}) \in A \to \pi_0 \vDash \varphi$ and $\pi_0(\mathscr{P}) \in B \to \pi_0 \vDash \neg\varphi$.

Lemma 3.2. *There exists a $g \in {}^{\omega\times\omega}\omega$ such that if $A, B \subseteq P(n)$ and $S(i, p, q, A, B)$, then we have*

(1) $S(i+1, 1-q, 1-p, B, A)$,

(2) $S(g(i, s), p^l, q^l, A, B)$ *for all* $s \in \omega$, $l \leq n^s$, $l \in \omega$,

(3) $S(i, p', q', A, B)$ *if* $p \geq p'$, $q \leq q'$.

Proof. (1) and (3) are trivial consequences of the definition. Let φ, $\mathscr{R}$, π be the same as in the definition of $S(i, p, q, A, B)$.

Let $\mathscr{I}$ be an s-ary and $\mathscr{N}$ a $k+s$-ary relation symbol. Let ψ' be the formula obtained by substituting $\mathscr{R}(t_0, \ldots, t_{k-1})$ for $\mathscr{N}(x_0, \ldots, x_{s-1}, t_0, \ldots, t_{k-1})$ at every occurrences of $\mathscr{R}$ in φ, where the variable $x_0, \ldots, x_{j-1}$ does not occur in φ. Let $\psi = \forall x_0, \ldots, x_{s-1}\, \mathscr{I}(x_0, \ldots, x_{s-1}) \to \psi'(x_0, \ldots, x_{s-1})$. If μ is an interpretation of $\mathscr{L}(\psi) - \mathscr{P} - \mathscr{N}$ with $\mu \supseteq \pi$ and $|\mu(\mathscr{I})| = l$, then we claim that ψ as φ, $\mathscr{N}$ as $\mathscr{R}$, μ as π satisfy the requirements of the definition of $S(g(i, s), p^l, q^l, A, B)$ for a suitable $g(i, s)$. Indeed if $\mu' \in \mathrm{Int}(\varphi, \mathscr{P}) - \mathscr{N}$ and $\mu'(\mathscr{P}) \in A$, then put

$$t = 2^{-n^{k+s}} |\{t \in \mathrm{Int}(\psi, \mathscr{P}, \mathscr{N})| \tau \supseteq \mu' \wedge \tau \vDash \psi\}|.$$

By the definition of ψ and (3.1)

$$t \geq 2^{-n^{k+s}} \Big(\prod_{\langle x_0,\ldots,x_{s-1}\rangle \in \mu(\mathcal{I})} 2^{n^k} \Big) \Big(\prod_{\langle x_0,\ldots,x_{s-1}\rangle \in \mu(\mathcal{I})} p \cdot 2^{n^k} \Big)$$
$$= 2^{-n^{k+s}} 2^{(n^s - l)n^k} p^l 2^{l \cdot n^k} = p^l.$$

We can prove the other inequality analogously. $\square$

Lemma 3.3. *Suppose* $A, B \subseteq P(n)$ $S(i, p, q, A, B)$ *and* $(1-p)+q < 2^{-(n+1)}$, *then* $S(i, 1, 0, A, B)$; *that is* A *and* B *are* i*-separable.*

Proof. Suppose φ, $\mathcal{R}$, π are the same as in the definition of $S(i, p, q, A, B)$. Then $(1-p)+q < 2^{-n+1}$ implies that $|A| \cdot (1-p) + |B| \cdot q < 1$, therefore there is a single $\bar{\tau} \in \mathrm{Int}(\mathcal{L}(\varphi, \mathcal{R}) - \mathcal{P})$, $\bar{\tau} \supseteq \pi$ which is a restriction of some τ in all of the sets defined in (3.1) and (3.2), that is for all $\tau \supseteq \bar{\tau}$, $\tau \in \mathrm{Int}(\mathcal{L}(\varphi, \mathcal{R}, \mathcal{P}))$ we have $\tau(\mathcal{P}) \in A \to \tau \models \varphi$ and $\tau(\mathcal{P}) \in B \to \tau \models \neg\varphi$. $\square$

Lemma 3.4. *If* n *is a power of* 2 *and* $0 \leq a \leq n$, $0 \leq b \leq n$, $A, B \subseteq P(n)$ *and* $\forall X \in A$ $|X| \geq a$, $\forall Y \in B$ $|Y| \leq b$, *then* $S(100, a/n, b/n, A, B)$.

The formula φ whose existence is stated in the lemma will contain the unary relation symbols $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{I}$ and the binary relation symbol $\mathcal{B}$. $\pi \in \mathrm{Int}(\mathcal{L}(\varphi) - \mathcal{P} - \mathcal{R})$ (from the definition of $S(i, p, q, A, B)$) will be an interpretation where $|\pi(\mathcal{I})| = \log_2 n$ and $\mathcal{B}$ codes a one-to-one mapping from the subsets of $\pi(\mathcal{I})$ onto n. The formula φ will be the following: "there exists an $x \in \pi(\mathcal{P})$ which corresponds to $\pi(\mathcal{R}) \cap \pi(\mathcal{I})$ at the mapping defined by $\mathcal{B}$". More precisely:

Proof. Let $\mathcal{I}$ be a unary relation symbol and $\mathcal{B}$ a binary one. There exists a $\pi \in \mathrm{Int}(\mathcal{L}(\mathcal{I}, \mathcal{B}))$ such that $n(\pi) = n$, $|\pi(\mathcal{I})| = \log_2 n$ and

$$\pi \models [\forall_1^1 C\, \exists! x\, \forall y\, \mathcal{I}(y) \to (C(y) \leftrightarrow \mathcal{B}(y, x))]$$
$$\wedge [\forall x\, \exists_1^1 C\, \forall y\, \mathcal{I}(y) \to (C(y) \leftrightarrow \mathcal{B}(y, x))]$$

(that is the elements of $n(\pi)$ are coded by the subsets of $\pi(\mathcal{I})$).

Let $\mathcal{R}$ be a unary relation symbol and

$$\varphi = \exists x\, \mathcal{P}(x) \wedge [\forall y\, \mathcal{I}(y) \to (\mathcal{R}(y) \leftrightarrow \mathcal{B}(y, x))].$$

Suppose $\pi' \in \mathrm{Int}(\mathcal{L}(\varphi, \mathcal{P}) - \mathcal{R})$, $\pi' \supseteq \pi$ and $\pi'(\mathcal{P}) - A$. Then

$$2^{-n} \, |\{\tau \in \mathrm{Int}(\varphi, \mathcal{P}, \mathcal{R}) \mid \tau \supseteq \pi' \wedge \tau \models \varphi\}|$$
$$= 2^{-n} \, |\pi'(\mathcal{P})| \, 2^{n - |\pi(\mathcal{I})|} = 2^{-|\pi(\mathcal{I})|} \, \pi'(\mathcal{P})| \geq \frac{a}{n}. \quad \square$$

Lemma 3.5. *There exists a function $f:\omega\to\omega$ such that if $i, j\in\omega$, $i\geq 1$ and $n\in\omega$ is sufficiently large, then $A, B\subseteq P(n)$,*

$$S\left(j, \left(1+\frac{1}{(\log n)^i}\right)p, p, A, B\right), \quad \frac{1}{n^4}\leq p\leq\frac{1}{n^2}$$

implies that there exists a p', $1/n^4\leq p'\leq 1/n^2$ with

$$S\left(f(j), \left(1+\frac{1}{(\log n)^{i-1}}\right)p', p', B, A\right). \tag{$*$}$$

Proof. We will prove that if we apply the rules (1), (2) with $l=[(3/p)\log n]\leq n^5$ and (3) with a suitable p'; we get ($*$). Indeed, applying (1) we get

$$S\left(j', 1-p, 1-\left(1+\frac{1}{(\log n)^i}\right)p, B, A\right).$$

Using that

$$\left(1-\frac{c}{x}\right)^x = \mathrm{e}^{-c}+\mathrm{o}_n\left(\frac{1}{n}\right) \quad \text{if } x\geq n^2, \quad 1\leq c\leq 2$$

and

$$\mathrm{e}^{-1-x}\leq \mathrm{e}^{-1}\left(1-\frac{x}{2}\right) \quad \text{if } 0<x<1$$

we get

$$q=(1-p)^l=\left(\left(1-\frac{1}{\frac{1}{p}}\right)^{1/p}\right)^{p[(3/p)\log n]}=\left(\mathrm{e}^{-1}+\mathrm{o}\left(\frac{1}{n}\right)\right)^{3\log n+\mathrm{o}(1/n)}$$
$$=\frac{1}{n^3}\left(1+\mathrm{o}\left(\frac{1}{\sqrt{n}}\right)\right)$$

and

$$p'=\left(1-\left(1+\frac{1}{(\log n)^i}\right)\right)=\left(\mathrm{e}^{-1-(1/(\log n)^i)}+\mathrm{o}\left(\frac{1}{n}\right)\right)^{3\log n+\mathrm{o}(1/n)}$$
$$\leq\left(\mathrm{e}^{-1}\left(1-\frac{1}{2(\log n)^i}\right)+\mathrm{o}\left(\frac{1}{n}\right)\right)^{3\log n+\mathrm{o}(1/n)}$$
$$=\frac{1}{n^3}\left(1-\frac{3}{2(\log n)^{i-1}}\right)\left(1+\mathrm{o}\left(\frac{1}{\sqrt{n}}\right)\right).$$

So we have $q\geq p'(1-3/2(\log n)^{i-1})^{-1}(1+\mathrm{o}(1/\sqrt{n}))$ that is $q\geq(1+(1/(\log n)^{i-1}))p'$ if n is sufficiently large. $\square$

Proof of Theorem 3.1. Let $i\in\omega$. Throughout the proof we will suppose that n is sufficiently large compared to i. It does not mean a real restriction because for any fixed n there exists a $j(n)$ (trivially) so that for all $E, F\subseteq P(n)$, $E\cap F=0$ we have $S(j(n), E, F)$. (The separating formula φ may list all of the elements of E and F.)

First we prove the theorem supposing that n is a power of 2. By Lemma 3.4 we have

$$S\left(100, \frac{a}{n}, \frac{b}{n}, A, B\right).$$

(3) of Lemma 3.2 and $a>(1+1/(\log n)^i)b$ implies that $S(100,(1+1/(\log n)^i)b/n, b/n, A, B)$. Let r be the smallest positive integer with $(b/n)^r \leqslant 1/n^2$. We may suppose that $1/n \leqslant b/n \leqslant (n-1)/n$ since in the case $b=0$ or $b=n$ the lemma trivially holds. Hence $r<n^2$ and $1/n^4 \leqslant (b/n)^r \leqslant 1/n^2$. Let us apply (2) of Lemma 3.2 with $l=r$, and put $p=(b/n)^r$. We have

$$S\left(j, \left(1+\frac{1}{(\log n)^i}\right)^r p, p, A, B\right)$$

and applying (3) again:

$$S\left(j, \left(1+\frac{1}{(\log n)^i}\right)p, p, A, B\right)$$

where j depends only on i.

