Possible Topological Quantum Computation via Khovanov Homology: D-Brane Topological Quantum Computer

Mario Vélez and Juan Ospina
Logic and Computation Group
Physical Engineering Program
School of Sciences and Humanities
EAFIT University
{mvelez, jospina}@eafit.edu.co

Abstract. A model of a D-Brane Topological Quantum Computer (DBTQC) is presented and sustained. The model is based on four-dimensional TQFTs of the Donaldson-Witten and Seiber-Witten kinds. It is argued that the DBTQC is able to compute Khovanov homology for knots, links and graphs. The DBTQC physically incorporates the mathematical process of categorification according to which the invariant polynomials for knots, links and graphs such as Jones, HOMFLY, Tutte and Bollobás-Riordan polynomials can be computed as the Euler characteristics corresponding to special homology complexes associated with knots, links and graphs. The DBTQC is conjectured as a powerful universal quantum computer in the sense that the DBTQC computes Khovanov homology which is considered like powerful that the Jones polynomial.

Keywords: D-Brane Topological Quantum Computation, Khovanov Homology, 4d-TQFTs, Surface Operators.

1. Introduction

A standard quantum computer (SQC) is able to factor high numbers and an "anyon topological quantum computer (ATQC)" is able to compute invariant polynomials for knots, links and graphs, such as Jones, HOMFLY and Tutte polynomials [1,2,3]. Both the SQC and the ATQC are based on the application of zero-dimensional objects with the aim to codified the quantum information. In the SQC the quantum gates are directly extracted from the unitary group and in the ATQC the quantum gates are realized using unitary representations of braid words.

Given that the elementary objects in nature are expected to be extended (like strings, membranes, p-branes, D-branes) is very relevant to consider the brane effects in the case of the topological quantum computation. In [4] at first step was made and the notion of a "String Topological Quantum Computer (STQC)" was sustained. In the present works we make the second step and the notion of a "D-brane Topological Quantum Computer (DBTQC)" will be introduced and sustained. The object of the present work is to show that the DBTQC is more powerful than the ATQC in the sense that the DBTQC is able to compute a mathematical structure more general than the invariant polynomials, being such structure the so-called Khovanov homology.

As it is known, the ATQC is based on the "three-dimensional topological quantum field theories (3d-TQTs)" of the Chern-Simons kind [1,2,3], classified by modular categories . In [4] a model of STQC was proposed using the two-dimensional TQTs corresponding to the topological strings classified by Frobenius algebras. In the present work a model of DBTQC is introduced using four-dimensional TQFs of the Donalson-Witten and Seiberg-Witten kinds [5]. As a mathematical prelude to the DBTQC the super-symmetric quantum mechanics [6] is consider like a certain kind of topological quantum computer which is able to compute Euler characteristics for manifolds.

The main mathematical ingredient to use in this work is the concept of categorification [7] according to which the invariant polynomials like Jones, HOMFLY, Tutte and Bollobás-Riordan-Tutte can be expressed as the Euler characteristic for a certain homology complex associated with knots, links and graphs. The heuristic idea in this work will consist in to obtain physical referents described by TQFTs for which the computation of Euler characteristics in singular and Khovanov homology is permanently realized due to internal symmetries and dynamics.

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2. Topological Quantum Algorithms for knot (Jones) and graph (Tutte) polynomials

Recently various topological quantum algorithms were proposed for computation of knot polynomials like Jones, HOMFLY and graph polynomials like Tutte and Bollobás-Riordan. In this section briefly descriptions of such algorithms will be presented.

2.1. Aharonov-Jones-Landau Algoritm for the Jones Polynomial of a knot

The knots and links correspond to the embeddings of S^1 in a three-dimensional manifold M^3 (usually R^3 or S^3). The knots and links can be conveniently characterized via algebraic structures like invariant polynomials. The more famous of such invariant polynomials is the Jones polynomial which can be defined using different ways including both the application of Temperley-Lieb algebras and Markov traces as the application of the so-called Kauffman bracket. In this last case we have the following definition for the Jones polynomial:

$$V_K(t) = f_K(t^{-\frac{1}{4}}), \qquad f_K(A) = (-A^3)^{-w(K)} < K > (A), \qquad < K > = \sum_S < K|S > \delta^{||S||-1}.$$

In these equations, K is a given knot or link, < K > represents the unnormalized Kauffman bracket for K, A is the skein variable, w(K) is the "writhe" of K , fK(A) is the normalized Kauffman bracket for K , S represents a particular Kauffman state for K, $< K|S>$ is the combinatorial weight for the state S , δ is the "loop" parameter, $\|S\|$ is the number "loops" in the state S , VK(t) represents the Jones polynomial for K and t is the Jones variable.
The Jones polynomial can be computed exactly for knots and links until 11 crossings but the exact computation of the Jones polynomial for every knot is a #P-hard problem [8]. The problem of classification of the all knots can not be solved using only the Jones polynomial but this polynomial provides a very powerful tool to make such classification.
As it was said the problem to exactly compute the Jones polynomial for every knot or link I is a #P-hard problem but the approximated quantum computation of this polynomial is a BQP-hard and BQP-complete problem; it is to say the approximated computation of the Jones polynomial demands the totality of the computational power that a quantum computer is able to provide; and at same time, every one problem in quantum computation can be reduced to an instance of approximated computation of a Jones polynomial.
The celebrated topological quantum algorithm proposed by Aharonov-Jones-Landau [1], computes an approximation of the Jones polynomial and it has the following structure:

1. Introduce the knot or link for computation.
2. Represent the introduced knot or link as the closure of a certain braid .
3. Represent the braid in terms of the braid generators.
4. Using the Jones-Wenzl representation of the braid group in the Temperlery-Lieba algebra, obtain a Temperley-Lieb representation ot the relevant braid.
5. Write the Jones polynomial in terms of the Kauffman bracket.
6. Write the Kauffman bracket in terms of Markov traces for the resulting Temperlye-Lieb operators.
7. Obtain the unitary representations of the relevant Temperley-Lieb operators.
8. Compute the Markov trace for the relevant Temperley-Lieb operador, using the Hadamard test.

2.2. Wocjan-Yard Algorithm for the HOMFLY Polynomial of a link

The so-called HOMFLY-PT polynomial is a generallization of the Jones polynomial and can be defined using the Iwahori-Hecke algebra and the associated Markov traces. The HOMFLY-PT polynomial also can be defined via the following skein relations (recurrence relations):

$$P\left(\bigcirc\right) = \frac{a-a^{-1}}{q-q^{-1}},$$

where P denotes the HOMFLY-PT polynomial and being *a* and *q* are the variables of the polynomial.
The HOMFLY polynomial, admits the following one-variable specialization

$$P_n(L) := P_{a=q^n}(L)\,,$$

in such way that we have the following particular cases: $P_0(L)$ is the classic Alexander polynomial for the link *L*; $P_1(L)=1$ for all link *L* and for hence is a trivial invariant; $P_2(L)=J(L)$ is the Jones polynomial for the link *L*.
The Wocjan-Yard (WY) algorithm [2] for the HOMFLY-PT polynomial is a generalization and extension of the AJL algorithm for the Jones Polynomial. The WY has the following structure:

1. Introduce the knot or link to consider.
2. Represent the introduced knot or link as the closure of a certain braid..
3. Write the braid in terms of the braid generators.
4. Use a representation of the braid group in the Iwahori-Hecke algebra with the aim to obtain an Iwahori-Hecke representation of the relevant braid.
5. Write the HOMFLY-PT polynomial in terms of the Markov trace for Iwahori-Hecke operators.
6. Obtain unitary representations of the relevant Iwahori-Hecke operators.
7. Compute the Markov trace of the resulting Iwahori-Hecke operator using the Hadamard test.

We observe that the WY algorithm results form the AJL algorithm when the Temperley-Lieb algebra is replaced by the Iwahori-Hecke algebra. Moreover it is possible to think about a quantum algorithm which is able to compute directly the Kauffman bracket by applying the more general algebra named Birman-Wenzl-Murakami and the corresponding Markov traces.

2.3. Aharonov-Arad-Eban-Landau (AAEL)Algorithm for the Tutte polynomial of a graph

The Tutte poloynomial for a Graph G is defined as

$$T(G;x,y)=\sum_{s\subset E(G)}(x-1)^{r(E)-r(s)}(y-1)^{|s|-r(s)}\,,$$

and the following relation between the Tutte polynomial and the Jones polynomial is valid

$$V(L;t)=(-t^{\frac{3}{4}})^{w(L)}t^{\frac{-(r-n)}{4}}T(G(L);-t,-t^{-1}),$$

The AAEL algorithm [3] gives an approximation of the Tutte polynomial for almost any pair of values for (x , y) but the algorithm does not provide the explicit or symbolic form of the Tutte polynomial. In all case but at least in theory, the AAEL algorithm is more efficient than the more efficient classical algorithms when very large graphs are involved. More commentaries about the AAEL algorithm can be observed in [3,4] but the basic structure is as follows.

1. Introduce a graph G as the input
2. Obtain the link L_G associated with the graph G.
3. Write the link L_G as a word in the braid generators.
4. Construct a representation of the braid algebra in the virtual Temperley-Lieb algebra..
5. Obtain the Temperley-Lieb representation of the braid word for L_G.
6. Determine the quantum charge according with:
$$Q=\rho(\Psi(L_G))$$
7. Use a standard quantum computer to determine $\langle 1|Q|1\rangle$.
8. Apply $K(L_G;d,\vec{u})=\langle 1|Q|1\rangle$ to obtain the Kauffman bracket for L_G.
9. Apply $K(L_G;d,\vec{u})=\sum_s d^{|s|}\prod u_e^{s(e)}$ to obtain $Z(G;d^2,d\vec{u})$.
10. Apply $Z_{Potts}=Z(G;q,\vec{v})$ to obtain Z_{Pottss}.

11. Apply $T(G;x,y)=(x-1)^{-k(E)}(y-1)^{-|V|}Z(G;(x-1)(y-1),\overrightarrow{y-1})$
to obtain the Tutte polynomial T(G; x , y).

