The only noncontextual model of the stabilizer subtheory is Gross'

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We prove that there is a unique nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory in odd dimensions, namely Gross' discrete Wigner function. This representation is equivalent to Spekkens' epistemically restricted toy theory, which is consequently singled out as the unique noncontextual ontological model for the stabilizer subtheory. Strikingly, the principle of noncontextuality is powerful enough (at least in this setting) to single out *one particular* classical realist interpretation. Our result explains the practical utility of Gross' representation, e.g. why (in the setting of the stabilizer subtheory) negativity in this particular representation implies generalized contextuality, and hence sheds light on why negativity of this particular representation is a resource for quantum computational speedup.

I. INTRODUCTION

Quantum computers have the potential to outperform classical computers at many tasks. One of the major outstanding problems in quantum computing is to understand what physical resources are necessary and sufficient for universal quantum computation. These resources may depend on one's model of computation [1–3], and in some cases it seems that neither entanglement nor even coherence is required in significant quantities [2].

The primary obstacle to building a quantum computer is that implementing low-noise gates is difficult in practice. While there are no gate sets which are easy to implement and also universal [4], the entire stabilizer subtheory [5, 6] can in fact be implemented in a fault-tolerant manner relatively easily. To promote the stabilizer subtheory to universal quantum computation, one must supplement it with additional nonstabilizer (or 'magic') processes. Because these nonstabilizer resources do not have a straightforward fault-tolerant implementation, they are typically noisy. To get around this problem, Bravyi and Kitaev [7] introduced the magic state distillation scheme, whereby fault-tolerant stabilizer operations are used to distill pure resource states out of the initially noisy resources. However, not every nonstabilizer resource can be distilled in this fashion to generate a state which promotes the stabilizer subtheory to universal quantum computation. It is a major open question to determine which states are in fact necessary and sufficient for this purpose.

Quasiprobability representations are a central tool for making progress on these and related problems. For finite-dimensional quantum systems, a number of quasiprobability representations have been studied. For example, Gibbons, Hoffman, and Wootters identified a family of representations on a discrete phase space [8], and Gross then singled out one of these with a higher degree of symmetry [9], by virtue of satisfying a property known as Clifford covariance. All of these have been used to study quantum computation [10–17].

Gross' representation in particular has been the most useful in understanding the resources required for computation. For instance, Ref. [12] extended the Gottesman-Knill theorem [6] by devising an explicit simulation protocol for quantum circuits composed of Clifford gates supplemented with arbitrary states and measurements that have nonnegative Gross' representation. Ref. [12] also proved that every state which is useful for magic state distillation necessarily has negativity in its Gross' representation. In Ref. [14], this result was leveraged to prove that every state that promotes the stabilizer subtheory to universal quantum computation via magic state distillation must also exhibit Kochen-Specker contextuality [18]. In recognition that negativity in Gross' representation is a resource for quantum computation in this sense, Ref. [13] introduced an entire resource theory [19] of Gross' negativity.

From a foundational perspective, it is surprising that any *particular* quasiprobability representation plays such a central role. As argued in Ref. [20], negativity of any one quasiprobability representation is not sufficient to establish nonclassicality in general scenarios. So how can it be that Gross' representation plays such an important role, e.g. that negativity in it is associated to a strong form of nonclassicality, namely computational speedups? Although Gross' representation is uniquely singled out from the family of GHW representations by Clifford covariance, it has previously been unclear what this property has to do with nonclassicality (not to mention why one would restrict one's attention to the family of GHW representations in the first place).

In this paper, we resolve this mystery by showing that the *only* nonnegative and diagram-preserving [21] quasiprobability representation of the stabilizer subtheory is Gross'. In the setting of the full stabilizer subtheory, this proves that negativity *of this particular* quasiprobability representation is a rigorous signature of nonclassicality, i.e., the failure of generalized noncontextuality. *Generalized noncontextuality* is a principled [22–24], useful [25–36], and operational [37–41] notion of classicality. If one's process has negativity in Gross' representation, then our result establishes that there is no nonnegative representation of the full stabilizer subtheory together with that process. Since nonnegative quasiprobability representations are in one-to-one correspondence with generalized noncontextual ontological models [20, 21, 23], this means that there is no noncontextual representation for the scenario, and hence no classical explanation of it.[1]

Given the known links between resources for quantum computation and negativity in Gross' representation, together with our result connecting such negativity to the failure of generalized noncontextuality, one can then derive connections between resources for quantum computation and generalized noncontextuality.

We illustrate this by proving two such results. First, we give an analogue of the celebrated result in Ref. [14]: namely, we prove that generalized contextuality is necessary for universal quantum computation. Second, we prove that a sufficient condition for any unitary to promote the stabilizer subtheory to universal quantum computation is that it have negativity in Gross' representation. This is in analogy with the fact that a sufficient condition for any pure state to promote the stabilizer subtheory to universal quantum computation via magic state distillation is that it have negativity in Gross' representation [12, 43].

Finally, we note that our main result demonstrates that the principle of generalized noncontextuality is a much stronger principle than was previously recognized, at least in some settings. This is exemplified by the fact that for stabilizer theories in odd dimensions, it does not merely provide constraints on ontological representations, it *completely fixes* the ontological representation. This offers some hope that if the notion of a generalized noncontextual model can be relaxed in such a way [42] that lifts the obstructions to modelling the entirety of quantum theory, such a model of the full theory might also be unique. In our view such a uniqueness result would offer a compelling reason to take the identified ontology seriously.