Now we may apply Lemma 3.5 repeatedly and after the ith step we get

$$S(j', 2p', p', X, Y) \quad \text{where } \{X, Y\}=\{A, B\}$$

and j' depends only on i and $n^{-4} \leqslant p' \leqslant n^{-2}$. Let δ_i $(i=1,2,3,4)$ be real numbers with $-\frac{1}{2}<\delta_i<\frac{1}{2}$ and

$$(1-p')^{[(3/p')\log n]} \geqslant n^{-3+\delta_1},$$
$$(1-2p')^{[(3/p')\log n]} \leqslant n^{-6+\delta_2},$$
$$(1-n^{-3+\delta_1})^{[n^{4+\delta_1}]} \leqslant \mathrm{e}^{-(1+\delta_3)n},$$
$$(1-n^{-6+\delta_2})^{[n^{4+\delta_1}]} \geqslant 1-n^{-2+\delta_4}.$$

We may choose the δ_i's sucessively always taking a δ_i with minimal $|\delta_i|$ so that $\delta_1, \ldots, \delta_4$ satisfy the corresponding inequalities.

Starting with $S(j, 1-p', 1-2p', X, Y)$ and using Lemma 3.2 repeatedly, we get a sequence $S(u_\mu, p_\mu, q_\mu, Z_\mu, W_\mu)$ where $\{Z_\mu, W_\mu\}=\{A, B\}$, such that every element of the sequence can be derived from the previous one by applying one of the rules (1), (2), (3) of Lemma 3.2.

We only give the pair p_μ, q_μ and we write after the pair the numbers of the rules applied at that pair. If we apply (3) we use the inequalities containing the δ_i's, except in the last case $(1-p', 1-2p')$, (2) $l=[(3/p')\log n]$, (3) $(n^{-3+\delta_1}, n^{-6+\delta_2})$, (1) $(1-n^{-6+\delta_2}, 1-n^{-3+\delta_1})$, (2) $l=[n^{4+\delta_1}]$, (3) $(1-n^{-2+\delta_4}, \mathrm{e}^{-(1+\delta_3)n})$, (1) $1-\mathrm{e}^{-(1+\delta_3)n}, n^{-2+\delta_4})$ (2) $l=n$ (3) $(1-2n\mathrm{e}^{-(1+\delta_3)n}, n^{-n(2-\delta_4)})=(p_0, q_0)$.

We have $S(j, p_0, q_0, Z, W)$ where j depends only on i and $1-p_0+q_0<2^{-n+1}$. Thus Lemma 3.3 implies that A, B are j separable for a j depending only on i.

Now if n is not a power of 2 let m be the smallest power of 2 greater than n.

$A, B \subseteq P(m)$ too, and they satisfy the conditions of the theorem with m (instead of n). Let φ be the formula which separates A, B on m. Applying Lemma 2.5 we get a formula ψ (whose length depends only on the length of φ and therefore only on i) which separates A and B on n. $\square$

Section 4

Let P' be the set-theoretical version of Peano arithmetic (that is the theory of hereditarily finite sets with the axiom of complete induction.) Suppose that P is a recursively enumerable theory and the hereditarily finite sets can be defined in P (e.g. $P = \mathrm{ZFC}$), that is there exist formulae $\lambda(x)$, $\mu(x, y)$ of P such that for any model M of P

$$\langle \{a \in M \mid \lambda(x)\}, \{\langle a, b\rangle \in M \times M \mid \mu(a, b), =\rangle$$

is a model of P'. We will suppose in the following that a pair λ, μ is fixed for every P, and we will use the notation $x \in y$ instead of $\mu(x, y)$ and the expression "x is hereditarily finite" instead of $\lambda(x)$. We will always assume that if M is a model of P, then the standard part of the hereditarily finite part of M is identical to the structure of hereditarily finite sets.

Let $\mathcal{M}$ be a language with the relation symbols $Q_0, \ldots, Q_{k-1}$ where the arity of Q_i is j_i. We may assume that $\mathcal{M}$ is a hereditarily finite set and therefore every interpretation of $\mathcal{M}$ is hereditarily finite as well). Let $\Phi(x)$ be the following formula of P: $\Phi(x) \equiv$ "x is an interpretation of $\mathcal{M}$, $n(x) \in \omega$ and $X(Q_0), X(Q_1), X(Q_2)$ are the restrictions of the relations $\leqslant$, $+$, $\cdot$ to $n(x)$".

If $M \vDash P$, $\pi \in M$ and $M \vDash \Phi(\pi)$ let us denote by π^{-M} the structure with the underlying set $\{x \in M \mid M \vDash x \in n(\pi)\}$ and with the relations

$$\{\langle x_0, \ldots, x_{j_i}\rangle \mid \langle x_0, \ldots, x_{j_i}\rangle \in M \wedge M \vDash \langle x_0, \ldots, x_{j_i}\rangle \in \pi(Q_i)\} \quad i = 0, 1, \ldots, k-1.$$

Let φ be a first-order formula of $\mathcal{M}$ of the form

$$\forall x_0\, \exists x_1 \cdots \forall x_{2m-2}\, \exists x_{2m-1}\, \psi(x_0, \ldots, x_{2m-1})$$

and let G_φ be a game where the players I, II alternately choose m, m elements from $n(\pi)$ (I starts). If $c_0, c_2, \ldots, c_{2m-2}$ are the elements chosen by I and $c_1, c_3, \ldots, c_{2m-1}$ are the elements chosen by II, then

$$\text{II wins} \quad \text{iff } \pi \vDash \psi(c_0, \ldots, c_{2m-1}).$$

Clearly $\pi \vDash \varphi \leftrightarrow$ (II has a winning strategy).

Suppose now that $M \vDash P$ and $m, \varphi, \pi \in M$ and m is nonstandard. In that case $M \vDash (\pi \vDash \varphi)$ implies that $M \vDash$"II has a winning strategy". However, if we consider the game G_φ^i where the players choose only i, i elements and II wins iff

$$\forall x_{2i}\, \exists x_{2i+1} \cdots \forall x_{2m-2}\, \exists x_{2m-1}\, \psi(c_0, \ldots, c_{2i-1}, x_{2i}, \ldots, x_{2m-1}),$$

then, if i is standard $M \models (\pi \models \varphi)$ will imply that II has a 'real' winning strategy (not only in M) moreover these strategies for various i's are compatible, that is there is a single strategy whose restrictions give winning strategy for all $i \in \omega$.

The next lemma expresses this assertion without referrring to games.

Lemma 4.1. *Suppose that $M \models P$ and $m, \varphi, \pi \in M$, $M \models (\varphi$ is a first-order formula of $\mathcal{M}$ of the form*

$$\forall x_0 \, \exists x_1 \cdots \forall x_{2m-2} \, \exists x_{2m-1} \, \psi(x_0, \ldots, x_{2m-1})$$

where $m \in \omega$ and $\pi \models \varphi$). If m is nonstandard, then there exists a function $g \in {}^M M$ with $(\forall x \in M \, M \models g(x) \in n(\pi))$ such that for any function $h \in {}^M M$ with $(\forall x \in M \, M \models h(x) \in n(\pi))$ we have that for all $i \in \omega$ if $a_0, \ldots, a_{2i-1} \in M$ so that $a_{2j} = h(\langle a_0, \ldots, a_{2j-1} \rangle)$ and $a_{2j+1} = g(\langle a_0, \ldots, a_{2j} \rangle)$ for all $j \in i$, then

$$M \models (\pi \models \forall x_{2i} \, \exists x_{2i+1} \cdots \forall x_{2m-2} \, \exists x_{2m-1} \, \psi(a_0, \ldots, a_{2i-1}, x_{2i}, \ldots, x_{2m-1})).$$

Proof. We have to define g only on the finite sequences of odd length, on other elements of M we may define it arbitrarily. We give the definition by recursion on the length of the sequence. Suppose that $i \in \omega$ and g is defined for all sequence of the type $\langle a_0, \ldots, a_{2j} \rangle$, $a_0, \ldots, a_{2j} \in M$, $j \in i$ and for all $h \in {}^M M$ with $\forall x \in M$ $M \models h(x) \in n(\pi)$, the conclusion of the lemma holds. Suppose that

$$M \models (\pi \models x_{2i} \, \exists x_{2i+1} \cdots \forall x_{2m-2} \, \exists x_{2m-1} \, \psi(b_0, \ldots, b_{2i-1}, x_{2i}, \ldots, x_{2m-1})). \quad (4.1)$$

Then there exists a $g'_b \in M$ where $b = \langle b_0, \ldots, b_{2i-1} \rangle$ so that

$$M \models \text{``} g'_b \in {}^{n(\pi)} n(\pi) \wedge (\pi \in \forall x_{2i} \, \forall x_{2i+2} \, \exists x_{2i+3} \cdots \forall x_{2m-2} \, \exists x_{2m-1} \, \psi(b_0, \ldots, b_{2i-1}, x_{2i}, g'_b(x_{2i}), x_{2i+2}, \ldots, x_{2m-1}))\text{''}. \quad (4.2)$$

Let f_b be a function defined by

$$f_b(x) = y \leftrightarrow (x, y \in M \wedge M \models \text{``} g'_b(x) = y \text{''})$$

and set for arbitrary $b_0, \ldots, b_{2i}$, $g(\langle b_0, \ldots, b_{2i} \rangle) = f_b(b_{2i})$ if (4.1) holds, $g(\langle b_0, \ldots, b_{2i} \rangle)$ is an arbitrary element of M otherwise. (4.2) and the inductive assumption implies that g meets the requirements of our lemma. □

Suppose that $\nu(x)$ is a first-order formula of P and M is a model of π, $\pi \in M$, $M \models \phi(\pi) \wedge \nu(\pi)$. We want to know whether ν, M, π have the following property:

(A) *There exists a countable model N of P and a $\tau \in N$ such that $N \models$ "τ is an interpretation of $\mathcal{M}$", $N \models \neg \nu(\tau)$ and $\pi^{-M} \cong \tau^{-N}$.*

The following theorem shows that (A) is an inner property of M, in the sense that there exists a sequence $\langle \varphi_n \rangle_{n \in \omega}$ of first-order formulae of $\mathcal{M}$ such that

$$(\mathrm{A}) \leftrightarrow \forall n \in \omega \, M \models (\pi \models \varphi_n).$$

Theorem 4.2. *Let $\nu(x)$ be a first-order formula of P with the free variable x. Then there exists a recursive sequence of first-order closed formulae $\langle \varphi_i \mid i \in \omega \rangle$ of $\mathcal{M}$ such that if M is a countable nonstandard model of P, then for all $\pi' \in M$*

$$(0\nu) \qquad (M \vDash \phi(\pi') \wedge \neg \nu(\pi')) \rightarrow \forall\, n \in \omega\, M \vDash (\pi' \vDash \varphi_n);$$

and if $\pi \in M$, $M \vDash \phi(\pi) \wedge \nu(\pi)$, then

$$(1\nu) \qquad \forall\, n \in \omega\, M \vDash (\pi \vDash \varphi_{n+1}) \rightarrow (\pi \vDash \varphi_n)$$

and the following assertions are equivalent:

(A) *There exists a countable model N of P and a $\tau \in N$ such that $N \vDash$ "τ is an interpretation of $\mathcal{M}$", $N \vDash \neg \nu(\tau)$ and $\tau^{-M} \cong \tau^{-M}$.*

(B) $\forall\, n \in \omega\, M \vDash$ "$\tau \vDash \varphi_n$".

(B′) *There exists a nonstandard natural number m in M such that $M \vDash$ "$\pi \vDash \varphi_m$".*

Proof. We may suppose that P has definable Skolem functions (we may add function symbols to P if it is necessary). By (1ν), (B) and (B′) are clearly equivalent for any $\langle \varphi_n \rangle$.

