Riemannian Geometry of Quantum Computation

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ABSTRACT. An introduction is given to some recent developments in the differential geometry of quantum computation for which the quantum evolution is described by the special unitary unimodular group $SU(2^n)$. Using the Lie algebra $su(2^n)$, detailed derivations are given of a useful Riemannian geometry of $SU(2^n)$, including the connection, curvature, and the geodesic equation for minimal complexity quantum computations.

1. INTRODUCTION

Any quantum computation can be ideally represented by a unitary transformation acting in the Hilbert space of the computational degrees of freedom of the quantum computer, and any unitary transformation can be faithfully represented by a network of universal quantum gates, such as two-qubit controlled-NOT gates and single-qubit gates. This is the basis of the quantum circuit model of quantum computation [1]. An important measure of the difficulty of performing a quantum computation is the number of quantum gates needed. A quantum algorithm is considered efficient if the number of required gates scales only polynomially (not exponentially) with the size of the problem. Quantum circuit networks are usually analyzed using discrete methods, however potentially powerful continuous differential geometric methods are under development, using sub-Riemannian [2]-[4], Riemannian [5]-[8], and also Finsler [5], and sub-Finsler [9] geometries. Since unitary transformations are themselves continuous, this is perhaps not a surprising development. Using these differential geometric methods, optimal paths may be sought in Hilbert space for executing a quantum computation.

A new innovative approach to the differential geometry of quantum computation and quantum circuit complexity was recently introduced by Nielsen and collaborators [5]-[8]. A Riemannian metric was formulated on the special unitary unimodular group manifold of multi-qubit unitary transformations, such that the

metric distance between the identity and the desired unitary operator, representing
the quantum computation, is equivalent to the number of quantum gates needed
to represent that unitary operator, thereby providing a measure of the complexity
associated with the corresponding quantum computation. The Riemannian metric
was defined as a positive-definite bilinear form defined in terms of the multi-qubit
Hamiltonian. The analytic form of the metric was chosen to penalize all directions
on the manifold not easily simulated by local gates. In this way, basic differential
geometric concepts such as the Levi-Civita connection, geodesic path, Riemann-
ian curvature, Jacobi fields, and conjugate points can be associated with quantum
computation. Equations for the Levi-Civita connection on the Riemannian man-
ifold can be obtained, as well as the characteristic curvature of the manifold. In
accord with the Schrödinger equation, the unitary transformation expressing the
quantum evolution is an exponential involving the Hamiltonian. The Hamilton-
ian can be expressed in terms of tensor products of the Pauli matrices which act
on the qubits. The Riemann curvature tensor can then be constructed from the
Christoffel symbols and their ordinary derivatives. The geodesic equation on the
manifold follows from the connection and determines the local optimal Hamilton-
ian evolution corresponding to the unitary transformation representing the desired
quantum computation. The optimal unitary evolution may follow by solving the ge-
odesic equation. Useful upper and lower bounds on the associated quantum circuit
complexity may be obtained. Such differential geometric approaches to quantum
computation are currently preliminary and many details remain to be worked out.

The present work presents an expository review of the Riemannian geometry
of the special unitary unimodular group manifold associated with quantum compu-
tation, and detailed derivations are presented of a suitable connection, curvature,
and geodesic equation expressed in terms of the tensor products of Pauli matrices
appearing in the Hamiltonian and representing gate operations. Also, examples of
some solutions to the geodesic equation are elaborated.

2. METRIC

A Riemannian metric is first chosen on the manifold of the Lie Group $SU(2^n)$
(special unitary group) of n-qubit unitary operators with unit determinant [10]-
[22]. The traceless Hamiltonian serves as a tangent vector to a point on the group
manifold of the n-qubit unitary transformation U. The Hamiltonian H is an element
of the Lie algebra $su(2^n)$ of traceless $2^n \times 2^n$ Hermitian matrices [20]-[22] and is
tangent to the evolutionary curve $e^{-iHt}U$ at $t = 0$. (Here and throughout, units
are chosen such that Planck's constant divided by 2π is $\hbar = 1$.)

Independent of U, the Riemannian metric (inner product) $\langle ., . \rangle$ is taken to be
a right-invariant positive definite bilinear form $\langle H, J \rangle$ defined on tangent vectors
(Hamiltonians) H and J. Following [8], the n-qubit Hamiltonian H can be divided
into two parts $P(H)$ and $Q(H)$, where $P(H)$ contains only one and two-body terms,
and $Q(H)$ contains more than two-body terms. Thus:

$$(2.1) \qquad\qquad H = P(H) + Q(H),$$

in which P and Q are superoperators (matrices) acting on H, and obey the following
relations:

$$(2.2) \qquad P + Q = I, \quad PQ = QP = 0, \quad P^2 = P, \quad Q^2 = Q,$$

where I is the identity. Letting H_m denote the m-body part of H, then

$$(2.3) \qquad P(H) = H_1 + H_2,$$

and

$$(2.4) \qquad Q(H) = \sum_{m=3}^{n-1} H_m.$$

For example, in the case of a 3-qubit Hamiltonian, for Pauli matrices σ_1, σ_2, and σ_3 (see Appendix A) [1], one has

$$
\begin{aligned}
P(H) =\ & x^1 \sigma_1 \otimes I \otimes I + x^2 \sigma_2 \otimes I \otimes I + x^3 \sigma_3 \otimes I \otimes I \\
& + x^4 I \otimes \sigma_1 \otimes I + x^5 I \otimes \sigma_2 \otimes I + x^6 I \otimes \sigma_3 \otimes I \\
& + x^7 I \otimes I \otimes \sigma_1 + x^8 I \otimes I \otimes \sigma_2 + x^9 I \otimes I \otimes \sigma_3 \\
& + x^{10} \sigma_1 \otimes \sigma_2 \otimes I + x^{11} \sigma_1 \otimes I \otimes \sigma_2 + x^{12} I \otimes \sigma_1 \otimes \sigma_2 \\
& + x^{13} \sigma_2 \otimes \sigma_1 \otimes I + x^{14} \sigma_2 \otimes I \otimes \sigma_1 + x^{15} I \otimes \sigma_2 \otimes \sigma_1 \\
& + x^{16} \sigma_1 \otimes \sigma_3 \otimes I + x^{17} \sigma_1 \otimes I \otimes \sigma_3 + x^{18} I \otimes \sigma_1 \otimes \sigma_3 \\
& + x^{19} \sigma_3 \otimes \sigma_1 \otimes I + x^{20} \sigma_3 \otimes I \otimes \sigma_1 + x^{21} I \otimes \sigma_3 \otimes \sigma_1 \\
& + x^{22} \sigma_2 \otimes \sigma_3 \otimes I + x^{23} \sigma_2 \otimes I \otimes \sigma_3 + x^{24} I \otimes \sigma_2 \otimes \sigma_3 \\
& + x^{25} \sigma_3 \otimes \sigma_2 \otimes I + x^{26} \sigma_3 \otimes I \otimes \sigma_2 + x^{27} I \otimes \sigma_3 \otimes \sigma_2 \\
& + x^{28} \sigma_1 \otimes \sigma_1 \otimes I + x^{29} \sigma_2 \otimes \sigma_2 \otimes I + x^{30} \sigma_3 \otimes \sigma_3 \otimes I \\
& + x^{31} \sigma_1 \otimes I \otimes \sigma_1 + x^{32} \sigma_2 \otimes I \otimes \sigma_2 + x^{33} \sigma_3 \otimes I \otimes \sigma_3 \\
(2.5) \qquad & + x^{34} I \otimes \sigma_1 \otimes \sigma_1 + x^{35} I \otimes \sigma_2 \otimes \sigma_2 + x^{36} I \otimes \sigma_3 \otimes \sigma_3,
\end{aligned}
$$

in which $\otimes$ denotes the tensor product [23], and the n in x^n is a superscript,

$$
\begin{aligned}
Q(H) =\ & x^{37} \sigma_1 \otimes \sigma_2 \otimes \sigma_3 + x^{38} \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \\
& + x^{39} \sigma_2 \otimes \sigma_1 \otimes \sigma_3 + x^{40} \sigma_2 \otimes \sigma_3 \otimes \sigma_1 \\
& + x^{41} \sigma_3 \otimes \sigma_1 \otimes \sigma_2 + x^{42} \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \\
& + x^{43} \sigma_1 \otimes \sigma_1 \otimes \sigma_2 + x^{44} \sigma_1 \otimes \sigma_2 \otimes \sigma_1 + x^{45} \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \\
& + x^{46} \sigma_1 \otimes \sigma_1 \otimes \sigma_3 + x^{47} \sigma_1 \otimes \sigma_3 \otimes \sigma_1 + x^{48} \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \\
& + x^{49} \sigma_2 \otimes \sigma_2 \otimes \sigma_1 + x^{50} \sigma_2 \otimes \sigma_1 \otimes \sigma_2 + x^{51} \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
& + x^{52} \sigma_2 \otimes \sigma_2 \otimes \sigma_3 + x^{53} \sigma_2 \otimes \sigma_3 \otimes \sigma_2 + x^{54} \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \\
& + x^{55} \sigma_3 \otimes \sigma_3 \otimes \sigma_1 + x^{56} \sigma_3 \otimes \sigma_1 \otimes \sigma_3 + x^{57} \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \\
& + x^{58} \sigma_3 \otimes \sigma_3 \otimes \sigma_2 + x^{59} \sigma_3 \otimes \sigma_2 \otimes \sigma_3 + x^{60} \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \\
(2.6) \qquad & + x^{61} \sigma_1 \otimes \sigma_1 \otimes \sigma_1 + x^{62} \sigma_2 \otimes \sigma_2 \otimes \sigma_2 + x^{63} \sigma_3 \otimes \sigma_3 \otimes \sigma_3.
\end{aligned}
$$

Here, all possible tensor products [23] of one and two-qubit Pauli matrix operators on three qubits appear in $P(H)$, and analogously, all possible tensor products of three-qubit operators appear in $Q(H)$. Tensor products including only the identity are excluded because the Hamiltonian is taken to be traceless. Each of the terms in Eqs. (2.5) and (2.6) is an 8×8 matrix. The various tensor products of Pauli matrices such as those appearing in Eqs. (2.5) and (2.6) are referred to as generalized Pauli matrices. In the case of an n-qubit Hamiltonian, there are $4^n - 1$ possible traceless tensor products (corresponding to the dimension of the $SU(2^n)$ tangent space), and each term is a $2^n \times 2^n$ matrix.

The right-invariant [10]-[13] Riemannian metric for tangent vectors H and J is given by [8]

$$(2.7) \qquad \langle H, J \rangle \equiv \frac{1}{2^n} \mathrm{Tr} \left[HP(J) + qHQ(J) \right].$$

Here q is a large penalty parameter which taxes many-body ($m > 2$) terms. The length l of an evolutionary path on the $SU(2^n)$ manifold is given by the integral over time t from an initial time t_i to a final time t_f, namely,

$$(2.8) \qquad l = \int_{t_i}^{t_f} dt \left(\langle H(t), H(t) \rangle \right)^{1/2},$$

and is a measure of the cost of applying a control Hamiltonian $H(t)$ along the path. Justification for the form of the metric, Eq. (2.7), is given in references [5], [8].

3. CONNECTION

In order to obtain the Levi-Civita connection, it is necessary to exploit the Lie algebra $su(2^n)$ associated with the group $SU(2^n)$. Because of the right-invariance of the metric, if the Christoffel symbols are calculated at the origin, the same expression applies everywhere on the manifold. Following [8], consider the unitary transformation

$$(3.1) \qquad U = e^{-iX}$$

in the neighborhood of the identity $I \in SU(2^n)$ (or equivalently in the neighborhood of the origin of the tangent space manifold) with

$$(3.2) \qquad X = x \cdot \sigma \equiv \sum_{\sigma} x^{\sigma} \sigma,$$

which expresses symbolically terms like those in Eqs. (2.5) and (2.6) generalized to 2^n dimensions. In Eqs. (3.1) and (3.2), X is defined using the standard branch of the logarithm with a cut along the negative real axis. In Eq. (3.2), for the general case of n qubits, x_{σ} represents the set of real $(4^n - 1)$ coefficients of the generalized Pauli matrices σ which represent all of the n-fold tensor products. It follows from Eq. (3.2) that the factor x^{σ} multiplying a particular term σ is given by

$$(3.3) \qquad x^{\sigma} = \frac{1}{2^n} \mathrm{Tr}(X\sigma).$$

These are so-called Pauli coordinates. (In the neighborhood of the origin, X will be represented as $X = \Delta x^{\mu} \mu$ for infinitesimal Δx^{μ} and generalized Pauli matrix μ, where the Einstein sum convention summing over μ is to be understood. See Eq. (3.32) below.)

