

Kitaev's quantum double model as an error correcting code

Shawn X. Cui,^{1,2*} Dawei Ding,¹ Xizhi Han,¹ Geoffrey Penington,¹ Daniel Ranard,¹ Brandon C. Rayhaun,¹ Zhou Shangnan¹

¹*Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305*

²*Department of Mathematics, Virginia Tech, Blacksburg, VA, 24061*

cui177@purdue.edu, dding@stanford.edu, hanxzh@stanford.edu,
geoffp@stanford.edu, dranard@stanford.edu, brayhaun@stanford.edu,
snzhou@stanford.edu

Abstract

Kitaev's quantum double models in 2D provide some of the most commonly studied examples of topological quantum order. In particular, the ground space is thought to yield a quantum error-correcting code. We offer an explicit proof that this is the case for arbitrary finite groups, which appears to be previously lacking in the literature. Actually a stronger claim is shown: any two states with zero energy density in some contractible region must have the same reduced state in that region. Alternatively, the local properties of a gauge-invariant state are fully determined by specifying that its holonomies in the region are trivial. We contrast this result with the fact that local properties of gauge-invariant states are not generally determined by specifying all of their non-Abelian fluxes — that is, the Wilson loops of lattice gauge theory do not form a complete commuting set of observables. We also note that the methods developed by P. Naaijken (PhD thesis, 2012) under a different context can be adapted to provide another proof of the error correcting property of Kitaev's model.

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*Present Affiliation: Departments of Mathematics, Physics and Astronomy, Purdue University, West Lafayette, IN 47907

1 Introduction

Topological phases of matter (TPMs) in two spatial dimensions are gapped quantum liquids that exhibit exotic properties such as stable ground state degeneracy, stable long-range entanglement, existence of quasi-particle excitations, (possibly) non-Abelian exchange statistics, etc. These phases are characterized by a new type of order, topological quantum order (TQO),¹ that is beyond Landau's theory of symmetry breaking. An important application of TQO is in topological quantum computing [6, 5], where information is encoded in non-local degrees of freedom and processed by manipulating quasi-particle excitations.

A large class of TQOs can be realized as lattice models in quantum spin systems where the Hamiltonian is given as a sum of pairwise commuting and geometrically local projectors. Examples of such constructions include the Levin-Wen string-net lattice models [8] and Kitaev's quantum double models [6]. In [2, 1], the authors gave a mathematically rigorous proof of gap stability under weak perturbations for quantum spin Hamiltonians satisfying two physically plausible conditions, TQO-1 and TQO-2. Roughly, TQO-1 states that the ground state space is a quantum error correcting code with a macroscopic distance, and TQO-2 means that the local ground state space coincides with the global one. See [2, 1] or §2.2 for a formal definition.

It is widely believed that both the Levin-Wen and Kitaev's quantum double models satisfy TQO-1 and TQO-2. However, a mathematical proof of this fact is missing to the best of our knowledge. See [7] for partial results in this direction. In this paper, we provide a rigorous proof for Kitaev's quantum double models. In fact, we prove a stronger property for Kitaev's model that simultaneously implies TQO-1 and TQO-2. Our result can be informally stated as:

States with locally zero energy density are locally indistinguishable.

See Theorem 3.1 for a formal statement.

The Levin-Wen models actually include Kitaev's models as special cases. Originally, Kitaev's models were only defined for finite groups. However this construction was generalized to finite dimensional Hopf \mathbb{C}^* -algebras in [3], and then further generalized to weak Hopf \mathbb{C}^* -algebras (or unitary quantum groupoids) in [4]. On the other hand, the Levin-Wen model takes as input any unitary fusion category. In [4], it was proved that the Levin-Wen model associated to a fusion category \mathcal{C} is equivalent to the generalized Kitaev model based on the weak Hopf algebra $H_{\mathcal{C}}$ reconstructed from \mathcal{C} such that $\text{Rep}(H_{\mathcal{C}}) \simeq \mathcal{C}$. Thus, the Levin-Wen models and the generalized Kitaev models are essentially equivalent.

It is an interesting question whether or not our current proof for the case of finite groups can be adapted to the case of Hopf algebras and/or to weak Hopf algebras. For finite groups, there are well-defined notions of local gauge

¹We use the terminology TPM and TQO interchangeably.

transformations and holonomy which allow us to obtain an explicit characterization of the ground states, though this is not necessary for the proof of our main result. In the general case, such notions are not as clear. We leave these questions for future study.

2 Background

In this section, we give a minimal review of a few preliminary notions which are necessary for understanding the proof of our main theorem. We begin by discussing generalities related to error correcting codes, topological quantum order, and the relationship between them, and then describe the particular models which we will be studying.

2.1 Error correcting codes

We provide a very brief introduction to quantum error correcting codes (QECCs), mainly to set up the conventions that will be used later. For a detailed account of the theory of QECCs, we recommend [10].

To protect quantum information against noise, a common strategy is to embed states $|\psi\rangle$ which contain information into a subspace \mathcal{C} , called the code subspace, of a larger Hilbert space \mathcal{H} . Quantum processing of the state is then modeled as a noisy quantum channel \mathcal{E} , which is a completely positive, trace preserving map on the density matrices living in \mathcal{H} . It is possible to successfully retrieve the information contained in $|\psi\rangle$ if there is another recovery quantum channel \mathcal{R} such that

$$(\mathcal{R} \circ \mathcal{E})(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi| \text{ for any } |\psi\rangle \in \mathcal{C}. \quad (1)$$

The recovery only needs to be perfect for states in the code subspace, and the larger Hilbert space acts as a resource of redundancy that makes the recovery possible.

Any quantum channel \mathcal{E} can be written as the composition of an isometry $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_E$ together with a partial trace over the ‘ancilla’ degrees of freedom \mathcal{H}_E as

$$\mathcal{E}(\rho) = \text{Tr}_E(V\rho V^\dagger). \quad (2)$$

This representation is unique up to isomorphisms of \mathcal{H}_E . If we choose some computational basis $\{|i\rangle \in \mathcal{H}_E\}$ for the ancilla system and make the partial trace explicit, we obtain an ‘operator-sum representation’ (or Kraus decomposition) for the quantum channel \mathcal{E} , given by

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad (3)$$

where the *operation elements* $E_i \in \text{End}(\mathcal{H})$ are defined by $E_i = \langle i| V$. The isometry condition $V^\dagger V = I$ becomes

$$\sum_i E_i^\dagger E_i = I.$$

For a noisy quantum channel, the E_i can be thought of as the operators that create errors. A general theorem concerning the existence of recovery channels can be found in 10.3 of [10], which we reproduce below:

Theorem 2.1. *Let $P \in \text{End}(\mathcal{H})$ be the projection onto the code subspace \mathcal{C} , and \mathcal{E} a quantum channel with operation elements $\{E_i\}$. A necessary and sufficient condition for the existence of a recovery channel \mathcal{R} correcting \mathcal{E} on \mathcal{C} is that*

$$PE_i^\dagger E_j P = \alpha_{ij} P, \quad (4)$$

for some Hermitian matrix α of complex numbers.