First we define φ_n. Let us add a new constant symbol p to P. P^+ will be the extended theory. We will suppose that $P^+ + \neg \nu(p)$ has definable Skolem functions. Again we may add new function symbols to the language if it is necessary.

Suppose that $\tau \in M$, $M \vDash P \wedge \phi(\pi)$. If $i \in \omega$ and $M \vDash$ "$a_0, \ldots, a_{i-1} \in n(\pi)$", then let $\operatorname{diag}_\pi(a_0, \ldots, a_{i-1})$ be the following set of formulae of P: $\{\varphi(x_0, \ldots, x_{i-1}) \mid x_0, \ldots, x_{i-1}\}$ are the free variables of φ and there exists a relation Q_j of the language $\mathcal{M}$ and a sequence $m_0, \ldots, m_{t-1} \in i$ where t is the arity of Q_j, such that $M \vDash (\pi(Q_j))(a_{m_0}, \ldots, a_{m_{t-1}})$ and

$$\varphi(x_0, \ldots, x_{i-1})\text{“}\phi(p) \wedge x_0, \ldots, x_{i-1} \in n(p) \wedge (\pi(Q_j))(x_{m_0}, \ldots, x_{m_{t-1}})\text{”}.$$

Let $\langle r_i(x_0, \ldots, x_{2i-1}) \mid i \in \omega \rangle$ be a recursive sequence of P^+ formulae such that any first-order formula whose free variables are $x_0, \ldots, x_{2i-t}$ occurs infinitely many in the sequence. For any $i \in \omega$, $a_0, a_1, \ldots, a_{2i-1} \in n(\pi)$ let $T_i'(\pi, a_0, \ldots, a_{2i-1})$ be the following theory:

$$P^+ + \exists\, x_0, \ldots, x_{2i-1}\, (\operatorname{diag}_\pi(a_0, \ldots, a_{2i-1})$$
$$\wedge \bigwedge_{j \in i} \exists\, z \in n(\pi)\, r_j(x_0, \ldots, x_{2j-2}, z) \rightarrow r_j(x_0, \ldots, x_{2j-2}, x_{2j-1})) + \neg \nu(p).$$

Set

$$T_i(\pi, a_0, \ldots, a_{2i-1}) = T_0'(\pi) + \cdots + T_j'(\pi, a_0, \ldots, a_{2j-1}) + \cdots + T_i(\pi, a_0, \ldots, a_{2i-1}).$$

Now we can define the sequence φ_n.

By the definition of $\phi(x)$ there exists a recursive sequence $\langle \varphi_i \mid i \in \omega \rangle$ of first-order closed formulae of $\mathcal{M}$ such that if $i \in \omega$ and M is a nonstandard model of P with $\pi \in M$, $M \in \phi(\pi)$, then

$$(M \vDash (\pi \vDash \varphi_i)) \leftrightarrow$$
$$M \vDash \forall\, a_0 \in n(\pi)\, \exists\, a_1 \in n(\pi) \cdots \forall\, a_{2i-1} \in n(\pi)\, \exists\, a_{2i-1} \in n(\pi)$$

[there is no contradiction shorter than i in the theory $T_i(\pi, a_0, \ldots, a_{2i-1})$].

Let N be a model of P^+ with the properties given in (A), and let $i \in \omega$. We may suppose that the formula φ_n is a hereditarily finite set so it is an element of N. If f is the isomorphism between π^{-M} and τ^{-N}, then for all $a_0, \ldots, a_{2i-1} \in n(\pi)$ we have

$$M \vDash \text{"in } T_i(\pi, a_0, \ldots, a_{2i-1}) \text{ there is no contradiction shorter than } i\text{"}$$

iff

$$N \vDash \text{"in } T_i(\tau, f(a_0), \ldots, f(a_{2i-1}) \text{ there is no contradiction shorter than } i\text{"}.$$

(Here we use that i is standard.) Hence $M \vDash (\pi \vDash \varphi_i) \leftrightarrow N \vDash (\tau \vDash \varphi_i)$. Let

$$\varphi_i = \forall z_0 \, \exists z_1 \cdots \forall z_{2i-2} \, \exists z_{2i-1} \, \psi_i(z_0, \ldots, z_{2i-1}).$$

We fix the Skolem functions $s_j(z_0, \ldots, z_{2j-2})$ for the $2j-1$th quantor (in N): If $\exists \, z \in n(\tau) \; r_j(z_0, \ldots, s_{j-1}, z_{2j-2}, z)$, then let $s_j(z_0, \ldots, z_{2j-2}) \in n(\tau)$ arbitrary with $r_i(z_0, \ldots, s_{j-1}, z_{2j-2}, s_j(z_0, \ldots, z_{2j-2}))$. Now for arbitrary $z_0, z_2, \ldots, z_{2i-2} \in N$, $\tau \vDash \psi_i(z_0, s_0, \ldots, z_{2i-2}, s_j)$ since N is a model of $T(\tau, z_0, s_0, \ldots, z_{2i-2}, s_j)$ so it cannot have a contradiction of standard length.

(B$'$) $\rightarrow$ (A). Apply Lemma 4.1 for $\varphi = \varphi_m$. Let $h \in {}^M M$ with $\forall \, x \in M \; M \vDash h(x) \in n(\pi)$ and

$$\forall a (a \in M \wedge M \vDash a \in n(\pi)) \rightarrow \exists \, i \in \omega \, h(i) = a.$$

The definition of φ_i implies that if a_i is defined as in Lemma 4.1, then the following theory is consistent ($c_0, c_1, \ldots$ are new constant symbols added to P^+):

$$(2\nu) \qquad T = \Big(P^+ + \sum_{i \in \omega} \Big[\operatorname{diag}_\pi(a_0, \ldots, a_{2i-1}) \wedge \bigwedge_{j \in i} \exists \, z \in n(\pi) \, r_i(x_0, \ldots, x_{2i-2}, z) \rightarrow r_i(x_0, \ldots, x_{2i-2}, x_{2i-1})\Big]_{x_i \rightarrow c_i} + \neg \nu(p).$$

where $[\cdots]_{x_i \rightarrow c_i}$ means that all of the free variables of the type x_j is substituted for the corresponding c_j.

Let N' be a model of T. If x is a constant symbol let $\bar{x}$ be the corresponding element of N'. N' is obviously a model of $P^+ + \neg \nu(p)$. $P^+ + \neg \nu(p)$ has definable Skolem functions therefore the definable hull of the set $\{\bar{c}_i \mid i \in \omega\}$ will be a model N of P^+. Let $\tau = \bar{p}$. We claim that N, τ satisfies (A). Indeed (2ν) and the definability of the elements of N implies:

$$\forall \, x \in N \, \exists \, i \in \omega \; (N \vDash x \in n(\tau) \rightarrow c_i = x)$$

and so $a_i \rightarrow \bar{c}_i$ is an isomorphism of π^{-M} onto τ^{-N}. $\square$

Suppose now that one of the relation symbols of $\mathcal{M}$ *is* $\mathcal{P}$ (unary) the others are $Q_0, Q_1, Q_2, \ldots, Q_k$. The following theorem will imply Theorem 4.3 and Theorem 4.4 mentioned in the introduction.

Theorem 4.5. *Suppose that $\nu(x)$ is a first-order formula of P with the free variable x and r is a recursive function and $P \vdash$ "if φ is a first-order sentence of $\mathcal{M}$ and $n > r(\varphi)$ and π_0 is an interpretation of $\mathcal{M} - \mathcal{P}$ with $n(\pi_0) = n$, then there exists $\pi_1, \pi_2 \in \mathrm{Int}(\mathcal{M})$ so that $\pi_1 \supseteq \pi_0$, $\pi_2 \supseteq \pi_0$, $(\pi_1 \vDash \varphi) \leftrightarrow (\pi_2 \vDash \varphi)$ but $\nu(\pi_1) \wedge \neg \nu(\pi_2)$". Then for all nonstandard n is an arbitrary countable nonstandard model M of P and $\pi_0 \in M$ there exists a $\pi \in M$ so that*

$$M \vDash \text{“}(\pi_0 \in \mathrm{Int}(\mathcal{M} - \mathcal{P}) \wedge n(\pi_0) = n) \to (\pi \supseteq \pi_0 \wedge \nu(\pi))\text{”}$$

and π satisfies condition (A) *of Theorem* 4.2.

Proof. Let n be a nonstandard natural number of M,

$$\pi_0 \in M \vDash \pi_0 \in \mathrm{Int}(\mathcal{M} - \mathcal{P}) \wedge n(\pi_0) = n.$$

Since r is a standard recursive function there exists a nonstandard $m \in N$ with $r(\varphi_m) < n$.

Moreover (0ν) implies that we may suppose that

$$(0\nu') \qquad M \vDash \forall x(\phi(x) \wedge \neg \nu(x)) \to \varphi_m.$$

According to the assumption of the theorem (with $\varphi = \varphi_m$), for all $\pi_0 \in M$ there exist $\pi_1, \pi_2 \in M$ with the properties (in M) described in the theorem. By $(0\nu')$ $\neg \nu(\pi_2)$ implies that $\pi_2 \vDash \varphi_m$, therefore $\pi_1 \vDash \varphi_m$ that is, we have (B′) and thus by Theorem 4.2, (A) too. □

Proof of Theorem 4.3. Let

$$\nu(x) \equiv \text{“}x(\mathcal{P}) \text{ is even”}.$$

Corollary 1.5 implies that ν meets the requirements of Theorem 4.5.

Let N, n, R, k as in Theorem 4.3. Let us apply Theorem 4.5 with $M \to N$, $\mathcal{M} = \mathcal{L}(Q_0, Q_1, Q_2, \mathcal{R})$ where $\mathcal{R}$ is a k-ary relation symbol and

$$N \vDash \phi(\pi_0) \wedge n(\pi_0) = n \wedge \pi_0(\mathcal{R}) = R.$$

Theorem 4.5 implies the existence of a $\pi \in M$ with

$$M \vDash \pi \in \mathrm{Int}(\mathcal{M} \cup \mathcal{P}) \wedge \pi \supseteq \pi_0 \wedge \nu(\pi)$$

and property (A). Let $P = \pi(\mathcal{P})$. Property (A) is obviously equivalent to the conclusion of Theorem 4.3. □

Proof of Theorem 4.4. Let N, n, k, R, l as in Theorem 4.4. Let

$$\mathcal{M} = \mathcal{L}(Q_0, Q_1, Q_2, \mathcal{R}, \mathcal{A}_1, \ldots, \mathcal{A}_e, c, \mathcal{P})$$

where $\mathcal{R}$ is k-ary, $\mathcal{A}_i$'s are unary, relation symbols, c is a constant symbol, and $\mathcal{P}$ is a binary relation symbol. Suppose that $\nu(x) =$ "$|\{y \mid \bar{\mathcal{P}}(c, y)\}|$ is odd" and let $\langle \varphi_i \rangle$ be the corresponding sequence defined in Theorem 4.2.

For all φ_i let $\varphi_i'(x)$ be the formula obtained from φ_i by substituting c for x and $\mathscr{A}_j(t)$ for $X_j(t)$, $j=1,\ldots,l$, at all occurrences of the constant symbol c and the relation symbols $\mathscr{A}_j$ where x is a first-order variable and X_j's are second-order unary variables not occurring in φ_i. Now we will work in N. Let Ψ be the Σ_1^1 formula

$$\exists_1^1 X_1,\ldots,\exists_1^1 X_l\, \forall x\, \neg\varphi_m'(x, X_1,\ldots,X_l)$$

where m is nonstandard with $r(\Psi)<n$ (r is from Corollary 2.7) and $(0\nu')$.

