Complex manifolds

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1 Motivation

1.1 Real and complex differentiability

This course is concerned with the theory of complex manifolds. On the surface, their definition is identical to the one of smooth manifolds. It is only necessary to replace open subsets of $\mathbb{R}^n$ by open subsets of $\mathbb{C}^n$, and smooth functions by holomorphic functions.

This might lead you to think that the notion of complex manifolds has been created as a mere intellectual exercise, which cannot hold any surprises for those already versed in the classical theory. It is the main goal of this course to show by how far this hypothesis fails. The complex and real worlds are fundamentally different. The complex analogues of "real" definitions often yields more restrictive, and rigid mathematical objects.

To illustrate this principle we take a look at the notion of differentiability. Let $U \subset \mathbb{R}$ be some open subset, and $f\colon U \longrightarrow \mathbb{R}$ a function. We say that f is *differentiable*, if for every $x \in U$ the limit

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \tag{1}$$

exists. It is well-known that the thereby defined derivative $f'(x)$ can be quite pathological, in particularly it doesn't even have to be continuous (for instance the continuous extension of the function $x \mapsto x^2 \sin(\frac{1}{x})$ for $x \neq 0$ has a non-continuous derivative).

Assume now that $U \subset \mathbb{C}$ is an open subset of the complex numbers, and $f\colon U \longrightarrow \mathbb{C}$ a complex-valued function. The limit in (1) makes sense as it stands, for complex variables, and we say that f is complex differentiable, or *holomorphic*, if and only if the aforementioned limit exists for every $x \in U$. In contrast to the example of a real-valued, only once differentiable function given above, we have that the derivative of a complex-differentiable function is always continuous, in fact the following is true.

Theorem 1.1. *Every holomorphic function f is infinitely many times complex differentiable.*

This theorem is easily deduced from an even stronger result.

Theorem 1.2. *Every holomorphic function f can be locally expressed by a converging power series. That is, for every $z_0 \in U$ there exists $\epsilon > 0$, and $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}$, such that $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ for $|z-z_0| < \epsilon$.*

These two theorems demonstrate the philosophical principle mentioned earlier: *"The complex analogues of "real" definitions often yields more restrictive, and rigid mathematical objects."* While analytic functions (that is, those locally expressible by converging power series) are examples of differentiable functions in both the real and complex world, they are revealed to be the only complex example thereof, while being considered *special* amongst real differentiable functions.

Pretty much everything that is known about holomorphic functions is a consequence of an identity of Cauchy. For a holomorphic function f as above, $z_0 \in U$, and $\epsilon > 0$, such that $U_\epsilon = \{z|\ |z-z_0| < \epsilon\} \subset U$, we have

$$\int_{|z-z_0|=\epsilon} f(z)dz = 0. \tag{2}$$

1.2 A proof of the Cauchy formula using "real techniques"

Despite the differences between real and complex analysis, it is possible to use the real-valued theory to say something about the complex theory. In this paragraph we will deduce Cauchy's formula (2) from the so-called Stokes Theorem, which should be familiar from last term's manifolds lecture.

Theorem 1.3 (Stokes)**.** *Let M be a bounded compact n-dimensional manifold with boundary ∂M. For every smooth $(n-1)$-form ω we have the identity*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We will consider the disc $\mathbb{D}_\epsilon(z_0) = \{z \in \mathbb{C}|\ |z-z_0| \leq \epsilon\}$ as a bounded manifold, such that the boundary $\mathbb{S}^1$ is oriented in a counterclockwise manner. We will then attach to a holomorphic function $f(z)$ a complex 1-form $\omega = f(z)dz$, and apply Stokes Theorem in order to deduce Cauchy's formula (2). Before getting there, we need some preparatory work.

We use the standard bijection $\mathbb{C} \cong \mathbb{R}^2$, obtained by decomposing a complex number $z = x + iy$ into real and imaginary part, $z \mapsto (x, y)$. In other words, we view $\mathbb{C}$ as a smooth manifold of dimension 2, and similarly this allows us to view $\mathbb{D}_\epsilon(z_0)$ as a smooth manifold of dimension 2 with boundary.

Similarly we may decompose f into real and imaginary parts, $f(x+iy) = u(x,y) + iv(x,y)$, yielding real-valued functions u and v. Holomorphicity of f can be read off the functions u and v.

Theorem 1.4 (Cauchy-Riemann)**.** *The function $f: U \longrightarrow \mathbb{C}$ is holomorphic, if and only if u and v are differentiable and satisfy the Cauchy–Riemann differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.*

This theorem follows easily from the following observation in linear algebra. We can embed the field of complex numbers $\mathbb{C}$ into the ring of (2×2)-matrices, by assigning to $a + ib$ the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The Cauchy–Riemann Equations can now be rephrased as stipulating that the Jacobi matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is of the shape above, that is, corresponds to multiplication of $(x + iy)$ by a complex number.

On the manifold with boundary $\mathbb{D}_\epsilon(z_0)$ we now introduce the complex 1-form $dz = dx+idy$. It is a formal sum of a real and imaginary part, both of which are smooth 1-forms in the usual sense. We define a second complex 1-form by formal complex multiplication $\omega = f(z)dz = (u+iv)(dx+idy) = (udx - vdy) + i(udy + vdx)$.

Lemma 1.5. *A complex-valued function f, such that real part u and imaginary part v are differentiable, satisfies the Cauchy–Riemann equation, if and only if $\omega = f(z)dz$ satisfies $d\omega = 0$.*

Proof. We compute $d\omega$ by independently differentiating real and imaginary part:

$$d[(udx - vdy) + i(udy + vdx)] = \frac{\partial u}{\partial y} dy \wedge dx - \frac{\partial v}{\partial x} dx \wedge dy + i\frac{\partial u}{\partial x} dx \wedge dy + i\frac{\partial v}{\partial y} dy \wedge dx.$$

Using that $dx \wedge dy = -dy \wedge dx$, we obtain the following result:

$$d\omega = [(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})]dx \wedge dy.$$

The equation $d\omega = 0$ is equivalent to vanishing of both the real and imaginary part, which corresponds to the Cauchy–Riemann Equations. □

Now we apply the Stokes Theorem to the manifold with boundary $\mathbb{D}_\epsilon(z_0)$, and the complex 1-form $\omega = f(z)dz$. For f a holomorphic function, we obtain

$$\int_{\partial \mathbb{D}_\epsilon(z_0)} f(z)dz = \int_{\mathbb{D}_\epsilon(z_0)} d\omega = 0.$$

1.3 Holomorphic functions on complex manifolds

In this paragraph we will give a sloppy definition of complex manifolds, which will be revised at a later point. We will ignore some technical points, like that the topological space underlying a complex manifold should be Hausdorff and second-countable. Moreover, we pretend that we already know what a holomorphic function in several variables is. Moreover, we will ignore differences between atlases and maximal atlases, etc.

Definition 1.6. *Let X be a topological space, together with an open covering $\{U_i\}_{i\in I}$, and homeomorphisms $\phi_i\colon U_i \xrightarrow{\simeq} U_i'$, where $U_i' \subset \mathbb{C}^n$ is an open subset. If for every pair (i, j) the induced bijection*

$$\phi_i(U_i \cap U_j) \xrightarrow{\phi_j \circ \phi_i^{-1}} \phi_j(U_i \cap U_j)$$

is holomorphic, we say that X is endowed with the structure of a complex manifold *of complex dimension n.*

The pairs (U_i, ϕ_i) are called charts. The covering $X = \bigcup_{i\in I} U_i$ introduces a system of locally defined complex coordinates. A complex manifold of complex dimension $n = 1$ is also called a *Riemann surface.*

Definition 1.7. *Let X be a continuous manifold, and $f\colon X \to \mathbb{C}$ be a continuous map. We say that f is holomorphic, if for every chart (U, ϕ) as above, the composition $f \circ \phi^{-1}\colon U' \longrightarrow \mathbb{C}$ is a holomorphic function on U'.*

To emphasise the difference between complex and real manifolds, we will show that a compact complex manifold has only very few holomorphic functions. We will deduce this from a lemma about holomorphic functions, which states that for a non-constant holomorphic function g defined on an open subset $U' \subset \mathbb{C}^n$, the image $g(U') \subset \mathbb{C}$ is an open subset.

Lemma 1.8. *If X is a compact, connected, complex manifold, then every holomorphic function $f\colon X \longrightarrow \mathbb{C}$ is constant.*

Proof. Let us assume that f is not constant. For every chart (U_i, ϕ_i) we obtain a holomorphic function $g_i = f \circ \phi_i$. We see that $f(X) = \bigcup_{i\in I} g(U_i') \subset \mathbb{C}$ is an open subset. On the other hand, since X is compact, the image $f(X) \subset \mathbb{C}$ is compact as well, thus in particularly closed. The only open and closed subsets of $\mathbb{C}$ are $\emptyset$ and $\mathbb{C}$. The second case is not possible, because $\mathbb{C}$ is not compact. $\square$

Corollary 1.9. *A compact complex manifold X cannot be embedded into any $\mathbb{C}^n$, unless X is 0-dimensional.*

Proof. Let us assume that we have an injective holomorphic map $h\colon X \hookrightarrow \mathbb{C}^n$. Let $p_i\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ the projection to the i-th coordinate, that is, the map sending $(z_0, \ldots, z_n) \in \mathbb{C}^n$ to z_i. The maps $p_i \circ h$ are holomorphic, and therefore constant on every connected component of X. This implies that h can only be injective, if X is a disjoint union of points. $\square$

So far this is probably the most striking difference with the theory of real manifolds. Every smooth manifold can be embedded into $\mathbb{R}^n$, for n large enough!

1.4 Complex projective space

In the preceding paragraph we saw that a compact complex manifold cannot be holomorphically embedded into $\mathbb{C}^n$. This means the theory of smooth manifolds favourite strategy to produce examples, an application of the Regular Value Theorem to produce smooth submanifolds of $\mathbb{R}^n$, will not directly apply to the theory of complex manifolds. However, at this stage we haven't even seen a single example of a compact complex manifold.

Example 1.10. *Consider the 2-sphere $\mathbb{S}^2$ as a topological space, and let $x \in \mathbb{S}^2$ be an arbitrary point. We denote by $N = (0, 0, 1)$ the "north pole", and by $S = -N$ the "south pole". In the manifolds course, stereographic projection was used to produce a homeomorphism between $\mathbb{S}^2 \setminus x$ with $\mathbb{R}^2$. If we further use the standard identification $\mathbb{C} \cong \mathbb{R}^2$, we obtain a homeomorphism (or chart) $\phi_1\colon \mathbb{S}^2 \setminus N \xrightarrow{\simeq} \mathbb{C}$. We also denote $\mathbb{S}^2$ by $\overline{\mathbb{C}}$, and refer to it as the* Riemann sphere, *and N as ∞.*

We will now construct a second chart, endowing $\overline{\mathbb{C}}$ with the structure of a complex manifold. Since the underlying topological space is homeomorphic to $\mathbb{S}^2$, it is our first example of a compact complex manifold.

Lemma 1.11. *Consider the map $z \mapsto \frac{1}{z}$, which is well-defined on $\mathbb{C} \setminus 0$. It extends to a homeomorphism of $i \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, by sending 0 to ∞, and ∞ to 0.*

Proof. We begin by proving that the map i is continuous. This is certainly the case for every point belonging to $\mathbb{C} \setminus 0 \cong \mathbb{S}^2 \setminus \{N, S\}$, so it remains to verify continuity at 0 and ∞. We use that 0 has a neighbourhood basis given by ϵ-neighbourhoods $U_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$, while ∞ has a neighbourhood basis by $V_R = \{z \in \mathbb{C} \mid |z| > R\} \cup \{\infty\}$. So to show continuity at 0, we need to show that for every $R > 0$, there exists an $\epsilon > 0$, such that $i(U_\epsilon) \subset V_R$. This is automatically satisfied for $z = 0$, so we may focus on $z \in U_\epsilon \setminus 0$. Since $|i(z)| = |\frac{1}{z}| = \frac{1}{|z|}$, we have that for $\epsilon = \frac{1}{R}$ the property $i(U_\epsilon) = V_R$ is satisfied. Continuity at ∞ is verified similarly.

We have $i^2 = \mathrm{id}_{\overline{\mathbb{C}}}$, since this property is satisfied in the dense subset $\mathbb{C} \setminus 0$, and the functions i^2 and $\mathrm{id}_{\overline{\mathbb{C}}}$ are continuous. Therefore, we see that i is its own continuous inverse, in particular it is a homeomorphism. □

This lemma now directly implies that the Riemann sphere $\overline{\mathbb{C}}$ is a complex manifold of dimension 1, that is, it is a Riemann surface. As charts we use $U_0 = \mathbb{C} \subset \overline{\mathbb{C}}$, with the identity map $\phi \colon U = \mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}$, and $V = \overline{\mathbb{C}} \setminus 0$ with the map $\psi = i|_V \colon V \to \mathbb{C}$. Since $V = i(U)$, and $\psi = \phi \circ i$, we obtain from Lemma 1.11 that ψ is a homeomorphism.

The change of coordinates map is given by

$$\begin{tikzcd} & U \cap V \arrow[dr, "\psi"] & \\ \mathbb{C} \setminus 0 \arrow[ur, "\phi^{-1}"] \arrow[rr, "z \mapsto \frac{1}{z}"] & & \mathbb{C} \setminus 0 \end{tikzcd}$$

which is a bijective holomorphic map, with holomorphic inverse (that is, a *biholomorphic* map). This implies that $\overline{\mathbb{C}}$ is a complex manifold.

The Riemann sphere $\overline{\mathbb{C}}$ is also often denoted by $\mathbb{P}^1$. It belongs to a family of complex manifolds $\mathbb{P}^n$, known as complex projective spaces.

Definition 1.12. *For $n \geq 0$, we denote by $\mathbb{P}^n$ the set of equivalence classes in $\mathbb{C}^{n+1} \setminus 0$, with respect to the equivalence relation $(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$, if and only if there exists $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus 0$, such that $(z_0, \dots, z_n) = (\lambda w_0, \dots, \lambda w_n)$. We have a canonical map $\mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$. We endow $\mathbb{P}^n$ with the quotient topology, that is, consider a subset $U \subset \mathbb{P}^n$ as open, if and only if the preimage $p^{-1}(U) \subset \mathbb{C}^{n+1} \setminus 0$ is open.*

The equivalence class $[(z_0, \dots, z_n)]$ will be denoted by $(z_0 : \dots, z_n)$. As a next step, we construct charts on $\mathbb{P}^n$, and verify that they define the structure of a complex manifold.

For $0 \leq i \leq n$ we let $U_i \subset \mathbb{P}^n$ be the subset consisting of all $(z_0 : \dots : z_n)$, such that $z_i \neq 0$. The preimage $p^{-1}(U_i)$ is $\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} | z_i \neq 0\}$ is an open subset, hence U_i is by definition open in the quotient topology. Moreover, we know that $\bigcup U_i = \mathbb{P}^n$, since for every $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus 0$, at least one coordinate z_i must be non-zero. Hence, we have defined an open covering $(U_i)_{i=0,\dots,n}$ of $\mathbb{P}^n$.

We define a continuous map $\widetilde{\phi}_i\colon p^{-1}(U_i)\longrightarrow \mathbb{C}^n$, by sending $(z_0,\ldots,z_n)$ to $\frac{1}{z_i}(z_0,\ldots,\widehat{z_i},\ldots,z_n)$ (omitting the i-th component). Since the value of $\widetilde{\phi}_i$ is the same for every representative of a given equivalence class $(z_0:\ldots:z_n)\in U_i$, we obtain a well-defined continuous map $\phi_i\colon U_i\longrightarrow \mathbb{C}^n$.

We have a continuous map $\widetilde{\psi}_i\colon \mathbb{C}^n\longrightarrow p^{-1}(U_i)$, sending $(w_1,\ldots,w_n)$ to $(z_0,\ldots,z_n)$, where $z_j=w_{j+1}$ for $0\le j\le i-1$, and $z_i=1$, and $z_j=w_j$ for $j\ge i+1$. The composition $p\circ\widetilde{\psi}_i$ defines a continuous map $\psi_i\colon \mathbb{C}^n\to U_i$. Since $\phi_i\circ\psi_i=\mathrm{id}_{\mathbb{C}^n}$, and $\psi_i\circ\phi_i=\mathrm{id}_{U_i}$, we have that ϕ_i is a homeomorphism.

The only fact that remains to be verified, is that the change of coordinates maps are holomorphic. We will do this for $i<j$, and the charts (U_i,ϕ_i), (U_j,ϕ_j). The case $j>i$ follows by relabelling coordinates. Let's assume that $(z_0:\ldots:z_n)$ belongs to $U_i\cap U_j$, that is, we have $z_i\neq 0$ and $z_j\neq 0$. The image $\phi_i(U_i\cap U_j)$ is equal to $\{(w_1,\ldots,w_n)\in\mathbb{C}^n\,|w_j\neq 0\}$. The resulting change of coordinates map is now given by the map

$$\phi_i(U_i\cap U_j)=\{(w_1,\ldots,w_n)\in\mathbb{C}^n\,|w_j\neq 0\}\longrightarrow\mathbb{C}^n,$$

sending $(w_1,\ldots,w_n)$ to $\phi_j(\psi_i(w_1,\ldots,w_n))=(\frac{w_1}{w_j},\ldots,\frac{w_i}{w_j},\frac{1}{w_j},\frac{w_{i+1}}{w_j},\ldots,\frac{w_{j-1}}{w_j},\frac{w_{j+1}}{w_j},\ldots,\frac{w_n}{w_j})$. Since the components of this map are rational functions, they are in particular holomorphic. Therefore, we have established the following result.

Proposition 1.13. *The topological space $\mathbb{P}^n$, together with the charts defined above, is a complex manifold.*

One can also show that the topological space underlying the complex manifold $\mathbb{P}^n$ is compact. The easiest way to see this is by noting that every equivalence class $[z_0:\ldots:z_n]$ has a representative $(z_0,\ldots,z_n)$ satisfying $|z_0|^2+\cdots+|z_n|^2=1$ (simply rescale appropriately). The subset $\{(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}\mid |z_0|^2+\cdots+|z_n|^2=1\}$ is by definition equivalent to $\mathbb{S}^{2n+1}$, that is, defines a compact topological space. Since $\mathbb{P}^n=p(\mathbb{S}^{2n+1})$, we see that $\mathbb{P}^n$ is the image of a compact topological space, that is, compact itself.

1.5 Complex projective manifolds

Let $G=G(z_0,\ldots,z_n)$ be a polynomial in $n+1$ variables. We represent it as a finite sum $\sum_{i_0,\ldots,i_n\geq 0}a_{i_1\ldots i_n}z_0^{i_0}\cdots z_n^{i_n}$, and refer to $\max(i_0+\cdots+i_n|a_{i_0\ldots i_n}\neq 0)$ as the degree $\deg G$ of G.

Definition 1.14. *The polynomial G is called* homogeneous *of degree d if for every $\lambda\in\mathbb{C}$, and every $z=(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}$ we have $G(\lambda z)=\lambda^d G(z)$.*

This definition can be reformulated in more explicit terms, as the following lemma shows. The proof is left as an exercise.

Lemma 1.15. *A polynomial $G=\sum_{i_0,\ldots,i_n\geq 0}a_{i_1\ldots i_n}z_0^{i_0}\cdots z_n^{i_n}$ is homogeneous of degree d if and only if $a_{i_1\ldots i_n}=0$ for $i_0+\ldots i_n\neq d$. In particular we see that for a non-zero homogeneous polynomial of degree d, we have $d=\deg G$.*

Note that homogeneous polynomials are quite a general class of polynomials. In fact, if $g(z_0,\ldots,z_{n-1})$ is an arbitrary polynomial of degree d, then there exists a unique homogeneous polynomial $G(z_0,\ldots,z_n)$ of degree d, such that $g(z_0,\ldots,z_{n-1})=G(z_0,\ldots,z_{n-1},1)$. Take for example $g(z_0,z_1)=z_0^2z_1+$

$5z_0 - 3z_1^2$. It is easy to see that $G(z_0, z_1, z_2) = z_0^2 z_1 + 5z_0 z_2^2 - 3z_1^2 z_2$ satisfies the requirement. We say that G is the *homogenisation* of g.

The reason to introduce homogeneous polynomials is that they can be used to define subsets of $\mathbb{P}^n$. For instance, if G is a non-zero homogeneous polynomial of degree d, then the subset $\{(z_0 : \ldots : z_n) | G(z_0, \ldots, z_n) = 0\}$ is well-defined. Indeed, if $(w_0 : \ldots : w_n) = (z_0 : \ldots : z_n)$, then there exists $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus 0$, such that $(w_0, \ldots, w_n) = \lambda(z_0, \ldots, z_n)$. Thus, we have $G(w_0, \ldots, w_n) = \lambda^d G(z_0, \ldots, z_n)$; and $G(z_0, \ldots, z_n)$ is zero, if and only if $G(w_0, \ldots, w_n)$ is zero.

Definition 1.16. *Let $G_1, \ldots, G_k$ be non-zero homogeneous polynomials of degrees d_i. The subset $\{(z_0 : \ldots : z_n) \in \mathbb{P}^n \,| G_i(z_0, \ldots, z_n) = 0 \;\forall i = 1, \ldots, k\}$ is called a complex projective subvariety of $\mathbb{P}^n$, and will be denoted by $V(G_0, \ldots, G_k)$*

It is not true that $V(G_0, \ldots, G_k)$ is always a complex manifold. However, we will see that this is true generically. The criterion for it to be a manifold is reminiscent of the Regular Value Theorem in the theory of smooth manifolds and will be proven using an analogue of this result for holomorphic functions.

Proposition 1.17. *Let $G_1, \ldots, G_k$ be non-zero homogeneous polynomials of degrees d_i, and assume that the Jacobi matrix $(\frac{\partial G_i}{\partial z_j})_{i,j}$ is regular for every value of $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus 0$, such that $G_i(z_0, \ldots, z_n) = 0$ for all $i = 1, \ldots, k$. Then there exists a natural structure of a complex manifold on $V(G_0, \ldots, G_k)$.*

We can now state one of the first big theorems. We will learn more about its proof at a later point.

Theorem 1.18 (Chow)**.** *Let X be a compact complex manifold, such that there exists a holomorphic map $f \colon X \longrightarrow \mathbb{P}^n$. Then, there exist non-zero homogeneous polynomials $G_1, \ldots, G_k$, such that $f(X) = V(G_0, \ldots, G_n)$.*

Chow's theorem implies in particular that every compact complex submanifold of $\mathbb{P}^n$ can be described in terms of purely algebraic data, such as a finite number of homogeneous polynomials. This is a surprising result, because the class of holomorphic functions is in general much bigger than just algebraic ones. At a later point we will deduce this result from a theorem of Serre, which is referred to as *GAGA* (abbreviating the title of Serre's article *Géométrie algébrique et géométrie analytique*). Stating Serre's result properly requires the language of sheaves, and cohomology. For now let us just explain its meaning by the cryptic remark that every holomorphic object (submanifolds, vector bundles, etc.), which we can define on $\mathbb{P}^n$ is in fact *algebraic*, that is, defined in terms of polynomials and rational functions.

1.6 Complex tori

In the last paragraph we saw how systems of homogeneous polynomials can be used to produce examples of compact complex manifolds. There is another, even easier strategy to produce a lot of examples.

Definition 1.19. *A lattice in $V = \mathbb{R}^n$ is a subgroup $\Gamma \subset (\mathbb{R}^n, +)$, such that V/Γ is a compact topological space, and $\Gamma \subset V$ is a discrete topological space.*

For example, if $V = \mathbb{R}$ is a 1-dimensional vector space, then $\mathbb{Z} \subset \mathbb{R}$ is a lattice. Indeed, the complex exponential map induces a homeomorphism $\mathbb{R} / \mathbb{Z} \subset \mathbb{S}^1$, and the circle is known to be a compact topological space. There is a more general class of example.

Example 1.20. *Let V be a finite-dimensional vector space with a basis $\omega_1, \ldots, \omega_n$. Then the subgroup $\Gamma(\omega_1, \ldots, \omega_n) = \mathbb{Z}\,\omega_1 + \cdots + \mathbb{Z}\,\omega_n$ is a lattice.*

It is clear that the subgroup defined above is discrete, in the lemma below we show that the quotient is indeed compact.

Lemma 1.21. *Let V be a finite-dimensional real vector space, and $\Gamma = \Gamma(\omega_1, \ldots, \omega_n) \subset V$ a lattice as in the example above. Then, the topological space V/Γ is a torus, that is, homeomorphic to a finite product of copies of $\mathbb{S}^1$ (and in particular it is compact).*

Proof. The complex exponential map induces a homeomorphism $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. We choose a basis $\omega_1, \ldots, \omega_n$, such that $\Gamma = \Gamma(\omega_1, \ldots, \omega_n)$. Choose the unique isomorphism of vector spaces $V \cong \mathbb{R}^n$, transforming (ω_i) to the standard basis (e_i). This induces a homeomorphism $V/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{S}^1)^n$. ☐

We will show first that every lattice Γ can be written as $\Gamma(\omega_1, \ldots, \omega_n) = \{\lambda_1\omega_1 + \cdots + \lambda_n\omega_n | (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n\}$,where $(\omega_1, \ldots, \omega_n)$ is a basis of $V = \mathbb{R}^n$ viewed as a real vector space of dimension n. Afterwards, we will verify that V/Γ is a complex manifold, if V is a complex vector space.

Lemma 1.22. *Let $\Gamma \subset V \cong \mathbb{R}^n$ be a lattice, then there exists a real basis $\omega_1, \ldots, \omega_n$, such that $\Gamma = \Gamma(\omega_1, \ldots, \omega_n)$.*

In the proof of this lemma we will use a theorem from algebra, which follows from the classification of finitely generated abelian groups.

Theorem 1.23. *Let A be a finitely generated torsion-free abelian group, then A is free. That is, there exists an isomorphism $A \cong \mathbb{Z}^n$.*

Proof of Lemma 1.22. Consider the linear span $W = \mathbb{R} \cdot \Gamma \subset V$, that is, the smallest real subspace, containing Γ. We claim that $W = V$. This follows from the fact that we have a surjective continuous map $V/\Gamma \longrightarrow V/W$, sending $[v]_\Gamma$ to $[v]_W$. The right hand side V/W is also a finite-dimensional vector space. Therefore, not compact, unless $V/W = 0$. Since images of compact spaces under continuous maps are compact, we must have $V/W = 0$, which implies $W = V$.

Linear algebra tells us, that the set Γ^n contains tuples $(v_1, \ldots, v_n)$, forming a real basis of the vector space V. We denote this subset of Γ^n by B_Γ.

We know that $\Gamma \supset \Gamma(\omega_1, \ldots, \omega_n)$. Consider now the quotient $\Gamma/\Gamma(\omega_1, \ldots, \omega_n) \subset V/\Gamma(\omega_1, \ldots, \omega_n)$, which is closed with respect to the quotient topology, since its preimage $p^{-1}(\Gamma/\Gamma(\omega_1, \ldots, \omega_n) = \Gamma$ in V is closed as a discrete subspace of V. Closed subsets of compact topological spaces are themselves compact, this implies that $\Gamma/\Gamma(\omega_1, \ldots, \omega_n)$ is compact.

On the other hand Γ is a discrete topological space, that is, every subset is open. This implies that every subset of $\Gamma/\Gamma(\omega_1, \ldots, \omega_n$ is open with respect to the quotient topology. Therefore, $\Gamma/\Gamma(\omega_1, \ldots, \omega_n$ is both compact and discrete, that is, it must be a finite set. Choosing representatives for elements of this finite sets, we obtain that Γ is a finitely generated abelian group.

We also know that $\Gamma \subset V$ is torsion free. By Theorem 1.23, torsion free finitely generated abelian groups are free. Hence there exists $(\omega_1, \ldots, \omega_n) \in B_\Gamma$, such that $\Gamma = \Gamma(\omega_1, \ldots, \omega_n)$. ☐

Now let V be a finite-dimensional complex vector space of dimension n. We can also treat V as a real vector space of dimension $2n$. If $\Gamma \subset V$ is a lattice in V as a real vector space, then V/Γ has the structure of a complex manifold. In order to define this structure, we need to exhibit complex charts, and verify that the change of coordinates maps are holomorphic.

In order to define the charts, we choose $\epsilon > 0$, such that $v \in \Gamma$ and $|v| < 2\epsilon$ implies $v = 0$. This is possible, because $\Gamma \subset V$ is discrete, that is, $0 \in \Gamma$ must possess a neighbourhood, which doesn't contain another element of Γ.

Let $p\colon V \to V/\Gamma$ be the canonical projection. If we restrict p to $U_\epsilon(v) = \{w \in V |\ |w - v| < \epsilon\}$, we obtain an injective map. Indeed if $p(w_1) = p(w_2)$, then $w_1 - w_2 \in \Gamma$. But this is impossible for $w_1,\ w_2 \in U_\epsilon(v)$, since we also have $|w_1 - w_2| < 2\epsilon$ by the triangle inequality.

Moreover, for every open subset $V \subset U_\epsilon(v)$, the image $p(V)$ is open in the quotient topology, since $p^{-1}(p(V)) = \bigcup_{w\in\Gamma}(w + V)$ is a union of open subsets. We have therefore defined a bijective continuous map $\psi_{\epsilon,v}\colon U_\epsilon(v) \longrightarrow p(U_\epsilon(v))$, mapping open sets to open sets. Hence $\psi_{\epsilon,v}$ is a homeomorphism with continuous inverse denoted by $\phi_{\epsilon,v}$.

We define charts by $(p(U_\epsilon(v)), \phi_{\epsilon,v})$, choosing ϵ once and for all. It only remains to verify that the change of coordinates maps are holomorphic. Let $v,\ w \in V$ be two element of V, the composition

$$\phi_v(U_\epsilon(v) \cap U_\epsilon(w)) \xrightarrow{\phi_v^{-1}} U_\epsilon(v) \cap U_\epsilon(w) \xrightarrow{\phi_w} \phi_w(U_\epsilon(v) \cap U_\epsilon(w))$$

is equivalent to the translation $z \mapsto z + c$, for some $c \in V$. Hence, it is holomorphic.

Proposition 1.24. *Let V and W be two finite-dimensional complex vector spaces, and $\Gamma \subset V$ and $\Gamma' \subset W$ be lattices. Then every holomorphic map $f\colon V/\Gamma \longrightarrow W/\Gamma'$, is induced by a complex linear map $\phi\colon V \longrightarrow \mathcal{W}$, such that $\phi(\Gamma) \subset \Gamma'$, such that $f([v]) = [Av + b]$ for some $b \in W$ and all $v \in V$.*

In the rest of the subsection we provide a sketch of the proof. A sketch because we haven't yet given a complete definition of the notions of complex manifolds, and holomorphic maps between them.

Proof. We choose complex isomorphisms $V \cong \mathbb{C}^n$ and $W \cong \mathbb{C}^m$. We denote by $\mathbb{C}^{m\times n}$ the complex vector space of linear maps $\mathbb{C}^n \to \mathbb{C}^m$. At first we will show that we have a well-defined holomorphic map $Df\colon V/\Gamma \to \mathbb{C}^{m\times n}$.

This map is defined as follows: for $[v] \in V/\Gamma$, we choose a representative $v \in V$, and also a representative w for $f([v])$. Choosing ϵ and δ appropriately, we may assume that for every $v \in V$ we have $f(U_\epsilon(v)) \subset U_\delta(w)$. We define $Df(v)$ to be the Jacobi matrix of $\phi_w \circ f \circ \phi_v^{-1}$. This is well-defined, since for another choice of representatives v' and w', the map $\phi_{w'} \circ f \circ \phi_{v'}^{-1}$ differs from the above one by pre- and postcomposition of translations, which have zero derivative. Therefore, the chain rule implies that the Jacobi matrices are independent of these choices. By definition of the function Df we have for the chart $(p(U_\epsilon(v)), \phi_v)$ that $Df \circ \phi_v^{-1}$ is the Jacobi matrix of the holomorphic map $U_\epsilon(v) \to U_\delta(w)$. In particular, it is a holomorphic function.

Using that V/Γ is compact and connected, it follows from Lemma 1.8 that Df is in fact a constant function. We denote the corresponding matrix by A. This implies that $\phi_w \circ f \circ \phi_v^{-1}$ is equivalent to $z \mapsto Az + b_{v,w}$, where $b_{v,w}$ is a constant depending on the charts labelled by v respectively w.

If $U_\epsilon(v) \cap U_\epsilon(v') \neq \emptyset$, then $b_{vw} - b_{v'w} \in \Gamma'$. This follows from $[Az + b_{vw}] = [Az + b_{v'w}]$ for $z \in U_\epsilon(v) \cap U_\epsilon(v')$. Similarly, if $U_\delta(w) \cap U_\delta(w') \neq \emptyset$, we must have $b_{vw} - b_{vw'} \in \Gamma'$. Since for every $v,\ v' \in V$, there exists a sequence of points $v_0, \dots, v_k \in V$, such that $v_0 = v$, $v_k = v'$, and $U_\epsilon(v_i) \cap U_\epsilon(v_{i+1}) \neq \emptyset$, we see that the element $[b] = [b_{v,w}] \in W/\Gamma'$ is well-defined, that is, independent of the choice of v and w.

This implies that $f(z) = [Az + b]$ for any $z \in V$. Since $z \in \Gamma$ implies $[v] = 0$, we must have $f(z) = [Az + b] = f(0) = [b]$. Hence, $f(z) \in \Gamma'$. $\square$

Corollary 1.25. *If two complex tori V/Γ and W/Γ are biholomorphic complex manifolds, then there exists a complex linear isomorphism $\phi\colon V \xrightarrow{\simeq} W$, such that $\phi(\Gamma) = \Gamma'$.*

It is an interesting question when a complex torus V/Γ can be realised as a complex submanifold of a projective space $\mathbb{P}^n$. For $\dim_{\mathbb{C}} V = 1$ this turns out to be always the case. Those complex tori are also known as elliptic curves. For $\dim_{\mathbb{C}} V \geq 2$ this is in general not possible.

2 Holomorphic functions of several variables

2.1 The basics

In the study of holomorphic functions in several variables we will often be inspired by methods of the 1 variable case. The most important example being Cauchy's formula below. However, there are also some genuinely new phenomena, as Hartog's Extension Theorem 2.14.

Proposition 2.1 (Cauchy)**.** *Let $U \subset \mathbb{C}$ be an open subset, and $f\colon U \longrightarrow \mathbb{C}$ a holomorphic function. If $\mathbb{D}_\epsilon(z_0) := \{z \in \mathbb{C} \,|\, |z - z_0| < \epsilon\} \subset U$ is a closed disc fully contained in U, then for every $z \in U_\epsilon(z_0)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\,\mathbb{D}_\epsilon(z_0)} \frac{f(w)}{w - z} dw.$$

Proof. We will use the same strategy as in the proof of another identity of Cauchy in 1.2. That is, we apply the Stokes Theorem to an appropriate manifold with boundary. We denote by $A_{\epsilon,\delta}$ the annulus $\mathbb{D}_\epsilon(z_0) \setminus U_\delta(z)$, for δ small enough. It is a manifold with boundary being a union of two circles.

As in 1.2, we define a complex 1-form $\frac{f(z)}{z-z_0}dz$ by decomposing into real and imaginary parts: $f = u + iv$, and $dz = dx + idv$. This 1-form is well-defined on $A_{\epsilon,\delta}$, because the singularity at $z = z_0$ lies outside of $A_{\epsilon,\delta}$. As in 1.2 we have that $d(\frac{f(z)}{z-z_0}dz) = 0$, since $\frac{f(z)}{z-z_0}$ is holomorphic. Therefore the Stokes Theorem implies

$$\int_{\partial\,\mathbb{D}_\epsilon} \frac{f(w)}{w - z} dw - \int_{\partial\,\mathbb{D}_\delta} \frac{f(w)}{w - z} dw = \int_{\partial A_{\epsilon,\delta}} \frac{f(w)}{w - z} dw = 0.$$

This also implies that the value of $\int_{\partial\,\mathbb{D}_\delta(z)} \frac{f(w)}{w-z} dw$ does not depend on δ. We will therefore study what happens when $\delta \to 0$.

To compute the integral we parametrise $\partial\,\mathbb{D}_\epsilon(z_0)$ by the path $\gamma\colon [0, 2\pi] \longrightarrow \mathbb{C}$, which sends $t \mapsto z + \delta e^{it}$. By substitution we obtain

$$\int_{\partial\,\mathbb{D}_\epsilon} \frac{f(w)}{w - z} dw = \int_0^{2\pi} \frac{f(z + \delta e^{it})}{\delta e^{it}} \delta i e^{it} dt = \int_0^{2\pi} f(z + \delta e^{it}) i dt$$

which tends to $2\pi i f(z)$, as required. $\square$

This formula implies directly that holomorphic functions are analytic. Let's take a look at the strategy, because it will also serve us well in the higher variable case. Recall that we have for $|z - z_0| < |w - z_0|$

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)(1-\frac{z-z_0}{w-z_0})} = \frac{1}{(w-z_0)}\sum_{k\geq n}(\frac{z-z_0}{w-z_0}))^k.$$

Therefore, we deduce from Cauchy's formula that

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\epsilon(z_0)}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\epsilon(z_0)}\sum_{k\geq 0}f(w)\frac{(z-z_0)^k}{(w-z_0)^k+1}) = \sum_{k\geq 0}[\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\epsilon(z_0)}\frac{f(w)}{(w-z)^{k+1}}dw](z-z_0)^k,$$

which converges uniformly for $\frac{|z-z_0|}{|w-z_0|} \leq \epsilon' < 1$.

The most direct way to define what it means for a function in several variables to be holomorphic is the following.

Definition 2.2. *Let $U \subset \mathbb{C}^n$ be an open subset, and $f\colon U \longrightarrow \mathbb{C}$ a continuous function. We say that f is holomorphic, if for each $(z_1, \dots, z_n) \in U$, and each $i = 1, \dots, n$, the function $f(z_1, \dots, z_{i-1}, w, z_{i+1}, \dots, z_n)$ is holomorphic in the variable w, defined in a small neighbourhood of $w = z_i$.*

As a first step, we will derive a multivariable version of Cauchy's integral formula.

Proposition 2.3. *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function, defined on an open subset $U \subset \mathbb{C}^n$. For a point $z = (z_1, \dots, z_n) \in U$ we choose $\epsilon_1, \dots, \epsilon_n > 0$, such that the* polydisc *$\mathbb{D}_{\epsilon_1,\dots,\epsilon_n}(z) = \mathbb{D}_{\epsilon_n}(z_1) \times \cdots \times \mathbb{D}_{\epsilon_n}(z_n)$ is a subset of U. Then, for each $w = (w_1, \dots, w_n) \in \mathbb{D}_{\epsilon_1,\dots,\epsilon_n}(z)$ we have*

$$f(w_1', \dots, w_n') = \frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}_{\epsilon_1}(z_1)}\cdots\int_{\partial\mathbb{D}_{\epsilon_n}(z_n)}\frac{f(w)}{(w_1-w_1')\cdots(w_n-w_n')}dw_1\cdots dw_n.$$

Proof. We prove the formula by induction on n, the anchor case being $n = 1$, for which it is the classical integral formula of Proposition 2.1.

By induction we assume that the statement of the proposition has already been proven for holomorphic functions in $n - 1$ variables.

The definition of holomorphic functions in several variables tells us that for a fixed tuple $w_1', \dots, w_{n-1}'$, the function $w_1 \mapsto f(w_1, w_2', \dots, w_n')$ is holomorphic in w_1. Therefore, applying the classical Cauchy formula, we obtain

$$f(w_1', \dots, w_n') = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_{\epsilon_1}(z_1)}\frac{f(w_1, w_2', \dots, w_n')}{(w_1 - w_1')}dw_1'.$$

Inserting the induction hypothesis for the holomorphic function in $n-1$ variables $(w_2', \dots, w_n') \mapsto f(w_1, w_2', \dots, w_n')$, we obtain the asserted identity. □

Corollary 2.4. *A holomorphic function $f\colon U \longrightarrow \mathbb{C}$, for $U \subset \mathbb{C}^n$ an open subset, is analytic. That is, for a small enough open neighbourhood V of each point $(z_1, \dots, z_n) \in U$, the there exists a power sum $\sum_{m_1,\dots,m_n\geq 0} a_m(w_1 - z_1)^{m_1}\cdots(w_n - z_n)^{m_n}$, convergent for $w \in V$, such that $f(w) = \sum_{m_1,\dots,m_n\geq 0} a_m(w_1 - z_1)^{m_1}\cdots(w_n - z_n)^{m_n}$ for each point w of V.*

The strategy of the proof is the same as in the 1-dimensional case, that is, uses the geometric series in each variable. We avoid proliferation of indices by leaving the details to the reader.

Given an open subset $U \subset \mathbb{C}^n$, we may view it as an open subset of $\mathbb{R}^{2n}$. With respect to this transition, the complex coordinates $(z_1, \dots, z_n)$ get replaced by $2n$ real coordinates $(x_1, y_n, \dots, x_n, y_n)$, satisfying the relation $z_i = x_i + iy_i$.

Let $A\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a complex linear function, given by an $(n \times n)$-matrix also denoted by $A = (a_{ij})_{i,j=1,\dots,n}$. The corresponding real linear function will be denoted by $\widetilde{A}$. It is represented by a $(2n \times 2n)$-matrix $\widetilde{A} = (\tilde{a}_{k,j})_{k,j=1,\dots,2n}$.

Remark 2.5. *The map $A \mapsto \widetilde{A}$ defines an injective morphism of rings $\mathsf{End}_{\mathbb{C}}(\mathbb{C}^n) \longrightarrow \mathsf{End}_{\mathbb{R}}(\mathbb{R}^{2n})$. This implies that if $B = \widetilde{A}$ is invertible, then also its inverse B^{-1} is induced by a complex matrix (namely A^{-1}). A real $(2n \times 2n)$-matrix $C = (c_{k,l})_{k,j=1,\dots,2n} \in \mathsf{End}_{\mathbb{R}}(\mathbb{R}^{2n})$ lies in the image, if and only if for every $i, j = 1, \dots, n$, the "linear Cauchy–Riemann equations" are satisfies, that is $c_{2i,2j} = c_{2i+1,2j+1}$, and $c_{2i,2j+1} = -c_{2i+1,2j}$.*

A (real) differentiable function $f = u + iv\colon U \longrightarrow \mathbb{C}$ is holomorphic, if the partial derivatives with respect to the real variables x_i, y_i satisfy for each $i = 1, \dots, n$ the Cauchy–Riemann equations $\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}$, and $\frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}$ (see 1.4 for more details on the Cauchy–Riemann equations).

Given a partially differentiable function $f = (f_1, \dots, f_k)\colon V \longrightarrow \mathbb{R}^k$, for $V \subset \mathbb{R}^m$ an open subset, we denote the Jacobi matrix by $D^{\mathbb{R}} f$. Likewise, if $U \subset \mathbb{C}^n$ is an open subset, and $f = (f_1, \dots, f_k)\colon U \longrightarrow \mathbb{C}^k$ we have a Jacobi matrix of complex partial derivatives $Df = \left(\frac{\partial f_i}{\partial z_j}\right)_{j=1,\dots,n}^{i=1,\dots,k}$. With respect to the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we have the following remark.

Remark 2.6. *A function $f\colon U \longrightarrow \mathbb{C}^k$ is holomorphic, if and only if it is differentiable as a function defined on an open subset of $\mathbb{R}^{2n}$, and for every $z \in U$ the real Jacobi matrix $D^{\mathbb{R}} f(z)$ is of the shape $\widetilde{A}$ for $A \in \mathsf{End}_{\mathbb{C}}(\mathbb{C}^n)$. We then have $A = Df(z)$.*

2.2 The Inverse Function Theorem

We'll use the insight above to deduce a holomorphic version of the Implicit Function Theorem.