In later sections, we will prove that the ground state space of Kitaev's quantum double model is a quantum error correcting code by showing that (4) holds.

2.2 Topological quantum order

We now review the definition of topological quantum order (TQO) introduced in [2]. Let $\Lambda = (V(\Lambda), E(\Lambda), F(\Lambda))$ be an $L \times L$ lattice with periodic boundary conditions. The requirement that the lattice has periodic boundary is purely for the sake of simplicity. In general, one can take any lattice of linear size L that lives on a surface of arbitrary genus. In Kitaev's quantum double model, the qudits are conventionally defined to live on the edges of Λ instead of the vertices; for simplicity we use the same convention here.² We therefore associate to each edge $e \in E(\Lambda)$ a qudit $\mathcal{H}_e = \mathbb{C}^d$, and take the total Hilbert space to be $\mathcal{H} = \bigotimes_{e \in E(\Lambda)} \mathcal{H}_e$. We consider Hamiltonians of the form

$$H = \sum_{v \in V(\Lambda)} (1 - P_v) + \sum_{f \in F(\Lambda)} (1 - P_f), \quad (5)$$

where P_v is a projector that acts non-trivially only on edges which meet the vertex v , and P_f is a projector that acts non-trivially only on the boundary edges of the plaquette f . We further demand that the P_v 's and the P_f 's mutually commute and that the Hamiltonian be frustration free, i.e. that the ground states of H are stabilized by each P_v and each P_f :

$$V_{\text{g.s.}} = \{|\psi\rangle \in \mathcal{H} : P_v |\psi\rangle = |\psi\rangle \text{ and } P_f |\psi\rangle = |\psi\rangle, \forall v \in V(\Lambda), f \in F(\Lambda)\}. \quad (6)$$

Denote the projection onto $V_{\text{g.s.}}$ by P , which can be written as

$$P = \prod_{v \in V(\Lambda)} P_v \prod_{f \in F(\Lambda)} P_f. \quad (7)$$

Let A be a sublattice of Λ of size $\ell \times \ell$, denote by $V(A)^\circ$ the subset of $V(A)$ that are in the interior of A (which is of size $(\ell - 2) \times (\ell - 2)$), and define

$$P_A = \prod_{v \in V(A)^\circ} P_v \prod_{f \in F(A)} P_f. \quad (8)$$

²The choice of whether the qudits live on the edges or vertices of the lattice is arbitrary and makes no difference to the definition.

We can now state the definition of TQO that we will use.

Definition 2.2 (Topological Quantum Order [2]). A Hamiltonian which is frustration-free is said to have topological quantum order (TQO) if there is a constant $\alpha > 0$ such that for any $\ell \times \ell$ sublattice A with $\ell \leq L^\alpha$, the following hold.

- TQO-1: For any operator O acting on A ,

$$POP = c_O P, \quad (9)$$

where c_O is some complex number.

- TQO-2: If B is the smallest square lattice whose interior properly contains A ,³ then $\text{Tr}_{\bar{A}}(P)$ and $\text{Tr}_{\bar{A}}(P_B)$ have the same kernel, where \bar{A} is the complement of A in Λ .

TQO-1 heuristically corresponds to the statement that a sufficiently local operator cannot be used to distinguish between two orthogonal ground states because they differ only in their global, “topological” properties. Furthermore, ground state degeneracy is “topologically protected” in systems satisfying TQO-1 in the sense that perturbations by local operators can induce energy level splitting only non-perturbatively, or at some large order in perturbation theory which increases with the size of the lattice. It is straightforward to show that TQO-1 is equivalent to the condition that all normalized ground states $|\psi\rangle \in V_{\text{g.s.}}$ have the same reduced density matrix on A .

TQO-2 is the statement that the local ground state spaces and the global one should agree. We emphasize that TQO-2 can be violated at regions with non-trivial topologies, which is why one restricts to square lattices.

Remark 2.3. For our purposes, TQO-1 and QECC are morally interchangeable. Indeed, if H is any Hamiltonian⁴ with P the projection onto the ground space $V_{\text{g.s.}}$, then the following are equivalent.

1. The Hamiltonian H has TQO-1.
2. The Hamiltonian H provides a QECC with code subspace $V_{\text{g.s.}}$. There exists an $\alpha > 0$ such that the code can correct any error $\rho \mapsto \sum_i E_i \rho E_i^\dagger$ for which every combination $E_i^\dagger E_j$ is supported on an $\ell \times \ell$ sublattice A with $\ell \leq L^\alpha$.

In §3.2, we will prove a theorem for Kitaev’s finite group models which simultaneously implies TQO-1 and TQO-2, and so by the above remark also implies that the model furnishes a QECC.

³So B has size $(\ell + 2) \times (\ell + 2)$.

⁴For defining TQO-1, we do not need that H is frustration free.

2.3 Kitaev's finite group lattice model

We now turn to Kitaev's finite group lattice models [6], which we will see instantiate the concepts of the previous sections. Let G be a finite group, Σ be an oriented 2D surface with no boundary, and $\Lambda = (V, E, F)^5$ be an arbitrary oriented lattice on Σ , where V , E , and F are the sets of vertices, oriented edges, and plaquettes of the lattice, respectively. Then, for every $e \in E$, set $\mathcal{H}_e = \mathbb{C}[G]$ the group algebra of G , i.e. \mathcal{H}_e is spanned by the basis $\{|g\rangle : g \in G\}$. The overall Hilbert space is given by $\mathcal{H} \equiv \bigotimes_{e \in E} \mathcal{H}_e$. A natural basis for this Hilbert space consists of tensor products of the form $|g\rangle \equiv \bigotimes_{e \in E} |g_e\rangle$; we refer to this as the *group basis*.