[1] Note that Ref. [42] introduced a more refined framework for studying ontological models and noncontextuality, and argued that better terminology for these are 'classical realist representations' and 'Leibnizianity', respectively. We do not use this framework or terminology here only so that our results are easier to parse for readers who have not read Ref. [42].

II. THE STABILIZER SUBTHEORY

The stabilizer subtheory is one of the most important subtheories of quantum theory in the field of quantum information, playing an important role in quantum computing [5–7, 14, 44, 45], quantum error correction [5, 6, 46–48], and quantum foundations [49–54].

The stabilizer subtheory is built around the Clifford group, whose elements will be referred to as Clifford unitaries. To define these, we first introduce the *Weyl operators* (also called generalized Pauli operators) for odd dimension d. Consider a d-dimensional quantum system, and define the computational basis $\{|0\rangle, \ldots, |d-1\rangle\}$ in its Hilbert space $\mathcal{H}$. Each basis element is labelled by an element of $\mathbb{Z}_d$ [2], which we refer to as the configuration space. Writing $\omega = \exp(\frac{2\pi i}{d})$, we define the translation operator X and boost operator Z via

$$X|x\rangle = |x+1\rangle \tag{1}$$
$$Z|x\rangle = \omega^x |x\rangle . \tag{2}$$

Note that here and throughout, all arithmetic is within $\mathbb{Z}_d$. These can be viewed as discrete position and momentum translation operators, respectively, for a particle on a ring. From these, the single-system Weyl operators are defined as

$$W_{p,q} = \omega^{-2^{-1}pq} Z^p X^q, \tag{3}$$

where $p, q \in \mathbb{Z}_d$, and 2^{-1} is the inverse of 2 in $\mathbb{Z}_d$.

The Weyl operators are unitaries that form a group with composition law

$$W_{p,q} W_{p',q'} = \omega^{2^{-1}\left[\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p' \\ q' \end{pmatrix}\right]} W_{p+p',q+q'}, \tag{4}$$

where $[\cdot,\cdot]$ denotes the symplectic product

$$\left[\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p' \\ q' \end{pmatrix}\right] := pq' - qp'. \tag{5}$$

The inverse of a Weyl operator is given by

$$W_{p,q}^{-1} = W_{p,q}^{\dagger} = W_{-p,-q}. \tag{6}$$

Hence, it is clear that for $d > 2$ the Weyl operators are not Hermitian.

[2] When d is prime, $\mathbb{Z}_d$ has the structure of a finite algebraic field. When d is odd but not prime, things are somewhat more complicated [9], but the results in this work still hold.

Note that the Weyl operators form an orthonormal basis for the complex vector space of linear operators on the Hilbert space, where orthonormality is with respect to a rescaled Hilbert-Schmidt inner product:

$$\frac{1}{d}\,\mathsf{tr}[W_{p,q}W^{\dagger}_{p',q'}] = \delta_{p,p'}\delta_{q,q'}. \tag{7}$$

The Clifford unitaries are defined as unitaries which—up to a phase—map Weyl operators to other Weyl operators under conjugation. That is, U is a Clifford unitary if for every p, q, one has

$$UW_{p,q}U^{\dagger} = \exp(i\phi)\,W_{p',q'} \tag{8}$$

for some ϕ, p', q' (which depend on p, q, and U).

Let us now define the *phase space* $V := \mathbb{Z}_d \times \mathbb{Z}_d$, which is a module[3] equipped with the symplectic product given by Eq. (5). Note that each Weyl operator is labeled by a phase space point $(p,q) = a \in V$. A function $f : V \to V$ is said to be linear if $f(\lambda a + b) = \lambda f(a) + f(b)$, for $\lambda \in \mathbb{Z}_d$, $a, b \in V$. A function $S : V \to V$ is called symplectic if it is linear and preserves the symplectic product, i.e. $[S\cdot, S\cdot] = [\cdot,\cdot]$. Note that the symplectic functions form a group. A transformation of the form $S \cdot +a$ where S is symplectic and $a \in V$ is called a symplectic affine transformation. Note that the symplectic affine transformations also form a group.

As shown in Ref. [9], every Clifford operation is of the form $W_a M_S$, where W_a is a Weyl operator labelled by $a \in V$, $S : V \to V$ is a symplectic function, M is a projective representation (called the Weil or metaplectic representation) of the symplectic group (i.e. $M_S M_T = \exp(i\phi) M_{ST}$ for some ϕ), and where $M_S W_v M_S^{\dagger} = W_{Sv}$ for any symplectic function S and for all $v \in V$.

Hence, each Clifford operation can be indexed by a phase space vector a and a symplectic map S, and so we will denote them by $C_{a,S} := W_a M_S$. Clearly, a Weyl operator $W_{p,q}$ is a Clifford unitary $C_{a,S}$, where $a = (p,q)$ and $S = \mathbb{1}$. Furthermore, the mapping $S \cdot +a \mapsto W_a M_S$ is a projective representation of the group of symplectic affine transformations [9].

The Clifford unitaries form a group, often termed the Clifford group, with composition rule

$$C_{a,S}C_{b,T} = C_{Sb+a,ST}. \tag{9}$$

The inverse of a Clifford unitary is

$$C^{-1}_{a,S} = C^{\dagger}_{a,S} = C_{-S^{-1}a,S^{-1}}. \tag{10}$$

It is therefore clear that the Clifford group in dimension d and the symplectic affine group for $\mathbb{Z}_d \times \mathbb{Z}_d$ are isomorphic groups.

