GEOMETRIC COMPLEXITY THEORY IV: NONSTANDARD QUANTUM GROUP FOR THE KRONECKER PROBLEM

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Dedicated to Sri Ramakrishna

Abstract. The Kronecker coefficient $g_{\lambda\mu\nu}$ is the multiplicity of the $GL(V) \times GL(W)$-irreducible $V_\lambda \otimes W_\mu$ in the restriction of the $GL(X)$-irreducible X_ν via the natural map $GL(V) \times GL(W) \to GL(V \otimes W)$, where V, W are $\mathbb{C}$-vector spaces and $X = V \otimes W$. A fundamental open problem in algebraic combinatorics is to find a positive combinatorial formula for these coefficients.

We construct two quantum objects for this problem, which we call the nonstandard quantum group and nonstandard Hecke algebra. We show that the nonstandard quantum group has a compact real form and its representations are completely reducible, that the nonstandard Hecke algebra is semisimple, and that they satisfy an analog of quantum Schur-Weyl duality.

Using these nonstandard objects as a guide, we follow the approach of Adsul, Sohoni, and Subrahmanyam [1] to construct, in the case $\dim(V) = \dim(W) = 2$, a representation $\check{X}_\nu$ of the nonstandard quantum group that specializes to $\mathrm{Res}_{GL(V)\times GL(W)} X_\nu$ at $q = 1$. We then define a global crystal basis $+\mathrm{HNSTC}(\nu)$ of $\check{X}_\nu$ that solves the two-row Kronecker problem: the number of highest weight elements of $+\mathrm{HNSTC}(\nu)$ of weight (λ, μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$. We go on to develop the beginnings of a graphical calculus for this basis, along the lines of the $U_q(\mathfrak{sl}_2)$ graphical calculus from [19], and use this to organize the crystal components of $+\mathrm{HNSTC}(\nu)$ into eight families. This yields a fairly simple, positive formula for two-row Kronecker coefficients, generalizing a formula in [15]. As a byproduct of the approach, we also obtain a rule for the decomposition of $\mathrm{Res}_{GL_2\times GL_2 \rtimes \mathcal{S}_2} X_\nu$ into irreducibles.

Contents

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1. Introduction

1.1. **The Kronecker problem.** This is a continuation of the series of articles [47, 48, 44] on geometric complexity theory (GCT), an approach to **P** vs. **NP** and related problems using geometry and representation theory. A basic philosophy of this approach is called the *flip*; see [46, 42, 43] for its detailed exposition. The flip suggests that separating the classes **P** and **NP** will require solving difficult positivity problems in algebraic geometry and representation theory. A central positivity problem arising here is the following fundamental problem in the representation theory of the symmetric group.

Let $\mathcal{S}_r$ denote the symmetric group on r letters and let M_ν denote the $\mathcal{S}_r$-irreducible corresponding to the partition ν. Given three partitions λ, μ, ν of r, the Kronecker coefficient $g_{\lambda\mu\nu}$ is defined to be the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$. As explained in §1.2, this is also equal to the multiplicity of the $GL(V) \times GL(W)$-irreducible $V_\lambda \otimes W_\mu$ in the restriction of the $GL(X)$-irreducible X_ν via the natural map $GL(V) \times GL(W) \to GL(X)$, where $X = V \otimes W$.

Problem 1.1 (Kronecker problem)**.** *Find a positive combinatorial formula for the Kronecker coefficients $g_{\lambda\mu\nu}$.*

There are two precise related problems in complexity theory that arise in the flip: (1) find a (positive) #**P** formula for Kronecker coefficients, and harder, (2) find a polynomial time algorithm to determine whether a Kronecker coefficient is zero.

Although the Kronecker problem has been studied since the early twentieth century, its general case still seems out of reach. A combinatorial interpretation for Kronecker coefficients in the case that two of the partitions are hooks was first given by Lascoux [34], and other formulae were later given by Remmel [52] and Rosas [56]. An explicit combinatorial formula for Kronecker coefficients in the case that λ and μ have at most two rows, which we refer to as the two-row case, was given by Remmel and Whitehead in [53]. Later, a formula for this case, not obviously equivalent to Remmel and Whitehead's, was given by Rosas [56]. Using Rosas's work, Briand, Orellana, and Rosas give a piecewise quadratic quasipolynomial formula for the two-row case [13]. Though these formulae for the two-row case are quite explicit, none of them is positive and hence do not solve the Kronecker problem in this case. Briand-Orellana-Rosas [13, 14] and Ballantine-Orellana [6] have also made progress on the Kronecker problem for the special case of reduced Kronecker coefficients, sometimes called the stable limit, in which the first part of the partitions λ, μ, ν is large.

In addition to the connections to complexity theory discussed in [47, 48, 44], the Kronecker problem also has connections to quantum information theory [12, 15] and the geometry of the $GL_a \times GL_b \times GL_c$-variety $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ [31, 32, 17, 2]. See [36, 58, 57] for more on its history and significance.

In this paper we focus on a stronger, basis-theoretic version of the Kronecker problem which, to our knowledge, has not been studied in the literature (even in the two-row case). As will be described in more precision and detail in §1.2–1.5, this version asks for a canonical basis for $\mathrm{Res}_{GL(V)\times GL(W)} X_\nu$ (actually, a quantization of this module) such that the labels of the highest weight basis elements of weight (λ, μ) give a combinatorial formula for $g_{\lambda\mu\nu}$. We believe this basis-theoretic strengthening to be important because (1) it is what is ultimately needed in GCT (see [43]), (2) it may be useful for better understanding the $GL_a \times GL_b \times GL_c$-variety $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, (3) the structure coefficients for the action of the Chevalley generators on the basis may have certain positivity properties and an interesting geometric interpretation, and (4) making more demands on combinatorial objects that count Kronecker coefficients may make them easier to find.

In this paper we give an approach to the basis-theoretic version of the Kronecker problem and implement it successfully in the two-row case. The approach uses two new quantum objects, the nonstandard quantum group and nonstandard Hecke algebra. In the two-row case, we construct a representation $\check{X}_\nu$ of the nonstandard quantum group that specializes to $\text{Res}_{GL(V)\times GL(W)}X_\nu$ at $q = 1$. We then define a canonical basis for $\check{X}_\nu$ and use this to obtain an explicit formula for two-row Kronecker coefficients. Much of the machinery developed here extends to more general cases than the two-row case. Additionally, the sequels [41, 40] describe a nonstandard quantum group, nonstandard Hecke algebra, and a conjectural scheme for constructing positive canonical bases of their representations for the more general plethysm problem [36, 58] of which the Kronecker problem considered in this article is a special case. We have not yet been able to use the machinery developed in this paper or the sequels to solve the Kronecker problem outside the two-row case or the plethysm problem because explicit computation of the canonical bases is much harder than in the two-row case. Nonetheless, we hope that the concrete implementation in the two-row case illustrates and supports the approach in general. The remainder of the introduction summarizes the approach and its implementation in the two-row case.

1.2. **The basis-theoretic version of the Kronecker problem.** Let V, W be $\mathbb{Q}$-vector spaces of dimensions d_V, d_W, respectively, considered as the natural representations of $U(\mathfrak{g}_V), U(\mathfrak{g}_W)$, respectively, where $\mathfrak{g}_V$ denotes the Lie algebra $\mathfrak{gl}(V)$. Set $X = V \star W$, where $\star$ is the symbol we use for tensor product between objects associated to V and objects associated to W, to distinguish these from other tensor products. There is a natural algebra homomorphism

$$U(\mathfrak{g}_V \oplus \mathfrak{g}_W) = U(\mathfrak{g}_V) \star U(\mathfrak{g}_W) \to U(\mathfrak{g}_X) \tag{1}$$

corresponding to the group homomorphism $GL(V) \times GL(W) \to GL(X),\ (g, g') \mapsto g \star g'$.

The vector space $X^{\otimes r}$ becomes a left $U(\mathfrak{g}_X)$-module via the coproduct of $U(\mathfrak{g}_X)$ and this left action commutes with the right action of $\mathcal{S}_r$ given by permuting tensor factors. Schur-Weyl duality says that, as an $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule,

$$X^{\otimes r} \cong \bigoplus_{\nu \vdash_{d_X} r} X_\nu \otimes M_\nu, \tag{2}$$

where X_ν is the irreducible $U(\mathfrak{g}_X)$-module of highest weight ν and $\nu \vdash_{d_X} r$ means that ν is a partition of r with at most $d_X := d_V d_W$ parts. We can also apply Schur-Weyl duality for $V^{\otimes r}$ and $W^{\otimes r}$ to obtain

$$V^{\otimes r} \star W^{\otimes r} \cong \bigoplus_{\substack{\lambda \vdash_{d_V} r \\ \mu \vdash_{d_W} r}} (V_\lambda \otimes M_\lambda) \star (W_\mu \otimes M_\mu) \cong \bigoplus_{\substack{\lambda \vdash_{d_V} r,\, \mu \vdash_{d_W} r \\ \nu \vdash_{d_X} r}} (V_\lambda \star W_\mu \otimes M_\nu)^{\oplus g_{\lambda\mu\nu}}. \tag{3}$$

Putting (2) and (3) together, we obtain the $(U(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathbb{Q}\mathcal{S}_r)$-bimodule isomorphism

$$\bigoplus_{\nu \vdash_{d_X} r} X_\nu \otimes M_\nu \cong \bigoplus_{\substack{\lambda \vdash_{d_V} r,\, \mu \vdash_{d_W} r \\ \nu \vdash_{d_X} r}} (V_\lambda \star W_\mu \otimes M_\nu)^{\oplus g_{\lambda\mu\nu}}. \tag{4}$$

Thus it is easily seen here that the Kronecker coefficient $g_{\lambda\mu\nu}$ is also the multiplicity of $V_\lambda \star W_\mu$ in $\mathrm{Res}_{U(\mathfrak{g}_V \oplus \mathfrak{g}_W)} X_\nu$, where the restriction is via the map (1).

We have decided that the isomorphism (4) coming from Schur-Weyl duality is a good setting to study the Kronecker problem because it allows both descriptions of Kronecker coefficients to be seen simultaneously. It also suggests a way to make more demands on a combinatorial formula for Kronecker coefficients—*in the hopes that demanding more structure on the combinatorial objects will make them easier to find.* We would like to obtain, not only a set of objects that count Kronecker coefficients, but stronger, a bijection between objects indexing both sides of (4), which amounts to a bijection

$$\bigsqcup_{\nu} SSYT_{d_X}(\nu) \times SYT(\nu) \cong \bigsqcup_{\lambda,\mu,\nu} SSYT_{d_V}(\lambda) \times SSYT_{d_W}(\mu) \times SYT(\nu) \times [g_{\lambda\mu\nu}], \qquad (5)$$

where $[k]$ denotes the set $\{1, \ldots, k\}$, $\mathrm{SSYT}_l(\nu)$ denotes the set of semistandard Young tableaux of shape ν and with entries in $[l]$, and $\mathrm{SYT}(\nu)$ denotes the set of standard Young tableaux of shape ν. Stronger still, we would like to find a basis for $X^{\otimes r}$ whose cells (cells are defined as a general notion for any module with basis in §2.4) correspond to the decompositions in (4) and whose labels are indexed by either side of (5); this is explained in more detail in the next subsections.