We define the *sites* of Λ to be the set of pairs $s = (v, p) \in V \times F$ such that p is adjacent to v . Given a site $s = (v, p)$ and two elements g, h in G , we define two sets of operators: gauge transformations $A_s(g)$ and magnetic operators $B_s(h)$. Their action is most readily seen in the group basis. For example, $A_{(v,p)}(g)$ acts on the edges which touch v by multiplication by g on the left, or multiplication by g^{-1} on the right, depending on whether the edge is oriented away from or towards v . The magnetic operator $B_{(v,p)}(h)$ computes the product of the group elements sitting on the edges of p , and compares it to h , annihilating the state if there is a discrepancy, while stabilizing it if the group elements agree. The prescription for computing the product is to start at v and move around p counter-clockwise, inverting the group element associated to an edge if that edge is oriented opposite relative to the direction of travel. For example,

$$\begin{aligned}
 A_s(g) \left| \begin{array}{c} \downarrow g_4 \\ \leftarrow g_1 \quad v \quad \rightarrow g_3 \\ \downarrow g_2 \end{array} \right\rangle &\equiv \left| \begin{array}{c} \downarrow g_4 g^{-1} \\ \leftarrow g g_1 \quad v \quad \rightarrow g g_3 \\ \downarrow g g_2 \end{array} \right\rangle \\
 B_s(h) \left| \begin{array}{c} \xleftarrow{h_3} \\ h_4 \downarrow \quad p \quad \uparrow h_2 \\ \xrightarrow{h_1} \\ v \end{array} \right\rangle &\equiv \delta_{h, h_1 h_2^{-1} h_3 h_4} \left| \begin{array}{c} \xleftarrow{h_3} \\ h_4 \downarrow \quad p \quad \uparrow h_2 \\ \xrightarrow{h_1} \\ v \end{array} \right\rangle
 \end{aligned} \tag{10}$$

where $\delta_{g,h}$ is the Kronecker delta symbol. Note also that $A_s(g)$ does not depend on the plaquette p , so we may write it more conveniently as $A_v(g)$. Some basic facts follow:

$$\begin{aligned}
 A_s(g) A_s(h) &= A_s(gh) \\
 B_s(g) B_s(h) &= \delta_{g,h} B_s(h) \\
 A_s(g) B_s(h) &= B_s(ghg^{-1}) A_s(g).
 \end{aligned} \tag{11}$$

⁵We abbreviate $V \equiv V(\Lambda)$, $E \equiv E(\Lambda)$, and $F \equiv F(\Lambda)$.

We can now define the vertex and plaquette operators as

$$\begin{aligned} A_v &\equiv \frac{1}{|G|} \sum_{g \in G} A_v(g) \\ B_p &\equiv B_{(v_p, p)}(\mathbb{1}), \end{aligned} \quad (12)$$

where v_p is any vertex adjacent to p and $\mathbb{1} \in G$ is the identity element.⁶ It is easily verified that for all $v \in V$, $p \in F$, A_v and B_p are commuting projectors. The Hamiltonian of this system is defined in terms of these projectors:

$$H = \sum_{v \in V} (1 - A_v) + \sum_{p \in F} (1 - B_p). \quad (13)$$

This Hamiltonian is frustration-free and the ground space is simply given by

$$V_{\text{g.s.}} \equiv \{|\psi\rangle \in \mathcal{H} : A_v |\psi\rangle = B_p |\psi\rangle = |\psi\rangle, \forall v \in V, p \in F\}. \quad (14)$$

In gauge-theoretic language, where we think of a state as specifying the field configuration of a G vector potential, the condition that $A_v |\psi\rangle = |\psi\rangle$ means that $|\psi\rangle$ is gauge invariant, while $B_s(h) |\psi\rangle = |\psi\rangle$ means that the connection is flat. Now, due to the identities

$$\begin{aligned} A_v(g) A_v &= A_v \\ B_{(v, p)}(h) B_p &= \delta_{h, \mathbb{1}} B_p, \end{aligned} \quad (15)$$

the action of the $A_s(g)$ and $B_s(h)$ operators on the ground space is simply

$$\begin{aligned} A_s(g) |\psi\rangle &= |\psi\rangle \\ B_s(h) |\psi\rangle &= \delta_{h, \mathbb{1}} |\psi\rangle \end{aligned} \quad (16)$$

for all $|\psi\rangle \in V_{\text{g.s.}}$. In Section 2.4, we show that the dimension of $V_{\text{g.s.}}$ is the number of orbits of $\text{Hom}(\pi_1(\Sigma), G)$ under the action of G by conjugation, where $\pi_1(\Sigma)$ is the fundamental group of Σ .

We recall that the toric code is the ground space of the above Hamiltonian for $\Sigma = T^2$ the two-torus, Λ an $L \times L$ periodic square lattice, and $G = \mathbb{Z}_2$. In this case, the orientations of the edges in E does not matter and we can identify $\mathbb{C}[G]$ with a qubit, with the two elements $0, 1$ of \mathbb{Z}_2 corresponding to $|0\rangle, |1\rangle$ of the computational basis. It is easy to check that

$$A_v = \frac{1 + X_v}{2}, \quad B_p = \frac{1 + Z_p}{2}, \quad (17)$$

where X_v is the tensor product of Pauli X operators on all the Hilbert spaces in the edges incident to v , and Z_p is the tensor product of Pauli Z operators on the edges on the boundary of p . The ground space is spanned by states corresponding to homology classes of loops on a torus. This is a consequence of the explicit characterization of the ground space corresponding to any finite group G in the next section.

⁶When g is the identity element, the definition of $B_s(g)$ depends only on the plaquette in s , not the vertex.

2.4 Ground state space of Kitaev's model

In this subsection, we discuss some properties of the ground state and count the ground state degeneracy. This result is known to experts in relevant areas. However, we did not find a reference that addresses it explicitly. Therefore, we think it is beneficial to the readers to provide a detailed and elementary derivation. We follow the notations from the previous subsection. There is an action of G on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation: for $g \in G$ and $\phi \in \text{Hom}(\pi_1(\Sigma), G)$, we set $(g \cdot \phi)(\cdot) \equiv g\phi(\cdot)g^{-1}$.

Theorem 2.4. *The dimension of $V_{\text{g.s.}}(\Sigma)$ is equal to the number of orbits in $\text{Hom}(\pi_1(\Sigma), G)$ under the G -action.*

Proof. A basis element $|g\rangle = \bigotimes_{e \in E} |g_e\rangle$ of the total Hilbert space is an assignment of a group element g_e to each edge $e \in E$. Let γ be any oriented path in the lattice, which can be thought of as a sequence of connected edges. The group element obtained by multiplying the group elements along the path is denoted by g_γ . If one edge is oriented opposite to the path, then we multiply the inverse of the group element of that edge.

The constraint $B_p|g\rangle = |g\rangle$ is equivalent to the condition that $g_{\partial p} = \mathbb{1}$, where ∂p is the boundary of p oriented counterclockwise, thought of as a path.⁷ Hence, the subspace fixed by all the B_p 's is spanned by the following set:

$$\begin{aligned} S &= \{|g\rangle : g_{\partial p} = \mathbb{1}, \forall p \in F\} \\ &= \{|g\rangle : g_\gamma = \mathbb{1}, \text{ for any contractible, closed } \gamma\} \end{aligned} \quad (18)$$

For any $h \in G$, we call the operator $A_v(h)$ a gauge transformation at the vertex v . For two basis elements $|g\rangle, |g'\rangle \in S$, we call $|g\rangle$ and $|g'\rangle$ gauge equivalent if $|g'\rangle$ can be obtained from $|g\rangle$ by applying some gauge transformations at several vertices, denoted by $|g\rangle \sim |g'\rangle$. Gauge equivalence defines an equivalence relation on S . We denote the set of equivalence classes by $[S]$.