For a fixed dimension, the Clifford group is generated by the generalized Hadamard gate H and the generalized phase gate S [55], defined respectively by

$$H\,|x\rangle = \frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d} \omega^{xk}\,|k\rangle\,, \tag{11}$$

$$S\,|x\rangle = \omega^{2^{-1}x(x-1)}\,|x\rangle\,. \tag{12}$$

Note also that

$$HW_{p,q}H^{\dagger} = W_{q,-p}. \tag{13}$$

The stabilizer subtheory in dimension d is defined as the set of processes which can be generated by sequential composition of eigenstates of the Weyl operators, projective measurements in the eigenbases of the Weyl operators, and Clifford unitaries on the associated Hilbert space, as well as appropriate convex mixtures of such processes.

This construction is easily generalized to allow parallel composition (i.e. multiple d-dimensional systems) as well, by defining the multiparticle Weyl operators as tensor products of those defined above; see Ref. [9] for more details.

III. QUASIPROBABILITY REPRESENTATIONS

A *quasiprobability representation* [21, 56] is akin to a mathematical representation of quantum[4] processes as stochastic processes on a sample space, except that the representation may take negative values. For the reasons laid out in Refs. [21, 42], we are only interested in quasiprobability representations that satisfy the assumption of *diagram preservation* [21, 42]—namely, that the representation of a composite process is equal to the composition of the representations of its component processes. This assumption is satisfied by most of the useful quasiprobability representations considered in the literature, including the standard (continuous-dimensional) Wigner function and Gross' representation.

The arguments of Ref. [21] imply that every diagram-preserving quasiprobability representation of a full dimensional subtheory[5] of quantum theory can be written as an exact frame representation [56], constructed

[3] If d is a prime power $d = p^k$ then this is moreover a finite vector space.

[4] Quasiprobability representations can also be defined for generalized probabilistic theories [57, 58] (GPTs) beyond quantum theory [21, 59], but we are here only interested in the case of quantum theory and its subtheories.

[5] That is, in which the states span the quantum state space and the effects span the quantum effect space. Note that the stabilizer subtheory is such a theory, which can be seen by noting that the Weyl operators span the space of Hermitian operators, and hence, so do their eigenstates.

as follows. One first associates to each system a basis $\{F_\lambda\}_\lambda$ for the real vector space $\mathsf{Herm}(\mathcal{H})$ of Hermitian operators on the associated Hilbert space $\mathcal{H}$, where

$$\mathsf{tr}[F_\lambda] = 1, . \tag{14}$$

Every basis has a unique dual basis, $\{D_\lambda\}_\lambda$, as proved in Lemma A.1, where

$$\sum_\lambda D_\lambda = \mathbb{1}, \tag{15}$$

and

$$\mathsf{tr}[D_{\lambda'} F_\lambda] = \delta_{\lambda\lambda'}. \tag{16}$$

In this representation, a completely-positive trace-preserving map [60, 61] $\mathcal{E}$ is represented by a quasistochastic map defined by

$$\xi_\mathcal{E}(\lambda'|\lambda) = \mathsf{tr}[D_{\lambda'}\mathcal{E}(F_\lambda)]. \tag{17}$$

As special cases, the representation of a state ρ is given by

$$\xi_\rho(\lambda) = \mathsf{tr}[D_\lambda \rho] \tag{18}$$

and the representation of an effect E is given by

$$\xi_E(\lambda) = \mathsf{tr}[F_\lambda E]. \tag{19}$$

Note that for a set of effects that sum to the identity, Eq. (14) ensures that $\sum_E \xi_E(\lambda) = 1$.

The quantum probabilities are recovered as

$$\mathsf{tr}[E\mathcal{E}(\rho)] = \sum_{\lambda',\lambda} \xi_E(\lambda')\xi_\mathcal{E}(\lambda'|\lambda)\xi_\rho(\lambda). \tag{20}$$

Note that this is an instance of diagram preservation, wherein one decomposes the probability into the composition of the representations of the state, channel, and effect.

A quasiprobability representation is said to be *nonnegative* if for every process $\mathcal{E}$, $0 \leq \xi_\mathcal{E}(\lambda'|\lambda) \leq 1$ for every λ, λ'. In this case, the representation is in one-to-one correspondence with a noncontextual ontological representation [22, 42].

The particular quasiprobability representation introduced by Gross [9] takes the sample space to be a discrete classical phase space V, and so its elements will be labelled by position and momentum, i.e $a := (p, q) \in V$, rather than λ. Hence, the basis operators in Gross' representation are indexed by $a \in V$, and we will denote them by A_a. They are defined as

$$\{A_a\}_a := \left\{ \frac{1}{d} \sum_b \exp([a, b]) W_b^\dagger \right\}_a , \tag{21}$$

and we will sometimes refer to them as phase space point operators. These operators form an orthonormal basis for $\mathsf{Herm}(\mathcal{H})$, and so the basis is essentially self-dual, so that both $\{F_\lambda\}$ and $\{D_\lambda\}$ are proportional to $\{A_a\}$, with $D_\lambda = \frac{1}{d}F_\lambda$. They moreover satisfy a number of useful properties (see, e.g., Lemma 29 of Ref. [9]) including

$$A_a = W_a A_0 W_a^\dagger. \tag{22}$$

Eq. (22) is a special case of a key feature of Gross' representation, namely *Clifford covariance* [9]:

$$C_{a,S} A_b C_{a,S}^\dagger = W_a M_S A_b M_S^\dagger W_a^\dagger = A_{Sb+a} \tag{23}$$

for any Clifford unitary $C_{a,S}$. This property implies, for example, that when one transforms a quantum state under a given Clifford unitary, the representation of the state transforms under the associated symplectic affine map, i.e.