However, nothing easy seems to work. One difficulty is that there does not seem to be a way to obtain a bijection between the weight basis $x_1, \ldots, x_{d_X}$ of X and the weight basis $\{v_i \star w_j\}_{i\in[d_V], j\in[d_W]}$ of $V \star W$ that is compatible with the Kronecker problem. The approach seems to be lost without some additional structure. So to aid it, we add structure from quantum groups and Hecke algebras, and try to apply the theory of canonical bases.

1.3. **Canonical bases connect quantum Schur-Weyl duality with RSK.** To get an idea of the basis-theoretic solution to the Kronecker problem we are after, let us see how the canonical basis of $V^{\otimes r}$ nicely connects quantum Schur-Weyl duality with the RSK correspondence. From this picture, we can also see two different ways that canonical bases yield a combinatorial formula for Littlewood-Richardson coefficients, which is another reason we have turned to canonical bases for a solution to the Kronecker problem.

Let $U_q(\mathfrak{g}_V)$ be the quantized enveloping algebra over $K = \mathbb{Q}(q)$ and $\mathscr{H}_r$ the type A_{r-1} Hecke algebra over $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$ (see §4.1 and §3 for precise definitions and conventions). From now on, we write V_λ (resp. M_λ) for the irreducible $U_q(\mathfrak{g}_V)$-module (resp. $K\mathscr{H}_r$-module) corresponding to λ and let $V_\lambda|_{q=1}$ (resp. $M_\lambda|_{q=1}$) denote the corresponding $U(\mathfrak{g}_V)$-module (resp. $\mathbb{Q}\mathcal{S}_r$-module). Schur-Weyl duality generalizes nicely to the quantum setting:

Theorem 1.2 (Jimbo [25])**.** *As a $(U_q(\mathfrak{g}_V), K\mathscr{H}_r)$-bimodule, $V^{\otimes r}$ decomposes into irreducibles as*

$$V^{\otimes r} \cong \bigoplus_{\lambda \vdash_{d_V} r} V_\lambda \otimes M_\lambda. \qquad (6)$$

This algebraic decomposition has a combinatorial underpinning, which is the bijection

$$[d_V]^r \cong \bigsqcup_{\lambda \vdash_{d_V} r} \mathrm{SSYT}_{d_V}(\lambda) \times \mathrm{SYT}(\lambda),\ \mathbf{k} \mapsto (P(\mathbf{k}), Q(\mathbf{k})), \tag{7}$$

given by the RSK correspondence, where $P(\mathbf{k})$ (resp. $Q(\mathbf{k})$) denotes the insertion (resp. recording) tableau of the word $\mathbf{k}$.

Now the upper canonical basis $B_V^r := \{c_{\mathbf{k}} : \mathbf{k} \in [d_V]^r\}$ of $V^{\otimes r}$ can be defined by $c_{\mathbf{k}} := v_{k_1} \heartsuit \ldots \heartsuit v_{k_r}$, $\mathbf{k} = k_1, \ldots, k_r \in [d_V]^r$, where $\heartsuit$ is like the $\diamond$ of [35] for tensoring based modules, adapted to upper canonical bases, as explained in [9, 16] and reviewed in §6.2.

The basis B_V^r has cells corresponding to the decomposition (6) and labels to (7), as the following theorem makes precise.

Theorem 1.3 ([22] (see [9, Corollary 5.7] and Theorem 6.5))**.**

(i) *The $\mathscr{H}_r$-module with basis $(V^{\otimes r}, B_V^r)$ decomposes into $\mathscr{H}_r$-cells as*

$$B_V^r = \bigsqcup_{\lambda \vdash_{d_V} r,\ T \in SSYT_{d_V}(\lambda)} \Gamma_T, \quad \textit{where}\ \ \Gamma_T := \{c_{\mathbf{k}} : P(\mathbf{k}) = T\}.$$

(ii) *The $\mathscr{H}_r$-cellular subquotient spanned by Γ_T is isomorphic to $M_{\mathrm{sh}(T)}$, where* $\mathrm{sh}(T)$ *denotes the shape of T.*

(iii) *The $U_q(\mathfrak{g}_V)$-module with basis $(V^{\otimes r}, B_V^r)$ decomposes into $U_q(\mathfrak{g}_V)$-cells as*

$$B_V^r = \bigsqcup_{\lambda \vdash_{d_V} r,\ T \in SYT(\lambda)} \Lambda_T, \quad \textit{where}\ \ \Lambda_T = \{c_{\mathbf{k}} : Q(\mathbf{k}) = T\}.$$

(iv) *The $U_q(\mathfrak{g}_V)$-cellular subquotient spanned by Λ_T is isomorphic to $V_{\mathrm{sh}(T)}$.*

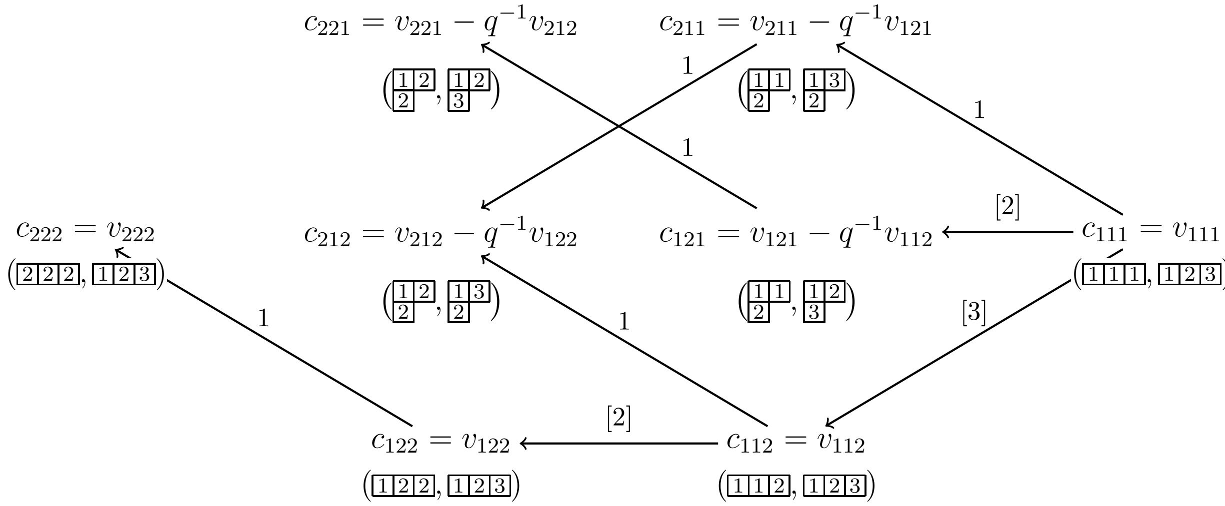


Figure 1: An illustration of Theorem 1.3 for $r = 3,\ d_V = 2$. The notation $v_{\mathbf{k}}$, $\mathbf{k} \in [d_V]^r$, denotes the tensor monomial $v_{k_1} \otimes \cdots \otimes v_{k_r}$. The pairs of tableaux are of the form $(P(\mathbf{k}), Q(\mathbf{k}))$. The arrows and their coefficients give the action of $F \in U_q(\mathfrak{g}_V)$ on the upper canonical basis B_V^3, where $[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$.

We can also use Theorem 1.3 to obtain two formulae for the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\,\mu}$: one comes from reading off the $U_q(\mathfrak{g}_V)$-cells of shape ν in a tensor product $K\Lambda_{Q^1} \otimes K\Lambda_{Q^2}$, $\mathrm{sh}(Q^1) = \lambda,\ \mathrm{sh}(Q^2) = \mu$, and the other from reading off the $\mathscr{H}_r$-cells of shape ν in an induced module $\mathscr{H}_r \otimes_{\mathscr{H}_k \otimes \mathscr{H}_{r-k}} (\mathbf{A}\Gamma_{P^1} \otimes \mathbf{A}\Gamma_{P^2})$, where $\lambda \vdash k,\ \mu \vdash r - k$, $\mathrm{sh}(P^1) = \lambda$, and $\mathrm{sh}(P^2) = \mu$ (see [10, §4.2]).

Our refined goal is now to find a canonical basis of $X^{\otimes r}$ with cells corresponding to (4) and labels to (5), but now X will be a K-vector space so that the basis can perhaps be defined by a globalization procedure like that used for quantum groups in [26, 27]. To do this, we first need to give $X^{\otimes r}$ the structure of a bimodule for quantum objects that are suited to this problem. This is addressed in the next subsection. Then in §1.5 and §1.7, we return to the construction of the basis.

1.4. **The nonstandard quantum group and Hecke algebra.** We seek quantum objects that play an analogous role for $X^{\otimes r}$ that $\mathscr{H}_r$ and $U_q(\mathfrak{g}_V)$ do for $V^{\otimes r}$. We also require that these objects be compatible with the commuting actions of $\mathscr{H}_r \otimes \mathscr{H}_r$ and $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ on $X^{\otimes r}$. The resulting quantized objects we have arrived at are the nonstandard Hecke algebra $\check{\mathscr{H}}_r$ and the nonstandard quantum group $GL_q(\check{X})$. These objects are, in a certain sense, the best possible quantizations satisfying these requirements.

We point out that new quantum objects are necessary for this problem. The commuting actions of $\mathscr{H}_r$ and $U_q(\mathfrak{g}_X)$ on $X^{\otimes r}$ are not satisfactory quantizations of the commuting $\mathcal{S}_r$ and $U(\mathfrak{g}_X)$ actions, given the compatibility requirements just mentioned. On the Hecke algebra side, this is because the Hecke algebra is not a Hopf algebra in any natural way.

Similarly, on the quantum group side, it can be shown [23] that the homomorphism (1) cannot be quantized in the category of Drinfel′d-Jimbo quantum groups.

The *nonstandard Hecke algebra* $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by the elements

$$\mathcal{P}_i := C'_{s_i} \otimes C'_{s_i} + C_{s_i} \otimes C_{s_i},\ i \in [d_V - 1], \tag{8}$$

where C'_{s_i} and C_{s_i} are the simplest lower and upper Kazhdan-Lusztig basis elements, which are proportional to the trivial and sign idempotents of the parabolic sub-Hecke algebra $(\mathscr{H}_r)_{\{s_i\}}$. We think of the inclusion $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ as a deformation of the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r} : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \otimes w$. As is explained more precisely in Remark 11.4, the nonstandard Hecke algebra is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ making $\check{\Delta}$ as close as possible to $\Delta_{\mathbb{Z}\mathcal{S}_r}$ at $q = 1$.