For each $[g] \in [S]$, define

$$|[g]\rangle := \sum_{|g\rangle \sim |g'\rangle} |g'\rangle \quad (19)$$

Since

$$A_v(h)|[g]\rangle = \sum_{|g\rangle \sim |g'\rangle} A_v(h)|g'\rangle = \sum_{|g\rangle \sim |g''\rangle} |g''\rangle = |[g]\rangle, \quad (20)$$

this implies that $|[g]\rangle$ is stabilized by A_v ,

$$A_v|[g]\rangle = \frac{1}{|G|} \sum_{h \in G} A_v(h)|[g]\rangle = |[g]\rangle. \quad (21)$$

We conclude that $|[g]\rangle \in V_{\text{g.s.}}(\Sigma)$. It is direct to check that $\{|[g]\rangle : [g] \in [S]\}$ forms a basis of $V_{\text{g.s.}}(\Sigma)$.

⁷For testing whether or not $g_{\partial p} = \mathbb{1}$, it does not matter which vertex we think of ∂p as starting at.

We now build a correspondence between $[S]$ and orbits in $\text{Hom}(\pi_1(\Sigma), G)$. Choose any vertex v_0 as a base point of Λ and choose a maximal spanning tree T containing v_0 . By definition, a maximal spanning tree is a maximal subgraph of the lattice Λ that does not contain any loops. Hence, any maximal spanning tree contains exactly $m := |V| - 1$ edges.

We define a map

$$\Phi : S \longrightarrow \text{Hom}(\pi_1(\Sigma), G) \quad (22)$$

as follows. Let γ be any closed path starting and ending at v_0 . For any $|g\rangle \in S$, define $\Phi(|g\rangle)([\gamma]) := g_\gamma$. Namely, $\Phi(|g\rangle)$ maps a closed path γ to the product of the group elements on it. The fact that $g_{\gamma_0} = \mathbb{1}$ for any contractible loop γ_0 implies that $\Phi(|g\rangle)([\gamma])$ only depends on the homotopy class of γ . Hence, $\Phi(|g\rangle)$ is a well defined map from $\pi_1(\Sigma, v_0)$ to G .⁸ It is clear that it is also a group homomorphism, so

$$\Phi(|g\rangle) \in \text{Hom}(\pi_1(\Sigma), G) \quad (23)$$

Now we show that Φ is onto and in fact $|G|^m$ -to-1.

Given any $\phi \in \text{Hom}(\pi_1(\Sigma), G)$, we construct a preimage $|g\rangle$ of ϕ as follows. The idea is that the group elements on the edges of the maximal spanning tree T are arbitrary, but the group elements on the rest of the edges are completely determined in terms of these and ϕ . For any edge e not in T , let $\partial_0 e$ and $\partial_1 e$ be the two end vertices of e . By construction, there is a unique path γ_i in T connecting v_0 to $\partial_i e$, where $i = 0, 1$. Let $\bar{\gamma}_1$ be the path γ_1 with reversed direction, then $\gamma = \gamma_0 e \bar{\gamma}_1$ is a closed path. An intuitive picture is that γ reaches $\partial_0 e$ along γ_0 from v_0 , travels through the edge e , and then goes back to v_0 along $\bar{\gamma}_1$. There exists a unique group element g_e such that

$$g_{\gamma_0} g_e g_{\bar{\gamma}_1} = \phi(\gamma) \quad (24)$$

It can be checked that $|g\rangle \in S$ and $\Phi(|g\rangle) = \phi$. Since we have $|G|^m$ choices of group elements to put on the spanning tree T when defining $|g\rangle$, the map Φ is $|G|^m$ -to-1.

On the other hand, for each given $|g\rangle$, if we are only allowed to apply gauge transformations on $|g\rangle$ at vertices other than v_0 , there are in total $|G|^m$ such transformations. These transformations are all different from each other acting of a fixed $|g\rangle$. If two basis elements $|g\rangle$ and $|g'\rangle$ are related by gauge transformations at vertices other than v_0 , then $\Phi(|g\rangle) = \Phi(|g'\rangle)$. We conclude that the preimage of ϕ contains precisely those $|g\rangle$'s that are related by gauge transformations at vertices other than v_0 . If we perform a gauge transformation $A_{v_0}(h)$ at v_0 to $|g\rangle$, then it is obvious that $\Phi(A_{v_0}(h)|g\rangle) = h\Phi(|g\rangle)h^{-1}$. Thus we have a one-to-one correspondence between gauge classes in S and orbits in $\text{Hom}(\pi_1(\Sigma), G)$. □

⁸As is standard in algebraic topology, the choice of basepoint is immaterial in defining the fundamental group up to isomorphism, so we suppress it from the notation from now on.

3 Main results

We now move on to the statement of our main theorem, which implies both TQO-1 and TQO-2.

Theorem 3.1. *Let H be the Hamiltonian of Kitaev's lattice model associated to any finite group G , closed surface Σ , and lattice Λ on Σ . Let $A \subset B \subset \Lambda$ be two rectangular sublattices contained in contractible subregions such that $V(A) \subset V(B)^\circ$, and denote*

$$\mathcal{H}_B = \{|\psi\rangle \in \mathcal{H} : A_v |\psi\rangle = B_p |\psi\rangle = |\psi\rangle, \forall v \in V(B)^\circ, p \in F(B)\}. \quad (25)$$

Then there exists a density matrix ρ_A on A such that

$$\text{Tr}_{\bar{A}} |\psi\rangle \langle \psi| = \rho_A, \quad (26)$$

for all $|\psi\rangle \in \mathcal{H}_B$ such that $\langle \psi | \psi \rangle = 1$.

Remark 3.2. Upon the completion of this work, we noticed that the operator algebra methods developed by P. Naaijken [9] while studying translation invariant Kitaev models in the infinite plane can be adapted to give an alternative proof of Theorem 3.1. Our proof is more explicit and elementary.

After warming up by proving that the toric code is a QECC in §3.1, the main theorem is proved in Section §3.2. In §3.3, we point out a subtlety: we show that there exist choices of gauge groups for which the magnetic flux operators are insufficient data for specifying a gauge-invariant state, contrary to intuition from gauge theory based on e.g. special unitary groups.