Suppose now that λ is an interpretation of $\mathscr{M}-\{\mathscr{A}_1,\ldots,\mathscr{A}_l, c\}$ with $n(\lambda)=n$ $\lambda(\mathscr{R})=R$, $\lambda\models\psi$. $(0\nu')$ implies that

$$S(\lambda(\mathscr{P}))\equiv\text{“}\forall\, u\in n(\lambda)\, |\{y\in n(\lambda)\mid \lambda(\mathscr{P})(u,y)\}| \text{ is even”}.$$

Indeed let $z_1,\ldots,z_l\subseteq n(\lambda)$ with $\lambda\models\forall x\,\neg\varphi_m'(x, z_1,\ldots,z_l)$. If u is any fixed element of $n(\lambda)$, then $\lambda\models\neg\varphi_m'(u, z_1,\ldots,z_m)$ that is if $\mu\supseteq\lambda$ with $\mu(\mathscr{A}_j)=z_j$ and $\mu(c)=u$ then we have $\mu\models\neg\varphi_m$ and hence by $(0\nu')$ $|\{\mu(\mathscr{P})(u,y)\mid y\in n(\lambda)\}|$ is even.

Corollary 2.7 implies that there exists a $\lambda'\in\operatorname{Int}(\mathscr{M}-\{\mathscr{A}_1,\ldots,\mathscr{A}_l, c\})$ with $\lambda'(\mathscr{R})=R$, $n(\lambda')=n$, $S(\lambda'(\mathscr{P}))$ but $\lambda'\models\neg\Psi$. This means that e.g.

$$\lambda'\models\neg\forall x\,\neg\varphi_m'(x, A_1,\ldots,A_l)$$

that is there is a $v\in n(\lambda)$ with $\lambda'\models\varphi_m'(v, A_1,\ldots,A_l)$. Now if $\pi\supseteq\lambda'$ and $\pi(x)=v$, $\pi(A_j)=A_j$, then $\pi\models\varphi_m$ that is by Theorem 4.2 there exists a countable model N' of P and a $\tau\in N'$ so that

$$N'\models\text{“}|\{y\mid \tau(\mathscr{P})(\tau(c),y)\}| \text{ is odd”}$$

and $\tau^{-N'}\cong\pi^{-N}$. $\square$

Section 5

We accept the following well-known results of probability theory (or combinatorics) without their proof (Lemma 5.2 is only a reformulation of Lemma 5.1).

Lemma 5.1. *$\forall\, m\in\omega\,\exists\, c>0\,\forall\, r\in\omega$ if S is the set of all $f\in {}^r m$ with*

$$|\{i\mid f(i)=0\}|<\frac{1}{2m}\,r$$

(that is the number of the roots of f is less than a half of its expected value), then

$$m^{-r}\,|S|\leqslant 2^{-cr}.$$

Lemma 5.2. *$\forall\, m\in\omega\,\exists\, c>0\,\forall\, r\in\omega$ if g is a function on $\{f\restriction i\mid f\in {}^r m, i\in m\}$ with* *values in m and S is the set of all* $f \in {}^r m$ *with*

$$|\{i \mid f(i) = g(f \upharpoonright i)\}| < \frac{1}{2m} r,$$

then $m^{-r} |S| \leq 2^{-cr}$.

In this section we prove Lemma 5.7. We have given in Section 1 a proof of a weaker version of the lemma. That proof was based on an approximation of ${}^n 2 - \bigcup X$ (where $\|X\| \leq k$) given in Lemma C.1. This approximation, however, is not good enough for our present purposes.

So we will do the following (simultaneously with a set of X's) we take a complete set of cylinders T so that for each fixed X and $D \in T$ $D \cap \bigcup X$ can be approximated with $\bigcup Y(X, D)$ where $Y(X, D)$ is a set of cylinders $\|Y(X, D)\| \leq k$ and the cylinders of $Y(X, D)$ (for fixed X, D) are concentrated on a small subset of n. More precisely the following lemma holds:

Lemma 5.3. $\forall$ *j, k,* $l \in \omega$ $\exists$ *u,* $v \in \omega$ *such that if n is large enough and W is a set of systems of cylinders,* $|W| = n^j$ *and* $C \in X \in W$ *implies that C is a cylinder on n and* $\|C\| \leq k$, *then there exists a complete set of cylinders T with* (a)$\wedge$(b)$\wedge$(c) *where:*

(a) $l(T) \leq n/(\log n)^l$,

(b) *for all* $X \in W$ *and* $D \in T$ *there exists a set of cylinders* $Y(X, D) \subseteq X$ *such that* $\|Y(X, D)\| \leq k$ *and*

$$\forall X, D\ (|\bigcup \{s(C) \mid C \in Y(X, D))\}| \leq (\log n)^u),$$

(c) *for all* $X \in W$

$$2^{-n} \left| \bigcup X \,\Delta \bigvee_D (T, Y(X, D)) \right| \leq 2^{-n/(\log n)^v}.$$

Remarks. (1) For any fixed X, D the complement of $Y(X, D)$ can be written in the form $\bigcup V(X, D)$ where $V(X, D)$ is a set of cylinders with

$$s(V(X, D)) = \bigcup_{C \in Y(X, D)} s(C)$$

and so $\|V(X, D)\| \leq (\log n)^u$. That is the conclusion of the lemma may be written in the following form:

$$2^{-n} \left| ({}^n 2 - \bigcup X) \,\Delta \bigvee_D (T, V(X, D)) \right| \leq 2^{-n/(\log n)^v},$$

So using this lemma we may conclude the proof of Lemma 5.7 essentially the same way as we did in the case of Lemma C.3. (We give this proof at the end of this section.)

(2) The complete set of cylinders T whose existence stated in this lemma cannot be given in the form $T = \{D \mid s(D) = B\}$ where B is a fixed subset of n.

To show the idea of the proof of Lemma 5.3 first we sketch the proof of the special case $k=1$. (The proof of this case is included in the detailed proof as well.)

We define a sequence of complete sets of cylinders T_i with $l(T_i)=i$, and $T_{[n/(\log n)^{l}]}$ will be T. Let $T_0=\{^n 2\}$. Suppose T_i is defined already. For every $D\in T_i$ let

$$W_D=\left\{X\in W \mid D\not\subseteq \bigcup X \wedge \left|\bigcup_{C\in X} s(C)-s(D)\right|>(\log n)^u\right\}$$

and $y_D\in n-s(D)$ so that

$$m_D=\left|\left\{X\in W_D \mid y_D\in \bigcup_{C\in X} s(C)-s(D)\right\}\right|$$

is maximal. The definition of W_D implies that

$$m_D\geqslant(\log n)^u\frac{|W_D|}{n}.$$

Let

$$T_{i+1}=\{F \mid \|F\|=i+1\wedge\exists\, D\in T\,(b(F)\supseteq b(D)\wedge y_D\in s(F))\}.$$

The definition of y_D and W_D implies that for all $D\in T_i$ there exists an $F\in T_{i+1}$, $F\subseteq D$ with

$$|W_F|\leqslant\left(1-\frac{(\log n)^u}{2\cdot n}\right)|W_D|.$$

So Lemma 5.2 implies that for almost all $D\in T_i$ (except $|T_i|\,2^{-ci}$ for some constant $c>0$) we have

$$|W_D|\leqslant\left(1-\frac{(\log n)^u}{2n}\right)^{i/4}|W|$$

so if

$$\frac{i}{4}>\frac{2n}{(\log n)^u}\cdot j\cdot\log n$$

(where $|W|=n^j$), then $|W_D|=0$ for almost all $D\in T_i$ which implies the lemma with $k=1$.

In the case $k>1$ the construction is similar if there is enough $X\in W_D$ with

$$\left|\bigcup\{s(C)-s(D) \mid C\in X\wedge C\cap D=0\wedge s(C)-s(D)=1\}\right|\geqslant(\log n)^u$$

(that is X contains enough cylinders which has a one element base inside D), then we may choose y_D as before. If not, then we choose y_D so that the numbers of X's with enough cylinders with one element base inside $F(D\supseteq F\in T_{i+1})$ be greater by a factor of $1+(\log n)^u/2n$. If it is not possible we try to increase the number of X's with enough cylinders with two element base inside F by a similar (u is bigger) factor etc.

If we have already enough cylinders with r element base inside F, then in the next step we will be able to increase the corresponding number for $r-1$. In order to be able count the cylinders with various sizes we will use only cylinders with pairwise disjoint $s(C)$'s from a fixed X.

Lemma 5.5 shows that if the number of cylinders with pairwise disjoint $s(C)$'s is small ($\leqslant(\log n)^c$), then we may reduce k. Lemma 5.4 deals with the case when $|W|$ is small.

Lemma 5.4. *$\exists\, m \in \omega$ such that Lemma 5.3 is true if*

$$|W| \leqslant \frac{n}{(\log n)^m}$$

(where m may depend on j, k, l but not on n).

Moreover $\forall\, k, l, \exists\, m, v$ such that if n is large enough and W is a set of systems of cylinders with

$$|W| \leqslant n/(\log n)^m, \quad X \in W \to \|X\| \leqslant k,$$

then there exists a complete set of cylinders T with

(α) $\quad l(T) = [n/(\log n)^l]$;

(β) $\quad \dfrac{1}{|T|}\left|\{D \in T \mid \exists\, X \in W \;\; D \neq D \cap \bigcup X \neq 0\}\right| \leqslant 2^{-n/(\log n)^v}$.

Proof of Lemma 5.4. We will define a sequence of complete sets of cylinders T_i, $0 \leqslant i \leqslant t/k$ where $t = [n/(\log n)^l]$. Let $T_0 = \{{}^n2\}$. Suppose T_i is defined and $D \in T_i$. If

$$\exists\, X \in W \quad D \neq D \cap \bigcup X \neq 0, \tag{5.1}$$

then let X_D be an element of W with this property, and

$$H_D \supseteq n - s(D) \quad \text{with } |H_D| = k \tag{5.2}$$

and

$$\exists\, C_D \in X_D \quad C_D \cap D \neq 0 \wedge s(C_D) - s(D) \supseteq H(D).$$

If (5.1) does not hold, then we define $H(D)$ arbitrarily with $H_D \supseteq n - s(D)$, $|H_D| = k$.

For later use let g_D be a one-to-one mapping of 2^k onto ${}^{(H_D)}2$ so that if D satisfies (5.1), then $g_D(0) = b(C_D) - b(D)$.

Put $T_{i+1} = \{F \mid \exists\, D \in T_i \;\; F$ is a cylinder $F \cap D \neq 0 \wedge s(F) = s(D) \cup H_D\}$. Clearly $F \in T_i \to \|F\| = ki$. Let $q = [t/k]$.

We prove that $T' = T_q$ satisfies (β). Clearly for every $D \in T'$ and $i \leqslant q$ there exists exactly one $D_i \in T_i$ with $D_i \supseteq D$.

Let $f_D(i) = g_{D_i}^{-1}(b(D) \restriction H(D))$. Clearly $D \to f_D$ is a one-to-one mapping between T' and ${}^q(2^k)$.

So Lemma 5.1 implies that the number of D's with $D\in T'$ and

$$(*)\qquad |\{i\in q \mid f_D(i)=0\}|<\frac{1}{2^{k+1}}q$$

is less than $|T'|\,2^{-c(k)q}$. Let $E\subseteq T'$ be the set of D's with $(*)$.