$$\xi_\rho(b) = \xi_{C_{a,S}\rho C_{a,S}^\dagger}(Sb + a). \tag{24}$$

To see this, note that the representation of the state is given by Eq. (18), which in this instance means that $\xi_\rho(b) := \mathsf{tr}[A_b \rho]$. Using this definition and the properties that we have so far introduced, we obtain:

$$\begin{aligned}
\xi_{C_{a,S}\rho C_{a,S}^\dagger}(Sb + a) &:= \mathsf{tr}[A_{Sb+a} C_{a,S} \rho C_{a,S}^\dagger] && (25)\\
&= \mathsf{tr}[C_{a,S}^\dagger A_{Sb+a} C_{a,S} \rho] && (26)\\
&\overset{(10)}{=} \mathsf{tr}[C_{-S^{-1}a,S^{-1}} A_{Sb+a} C_{-S^{-1}a,S^{-1}}^\dagger \rho] && (27)\\
&\overset{(23)}{=} \mathsf{tr}[A_{S^{-1}(Sb+a)-S^{-1}a} \rho)] && (28)\\
&= \mathsf{tr}[A_b \rho] && (29)\\
&=: \xi_\rho(b). && (30)
\end{aligned}$$

IV. UNIQUENESS OF GROSS' REPRESENTATION

Our main result is the following.

Theorem IV.1. *The unique nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory in odd dimensions is Gross' representation.*

The proof is given in Appendix B. The proof proceeds as follows. First, we apply the structure theorem in Ref. [21] to show that any nonnegative and diagram-preserving representation of the stabilizer subtheory must be an exact frame representation. Next, we leverage the fact that noncontextuality implies outcome determinism to find a privileged labeling of the ontic states as points in a phase space. We show that this implies translational covariance: that is, Clifford covariance for all Weyl operators. We then show that this implies that measurements of Weyl operators must be represented as in Gross' representation, up to an offset.[6]

[6] We believe, but have not shown, that distinct GHW representations differ by exactly these offsets.

By similar arguments, we show that the representation of the Hadamard satisfies Clifford covariance, and that this additional fact implies that the offset must be zero. This completely fixes the representations of measurements of Weyl operators to be as in Gross' representation. Since these form a basis for the complex vector space of linear operators on the Hilbert space, it follows that the basis operators in the frame representation are exactly Gross' phase space point operators.

As shown in Ref. [49, 51], Gross' representation is identical to Spekkens' epistemically restricted toy theory [62] for odd dimensions [49]. Furthermore, it is shown in Ref. [21] that noncontextual ontological models of an operational theory are in one-to-one correspondence with ontological models of the GPT defined by the operational theory, and also in one-to-one correspondence with diagram-preserving and nonnegative quasiprobability representations of the GPT defined by the operational theory. Through these equivalences, our result can be stated in a number of equivalent ways (e.g., depending on whether one views the stabilizer subtheory as an operational theory or as a GPT). Perhaps the most natural equivalent statement of Theorem IV.1 is the following: For odd dimensions, the unique noncontextual representation of the stabilizer subtheory is Spekkens' epistemically restricted toy theory.

There are several senses in which our uniqueness result is stronger than that proven by Gross [9]. Most importantly, the principle of generalized noncontextuality is a well-established notion of classicality, while the notion of covariance is not. Additionally, our result starts from the very weak assumption of classical realism [42]—that is, the ontological models framework—while Gross' result requires two additional assumptions beyond this, namely that the representation is on a $d \times d$ phase space and gives the correct marginal probabilities. In our approach, both of these are derived. Finally, our uniqueness result holds in all odd dimensions, while Gross' uniqueness result was proven only for odd prime dimensions.

In even dimensions, the common wisdom is that the stabilizer subtheory does not admit of *any* noncontextual model. For $d = 2$ in particular, there are known proofs of contextuality, e.g. in Ref. [53]. For even dimensions $d > 2$, the presence or absence of contextuality may depend on one's definition of the stabilizer subtheory. For example, one could consider such a system as a composite containing a qubit together with a system of dimension $\frac{d}{2}$; this is an instance of what Gross calls the 'multi-particle' view. Alternatively, one could consider it as a single, monolithic system; Gross calls this the 'single-particle' view. These two options lead one to distinct operational theories; e.g., Gross has shown that there are many more multi-particle stabilizer states than there are single-particle stabilizer states [9]. Each of these might sensibly be termed a 'stabilizer subtheory of dimension d'. It is clear that the subtheory defined by the multiparticle view above does in fact exhibit contextuality, since it contains the single-qubit stabilizer theory as a subtheory. In the single-particle view (which we, like Gross, have focused on here), this is not the case, and (to our knowledge) it remains an open question whether or not there exist proofs of contextuality in all even dimensions.

Conjecture 1. *Every stabilizer subtheory of even dimension admits a proof of generalized contextuality.*

Such a result, together with our main theorem, would constitute a complete characterization of the (non)classicality of the stabilizer subtheories in every dimension.

V. GENERALIZED CONTEXTUALITY AS A RESOURCE FOR QUANTUM COMPUTATION

The stabilizer subtheory is efficiently simulable [6]. However, if one supplements it with appropriate nonstabilizer states, one can achieve universal quantum computation through magic state distillation [7].