3.1 The toric code is QECC: a warm up

In this section, we warm up by proving that the toric code is a QECC, which was shown in [6]. The toric code is a special case of Kitaev's models, and we are proving a weaker result than Theorem 3.1, but we will improve upon both of these points in the next section.

We proceed by showing that the toric code obeys the Knill-Laflamme conditions, which state that a set of errors $\mathcal{E} = \{E_i\}$ is correctable by an error correcting code represented by a projector P onto the code subspace if and only if

$$PE_i^\dagger E_j P = \alpha_{ij} P \quad (27)$$

where α_{ij} form the entries of a Hermitian matrix (see Theorem 2.1). The projection operator for the toric code is given by

$$P = \prod_{v \in V} \frac{1 + X_v}{2} \prod_{p \in F} \frac{1 + Z_p}{2}. \quad (28)$$

Now, consider a general error on $k \in \mathbb{N}$ qubits. Since tensor products of Pauli operators span all possible operators, it is sufficient to consider the errors

$$\mathcal{E}(k) \equiv \left\{ \bigotimes_{e \in S_k} \sigma_e : S_k \subseteq E, |S_k| \leq k, \sigma_e \in \{X, Y, Z\} \right\}. \quad (29)$$

We claim that if $k > \lfloor \frac{L-1}{2} \rfloor$, where L is the size of the lattice, then $\mathcal{E}(k)$ is not correctable. To see this, first note $k > \lfloor \frac{L-1}{2} \rfloor$ implies $k \geq \lceil \frac{L}{2} \rceil$. Thus, we can form the operator $E_i^\dagger E_j$ that is a tensor product of X along a noncontractible loop by choosing appropriate E_i, E_j . Then, $E_i^\dagger E_j$ transforms two orthogonal states in the codespace into each other and therefore $PE_i^\dagger E_j P \not\propto P$.

Now suppose $k \leq \lfloor \frac{L-1}{2} \rfloor$. We first compute the commutation relations

$$\begin{aligned} (X \otimes I \otimes \cdots \otimes I) \frac{1 + X_v}{2} &= \frac{1 + X_v}{2} (X \otimes I \otimes \cdots \otimes I) \\ (X \otimes I \otimes \cdots \otimes I) \frac{1 + Z_p}{2} &= \frac{1 + (-1)^{i(p)} Z_p}{2} (X \otimes I \otimes \cdots \otimes I), \end{aligned} \quad (30)$$

where I is the identity operator on $\mathbb{C}[\mathbb{Z}_2]$ and $i(p)$ is an indicator for whether the first edge is on the boundary of p . Similarly,

$$\begin{aligned} (Z \otimes I \otimes \cdots \otimes I) \frac{1 + X_v}{2} &= \frac{1 + (-1)^{i(v)} X_v}{2} (Z \otimes I \otimes \cdots \otimes I) \\ (Z \otimes I \otimes \cdots \otimes I) \frac{1 + Z_p}{2} &= \frac{1 + Z_p}{2} (Z \otimes I \otimes \cdots \otimes I). \end{aligned} \quad (31)$$

Now, we can represent, up to a phase, $E_i^\dagger E_j$ as a product of Pauli's of the form $I \otimes \cdots \otimes I \otimes \sigma \otimes I \otimes \cdots \otimes I$, where $\sigma \in \{X, Z\}$. We then commute P across each of the factors. We first consider the edges on which a Pauli Z is acted on. Then, unless every vertex is incident to an even number of them, there will exist a vertex v for which $c(v) = 1$, which would imply $PE_i^\dagger E_j P = 0$. Otherwise, the edges form loops. Since there are at most $2k \leq L - 1$ edges acted on, the loops must be contractible. A similar argument holds for Pauli X where there we work in the dual lattice. We conclude that $PE_i^\dagger E_j P = 0$ or $E_i^\dagger E_j$ is, up to a phase, a product of X_v, Z_p , which act trivially on the ground space. Hence the Knill-Laflamme condition is satisfied.

3.2 States with locally zero energy density are locally indistinguishable

We now give a proof of Theorem 3.1.

Consider a rectangular sublattice A , contained in a simply connected region of the surface. (The rectangular assumption could be relaxed, at the cost of more complicated exposition.)

Assume that some state $|\psi\rangle$ on the entire lattice is invariant under all A_v, B_p operators whose support intersects with A . That is, we assume

$$\begin{aligned} A_v|\psi\rangle &= |\psi\rangle \\ B_p|\psi\rangle &= |\psi\rangle \end{aligned} \tag{32}$$

for all A_v, B_p operators such that $v \in V(A)$ or $\partial p \cap E(A) \neq \emptyset$. One can think of such a state as “zero-energy density” on the region A , where the energy density is given by the quantum double Hamiltonian.

We will show that all such states $|\psi\rangle$ have the same reduced density matrix ρ_A on the region A , and we will explicitly construct ρ_A .

Write $|\psi\rangle$ in the group basis. Because it is invariant under the B_p operators intersecting A , the only product states in this expansion will be the ones with trivial holonomy on all closed loops in A . Therefore we can write $|\psi\rangle$ as

$$|\psi\rangle = \sum_{\substack{g_A \text{ with trivial} \\ \text{holonomies on } A}} |g_A\rangle_A |\phi_{g_A}\rangle_{\bar{A}}, \tag{33}$$

where the sum is over all assignments $g_A = (g_e)_{e \in E(A)}$ of group elements to edges in A , such that all of the holonomies on A are trivial. The states $|\phi_{g_A}\rangle_{\bar{A}}$ are some set of states on \bar{A} depending on g_A . Note that the states $|g_A\rangle_A$ are orthonormal, while the states $|\phi_{g_A}\rangle_{\bar{A}}$ are not normalized and not necessarily orthogonal.

Next we will need the following result: For any two product states of group elements $|g_A\rangle$ and $|g'_A\rangle$ with trivial holonomies on A , and with the same group elements on the boundary ∂A , there exists a gauge transformation acting only on the vertices $V(A)^\circ$ in the interior of A which transforms $|g_A\rangle_A$ to $|g'_A\rangle_A$. That is, there is some gauge transformation U_{int} supported on the interior of A such that

$$U_{\text{int}}|g_A\rangle_A = |g'_A\rangle_A. \tag{34}$$

To build such a gauge transformation U_{int} that takes $|g_A\rangle_A$ to $|g'_A\rangle_A$ consider the internal vertices of A , ordered from left to right, then top to bottom. Start at the top left internal vertex v_0 , and choose the unique g_0 such that $A_{v_0}(g_0)|g_A\rangle_A$ matches $|g'_A\rangle_A$ on the entire top left plaquette. Now move one vertex rightward, to internal vertex v_1 , and choose the unique g_1 such that $A_{v_1}(g_1)A_{v_0}(g_0)|g_A\rangle_A$ matches $|g'_A\rangle_A$ on the top left two plaquettes. Continue in this manner until we have found a gauge transformation on the top row of internal vertices such that both states match elements on the entire top row of plaquettes. Repeat this procedure for the next row, and so on, until all internal vertices have been considered. Then we have constructed the (unique) desired U_{int} .