For a fixed $D\notin E$ $f_D(i)=0$ implies $C_{D_i}\supseteq D$ or $\neg((5.1)$ for $D_i)$. In the latter case we have $\neg((5.1)$ for $D)$ that is for all $X\in W$ $D=D\cap\bigcup X$ or $D\cap\bigcup X=0$. If the first case holds for two different i's, then the definition of C_D implies that the corresponding X_{D_i}'s are also different. Hence if we choose m with

$$\frac{n}{(\log n)^m}<\frac{1}{2^{k+1}}q=\frac{1}{2^{k+1}}\left[\frac{1}{k}\left[\frac{n}{(\log n)^l}\right]\right],$$

then we have $|W|<(1/2^{k+1})q$, and so if $D\notin E$ then there exists an i with $\neg((5.1)$ for $D_i)$. We have that the set defined in (β) with $(T=T')$ is a subset of E which implies (β) with $v=l+1$.

An arbitrary complete refinement T of T' with $l(T)=[n/(\log n)^l]$ will satisfy (α) as well.

The first assertion of the lemma is a consequence of the second one since we may define $Y(X,D)=0$ if $\bigcup X\cap D=0$ and $Y(X,D)=\{A\}$ (where A is the cylinder with $b(A)=0$) if $D=D\cap\bigcup X$ ($Y(X,D)=0$ otherwise).

Let us write the assertion of Lemma 5.3 in the following form:

$$\forall\, j,k,l\in\omega\ \exists\, u,v,n_0\ \forall\, n>n_0,\ W\ \ \Phi(j,k,l,u,v,nW).$$

Lemma 5.5. *Suppose that Lemma* 5.3 *holds for some fixed* k (*that is* $\forall j,l\ \exists\, u,v,n_0\ \forall\, n>n_0,\ W\,\Phi(j,k,l,u,v,n,W)$) *and for all* $s\in\omega$ *let* $\Psi(W,s)$ *the following property of* W *and* s:

$$\Psi(W,s)\equiv(\textit{for all } X\in W \textit{ at most } (\log n)^s \textit{ cylinders } C \textit{ of } X \textit{ can be given simultaneously with pairwise disjoint } s(C)\textit{'s and } \|C\|=k+1).$$

Then for all s *if* n *is large enough Lemma* 5.3 *holds with* $k+1$ *if we restrict our attention to the* W*'s with* $\Psi(W,s)$, *that is*

$$\forall\, s,j,l\in\omega\ \exists\, u,v,n_0\in\omega\ \forall\, n>n_0,\ W\,\Psi(W,s)\to\Phi(j,k+1,l,u,v,n,W).$$

Proof. Let us choose for all $X\in W$ a maximal $V_X\subseteq X$ with the property

$$C_1,C_2\in V_X\to(\|C_1\|=k+1\wedge s(C_1)\cap s(C_2)=0).$$

Our assumption imply that $|V_x|\leq(\log n)^s$. Let $H_x=\bigcup\{s(C)\mid C\in V_x\}$. For every cylinder $c\in X$ let C_x be the cylinder with $b(C_x)=b(C)\restriction(s(C)-H_x)$. If K is a cylinder and $s(K)\subseteq H_x$, then let

$$X[K]=\{C_x \mid b(C)\restriction H_x=b(K)\}.$$

Let $W'=\{X[K]\mid X\in W, s(K)\subseteq H_x, \|K\|<k+1\}$. Apply Lemma 5.3 for k and

$W = W'$. Let T, Y be the sets guaranteed by Lemma 5.3 and for all $X \in W$, $D \in T$ put

$$Y'(X, D) = \{C \in X \mid \exists K\, s(K) \subseteq H_x \wedge C_x \in Y(X[K], D) \wedge \|K\| \leq k+1\}. \tag{5.3}$$

Clearly

$$\bigcup X \supseteq \bigvee_D (T, Y'(X, D) = S.$$

We have to estimate $|\bigcup X - S|$. Suppose $f \in \bigcup X - S$. Then there is a $C \in X$ such that $f \in C$. Let K be a cylinder with $b(K) = b(C) \restriction H_x$. Clearly $f \in \bigcup X[K]$. Let $D \in T$, $f \in D$. We claim that $f \notin \bigcup Y(X[K], D)$. Suppose not. Then (5.3) implies (using C and K defined above) that $f \in \bigcup Y'(X, D)$ that is $f \in S$ in contradiction to $f \in \bigcup X - S$.

Therefore, we have

$$\bigcup X - S \subseteq \bigcup_K (X[K] - \bigcup\{\bigcup Y(X[K], D) \mid D \in T\}),$$

where $s(K) \subseteq H_x$. For K we have at most $(k+2)(2(\log n)^s)^{k+1} \leq n$ possible choices that is

$$|\bigcup X - S| \leq n \cdot 2^n \cdot 2^{-n/(\log n)^v} \leq 2^n \cdot 2^{-n/(\log n)^{v+1}}. \quad \square$$

Lemma 5.6. *$\forall\, k, t \in \omega\ \exists\, v', p \in \omega$ such that if n is large enough and W is the same as in Lemma 5.3, then there exists a complete set of cylinders T with the following properties:*

(I) $l(T) = [n/(\log n)^t]$;

(II) $$2^{-z}\, |\{D \in T \mid \forall\, X \in W\, \forall\, V \subseteq X\, [((\forall\, C_1, C_2 \in V\, s(C_1) \cap s(D) = 0 \wedge \|C_1\| = k \wedge s(C_1) \cap s(C_2) = 0) \rightarrow |V| \leq (\log n)^p) \vee D \subseteq \bigcup X]\}| \geq 1 - 2^{n/(\log n)^{v'}}$$

where $z = [n/(\log n)^t]$.

Proof of Lemma 5.6. First we will prove Lemma 5.6 with (II′) instead of (II):

(II′) $$\exists\, \varepsilon > 0\quad 2^{-z}\, |\{D \in T \mid |\{X \in W \mid \forall\, V \subseteq X\, (\forall\, C_1, C_2, \in V\, s(C_1) \cap s(D) = 0 \wedge \|C_1\| = k \wedge s(C_1) \cap s(C_2) = 0) \rightarrow |V| \leq (\log n)^p\}| \geq \varepsilon\, |W| \vee |\{X \in W \mid D \subseteq \bigcup X\}| \geq \varepsilon\, |W|\}| \geq 1 - 2^{-n/(\log n)^{v'}}$$

where ε is an absolute constant.

By Lemma 5.4 we may suppose that $n/|W| \geq (\log n)^m$ for some fixed m. We will give T in the following form:

Let $G = \{g \in {}^i 2 \mid i \leq z\}$. We will define for every $g \in G$ a cylinder $T(g)$ such that we have$(*) \equiv (*1) \wedge \cdots \wedge (*5)$ where

$(*1)$ $\quad |g| = \|T(g)\|$,

$(*2)$ $\quad g \subseteq g' \rightarrow T(g) \supseteq T(g')$,

$(*3)$ $\quad (g \in {}^{i}2 \wedge g' \in {}^{i+1}2 \wedge g \subseteq g' \wedge \{y\}$

$$= s(T(g')) - s(T(g))) \rightarrow b(T(g'))(y) = g'(i),$$

$(*4)$ $\quad (g, g' \in {}^{i}2 \wedge g \restriction i-1 = g' \restriction i-1) \rightarrow s(T(g)) = s(T(g'))$,

$(*5)$ $\quad T(0) = {}^{n}2$.

T will be defined as $T = \{T(f) \mid f \in {}^{z}2\}$. We will give the function $g \rightarrow T(g)$ by recursion on $|g|$. Let $p_1, \ldots, p_k$ be a sequence of integers with $p_1 > t+1$, $p_j > 2p_{j-1} + t + 2$ for $j = 2, \ldots, k$, $p = p_k + 1$ and $0 < \varepsilon < 1/100$.

Together with $T(g)$ we define for all $g \in G$ with $k \mid |g|$, and $1 \leqslant \alpha \leqslant k$ the set $K(\alpha, g) \subseteq \{\langle C, X\rangle \mid X \in W \wedge C \in X\}$ such that we have $(A) \vee (P_1 \wedge \cdots \wedge P_8)$ where

$$(A) \equiv |\{X \in W \mid \forall\, V \subseteq X \operatorname{dis}(V) \rightarrow |V| \leqslant (\log n)^p\}| \geqslant \varepsilon\, |W|$$
$$\vee |\{X \in W \mid T(g) \subseteq \bigcup X\}| \geqslant \varepsilon\, |W|$$

where

$$\operatorname{dis}(V) \equiv \forall\, C_1\, C_2 \in V\, s(C_1) \cap T(g) = 0 \wedge \|C_1\| = k \wedge s(C_1) \cap s(C_2) = 0.$$

We will use the following notations in the formulation of $(P_1) \cdots (P_8)$ if $X \in W$

$$K_x(\alpha, g) = \{C \mid \langle C, X\rangle \in K(\alpha, g)\},$$
$$F(\alpha, g) = \{X \in W \mid |K_x(\alpha, g)| \geqslant 5k^3 (\log n)^{p_\alpha}\}$$

g will denote an arbitrary $g \in G$ with $k \mid |g|$.

(P$_1$) $1 \leqslant \alpha < k$ implies that

$$|F(\alpha, g)| < \frac{1}{5k^2}\, |W|.$$

(P$_2$)

$$|K(k, g)| \geqslant \tfrac{1}{2}\, |W|\, (\log n)^{p_k}.$$

(P$_3$) for all $1 \leqslant \alpha \leqslant k$, $X \in W$,

$$|K_x(\alpha, g)| \leqslant 6k^3 (\log n)^{p_\alpha}.$$

(P$_4$) if $1 \leqslant \alpha, \beta \leqslant k$, $X \in W$, then

$$C_1 \in K_x(\alpha, g) \wedge C_2 \in K_x(\beta, g) \wedge C_1 \neq C_2$$

implies that $(s(C_1) \cap s(C_2)) - s(T(g)) = 0$.

(P$_5$) $\quad \langle C, X\rangle \in K(\alpha, g) \rightarrow (|s(C) - s(T(g))| = \alpha \wedge C \cap T(g) \neq 0)$.

(P$_6$) for all $y \in n - s(T(g))$, $1 \leqslant \alpha < k$

$$|\{\langle C, X\rangle \in K(\alpha, g) \mid y \in s(C)\}| \leqslant (\log n)^{p_\alpha} \frac{|W|}{n}.$$

(P_7) $\alpha < \beta$ implies that

$$X \in F(\alpha, g) \to K_x(\beta, g) = 0.$$

(P_8) $\bigcup X \supseteq T(g)$ implies that

$$K_x(\alpha, g) = 0 \quad \text{for all } 1 \leq \alpha < k.$$

Let $T(0) = {}^n2$. For all $\alpha < k$ let $K(\alpha, 0) = 0$. If $T(0)$ satisfies (A), then put $K(k, 0) = 0$. Suppose $\neg$(A).

Then there exists a $H \subseteq W, |H| \geq (1-2\varepsilon)|W|$ such that $X \in H \to T(0) \not\subseteq \bigcup X$ and a function $X \to V_x (X \in H)$ such that we have $\operatorname{dis}(V_x)$ (with $g = 0$) and

$$(\log n)^{p_k} \leq |V_x| \leq 2(\log n)^{p_k}.$$

Let

$$K(k, 0) = \bigcup_{X \in H} \{\langle C, X\rangle \mid C \in V_x\}.$$

Properties (P_1), . . . , (P_8) are trivially satisfied in this case.