To define the nonstandard quantum group $GL_q(\check{X})$, we follow the approach of [54, 30] to quantum groups. We now recall a few of the relevant concepts, leaving a thorough review to §4.2–4.6. The quantum group $GL_q(V)$ is not an actual group, but just a virtual object associated to the quantized enveloping algebra $U_q(\mathfrak{g}_V)$ and the quantum coordinate algebra $\mathscr{O}(GL_q(V))$. These are dually paired Hopf algebras, which implies that any $\mathscr{O}(GL_q(V))$-comodule is a $U_q(\mathfrak{g}_V)$-module. In this paper, we are only interested in the V_λ for partitions λ, which are both $U_q(\mathfrak{g}_V)$-modules and $\mathscr{O}(GL_q(V))$-comodules, so (at least for our purposes) $U_q(\mathfrak{g}_V)$ and $\mathscr{O}(GL_q(V))$ provide dual approaches to the same objects. The quantum coordinate algebra $\mathscr{O}(M_q(V))$ is defined to be the FRT-algebra $A(\mathscr{R}_{V,V})$ [54] associated to the $\mathscr{R}$-matrix $\mathscr{R}_{V,V} \in \mathrm{End}(V^{\otimes 2})$, which is a quotient of the tensor bialgebra $T(U) = \bigoplus_{r \geq 0} U^{\otimes r}$, $U := V \otimes V^*$, that specializes to $\mathscr{O}(M(V))$ at $q = 1$. The Hopf algebra $\mathscr{O}(GL_q(V))$ is then defined from $\mathscr{O}(M_q(V))$ by inverting the quantum determinant.

The nonstandard quantum group $GL_q(\check{X})$ is a virtual object associated to the *nonstandard coordinate algebra* $\mathscr{O}(GL_q(\check{X}))$ (we have yet to construct a nonstandard enveloping algebra dual to $\mathscr{O}(GL_q(\check{X}))$, but we think this is possible). To define $\mathscr{O}(GL_q(\check{X}))$, we first define the *nonstandard coordinate algebra* $\mathscr{O}(M_q(\check{X}))$. This is most quickly defined as the FRT-algebra $A(P_+^{\check{X}})$, where $P_+^{\check{X}} \in \mathrm{End}(\check{X}^{\otimes 2})$ is equal to the action of $\frac{1}{[2]^2}\mathcal{P}_1$ on $\check{X}^{\otimes 2}$ ($\mathcal{P}_1$ acts on $\check{X}^{\otimes 2}$ by quantum Schur-Weyl duality for $V^{\otimes 2}$ and $W^{\otimes 2}$); see §8.1 for details. Here and throughout the paper, $\check{X}$ is the same as X, the decoration indicating that it is associated to a nonstandard object.

Much of the abstract theory of the standard quantum group $GL_q(V)$ can be replicated in the nonstandard case, but explicit computations become significantly harder. For instance, we can define nonstandard symmetric and exterior algebras $\check{S}(\check{X})$ and $\check{\Lambda}(\check{X})$ (§8.2), which are $\mathscr{O}(M_q(\check{X}))$-comodule algebras and specialize to the symmetric and exterior algebras of X at $q = 1$. However, $\check{S}(\check{X})$ is already isomorphic to $\mathscr{O}(M_q(V))$ when $W = V^*$, and thus explicitly determining the multiplication in this algebra, in terms of the Gelfand-Tsetlin basis, say, has been intensively studied and is still not completely understood (see [30, 60] and §8.3). Understanding $\mathscr{O}(M_q(\check{X}))$ explicitly is yet another level of difficulty beyond this. We show (Appendix A) that a natural reduction system for this coordinate algebra does not satisfy the diamond property.

Let $\check{\Lambda}^r \check{X}$ denote the degree r part of $\check{\Lambda}(\check{X})$. The *nonstandard determinant* $\check{D}$ is defined to be the matrix coefficient of the comodule $\check{\Lambda}^{d_X} \check{X}$. This object is somewhat mysterious in that we do not understand it explicitly (in the monomial basis, say). Nonetheless, we show (§10) that $\mathscr{O}(M_q(\check{X}))[\frac{1}{\check{D}}]$ can be given a Hopf algebra structure. The result is the *nonstandard coordinate algebra* $\mathscr{O}(GL_q(\check{X}))$. We now state our main theorem about this object, which is proved in §9 and §10.

Theorem 1.4 (Theorem 10.7). *Assume that all objects are over $\mathbb{C}$ and q is real and transcendental. Then*

(a) The Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ can be made into a Hopf $*$-algebra. This is considered to be the coordinate ring of the compact real form of the nonstandard quantum group $GL_q(\check{X})$. This virtual compact real form is denoted $\mathtt{U}_q(\check{X})$, which is a compact quantum group in the sense of Woronowicz [62].

(b) There is a Hopf $*$-algebra homomorphism

$$\tilde{\psi} : \mathscr{O}(GL_q(\check{X})) \to \mathscr{O}(GL_q(V)) \otimes \mathscr{O}(GL_q(W)),$$

(c) Every finite-dimensional representation of $\mathtt{U}_q(\check{X})$ is unitarizable, and hence, is a direct sum of irreducible representations.

(d) An analog of the Peter-Weyl theorem holds:

$$\mathscr{O}(GL_q(\check{X})) = \bigoplus_{\alpha \in \check{\mathscr{P}}} \check{\mathcal{X}}_\alpha^* \otimes \check{\mathcal{X}}_\alpha,$$

where $\check{\mathscr{P}}$ is an index set for the irreducible right comodules of $\mathscr{O}(GL_q(\check{X}))$ and $\check{\mathcal{X}}_\alpha$ is the comodule labeled by α.

In a similar spirit, we show (Proposition 11.8) that the nonstandard Hecke algebra $K\check{\mathscr{H}}_r$ is semisimple. As far as the representation theory of $\check{\mathscr{H}}_r$ and $\mathscr{O}(GL_q(\check{X}))$ are concerned, we have had more luck understanding that of $\check{\mathscr{H}}_r$. Fortunately, as the next result shows, we can transfer our knowledge of $K\check{\mathscr{H}}_r$-irreducibles to $\mathscr{O}(GL_q(\check{X}))$-irreducibles.

Just as in the standard case, there are commuting actions of the nonstandard Hecke algebra and nonstandard quantum group on $\check{X}^{\otimes r}$. Since we do not yet have a nonstandard enveloping algebra dual to $\mathscr{O}(GL_q(\check{X}))$, we instead work with the *nonstandard Schur algebra*, denoted $K\check{\mathscr{S}}(\check{X}, r)$, which is defined to be the algebra dual to the coalgebra $\mathscr{O}(M_q(\check{X}))_r$. We have the following nonstandard analog of quantum Schur-Weyl duality.

Theorem 1.5 (Theorem 12.1). *As a $(K\check{\mathscr{S}}(\check{X}, r), K\check{\mathscr{H}}_r)$-bimodule, $\check{X}^{\otimes r}$ decomposes into irreducibles as*

$$\check{X}^{\otimes r} \cong \bigoplus_{\alpha \in \check{\mathscr{P}}_r} \check{\mathcal{X}}_\alpha \otimes \check{M}_\alpha, \tag{9}$$

where $\check{\mathscr{P}}_r$ is an index set so that $\check{\mathcal{X}}_\alpha$ ranges over $K\check{\mathscr{S}}(\check{X}, r)$-irreducibles and $\check{M}_\alpha$ ranges over $\check{\mathscr{H}}_r$-irreducibles.

We deduce that there are nonnegative integers $n_\alpha^{\lambda,\mu} = n^\alpha_{\lambda,\mu}$ that correspond to the multiplicities in the following two decomposition problems:

$$\check{\mathscr{X}}_\alpha \cong \bigoplus_{\lambda,\mu} (V_\lambda \star W_\mu)^{\oplus n_\alpha^{\lambda,\mu}}, \quad \operatorname{Res}_{\check{\mathscr{H}}_r} M_\lambda \star M_\mu \cong \bigoplus_\alpha \check{M}_\alpha^{\oplus n^\alpha_{\lambda,\mu}}. \tag{10}$$

The representation theory of the nonstandard Hecke algebra and quantum group thus decompose the Kronecker problem into two steps:

(i) Determine the multiplicity $n^\alpha_{\lambda,\mu}$ of the irreducible $\check{\mathscr{H}}_r$-module $\check{M}_\alpha$ in the tensor product $M_\lambda \otimes M_\mu$. Equivalently, determine the multiplicity $n_\alpha^{\lambda,\mu}$ of the irreducible $\mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$-comodule $V_\lambda \star W_\mu$ in the irreducible $\mathscr{O}(M_q(\check{X}))$-comodule $\check{\mathscr{X}}_\alpha$.

(ii) Determine the multiplicity $m_{\alpha\nu}$ of the $\mathcal{S}_r$-irreducible $M_\nu|_{q=1}$ in $\check{M}_\alpha|_{q=1}$.

The resulting formula for Kronecker coefficients is

$$g_{\lambda\mu\nu} = \sum_\alpha n^\alpha_{\lambda,\mu} m_{\alpha\nu}. \tag{11}$$

Thus a positive combinatorial formula for $n^\alpha_{\lambda,\mu}$ and $m_{\alpha\nu}$ would yield one for Kronecker coefficients.

Unfortunately, this does not get us very far. Despite its being as small as possible, $\check{\mathscr{H}}_r$ has dimension much larger than that of $\mathcal{S}_r$. Similarly, despite its being as large as possible, $\mathscr{O}(M_q(\check{X}))_r$ has dimension much smaller than that of $\mathscr{O}(M(X))_r$ (see Remark 8.5 and Proposition 8.11). It turns out that in the two-row ($d_V = d_W = 2$) case, $\check{\mathscr{H}}_{r,2}$ is quite close to $S^2\mathscr{H}_r$ and the nonstandard coordinate algebra $\mathscr{O}(GL_q(\check{X}))$ is close to the smash coproduct $\mathscr{O}_q^\tau := \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \rtimes \mathscr{F}(\mathcal{S}_2)$, where $\check{\mathscr{H}}_{r,2}$ is the image of $\check{\mathscr{H}}_r$ in $\operatorname{End}(\check{X}^{\otimes r})$ when $d_V = d_W = 2$ and $\mathscr{F}(\mathcal{S}_2)$ is the Hopf algebra of functions on $\mathcal{S}_2$; see [11], §13.6, and Appendix B for details. Thus most of the work is left to determining the multiplicities in (ii).