Any state which promotes the stabilizer subtheory to universal quantum computation must have negativity in its Gross' representation [12]. Ref. [14] further showed that every such state can be used to generate state-dependent proofs of Kochen-Specker contextuality using stabilizer measurements [14], and hence that contextuality is necessary for universality in this model of quantum computation.

The key argument of Ref. [14] was a graph-theoretic proof that if a state is negative in Gross' representation, then it admits a (state-dependent) proof of Kochen-Specker contextuality using only stabilizer measurements. Our main theorem, Theorem IV.1, is analogous, establishing that if a state is negative in Gross' representation, then it admits a proof of *generalized* contextuality.

Hence, we arrive at a result akin to that of Ref. [14]: generalized contextuality is necessary for universality in the state injection model of quantum computation.

Theorem V.1. *Consider any state ρ which promotes the stabilizer subtheory to universal quantum computation. There is no generalized noncontextual model for the stabilizer subtheory together with ρ.*

This follows immediately from the fact that negativity in a state's Gross' representation is necessary for it to promote the stabilizer subtheory to universal quantum computation [12], together with our result that negativity in Gross' representation implies generalized contextuality.

One may think that one could prove this without reference to our Theorem IV.1 by leveraging the fact that the impossibility of any nonnegative quasiprobability representation is equivalent to the failure of generalized noncontextuality [20, 21, 23]. However, it is not clear that every quasiprobability representation is efficiently computable, and hence, it is not clear that a nonnegative quasiprobability representation would describe an efficient simulation of a quantum computation.

A. On the sufficiency of generalized contextuality for universal quantum computation

Thus far we have focused on the necessity of contextuality for quantum computation. However, the fact that Gross' representation provides the unique generalized noncontextual representation of the stabilizer subtheory will likely also be useful for discovering in what sense (if any) generalized contextuality is *sufficient* for quantum computation.

Without any caveats, generalized contextuality is clearly not sufficient for universal quantum computation. This can be seen by the example of the stabilizer theory in dimension 2, which admits proofs of contextuality [53] and yet is efficiently simulable [6].

Still, it is conceivable that there is a more nuanced sufficiency result relating contextuality and computation, e.g. by leveraging quantitative measures of generalized contextuality [63] or by focusing on particular dimensions and models of quantum computation.

We now prove a related result (which does not explicitly rely on our main theorem).

From Ref. [12, 43], we know that access to enough copies of any nonstabilizer pure state promotes the stabilizer subtheory to universal quantum computation. Similarly, access to enough copies of any nonstabilizer unitary promotes the stabilizer subtheory to universal quantum computation, since the Clifford unitaries together with any other unitary gate forms a universal gate set [64, 65].

It is well known that every pure nonstabilizer state is negatively represented in Gross' representation [9]. Additionally, it is not hard to see that every nonstabilizer unitary gate is negatively represented in Gross' representation. Since Gross' representation is diagram-preserving, a composite process is represented negatively only if at least one of its component processes is represented negatively. But by the universal gate set property, combining the positively represented Clifford with the given nonstabilizer gate allows the approximation of any unitary. In particular, it can approximate some unitary that maps stabilizer states to a nonstabilizer state that is far from any stabilizer state. Since, as we just noted, the non-stabilizer state is negatively represented, so must the unitary that maps to it from a (positively represented) stabilizer state. Hence, we obtain the following theorem:

Theorem V.2. *A sufficient condition for any pure state or unitary to promote the stabilizer subtheory to universal quantum computation is that it be negatively represented in Gross' representation.*

For the case of pure states, this result was pointed out in Refs. [12, 43].

Perhaps the most important open question that remains is whether an analogous sufficiency result holds for mixed quantum states and generic quantum channels.

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Appendix A: Useful Preliminaries

It is well-known that a basis of a vector space uniquely defines a dual basis in the dual vector space (i.e. the space of functionals on the vector space). We will leverage this fact, but in a slightly different form:

Lemma A.1. *Given any basis $\{F_\lambda\}_\lambda$ for a d^2-dimensional real vector space $\mathsf{Herm}(\mathcal{H})$ of Hermitian operators on a Hilbert space $\mathcal{H}$, there is a unique set $\{D_\lambda\}_\lambda$ of d^2 Hermitian operators satisfying*

$$\mathsf{tr}(D_{\lambda'} F_\lambda) = \delta_{\lambda,\lambda'}, \tag{A1}$$

and $\{D_\lambda\}_\lambda$ also forms a basis for $\mathsf{Herm}(\mathcal{H})$.

Proof. Consider any basis $\{F_\lambda\}_\lambda$ of $\mathsf{Herm}(\mathcal{H})$. It uniquely specifies a basis $\{\mathcal{D}_\lambda\}_\lambda$ of the dual vector space $\mathsf{Herm}(\mathcal{H})^*$, where $\{\mathcal{D}_\lambda\}_\lambda$ are linear functionals satisfying $\mathcal{D}_{\lambda'}(F_\lambda) = \delta_{\lambda,\lambda'}$.[7] Now, in order to obtain again a set of Hermitian operators $\{D_\lambda\}_\lambda$, we use the Riesz representation theorem [66], which states that each of these functionals $\mathcal{D}_\lambda$ can be written as the Hilbert-Schmidt inner product with a unique Hermitian operator D_λ, namely

$$\mathcal{D}_\lambda(\cdot) = \mathsf{tr}[(\cdot) D_\lambda]. \tag{A2}$$

[7] To see that this is unique, consider a linear functional $\mathcal{D}'_{\lambda'}$ satisfying $\mathcal{D}'_{\lambda'}(F_\lambda) = \delta_{\lambda,\lambda'}$ for all λ. Since a linear functional is fully specified by its action on a basis, $\mathcal{D}'_{\lambda'}$ is the exact same functional as $\mathcal{D}_{\lambda'}$.