Theorem 2.7 (Inverse Function Theorem)**.** *Let $U \subset \mathbb{C}^n$ be an open subset, and $f = (f_1, \dots, f_n)\colon U \to \mathbb{C}^n$ a holomorphic function. Assume that $w \in U$ is a point, such that the complex Jacobi matrix $(\frac{\partial f_i}{\partial z_j})$ is invertible at $z = w$. Then, there exist open neighbourhoods V of w, and W of $f(w)$, as well as a holomorphic function $g\colon W \longrightarrow V$, such that $g \circ f = \mathrm{id}_V$, and $f \circ g = \mathrm{id}_W$.*

Proof. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and apply the Inverse Function Theorem for smooth functions, to deduce the existence of neighbourhoods V and W as above, and a smooth function $g\colon W \longrightarrow V$ with the required property. It only remains to show that g is actually holomorphic.

The chain rule implies that the Jacobi-matrix of g, with respect to $(x_1, y_1, \dots, x_n, y_n)$ is equivalent to the inverse of the Jacobi-matrix of f with respect to $(x_1, y_1, \dots, x_n, y_n)$. Since the Jacobi-matrix of f corresponds to a complex matrix, by means of the isomorphism $\mathbb{C}^n \cong \mathbb{R}^n$, this must also be true for the Jacobi-matrix of g (by Remark 2.5). Therefore we deduce from Remark 2.6 that g is holomorphic. $\square$

2.3 The Implicit Function Theorem

The Implicit Function Theorem gives us a sufficient condition, when the solution set of a system of holomorphic equations, can be parametrised by holomorphic functions, near a given solution. We will use it in the future to construct several examples of complex manifolds, for instance smooth complex projective manifolds, which arise as simultaneous zero sets to a system of homogeneous polynomials.

Theorem 2.8 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{C}^n$ be an open subset, and $f = (f_1, \ldots, f_k)\colon U \longrightarrow \mathbb{C}^k$ a holomorphic function. If $f(0) = 0$, and the determinant $|\frac{\partial f_i}{\partial z_j}(0)|_{0 \leq i,j \leq k} \neq 0$ is non-zero. Then there exists a holomorphic function $g = (g_1, \ldots, g_k)\colon V \longrightarrow \mathbb{C}^k$, for some open subset $V \subset \mathbb{C}^{n-k}$, such that for some neighbourhood W of 0 in U we have $f(z) = 0$ is equivalent to $z_i = g_i(z_{k+1}, \ldots, z_n)$ for $0 \leq i \leq k$, and $z \in W$.*

Proof. Consider the function $F\colon U \longrightarrow \mathbb{C}^n$, given by $(z_1, \ldots, z_n) \mapsto (f_1, \ldots, f_k, z_{k+1}, \ldots, z_n)$. Since all components are holomorphic functions, F is holomorphic as well. Moreover, we have $|\frac{\partial F_i}{\partial z_j}(0)|_{0 \leq i,j \leq n} = |\frac{\partial f_i}{\partial z_j}(0)|_{0 \leq i,j \leq k} \neq 0$, since the Jacobi matrix of F has a block upper triangular shape, with invertible diagonal blocks

$$\left(\begin{array}{c|c} \left(\frac{\partial f_i}{\partial z_j}(0)\right)_{i,j=1,\ldots,k} & * \\ \hline 0 & 1 \end{array}\right).$$

We can therefore apply the Inverse Function Theorem 2.7 to deduce the existence of a locally defined holomorphic inverse $G\colon V' \longrightarrow U$, where $V' \subset \mathbb{C}^n$ is an open neighbourhood of $F(0) = 0$ (by assumption). We denote the components of G by $(g_1', \ldots, g_n')$. We let $g_1, \ldots, g_n$ be the holomorphic functions on $V = V' \cap \{0\}^k \times \mathbb{C}^{n-k}$ obtained by restriction. It follows from the definition of F, and the fact that G is an inverse, that $g_{k+1}, \ldots, g_n$ are just the functions $(z_{k+1}, \ldots, z_n) \mapsto z_i$ for $i = k+1, \ldots, n$

The relation $F \circ G = \mathrm{id}$ implies that for $(z_{k+1}, \ldots, z_n) \in V$ we have $f_i(g_1(z_{k+1}, \ldots, g_n), \ldots, g_k(z_{k+1}, \ldots, g_n)) = 0$.

Similarly, for $W = G(V')$, and $z \in W$, such that $f(z) = 0$, we obtain from $G \circ F = \mathrm{id}$ the relation $(z_1, \ldots, z_n) = (g_1(z_{k+1}, \ldots, z_n), \ldots, g_k(z_{k+1}, \ldots, z_n), z_{k+1}, \ldots, z_n)$. This concludes the proof. □

Although the statement of the Implicit Function Theorem is already quite general, it is important to remember the existence of a biholomorphic map G as constructed in the proof.

Porism 2.9. *Let $U \subset \mathbb{C}^n$ be an open subset, and $f = (f_1, \ldots, f_k)\colon U \longrightarrow \mathbb{C}^k$ a holomorphic function as in the statement of Theorem 2.8. Then there exist open neighbhourhoods $W, W' \subset \mathbb{C}^n$ of 0, such that $W \subset U$, and a biholomorphic map $G\colon W' \longrightarrow W$, such that $(f^1, \ldots, f^k) \circ G(z^1, \ldots, z^n) = (z^1, \ldots, z^k)$.*

2.4 $\overline{\partial}$-derivatives

There is a formal device, which simplifies many arguments concerning multivariable holomorphic functions. We fix an open subset $U \subset \mathbb{C}^n$, and a continuous function $f\colon U \longrightarrow \mathbb{C}$. Using the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, which on the level of coordinates works by decomposing each complex component z_i into real and imaginary part, $z_i = x_i + iy_i$, we may ask that f is partially differentiable with respect to the $2n$ real coordinates x_i and y_i.

Definition 2.10. *For f as above, we define a differential operator*

$$\frac{\partial f}{\partial \overline{z}_i} = \frac{1}{2}(\frac{\partial f}{\partial x_i} + i\frac{\partial f}{\partial y_i}).$$

We also define

$$\frac{\partial f}{\partial z_i} = \frac{1}{2}(\frac{\partial f}{\partial x_i} - i\frac{\partial f}{\partial y_i}).$$

Using this notation we can repackage the Cauchy–Riemann equations.

Lemma 2.11. *A function $f = u + iv$ as above, satisfies the Cauchy–Riemann equations for the coordinates (x_i, y_i) if and only if $\frac{\partial f}{\partial \overline{z}_i} = 0$.*

Proof. Let us compute $\frac{\partial f}{\partial \overline{z}_i}$, with respect to the decomposition $f = u + iv$. We have

$$\frac{1}{2}(\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial y_i}) = \frac{1}{2}(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial y_i} + i\frac{\partial u}{\partial y_i} + i\frac{\partial v}{\partial x_i}).$$

And by considering real and imaginary part separately, we see that the Cauchy–Riemann equations with respect to (x_i, y_i) are satisfied, if and only if $\frac{\partial f}{\partial \overline{z}_i} = 0$. $\square$

This differential operator allows us to check that certain functions, defined by integration, are holomorphic.

Lemma 2.12. *For a smooth function $f\colon U \longrightarrow \mathbb{C}$, with $U \subset \mathbb{C}^2$ and open subset, and $\gamma\colon [0,1] \longrightarrow \mathbb{C}$ a continuous path, we have*

$$\frac{\partial}{\partial \overline{w}} \int_\gamma f(z,w)dz = \int_\gamma \frac{\partial}{\partial \overline{w}} f(z,w)dz,$$

whenever well-defined. In particular, if f is holomorphic in w, then so is $w \mapsto \int_\gamma f(z,w)dz$, whenever well-defined.

2.5 Hartog's Extension Theorems

The result of this paragraph highlights an interesting difference between holomorphic functions in 1 and several variables. If f is a holomorphic function on an open subset of 2-dimensional complex space, defined away from a point, then it extends to a holomorphic function on that point. This means that the set of singularities of a holomorphic function in 2-variables is 1-dimensional or empty. Moreover, the proof given below, also gives us an explicit method to construct a holomorphic extension.

In the proof we will use the following lemma, the proof of which is left to the reader.

Lemma 2.13 (Principle of analytic continuation)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function, and $U \subset \mathbb{C}^n$ a connected open subset. If there exists an open subset $\emptyset \neq V \subset U$, such that $f|_V = 0$, then $f = 0$.*

We denote by $U_{\epsilon_1,\epsilon_2} = \{(z,w) \in \mathbb{C}^2 \mid |z| < \epsilon_1, |w| < \epsilon_2\}$ the polydisk and by $\mathbb{D}_{\epsilon_1,\epsilon_2}$ its closure.

Theorem 2.14 (Hartog)**.** *Let $\epsilon_i > \delta_i > 0$ for $i = 1, 2$, and $f : U_{\epsilon_1,\epsilon_2} \setminus \mathbb{D}_{\delta_1,\delta_2} \longrightarrow \mathbb{C}$ be a holomorphic function, then f extends to a holomorphic function on $U_{\epsilon_1,\epsilon_2}$.*

Proof. Choose r between ϵ_2 and δ_2, and define $F : U_\epsilon$ to be

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w_2|=r} \frac{f(z_1, w_2)dw_2}{w_2 - z_2},$$

which is well-defined for all $(z_1, z_2) \in U_\epsilon(0)$. Moreover, F agrees with f, whenever $|z_i| > \delta_i$, by Cauchy's integral formula. A 3-variable version of Lemma 2.12 tells us that $\frac{\partial}{\partial z_i}$ applied to the above integral is zero, because we may exchange the order of this differential operator with the integral. $\square$

2.6 Complex vector fields

Let's fix an open subset $U \subset \mathbb{C}^n$ of a complex space. Our goal in this paragraph is to define complex vector fields on U. A careful study of this local situation will allow us later to give a definition of complex vector fields on complex manifolds. From now on we modify our notation slightly, turning some subscripts into superscripts, to be consistent with Ricci index notation.

Definition 2.15. *A complex vector field $\mathcal{V}$ on U is a first order differential operator of the shape*

$$\sum_{i=1}^{n} \mathcal{V}^i \frac{\partial}{\partial z^i}, \tag{3}$$

where $(\mathcal{V}^1, \ldots, \mathcal{V}^n) : U \longrightarrow \mathbb{C}^n$ is a holomorphic function on U.

Equivalently, we could have defined a complex vector field $\mathcal{V}$ as a holomorphic function $U \longrightarrow \mathbb{C}^n$. Since the differential operators $\frac{\partial}{\partial z^i}$ are linearly independent (which can be seen by using the relation $\frac{\partial z^i}{\partial z^j} = \delta^i_j$), no information is lost by forming the linear combination as in (3).

The advantage of thinking of $\mathcal{V}$ in terms of the differential operator is that it'll become clear how biholomorphic changes of coordinates affect complex vector fields. It is instructive to think this through in some detail: consider a biholomorphic map $\phi : V \longrightarrow U$, with inverse $\psi : U \longrightarrow V$. A holomorphic function $g : U \longrightarrow \mathbb{C}^k$ induces a holomorphic function $g \circ \phi$ on V by composition. So, given a vector field $\mathcal{V}$ on U, represented by an n-tuple of holomorphic functions $(\mathcal{V}^1, \ldots, \mathcal{V}^n)$, we could be tempted to simply consider the composition $(\mathcal{V}^1 \circ \phi, \ldots, \mathcal{V}^n \circ \phi)$. This however would be inconsistent with the viewpoint of a complex vector field $\mathcal{V}$ as a differential operator.

Indeed, if $U' \subset U$ is an open subset, and $f : U \longrightarrow \mathbb{C}$ a holomorphic function, then we obtain a holomorphic function $h = f \circ \phi : \psi(U') \longrightarrow \mathbb{C}$. Equivalently, we can write $f = h \circ \psi$, and use the chain rule to compute

$$\mathcal{V} f(z_0) = \mathcal{V}(h \circ \psi)(z_0) = \sum_{j=0}^{n} \mathcal{V}^j \frac{\partial(h \circ \psi)}{\partial z^j}(z_0) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathcal{V}^j \frac{\partial h}{\partial w^i}(\psi(z_0)) \frac{\partial \psi^j}{\partial z^j}(z_0).$$

This defines a holomorphic function on U', but by composing with ϕ, we see that we have defined a first order differential operator

$$h \mapsto \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{\partial \psi^i}{\partial z^j} (\mathcal{V}^j \circ \phi) \circ \phi \right) \frac{\partial h}{\partial w^i}.$$

Thinking of the complex coordinates of V, denoted by $(w^1, \ldots, w^n)$ in biholomorphic dependence of the complex coordinates $(z^1, \ldots, z^n)$ in U, we can rewrite this as the suggestive identity

$$\sum_{i=1}^{n} \mathcal{V}^i \frac{\partial}{\partial z^i} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathcal{V}^j \circ \phi) \frac{\partial w^i}{\partial z^j} \frac{\partial}{\partial w^i}.$$

This inspires the following definition.

Definition 2.16. *For open subset $U, V \subset \mathbb{C}^n$, a biholomorphic map $\psi \colon U \longrightarrow V$, and a complex vector field $\mathcal{V} = \sum_{i=1}^{n} \mathcal{V}^i \frac{\partial}{\partial z^i}$ on U, we define*

$$\psi_* \mathcal{V} = D\psi \begin{pmatrix} \mathcal{V}^1 \circ \phi \\ \vdots \\ \mathcal{V}^n \circ \phi \end{pmatrix} = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{\partial \psi^i}{\partial z^j} (\mathcal{V}^j \circ \phi) \right) \frac{\partial}{\partial w^i}.$$

Similarly to our treatment of complex vector fields we can study complex analogues of n-forms introduced in the smooth manifolds lecture.

2.7 Complex differential forms

As before we fix an open subset $U \subset \mathbb{C}^n$. A complex m-form ω on U is an object, which assigns to m complex vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_m$, defined on an open subset $U' \subset U$, a holomorphic function $\omega(\mathcal{V}_1, \ldots, \mathcal{V}_m)$ on U'. Several properties need to be satisfied. For instance we want this construction to be compatible with restriction, that is, for an open subset $U'' \subset U'$ we want

$$\omega(\mathcal{V}_1 |_{U''}, \ldots, \mathcal{V}_m |_{U''}) = \omega(\mathcal{V}_1, \ldots, \mathcal{V}_m)|_{U''}. \tag{4}$$

If $\mathcal{W}$ is another complex vector field on U', and f a holomorphic function on U', then we have that the following multilinearity condition is satisfied for each index $i = 1, \ldots, m$

$$\omega(\mathcal{V}_1, \ldots, \mathcal{V}_{i-1}, \mathcal{V}_i + f\mathcal{W}, \mathcal{V}_{i+1}, \ldots, \mathcal{V}_n) = \omega(\mathcal{V}_1, \ldots, \mathcal{V}_{i-1}, \mathcal{V}_i, \mathcal{V}_{i+1}, \ldots, \mathcal{V}_n) + f\omega(\mathcal{V}_1, \ldots, \mathcal{V}_{i-1}, \mathcal{W}, \mathcal{V}_{i+1}, \ldots, \mathcal{V}_n). \tag{5}$$

Moreover, for every pair of indices $1 \leq i < j \leq m$ an m-form satisfies

$$\omega(\mathcal{V}_1, \ldots, \mathcal{V}_{i-1}, \mathcal{V}_j, \mathcal{V}_{i+1}, \ldots, \mathcal{V}_{j-1}, \mathcal{V}_i, \mathcal{V}_{j+1}, \ldots, \mathcal{V}_m) = -\omega(\mathcal{V}_1, \ldots, \mathcal{V}_m). \tag{6}$$

For every ordered m-tuple $1 \leq i_1 < \cdots < i_m \leq n$ we denote by $\omega_{i_1 \ldots i_m}$ the holomorphic function on U given by

$$\omega(\frac{\partial}{\partial z^{i_1}}, \ldots, \frac{\partial}{\partial z^{i_m}}).$$

Since every complex vector field can be expressed as a linear combination (3) of the coordinate vector fields $\frac{\partial}{\partial z^i}$, the multilinearity (5) and anti-symmetry conditions (6) of m-forms guarantee that those holomorphic functions determine ω completely.

We write $dz^{i_1} \wedge \cdots \wedge dz^{i_m}$ for the holomorphic m-form which satisfies for

$$1 \leq j_1 < \cdots < j_m \leq n$$

the condition

$$(dz^{i_1} \wedge \cdots \wedge dz^{i_m})(\frac{\partial}{\partial z^{j_1}}, \ldots, \frac{\partial}{\partial z^{j_m}}) = \begin{cases} 1, \text{ if } (i_1, \ldots, i_m) = (j_1, \ldots, j_n) \\ 0, \text{ else.} \end{cases}$$

It is then clear that we have

$$\omega = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \omega_{i_1 \ldots i_m} dz^{i_1} \wedge \cdots \wedge dz^{i_m} \tag{7}$$

for every complex m-form.

Definition 2.17. *The complex vector space of complex m-forms on $U \subset \mathbb{C}^n$ will be denoted by $\Omega^m(U)$. By convention we agree that a complex 0-form is a holomorphic function. We will denote the complex vector space of holomorphic functions on U by $\mathcal{O}(U)$.*

Exercise 2.18. *For an inclusion of open subsets $U \subset V$ define a restriction map $\Omega^m(V) \longrightarrow \Omega^m(U)$.*

The collection of complex m-forms for varying m interact with each other by means of the exterior, or *wedge product* $\wedge$, and the *exterior differential* d.

Definition 2.19. *The wedge product $\wedge \colon \Omega^m(U) \times \Omega^k(U) \longrightarrow \Omega^{m+k}(U)$ is defined to be the unique collection of maps satisfying for $\omega, \omega' \in \Omega^m(U)$, $\eta \in \Omega^k(U)$, and $f \in \mathcal{O}(U)$*

(a) $\omega \wedge \eta = (-1)^{mk} \eta \wedge \omega$

(b) $(\omega + \omega') \wedge \eta = \omega \wedge \eta + \omega' \wedge \eta$

(c) $(f\omega) \wedge \eta = f(\omega \wedge \eta)$.

(d) for $1 \leq i_1 < \ldots i_m \leq n$, $(dz^{i_1}) \wedge \cdots \wedge (dz^{i_m})$ agrees with the m-form $dz^{i_1} \wedge \cdots \wedge dz^{i_m}$ defined above.

(e) For $U' \subset U$ an open subset we have $(\omega \wedge \eta)|_{U'} = (\omega|_{U'}) \wedge (\eta|_{U'})$.

The exterior differential is a linear map $\Omega^m \longrightarrow \Omega^{m+1}$, uniquely characterised by a few axioms.

Definition 2.20. *The exterior differential is the unique collection of maps $\Omega^m(U) \longrightarrow \Omega^{m+1}(U)$, which satisfies for $\omega, \omega' \in \Omega^m(U)$, $\eta \in \Omega^k(U)$, and $f \in \mathcal{O}(U)$*

(a) $d(\omega + \omega') = d\omega + d\omega'$

(b) $d(f\omega) = \sum_{i=1}^n \frac{\partial f}{\partial z^i} dz^i \wedge \omega + f d\omega$

(c) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^m \omega \wedge d\eta$.

(e) For an open subset $U' \subset U$ we have $(d\omega)|_{U'} = d(\omega|_{U'})$.

Differentiating a complex m-form twice leads to a vanishing $(m+2)$-form.

Lemma 2.21. *For a complex m-form $\omega \in \Omega^m(U)$ defined on an open subset $U \subset \mathbb{C}^n$ we have $d^2\omega = 0$.*

Proof. Since ω can be written as $\sum_{1\leq i_1<\cdots<i_m\leq n}\omega_{i_1\dots i_m}dz^{i_1}\wedge\cdots\wedge dz^{i_m}$ it suffices to check this identity for 0-forms, that is a holomorphic function $f\in\mathcal{O}(U)$. We have

$$d^2 = d\sum_{i=1}^{n}\frac{\partial f}{\partial z^i}dz^i = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial z^j\partial z^i}dz^j\wedge dz^i.$$

Since $dz^i\wedge dz^j = -dz^j\wedge dz^i$, and $dz^i\wedge dz^i = 0$, the sum on the right hand side consists of $n(n-1)$ terms, which can be paired to cancel each other out. □

This discussion shows that as far as computations are concerned, a complex m-form is just a collection of $\binom{n}{m}$ holomorphic functions. However, their definition as alternating multi-linear forms on holomorphic vector fields, and equation (7) are more than mere bookkeeping devices. They help us to pin down the behaviour of m-forms with respect to biholomorphic coordinate changes. The key difference to the theory of complex vector fields is that complex m-forms can be pulled back along an arbitrary holomorphic map, without imposing the existence of a holomorphic inverse.

Definition 2.22. *For open subsets $U\subset\mathbb{C}^{n_1}$, $V\subset\mathbb{C}^{n_2}$, a holomorphic map $\psi\colon U\longrightarrow V$, and a complex m-form ω on V, we define*

$$\psi^*\omega = \sum_{1\leq i_1<\cdots<i_m\leq n_2}(\omega_{i_1<\cdots<i_m}\circ\psi)(\psi^*dz^1)\wedge\cdots\wedge(\psi^*dz^m) = \sum_{1\leq i_1<\cdots<i_m\leq n_2}(\omega_{i_1<\cdots<i_m}\circ\psi)d\psi^1\wedge\cdots\wedge d\psi^m.$$

One sees easily that $\psi^*(\omega_1+\omega_2)=\psi^*\omega_1+\psi^*\omega_2$, $\psi^*(f\omega)=(f\circ\psi)f^*\omega$, and $\psi^*(\omega\wedge\eta)=\psi^*\omega\wedge\psi^*\eta$. Moreover, pullback is compatible with exterior differentiation $\psi^*d\omega = d(\psi^*\omega)$.

Lemma 2.23. *Let $U\subset\mathbb{C}^{n_1}$, $V\subset\mathbb{C}^{n_2}$ be open subsets, and $\omega\in\Omega^m(V)$. For a holomorphic function $\phi\colon U\longrightarrow V$ we have $d\phi^*\omega=\phi^*d\omega$.*

Proof. Since ϕ^* commutes with sums and wedge products, and d is compatible with wedge products by means of a sort of Leibniz rule (see (c) of Definition 2.20), it suffices to prove the identity for complex 1-forms (since every general complex m-form can be written as a finite sum of m-forms obtained by wedge products of complex 1-forms). Henceforth, we assume $m=1$, and write $\omega=\sum_{i=1}^{n_2}\omega_i dz^i$. We have $\phi^*\omega=\sum_{i=1}^{n_2}(\omega_i\circ\phi)d\phi^i$. Therefore, $d\phi^*\omega=\sum_{i=1}^{n_2}(d(\omega_i\circ\phi)\wedge d\phi^i + (\omega_i\circ\phi)\wedge d^2\phi^i)=\phi^*(\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}d\omega_i\wedge dz^j)$. In our computation we used the identity $d^2=0$ from Lemma 2.21. □

Lemma 2.24. *For open subsets $U_i\subset\mathbb{C}^{n_i}$ for $i=1,2,3$, and holomorphic maps $\phi\colon U_1\longrightarrow U_2$, $\psi\colon U_2\longrightarrow U_3$, we have an equality of linear maps*

$$(\psi\circ\phi)^*=\phi^*\psi^*\colon\Omega^m(U_3)\longrightarrow\Omega^m(U_1).$$

Proof. Since pullback commutes with d and $\wedge$, it suffices to prove this assertion for $m=0$, that is, holomorphic functions $f\in\mathcal{O}(U_3)$. We have $(\psi\circ\phi)^*f = f\circ(\psi\circ\phi)=(f\circ\psi)\circ\phi=\phi^*(f\circ\psi)=\phi^*\psi^*f$. □

Remark 2.25. *For $m>n$ and $U\subset\mathbb{C}^n$, ever complex m-form ω on U vanishes. As we have seen in (7) we have $\omega=\sum_{1\leq i_1<\cdots<i_m\leq n}\omega_{i_1\dots i_m}dz^{i_1}\wedge\cdots\wedge dz^{i_m}$. But for $m>n$ there is no ordered tuple $1\leq i_1<\cdots<i_m\leq n$.*

3 Complex manifolds

In this section we introduce the principal object of interest of this course: a complex manifold. A (real) manifold was defined to be a topological space X, satisfying mild technical conditions, endowed with a so-called atlas. An atlas is a collection of open subsets $U \subset X$, and homeomorphisms $\phi\colon U \xrightarrow{\simeq} U'$, where $U' \subset \mathbb{R}^n$ is an open subset of Euclidean space. A pair (U, ϕ) is also called a *chart*. Given two charts (U, ϕ) and (V, ψ), we may define the so-called change-of-coordinates map. It's the horizontal arrow in the commutative diagram below.

$$\begin{tikzcd} & U \cap V \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\ \psi(U \cap V) \arrow[rr, "\phi|_{U\cap V} \circ \psi^{-1}|_{\psi(U\cap V)}"] & & \phi(U \cap V) \end{tikzcd}$$

In the definition of a (real) manifold one demands that the transition map $\phi|_{U\cap V} \circ \psi^{-1}|_{\psi(U\cap V)}$ is smooth, that is infinitely many times differentiable.

The definition of a complex manifold is similar. We demand that each chart (U, ϕ) defines a homeomorphism between U and an open subset $U' \subset \mathbb{C}^n$. The transition map above is required to be *holomorphic.*

3.1 The definition

Let's repeat the definition of a complex manifolds with the necessary technical details. To prepare the ground we need to introduce the notion of an atlas.

Definition 3.1. *Let X be a connected topological space.*

(a) A (complex n-dimensional) chart is a pair (U, ϕ), where $U \subset X$ is an open subset, and $\phi\colon U \xrightarrow{\simeq} U'$, where $U' \subset \mathbb{C}^n$ is an open subset of a complex space.

(b) Two charts (U, ϕ), (V, ψ) are called compatible*, if the* change-of-coordinates *map*

$$\phi|_{U\cap V} \circ \psi^{-1}|_{\psi(U\cap V)}\colon \psi(U \cap V) \to \phi(U \cap V)$$

is holomorphic.

(c) A (complex) atlas $\mathcal{U}$ is a set of charts covering X, such that every pair of charts in $\mathcal{U}$ is compatible.

(d) An atlas is called maximal, if it is not strictly contained in another atlas (that is, it contains every chart (U, ϕ), which is compatible with all charts in $\mathcal{U}$).

Now we are ready to give the definition of a complex manifold.

Definition 3.2. *A complex manifold is a second countable, Hausdorff topological space X, together with a maximal complex atlas $\mathcal{U}$ for every connected component of X.*

In practice the definition above is impractical. It is next to impossible to "write down" a maximal atlas, trivial cases excluded. In concrete examples, we define the structure of a complex manifold on a topological space X by defining an atlas, and refining it to a maximal atlas.

Exercise 3.3. *Show that every atlas is contained in a maximal atlas. Do you need the axiom of choice?*

Let's take another look at the examples covered in Section 1. We have defined atlases for $\mathbb{P}^n$ and V/Γ, where $\Gamma \subset V$ is a lattice in a complex vector space. To complete the proof that these are complex manifolds it remains to verify that the topological spaces are Hausdorff and second countable.

Lemma 3.4. *If $X = \bigcup_{i \in \mathbb{N}} U_i$ is a countable covering by open subsets, such that each U_i is second countable, then X is second countable as well.*

Proof. Recall that U_i is second countable, if and only if there exists a sequence of open subsets $\{V_{ij}\}_{j \in \mathbb{N}}$, such that every open subset $W \subset U_i$ can be expressed as the union $\bigcup_{V_{ij} \subset W} W = W$. Since $\mathbb{N} \times \mathbb{N}$ is countable, we have produced a countable collection of open subsets $\{V_{ij}\}_{(i,j) \in \mathbb{N}^2}$ of X with an analogous property: let $W \subset X$ be an arbitrary open subset, then we have that

$$\bigcup_{V_{ij} \subset W} V_{ij} = \bigcup_{i \in \mathbb{N}} \bigcup_{V_{ij} \subset W \cap U_i} V_{ij} = \bigcup_{i \in \mathbb{N}} (W \cap U_i) = W.$$

This shows that X is also second countable. □

As we know from our construction of the charts, $\mathbb{P}^n$ admits a finite covering my topological spaces homeomorphic to $\mathbb{C}^n$, that is, $\mathbb{P}^n$ is second countable by the Lemma above. Similarly, we have remarked that the spaces V/Γ are compact and covered by topological spaces homeomorphic to open subsets of $\mathbb{C}^n$. Compactness tells us that a finite number of these is already a covering, and the Lemma above implies that complex tori are second countable too. Since V/Γ is homeomorphic to $(\mathbb{S}^1)^{2n}$, we see that complex tori are also Hausdorff spaces. It remains to conclude that $\mathbb{P}^n$ is a complex manifold.

Exercise 3.5. *The topological space $\mathbb{P}^n$ is Hausdorff.*

3.2 Holomorphic functions, complex vector fields, and forms

The definition of a complex manifold has been set up in a way that it is meaningful to define holomorphic functions on them. From now on we fix a complex manifold X, and denote the maximal atlas, inducing the complex structure on X, by $\mathcal{U}$.

Definition 3.6. *A continuous function $f \colon X \longrightarrow \mathbb{C}^k$ is called holomorphic, if and only if for every chart $(U, \phi) \in \mathcal{U}$ we have that $f \circ \phi^{-1}$ is holomorphic.*

In fact we could have stated the definition above in a slightly different way. We could also say that f is holomorphic, if and only if for every $x \in X$, there exists a chart (U, ϕ), containing x, such that $f \circ \phi^{-1}$ is holomorphic at $\phi(x)$.

However, if (U_1, ϕ_1) and (U_2, ϕ_2) are two charts containing x, it follows from the definition of a complex manifold that $f \circ \phi_1^{-1}$ is holomorphic at $\phi_1(x)$ if and only if $f \circ \phi_2^{-1}$ is holomorphic at $\phi_2(x)$. Indeed, we have $f \circ \phi_1^{-1} = (f \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$, and the change of coordinates map $\phi_2 \circ \phi_1^{-1}$ is a biholomorphic map (where it's defined). This implies the assertion.

Similarly we can define complex vector fields on a manifold. The definition is inspired by the one above.

Definition 3.7. *A complex vector field on X is a rule which assigns to every chart $(U,\phi) \in \mathcal{U}$ a complex vector field $\mathcal{V}_{(U,\phi)}$ on $U' \subset \mathbb{C}^n$, such that for $(U_1,\phi_1),(U_2,\phi_1) \in \mathcal{U}$ we have*

$$(\phi_{21})_* \mathcal{V}_{(U_1,\phi_1)} |_{\phi_1(U_1 \cap U_2)} = \mathcal{V}_{(U_2,\phi_2)} |\phi_2(U_1 \cap U_2),$$

where $\phi_{21} \colon \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$ is the biholomorphic change of coordinates map $\phi_2 \circ \phi_1^{-1}$.

Let's unravel this definition a little bit. The expression $(\phi_{21})_*$ refers to the push-forward operation of complex vector fields that we introduced in Definition 2.16. More concretely, we demand that

$$\mathcal{V}_{(U_2,\phi_2)} |_{\phi_2(U_1\cap U_2)} = D\phi_{21} \begin{pmatrix} \mathcal{V}^1_{(U_1,\phi_1)} \circ \phi_{21}^{-1} \\ \vdots \\ \mathcal{V}^n_{(U_1,\phi_1)\circ\phi_{21}^{-1}} \end{pmatrix} |_{\phi_2(U_1\cap U_2)} = \sum_{i=0}^n \left(\sum_{j=0}^n (\frac{\partial \phi_{21}^i}{\partial z^j} \mathcal{V}^j \circ \phi_{21}^{-1}) \right) \frac{\partial}{\partial w^i}.$$

Unlike in the definition of a holomorphic function, this definition produces a global object from scratch, by assembling local data (the complex vector fields $\mathcal{V}_{(U,\phi)}$) in a consistent way. When we will have introduced the notion of sheaves, we will see that this definition fits very well into a sheaf-theoretic framework.

Definition 3.8. *A complex m-form on X is a rule which assigns to a chart (U,ϕ) an m-form $\omega_{(U,\phi)}$, such that for every pair of charts $(U_1,\phi_1),(U_2,\phi_2)$, and the change of coordinates map $\phi_{21} \colon \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$ we have*

$$(\phi_{21})^* \omega_{(U_2,\phi_2)}|_{\phi_2(U_1\cap U_2)} = \omega_{(U_1,\phi_1)}|_{\phi_1(U_1\cap U_2)}.$$

We remind the reader of the operation $\psi^*\omega$ defined in Definition 2.22. Formally it is characterised by the fact that $\psi^* d\omega = d\psi^*\omega$, $\psi^*(\omega \wedge \eta) = \psi^*\omega \wedge \psi^*\eta$, and for 0-forms (that is, holomorphic functions) f we have $\psi^* f = f \circ \psi$.

Maybe you're worried that a maximal atlas is too big as a collection of charts (U,ϕ) to work with complex m-forms or vector fields. The next lemma addresses this question, and as we'll see in its proof, holomorphic compatibility implies that we don't have to work with all possible charts to define those objects.

Lemma 3.9. *Let $\bigcup_{i\in I} U_i = X$ be an open covering of X, such that $(U_i,\phi_i) \in \mathcal{U}$ are charts. Assume that for every $i \in I$ we have a complex m-form $\omega_i \in \Omega^m(U_i')$ (respectively a complex vector field $\mathcal{V}_i$ on U_i'), such that for $i,j \in I$ the compatibility condition*

$$(\phi_{ji})^* \omega_2|_{\phi_j(U_i\cap U_j)} = \omega_i|_{\phi_i(U_i\cap U_j)},$$

respectively

$$(\phi_{ji})_* \mathcal{V}_1 |_{\phi_i(U_i\cap U_j)} = \mathcal{V}_2 |\phi_j(U_i \cap U_j)$$

is satisfied. Then, there exists a unique complex m-form ω on X (respectively a complex vector field $\mathcal{V}$), such that $\omega_{(U_i,\phi_i)} = \omega_i$ for all $i \in I$ (respectively $\mathcal{V}_{(U_i,\phi_i)} = \mathcal{V}_i$).

Proof. We will only discuss the case of m-forms, leaving the analogous situation of complex vector fields to the reader. Given an arbitrary chart (U,ϕ), we need to define a complex m-form $\omega_{(U,\phi)}$. To do so, we observe that $V_i = \phi(U_i \cap U) \subset U'$ defines an open covering of $U' \subset \mathbb{C}^n$, and that the restrictions $\omega_i|_{V_i}$ still define a compatible collection of locally defined m-forms for U'. Therefore, in order to define $\omega_{(U,\phi)} \in \Omega^m(U')$, we may temporarily assume that $X = U' \subset \mathbb{C}^n$ is an open subset of X.

Using the temporary assumption that $X \subset \mathbb{C}^n$, we define for every $1 \leq j_1 < \cdots < j_m \leq n$, and every $i \in I$, a holomorphic function $\omega_{j_1 \dots j_m;i} \in \mathcal{O}(V_i)$. The functions are defined by means of the decomposition of the complex m-form on V_i

$$\phi_i^* \omega_i = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \omega_{j_1 \dots j_m;i} dz^{j_1} \wedge \cdots \wedge dz^{j_m}.$$

The compatibility condition $(\phi_{ji})^* \omega_2|_{\phi_j(U_i \cap U_j)} = \omega_i|_{\phi_i(U_i \cap U_j)}$, implies that for $i, i' \in I$ we have

$$\omega_{j_1 \dots j_m;i}|_{V_i \cap V_{i'}} = \omega_{j_1 \dots j_m;i'}.$$

Or, in other words, the family of holomorphic functions $\omega_{j_1,\dots,j_m;i}$ agree on overlaps of the open covering $\{V_i\}_{i \in I}$ of $X \subset \mathbb{C}^n$. Therefore, there exist holomorphic functions $\omega_{j_1 \dots j_m}$ on X, such that $\omega_{j_1 \dots j_m}|_{V_i} = \omega_{j_1 \dots j_m;i}$. We let $\omega = (\omega_{U,\phi})$ be $\sum_{1 \leq j_1 < \cdots < j_m \leq n} \omega_{j_1 \dots j_m} dz^{j_1} \wedge \cdots \wedge dz^{j_m}$.

From now on we discard the temporary assumption $X \subset \mathbb{C}^n$, and let X be a general complex manifold, like in the statement of the Lemma. For every $(U, \phi) \in \mathcal{U}$ we have defined an $\omega_{(U,\phi)} \in \Omega^m(U')$, and it remains to check compatibility of the complex m-forms.

For $(U_1, \phi_1), (U_2, \phi) \in \mathcal{U}$ we denote the biholomorphic change of coordinates map by

$$\phi_{21} = \phi_2 \circ \phi_1^{-1} \colon \phi_1(U_1) \longrightarrow \phi_2(U_2).$$

We have to show that

$$\phi_{21}^* \omega_{(U_2,\phi)}|_{\phi_1(U_1 \cap U_2)} = \omega_{(U_1,\phi_1)}|_{\phi_2(U_1 \cap U_2)}. \tag{8}$$

It suffices to establish (8) for an open neighbourhood U of every $x \in U_1 \cap U_2$. Choose $i \in I$, such that $x \in U_i$, and consider the chart $(U_3, \phi_3) = (U_i \cap U_1 \cap U_2, \phi_i|_{U_i \cap U_1 \cap U_2})$. We now obtain a commutative triangle of change of coordinates maps

$$\begin{tikzcd}
\phi_1(U_1 \cap U_2 \cap U_3) \arrow[rr, "\phi_{31}"] \arrow[dr, "\phi_{21}"'] & & \phi_3(U_1 \cap U_2 \cap U_3) \\
& \phi_2(U_1 \cap U_2 \cap U_3). \arrow[ur, "\phi_{32}"'] &
\end{tikzcd}$$

By definition of ω_{U_1,ϕ_1} we have $\omega_{(U_1,\phi_1)}|_{\phi_1(U_1 \cap U_2 \cap U_3)} = \phi_{31}^* \omega_3$, where $\omega_3 = \omega_i|_{\phi_1(U_1 \cap U_2 \cap U_3)}$. And similarly, $\omega_{(U_2,\phi_2)}|_{\phi_2(U_1 \cap U_2 \cap U_3)} = \phi_{21}^* \omega_3$. This implies that $\phi_{21}^* \omega_2 = \omega_1$, since $\phi_{31}^* = (\phi_{32} \circ \phi_{21})^* = \phi_{21}^* \circ \phi_{23}^*$ by Lemma 2.24. $\square$

3.3 Smooth complex projective varieties

We have already seen in Corollary 1.9 that a compact complex manifold cannot be embedded into $\mathbb{C}^n$. Nonetheless, the notion of complex submanifolds plays as much of a role in the theory of complex manifolds to construct examples, as it does in the theory of smooth manifolds. The formal definition of complex submanifolds is as follows.

Definition 3.10. *An m-dimensional complex submanifold of an n-dimensional complex manifold X is a subset $Y \subset X$, such that for every $y \in Y$ there exists a chart (U, ϕ) of the complex manifold X, satisfying the condition*

$$\phi|_{Y\cap U} \colon Y \cap U \xrightarrow{\simeq} U' \cap \{0\}^{n-m} \times \mathbb{C}^m = \{(z^1, \ldots, z^n) \in U' | z_{n-m+1} = \cdots = z_n = 0\}.$$

Every complex submanifold $Y \subset X$ is of course a complex manifold itself. The pairs $(Y \cap U, \phi|_{Y\cap U})$ for complex charts (U, ϕ) of X as in the definition above, give rise to charts for Y. We only need to consider $\phi|_{Y\cap U}$ as a homeomorphism from $Y \cap U$ to an open subset of $\mathbb{C}^m$. This gives us a covering of compatible charts for Y, and we simply pass to the maximal atlas containing these charts.

Since X is a complex manifold, the underlying topological space of X is Hausdorff and second countable. This implies that those topological properties are also held by the underlying space of Y. Summarising this discussion we see that a complex submanifold carries indeed the structure of a complex manifold.

We fix k homogeneous polynomials $G_i = G_i(z_0, \ldots, z_n)$ of degree d_i in $n+1$ variables. Moreover, we assume that $w = (w_0, \ldots, w_n) \in \mathbb{C}^{n+1} \backslash 0$ is a vector, such that $G_i(z_0, \ldots, z_n) = 0$ for every $i = 1, \ldots, k$, the condition

$$\left(\frac{\partial G_i}{\partial z^j}(w)\right)_{j=0,\ldots,m}^{i=,1,\ldots,k} \text{ is a surjective linear map} \tag{9}$$

is satisfied. Recall that we denote by $V(G_1, \ldots, G_k) \subset \mathbb{P}^n$ the subset given by tuples $(z_0 : \cdots : z_n) \in \mathbb{P}^n$, such that $G_i(z_0, \ldots, z_n) = 0$ for all $i = 1, \ldots, k$.

Proposition 3.11. *If the homogeneous polynomials $G_1, \ldots, G_k$ satisfy condition* (9) *for every $w \in V(G_1, \ldots, G_k)$, then $V(G_1, \ldots, G_k)$ is a complex submanifold*

We will deduce this proposition as a special case of a holomorphic version of the Regular Value Theorem. At first we need to explain what it means for a map between complex manifolds to be holomorphic.

If X and Y are complex manifolds, and $f \colon X \longrightarrow Y$ is a continuous map we are going to define next what it means for f to be holomorphic. For a chart (V, ψ) of Y, we consider a chart (U, ϕ), such that $f(U) \subset V$. This is possible because f is continuous. With respect to these choice we obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ U' \arrow[r, "\psi\circ f\circ\phi^{-1}"'] & V'. \end{tikzcd}$$

Definition 3.12. *A continuous map $X \xrightarrow{f} Y$ between two complex manifolds is called* holomorphic*, if and only if $\psi \circ f \circ \phi^{-1}$ is holomorphic for every pair of charts (U, ϕ) of X and (V, ψ) of Y, such that $f(U) \subset V$.*

Since the transition function between overlapping charts is a biholomorphic function, the definition above has a lot of redundancy built in, which makes it feasible to verify in explicit situations that a continuous function is holomorphic.

In practice it is sufficient to choose charts $\{(U_i\phi_i)\}_{i\in I}$ for X, and $\{(V_j,\psi_j)\}_{j\in J}$ for Y, such that $\bigcup_{i\in I} U_i = X$, $\bigcup_{j\in J} V_j = Y$, and the image $f(U_i)$ of every U_i is contained in some $V_{j(i)}$. One is then only required to show that for every $i \in I$ that the resulting map $\psi_{j(i)} \circ f \circ \phi_i^{-1}$ is holomorphic.

Theorem 3.13 (Regular Value Theorem)**.** *Let $X \xrightarrow{f} Y$ be a holomorphic map between complex manifolds, and $y \in Y$ a so-called regular value, that is, for every $x \in f^{-1}(y)$, and every chart (U,ϕ) containing x, we have that $(\frac{\partial(f\circ\phi^{-1})^i}{\partial z^j})_{i,j}$ is a surjective linear map. Then, $f^{-1}(y) \subset X$ is a complex submanifold.*

Proof. Without loss of generality one can assume that $Y = \mathbb{C}^k$ is a complex space. This is possible because $y \in Y$ belongs to a chart (V,ψ), which is by means of ψ biholmorphically equivalent to an open subset $V' \subset \mathbb{C}^k$. The preimage $f^{-1}(y) \subset X$ only depends on $f^{-1}(V) \xrightarrow{f|_{f^{-1}(V)}} V$. Henceforth, we replace X by the open subset $f^{-1}(V)$ and Y by $\mathbb{C}^k$.

Let (U,ϕ) be a complex chart containing $x \in f^{-1}(x)$. We have a holomorphic function $f \circ \phi^{-1}$ on $U' \subset \mathbb{C}^n$. We use the Implicit Function Theorem 2.8 to construct a smaller open neighbourhood $W \subset U$ of x, belonging to a chart (W,ψ), such that we have a homeomorphism $\psi\colon f^{-1}(y)\cap W$ with $\{0\}^k \times \mathbb{C}^{n-k} \cap W'$ for W' an open subset of $\mathbb{C}^n$.