2.4. Quantum Algorithms for the Bollobás-Riordan-Tutte polynomial of a ribbon graph

The Bollobás-Riordan-Tutte polynomial of a ribbon graph is defined as [4]

$$R(G;x,y,z)=\sum_{F\in F(G)} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-bc(F)+n(F)}.$$

The following equations give the relations between the Bollobás-Riordan-Tutte polynomial and the Tutte, Jones and HOMFLY polynomials [4]:

$$R(G;x-1,y-1,z)=T(\Gamma(G);x,y),$$

$$V_L(t)=(-t^{\frac{3}{4}})^{w(L)}t^{\frac{-(r(\hat{G})-n(\hat{G}))}{4}}(-t^{1/2}-t^{-1/2})^{k(\hat{G})-1}R_{\hat{G}}(-t-1,-t^{-1}-1,\frac{1}{-t^{1/2}-t^{-1/2}}),$$

$$H(L(G);x,y)=(\frac{1}{xy})^{v(G)-1}(\frac{y}{x})^{e(G)}(x^2-1)^{k(G)-1}R(G;x^2-1,\frac{x-x^{-1}}{xy^2},\frac{y}{x-x^{-1}}).$$

In [4] three possible quantum algorithms for the computation of the BRT polynomial of a given ribbon graph were proposed. The first algorithm was based on the quasi-tree expansion for the BRT polynomial [4] and on the quantum computing of the Binary Decision Diagram [4] corresponding to the set of all quasi-trees for the considered ribbon graph. The second algorithm was based on the relationship among the BRT polynomial and the Kauffman bracket for the particular link associated with the given ribbon graph [4] and doing use of a modified version of the AAEL algorithm. The third algorithm was based on the relation between the BRT polynomial and the HOMFLY polynomial [4] and using a modified version of the WY algorithm[2,4].

3. Knot and Graph Polynomials via Categorification

Recently, M Khovanov [7] was able to obtain a procedure called categorification for invariant polynomials corresponding to structures of topological entanglement such as knots, links, tangles and graphs. The main idea of Khovanov consists in to associate homology complexes with knots, links and graphs in such way that the invariant polynomials, like Jones, HOMFLY , Tutte and Bollobás-Riordan are represented as the Euler characteristics for the corresponding complexes. The categorification process had been realized for the previously mentioned polynomials and in this section we briefly resume the main results in categorification.

3.1. Categorification of the Jones Polynomial

The central idea of Khovanov [7] consists in to associate with every know, link or graph a homology complex according with the scheme:

$$D \xrightarrow{\text{Khovanov}} C^{*,*}(D), \quad C^{*,*}(D) \xrightarrow{\text{Homology}} KH^{*,*}(D), \quad KH^{*,*}(D)\cong KH^{*,*}(D').$$

The graded Euler characteristic is given by

$$\sum_{i,j\in\mathbb{Z}}(-1)^i q^j \dim(KH^{i,j}(D))=\hat{J}(D).,$$

where $\hat{J}(D)$ represents the Jones polynomial for the know or link D.

The Knovanov categorification of the Jones polynomial permits to construct a Hilbert polynomial associated with knots and links which is called the Khovanov polynomial, being this last polynomial more powerful than the Jones polynomial to classify knots and links; but the Khovanov polynomial is not sufficiently powerful to completely solve the problem of classification of knots.

3.2. Categorification of the HOMFLY Polynomial

The categorification of the HOMFLY-PT polynomial is a generalization of the categorification of the Jones Polynomial.. Again the idea is to associate with a given link K a multi-graded homology complex defined by

$$H_n(L) = \bigoplus_{i,j\in\mathbb{Z}} H_n^{i,j}(L), \quad n > 1,$$

for which the multi-graded Euler characteristcs is given by

$$P_n(L) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim H_n^{i,j}(L)$$

,

where $P_n(L)$ is the HOMFLY-PT polynomial.

The categorification of the HOMFLY-PT polynomial, is called the Khovanov-Rozansky (KR) [9] homology and it had been applied recently in mathematical and theoretical physics in the ambit of topological strings. In this work we argue that the KR homology could be used to construct a model of string topological quantum computer. More details about this idea will be presented in section 4.

3.3. Categorification of the Tutte Polynomial

The categorification process was recently extended from the knots and links to graphs and matroids.. For example the Tutte polynomial has the following categorification [10]

$$\chi(C(G),q) = \sum_{0\leq 1\leq n} (-1)^i q \dim(H^i) = T(G;x,y),$$

it is to say the Tutte polynomial of a given graph can be computed as the Euler characteristic for the graph homology. Moreover, the categorification of the Tutte polynomial made possible to construct a Hilbert polynomial for graphs being this Hilbert polynomial more powerful than the Tutte polynomial concerning the problem of classification of graphs.

3.4. Categorification of the Bollobás-Riordan-Tutte Polynomial.

Actually the categorification of the Bollobás-Riordan-Tutte (BRT) polynomial is an open problem in geometric topology and combinatorics. Only very recently a categorification for certain uni-variable BRT polynomial was presented.