Consider a curve $e^{-iHt}e^{-iX}$ in the $SU(2^n)$ group manifold, evolving from a point $U = e^{-iX}$, and representing a system with initial action X acted on by a control Hamiltonian H. For the point U on the $SU(2^n)$ manifold with tangent vector H to the curve, one has in the neighborhood of the identity,

$$(3.4) \qquad e^{-iHt}e^{-iX} = e^{-i(X+Jt)} + 0(t^2)$$

to second-order in the time t. This follows from the Baker-Campbell-Hausdorff formula [15], [18], [24]-[29]. The right side of Eq. (3.4) contains the resulting total action $(X + Jt)$. Explicitly, the matrix J, the so-called Pauli representation

of the tangent vector in the Pauli-coordinate representation in the tangent space $T_U SU(2^n)$, is related to H, the Hamiltonian representation of the tangent vector, by

$$(3.5) \qquad H = E_X(J),$$

in which the linear superoperator E_X is given by

$$(3.6) \qquad E_X = i\mathrm{ad}_X^{-1}(e^{-i\mathrm{ad}_X} - I),$$

where I is the identity, a power series expansion is to be understood since the operator ad_X is not invertible, and $\mathrm{ad}_X(Y)$ is the Lie bracket, defined by the ordinary matrix commutator

$$(3.7) \qquad \mathrm{ad}_X(Y) \equiv [X, Y].$$

The power series expansion of E_X is

$$(3.8) \qquad E_X = \sum_{j=0}^{\infty} \frac{(-i\mathrm{ad}_X)^j}{(j+1)!}.$$

Near the origin, E_X is invertible, and one has

$$(3.9) \qquad J \equiv D_X(H) = E_X^{-1}(H).$$

It then follows from Eqs. (3.8) and (3.9) near the origin that

$$(3.10) \qquad E_X = I - \frac{i}{2}\mathrm{ad}_X + O(X^2),$$

and

$$(3.11) \qquad D_X = I + \frac{i}{2}\mathrm{ad}_X + O(X^2).$$

One also has the adjoint relations with respect to the trace inner product (see Appendix B):

$$(3.12) \qquad E_X^{\dagger} = E_{-X},$$

$$(3.13) \qquad D_X^{\dagger} = D_{-X}.$$

Next, the right-invariant metric, Eq. (2.7), in the so-called Hamiltonian representation can be written as

$$(3.14) \qquad \langle H, J \rangle = \frac{1}{2^n}\mathrm{Tr}(HG(J)),$$

in which the positive self-adjoint superoperator G is given by

$$(3.15) \qquad G = P + qQ.$$

It is also useful to define a Hermitian matrix L, dual to the Hamiltonian H,

$$(3.16) \qquad L = G(H),$$

so that Eq. (3.14) can also be written as

$$(3.17) \qquad \langle H, J \rangle \equiv \frac{1}{2^n}\mathrm{Tr}(LJ).$$

Now consider the metric $\langle Y, Z \rangle$ for tangent vector fields Y and Z in the neighborhood of the origin at point $U = e^{-iX}$ (See Eq. (3.32)). By Eq. (3.5), the

so-called Hamiltonian representations $\{Y^H, Z^H\}$ of the vector fields is related to their so-called Pauli representations $\{Y^P, Z^P\}$ by

$$(3.18) \qquad Y^H = E_X(Y^P), \qquad Z^H = E_X(Z^P).$$

Substituting Eqs. (3.18) in Eq. (3.14), one obtains

$$(3.19) \qquad \langle Y, Z \rangle = \frac{1}{2^n} \mathrm{Tr}(Y^H G(Z^H)) = \frac{1}{2^n} \mathrm{Tr}(E_X(Y^P) G \circ E_X(Z^P)),$$

or

$$(3.20) \qquad \langle Y, Z \rangle = \frac{1}{2^n} \mathrm{Tr}(Y^P E_X^\dagger \circ G \circ E_X(Z^P)).$$

Equivalently,

$$(3.21) \qquad \langle Y, Z \rangle = \frac{1}{2^n} \mathrm{Tr}(Y^P G_X(Z^P)),$$

where

$$(3.22) \qquad G_X \equiv E_X^\dagger \circ G \circ E_X.$$

The metric can be rewritten in the familiar Riemannian tensor form $g_{\sigma\tau}$, in a coordinate basis, as follows. The vectors Y^P and Z^P in the Pauli representation can be written as

$$(3.23) \qquad Y^P = \sum_\sigma y^\sigma \sigma, \qquad Z^P = \sum_\sigma z^\sigma \sigma$$

with Pauli coordinates y^σ and z^σ. Here σ, as an index, is used to refer to a particular tensor product appearing in the generalized Pauli matrix σ. This index notation, used throughout, is a convenient abbreviation for the actual numerical indices (e.g. in Eq. (2.5), the number 22 appearing in x^{22}, the coefficient of $\sigma_2 \otimes \sigma_3 \otimes I$). Then substituting Eqs. (3.23) in Eq. (3.21), one obtains

$$(3.24) \qquad \langle Y, Z \rangle = \frac{1}{2^n} \mathrm{Tr}(\sum_\sigma y^\sigma \sigma G_X(\sum_\tau z^\tau \tau)),$$

or

$$(3.25) \qquad \langle Y, Z \rangle = \sum_{\sigma\tau} g_{\sigma\tau} y^\sigma z^\tau,$$

in which the Pauli-coordinate representation of the metric tensor $g_{\sigma\tau}$ is given by

$$(3.26) \qquad g_{\sigma\tau} = \frac{1}{2^n} \mathrm{Tr}(\sigma G_X(\tau)).$$

According to Eqs. (3.22), (3.10), and Eq. (12.13) of Appendix B, one has in the neighborhood of the origin:

$$(3.27) \qquad G_X = \left(1 + \frac{i}{2} \mathrm{ad}_X\right) \circ G \circ \left(1 - \frac{i}{2} \mathrm{ad}_X\right) + 0(X^2),$$

or equivalently,

$$(3.28) \qquad G_X = G + \frac{i}{2} [\mathrm{ad}_X, G] + 0(X^2).$$

Next one has for the partial derivative of $g_{\sigma\tau}$ with respect to x^μ:

$$(3.29) \qquad g_{\sigma\tau,\mu} = \lim_{\Delta x \to 0} \frac{g_{\sigma\tau}(x + \Delta x^\mu) - g_{\sigma\tau}(x)}{\Delta x^\mu},$$

where the comma followed by μ denotes the partial derivative $\partial/\partial x^\mu$. Using Eqs. (3.26) and (3.28), then Eq. (3.29) becomes

$$(3.30) \qquad g_{\sigma\tau,\mu} = \lim_{\Delta x \to 0} \frac{1}{2^n} \text{Tr} \frac{\sigma\left(G + \frac{i}{2}[\text{ad}_X, G]\right)(\tau) - \sigma G(\tau)}{\Delta x^\mu},$$

or

$$(3.31) \qquad g_{\sigma\tau,\mu} = \lim_{\Delta x \to 0} \frac{i}{2^{n+1}} \text{Tr} \frac{\sigma[X, G(\tau)] - \sigma[G(\tau), X]}{\Delta x^\mu}.$$

In the neighborhood of the origin, one has for infinitesimals Δx^μ, using the Einstein sum convention for repeated upper and lower indices,

$$(3.32) \qquad X = \Delta x^\mu \mu,$$

and when the μ-component is substituted in Eq. (3.31), one obtains

$$(3.33) \qquad g_{\sigma\tau,\mu} = \frac{i}{2^{n+1}} \text{Tr} \left(\sigma[\mu, G(\tau)] - \sigma[G(\tau), \mu]\right).$$

Next expanding the commutators, and using the cyclic property of the trace, one obtains

$$(3.34) \qquad g_{\sigma\tau,\mu} = \frac{i}{2^{n+1}} \text{Tr} \left(2(G(\tau)\sigma\mu - \sigma G(\tau)\mu)\right),$$

or equivalently,

$$(3.35) \qquad g_{\sigma\tau,\mu} = \frac{i}{2^{n+1}} \text{Tr} \left(2[G(\tau), \sigma]\mu\right).$$

However because any Riemannian metric tensor is symmetric, one has

$$(3.36) \qquad g_{\sigma\tau,\mu} = \frac{1}{2} \left(g_{\sigma\tau,\mu} + g_{\tau\sigma,\mu}\right),$$

and substituting Eq. (3.35) in Eq. (3.36), one obtains [8]

$$(3.37) \qquad g_{\sigma\tau,\mu} = \frac{i}{2^{n+1}} \text{Tr} \left\{\left([G(\sigma), \tau] + [G(\tau), \sigma]\right)\mu\right\}.$$

The familiar form of the Levi-Civita connection of Riemannian geometry, in a coordinate basis, is given by the Christoffel symbols of the first kind, namely, [17]

$$(3.38) \qquad \Gamma_{\mu\sigma\tau} = \frac{1}{2}(g_{\mu\sigma,\tau} + g_{\mu\tau,\sigma} - g_{\sigma\tau,\mu}).$$

Substituting Eq. (3.37) in Eq. (3.38), one obtains

$$
\begin{aligned}
(3.39) \qquad \Gamma_{\mu\sigma\tau} = \quad & \frac{1}{2}\frac{i}{2^{n+1}} \text{Tr} \left(\left([G(\mu), \sigma] + [G(\sigma), \mu]\right)\tau \right. \\
& + \left([G(\mu), \tau] + [G(\tau), \mu]\right)\sigma \\
& \left. - \left([G(\sigma), \tau] + [G(\tau), \sigma]\right)\mu\right),
\end{aligned}
$$

and expanding the commutators, using the cyclic property of the trace, and simplifying, this becomes

$$(3.40) \qquad \Gamma_{\mu\sigma\tau} = \frac{i}{2^{n+1}} \text{Tr}((\tau G(\sigma) - G(\sigma)\tau)\mu + (\sigma G(\tau) - G(\tau)\sigma)\mu),$$

or

$$(3.41) \qquad \Gamma_{\mu\sigma\tau} = \frac{i}{2^{n+1}} \text{Tr}(([\sigma, G(\tau)] + [\tau, G(\sigma)])\mu),$$

and again using the cyclic property of the trace, one obtains

$$(3.42) \qquad \Gamma_{\mu\sigma\tau} = \frac{i}{2^{n+1}} \mathrm{Tr}(\mu([\sigma, G(\tau)] + [\tau, G(\sigma)])).$$

The inverse metric is given by (see Appendix C):

$$(3.43) \qquad g^{\sigma\tau} = \frac{1}{2^n} \mathrm{Tr}(\sigma F(\tau)).$$

It then follows that the Christoffel symbols of the second kind [**17**] are given by (see Appendix D) [**8**]

$$(3.44) \qquad \Gamma^{\rho}_{\sigma\tau} = \frac{i}{2^{n+1}} \mathrm{Tr}\left(F(\rho)\left([\sigma, G(\tau)] + [\tau, G(\sigma)]\right)\right),$$

in which one defines

$$(3.45) \qquad F(\rho) \equiv G^{-1}(\rho).$$

Next, for a generic Riemannian connection Γ^j_{kl} and vectors Z and Y, written in a coordinate basis, one has the familiar equation for the covariant derivative of Z along Y:

$$(3.46) \qquad (\nabla_Y Z)^j = \frac{\partial z^j}{\partial x^k} y^k + \Gamma^j_{kl} \, y^k z^l,$$

in which the Einstein convention of summing over repeated indices is implicit. Replacing indices (j, k, l) by (σ, τ, λ), multiplying both sides of Eq. (3.46) by σ, and summing over σ yields

$$(3.47) \qquad \sum_{\sigma} \sigma(\nabla_Y Z)^{\sigma} = \sum_{\sigma\tau} y^{\tau} \sigma \frac{\partial z^{\sigma}}{\partial x^{\tau}} + \sum_{\sigma\tau\lambda} \sigma \Gamma^{\sigma}_{\tau\lambda} y^{\tau} z^{\lambda},$$

and substituting Eqs. (3.23) and (3.44) in Eq. (3.47), one obtains

$$(3.48) \quad (\nabla_Y Z)^P \equiv \sum_{\tau} y^{\tau} \frac{\partial Z^P}{\partial x^{\tau}} + \sum_{\sigma\tau\lambda} \sigma \frac{i}{2^{n+1}} \mathrm{Tr}\left\{F(\sigma)\left([\tau, G(\lambda)] + [\lambda, G(\tau)]\right)\right\} y^{\tau} z^{\lambda}.$$

The following identity is true (see Appendix E):

$$(3.49) \qquad \sum_{\sigma} \sigma \mathrm{Tr}\left\{F(\sigma)[\tau, G(\lambda)]\right\} = 2^n F([\tau, G(\lambda)]),$$

so that Eq. (3.48) becomes

$$(3.50) \quad (\nabla_Y Z)^P \equiv \sum_{\tau} y^{\tau} \frac{\partial Z^P}{\partial x^{\tau}} + \sum_{\tau\lambda} \frac{i}{2^{n+1}} 2^n \left(F([\tau, G(\lambda)]) + F([\lambda, G(\tau)])\right) y^{\tau} z^{\lambda}.$$

Then substituting Eqs. (3.23) in Eq. (3.50), and using the Einstein sum convention, one obtains the Pauli representation of the connection evaluated at the origin with the vector fields given in the Pauli representation, namely, [**8**]:

$$(3.51) \qquad (\nabla_Y Z)^P \equiv y^{\tau} \frac{\partial Z^P}{\partial x^{\tau}} + \frac{i}{2} \left(F([Y^P, G(Z^P)]) + F([Z^P, G(Y^P)])\right).$$