However, we have gained *something*. In addition to the slight help that (11) provides, finding a basis for $\check{M}_\alpha$ whose cells are compatible with the decomposition $\check{M}_\alpha|_{q=1} \cong \bigoplus_\nu (M_\nu|_{q=1})^{\oplus m_{\alpha\nu}}$ has significantly more structure than finding a basis for $M_\lambda \otimes M_\mu|_{q=1}$ compatible with the decomposition $M_\lambda \otimes M_\mu|_{q=1} \cong \bigoplus_\nu (M_\nu|_{q=1})^{\oplus g_{\lambda\mu\nu}}$, despite the fact that $\check{M}_\alpha$ is typically equal to some $\operatorname{Res}_{\check{\mathscr{H}}_r} M_\lambda \otimes M_\mu$; that is, finding a basis *before* specializing $q = 1$ has more structure than finding it *after* specializing.

Also, it is shown in [11] that the restriction of an $\check{\mathscr{H}}_{r,2}$-irreducible to $\check{\mathscr{H}}_{r-1,2}$ is multiplicity-free. The seminormal basis (in the sense of [51]) of some $\check{M}_\alpha = \operatorname{Res}_{\check{\mathscr{H}}_{r,2}} M_\lambda \otimes M_\mu$ coming from restricting along the chain $\check{\mathscr{H}}_{1,2} \subseteq \cdots \subseteq \check{\mathscr{H}}_{r-1,2} \subseteq \check{\mathscr{H}}_{r,2}$ is significantly different from the one coming from the chain $\mathscr{H}_{1,2} \otimes \mathscr{H}_{1,2} \subseteq \cdots \subseteq \mathscr{H}_{r-1,2} \otimes \mathscr{H}_{r-1,2} \subseteq \mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2}$. Here, $\mathscr{H}_{r,2}$ denotes the Temperley-Lieb algebra (see §13.6). The article [40] suggests a conjectural scheme for constructing a canonical basis of $\check{M}_\alpha$ using the former chain, but we have not been able to prove its correctness. We therefore follow a different path to construct a canonical basis for the two-row Kronecker problem, described below. Along similar lines, it is illustrated in §15.2 that, though the difference between $\mathscr{O}_q^\tau$ and $\mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$

is small, the little bit of extra structure added by considering $\mathscr{O}_q^\tau$-comodules can be quite important.

1.5. **Towards an upper canonical basis for $\check{X}^{\otimes r}$.** Our goal is now to construct a basis of $\check{X}^{\otimes r}$ with $K\check{\mathscr{S}}(\check{X}, r)$-cells and $\check{\mathscr{H}}_r$-cells that are compatible with the decomposition

$$\check{X}^{\otimes r} \cong \bigoplus_{\alpha \in \check{\mathscr{P}}_r} \check{\mathcal{X}}_\alpha \otimes \check{M}_\alpha, \tag{12}$$

and so that after specializing $q = 1$, the cells are compatible with the decomposition

$$\bigoplus_{\nu \vdash_{d_X} r} X_\nu|_{q=1} \otimes M_\nu|_{q=1}. \tag{13}$$

See Conjecture 19.1 for a more precise and detailed statement. After many failed attempts, we succeeded in constructing such a basis in the two-row case for r up to 4 (see Examples 19.2 and 19.3) and for some highest weight spaces of $\check{X}^{\otimes 5}$ and $\check{X}^{\otimes 6}$. We are therefore quite hopeful that such a basis exists in general. However, the construction of global crystal bases from a balanced triple [26, 27] and the similar theory of based modules [35] does not seem to be enough here.

Though the construction of a basis for all of $\check{X}^{\otimes r}$ remains unfinished, we have been able to construct one so-called *fat cell* $\check{X}_\nu$ of $\check{X}^{\otimes r}$, which is defined in Conjecture 19.1 to be a union of $K\check{\mathscr{S}}(\check{X}, r)$-cells that corresponds to a copy of $X_\nu|_{q=1}$ in (13). Let ν' be the conjugate of the partition ν. The fat cell $\check{X}_\nu$ we can construct is the one corresponding to the recording SYT $(Z^*_{\nu'})^T$ in the left-hand side of (5), where $(Z^*_{\nu'})^T$ is the SYT with $1, \ldots, \nu'_1$ in its first column, $\nu'_1 + 1, \ldots, \nu'_1 + \nu'_2$ in its second column, etc. This is enough to give a nice basis-theoretic solution to the two-row Kronecker problem.

1.6. **The approach of Adsul, Sohoni, and Subrahmanyam.** Our construction of $\check{X}_\nu$ follows the construction of Adsul, Sohoni, and Subrahmanyam [1] of a similar quantum object for the Kronecker problem. Their construction can be viewed as a quantum version of the robust characteristic-free definition of Schur modules from [3] (see [61, §2.1]). We next recall this definition.

Let X be a free module over a commutative ring, $\nu' \vdash_l r$, and set

$$Y_{\nu'} := \Lambda^{\nu'_1} X \otimes \cdots \otimes \Lambda^{\nu'_l} X.$$

The Schur module $L_{\nu'} X$ [61, §2.1] is first defined in the $l = 2$ case to be $Y_{\nu'}/Y_{\triangleright \nu'}$ where $Y_{\triangleright \nu'}$ is defined in terms of the product and coproduct on the exterior algebra $\Lambda(X)$—we do not need to know the details for our application except that $L_{\nu'} X$ agrees with what we have been calling $X_\nu|_{q=1}$ in this $l = 2$ case. In general, $L_{\nu'} X$ is defined to be the quotient of $Y_{\nu'}$ by the (generally, not direct) sum over all $i \in [l-1]$ of

$$Y_{\triangleright^i \nu'} := Y_{(\nu'_1, \ldots, \nu'_{i-1})} \otimes Y_{\triangleright(\nu'_i, \nu'_{i+1})} \otimes Y_{(\nu'_{i+2}, \ldots, \nu'_l)}. \tag{14}$$

In the case that $X = \mathbb{Q}^{d_X}$, the Schur module $L_{\nu'} X$ is equal to the $U(\mathfrak{g}_X)$-irreducible $X_\nu|_{q=1}$.

The $\mathscr{O}(GL_q(\check{X}))$-comodule $\check{X}_\nu$ is defined in a similar way. Although the irreducible $\mathscr{O}(GL_q(\check{X}))$-comodules are in general much smaller than those of $\mathscr{O}(GL(X))$, the nonstandard exterior algebra $\check{\Lambda}(\check{X})$ specializes to $\Lambda(X)$ at $q = 1$. So, as above, let $\nu' \vdash_l r$ and define

$$\check{Y}_{\nu'} := \check{\Lambda}^{\nu'_1}\check{X} \otimes \check{\Lambda}^{\nu'_2}\check{X} \otimes \ldots \otimes \check{\Lambda}^{\nu'_l}\check{X}. \tag{15}$$

Next, we restrict to the two-row ($d_V = d_W = 2$) case, and define the submodule $\check{Y}_{\triangleright\nu'}$ of $\check{Y}_{\nu'}$ "by hand" for $l = \ell(\nu') = 2$ (the reader may now wish to take a look at Figures 4–11, where the $\check{Y}_{\triangleright\nu'}$ are defined). Then for $\nu' \vdash_l r$, $\check{Y}_{\triangleright^i\nu'}$ is defined just as in (14) and

$$\check{X}_\nu := \check{Y}_{\nu'} / \left(\sum_{i=1}^{l-1} \check{Y}_{\triangleright^i\nu'}\right).$$

This is a $\mathscr{O}(GL_q(\check{X}))$-comodule and therefore a $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module, and it specializes to $\mathrm{Res}_{U(\mathfrak{g}_V\oplus\mathfrak{g}_W)}(X_\nu|_{q=1})$ at $q = 1$, though some care is required to define the correct integral form of $\check{X}_\nu$ to make sense of this specialization.

We point out that it is possible to define a version of $\check{X}_\nu$ for general d_V, d_W and $l = 2$ (see §19.2). Extending this to $l > 2$ yields $\mathscr{O}(GL_q(\check{X}))$-comodules $\check{X}_\nu$ that in general have K-dimension less than $\dim_{\mathbb{Q}}(X_\nu|_{q=1})$. Similar difficulties are encountered in [1]. Berenstein and Zwicknagl [7, 63] investigate a quantum approach to the plethysms S^rV_λ, Λ^rV_λ and encounter similar difficulties.

1.7. **A global crystal basis for $\check{X}_\nu$.** Now we come to our new results in crystal basis theory and combinatorics. To define a basis of $\check{X}_\nu$, we first define (§14) a global crystal basis of $\check{\Lambda}^r\check{X}$, whose elements are labeled by what we call nonstandard columns of height-r (NSC^r). We then define a canonical basis of $\check{Y}_\alpha$ by putting the bases NSC$^{\alpha_i}$ together using Lusztig's construction for tensoring based modules [35, Theorem 27.3.2]. This basis is labeled by nonstandard tabloids (NST), which are just sequences of nonstandard columns. We show that the image of (a rescaled version of) a certain subset of NST(ν') in $\check{X}_\nu$ yields a well-defined basis +HNSTC(ν) of $\check{X}_\nu$, thus obtaining

Theorem 1.6. *The set +HNSTC(ν) is a global crystal basis of $\check{X}_\nu$ that solves the two-row Kronecker problem: the number of highest weight elements of +HNSTC(ν) of weight (λ, μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$.*

We comment here that for this part of the paper, we mostly work with $U_q^\tau := U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W) \rtimes \mathcal{S}_2$ modules instead of $\mathscr{O}(GL_q(\check{X}))$-comodules. We do not lose much and gain convenience by doing this because $\mathscr{O}_q^\tau$ is close to $\mathscr{O}(GL_q(\check{X}))$ in the two-row case and U_q^τ is Hopf dual to $\mathscr{O}_q^\tau$. Moreover, with slight modifications of the usual theory, we have a theory of based modules for U_q^τ. The $\mathscr{O}(GL_q(\check{X}))$-comodule $\check{X}_\nu$ is a U_q^τ-module, so in addition to obtaining a rule for two-row Kronecker coefficients, we also obtain a rule for what we call the symmetric and exterior two-row Kronecker coefficients—the *symmetric* (resp. *exterior*) *Kronecker coefficient* $g_{+\lambda\nu}$ (resp. $g_{-\lambda\nu}$) is the multiplicity of $M_\nu|_{q=1}$ in $S^2M_\lambda|_{q=1}$ (resp. $\Lambda^2M_\lambda|_{q=1}$). See Theorem 15.21, the stronger and more technical version Theorem 1.6, and (16), below, for this rule.