4. Topological Quantum Computation via Khovanov Homology

In the previous section the knot and graph polynomials were reduced to Euler characteristics corresponding with various kinds of homology complexes. In the present section, we present some physical systems which naturally are able to compute Euler characteristics. The first example is given by the super-symmetric quantum mechanics and its relationship with the Atiyah-Singer Index theorem. The second example is given by the two-dimensional topological quantum field theories associated with the topological strings. The final example is given by the four-dimensional topological quantum field theories like Donaldson-Witten and Seiberg-Witten. As it will be seen, all these physical systems are permanently computing homological invariants and them can be used as the “hardware” for the so-called D-brane topological quantum computers.

4.1. Super-Symmetric Quantum Mechanics and the Atiyah-Singer Index Theorem

In [11] was showed that the jones polynomial of a given knot or link can be represented as the path integral for a topological quantum field theory of the Chern-Simons kind. In [12] the results of [11] were applied to the construction of a model of topological quantum computation ; and in [1] a quantum algorithm was given for the computation of the jones polynomial . More over in [3] a quantum algorithm was given for the computation of the tutte polynomial of a

given planar graph. From other side, recently was discovered that both the jones polynomial as the tutte polynomial can be represented as the generalized euler characteristics corresponding to the Khovanov homology for knots, links and graphs [7]. Now, in [6] was showed that euler characteristic in singular homology can be expressed as the path integral for a super-symmetric quantum mechanics. The idea here, is to combine the results in [7] and [6], with the aim to sketch a model of topological super-symmetric quantum computation. This model could compute euler characteristics of Khovanov homology.
In singular homology, the Euler-Poincare chatacteristic is defined as [6]

$$\sum(-1)^i \dim(H_i(M;\mathbb{Q})) = \chi(M).$$

where *M* is a differential manifold and $H_i(M;Q)$ represents the homology group of M with order i. From other side, in super-symmetric quantum mechanics is possible to define the Witten index which is able to detect the asymmetry between bosons and fermions of zero level, according to [6]:

$$\mathrm{Tr}(-1)^F e^{-\beta H} = n_B^{E=0} - n_F^{E=0},$$

where F is the fermionic number, H is the hamiltonian, $\beta = 1/KT$, being K the Boltzman constant and T the temperature.
The Witten index can be expressed as a Feynman path integral with periodic boundary conditions according with [6]

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int_{\mathbf{PBC}} d\phi(t)\, d\psi(t) \exp - S_E(\phi, \psi),$$

where $\phi(t)$ is the bosonic field, $\Psi(t)$ is the fermionic field and $S_E(\phi,\Psi)$ is the action. The relevant lagrangian is given by [16]

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\bar{\psi}^i\gamma^0 \frac{D}{dt}\psi^j + \frac{1}{12} R_{ijk\ell}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^\ell, \qquad \frac{D}{dt}\psi^i = \frac{d}{dt}\psi^i + \Gamma^i{}_{jk}\dot{\phi}^j\psi^k,$$

where $g_{i,j}$ is the metric on M with its corresponding Christoffel symbols, Dirac gamma matrices and Riemann curvature tensor.
Using all ingredients is possible to derive that [6]

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H} = \frac{(-1)^{d/2}}{2^d \left(\frac{d}{2}\right)!\, \pi^{d/2}} \int d(\mathrm{Vol}) \varepsilon^{i_1 j_1 \dots i_n j_n} \cdot \varepsilon^{k_1 \ell_1 \dots k_n \ell_n} R_{i_1 j_1 k_1 \ell_1} \cdots R_{i_n j_n k_n \ell_n},$$

and in the particular case of a tour-dimensional manifold we have [6]

$$\chi(M) = \frac{1}{32\pi^2} \int_M \varepsilon_{abcd} R_{ab} \wedge R_{cd}.$$

These equations are showing that the super-symmetric quantum mechanics, by itself, is permanently computing euler characteristics in singular homology. We conjecture that using a modified supersymmetric quantum mechanics will be possible to compute euler characteristics in Khovanov homology and then will be possible to compute link and graph polynomials in an efficient way.

4.2. D-Brane Topological Quantum Computation of the Knot-Graph Homology

In mathematics, a natural number can be considered as the dimension of certain vector space and this last can be considered as the categorification of such number. Similarly, a vector space can be considered as the Grothendieck group of a category and this last algebraic structure can be considered as the categorification of the such vector space.
According with Khovanov, the knot, link and graph polynomials (Jones, HOMLFPY, Tutte, Bollobás-Riordan) can categorificated as the graded euler characteristic of a certain multi-graded knot or link or graph homology complex, according with

$$P(q) = \sum_{i,j} (-1)^t q^j \dim(H_{i,j}) .$$

For example when the relevant lie algebra es gl(1,1), the Alexander polynomial is categorified using the so called knot Floer homology denoted HFK(K). Similarly, when the lie algebra is sl(1), a numerical invariant is categorified using the so called Lee's deformed theory $H^{'}(K)$. More interesting, when the sl(2) is the relevant algebra, the Jones polynomial is categorified using the celebrated Khovanov homology $H^{Kh}(K)$. Moreover, when the lie algebra is sl(N), the HOMFLY polynomial is categorified using the so called Khovanov-Rozansky homology $HKR^N(K)$. Similarly for the Tutte and Bollobás-Riordan polynomials, which were categorified very recently using Graph homology H(G).
As it is well known, the knot, link and graphs polynomials arise in the Chern-Simons theory as vacuum expected values for Willson lines of the form

$$W_R(K) = \mathrm{Tr}_R\Big(P \exp \oint_K A\Big)$$
.