Suppose now that $k \mid i$, $T(g)$ is defined for all $g, |g| \leq i$ with property (*) and $K(\alpha, g)$ is defined for all g with $k \mid |g| \leq i$ so that $(A) \vee ((P_1) \wedge \cdots \wedge (P_8))$ holds.

Let $h \in {}^i2$. We will define $T(g)$ on the extentions g of h with $|g| \leq i+k$ and $K(\alpha, g)$ for all $g \supset h$, $|g| = i + k$, $1 \leq \alpha \leq k$. If h satisfies (A), then we define $T(g)$ arbitrarily with (*) (obviously if $g' \supseteq h$, $g' \in G$, then g' also satisfies (A)), and in this case let $K(\alpha, g) = 0$. Suppose $\neg$(A). Let $Q(Y, \alpha)$ be a relation defined by

$$Q(Y, \alpha) \quad \text{iff ``} Y \subseteq n - s(T(h)), |Y| = k, 1 \leq \alpha \leq k,$$

$$\left|\left\{\langle C, X\rangle \in \bigcup_{\alpha \leq \gamma \leq k} K(\gamma, h) \,\middle|\, |s(C) - (Y \cup s(T(h)))| = \alpha - 1\right\}\right|$$

$$\geq \frac{1}{2k}\frac{|W|}{n}(\log n)^{p_\alpha}\text{''}.$$

(P_2) implies that $|K(k, h)| \geq \frac{1}{2}|W|(\log n)^{p_k}$, so there is an $y \in n - s(T(h))$ with

$$|\{\langle C, X\rangle \in K(k, h) \mid y \in s(C)\}| \geq \frac{1}{2}\frac{|W|}{n}(\log n)^{p_k}.$$

Therefore if $Y \subseteq n - s(T(h))$, $y \in Y$, $|Y| = k$, then we have $Q(Y, \alpha)$ for some α.

Let δ be the smallest integer with $\exists Y\, Q(Y, \delta)$ and let Z be a fixed set with $Q(Z, \delta)$. (Z and δ depends on h) and put $Z = \{z_1, \ldots, z_k\}$.

Assume now that $g \supset h$, $|g| \leq i + k$. We have to define the cylinder $T(g)$.

Let

$$s(T(g)) = s(T(h)) \cup \{z_1, \ldots, z_{|g|-i}\}$$

and

$$b(T(g))(x) = b(T(h))(x) \quad \text{if } x \in s(T(h))$$

$$b(T(g))(x) = g(j + i - 1) \quad \text{if } x = z_j,\ j = 1, \ldots, k.$$

Obviously $T(g)$ satisfies (*) with this definition moreover for a fixed h and $h \subset g$, $|g| \leq i + k$ $s(T(g))$ depends only on $|g|$.

Let

$$\hat{K} = \bigcup_{1\le\alpha\le k} K(\alpha, h),$$

$$K_1(g) = \{\langle C, X\rangle \in \hat{K} \mid s(C)\cap Y \ne 0\},$$

$$K_2(g) = \{\langle C, X\rangle \in \hat{K} \mid T(g) \subseteq \bigcup X\}.$$

Now we define $K(\alpha, g)$ for $g \supset h$, $|g| = i+k$. For all $1\le\alpha\le k$ let

$$K'(\alpha, g) = K(\alpha, h) - (K_1(g)\cup K_2(g)).$$

If $1\le\alpha<\delta-1$, then let

$$K(\alpha, g) = K'(\alpha, g).$$

If $1\le\alpha=\delta-1$, then let

$$K(\delta-1, g) = K'(\alpha, g)\cup(\{\langle C, X\rangle\in\hat{K} \mid C\cap T(g)\ne 0 \wedge |s(C)-(Y\cup s(T(h)))| = \delta-1\} - K_2(g)). \tag{5.4}$$

For $\delta-1<\alpha<k$ let

$$K(\alpha, g) = K'(\alpha, g) - \{\langle C, X\rangle \mid X\in F(\delta-1, g)\}. \tag{5.5}$$

If g does not satisfy (A), then there exists a $H\subseteq W$ $|H|\ge(1-2\varepsilon)|W|$ such that $X\in H\to T(g)\nsubseteq\bigcup X$ and a function $X\to V_x$ $(X\in H)$ such that $\mathrm{dis}(V_x)$ and

$$(\log n)^{p_k} \le |V_x| \le 2(\log n)^{p_k}.$$

Let

$$K(k, g) = \{\langle C, X\rangle \mid C\in V_x\wedge X\in H\wedge C\in X\wedge\forall\,\alpha<k, C_1\, X\notin F(\alpha, g) \wedge[\langle C_1, X\rangle\in K(\alpha, g)\to((s(C)\cap s(C_1)) - s(T(g)) = 0)]\}. \tag{5.6}$$

We have to prove that if g does not satisfy (A), then $(\mathrm{P}_1)\wedge\cdots\wedge(\mathrm{P}_8)$ holds.

Suppose $\neg$(A),

(P_1): If $k\ne\alpha\ne\delta-1$, then $K(\alpha, g)\subseteq K(\alpha, h)$ and so $F(\alpha, g)\subseteq F(\alpha, h)$ so (P_1) (h) implies (P_1) (g).

Assume $\alpha=\delta-1$. According to the definition of $K(\delta-1, g)$, $\langle C, X\rangle\in K(\delta-1, g)-K(\delta-1, h)$ implies that $\langle C, X\rangle\in K(\beta, h)$ for some β, and $s(C)\cap Y\ne 0$. Thus by (P_4) (h) for any fixed $X\in W$ $|K_x(\delta-1, g)-K_x(\delta-1, h)|\le k$. So

$$|F(\delta-1, g)| \ge \frac{1}{5k^2}|W|$$

would imply

$$|\{X \mid K_x(\delta-1, h)\ge 5k^3(\log n)^{p_{\delta-1}}-k\}| \ge \frac{1}{5k^2}|W|.$$

In this case, however, there is an $y\in n-s(T(h))$ with

$$|\{\langle C, X\rangle\in K(\delta-1, h) \mid y\in s(C)\}| \ge \frac{5k^3(\log n)^{p_{\delta-1}}-k}{5k^2}\frac{|W|}{n} \ge \frac{1}{2}\frac{|W|}{n}(\log n)^{p_{\delta-1}}$$

so taking an $Y \subseteq n - s(T(h))$, $y \in Y$ $|Y| = k$ we get that there is a $\gamma \leq \delta - 1$ with $Q(Y, \gamma)$ which contradicts to the minimality of δ.

(P_3): For $\alpha \neq \delta - 1$, $\alpha \neq k$ $K_x(\alpha, g) \subseteq K_x(\alpha, h)$ that is (P_3) $(h) \to$ (P_3) (g).

$\alpha = k$. $K_x(k, g) \subseteq V_x$ and $|V_x| \leq 2(\log n)^{p_k}$ by definition.

$\alpha = \delta - 1$. If $x \in F(\delta - 1, h)$, then by (P_7) (h) $K_x(\beta, h) = 0$ for all $\beta > \alpha$, and so by (P_5) (h) and the definition of $K(\delta - 1, g)$ we have that $K_x(\delta - 1, g) \subseteq K_x(\delta - 1, h)$ so (P_3) $(h) \to$ (P_3) (g).

Suppose $X \notin F(\delta - 1, h)$. Then

$$|K_x(\delta - 1, h)| < 5k^3(\log n)^{p_{\delta-1}}.$$

We have seen in the proof of (P_1) that

$$|K_x(\delta - 1, g) - K_x(\delta - 1, h)| \leq k$$

so we have

$$|K_x(\delta - 1, g)| \leq 6k^3(\log n)^{p_{\delta-1}}$$

(P_2): According to the definition of $K(k, g)$ it can be written in the form

$$K(k, g) = J_1 - (J_2 \cup J_3)$$

where

$$J_1 = \{\langle C, X\rangle \mid C \in V_x \wedge X \in H\},$$

$$J_2 = \bigcup_{\alpha<k} \{\langle C, X\rangle \mid C \in V_X \wedge X \in F(\alpha, g)\},$$

$$J_3 = \{\langle C, X\rangle \mid C \in V_x \wedge \exists\, \alpha < k,\ C_1 \in K_x(\alpha, g)\ s(C) \cap s(C_1) \neq 0\}.$$

$H \geq (1 - 2\varepsilon)|W|$ and $(\log n)^{p_k} \leq |V_x|$ implies that $|J_1| \geq (1 - 2\varepsilon)|W|(\log n)^{p_k}$. By ($P_1$) (g) and $|V_x| \leq 2(\log n)^{p_k}$,

$$|J_2| \leq \frac{k}{5k^2}|W|\, 2(\log n)^{p_k} \leq \tfrac{2}{5}|W|(\log n)^{p_k}.$$

The definition of V_x, (P_3) (g) implies that

$$|J_3| \leq k \sum_{\alpha<k} |K_x(\alpha, g)| \leq 6 \cdot k^4\Big(\sum_{\alpha<k} (\log n)^{p_\alpha}\Big)|W|$$
$$\leq 6 \cdot k^4(\log n)^{p_k - 1}|W| \leq \tfrac{1}{20}(\log n)^{p_k}|W|.$$

So we have

$$|K(k, g)| \geq |J_1| - |J_2| - |J_3| \geq \tfrac{1}{2}|W|(\log n)^{p_k}.$$

(P_4): $\bigcup_{\gamma<k} K_x(\gamma, g) \subseteq \bigcup_{\gamma\leq k} K_x(\gamma, h)$ according to the definition of K so $s(T(g)) \supseteq s(T(h))$ implies that for $\alpha \neq k \neq \beta$ (P_4) $(h) \to$ (P_4) (g) if $\alpha = k$ or $\beta = k$, then the definition of V_x and $K(k, g)$ implies the assertion.

(P_5): If $\alpha \neq \delta - 1$. $\alpha \neq k$, then $K(\alpha, g) \subseteq K(\alpha, h)$ so (P_5) $(h) \to$ (P_6) (g) for $\alpha = \delta - 1$, $K'(\alpha, g) \subseteq K(\alpha, h)$, $s(T(g)) = s(T(h)) \cup Y$ and (5.4) implies the assertion.

If $\alpha = k$, then (P_5) is a consequence of

$$C \in V_x \to \|C\| = k \wedge (s(C) \cap s(T(g)) = 0.$$

(P_6): For $\alpha \neq \delta-1$ $K(\alpha, g) \subseteq K(\alpha, h)$ $s(T(g)) \supseteq s(T(h))$ and $(P_6)(h)$ implies $(P_6)(g)$.

Assume $\alpha = \delta - 1$, $y \in n - s(T(g))$,

$$|\{\langle C, X\rangle \in K(\delta-1, g) \mid y \in s(C)\}| \geqslant (\log n)^{p_\alpha} \frac{|W|}{n}$$

$|s(C) - s(T(h))| = \delta - 1 < k$ for all $C \in K_x(\delta-1, g)$ implies that there is a $z \in Z$ such that

$$|\{\langle C, X\rangle \in K(\delta-1, g) \mid y \in s(C) \wedge z \notin s(C)\}| \geqslant \frac{1}{k}(\log n)^{p_\alpha} \frac{|W|}{n}.$$

Let $Y = (Z - \{z\}) \cup \{y\}$. Since $K(\delta-1, g) \subseteq \bigcup_\gamma K(\gamma, h)$ the last inequality implies that for some $\gamma < \delta$ we have $Q(Y, \gamma)$ which contradicts the minimality of δ.