There are some subtleties that arise in the construction of +HNSTC(ν) and the proof of this theorem. In order to obtain the global crystal basis +HNSTC(ν), the rescaling of the NST(ν') must be chosen carefully. Each NST T of size r has a V-column (resp. W-column) reading word $\mathbf{k} \in \{1,2\}^r$ (resp. $\mathbf{l} \in \{1,2\}^r$). The word $\mathbf{k}$ (resp. $\mathbf{l}$) is naturally associated to the canonical basis element $c_{\mathbf{k}} \in B_V^r \subseteq V^{\otimes r}$ (resp. $c_{\mathbf{l}} \in B_W^r \subseteq W^{\otimes r}$). These basis elements are nicely depicted as a diagram of arcs according to the $U_q(\mathfrak{sl}_2)$ graphical calculus of [19]. We define the degree $\deg(T)$ of an NST T in terms of the diagrams of its V and W-column reading words (for the full definition, see Definition 15.1). The rescaled version of T is then $(-\frac{1}{[2]})^{\deg(T)}T$. Experts on global crystal bases may find this to be the most interesting part of the paper. Another difficulty (which is closely related to the need for rescaling) is that $\check{Y}_{\triangleright i_{\nu'}}$ is not easily expressed in terms of the basis NST(ν'). To remedy this we define a canonical basis for $\check{Y}_{\triangleright i_{\nu'}}$ and prove some general results about how tensoring based modules is compatible with projections.

In §17.1 we give a description of the crystal components of $(\check{X}_\nu, +\text{HNSTC}(\nu))$ in terms of arcs of the reading words of NST, which is independent of a +HNSTC in the component and the rescaled NST representing the +HNSTC. This graphical description of the crystal components helps us organize and count them. We show that the degree 0 crystal components (degree for NST gives rise to a well-defined notion of degree for crystal components) can be grouped into eight different one-parameter families depending on the heights of the columns that the arcs connect (see Figure 18), and counting crystal components easily reduces to the degree 0 case. This description helps us obtain explicit formulae for Kronecker coefficients. We also use it to write down explicitly (Theorem 17.7) all the structure coefficients for the action of the Chevalley generators on +HNSTC; we observe that these satisfy a certain positivity property.

Finally, in §18 we show that Theorem 1.6 actually produces a fairly simple positive formula for two-row Kronecker coefficients. For example, define the *symmetric* (resp. *exterior*) *Kronecker generating function*

$$g_{\varepsilon\nu}(x) := \sum_{\lambda \vdash_2 r} g_{\varepsilon\lambda\nu} x^{\lambda_1 - \lambda_2}, \quad \varepsilon = + \text{ (resp. } \varepsilon = -).$$

Here ν is any partition of r of length at most 4; let n_i be the number of columns of the diagram of ν of height i. For $k \in \mathbb{Z}$, define $[\![k]\!] = x^k + x^{k-2} + \cdots + x^{k'}$, where k' is 0 (resp. 1) if k is even (resp. odd); if $k < 0$, then $[\![k]\!] := 0$. The symmetric and exterior Kronecker generating functions are given by

$$g_{\varepsilon\nu}(x) = \begin{cases} [\![n_1]\!][\![n_2]\!][\![n_3]\!] & \text{if } \quad (-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-1]\!][\![n_3]\!]x & \text{if } -(-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1]\!][\![n_2-1]\!][\![n_3-1]\!]x & \text{if } -(-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-2]\!][\![n_3-1]\!]x^2 & \text{if } \quad (-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \end{cases} \tag{16}$$

where we have identified the values $+, -$ for ε with $+1, -1$. We also easily recover a nice formula for certain two-row Kronecker coefficients from [15] as well as the exact conditions for two-row Kronecker coefficients to vanish, from [13].

1.8. **Organization.** Sections 2–7 are preparatory. We fix conventions for the Hecke algebra $\mathscr{H}_r$ and its Kazhdan-Lusztig basis (§3) and for the quantized enveloping algebra $U_q(\mathfrak{g}_V)$ and the quantum coordinate algebra $\mathscr{O}(GL_q(V))$ (§4). Subsections 4.2–4.6 explain the quantum coordinate algebras $\mathscr{O}(M_q(V))$ and $\mathscr{O}(GL_q(V))$ in a way that prepares for the definitions of the corresponding nonstandard objects. We review (§5) global crystal bases from [26, 27], based modules and tensoring based modules from [35], and projected canonical bases from [9]. Section 6 contains more details about the upper canonical basis of $V^{\otimes r}$. In §6.3, we review the $U_q(\mathfrak{sl}_2)$ graphical calculus from [19].

The first part of new material in this paper (§8–13) defines the nonstandard objects and develops their representation theory: in §8–10 we define the nonstandard quantum group $GL_q(\check{X})$ and prove Theorem 1.4. We give explicit examples for $\mathscr{O}(M_q(\check{X}))$ (§8.4) and for nonstandard minors (§9.2). Then in §11 we define the nonstandard Hecke algebra and establish some of its basic properties and representation theory. The algebra $\check{\mathscr{H}}_3$ is treated in detail in §11.6–11.7. In §12 we prove the nonstandard analog of Schur-Weyl duality (Theorem 1.5) and go over the two-row, $r = 3$ example in detail. In §13 we discuss the approximations $\mathscr{O}_q^\tau$ and U_q^τ to $\mathscr{O}(GL_q(\check{X}))$ and give a complete description of the representation theory of the nonstandard Hecke algebra and quantum group in the two-row case.

The second part of the new material (§14–18) contains a proof of Theorem 1.6 and consequences of this theorem. The bulk of the proof, particularly the necessary canonical basis theory, is contained in §15, and the necessary combinatorics is worked out in §16–17. Section 17 develops the beginnings of a graphical calculus for the basis +HNSTC, and section 18 gives explicit formulae for Kronecker coefficients. Finally, §19 gives more details about the conjectural basis of $\check{X}^{\otimes r}$.

2. Basic concepts and notation

We introduce our basic notation and conventions for ground rings, tensor products, and type A combinatorics for the weight lattice, partitions, words, and tableaux. We also define cells in the general setting of modules with basis, rather than only for W-graphs, and recall some basic notions about comodules and Hopf algebras.

2.1. **General notation.** We work primarily over the ground rings $K = \mathbb{Q}(q)$, $\mathbb{C}$, and $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. Define K_0 (resp. K_∞) to be the subring of K consisting of rational functions with no pole at $q = 0$ (resp. $q = \infty$). For the parts of the paper involving Gelfand-Tsetlin bases and the quantum unitary group $\mathtt{U}_q(V)$, we work over the complex numbers $\mathbb{C}$ and in this context q is taken to be a real number not equal to $0, \pm 1$, rather than an indeterminate.

Let $\bar{\cdot}$ be the involution of K determined by $\overline{q} = q^{-1}$; it restricts to an involution of $\mathbf{A}$. For a nonnegative integer k, the $\bar{\cdot}$-invariant quantum integer is $[k] := \frac{q^k - q^{-k}}{q - q^{-1}} \in \mathbf{A}$ and the quantum factorial is $[k]! := [k][k-1] \dots [1]$. If $N_{\mathbf{A}}$ is an $\mathbf{A}$-module, then the $q = a$ *specialization* $N|_{q=a}$, $a \in \mathbb{Q}$, is defined to be $\mathbb{Q} \otimes_{\mathbf{A}} N_{\mathbf{A}}$, the map $\mathbf{A} \to \mathbb{Q}$ given by $q \mapsto a$; in a couple places we will also use this notation with $\mathbb{Z}[\frac{1}{2}]$ in place of $\mathbb{Q}$.

The notation $[k]$ also denotes the set $\{1, \dots, k\}$ in addition to the quantum integer, but these usages should be easy to distinguish from context. The notation Ω_r^n denotes the set of subsets of $[n]$ of size r.

Throughout the paper V, W, and $X = V \otimes W$ will denote vector spaces of dimensions d_V, d_W, d_X, respectively. These will be over the field K or $\mathbb{C}$. For an R-module X, set $X^* = \mathrm{Hom}_R(X, R)$. See §2.5 for important conventions about duals.

If R is a ring and B is a subset of an R-module N, then RB denotes the R-span of B.

Let (W, S) be a Coxeter group with length function ℓ and Bruhat order $<$. If $\ell(vw) = \ell(v) + \ell(w)$, then $vw = v \cdot w$ is a *reduced factorization.* The *right descent set* of $w \in W$ is $R(w) = \{s \in S : ws < w\}$. The type A_{r-1} Coxeter group is denoted $(\mathcal{S}_r, S)$, the symmetric group on r letters with simple reflections $S = \{s_1, \ldots, s_{r-1}\}$.

For any $J \subseteq S$, the *parabolic subgroup* W_J is the subgroup of W generated by J. Each right coset $W_J w$ contains a unique element of minimal length called a minimal coset representative. The set of all such elements is denoted ${}^J W$.

2.2. **Tensor products.** Since we will be working with complicated tensor products of many modules in this paper, we use three different symbols for tensor products depending on the context. The symbol $*$ is used for tensor products between an object and its dual, the symbol $\star$ for tensor products of objects involving V with objects involving W, and the symbol $\otimes$ for all other tensor products.

So, for instance, we write $V * V^*$ for $V \otimes V^*$ and $X = V \star W$ for $V \otimes W$. We will come across expressions like

$$(X * X^*)^{\otimes r} \cong X^{\otimes r} * (X^*)^{\otimes r} \cong (V * V^*)^{\otimes r} \star (W * W^*)^{\otimes r} \cong (U^V)^{\otimes r} \star (U^W)^{\otimes r},$$

where $U^V = V * V^*$, $U^W = W * W^*$. This will make it more clear where different elements lie in expressions like

$$z_{\rho(i,j)}^{\rho(k,l)} = y_{\rho(i,j)} * y^{\rho(k,l)} = x_{\substack{i\\j}} * x^{\substack{k\\l}} = u_i^k \star u_j^l.$$

2.3. **Words and tableaux.** In this paper we work almost entirely in type A. The *weight lattice* $\mathbf{X}(\mathfrak{g}_V)$ of the Lie algebra $\mathfrak{g}_V := \mathfrak{gl}(V)$ is $\mathbb{Z}^{d_V}$ with standard basis $\epsilon_1, \ldots, \epsilon_{d_V}$. Its dual, $\mathbf{X}(\mathfrak{g}_V)^*$, has basis $\epsilon^1, \ldots, \epsilon^{d_V}$, dual to the standard. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}, i \in [d_V - 1]$.

We write $\lambda \vdash_l r$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of size $r = |\lambda| := \sum_{i=1}^{l} \lambda_i$. A partition $\lambda \vdash_{d_V} r$ is identified with the weight $\lambda_1\epsilon_1 + \cdots + \lambda_{d_V}\epsilon_{d_V} \in \mathbf{X}(\mathfrak{g}_V)$. We also write $\lambda = [n_l, \ldots, n_1]$ as an alternative notation for the partition $(n_l + \cdots + n_1, n_l + \cdots + n_2, \ldots, n_l)$; note that n_i is the number of columns of the diagram of λ of height i. Let $\mathscr{P}_r$ denote the set of partitions of size r and $\mathscr{P}_{r,l}$ the set of partitions of size r with at most l parts; let $\mathscr{P}'_r$ (resp. $\mathscr{P}'_{r,l}$) be the subset of $\mathscr{P}_r$ (resp. $\mathscr{P}_{r,l}$) consisting of those partitions that are not a single row or column shape.