This picks out a unique basis $\{D_\lambda\}_\lambda$ which satisfies Eq. (A1). □

Note that the operators $\{F_\lambda\}_\lambda$ and $\{D_\lambda\}_\lambda$ are both in $\mathsf{Herm}(\mathcal{H})$. For a basis $\{F_\lambda\}_\lambda$, we refer to the set $\{D_\lambda\}_\lambda$ constructed using this lemma as the *dual basis*.

Another useful lemma we will require is the following.

Lemma A.2. *A nonnegative and diagram-preserving quasiprobabilistic representation of any unitary superoperator $\mathcal{U}(\cdot) := U(\cdot)U^\dagger$ is given by a permutation; that is, by a conditional probability distribution*

$$\xi_{\mathcal{U}}(\lambda'|\lambda) = \delta_{\sigma_U(\lambda'),\lambda} \tag{A3}$$

for some permutation $\sigma_U : \Lambda \to \Lambda$.

Proof. By definition, a nonnegative quasiprobabilistic representation ξ represents every unitary superoperator $\mathcal{U}$ as a stochastic map from Λ to itself, so $\xi_{\mathcal{U}}$ and $\xi_{\mathcal{U}^\dagger}$ are stochastic maps. By diagram preservation, it holds that $\xi_{\mathcal{U}\mathcal{U}^\dagger} = \xi_{\mathcal{U}} \circ \xi_{\mathcal{U}^\dagger}$. But $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$, and hence $\xi_{\mathcal{U}\mathcal{U}^\dagger} = \xi_{\mathbb{1}}$, where (by diagram preservation) $\xi_{\mathbb{1}}$ must be the identity matrix. Therefore $\xi_{\mathcal{U}^\dagger} \circ \xi_{\mathcal{U}}$ is the identity matrix, so $\xi_{\mathcal{U}^\dagger}$ is the left inverse of $\xi_{\mathcal{U}}$, and so (by the fact that they are square matrices) $\xi_{\mathcal{U}}$ and $\xi_{\mathcal{U}^\dagger}$ are inverses. But the only (square) stochastic matrices whose inverses are stochastic are permutations. Hence $\xi_{\mathcal{U}}$ is a permutation for every unitary U. □

A final useful lemma is a well-known result from Ref. [22]:

Lemma A.3. *Projective measurements have an outcome-deterministic representation in any noncontextual ontological model. That is, representation of the projectors in a projective measurement are conditional probability distributions valued in $\{0,1\}$. Furthermore, every ontic state is in the support of the representations of one and only one of eigenstates in any given projective measurement.*

This lemma was originally proven for full quantum theory, but it immediately generalizes to the stabilizer subtheory.

Appendix B: Proof of Theorem IV.1

We start from the assumption that we have *some* nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory in some odd dimension d. Note that this subtheory is tomographically local, and has GPT dimension d^2. Hence, Corollary VI.2 of Ref. [21] implies that the number of elements in the sample space is exactly d^2. Since a nonnegative and diagram-preserving quasiprobability representation is equivalent to a noncontextual ontological representation, we will refer to the elements of the sample space as 'ontic states'.

The structure theorems in Ref. [21] (in particular, Corollary VI.2) imply that this representation is an *exact frame representation* [56] composed of a basis $\{F_\lambda\}_\lambda$ and its dual $\{D_\lambda\}_\lambda$ (in the sense of Lemma A.1,) such that the representation of a completely positive trace preserving map $\mathcal{E}$ is given by the conditional quasiprobability distribution

$$\xi_{\mathcal{E}}(\lambda'|\lambda) = \mathsf{tr}[D_{\lambda'}\mathcal{E}(F_\lambda)]. \tag{B1}$$

Here, $\{F_\lambda\}_\lambda$ is a spanning and linearly independent set of d^2 Hermitian operators, as is $\{D_\lambda\}_\lambda$, where these satisfy

$$\mathsf{tr}[F_\lambda] = 1, \tag{B2}$$

$$\sum_\lambda D_\lambda = \mathbb{1}, \tag{B3}$$

and

$$\mathsf{tr}[D_{\lambda'}F_\lambda] = \delta_{\lambda\lambda'}. \tag{B4}$$

(Note, however, that the elements of each basis need not be pairwise orthogonal.)

Consider in particular the two stabilizer measurements corresponding to the $X^\dagger$ and Z operators.[8] If we label the outcome of $X^\dagger$ by $p \in \mathbb{Z}_d$ and the outcome of Z by $q \in \mathbb{Z}_d$, then by outcome determinism (Lemma A.3), each ontic state corresponds to an ordered pair (p,q). In fact, this correspondence is bijective, and hence we can choose a useful labelling of the ontic states, i.e. $\lambda \mapsto (p,q)$ (so that measurements of $X^\dagger$ reveal p and measurements of Z reveal q). To see that the correspondence is surjective, consider an eigenstate of X with eigenvalue ω^{-p_1}. The ontic states in the support of its representation must have $p = p_1$ so that the outcome of an X measurement is always p_1. Furthermore, a measurement of Z on this eigenstate gives a uniformly random outcome q, and so the ontic states in the support of its representation must include *every* ontic state of the form (p_1, q), for arbitrary $q \in \mathbb{Z}_d$. This holds for all d eigenstates of X, and thus for all $p_1 \in \mathbb{Z}_d$. So for every pair p, q, there exists some ontic state (in the support of one of the eigenstates of X) which has (p,q) as its label. This establishes surjectivity. Since the number (d^2) of ontic states is the same as the number of pairs (p,q), surjectivity implies bijectivity.