Using the chart ϕ we identify U with $U' \subset \mathbb{C}^n$. To simplify the notation we will refrain from distinguishing U and U'. By translating the open subset we may assume without loss of generality that $x = 0 \in U$ and $f(0) = 0$. We now have an open neighbourhood of $0 \in U \subset \mathbb{C}^n$ and a holomorphic function $f\colon U \longrightarrow \mathbb{C}^k$, such that the Jacobi matrix $\left(\frac{\partial f^i}{\partial z^k}(0)\right)_{j=1,\dots,n}^{i=1,\dots,k}$ is a surjective linear map. This implies that there exist $0 \le i_1 < \cdots < i_k \le n$, such that $\left(\frac{\partial f^i}{\partial z^{i_j}}(0)\right)_{j=1,\dots,k}^{i=1,\dots,k}$ is surjective. By permuting coordinates we may assume $i_j = j$ for every $j = 1,\dots,k$. By Porism 2.9 we see that there exist open neighbhourhoods W, W' of 0 in $\mathbb{C}^n$, and a biholomorphic map $h\colon W \longrightarrow W'$, such that $(f\circ\phi)(z^1,\dots,z^n) = (z^1,\dots,z^k)$. Thus, by restriction we have a homeomorphism

$$h\colon f^{-1}(0)\cap W \xrightarrow{\simeq} \{(z^1,\dots,z^n) \in W' | z_1 = \cdots = z_k = 0\}.$$

This allows one to conclude that $f^{-1}(y) \subset X$ is a complex submanifold. □

We cannot directly apply Theorem 3.13 to the system of homogeneous equations $G_1,\dots,G_k$. Despite having holomorphic functions $G_i\colon\ \mathbb{C}^{n+1}\setminus 0 \longrightarrow \mathbb{C}$, we cannot directly define a holomorphic function $G_i\colon (z^1 : \cdots : z^n) \mapsto G_i(z^0,\dots,z^n)$, because choosing a different representative (e.g. $\lambda(z^0,\dots,z^n)$) modifies the value of G_i by λ^{d_i}.

Proof of Proposition 3.11. Let G be a homogeneous polynomial of degree d. By expressing G as a linear combination of degree d monomials $(z^0)^{d_0}\cdots(z^n)^{d_n}$, we see that

$$\sum_{i=0}^{n} z^i \frac{\partial G}{\partial z^i} = d\cdot G(z^0,\dots,z^n). \tag{10}$$

In particular if $(z^0,\cdots,z^n) \in \mathbb{C}^{n+1}\setminus 0$ is a non-trivial zero of G, that is $G(z_0,\dots,z_n) = 0$, then $\sum_{i=0}^{n} z^i \frac{\partial G}{\partial z^i} = d\cdot G(z^0,\dots,z^n) = 0$.

Recall the charts (U_i, ϕ) of $\mathbb{P}^n$, and the inverses $\psi_i \colon \mathbb{C}^n \longrightarrow \mathbb{P}^n$, presented by a map $\widetilde{\psi}_i \colon \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1} \setminus 0$. We will apply the Regular Value Theorem 3.13 to $(G_1, \dots, G_k) \circ \widetilde{\psi}_i$, to see that $V(G_1, \dots, G_k) \cap U_i$ is a submanifold for every $i = 0, \dots, n$. This will imply that $V(G_1, \dots, G_k) \subset \mathbb{P}^{n+1}$ is a submanifold. Without loss of generality we may assume $i = 0$, because the general case may be reduced to this one by relabelling the coordinates $z^0, \dots, z^n$.

The map $\widetilde{\psi}_0 \colon \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is given by $(w^1, \dots, w^n) \mapsto (1, w^1 : \cdots, w^n)$. Hence we have

$$(G_1, \dots, G_k) \circ \widetilde{\psi}_i \colon (w^1, \dots, w^n) \mapsto (G_1(1, w^1, \dots, w^n), \dots, G_k(1, w^1, \dots, w^n)).$$

The chain rule implies

$$\left(\frac{\partial (G_i \circ \widetilde{\psi}_0)}{\partial w^j} \right)_{j=0,\dots,n}^{i=1,\dots,k} = \left(\frac{\partial G_i}{\partial z^j} \right)_{j=0,\dots,n}^{i=1,\dots,k} \cdot \left(\frac{\partial (\widetilde{\psi}_0)^i}{\partial w^j} \right)_{\ell=1,\dots,n}^{j=0,\dots,n}.$$

This implies that the matrix on the left hand side is obtained from the Jacobi matrix $A(z^0, \dots, z^n) = \left(\frac{\partial G_i}{\partial z^j} \right)_{j=0,\dots,n}^{i=1,\dots,k}$ by setting the variable $z^0 = 0$ and omitting the 0-th column. Omitting the 0-th column corresponds to considering a complex linear subspace $\mathbb{C}^n$ with the basis $(e_1, \dots, e_n)$ rather than $(e_0, \dots, e_n)$ for $\mathbb{C}^{n+1}$. It remains to show that the resulting linear map $A|_{\mathbb{C}^n}$ is still surjective.

To see this we consider the basis of $\mathbb{C}^{n+1}$

$$\left(\begin{pmatrix} 1 \\ w^1 \\ \vdots w^n \end{pmatrix}, e_1, \dots, e_n \right).$$

As we have seen above, we have $\sum_{i=0}^n z^i \frac{\partial G}{\partial z^i}|_{z^0=1, z^i=w^i} = d \cdot G(1, w^1, \dots, w^n) = 0$, and therefore the matrix A vanishes on the first vector of this basis. Since A is surjective, we see that $A|_{\mathbb{C}^n}$ has to be surjective as well, which is what we wanted to check. $\square$

3.4 Holomorphic maps between manifolds

It is time to study more examples of holomorphic maps between manifolds.

Example 3.14. *The canonical map $p \colon \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$, sending $(z^0, \dots, z^n)$ to $(z^0 : \dots : z^n)$ is holomorphic.*

Complex projective space $\mathbb{P}^n$ is covered by the $n + 1$ standard charts (U_i, ϕ_i). The preimage $p^{-1}(U_i)$ is by definition equal to the open subset $\{(z^0, \dots, z^n) | z^i \neq 0\}$. We choose the family of charts $\{(p^{-1}(U_i), \mathrm{id})\}_{i=0,\dots,n}$ for $\mathbb{C}^{n+1} \setminus 0$, and $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ for $\mathbb{P}^n$. The map $\phi_i \circ p \circ \mathrm{id}^{-1}$ is given by $(z^0, \dots, z^n) \mapsto (\frac{z^0}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i})$, which is indeed holomorphic for $z_i \neq 0$.

Definition 3.15. *Let $X \xrightarrow{f} Y$ be a holomorphic map between complex manifolds. For $\omega \in \Omega^m(Y)$ we define $f^*\omega$ to be the complex m-form, which assigns to every chart (U, ϕ) of X with image $f(U) \subset V$ for a chart (V, ψ) of Y, the form $(f^*\omega)_{(U,\phi)} = (\psi \circ f \circ \phi^{-1})^* \omega_{(V,\psi)}$.*

In principle it is possible that X has a chart (U, ϕ), such that $f(U) \not\subset V$ for every complex chart (V, ψ) of Y. Take for instance the holomorphic map $p \colon \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$. The left hand side is already an open subset of $\mathbb{C}^{n+1}$, that is, we have a tautological chart $(U, \phi) = (\mathbb{C}^{n+1} \setminus 0, \mathrm{id})$. But the image $p(U) = \mathbb{P}^n$ is too big to be contained in a single chart.

This illustrates that in Definition 3.15 we have a priori defined $(f^*\omega)_{(U,\phi)}$ only for a subset of the atlas $\mathcal{U}$. However, since f is continuous, the collection of these charts covers X. As we have checked in Lemma 3.9, this is sufficient to define a differential form.

In this section we will study an interesting example of a holomorphic map: a blow-up $\mathrm{Bl}_0(\mathbb{C}^n) \xrightarrow{f} \mathbb{C}^n$. The complex manifold $\mathrm{Bl}_0(\mathbb{C}^n)$ has the property that $f^{-1}(\mathbb{C} \setminus 0) \longrightarrow \mathbb{C} \setminus 0$ is a biholomorphic map, and $f^{-1}(0)$ is equivalent to a projective space $\mathbb{P}^{n-1}$. This sudden chance in dimension of the fibre is the reason why this construction is referred to as blow-up.

Definition 3.16. *Remember that as a set $\mathbb{P}^{n-1}$ is the set of 1-dimensional complex subspaces (or lines) $\ell \subset \mathbb{C}^n$. As a topological space, we define $\mathrm{Bl}_0(\mathbb{C}^n)$ to be the subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$, as defined by the incidence relation $z \in \ell$:*

$$\mathrm{Bl}_0(\mathbb{C}^n) = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \,|z \in \ell\}.$$

We let $f\colon \mathrm{Bl}_0(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$ be the map induced by the projection $(z, \ell) \mapsto z$.

Since $\mathrm{Bl}_0(\mathbb{C}^n)$ is defined as a topological subspace of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ it is clear that it is Hausdorff and second countable. For the same reason we see that f is a continuous map.

There's another continuous map, $g\colon \mathrm{Bl}_0(\mathbb{C}^n) \longrightarrow \mathbb{P}^{n-1}$, mapping (z, ℓ) to $\ell \in \mathbb{P}^{n-1}$. By definition, we have the interesting relation $g^\ell = \ell \subset \mathbb{C}^n$ for every point of $\ell \in \mathbb{P}^{n-1}$.

We've already introduced standard charts (U_i, ϕ_i) for $\mathbb{P}^{n-1}$, and will use a similar construction to define charts for $\mathrm{Bl}_0(\mathbb{C}^n)$. Since g is continuous, $g^{-1}(U_i) \subset \mathrm{Bl}_0(\mathbb{C}^n)$ is open. Changing back to our usual notation $\ell = (z^0 : \cdots : z^{n-1})$ we have a continuous map $\phi'_i\colon g^{-1}(U_i) \longrightarrow \mathbb{C}^n$

$$(u, (z^0 : \cdots : z^{n-1})) \mapsto (u^i, \frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i})$$

with a continuous inverse $\mathbb{C}^n \longrightarrow g^{-1}(U_i)$

$$\left(w^0(w^1, \cdots, w^i, 1, w^{i+1}, \ldots, w^{n-1}), (w^1 : \cdots, w^i : 1 : w^{i+1} : \ldots, w^{n-1})\right).$$

Exercise 3.17. *Verify that coordinate changes are holomorphic, and hence conclude that $\mathrm{Bl}_0(\mathbb{C}^n)$ is a complex manifold.*

Using these charts we can check that $f\colon \mathrm{Bl}_0(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$ and $g\colon \mathrm{Bl}_0(\mathbb{C}^n) \longrightarrow \mathbb{P}^{n-1}$ are holomorphic maps. We have $f \circ (\phi'_i)^{-1}(w^0, \ldots, w^n) = w^0(w^1, \cdots, w^i, 1, w^{i+1}, \ldots, w^{n-1})$, which is certainly holomorphic. Similarly, $\phi_i \circ g \circ (\phi'_i)^{-1}(w^0, \ldots, w^n) = w^1, \cdots, w^i, 1, w^{i+1}, \ldots, w^{n-1}$ is a holomorphic map.

As we have observed above, the fibres of g are naturally 1-dimensional complex vector spaces. Moreover, we have produced biholomorphic maps

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[rr, "\simeq"] \arrow[dr, "g"'] & & U_i \times \mathbb{C} \arrow[dl, "p_1"] \\
& U_i &
\end{tikzcd}$$

which are fibrewise linear. As we will see in the next section, this is an example of a line bundle (that is, a complex vector bundle of rank 1).

4 Vector bundles

4.1 The basics of holomorphic vector bundles

We fix a complex manifold X and will study so-called *holomorphic vector bundles* on X. To simplify the notation we will often omit *holomorphic* and only speak of vector bundles.

Definition 4.1. *A (rank n) vector bundle E over X is a holomorphic map of complex manifolds $E \xrightarrow{\pi} X$, such that*

(a) every fibre $\pi^{-1}(x)$ has the structure of a finite-dimensional complex vector space,

(b) there exists an open covering $\bigcup_{i \in I} U_i = X$ and biholomorphic maps $\pi^{-1}(U_i) \xrightarrow{\alpha_i} U_i \times \mathbb{C}^n$,

(c) such that $p_1 \circ \alpha_i = \pi$, where $p_1 \colon U_i \times \mathbb{C}^n \longrightarrow U_i$ is the canonical projection to the second component. That is, the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr] \arrow[dr] & & U_i \times \mathbb{C}^n \arrow[dl] \\ & U_i, & \end{tikzcd}$$

(d) α_i is fibrewise linear, that is, $\forall x \in X$ the induced map $\pi^{-1}(x) \times \mathbb{C}^n$ is a linear isomorphism of complex vector spaces.

In heuristic terms we can think of a vector bundle $E \xrightarrow{\pi} X$ as a family of (complex) vector spaces over X. To each point $x \in X$ we associate a vector space $\pi^{-1}(X)$. This family is not supposed to behave pathologically, hence we stipulate that X can be covered by open subsets over which the vector bundle is *trivial*, that is, of the shape $U_i \times \mathbb{C}^n$. Vector bundles are *locally trivial*, but often allow to encode interesting *global* phenomena.

Let $E \xrightarrow{\pi} X$ be a vector bundle. The complex manifold E is called the *total space*, while X is referred to as the *base*. The holomorphic map π is often called the *structure map*. This terminology is useful, but distinguishing between the total space of a vector bundle and the vector bundle itself (which actually is a triple (E, X, π)) is too cumbersome in practice. We will therefore often refer to a vector bundle E on X, and suppress the structure map π from our terminology.

Definition 4.2. *A (rank n)* cocycle *datum on a complex manifold X is a pair $\big(\{U_i\}_{i \in I}, (\alpha_{ij})_{ij \in I^2}\big)$, where $X = \bigcup_{i \in I} U_i$ is an open covering for X, and $\alpha_{ij} \colon U_i \cap U_j \longrightarrow \mathrm{GL}_n(\mathbb{C}) \subset \mathbb{C}^{n \times n}$ is a holomorphic function, such that for every triple of indices $(i, j, k) \in I^3$ the so-called* cocycle identity

$$\alpha_{ik} = \alpha_{ij} \cdot \alpha_{jk}$$

is satisfied, where $\cdot$ denotes pointwise matrix multiplication.

Note that taking $(i, j, k) = (i, i, i)$ we obtain $\alpha_{ii} = \alpha_{ii}^2$, that is, since $\alpha_{ii} \in \mathrm{GL}_n(\mathbb{C})$, we must have $\alpha_{ii} = 1$. Similarly, taking $(i, j, k) = (i, j, i)$, we obtain $\alpha_{ii} = 1 = \alpha_{ij}\alpha_{ji}$, that is, $\alpha_{ji} = (\alpha_{ij})^{-1}$ (taking the inverse matrix).

If E is a vector bundle of rank n on X, then there exists an open covering $\bigcup_{i\in I} U_i = X$, and fibrewise linear biholomorphic maps $\alpha_i \colon \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times \mathbb{C}^n$. Over the intersections $U_{ij} = U_i \cap U_j$ we therefore obtain a fibrewise linear map $\alpha_i \circ \alpha_j^{-1} \colon U_{ij} \times \mathbb{C}^n \longrightarrow U_{ij} \times \mathbb{C}^n$, defined as the composition of the top row of the following commutative diagram

$$\begin{tikzcd}
U_{ij} \times \mathbb{C}^n \arrow[r, "\alpha_j^{-1}"] \arrow[dr, "p_1"'] & \pi^{-1}(U_{ij}) \arrow[r, "\alpha_i"] \arrow[d, "\pi"] & U_{ij} \times \mathbb{C}^n \arrow[dl, "p_1"] \\
& U_{ij}. &
\end{tikzcd}$$

By commutativity of this diagram, and fibrewise linearity, we see that $\alpha_i \circ \alpha_j^{-1}$ is of the shape $x \mapsto (x, \alpha_{ij}(x))$, where $\alpha_{ij} \colon U \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is a holomorphic function taking values in the open subset $\mathrm{GL}_n(\mathbb{C}) \subset \mathbb{C}^{n\times n}$. This process defines a cocycle datum. To see this, it suffices to check that for every triple of indices (i,j,k), the relation $\alpha_{ik} = \alpha_{ij} \cdot \alpha_{jk}$ is satisfied on $U_{ijk} = U_i \cap U_j \cap U_k$. This follows from the fact that $x \mapsto (x, \alpha_{ij})$ is by definition equal to $\alpha_i \circ \alpha_j^{-1}$. Hence, we have $\alpha_i \circ \alpha_j^{-1} \circ \alpha_j \circ \alpha_k^{-1} = \alpha_i \circ \alpha_k^{-1}$ on U_{ijk}, which implies the cocycle relation $\alpha_{ik} = \alpha_{ij} \cdot \alpha_{jk}$.

Lemma 4.3. *Every rank n cocycle datum on a complex manifold can be realised by a vector bundle $E \xrightarrow{\pi} X$.*

Proof. As a first step we have to construct a topological space E and a continuous map $\pi \colon E \longrightarrow X$. Next, we have to construct an atlas on E, and show that π is holomorphic, and construct bundle charts to show it is indeed a vector bundle.

We define E to be the quotient $(\coprod_{i\in I} U_i \times \mathbb{C}^n)/\simeq$, where (x,z) is equivalent to (y,w) if and only if $\exists (i,j) \in I^2$, such that $x \in U_i$, $y \in U_j$, $x = y$, and $\alpha_{ij}(w) = z$. The map π sends the equivalence class of (x,z) to x. The universal property of the quotient topology guarantees that it is continuous.

By construction, $\pi^{-1}(U_i)$ is in bijection with $U_i \times \mathbb{C}^n$, by means of the continuous map $U_i \times \mathbb{C}^n \longrightarrow \pi^{-1}(U_i)$ sending (x,z) to the equivalence class $[(x,z)]$. We call the inverse of this homeomorphism α_i (it's continuous by the universal property of the quotient topology).

The resulting topological space is second countable, because there exists a countable covering of X by open subsets: $X = \bigcup_{n\in\mathbb{N}} V_n$, such that every V_n is contained in one of the U_i (since X is second countable). The preimage $\pi^{-1}(V_n)$ is thus homeomorphic to $V_i \times \mathbb{C}^n$, and thus second countable itself. Lemma 3.4 implies that E is second countable.

Exercise 4.4. *Show that E is Hausdorff.*

It remains to construct an atlas for E. We may assume without loss of generality that every U_i belongs to a chart (U_i, ϕ_i) of the complex manifold X. This is achieved by replacing the open subsets U_i by smaller ones, if necessary. We then have homeomorphisms

$$\pi^{-1}(U_i) \longrightarrow U_i' \times \mathbb{C}^n = \phi_i(U_i) \times \mathbb{C}^n,$$

which serve as our charts. For $i, j \in I^2$ the change of coordinates maps are given by $(x,z) \mapsto (\phi_{ji}(x), \alpha_{ji} \cdot z)$, where ϕ_{ji} denotes the change of coordinates map of (U_i, ϕ_i) and (U_j, ϕ_j). Since this map is constructed from holomorphic functions, it is holomorphic itself. $\square$

We can now use abstract cocycle data to define new vector bundles.

Definition 4.5. *Let $(\{U_i\}_{i\in I}, (\alpha_{ij})_{(i,j)\in I^2})$ be a cocycle datum for a vector bundle E on a complex manifold X. Then, $(\{U_i\}_{i\in I}, ((\alpha_{ij}^{\mathsf{T}})^{-1})_{(i,j)\in I^2})$ is a cocycle datum for the dual vector bundle $E^\vee$.*

We leave it as an exercise to the reader to check that the isomorphism class of $E^\vee$ only depends on the isomorphism class of E.

Definition 4.6. *Let X be a complex manifold.*

(a) An isomorphism (or equivalence) between two vector bundle $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X$ is a pair of mutually inverse holomorphic maps $f\colon E \longrightarrow E'$, $g\colon E' \longrightarrow E$, which are fibrewise linear, such that

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

(b) The trivial rank n vector bundle on X is $X \times \mathbb{C}^n \xrightarrow{p_1} X$, where $p_1\colon (x,z) \mapsto x$ is the projection to the first component.

(c) We say that a rank n vector bundle E on X is trivial, if it is isomorphic to the trivial rank n vector bundle $X \times \mathbb{C}^n$.

We will see many examples of non-trivial vector bundles. But how can one actually prove that a vector bundle is non-trivial, or in general that two vector bundles are not isomorphic? As a first tool to do so we introduce the vector space of global sections of a vector bundle. It turns out to be an isomorphism invariant, and also explicitly computable. By computing the dimension of this vector space we can compare and distinguish between the examples introduced below.

Definition 4.7. *A* global section *of a vector bundle $E \xrightarrow{\pi} X$ is a holomorphic map $s\colon X \longrightarrow E$, such that $\pi \circ s = \mathrm{id}_X$, that is, the diagram*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "\mathrm{id}_X"'] & E \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

commutes. We denote the set of global sections of E by $\Gamma(X,E)$.

As we already hinted at above, the set of global sections is actually canonically endowed with the structure of a $\mathbb{C}$-vector space. The reason is that $s\colon X \longrightarrow E$ is a holomorphic map, which assigns to every $x \in X$ an element in the vector space $\pi^{-1}(x)$. Given two sections $s_1, s_2 \in \Gamma(X,E)$ we can therefore define $s_1 + s_2\colon x \mapsto s_1(x) + s_2(x) \in \pi^{-1}(x)$. In order for this to make sense however, we have to verify that the resulting map is holomorphic. Similarly, given a complex number $\lambda \in \mathbb{C}$ we will check that $\lambda \cdot s\colon x \mapsto \lambda \cdot s(x)$ is a holomorphic function.

Lemma 4.8. *Let E be a vector bundle on X, and $s_1, s_2 \in \Gamma(X,E)$ two global sections. Then the functions $s_1 + s_2 \colon X \longrightarrow E$ defined by fibrewise addition is holomorphic, that is, defines a global section of E. Similarly, for every $\lambda \in \mathbb{C}$, the function $\lambda \cdot s \colon X \longrightarrow E$ is holomorphic, and thus defines a global section of E. The triple $(\Gamma(X,E), +, \cdot)$ satisfies the axioms of a complex vector space.*

Proof. Since $E \xrightarrow{\pi} X$ is a vector bundle, every point $x \in X$ has an open neighbourhood $U \subset X$, such that $\pi^{-1}(U) \cong U \times \mathbb{C}^n$ is equivalent to the trivial vector bundle. The restriction of a section $s \in \Gamma(X,E)$ is given by a holomorphic function $U \mapsto^{s|_U} U \times \mathbb{C}^n$, which is of the shape $x \mapsto (x, f(x))$, where $f \colon U \longrightarrow \mathbb{C}^n$ is a holomorphic function. Therefore, we see that $s_1 + s_2$ is presented by the holomorphic function $x \mapsto (x, f_1(x) + f_2(x))$. The sum $f_1 + f_2$ of two holomorphic functions is of course again holomorphic. After all it can be written as a composition of the holomorphic maps $U \xrightarrow{(f_1,f_2)} \mathbb{C}^n \times \mathbb{C}^n$, and $+ \cdot \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$.

A similar analysis applies to the map $\lambda \cdot s$. It only remains to show that with respect to the operations $+$ and $\cdot$ the set $\Gamma(X,E)$ really is a vector space.

The zero element is represented by the zero section $0_X \colon X \longrightarrow E$, which maps $x \longrightarrow 0 \in \pi^{-1}(X)$. It is clear that the axioms of a vector space are satisfied, because they hold fibrewise by assumption. $\square$

In Definition 4.2 we introduced so-called cocycle data for vector bundles. One can also express global sections in terms of the cocycle data.

Lemma 4.9. *If $\left(\{U_i\}_{i\in I}, (\alpha_{ij})_{(i,j)\in I^2}\right)$ is a cocycle datum for a vector bundle E on X, then we have a canonical bijection between $\Gamma(X,E)$ and the collection of holomorphic functions $\{f_j \colon U_j \longrightarrow \mathbb{C}^n\}_{j\in I}$, satisfying the condition $f_i|_{U_i\cap U_j} = \alpha_{ij} f_j|_{U_i \cap U_j}$ for all $(i,j) \in I^2$.*

Proof. By assumption we have an isomorphism of vector bundles $\alpha_i \colon \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times \mathbb{C}^n$ for every $i \in I$. If $s \colon X \longrightarrow E$ is a global section, we can use the isomorphisms α_i to see that $s|_{U_i}$ is of the shape $x \mapsto (x, f_i(x))$, where $f_i \colon U_i \longrightarrow \mathbb{C}^n$ is a holomorphic function.

Since two trivialisations differ by α_{ij}, we obtain the relation $f_i(x) = \alpha_{ij}(x) f_j(x)$ for all $x \in U_i \cap U_j$.

Vice versa, given $\{f_j \colon U_j \longrightarrow \mathbb{C}^n\}_{j\in I}$, we can define a section $s \colon X \mapsto E$ by sending $x \in U_i$ to $\alpha_i^{-1}(f_i(x))$. This does not depend on the choice of $i \in I$, such that $x \in U_i$, due to the relation $f_i|_{U_i\cap U_j} = \alpha_{ij} f_j|_{U_i\cap U_j}$. Since the resulting map $X \longrightarrow E$ is holomorphic on each U_i, we see that it defines indeed a global section. $\square$

4.2 Line bundles

A vector bundle of rank 1 is called a *line bundle*. Here we use the convention that a line is a 1-dimensional complex vector space. A line bundle is therefore quite literally a bundle of lines over a complex manifold X.

Lemma 4.10. *We say that a section s of a line bundle L on X is* nowhere vanishing *if for all $x \in X$ we have $s(x) \neq 0$. There is a canonical bijection between nowhere vanishing sections and isomorphisms $\alpha \colon L \xrightarrow{\simeq} X \times \mathbb{C}$.*

Proof. The trivial bundle $X \times \mathbb{C}$ has a canonical section given by the constant map $x \mapsto 1$. It is clear that this section is nowhere vanishing. Hence, if $\alpha \colon L \xrightarrow{\simeq} X \times \mathbb{C}$ is an isomorphism with the trivial bundle, then we obtain an induced section of L.

Vice versa, if $s \in \Gamma(X, L)$ is nowhere vanishing, then every $y \in \pi^{-1}(x)$ is a multiple of $s(x)$; $y = \lambda s(x)$. Sending y to (x, λ) defines a bijective map $\alpha \colon L \longrightarrow X \times \mathbb{C}$. To see that this map is biholomorphic, we use that every line bundle is locally trivial, and sections are given by holomorphic functions (the details are left to the reader). □

In Definition 4.2 we have defined cocycle data to describe vector bundles. The special case of rank 1 is given by pairs $\left(\{U_i\}_{i\in I}, (\alpha_{ij})_{i,j\in I^2}\right)$, where $\alpha_{ij} \colon U_i \cap U_j \longrightarrow \mathbb{C}^\times$ is a holomorphic function which is nowhere zero.

Definition 4.11. *Let L and L' be line bundles on X, given by the cocycle data $\left(\{U_i\}_{i\in I}, (\alpha_{ij})_{i,j\in I^2}\right)$ and $\left(\{U_i\}_{i\in I}, (\alpha'_{ij})_{i,j\in I^2}\right)$. We define the tensor product $L \otimes L'$ to be the line bundle given by the cocycle datum $\left(\{U_i\}_{i\in I}, (\alpha_{ij}\alpha'_{ij})_{i,j\in I^2}\right)$*

Our current definition of tensor products is depending on the representation of a line bundle through a cocycle datum. Moreover, it doesn't directly expand to higher ranks.

Exercise 4.12. *Verify that the isomorphism class of the tensor product of two line bundles is independent of the chosen cocycle datum. Let $\left(\{U_i\}_{i\in I}, (\alpha_{ij})_{i,j\in I^2}\right)$, $\left(\{U_i\}_{i\in I}, (\alpha'_{ij})_{i,j\in I^2}\right)$ be cocycle data for rank n vector bundles. Explain why in general $\left(\{U_i\}_{i\in I}, (\alpha_{ij}\alpha'_{ij})_{i,j\in I^2}\right)$ won't be a cocycle datum, unless $n = 1$. Can you think of an alternative way to define a tensor product for higher rank vector bundles?*

After having defined sheaves we will find a more general definition of tensor products, which is not depending on any choices, and works for vector bundles of general rank.

Definition 4.13. *For a complex manifold X one denotes by $\mathsf{Pic}(X)$ the set of isomorphism classes of line bundles on X. With respect to the operation given by $\otimes$, $\mathsf{Pic}(X)$ can be viewed as a group. The inverse of a line bundle L is the dual $L^\vee$, and the unit is represented by the trivial line bundle $X \times \mathbb{C}$.*

This group is an important invariant of complex manifolds. The next section is devoted to constructing line bundle L_d on complex projective spaces $\mathbb{P}^n$, such that we get a group homomorphism $\mathbb{Z} \longrightarrow \mathsf{Pic}(X)$, sending d to the isomorphism class of L_d. For $n \geq 1$ we will show that this is really an injective map. Later, after we developed sheaf cohomology, we will be able to prove that this group homomorphism is in fact an isomorphism, and therefore that $\mathsf{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ for $n \geq 1$.

4.3 Line bundles on projective space

For every integer $d \in \mathbb{Z}$ we will define a line bundle L_d on complex projective space $\mathbb{P}^n$, $n \geq 1$. Recall that $\mathbb{P}^n$ can be described as a quotient $(\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^\times$. We will therefore define L_d as a quotient $\left((\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}\right)/\mathbb{C}^\times$.

Definition 4.14. *Consider the topological space $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}$ with the action of $\mathbb{C}^\times$ given by $\lambda \cdot (z^0, \dots, z^n; w) = (\lambda z^0, \dots, \lambda z^n; \lambda^d w)$. We define L_d to be the quotient $(\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^\times$, and $\pi \colon L_d \longrightarrow \mathbb{P}^n$ the continuous map induced by $(z^0, \dots, z^n; w) \mapsto (z^0, \dots, z^n)$.*

Recall that we have standard charts (U_i, ϕ_i) for $\mathbb{P}^n$. We define a homeomorphism

$$\alpha_i \colon \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}$$

by the formula

$$[(z^0,\ldots,z^n;w)] \mapsto \left((z^0:\cdots:z^n),\frac{w}{(z^i)^d}\right).$$

Using the definition of the quotient topology it is easy to see that this is a well-defined continuous map. The inverse map α_i^{-1} sends $\big((z^0:\ldots:z^n),u\big)$ to $[(z^0,\ldots,z^n,(z^i)^d u)]$. As before, the universal property of the quotient topology implies that this map is continuous. This shows that α_i is indeed a homeomorphism.

It remains to check that the transition functions $\alpha_{ij}=\alpha_i\circ\alpha_j\colon (U_i\cap U_j)\times\mathbb{C}\longrightarrow(U_i\cap U_j)\times\mathbb{C}$ are holomorphic, and linear in the second component $\mathbb{C}$. Using the formulae given above, we see that $\alpha_{ij}\left((z^0:\ldots:z^n),u\right)=\left((z^0:\ldots:z^n),\left(\frac{z^j}{z^i}\right)^d u\right)$. Composing with $(\phi_i)^{-1}=\psi_i\colon\mathbb{C}^n\xrightarrow{\simeq}U_i$ from the right and ϕ_i from the left, we obtain the function

$$(w^1,\ldots,w^n;u)\mapsto(w^1,\ldots,w^n;w_j^d u),$$

if $i<j$. This is certainly a holomorphic function, which is linear in the variable u.

We will now prove the following theorem. It's another example of Serre's GAGA principle.

Theorem 4.15. *The vector space of holomorphic sections of L_d is canonically equivalent to $\mathbb{C}[z^0,\ldots,z^n]_d$, that is, the vector space of degree d homogeneous polynomials in $n+1$ variables. In particular, there is no non-zero holomorphic section of L_d for $d<0$.*

We will deduce the theorem from a lemma, which is intuitively clear.

Lemma 4.16. *A holomorphic section $s\colon\mathbb{P}^n\longrightarrow L_d$ corresponds to a holomorphic section*

$$(\mathbb{C}^{n+1}\setminus 0)\xrightarrow{\tilde{s}}\left((\mathbb{C}^{n+1}\setminus 0)\times\mathbb{C}\right),$$

which is $\mathbb{C}^\times$-equivariant.

Proof of Theorem 4.15 using Lemma 4.16. Let's determine the $\mathbb{C}^\times$-equivariant sections of $\left((\mathbb{C}^{n+1}\setminus 0)\times\mathbb{C}\right)$. By definition, $\left((\mathbb{C}^{n+1}\setminus 0)\times\mathbb{C}\right)$ is the trivial line bundle on $(\mathbb{C}^{n+1}\setminus 0)$. Its holomorphic sections are therefore the same as holomorphic functions f on $(\mathbb{C}^{n+1}\setminus 0)$. By Hartog's Extension Theorem 2.14, a holomorphic function on $(\mathbb{C}^{n+1}\setminus 0)$ extends uniquely to a holomorphic function on $\mathbb{C}^{n+1}$, since we assumed $n\geq 1$.

We know that holomorphic functions are analytic, that is, there exists an ϵ-neighbourhood of $0\in\mathbb{C}^{n+1}$, where f can be represented by a converging power series

$$f(z)=\sum_{d_0\ldots,d_n\geq 0} a_{d_0\ldots d_n}(z^0)^{d_0}\cdots(z^n)^{d^n},$$

for $z\in U_\epsilon(0)$. The condition that the section is $\mathbb{C}^\times$-equivariant translates into the property $f(\lambda z)=\lambda^d f(z)$ for $\lambda\in\mathbb{C}^\times$. This implies that the coefficients above vanish for $d_0+\cdots+d_n\neq d$, that is, f is a homogeneous polynomial of degree d. □

It remains to prove the lemma.

Proof of Lemma 4.16. Let s be a holomorphic section of $L_d \xrightarrow{\pi} \mathbb{P}^n$. Using the bundle charts $\alpha_i\colon \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}$ we obtain a holomorphic section $\alpha_i(s)$ of the trivial bundle $U_i \times \mathbb{C}$, corresponding to a holomorphic function f_i. On the overlaps $U_i \cap U_j$ the resulting holomorphic functions are related according to

$$f_i = \left(\frac{z_j}{z_i}\right)^d f_j,$$

or equivalently $(z^i)^d f_i \circ p = (z^j)^d f_j \circ p$ on $p^{-1}(U_i \cap U_j)$, where $p\colon \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$ is the map $(z^0, \ldots, z^n) \mapsto (z^0 : \ldots : z^n)$.

This implies that there exists a well-defined holomorphic function $f\colon \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{C}$, satisfying $f|_{p^{-1}(U_i)} = (z^i)^d f_i \circ p$. Multiplying $(z^0, \ldots, z^n)$ with λ therefore has to change the value of f with λ^d, that is, f defines a $\mathbb{C}^\times$-equivariant section of $\big((\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}\big) \longrightarrow (\mathbb{C}^{n+1} \setminus 0)$.

Vice versa, any $\mathbb{C}^\times$-equivariant section corresponds to a holomorphic function $f\colon \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{C}$, satisfying $f(\lambda z) = \lambda^d f(z)$ for $\lambda \in \mathbb{C}^\times$. We have a function $f_i\colon U_i \longrightarrow \mathbb{C}$, which sends $(z^0 : \cdots : z^n)$ to $\frac{1}{(z^i)^d} f(z^0, \ldots, z^n)$. This is a well-defined continuous function by definition of the quotient topology. Precomposing it with $\phi_i^{-1} = \psi_i$ we obtain $f_i \circ \psi_i(w^1, \ldots, w^n) = \frac{1}{1^d} f(w^1, \ldots, 1, \ldots, w^n)$, which is by definition a holomorphic function. For the intersection $U_i \cap U_j$ of two charts we have the relation $f_i = \left(\frac{z_j}{z_i}\right)^d f_j$, and hence gives rise to a section of $L_d \longrightarrow \mathbb{P}^n$. $\square$

Corollary 4.17. *For $d \neq d'$ the line bundles L_d and $L_{d'}$ on $\mathbb{P}^n$ are not equivalent.*

Proof. For $d \geq 0$ we have determined the vector space of global sections $\Gamma(\mathbb{P}^n, L_d) \cong \mathbb{C}[z^0, \ldots, z^n]_d$ in Theorem 4.15. For $0 \leq d < d'$ we have a non-canonical embedding of vector spaces $\mathbb{C}[z^0, \ldots, z^n]_d \hookrightarrow \mathbb{C}[z^0, \ldots, z^n]_{d'}$, which sends a degree d homogeneous polynomial G to $(z^0)^{d'-d} G$. This map is injective, but cannot be surjective, since the degree d' homogeneous polynomial $(z^1)^{d'}$ doesn't lie in its image. This implies that for non-negative numbers $d \neq d'$ the vector space of global sections are of different dimensions, therefore $L_d \ncong L_{d'}$.

We have seen that L_d is given by the cocycle $(\alpha_{ij}) = \left(\frac{z^j}{z^i}\right)^d$. This implies that L_{-d} is equivalent to the dual of L_d (see Definition 4.5). This implies that $L_d \ncong L_{d'}$ for negative integers $d \neq d'$.

For $d < 0 < d'$ we know that $L_d \cong L'_d$ is impossible, because L_d does not have any global sections. $\square$

After having developed the appropriate tools we will show that every line bundle on $\mathbb{P}^n$ is equivalent to one of the line bundles L_d. For $\mathbb{P}^1$ even more is true.

Theorem 4.18 (Birkhoff, Grothendieck)**.** *Every rank m vector bundle E on $\mathbb{P}^1$ can be uniquely written as a direct sum of line bundle $L_{d_1} \oplus \cdots \oplus L_{d_m}$.*

Birkhoff didn't formulate his theorem in terms of line bundles on projective space. He proved a factorisation theorem for matrices in Laurent polynomials, known as Birkhoff factorisation. Grothendieck recognised the geometric content of Birkhoff's result and reproved the result (in higher generality) using sheaf cohomology.

4.4 Line bundles and maps to projective space

Now that we have studied line bundles on complex projective space $\mathbb{P}^n$, we will try a different approach. Starting with a line bundle L on a complex manifold X, and sections $s_0, \ldots, s_n$ of L, we will use this datum to construct a holomorphic map $X \longrightarrow \mathbb{P}^n$.

Definition 4.19. *For a line bundle L on a complex manifold X we say that a tuple of sections $(s_0, \ldots, s_n) \in \Gamma(X, L)$ generates L, if for every $x \in X$ there exists at least one $i \in \{0, \ldots, n\}$, such that $s_i(x) \neq 0$. In this case, we define a map $f\colon X \longrightarrow \mathbb{P}^n$ by stipulating $x \mapsto (s_0(x) : \cdots : s_n(x))$.*

This construction is as simple, as it is brilliant. It doesn't make any sense to think of $s_i(x)$ as an element of $\mathbb{C}$, but rather $s_i(x)$ belongs to the 1-dimensional complex vector space $\pi^{-1}(x)$. But all 1-dimensional complex vector spaces are isomorphic to $\mathbb{C}$, and the set of automorphisms of $\mathbb{C}$ is $\mathbb{C}^\times$. In order to define $(s_0(x) : \cdots : s_n(x))$ we are formally forced to choose an arbitrary identification of $\pi^{-1}(x)$ with $\mathbb{C}$. However, as we are only interested in $(s_0(x), \ldots, s_n(x))$ up to rescaling, we obtain a well-defined element of $\mathbb{P}^n$. The assumption that the tuple $(s_0, \ldots, s_n)$ generates L was needed, because $(0 : \cdots : 0)$ does not define a point in $\mathbb{P}^n$.

Lemma 4.20. *The map f defined above in 4.19 is holomorphic.*

Proof. Let (V, α) be a bundle chart of L. That is, $\alpha\colon \pi^{-1}(V) \xrightarrow{\simeq} V \times \mathbb{C}$ as a line bundle bundle. The sections $(\alpha(s_j))_{j=0,\ldots,n}$ can then be understood to be holomorphic functions $g_0, \cdots, g_n$, that is, in particular we have a holomorphic map $g = (g_0, \ldots, g_n)\colon V \longrightarrow \mathbb{C}^{n+1}$. By assumption, the image of this map is contained in $\mathbb{C}^{n+1} \setminus 0$, and it is therefore meaningful to form the composition $p \circ g$, where $p\colon\ \mathbb{C}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$ is the canonical projection. Since the composite of two holomorphic maps is again holomorphic, we see that $p \circ g$ is holomorphic. However, by definition, $p \circ g = f|_V$. The complex manifold X can be covered by bundles charts (V, α), and this implies that $f\colon X \longrightarrow \mathbb{P}^n$ is holomorphic everywhere. □

Lemma 4.21. *Consider the line bundle L_1 on $\mathbb{P}^n$ and recall that $\Gamma(\mathbb{P}^n, L_1) \cong \mathbb{C}[z^0, \ldots, z^n]_1$ is equivalent to the space of homogeneous polynomials in $n+1$ variables. The tuple of sections $(z^0, \ldots, z^n)$ is generating and the resulting holomorphic map $f\colon\ \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is the identity map $\mathrm{id}_{\mathbb{P}^n}$.*

Proof. We can use the local computations above to identify the map $f \circ \psi_0$, where $\psi_0 = \phi_0^{-1}\colon\ \mathbb{C}^n \xrightarrow{\simeq} U_0 \subset \mathbb{P}^n$. The sections $(z^0, \ldots, z^n)$ correspond to the holomorphic functions $(w^1, \ldots, w^n) \mapsto (1, w^1, \ldots, w^n)$. And since the constant function 1 is never 0, we see that at least for every point $x \in U_0$ there exists one section which is never 0. Permuting coordinates we see that this is true for any chart (U_i, ϕ_i) of $\mathbb{P}^n$.

The holomorphic map $f \circ \psi_0$ is given by $(w^1, \ldots, w^n) \mapsto (1 : w^1 : \cdots : w^n)$, which is equal to ψ_0. This implies that $f|U_0$ is equal to the identity map. Since U_0 is dense in $\mathbb{P}^n$ (or by permuting coordinates), we obtain that $f = \mathrm{id}_{\mathbb{P}^n}$. □

Vice versa, any holomorphic map $X \xrightarrow{f} \mathbb{P}^n$ induces a line bundle $L = f^*L_1$, generated by $n+1$ sections $s_i = f^*x_i$. And of course, f is precisely the holomorphic map corresponding to this datum. The construction fits into a more general context of pulling back vector bundles and sections. We begin with the general notion of fibre products.

Definition 4.22. *Let $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} Y$ be continuous maps of topological spaces. The fibre product $X \times_Y Z$ is defined to be the following subset of the product*

$$\{(x, y) \in X \times Y | f(x) = g(z)\}$$

with the subspace topology.

The projections $p_1 \colon X \times Z \longrightarrow X$ and $p_2 \colon X \times Z \longrightarrow Z$ restrict to maps from $X \times_Y Z$ to X respectively Y. By construction of the fibre product they belong to a commutative square

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\
X \arrow[r] & Y.
\end{tikzcd}$$

In fact, this commutative square is the universal such diagram. That is, if you have a space W and maps $W \xrightarrow{h} X$, $W \xrightarrow{k} Y$, such that $f \circ h = g \circ k$, then there exists a unique map $W \longrightarrow X \times_Z Y$, $w \mapsto (h(w), k(w))$, fitting into a commutative diagram

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y.
\end{tikzcd}$$

The intuition behind fibre products or pullbacks is as follows: $X \times_Y Z \longrightarrow X$ is a space with "the same fibres" as $Z \longrightarrow Y$. That is, for $x \in X$, the fibre of $p_1^{-1}(x)$ is the same as $g^{-1}(f(x))$. This follows directly from the set-theoretic definition of the fibre product (but also from the universal property). Since fibre products leave the fibres of a map un-affected, they are the perfect device to pullback vector bundles! If the fibres of $Z \longrightarrow Y$ have a vector space structure, the same is true for the fibres of $X \times_Y Z \longrightarrow X$, because every fibre of the second map is a fibre of the first one.

Lemma 4.23. *For $\pi \colon E \longrightarrow Y$ a vector bundle over a complex manifold Y, and $f \colon X \longrightarrow Y$ a holomorphic map, we define the pullback f^*E to be the fibre product $X \times_Y E$ together with the continuous map $\pi_{f^*E} \colon X \times_Y E \longrightarrow X$. This construction can be refined to define a holomorphic vector bundle on X.*

Proof. The proof can be divided into several steps, using only the formal definition of fibre products.