Consider any two terms g_A, g'_A that appear in the decomposition (33). By equation (34), there exists some gauge transformation U_{int} satisfying $U_{\text{int}}|g_A\rangle_A = |g'_A\rangle_A$. Since by assumption on the state $|\psi\rangle$, this gauge transformation leaves

$|\psi\rangle$ invariant, we have

$$\begin{aligned}
|\phi_{g'_A}\rangle_{\bar{A}} &= {}_A\langle g'_A|\psi\rangle \\
&= {}_A\langle g'_A|U_{\text{int}}|\psi\rangle \\
&= {}_A\langle g_A|\psi\rangle \\
&= |\phi_{g_A}\rangle_{\bar{A}}.
\end{aligned} \tag{35}$$

So $|\phi_{g'_A}\rangle_{\bar{A}} = |\phi_{g_A}\rangle_{\bar{A}}$ for any g_A, g'_A with the same boundary elements. We therefore take the subscript of ϕ to be an assignment of group elements to the edges of ∂A , e.g. $\phi_{g_{\partial A}}$ where $g_{\partial A} = (g_e)_{e \in E(\partial A)}$. Then, we can further refine the decomposition (33) as

$$|\psi\rangle = \sum_{\substack{g_{\partial A} \text{ with trivial} \\ \text{holonomy on } \partial A}} |\xi_{g_{\partial A}}\rangle_A |\phi_{g_{\partial A}}\rangle_{\bar{A}}, \tag{36}$$

where the sum is over all assignments $g_{\partial A}$ of group elements to edges on the boundary ∂A , such that the holonomy on the entire boundary ∂A is trivial. The above decomposition uses the state

$$|\xi_{g_{\partial A}}\rangle_A \equiv \sum_{\substack{g_A \text{ with trivial} \\ \text{holonomy on } A \\ \text{s.t. } (g_A)|_{\partial A} = g_{\partial A}}} |g_A\rangle_A \tag{37}$$

where the sum is over all g with trivial holonomy on A whose elements on the boundary ∂A match the assignment $g_{\partial A}$.

We will show that the decomposition of (36) is actually a Schmidt decomposition, with uniform Schmidt coefficients, which have been absorbed into the non-normalized states $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$. We will show this by showing the states $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ are all orthogonal and equal norm.

First, note that the states $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ are orthogonal for two distinct assignments $g_{\partial A}$ and $g'_{\partial A}$ of group elements to ∂A . To see this, consider an edge $E_0 \in E(\partial A)$ on which $g_{\partial A}$ and $g'_{\partial A}$ differ. We use the invariance of $|\psi\rangle$ under the operator B_{P_0} for the plaquette P_0 that intersects A precisely at this boundary edge E_0 . This invariance implies that both $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ must be composed completely of product states in the group basis on \bar{A} whose holonomy on P_0 (including the edge E_0) is trivial. But because $g_{\partial A}$ and $g'_{\partial A}$ differ at E_0 , any two product states in the expansion of $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ respectively must differ at some edge of P_0 in \bar{A} . Thus $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ must be orthogonal.

Next we will need the fact that for any two assignments $g_{\partial A}$ and $g'_{\partial A}$ of group elements to ∂A with trivial holonomy on ∂A , there exists a gauge transformation $U_{\partial A}$ acting only on the vertices of ∂A that brings $g_{\partial A}$ to $g'_{\partial A}$. To build such a gauge transformation $U_{\partial A}$, choose a contiguous ordering of the L vertices of ∂A , starting with some vertex v_0 . Choose the unique g_1 such that $A_{v_1}(g_1)$ acting on the assignment $g_{\partial A}$ will match the assignment $g'_{\partial A}$ on the boundary

edge from v_0 to v_1 . Next, choose the unique g_2 such that $A_{v_2}(g_2)A_{v_1}(g_1)$ acting on the assignment $g_{\partial A}$ will match the assignment $g'_{\partial A}$ on the boundary edges from v_0 to v_2 . Proceed in this way until finding a boundary gauge transformation $A_{v_{L-1}}(g_{L-1}) \cdots A_{v_2}(g_2)A_{v_1}(g_1)$ acting on the assignment $g_{\partial A}$ matches the assignment $g'_{\partial A}$ on all the edges from v_0 to v_{L-1} . Then these two assignments will also match on the final edge from v_{L-1} to v_0 , using the fact that both $g_{\partial A}$ and $g'_{\partial A}$ were assumed to have trivial holonomy along the boundary.

To see that the states any two states $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ have equal norm, consider the gauge transformation $U_{\partial A}$ mentioned above, taking $g_{\partial A}$ to $g'_{\partial A}$. We can factor $U_{\partial A}$ as a product of a unitary acting on A and a unitary acting on \bar{A} , i.e. $U_{\partial A} = V_A V_{\bar{A}}$. From the definition of $|\xi_{g_{\partial A}}\rangle_A$, we can see

$$V_A |\xi_{g_{\partial A}}\rangle_A = |\xi_{g'_{\partial A}}\rangle_A. \quad (38)$$

Then from the invariance of $|\psi\rangle$ under $U_{\partial A}$, and using decomposition (36), we have

$$\begin{aligned} |\phi_{g'_{\partial A}}\rangle_{\bar{A}} &= {}_A \langle \xi_{g'_{\partial A}} | \psi \rangle \\ &= {}_A \langle \xi_{g'_{\partial A}} | U_{\partial A} | \psi \rangle \\ &= V_{\bar{A}} ({}_A \langle \xi_{g'_{\partial A}} | V_A | \psi \rangle) \\ &= \langle V_{\bar{A}} \xi_{g_{\partial A}} | \psi \rangle \\ &= V_{\bar{A}} |\phi_{g_{\partial A}}\rangle_{\bar{A}}. \end{aligned} \quad (39)$$

Because $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$ and $|\phi_{g'_{\partial A}}\rangle_{\bar{A}}$ are related by a unitary $V_{\bar{A}}$, it follows they must have the same norm.