Next, we show that the assumed labelling forces the representation to manifestly satisfy translational covariance: that is, the Weyl unitaries must be represented in a Clifford covariant manner, so that the unitary superoperator $\mathcal{W}_{p_1,q_1}(\cdot) := Z^{p_1}X^{q_1}(\cdot)(X^{q_1})^\dagger(Z^{p_1})^\dagger$ is represented by the permutation $(p,q) \to (p+p_1, q+q_1)$. To see

[8] Note that although the Weyl operators are not Hermitian operators, they *are* normal operators, and hence have a spectral decomposition, which implies one can carry out a projective measurement in the eigenbasis of each.

this, first recall that the representation of a unitary superoperator is necessarily a permutation, as shown in Lemma A.2. Next, we determine the representation of the unitary superoperator $\mathcal{X}(\cdot) := X(\cdot)X^\dagger$. Consider an eigenstate of X with eigenvalue ω^{-p_1}. We argued above that the ontic states in the support of its representation must have $p = p_1$. Because the state is invariant under the unitary superoperator $\mathcal{X}$, the value of p must be unchanged by it. Similarly, consider an eigenstate of Z with eigenvalue ω^{q_1}. The ontic states in the support of its representation must have $q = q_1$. Applying the unitary superoperator $\mathcal{X}$ increments the Z eigenstate and corresponding eigenvalue by one, so that the value of q is transformed to $q_1 + 1$. Hence, we see that the representation of the unitary superoperator $\mathcal{X}$ takes $p \to p$ and $q \to q + 1$, which fully specifies its action as a permutation on the ontic states. (Note that this argument only holds for ontic states in the support of one of the X eigenstates and also in the support of one of the Z eigenstates. But by Lemma A.3, every ontic state is of this sort.) By a similar argument, the representation of the unitary superoperator $\mathcal{Z}(\cdot) := Z(\cdot)Z^\dagger$ takes $p \to p+1$ and $q \to q$. Since all Weyl unitary superoperators can be generated by composing $\mathcal{X}$ and $\mathcal{Z}$, and since the representation is diagram-preserving, this fully specifies the permutations representing all of the Weyl unitary superoperators. In particular, the unitary superoperator generated by $W_{p_1,q_1} = \omega^{-2^{-1}p_1q_1} Z^{p_1} X^{q_1}$ is indeed represented by the permutation $(p, q) \mapsto (p + p_1, q + q_1)$.

By a similar argument, we can deduce the representation of the Hadamard unitary superoperator $\mathcal{H}(\cdot) := H(\cdot)H^\dagger$, where H is defined in Eq. (11). In particular, if we start in the eigenstate of X with eigenvalue ω^{p_1}, then $p = -p_1$, and the Hadamard maps this to the eigenstate of Z with eigenvalue ω^{p_1}, for which $q = p_1$. So we see that the permutation representing the Hadamard superoperator results in a final value for q equal to the initial value of $-p$. Similarly, for the eigenstate of Z with eigenvalue ω^{q_1}, one has $q = q_1$, and this is mapped to the eigenstate of X with eigenvalue ω^{-q_1}, for which $p = q_1$. So we see that the permutation representing the Hadamard superoperator also results in a final value for p equal to the initial value of q. This fully specifies its action as a permutation on the ontic states, namely $(p, q) \mapsto (q, -p)$.

Next, we find strong constraints on the representation of measurements of arbitrary Weyl operators (not just X and Z measurements, whose representation was already determined above). In particular, we will show that translation covariance implies that every measurement of a Weyl operator is represented as in Gross' representation, up to an offset. Consider a measurement of a given Weyl operator W_{p_1,q_1} when the ontic state happens to be $(0, 0)$. By outcome determinism, we will always get a particular outcome, which we will label v_{p_1,q_1}. From this, together with the fact that every ontic state is in the support of *one* of the eigenstates of W_{p_1,q_1}, (in accordance with Lemma A.3,) it follows that the $(0, 0)$ ontic state must be in the support of the eigenstate of W_{p_1,q_1} which has eigenvalue $\omega^{v_{p_1,q_1}}$. (The actual quantum state prepared may not have been this eigenstate, but could be any state whose support contains the $(0, 0)$ ontic state.) Now, if we consider the application of the Weyl unitary superoperator $\mathcal{W}_{p_2,q_2}$ (to the quantum state, whatever it was), translational covariance means that the ontic state will be mapped from $(0, 0)$ to (p_2, q_2). From the composition rule of the Weyl operators in Eq. (4), i.e.

$$W_{p_1,q_1} W_{p_2,q_2} = \omega^{2^{-1}(p_1q_2-q_1p_2)} W_{p_1+p_2,q_1+q_2}, \tag{B5}$$

one can show that the unitary superoperator $\mathcal{W}_{p_2,q_2}$ maps the eigenstate of W_{p_1,q_1} with eigenvalue $\omega^{v_{p_1,q_1}}$ to the eigenstate of W_{p_1,q_1} with $\omega^{(v_{p_1,q_1}+p_1q_2-q_1p_2)}$. So the ontic state (p_2, q_2) must be in the support of this latter eigenstate, and hence always gives outcome $v_{p_1,q_1} + p_1q_2 - q_1p_2$ for a measurement of W_{p_1,q_1}. In Gross' representation, the outcome of a measurement of W_{p_1,q_1} on ontic state (p_2, q_2) is always $p_1q_2 - q_1p_2$, so we see that translational covariance gets us to Gross' representation, except with a possible (arbitrary but fixed) shift v_{p_1,q_1}.