(1) The pullback of the trivial vector bundle: $X \times_Y (Y \times \mathbb{C}^n) \cong X \times \mathbb{C}^n$. This follows directly from the definition of the fibre product: $X \times_Y (Y \times \mathbb{C}^n) = \{(x, y, z) \in X \times Y \times \mathbb{C}^n \,| f(x) = y\} \cong X \times Y$, because $y = f(x)$ means that y is redundant.

(2) $X \times_Y (?)$ defines a functor from topological spaces with a map to Y, $Z \longrightarrow Y$. That is, in particular if $Z \xrightarrow{\simeq} Z'$ is a homeomorphism fitting into a commutative diagram

$$\begin{tikzcd}
Z \arrow[rr, "\simeq"] \arrow[dr] & & Z' \arrow[dl] \\
& Y, &
\end{tikzcd}$$

then we get an induced homeomorphism $X \times_Y Z \xrightarrow{\simeq} X \times_Y Z'$.

(3) If $(\{U_i\}_{i\in I}, (\alpha_i)_{i\in I})$ is a cocycle datum for $E \xrightarrow{\pi} Y$, that is, we have that $\alpha_{ij} = \alpha_i \circ \alpha_j$ is a holomorphic function $U_i \cap U_j \longrightarrow \mathrm{GL}_n(\mathbb{C})$, then we want to obtain from this a cocycle datum for f^*E. Since $\bigcup_{i\in I} U_i = Y$ we see that $\bigcup_{i\in I} f^{-1}(U_i) = X$. The definition of fibre products readily implies that $\pi^{-1}_{f^*E}(f^{-1}(U_i)) = f^{-1}(U_i) \times_{U_i} \pi^{-1}(E)$. By (2), the homeomorphism $\alpha_i \colon f^{-1}(U_i) \xrightarrow{\simeq} U_i \times \mathbb{C}^n$ induces therefore a homeomorphism $f^*\alpha_i \colon f^{-1}(U_i) \xrightarrow{\simeq} f^{-1}(U_i) \times \mathbb{C}^n$.

It remains to check that the maps $f^*\alpha_{ij} = f^*\alpha_i \circ f^*\alpha_j$ are holomorphic maps $f^{-1}(U_i) \cap f^{-1}(U_i) = f^{-1}(U_i \cap U_j) \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Since $f^*\alpha_{ij} = \alpha_{ij} \circ f$, and α_{ij} and f are supposed to be holomorphic, we are done. $\square$

We know that holomorphic sections $s \in \Gamma(Y, E)$ correspond to families of holomorphic functions $h_i \colon U_i \longrightarrow \mathbb{C}^n$, such that $\alpha_{ij} \cdot (h_j|_{U_i \cap U_j}) = h_i|_{U_i \cap U_j}$. The compositions $h_i \circ f \colon f^{-1}(U_i) \longrightarrow \mathbb{C}^n$ define holomorphic functions, satisfying the relations $(\alpha_{ij} \circ f) \cdot (h_j \circ f|_{f^{-1}(U_i \cap U_j)}) = h_i \circ f|_{f^{-1}(U_i \cap U_j)}$. Hence we obtain a holomorphic section $f^*s \in \Gamma(X, f^*E)$.

Using the same arguments as in the proof above, one sees that the continuous map $X \xrightarrow{f^*s} f^*E = X \times_Y E$ is given by $x \mapsto (f(x), s(f(x)))$. In particular, the section f^*s does not depend on the choice of a cocycle datum for E.

Definition 4.24. *Let L_1, L_2 be line bundles on a complex manifold X, and $(s^i_0, \dots, s^i_n) \in \Gamma(X, L_i)$ be a generating set of sections. We say that the tuples $(L_1, s^1_0, \dots, s^1_n)$ and $(L_2, s^2_0, \dots, s^2_n)$ are equivalent if there exists an isomorphism of line bundles $L_1 \cong L_2$, such that the induced isomorphism $\Gamma(X, L_1) \cong \Gamma(X, L_2)$ maps $(s^1_0, \dots, s^1_n)$ to $(s^2_0, \dots, s^2_n)$.*

Theorem 4.25. *Let X be a complex manifolds. The set of holomorphic maps $X \xrightarrow{f} \mathbb{P}^n$ is in bijection with the set of equivalence classes of tuples $(L, s_0, \dots, s_n)$, where L is a line bundle on X, and $(s_0, \dots, s_n)$ is a generating tuple of sections of L.*

Proof. Let's denote the set of holomorphic maps $X \longrightarrow \mathbb{P}^n$ by $\mathsf{Hom}(X, \mathbb{P}^n)$, and the set of isomorphism classes of tuples $(L, s_0, \dots, s_n)$ by $\mathcal{L}_{n+1}(X)$.

We have already constructed a map $A \colon \mathcal{L}_{n+1}(X) \longrightarrow \mathsf{Hom}(X, \mathbb{P}^n)$, by sending $(L, s_0, \dots, s_n)$ to the map $f \colon x \mapsto (s_0(x) : \dots : s_n(x))$.

There's also a map $B\colon \mathsf{Hom}(X, \mathbb{P}^n) \longrightarrow \mathcal{L}_{n+1}(X)$, which sends $X \xrightarrow{f} \mathbb{P}^n$ to the tuple $(f^*L_1, f^*(z^0), \dots, f^*(z^n))$. To decipher this notation, recall that $\Gamma(X, L_1)$ is equivalent to the vector space of degree 1 homogeneous polynomials. That is, every $(z^i) \in \mathbb{C}[z^0, \dots, z^n]_1$ gives rise to a global section of L_1.

It remains to verify that A and B are mutually inverse maps. We'll first take a look at $A \circ B = \mathrm{id}_{\mathsf{Hom}(X,\mathbb{P}^n)}$. Let $f \colon X \longrightarrow \mathbb{P}^n$ be a holomorphic map. We need to check that f is equal to the holomorphic map g assigned to the tuple $(f^*L_1, s_0, \dots, s_n)$. By definition, $g \colon x \mapsto (f^*(z^0)(x) : \cdots : f^*(z^n)(x))$.

We have seen above that $f^*(z^i) \colon X \longrightarrow f^*L_1$ is equal to the map $x \mapsto (f(x), (z^i \circ f))$. In Lemma 4.21 we showed that the holomorphic map $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ associated to the tuple $(L_1, z^0, \dots, z^n)$ is the identity map. This implies that this map is equal to f.

It remains to show that $B \circ A = \mathrm{id}_{\mathcal{L}_{n+1}(X)}$. Let $(L, s_0, \dots, s_n) \in \mathcal{L}_{n+1}(X)$, and $X \xrightarrow{f} \mathbb{P}^n$ the associated holomorphic map. We have to prove that $f^*L_1 \cong L$, and with respect to this identification we have $s_i = f^*(z^i)$.

Consider the standard open covering $\bigcup_{i=0}^{n} U_i = \mathbb{P}^n$. We can define U_i as the locus where the section $z^i \neq 0$ of L_1. Therefore, the preimage $f^{-1}(U_i)$ is equal to the open subset where $s_i \neq 0$, respectively $f^*(z^i) \neq 0$. Using Lemma 4.10, we see that we have trivialisations $L|_{f^{-1}(U_i)} \xrightarrow{\simeq} f^{-1}(U_i) \times \mathbb{C} \times \mathbb{C}$ and $f^*L_1|_{f^{-1}(U_i)} \xrightarrow{\simeq} f^{-1}(U_i) \times \mathbb{C}$, induced by the nowhere vanishing sections s_i respectively $f^*(z^i)$. By composition we obtain an isomorphism $\beta_i \colon L|_{f^{-1}(U_i)} \cong f^*L_1|_{f^{-1}(U_i)}$.

It suffices to check that on $U_i \cap U_j$ the two isomorphisms β_i and β_j agree. Replacing the section s_i by the section s_j introduces the factor $\frac{s_j}{s_i}$. Similarly, changing $f^*(z^i)$ by $f^*(z^j)$ the factor $\frac{z^i \circ f}{z^j \circ f} = \frac{s_i}{s_j}$. And we see that both changes cancel each other out.

□

Definition 4.26. *Let X be a complex manifold and L a line bundle on X. We say that L is* very ample*, if there exist sections $s_0, \dots, s_n \in \Gamma(X, L)$ that generate L and yield an* embedding *into projective space $X \hookrightarrow \mathbb{P}^n$ as a submanifold.*

5 Sheaves

Few, light, and worthless,-yet their trifling weight
Through all my frame a weary aching leaves;
For long I struggled with my hapless fate,
And staid and toiled till it was dark and late,
Yet these are all my sheaves.

Elizabeth Akers Allen

This section is devoted to determining the common denominator of the following constructions we have encountered in this class: holomorphic functions, complex vector fields, complex m-forms, and sections of vector bundles. Locally, all these objects can be described in terms of (a tuple of) holomorphic functions on an open subset of a complex vector space. However, globally, they exhibit novel features, and for example the section of a line bundle should not be confused with a holomorphic function. We have seen in Theorem 4.15 that for $d \geq 0$, the vector space of global sections of L_d on $\mathbb{P}^1$ is at least 3-dimensional. That's quite different from the 1-dimensional vector space of holomorphic functions on $\mathbb{P}^1$ (all of which are constant).

For a complex manifold X, equipped with a line bundle L, we temporarily denote for an open subset $U \subset X$ the vector space of holomorphic functions, sections of line bundles, complex m-forms, or complex vector fields on U by $\mathcal{F}(U)$. For $U \subset V$ we have a linear restriction map $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$, which takes a element $f \in \mathcal{F}(V)$ (think of f as a holomorphic function, section of a line bundle, complex vector field, or complex m-form on V), and sends it to the restriction $f|_U$.

Restriction of functions, etc., satisfies a few unsurprising identities. For example, considering the tautological inclusion $U \subset U$, the map $f \mapsto f|_U$ is the identity map. Similarly, for a triple of inclusion $U \subset V \subset W$, we have $(f|_V)|_U = f|_U$.

We can summarise our discussion so far as observing that functions, sections of line bundles, etc. have in common that they can be restricted to subsets of their domain of definitions. Using technical jargon, we have just seen our first examples of *presheaves*.

However, there are a lot of examples of presheaves, which don't possess the same formal properties as holomorphic functions and sections of line bundles. To sift chaff from the wheat[1], we

[1] Agricultural terminology is very popular in this area.

stipulate one further condition in order to define sheaves. It's known as the *sheaf condition*, but might as well be called the *jigsaw puzzle principle*: matching local fragments can be assembled into a global picture.

Given an open covering $\{U_i\}_{i\in I}$ of X, and elements $f_i \in \mathcal{F}(U_i)$ for every $i \in I$ (that is, holomorphic functions, sections of line bundles, etc.), such that on the overlaps $U_{ij} = U_i \cap U_j$ we have $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, there exists a *unique* $f \in \mathcal{F}(X)$, such that $f|_{U_i} = f_i$.

5.1 Presheaves

The notion of sheaves intends to provide an abstract generalisation of functions on a topological space. As a predecessor one has to study presheaves, which aim to extract the most basic property of functions.

Definition 5.1. *Let X be a topological space, a* presheaf *$\mathcal{F}$ on X consists of the following data:*

- *an abelian group $\mathcal{F}(U)$ for every open subset $U \subset X$,*
- *a homomorphism $r_U^V\colon \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ for every inclusion of open subsets $U \subset V \subset X$,*
- *such that for every triple of inclusions of open subsets $U \subset V \subset W \subset X$ we have a commutative diagram*

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "r_V^W"] \arrow[dr, "r_U^W"'] & \mathcal{F}(V) \arrow[d, "r_U^V"] \\ & \mathcal{F}(U), \end{tikzcd}$$

that is, we have $r_U^W = r_V^W \circ r_U^V$.

We will often denote the effect of r_U^V by using classical notation for restriction of functions. E.g., for $s \in \mathcal{F}(V)$ we write $r_U^V(s) = s|_U$.

In order to put this definition into context, let us observe that functions on a topological space X form a presheaf. If $V \subset X$ is an open subset, and $f\colon V \longrightarrow \mathbb{C}$ a continuous function, then for every open subset $U \subset X$ we obtain a continuous function $f|_U$.

If $f\colon W \longrightarrow \mathbb{C}$ is continuous, and we have $U \subset V \subset W$, then by definition

$$f|_U = (f|_V)|_U.$$

There are a lot more examples, using complex-valued functions satisfying stricter conditions than just continuity.

Example 5.2. *We fix a topological space X. In some of the examples, e.g. (a), (d), (e), (f), and (g) we will assume that X is endowed with the structure of a complex manifold.*

(a) (The presheaf of holomorphic functions on $\mathbb{C}$.) For $U \subset X$ define $\mathcal{O}_X(U)$ to be the abelian group of holomorphic functions $f\colon U \longrightarrow \mathbb{C}$. For $U \subset V \subset \mathbb{C}$ define $r_{V,U}\colon \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$ to be the map sending $f\colon V \longrightarrow \mathbb{C}$ to $f|_U$.

(b) We denote by $\underline{\mathbb{Z}}^{pre}$ the presheaf of constant, integer-valued functions. That is, to an open subset $U \subset X$ we assign the abelian group $\mathbb{Z}$, whose elements we interpret as constant functions on U.

(c) *We denote by* $\underline{\mathbb{Z}}$ *the presheaf of* locally constant *integer-valued functions. To an open subset* $U \subset X$ *we assign the abelian group of continuous functions* $U \xrightarrow{f} \mathbb{Z}$*. Since* $\mathbb{Z}$ *is a discrete topological space, we see that every* $x \in X$ *has an open neighbourhood* $U = f^{-1}(f(x))$*, such that* $f|_U$ *is constant. However,* f *will not be globally constant. For example, if* $X = [0,1] \cup [2,3]$*, we could consider the function, which is equal to* 1 *on* $[0,1]$*, and equal to* 2 *on* $[2,3]$*.*

(d) *We also have the presheaf* $\mathcal{O}_X^{\times}$*, which assigns to* $U \subset X$ *the abelian group of holomorphic functions* $f \colon U \longrightarrow \mathbb{C}$*, which are nowhere zero.*

(e) *We denote by* Ω_X^m *the presheaf of complex* m*-forms, that is, to an open subset* $U \subset X$ *we assign the vector space of complex* m*-forms.*

(f) *The presheaf of complex vector fields on* X *is denote by* $\mathcal{V}_X$*.*

(g) *Let* E *be a vector bundle on* X*. We have a presheaf of sections* $\underline{E}$*, which maps an open subset* $U \subset X$ *to the vector space of sections* s *defined over* U*:*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, hook] & X. \end{tikzcd}$$

Presheaves on a fixed topological space X form a category. In order to see this we have to define first what we mean by a morphism of presheaves.

Definition 5.3. *Let* X *be a topological space, and* $\mathcal{F}$ *and* $\mathcal{G}$ *be presheaves on* X*. A* morphism *of presheaves* $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *is given by a homomorphism of abelian groups* $\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U)$ *for every open subset* $U \subset X$*, such that for every inclusion* $U \subset V \subset X$ *of open subsets, we have a commutative diagram*

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "f_V"] \arrow[d, "r_U^V"'] & \mathcal{G}(V) \arrow[d, "r_U^V"] \\ \mathcal{F}(U) \arrow[r, "f_U"] & \mathcal{G}(U). \end{tikzcd}$$

For instance we have a morphism $\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$, which takes a locally-constant $\mathbb{Z}$-valued function, and views it as a holomorphic function on X. Or we could take $f \in \mathcal{O}_X(U)$ and send it to the nowhere vanishing holomorphic function $\exp(f) \in \mathcal{O}_X^{\times}(U)$. This defines a morphism $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^{\times}$.

Definition 5.4. *A morphism* $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *of presheaves on a topological space* X *is called an* isomorphism*, if there exists a morphism* $g \colon \mathcal{G} \longrightarrow \mathcal{F}$*, such that* $f \circ g = \mathrm{id}_{\mathcal{G}}$*, and* $g \circ f = \mathrm{id}_{\mathcal{F}}$*.*

As we are already used to from the world of sets, groups, and vector spaces, we can describe isomorphisms also as *bijective* morphisms of presheaves.

Definition 5.5. *We denote by* $\mathcal{F} \xrightarrow{f} \mathcal{G}$ *a morphism of presheaves on* X*.*

(a) *We say that* f *is* injective*, if for every open* $U \subset X$*, the map* $f_U \colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ *is injective.*

(b) ... f *is* surjective, *if ... the map* f_U *is surjective.*

(c) *We say that* f *is* bijective *if it is injective and surjective.*

It is clear that if f is bijective, then for every open $U \subset X$ the map $\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U)$ is bijective. In particular, there exists an inverse map f_U^{-1}, and it remains to see that the collection of such maps forms a morphism of presheaves. For every inclusion of open subsets $U \subset V \subset X$, we have that $r_U^V \circ f_V = f_U \circ r_U^V$. Applying f_V^{-1} from the right, and f_U^{-1} from the left, we obtain the relation $f_U^{-1} \circ r_U^V = r_U^V \circ f_V^{-1}$. That is, f^{-1} is indeed a morphism of presheaves.

5.2 Sheaves

A presheaf is called a sheaf, it a compatible collection of local sections can be assembled into a global section.

Definition 5.6. *Let* F *be a presheaf on a topological space* X*, such that the following condition is satisfied: for every open covering* $U = \bigcup_{i \in I} U_i$*, and a collection of sections* $s_i \in \mathcal{F}(U_i)$*, such that* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ *for every pair of indices* (i, j)*, there exists a* unique *section* $s \in \mathcal{F}(U)$*, such that* $s|_{U_i} = s_i$*. We then say that* $\mathcal{F}$ *is a* sheaf *on* X*.*

The definition of a sheaf is also often stated by imposing the following two conditions. As before $U = \bigcup_{i \in I} U_i$ is an open covering.

(S1) If $s, t \in \mathcal{F}(U)$ are two sections, such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(S2) If $s_i \in \mathcal{F}(U_i)$ is a collection of sections, such that $s_i|_{U_i \cap U_i} = s_j|_{U_i \cap U_j}$ for all pairs of indices, then there exists a sections $s \in \mathcal{F}(U)$, such that $s|_{U_i} = s_i$ for all $i \in I$.

The condition (S2) corresponds to $\exists$ in our definition of a sheaf, while (S1) is equivalent to unicity in the definition above.

Most of the examples of presheaves we've encountered so far, are actually sheaves.

Example 5.7. We fix a topological space X. In some of the examples, e.g. (a), (d), (e), (f), and (g) we will assume that X is endowed with the structure of a complex manifold.

(a) The presheaf $\mathcal{O}_X$ of holomorphic functions on X is a sheaf. Indeed, if $U \subset X$ is an open subset, and $U = \bigcup_{i \in I} U_i$ an open covering of U, and $f_i : U_i \longrightarrow \mathbb{C}$ are holomorphic functions, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there exists a unique holomorphic function $U \xrightarrow{f} \mathbb{C}$, such that $f|_{U_i} = f_i$. Indeed, we can define f by the following rule: $f(x) = f_i(x)$, if $x \in U_i$. It only remains to check that this expression does not depend on the choice of an U_i containing x. If $x \in U_i$ and $x \in U_j$, then $x \in U_{ij} = U_i \cap U_j$. That is, we have $f_i(x) = f_j(x)$. This yields a well-defined map of sets $f : U \longrightarrow \mathbb{C}$. Since holomorphicity is a local property, and $f|_{U_i}$ is holomorphic for every $i \in I$, we see that f is locally holomorphic, hence holomorphic.

(b) The presheaf of constant integer-valued functions $\underline{\mathbb{Z}}^{pre}$ is *not* a sheaf in general. To see this for a generic topological space, we choose an open subset $U \subset X$, such that $U = U_0 \cup U_1$, with $U_0 \cap U_1 = \emptyset$. In other words, U has (at least) two connected components. We define $f_0 : U_0 \longrightarrow \mathbb{Z}$ to be the constant function with value 0, and $f_1 : U_1 \longrightarrow \mathbb{Z}$ to be the constant function with value 1. If $\underline{\mathbb{Z}}^{pre}$ was a sheaf, there would be a *constant* function $f : U \longrightarrow \mathbb{Z}$, such that $f|_{U_0} = f_0$ and $f|_{U_1} = f_1$. Since f has to be constant, this would imply $0 = 1$.

(c) In (b) we've seen a first example of a presheaf which is not a sheaf. However, it is clear that this can be remedied by considering the presheaf of *locally constant* integer-valued functions $\underline{\mathbb{Z}}$, which is always a sheaf. Indeed, a function $U \longrightarrow \mathbb{Z}$ is locally constant, if and only if it is continuous, where we endow $\mathbb{Z}$ with the discrete topology. One can then argue similarly to (a) to verify the sheaf condition.

(d) The presheaf of nowhere zero holomorphic functions $\mathcal{O}_x^\times$ is a sheaf. If $f_i : U_i \longrightarrow \mathbb{C}$ are nowhere zero holomorphic functions, such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, then we've seen in (a) that there exists a unique holomorphic function $f : U \longrightarrow \mathbb{C}$, agreeing with the locally defined functions f_i on each U_i. Since the functions f_i is nowhere zero, we obtain $f(x) \neq 0$ for every x, which belongs to a U_i. But $U = \bigcup_{i \in I} U_i$, hence f is also nowhere 0.

(e) The presheaf Ω_X^m of complex m-forms is a sheaf. We have verified the sheaf condition in Lemma 3.9.

(f) The presheaf of complex vector fields $\mathcal{V}_X$ on X is a sheaf. This has been shown in Lemma 3.9.

(g) Let E be a vector bundle on X. The presheaf of sections $\underline{E}$, which maps an open subset $U \subset X$ to the vector space of sections s defined over U, is a sheaf. To see this, we recall that a section is given by a holomorphic map $U \longrightarrow E$, fitting into a commutative diagram.

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, hook] & X. \end{tikzcd}$$

If $s_i : U_i \longrightarrow E$ is a collection of sections, such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, then we can define $s : U \longrightarrow E$ by $s(x) = s_i(x)$, if $x \in U_i$. As in (a) we see that s is a well-defined holomorphic function $U \xrightarrow{s} E$. We have $\pi(s(x)) = \pi(s_i(x)) = x$, hence the diagram above commutes, and s is indeed a section.

5.3 Locally free sheaves and vector bundles

In this paragraph we'll see that vector bundles on complex manifolds can be described as so-called *locally free* sheaves (of finite rank). In order to justify the nomenclature we recall the notion of an R-module, and what it means for an R-module M to be free, and finitely generated.

Definition 5.8. *Let R be a ring (with a unit $1 \in R$). An R-module M consists of an abelian group $(M, +)$, together with a scalar multiplication $\cdot : R \times M \longrightarrow M$, such that for every $\lambda \in R$, the map $m \mapsto \lambda \cdot m$ defines a homomorphism of abelian groups, and the resulting map $R \longrightarrow \mathsf{End}(M)$ is a ring homomorphism. That is, we have $\lambda \cdot (m + n) = \lambda \cdot m + \lambda \cdot n$, $1 \cdot m = m$ for every $m \in M$, and $(\lambda + \mu) \cdot m = \lambda m + \mu \cdot m$, as well as $(\lambda\mu) \cdot m = \lambda \cdot (\mu \cdot m)$.*

If k is a field, then a k-module is simply a k-vector space.

For every non-negative integer $n \geq 0$, we have the R-module R^n. We say that an R-module M is free of rank n, if it is isomorphic as an R-module to R^n. We say that an R-module M is finitely generated, if there exists a surjection of R-modules $R^m \twoheadrightarrow M$.

Every free R-module of rank n is finitely generated, but the converse is not true in general (unless R is a field). Let R be a commutative ring with a unit, which is not a field. Then there exists a non-zero ideal $I \subset R$. The quotient R/I is a finitely generated R-module, since we have a surjection $R \twoheadrightarrow R/I$. We leave it to the reader to conclude the proof that R/I cannot be free.

Now we are ready to define sheaf of rings, and sheaves of modules. In order to give compact definitions (in a non-topological sense), we use that there is a natural notion of products of two presheaves $\mathcal{F} \times \mathcal{G}$ on X, which respects the sheaf condition.

Definition 5.9. *Let X be a topological space, and $\mathcal{F}, \mathcal{G}$ two presheaves on X. We define $\mathcal{F} \times \mathcal{G}$ to be the presheaf which assigns to an open subset $U \subset X$ the abelian group $\mathcal{F}(U) \times \mathcal{G}(U)$.*

The product of two sheaves is again a sheaf. This follows from the definition, since the sheaf condition holds componentwise. And sections of $(\mathcal{F} \times \mathcal{G})(U)$ correspond to an ordered tuple (s, t), where $s \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$.

Definition 5.10. *Let X be a topological space.*

(a) A sheaf of rings $\mathcal{R}$ on X is given by a sheaf $\mathcal{R}$, together with a multiplication map $\cdot \, \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ of sheaves, such that for every open subset $U \subset X$ the abelian group $\mathcal{R}(U)$ is a ring, with respect to the multiplication $\cdot\colon \mathcal{R}(U) \times \mathcal{R}(U) \longrightarrow \mathcal{R}(U)$.

(b) Let $\mathcal{R}$ be a sheaf of rings on X. A sheaf of $\mathcal{R}$-modules $\mathcal{M}$ is a sheaf $\mathcal{M}$ on X, together with a multiplication map $\cdot\colon \mathcal{R} \times \mathcal{M} \longrightarrow \mathcal{M}$ of sheaves, such that for every $U \subset X$, the map $\mathcal{R}(U) \times \mathcal{M}(U) \longrightarrow \mathcal{M}(U)$ endows $\mathcal{M}(U)$ with the structure of an $\mathcal{R}(U)$-module.

It's illustrative to unpack the definition of sheaves of rings, and sheaves of modules. A sheaf of rings $\mathcal{R}$ is given by a sheaf, such that for every open $U \subset X$, the abelian group $\mathcal{R}(U)$ is additionally endowed with the structure of a ring, and all the restriction maps $r_U^V\colon \mathcal{R}(V) \longrightarrow \mathcal{R}(U)$, for every inclusion of open subsets $U \subset V$, are ring homomorphisms.

If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules, then for every open $U \subset X$ we have the structure of an $\mathcal{R}(U)$-module on $\mathcal{M}(U)$. Moreover, if $U \subset V$ is an inclusion of open subsets, we have for $f \in \mathcal{R}(V)$, and $m \in \mathcal{M}(V)$ the compatibility condition

$$(f \cdot m)|_U = (f|_U) \cdot (m|_U).$$

We are now almost ready to define locally free sheaves of modules. The definition below requires us to restrict sheaves on X to open subsets $U \subset X$. This makes sense, because an open subset $U' \subset U$ of an open subset $U \subset X$, also defines an open subset $U' \subset X$ of X. A sheaf $\mathcal{F}$ on X gives therefore rise to a sheaf $\mathcal{F}|_U$ on every open subset $U \subset X$.

Definition 5.11. *Let X be a topological space, and $\mathcal{R}$ a sheaf of rings on X. A sheaf of $\mathcal{R}$-modules $\mathcal{M}$ is said to be* locally free *of rank n, if there exists an open covering $X = \bigcup_{i \in I} U_i$, such that $\mathcal{M}|_{U_i} \cong \mathcal{R}^n|_{U_i}$ as a sheaf of $\mathcal{R}$-modules.*

The goal of this paragraph is to prove the following theorem.

Theorem 5.12. *Suppose that X is a complex manifold. There is a one-to-one correspondence between* complex vector bundles *E of rank n on X, and* locally free sheaves *of $\mathcal{O}_X$-modules $\mathcal{M}$ of rank n.*

The proof is broken down into several steps. At first we have to explain how one can pass from a complex vector bundle E to a locally free sheaf of $\mathcal{O}_X$-modules.

Lemma 5.13. *The sheaf of sections $\underline{E}$ of a rank n complex vector bundle is a locally free $\mathcal{O}_X$-module of rank n.*

Proof. If $U \subset X$ is an open subset, and $s\colon U \longrightarrow E$ a section of E over U, then for every holomorphic function $f \in \mathcal{O}_X(U)$ we have a section $f \cdot s\colon U \longrightarrow E$. For every $x \in U$, $(f \cdot s)(x) = f(x)s(x)$, and it's not difficult to check that this defines a holomorphic section (see for instance the proof of Lemma 4.8 for similar arguments). Hence $\underline{E}$ is naturally endowed with the structure of an $\mathcal{O}_X$-module.

By definition, there exists an open covering $X = \bigcup_{i \in I} U_i$, such that $E|_{U_i} \cong U_i \times \mathbb{C}^n$ as a vector bundle. This implies that $\underline{E}|_{U_i}$ is equivalent to $\mathcal{O}_X^n$, because a section of a trivial vector bundle $U \times \mathbb{C}^n \longrightarrow U$, corresponds to an n-tuple of holomorphic functions over U. $\square$

Lemma 5.14. *The set of isomorphisms of sheaves of $\mathcal{O}_X$-modules, $\mathcal{O}_X^n \longrightarrow \mathcal{O}_X^n$ is equivalent to the set of holomorphic maps $X \longrightarrow \mathrm{GL}_n(\mathbb{C})$.*

Proof. We begin by considering the abstract situation of automorphisms of free R-modules, where R is a ring with unit. Every R-linear map $R^n \xrightarrow{f} R^n$ is given by an $(n \times n)$-matrix $A \in R^{n \times n}$. It is an isomorphism, if and only if $A \in \mathrm{GL}_n(R)$. Hence, for every open subset $U \subset X$, we obtain a matrix $A_U \in \mathrm{GL}_n(\mathcal{O}_X(U))$. Since A_U represents f_U, where $f\colon \mathcal{O}_X^n \longrightarrow \mathcal{O}_X^n$ is a map of sheaves, we have for every inclusion $U \subset V$ of open subsets the relation $A_V|_U = A_U$. The matrix A_X corresponds to a holomorphic function $X \longrightarrow \mathrm{GL}_n(\mathbb{C})$, and as we have seen, it defines the map of sheaves $f\colon \mathcal{O}_X^n \longrightarrow \mathcal{O}_X^n$. $\square$

Proof of Theorem 5.12. We have already seen that the sheaf of sections $\underline{E}$ of a complex vector bundle of rank n is a locally free $\mathcal{O}_X$-module of rank n. As a next step, we have to show that every rank n locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is equivalent to $\underline{E}$, for some rank n complex vector bundle E.

Given $\mathcal{M}$, we choose an open covering $X = \bigcup_{i \in I} U_i$, and isomorphisms of sheaves $\phi_i\colon \mathcal{M}|_{U_i} \xrightarrow{\simeq} (\mathcal{O}_{U_i})^n$. Over the overlaps $U_{ij} = U_i \cap U_j$ we obtain an isomorphism of free sheaves $\phi_{ij}\colon \mathcal{O}_{U_{ij}}^n \xrightarrow{\phi_i \circ \phi_j^{-1}} \mathcal{O}_{U_{ij}}^n$. By virtue of Lemma 5.14, ϕ_{ij} corresponds to a holomorphic function $U_{ij} \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Moreover, the definition of ϕ_{ij} implies immediately that the cocycle condition $\phi_{ij} \cdot \phi_{jk} = \phi ik$ is satisfied on triple intersections U_{ijk}. We have therefore constructed a cocycle datum $\big(\{U_i\}_{i \in I}, (\phi_{ij})_{(i,j) \in I^2}\big)$. Let us denote by E the corresponding complex vector bundle.

For an open subset $V \subset X$, a section $s\colon V \longrightarrow E$ corresponds to a tuple of holomorphic functions $f_i\colon V \cap U_i \longrightarrow \mathbb{C}^n$, satisfying $f_i = \phi_{ij} f_j$ for every $(i,j) \in I^2$ (see Lemma 4.9). However, such a tuple, also defines an element of $\mathcal{M}(V)$: indeed, we have $f_i \in \mathcal{O}_{U_i}^n$, and hence $s_i = \phi_i^{-1}(f_i) \in \mathcal{M}(U_i)$. On the overlaps U_{ij}, we have $s_i|_{U_{ij}} = \phi_i^{-1}(f_i)|_{U_{ij}} = \phi_i^{-1}(\phi_{ij} f_j)|_{U_{ij}} = \phi_i^{-1}(\phi_i \circ \phi_j^{-1} f_j)|_{U_{ij}} = s_j|_{U_{ij}}$. By the sheaf condition, there exists a unique section $s \in \mathcal{M}(U_i)$, such that $s|_{U_i} = s_i$. This shows $\mathcal{M} \cong \underline{E}$.

To conclude the proof, it remains to show that for two vector bundle E, F on X. the existence of an isomorphism of $\mathcal{O}_X$-modules $\underline{E} \cong \underline{F}$ implies the existence of an isomorphism of vector bundles $E \cong F$.

Indeed, let $X = \bigcup_{i \in I} U_i$ be an open covering of X, such that both E and F can be trivialised over the open subsets U_i. We denote the cocycle datum for E by (ϕ_{ij}), and for F by (ψ_{ij}). An isomorphism of $\mathcal{O}_X$-modules $\underline{E} \xrightarrow{\simeq} \underline{F}$ yields for every U_i:

$$\mathcal{O}_{U_i}^n \xrightarrow{\beta_i} \mathcal{O}_{U_i}^n .$$

By Lemma 5.14, we may understand β_i to be a holomorphic function $U_i \longrightarrow \mathrm{GL}_n(\mathbb{C})$. The definition of β_i implies the relation $\phi_{ij}\beta_j = \beta_i\phi_{ij}$ on every overlap U_{ij}. Using a construction similar to Lemma 4.9 we obtain a well-defined isomorphism of vector bundles $E \longrightarrow F$. The details are left to the reader as an exercise. □

Corollary 5.15. *Let X be a complex manifold of (complex) dimension n. There exists a rank n vector bundle $TX \longrightarrow X$, called the tangent bundle, such that the sheaf of sections $\underline{TX}$ is equivalent to the sheaf of complex vector fields $\mathcal{V}_X$. Similarly, there exists a rank $\binom{n}{m}$ vector bundle $\bigwedge^m T^*X \longrightarrow X$, such that $\underline{\bigwedge^m T^*X} \longrightarrow X$ is equivalent to the sheaf of complex m-forms Ω_X^m.*

Proof. We only have to show that the sheaves $\mathcal{V}_X$ and Ω_X^m are locally free. It is clear that they are $\mathcal{O}_X$-modules, because vector fields and m-forms can be multiplied with a holomorphic function.

By definition, a complex manifold X is covered by open subsets, which are biiholomorphic to open subsets U of $\mathbb{C}^n$. We may therefore without loss of generality assume that $X = U$ is an open subset of $\mathbb{C}^n$ (due to the local nature of the statement we are interested in). We have seen in Definition 2.15 that a complex vector field on U is given by an n-tuple of holomorphic functions

$$\mathcal{V} = \sum_{i=1}^{n} \mathcal{V}^i \frac{\partial}{\partial z^i}.$$

In (7) we expressed an m-form by means of an $\binom{n}{m}$-tuple of holomorphic functions

$$\omega = \sum_{0 \le i_0 < \cdots < i_m \le n} \omega_{i_0 \cdots i_m} dz^{i_0} \wedge \cdots \wedge dz^{i_m}.$$

These considerations show that the sheaves $\mathcal{V}_X$ and Ω_X^m are locally free. □

5.4 Exact sequences of sheaves

In this paragraph we denote by $\mathcal{F} \xrightarrow{f} \mathcal{G}$ a map of sheaves on a topological space X. We will study the local nature of injectivity and surjectivity. It will turn out that injectivity of f is a local property, while the same cannot be said for surjectivity.

Lemma 5.16. *If every $x \in X$ has an open neighbourhood U_x, such that $f_U\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective, then the map $\mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ is injective.*

Proof. Recall that a map of abelian groups $A \xrightarrow{g} B$ is injective, if and only if $g(x) = 0$ is equivalent to $x = 0$. Translated to our situation this means that we want to show for every $s \in \mathcal{F}(X)$, such that $f_X(s) = 0$, we already have $s = 0$.

By assumption we have an open covering $\{U_x\}_{x \in X}$ of X, such that $f_{U_x}\colon \mathcal{F}(U_x) \longrightarrow \mathcal{G}(U_x)$ is injective for every $x \in X$. The assumption $f_X(s) = 0$ implies $0 = f_X(s)|_{U_s} = f_{U_x}(s|_{U_x})$. Hence, we have $s|_{U_x} = 0$ for every $x \in X$. The sheaf property implies now that $s = 0$, since the two sections s and 0 both solve the glueing problem for $0 \in \mathcal{F}(U_{xy})$ for $(x, y) \in X^2$. □

Definition 5.17. *We say that f is* locally surjective*, if for every open subset $U \subset X$, and every $s \in \mathcal{G}(U)$, there exists an open covering $U = \bigcup_{i\in I} U_i$, and sections $t_i \in \mathcal{F}(U_i)$, such that $f_{U_i}(t_i) = s|_{U_i}$.*

It is important to emphasise the difference between local surjectivity and surjectivity. The definition above does not imply that there exists a section $t \in \mathcal{F}(U)$, such that $f_U(t) = s$. Indeed, we know that the equation $f(t) = s$ can only locally be solved in t. The problem is that there is no unique element in the preimage, and hence, the local solutions t_i, don't glue - that is, they won't necessarily agree on the overlaps: $t_i|_{U_{ij}} \neq t_j|_{U_{ij}}$ in general. As we have seen in Lemma 5.16, the same problem does not arise for injective maps.

Probably the first example, where this phenomenon has been observed in the history of mathematics, is in the study of complex logarithms.

Example 5.18. Let $X = \mathbb{C}\setminus\{0\}$, we have a morphism of sheaves $\exp\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times$, which sends a holomorphic function $U \xrightarrow{f} \mathbb{C}$ to the invertible holomorphic function $\exp(f)$.

If $g\colon X \longrightarrow \mathbb{C}^\times$ is an invertible holomorphic function, we may ask if there exists a holomorphic $f\colon X \longrightarrow \mathbb{C}^\times$, such that $\exp(f) = g$. It is clear that this equation doesn't have a unique solution. Indeed, for every such f, also $f + 2\pi i k$, with $k \in \mathbb{Z}$ is a solution.

Nonetheless, there exists an open covering $\{U_i\}_{i\in I}$ of X, such that we have a holomorphic function $f_i\colon U_i \longrightarrow \mathbb{C}$, satisfying $\exp(f_i) = g|_{U_i}$ for every $i \in I$. To construct such an open covering can be constructed as follows. Observe that every $z \in \mathbb{C}^\times$ has a small ϵ-neighbhourhood $U_\epsilon(z)$, where a complex logarithm $\widetilde{\log}\colon U_\epsilon(z) \longrightarrow \mathbb{C}$ can be defined (non-canonically). Hence, over $U_i = g^{-1}(U_{\epsilon_i}(z_i))$ it makes sense to define $f_i = \widetilde{log} \circ g|_{U_i}$, where we assume that z_i is a family of points in X, such that $\bigcup_{i\in I} U_{\epsilon_i}(Z_i) = \mathbb{C}^\times$. We have $\exp(f_i) = \exp(\widetilde{\log} \circ g|_{U_i}) = g_{U_i}$, and hence have shown that $\exp\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times$ is a locally surjective morphism of sheaves.

However, it is clear that $\mathcal{O}_X(X) \xrightarrow{\exp} \mathcal{O}_X^\times(X)$ is not a surjective map. Indeed, let $g\colon X \longrightarrow \mathbb{C}^\times$ be the holomorphic function $z \mapsto z$. If there was a holomorphic function $f\colon X \longrightarrow \mathbb{C}$, such that $\exp(f) = g$, we would have $f'(z) = \frac{g'}{g}(z) = \frac{1}{z}$ (this is a consequence of the chain rule). Therefore, the integral

$$\int_{\partial \mathbb{D}_\epsilon(0)} \frac{dz}{z}$$

would be zero. But we know that this is not true, since the value of this integral can be computed to be $2\pi i$.

A pessimist would probably abandon the theory of sheaves at this point. Next week, we will carefully study the obstruction for a locally surjective map of sheaves to be surjective. The answer turns out to be given by *sheaf cohomology.* As a preparation we need to define exact sequences of sheaves.

Definition 5.19. *Let A, B, C be abelian groups, and $A \xrightarrow{f} B \xrightarrow{g} C$ homomorphisms. We say that this sequence of maps is exact, if* $\ker g = \operatorname{image} f$.

A more concrete definition of exactness in terms of elements is the following: for $b \in B$, we have $g(b) = 0$ if and only if $\exists a \in A$, such that $f(a) = b$. This motivates the following definition of exactness for sheaves.

Definition 5.20. *A sequence of sheaves* $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ *on a topological space* X *is called exact, if and only if for every open* $U \subset X$*, and* $b \in \mathcal{G}(U)$*, we have* $g(b) = 0$*, if and only if there exists an open covering* $U = \bigcup_{i \in I} U_i$*, and sections* $a_i \in \mathcal{F}(U_i)$*, such that* $f(a_i) = b|_{U_i}$ *for all* $i \in I$*.*

Our definition of an exact sequence of sheaves is inspired by the definition of locally surjective maps of sheaves. Indeed, we see that a map of sheaves $\mathcal{G} \longrightarrow \mathcal{H}$ is locally surjective, if and only if $\mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is exact.

Lemma 5.21 (Exponential sequence)**.** *Let* X *be a complex manifold, the sequence of sheaves*

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \cdot} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

is exact.

Proof. The statement above can be broken down into three assertions.

Exactness at $\underline{\mathbb{Z}}$: we have to show that the map $\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ is injective. But this is clear, since $2\pi i n = 0$ is equivalent to $n = 0$.

Exactness at $\mathcal{O}_X$: If $f \in \mathcal{O}_X(U)$ is a holomorphic function on $U \subset X$, such that $\exp(f) = 1$, then $f(z) = 2\pi n(z)$, where $n\colon U \longrightarrow \mathbb{Z}$ is a holomorphic function, taking values in $\mathbb{Z}$. Since holomorphic functions are continuous, we see that n is locally constant. Thus, it defines a section $n \in \underline{\mathbb{Z}}(U)$.

Exactness at $\mathcal{O}_X^\times$: We have to show that the map exp is locally surjective. We cover $\mathbb{C}^\times$ by an open covering $\bigcup_{i \in I} V_i = \mathbb{C}^\times$, such that each V_i supports a branch of complex logarithm $\widetilde{\log}\colon V_i \longrightarrow \mathbb{C}$. Consider a holomorphic function $g\colon U \longrightarrow \mathbb{C}^\times$, where $U \subset X$ is an open subset. On $U_i = g^{-1}(V_i)$ we have well-defined holomorphic functions $f_i = \widetilde{\log} \circ g|_{U_i}$. By definition, $\exp(f_i) = g|_{U_i}$. □

Definition 5.22. *Let* X *be a topological space, and* $\mathcal{F}$ *a sheaf on* X*. We write* $\Gamma(X, \mathcal{F})$ *for the abelian group* $\mathcal{F}(X)$ *of global sections.*

We have seen that an exact sequence of sheaves does not induce an exact sequence of global sections in general. However, the following is true.

Lemma 5.23. *Let* $0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \longrightarrow 0$ *be a short exact sequence of sheaves. The sequence of abelian groups*

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H})$$

is exact.

Proof. Exactness at $\Gamma(X, \mathcal{F})$ is equivalent to injectivity of the map $\mathcal{F}(X) \longrightarrow \mathcal{G}(X)$. We have seen this in Lemma 5.16.

Exactness at $\Gamma(X, \mathcal{G})$ can be shown as follows: let $s \in \mathcal{G}(X)$, such that $g(s) = 0$. By assumption there exists an open covering $X = \bigcup_{i \in I} U_i$, and sections $t_i \in \mathcal{F}(U_i)$, such that $f(t_i) = s|_{U_i}$. We claim that the glueing condition is satisfied, that is $t_i|_{U_{ij}} = t_j|_{U_{ij}}$ for all $(i,j) \in I^2$. This is true, since $f(t_i|U_{ij}) = s|_{U_{ij}} = f(t_j|_{U_{ij}})$, but f is injective, which implies what we want. Therefore, there exists a section $t \in \mathcal{F}(X)$, such that $f(t)|_{U_i} = s|_{U_i}$ for all $i \in I$. We conclude that $f(t) = s$. □

6 Sheaf cohomology

In this section we will introduce sheaf cohomology. We will take an axiomatic approach, and assume at the beginning that a theory with such properties exists. At the end of the section we will verify by a simple inductive procedure that this is indeed the case.