(P_7): For $\beta = k$ it is a consequence of (5.6). Suppose $\beta < k$.

If $\alpha \neq \delta - 1$, then by $K_x(\alpha, g) \subseteq K_x(\alpha, h)$ we have $X \in F(\alpha, h)$ and so by $(P_7)(h)$ $K_x(\gamma, g) = 0$ for all $\gamma > \alpha$. By the definition of K,

$$K_x(\beta, g) \subseteq \bigcup_{\gamma \geqslant \beta} K_x(\gamma, h) = 0.$$

If $\alpha = \delta - 1$, then (5.5) implies the assertion.

(P_8): $\bigcup X \supseteq T(g)$ implies that

$$K_2(g) \supseteq \bigcup_{1 \leqslant \gamma \leqslant k} K_x(\gamma, h)$$

thus if $\alpha < k$

$$K_x(\alpha, g) \subseteq \bigcup_{1 \leqslant \gamma \leqslant k} K_x(\gamma, h) - K_2(g) = 0.$$

For $\alpha = k$ the definition of H implies the assertion.

We prove three more properties of K.

(P_9): If h does not satisfy (A) and $\delta = 1$, then there is a $g \supseteq h$ $|g| = i + k$ so that

$$|L(g) - L(h)| \geqslant 2^{-k-1} k^{-2} \frac{|W|}{n} (\log n)^{p_1}$$

where $L(f) = \{X \in W \mid T(f) \subseteq \bigcup X\}$.

Proof. $\delta = 1$ implies that $Q(Z, 1)$, that is if

$$\{\langle C, X\rangle \in \bigcup_{1 \leqslant \gamma \leqslant k} K(\gamma, h) \mid s(C) \subseteq Z \cup s(T(h))\} = J,$$

then

$$|J| \geqslant \frac{1}{2k} \frac{|W|}{n} (\log n)^{p_1}.$$

$\langle C, X\rangle \in J$ implies by $(P_5)(h)$ and $Z \subseteq n - s(T(h))$ that $s(C) \cap Z \neq 0$, thus by $(P_4)(h)$ and $|Z| = k$ for a fixed X there are at most k different C with $\langle C, X\rangle \in J$.

Therefore if $\bar{J}=\{X \mid \exists\, C\langle C, X\rangle\in J\}$, then

$$|\bar{J}|\geqslant\frac{1}{2k^2}\frac{|W|}{n}(\log n)^{p_1}$$

If $r\in {}^Z2$, then let

$$\bar{J}_r=\{X \mid \exists C \text{ “}b(C)\text{ and } r \text{ are compatible”}\wedge\langle C, X\rangle\in J\}.$$

Clearly

$$|\bar{J}_r|\geqslant 2^{-k}\frac{1}{2k^2}\frac{|W|}{n}(\log n)^{p_1}\quad\text{for some } r\in {}^Z2.$$

Let $g=h\cup r$. If $\langle C, X\rangle\in J$, then $s(C)\subseteq Z\cup s(T(h))$. By (P_5) (h) $C\cap T(h)\neq 0$ so if $b(C)$ and r are compatible, then $b(C)\subseteq g$ that is $T(g)\subseteq C\subseteq\bigcup X$. We have $\bar{J}_r\subseteq L(g)$. On the other hand (P_8) (h) and $\bigcup_\alpha K_x(\alpha, h)\neq 0$ for any $X\in\bar{J}_r$ implies that $\bar{J}_r\cap L(h)=0$, that is $L(g)-L(h)\supseteq\bar{J}_r$.

(P_{10}): If h does not satisfy (A) and $\delta>1$, then there is a $g\supseteq h$ $|g|=i+k$ so that

$$|K(\delta-1, g)-K(\delta-1, h)|$$
$$+6k^3|L(g)-L(h)|(\log n)^{p_{\delta-1}}\geqslant 2^{-k}\frac{1}{2k^2}\frac{|W|}{n}(\log n)^{p_\delta}.$$

Proof. Let

$$J=\left\{\langle C, X\rangle\in\bigcup_{\delta\leqslant\gamma\leqslant k}K(\gamma, h)\;\middle|\;|s(C)-(Z\cup s(T(h)))|=\delta-1\right\}.$$

By the same argument as in the proof of (P_9) we get that

$$\bar{J}_r\geqslant 2^{-k}\frac{1}{2k^2}\frac{|W|}{n}(\log n)^{p\delta}\quad\text{for some } r\in {}^Z2$$

where the definition of $\bar{J}_r$ is the same as in the proof of (P_{10}) (using the J defined here).

Put $J_r=\{\langle C, X\rangle\in J \mid b(C)$ and r are compatible$\}$. Clearly $|J_r|\geqslant|\bar{J}_r|$.

Let $g=h\cup r$.

By (5.4) $J_r-K_2(g)\subseteq K(\delta-1, g)$. $J_r\subseteq\bigcup_{\delta\leqslant\gamma\leqslant k}K(\gamma, h)$ and (P_5) (h) implies that $J_r\cap K(\delta-1, h)=0$.

So we have

$$K(\delta-1, g)-K(\delta-1, h)\supseteq J_r-K_2(g).$$

According to the definition of $K_2(g)$, $L(g)$, (P_8) (h) and (P_3) (g) we have $|K_2(g)|\leqslant 6k^3(\log n)^{p_{\delta-1}}|L(g)-L(h)|$ that is

$$|K(\delta-1, g)-K(\delta-1, h)|\geqslant 2^{-k}\frac{1}{2k^2}\frac{|W|}{n}(\log n)^{p_\delta}$$
$$-6k^3(\log n)^{p_{\delta-1}}|L(g)-L(h)|.$$

(P_{11}): If g does not satisfy (A), then for all $1\leqslant\alpha<k$

$$|K(\alpha, h)-K(\alpha, g)|\leqslant 6k^3(\log n)^{p_\alpha}$$
$$\times\left[\left(\sum_{1\leqslant\gamma<\alpha}|K(\gamma, g)-K(\gamma, h)|\right)+|L(g)-L(h)|+\frac{|W|}{n}\right].$$

Proof. If $\alpha \leqslant \delta - 1$, then by the definition of $K(\alpha, g)$ we have $K(\alpha, h) - K(\alpha, g) \subseteq K_1^\alpha(g) \cup K_2^\alpha(g)$ where $K_i^\alpha(g) = K_i(g) \cap K(\alpha, h)$. By $(P_8)(h)$

$$K_2(g) \subseteq \{\langle C, X\rangle \in \hat{K} \mid X \in L(g) - L(h)\}$$

and so, according to $(P_3)(g)$:

$$|K_2^\alpha(g)| \leqslant 6k^3\, |L(g) - L(h)|\, (\log n)^{p_\alpha}.$$

By $(P_6)(h)$ and $(P_4)(h)$

$$|K_1^\alpha(g)| \leqslant k(\log n)^{p_\alpha} \frac{|W|}{n}$$

that is

$$|K(\alpha, h) - K(\alpha, g)| \leqslant (\log n)^{p_\alpha}\left(6k^3\, |L(g) - L(h)| + k\frac{|W|}{n}\right).$$

If $\alpha > \delta - 1$, then again by the definition of $K(\alpha, g)$ we have

$$K(\alpha, h) - K(\alpha, g) \subseteq K_1^\alpha(g) \cup K_2^\alpha(g) \cup (K(\alpha, h) \cap \{\langle C, X\rangle \mid X \in F(\delta - 1), g)\}).$$

We may estimate $|K_1^\alpha(g)|$ and $|K_2^\alpha(g)|$ as in the previous case.

Suppose $\langle C, X\rangle \in K(\alpha, h)$ and $X \in F(\delta - 1, g)$. $(P_5)(h)$ implies that $X \notin F(\delta - 1, h)$, that is $|K_x(\delta - 1, g) - K_x(\delta - 1, h)| \geqslant 1$. According to $(P_3)(h)$ $|K_x(\alpha, h)| \leqslant 6k^3(\log n)^{p_\alpha}$ so we have

$$\begin{aligned}
&|K(\alpha, h) \cap \{\langle C, X\rangle \mid X \in F(\delta - 1, g)\}| \\
&\quad\leqslant \sum_{\substack{X \in F(\delta - 1, g),\\ X \notin F(\delta - 1, h)}} |K_x(\delta - 1, g) - K_x(\delta - 1, h)| \cdot 6k^3(\log n)^{p_\alpha} \\
&\quad\leqslant 6k^3(\log n)^{p_\alpha}\, |K(\delta - 1, g) - K(\delta - 1, h)| \\
&\quad\leqslant 6k^3(\log n)^{p_\alpha} \sum_{\gamma < \alpha} |K(\gamma, g) - K(\gamma, h)|.
\end{aligned}$$

Notation. For all h we will denote by $g(h)$ a fixed g with the properties given in (P_9) or (P_{10}).

Let

$$S = \{f \in {}^z2 \mid \forall\, i \in z\ f \restriction i \text{ does not satisfy (A)}\}.$$

We want to prove that $2^{-z}\, |S| \leqslant 2^{-CZ}$ for some $c > 0$.

Suppose not. If $f \in s$, then let

$$H_f = \{i \in Z \mid k \mid i \wedge f \restriction (i + k) = g(f \restriction i)\}.$$

Lemma 5.2 implies that

$$2^{-z}|\{f \in S \mid |H_f| \leqslant 2^{-k-1}Z\}| \leqslant 2^{-c'z} \quad \text{for some } C' > 0.$$

So our indirect assumption implies that $S' = \{f \in S \mid |H_f| > 2^{-k-1}Z\}$ is not empty.