Similarly, the homological knot, link and graph invariants have a realization in topological string theory as a two-dimensional TQFTs classified by a Frobenius algebras. Actually is expected that the knot-homology will have a physical realization also in four-dimensional TQFTs such as Donaldson-Witten and Seiberg-Witten [5].

The model of Topological Quantum Computation proposed here is based on the N=4 topological super-Yang-Mills theory in four dimensions with surface operators [5]. The resulting topological quantum computer will be able to compute homological invariants H_Y for a closed 3-manifold Y given like an input. More over the resulting topological quantum computer will be able to compute knot-graph homology $H_{Y:K}$ for a 3-manifold Y with a knot (link or graph) K. This topological quantum computer makes the computations using both the Heegard decomposition of the input Y as the braid group action on branes on the moduli space. Given that the topological quantum computer proposed here uses D-branes for computations, we coin here the name D-brane Topological Quantum Computer (DTQC). Specifically the DTQC is constructed starting from a four-dimensional topological field theory with boundaries and corners; and both the links as the link cobordisms can be introduced via surface operators in the four dimensional gauge theory. In contrast the "standard" Anyonic Topological Quantum Computer (ATQC) is constructed via the three-dimensional topological field theory of the Chern-Simons kind. The DTQC is constructed by exploiting the fact that every topological gauge theory which admits surface operators gives rise origin to an action of the braid group on D-branes. Justly the D-branes contain and codified the topological qubits over which the quantum information is registered and the surface operators with the corresponding braid representations, are configuring the topological quantum gates which are able to process the quantum information. Concretely for the construction of the DTQC the Donaldson-Witten theory, the Seiberg-Witten theory , the N=2 supersymmetric gauge theory and the N=4 super-Yang-Mills theory are applied and as it was said before, the DTQC is able to compute knot-graph homologies and for hence the corresponding knot-graph polynomials such as Jone and Tutte polynomials.
The mathematical possibility of the DTQC is strongly linked with the fact that many knot homologies can be extended to a functor F from the category of 3-manifolds with links and cobordisms to the category of graded vector spaces and homomorphism, formally [5]

$$F(Y;K) = H_{Y;K} \quad , \quad F(X;D) : H_{Y;K} \to H_{Y';K'} ;$$

where Y is a closed 3-manifold with a knot (link or graph) K; $H_{Y;K}$ is a knot homology; X is a 4-manifold with boundary representing a cobordism between Y and $Y^{'}$; and D is link cobordims between K and $K^{'}$ In this formalism, $H_{Y;K}$ is the Hilbert space for the topological qubits and F(X;D) represents a topological quantum gate or a topological quantum circuit or a topological quantum program. As it was said before, this mathematical formalism for the DTQC can be realized in the context of the four-dimensional topological quantum theories which admit surface operators.
In the case of the Donaldson-Witten (DW) theory, the computation of topological invariants is as follows: the Hilbert space for the degrees of freedom is given by [5]

$$\mathcal{H}_Y = HF_*^{\mathrm{inst}}(Y)$$
,

the corresponding euler character is [5]

$$\chi(\mathcal{H}_Y) = 4\lambda_G(Y)$$
;

and the physical partition function is [5]

$$Z_{DW}(\mathbf{S}^1 \times Y) = 4\lambda$$

which is showing that the physical system described by the DW theory is permanently computing the topological invariant λ.
Similarly, the categorified computation of knot-homology invariants runs as follows for the case of the DW theory with surfaces operators: the Hilbert space is now [5]

$$\mathcal{H}_{Y;K,\alpha} = HF_*^{\text{inst}}(Y;K,\alpha)$$
,

the corresponding euler character takes the form [5]

$$\chi(\mathcal{H}_{Y;K,\alpha}) = \lambda_\alpha(Y;K)$$
;

and the physical partition function is given in terms of the knot-homology invariant which is showing that the physical systems described by the DW theory is permanently computing the knot-homology invariant λ_α .
Analogous results are obtained via the Seiberg-Witten (SW) Theory and the relevant equations are collected in the following table [5]

$\mathcal{H}_Y = HM_*(Y)$,	$HM_*(Y) \cong HF_*(Y)$,	$Z_{SW}(Y) = \lambda(Y)$
$\mathcal{H}_{Y_0;L;x_i,\alpha_i} = HM_*(Y;x_i,\alpha_i)$,	$\tau_{\alpha_i}(Y;q_i) := \sum_{x\in H(Y)} \chi\left(HM_*(Y;x_i,\alpha_i)\right)\cdot q^x$	
$\mathcal{H}_{L;x} = HFL_*(L;x)$,	$\sum_{x\in H(Y)} \chi\left(\mathcal{H}_{L;x}\right)\cdot q^x = \Delta(L;q)$	