To obtain the Hamiltonian representation of the connection, one has according to Eqs. (3.9) and (3.11) near the origin,

$$(3.52) \qquad Z^P = D_X(Z^H) = \left(1 + \frac{i}{2} ad_X\right)(Z^H).$$

Also, clearly,

$$(3.53) \qquad \frac{\partial Z^P}{\partial x^\sigma} = \lim_{\Delta x \to 0} \frac{Z^P(x + \Delta x) - Z^P(x)}{\Delta x^\sigma},$$

and substituting Eq. (3.52) in Eq. (3.53), then

$$(3.54) \qquad \frac{\partial Z^P}{\partial x^\sigma} = \lim_{\Delta x \to 0} \frac{\left(1 + \frac{i}{2} ad_X\right)\left(Z^H(x + \Delta x)\right) - Z^H}{\Delta x^\sigma},$$

or substituting Eqs. (3.7) and (3.32), and dropping the Einstein sum convention here only, then

$$(3.55) \qquad \frac{\partial Z^P}{\partial x^\sigma} = \lim_{\Delta x \to 0} \frac{\frac{i}{2}\left[\Delta x^\sigma \sigma, Z^H\right] + \frac{\partial Z^H}{\partial x^\sigma}\Delta x^\sigma}{\Delta x^\sigma} = \frac{i}{2}\left[\sigma, Z^H\right] + \frac{\partial Z^H}{\partial x^\sigma}.$$

Thus

$$(3.56) \qquad Z^P_{,\sigma} = \frac{i}{2}\left[\sigma, Z^H\right] + Z^H_{,\sigma},$$

or multiplying by y^σ, using Eq. (3.23), and restoring the Einstein sum convention, one has

$$(3.57) \qquad y^\sigma Z^P_{,\sigma} = y^\sigma Z^H_{,\sigma} + \frac{i}{2}\left[Y^P, Z^H\right].$$

Next substituting Eq. (3.57) in Eq. (3.51), one obtains

$$(3.58) \quad \left(\nabla_Y Z\right)^P = y^\sigma Z^H_{,\sigma} + \frac{i}{2}\left[Y^P, Z^H\right] + \frac{i}{2} F\left(\left[Y^P, G(Z^P)\right] + \left[Z^P, G(Y^P)\right]\right).$$

But at the origin, it is true that

$$(3.59) \qquad \left(\nabla_Y Z\right)^H = \left(\nabla_Y Z\right)^P, \quad Y^H = Y^P, \quad Z^H = Z^P,$$

and the components y^σ of Y are the same in both representations. Therefore using Eqs. (3.58) and (3.59), one obtains the Hamiltonian representation of the connection at the origin [8]:

$$(3.60) \quad \left(\nabla_Y Z\right)^H = y^\sigma Z^H_{,\sigma} + \frac{i}{2}\left(\left[Y^H, Z^H\right] + F\left(\left[Y^H, G(Z^H)\right] + \left[Z^H, G(Y^H)\right]\right)\right),$$

in which $Z^H_{,\sigma} \equiv \frac{\partial Z^H}{\partial x^\sigma}$. Equation (3.60) gives the covariant derivative of the vector Z^H along the vector Y^H.

4. GEODESIC EQUATION

Next consider a curve passing through the origin with tangent vector Y^H and $y^\sigma = dx^\sigma/dt$. Then according to Eq. (3.60) and the chain rule, the covariant derivative along the curve in the Hamiltonian representation is given by

(4.1)
$$\left(D_t Z\right)^H \equiv \left(\nabla_Y Z\right)^H = \frac{dZ^H}{dt} + \frac{i}{2}\left(\left[Y^H, Z^H\right] + F\left(\left[Y^H, G(Z^H)\right] + \left[Z^H, G(Y^H)\right]\right)\right).$$

Because of the right-invariance of the metric, Eq. (4.1) is true on the entire manifold. Furthermore, for a right-invariant vector field Z^H, one has

$$(4.2) \qquad \frac{dZ^H}{dt} = 0,$$

and substituting Eq. (4.2) in Eq. (4.1), one obtains

(4.3) $\qquad (\nabla_Y Z)^H = \dfrac{i}{2} \left\{ \left[Y^H, Z^H \right] + F\left(\left[Y^H, G(Z^H) \right] + \left[Z^H, G(Y^H) \right] \right) \right\},$

which is also true everywhere on the manifold.

One can next proceed to obtain the geodesic equation. A geodesic in $SU(2^n)$ is a curve $U(t)$ with tangent vector $H(t)$ parallel transported along the curve, namely,

(4.4) $\qquad\qquad\qquad\qquad\qquad D_t H = 0.$

However, according to Eq. (4.1) with $Y^H = Z^H = H$, one has

(4.5) $\qquad D_t H = \dfrac{dH}{dt} + \dfrac{i}{2}([H, H] + F\left([H, G(H)] + [H, G(H)] \right)),$

which when substituting Eq. (4.4) becomes [8]

(4.6) $\qquad\qquad\qquad\qquad \dfrac{dH}{dt} = -iF\left([H, G(H)] \right).$

One can rewrite Eq. (4.6) using Eqs. (3.16) and (3.45),

(4.7) $\qquad\qquad\qquad\qquad L \equiv G(H) = F^{-1}(H),$

and then noting that

(4.8) $\qquad\qquad\qquad \dfrac{dL}{dt} = \dfrac{d}{dt}\left(F^{-1}(H) \right) = F^{-1}\left(\dfrac{dH}{dt} \right).$

Thus substituting Eq. (4.6) in Eq. (4.8), one obtains

(4.9) $\qquad\qquad\qquad \dfrac{dL}{dt} = -iF^{-1}\left(F([H, G(H)]) \right),$

or

(4.10) $\qquad\qquad\qquad\qquad \dfrac{dL}{dt} = -i[H, G(H)],$

and again using Eq. (4.7), Eq. (4.10) becomes

(4.11) $\qquad\qquad\qquad \dfrac{dL}{dt} = -i[H, L] = i[L, H].$

Furthermore substituting Eq. (4.7) in Eq. (4.11), one obtains the sought geodesic equation [8]:

(4.12) $\qquad\qquad\qquad\qquad \dfrac{dL}{dt} = i[L, F(L)].$

Equation (4.12) is a Lax equation. a well-known nonlinear differential matrix equation, and the operators L and $iF(L)$ are Lax pairs [31]-[36].

An alternate form for the geodesic equation can be obtained by first substituting Eq. (3.15) in Eq. (3.45), obtaining

(4.13) $\qquad\qquad\qquad\qquad F = P + q^{-1}Q.$

Equation (4.13) follows since one then has according to Eqs. (4.7), (3.15), and (4.13), that

(4.14) $\qquad\qquad G^{-1}G = FG = \left(P + q^{-1}Q \right)(P + qQ),$

or

(4.15) $\qquad\qquad G^{-1}G = P^2 + qPQ + q^{-1}QP + Q^2,$

and using Eqs. (2.2), this becomes

(4.16) $$G^{-1}G = I,$$

as it must. Then substituting Eq. (4.13) in Eq. (4.12), one obtains

(4.17) $$\frac{dL}{dt} = i[L, P(L) + q^{-1}Q(L)].$$

Using Eq. (2.2) in Eq. (4.17), one has

(4.18) $$\frac{dL}{dt} = i\left[L, P(L) + q^{-1}(L - P(L))\right],$$

or

(4.19) $$\frac{dL}{dt} = iq^{-1}[L, L] + i(1 - q^{-1})[L, P(L)].$$

Finally then Eq. (4.19) becomes [8]

(4.20) $$\frac{dL}{dt} = i(1 - q^{-1})[L, P(L)].$$

It follows that if $[L, P(L)] = 0$, or equivalently if H_1 and H_2 commute with H_m for $m \geq 3$, then $dL/dt = 0$, or using Eq. (4.7), then also $dH/dt = 0$, namely the Hamiltonian is constant, and therefore, using the Schrödinger equation, it follows that the geodesic path becomes simply e^{-iHt}. The latter is also the case if only one and two-body terms appear in the Hamiltonian.

Yet another useful form for the geodesic equation follows by first defining

(4.21) $$M = (1 - q^{-1})L, \quad q \neq 1.$$

Then

(4.22) $$\frac{dM}{dt} = (1 - q^{-1})\frac{dL}{dt},$$

and substituting Eqs. (4.20) and (4.21) in Eq. (4.22), then

(4.23) $$\frac{dM}{dt} = i(1 - q^{-1})^2 \left[L, \frac{1}{(1 - q^{-1})}P(M)\right],$$

or equivalently [8],

(4.24) $$\frac{dM}{dt} = i[M, P(M)], \quad q \neq 1,$$

independent of q, provided $q \neq 1$. Equations (4.7) and (4.21) imply

(4.25) $$H = G^{-1}(L) = \frac{1}{(1 - q^{-1})}G^{-1}(M),$$

and therefore solving Eq. (4.24) for M yields the Hamiltonian H producing the geodesic path.

5. CONSTANTS OF MOTION

Constants of the motion for the geodesic Eq. (4.12) are readily obtained as follows. For arbitrary constant L_0 and unitary transformation $U(t)$, define the function $L(t)$ by

$$(5.1) \qquad L(t) = U(t)L_0 U^\dagger(t) \; .$$

Then

$$(5.2) \qquad \frac{dL(t)}{dt} = \frac{dU(t)}{dt} L_0 U^\dagger(t) + U(t)L_0 \frac{dU^\dagger(t)}{dt} \; .$$

Also, for a state $|\Psi\rangle$ given by

$$(5.3) \qquad |\Psi\rangle = U(t) \; |\Psi_0\rangle \; ,$$

one has

$$(5.4) \qquad \frac{d\,|\Psi\rangle}{dt} = \frac{dU(t)}{dt} |\Psi_0\rangle \; .$$

But the Schrödinger equation is

$$(5.5) \qquad i\hbar \frac{d\,|\Psi\rangle}{dt} = H\,|\Psi\rangle = HU\,|\Psi_0\rangle \; ,$$

and substituting Eq. (5.4) in Eq. (5.5), one obtains (letting $\hbar = 1$):

$$(5.6) \qquad \frac{dU(t)}{dt} = \frac{1}{i} HU(t)$$

and therefore

$$(5.7) \qquad \frac{dU^\dagger(t)}{dt} = -\frac{1}{i} U^\dagger(t)H.$$

Next substituting Eqs. (5.6) and (5.7) in Eq. (5.2), one has

$$(5.8) \qquad \frac{dL(t)}{dt} = \frac{1}{i} HU(t)L_0 U^\dagger(t) - \frac{1}{i} U(t)L_0 U^\dagger(t)H \; ,$$

or

$$(5.9) \qquad \frac{dL(t)}{dt} = \frac{1}{i}[H, U(t)L_0 U^\dagger(t)] \; .$$

Also, according to Eq. (4.7), one has

$$(5.10) \qquad H = G^{-1}(L) = F(L) \; .$$

Next, substituting Eqs. (5.1) and (5.10) in Eq. (5.9), one obtains

$$(5.11) \qquad \frac{dL}{dt} = \frac{1}{i}[F(L), L] \; ,$$

or

$$(5.12) \qquad \frac{dL}{dt} = i[L, F(L)] \; .$$

Thus $L(t)$ given by Eq. (5.1) satisfies the geodesic equation, Eq. (4.12), for any $L_0 = G(H_0)$, in which H_0 is some constant Hamiltonian. Next it follows from Eq. (5.1) that

$$(5.13) \qquad U^\dagger(t)L(t)U(t) = U^\dagger(t) \; U(t) \; L_0 U^\dagger(t)U(t),$$

which by unitarity,

$$U^\dagger(t)U(t) = 1, \tag{5.14}$$

becomes

$$U^\dagger(t)L(t)U(t) = L_0, \tag{5.15}$$

a matrix-valued constant of geodesic motion which completely determines the system geodesics.

One can also show that one-body terms are constants of the motion. Let $S(X)$ map an n-body matrix X into 1-body terms. Then Eq. (4.20) implies:

$$\frac{dS(L)}{dt} = S\frac{dL}{dt} = i(1 - q^{-1})S([L, P(L)]) , \tag{5.16}$$

or using Eq. (2.2), then

$$\frac{dS(L)}{dt} = i(1 - q^{-1})S([P(L) + Q(L), P(L)]), \tag{5.17}$$

or

$$\frac{dS(L)}{dt} = i(1 - q^{-1})S([Q(L), P(L)]). \tag{5.18}$$

Next letting T map into two-body terms, one has

$$[Q(L), P(L)] = [Q(L), S(L) + T(L)] = [Q(L), S(L)] + [Q(L), T(L)] , \tag{5.19}$$

but the commutator of $Q(L)$ with one-body terms in $P(L)$ yields three- or more-body terms. For example,

$$
\begin{aligned}
[Q(L), S(L)] \quad &\backsim \quad [\sigma_i \otimes \sigma_j \otimes \sigma_k, I \otimes I \otimes \sigma_l] \\
&= \quad (\sigma_i \otimes \sigma_j \otimes (\sigma_k\sigma_l) - \sigma_i \otimes \sigma_j \otimes (\sigma_l\sigma_k)) \\
&= \quad i\varepsilon_{klm}\sigma_i \otimes \sigma_j \otimes \sigma_m - i\varepsilon_{lkm}\sigma_i \otimes \sigma_j \otimes \sigma_m \\
&= \quad 2i\varepsilon_{klm}\sigma_i \otimes \sigma_j \otimes \sigma_m.
\end{aligned}
\tag{5.20}
$$

Thus the commutator consists of three- or more-body terms, since $Q(L)$ generally contains three- or more-body terms. Also in Eq. (5.19), the commutator of $Q(L)$ with two-body terms in $P(L)$ yields two- and more-body terms. For example,

$$
\begin{aligned}
[Q(L), T(L)] \quad &\backsim \quad [\sigma_i \otimes \sigma_j \otimes \sigma_k, I \otimes \sigma_j \otimes \sigma_i] \\
&= \quad (\sigma_i \otimes I \otimes i\varepsilon_{kim}\sigma_m - \sigma_i \otimes I \otimes i\varepsilon_{ikm}\sigma_m) \\
&= \quad 2i\varepsilon_{kim}\sigma_i \otimes I \otimes \sigma_m.
\end{aligned}
\tag{5.21}
$$

Thus the commutator consists of two- or more-body terms. So one has

$$S([Q(L), P(L)]) = 0, \tag{5.22}$$

and substituting Eq. (5.22) in Eq. (5.18), one obtains

$$\frac{dS(L)}{dt} = 0 , \tag{5.23}$$

and one concludes that

$$S(L) = S_0 \tag{5.24}$$

is constant.