The partial order $\trianglelefteq, \triangleleft$ on $\mathbf{X}(\mathfrak{g}_V)$ is defined by $\lambda \trianglelefteq \mu$ if $\mu - \lambda$ is a nonnegative sum of simple roots. In the case $\lambda, \mu \vdash r$, this corresponds to the usual dominance order on partitions. The conjugate partition λ' of a partition λ is the partition whose diagram is the transpose of the diagram of λ.

We let $\alpha \vDash_l^{d_X} r$ denote a composition $\alpha = (\alpha_1, \dots, \alpha_l)$ of r with $\alpha_i \in [d_X]$. For $\zeta = (\zeta_1, \dots, \zeta_l)$ a weak composition of r (i.e. $\zeta_i \geq 0$), let B_j be the interval $[\sum_{i=1}^{j-1} \zeta_i + 1, \sum_{i=1}^{j} \zeta_i]$, $j \in [l]$. Define $J_\zeta = \{s_i : i, i+1 \in B_j \text{ for some } j\}$ so that $(\mathcal{S}_r)_{J_\zeta} \cong \mathcal{S}_{\zeta_1} \times \cdots \times \mathcal{S}_{\zeta_l}$.

Let $\mathbf{k} = k_1 k_2 \dots k_r \in [d_V]^r$ be a word of length r in the alphabet $[d_V]$. The *content* of $\mathbf{k}$ is the tuple $(\zeta_1, \dots, \zeta_{d_V})$ whose i-th entry ζ_i is the number of i's in $\mathbf{k}$. The symmetric group $\mathcal{S}_r$ acts on $[d_V]^r$ on the right by $\mathbf{k} s_i = k_1 \dots k_{i-1}\, k_{i+1}\, k_i\, k_{i+2} \dots k_r$. Define $\text{sort}(\mathbf{k})$ to be the tuple obtained by rearranging the k_j in weakly increasing order. For a word $\mathbf{k}$ of content ζ, define $d(\mathbf{k})$ to be the element w of ${}^{J_\zeta}\mathcal{S}_r$ such that $\text{sort}(\mathbf{k}) w = \mathbf{k}$.

The set of standard Young tableaux is denoted SYT and the subset of SYT of shape λ is denoted $\text{SYT}(\lambda)$. The set of semistandard Young tableaux of size r with entries in $[l]$ is denoted SSYT_l^r and the subset of SSYT_l^r of shape $\lambda \vdash r$ is $\text{SSYT}_l(\lambda)$. Tableaux are drawn in English notation, so that entries of an SSYT strictly increase from north to south along columns and weakly increase from west to east along rows. For a tableau T, $|T|$ is the number of squares in T and $\text{sh}(T)$ its shape. The *content* of a tableau T is the content of any word with insertion tableau T.

We let $P(\mathbf{k}), Q(\mathbf{k})$ denote the insertion and recording tableaux produced by the Robinson-Schensted-Knuth (RSK) algorithm applied to the word $\mathbf{k}$. We abbreviate $\text{sh}(P(\mathbf{k}))$ simply by $\text{sh}(\mathbf{k})$. Let Z_λ be the superstandard tableau of shape and content λ—the tableau whose i-th row is filled with i's. Let Z_λ^* be the SYT of shape λ with $1, \dots, \lambda_1$ in the first row, $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ in the second row, etc. The notation Q^T denotes the transpose of an SYT Q, so that $\text{sh}(Q^T) = \text{sh}(Q)'$.

For an SYT Q, let $\ell(Q)$ denote the distance between Q and Z_λ^* in the dual Knuth equivalence graph on $\text{SYT}(\lambda)$ (for a definition of this graph, see [5]). It is not hard to show that for any $P \in \text{SYT}(\lambda)$, $\ell(Q) \equiv \ell(w) - \ell(z) \mod 2$, where $w = \text{RSK}^{-1}(P, Q)$, $z = \text{RSK}^{-1}(P, Z_\lambda^*)$.

2.4. **Cells.** We define cells in the general setting of modules with basis, following [9]. Let H be an R-algebra for some commutative ring R. Let M be a left H-module and Γ an R-basis of M. The preorder $\leq_\Gamma$ (also denoted $\leq_M$) on the vertex set Γ is generated by the relations

$$\delta \preceq_\Gamma \gamma \quad \begin{array}{c}\text{if there is an } h \in H \text{ such that } \delta \text{ appears with nonzero} \\ \text{coefficient in the expansion of } h\gamma \text{ in the basis } \Gamma.\end{array} \tag{17}$$

Equivalence classes of $\leq_\Gamma$ are the *left cells* of (M, Γ). The preorder $\leq_M$ induces a partial order on the left cells of M, which is also denoted $\leq_M$.

A *cellular submodule* of (M, Γ) is a submodule of M that is spanned by a subset of Γ (and is necessarily a union of left cells). A *cellular quotient* of (M, Γ) is a quotient of M by a cellular submodule, and a *cellular subquotient* of (M, Γ) is a cellular quotient of a cellular submodule. We denote a cellular subquotient $R\Gamma'/R\Gamma''$ by $R\Lambda$, where $\Gamma'' \subseteq \Gamma' \subseteq \Gamma$ span cellular submodules and $\Lambda = \Gamma' \setminus \Gamma''$. We say that the left cells Λ and Λ' are isomorphic if $(R\Lambda, \Lambda)$ and $(R\Lambda', \Lambda')$ are isomorphic as modules with basis.

Sometimes we speak of the left cells of M, cellular submodules of M, etc. or left cells of Γ, cellular submodules of Γ, etc. if the pair (M, Γ) is clear from context. For a right

H-module M, the *right cells*, *cellular submodules*, etc. of M are defined similarly with γh in place of $h\gamma$ in (17). We also use the terminology H-cells, H-cellular submodules, etc. to make it clear that the algebra H is acting, and we omit left and right when they are clear.

If (M, Γ) is as above and $M \cong \bigoplus_{i \in I} M_i$ is a decomposition of M as a direct sum of H-modules, then we say that (M, Γ) is *compatible* with the decomposition if every cellular submodule of M is of the form $\bigoplus_{i \in J} M_i$ for some $J \subseteq I$.

2.5. **Comodules.** In the next two subsections, we fix some notation regarding comodules and dual pairings of Hopf algebras, mostly following [30, Chapters 1,11]; this reference contains a good introduction to these generalities.

Let $\mathscr{A}$ be a coalgebra over a field K, N a K-vector space with basis $e_1, \ldots, e_n$, and $\varphi : N \to A \otimes N$ the left corepresentation of A on N given by

$$\varphi(e_j) = \sum_{k=1}^{n} m_j^k \otimes e_k. \tag{18}$$

The matrix $(m_j^k)_{j,k \in [n]}$ is called the *coefficient matrix* of φ (or of N) with respect to the basis $e_1, \ldots, e_n$ and its entries $\{m_j^k : j, k \in [n]\}$ are the *matrix coefficients* of φ (or of N) with respect to $e_1, \ldots, e_n$. Coefficient matrices and matrix coefficients are defined similarly for right corepresentations.

Remark 2.1. In this paper we identify the endomorphism algebra $\mathrm{End}(V)$ with $V^* \otimes V = V^* * V$ and the coalgebra dual to $\mathrm{End}(V)$ with $V * V^*$. We adopt the convention that dual objects take upper indices and ordinary objects take lower indices. Thus for algebras, upper indices correspond to rows and lower indices to columns, and for coalgebras, lower indices correspond to rows and upper indices to columns. Dual objects will typically correspond to right corepresentations and ordinary objects to left corepresentations. In §4.2–4.6, §8–10, and Appendices A and B, we will work with left and right corepresentations, and we are careful to distinguish between the two. For the remainder of the paper, such care is not necessary and we will typically work with left modules and right comodules, but will write V_λ in place of V_λ^*, X in place of X^*, etc. to avoid extra symbols.

Corresponding to the left corepresentation φ above, there is a right corepresentation $(\varphi)_R : N^* \to N^* \otimes A$ of A on $N^* = \mathrm{Hom}_K(N, K)$ given by

$$(\varphi)_R(e^k) = \sum_{j=1}^{n} e^j \otimes m_j^k, \tag{19}$$

where $e^1, \ldots, e^n$ is the basis dual to $e_1, \ldots, e_n$. We will write $(N)_R$ for the right comodule on N^* corresponding to φ_R. This construction is independent of the basis $\{e_i\}$. The comodules N and $(N)_R$ share the same coefficient matrix with respect to any basis of N and corresponding dual basis of $(N)_R$.

Similarly, given a K-vector space N' with basis $e^1, \ldots, e^n$, and $\varphi' : N' \to N' \otimes A$ the right corepresentation of A on N' given by

$$\varphi'(e^k) = \sum_{j=1}^{n} e^j \otimes m_j^k, \tag{20}$$

there is a left corepresentation of A on N'^* given by

$$(\varphi')_L(e_j) = \sum_{k=1}^{n} m_j^k \otimes e_k, \tag{21}$$

where $e_1, \ldots, e_n$ is the basis of N'^* dual to $e^1, \ldots, e^n$. The corresponding left comodule on N'^* is denoted $(N')_L$. The coefficient matrix (m_j^k) of N' with respect to $e^1, \ldots, e^n$ is the same as the coefficient matrix of $(N')_L$ with respect to $e_1, \ldots, e_n$.

2.6. **Dually paired Hopf algebras.** Given two K-bialgebras $\mathcal{U}$ and $\mathscr{A}$, a bilinear map $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathscr{A} \to K$ is a *dual pairing of bialgebras* if

$$\begin{aligned} \langle \Delta_{\mathcal{U}}(f), a_1 \otimes a_2 \rangle &= \langle f, a_1 a_2 \rangle, \quad \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta_{\mathscr{A}}(a) \rangle, \\ \langle f, 1_{\mathscr{A}} \rangle &= \epsilon_{\mathcal{U}}(f), \qquad\quad \langle 1_{\mathcal{U}}, a \rangle = \epsilon_{\mathscr{A}}(a) \end{aligned}$$

for all $f, f_1, f_2 \in \mathcal{U}$ and $a, a_1, a_2 \in \mathscr{A}$. If $\mathcal{U}$ and $\mathscr{A}$ are Hopf algebras, then compatibility with the antipode is automatic [30, Proposition 9, Chapter 1]:

$$\langle S_{\mathcal{U}}(f), a \rangle = \langle f, S_{\mathscr{A}}(a) \rangle, \quad f \in \mathcal{U},\ a \in \mathscr{A},$$

and in this case $\langle \cdot, \cdot \rangle$ is a *dual pairing of Hopf algebras.*

If $\mathcal{U}$ and $\mathscr{A}$ are dually paired bialgebras, then to any right $\mathscr{A}$-corepresentation $\varphi : N \to N \otimes \mathscr{A}$, there corresponds a left $\mathcal{U}$-representation $\hat{\varphi} : \mathcal{U} \to \mathrm{End}(N)$ given by

$$\hat{\varphi}(f)x = ((id \otimes f) \circ \varphi)(x) = \sum x_{(0)} \langle f, x_{(1)} \rangle, \quad f \in \mathcal{U},\ x \in N,$$

where $\varphi(x) = \sum x_{(0)} \otimes x_{(1)}$ expresses φ in Sweedler notation. Note however that for dually paired bialgebras, a representation of one does not, in general, come from a corepresentation of the other, even if they are Hopf algebras and the pairing is nondegenerate.