Let f be a fixed element of S'. According to the definition of S there is an $\beta < k$, $\beta \geqslant 1$ so that

$$|\{i \in Z \mid k \mid i \wedge \delta(f \restriction i) = \beta\}| \geqslant \frac{1}{2k^2} Z.$$

If $\beta = 1$, then according to (P_9)

$$M = \sum_i (L(f \restriction i) - L(f \restriction i-k)| \geqslant c_1(k)\frac{|W|}{n}(\log n)^{p_1}Z.$$

$Z = [n/(\log n)t]$ and $p_1 > t+1$ implies that $M > |W|$ which is impossible since $L(f \restriction i-k) \subseteq L(f \restriction i) \subseteq W$ for all $i \in Z$, $k \mid i$. Assume now that $\beta > 1$. In this case (P_{10}) implies that

$$\begin{aligned} M'_\beta &= \sum_{k \mid i} |K(\beta-1, f \restriction i) - K(\beta-1, f \restriction i-k)| \\ &\geqslant Zc_2(k)\frac{|W|}{n}(\log n)^{p_\beta} - c_3(k)\sum_i |L(f \restriction i) - L(f \restriction i-k)|\,(\log n)^{p_{\beta-1}}. \end{aligned}$$

Using again that $\sum_i |L(f \restriction i) - L(f \restriction i-k)| \leqslant |W|$ and $Z = n/(\log n)^t$ and $p_{\beta-1} < \frac{1}{2}p_\beta < p_{\beta-t-1}$ we get

$$M'_\beta \geqslant (\log n)^{p_{\beta-t-1}}|W| \geqslant (\log n)^{p_\beta/2}|W|.$$

Let $\gamma > 1$ be the smallest integer with

$$M'_\gamma > (\log n)^{p_\gamma/2}|W|.$$

(P_{11}) implies that

$$\begin{aligned} M''_\gamma = \sum_i |K(\gamma-1, f \restriction i-k) - K(\gamma-1, f \restriction i)| &\leqslant 6k^3(\log n)^{p_{\gamma-1}} \\ \times\Bigg[\Bigg(\sum_{1 \leqslant \alpha < \gamma-1}\ \sum_i |K(\alpha, f \restriction i) - K(\alpha, f \restriction i-k)|\Bigg) \\ + \sum_i |L(f \restriction i) - L(f \restriction i-k)| + Z\frac{|W|}{n}\Bigg]. \end{aligned}$$

$Z = n/(\log n)^t$, $\sum_i |L(f \restriction i) - L(f \restriction i-k)| \leqslant |W|$ and the minimality of γ implies that

$$\begin{aligned} M''_\gamma \leqslant 6k^3(\log n)^{p_{\gamma-1}}\Big[&k(\log n)^{p_{\gamma-2}/2} \\ &+ \sum_i |K(1, f \restriction i) - K(1, f \mid i-k)| + |W| + (\log n)^{-t}|W|\Big]. \end{aligned}$$

Applying (P_{11}) in the case $\alpha = 1$ we get

$$\begin{aligned} \sum_i |K(1, f \restriction i-k) - K(f \restriction i)| &\leqslant 6k^3(\log n)^{p_1}(|W| + (\log n)^{-t}|W|) \\ &\leqslant 12k^3(\log n)^{p_1}|W|. \end{aligned}$$

Since $|K(1, f \restriction i)| \leqslant 6k^3(\log n)^{p_1}|W|$ for all i (as a consequence of (P_3)) we have that

$$\begin{aligned} \sum_i |K(1, f \restriction i) - K(1, f \restriction i-k)| &\leqslant \sum_i |K(1, f \restriction i-k) - K(1, f \restriction i)| \\ &\quad + 2 \cdot 6k^3(\log n)^{p_1}|W| \leqslant 24k^3(\log n)^{p_1}|W|. \end{aligned}$$

Thus we get the following upper bound for $|M''_\gamma|$

$$M''_\gamma \leqslant c_4(k)(\log n)^{2p_{\gamma-1}}|W|.$$

Clearly (using (P_3))

$$\begin{aligned} M'_\gamma &\leqslant M''_\gamma + |K(\gamma-1, f \restriction 0)| + \left|K\left(\gamma-1, f \restriction k\left[\frac{Z}{k}\right]\right)\right| \\ &\leqslant c_4(k)((\log n)^{2p_{\gamma-1}} + 12k^3(\log n)^{p_{\gamma-1}})\,|W| \\ &\leqslant c_5(k)(\log n)^{2p_{\gamma-1}}\,|W| \end{aligned}$$

which contradicts $M'_\gamma \geqslant (\log n)^{p_\gamma/2}\,|W|$. So we have proved $2^{-Z}\,|S| \leqslant 2^{-cZ}$ for some $c>0$, which implies (II′) with $v'=t$. Now we prove that (II′) $\to$ (II). First note that if the lemma is true for some t, then it is true for all smaller t's as well, since we may take an arbitrary complete refinement of T and it will satisfy (II) as well. So let $t' > c_1 j + t$ where $|W| \leqslant n^j$ and c_1 is a constant whose value will be fixed later.

By recursion on i we define a sequence of complete cylinders $T^0, T^1, \ldots, T^{\lceil c\cdot j\log n\rceil}$ and a sequence of functions $W^0(D), W^1(D), \ldots$ whose values are subsets of W.

The definition of these sequence is the following $T^0 = \{{}^n2\}$, $W^0({}^n2) = W$. Suppose T^i, W^i are defined. If X is a set of cylinders and D is a cylinder on n, then let

$$\begin{aligned} [X]_D = \{C \mid {}&\text{“}C \text{ is a cylinder on } n-s(D)\text{”} \\ &\wedge \exists\, F \in X\, F \cap D \neq 0 \wedge b(F) \vdash (n-s(D)) = b(C)\} \end{aligned}$$

If $D \in T^i$ let $\bar{W}^i(D) = \{[X]_D \mid X \in W^i(D)\}$. Let us apply (II′) with $W \to \bar{W}^i(D)$, $t \to t'$, $n \to (n-s(D))$. Let T^i_D the corresponding complete set of cylinders on $n-s(D)$. Now we define T^{i+1} by

$$\begin{aligned} T^{i+1} = \{E \mid {}&\text{“}E \text{ is a cylinder on } n\text{”} \\ &\wedge \exists\, D \in T^i\, K \in T^i_D\; b(E) = b(D) \cup b(K)\}. \end{aligned}$$

If $E \in T^{i+1}$, $b(E) = b(D) \cup b(K)$, then let

$$W^{i+1}(E) = \{X \in W^i \mid \neg(\mathcal{P}(K, [X]_D) \vee K \subseteq \bigcup [X]_D)\}$$

where

$$\begin{aligned} \mathcal{P}(F, Y) \leftrightarrow \forall\, V \subseteq Y\, (\forall\, C_1, C_2 \in V\;\; & s(C_1) \cap s(F) = 0 \\ & \wedge \|C_1\| = k \wedge s(C_1) \cap s(C_2) = 0) \to |V| \leqslant (\log n)^p. \end{aligned}$$

Our definition imply that

$$l(T^i) \leqslant i\frac{n}{(\log n)^{t'}}$$

and $|W^{i+1}(E)| \leqslant (1-\varepsilon)^i\,|W|$ for all $E \in T^{i+1}$ except a subset $S^{i+1} \subseteq T^{i+1}$ with $|S^{i+1}| \leqslant 2^{l(T^i)} \cdot 2^{-n/(\log n)^{v'}}$. Let $T = T^{i_0}$ where $i_0 = c \cdot j \log n$ and $(1-\varepsilon)^{i_0}\,|W| < 1$. If $E \in T^{i_0}$ let E_i be the unique element of T^i with $E_i \supseteq E$.

If E_i is not among the exceptions that is $E_i \notin S_i$ for all $i \leqslant i_0$, then $W^{i_0}(E) = 0$. In that case for all $X \in W$ there exists an $i \leqslant i_0$ with $X \in W_i[E_i]$, $X \notin W_{i+1}[E_{i+1}]$. The definition of $W_{i+1}[E_{i+1}]$ implies that $\mathscr{P}(E, X) \vee E \subseteq \bigcup X$ for all E with $\forall i \leq i_0$ $\exists i \notin S_i$.

$$\begin{aligned}|\{E \in T \mid \exists\, i < i_0\; E_i \in S_i\}| &\leq \sum_{i \leq i_0} |\{E \in T \mid E_i \in S_0\}| \\ &\leq i_0 2^{l(T)} 2^{-n/(\log n)^{v'}} \\ &\leq 2^{l(T)} 2^{-n/(\log n)^{v'+1}}\end{aligned}$$

which proves (II). □

Now using Lemmas 5.6 and 5.5 we may prove Lemma 5.3 by induction on k. Indeed let $\bar{T}$ be a complete set of cylinders with (I) and (II) (with $k+1$ instead of k, $t = k+1$, $v' = v+1$).

Let $T' \subseteq \bar{T}$ be the set defined in (II). If $D \in T'$ let $W_D = \{X_D \mid X \in W \wedge D \not\subseteq \bigcup X\}$ where

$$X_D = \{F \mid \text{“}F \text{ is a cylinder on } n - s(D)\text{”} \wedge \exists\, C \in X \quad C \cap D \neq 0 \wedge b(C) \restriction n - s(D) = b(F)\}.$$

Clearly for all $D \in T'$ we have $\Psi(W_{D,p})$ (cf. Lemma 5.5). Now we may apply Lemma 5.5 ($l \to l+1, v \to v+1$) and we get a complete set of cylinders R_D on $n - s(D)$ and a function $Y^D(X, F)$ for all $D \in T'$. (Let R_D be an arbitrary complete set of cylinders on $n - s(D)$ with $l(R_D) = l+1$ if $D \in \bar{T} - T'$.)

Let

$$T = \{E \mid \text{“}E \text{ is a cylinder on } n\text{“} \wedge \exists\, D \in \bar{T}, F \in R_D\; b(E) = b(D) \cup b(F)\}.$$

and if $b(E) = b(D) \cup b(F)$ let

$$Y(X, E) = \{C \mid C \in X \wedge \exists\, C' \in Y^D(X, F)\; b(C) \restriction (n - s(D)) = b(C)\}$$

if $D \in T'$, $X \in W_D$; $Y(x, E) = \{C\}$ for some $C \in X$ with $D \subseteq C$ if $D \in T'$ $X \in W - W_D$; $Y(X, E) = 0$ if $D \notin T'$. Clearly T and $Y(x, E)$ satisfy the requirements of Lemma 5.3.

Lemma 5.7. $\forall\, \varepsilon > 0$, $k, j \in \omega$ $\exists\, t, v \in \omega$; *such that if n is large enough and $\mathscr{B}$ is a set of systems of cylinders with $|\mathscr{B}| = n^j$ and $C \in X \in \mathscr{B} \to \|C\| \leq k$, then there exists a complete set of cylinders R with*

(a) $l(D) = [n - n^{1-\varepsilon}]$

(b) *for all $X \in \mathscr{B}$, $E \in R$ there exists a set of cylinders $Z(X, E)$ such that*

$$\|Z(X, E)) \leq t$$

and

$$2^{-n} \left|({}^{n}2 - \bigcup X)\, \Delta \bigvee_{E} (R, Z(X, E))\right| \leq 2^{-n/(\log n)^v}.$$

Proof of Lemma 5.7. Let us apply Lemma 5.3 in the case $W = \mathscr{B}$, $l = 1$. Then we get a complete set of cylinders T. Let $D \in T$. Applying Lemma C.2 we get a $t \in \omega$

and a $H_D \subseteq n$, $H_D \supseteq s(D)$, $|H_D| = [n - n^{1-\varepsilon}]$ with

$$|(n - H_D) \cap \bigcup \{s(C) \mid C \in Y(X, D)\} \leq t \tag{5.7}$$

for all $X \in W$, $D \in T$. (t does not depend on D or n.)

Let

$$R = \{E \mid E \text{ is a cylinder on } n \wedge (\exists\, D \in T \;\; E \subseteq D \wedge s(E) = H_D)\}.$$

For an $E \in R$ we will denote the corresponding D by $D(E)$. If $X \in \mathscr{B}$, $E \in R$, then let us define $Z(X, E)$ as follows: first set

$$S(E, X) = (n - H_{D(E)}) \cap \bigcup \{s(C) \mid C \in Y(X, D(E))\}.$$

By (5.7) $|S(X, E)| \leq t$. Now let

$$Z(X, E) = \{C \mid C \text{ is a cylinder on } n \wedge s(C) = S(X, E) \wedge \forall\, C' \in Y(X, D(E))\; E \cap C \nsubseteq E \cap C'\}.$$

The definition of $Z(X, E)$ implies that for all $X \in \mathscr{B}$, $D \in T$

$$\bigcup \{\bigcup \{C \cap E \mid C \in Z(X, E)\} \mid D(E) = D \wedge E \in R\} = {}^n 2 - \bigcup \{C \cap D \mid C \in Y(X, D)\}.$$

Taking the union of both sides of this equation for all $D \in T$, by Lemma 5.3 we get the required inequality. □

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