This table shows that a physical system described by the SW theory is permanently computing knot-homology invariants.
Finally, for the case of the N=4 four dimensional Super-Yang-Mills theory, the basic equations are

$$\mathcal{H}_{Y;K} = HF_*^{\text{symp}}(\mathcal{M}_H;\mathcal{B}_1,\mathcal{B}_2) \quad , \quad Z = \lambda_{G_{\mathbf{C}}}(Y;K) \ ;$$

and these equations shows that a physical system described by the N=4 gauge theory is permanently computing knot-homology invariants
From other side, in the two-dimensional TQFTs which are describing the topological string theory, the amplitudes for topological strings can be rewritten in terms of integer enumerative invariants which are counting the degeneracy of states in Hilbert spaces, it is to say approximately the number of holomorphic branes ending on Lagrangian branes. In topological string theory, we obtain in consequence a physical reformulation of the polynomial link invariants (Jones, HOMFLY) in terms of the so-called Ooguri-Vafa invariants which approximately determine the Euler characteristic of the Q-cohomology, it is to say, the cohomology with respect to the nilpotent components of the supercharge.

Moreover, recently was proposed that the homological grading of link homologies is related to the extra charge in certain extension of topological string [5]. Particularly, the supersymmetric states of the holomorphic branes that are ending on Lagrangian branes, labeled by all physical quantum charges, should compute homological invariants of knots and links according with H(L)= H_{BPS} and [5]

$$\mathcal{H}^{\mathfrak{g},R}(\text{unknot}) \cong \mathcal{J}(W_{\mathfrak{g},R}(x_i)) , \qquad \mathcal{H}^{sl(N),\Lambda^k}(\text{unknot}) \cong H^*(Gr(k,N)) .$$

It is possible to compute the homology groups $H^{g,R}$ directly from the string theory using the fact that the topological vertex, which is able to compute topological string amplitudes in toric geometries, can be refined to compute refined BPS invariants. In consequence, given that the topological vertex formalism is structured from open string amplitudes, the refinement together with certain conjecture implies that the refined topological vertex will be able to compute homological link invariants, at least for the class of links which can be designed in terms of local toric geometries.
In general the refined vertex is able to compute all homological invariants of a huge class of links colored by arbitrary representations, it is to say, the refined vertex is able to compute $H^{sl(N);R_1,\dots,R_l}$.
For example the colored link polynomials are computed according with the equations in the following table [5]

$\overline{P}_{sl(N);R_1,\dots,R_\ell}(q)$,	$\overline{P}_N(q) \equiv \overline{P}_{sl(N);\square,\cdots,\square}(q)$
$W(L) = W_{R_1,\cdots,R_\ell}(L)$,	$W(L) = W_{\square,\cdots,\square}(L)$
$\bar{P}_N(L) = q^{-2N\mathrm{lk}(L)}\langle W(L)\rangle$,	$\mathrm{lk}(L) = \sum_{a<b}\mathrm{lk}(K_a,K_b)$

In the case of the corresponding colored homological invariants, we start from the doubly-graded homology theory denoted $H_{i,j}^{sl(N);R_1,\dots R_l}(L)$ whose graded Euler characteristic is the polynomial given by [5]

$$\overline{P}_{sl(N);R_1,\dots,R_\ell}(q) = \sum_{i,j\in\mathbb{Z}} (-1)^j q^i \dim \mathcal{H}_{i,j}^{sl(N);R_1,\dots,R_\ell}(L) ,$$

and the graded Poincaré polynomial is defined as [5]

$$\overline{P}_{sl(N);R_1,\dots\dots R_\ell}(q,t) = \sum_{i,j\in Z} q^i t^j \dim(H_{i,j}^{sl(N);R_1,\dots R_\ell}(L)) ..$$

It is observed that when t = -1 the graded Poincaré polynomial is reduced to the graded Euler characteristic.
Now, when $R_\alpha = \square$ for all $\alpha = 1,\dots,l$, the homology $H_{i,j}^{sl(N);R_1,\dots R_l}(L)$ is precisely the Khovanov-Rozansky homology, denoted $HKR_{i,j}^N(L)$ for which we have the following graded Poincaré polynomial [5]

$$KhR_N(q,t) \equiv \overline{P}_{sl(N);\lozenge,\dots,\lozenge}(q,t) = \sum_{i,j\in Z} \dim(HKR_{i,j}^N(L)) .$$

Specifically in the case of a link with two components the following equations give the computation of the link-homology via BPS states and the graduation given by the quantum charges [5]:

$$D_{J,s,r} := (-1)^F \dim \mathcal{H}_{BPS}^{F,J,s,r}$$

$$\overline{KhR}_N(L) = q^{-2N\mathrm{lk}(L)}\left[t^{\alpha}\overline{KhR}_N(K_1)\overline{KhR}_N(K_2) + \frac{1}{q-q^{-1}}\sum_{J,s,r\in\mathbb{Z}} D_{J,s,r}q^{NJ+s}t^r\right]$$