3. Hecke algebras and canonical bases

The *Hecke algebra* $\mathscr{H}(W)$ of (W, S) is the free $\mathbf{A}$-module with standard basis $\{T_w : w \in W\}$ and relations generated by

$$\begin{array}{ll} T_v T_w = T_{vw} & \text{if } vw = v \cdot w \text{ is a reduced factorization,} \\ (T_s - q)(T_s + q^{-1}) = 0 & \text{if } s \in S. \end{array} \tag{22}$$

We remark that the q here is frequently $q^{1/2}$ in the literature on Hecke algebras, as it is, for instance, in [28]. We have chosen this convention so that in quantum Schur-Weyl duality, the Hecke algebra q matches the usual notation (as in [26, 27, 30]) for the quantum group q.

For each $J \subseteq S$, $\mathscr{H}(W)_J$ denotes the subalgebra of $\mathscr{H}(W)$ with $\mathbf{A}$-basis $\{T_w : w \in W_J\}$, which is isomorphic to $\mathscr{H}(W_J)$.

In this section we recall the definition of the Kazhdan-Lusztig basis elements C_w of [28] and some of their basic properties. Then we specialize to type A and review the beautiful connection between cells and the RSK algorithm.

3.1. **The upper canonical basis of $\mathscr{H}(W)$.** The *bar-involution*, $\bar{\cdot}$, of $\mathscr{H}(W)$ is the additive map from $\mathscr{H}(W)$ to itself extending the $\bar{\cdot}$-involution of $\mathbf{A}$ and satisfying $\overline{T_w} = T_{w^{-1}}^{-1}$. Observe that $\overline{T_s} = T_s^{-1} = T_s + q^{-1} - q$ for $s \in S$. Some simple $\bar{\cdot}$-invariant elements of $\mathscr{H}(W)$ are $C'_{\mathrm{id}} := T_{\mathrm{id}}$, $C_s := T_s - q = T_s^{-1} - q^{-1}$, and $C'_s := T_s + q^{-1} = T_s^{-1} + q$, $s \in S$.

Define the lattice $(\mathscr{H}_r)_{\mathbb{Z}[q]} := \mathbb{Z}[q]\{T_w : w \in W\}$ of $\mathscr{H}_r$.

For each $w \in W$, there is a unique element $C_w \in \mathscr{H}(W)$ such that $\overline{C_w} = C_w$ and C_w is congruent to $T_w \mod q(\mathscr{H}_r)_{\mathbb{Z}[q]}$. (23)

The $\mathbf{A}$-basis $\Gamma_W := \{C_w : w \in W\}$ is the *upper canonical basis* of $\mathscr{H}(W)$ (we use this language to be consistent with that for crystal bases).

The coefficients of the upper canonical basis in terms of the standard basis are essentially the *Kazhdan-Lusztig polynomials* $P_{x,w}$:

$$C_w = \sum_{x \in W} P^-_{x,w} T_x. \tag{24}$$

The $P^-_{x,w}$ are related to the $P_{x,w}$ defined in [28] by $P^-_{x,w}(q) = \iota(q_{\mathrm{KL}}^{(\ell(x)-\ell(w))/2} P_{x,w}(q_{\mathrm{KL}}))$, where ι is the involution of $\mathbf{A}$ defined by $\iota(q) = -q^{-1}$ and q_{KL} is the q used in [28], related to ours by $q_{\mathrm{KL}}^{1/2} = q$. Now let $\mu(x, w) \in \mathbb{Z}$ be the coefficient of q^{-1} in $\iota(P^-_{x,w})$ (resp. $\iota(P^-_{w,x})$) if $x \leq w$ (resp. $w \leq x$). Then the right regular representation in terms of the upper canonical basis of $\mathscr{H}_r$ takes the following simple form:

$$C_w C_s = \begin{cases} -[2] C_w & \text{if } s \in R(w), \\ \displaystyle\sum_{\{w' \in W : s \in R(w')\}} \mu(w', w) C_{w'} & \text{if } s \notin R(w). \end{cases} \tag{25}$$

The simplicity and sparsity of this action along with the fact that the right cells of Γ_W often give rise to $\mathbb{C}(q) \otimes_{\mathbf{A}} \mathscr{H}(W)$-irreducibles are among the most amazing and useful properties of canonical bases.

3.2. **Cells in type A.** Let $\mathscr{H}_r = \mathscr{H}(\mathcal{S}_r)$ be the type A Hecke algebra. It is well known that $K\mathscr{H}_r := K \otimes_{\mathbf{A}} \mathscr{H}_r$ is semisimple and its irreducibles in bijection with partitions of r; let M_λ and $M_\lambda^{\mathbf{A}}$ be the $K\mathscr{H}_r$-irreducible and Specht module of $\mathscr{H}_r$ of shape $\lambda \vdash r$ (hence $M_\lambda \cong K \otimes_{\mathbf{A}} M_\lambda^{\mathbf{A}}$).

The work of Kazhdan and Lusztig [28] shows that the decomposition of $\Gamma_{\mathcal{S}_r}$ into right cells is $\Gamma_{\mathcal{S}_r} = \bigsqcup_{\lambda \vdash r,\, P \in \mathrm{SYT}(\lambda)} \Gamma_P$, where $\Gamma_P := \{C_w : P(w) = P\}$. Moreover, the right cells $\{\Gamma_P : \mathrm{sh}(P) = \lambda\}$ are all isomorphic, and, denoting any of these cells by Γ_λ, $\mathbf{A}\Gamma_\lambda \cong M_\lambda^{\mathbf{A}}$. A combinatorial discussion of left cells in type A is given in [10, §4].

We refer to the basis Γ_λ of $M_\lambda^{\mathbf{A}}$ as the *upper canonical basis of* M_λ and denote it by $\{C_Q : Q \in \mathrm{SYT}(\lambda)\}$, where C_Q corresponds to C_w for any (every) $w \in \mathcal{S}_r$ with recording tableau Q. Note that with these labels the action of C_s on the upper canonical basis of

M_λ is similar to (25), with $\mu(Q', Q) := \mu(w', w)$ for any w', w such that $P(w') = P(w)$, $Q' = Q(w'), Q = Q(w)$, and right descent sets

$$R(C_Q) = \{s_i : i + 1 \text{ is strictly to the south of } i \text{ in } Q\}. \tag{26}$$

Example 3.1. The integers $\mu(Q', Q)$ for the upper canonical basis of $M_{(3,1)}$ are given by the following graph (μ is 1 if the edge is present and 0 otherwise), and descent sets are shown below each tableau.

$$\begin{array}{ccccc}
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} & \text{———} & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} & \text{———} & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array} \\
\{s_3\} & & \{s_2\} & & \{s_1\}
\end{array}$$

4. The quantum group $GL_q(V)$

The quantum group $GL_q(V)$ is a virtual object associated to two Hopf algebras—the Drinfel′d-Jimbo quantized enveloping algebra $U_q(\mathfrak{g}_V)$ and the quantum coordinate algebra $\mathscr{O}(GL_q(V))$. These are dually paired Hopf algebras, and this connects the corepresentation theory of $\mathscr{O}(GL_q(V))$ to the representation theory of $U_q(\mathfrak{g}_V)$. In this section we recall the definition of $U_q(\mathfrak{g}_V)$, following [26, 24], and of $\mathscr{O}(GL_q(V))$, following [30, 54]. Our treatment of $\mathscr{O}(M_q(V))$ and $\mathscr{O}(GL_q(V))$ here will prepare us for the construction of the corresponding nonstandard objects in §8–10. In §4.7, we fix notation regarding representations of $GL_q(V)$.

4.1. **The quantized enveloping algebra $U_q(\mathfrak{g}_V)$.** The *quantized universal enveloping algebra* $U_q(\mathfrak{g}_V)$ is the associative K-algebra generated by $q^h, h \in \mathbf{X}(\mathfrak{g}_V)^*$ (set $K_i = q^{\epsilon^i - \epsilon^{i+1}}$) and $E_i, F_i, i \in [d_V - 1]$ with relations

$$\begin{array}{ll}
q^0 = 1, & q^h q^{h'} = q^{h+h'}, \\
q^h E_i q^{-h} = q^{\langle \alpha_i, h\rangle} E_i, & q^h F_i q^{-h} = q^{-\langle \alpha_i, h\rangle} F_i, \\
E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, & \\
E_i E_j - E_j E_i = F_i F_j - F_j F_i = 0 & \text{for } |i-j| > 1, \\
E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 & \text{for } |i-j| = 1, \\
F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 & \text{for } |i-j| = 1.
\end{array} \tag{27}$$

The *bar-involution*, $\bar{\cdot} : U_q(\mathfrak{g}_V) \to U_q(\mathfrak{g}_V)$, is the $\mathbb{Q}$-linear automorphism extending the involution $\bar{\cdot}$ on K and satisfying

$$\overline{q^h} = q^{-h}, \ \overline{E}_i = E_i, \ \overline{F}_i = F_i. \tag{28}$$

Let $\varphi : U_q(\mathfrak{g}_V) \to U_q(\mathfrak{g}_V)$ be the algebra antiautomorphism determined by

$$\varphi(E_i) = F_i, \quad \varphi(F_i) = E_i, \quad \varphi(K_i) = K_i.$$

The algebra $U_q(\mathfrak{g}_V)$ is a Hopf algebra with coproduct Δ given by

$$\Delta(q^h) = q^h \otimes q^h, \ \ \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \ \ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \tag{29}$$

This is the same as the coproduct used in [16, 27, 24]; it differs from the coproduct of [35] by $(\varphi \otimes \varphi) \circ \Delta \circ \varphi$ and from that of [30] by $(\bar{\cdot} \otimes \bar{\cdot}) \circ \Delta \circ \bar{\cdot}$.

4.2. **FRT-algebras.** The quantum coordinate algebra $\mathscr{O}(M_q(V))$ and nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ will be defined in the generality of FRT-algebras [54] (see also [30, Chapter 9]).