6.1 Overview: why do we need sheaf cohomology?

We have seen in Lemma 5.23 that the functor $\Gamma(X,-)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$ doesn't send short exact sequences of sheaves to short exact sequences of abelian groups. Instead,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

gives rise to an exact sequence falling short of being short exact.

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H}).$$

The example of the exponential sequence $0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$ shows that the map $\Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H})$ is not surjective in general. Our best hope is therefore that we can continue the above sequence to a long exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \xrightarrow{g} \Gamma(X,\mathcal{H}) \xrightarrow{\delta} H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H})$$

with the help of mysterious functors $H^1(X,-)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$.

If we had such a formalism at our disposal, we could analyse precisely which global sections $s \in \Gamma(X,\mathcal{H}) = \mathcal{H}(X)$ can be lifted to global section t of $\mathcal{G}$. Indeed, exactness of the sequence above asserts that there exists a $t \in \mathcal{G}(X)$, such that $g(t) = s$, if and only if $\delta(s) = 0$. That is, $\delta(s) \in H^1(X,\mathcal{F})$ measures the obstruction for a global section of $\mathcal{H}$ to stem from a global section of $\mathcal{G}$.

Definition 6.1 (Sheaf cohomology)**.** *A* sheaf cohomology theory *is a collection of functors*

$$H^i(X,-)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$$

for $i \geq 0$, such that:

(A1) $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ as functors,

(A2) for every short exact sequence of sheaves $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ we have a long exact sequence

$$\cdots \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G}) \longrightarrow H^i(X,\mathcal{H}) \longrightarrow H^{i+1}(\mathcal{F}) \longrightarrow \cdots$$

(A3) (...)

The third axiom will be stated at a later point. We will see that there is essentially only one theory of sheaf cohomology up to a unique isomorphism. At first we take another look at our favourite example, and try to guess what the space of obstructions $H^1(X,\underline{\mathbb{Z}})$ amounts to in geometric terms.

6.2 The exponential sequence revisited

For the purpose of this discussion we let X be $\mathbb{C}\setminus 0$. The motivated reader may work with an arbitrary complex manifold X. We consider the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i\cdot} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0,$$

and ask when a nowhere vanishing holomorphic function $f\colon X \longrightarrow \mathbb{C}^\times$ has a global logarithm $\widetilde{\log} f$. That is, when does there exists a holomorphic function $g\colon X \longrightarrow \mathbb{C}$ such that $f = \exp(g)$?

We know that for for $f(z) = z$ such a function $g = \widetilde{\log} z$ cannot exist. However, if $f(z) = \text{const} = c \neq 0$, we can choose $d \in \mathbb{C}$, such that $\exp(d) = c$, and define $g(z) = \text{const} = d$. How can we decide if a general $f \in \Gamma(X, \mathcal{O}_X^\times)$ admits a global logarithm? We will utilise a notion from topology: covering spaces. As the attentive reader will confirm, their definition is reminiscent of the definition of vector bundles.

Definition 6.2. *Let G be a group, endowed with the discrete topology. A continuous map $\pi\colon Y \longrightarrow X$ is called a G-*Galois covering*, if*

(a) the group G acts on Y by homeomorphisms, and the map π is G-invariant, that is $\pi(gy) = \pi(y)$ for all $g \in G$, and $y \in Y$,

(b) for every $x \in X$, there exists a $y \in \pi^{-1}(x)$, such that $G \longrightarrow \pi^{-1}(x)$, $g \mapsto g \cdot y$ is a bijection,

(c) every $x \in X$ has an open neighbourhood U, such that there exists a G-equivariant homeomorphism $\phi\colon \pi^{-1}(U) \xrightarrow{\simeq} U \times G$, fitting into a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_1"] \\ & U. & \end{tikzcd}$$

Recall that a map of spaces $U \longrightarrow V$ carrying a G-action is called G-equivariant, if $f(gu) = gf(u)$ for all $g \in G$, and $u \in U$.

An example of a $\mathbb{Z}$-Galois covering is the exponential map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$. This is the reason, why we introduced this notion.

There is a correspondence between G-Galois coverings of X, and group homomorphisms $\pi_1(X, x) \longrightarrow G$, as long as X is a nice enough space (for instance, a manifold). The idea is the following: given a G-Galois covering as above, and a closed path $\gamma\colon [0,1] \longrightarrow X$, with $\gamma(0) = \gamma(1) = x$. We can construct a continuous path $\tilde{\gamma}\colon [0,1] \longrightarrow Y$, such that $\gamma(0) = y \in \pi^{-1}(Y)$, and $\pi \circ \tilde{\gamma} = \gamma$. There is a unique such path $\tilde{\gamma}$, since every $x' \in X$ has an open neighbourhood U, over which $\pi^{-1}(U) \cong U \times G$ is a trivial fibration (hence lifts exist, and are unique if one point on the curve is specified as a sort of boundary condition).

Take for instance $\gamma(t) = e^{2\pi i t}$ as a path in $X = \mathbb{C}^\times$. By definition a lift is given by $\tilde{\gamma}(t) = 2\pi i t$. And we see that $0 = \tilde{\gamma}(0) \neq 2\pi i = \tilde{\gamma}(1)$.

However, since $\pi(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = \pi(\tilde{\gamma}(1))$, we have that $\tilde{\gamma}(1) \in \pi^{-1}(x)$. The definition of a G-Galois covering implies now that there exists a unique $g \in G$, such that $\tilde{\gamma}(1) = g \cdot y = \tilde{\gamma}(0)$. Moreover, this g depends only on the homotopy class of the path γ, leaving the end points fixed. This allows one to associate to a G-Galois covering $\pi\colon Y \longrightarrow X$ a group homomorphism $\pi_1(X, x) \longrightarrow G$. It follows from basic theory of covering spaces that every homomorphism $\pi_1(X, x) \longrightarrow G$ arises in this way. The details are left to the reader (see exercise sheet).

Definition 6.3. *A G-Galois covering $\pi\colon Y \longrightarrow X$ is* trivial*, if there exists a G-equivariant homo-*

morphism $Y \cong X \times G$, such that

$$\begin{tikzcd} \pi^{-1}(X) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & X \times G \arrow[dl, "p_1"] \\ & X. & \end{tikzcd}$$

A G-Galois covering is trivial, if and only if the corresponding homomorphism $\pi_1(X,x) \longrightarrow G$ is the constant map to the unit of G.

We will be interested in the case $G = \mathbb{Z}$, since the existence of logarithms is closely related to $\mathbb{Z}$-Galois coverings.

Proposition 6.4. *Let X be a complex manifold, and $f\colon X \longrightarrow \mathbb{C}^\times$ be a nowhere vanishing holomorphic function. The fibre product*

$$Y_f = X \times_{\mathbb{C}^\times} \mathbb{C} = \{(x,z) \in X \times \mathbb{C} \,|\, f(x) = \exp(z)\} \longrightarrow X$$

is a $\mathbb{Z}$-Galois covering, with respect to the action $k\cdot(x,z) = (x, 2\pi i k + z)$. There exists a holomorphic function $g\colon X \longrightarrow \mathbb{C}$, such that $f = \exp(g)$, if and only if the $\mathbb{Z}$-Galois covering constructed above is trivial.

Proof. Let's show first that $Y_f \xrightarrow{\pi}$ is a $\mathbb{Z}$-Galois covering. There exists an open covering $\bigcup_{i\in I} U_i = \mathbb{C}^\times$, such that we have holomorphic functions $\widetilde{\log}_i\colon U_i \longrightarrow \mathbb{C}$, such that $\exp(\widetilde{\log}_i) = \mathrm{id}_{U_i}$. Therefore, $\bigcup_{i\in I} f^{-1}(U_i) = X$ is an open covering of X, and the map

$$(x,z) \mapsto (x, \frac{z - \widetilde{\log}_i(z)}{2\pi i})$$

defines a $\mathbb{Z}$-equivariant homeomorphism between $\pi^{-1}(f^{-1}(U_i))$ and $f^{-1}(U_i) \times \mathbb{Z}$. This proves that $Y_f \xrightarrow{\pi} X$ is a $\mathbb{Z}$-Galois covering.

Next we will show that $Y_f \longrightarrow X$ is the trivial $\mathbb{Z}$-Galois covering, if and only if $f = \exp(g)$, for some continuous function $g \longrightarrow \mathbb{C}$ (it is then clear that g is also holomorphic). The trivial $\mathbb{Z}$-Galois covering $X \times \mathbb{Z} \longrightarrow X$ has at least one continuous section $s\colon X \longrightarrow X \times \mathbb{Z}$, for instance $x \mapsto (x,0)$. If $Y_f \longrightarrow X$ is the trivial covering, then there exists a continuous map $s\colon X \longrightarrow Y_f$, such that $\pi \circ s = \mathrm{id}_X$. By definition of $Y = X \times_{\mathbb{C}^\times} \mathbb{C}$, we see that $g = p_2 \circ s\colon X \longrightarrow \mathbb{C}$ is a continuous map, such that $\exp(g) = f$.

Vice versa, assume that there's a holomorphic map $f\colon X \longrightarrow \mathbb{C}$, with the property $\exp(f) = g$. Then, we can define a $\mathbb{Z}$-equivariant homomorphism $Y_f \longrightarrow X \times \mathbb{Z}$ by the formula

$$(x,z) \mapsto (x, \frac{z - f(x)}{2\pi i}).$$

This concludes the proof. □

Summarising the discussion above, we see that we associated to every nowhere vanishing holomorphic function $f\colon X \longrightarrow \mathbb{C}^\times$ a $\mathbb{Z}$-Galois covering, which is trivial, if and only if $f = \exp(g)$, for some holomorphic function $g\colon X \longrightarrow \mathbb{C}$. The set of isomorphism classes of $\mathbb{Z}$-Galois covering is equal to the set of group homomorphisms $\rho\colon \pi_1(X,x) \longrightarrow \mathbb{Z}$. Since two $\mathbb{Z}$-valued functions can be multiplied, we see that this set actually carries a natural structure of an abelian group, which we will denote by $\check{H}^1(X, \underline{\mathbb{Z}})$. That is, we've verified the following theorem.

Theorem 6.5. *Let X be a connected complex manifold, there exists an exact sequence*

$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X^\times) \xrightarrow{\delta} \check{H}^1(X, \underline{\mathbb{Z}}).$$

We see that the map δ fulfils the goal of determining the obstruction to the existence of a holomorphic function g with $\exp(g) = f$.

6.3 Sheaf cohomology as cohomology

Probably you've already encountered the notions of cohomology (or homology) of chain complexes in other lectures (differential or algebraic topology). In those courses, cohomology refers in first instance to the cohomology of a chain complex of abelian groups. Recall that a chain complex is a sequence of homomorphisms of abelian groups

$$A^\bullet = [A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^3 \xrightarrow{d^3} \cdots],$$

such that the relation $d^{i+1} \circ d^i = 0$ is satisfied for all $i \geq 0$. The abelian group $H^n(A^\bullet)$ is defined to be

$$\frac{\ker d^n}{\text{image } d^{n-1}},$$

and referred to as *degree n cohomology* of the chain complex $A^\bullet$. Note that $A^\bullet$ is exact at A^n, if and only if $H^n(A^\bullet) = 0$. Thus, we think of $H^n(A^\bullet)$ as measuring the failure of $A^\bullet$ being exact at A^n.

We have also seen that sheaf cohomology measures the failure of exactness of the global sections functor $\Gamma(X, -)$. It is therefore not unreasonable to ask, if the cohomology groups $H^i(X, \mathcal{F})$ of a sheaf $\mathcal{F}$ on X can be computed as cohomology groups of a chain complex. It turns out that the answer is yes. There will be more than one such complex computing sheaf cohomology, and every instance of this phenomenon translates into an interesting comparison theorem. For example:

Theorem 6.6 (de Rham)**.** *Let X be a smooth manifold, we denote by $H^i_{\mathrm{sing}}(X, \mathbb{R})$ the singular cohomology of X with real coefficients, and by $H^i_{\mathrm{dR}}(X, \mathbb{R})$ the so-called de Rham cohomology of X. There is a canonical isomorphism*

$$H^i_{\mathrm{sing}}(X, \mathbb{R}) \cong H^i_{\mathrm{dR}}(X, \mathbb{R}).$$

We will recall the definitions of singular and de Rham cohomology in more detail. For singular cohomology this requires some preparation, but the rough idea is to cove X by simplices (basic geometric shapes, generalising an interval of bounded length, a triangle, a tetrahedron, etc), and construct a chain complex from these decompositions. De Rham cohomology is computed using the real vector spaces of smooth m-forms $\Omega^m_{\mathbb{R}}(X)$, and the fact that exterior differentiation $d\colon \Omega^m_{\mathbb{R}}(X) \longrightarrow \Omega^{m+1}_{\mathbb{R}}(X)$ satisfies the relation $d \circ d = 0$. Hence, differential forms give rise to a chain complex in vector spaces, and the resulting cohomology groups, are denoted by $H^i_{\mathrm{dR}}(X, \mathbb{R})$.

Example 6.7. Assume that X is an open subset of $\mathbb{R}^3$. A smooth (real) m-form on X is then expressible as a linear combination $\sum_{0 \leq i_1 < \cdots < i_m \leq m} f_{i_1 \ldots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}$. Only the cases $m = 0, 1, 2, 3$ are non-trivial. A 0-form or 3-form corresponds to a single smooth function f, and 1-forms and 2-forms correspond to a triple of smooth functions (f_1, f_2, f_3). Analysing the formulae for exterior differentiation, and battling with signs, one sees that $d^0\colon \Omega^0_{\mathbb{R}}(X) \longrightarrow \Omega^1(X)$ is basically computing the gradient vector field of a smooth function, while d^1 corresponds to the curl operator, and d^2 to divergence of vector fields.

It is well-known that gradient vector fields are curl-free, but what about the converse? De Rham's theorem tells us that the quotient vector space

$$\frac{\{V \text{ smooth vector field on } X | \nabla \times V = 0\}}{\{\nabla \cdot f | f \text{ smooth function on } X\}} = H^1_{\mathrm{sing}}(X, \mathbb{R})$$

is only depending on the topology of X. Hence, if there are curl-free vector fields, only the topology of X obstructs us from expressing them globally as a gradient.

Exercise 6.8. *(a) In the movie "A beautiful mind", Russell Crowe asks the students of his multivariable calculus class to solve a supposedly very difficult problem about vector fields. Solve it or argue why Russell's exercise might have been too difficult for an introductory course.*

(b) Remind yourself that on a star-shaped open subset of $\mathbb{R}^3$, every curl-free smooth vector field can be written as a gradient of a smooth function. Deduce that the following is a short exact sequence of sheaves $0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty_X \xrightarrow{\nabla\cdot} \mathcal{V}^\infty_X \xrightarrow{\nabla\times} 0$ on X. Here, C^∞_X is the sheaf, assigning to an open subset U, the vector space of smooth $\mathbb{R}$-valued functions, and $\mathcal{V}^\infty_X$ denotes the sheaf of smooth real vector fields on X. Mimic our analysis of the existence of complex logarithms (Theorem 6.5), using $\mathbb{R}$-Galois covers, to understand when a curl-free vector field on X can be written as a gradient of a smooth function.

The comparison between de Rham and singular cohomology is achieved by comparing both sides to the cohomology of the sheaf $\underline{\mathbb{R}}$ on X, which assigns to an open subset $U \subset X$ the real vector space of *locally constant* maps $U \longrightarrow \mathbb{R}$ (we could also view $\mathbb{R}$ as a discrete topological space, and look at continuous maps $U \longrightarrow \mathbb{R}^{\mathrm{disc}}$).

The so-called de Rham complex $(\Omega^\bullet_{\mathbb{R}}(X), d)$ can be viewed as the chain complex of global sections $\Gamma(X, -)$ of an exact sequence in sheaves. Let us denote by Ω^m_X the sheaf, which assigns to an open subset $U \subset X$ the sheaf of smooth (real) m-forms on U. In differential geometry or topology, the following lemma should have been proven.

Lemma 6.9. *The sequence of sheaves $0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \longrightarrow \cdots$ is exact.*

We will deduce from this lemma, and the Universal Comparison Theorem below that $H^i_{dR}(X, \mathbb{R})$ agrees with the sheaf cohomology group $H^i(X, \underline{\mathbb{R}})$.

Theorem 6.10 (Universal Comparison Theorem)**.** *Let $\mathcal{F}$ be a sheaf on a topological space X, and*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2 \xrightarrow{d^2} \longrightarrow \cdots$$

an exact sequence of sheaves, such that $H^i(X, \mathcal{G}^j) = 0$ for all $i \geq 1$ (one also says that the sheaf $\mathcal{G}^j$ is acyclic). Then, $H^i(X, \mathcal{F})$ is isomorphic to the degree i cohomology group of the chain complex

$$\Gamma(X, \mathcal{G}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{G}^1) \xrightarrow{d^1} \Gamma(X, \mathcal{G}^2) \xrightarrow{d^2} \cdots .$$

Proof. We will prove this statement by induction on i. In the course of the proof, we denote by $K^i = \ker d^i$, the subsheaf of $\mathcal{G}^i$, assigning to an open subset $U \subset X$ the subgroup $\{s \in \mathcal{G}^i(U) | d^i(s) = 0\}$.

Our proof is anchored to the case $i = 0$. Exactness of $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1$ yields a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow K^1 \longrightarrow 0.$$

Applying $\Gamma(X, -)$, we obtain from Lemma 5.23 that $0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}^0) \longrightarrow \Gamma(X, K^1)$ is exact. We have a commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{G}^0) \arrow[r] \arrow[dr, "d^0"'] & \Gamma(X, K^1) \arrow[d, hook] \\
& \Gamma(X, \mathcal{G}^1),
\end{tikzcd}$$

where injectivity of the vertical arrow follows from Lemma 5.23 applied to the short exact sequence $0 \longrightarrow K^1 \longrightarrow \mathcal{G}^1 \longrightarrow K^2 \longrightarrow 0$ of sheaves (also a consequence of exactness). Therefore, we have

$$H^0(\Gamma(X, \mathcal{G}^\bullet)) = \ker(d^0 \colon \Gamma(X, \mathcal{G}^0) \longrightarrow \Gamma(X, \mathcal{G}^1)) = \Gamma(X, \mathcal{F}).$$

Let's assume that the claim had already been proven for a given $i \geq 0$, and see why the assertion must also hold for degree $i + 1$ cohomology groups. By exactness of the sequence $\mathcal{G}^\bullet$ we have a short exact sequence

$$0 \longrightarrow K^i \longrightarrow \mathcal{G}^i \xrightarrow{d^i} L^{i+1} \longrightarrow 0.$$

The associated long exact sequence in cohomology contains the following interesting part

$$H^0(X, \mathcal{G}^i) \longrightarrow H^0(X, K^{i+1}) \longrightarrow H^1(X, K^i) \longrightarrow H^1(X, \mathcal{G}^i) = 0.$$

Exactness implies that $H^1(X, \mathcal{K}^i) = \frac{\Gamma(X, \mathcal{K}^{i+1})}{\mathrm{image}(\Gamma(X, \mathcal{G}^i) \longrightarrow \Gamma(X, K^{i+1}))} = H^{i+1}(\Gamma(X, \mathcal{G}^\bullet))$. The short exact sequences $0 \longrightarrow K^j \longrightarrow \mathcal{G}^j \longrightarrow K^{j+1} \longrightarrow 0$ imply now inductively that we have

$$H^1(X, K^i) = H^2(X, K^{i-1}) = \cdots = H^{i+1}(X, \mathcal{F}).$$

This concludes the proof of the argument. $\square$

The Universal Comparison Theorem implies de Rahm's theorem, once we've shown that the sheaves of smooth m-forms don't have any higher cohomology; and we found a way of expressing singular cohomology via a resolution of sheaves. We defer the proof of the vanishing of higher cohomology to a later section, and devote the rest of this paragraph to an explanation how singular cohomology arises from a resolution of sheaves.

Definition 6.11. *(a) For every positive integer $n \geq 0$, we denote by $\Delta^n = \{(\lambda_0, \ldots, \lambda_n) \in [0,1]^{n+1} | \lambda_0 + \cdots + \lambda_n = 1\} \subset \mathbb{R}^{n+1}$, and view it as a topological space with respect to the subset topology.*

(b) Let X be a topological space. A singular n-simplex is a continuous map $\sigma \colon \Delta^n \longrightarrow X$. The set of singular n-simplices on X will be denoted by $S_n(X)$.

(c) For $0 \leq i \leq n$ we denote by $\partial_i \colon \Delta^{n-1} \longrightarrow \Delta^n$, the unique affine map (linear + translation), which preserves the vertices with respect to their standard ordering:

$$e_0 \mapsto e_0,\ e_1 \mapsto e_1, \cdots, e_{i-1} \mapsto e_{i-1},\ e_i \mapsto e_{i+1}, \cdots e_{n-1} \mapsto e_n.$$

(d) For an abelian group A, we let $C^n(X, A)$ be the set of maps of sets $S_n(X) \xrightarrow{c} A$. We will denote $c(\sigma)$ by $\int_\sigma c$, and define an A-linear map by

$$d^n \colon C^n(X, A) \longrightarrow C^{n+1}(X, A)$$

by the formula $\int_\sigma d(c) = \int_{\partial\sigma} c = \sum_{i=0}^{n+1} (-1)^i \int_{\sigma \circ \partial_i} c$.

It is an important exercise to check that $d^{n+1} \circ d^n = 0$ for every $n \geq 0$. Therefore, we have constructed a chain complex

$$0 \longrightarrow C^0(X, A) \longrightarrow C^1(X, A) \longrightarrow C^2(X, A) \longrightarrow \cdots .$$

The degree i cohomology of this chain complex is called the i-th singular cohomology group $H^i_{\mathrm{sing}}(X, A)$ with coefficients in A.

Theorem 6.12. *If X is a locally contractible topological space, then $H^i_{\mathrm{sing}}(X, A)$ is equivalent to $H^i(X, \underline{A})$, where $\underline{A}$ is the sheaf of locally constant A-valued functions on X.*

Sketch. The main idea is to consider the presheaf C^n_X, assigning to an open subset $U \subset X$ the abelian group

$$U \mapsto C^n(U, A).$$

We have a map of presheaves $\underline{A} \longrightarrow C^n_X$, which assigns to a locally constant function $f \colon U \longrightarrow A$, the element $c \in C^0(U, A)$, given by $\int_\sigma c = f(\sigma(1))$ (a singular 0-simplex is just a single point in X). If U is contractible as a topological space, it is shown in algebraic topology that the complex $0 \longrightarrow A \longrightarrow C^0(U, A) \longrightarrow C^1(U, A) \longrightarrow \cdots$ is exact. If there was any justice in the world, we could pretend that C^n_X was a sheaf, and hence assert that

$$0 \longrightarrow \underline{A} \longrightarrow C^0_X \longrightarrow C^1_X \longrightarrow \cdots$$

was a resolution by sheaves, and that $H^i(X, C^j_X) = 0$ for $i \geq 1$. Unfortunately, C^j_X is not a sheaf, and one needs to invoke the sheafification process discussed in the exercises. Let us denote by $\underline{C}^n_X$ the sheafification. What saves the day is that $K^j = \ker(d^j : \underline{C}^j_X \longrightarrow \underline{C}^{j+1}_X) = \ker(d^j : C^j_X \longrightarrow C^{j+1}_X)$, and the proof of the universal comparison theorem still applies to our situation pre-sheafification. $\square$

Exercise 6.13. *Turn the sketch above into a complete proof.*

6.4 Construction of sheaf cohomology

We start with some diagram chasing lemmas. Recall that for a morphism $f \colon A \longrightarrow D$ we denote by coker f the quotient $D/\text{image } f$.

Exercise 6.14. *(Snake Lemma) Given a commutative diagram*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F & \end{tikzcd}$$

with exact rows, show that there is an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

An important corollary of the Snake Lemma is the tic-tac-toe lemma.

Lemma 6.15 (Tic-Tac-Toe)**.** *Assume that the following diagram is commutative, and its columns are exact.*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

If the first two rows are exact, then so is the third row.

Proof. Take a thin strip of paper and cover the last row of the commutative diagram above. The resulting diagram contains the one of the Snake Lemma as a subdiagram. The morphism $C \longrightarrow F$ has kernel 0 and cokernel I. Therefore we obtain directly that

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow G \longrightarrow H \longrightarrow I$$

is exact. it remains to show that $H \longrightarrow I \longrightarrow 0$ is exact, that is $H \longrightarrow I$ is surjective. To see this, we zoom into the commutative diagram above, to isolate the commuting square

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[d] & F \arrow[d, two heads] \\ H \arrow[r] & I. \end{tikzcd}$$

The arrows marked by $\twoheadrightarrow$ are surjective by assumption. Commutativity of the diagram implies that $H \longrightarrow I$ is surjective too. $\square$

The tic-tac-toe lemma also holds for sheaves on a topological space X. This is true, because exactness of a sequence of sheaves $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ can be tested by verifying that for every $x \in X$ the sequence of stalks $\mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x$ is an exact sequence of abelian groups (see E7).

As a next step we have to define *injective sheaves*. As warm-up we discuss injectivity for abelian groups.

Definition 6.16. *An abelian group I is called* injective*, if the following is true: let $A \hookrightarrow B$ be an injective morphism of abelian groups, and $f\colon A \longrightarrow I$ a homomorphism, then there exists a homomorphism $B \longrightarrow I$, such that the diagram*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd}$$

commutes.

It's important to emphasise that this is *not* a universal property. An extension $B \longrightarrow I$ as the one above, is in general not unique. The next lemma shows that injectivity is quite easy to verify for abelian groups, and implies that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Lemma 6.17. *An abelian group M is called* divisible*, if for every $m \in M$, and every integer $n \geq 1$, there exists an $m' \in M$, such that $n \cdot m' = m$. An abelian group is injective if and only if it is divisible.*

Proof. Let's assume first that I is injective, and show it's divisible. For $m \in I$ and an integer $n \geq 1$, we consider the map $n\,\mathbb{Z} \longrightarrow M$

$$\begin{tikzcd} n\,\mathbb{Z} \arrow[r, hook] \arrow[d, "n \mapsto m"'] & \mathbb{Z} \arrow[dl, dashed, "\exists ?"] \\ M & \end{tikzcd}$$

which sends the generator n to m. Since $n\,\mathbb{Z} \subset \mathbb{Z}$ is an injective homomorphism of abelian groups, and I is injective, there exists a homomorphism $g\colon \mathbb{Z} \longrightarrow I$, such that $g|_{n\,\mathbb{Z}} = f$. Therefore, $m' = f(1)$ satisfies the property $n \cdot m' = n \cdot g(1) = g(n) = f(n) = m$. This shows that I is divisible.

Vice versa, let M be a divisible abelian group. We have to show that M is injective. We will prove this using the axiom of choice. Let $A \hookrightarrow B$ be an injective homomorphism of abelian groups, and $f\colon A \longrightarrow M$ a homomorphism. We have to prove that f can be extended to B.

We consider the set of pairs (A', g), where $A \subset A' \subset B$ is a subgroup, and $g\colon A' \longrightarrow M$ is a homomorphism, such that $g|_{A'} = f$. There is an ordering on this set. We say $(A', g) \leq (A'', h)$ if $A' \subset A''$, and $h|_{A'} = g$. This set is partially ordered, because if $I = \{(A_i, f_i)\}$ is totally ordered subset, then there exists an upper bound, given by $(\bigcup_{i \in I} A_i, g)$, with $g(a) = f_i(a)$ if $a \in A_i$.

Zorn's lemma implies that there exists a maximal element $(\bar{A}, \bar{f})$. It remains to show that $B = \bar{A}$. Assume by contradiction that there exists $x \in B \setminus \bar{A}$. We claim that $\bar{f}$ can be extended to the subgroup $\bar{A} + \mathbb{Z}\,x \longrightarrow M$. We only need to define $f(x)$ in a suitable way, in order to make sense of it. If $\mathbb{Z}\,x \cap (\bar{A} \setminus 0) = \emptyset$, then we can define $f(x) = 0$ or any other element of M. If the intersection is not empty, then there exists a minimal integer $n \geq 1$, such that $nx \in \bar{A}$. Since M is divisible, there exists an $m' \in M$, such that $nm' = \bar{f}(nx)$. We define $f(x) = m'$. This is a contradiction, and we conclude $\bar{A} = B$. $\square$

Definition 6.18. *A sheaf $\mathcal{I}$ is called* injective*, if the following is true: let $\mathcal{F} \hookrightarrow \mathcal{G}$ be an injective morphism of sheaves, and $f\colon \mathcal{F} \longrightarrow \mathcal{I}$ a morphism of sheaves. Then there exists a momorphism $\mathcal{G} \longrightarrow \mathcal{I}$, such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{G} \arrow[dl, dashed, "\exists"] \\ \mathcal{I} & \end{tikzcd}$$

commutes.

Again we emphasise that injectivity is a property, and not a universal property! We are now ready to state the third axiom of sheaf cohomology, and prove the existence of such a theory.

Definition 6.19 (revised version of 6.1)**.** *A* sheaf cohomology theory *is a collection of functors*

$$H^i(X,-)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$$

for $i \geq 0$, such that:

(A1) *$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ as functors,*

(A2) *for every short exact sequence of sheaves $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ we have a long exact sequence*

$$\cdots \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G}) \longrightarrow H^i(X,\mathcal{H}) \longrightarrow H^{i+1}(\mathcal{F}) \longrightarrow \cdots .$$

And for every commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{G}_2 \arrow[r] & \mathcal{H}_2 \arrow[r] & 0 \end{tikzcd}$$

we have a commutative diagram whose rows are the aforementioned long exact sequences.

$$\begin{tikzcd} \cdots \arrow[r] & H^i(X,\mathcal{F}_1) \arrow[r] \arrow[d] & H^i(X,\mathcal{G}_1) \arrow[r] \arrow[d] & H^i(X,\mathcal{H}_1) \arrow[r] \arrow[d] & H^{i+1}(X,\mathcal{F}_1) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^i(X,\mathcal{F}_2) \arrow[r] & H^i(X,\mathcal{G}_2) \arrow[r] & H^i(X,\mathcal{H}_2) \arrow[r] & H^{i+1}(X,\mathcal{F}_2) \arrow[r] & \cdots \end{tikzcd}$$

(A3) *If $\mathcal{I}$ is an injective sheaf, then $H^i(X,\mathcal{I}) = 0$ for $i \geq 1$.*

What did (A3) buy us? We've shown in the exercises (see E7) that every sheaf $\mathcal{F}$ can be embedded into an injective sheaf $\mathcal{I}$. Therefore, there exists a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}\,/\,\mathcal{F} \longrightarrow 0.$$

The stalks of the sheaf $(\mathcal{I}\,/\,\mathcal{F})_x$ are equal to the cokernel $\mathcal{I}_x\,/\,\mathcal{F}_x$. The sections of $\mathcal{I}\,/\,\mathcal{F}(U)$ are given by elements $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{I}_x\,/\,\mathcal{F}_x$, such that there exists an open covering $U = \bigcup_{i\in I} U_i$, sections $t_i \in \mathcal{I}(U_i)$, such that $s_x = [(t_i)_x]$ for every $x \in U_i$.

Assuming that there is a theory of sheaf cohomology satisfying the axioms (A1-3), we obtain for $i \geq 0$ an exact sequence

$$\cdots \longrightarrow H^i(X, \mathcal{I}) \longrightarrow H^i(X, \mathcal{I} / \mathcal{F}) \longrightarrow H^{i+1}(\mathcal{F}) \longrightarrow 0 = H^{i+1}(X, \mathcal{I}),$$

where we use that $H^{i+1}(X, \mathcal{I}) = 0$, since $\mathcal{I}$ is injective. This implies

$$H^1(X, \mathcal{F}) \cong \operatorname{coker}(H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{I} / \mathcal{F})),$$

and for $i \geq 1$:

$$H^{i+1}(X, \mathcal{F}) \cong H^i(X, \mathcal{I} / \mathcal{F}).$$

We will turn this observation into an inductive definition. For this to make sense, we have to verify the resulting higher cohomology groups, are independent of the choice of the embedding $\mathcal{F} \hookrightarrow \mathcal{I}$.

Lemma 6.20. *The following defines a functor $H^1(X, -)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$. For every $\mathcal{F} \in \mathsf{Sh}(X)$ we choose an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$, where $\mathcal{I}$ is an injective sheaf, and define*

$$H^1(X, \mathcal{F}) = \operatorname{coker}(H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{I} / \mathcal{F})).$$

For a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ we choose a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d, dashed, "\exists"] \\ \mathcal{G} \arrow[r, hook] & \mathcal{J} \end{tikzcd}$$

and define $H^1(f)$ to be the map

$$\begin{tikzcd} H^0(X, \mathcal{I}) \arrow[r] \arrow[d] & H^0(X, \mathcal{I} / \mathcal{F}) \arrow[r] \arrow[d] & H^1(X, \mathcal{F}) \arrow[r] \arrow[d, dashed, "H^1(f)"] & 0 \\ H^0(X, \mathcal{J}) \arrow[r] & H^0(X, \mathcal{J} / \mathbb{G}) \arrow[r] & H^1(X, \mathcal{G}) \arrow[r] & 0. \end{tikzcd}$$

The resulting functor is independent (up to a unique natural isomorphism) of the chosen embeddings $\mathcal{F} \longrightarrow \mathcal{I}$.

Proof. We begin the proof by verifying that the resulting map $H^1(f)$ is independent of the choices. That is, if we have two morphisms g and h giving rise to a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\ \mathcal{G} \arrow[r, hook] & \mathcal{J} \end{tikzcd}$$

we want to prove that the induced maps $\alpha, \beta\colon H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G})$ agree. We will show that $\alpha - \beta = 0$.

$$\begin{tikzcd} H^0(X, \mathcal{I}) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] & H^0(X, \mathcal{I} / \mathcal{F}) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] & H^1(X, \mathcal{F}) \arrow[r] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & 0 \\ H^0(X, \mathcal{J}) \arrow[r] & H^0(X, \mathcal{J} / \mathcal{F}) \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & 0. \end{tikzcd}$$

The map $g - h\colon \mathcal{I} \longrightarrow \mathcal{I}$ satisfies $(g-h)|_{\mathcal{F}} = 0$ by definition. Therefore, we obtain a factorisation as indicated by the dotted arrow

$$\begin{tikzcd}
\mathcal{I} \arrow[r] \arrow[d, "g-h"'] & \mathcal{I}\,/\,\mathcal{F} \arrow[d] \arrow[dl, dotted] \\
\mathcal{J} \arrow[r] & \mathcal{J}\,/\,\mathcal{G},
\end{tikzcd}$$

and we see that $g-h\colon H^0(X,\mathcal{I}\,/\,\mathcal{F}) \longrightarrow H^0(X,\mathcal{I}\,/\,\mathcal{F})$ factors through $H^0(X,\mathcal{J})$. Since $H^1(X,\mathcal{G}) = \operatorname{coker}(H^0(X,\mathcal{J}) \longrightarrow H^0(X,\mathcal{J}\,/\,\mathcal{G}))$, we obtain $\alpha - \beta = 0$.

Applying this to the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\
\mathcal{F} \arrow[r, hook] & \mathcal{I},
\end{tikzcd}$$

we see that $H^1(\mathrm{id}_{\mathcal{F}})$ is the identity map of $H^1(X,\mathcal{F})$. Applying the observation to

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d] \\
\mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{J} \arrow[d] \\
\mathcal{F} \arrow[r, hook] & \mathcal{I},
\end{tikzcd}$$

(and also switching the roles of $\mathcal{I}$ and $\mathcal{J}$) we see that the abelian group $H^1(X,\mathcal{F})$ is independent of the chosen embedding $\mathcal{F} \hookrightarrow \mathcal{I}$.

It remains to see that for composable morphisms of sheaves $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ we have $H^1(g \circ f) = H^1(g) \circ H^1(f)$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d, dashed, "\exists"] \\
\mathcal{G} \arrow[r, hook] \arrow[d] & \mathcal{J} \arrow[d, dashed, "\exists"] \\
\mathcal{H} \arrow[r, hook] & \mathcal{K}\,.
\end{tikzcd}$$

We know that the induced maps $H^1(f), H^1(g), H^1(g\circ)$ are independent of the chosen extensions presented by the dashed arrows. Therefore $H^1(g \circ f) = H^1(g) \circ H^1(f)$, because we can simply form the composition of two successive extension. $\square$

The next lemma can be understood as a consistency check of the third axiom of sheaf cohomology (A3). The condition $H^1(X,\mathcal{I}) = 0$ for an injective sheaf $\mathcal{I}$ implies in particular (using the long exact sequence) that every short exact sequence of sheaves $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ gives rise to a short exact sequence of global sections. This can be verified directly, and will be used in the proof of existence of sheaf cohomology.

Lemma 6.21. *Let* $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *be a short exact sequence of sheaves, such that* $\mathcal{I}$ *is injective. Then the sequence of global sections*

$$0 \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$

is exact.

Proof. Using injectivity of $\mathcal{I}$, we obtain a morphism of sheaves $r\colon \mathcal{F} \longrightarrow \mathcal{I}$, such that

$$\begin{tikzcd} \mathcal{I} \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & \mathcal{G} \arrow[dl, dashed, "\exists s"] \\ \mathcal{I} & \end{tikzcd}$$

commutes. This implies that the short exact sequence $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ splits (see E5, 1(d) for a similar exercise), that is $\mathcal{G} \cong \mathcal{I} \oplus \mathcal{H} = \mathcal{I} \times \mathcal{H}$. But $H^0(X, \mathcal{I} \oplus \mathcal{H}) \cong H^0(X, \mathcal{I}) \oplus H^0(X, \mathcal{H})$, and therefore we see that the map $H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H})$ is indeed surjective, and the sequence above thus exact. $\square$

Lemma 6.22. *For every short exact sequence of sheaves* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *we have a long exact sequence* $0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H})$, *where* $H^1(X, -)$ *is the functor defined in Lemma 6.20.*

Proof. In the exercises (see E7, ex. 5) it is shown that there exist embeddings into injective sheaves $\mathcal{F} \hookrightarrow \mathcal{I}$, $\mathcal{G} \hookrightarrow \mathcal{J}$, and $\mathcal{H} \hookrightarrow \mathcal{K}$, such that we have the following commutative diagram with exact rows and exact columns.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{J} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}/\mathcal{F} \arrow[r] \arrow[d] & \mathcal{J}/\mathcal{G} \arrow[r] \arrow[d] & \mathcal{K}/\mathcal{H} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Applying the functor $H^0(X, -)$ to the lower two rows we obtain the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{I}) \arrow[r] \arrow[d] & H^0(X, \mathcal{J}) \arrow[r] \arrow[d] & H^0(X, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(X, \mathcal{I}/\mathcal{F}) \arrow[r] & H^0(X, \mathcal{J}/\mathcal{G}) \arrow[r] & H^0(X, \mathcal{K}/\mathcal{H}). & \end{tikzcd}$$

The first row is exact by virtue of Lemma 6.21. The Snake Lemma gives rise to the required long exact sequence $0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \longrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H})$.
$\square$

Theorem 6.23. *There exists a unique (up to a unique isomorphism) formalism of sheaf cohomology as in Definition 6.19.*

Proof. We will prove by induction on the degree i, that there exists a family of functors $H^i(X,-)\colon \mathsf{Sh}(X) \longrightarrow \mathsf{AbGrp}$, verifying the axioms $(A1-3)$ up to the given degree. For $i \leq 1$ we know that such a family exists by virtue of Lemma 6.20. In this lemma we verified explicitly that (A1-2) are satisfied, and (A3) holds by definition of the functor $H^1(X,-)$. Indeed, if $\mathcal{I}$ is an injective sheaf, we may consider the trivial embedding $\mathrm{id}_{\mathcal{I}}\colon \mathcal{I} \hookrightarrow \mathcal{I}$, and hence obtain $H^1(X,\mathcal{I}) = \mathrm{coker}(H^1(X,\mathcal{I}) \xrightarrow{\mathrm{id}} H^0(X,\mathcal{I})) = 0$.

For $i \geq 1$ we define $H^{i+1}(X,\mathcal{F}) = H^i(X,\mathcal{I}/\mathcal{F})$, where $\mathcal{F} \hookrightarrow \mathcal{I}$ is an embedding into an injective sheaf (according to E7, ex. 5 this is always possible). For a morphism of sheaves $\mathcal{F} \xrightarrow{f} \mathcal{G}$ we choose a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I} \arrow[d, dashed] \\ \mathcal{G} \arrow[r, hook] & \mathcal{J}. \end{tikzcd} \tag{11}$$

as in our construction of the functor $H^1(X,-)$ in Lemma 6.20. We will use the same strategy as in *loc. cit.* to verify that the induced map

$$\begin{tikzcd} H^{i+1}(X,\mathcal{F}) \arrow[r] \arrow[d, equal] & H^{i+1}(X,\mathcal{G}) \arrow[d, equal] \\ H^i(X,\mathcal{I}/\mathcal{F}) \arrow[r] & H^i(X,\mathcal{J}/\mathcal{G}) \end{tikzcd}$$

is independent of the choice of the dashed morphism. Again we denote by $g,h\colon \mathcal{I} \longrightarrow \mathcal{J}$ two morphisms of sheaves, fitting into the commutative diagram (11). Their difference $g-h$ satisfies $(g-h)|_{\mathcal{F}} = 0$ by commutativity. Therefore, we obtain a factorisation

$$\begin{tikzcd} \mathcal{I} \arrow[r] \arrow[d, "g-h"'] & \mathcal{I}/\mathcal{F} \arrow[d, "g-h"] \arrow[dl, dotted] \\ \mathcal{J} \arrow[r] & \mathcal{J}/\mathcal{G}. \end{tikzcd} \tag{12}$$

as indicated by the dotted arrow. Applying the functor $H^i(X,-)$, we obtain a commutative diagram

$$\begin{tikzcd} H^(X,\mathcal{I}) \arrow[r] \arrow[d, "g-h"'] & H^i(\mathcal{I}/\mathcal{F}) \arrow[d, "g-h"] \arrow[dl, dotted] \\ H^i(X,\mathcal{J}) \arrow[r] & H^i(X,\mathcal{J}/\mathcal{G}). \end{tikzcd} \tag{13}$$

By the induction hypothesis we have $H^i(X,\mathcal{J}) = 0$. Therefore, we see that $g-h$ induces the zero morphism. As in the proof of Lemma 6.20 we conclude that $H^{i+1}(X,-)$ is a functor.

It remains to verify the axioms $(A2-3)$. If $\mathcal{I}$ is an injective sheaf, then we obtain for $i \geq 1$, $H^{i+1}(X,\mathcal{I}) = H^i(X,\mathcal{I}/\mathcal{I}) = 0$. This shows that our family of functors satisfies (A3).

Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be a short exact sequence of sheaves. It has been shown in the exercises (E7, ex. 5) that there exists a commutative diagram with exact rows and columns, such that $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are injective sheaves.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{J} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}/\mathcal{F} \arrow[r] \arrow[d] & \mathcal{J}/\mathcal{G} \arrow[r] \arrow[d] & \mathcal{K}/\mathcal{H} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{14}$$

For $i \geq 2$ we may simply apply the existence of the long exact sequence for $H^{i-1}(X,-)$ and $H^i(X,-)$, to obtain

$$\begin{tikzcd}
H^{i-1}(X,\mathcal{I}/\mathcal{F}) \arrow[r] \arrow[d, equal] & H^{i-1}(X,\mathcal{J}/\mathcal{G}) \arrow[r] \arrow[d, equal] & H^{i-1}(X,\mathcal{K}/\mathcal{H}) \arrow[r] \arrow[d, equal] & H^i(X,\mathcal{I}/\mathcal{F}) \arrow[r] \arrow[d, equal] & H^i(X,\mathcal{J}/\mathcal{G}) \arrow[r] \arrow[d, equal] & H^i(X,\mathcal{K}/\mathcal{H}) \arrow[d, equal] \\
H^i(X,\mathcal{F}) \arrow[r] & H^i(X,\mathcal{G}) \arrow[r] & H^i(X,\mathcal{H}) \arrow[r] & H^{i+1}(X,\mathcal{F}) \arrow[r] & H^{i+1}(X,\mathcal{G}) \arrow[r] & H^{i+1}(X,\mathcal{H}).
\end{tikzcd}$$

For $i = 1$ we have to be more careful, because $H^1(X,\mathcal{F})$ is not equal to $H^0(X,\mathcal{I}/\mathcal{F})$ in general, but rather to $\operatorname{coker}(H^0(X,\mathcal{I}) \longrightarrow H^0(X,\mathcal{I}/\mathcal{F}))$. We obtain the corresponding part of the long exact sequence as follows. Consider the bottom two rows of (14). Lemma 6.20 implies that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{I}) \arrow[r] \arrow[d, "\alpha"] & H^0(X,\mathcal{J}) \arrow[r] \arrow[d, "\beta"] & H^0(X,\mathcal{K}) \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H^0(X,\mathcal{I}/\mathcal{F}) \arrow[r] & H^0(X,\mathcal{J}/\mathcal{G}) \arrow[r] & H^0(X,\mathcal{K}/\mathcal{H}) \arrow[r] & H^1(X,\mathcal{I}/\mathcal{F}) \arrow[r] & H^1(X,\mathcal{J}/\mathcal{G}) \arrow[r] & H^1(X,\mathcal{K}/\mathcal{H})
\end{tikzcd}$$

with exact rows. Taking cokernels of the vertical arrows, a simple diagram chase verifies that the sequence

$$\begin{tikzcd}
\operatorname{coker}\alpha \arrow[r] \arrow[d, equal] & \operatorname{coker}\beta \arrow[r] \arrow[d, equal] & \operatorname{coker}\gamma \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{I}/\mathcal{F}) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{J}/\mathcal{G}) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{K}/\mathcal{H}) \arrow[d, equal] \\
H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{G}) \arrow[r] & H^1(X,\mathcal{H}) \arrow[r] & H^2(X,\mathcal{F}) \arrow[r] & H^2(X,\mathcal{G}) \arrow[r] & H^2(X,\mathcal{H})
\end{tikzcd}$$

is exact as required. $\square$

7 GAGA

In this section we will finally state and (partially) prove Serre's GAGA theorems. GAGA, the abbreviation of *Géométrie algébrique et géométrie analytique*, usually refers to the following three theorems (which we state in vague terms, and in a slightly less general form for the purpose of this introduction) and their corollaries.

Theorem 7.1 (Serre's GAGA)**.** *Let X be a complex projective manifold.*

(a) Let E be an algebraic vector bundle on X, then every global holomorphic section of E is algebraic.

(b) If E and F are algebraic vector bundles on X, then every fibrewise linear holomorphic morphism of vector bundles $E \longrightarrow F$ is algebraic.

(c) Every holomorphic vector bundle E on X arises from an algebraic vector bundle in an essentially unique way.

We will prove (a) and (b) completely in this class. The third result (c) requires deeper analytic results, but we will nonetheless try to gain an understanding why it is true. Combining the three statements above, one sees that the category of algebraic and holomorphic vector bundles on X are equivalent. This has the curious interpretation that somehow that compactness of a complex projective manifold X forces holomorphic vector bundles and their sections to be algebraic.

In order to understand this theorem we have to make precise what we mean by *algebraic*. That is the content of the first paragraph below. We will show that for so-called vector bundles E, there exist complex vector spaces $H^i_{\mathrm{alg}}(X, E)$ of algebraic degree i cohomology classes. For $i = 0$ this vector space agrees with the space of algebraic (or regular) sections of E. Serre proves the first part of the theorem above by a downward induction on the degree i. We will show that for $i > \dim_{\mathbb{C}} X$ the cohomology spaces $H^i(X, E)$ and $H^i_{\mathrm{alg}}(X, E)$ will be both 0. The inductive comparison is anchored to this case. The long exact sequence of sheaf cohomology, and an ingenious application of the Five Lemma allow us to deduce that $H^i(X, E) \cong H^i_{\mathrm{alg}}(X, E)$ the two types of cohomology groups agree canonically.

The argument contains a second inductive strand: over the complex dimension. In fact, it turns out to be easier to prove a more general comparison theorem, where vector bundles are replaced by so-called *coherent sheaves*. This way one can reduce everything to the case $X = \mathbb{P}^n$. Recall that vector bundles on X correspond to locally free sheaves of $\mathcal{O}_X$-modules on X. A coherent sheaf is a locally finitely generated sheaf of $\mathcal{O}_X$-modules. If E is a vector bundle on X, and $i\colon X \hookrightarrow \mathbb{P}^n$ denotes the embedding into projective space, then the sheaf $i_*\underline{E}$ is not locally free, unless $X = \mathbb{P}^n$ or $X = \emptyset$. However, $i_*\underline{E}$ is always coherent. While this argument is formally very pleasing, we'll try to stay within the realm of complex manifolds and vector bundles as long as possible.

7.1 Projective algebraic geometry in a nutshell

Let $G_1, \dots, G_k \in \mathbb{C}[z_0, \dots, z_n]$ be homogeneous polynomials of degree d_i in $n{+}1$ variables $z_0, \dots, z_n$. We denote by $V(G_1, \dots, G_k) \subset \mathbb{P}^n$ the subset

$$\{(z_0 : \dots : z_n) \in \mathbb{P}^n \,|\, G_i(z_0, \dots, z_n) = 0 \; \forall i = 1, \dots, k\},$$

and refer to subsets of $\mathbb{P}^n$ of this type as *complex projective varieties.* In Proposition 3.11 we proved that if

$$\left(\frac{\partial G_i}{\partial z^j}(w)\right)_{j=0,\ldots,m}^{i=,1,\ldots,k} \text{ is a surjective linear map,} \tag{15}$$

then $V(G_1,\ldots,G_k)$ is a complex submanifold.

In this section we aim to turn the subset $V(G_1,\ldots,G_k) \subset \mathbb{P}^n$ into a mathematical object (an algebraic variety) itself. We will see that there exists a topology (the *Zariski topology*) on $V(G_1,\ldots,G_k)$, which is coarser than the standard topology on subsets of $\mathbb{P}^n$, but better adapted to algebraic geometry. The Zariski topology carries a sheaf of rings $\mathcal{O}_X^{\mathrm{reg}}$, the so-called regular sections of $\mathcal{O}_X$. Regular sections can be described in terms of polynomials and rational functions, and are therefore of algebraic nature.

Definition 7.2. *Let $V = V(G_1,\ldots,G_k)$ be a projective variety. A subset $U \subset V$ is called* Zariski open*, if there exists a projective variety $W = W(H_1,\ldots,H_\ell)$, such that $U = V \setminus W$.*

Since a projective variety $W(H_1,\ldots,H_\ell) \subset \mathbb{P}^n$ is closed in the standard topology, every Zariski open subset is open in the standard topology. However, Zariski open subsets are called open in their own right.

Lemma 7.3. *The set of Zariski open subsets defines a topology on the set $V = V(G_1,\ldots,G_k) \subset \mathbb{P}^n$.*

Proof. The empty subset $\emptyset \subset V$ is Zariski open, because $\emptyset = V \setminus V$. Similarly, we have that V is Zariski open, because $V = V \setminus \emptyset = V \setminus V(1)$, where 1 denotes the constant homogeneous polynomial of degree 0.

If U_1, U_2 are Zariski open subsets, that is, $U_i = V \setminus W_i$, then $U_1 \cap U_2 = V \setminus (W_1 \cup W_2)$. Writing $W_i = V(H_{1i},\ldots,H_{\ell i})$ (possibly repeating polynomials several times), we have $W_1 \cup W_2 = V(H_{i1} \cdot H_{j2})_{i,j=1,\ldots,\ell}$.

For $i \in I$ we assume that $U_i = V \setminus W_i = V(H_{1i},\ldots,H_{\ell_i i})$ is a Zariski open subset. Then $\bigcup_{i\in I} U_i = V \setminus \bigcap_{i\in I} W_i$. If I is a finite set, then it is clear that

$$\bigcap_{i\in I} W_i = V(H_{1i},\ldots,H_{\ell_i i})_{i\in I}$$

is again a projective variety. For the general case, we have to use that the ring $\mathbb{C}[z_0,\ldots,z_n]$ is Noetherian, that is, every ideal is finitely generated. We consider the ideal J generated by $(H_{1i},\ldots,H_{\ell_i i})_{i\in I}$, and choose finitely many homogeneous generators $H_1,\ldots,H_\ell$. We then have $\bigcap_{i\in I} W_i = V(H_1,\ldots,H_\ell)$. $\square$

If $X = V(G_1,\ldots,G_k)$ is a projective variety, we denote the topological space induced by the Zariski topology by X_{Zar}. There is a natural sheaf of rings on X_{Zar}.

Definition 7.4. *Let X_{Zar} be the topological space associated to a projective variety, let $U \subset X$ be a Zariski open subset. A function $U \xrightarrow{f} \mathbb{C}$ is called* regular*, if there exists an open covering $U = \bigcup_{i\in I} U_i$, such that for every $i \in I$, there exist homogeneous polynomials G, $H \in \mathbb{C}[z_0,\ldots,z_n]_d$ of degree d ($H \neq 0$), such that $\forall (z_0 : \ldots : z_n) \in U_i$ we have $f(z_0 : \ldots : z_n) = \frac{G}{H}(z_0,\ldots,z_n)$. We denote the sheaf of regular functions on X_{Zar} by $\mathcal{O}_X^{\mathrm{reg}}$.*

A regular function is precisely a holomorphic function, which can be locally expressed in terms of rational functions. It is important to emphasise that f is not strictly meromorphic, that is, doesn't have any poles. The domain of definition of f excludes by definition the closed subvariety of zeroes of the denominator H.

We have seen that every Zariski open subset $U \subset X_{\mathrm{Zar}}$ is also open with respect to the standard (or manifold topology if X is smooth) of $X \subset \mathbb{P}^n$. On U we can compare the rings $\mathcal{O}_X(U)$ and $\mathcal{O}^{\mathrm{reg}}_{X_{\mathrm{Zar}}}(U)$. By definition, $\mathcal{O}^{\mathrm{reg}}_{X_{\mathrm{Zar}}}(U) \subset \mathcal{O}_X(U)$ is a subring. However, we are not allowed to call $\mathcal{O}^{\mathrm{reg}}_{X_{\mathrm{Zar}}}$ a subsheaf of $\mathcal{O}_X$, since the sheaves live on different topological spaces.

Definition 7.5. *An* algebraic vector bundle *of rank n on a complex projective manifold X is a locally free $\mathcal{O}_{X^{\mathrm{reg}}_{\mathrm{Zar}}}$-module $\mathcal{E}$ of rank n.*

This definition is inspired by Theorem 5.12, which showed that the theory of holomorphic vector bundles is equivalent to locally free sheaves of $\mathcal{O}_X$-modules of rank n. There is another definition, which justifies calling them vector bundles.

Exercise 7.6. *Show that every locally free $\mathcal{O}_{X^{\mathrm{reg}}_{\mathrm{Zar}}}$-module $\mathcal{E}^{\mathrm{reg}}$ of rank n, is given by the sheaf of* regular *sections of a complex vector bundle $E \xrightarrow{\pi} X$, which is algebraic in the following sense. There exists a cocycle datum $(\{U_i\}_{i \in I}, (\phi_{ij})_{(i,j) \in I^2})$, such that every U_i is a Zariski open subset, and $\phi_{ij} \colon U_{ij} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is a regular morphism. A section s of E is then called* regular*, if it corresponds to a tuple of regular functions $f_i \colon U_i \longrightarrow \mathbb{C}^n$, such that $\phi_{ij} f_j = f_i$ for all $i, j \in I$.*

An algebraic vector bundle of rank 1 is also called an algebraic line bundle. The most important example of algebraic line bundles are the twisting line bundles on $\mathbb{P}^n$.

Definition 7.7. *Recall that the standard atlas of projective space consists only of Zariski open subsets $U_i = \{(z_0 : \ldots : z_n) \in \mathbb{P}^n \,|\, z_i \neq 0\}$. Moreover, the cocycle datum, $(\{U_i\}_{i=0,\ldots,n}, (\frac{z_j^d}{z_i^d})_{i,j=1,\ldots,n})$ of the line bundles L_d, is the cocycle datum of an algebraic line bundle on $\mathbb{P}^n$. We denote its sheaf of regular sections by $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)$, and its sheaf of holomorphic sections by $\mathcal{O}_{\mathbb{P}^n}(d)$.*

We have already computed the complex vector space of global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, and have shown that it is canonically equivalent to the complex vector space of homogeneous degree d polynomials. Given the algebraic nature of homogeneous polynomials, one suspects immediately that that is a first instance of the GAGA principle. To verify this suspicion we will compute the vector space of global sections $\Gamma(\mathbb{P}^n, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d))$, which are regular.

Note that a holomorphic section $s \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^d}(d))$ corresponds to a tuple of holomorphic functions $f_i \colon U_i \longrightarrow \mathbb{C}$, for $i = 0, \ldots, n$, such that $f_i = \frac{z_j^d}{z_i^d} f_j$ for all i, j. A holomorphic function is regular, if the functions f_i are regular.

Lemma 7.8. *The canonical map $\Gamma(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)) \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ is an equivalence.*

Proof. It is clear that this map is injective. It remains therefore to show that it is surjective. We have computed the space of holomorphic global sections in Theorem 4.15. It is therefore sufficient to show that every holomorphic global section of Theorem 4.15 is regular.

Given a degree d homogeneous polynomial $G = G(z_0, \ldots, z_n)$, one associates to G a holomorphic section $s \colon \mathbb{P}^n \longrightarrow L_d$ as follows. Recall that L_d was defined to be the quotient space of $(\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}$ by the action of $\mathbb{C}^\times$, where

$$\lambda \cdot (z_0, \ldots, z_n, w) = (\lambda z_0, \ldots, \lambda z_n, \lambda^d w).$$

The map $s_G\colon \mathbb{P}^n \longrightarrow L_d$ sends $(z_0 : \ldots : z_n)$ to $[(z_0, \ldots, z_n, G(z_0, \ldots, z_n))]$. Let us see what holomorphic function this section corresponds to with respect to the trivialisation $\alpha_i\colon L_d|_{U_i} \xrightarrow{\simeq} U_i \times \mathbb{C}$, which sends $[(z_0, \ldots, z_n, w)]$ to $[((z_0 : \ldots : z_n), \frac{w}{z_i^d})]$. We obtain $\alpha_i(s_G)(z_0 : \ldots : z_n) = \frac{G(z_0,\ldots,z_n)}{z_i^d}$. But these are regular functions. This concludes the proof. $\square$

The next construction we will take a look at is twisting $E(d)$ of algebraic vector bundles E. The reason is that an algebraic vector bundle $E(d)$ might not have any (regular nor holomorphic) sections at all. By sufficient twisting of the vector bundle, we can ensure to obtain a lot of global sections. We begin a definition using cocycles, and subsequently we will reformulate it in terms of tensor products (which has the advantage of being formally better behaved).

Definition 7.9. *Let E be an algebraic vector bundle on $\mathbb{P}^n$, corresponding to a cocycle datum $(\{U_i\}_{i=0,\ldots,n}, (\phi_{i,j})$, where $\{U_i\}_{i=0,\ldots,n}$ is the standard open covering of $\mathbb{P}^n$ (it is a priori not clear that such a presentation always exist, yet it is true). We define $E(d)$ to be the algebraic vector bundle corresponding to the twisted cocycle datum $(\{U_i\}_{i=0,\ldots,n}, \frac{z_j^d}{z_i^d}\phi_{ij}))$.*

As we remarked above, it is a priori not clear that this definition can be applied to every algebraic vector bundle. Moreover, it is not clear that it is independent of the chosen presentation via coycles. We will remedy this drawback by making a small detour through the world of sheaves and tensor products.

Lemma 7.10. *Let R be a (commutative) ring (with unit), and M, N be R-modules. There exists an R-module $M \otimes_R N$, and a bilinear map $- \otimes -\colon M \times N \longrightarrow M \otimes_R N$, such that for every bilinear map of R-modules $f\colon M \times N \longrightarrow P$, there exists a unique linear map $M \otimes_R N \longrightarrow P$, such that*

$$\begin{tikzcd} M \times N \arrow[d, "-\otimes-"'] \arrow[dr, "f"] & \\ M \otimes_R N \arrow[r, dashed, "\exists!"] & P \end{tikzcd}$$

commutes.

Proof. We define $M \otimes_R N$ as a quotient F/S, where F is the free R-module generated by formal symbols $m \otimes n$ for every pair $(m, n) \in M \times N$. The R-module S is defined to be the submodule of F, generated by elements $(m + m') \otimes n - m \otimes n - m' \otimes n$, $m \otimes (n + n') - m \otimes n - m \otimes n'$, $(\lambda m) \otimes n - m \otimes (\lambda n)$, and $\lambda(m \otimes n) - (\lambda m) \otimes n$ for every possible choice of elements $m, m' \in M$, $n, n' \in N$, and $\lambda \in R$. The map $M \times N \longrightarrow F/S = M \otimes_R S$, sending (m, n) to $m \otimes n$ is by definition bilinear. Moreover, if $f\colon M \times N \longrightarrow P$ is a bilinear map, then we obtain a linear map $F \longrightarrow P$, by sending $m \otimes n \mapsto f(m, n)$. Since $f|_S = 0$, there is an induced linear map $g\colon F/S = M \otimes_R N \longrightarrow P$. By construction it fits into the commutative diagram above, that is, we have $g(m \otimes n) = f(m, n)$. This takes care of the existence part of the assertion. Uniqueness is also clear, since commutativity of the diagram enforces the relation $g(m \otimes n) = f(m, n)$, and every element in $M \otimes_R N$ can be written as an R-linear combination of symbols $m \otimes n$. $\square$

Corollary 7.11. *The tensor product of free R-modules of ranks m and n is free of rank mn.*

Proof. Since objects satisfying universal properties are unique up to a unique isomorphism, it suffices to show that we have a universal bilinear map $- \otimes -\colon R^m \times R^n \longrightarrow R^{mn}$. We will denote the standard basis of R^m by $e_1, \dots, e_n$, and of R^n by $f_1, \dots, f_n$. A basis of R^{mn} is given by formal symbols $e_i \otimes f_j$, and we define a bilinear map $R^m \times R^n$, by sending (e_i, f_j) to $e_i \otimes f_j$.

Given a bilinear map $f\colon R^m \times R^n \longrightarrow P$, where P is an arbitrary R-module, we use that f is uniquely determined by the values $f(e_i, f_j)$. We have a linear map $g\colon R^{mn} \longrightarrow P$, by sending $e_i \otimes f_j$ to $f(e_i, f_j)$. By construction it verifies commutativity as above and is the only linear map $R^{mn} \longrightarrow P$ with this property. $\square$

As a corollary we obtain a tensor product operation for sheaves of modules, preserving locally free ones of finite rank. For locally free sheaves of $\mathcal{O}_X$-modules of finite rank this can be understood to be a tensor product operation for holomorphic vector bundles.

Lemma 7.12. *Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X (commutative and with unit). For two sheaves of $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$, there exists a unique (up to a unique isomorphism) $\mathcal{R}$-module $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$, and a bilinear sheaf morphism $- \otimes -\colon \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$, such that for every bilinear map of sheaves $f\colon \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{P}$, there exists a unique linear sheaf morphism $g\colon \mathcal{M} \otimes_{\mathcal{R}} \mathcal{N} \longrightarrow \mathcal{P}$, such that the diagram*

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{N} \arrow[d, "- \otimes -"'] \arrow[dr, "f"] & \\
\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N} \arrow[r, dashed, "\exists!"] & \mathcal{P}
\end{tikzcd}$$

commutes. If $\mathcal{M}$ and $\mathcal{N}$ are locally free of finite rank, then so is $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$.

Proof. We define a presheaf $(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})^{\mathrm{pre}}$, which sends an open subset $U \subset X$ to the $\mathcal{R}(U)$-module $\mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{N}(U)$. Applying Lemma 7.10 for every $U \subset X$, we see that we obtain a unique linear maps $g_U\colon \mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{N}(U) \longrightarrow \mathcal{P}(U)$, such that $g_U(m \otimes n) = f_U(m, n)$ for sections $m \in \mathcal{M}(U)$, $n \in \mathcal{N}(U)$. For an inclusion of open subsets $U \subset V$, we therefore see that the two maps $r_U^V \circ g_V$ and g_U must agree, because they both satisfy the commutativity condition $(r_U^V \circ g_V)(m \otimes n) = (r_U^V \circ f_V)(m, n) = f_U(m, n)$, and $g_U(m \otimes n) = f_U(m, n)$. Therefore we obtain a commutative diagram of presheaves

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{N} \arrow[d, "- \otimes -"'] \arrow[dr, "f"] & \\
(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})^{\mathrm{pre}} \arrow[r, dashed, "\exists!"] & \mathcal{P}.
\end{tikzcd}$$

Defining $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ to be the sheafification of $(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})^{\mathrm{pre}}$, and using the universal property of sheafification, we see that we there is a unique morphism of sheaves $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N} \longrightarrow \mathcal{P}$, such that the following diagram of presheaves

$$\begin{tikzcd}
& \mathcal{M} \times \mathcal{N} \arrow[dl, "- \otimes -"'] \arrow[d, "- \otimes -"] \arrow[dr, "f"] & \\
(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})^{\mathrm{pre}} \arrow[r, dashed, "\exists!"] & \mathcal{M} \otimes_{\mathcal{R}} \mathcal{N} \arrow[r, dashed, "\exists!"] & \mathcal{P}.
\end{tikzcd}$$

commutes.

It remains to show that $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ is locally free of finite rank, when $\mathcal{M}$ and $\mathcal{N}$ are. Let $U \subset X$ be an open subset, such that $\mathcal{M}|_U \cong (\mathcal{R}|_U)^m$ and $\mathcal{N}|_U \cong (\mathcal{R}|_U)^n$. We know that X has an open covering consisting of open subsets U with this property. By corollary 7.11 we have that $(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})^{\mathrm{pre}}|_U \cong (\mathcal{R}|_U)^{mn}$. Since the right hand side is already a sheaf, it agrees with its sheafification $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$. This shows that $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ is locally free of finite rank. □

? The reader might be curious if sheafification was really necessary. To see that this is indeed the case, consider the example $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. The cocycle pictures implies that this sheaf is equivalent to $\mathcal{O}_{\mathbb{P}^1}$. Hence its space of global sections is 1-dimensional. However, $\mathcal{O}_{\mathbb{P}^1}(-1)$ is known not to have any non-zero global sections. Therefore, the tensor product $\mathcal{O}_{\mathbb{P}^1}(-1)(\mathbb{P}^1) \otimes_{\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)} \mathcal{O}_{\mathbb{P}^1}(1) = 0$.

Definition 7.13. *Let E be an algebraic vector bundle on $\mathbb{P}^n$, that is a locally free sheaf of $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-modules of finite rank. For an integer $d \in \mathbb{Z}$ we define $E(d)$ to be the tensor product $E \otimes_{\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}} \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)$.*

The reason to introduce this twisting operation is the following important result.

Theorem 7.14. *Let E be an algebraic vector bundle on $\mathbb{P}^n$. There exists an integer $d_0 \in \mathbb{Z}$, such that for $d \geq d_0$ the algebraic vector bundle $E(d)$ is generated by finitely global (regular) sections $s_1, \ldots, s_m \in \Gamma^{\mathrm{reg}}(\mathbb{P}^n, E(d))$. That is, the map of sheaves $\mathcal{O}^m \longrightarrow E(d)$, sending a tuple of regular functions $(f_1, \ldots, f_m)$ to $f_1 s_1 + \cdots + f_m s_m$ is a surjective morphism of sheaves.*

Proof. Since E is a locally free sheaf of $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-modules, there exist Zariski open subsets $\bigcup_{i \in I} V_i = \mathbb{P}^n$, such that $E|_{V_i} \cong (\mathcal{O}^{\mathrm{reg}}_{V_i})^m$, for some integer m (the rank of E). The topological space $\mathbb{P}^n$ is compact, hence we may assume that I is a finite set. For every $i \in I$ we choose sections $s^i_1, \ldots, s^i_m$ which form a basis for $E|_{U_i}$.

We will show that for each s^i_j there exists a homogeneous polynomial $G(z_0, \ldots, z_n)$ of degree $d >> 0$, such that the section $s^i_j \otimes s_G$ of $E(d)$ extends from U_i to all of $\mathbb{P}^n$. The intuitive idea is that locally s^i_j has a presentation as a rational function, and the zeroes of the denominator obstruct us from extending s^i_j beyond U_i. Since we are only dealing with finitely many sections, we can choose d big enough, to find homogeneous polynomials G^i_j of degree d, such that $s^i_j \otimes s_{G^i_j}$ always extends to a global section of $E(d)$.

So let $s \in E(U)$, where U is a Zariski open subset of $\mathbb{P}^n$. We will show that there exists a homogeneous polynomial G, such that $s \otimes s_G$ extends to a global section of $E(d)$. Let $x \in \mathbb{P}^n \setminus U$, and choose Zariski open neighbourhood $V \subset \mathbb{P}^n$ of x. By making V smaller, we can assume that $E|_V \cong (\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n})^m$ is a free $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-module. The section $s|_V$ can therefore be represented as a tuple of regular functions $(f_1, \ldots, f_m) = (\frac{F_1}{G_1}, \ldots, \frac{F_m}{G_m})$. This shows that $s \otimes s_{G_1 \cdots G_m}$ extends uniquely from a section over $U \cap V$ to a section over V.

Using compactness of $\mathbb{P}^n$, one covers $\mathbb{P}^n$ by finitely many open subsets where an extension across U exists, after tensoring by a homogeneous polynomial. Taking the maximal degree, we can patch these local extensions together. □

At this point we can take a glimpse at the proof of the first assertion of GAGA. We will focus on the case of algebraic vector bundles on $\mathbb{P}^n$. Let's take the following facts for granted.

(a) For every algebraic vector bundle E on $\mathbb{P}^n$ we have a natural map $H^i(\mathbb{P}^n_{\mathrm{Zar}}, E) \longrightarrow H^i(\mathbb{P}^n, \underline{E})$, mapping from the cohomology of the sheaf of regular sections to the cohomology of the sheaf of holomorphic sections. (GAGA asserts that this map is an isomorphism.)

(b) For the line bundles $\mathcal{O}^{\text{reg}}_{\mathbb{P}^n}$ the map from (a) is an isomorphism (GAGA holds for these line bundles).

(c) For every short exact sequence $0 \longrightarrow K \longrightarrow E \longrightarrow F \longrightarrow 0$, with E, F, algebraic vector bundles, K is an algebraic vector bundle too.

(d) Both algebraic, and holomorphic cohomology groups of algebraic vector bundles vanish for degrees $i > m$.

If E is an arbitrary algebraic vector bundle on $\mathbb{P}^n$, then we have seen that there exists a $d \in \mathbb{Z}$, such that we have a surjective map of sheaves $(\mathcal{O}^{\text{reg}}_{\mathbb{P}^n})^m \twoheadrightarrow E(d)$. Tensoring with $\mathcal{O}^{\text{reg}}_{\mathbb{P}^n}(-d)$ we obtain a surjective morphism of sheaves $(\mathcal{O}^{\text{reg}}_{\mathbb{P}^n}(-d))^m \twoheadrightarrow E$. Let's denote the kernel by K. By fact (c) we have a short exact sequence of algebraic vector bundles

$$0 \longrightarrow K \longrightarrow (\mathcal{O}^{\text{reg}}_{\mathbb{P}^n}(-d))^m \longrightarrow E \longrightarrow 0.$$

Let's take a look at the long exact sequences for cohomology of regular sections and holomorphic sections:

$$\begin{tikzcd}
H^i(\mathbb{P}^n_{\text{Zar}}, K) \arrow[r] \arrow[d, "f"] & H^i(\mathbb{P}^n_{\text{Zar}}, (\mathcal{O}^{\text{reg}}_{\mathbb{P}^n}(-d))^m) \arrow[r] \arrow[d, "g"] & H^i(\mathbb{P}^n_{\text{Zar}}, E) \arrow[r] \arrow[d, "h"] & H^{i+1}(\mathbb{P}^n_{\text{Zar}}, E) \arrow[r] \arrow[d, "k"] & H^{i+1}(\mathbb{P}^n_{\text{Zar}}, K) \arrow[d, "\ell"] \\
H^i(\mathbb{P}^n, K) \arrow[r] & H^i(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}(-d))^m) \arrow[r] & H^i(\mathbb{P}^n, E) \arrow[r] & H^{i+1}(\mathbb{P}^n, E) \arrow[r] & H^{i+1}(\mathbb{P}^n, K)
\end{tikzcd}$$

For very high degrees both types of cohomology groups vanish. We will therefore prove by descending induction on i that $H^i(\mathbb{P}^n_{\text{Zar}}, E) \longrightarrow H^i(\mathbb{P}^n, E)$ is an isomorphism. Let's assume that this has been proven for $i+1$ already. That is, the maps k and ℓ are isomorphisms. Since g is by assumption also bijective, we see from the Five Lemma that h is surjective.

But K is also a vector bundle, therefore we see that the map f has to be surjective too. Another application of the Five Lemma gives us that h is an isomorphism.

7.2 Coherent sheaves

In this paragraph we discuss the theory algebraic *coherent* sheaves. This will allow us to reduce the entire statement of GAGA to a statement about sheaves on $\mathbb{P}^n$ rather than a complex submanifold $X \subset \mathbb{P}^n$ given by a smooth projective variety X. Let $i \colon X \hookrightarrow \mathbb{P}^n$ be the canonical inclusion map. A vector bundle on X can be understood as a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on X of finite rank. We can push the sheaf $\mathcal{E}$ forward to $\mathbb{P}^n$, that is, consider $(i_* \mathcal{E})(U) = \mathcal{E}(i^{-1}(U)) = \mathcal{E}(U \cap X)$.

We have verified on E6 that this construction defines a sheaf. We know that $i_* \mathcal{E}$ cannot be locally free in general, since for $V \subset \mathbb{P}^n \setminus X$ we have $(i_* \mathcal{E})(V) = \mathcal{E}(\emptyset) = 0$. Therefore, $i_* \mathcal{E}\,|_{\mathbb{P}^n \setminus X} = 0$, which could only happen for a locally free sheaf, if it was of rank 0.

However, the pushforward $i_* \mathcal{E}$ turns out to be coherent. Versions of the theory of coherent sheaves exist in both algebraic and analytic geometry. We will content ourself with establishing the basic properties of algebraic coherent sheaves. The analytic case requires more care.

Before delving into this theory, we prove the following lemma, which will allow us to reduce the proof of GAGA to the case of $\mathbb{P}^n$.

Proposition 7.15. *Let $X \subset Y$ be a closed subspace of a topological space Y, we denote the inclusion map $X \hookrightarrow Y$ by i. For every sheaf $\mathcal{F}$ on Y we have a canonical equivalence*

$$H^i(X, \mathcal{F}) \cong H^i(Y, i_* \mathcal{F}).$$

The proof relies on a couple of lemmas.

Lemma 7.16. *For i as above, the pushforward functor i_* is exact, that is that it sends a short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ on X to a short exact sequence on Y.*

Proof. Recall that a sequence of sheaves on Y is exact, if and only if the sequence of stalks at y is exact for every $y \in Y$. So let's compute the stalks of $(i_* \mathcal{F})_y$. We claim that for $y \notin X$ the stalk is 0, and for $x \in X$, the stalk equals $\mathcal{F}_x$. This then immediately verifies the claim that the pushforward (along the embedding of a closed subspace) of an exact sequence is still exact.

It remains to compute the stalks of $i_* \mathcal{F}$. Recall that the element of the stalk of a sheaf $\mathcal{K}$ at y, are given by equivalence classes of pairs (U, s), where $U \subset Y$ is an open neighbourhood of x, and $s \in \mathcal{K}(U)$ is a section. Two pairs (U, s) and (V, t) are equivalent if $\exists x \in W \subset U \cap V$, with W open, such that $s|_W = t|_W$.

For $y \notin X$, we can choose a neighbourhood $V \subset Y \setminus X$ of y. Since every $s \in (i_* \mathcal{F})(U)$ equals 0. We conclude that $(i_* \mathcal{F})_x = 0$.

For $x \in X$, every open neighbourhood $U \subset X$ of x in X, can be realised as an intersection $U = U' \cap X$, where U' is an open neighbourhood of x in Y. The resulting abelian group $(i_* \mathcal{F})(U') = \mathcal{F}(U)$ is independent of the chosen U' with this property. This implies directly that $(i_* \mathcal{F})_x = \mathcal{F}_x$. $\square$

It's important to emphasise that for a general continuous map $f \colon X \longrightarrow Y$, the pushforward functor f_* is **not** exact. For $Y = \{pt\}$ a singleton, $\mathsf{Sh}(Y) = \mathsf{AbGrp}$, and f_* agrees with the global sections functor $\Gamma(X, -)$ with respect to this identification. We know from the example of the exponential sequence that this functor is not exact!

Lemma 7.17. *For a continuous map $f \colon X \longrightarrow Y$, the functor $f_* \colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y)$ has a left adjoint $f^{-1} \colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X)$. That is, for every sheaf $\mathcal{F} \in \mathsf{Sh}(Y)$ and $\mathcal{G} \in \mathsf{Sh}(X)$ we have*

$$\mathit{Hom}_{\mathsf{Sh}(X)}(f^{-1} \mathcal{F}, \mathcal{G}) \longrightarrow \mathit{Hom}_{\mathsf{Sh}(Y)}(\mathcal{F}, f_* \mathcal{G}).$$

Proof. We would like to define $(f^{-1} \mathcal{F})(U) = \mathcal{F}(f(U))$. But unless f is an open map, this doesn't make any sense a priori. We will therefore interpret this formula by taking the stalk of $\mathcal{F}$ around $f(U)$. That is we consider equivalence classes of pairs (V, s), where $V \subset Y$ is an open subset containing $f(U)$, and $s \in \mathcal{F}(V)$. The equivalence relation is defined as for stalks: if there exists an open subsets $W \subset V \cap V'$, such that $f(U) \subset W$, such that $t|_W = s|_W$, then (V, s) and (V', t) are equivalent.

The resulting presheaf $f^{-1}_{pre} \mathcal{F}$ is not a sheaf in general. But we know that this can be remedied by sheafification. This is how $f^{-1} \mathcal{F}$ is defined.

Let's check that every map $\mathcal{F} \xrightarrow{g} f_* \mathcal{G}$ gives rise to a map $f^{-1} \mathcal{F} \xrightarrow{h} \mathcal{G}$. Given g, we have for every $V \subset Y$ a homomorphism $\mathcal{F}(V) \xrightarrow{g_V} \mathcal{G}(f^{-1}(V))$. For an open subset $V \subset Y$, containing $f(U)$, and an element $[(V, s)] \in f^{-1} \mathcal{F}(U)$ we can therefore look at the image $g_V(s) \in \mathcal{G}(f^{-1}(V))$, and restrict it further down to U. This defines a morphism of presheaves $f^{-1}_{pre} \mathcal{F} \longrightarrow \mathcal{G}$. The universal property of sheafification yields a unique morphism $f^{-1} \mathcal{F} \longrightarrow \mathcal{G}$.

Vice versa, given h, we can construct a morphism of sheaves $\mathcal{F} \xrightarrow{g} f_* \mathcal{G}$. This is left as an exercise to the reader. $\square$

Proof of Proposition 7.15. We will show that for every short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{I} / \mathcal{F} \longrightarrow 0$ with $\mathcal{I}$ injective, the pushforward of this sequence

$$0 \longrightarrow i_* \mathcal{F} \longrightarrow i_* \mathcal{I} \longrightarrow i_*(\mathcal{I} / \mathcal{F}) \longrightarrow 0$$

is also a short exact sequence with $i_* \mathcal{I}$ injective. Exactness has already been verified in Lemma 7.16. Let's show that $i_* \mathcal{I}$ is injective.

A diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{N} \\ i_* \mathcal{I} & \end{tikzcd}$$

corresponds to a diagram

$$\begin{tikzcd} i^{-1} \mathcal{M} \arrow[r, hook] \arrow[d] & i^{-1} \mathcal{N} \arrow[dl, dashed, "\exists"] \\ \mathcal{I} & \end{tikzcd}$$

by Lemma 7.17. Since $\mathcal{I}$ is injective, there exists a dashed arrow as in the diagram above, rendering the triangle commutative. Applying Lemma 7.17 again, we obtain

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{N} \arrow[dl, dashed, "\exists"] \\ i_* \mathcal{I}, & \end{tikzcd}$$

and therefore see that $i_* \mathcal{I}$ is injective. $\square$

Coherent sheaves on $\mathbb{P}^n_{\mathrm{Zar}}$ are certain sheaves of $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-modules. The following examples are the most important ones for us: locally free sheaves of finite rank $\mathcal{E}$ (algebraic vector bundles), and also for every inclusion of a complex projective manifold $i \colon X \hookrightarrow \mathbb{P}^n$ the sheaves $i_* \mathcal{E}$ are coherent, where $\mathcal{E}$ is an algebraic vector bundle on X. There are however more examples of coherent sheaves, that is, not every coherent sheaf on $\mathbb{P}^n$ is of the form $i_* \mathcal{E}$. At first we define a related notion, the one of quasi-coherent sheaves.

Although $\mathbb{C}^n$ is not a complex projective manifold, it makes sense to speak of the Zariski topology on $\mathbb{C}^n$, and the sheaf of regular functions on $\mathbb{C}^n$. The reason is that we can realise $\mathbb{C}^n$ as a Zariski-open subset of $\mathbb{P}^n$. In fact, any of the standard charts $U_i \subset \mathbb{P}^n$ does the trick. The Zariski topology on $\mathbb{P}^n$ induces therefore a topology on the subsets $U_i = \mathbb{C}^n$. Restriction of sheaves yields a sheaf of regular functions $\mathcal{O}^{\mathrm{reg}}$. The resulting topological space, with a sheaf of rings $\mathcal{O}^{\mathrm{reg}}$ is often denoted by $\mathbb{A}^n$, to distinguish it from $\mathbb{C}^n$ with the standard topology and the sheaf of holomorphic functions. It is called n-dimensional affine space. Its main premise is that $\mathbb{A}^n$ can be defined without reference to Zariski open subsets of $\mathbb{P}^n$, and regular functions on $\mathbb{P}^n$.

Lemma 7.18. *A subset $U \subset \mathbb{C}^n$ is Zariski-open, if and only if there exist polynomials $h_1, \ldots, h_k \in \mathbb{C}[x_1, \ldots, x_n]$, such that $U = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, \exists i \in \{1, \ldots, k\} \colon h_i(z_1, \ldots, z_n) \neq 0\}$. A function $f \colon U \longrightarrow \mathbb{C}$ is regular, if and only if there exists a Zariski-open covering $U = \bigcup_{i=1}^m U_i$, such that $f|_{U_i} = \frac{g_i}{h_i}$ for polynomials $g_i, h_i \in \mathbb{C}[x_1, \ldots, x_m]$, $h_i \neq 0$.*

Proof. We leave the details of the proof as an exercise to the reader. The main idea of the construction is the following. We can pass to rational functions $\frac{g}{h}$ on $\mathbb{C}^n$ to rational functions on $\mathbb{P}^n$ by homogenisation. Similarly, given a rational function $\frac{G}{H}$ on $\mathbb{P}^n$ with G and H two homogeneous polynomials in $n+1$ variables of degree d. The restriction $\frac{G}{H}|_{U_i}$ depends solely on the dehomogenisations of G and H, obtained by specialising the variable z_i to 1. Using this correspondence, and the fact that both the Zariski topology, and the sheaf of regular functions, are defined in terms of rational functions, one obtains the statement of the lemma. □

Lemma 7.19. *We denote by R as always a commutative unital ring. Let M be an R-module, and $f \in R$ an element. The localisation M_f of M at f is the R-module given by formal symbols $\frac{m}{f^n}$, where $m \in M$, and $n \geq 0$. We stipulate that $\frac{m}{f^n} = \frac{m'}{f^{n'}}$ if and only if $f^d(f^{n'}m - f^n m') = 0$ for $d >> 0$.*

Note that R_f is a ring itself, which could be defined to be $R[x]/(xf-1)$. It's the ring obtained from R by formally adjoining an inverse f^{-1} to f. The R-module M_f is in fact naturally an R_f-module.

We can now define quasi-coherent sheaves, the immediate predecessor of coherent sheaves. There definition is inspired by a property we observed for algebraic vector bundles $\mathcal{E}$ on $\mathbb{P}^n$. If $H \in \mathbb{C}[z_0, \dots, z_n]_d$ is a non-zero homogeneous polynomial of degree d, then we have a Zariski open subset $V \subset \mathbb{P}^n$ given by the locus where H is non-zero. Let $U \subset \mathbb{P}^n$ be another Zariski open subset. A regular section $s \in \mathcal{E}(U \cap V)$ can be written as $\frac{t}{H^k}$, where $k \in \mathbb{N}$ and $t \in \mathcal{E}(U)$ is a regular section over U. It is precisely the appearance of H in the denominator which stops us from extending s across the complement $U \setminus V \subset U$.

Definition 7.20. *(a) A sheaf $\mathcal{F}$ of $\mathcal{O}^{\mathrm{reg}}$-modules on $\mathbb{A}^n$ is called quasi-coherent, if for every Zariski open subset $U_f = \{z \in \mathbb{C}^n \,|\, f(z) \neq 0\} \subset \mathbb{A}^n$ we have $\mathcal{F}(U_f) = \mathcal{F}(\mathbb{A}^n)_f = \mathcal{F}(\mathbb{A}^n)[f^{-1}]$.*

(b) A sheaf $\mathcal{F}$ of $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-modules on $\mathbb{P}^n_{\mathrm{Zar}}$ is called quasi-coherent*, if for every $i \in \{0, \dots, n\}$ the sheaf $\mathcal{F}|_{U_i}$ is quasi-coherent.*

Next we classify quasi-coherent sheaves on $\mathbb{A}^n$. The proposition below tells us that the functor, sending a quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{A}^n$ to the $\mathbb{C}[x_1, \dots, x_n]$-module $\mathcal{F}(\mathbb{A}^n)$ is an equivalence of categories.

Proposition 7.21. *We have an equivalence of categories $\Gamma(\mathbb{A}^n, -)\colon \mathsf{QCoh}(\mathbb{A}^n) \longrightarrow \mathit{Mod}(\mathbb{C}[x_1, \dots, x_n])$, sending exact sequences of sheaves to exact sequences of modules. In other words, for every quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{A}^n$ there exists a unique $\mathbb{C}[x_1, \dots, x_n]$-module M, such that $\mathcal{F}(U_f) = M_f$, every module M arises in this way, and $\mathcal{O}^{\mathrm{reg}}$-linear morphisms of quasi-coherent sheaves, correspond to morphisms of $\mathbb{C}[x_1, \dots, x_n]$-modules.*

Proof. The proof is formally very similar to the one of Proposition 7.26, which we will discuss below. It will be broken down into several easy steps on an exercise sheet, and left to the reader as an exercise. □

The fact that the global sections functor for quasi-coherent sheaves on $\mathbb{A}^n$ is exact, can be strengthened[2]:

[2]Possibly we will include a proof of this Corollary in a non-examinable appendix in the future.