We conclude that decomposition of (36) is a Schmidt decomposition, with uniform Schmidt coefficients that have been absorbed into the non-normalized states $|\phi_{g_{\partial A}}\rangle_{\bar{A}}$. Thus we can immediately calculate the reduced density matrix

$$\begin{aligned} \rho_A &= \text{Tr}_{\bar{A}} |\psi\rangle\langle\psi| \\ &\propto \sum_{\substack{g_{\partial A} \text{ with trivial} \\ \text{holonomy on } \partial A}} |\xi_{g_{\partial A}}\rangle_A \langle \xi_{g_{\partial A}} |_A. \end{aligned} \quad (40)$$

This reduced state on A is manifestly invariant of the state $|\psi\rangle$, depending only on our original assumptions that $|\psi\rangle$ has zero energy-density on A .

3.3 Wilson loops are not a complete set of observables

It is standard in gauge theory to think of Wilson loop operators as the basic gauge-invariant observables. In typical models, such as gauge theory based on special unitary groups, or in Kitaev's lattice model based on the group \mathbb{Z}_2 , these observables are sufficient to completely characterize a gauge-invariant state, see [11] for a discussion of these issues. It is therefore tempting to think that this holds quite generally, e.g. for lattice gauge theory or Kitaev's lattice model based on any finite group G . Such a result would seem to suggest that Theorem 3.1

is “morally obvious”: if gauge invariant states are determined by their Wilson loops, our main result would simply be an easy corollary of a local version of this statement.

In fact, we will show that this naive intuition fails for certain choices of G . That is, for judiciously chosen G , we will exhibit a pair of *orthogonal* gauge-invariant states with the same Wilson loops. This result emphasizes that it is a property only of the ground space—where the Wilson loops are not only locally the same, but also locally trivial—that states are determined by their non-Abelian fluxes.

Let us state our claim more precisely. We will work in the gauge invariant subspace

$$\mathcal{H}_{\text{gauge}} = \{|\psi\rangle \in \mathcal{H} \mid A_v|\psi\rangle = |\psi\rangle, \forall v \in V\}. \quad (41)$$

The magnetic plaquette operators $B_s(h)$ do not in general preserve the gauge invariant subspace, so we will consider combinations which do:

$$B_s([h]) \equiv \frac{|[h]|}{|G|} \sum_{g \in G} B_s(ghg^{-1}) \quad (42)$$

where $|[h]|$ is the order of the conjugacy class of h . These operators depend on h only through its conjugacy class; heuristically, they compute the product of group elements around the plaquette and check whether or not that product is conjugate to h , annihilating the state if it is not, and stabilizing the state if it is. We are free to define more general magnetic operators $B_\gamma([h])$ for any closed loop γ , defined in the obvious way. To avoid confusion, we will refer to the product of group elements around a path (which is measured by $B_\gamma(h)$ and is not gauge-invariant) as a *holonomy*; the conjugacy class of this product (which is measured by $B_\gamma([h])$ and is gauge-invariant) will be referred to as a *Wilson loop*.

Using e.g. equations (11), it is straightforward to show that the $B_\gamma([h])$ commute with every A_v and thus map $\mathcal{H}_{\text{gauge}} \rightarrow \mathcal{H}_{\text{gauge}}$. Moreover, they all commute with one another, and so we can work with a basis of the gauge-invariant subspace consisting of simultaneous eigenstates of the $B_\gamma([h])$. Our claim is then the following.

Proposition 3.3. *There exist Kitaev lattice models (Σ, Λ, G) as well as an orthonormal pair of gauge-invariant states $|\psi\rangle, |\chi\rangle \in \mathcal{H}_{\text{gauge}}$ which are eigenstates of every Wilson loop operator $B_\gamma([h])$ with identical eigenvalues,*

$$\langle\psi|B_\gamma([h])|\psi\rangle = \langle\chi|B_\gamma([h])|\chi\rangle. \quad (43)$$

Thus, $|\psi\rangle$ and $|\chi\rangle$ are gauge-invariant states which cannot be distinguished by Wilson loop observables. The rest of this section is dedicated to the proof of this proposition.

The key ingredient which enters our construction is the existence of finite groups which admit *outer class automorphisms*.⁹ An automorphism $\phi : G \rightarrow G$

⁹As we will see, this constitutes a sufficient condition on G so that, for Λ large enough, (Σ, Λ, G) will satisfy Proposition 3.3.

is said to be outer if it is not of the form $\phi(g) = hgh^{-1}$ for any h in G ; it is a class automorphism if it preserves conjugacy classes, i.e. if g is conjugate to $\phi(g)$ for every g in G . We will not need any examples of such groups for our proof, so for the remainder of this section take their existence for granted. The interested reader is encouraged to consult e.g. [12] for examples of explicit constructions.

Now, take Σ any closed 2D surface, $G = \{h_1, \dots, h_N\}$ any finite group which admits an outer class automorphism ϕ , and Λ any lattice on Σ which has at least an $N \times N$ square sublattice A , where $N := |G|$. We start by constructing a state of \mathcal{H} in the group basis which features every possible holonomy. In other words, we want a state $|g\rangle = \bigotimes_{e \in E(\Lambda)} |g_e\rangle$ such that, for each h in G , there is some loop γ such that $B_\gamma(h)|g\rangle = |g\rangle$. This is easy to achieve. Let $A_n \subset A$ for $n = 1, \dots, |G|$ be the square of side-length n whose lower-left corner sits at the lower left corner of A . Assign group elements to the edges of A_1 in such a way that the product of elements around A_1 (starting at the lower-left corner of A) is equal to h_1 . Proceed inductively by choosing group elements associated to the unassigned edges of A_n in such a way that the holonomy around A_n is equal to h_n . Finally, assign group elements to any remaining edges however you would like. The state $|g\rangle$ so constructed satisfies that $B_{A_n}(h_n)|g\rangle = |g\rangle$, and moreover $B_{A_n}([h_n])|g\rangle = |g\rangle$. Now, define $|\psi\rangle$ to be the “gauge-symmetrization” of $|g\rangle$,

$$|\psi\rangle = \mathcal{N}_\psi \sum_{|g'\rangle \sim |g\rangle} |g'\rangle \quad (44)$$

where \sim denotes gauge equivalence, and \mathcal{N}_ψ is chosen to normalize $|\psi\rangle$. Gauge transformations at most change holonomies by conjugation, so $|\psi\rangle$ has the same Wilson loops as the state $|g\rangle$ from which it was constructed. We can do the same for the state $|\phi(g)\rangle = \bigotimes_{e \in E(\Lambda)} |\phi(g_e)\rangle$, and define

$$|\chi\rangle = \mathcal{N}_\chi \sum_{|g'\rangle \sim |\phi(g)\rangle} |g'\rangle \quad (45)$$

Since ϕ preserves conjugacy classes, $|\phi(g)\rangle$ and $|g\rangle$ have the same Wilson loops, and so it follows that $|\psi\rangle$ and $|\chi\rangle$ have the same Wilson loops as well. It remains only to show that these two states are orthogonal. We will do this by showing that $|g\rangle$ is gauge-inequivalent to $|\phi(g)\rangle$, from which it follows that every term in the sum in equation (44) is orthogonal to every term in the sum in equation (45). For this we need the following lemma.