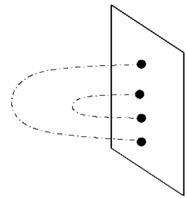
$$\bar{P}_N(L) = q^{-2N\mathrm{lk}(L)}\left[\bar{P}_N(K_1)\bar{P}_N(K_2) + \sum_{J,s} N_{(\Box,\Box),J,s}q^{NJ+s}\right]$$

The physical systems governed by four-dimensional TQFTs or two-dimensional TQFTs (topological strings) can be efficiently simulated on a quantum computer and reciprocally, the four-dimensional and two-dimensional TQTFs can be used to construct a model of computation by at least so strong as the standard quantum model BQP. All this shows that the four-dimensional and two-dimensional TQFTs provide an alternative way of looking at quantum computation respect to the way which is provided by the three-dimensional TQFTS. The mathematical structure of the four-dimensional and two-dimensional TQFTs and their deepest relationships with the link-graph homologies suggest the existence of new quantum algorithms for the computation of link-graph homology , which contain as particular cases the previously know topological quantum algorithms such as AJL, WJ and AAEL algorithms for Jones, HOMFLY and Tutte polynomials respectively.
A topological quantum field theory (TQFT) is a mathematical construction which unifies topological themes in conformal field theory, Chern-Simons theory (three-dimensional TQFT), topological strings (two-dimensional TQFT), Donaldson-Witten theory (four-dimensional TQFT) and Seiberg-Witten theory (four-dimensional TQFT).

For the Anyon Topological Quantum Computation, based on the Chern-Simons theory (three-dimensional TQFT), the action of the braid group on anyons is induced by braiding the corresponding worldliness. In the case of the D-Brane Topological Quantum Computation, based on the four-dimensional TQFTs (Donaldson-Witten, Seiberg-Witten) the action of the braid group on branes is induced by braiding of the corresponding surface operators. In general, the mapping class group of the surface Σ acts on branes on *M*. In particular when the surface Σ is plane with *n* punctures, the moduli space *M* is represented as fibered over the configuration space denoted $\mathrm{Conf}^n(X)$ of *n* unordered points in X, namely $M \to \mathrm{Conf}^n(X)$; and the braid grop defined as $B_n = \pi_1(\mathrm{Conf}^n(X))$, it is to say, the mapping class group of the n-punctured disk, acts on the category $F(\Sigma)$; being this last category considered as the category of D-branes in the topological sigma-model obtained via the dimensional reduction of gauge theory on Σ. The objects of the category $F(\Sigma)$ are describing BRST-invariant boundary conditions in the four-dimensional TQFT on 4-manifolds X with corners and being such manifolds locally represented by $X = \mathbf{R} \times \mathbf{R}_+ \times \Sigma$.
In general every braid can be represented as a non-contractible loop in the configuration space $\mathrm{Conf}^n(X)$ and when we run around the loop, the fibration $M \to \mathrm{Conf}^n(X)$ picks out a monodromy which is able to act on the category of branes $F(\Sigma)$ as an auto-equivalence, formally $B_n \to \mathrm{Auteq}(F(\Sigma))$ and for hence $\beta \to \phi_\beta$, for all $\beta \in B_n$. A simple example occurs when *M* contains one A_{n-1} chain of Lagrangian spheres and in this case the action of the braid group B_n is justly over A-branes. Other simple example occurs when *M* contains one A_{n-1} chain of spherical objects. In other words, these configurations occur when *M* can be converted into a manifold with singularity of type A_{n-1} . Specifically, the so called Dehn twists along A_{n-1} chains of Lagrangian spheres can be considered as the braid generators of B_n and for hence we obtain an action of the braid group B_n with *n* strands on the category of A-branes, denoted **Fuk**(M).
Similarly, we obtain the action of the braid group on B-branes when the braid group B_n is generated by the so called twist functors along spherical objects. In general, many examples of braid group actions on branes can be obtained by considering four-dimensional topological gauge theory with surface operators.

In the model of D-brane Topological Quantum Computation proposed in this work, the action of the braid group on the branes (A-branes and B-branes) is exploited to efficiently compute the knot-homology H_K , of a given knot K represented as the closure of certain braid. More explicitly we propose the following D-brane Topological Quantum Algorithm:

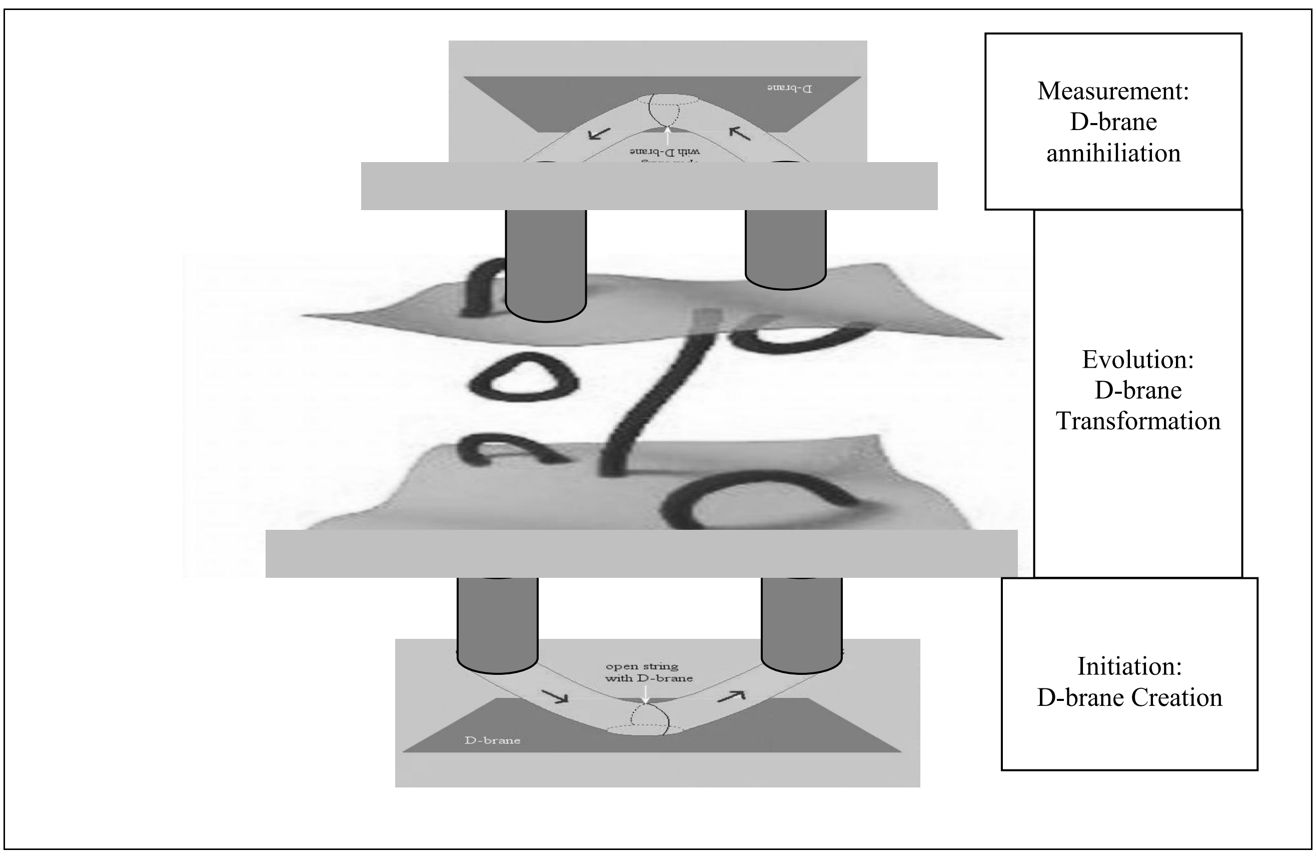
1. Introduce a knot K.
2. Write the knot as the closure of a braid β



3. Represent the space of quantum ground states, denoted H_K, in the four-dimensional gauge theory with a surface operator on D = **R** x K, as the space of open string states between branes B and B' = ϕ_β (B); being B the basic brane and being B' the brane obtained from B by applying the fuctor ϕ_β
4. Make a unitary representation for ϕ_β (B) denoted U (ϕ_β (B)).
5. Implement U (ϕ_β (B)) in a standard quantum computer.
6. Construct the output as [5]

$$H_K = \begin{cases} HF_*^{symp}(M; B; U(\phi_\beta(B)) & for\ A - \mathrm{mod}\,el \\ Ext^*(B, U(\phi_\beta(B))) & for\ B - \mathrm{mod}\,el \end{cases}$$

The following figure shows a D-brane Topological Quantum Computer while computing a Knot-homology.



A diagram of a D-Brane Topological Quantum Computer computing Knot Homologies.

5. Conclusions

A model of a D-Brane Topological Quantum Computer (DBTQC) was presented and sustained. The model was based on four-dimensional TQFTs such as Donaldson-Witten and Seiberg-Witten. The resulting DBTQC is able to compute Khovanov homology and its generalizations for links and graphs. The general algorithm executed naturally by the DBTQC contains as particular cases the AJL, WY and AAEL algorithms for the Jones, HOMFLY and Tutte polynomials respectively. The main mathematical ingredient for the DBTQC was the categorification process and the main physical ingredient for the DBTQC was the surface operator, which is a multi-dimensional generalization of the Wilson loop. In general the DBTQC is able to compute Euler characteristics for various kinds of homology complexes and in the sense of categorification, the DBTQC is a powerful universal quantum computer. It is expected that the DBTQC can be implemented using the new phases of condensed matter recently discovered like the string-net matter and the grapheme. In all case the DBTQC appears like an alternative and a generalization of the ATQC. For a future research the effects of the brane-world on the DBTQC must be considered. This work was supported partly by EAFIT University.

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