Corollary 7.22. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbb{A}^n$. Then for every $i > 0$ the cohomology groups $H^i(\mathbb{A}^n, \mathcal{F})$ vanish.*

Lemma 7.23. *A quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{A}^n$ is called coherent, if the corresponding $\mathbb{C}[x_1, \ldots, x_n]$-module M is finitely generated. This is a local property. That is, if there exists a finite Zariski open covering $\mathbb{A}^n = \bigcup_{j=1}^k U_{f_j}$ by Zariski open subsets U_f, such that $\mathcal{F}(U_j)$ is a finitely generated $\mathbb{C}[x_1, \ldots, x_n]_{f_j}$-module for every j, then $\mathcal{F}$ is coherent.*

Proof. For every $j = 1, \ldots, k$ we choose elements $s_1^j, \ldots, s_{m_j}^j$ generating $\mathcal{F}(U_{f_j})$. Since $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F}(U_{f_j}) = \mathcal{F}(\mathbb{A}^n)_{f_j}$, and therefore there exists $d >> 0$, such that we can write $s_i^j = \frac{t_i^j}{f_j^d}$ with $t_i^j \in \mathcal{F}(M)$. Consider the free R-module F of finite rank m generated by symbols e_j^i for $j = 1, \ldots, k$ and $i = 1, \ldots, m_j$. Let $F \longrightarrow M$ be the map sending e_i^j to t_i^j. By virtue of Proposition 7.21 we obtain a well-defined map $(\mathcal{O}_{\mathbb{A}^n}^{\text{reg}})^m \longrightarrow \mathcal{F}$. Its cokernel has to be 0, since it is zero with respect to the open covering U_{f_j}. This implies that the map is surjective, and therefore M is finitely generated, and $\mathcal{F}$ is coherent. □

Definition 7.24. *A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}^{\text{reg}}$-modules on $\mathbb{P}^n$ is called* coherent*, if for every $i = 0, \ldots, n$ the restriction $\mathcal{F}|_{U_i}$ is a coherent sheaf on affine space.*

Sometimes the standard covering $\{U_i\}_{i \in I}$ is too fine to check coherence. In these cases, it is useful to remember Lemma 7.23, which allows one to check coherence on a finer Zariski open covering. The basic properties of coherent sheaves listed below will be useful when proving GAGA.

Lemma 7.25. *(a) The direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ of two coherent sheaves on $\mathbb{P}^n_{\text{Zar}}$ is again coherent.*

(b) For a morphism of coherent sheaves $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$, kernel and cokernel are coherent.

(c) For every complex projective submanifold $i \colon X \hookrightarrow \mathbb{P}^n$, and an algebraic vector bundle $\mathcal{E}$ on X, $i_* \mathcal{E}$ is a coherent sheaf on $\mathbb{P}^n_{\text{Zar}}$. In particular, ever algebraic vector bundle on $\mathbb{P}^n_{\text{Zar}}$ is coherent.

Proof. Assertion (a) is clear, its proof boils down to the fact that the direct sum of two finitely generated modules is again finitely generated.

The same reasoning applies to the first part of assertion (b): if $M_1 \longrightarrow M_2$ is a map of finitely generated R-modules, then $\operatorname{coker} f = M_2/\operatorname{image} f$ is finitely generated as a consequence of the finite generation of M_2. Indeed, if $R^n \twoheadrightarrow M_2$ is a surjective map, then the composition $R^n \twoheadrightarrow M_2 \twoheadrightarrow \operatorname{coker} f$ is surjective and proves finite generation of $\operatorname{coker} f$. The assertion that $\ker f$ is finitely generated is not true for arbitrary commutative rings R (with unit). However, it is true for so-called Noetherian ones, that is rings for which every ideal is finitely generated. Noetherian rings satisfy the property that every submodule of a finitely generated module is finitely generated (the proof is a simple inductive argument on the minimal number on generators). Since polynomial rings happen to be Noetherian, we deduce that $\ker f$ is finitely generated too.

In order to prove (c) it suffices to check that $i_* \mathcal{O}_X^{\text{reg}}$ is coherent on $\mathbb{P}^n_{\text{Zar}}$. Indeed, if $\mathcal{E}$ is locally free of rank m, then there would exists a Zariski open covering $\mathbb{P}^n = \bigcup_{j=1}^k V_j$ of $\mathbb{P}^n$, such that $\mathcal{E}|_{V_j \cap X}$ is free of rank m, that is isomorphic to $(\mathcal{O}_{V_j \cap X}^{\text{reg}})^m$ (we can produce a finite cover, using compactness of $\mathbb{P}^n$). Therefore we see that $(i_* \mathcal{E})|V_i \cong (i_* \mathcal{O}_{V_j \cap X})^m$. Since coherence is a local property by Lemma 7.23, and we assume that $i_* \mathcal{O}_{V_j \cap X}$ is coherent, we are done. It remains to show that $i_* \mathcal{O}_X^{\text{reg}}$ is indeed coherent.

If $X = V(G_1, \dots, G_k)$, then for every $i = 0, \dots, n$ we have that $X \cap U_j$ is the zero set of the system of polynomial equations $g_i(w_1, \dots, w_n) = G_i(w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n)$. The so-called Nullstellensatz in commutative algebra then implies that the ring of regular functions on $X \cap U_j$ is equal to $\mathbb{C}[w_1, \dots, w_n]/(g_1, \dots, g_n)$, which is a module generated by one element. $\square$

Proposition 7.26. *For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_{\mathrm{Zar}}$, there exists a $d \in \mathbb{Z}$, such that $\mathcal{F}(d) = F \otimes_{\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}} \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)$ is globally generated, that is, there exists a surjection of $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}$-modules $(\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n})^m \twoheadrightarrow \mathcal{F}(d)$.*

Proof. For $i = 0, \dots, n$ the restriction $\mathcal{F}|_{U_i}$ corresponds to a finitely generated $\mathbb{C}[w_1, \dots, w_n] = \mathbb{C}[(\frac{z_j}{z_j})_{j \neq i}]$-module M_i. Since $\mathcal{F}$ is quasi-coherent, we can express an element in $M = \mathcal{F}(U_i)$ over $U_j \cap U_i$ for $j \neq i$ as a fraction $s = \frac{t_j}{w_{i(j)}^{d_j}} = \frac{z_j^{d_j} t_j}{z_i^{d_j}}$, where $t_j \in \mathcal{F}(U_j)$ and $w_{i(j)} = w_i = \frac{z_i}{z_j}$ if $i > j$, and w_{j+1} if $j < i$.

On $U_{ij\ell}$ we have $s|_{U_{ij\ell}} = \frac{z_j^{d_k} t_j}{z_i^{d_k}}$ and $\frac{z_\ell^{d_\ell} t_\ell}{z_i^{d_\ell}}$ that we want to compare. We can assume $d_k = d_\ell = d'$ by choosing the maximum. The definition of localisation implies that there exists an $m \geq 0$, such that $z_i^{m+d'} t_j = \frac{z_\ell^{m+d'}}{z_j^{m+d'}} z_i^{m+d'} t_\ell$. Replacing t_j by $z_i^{m+d_{i\ell}} t_j$, and t_ℓ by $z_i^{m+d_{ik}} t_\ell$, we can therefore assume that $t^i|_{U_{ij\ell}}$ and $t^i_\ell|_{U_{ij\ell}}$ assume as sections of $\mathcal{F}(d)$, where $d = m + d'$. The sheaf property of $\mathcal{F}$ allows one to patch the collection $(t_j)_{j \neq i}$, together with $z_i^d s$ to a global section $t \in \mathcal{F}(d)(X)$, such that $s = \frac{t}{z_i^d}$.

We choose sections $s^i_1, \dots, s^i_{m_i}$ (respectively elements of M_i), which generate $\mathcal{F}|_{U_i}$. For every element of this list we choose an extension $\tilde{s}^i_k$ to a global section of $\mathcal{F}(d)$. The collection $(\tilde{s}^i_k)^{j=1,\dots,m_i}_{i=0,\dots,n}$ generates $\mathcal{F}(d)$ as a sheaf, because for every U_i we have that $\mathcal{F}(d)|_{U_i}$ is generated by the elements $(\tilde{s}^k_i)^{k=1,\dots,m_i}$, because $\tilde{s}^k_i|_{U_i} = s^k_i$ generates $\mathcal{F}|_{U_i}$ by assumption. Here we use the natural trivialisation $\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)|_{U_i} \cong \mathcal{O}^{\mathrm{reg}}_{U_i}$. $\square$

7.3 A short exact sequence for twisting sheaves

In a non-canonical manner, $(n-1)$-dimensional complex projective space $\mathbb{P}^{n-1}$ is a complex submanifold of $\mathbb{P}^n$. Indeed, we can realise $\mathbb{P}^{n-1}$ as the complex projective variety, corresponding to the linear homogeneous equation $z_0 \neq 0$. Following Serre we refer to this particular copy of $\mathbb{P}^{n-1}$ inside $\mathbb{P}^n$ with $E = V(z_0) \subset \mathbb{P}^n$. We write $i \colon E \hookrightarrow \mathbb{P}^n$ for the inclusion morphism. The following short exact sequence will enable us to prove GAGA for the sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ by induction on n.

Lemma 7.27. *For every $d \in \mathbb{Z}$ we have a short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow i_* \mathcal{O}_E(d) \longrightarrow 0,$$

respectively

$$0 \longrightarrow \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d) \longrightarrow i_* \mathcal{O}^{\mathrm{reg}}_E(d) \longrightarrow 0.$$

Proof. We give the proof for sheaves of holomorphic sections (the first short exact sequence), since the proof for regular sections is the same. At first we prove the assertion for $d = 0$. We begin by constructing a locally surjective morphism of sheaves $p \colon \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow i_* \mathcal{O}_E$. We let p be the restriction morphism, which sends a holomorphic function $f \colon U \longrightarrow \mathbb{C}$, where $U \subset \mathbb{P}^n$ is an open subset, to $f|_E$.

In order to show that p is locally surjective, consider one the charts (U_i, ϕ_i). We will show that the maps $\mathcal{O}_{\mathbb{P}^n}(U_i) \longrightarrow i_* \mathcal{O}_E(U_i)$ are surjective. For $i = 0$, the right hand side is equal to 0, because $E = \mathbb{P}^n \setminus U_0$, and thus $i_* \mathcal{O}_E(U_0) = \mathcal{O}_E(E \cap U_0) = \mathcal{O}_E(\emptyset) = 0$.

For $i \neq 0$, $E \cap U_i$ corresponds to the closed subset of $\mathbb{C}^n$ given by $\{w_1 = 0\}$. Therefore, $i_* \mathcal{O}_E(U_i) = \{f(w_2, \dots, w_n) \text{ holomorphic function})\}$. But it is clear that we can extend f to a holomorphic function in $(w_1, \dots, w_n)$ by defining $g(w_1, \dots, w_n) = f(w_2, \dots, w_n)$.

What's the kernel of the morphism of sheaves p? It is given by the map $\mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{z_0 \cdot} \mathcal{O}_{\mathbb{P}^n}$, which multiplies a local section $s \in \mathcal{O}_{\mathbb{P}^n}(U)$ with the restriction of $z_0 \in \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$. To see that this makes sense, recall our definition of the line bundles L_d, or use the identity $\mathcal{O}_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^n}$.

As before, we can check that this is the kernel of p, after restricting to the charts U_i. Over U_0, z_0 corresponds to the constant holomorphic function 1, and we see that $\mathcal{O}_{\mathbb{P}^n}(-1)|_{U_0} \xrightarrow{z_0 \cdot} \mathcal{O}_{\mathbb{P}^n}|_{U_0}$ is an isomorphism of sheaves, as we would have expected from $\mathcal{O}_E|_{U_0} = 0$. For $i \neq 0$, z_0 corresponds to the holomorphic function $(w_1, \dots, w_n) \mapsto w_1$. We have to show that every holomorphic function f on U_i, satisfying $f|_E = 0$, can be written as $f = w_1 g$, where g is a holomorphic function. It suffices to show that $\frac{f}{w_1}$ is locally holomorphic. In order to see this we apply the Weierstrass division theorem, to locally obtain $f = h w_1 + r$, where r is a polynomial in w_1 of degree 0, that is, it's a holomorphic function in the variables $(w_2, \dots, w_n)$. Since $f|_E = w_1|_E = 0$, we obtain $r|_E = 0$. Since r is independent of w_1, we see that $r = 0$. This proves the assertion.

The general case is obtained by tensoring the short exact sequence above with $\mathcal{O}_{\mathbb{P}^n}(d)$, and using the rule $\mathcal{O}_{\mathbb{P}^n}(d_1) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d_2) \cong \mathcal{O}_{\mathbb{P}^n}(d_1 + d_2)$. The only remaining step is to check that $(i_* \mathcal{O}_E)(d) \cong i_*(\mathcal{O}_E(d))$. This follows by recalling that $\mathcal{O}_{\mathbb{P}^n}(d)$ is given by the cocycle $(\frac{z_j^d}{z_i^d})_{ij}$, which restricts to the cocycle datum for $\mathcal{O}_E(d)$ on E (we're just omitting the variable z_0). $\square$

These short exact sequences are relevant, because we've seen that $H^i(E, \mathcal{F}) \cong H^i(\mathbb{P}^n, i_* \mathcal{F})$. The corresponding long exact sequence in cohomology will enable us to prove comparison results for the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ by induction on n. Before getting to this, we need to introduce the comparison maps $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)) \longrightarrow H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}_{\mathbb{P}^n}(d))$. Before getting to this, we make a little detour via the Riemann-Roch Theorem for Riemann surfaces (that is, compact complex manifolds of complex dimension 1). We will use (without proof) that cohomology groups of vector bundles on projective complex manifolds are finite dimensional vector spaces.

Theorem 7.28 (Riemann-Roch for $\mathbb{P}^1$)**.** *Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^1$, we denote by $\chi(F) \in \mathbb{Z}$ the integer given by* $\dim H^0(\mathbb{P}^1, \mathcal{F}) - \dim H^0(\mathbb{P}^1, \mathcal{F})$ *(assuming it is well-defined). We have $\chi(\mathcal{O}_{\mathbb{P}^n}(d)) = d + 1$.*

Sketch. If $0 \longrightarrow A_0 \xrightarrow{d_0} \cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} \longrightarrow A_n \longrightarrow 0$ is an exact sequence of vector spaces, then the alternating sum of dimensions $\sum_{i=0}^n (-1)^i \dim A_i = 0$. This can be proven by induction on n. The anchoring cases of $n = 0$, or $n = 1$ are easy to prove. We also have an exact sequence $0 \longrightarrow A_0 \xrightarrow{d_0} \cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} \longrightarrow \ker d_{n-1} \longrightarrow 0$, of length $n = n + 1 - 1$. Therefore, by virtue of induction, $\sum_{i=0}^{n-2} (-1)^i \dim A_i + (-1)^{n-1} \dim \ker d_{n-1} = 0$. Since $d_n \colon A_{n-1} \longrightarrow A_n$ is surjective, we have $\dim \ker d_{n-1} = \dim A_{n-1} - \dim A_n$, which concludes the proof of the claim.

We apply this to the long exact sequence corresponding to the short exact sequence $0 \longrightarrow \mathcal{O}(d-1) \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}_E(d) \longrightarrow 0$. Note that E is a point, therefore $H^0(E, \mathcal{O}_E(d)) = \mathbb{C}$ is the only non-zero cohomology group. Using the cohomology vanishing property (which we will discuss in the future) that $H^2(\mathbb{P}, \mathcal{O}(d)) = 0$, we obtain

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(d-1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(d)) \longrightarrow \mathbb{C} \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(d-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(d)) \longrightarrow 0 \longrightarrow 0.$$

The alternating sum of dimensions can be re-arranged into the identity $\chi(\mathcal{O}(d)) = \chi(\mathcal{O}(d-1)) + 1$. Therefore it remains to prove the assertion for a single $d \in \mathbb{Z}$. We already know that $H^0(\mathbb{P}^1, \mathcal{O}) = 1$, next week we will show that it's also true that $H^1(\mathbb{P}^1, \mathcal{O}) = 0$. □

Together with our previous computation that $\dim H^0(\mathbb{P}^1, \mathcal{O}(d)) = d+1$, if $d \geq 0$, the Riemann-Roch formula determines the dimensions of the cohomology groups of the sheaves $\mathcal{O}(d)$ on $\mathbb{P}^1$ completely. We can also observe the following interesting symmetry

$$\dim H^0(\mathbb{P}^1, \mathcal{O}(d)) = \dim H^1(\mathbb{P}^1, \mathcal{O}(-2-d)).$$

This fascinating identity can be explained through so-called *Serre duality*, the counterpart of Poincaré duality for coherent sheaves. One can show that on a smooth projective manifold X of dimension n, and a vector bundle E, we have a canonical isomorphism of vector spaces

$$H^i(X, \underline{E}) \cong H^{n-i}(X, \underline{E}^\vee \otimes \Omega^n_X).$$

The Riemann-Roch theorem can be generalised to smooth projective manifolds X, but even stating it properly would take us too far afield. For X a Riemann surface, this is possible. For every line bundle L on X, one can define its degree $\deg L \in \mathbb{Z}$. For $X = \mathbb{P}^1$ we have $\deg \mathcal{O}(d) = d$. The underlying topological space of X is an orientable surface of genus g. The Riemann-Roch formula reads

$$\chi(\underline{L}) = 1 - g + \deg L.$$

7.4 The proof of the first GAGA theorem (modulo basic cohomology computations)

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathrm{Zar}}$, for example $\mathcal{F}$ could be a locally free sheaf of $\mathcal{O}^{\mathrm{reg}}$-modules, corresponding to an algebraic vector bundle. Serre defines a sheaf of holomorphic sections $\mathcal{F}^h$ on $\mathbb{P}^n$. For E an algebraic vector bundle, given by a regular cocycle datum $(\{U_i\}_{i \in I}, (\phi_{ij}))$, E^h is equal to the sheaf of holomorphic sections $\underline{E}$. That is, to an open subset $V \subset \mathbb{P}^n$, we assign the collection of holomorphic functions $f_i \colon U_i \cap V \longrightarrow \mathbb{C}$, such that $f_i = \phi_{ij} f_j$. This functor can be generalised to coherent sheaves, by using the fact that Zariski locally, every coherent sheaf can be written as a cokernel of a map of locally free sheaves:

$$(\mathcal{O}^{\mathrm{reg}})^m \longrightarrow (\mathcal{O}^{\mathrm{reg}})^k \longrightarrow \mathcal{F} \longrightarrow 0.$$

By definition, $(\mathcal{O}^{\mathrm{reg}})^h = \mathcal{O}$, hence it makes sense to write $\mathcal{F}^h = \mathrm{coker}(\mathcal{O}^m \longrightarrow \mathcal{O}^k)$. It is important to check that this does not depend on the presentation of $\mathcal{F}$ as a quotient. We will say more about this in a lecture devoted entirely to this process of taking holomorphic sections. The following lemma is essential to the proof of GAGA. It asserts the existence of a comparison map between regular and holomorphic cohomology for coherent sheaves.

Lemma 7.29. *(a) The functor $\mathcal{F} \mapsto \mathcal{F}^h$ from coherent sheaves on $\mathbb{P}^n_{\mathrm{Zar}}$ to sheaves on $\mathbb{P}^n$ sends exact sequences of coherent sheaves to exact sequences.*

(b) For every $i \in \mathbb{N}$, and coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there exists a natural map $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}^h)$. Naturality asserts that for every morphism $\mathcal{F} \longrightarrow \mathcal{G}$ of coherent sheaves we have a commutative

diagram

$$\begin{tikzcd}
H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{G}) \arrow[d] \\
H^i(\mathbb{P}^n, \mathcal{F}^h) \arrow[r] & H^i(\mathbb{P}^n, \mathcal{G}^h),
\end{tikzcd}$$

and for every short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *of coherent sheaves we have a commutative diagram*

$$\begin{tikzcd}
H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{H}) \arrow[r] \arrow[d] & H^{i+1}(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \arrow[d] \\
H^i(\mathbb{P}^n, \mathcal{H}^h) \arrow[r] & H^{i+1}(\mathbb{P}^n, \mathcal{F}^h).
\end{tikzcd}$$

The proof of this lemma is deferred to another lecture. The proof of (a) makes use of results related to Weierstrass preparation, and is mostly algebraic. The proof of (b) relies on the fact that for every coherent sheaf $\mathcal{F}$ on $\mathbb{A}^n$, the higher cohomology groups $H^i(\mathbb{C}^n, \mathcal{F}^h)$ vanish (here we use the fact that the sheaf of holomorphic sections can be defined locally on $\mathbb{P}^n$, and that $\mathbb{P}^n$ can be covered by Zariski open subsets equivalent to $\mathbb{A}^n$).

Next week we will learn the techniques to prove $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $i \geq 1$. Taking this for granted, we obtain the following interesting special case of GAGA.

Proposition 7.30. *If the natural map* $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O})$ *is an isomorphism for every* $i \in \mathbb{N}$*, and every* $n \in \mathbb{N}$*, then we have that also* $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}(d)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O}(d))$ *is an isomorphism for every* $d \in \mathbb{Z}$.

Proof. This follows from the short exact sequence $0 \longrightarrow \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d) \longrightarrow i_* \mathcal{O}^{\mathrm{reg}}_E(d) \longrightarrow 0$. Applying the functor given by holomorphic sections, the resulting long exact sequence in cohomology gives us the following commutative diagram with exact rows

$$\begin{tikzcd}
H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d-1)) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)) \arrow[r] \arrow[d] & H^i(E_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_E(d)) \arrow[r] \arrow[d] & H^{i+1}(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d \arrow[d] \\
H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(d)) \arrow[r] & H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \arrow[r] & H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] & H^i(E, \mathcal{O}_E(d)) \arrow[r] & H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d -
\end{tikzcd}$$

and the Five Lemma implies together with induction on n, that $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}(d)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O}(d))$ is an isomorphism, if it is an isomorphism for $d-1$. Similarly, we can show that the case of d, implies the one of $d-1$. This reduces the entire assertion of GAGA for the sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ to the case $d = 0$. $\square$

We are now ready to prove the entire proof of part (a) of the GAGA theorem, to the assertion that $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is an isomorphism. For $i = 0$ we already know this to be true. Next week we will show that the higher cohomology groups $H^i(\mathbb{P}^n, \mathcal{O})$ vanish. Together with the descending induction argument, this concludes the proof of GAGA (part (a)).

Proposition 7.31. *Assume that the natural map* $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O})$ *is an isomorphism for every* $i \in \mathbb{N}$*, and assume that higher cohomology groups of coherent sheaves on* $\mathbb{P}^n$ *vanish for* $i \geq n+1$*. Then, for every coherent sheaf* $\mathcal{F}$ *on* $\mathbb{P}^n_{\mathrm{Zar}}$*, the map* $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}^h)$ *is an isomorphism.*

Corollary 7.32. *Let X be a projective complex manifold, and E an algebraic vector bundle on X. Then the vector space of holomorphic section of E is isomorphic to the vector space of regular sections of E (assuming GAGA(a) for $\mathcal{O}_{\mathbb{P}^n}$).*

Proof. Let $\mathcal{E}$ be the sheaf of regular sections of E on X_{Zar}. We denote by $i\colon X \hookrightarrow \mathbb{P}^n$ an inclusion of X into complex projective space. We have $H^0(X_{\mathrm{Zar}}, \mathcal{E}) = H^0(\mathbb{P}^n, i_* \mathcal{E})$. By GAGA part (a) this vector space is equivalent to $H^0(X, i_* \underline{E})$, which agrees with the space of holomorphic sections of E. $\square$

Proof of Proposition 7.31. According to Proposition 7.26 we can choose $d >> 0$, such that there is a surjection of sheaves $(\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n})^m \twoheadrightarrow \mathcal{F}(d)$. Untwisting, we obtain a surjection of sheaves $(\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(-d))^m \twoheadrightarrow \mathcal{F}$. Let K be the sheaf given by the kernel of this morphism of sheaves. The resulting short exact sequence of sheaves yields the following commutative diagram with exact rows.

$$\begin{tikzcd}
H^i(\mathbb{P}^n_{\mathrm{Zar}}, K) \arrow[r] \arrow[d, "f"] & H^i(\mathbb{P}^n_{\mathrm{Zar}}, (\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(-d))^m) \arrow[r] \arrow[d, "g"] & H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \arrow[r] \arrow[d, "h"] & H^{i+1}(\mathbb{P}^n_{\mathrm{Zar}}, K) \arrow[r] \arrow[d, "k"] & H^{i+1}(\mathbb{P}^n_{\mathrm{Zar}}, (\mathcal{O}^{\mathrm{reg}}_{\mathbb{P}^n}(-d))^m) \arrow[d, "\ell"] \\
H^i(\mathbb{P}^n, K^h) \arrow[r] & H^i(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}(-d))^m) \arrow[r] & H^i(\mathbb{P}^n, \mathcal{F}^h) \arrow[r] & H^{i+1}(\mathbb{P}^n, K^h) \arrow[r] & H^{i+1}(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}(-d))^m)
\end{tikzcd}$$

As planned, we argue by descending induction on i. For $i > n$, the cohomology groups vanish, and therefore the map comparing regular and holomorphic cohomology is an isomorphism. We also know that the maps g and ℓ are isomorphisms for all values of i.

By descending induction we can assume that k is an isomorphism. The Five Lemma now implies that h is surjective. Since K is also a coherent sheaf this implies that f is surjective too, and hence a second application of the Five Lemma yields that h is bijective. $\square$

Part (a) of GAGA quickly implies (b). We will also explain the idea underlying the proof of (c), but since the proof uses a lot more analysis, we will not be able to explain all the details for time reasons.

Corollary 7.33. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two algebraic vector bundles on $\mathbb{P}^n$, or more generally a complex projective manifold X. Then, every $\mathcal{O}_X$-linear map $\mathcal{E}_1^h \longrightarrow \mathcal{E}_2^h$ is induced by a unique regular morphism $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$.*

Proof. The sheaves $\mathcal{E}_1$, and $\mathcal{E}_2$ are $\mathcal{O}^{\mathrm{reg}}_X$-locally free of finite ranks n_1 and n_2. Let $\underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{E}_2)$ be the sheaf of $\mathcal{O}^{\mathrm{reg}}_X$-modules, which assigns to a Zariski open subset $U \subset X$ the $\mathcal{O}^{\mathrm{reg}}_X$-module of linear maps $\mathsf{Hom}(\mathcal{E}_1(U), \mathcal{E}_2(U))$. We leave the standard verification that $\underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{E}_2)$ is a sheaf to the reader.

We claim that $\underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{E}_2)$ is locally free of rank $n_1 n_2$. By definition, its global sections are equivalent to the vector space of linear maps $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$. GAGA part (a) implies therefore that every linear map between $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$ is induced by a unique linear map $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$.

Let $U \subset X$ be an open subset, such that $\mathcal{E}_1|_U$ and $\mathcal{E}_2|_U$ are free of ranks n_1 and n_2. We have $\underline{\mathrm{Hom}}((\mathcal{O}^{\mathrm{reg}}_X)^{n_1}, (\mathcal{O}^{\mathrm{reg}}_X)^{n_2}) = (\mathcal{O}^{\mathrm{reg}}_X)^{n_1 \times n_2}$. $\square$

There is also a theory of analytic coherent sheaves on a complex manifold. An $\mathcal{O}_X$-linear sheaf $\mathcal{F}$ on X is called an *analytic coherent sheaf*, if there exists an open covering $X = \bigcup_{i \in I} U_i$, such that for every $i \in I$ we have an exact sequence

$$\mathcal{O}^m_{U_i} \longrightarrow \mathcal{O}^k_{U_i} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The proof of GAGA (c) follows now from this result, and the following important theorem. Serre deduces it from the fact that $\mathbb{P}^n$ is compact, and that the cohomology groups $H^i(\mathbb{P}^n, \mathcal{F})$ are finite-dimensional.

Theorem 7.34. *Let $\mathcal{F}$ be an analytic coherent sheaf on $\mathbb{P}^n$, then there exists $d >> 0$, such that we have a surjection $\mathcal{O}_{\mathbb{P}^n}^k \twoheadrightarrow \mathcal{F}(d)$.*

As before, this yields a short exact sequence $0 \longrightarrow K \longrightarrow (\mathcal{O}_{\mathbb{P}^n}(-d))^k \longrightarrow \mathcal{F} \longrightarrow 0$, and using the fact that K is also an analytic coherent sheaf, we can express $\mathcal{F}$ as a cokernel of a linear morphism $(\mathcal{O}_{\mathbb{P}^n}(-d'))^m \longrightarrow (\mathcal{O}_{\mathbb{P}^n}(d))^k$. We have just seen that every such morphism is regular, hence $\mathcal{F}$ must be equal to the sheaf of holomorphic sections of an algebraic coherent sheaf.

7.5 Holomorphic sections of coherent sheaves

In this paragraph we briefly discuss the definition of the functor $\mathcal{F} \mapsto \mathcal{F}^h$, discuss why it is exact, and define a natural morphism $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}^h)$. Let U be a Zariski open subset of $\mathbb{P}^n$. We will mostly be interested in the cases $U = \mathbb{P}^n$ or $U = U_i$. We denote by $\Phi\colon U \longrightarrow U_{\mathrm{Zar}}$ the continuous map, which is the identity on underlying sets. It is continuous, since a Zariski open subset is also open in the standard topology.

Definition 7.35. *For a sheaf $\mathcal{F}$ of $\mathcal{O}_U^{\mathrm{reg}}$-modules, we define $\mathcal{F}^h = \Phi^{-1}\,\mathcal{F} \otimes_{\Phi^{-1}\,\mathcal{O}_U^{\mathrm{reg}}} \mathcal{O}_U$.*

Recall that $\Phi^{-1}\colon \mathsf{Sh}(U_{\mathrm{Zar}}) \longrightarrow \mathsf{Sh}(U)$ is the functor we constructed in Lemma 7.17 for a general continuous map. We have a canonical map $\Phi^{-1}\,\mathcal{O}_U^{\mathrm{reg}} \longrightarrow \mathcal{O}_U$, which is adjoint to the map $\mathcal{O}_U^{\mathrm{reg}} \longrightarrow \Phi_*\,\mathcal{O}_U$ (see Lemma 7.17 for an explanation of the word *adjoint*).

Lemma 7.36. *Holomorphisation satisfies a universal property. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_U$-modules, and $\mathcal{F} \longrightarrow \Phi_*\,\mathcal{G}$ an $\mathcal{O}_U^{\mathrm{reg}}$-linear map (where $\Phi_*\,\mathcal{G}$ is viewed as a sheaf of $\mathcal{O}_U^{\mathrm{reg}}$ through the canonical map $\mathcal{O}_U^{\mathrm{reg}} \longrightarrow \Phi_*\,\mathcal{O}_U$). Then there exists a unique morphism of sheaves $\mathcal{F}^h \longrightarrow \mathcal{G}$, such that*

$$\begin{tikzcd} \mathcal{F} \arrow[d] \arrow[dr] & \\ \Phi_*\,\mathcal{F}^h \arrow[r, dashed] & \Phi_*\,\mathcal{G} \end{tikzcd}$$

commutes.

This universal property implies directly that holomorphisation commutes with cokernels, that is, $\mathrm{coker}(\mathcal{F}_1 \longrightarrow \mathcal{F}_2)^h \cong \mathrm{coker}(\mathcal{F}_1^h \longrightarrow \mathcal{F}_2^h)$. This shows that for a short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of coherent sheaves, we have an exact sequence $\mathcal{F}^h \longrightarrow \mathcal{G}^h \longrightarrow \mathcal{H}^h \longrightarrow 0$, but showing that the map $\mathcal{F}^h \longrightarrow \mathcal{G}^h$ is injective is less obvious. Serre proves this using commutative algebra, the proof is not difficult to understand, but unfortunately we don't have the time to explain it. In the end it boils down to the Weierstrass Preparation Theorem and formal computations with power series. The following assertion is more generally known to be true for analytic coherent sheaves on $\mathbb{C}^n$. This version is known as Cartan's Theorem B. The proof below uses that $H^i(\mathbb{C}^n, \mathcal{O}) = 0$ for $i \geq 1$. We will prove this next week, using the Dolbeault resolution.

Proposition 7.37 (Weak Cartan B)**.** *Let $\mathcal{F}$ be an algebraic coherent sheaf on $\mathbb{A}^n$, then for $i \geq 1$ we have $H^i(\mathbb{C}^n, \mathcal{F}^h) = 0$. The same statement holds when we replace $\mathbb{A}^n$ by a Zariski open subset of the shape $\{f \neq 0\}$, where $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial.*

Proof. We prove this by descending induction, using the fact that $H^i(\mathbb{C}^n, \mathcal{F}^h) = 0$ for $i > n$, and $H^i(\mathbb{C}^n, \mathcal{O}) = 0$ for $i \geq 1$. We will see a proof of these statements very soon, using the formalism of Dolbeault cohomology.

Since $\mathcal{F}$ is coherent, there exists a short exact sequence $0 \longrightarrow K \longrightarrow (\mathcal{O}_{\mathbb{A}^n}^{\mathrm{reg}})^m \longrightarrow \mathcal{F} \longrightarrow 0$, where K is also coherent. The long exact sequence in cohomology implies $H^i(\mathbb{C}^n, \mathcal{F}^h) = H^{i+1}(\mathbb{C}^n, K)$ for $i \geq 1$. Therefore, we obtain by descending induction on i the asserted vanishing statement. $\square$

We denote by $\Phi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n_{\mathrm{Zar}}$ the continuous map, which is the identity on underlying sets. It is continuous, because every Zariski open subset is also open in the standard topology. Part (b) of Lemma 7.29, which asserted the existence of a natural map $H^i(\mathbb{P}^n_{\mathrm{Zar}}, \mathcal{F}) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}^h)$ follows from the lemma below, and the fact that we have a natural map $\mathcal{F} \longrightarrow \Phi_* \mathcal{F}^h$.

Lemma 7.38. *For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_{\mathrm{Zar}}$, we have an equivalence $H^i(\mathbb{P}^n, \mathcal{F}^h) \cong H^i(\mathbb{P}^n_{\mathrm{Zar}}, \Phi_* \mathcal{F}^h)$, that is, it doesn't matter if we compute the cohomology of $\mathcal{F}^h$ with respect to the standard or the Zariski topology.*

Proof. Let $(\mathcal{I}^\bullet, d)$ be an injective resolution of $\mathcal{F}$, that is a complex of injective sheaves, exact in positive degrees, such that $\ker d_0 = \mathcal{F}^h$. The pushforward $\Phi_*(\mathcal{I}^\bullet)$ is also a chain complex. If we knew that the sheaves $\Phi_* \mathcal{I}^i$ were still acyclic (= no higher cohomology), then we would only have to check that this still defines a resolution of $\Phi_* \mathcal{F}$. The pushforward of injective sheaves is not injective, but can be shown to be flasque as follows. We have seen in ex. 1, E8 that injective sheaves are flasque, and that the pushforward of flasque sheaves is flasque. Hence, it remains to show that $\Phi_*(\mathcal{I}^\bullet)$ is exact in positive degrees. Since exactness for chain complexes of sheaves is a local property, it suffices to show that for every standard chart $U_i \subset \mathbb{P}^n$, the restriction $\Phi_*(\mathcal{I}^\bullet)|_{U_i}$ is exact in positive degrees. Let $V = (U_i)_f \subset U_i$ be a Zariski open subset of the form $\{f \neq 0\}$, for a polynomial f. The chain complex of global sections $\mathcal{I}^0(V) \longrightarrow \mathcal{I}^1(V) \longrightarrow \cdots$ computes $H^\bullet(V, \mathcal{F}|_V)$, which we have seen to vanish in positive degrees. Since every Zariski open subset of U_i is a union of subsets $(U_i)_f$, this implies the required exactness assertion in positive degrees. $\square$

8 Computing Cohomology: Dolbeault and Čech

In this section we will discuss two important tools to compute cohomology of sheaves. The Dolbeault resolution and the Čech complex.

8.1 The Dolbeault resolution

The Dolbeault resolution is to the sheaf of holomorphic functions $\mathcal{O}_X$ on a complex manifold, what the de Rham resolution is to the sheaf of locally constant functions $\underline{\mathbb{R}}$ on a smooth manifold. Before we can define it, we need to introduce some notation, and recall the differential operator $\frac{\partial}{\partial \bar{z}_i}$ from 2.4. For a C^∞-function $f\colon U \longrightarrow \mathbb{C}$, where $U \subset \mathbb{C}^n$ we defined

$$\frac{\partial f}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial f}{\partial x_i} + i\frac{\partial f}{y_i}).$$

The function f is holomorphic, if and only if for every $i = 0, \ldots, n$ we have $\frac{\partial f}{\partial \bar{z}_i} = 0$. This is simply a reformulation of the Cauchy-Riemann equations.

The idea underlying the Dolbeault resolution is that there should be a sheaf $\mathcal{A}^{0,1}$, and a map $C^\infty_X \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}$, such that $\ker \overline{\partial} = \mathcal{O}_X$. Here, C^∞_X denotes the sheaf, sending an open subset $U \subset X$ to the vector space of C^∞-functions $U \longrightarrow \mathbb{C}$.

Of course we cannot stop there, we have to keep going and produce a complex

$$\mathcal{A}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{A}^{0,2} \xrightarrow{\overline{\partial}} \mathcal{A}^{0,3} \xrightarrow{\overline{\partial}} \cdots.$$

Definition 8.1. *Let U be an open subset of $\mathbb{C}^n$. We define $\mathsf{A}^{p,q}(U)$ to be the complex vector space whose elements are formally denoted by*

$$\sum f_{i_1\ldots i_p;j_1\ldots j_q} dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q},$$

where the sum ranges over $0 \leq i_1 < \cdots < i_p \leq n$, $0 \leq j_1 < \cdots < j_q \leq n$, where $f_{i_1\ldots i_p;j_1\ldots j_q}$ denotes a complex-valued C^∞-function on U. The resulting sheaf on $\mathbb{C}^n$ will be denoted by $\mathcal{A}^{p,q}$

It is not difficult to see that $\mathcal{A}^{p,q}$ is closely related to the sheaf of smooth differential forms on $\mathbb{C}^n$.

Lemma 8.2. *Denote by $\mathcal{A}^m(U)$ the product $\prod_{p+q=m} \mathcal{A}^{p,q}(U)$. There is a canonical equivalence between $\mathcal{A}^m(U)$, and the complex vector space of $\mathbb{C}$-valued smooth m-forms*

$$\sum_{1\leq i_1<\cdots<i_n\leq 2m} f_{i_1\cdots i_{2m}} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \wedge \cdots \wedge dx^{i_m}$$

on U (as a $2n$-dimensional real manifold), where we use the notational shorthand $x^{n+j} = y^j$ for $j = 1, \ldots, n$.

Proof. An explicit isomorphism can be produced by making the substitutions: $dz^i = (dx^i + idy^i)$, $d\overline{z}^i = dx^i - idy^i$, $dx^i = \frac{1}{2}(dz^i + d\overline{z}^i)$, and $dy^i = \frac{1}{2i}(dz^i - d\overline{z}^i)$. $\square$

As for m-forms, we have a wedge product, $\mathcal{A}^{p,q} \times \mathcal{A}^{p',q'} \longrightarrow \mathcal{A}^{p+p',q+q'}$, which is given juxtaposition of terms like $dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}$, and $dz^{i_1} \wedge \cdots dz^{i_{p'}} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_{q'}}$, and re-arranging terms. As usual we have the rule $dz^i \wedge dz^i = 0$, and $d\overline{z}^i \wedge d\overline{z}^i = 0$.

Next we define the operators ∂ and $\overline{\partial}$.

Definition 8.3. *The linear maps $\partial^{p,q} \colon \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p+1,q}$ and $\overline{\partial}^{p,q} \colon \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q+1}$ are defined as the unique linear maps, such that*

$$d\colon \mathcal{A}^m \longrightarrow \mathcal{A}^{m+1}$$

for $m = p + q$ decomposes as $d = \sum_{p+q=m}(\partial^{p,q} + \overline{\partial}^{p,q})$.

This implies directly the important relations $\partial^2 = 0$, and $\overline{\partial}^2 = 0$.

To obtain an explicit formula for ∂ and $\overline{\partial}$, one uses that $\frac{\partial}{\partial x^i} = \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \overline{z}^i}$, and $\frac{\partial}{\partial y^i} = i(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \overline{z}^i})$. Since we have

$$d(f dx^{i_1} \wedge \cdots \wedge dx^{i_{m'}} \wedge dy^{i_{m'+1}} \wedge \cdots \wedge dy^{i_{2m}}) = \sum_{j=1}^{n} (\frac{\partial f}{\partial x^j} dx^j + \frac{\partial f}{\partial y^j} dy^j) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{m'}} \wedge dy^{i_{m'+1}} \wedge \cdots \wedge dy^{i_{2m}},$$

we obtain from the identity

$$\frac{\partial f}{\partial x^j}dx^j + \frac{\partial f}{\partial y^j}dy^j = \frac{1}{2}(\frac{\partial f}{\partial z^j} + \frac{\partial f}{\partial \overline{z}^j})(dz^j + d\overline{z}^j) + \frac{1}{2}(\frac{\partial f}{\partial z^j} - \frac{\partial f}{\partial \overline{z}^j})(dz^j - d\overline{z}^j) = \frac{\partial f}{\partial z^j}dz^j + \frac{\partial f}{\partial \overline{z}^j}d\overline{z}^j,$$

that

$$\partial^{p,q}(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}) = \sum_{j=1}^n \frac{\partial f}{\partial z^j} dz^j \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q},$$

and

$$\overline{\partial}^{p,q}(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}) = \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}.$$

Lemma 8.4 (Dolbeault–Grothendieck)**.** *The complex of sheaves* $\mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,2} \longrightarrow \cdots$ *is exact in positive degrees. The kernel* $\ker \overline{\partial^{0,0}}$ *is equivalent to the sheaf* $\mathcal{O}_{\mathbb{C}^n}$ *of holomorphic functions.*

The proof of this result is not entirely straightforward. We begin by examining a special case. We may assume $p = 0$, because the terms $dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ are merely decorative, as long as we only care about the $\overline{\partial}$-operators. The proof uses the general Cauchy integral formula.

Lemma 8.5. *Let $U \subset \mathbb{C}$ be an open subset, and $f \colon U \longrightarrow \mathbb{C}$ a C^∞-function. Then, we have for every $z_0 \in U$, and $\epsilon > 0$, such that $\overline{U}_\epsilon(z_0) \subset U$, the formula*

$$2\pi i f(z) = \int_{|w-z_0|=\epsilon} \frac{f(w)}{w-z}dw + \int_{\overline{U}_\epsilon(z_0)} \frac{\partial f}{\partial \overline{w}}(w)\frac{dw \wedge d\overline{w}}{w-z},$$

where $z \in U_\epsilon(z_0)$.

The proof of this lemma is similar to the way we deduced the Cauchy integral formula from the Stokes theorem for smooth manifolds. The details are left to the reader as an exercise.

Proof of the Dolbeault–Grothendieck Lemma for $n = 1$. The complex has precisely two non-zero terms:

$$\mathcal{A}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1} \longrightarrow 0 \longrightarrow \cdots.$$

We already know that for a C^∞-function $f \colon U \longrightarrow \mathbb{C}$ we have $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z} = 0$ if and only if f is holomorphic. It remains to show that for every $(0,1)$-form $f(z)d\overline{z}$ on U there exists an open covering $U = \bigcup_{i \in I} U_i$, such that $f(z)d\overline{z}|_{U_i} = \overline{\partial} g_i$, for a C^∞-function $g_i \colon U_i \longrightarrow \mathbb{C}$.

For every $z_0 \in U$, we choose an ϵ-neighbourhood, such that $\overline{U}_\epsilon(z_0) \subset U$. We may assume that for $|z - z_0| \geq \frac{\epsilon}{2}$ we have $f(z) = 0$, by multiplying f with a suitable bump function. This does not restrict the generality, since we only care about producing g, such that $f = \overline{\partial} g = f$, locally around each $z_0 \in U$. It has the added advantage that f can now be considered as a C^∞-function on all of $\mathbb{C}$ (it's some kind of bump function itself).

For $z \in U_\epsilon(z_0)$ we define

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(w)}{w-z} dw \wedge d\overline{w}.$$

Substituting $w = w' + z$, we obtain

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(w'+z)}{w'} dw' \wedge d\overline{w}'.$$

We leave the straight-forward verification that g is of class C^∞ to the reader. Using the last formula, we obtain

$$2\pi i \frac{\partial}{\partial \overline{z}} g(z) = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}}(w'+z) \frac{1}{w'} dw' \wedge d\overline{w}' = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}}(w) \frac{1}{w-z} dw \wedge d\overline{w} = 2\pi i f(z).$$

And hence we see that $\overline{\partial} g = f$, as required. □

We apply the same strategy variable-by-variable to prove the n-dimensional version of the Grothendieck–Dolbeault Lemma.