Now, the Weyl operators are a basis for the complex vector space of linear operators on the Hilbert space, we can decompose the operator $F_{0,0}$ (namely, the element of the basis $\{F_\lambda\}_\lambda$ with $\lambda = (0, 0)$) as

$$F_{0,0} = \frac{1}{d}\sum_{p,q} f_{p,q} W_{p,q} \tag{B6}$$

for some complex coefficients $f_{p,q}$, where the fact that $F_{0,0}$ is Hermitian implies that

$$f_{p,q} = f^*_{-p,-q}. \tag{B7}$$

Any given Weyl operator W_{p_1,q_1} has a spectral decomposition $\sum_\alpha \omega^\alpha \Pi^{p_1,q_1}_\alpha$ in terms of its eigenvalues ω^α for $\alpha \in \mathbb{Z}_d$ and the projectors Π^{p_1,q_1}_α onto the corresponding eigenvectors. Computing the quantity $\mathsf{tr}[F_{0,0}W_{p_1,q_1}]$, we obtain

$$\mathsf{tr}[F_{0,0}W_{p_1,q_1}] = \sum_\alpha \omega^\alpha \mathsf{tr}[F_{0,0}\Pi^{p_1,q_1}_\alpha]. \tag{B8}$$

But we know that $\mathsf{tr}[F_{0,0}\Pi^{p_1,q_1}_\alpha]$ is the probability of outcome α occurring in a measurement of W_{p_1,q_1} when the ontic state is $(0, 0)$, and we have already established that the outcome that must occur in this case is that corresponding to eigenvalue $\omega^{v_{p_1,q_1}}$. It follows that $\mathsf{tr}[F_{0,0}W_{p_1,q_1}] = \omega^{v_{p_1,q_1}}$.

But a substitution of Eq. (B6) into the left-hand side of Eq. (B8) also allows us to compute this value as

$$\mathsf{tr}[F_{0,0}W_{p_1,q_1}] = \frac{1}{d}\sum_{p,q} f_{p,q}\mathsf{tr}[W_{p,q}W_{p_1,q_1}] = f_{-p_1,-q_1}, \tag{B9}$$

where the last equality follows from Eq. (7). Hence, we see that

$$f_{-p_1,-q_1} = \omega^{v_{p_1,q_1}}, \tag{B10}$$

and combining this with Eq. (B7) gives

$$-v_{p,q} = v_{-p,-q} \tag{B11}$$

for any p, q. Substituting Eq. (B9) into Eq. (B6), we get

$$F_{0,0} = \frac{1}{d} \sum_{p,q} \omega^{-v_{p,q}} W_{p,q}. \tag{B12}$$

Next, we will show that the shifts must in fact vanish; that is, $v_{p,q} = 0$ for all p, q. First, note that

$$H W_{p,q} H^\dagger = W_{q,-p}, \tag{B13}$$

and so from Eq. (B12)

$$H F_{0,0} H^\dagger = \frac{1}{d} \sum_{p,q} \omega^{-v_{p,q}} H W_{p,q} H^\dagger \tag{B14}$$

$$= \frac{1}{d} \sum_{p,q} \omega^{-v_{p,q}} W_{q,-p} \tag{B15}$$

$$= \frac{1}{d} \sum_{p,q} \omega^{-v_{q,-p}} W_{p,q}. \tag{B16}$$

But because the Hadamard is represented covariantly and so maps (p, q) to $(q, -p)$, we have

$$H F_{0,0} H^\dagger = F_{0,0} = \frac{1}{d} \sum_{p,q} \omega^{-v_{p,q}} W_{p,q}, \tag{B17}$$

and hence

$$v_{p,q} = v_{q,-p} \tag{B18}$$

for all p, q. Applying Eq. (B18) twice gives $v_{p,q} = v_{q,-p} = v_{-p,-q}$. Combining this with Eq. (B11), it follows that $v_{p,q} = 0$ for all p, q.

At this point, we have proven that

$$F_{0,0} = \frac{1}{d} \sum_{p,q} W_{p,q}, \tag{B19}$$

which exactly coincides with Gross' phase point operator (which he labels $A(0, 0)$). Furthermore, we already argued that our representation must satisfy translation covariance, which is satisfied if and only if $F_{p,q} = W_{p,q} F_{0,0} W_{p,q}^\dagger$; then, by Eq. (22), each $F_{p,q}$ exactly coincides with Gross' corresponding phase point operator. Hence, the set of basis operators $\{F_\lambda\}_\lambda = \{F_{p,q}\}_{p,q}$ is exactly equal to the set of phase point operators in Gross' representation. It remains only to apply Lemma A.1 to construct the unique dual basis $\{D_\lambda\}_\lambda = \{D_{p,q}\}_{p,q}$; since Gross' set of phase point operators is self-dual, it follows that the dual to $\{F_{p,q}\}_{p,q}$ is itself.

Hence, any nonnegative and diagram-preserving quasiprobability representation for the stabilizer subtheory in odd dimensions is equivalent to Gross'.

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