6. GEODESICS FOR CONSTANT HAMILTONIAN

Using the geodesic equation in the form given by Eq. (4.20), it is also evident that when $q = 1$, L is constant along geodesics, namely,

$$(6.1) \qquad\qquad L(t) = L_0,$$

where L_0 is a constant matrix. It then follows from Eqs. (5.10) and (6.1), that the Hamiltonian is also constant, namely,

$$(6.2) \qquad\qquad H = G^{-1}(L_0) \equiv H_0.$$

Also, for large q, Eq. (4.20) becomes

$$(6.3) \qquad\qquad \frac{dL}{dt} = i[L, P(L)],$$

which is vanishing if

$$(6.4) \qquad\qquad [L, P(L)] = 0,$$

or equivalently, using Eq. (2.2),

$$(6.5) \qquad\qquad [Q(L), P(L)] = 0 .$$

One again concludes that if one- and two-body terms commute with three- and more-body terms, or if the Hamiltonian H contains only one- and two-body terms, then Eqs. (6.1) and (6.2) again hold, namely the Hamiltonian is constant. It then follows from the Schrödinger equation that the corresponding unitary evolution is given by the geodesic

$$(6.6) \qquad\qquad U(t) = e^{-iHt},$$

as one might expect.

7. THREE-QUBIT GEODESICS

Next consider the three-qubit case. For this case define

$$(7.1) \qquad\qquad G_3^s \equiv sS + T + qQ,$$

in which S, T, and Q are superoperators (matrices) mapping onto the subspace of three-qubit Hamiltonians containing only one-, two-, and three-body terms, respectively. Also in Eq. (7.1), s is a useful parameter. One has the following commutation relations between the matrix subspaces S, T, and Q:

$$(7.2) \qquad\qquad [S, T] \subseteq T,$$

$$(7.3) \qquad\qquad [S, Q] \subseteq Q,$$

$$(7.4) \qquad\qquad [T, Q] \subseteq T.$$

Examples supporting Eqs. (7.2)-(7.4) are as follows:

$$[S, T] \quad \backsim \quad [\sigma_1 \otimes I \otimes I, \sigma_2 \otimes \sigma_3 \otimes I] = (\sigma_1 \sigma_2) \otimes \sigma_3 \otimes I - (\sigma_2 \sigma_1) \otimes \sigma_3 \otimes I$$
$$(7.5) \qquad = \quad 2i\sigma_3 \otimes \sigma_3 \otimes I \ \subseteq T,$$

$$[S, Q] \quad \backsim \quad [I \otimes I \otimes \sigma_l, \sigma_i \otimes \sigma_j \otimes \sigma_k]$$
$$= (\sigma_i \otimes \sigma_j \otimes (\sigma_l \sigma_k) - \sigma_i \otimes \sigma_j \otimes (\sigma_k \sigma_l))$$
(7.6)
$$= 2i\varepsilon_{lkm}\sigma_i \otimes \sigma_j \otimes \sigma_m \subseteq Q,$$

$$[T, Q] \quad \backsim \quad [\sigma_n \otimes \sigma_m \otimes I, \sigma_1 \otimes \sigma_2 \otimes \sigma_3]$$
$$= ((\sigma_n \sigma_1) \otimes (\sigma_m \sigma_2) \otimes \sigma_3 - (\sigma_1 \sigma_n) \otimes (\sigma_2 \sigma_m) \otimes \sigma_3)$$
(7.7)
$$= i\varepsilon_{n1p}\sigma_p \otimes i\varepsilon_{m2q}\sigma_q \otimes \sigma_3 - i\varepsilon_{1np}\sigma_p \otimes i\varepsilon_{2mq}\sigma_q \otimes \sigma_3 = 0,$$

and

$$[T, Q] \quad \backsim \quad [\sigma_1 \otimes \sigma_3 \otimes I, \sigma_1 \otimes \sigma_2 \otimes \sigma_3]$$
$$= ((\sigma_1 \sigma_1) \otimes (\sigma_3 \sigma_2) \otimes \sigma_3 - (\sigma_1 \sigma_1) \otimes (\sigma_2 \sigma_3) \otimes \sigma_3)$$
$$= I \otimes i\varepsilon_{32p}\sigma_p \otimes \sigma_3 - I \otimes i\varepsilon_{23p}\sigma_p \otimes \sigma_3$$
(7.8)
$$= 2i\varepsilon_{32p}I \otimes \sigma_p \otimes \sigma_3 \subseteq T.$$

Next define

(7.9)
$$L = (S + T + Q)(L),$$

and in much of the following, at the risk of an ambiguous but convenient notation,

(7.10)
$$S \equiv S(L), \quad T \equiv T(L), \quad Q \equiv Q(L).$$

It then follows from the geodesic equation (5.12) and Eqs. (7.1), (5.10) and (7.2)-(7.4) that

$$\frac{dS(L)}{dt} = iS([(S + T + Q)(L), (s^{-1}S(L) + T(L) + q^{-1}Q(L)])$$
$$= iS([S, T] + q^{-1}[S, Q]$$
$$\quad + s^{-1}[T, S] + q^{-1}[T, Q] + s^{-1}[Q, S] + [Q, T])$$
$$= iS(\{\subseteq T\} + q^{-1}\{\subseteq Q\} + s^{-1}\{\subseteq T\} + q^{-1}\{\subseteq T\} + s^{-1}\{\subseteq Q\} + \{\subseteq T\})$$
(7.11)
$$= 0,$$

$$\frac{dT(L)}{dt} = iT([(S + T + Q)(L), (s^{-1}S(L) + T(L) + q^{-1}Q(L)])$$
$$= iT([S, T] + q^{-1}[S, Q] + s^{-1}[T, S]$$
$$\quad + q^{-1}[T, Q] + s^{-1}[Q, S] + [Q, T])$$
$$= iT(\{\subseteq T\} + q^{-1}\{\subseteq Q\} + s^{-1}\{\subseteq T\} + q^{-1}\{\subseteq T\} + s^{-1}\{\subseteq Q\} + \{\subseteq T\})$$
$$= i(([S, T]) + 0 + s^{-1}([T, S])$$
$$\quad + q^{-1}([T, Q]) + 0 + ([Q, T]))$$
(7.12)
$$= i((1 - s^{-1})[S, T] + (1 - q^{-1})[Q, T]),$$

and

$$\frac{dQ(L)}{dt} = iQ([S, T] + q^{-1}[S, Q] + s^{-1}[T, S]$$
$$\quad + q^{-1}[T, Q] + s^{-1}[Q, S] + [Q, T])$$
$$= i(q^{-1}[S, Q] + s^{-1}[Q, S])$$
(7.13)
$$= i(q^{-1} - s^{-1})[S, Q].$$

Thus one has [**8**]

$$(7.14) \qquad \frac{dS(L)}{dt} = 0,$$

$$(7.15) \qquad \frac{dT}{dt} = i([((1 - s^{-1})S + (1 - q^{-1})Q), T]),$$

and

$$(7.16) \qquad \frac{dQ}{dt} = i(q^{-1} - s^{-1})[S, Q].$$

From Eq. (7.14), it follows that

$$(7.17) \qquad S(t) = S_0,$$

where S_0 is a constant matrix.

Next substituting Eq. (7.17) in Eq. (7.16), and defining

$$(7.18) \qquad k = i(q^{-1} - s^{-1}),$$

one obtains

$$(7.19) \qquad \frac{dQ}{dt} = k[S_0, Q],$$

or equivalently,

$$(7.20) \qquad \frac{dQ}{dt} = kS_0 Q - kQS_0.$$

Next, in order to solve Eq. (7.20) for $Q(t)$, one may make the ansatz:

$$(7.21) \qquad Q(t) = f(S_0, t)\overline{Q}g(S_0, t),$$

in which $f(S_0, t)$ and $g(S_0, t)$ are matrix functions to be determined, and $\overline{Q}$ is a constant matrix. Then one has

$$(7.22) \qquad \frac{dQ(t)}{dt} = \frac{d}{dt}f(S_0, t)\overline{Q}g(S_0, t) + f(S_0, t)\overline{Q}\frac{d}{dt}g(S_0, t),$$

or equivalently,

(7.23)
$$\frac{dQ(t)}{dt} = \frac{d}{dt}f(S_0, t)f(S_0, t)^{-1}f(S_0, t)\overline{Q}g(S_0, t) + f(S_0, t)\overline{Q}g(S_0, t)g(S_0, t)^{-1}\frac{d}{dt}g(S_0, t),$$

and substituting Eq. (7.21) in Eq. (7.23), one obtains

$$(7.24) \qquad \frac{dQ(t)}{dt} = \frac{d}{dt}f(S_0, t)f(S_0, t)^{-1}Q(t) + Q(t)g(S_0, t)^{-1}\frac{d}{dt}g(S_0, t).$$

Next comparing Eqs. (7.20) and (7.24), it follows that

$$(7.25) \qquad \frac{d}{dt}f(S_0, t)f(S_0, t)^{-1} = -g(S_0, t)^{-1}\frac{d}{dt}g(S_0, t) = kS_0,$$

and therefore

$$(7.26) \qquad f(S_0, t) = c_f e^{kS_0 t},$$

and

$$(7.27) \qquad g(S_0, t) = c_g e^{-kS_0 t},$$

where c_f and c_g are constants. Therefore, substituting Eqs. (7.26), (7.27), and (7.18) in Eq. (7.21), using Eq. (7.17), and defining $Q_0 \equiv c_f c_g \overline{Q} = Q(0)$, one obtains [**8**]

$$(7.28) \qquad Q(t) = e^{it(q^{-1}-s^{-1})S_0} Q_0 e^{-it(q^{-1}-s^{-1})S_0}.$$

A check that Eq. (7.28) does indeed satisfy Eq. (7.16) is given in Appendix F.

Next substituting Eqs. (7.17) and (7.28) in Eq. (7.15), and defining

$$(7.29) \qquad k_1 = i(1 - s^{-1})$$

and

$$(7.30) \qquad k_2 = i(1 - q^{-1}),$$

one obtains

$$(7.31) \qquad \frac{dT}{dt} = k_1[S_0, T] + k_2[e^{kS_0 t} Q_0 e^{-kS_0 t}, T],$$

or

$$(7.32) \qquad \frac{dT}{dt} = (k_1 S_0 + k_2 e^{kS_0 t} Q_0 e^{-kS_0 t})T - T(k_1 S_0 + k_2 e^{kS_0 t} Q_0 e^{-kS_0 t}).$$

Next, making the ansatz

$$(7.33) \qquad T(t) = a(S_0, Q_0, t)\overline{T}b(S_0, Q_0, t),$$

in which $a(S_0, Q_0, t)$ and $b(S_0, Q_0, t)$ are matrix functions to be determined, and $\overline{T}$ is a constant matrix, then

$$(7.34) \qquad \frac{dT}{dt} = \frac{da(S_0, Q_0, t)}{dt}\overline{T}b(S_0, Q_0, t) + a(S_0, Q_0, t)\overline{T}\frac{db(S_0, Q_0, t)}{dt},$$

or equivalently

$$\frac{dT}{dt} = \frac{da(S_0, Q_0, t)}{dt}a^{-1}(S_0, Q_0, t)a(S_0, Q_0, t)\overline{T}b(S_0, Q_0, t)$$

$$(7.35) \qquad\qquad + a(S_0, Q_0, t)\overline{T}b(S_0, Q_0, t)b^{-1}(S_0, Q_0, t)\frac{db(S_0, Q_0, t)}{dt}.$$

But substituting Eq. (7.33) in Eq. (7.35), one obtains

$$(7.36) \quad \frac{dT}{dt} = \frac{da(S_0, Q_0, t)}{dt}a^{-1}(S_0, Q_0, t)T(t) + T(t)b^{-1}(S_0, Q_0, t)\frac{db(S_0, Q_0, t)}{dt}.$$

Comparing Eqs. (7.32) and (7.36), then

(7.37)
$$\frac{da(S_0, Q_0, t)}{dt}a^{-1}(S_0, Q_0, t) = -b^{-1}(S_0, Q_0, t)\frac{db(S_0, Q_0, t)}{dt} = k_1 S_0 + k_2 e^{kS_0 t} Q_0 e^{-kS_0 t}.$$

To solve Eq. (7.37), one makes the ansatz:

$$(7.38) \qquad a(S_0, Q_0, t) = c_a e^{k_3 S_0 t} e^{(k_4 S_0 + k_5 Q_0)t},$$

where c_a is a constant. Then one has

$$(7.39) \qquad \frac{da}{dt} = k_3 S_0 a + a(k_4 S_0 + k_5 Q_0),$$

and multiplying from the right with a^{-1}, one obtains

$$(7.40) \qquad \frac{da}{dt}a^{-1} = k_3 S_0 + a(k_4 S_0 + k_5 Q_0)a^{-1}.$$

Next substituting Eq. (7.38) in Eq. (7.40), one obtains

$$
\begin{aligned}
\frac{da}{dt}a^{-1} &= k_3 S_0 + e^{k_3 S_0 t}e^{(k_4 S_0 + k_5 Q_0)t}(k_4 S_0 + k_5 Q_0)e^{-(k_4 S_0 + k_5 Q_0)t}e^{-k_3 S_0 t} \\
&= k_3 S_0 + e^{k_3 S_0 t}(k_4 S_0 + k_5 Q_0)e^{-k_3 S_0 t} \\
&= k_3 S_0 + k_4 S_0 + k_5 e^{k_3 S_0 t}Q_0 e^{-k_3 S_0 t} \\
(7.41) \qquad &= (k_3 + k_4)S_0 + k_5 e^{k_3 S_0 t}Q_0 e^{-k_3 S_0 t}.
\end{aligned}
$$