Lemma 3.4. *Fix an arbitrary base-point v_0 in V . Two states $|g\rangle, |g'\rangle$ in the group basis are gauge equivalent if and only if their holonomies based at v_0 agree up to simultaneous conjugation, i.e. if and only if there is a single h in G such that $g'_\gamma = hg_\gamma h^{-1}$ for every loop γ which starts and ends at v_0 .*

Proof. In the forward direction, if $|g\rangle$ and $|g'\rangle$ are gauge-equivalent, they are by definition related by a product of gauge-transformations at different vertices, $|g'\rangle = \prod_{v \in V} A_v(h_v)|g\rangle$ for some h_v in G . The gauge-transformations away

from v_0 do not change the holonomies based at v_0 , while the single gauge-transformation $A_{v_0}(h_{v_0})$ at v_0 changes all holonomies by conjugation by h_{v_0} , i.e. $g'_\gamma = h_{v_0} g_\gamma h_{v_0}^{-1}$.

In the reverse direction, assume that the holonomies based at v_0 are simultaneously conjugate by an element h in G . We will specify a sequence of gauge transformations which transforms $|g\rangle$ into $|g'\rangle$. First, we specify the gauge transformation needed at the base-point. Since acting with a gauge transformation at v_0 conjugates all holonomies based at v_0 , we act with $A_{v_0}(h^{-1})$, so chosen because $A_{v_0}(h^{-1})|g\rangle$ will have the same based holonomies as $|g'\rangle$.

Lay down a maximal spanning tree T of Λ which contains v_0 . Recall that gauge transformations away from v_0 do not change holonomies based at v_0 , and note that states in the group basis are fully determined by their holonomies based at v_0 as well as the group elements assigned to the edges of T . With this in mind, we will apply our remaining gauge transformations to the vertices of T in order to make the states agree on the edges of T (and therefore on all of Λ).

We proceed inductively. Choose a path from v_0 to any leaf, and label the vertices which arise as $[v_0, v_1, \dots, v_r]$ and the edges between them as $[e_1, \dots, e_r]$. Compare the group element assigned to e_1 by $A_{v_0}(h^{-1})|g\rangle$ and $|g'\rangle$, and act with the unique gauge transformation $A_{v_1}(h_1)$ which makes $A_{v_1}(h_1)A_{v_0}(h^{-1})|g\rangle$ and $|g'\rangle$ agree at the edge e_1 . Inductively walk through the path, and at the n th step, apply the unique gauge transformation $A_{v_n}(h_n)$ which makes $A_{v_n}(h_n) \cdots A_{v_1}(h_1)A_{v_0}(h^{-1})|g\rangle$ agree with $|g'\rangle$ at the edge e_n , noting that application of $A_{v_n}(h_n)$ does not interfere with any of the previously assigned edges e_1, \dots, e_{n-1} since by assumption T is a tree. The two states one has at the end of this procedure agree at all the edges e_1, \dots, e_r .

One can continue to apply this protocol to any remaining paths from a vertex in T to one of its leaves which have not yet been traversed. Calling the overall gauge transformation one obtains \mathcal{G} , the net result is that $\mathcal{G}|g\rangle$ and $|g'\rangle$ agree at every edge in the spanning tree, and have identical holonomies. It follows that they agree on all of Λ , and so they are in fact equal. \square

Now let v_0 be the lower-left hand corner of A . Recall that by construction, $g_{A_n} = h_n$ and meanwhile $\phi(g)_{A_n} = \phi(h_n)$. These holonomies cannot be the same up to simultaneous conjugation: since every element of G is realized as a holonomy, this would imply that ϕ is not an outer automorphism, in contradiction with our assumption. Thus $|g\rangle$ is gauge-inequivalent from $|\phi(g)\rangle$, and so $|\psi\rangle$ is orthogonal to $|\chi\rangle$.

4 Conclusion

In this short note, we have shown that Kitaev's finite group models obey a theorem which, at the level of slogans, says that "states with locally zero energy density are locally indistinguishable." The theorem implies in particular that Kitaev's models have topological quantum order (TQO-1 and TQO-2) and moreover furnish a quantum error correcting code (QECC), a fact which, although

well-appreciated, appears not to have been proved rigorously in the literature. In contrast, we have demonstrated that an analogous result cannot hold for excited states. Namely, contrary to intuitions one might have from typical gauge theory models, Wilson loop operators do not form a complete set of commuting observables.

As was mentioned in the introduction, Kitaev's models can be generalized from finite groups to Hopf \mathbb{C}^* -algebras; the latter reduce to the former when the Hopf-algebra is taken to be the group algebra $\mathbb{C}[G]$ associated to G . It is interesting to ask to what extent the techniques we have used can be adapted to the Hopf-algebra case. Given the equivalence between these generalized Kitaev models and the Levin-Wen string net models, a successful generalization would therefore constitute a proof that the Levin-Wen models have TQO as well. We leave this question for future study.

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References

- [1] Sergey Bravyi and Matthew B Hastings. A short proof of stability of topological order under local perturbations. *Communications in mathematical physics*, 307(3):609, 2011.
- [2] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis. Topological quantum order: stability under local perturbations. *Journal of mathematical physics*, 51(9):093512, 2010.
- [3] Oliver Buerschaper, Juan Martín Mombelli, Matthias Christandl, and Miguel Aguado. A hierarchy of topological tensor network states. *Journal of Mathematical Physics*, 54(1):012201, 2013.
- [4] Liang Chang. Kitaev models based on unitary quantum groupoids. *Journal of Mathematical Physics*, 55(4):041703, 2014.
- [5] Michael H Freedman, Michael Larsen, and Zhenghan Wang. A modular functor which is universal for quantum computation. *Communications in Mathematical Physics*, 227(3):605–622, 2002.
- [6] A Yu Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, 2003.

- [7] Michael Levin and Xiao-Gang Wen. Detecting topological order in a ground state wave function. *Physical review letters*, 96(11):110405, 2006.
- [8] Michael A Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Physical Review B*, 71(4):045110, 2005.
- [9] Pieter Naaijken. *Anyons in infinite quantum systems: QFT in $d=2+1$ and the toric code*. PhD thesis, Radboud Universiteit Nijmegen, 2012.
- [10] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, New York, NY, USA, 10th edition, 2011.
- [11] Ambar Sengupta. Gauge invariant functions of connections. *Proceedings of the American Mathematical Society*, 121(3):897–905, 1994.
- [12] GE Wall. Finite groups with class-preserving outer automorphisms. *Journal of the London Mathematical Society*, 1(4):315–320, 1947.