Proof of the Grothendieck–Dolbeault Lemma. Let $\omega \in \mathcal{A}^{0,q}(U)$ be a $(0,q)$-form, such that $\overline{\partial}\omega = 0$. We claim that there for every $z_0 \in U$ there exists an $\epsilon > 0$, such that we have an $\phi \in \mathcal{A}^{0,q-1}(U_\epsilon(z_0)$ satisfying $\omega = \overline{\partial}\phi$.

The construction of ϕ is by induction. We assume that ω is built from the $(0,1)$-forms $d\overline{z}^1, \dots, d\overline{z}^k$, where $0 \leq k \leq n$, and we will prove the assertion by induction on k. If $k = 0$, then $\omega = 0$, and the assertion is true.

Let's assume that the lemma has already been proven for $k-1$. We can decompose ω as $\eta \wedge d\overline{z}^k + \omega'$, where η and ω' only contain the $(0,1)$-forms $d\overline{z}^1, \dots, d\overline{z}^{k-1}$. We write $\eta = \sum f_{i_1 \dots i_{q-1}} d\overline{z}^{i_1} \wedge \dots \wedge d\overline{z}^{i_{q-1}}$.

The relation $\overline{\partial}\omega = 0$ implies for $j > k$

$$\frac{\partial f_{i_1 \dots i_{q-1}}}{\partial \overline{z}^j} = 0. \tag{16}$$

We now use the same integration method as in the proof of the 1-dimensional case to deduce a smooth function $g_{i_1 \dots i_{q-1}}$, such that $\frac{\partial g_{i_1 \dots i_{q-1}}}{\partial \overline{z}^k} = f_{i_1 \dots i_{q-1}}$. Let $\phi' = \sum g_{i_1 \dots i_{q-1}} dz^{i_1} \wedge \dots \wedge dz^{i_{q-1}}$. By construction we have that $\omega - \overline{\partial}\phi'$ only contains the $(0,1)$-forms $d\overline{z}^1, \dots, d\overline{z}^{k-1}$. By induction, there exists a $\phi'' \in \mathcal{A}^{0,q-1}$ in some ϵ-neighbourhood of z_0, such that $\omega - \overline{\partial}\phi' = \overline{\partial}\phi''$. Hence, $\phi = \phi' + \phi''$ does the trick. □

There is an interesting refinement of the Grothendieck–Dolbeault Lemma, which applies to $(0,q)$-forms on $\mathbb{C}^n$.

Proposition 8.6. *Let $\omega \in \mathcal{A}^{0,q}(\mathbb{C}^m)$ be a $(0,q)$-form, such that $\overline{\partial}\omega = 0$. Then, there exists a $(0,q-1)$-form $\phi \in \mathcal{A}^{0,1}(\mathbb{C}^m)$, such that $\omega = \overline{\partial}\phi$.*

Proof. We begin with the case $q = 1$ which needs to be discussed separately. We write $\mathbb{C}^m = \bigcup_{n \geq 1} U_n(0)$, we claim that there exists a sequence of smooth functions $\phi_n \colon \mathbb{C}^m \longrightarrow \mathbb{C}$, such that for $\overline{\partial}\phi_n|_{U_{n+\frac{1}{2}}(0)} = \omega|_{U_{n+\frac{1}{2}}(0)}$, and for all $z \in \overline{U}_n(0)$ we have $|\phi_{n+1}(z) - \phi_n(z)| \leq \frac{1}{2^n}$. The existence of such a sequence follows inductively, the case $n = 1$ being a direct consequence of the proof of the Grothendieck–Dolbeault Lemma.

Let's assume by induction that $\phi_1, \dots, \phi_n$ have been constructed. We know that there exists ϕ'_{n+1}, such that $\overline{\partial}\phi'_{n+1}|_{U_{n+\frac{3}{2}}(0)} = \omega|_{U_{n+\frac{3}{2}}(0)}$. For $z \in U_n(0)$ we have $\overline{\partial}(\phi'_{n+1} - \phi)(z) = 0$ for

$z \in U_{n+\frac{1}{2}}(0)$, that is, the difference is holomorphic on $U_{n+\frac{1}{2}(0)}$. Choose a polynomial $g(z)$, such that $|\phi'_{n+1}(z) - \phi_n(z) - g(z)| < \frac{1}{2^n}$ for $z \in \overline{U}_n(0)$. Since g is everywhere holomorphic, we may define $\phi_{n+1} = \phi_n - g$, without affecting the property $\overline{\partial}\phi_{n+1}|_{U_{n+\frac{3}{2}}(0)} = \omega|_{U_{n+\frac{3}{2}}(0)}$.

Defining $\phi = \lim_{n\to\infty} \phi_n$ one sees directly that the resulting function is a continuous function, since the convergence is uniform on the subsets $\overline{U}_n(0)$. Writing $\phi = \phi_1 \sum_{n\geq 1}(\phi_{n+1} - \phi_n)$, one obtains a presentation of ϕ, which restricts on each $U_n(0)$ to a finite sum of smooth functions, ϕ_1, $\phi_2 - \phi_1, \ldots, \phi_n - \phi_{n-1}$, and a uniformly convergent infinite sum of holomorphic functions (thus it's holomorphic itself). We conclude that ϕ is a smooth function, and that $\overline{\partial}\phi = \omega$.

Let $q \geq 2$. We claim that there are $(0, q-1)$-forms ϕ_n, such that we have $\overline{\partial}\phi_n|_{U_{n+\frac{1}{2}}(0)} = \omega|_{U_{n+\frac{1}{2}}}$, and that $\phi_{n+1}|_{\overline{U}_n(0)} = \phi_n|_{\overline{U}_n(0)}$.

By induction we assume that $\phi_1, \ldots, \phi_n$ have been constructed. We choose ϕ'_{n+1}, such that $\overline{\partial}\phi'_{n+1}|_{U_{n+\frac{3}{2}}(0)} = \omega|_{U_{n+\frac{3}{2}}}$. The difference $\phi'_{n+1} - \phi_n$ is sent to 0 by $\overline{\partial}$ on the subset $U_{n+\frac{1}{2}}(0)$. Therefore, there exists a $(0, q-2)$-form ψ, such that

$$(\phi'_{n+1} - \phi_n)|_{U_{n+\frac{1}{4}}(0)} = \overline{\partial}\psi|_{n+\frac{1}{4}(0)}.$$

Therefore, we see that $\phi_{n+1} = \phi'_{n+1} - \overline{\partial}\psi$ satisfies $\phi_{n+1}|_{\overline{U}_n(0)} = \phi_n|_{\overline{U}_n(0)}$, and $\overline{\partial}\phi_{n+1} = \overline{\partial}\phi'_{n+1}$, because $\overline{\partial}^2 = 0$. $\square$

Definition 8.7. *Let X be a complex manifold. For an integer k we denote by $\mathcal{A}^k$ the sheaf of smooth, complex-valued k-forms on X (take the usual vector space of smooth k-forms and apply the tensor product functor $-\otimes_{\mathbb{R}} \mathbb{C}$). By choosing local coordinates, we define $\mathcal{A}_X^{p,q}$ (for $p+q=k$) to be the subsheaf consisting to k-forms which are locally expressible as*

$$\sum f_{i_1\ldots i_p;j_1\ldots j_q} dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}.$$

8.2 Partitions of unity and sheaf cohomology

In this paragraph we will show that the Dolbeault resolution that we introduced earlier is a so-called *acyclic resolution* of the sheaves Ω^p. This implies in particular that

$$H^q(X, \Omega^p) \cong \frac{\ker(\overline{\partial}\colon \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q+1})}{\mathrm{image}(\overline{\partial}\colon \mathcal{A}^{p,q-1} \longrightarrow \mathcal{A}^{p,q})}.$$

This follows at once from the Universal Comparison Theorem 6.10, once we have shown that the sheaves $\mathcal{A}^{p,q}$ don't have any higher cohomology themselves (that is, are *acyclic*). We will deduce this from the existence of partitions of unity.

Definition 8.8. *Let X be a topological space, and $\mathcal{R}$ a sheaf of rings on X. We denote by $U \subset X$ a closed subset of X, and $X = \bigcup_{i\in I} U_i$ an open covering of X.*

(a) The support, $\operatorname{supp} s$*, of a section $s \in \mathcal{R}(U)$ is defined to be the closure of the set $\{x \in X | s_x \in \mathcal{R}_x \setminus 0\}$. Alternatively, you can define* $\operatorname{supp} s$ *to be the complement of the maximal open subset $V \subset X$, such that $s|_{V\cap U} = 0$. By definition, we have* $\operatorname{supp} s \subset \overline{U}$.

(b) A partition of unity*, subordinate to $\{U_i\}_{i\in I}$ consists of a family of sections $(s_j)_{j\in J} \in \mathcal{R}(X)^J$, such that*

- *for every $j \in J$ there exists an $i \in I$, such that we have* $\operatorname{supp} s_j \subset U_i$,
- *every $x \in X$ has a neighbourhood V, such that only finitely many sections $s_j|_V$ are non-zero,*
- *and the identity $\sum_{j \in J} s_j = 0$ holds (the infinite sum makes sense because all but finitely many sections vanish locally).*

(c) If $\mathcal{R}$ is a sheaf of rings which admits a partition of unity subordinate to any cover then we say that $\mathcal{R}$ is fine[3].

There are many examples of fine sheaves of rings, in fact most examples the reader encountered previously to studying complex manifolds were probably fine. Partitions of unity exist for continuous functions on paracompact spaces. On smooth manifolds partitions of unity can be constructed using bump functions. Holomorphic partitions of unity don't exist however. The support condition $\operatorname{supp} s_j \subset U_i$ would imply $s = 0$.

Lemma 8.9. *Let $U \subset X$ be an open subset, and $s \in \mathcal{R}(X)$ a section, such that* $\operatorname{supp} s \subset U$. *Given any sheaf of $\mathcal{R}$-modules $\mathcal{M}$, and any section $m \in \mathcal{M}(U)$, there is a unique section m', denoted $s \cdot m$ in future, such that $m'|_U = s \cdot m$, and $m'|_{X \setminus \operatorname{supp} s} = 0$.*

Proof. This is a straight-forward application of the sheaf property for $\mathcal{M}$ and the open covering $X = U \cup (X \setminus \operatorname{supp} s)$. Since $(s \cdot m)|_{U \setminus \operatorname{supp} s} = 0$, we see that there exists a unique such section. □

Lemma 8.10. *If $\mathcal{R}$ is a fine sheaf of rings, then the higher cohomology groups of every sheaf of $\mathcal{R}$-modules vanish.*

Proof. We prove this by our usual inductive trick. Let $\mathcal{M}$ be an $\mathcal{R}$-module, and

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{I} \longrightarrow \mathcal{I} / \mathcal{M} \longrightarrow 0$$

a short exact sequence of $\mathcal{R}$-modules, such that $\mathcal{I}$ is injective. Analysing our construction of injective sheaves containing a given sheaf, one sees that this also works for sheaves of $\mathcal{R}$-modules. The long exact sequence (and vanishing of higher cohomology of injective sheaves) imply

$$H^i(X, \mathcal{M}) \cong H^{i-1}(X, \mathcal{I} / \mathcal{M})$$

for $i \geq 2$, and

$$H^1(X, \mathcal{M}) \cong \operatorname{coker}(H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{I} / \mathcal{M})).$$

By induction we have therefore reduced the proof to showing that the map $H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{I} / \mathcal{M})$ is surjective. Let $s \in \mathcal{I} / \mathcal{M}(X)$ be a section. By definition of the quotient sheaf $\mathcal{I} / \mathcal{M}$ there exists an open covering $X = \bigcup_{i \in I} U_i$, and sections $t_i \in \mathcal{I}(U_i)$, such that $s|_{U_i}$ equals the image of t_i in $\mathcal{I} / \mathcal{M}$.

Since $\mathcal{R}$ is a fine sheaf of rings, there exists a partition of unity $(e_j)_{j \in J}$, subordinate to $\{U_i\}_{i \in I}$. Choose for every $j \in J$ an $i(j) \in I$, such that $\operatorname{supp} e_j \subset U_{i(j)}$. Define the section

$$t = \sum_{j \in J} e_j \cdot t_{i(j)},$$

[3] In the literature one finds the definition of fine sheaves, which is similar but slightly different from the one given here. In the case of sheaves of rings it is an easy exercise to check that a sheaf is fine in our sense if and only if it is fine in the traditional one.

we have seen in Lemma 8.9 that this is well-defined. We can check stalk-by-stalk that the section t is mapped to the section s in the quotient $\mathcal{I}/\mathcal{M}$. This shows surjectivity of the map $H^0(X,\mathcal{I}) \longrightarrow H^0(X,\mathcal{I}/\mathcal{M})$ and hence provides an anchor for our inductive argument. $\square$

Corollary 8.11. *For any complex manifold X the sheaves $\mathcal{A}_X^{p,q}$ are acyclic. In particular, one obtains that the Dolbeault complex*

$$\mathcal{A}_X^{p,0}(X) \longrightarrow \mathcal{A}_X^{p,1}(X) \longrightarrow \cdots$$

computes the sheaf cohomology $H^q(X,\Omega_X^p)$.

As a another consequence we obtain that the de Rham complex of a smooth manifold X

$$\Omega^0_{X,\mathbb{R}} \longrightarrow \Omega^1_{X,\mathbb{R}} \longrightarrow \cdots$$

computes the cohomology of the sheaf of locally constant $\mathbb{R}$-valued functions $\underline{\mathbb{R}}_X$.

Corollary 8.12. *The sheaves $\mathcal{O}_{\mathbb{C}^n}$, and $\Omega^p_{\mathbb{C}^n}$ have vanishing higher cohomology groups.*

Proof. This is a direct consequence of Proposition 8.6. $\square$

This corollary can be generalised by replacing $\mathbb{C}^n$ by open subsets of the shape $(\mathbb{C}^\times)^r \times \mathbb{C}^s$.

8.3 Čech cohomology

In this paragraph we fix a topological space X, and a sheaf $\mathcal{F} \in \mathsf{Sh}(X)$. We assume that we have a open covering $X = \bigcup_{i\in I} U_i$. The Čech cohomology of $\mathcal{F}$ with respect to the covering $\{U_i\}_{i\in I}$ is defined to be the cohomology of the following chain complex:

$$\prod_{i\in I}\Gamma(U_i,\mathcal{F}) \longrightarrow \prod_{(i,j)\in I^2}\Gamma(U_{ij},\mathcal{F}) \longrightarrow \prod_{(i,j,k)\in I^3}\Gamma(U_{ijk},\mathcal{F}) \longrightarrow \cdots,$$

where the maps are given by

$$\delta\colon \check{C}^p(X,\mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}}\Gamma(U_{i_0\cdots i_p},\mathcal{F}) \longrightarrow \prod_{(j_0,\dots,j_{p+1})\in I^{p+2}}\Gamma(U_{j_0\dots j_{p+1}},\mathcal{F}) = \check{C}^{p+1}(X,\mathcal{F}),$$

which sends $(c_{i_0\cdots i_p})_{i_0\dots i_p}$ to the tuple of sections $(d_{j_0\dots j_{p+1}})$, such that

$$d_{j_0\dots j_{p+1}} = \sum_{i=0}^{p+1}(-1)^i(c_{j_0\dots\widehat{j_i}\dots j_{p+1}}|_{U_{j_0\cdots j_{p+1}}}).$$

One can verify by a direct computation that $\delta^2 = 0$, hence this defines a chain complex, known as the Čech complex. The sheaf property of $\mathcal{F}$ implies directly that $\check{H}^0_{\{U_i\}_{i\in I}}(X,\mathcal{F}) = \mathcal{F}(X)$.

Definition 8.13. *We will denote the cohomology group of this complex by $\check{H}^p_{\{U_i\}_{i\in I}}(X,\mathcal{F})$. It is called the degree p Čech cohomology of $\mathcal{F}$ with respect to the open covering $\{U_i\}_{i\in I}$.*

Elements of $H^p_{\{U_i\}_{i\in I}}(X, \mathcal{F})$ are therefore presented by so-called Čech p-cocycles. That is, a collection of local sections $(c_{i_0\cdots i_p})_{i_0\ldots i_p}$, satisfying $\delta((c_{i_0\cdots i_p})_{i_0\ldots i_p}) = 0$. For $p = 1$ we have already seen this in our discussion of line bundles and G-Galois covers.

In a way, Čech cohomology is amenable to direct computation, at least when working with a finite cover. In this case, one can also show that $\check{H}^i_{\{U_i\}_{i\in I}}(X, \mathcal{F})$ vanishes for $i > |I| - 1$.

Lemma 8.14. *On a complex manifold X, $\check{H}^1_{\{U_i\}_{i\in I}}(X, \mathcal{O}_X^\times)$ is canonically equivalent to the abelian group of isomorphism classes of complex line bundles L, which are trivialisable when restricted to the open subsets U_i for every $i \in I$.*

Proof. We denote this set of line bundles (defined in the statement of the lemma) by $\mathsf{Pic}_{\{U_i\}_{i\in I}}(X)$. By definition, every $L \in \mathsf{Pic}_{\{U_i\}_{i\in I}}(X)$ admits an abstract cocycle datum $\big(\{U_i\}_{i\in I}, (\phi_{ij})_{(i,j)\in I^2}\big)$. By definition, the collection of nowhere vanishing holomorpic functions $\phi_{ij} \in \mathcal{O}_X^\times(U_{ij})$ satisfies the cocycle condition

$$\phi_{ik} = \phi_{ij} \cdot \phi_{jk},$$

which (after switching from multiplicative to additive notation) amounts precisely to the condition that $(\phi_{ij}) \in \ker \delta$.

Two cocycle data for L with respect to the same covering $\{U_i\}_{i\in I}$ can differ only to the effect that different trivialisations $\phi_i\colon L|_{U_i} \xrightarrow{\simeq} U_i \times \mathbb{C}$ could have been chosen. Two such trivialisations can be related by multiplying with a nowhere vanishing function $\phi_i \in \mathcal{O}_X^\times(U_i)$. The resulting 1-cocycle would be $(\phi_i\phi_j^{-1})\phi_{ij}$. That is, differs by precisely a Čech coboundary from (ϕ_{ij}). $\square$

Virtually the same argument shows the following.

Lemma 8.15. *For a manifold X, and an abelian group A, $\check{H}^1_{\{U_i\}_{i\in I}}(X, \underline{A})$ is canonically equivalent to the abelian group of isomorphism classes of A-Galois covers of X, which are trivialisable when restricted to the open subsets U_i for every $i \in I$.*

Čech cohomology groups can also be defined without fixing a particular cover (or in other words, by considering all open coverings at once). In order to give this definition, we have to recall the notion of a filtered colimit. This construction is reminiscent of how we defined stalks of sheaves at a point $x \in X$.

Definition 8.16. *Let I be a partially ordered set, such that every 2-element subset $\{i, j\} \subset I$ has a common upper bound (we also say that I is filtered). Let $(A_i)_{i\in I}$ a family of abelian groups (or vector spaces), such that for every $i \le j$ we have a homomorphism $f^i_j\colon A_i \longrightarrow A_j$, satisfying the relation $f^i_i = \mathrm{id}_{A_i}$, and $f^j_k \circ f^i_j = f^i_k$ for every ordered triple $i \le j \le k$. The colimit (a. k. a. direct limit) of $(A_i)_{i\in I}$ is defined to be the abelian group (resp. vector space)*

$$\operatorname*{colim}_{i\in I} A_i = \varinjlim_{i\in I} A_i,$$

consisting of equivalence classes of pairs (i, a). Here, $i \in I$, and $a \in A_i$. And we say that (i, a) is equivalent to (j, b) if and only if there exists a common upper bound k of $\{i, j\}$, such that we have an element $c \in A_k$, satisfying $f^i_k(a) = c = f^j_k(b)$.

The set of open neighbourhoods U of $x \in X$, with respect to the partial ordering $U \le V \Leftrightarrow V \subset U$ is a filtered poset. Given a sheaf $\mathcal{F}$ on X, we obtain a directed system of abelian groups $\mathcal{F}(U)$, and the induced colimit is by definition equivalent to the stalk $\mathcal{F}_x$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$, and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X. We say that $\mathcal{U} \leq \mathcal{V}$, if for every $j \in J$, there exists an $i(j) \in I$, such that $V_j \subset U_i$. The resulting map $J \longrightarrow I$ allows us to define a map between the resulting Čech cohomology groups

$$\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) \longrightarrow \check{H}^p_{\mathcal{V}}(X, \mathcal{F}).$$

We define *the Čech cohomology* of $\mathcal{F}$ to be the direct limit of this system of cohomology groups, and denote it by $\check{H}^p(X, \mathcal{F})$.

As an immediate corollary of the lemmas about line bundles and A-Galois covers, we obtain the following assertions. We just use the fact that any line bundle, respectively A-Galois covering, is trivialised by some open covering, and hence they all get detected by the Čech cohomology group $\check{H}^1(X, -)$.

Corollary 8.17. *(a) On a complex manifold X, the abelian group $\check{H}^1(X, \mathcal{O}_X^\times)$ is equivalent to the abelian group of isomorphism classes of line bundles* $\mathsf{Pic}(X)$.

(b) On a connected manifold X, the abelian group $\check{H}^1(X, \underline{A})$ is equivalent to the abelian group of isomorphism classes of A-Galois covers $\mathit{Hom}(\pi_1(X), A)$.

Our next goal is to replace Čech cohomology in the statement above by actual sheaf cohomology. Therefore we embark on a more detailed study of the relation between sheaf cohomology and Čech cohomology. Note that if X is a compact topological space, then every open covering is $\leq$ then a finite covering. Hence it's sufficient to work with respect to finite coverings on compact spaces.

Proposition 8.18. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X. For every tuple $(i_0, \ldots, i_r) \in I^{r+1}$ we denote by $j_{i_0 \ldots i_r} \colon U_{i_0 \ldots i_r} = U_{i_0} \cap \cdots \cap U_{i_r} \hookrightarrow X$ the inclusion. Let $\mathcal{F}$ be a sheaf on X, such that for every $(i_0, \ldots, i_r) \in I^{r+1}$ we have that $(j_{i_0 \ldots i_r})_*(\mathcal{F}|_{U_{i_0 \cdots i_r}})$ is acyclic (that is, no higher cohomology), then $\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) \cong H^p(X, \mathcal{F})$.*

Proof. The Čech resolution also makes can also be viewed as a resolution of sheaves, that is, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{C}^0_{\mathcal{U}}(\mathcal{F}) \longrightarrow \check{C}^1_{\mathcal{U}}(\mathcal{F}) \longrightarrow \cdots,$$

where $\check{C}^p_{\mathcal{U}}(\mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in I^{p+1}} (j_{i_0 \ldots i_r})_*(\mathcal{F}|_{U_{i_0 \cdots i_r}})$. By assumption, $\check{C}^p_{\mathcal{U}}(\mathcal{F})$ is a product of acyclic sheaves, and thus is acyclic itself. The Universal Comparison Theorem 6.10 implies that $H^p(X, \mathcal{F})$ agrees with the degree p cohomology of the chain complex of global sections, which is the Čech complex with respect to $\mathcal{U}$. □

Corollary 8.19. *If $\mathcal{F}$ is a flasque sheaf on X, then its higher Čech cohomology vanishes with respect to any open covering $\mathcal{U}$. In particular $\check{H}^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. We have seen in the exercises that flasque sheaves are acyclic. Since restrictions to open subsets, and pushforwards of flasque sheaves are flasque, we obtain that the sheaves $\check{C}^p(\mathcal{F})$ are flasque. Therefore, they are acyclic, and hence assumptions of Proposition 8.18 are satisfied. We deduce that $0 = H^i(X, \mathcal{F}) = \check{H}^i_{\mathcal{U}}(X, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$. □

So far we have seen that Čech cohomology verifies the two axioms (A1) and (A3) of sheaf cohomology. In general it is not true that every short exact sequence of sheaves yields a long exact sequence of Čech cohomology groups, and hence sheaf cohomology will not always agree with Čech cohomology. However, whenever this missing axiom is satisfied by the Čech cohomology groups, we obtain directly from our axiomatic approach that Čech cohomology agrees with sheaf cohomology. Nonetheless, we always have a long exact sequence up to degree 1.

Lemma 8.20. *Given a short exact sequence of sheaves $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ on X, we have an exact sequence of Čech cohomology groups*

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^1(X,\mathcal{F}) \longrightarrow \check{H}^1(X,\mathcal{G}) \longrightarrow \check{H}^1(X,\mathcal{H}).$$

Proof. We only define the map δ, and check exactness of $\Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^1(X,\mathcal{F})$, and leave the remaining parts as an exercise to the reader.

We have a map $\delta \colon \Gamma(X,\mathcal{H}) \longrightarrow \check{H}^1(X,\mathcal{F})$, which is defined as follows. Let $s \in \Gamma(X,\mathcal{H})$ be a section, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering, such that we have sections $t_i \in \mathcal{G}(U_i)$, which map to the restrictions $s|_{U_i}$ for every $i \in I$. By exactness, $(t_i|_{U_{ij}}) - (t_j|_{U_{ij}}) = c_{ij}$, defines a section of $\mathcal{F}$ over U_{ij}. The construction as a difference, reveals directly that c_{ij} is a Čech 1-cocycle with respect to $\mathcal{U}$. We define $\delta(s) = [(c_{ij})]$.

If s has a global lift, then we can choose $\mathcal{U}$ to be the trivial cover $X = X$, and hence get the zero cocycle $s_{ii} = 0$. This implies that the composition $\Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^1(X,\mathcal{F})$ equals the zero map.

Vice versa, if $\delta(s) = 0$, then there exists a Čech 1-cocycle $(d_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i)$, such that $c_{ij} = (d_i - d_j)|_{U_{ij}}$. We can then define $t'_i = t_i - d_i$, and obtain $t'_i|_{U_{ij}} = t'_j|_{U_{ij}}$. Hence, the sheaf property implies that there exists a global section $t' \in \mathcal{G}(X)$, such that $t'|_{U_i} = t_i$. This concludes the proof of exactness. □

Corollary 8.21. *We have $H^1(X,\mathcal{F}) \cong \check{H}^1(X,\mathcal{F})$. In particular, if X is a complex manifold, then $\mathsf{Pic}(X) \cong H^1(X,\mathcal{O}_X^\times)$. Moreover, we have for X a connected manifold that $H^1(X,\underline{A}) = \mathsf{Hom}(\pi_1(X),A)$ (that is, isomorphism classes of A-Galois covers).*

Proof. Choose a short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{F} \longrightarrow 0$ with $\mathcal{I}$ injective. Since injective sheaves are flasque, we have $H^1(X,\mathcal{I}) = \check{H}^1(X,\mathcal{I}) = 0$. The long exact sequences imply

$$H^1(X,\mathcal{F}) = \operatorname{coker}(\Gamma(X,\mathcal{I}) \longrightarrow \Gamma(X,\mathcal{I}/\mathcal{F})) \cong \check{H}^1(X,\mathcal{F}).$$

The assertion about $\mathsf{Pic}(X)$ follows from Corollary 8.21 (a). □

However, in most cases (that is spaces or sheaves we actually care about), Čech cohomology gives the right answer, that is, agrees with sheaf cohomology. We have the following theorem:

Theorem 8.22 (Leray)**.** *Let $\mathcal{F}$ be a sheaf on X, and $\mathcal{U}$ an open covering, such that $H^j(U_{i_0\ldots i_r}, \mathcal{F}|_U) = 0$, for every tuple $(i_0,\ldots,i_r) \in I^{r+1}$, where $U_{i_0\ldots i_r} = U_{i_0} \cap \cdots \cap U_{i_r}$. Then $\check{H}^i_{\mathcal{U}}(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$.*

In the proof of this theorem, we have to produce a suitable long exact sequence for Čech cohomology groups of sheaves satisfying the condition of the theorem. This is possible thanks to the following lemma.

Lemma 8.23. *Let $C^\bullet$, $D^\bullet$, and $E^\bullet$ be chain complexes, and $0 \longrightarrow C^\bullet \longrightarrow D^\bullet \longrightarrow E^\bullet \longrightarrow 0$ a short exact sequence. Then, we have a long exact sequence of cohomology groups*

$$\cdots \longrightarrow H^i(C^\bullet) \longrightarrow H^i(D^\bullet) \longrightarrow H^i(E^\bullet) \longrightarrow H^{i+1}(C^\bullet) \longrightarrow \cdots.$$

Proof. For a chain complex $C^\bullet$ we write $B^i_{C^\bullet}$ for image$(d^{i-1}\colon C^{i-1}\longrightarrow C^i)$, and $Z^i_{C^\bullet} = \ker(d^i\colon C^i\longrightarrow C^{i+1})$. By definition, we have $H^i(C^\bullet) = Z^i(C^\bullet)/B^i(C^\bullet)$. We have a commutative diagram with exact rows

$$\begin{tikzcd} & C^i/B^i_{C^\bullet} \arrow[r] \arrow[d, "d^i"'] & D^i/B^i_{D^\bullet} \arrow[r] \arrow[d, "d^i"'] & E^i/B^i_{E^\bullet} \arrow[r] \arrow[d, "d^i"'] & 0 \\ 0 \arrow[r] & Z^{i+1}_{C^\bullet} \arrow[r] & Z^{i+1}_{D^\bullet} \arrow[r] & Z^{i+1}_{E^\bullet} & \end{tikzcd}$$

and the Snake Lemma yields a long exact sequence

$$H^i(C^\bullet)\longrightarrow H^i(D^\bullet)\longrightarrow H^i(E^\bullet)\longrightarrow H^{i+1}(C^\bullet)\longrightarrow H^{i+1}(D^\bullet)\longrightarrow H^{i+1}(E^\bullet),$$

where we have used that the kernel of $C^i/B_{C^\bullet} \xrightarrow{d^i} Z^{i+1}_{C^\bullet}$ is equivalent to $H^i(C^\bullet)$, and its cokernel to $H^{i+1}(C^\bullet)$. □

Proof of Theorem 8.22. We can now prove Leray's theorem. Let $0\longrightarrow \mathcal{F}\longrightarrow \mathcal{I}\longrightarrow \mathcal{I}/\mathcal{F}\longrightarrow 0$ be a short exact sequence with $\mathcal{I}$ exact. Since injective sheaves don't have higher cohomology, we see that $\mathcal{F}$, $\mathcal{I}$, and $\mathcal{I}/\mathcal{F}$ satisfy the assumptions of the theorem (the quotient as a consequence of the long exact sequence in cohomology). Moreover, vanishing of cohomology of these sheaves on the subsets $U_{i_0\cdots i_p}$ guarantees that we have a long exact sequence of chain complexes in abelian groups

$$0\longrightarrow \check{C}^\bullet(X,\mathcal{F})\longrightarrow \check{C}^\bullet(X,\mathcal{I})\longrightarrow \check{C}^\bullet(X,\mathcal{I}/\mathcal{F})\longrightarrow 0.$$

The previous lemma (8.23) implies that we have a long exact sequence of cohomology groups

$$\cdots\longrightarrow \check{H}^i(X,\mathcal{F})\longrightarrow \check{H}^i(X,\mathcal{I})\longrightarrow \check{H}^i(X,\mathcal{I}/\mathcal{F})\longrightarrow \check{H}^{i+1}(X,\mathcal{F})\longrightarrow\cdots.$$

Since injective sheaves are flasque, and flasque sheaves don't have any higher Čech cohomology groups by Corollary 8.19, we obtain

$$\check{H}^1(X,\mathcal{F}) \cong \operatorname{coker}(H^0(X,\mathcal{I})\longrightarrow H^0(X,\mathcal{I}/\mathcal{F})) \cong H^1(X,\mathcal{F}),$$

and for $i\geq 2$ the equivalence

$$\check{H}^i(X,\mathcal{F}) \cong \check{H}^{i-1}(X,\mathcal{I}/\mathcal{F}).$$

By induction we see that $\check{H}^i(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$. □

In the next section we will use Leray's theorem to compute the cohomology of $\mathcal{O}_{\mathbb{P}^n}$.

8.4 Cohomology of complex projective space

In this paragraph we will explain two important results for the cohomology of sheaves on $\mathbb{P}^n$.

Theorem 8.24 (Cohomology of $\mathbb{P}^n$)**.** *Recall that for an abelian group A we denote by $\underline{A}$ the sheaf of locally constant A-valued functions.*

(a) We have $H^{2i+1}(\mathbb{P}^n,\underline{A}) = 0$, and $H^{2i}(\mathbb{P}^n,\underline{A}) \cong A$, for $0\leq i\leq n$.

(b) The cohomology group $H^i(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}) = 0$ vanishes for $i\geq 1$, and $H^0(\mathbb{P}^n,\mathbb{C}) = \mathbb{C}$.

Assertion (a) for $2i+1 = 1$ follows from Corollary 8.21, because $\mathbb{P}^1$ is connected and 1-connected. The general case follows from a computation of the singular cohomology groups of $\mathbb{P}^n$, using that $\mathbb{P}^n$ has a CW decomposition with precisely one cell for every even number between 0 and $2n$.

We now turn to the details of the proof of (b). We can apply Leray's Theorem 8.22 to the standard covering $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ of projective space, since $\mathcal{O}_{\mathbb{P}^n}$ is known not to have any cohomology over the intersections $U_{i_0\dots i_p}$ by Cartan's Theorem B (see Proposition 7.37 which is covering the cases of relevance to us, and also the strengthening of the Grothendieck-Dolbeault Lemma 8.6).

Lemma 8.25. *Let $f\colon \mathbb{C}^\times \times U \longrightarrow \mathbb{C}$ be a holomorphic function in complex variables $(z, w_1, \dots, w_{n-1})$, where we assume $z \neq 0$, and that $U \subset \mathbb{C}^{n-1}$ is an open subset. There exist holomorphic functions $g\colon \mathbb{C} \times U \longrightarrow \mathbb{C}$, and $h\colon \mathbb{C} \times U \longrightarrow \mathbb{C}$, such that we have*

$$f(z, w_1, \dots, w_{n-1}) = g(z, w_1, \dots, w_n) - h(\frac{1}{z}, w_1, \dots, w_{n-1}),$$

for $(z, w_1, \dots, w_{n-1}) \in \mathbb{C}^\times \times \mathbb{C}^{n-1}$.

Proof. We define

$$g(z, w_1, \dots, w_{n-1}) = \frac{1}{2\pi i}\int_{|u|=R} \frac{f(u, w_1, \dots, w_{n-1})}{(u-z)} du,$$

where R is an arbitrary positive real number, satisfying $R > |z|$. Homotopy invariance of integrals of holomorphic functions implies that this is a well-defined holomorphic function on $\mathbb{C}^n$. We can compute $f - g$ for $z \neq 0$, by using the Cauchy-integral formula for f:

$$\frac{1}{2\pi i}\int_{|u|=R} \frac{f(u, w_1, \dots, w_{n-1})}{(u-z)} du - \frac{1}{2\pi i}\int_{|u-z|=\epsilon} \frac{f(u, w_1, \dots, w_{n-1})}{(u-z)} du,$$

where ϵ is a sufficiently small positive number. Using the Residue Theorem we obtain

$$f - g = \operatorname{res}_{u=0} \frac{f(u, w_1, \dots, w_{n-1})}{u-z} = -\frac{\operatorname{res}_{u=0} f}{z},$$

where $\operatorname{res}_{u=0} f$ is a holomorphic function in the variables $(w_1, \dots, w_{n-1})$. Since $(f-g)(\frac{1}{z}, w_1, \dots, w_{n-1})$ is defined for all values of z, we see that $h(z, w_1, \dots, w_{n-1}) = z \operatorname{res}_{u=0} f(u, w_1, \dots, w_{n-1})$ satisfies the requirements. ☐

As an immediate corollary we obtain vanishing of $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ for $i \geq 1$. For $i \geq 2$ this vanishing result follows from the Dolbeault resolution, hence it is sufficient to prove vanishing for $i = 1$ in this case.

Corollary 8.26. *We have $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.*

Proof. We can apply Leray's theorem to $X = \mathbb{P}^1$, and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}$ and the standard covering $U_0 \cup U_1 = \mathbb{P}^1$. The reason is that $U_i \cong \mathbb{C}$, and $U_0 \cap U_1 \cong \mathbb{C}^\times$, and by the Grothendieck–Dolbeault Lemma there the sheaf of holomorphic functions $\mathcal{O}$ is acyclic on $\mathbb{C}$, and $\mathbb{C}^\times$. Therefore, the cohomology group $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}})$ is equivalent to the degree i cohomology of the Čech complex:

$$\mathcal{O}(U_0) \times \mathcal{O}(U_1) \longrightarrow \mathcal{O}_{U_0} \times \mathcal{O}_{U_1} \times \mathcal{O}_{U_0 \cap U_1} \longrightarrow \cdots .$$

The definition of the boundary operator δ in Čech cohomology implies that $\delta(f, g) = (0, 0, f|_{U_{01}} - g|_{U_{01}})$, and that $\delta^1\colon \check{C}^1_{\mathcal{U}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}) \longrightarrow \check{C}^2_{\mathcal{U}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}})$ is the zero map, and similarly in higher degrees.

Therefore we are really computing the cohomology of the complex

$$\mathcal{O}(U_0) \times \mathcal{O}(U_1) \longrightarrow \mathcal{O}_{U_0 \cap U_1} \longrightarrow 0 \longrightarrow \cdots .$$

We already know $H^0(\mathbb{P}^1, \mathcal{O}) \cong \mathbb{C}$, and it remains to compute H^1 as the cokernel of the map $\mathcal{O}(U_0) \times \mathcal{O}(U_1) \longrightarrow \mathcal{O}_{U_0 \cap U_1}$, which sends (g,h) to $g|_{U_{01}} - h|_{U_{01}}$. But we have seen in Lemma 8.25 that every holomorphic function on U_{01} can be written as the difference of a holomorphic function on U_0 and on U_1. This shows that the cokernel is 0, and therefore $H^i(\mathbb{P}^1, \mathcal{O}) = 0$ for $i \geq 1$. ☐

In order to convey a feeling for the higher-dimensional case, we will show $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$.

Corollary 8.27. *We have $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$.*

Proof. We apply Leray's Theorem to the standard covering $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$. We have to show that a collection of holomorphic functions $(f_{012}, f_{013}, f_{023}, f_{123})$, satisfying

$$f_{123} - f_{023} + f_{013} - f_{012} = 0, \tag{17}$$

can be expressed as $(g_{12} - g_{02} + g_{01}, g_{13} - g_{03} + g_{01}, g_{23} - g_{03} + g_{02}, g_{23} - g_{13} + g_{12})$ for holomorphic functions g_{ij} on the double intersections U_{ij}.

We set $g_{01} = 0$, and choose g_{12}, g_{02}, such that $g_{12} - g_{02} = f_{012}$, using Lemma 8.25. Now we choose holomorphic functions g_{13} and g_{03}, such that $g_{13} - g_{03} = f_{013} - g_{01} = f_{013}$.

Next, we choose holomorphic functions g_{23}, g_{03}, such that $g_{23} - g_{03} = f_{023} - g_{02}$. It remains to check that we have $g_{23} - g_{13} + g_{12} = f_{123}$. But this follows directly from relation (17). ☐

8.5 Line bundles on complex manifolds

The cohomology computation of $\mathcal{O}_{\mathbb{P}^n}$ is an essential ingredient in the proof of GAGA, and allows us to classify line bundles on $\mathbb{P}^n$. Recall the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}^{\times}_{\mathbb{P}^n} \longrightarrow 0.$$

Using the equivalence $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n} \times) \cong \mathsf{Pic}(\mathbb{P}^n)$ of Corollary 8.21, we can state the associated long exact sequence as

$$\cdots \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \longrightarrow \mathsf{Pic}(\mathbb{P}^n) \longrightarrow H^2(\mathbb{P}^n, \underline{\mathbb{Z}}) \longrightarrow H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \longrightarrow \cdots .$$

We have seen that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $i \geq 2$, and conclude that $\mathsf{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \underline{\mathbb{Z}}) \cong \mathbb{Z}$. The last equivalence uses Theorem 8.24(a).

One immediately guesses that the equivalence $\mathsf{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ associated to the integer $d \in \mathbb{Z}$ the line bundle $\mathcal{O}(d)$. This is indeed the case, but since we haven't yet obtained a conceptual understanding of the group $H^2(\mathbb{P}^n, \underline{\mathbb{Z}})$ (e.g., we haven't discussed methods to construct/describe elements), it is not easy to prove this assertion at this point. Instead, we will refer once more to the GAGA Theorem, to prove that every line bundle on $\mathbb{P}^n$ is equivalent to a line bundle $\mathcal{O}(d)$. In order to keep our discussion as simple as possible, we limit ourselves to the case $n = 1$.

From now on we consider $\mathbb{P}^1$ with the Zariski topology. That is, a subset $U \subset \mathbb{P}^1$ is Zariski open, if and only if $\mathbb{P}^1 \backslash U$ consists of finitely many points (this is true, because we are working in complex dimension 1). We denote by $\mathcal{M}$ the sheaf of rational functions, that is to $U \subset \mathbb{P}^1$ we associate the vector space of all rational functions $V \subset \mathbb{C}$, where $V \subset U$. Let $\mathcal{M}^\times$ be the sheaf of non-zero rational functions, which form an abelian group with respect to multiplication.

We have an inclusion of sheaves $\mathcal{O}_{\mathrm{reg}}^\times \hookrightarrow \mathcal{M}^\times$, and denote the sheaf of quotients by Div (we call it the sheaf of *divisors*). A section $D \in \mathsf{Div}(U)$ can be represented by a formal linear combination $n_1 \cdot x_1 + \cdots + n_k \cdot x_k$, where $x_i \in U$ (this is what we call a divisor). We assign to a rational function f a divisor, by weighting every point $x \in U$ with the pole order of f at x.

The sheaves $\mathcal{M}^\times$ and Div are flasque, and hence the long exact sequence implies

$$\operatorname{coker}(\mathcal{M}^\times(X) \longrightarrow \mathsf{Div}(X)) \cong \mathsf{Pic}(\mathbb{P}^1).$$

The map $\mathsf{Div}(X) \longrightarrow \mathbb{Z}$ sending $\sum_{i=1}^k n_i x_i$ to $\sum_{i=1}^k n_i$ is surjective, because $1 \cdot x$ is sent to $1 \in \mathbb{Z}$. Moreover, since the sum of pole orders of a rational function f on $\mathbb{P}^1$ is 0, we get a well-defined map

$$\mathsf{Pic}(\mathbb{P}^n) \longrightarrow \mathbb{Z}\,.$$

Lemma 8.28. *Let $f\colon \mathbb{P}^1 \longrightarrow \overline{\mathbb{C}}$ be a non-zero meromorphic function on $\mathbb{P}^1$. The sum over all the pole orders, minus the zero orders is* 0.

Proof. Without loss of generality we can assume that f does not have a pole or zero at $\infty \in \overline{\mathbb{C}} \cong \mathbb{P}^1$. Otherwise, we apply a generic Möbius transformation to $\overline{\mathbb{C}}$ to perturb a possible pole or zero away from ∞. Choose a very small closed path γ around ∞, but which does not pass through ∞. We can view it as a non-trivial path in $\mathbb{C}$, non-trivially circling the poles and zeroes of f. The Residue Theorem implies that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = m \cdot \sum_x \operatorname{ord}_{z=x} f,$$

where m is the non-zero winding number of f. Since γ is a path on $\mathbb{P}^1$ which can be contracted to ∞, homotopy invariance of integration implies that the sum above vanishes. $\square$

Since the map $\mathsf{Pic}(\mathbb{P}^1) \longrightarrow \mathbb{Z}$ is surjective, and we know that $\mathsf{Pic}(\mathbb{P}^n)$ is an infinite cyclic group, we conclude that this map is an isomorphism. It remains to show that $\mathcal{O}(d)$ is sent to $d \in \mathbb{Z}$. Which allows us to conclude that every line bundle on $\mathbb{P}^n$ is isomorphic to a line bundle $\mathcal{O}(d)$. This is an easy computation, using the description of the boundary map in Čech cohomology.