Comparing Eqs. (7.37) and (7.41), it follows that

$$
(7.42) \qquad\qquad k_3 + k_4 = k_1,
$$

$$
(7.43) \qquad\qquad k_5 = k_2,
$$

and

$$
(7.44) \qquad\qquad k_3 = k.
$$

From Eqs. (7.42), (7.44), (7.29), (7.30), and (7.18), it follows that

$$
(7.45) \qquad k_4 = k_1 - k = i(1 - s^{-1}) - i(q^{-1} - s^{-1}) = i(1 - q^{-1}) = k_2,
$$

and Eq. (7.38) becomes

$$
(7.46) \qquad a(S_0, Q_0, t) = c_a e^{i(q^{-1} - s^{-1})S_0 t}e^{i(1 - q^{-1})(S_0 + Q_0)t}.
$$

Next make the ansatz:

$$
(7.47) \qquad b(S_0, Q_0, t) = c_b e^{(k_6 S_0 + k_7 Q_0)t}e^{k_8 S_0 t},
$$

where c_b is a constant. Then

$$
(7.48) \qquad \frac{db}{dt} = (k_6 S_0 + k_7 Q_0)b + b k_8 S_0,
$$

and therefore

$$
(7.49) \qquad b^{-1}\frac{db}{dt} = k_8 S_0 + b^{-1}(k_6 S_0 + k_7 Q_0)b.
$$

Then substituting Eq. (7.47) in Eq. (7.49), one obtains

$$
\begin{aligned}
b^{-1}\frac{db}{dt} &= k_8 S_0 + e^{-k_8 S_0 t}e^{-(k_6 S_0 + k_7 Q_0)t}(k_6 S_0 + k_7 Q_0)e^{(k_6 S_0 + k_7 Q_0)t}e^{k_8 S_0 t} \\
&= k_8 S_0 + k_6 S_0 + k_7 e^{-k_8 S_0 t}Q_0 e^{k_8 S_0 t} \\
(7.50) \qquad &= (k_8 + k_6)S_0 + k_7 e^{-k_8 S_0 t}Q_0 e^{k_8 S_0 t}.
\end{aligned}
$$

Comparing Eqs. (7.50) and (7.37), then

$$
(7.51) \qquad\qquad k_8 + k_6 = -k_1,
$$

$$
(7.52) \qquad\qquad k_7 = -k_2,
$$

$$
(7.53) \qquad\qquad k_8 = -k,
$$

and using Eqs. (7.45), (7.51), and (7.53), one also has

$$
(7.54) \qquad\qquad k_6 = -k_1 + k = -k_2,
$$

so that Eq. (7.47) becomes

$$
(7.55) \qquad b(S_0, Q_0, t) = c_b e^{-k_2(S_0 + Q_0)t}e^{-k S_0 t}.
$$

Then substituting Eqs. (7.30) and (7.18) in Eq. (7.55), one obtains

(7.56) $$b(S_0, Q_0, t) = c_b e^{-i(1-q^{-1})(S_0+Q_0)t} e^{-i(q^{-1}-s^{-1})S_0 t}.$$

Finally substituting Eqs. (7.56) and (7.46) in Eq. (7.33), and defining $T_0 \equiv c_a c_b \overline{T} = T(0)$, one concludes that the solution to Eq. (7.15) is [8]

(7.57) $$T(t) = e^{it(q^{-1}-s^{-1})S_0} e^{it(1-q^{-1})(S_0+Q_0)} T_0 e^{-it(1-q^{-1})(S_0+Q_0)} e^{-it(q^{-1}-s^{-1})S_0}.$$

A check that Eq. (7.57) does indeed satisfy Eq. (7.15) is given in Appendix F.

Next using Eqs. (7.1) and (7.10), the Hamiltonian is given by

(7.58) $$H(t) = G_3^{s-1}(L) = s^{-1}S(t) + T(t) + q^{-1}Q(t),$$

or substituting Eqs. (7.17), (7.28), and (7.57), one obtains the locally optimal Hamiltonian,

(7.59) $$\begin{aligned} H(t) \;=\; & s^{-1}S_0 + e^{it(q^{-1}-s^{-1})S_0} e^{it(1-q^{-1})(S_0+Q_0)} T_0 \\ & \times e^{-it(1-q^{-1})(S_0+Q_0)} e^{-it(q^{-1}-s^{-1})S_0} \\ & + q^{-1} e^{it(q^{-1}-s^{-1})S_0} Q_0 e^{-it(q^{-1}-s^{-1})S_0}. \end{aligned}$$

8. SOLUTION FOR LARGE PENALTY FACTOR

In this section, the solution to the geodesic equation is obtained for large penalty parameter q. One can assume the following normalization:

(8.1) $$\langle H(t), H(t) \rangle = 1,$$

and using Equations (3.14) and (8.1), one obtains (again for the three-qubit case)

(8.2) $$\frac{1}{2^3} \operatorname{Tr}(H(t) G_3^s(H(t))) = 1.$$

Next using Eqs. (3.16), (3.17), (7.9), (7.10), and (7.1) in Eq. (8.2), one has

$$\begin{aligned} 1 \;=\; & \frac{1}{2^3} \operatorname{Tr}(L(t)H(t)) = \frac{1}{2^3} \operatorname{Tr}(L(t) G_3^{s-1} L(t)). \\ =\; & \frac{1}{2^3} \operatorname{Tr}\{(S(L) + T(L) + Q(L))(s^{-1}S(L) + T(L) + q^{-1}Q(L)\} \\ =\; & \frac{1}{2^3} \operatorname{Tr}(s^{-1}S^2 + ST + q^{-1}SQ + s^{-1}TS + T^2 \\ & + q^{-1}TQ + s^{-1}QS + QT + q^{-1}Q^2) \end{aligned}$$

(8.3) $$\geq \frac{1}{2^3} \operatorname{Tr}(s^{-1}S^2),$$

or equivalently,

(8.4) $$\frac{1}{2^3} \operatorname{Tr}(S^2) \leq s.$$

Analogously,

(8.5) $$1 \geq \frac{1}{2^3} \operatorname{Tr}(T^2),$$

or

$$(8.6) \qquad \frac{1}{2^3}\operatorname{Tr}(T^2) \leq 1.$$

Also analogously, one has

$$(8.7) \qquad 1 \geq \frac{1}{2^3}\operatorname{Tr}(q^{-1}Q^2),$$

or

$$(8.8) \qquad \frac{1}{2^3}\operatorname{Tr}(Q^2) \leq q.$$

According to Eq. (8.8), one has that $q^{-1}Q \sim O(q^{-1}q^{1/2}) \sim O(q^{-1/2})$, and therefore $q^{-1}Q$ can be neglected for large q. According to Eq. (7.58), the resulting error in $U(t) = e^{-iHt}$ is $O(tq^{-1/2})$. Then using Eq. (7.57) and neglecting q^{-1} terms, one has

$$(8.9) \qquad T(t) \underset{q\to\infty}{\longrightarrow} = e^{-its^{-1}S_0}e^{it(S_0+Q_0)}T_0 e^{-it(S_0+Q_0)}e^{-its^{-1}S_0}.$$

The resultant error in $T(t) \leq t(q^{-1}s^{1/2} + q^{-1}q^{1/2}) \sim t(q^{-1}s^{1/2} + q^{-1/2})$, so that the resulting error in $U(t) \leq t^2(q^{-1}s^{1/2} + q^{-1/2})$. Using Eq. (7.58), one therefore has

$$(8.10) \qquad H(t) = s^{-1}S(t) + T(t) + q^{-1}Q(t) \approx \widetilde{H}(t) \equiv s^{-1}S(t) + T(t).$$

It then follows that the approximate Hamiltonian $\widetilde{H}(t)$ for large q is given by

$$(8.11) \qquad \widetilde{H}(t) \equiv s^{-1}S_0 + e^{-its^{-1}S_0}e^{it(S_0+Q_0)}T_0 e^{-it(S_0+Q_0)}e^{its^{-1}S_0}.$$

The resulting approximate solution $\widetilde{U}(t)$ to the Schrödinger equation then satisfies $\|U(t) - \widetilde{U}(t)\| \leq O(tq^{-1}q^{1/2} + t^2(s^{1/2}q^{-1} + q^{-1/2}))$. Next make a change of variables to

$$(8.12) \qquad \widetilde{V} \equiv e^{-it(S_0+Q_0)}e^{its^{-1}S_0}\widetilde{U}.$$

Then

$$(8.13) \qquad \frac{d\widetilde{V}}{dt} \equiv -i(S_0+Q_0)\widetilde{V} + is^{-1}e^{-it(S_0+Q_0)}S_0 e^{its^{-1}S_0}\widetilde{U} + e^{-it(S_0+Q_0)}e^{its^{-1}S_0}\frac{d\widetilde{U}}{dt}.$$

But using Eq. (5.6), one has

$$(8.14) \qquad \frac{d\widetilde{U}}{dt} \approx \frac{dU(t)}{dt} = \frac{1}{i}HU \approx \frac{1}{i}\widetilde{H}\widetilde{U},$$

so substituting Eqs. (8.14), (8.12), and (8.11) in Eq. (8.13), and simplifying, it follows that

$$
\begin{aligned}
\frac{d\widetilde{V}}{dt} &\approx -i(S_0+Q_0)\widetilde{V} + is^{-1}e^{-it(S_0+Q_0)}S_0 e^{its^{-1}S_0}e^{-its^{-1}S_0}e^{it(S_0+Q_0)}\widetilde{V}\\
&\quad + \frac{1}{i}e^{-it(S_0+Q_0)}e^{its^{-1}S_0}\\
&\quad \times \left(s^{-1}S_0 + e^{-its^{-1}S_0}e^{it(S_0+Q_0)}T_0 e^{-it(S_0+Q_0)}e^{its^{-1}S_0} \right)\\
&\quad \times e^{-its^{-1}S_0}e^{it(S_0+Q_0)}\widetilde{V}\\
&= -i(S_0+Q_0)\widetilde{V} + is^{-1}e^{-it(S_0+Q_0)}S_0 e^{it(S_0+Q_0)}\widetilde{V}\\
&\quad + \frac{1}{i}e^{-it(S_0+Q_0)}e^{its^{-1}S_0}\\
&\quad \times \left(s^{-1}S_0 e^{-its^{-1}S_0}e^{it(S_0+Q_0)} + e^{-its^{-1}S_0}e^{it(S_0+Q_0)}T_0 \right)\widetilde{V}\\
&= -i(S_0+Q_0)\widetilde{V} + is^{-1}e^{-it(S_0+Q_0)}\\
&\quad \times \left(S_0 e^{it(S_0+Q_0)} - S_0 e^{it(S_0+Q_0)} \right)\widetilde{V}\\
&\quad + \frac{1}{i}T_0\widetilde{V}\\
&= -i(S_0+T_0+Q_0)\widetilde{V}.
\end{aligned}
$$

$$(8.15)$$

Next integrating Eq. (8.15), and noting that $\widetilde{V}(0) = \widetilde{U}(0) = 1$, one obtains

$$(8.16) \qquad \widetilde{V} = e^{-it(S_0+T_0+Q_0)},$$

and substituting Eq. (8.16) in Eq. (8.12), one obtains for large q:

$$(8.17) \quad \widetilde{U} = e^{-its^{-1}S_0}e^{it(S_0+Q_0)}\widetilde{V} = e^{-its^{-1}S_0}e^{it(S_0+Q_0)}e^{-it(S_0+T_0+Q_0)}, \quad q \gg 1.$$

For the case $s \to 0$, there is negligible cost for single-qubit unitary operations, the S_0 term in the last two exponents in Eq. (8.17) can be neglected, and Eq. (8.17) becomes

$$(8.18) \qquad \widetilde{U} = e^{-its^{-1}S_0}e^{itQ_0}e^{-it(T_0+Q_0)}, \quad q \gg 1.$$

In the case of general s, according to Eq. (8.17), one has

$$(8.19) \qquad \widetilde{U} = e^{-its^{-1}S_0}e^{it(S_0+Q_0)}e^{-it(S_0+T_0+Q_0)}, \quad q \gg 1.$$

One might expect that $S_0 + Q_0 \gg T_0$ and that $S_0 + Q_0$ is nondegenerate. Then first-order perturbation theory implies [8]

$$(8.20) \qquad \widetilde{U} \approx e^{-its^{-1}S_0}e^{-itR_{S_0+Q_0}(T_0)}, \quad q \gg 1,$$

in which $R_{S_0+Q_0}(T_0)$ is the diagonal matrix remaining in the eigenbasis of $S_0 + Q_0$ and the off-diagonal terms of T_0 are removed. If Q_0 is nondegenerate as $s \to 0$, then one obtains

$$(8.21) \qquad \widetilde{U}(t) \approx e^{-its^{-1}S_0}e^{-itR_{Q_0}(T_0)}, \quad q \gg 1.$$

9. RIEMANN CURVATURE TENSOR

For a right-invariant vector field Z, one has after substituting

$$(9.1) \qquad Z = \sum_{\tau} z^{\tau}\tau, \quad Y = \sum_{\sigma} y^{\sigma}\sigma$$

in Eq. (4.3) (Here the Hamiltonian representation is to be understood):

$$(9.2) \qquad \sum_{\sigma\tau} \nabla_{\sigma}\tau y^{\sigma} z^{\tau} = \frac{i}{2}\sum_{\sigma\tau}\left([\sigma,\tau] + F([\sigma,G(\tau)] + [\tau,G(\sigma)])\right)y^{\sigma}z^{\tau},$$

and therefore

$$(9.3) \qquad \nabla_{\sigma}\tau = \frac{i}{2}\left([\sigma,\tau] + F([\sigma,G(\tau)] + [\tau,G(\sigma)])\right).$$

Evidently, using Eqs. (3.15), 13.3) and (13.4) of Appendix C, one has

$$(9.4) \qquad [\sigma,G(\tau)] = \begin{cases} [\sigma,\tau], & \tau \in S_{12} \cup S_0 \\ q[\sigma,\tau], & \tau \notin S_{12} \cup S_0 \end{cases},$$

and therefore

$$(9.5) \qquad F([\sigma,G(\tau)]) = \begin{cases} F([\sigma,\tau]), & \tau \in S_{12} \cup S_0 \\ qF([\sigma,\tau]), & \tau \notin S_{12} \cup S_0 \end{cases},$$

or using Eq. (4.13), this becomes

$$(9.6) \qquad F([\sigma,G(\tau)]) = \begin{cases} \frac{1}{q_{[\sigma,\tau]}}[\sigma,\tau], & \tau \in S_{12} \cup S_0 \\ \frac{q}{q_{[\sigma,\tau]}}[\sigma,\tau], & \tau \notin S_{12} \cup S_0 \end{cases},$$

where

$$(9.7) \qquad q_{[\sigma,\tau]} = 1 \text{ if } [\sigma,\tau]=0, \quad q_{[\sigma,\tau]} = q_{\lambda} \text{ if } [\sigma,\tau] \propto \lambda, \text{ and } q_{[\sigma,\tau]} = q_{[\tau,\sigma]},$$

and q_{λ} is defined by Eq. (13.8) of Appendix C. Equation (9.6) can be written as

$$(9.8) \qquad F([\sigma,G(\tau)]) = \frac{q_{\tau}}{q_{[\sigma,\tau]}}[\sigma,\tau].$$

Next substituting Eq. (9.8) in Eq. (9.3), one obtains

$$(9.9) \qquad \nabla_{\sigma}\tau = \frac{i}{2}\left(\left(1 + \frac{q_{\tau}}{q_{[\sigma,\tau]}}\right)[\sigma,\tau] + \frac{q_{\sigma}}{q_{[\tau,\sigma]}}[\tau,\sigma]\right),$$

or equivalently, using Eq.(9.7), this becomes

$$(9.10) \qquad \nabla_{\sigma}\tau = \frac{i}{2}\left(1 + \frac{q_{\tau}-q_{\sigma}}{q_{[\sigma,\tau]}}\right)[\sigma,\tau],$$

or

$$(9.11) \qquad \nabla_{\sigma}\tau = ic_{\sigma,\tau}[\sigma,\tau],$$

where

$$(9.12) \qquad c_{\sigma,\tau} = \frac{1}{2}\left(1 + \frac{q_{\tau}-q_{\sigma}}{q_{[\sigma,\tau]}}\right).$$

The Riemann curvature tensor with the inner-product (metric) Eq. (3.14) is given by [30]

$$(9.13) \qquad R(W,X,Y,Z) = \left\langle \nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{i[W,X]}Y, Z \right\rangle,$$

and after substituting the vector fields,

$$(9.14) \qquad W = \sum_\sigma w^\rho \rho, \quad X = \sum_\sigma z^\sigma \sigma, \quad Y = \sum_\tau y^\tau \tau, \quad Z = \sum_\mu z^\mu \mu,$$

Eq. (9.13) becomes

$$(9.15) \qquad R_{\rho\sigma\tau\mu} = \left\langle \nabla_\rho \nabla_\sigma \tau - \nabla_\sigma \nabla_\rho \tau - \nabla_{i[\rho,\sigma]} \tau, \mu \right\rangle.$$

Next, for three right-invariant vector fields X, Y, and Z, one has

$$(9.16) \qquad 0 = \nabla_Y \langle X, Z \rangle = \langle X, \nabla_Y Z \rangle + \langle \nabla_Y X, Z \rangle,$$

or

$$(9.17) \qquad \langle X, \nabla_Y Z \rangle = - \langle \nabla_Y X, Z \rangle,$$

and substituting Eqs. (9.14) in Eq. (9.17), one then has

$$(9.18) \qquad \langle \sigma, \nabla_\tau \mu \rangle = - \langle \nabla_\tau \sigma, \mu \rangle.$$

Therefore

$$(9.19) \qquad \langle \nabla_\rho \nabla_\sigma \tau, \mu \rangle = - \langle \nabla_\sigma \tau, \nabla_\rho \mu \rangle,$$

and

$$(9.20) \qquad \langle \nabla_\sigma \nabla_\rho \tau, \mu \rangle = - \langle \nabla_\rho \tau, \nabla_\sigma \mu \rangle.$$

Then substituting Eqs. (9.19) and (9.20) in Eq. (9.15), and interchanging the first and second terms, one obtains

$$(9.21) \qquad R_{\rho\sigma\tau\mu} = \langle \nabla_\rho \tau, \nabla_\sigma \mu \rangle - \langle \nabla_\sigma \tau, \nabla_\rho \mu \rangle - \langle \nabla_{i[\rho,\sigma]} \tau, \mu \rangle.$$

Also clearly

$$(9.22) \qquad \nabla_{iY} Z = i \nabla_Y Z,$$

so Eq. (9.21) can also be written as

$$(9.23) \qquad R_{\rho\sigma\tau\mu} = \langle \nabla_\rho \tau, \nabla_\sigma \mu \rangle - \langle \nabla_\sigma \tau, \nabla_\rho \mu \rangle - i \langle \nabla_{[\rho,\sigma]} \tau, \mu \rangle.$$

Next substituting Eq. (9.11) in Eq. (9.23), one obtain the following useful form for the Riemann curvature tensor [8], [37]-[39]:

$$(9.24)$$
$$R_{\rho\sigma\tau\mu} = c_{\rho,\tau} c_{\sigma,\mu} \langle i[\rho,\tau], i[\sigma,\mu] \rangle - c_{\sigma,\tau} c_{\rho,\mu} \langle i[\sigma,\tau], i[\rho,\mu] \rangle - c_{[\rho,\sigma],\tau} \langle i[i[\rho,\sigma],\tau], \mu \rangle.$$

The Riemannian curvature is important for investigations of the global characteristics of geodesic paths in the group manifold [14]

10. SECTIONAL CURVATURE

The sectional curvature spanned by orthonormal right-invariant orthonormal vector fields X and Y is defined by [13]

$$(10.1) \qquad K(X,Y) \equiv R(X,Y,Y,X).$$

From Eqs. (9.14) and (9.21), it immediately follows that

$$(10.2) \qquad R(W,X,Y,Z) = \langle \nabla_W Y, \nabla_X Z \rangle - \langle \nabla_X Y, \nabla_W Z \rangle - \langle \nabla_{i[W,X]} Y, Z \rangle,$$

and substituting Eq. (10.2) in Eq. (10.1), one obtains

$$(10.3) \qquad K(X,Y) = \langle \nabla_X Y, \nabla_Y X \rangle - \langle \nabla_Y Y, \nabla_X X \rangle - \langle \nabla_{i[X,Y]} Y, X \rangle.$$

Next it is useful to define

(10.4) $$B(X,Y) = F(i[G(X),Y]),$$

and using Eqs. (3.14) and Eq. (10.4), one obtains

(10.5) $$\langle B(X,Y), Z \rangle = \langle F(i[G(X),Y]), Z \rangle \,,$$

or equivalently, using Eq. (3.14), then

(10.6) $$\langle B(X,Y), Z \rangle = \frac{1}{2^n} \text{Tr}(F(i[G(X),Y])G(Z)).$$

Because the superoperator G is Hermitian, Eq. (10.6) can also be written as

(10.7) $$\langle B(X,Y), Z \rangle = \frac{1}{2^n} \text{Tr}(GF(i[G(X),Y])Z),$$

but according to Eq. (3.45) one has

(10.8) $$GF = I,$$

and therefore Eq. (10.7) becomes

(10.9) $$\langle B(X,Y), Z \rangle = \frac{1}{2^n} \text{Tr}(i[G(X),Y]Z).$$

Next expanding the commutator, Eq. (10.9) becomes

(10.10) $$\langle B(X,Y), Z \rangle = \frac{i}{2^n} \text{Tr}(G(X)YZ - YG(X)Z),$$

and using the cyclic property of the trace, one obtains

(10.11) $$\langle B(X,Y), Z \rangle = \frac{i}{2^n} \text{Tr}(G(X)YZ - G(X)ZY),$$

or, equivalently,

(10.12) $$\langle B(X,Y), Z \rangle = \frac{i}{2^n} \text{Tr}(G(X)[Y,Z]).$$

But since the superoperator G is Hermitian, Eq. (10.12) can also be written as

(10.13) $$\langle B(X,Y), Z \rangle = \frac{i}{2^n} \text{Tr}(XG([Y,Z])),$$

or equivalently, using Eq. (3.14), this becomes

(10.14) $$\langle X, i[Y,Z] \rangle = \langle B(X,Y), Z \rangle \,.$$

But according to Eqs. (3.14), it follows that for vectors X and Y one has

(10.15) $$\langle X, Y \rangle = \frac{1}{2^n} \text{Tr}(XG(Y)),$$

and because the superoperator G is Hermitian, this can also be written as

(10.16) $$\langle X, Y \rangle = \frac{1}{2^n} \text{Tr}(G(X)Y),$$

which by the cyclic invariance of the trace becomes

(10.17) $$\langle X, Y \rangle = \frac{1}{2^n} \text{Tr}(YG(X)),$$

or equivalently using Eq. (3.14), it follows that

(10.18) $$\langle X, Y \rangle = \langle Y, X \rangle \,,$$

consistent with the Riemannian symmetric metric.

Next, for a right-invariant field Y, one has, using Eq. (4.3),

(10.19) $$\nabla_X Y = \frac{1}{2}\left(i[X,Y] + F(i[X,G(Y)]) + F(i[Y,G(X)])\right),$$

or

(10.20) $$\nabla_X Y = \frac{1}{2}\left(i[X,Y] - F(i[G(Y),X]) - F(i[G(X),Y])\right).$$

Then substituting Eq. (10.4) in Eq. (10.20), one obtains

(10.21) $$\nabla_X Y = \frac{1}{2}\left(i[X,Y] - B(X,Y) - B(Y,X)\right).$$

Next, according to Eq. (10.3), it follows that
(10.22)
$$K(X,Y) = R(X,Y,Y,X) = \langle \nabla_X Y, \nabla_Y X \rangle - \langle \nabla_Y Y, \nabla_X X \rangle - i\langle \nabla_{[X,Y]} Y, X \rangle.$$

According to Eq. (10.21), one has

(10.23) $$\nabla_{[X,Y]} Y = \frac{1}{2}\left(i[[X,Y],Y] - B([X,Y],Y) - B(Y,[X,Y])\right).$$

Also using Eq. (10.21), one obtains
(10.24)
$$\langle \nabla_X Y, \nabla_Y X \rangle = \frac{1}{4}\left(\langle i[X,Y] - B(X,Y) - B(Y,X), i[Y,X] - B(Y,X) - B(X,Y)\rangle\right),$$

or, equivalently,
(10.25)
$$\langle \nabla_X Y, \nabla_Y X \rangle = \frac{1}{4}\left(\langle i[X,Y] - B(X,Y) - B(Y,X), -i[X,Y] - B(X,Y) - B(Y,X)\rangle\right),$$

or

$$\langle \nabla_X Y, \nabla_Y X \rangle = \frac{1}{4}\langle -i[X,Y], i[X,Y]\rangle$$
$$-\frac{1}{4}\langle i[X,Y], B(X,Y) + B(Y,X)\rangle + \frac{1}{4}\langle B(X,Y) + B(Y,X), i[X,Y]\rangle$$
(10.26) $$+\frac{1}{4}\langle B(X,Y) + B(Y,X), B(X,Y) + B(Y,X)\rangle.$$

Next using Eq. (10.18) in Eq. (10.26), then
(10.27)
$$\langle \nabla_X Y, \nabla_Y X \rangle = -\frac{1}{4}\langle i[X,Y], i[X,Y]\rangle + \frac{1}{4}\langle B(X,Y) + B(Y,X), B(X,Y) + B(Y,X)\rangle.$$

Also, one has

$$\langle \nabla_Y Y, \nabla_X X \rangle = \frac{1}{4}(\langle -i[Y,Y], i[X,X]\rangle - \langle i[Y,Y], 2B(X,X)\rangle + \langle 2B(Y,Y), i[X,X]\rangle$$
(10.28) $$+4\langle B(Y,Y), B(X,X)\rangle).$$

But, according to Eqs. (10.4), (3.15), and (2.2), one has

$$B(X,X) = F(i[G(X),X) = F(i(P + qQ)(X),(P+Q)(X))$$
$$= iF((P + Q + (q-1)Q)(X),(P+Q)(X))$$
(10.29) $$= iF(q-1)[Q(X),P(X)] = 0.$$

Then Substituting Eq. (10.29) in Eq. (10.28) and simplifying, one obtains

(10.30) $$\langle \nabla_Y Y, \nabla_X X \rangle = 0.$$

Next substituting Eqs. (10.27), (10.30), and (10.23) in Eq. (10.22), one has

$$K(X,Y) \;=\; -\frac{1}{4}\left\langle i[X,Y], i[X,Y]\right\rangle + \frac{1}{4}\left\langle B(X,Y)+B(Y,X), B(X,Y)+B(Y,X)\right\rangle$$

(10.31) $$\qquad -\frac{i}{2}\left\langle i[[X,Y],Y], X\right\rangle + \frac{i}{2}\left\langle B([X,Y],Y), X\right\rangle + \frac{i}{2}\left\langle B(Y,[X,Y]), X\right\rangle.$$

Expanding the third term of Eq. (10.31), one has, using Eq. (3.14),

$$-\frac{i}{2}\left\langle i[[X,Y],Y], X\right\rangle \;=\; \frac{1}{2}\left(\frac{1}{2^n}\right)\mathrm{Tr}\left([[X,Y],Y]G(X)\right)$$

(10.32) $$\qquad\qquad\qquad =\; \frac{1}{2}\left(\frac{1}{2^n}\right)\mathrm{Tr}\left(([X,Y]Y - Y[X,Y])G(X)\right),$$

and using the cyclic invariance of the trace, then

$$-\frac{i}{2}\left\langle i[[X,Y],Y], X\right\rangle \;=\; \frac{1}{2}\left(\frac{1}{2^n}\right)\mathrm{Tr}([X,Y]YG(X) - [X,Y])G(X)Y)$$

(10.33) $$\qquad\qquad\qquad =\; \frac{1}{2}\left(\frac{1}{2^n}\right)\mathrm{Tr}(i[X,Y]i[G(X),Y]).$$

Next using Eqs. (10.4) and (3.45) in Eq. (10.33), one obtains

(10.34) $$\qquad -\frac{i}{2}\left\langle i[[X,Y],Y], X\right\rangle = \frac{1}{2}\left\langle i[X,Y], B(X,Y)\right\rangle.$$

Next, in the fourth term of Eq. (10.31) one has, using Eq. (10.14),

(10.35) $$\qquad \left\langle B([X,Y],Y), X\right\rangle = \left\langle [X,Y], i[Y,X]\right\rangle = i\left\langle i[X,Y], i[X,Y]\right\rangle.$$

The fifth term of Eq. (10.31), using Eq. (10.14), is

(10.36) $$\qquad \left\langle B(Y,[X,Y]), X\right\rangle = \left\langle Y, i[[X,Y],X]\right\rangle,$$

or equivalently,

(10.37) $$\qquad \left\langle B(Y,[X,Y]), X\right\rangle = \left\langle Y, i[X,[Y,X]]\right\rangle,$$

and using Eq. (10.14), this becomes

(10.38) $$\qquad \left\langle B(Y,[X,Y]), X\right\rangle = -\left\langle B(Y,X), [X,Y]\right\rangle.$$

Next using Eq. (10.18), Eq. (10.38) becomes

(10.39) $$\qquad \left\langle B(Y,[X,Y]), X\right\rangle = -\left\langle [X,Y], B(Y,X)\right\rangle.$$

Next, substituting Eqs. (10.34), (10.35), and (10.39) in Eq. (10.31), one has

$$K(X,Y) \;=\; -\frac{1}{4}\left\langle i[X,Y], i[X,Y]\right\rangle + \frac{1}{4}\left\langle B(X,Y)+B(Y,X), B(X,Y)+B(Y,X)\right\rangle$$

(10.40) $$\qquad +\frac{1}{2}\left\langle i[X,Y], B(X,Y)\right\rangle - \frac{1}{2}\left\langle i[X,Y], i[X,Y]\right\rangle - \frac{1}{2}\left\langle i[X,Y], B(Y,X)\right\rangle,$$

and combining terms, this becomes [**8**], [**37**]-[**39**]

$$K(X,Y) \;=\; -\frac{3}{4}\left\langle i[X,Y], i[X,Y]\right\rangle + \frac{1}{4}\left\langle B(X,Y)+B(Y,X), B(X,Y)+B(Y,X)\right\rangle$$

(10.41) $$\qquad +\frac{1}{2}\left\langle i[X,Y], B(X,Y)-B(Y,X)\right\rangle.$$

11. APPENDIX A: PAULI MATRICES

The Pauli matrices are defined by [1]

(11.1)
$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

They are Hermitian,

(11.2)
$$\sigma_i = \sigma_i^\dagger, \qquad i = 1, 2, 3,$$

and, except for σ_0, they are traceless,

(11.3)
$$\mathrm{Tr}\sigma_i = 0, \quad i \neq 0.$$

Their products are given by

(11.4)
$$\sigma_i^2 = I,$$

and, using the Einstein sum convention for repeated indices (both lower in this case),

(11.5)
$$\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k, \quad i, j, k \neq 0,$$

expressed in terms of the totally antisymmetric Levi-Civita symbol with $\varepsilon_{123} = 1$.

12. APPENDIX B: SUPEROPERATOR ADJOINTS

The purpose of this Appendix is to derive Eqs. (3.12) and (3.13) for the adjoint superoperators $E_X^\dagger$ and $D_X^\dagger$ (at least to first order). For vectors X, Y, Z, consider the trace inner product

(12.1)
$$(Y, \mathrm{ad}_X Z) \equiv \mathrm{Tr}\left(Y^\dagger \mathrm{ad}_X Z\right),$$

or using Eq. (3.7), this becomes

(12.2)
$$(Y, \mathrm{ad}_X Z) \equiv \mathrm{Tr}\left(Y^\dagger [X, Z]\right),$$

or expanding the commutator, then

(12.3)
$$(Y, \mathrm{ad}_X Z) \equiv \mathrm{Tr}\left(Y^\dagger X Z - Y^\dagger Z X\right).$$

Next using the cyclic property of the trace, Eq. (12.3) becomes

(12.4)
$$(Y, \mathrm{ad}_X Z) \equiv \mathrm{Tr}\left(Y^\dagger X Z - X Y^\dagger Z\right),$$

or equivalently, again using the trace inner product,

(12.5)
$$(Y, \mathrm{ad}_X Z) \equiv -\mathrm{Tr}\left([X, Y^\dagger]Z\right) = -\left([X, Y^\dagger]^\dagger, Z\right),$$

or

(12.6)
$$(Y, \mathrm{ad}_X Z) \equiv -\left([Y, X^\dagger], Z\right) = \left([X^\dagger, Y], Z\right).$$

But by definition of the adjoint, one has

(12.7)
$$(Y, \mathrm{ad}_X Z) \equiv \left(\mathrm{ad}_X^\dagger Y, Z\right),$$

and comparing Eqs. (12.6) and (12.7), one obtains

(12.8)
$$\mathrm{ad}_X^\dagger Y = [X^\dagger, Y].$$

Next, if X is Hermitian, then $X = X^\dagger$, and Eq. (12.8) becomes

(12.9)
$$\mathrm{ad}_X^\dagger Y = [X, Y] = \mathrm{ad}_X Y,$$

and it follows that ad_X is Hermitian,

(12.10) $$\text{ad}_X^\dagger = \text{ad}_X.$$

Also, Eq. (3.7) implies

(12.11) $$\text{ad}_X Y = [X, Y] = -[-X, Y] = -\text{ad}_{-X} Y,$$

and therefore

(12.12) $$\text{ad}_X = -\text{ad}_{-X}.$$

Next, Eqs. (3.10) and (12.10) imply

(12.13) $$E_X^\dagger = I + \frac{i}{2}\text{ad}_X + O(X^2).$$

Also according to Eqs. (3.10) and (12.12), one has

(12.14) $$E_{-X} = I + \frac{i}{2}\text{ad}_X + O(X^2).$$

Comparing Eqs. (12.13) and (12.14), one has (at least to first order),

(12.15) $$E_X^\dagger = E_{-X}.$$

Also, according to Eqs. (3.9) and (3.10),

(12.16) $$D_X = E_X^{-1} = I + \frac{i}{2}\text{ad}_X + O(X^2).$$

Then

(12.17) $$D_X^\dagger = I - \frac{i}{2}\text{ad}_X^\dagger + O(x^2),$$

and substituting Eq. (12.10) and using Eq. (12.12), then

(12.18) $$D_X^\dagger = I + \frac{i}{2}\text{ad}_{-X} + O(x^2).$$

Also, Eq. (12.16) implies that

(12.19) $$D_{-X} = I + \frac{i}{2}\text{ad}_{-X} + O(x^2).$$

Comparing Eqs. (12.18) and (12.19), then (at least to first order) one has

(12.20) $$D_X^\dagger = D_{-X}.$$

13. APPENDIX C: INVERSE METRIC

From Eq. (11.3) and the property of the trace of a tensor product [23], it follows that for a generalized Pauli matrix $\sigma \notin \{I \otimes I \otimes ...\}$, one has

(13.1) $$\text{Tr}(\sigma) = \text{Tr}(I \otimes ..\sigma_i \otimes ..\sigma_j \otimes ..) = \text{Tr}(I)..\text{Tr}(\sigma_i)..\text{Tr}(\sigma_j).. = 0.$$

Also, using Eqs. (2.7) and (3.15), one has

(13.2) $$\langle \sigma, \tau \rangle \equiv \frac{1}{2^n}\text{Tr}\left(\sigma G(\tau)\right) = \frac{1}{2^n}\left\{\text{Tr}\left(\sigma P(\tau)\right) + q\text{Tr}\left(\sigma Q(\tau)\right)\right\}.$$

Next, denoting S_0 (not to be confused with S_0 in Section 7) as the set of generalized Pauli matrices containing only tensor products of the identity, and S_{12} as the set of generalized Pauli matrices containing only one and two body terms, that is

(13.3) $$S_0 \equiv \{I \otimes I \otimes ...\},$$

and

$$(13.4) \qquad S_{12} = \{I \otimes I \otimes ...\sigma_i \otimes I.., ...\} \cup \{I \otimes I \otimes ...\sigma_i \otimes I..\sigma_j \otimes I.., ...\},$$

then Eq. (13.2) becomes

$$(13.5) \quad \langle \sigma, \tau \rangle = \frac{1}{2^n} \left\{ \begin{array}{ll} \mathrm{Tr}\left(IP(\tau)\right) + q\mathrm{Tr}\left(IQ(\tau)\right), & \sigma \in S_0 \\ \mathrm{Tr}\left(P(\sigma)P(\tau)\right) + q\mathrm{Tr}\left(P(\sigma)Q(\tau)\right), & \sigma \in S_{12} \\ \mathrm{Tr}\left(Q(\sigma)P(\tau)\right) + q\mathrm{Tr}\left(Q(\sigma)Q(\tau)\right), & \sigma \notin S_0, S_{12} \end{array} \right. ,$$

and using Eqs. (11.3)-(11.5) and the Kronecker delta $\delta_{\sigma\tau}$, then Eq. (13.5) becomes

$$(13.6) \qquad \langle \sigma, \tau \rangle = \left\{ \begin{array}{ll} 0, & \sigma \in S_0 \\ \delta_{\sigma\tau}, & \sigma \in S_{12} \\ q\delta_{\sigma\tau}, & \sigma \notin S_0 \cup S_{12} \end{array} \right. ,$$

or equivalently

$$(13.7) \qquad \langle \sigma, \tau \rangle = q_\sigma \delta_{\sigma\tau},$$

where

$$(13.8) \qquad q_\sigma \equiv \left\{ \begin{array}{ll} 0, & \sigma \in S_0 \\ 1, & \sigma \in S_{12} \\ q, & \sigma \notin S_0 \cup S_{12} \end{array} \right. .$$

Next, according to Eqs. (3.26), (3.28), (13.2) and (13.6), the metric at the origin is given by

$$(13.9) \qquad g_{\tau\lambda} = \frac{1}{2^n}\mathrm{Tr}(\tau G(\lambda)) = \langle \tau, \lambda \rangle = q_\tau \delta_{\tau\lambda} = q_\lambda \delta_{\tau\lambda}.$$

The inverse metric is given by

$$(13.10) \qquad g^{\sigma\tau} = \frac{1}{2^n}\mathrm{Tr}(\sigma G^{-1}(\tau)),$$

which is justified by the following. Substituting Eqs. (3.45) and (4.13) in Eq. (13.10), then

$$(13.11) \qquad g^{\sigma\tau} = \frac{1}{2^n}\left\{ \mathrm{Tr}(\sigma P(\tau)) + \frac{1}{q}\mathrm{Tr}(\sigma Q(\tau)) \right\},$$

which, analogously to Eqs. (13.2), (13.5) and (13.6), becomes (ignoring S_0 which was assumed to be excluded)

$$(13.12) \qquad g^{\sigma\tau} = \left\{ \begin{array}{ll} \delta^{\sigma\tau}, & \sigma \in S_{12} \\ \frac{1}{q}\delta^{\sigma\tau}, & \sigma \notin S_0 \cup S_{12} \end{array} \right. ,$$

or equivalently,

$$(13.13) \qquad g^{\sigma\tau} = \frac{1}{q_\sigma}\delta^{\sigma\tau}.$$

Equations (13.13) and (13.9) with the Einstein convention, summing over repeated upper and lower indices, imply

$$(13.14) \qquad g^{\sigma\tau}g_{\tau\lambda} = \sum_\tau \frac{1}{q_\sigma}\delta^{\sigma\tau}q_\lambda\delta_{\tau\lambda} = \delta^\sigma_\lambda,$$

so in fact Eq. (13.10) is the valid inverse metric. Using Eq. (3.45), the inverse metric Eq. (13.10) can also be written as [**8**]

$$(13.15) \qquad g^{\sigma\tau} = \frac{1}{2^n}\mathrm{Tr}(\sigma F(\tau)).$$

14. APPENDIX D: RAISED CHRISTOFFEL SYMBOLS

The raised form of the Christoffel symbols (Christoffel symbols of the second kind) [**17**] is obtained by substituting Eqs. (13.15) and (3.42) in

$$(14.1) \qquad \Gamma^{\rho}_{\sigma\tau} = g^{\rho\lambda}\Gamma_{\lambda\sigma\tau},$$

and summing over λ (Einstein sum convention). Thus

$$(14.2) \qquad \Gamma^{\rho}_{\sigma\tau} = \sum_{\lambda} \frac{1}{2^n}\mathrm{Tr}(\rho F(\lambda))\frac{i}{2^{n+1}}\mathrm{Tr}\left(\lambda\left([\sigma, G(\tau)] + [\tau, G(\sigma)]\right)\right).$$

Next one notes using Eq. (4.13) that

$$(14.3) \qquad F(\lambda)\mathrm{Tr}(\lambda...) = \left(P + \frac{1}{q}Q\right)(\lambda)\mathrm{Tr}(\lambda...),$$

or

$$(14.4) \qquad F(\lambda)\mathrm{Tr}(\lambda...) = \begin{cases} P(\lambda)\mathrm{Tr}(P(\lambda)...), & \lambda \in S_{12} \\ \frac{1}{q}Q(\lambda)\mathrm{Tr}\left(Q(\lambda)...\right), & \lambda \notin S_0, S_{12} \end{cases},$$

or equivalently, taking account of Eq. (2.2) and using Eq. (4.13), then

$$(14.5) \qquad F(\lambda)\mathrm{Tr}(\lambda...) = \lambda\mathrm{Tr}((P + \frac{1}{q}Q)(\lambda)...) = \lambda\mathrm{Tr}(F(\lambda)...).$$

Then using Eq. (14.5) in Eq. (14.2), one obtains

$$(14.6) \qquad \Gamma^{\rho}_{\sigma\tau} = \sum_{\lambda} \frac{1}{2^n}\mathrm{Tr}(\rho\lambda)\frac{i}{2^{n+1}}\mathrm{Tr}\left(F(\lambda)\left([\sigma, G(\tau)] + [\tau, G(\sigma)]\right)\right).$$

From Eqs. (11.3)-(11.5) it follows that the generalized Pauli matrices are orthogonal with respect to the trace inner product, namely,

$$(14.7) \qquad \mathrm{Tr}(\rho\lambda) = 2^n\delta^{\rho\lambda},$$

and substituting Eq. (14.7) in Eq. (14.6), then

$$(14.8) \qquad \Gamma^{\rho}_{\sigma\tau} = \sum_{\lambda} \frac{1}{2^n}2^n\delta^{\rho\lambda}\frac{i}{2^{n+1}}\mathrm{Tr}\left(F(\lambda)\left([\sigma, G(\tau)] + [\tau, G(\sigma)]\right)\right),$$

or finally,

$$(14.9) \qquad \Gamma^{\rho}_{\sigma\tau} = \frac{i}{2^{n+1}}\mathrm{Tr}\left(F(\rho)\left([\sigma, G(\tau)] + [\tau, G(\sigma)]\right)\right).$$

15. APPENDIX E: AN IDENTITY

In this Appendix, the identity given by Eq. (3.49) is derived. It follows from Eq. (11.4) and (11.5) that for some coefficients $a_{\tau\lambda\sigma}$ and $b_{\tau\lambda}$ one has

$$(15.1) \qquad [\tau, G(\lambda)] = \sum_{\sigma} a_{\tau\lambda\sigma}\sigma + b_{\tau\lambda}I.$$

Then multiplying both sides of Eq. (15.1) by κ and taking the trace, one obtains

$$(15.2) \qquad \mathrm{Tr}\,(\kappa[\tau, G(\lambda)]) = \sum_{\sigma} a_{\tau\lambda\sigma}\mathrm{Tr}(\kappa\sigma) + b_{\tau\lambda}\mathrm{Tr}(\kappa).$$

Next substituting Eqs. (11.3) and (14.7) in Eq. (15.2), the latter becomes

$$(15.3) \qquad \mathrm{Tr}\,(\kappa[\tau, G(\lambda)]) = \sum_{\sigma} a_{\tau\lambda\sigma}2^{n}\delta_{\kappa\sigma} = 2^{n}a_{\tau\lambda\kappa},$$

and therefore

$$(15.4) \qquad a_{\tau\lambda\sigma} = \frac{1}{2^{n}}\mathrm{Tr}\,(\sigma[\tau, G(\lambda)]).$$

Next operating on both sides of Eq. (15.1) with F, and using Eq. (4.13), one has

$$(15.5) \qquad F\,([\tau, G(\lambda)]) = \sum_{\sigma} a_{\tau\lambda\sigma}F(\sigma),$$

and substituting Eq. (15.4) in Eq. (15.5), one obtains

$$(15.6) \qquad F\,([\tau, G(\lambda)]) = \sum_{\sigma} F(\sigma)\frac{1}{2^{n}}\mathrm{Tr}\,(\sigma[\tau, G(\lambda)]),$$

or

$$(15.7) \qquad 2^{n}F\,([\tau, G(\lambda)]) = \sum_{\sigma} F(\sigma)\mathrm{Tr}\,(\sigma[\tau, G(\lambda)]).$$

Then substituting Eq. (14.5) in Eq. (15.7), one obtains

$$(15.8) \qquad 2^{n}F\,([\tau, G(\lambda)]) = \sum_{\sigma} \sigma\mathrm{Tr}\,(F(\sigma)[\tau, G(\lambda)]),$$

or equivalently,

$$(15.9) \qquad \sum_{\sigma} \sigma\mathrm{Tr}\,(F(\sigma)[\tau, G(\lambda)]) = 2^{n}F\,([\tau, G(\lambda)]).$$

which is Eq. (3.49).

16. APPENDIX F: CHECKS OF THREE-QUBIT SOLUTIONS

In this Appendix, checks are given of the solution, Eq. (7.28) to Eq. (7.16),

$$(16.1) \qquad Q(t) = e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{-it(q^{-1}-s^{-1})S_0},$$

and the solution, Eq. (7.57) to Eq. (7.15),

$$(16.2) \quad T(t) = e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0}.$$

A check that Eq. (16.1) does indeed satisfy Eq. (7.16) proceeds by first calculating

$$(16.3) \qquad \begin{aligned} \frac{dQ}{dt} &= i(q^{-1}-s^{-1})S_0 Q(t) - i(q^{-1}-s^{-1})Q(t)S_0 \\ &= i(q^{-1}-s^{-1})[S_0, Q(t)], \end{aligned}$$

which agrees with Eqs. (7.16) and (7.17).

Proceeding to check that Eq. (16.2) satisfies Eq. (7.15), it follows from Eq. (16.2) that

$$
\begin{aligned}
\frac{dT}{dt} &= i(q^{-1} - s^{-1})S_0 T(t) - i(q^{-1} - s^{-1})T(t)S_0 + e^{it(q^{-1}-s^{-1})S_0} \\
&\quad \times \frac{d}{dt}\left(e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)}\right)e^{-it(q^{-1}-s^{-1})S_0} \\
&= i(q^{-1} - s^{-1})[S_0, T(t)] + e^{it(q^{-1}-s^{-1})S_0}\{i(1-q^{-1}) \\
&\quad \times[(S_0 + Q_0), e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)}]\} \\
&\quad \times e^{-it(q^{-1}-s^{-1})S_0}.
\end{aligned}
$$

(16.4)

The following commutation relation is true:

$$
(16.5) \quad [A, BC] = ABC - BCA = BAC - BCA + ABC - BAC = B[A, C] + [A, B]C,
$$

and using Eq. (16.5) twice, and noting that $(S_0 + Q_0)$ commutes with itself, Eq. (16.4) becomes

$$
\begin{aligned}
\frac{dT}{dt} &= i(q^{-1} - s^{-1})[S_0, T(t)] + e^{it(q^{-1}-s^{-1})S_0}i(1-q^{-1}) \\
&\quad \times \left\{ e^{it(1-q^{-1})(S_0+Q_0)}[(S_0 + Q_0), T_0 e^{-it(1-q^{-1})(S_0+Q_0)}] \right. \\
&\quad \left. + [(S_0 + Q_0), e^{it(1-q^{-1})(S_0+Q_0)}]T_0 e^{-it(1-q^{-1})(S_0+Q_0)} \right\} \\
&\quad \times e^{-it(q^{-1}-s^{-1})S_0},
\end{aligned}
$$

(16.6)

or

$$
\begin{aligned}
\frac{dT}{dt} &= i(q^{-1} - s^{-1})[S_0, T(t)] + e^{it(q^{-1}-s^{-1})S_0}i(1-q^{-1}) \\
&\quad \times \left\{ e^{it(1-q^{-1})(S_0+Q_0)}T_0[(S_0 + Q_0), e^{-it(1-q^{-1})(S_0+Q_0)}] \right. \\
&\quad \left. + e^{it(1-q^{-1})(S_0+Q_0)}[(S_0 + Q_0), T_0]e^{-it(1-q^{-1})(S_0+Q_0)} \right\} \\
&\quad \times e^{-it(q^{-1}-s^{-1})S_0},
\end{aligned}
$$

(16.7)

or equivalently,

$$
\begin{aligned}
\frac{dT}{dt} &= i(q^{-1} - s^{-1})[S_0, T(t)] \\
&\quad + e^{it(q^{-1}-s^{-1})S_0}i(1-q^{-1})e^{it(1-q^{-1})(S_0+Q_0)}[(S_0 + Q_0), T_0] \\
&\quad \times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0},
\end{aligned}
$$

(16.8)

or

$$
\begin{aligned}
\frac{dT}{dt} \;=\;& i(q^{-1}-s^{-1})[S_0,T(t)] + i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0} \\
& \times (S_0+Q_0)e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)} \\
& \times e^{-it(q^{-1}-s^{-1})S_0} \\
& -i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)} \\
& \times (S_0+Q_0)e^{-it(q^{-1}-s^{-1})S_0},
\end{aligned}
$$
(16.9)

or

$$
\begin{aligned}
\frac{dT}{dt} \;=\;& i(q^{-1}-s^{-1})[S_0,T(t)] + i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0} \\
& \times Q_0 e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0} \\
& -i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}Q_0 e^{-it(q^{-1}-s^{-1})S_0} \\
& +i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0} \\
& \times [S_0, e^{it(1-q^{-1})(S_0+Q_0)}T_0 e^{-it(1-q^{-1})(S_0+Q_0)}]e^{-it(q^{-1}-s^{-1})S_0}.
\end{aligned}
$$
(16.10)

Next, the right side of Eq. (7.15), using Eqs. (7.17), (7.28), and (7.57), becomes

$$
\begin{aligned}
& i[((1-s^{-1})S + (1-q^{-1})Q),T] \\
=\;& i(1-s^{-1})[S,T] + i(1-q^{-1})[Q,T] \\
=\;& i[S_0,T] - is^{-1}[S_0,T] + i(1-q^{-1})[Q,T] \\
=\;& i[S_0,T] - is^{-1}[S_0,T] + i(1-q^{-1}) \\
& \times [e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{-it(q^{-1}-s^{-1})S_0}, e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)} \\
& \times T_0 e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0}] \\
=\;& i(1-q^{-1})[S_0,T] + i(q^{-1}-s^{-1})[S_0,T] \\
& +i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0} \\
& -i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}Q_0 e^{-it(q^{-1}-s^{-1})S_0} \\
=\;& i(q^{-1}-s^{-1})[S_0,T] \\
& +i(1-q^{-1})[S_0, e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0}] \\
& +i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0} \\
& -i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0 \\
& \times e^{-it(1-q^{-1})(S_0+Q_0)}Q_0 e^{-it(q^{-1}-s^{-1})S_0},
\end{aligned}
$$
(16.11)

or

$$i[((1-s^{-1})S+(1-q^{-1})Q),T]$$

$$= \quad i(q^{-1}-s^{-1})[S_0,T]$$

$$+i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0}$$

$$-i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}Q_0 e^{-it(q^{-1}-s^{-1})S_0}$$

$$+i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}[S_0,e^{-it(q^{-1}-s^{-1})S_0}]$$

$$+i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}[S_0,e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}]e^{-it(q^{-1}-s^{-1})S_0}$$

$$= \quad i(q^{-1}-s^{-1})[S_0,T]$$

$$+i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}Q_0 e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}e^{-it(q^{-1}-s^{-1})S_0}$$

$$-i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}Q_0 e^{-it(q^{-1}-s^{-1})S_0}$$

$$+i(1-q^{-1})e^{it(q^{-1}-s^{-1})S_0}[S_0,e^{it(1-q^{-1})(S_0+Q_0)}T_0$$

$$\times e^{-it(1-q^{-1})(S_0+Q_0)}]e^{-it(q^{-1}-s^{-1})S_0}.$$

(16.12)

Comparing Eqs. (16.12), (16.10), and (7.15), one concludes that the left and right sides of Eq. (7.15) agree.

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