Let V be a K-vector space of dimension d_V, with standard basis $v_1, \ldots, v_{d_V}$. Let $U = V * V^*$ be the K-vector space with standard basis $\{u_i^j = v_i * v^j : i, j \in [d_V]\}$, where $v^1, \ldots, v^{d_V}$ is the basis of V^* dual to $v_1, \ldots, v_{d_V}$. We view U as the coalgebra dual to the endomorphism algebra $\mathrm{End}(V)$. In terms of the standard basis, the comultiplication and counit are given by

$$\Delta(u_i^j) = \sum_k u_i^k \otimes u_k^j, \quad \epsilon(u_i^j) = \delta_{ij},$$

or in matrix form

$$\Delta(\mathbf{u}) = \mathbf{u} \dot{\otimes} \mathbf{u}, \ \epsilon(\mathbf{u}) = \mathbf{I}, \tag{30}$$

where $\mathbf{u}$ is the $d_V \times d_V$ matrix (u_i^j) with entries in U and $\dot{\otimes}$ denotes matrix multiplication with tensor product in place of scalar multiplication. For coalgebras we adopt the convention that upper indices correspond to columns and lower indices to rows (see Remark 2.1).

The tensor algebra $K\langle u_i^j \rangle = T(U) = \bigoplus_{r \geq 0} U^{\otimes r}$ is a K-bialgebra with comultiplication and counit extending those of U in the unique way that makes them into algebra homomorphisms.

Let $\mathscr{R} = \mathscr{R}_{V,V} \in M_{d_V{}^2}(K)$ be a nonsingular matrix, identified with an element of $\mathrm{End}(V^{\otimes 2})$ via the standard basis, and let $\hat{\mathscr{R}}_{V,V} = \hat{\mathscr{R}} = \tau \circ \mathscr{R}$, where τ is the flip of $V \otimes V$. The *FRT-algebra* $A(\mathscr{R})$ [54] is the quotient algebra $K\langle u_i^j \rangle / \mathcal{I}_{\mathscr{R}}$, where $\mathcal{I}_{\mathscr{R}}$ is the two-sided ideal generated by certain degree two relations, which, in matrix form, are

$$\hat{\mathscr{R}}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})\hat{\mathscr{R}}, \tag{31}$$

where $\mathbf{u} \otimes \mathbf{u}$ is the $d_V{}^2 \times d_V{}^2$ matrix with the entry $u_i^j \otimes u_k^l$ in the ik-th row and jl-th column. This is to be interpreted as an equality of elements of $M_{d_V{}^2}(T(U))$, i.e., $d_V{}^4$ many equations, each requiring some linear combination of elements of $U^{\otimes 2}$ to be equal to another linear combination of elements of $U^{\otimes 2}$. For an explicit form of these relations in the case $\hat{\mathscr{R}}$ is given by (32), see (43) below. By [30, Proposition 9.1], $\mathcal{I}_{\mathscr{R}}$ is a coideal of $T(U)$, hence $A(\mathscr{R})$ is a bialgebra with coproduct and counit given by (30).

4.3. **The quantum coordinate algebra $\mathscr{O}(M_q(V))$.** The *quantum coordinate algebra* $\mathscr{O}(M_q(V))$ of the *standard quantum matrix space* $M_q(V)$ is the FRT-bialgebra corresponding to the $\hat{\mathscr{R}}$ given by

$$\hat{\mathscr{R}}_{V,V} = \sum_{i<j} (q - q^{-1}) v^{ij} * v_{ij} + \sum_{i \neq j} v^{ij} * v_{ji} + q \sum_i v^{ii} * v_{ii}, \tag{32}$$

where $v_{ij} = v_i \otimes v_j$, $v^{ij} = v^i \otimes v^j$. It is known that $\hat{\mathscr{R}}$ satisfies the quadratic equation

$$(\hat{\mathscr{R}} - qI)(\hat{\mathscr{R}} + q^{-1}I) = 0, \tag{33}$$

and has the spectral decomposition

$$\hat{\mathscr{R}} = qP_+ - q^{-1}P_-, \tag{34}$$

where the projections $P_+ = P_+^V$ and $P_- = P_-^V$ are

$$P_+ = \frac{1}{[2]}(\hat{\mathscr{R}} + q^{-1}I), \quad P_- = \frac{1}{[2]}(-\hat{\mathscr{R}} + qI), \tag{35}$$

so that $I = P_+ + P_-$ is the spectral decomposition of the identity.

These projections are quantum analogs of the symmetrization and antisymmetrization operators on $V^{\otimes 2}$. Specifically, let the *symmetric subspace* $S_q^2V := (V \otimes V)P_+$ be the image of P_+, and let the *antisymmetric subspace* $\Lambda_q^2 V := (V \otimes V)P_-$ be the image of P_-.

The *quantum symmetric algebra of* V, denoted $S_q(V)$, is the quotient algebra of $T(V)$ by the two-sided ideal generated by $\Lambda_q^2 V$. Explicitly, this is the algebra over the v_i's subject to the relations

$$v_j v_i = q^{-1} v_i v_j, \quad i < j. \tag{36}$$

These relations can also be put in matrix form, but we have found the above two descriptions more convenient. The *quantum exterior algebra of* V, denoted $\Lambda_q(V)$, is the quotient algebra of $T(V)$ by the two-sided ideal generated by $S_q^2 V$. Explicitly, this is the algebra over the v_i's subject to the relations

$$v_i^2 = 0, \text{ and } v_j v_i = -q v_i v_j, \quad i < j. \tag{37}$$

Let $S_q^r V$ and $\Lambda_q^r V$ be the degree r-components of $S_q(V)$ and $\Lambda_q(V)$, respectively.

We think of $S_q(V)$ as the coordinate algebra of a virtual symmetric quantum space V_{sym} with commuting coordinates (in the quantum sense), and $\Lambda_q(V)$ as the coordinate algebra of a virtual antisymmetric quantum space $V_\wedge$ with anti-commuting coordinates (in the quantum sense).

We can now give some other descriptions of $\mathscr{O}(M_q(V))$, which we have found to be more convenient than the matrix form (31). Both $S_q(V)$ and $\Lambda_q(V)$ are left $\mathscr{O}(M_q(V))$-comodule algebras via $v_i \mapsto \sum_j u_i^j \otimes v_j$ and $S_q(V^*)$ and $\Lambda_q(V^*)$ are right $\mathscr{O}(M_q(V))$-comodule algebras via $v^j \mapsto \sum_i v^i \otimes u_i^j$. In fact, it can be shown that $\mathscr{O}(M_q(V))$ is the largest bialgebra quotient of $T(U)$ such that $S_q^2 V$ is a left $\mathscr{O}(M_q(V))$-comodule and $S_q^2 V^*$ is a right $\mathscr{O}(M_q(V))$-comodule. Similarly, $\mathscr{O}(M_q(V))$ is the largest bialgebra quotient of $T(U)$ such that $\Lambda_q^2 V$ is a left $\mathscr{O}(M_q(V))$-comodule and $\Lambda_q^2 V^*$ is a right $\mathscr{O}(M_q(V))$-comodule. This view of the standard quantum group, emphasized by Manin [37], carries over nicely to the nonstandard setting.

Note that, by (35), the defining relations (31) of $\mathscr{O}(M_q(V))$ are equivalent to either of

$$P_+(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})P_+, \tag{38}$$

$$P_-(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})P_-. \tag{39}$$

It can also be shown that the two-sided ideal $\mathcal{I}_{\mathscr{R}}$ is that generated by

$$(V \otimes V * V^* \otimes V^*)(P_+^V * P_-^{V^*} + P_-^V * P_+^{V^*}) = S_q^2 V * \Lambda_q^2 V^* \oplus \Lambda_q^2 V * S_q^2 V^*, \tag{40}$$

where $P_\pm^{V^*}$ are defined the same as $P_\pm^V$, with the basis $v^1, \ldots, v^{d_V}$ in place of $v_1, \ldots, v_{d_V}$.

4.4. **The quantum determinant and the Hopf algebra $\mathscr{O}(GL_q(V))$.** Let

$$\varphi_r^R : \check{\Lambda}^r \check{V}^* \to \check{\Lambda}^r \check{V}^* \otimes \mathscr{O}(M_q(V)),$$
$$\varphi_r^L : \check{\Lambda}^r \check{V} \to \mathscr{O}(M_q(V)) \otimes \check{\Lambda}^r \check{V}$$

be the right and left corepresentations corresponding to the right $\mathscr{O}(M_q(V))$-comodule $\check{\Lambda}^r \check{V}^*$ and left $\mathscr{O}(M_q(V))$-comodule $\check{\Lambda}^r \check{V}$.

Recall that $\Omega_r^{d_V}$ denotes the set of subsets of $[d_V]$ of size r. For a subset $I \in \Omega_r^{d_V}$, with $I = \{i_1, \dots, i_r\}$, $i_1 < i_2 < \cdots$, let $v_I = v_{i_1} \cdots v_{i_r} \in \Lambda_q^r V$ and $v^I = v^{i_1} \cdots v^{i_r} \in \Lambda_q^r V^*$. The *standard monomial basis* of $\Lambda_q^r V$ (resp. $\Lambda_q^r V^*$) is $\{v_I : I \in \Omega_r^{d_V}\}$ (resp. $\{v^I : I \in \Omega_r^{d_V}\}$). It is known that the isomorphism $\Lambda_q^r V^* \cong (\Lambda_q^r V)_R$ of right $\mathscr{O}(M_q(V))$-comodules identifies the standard monomial basis of $\Lambda_q^r V^*$ with the basis dual to the standard monomial basis of $(\Lambda_q^r V)_R$ (here, $(\cdot)_R$ is the notation for dualizing comodules explained in §2.5). The *right quantum r-minors* $D_J^{I,R}(V)$ of $\mathscr{O}(M_q(V))$ are defined to be the matrix coefficients of the right corepresentation φ_r^R in the standard monomial basis. Explicitly, they are defined by

$$\varphi_r^R(v^I) = \sum_{J \in \Omega_r^{d_V}} v^J \otimes D_J^{I,R}(V), \;\; I \in \Omega_r^{d_V}.$$

The *left quantum r-minors* $D_J^{I,L}(V)$ are defined by

$$\varphi_r^L(v_J) = \sum_{I \in \Omega_r^{d_V}} D_J^{I,L}(V) \otimes v_I, \;\; J \in \Omega_r^{d_V}.$$

It is known that

$$D_J^I = D_J^{I,R}(V) = D_J^{I,L}(V) = \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)} u_{j_1}^{i_{\sigma(1)}} \cdots u_{j_r}^{i_{\sigma(r)}}, \tag{41}$$

where $\ell(\sigma)$ is the number of inversions of the permutation σ. The quantum determinant $D_q = D_q(V)$ of $\mathscr{O}(M_q(V))$ is defined to be D_J^I, with $I = J = [d_V]$. Explicitly,

$$D_q = \sum_{\sigma \in \mathcal{S}_{d_V}} (-q)^{\ell(\sigma)} u_1^{\sigma(1)} \cdots u_{d_V}^{\sigma(d_V)}. \tag{42}$$

The coordinate algebra $\mathscr{O}(GL_q(V))$ of the quantum group $GL_q(V)$ is obtained by adjoining the inverse D_q^{-1} to $\mathscr{O}(M_q(V))$. By applying the corepresentation maps to the nondegenerate pairings

$$\Lambda_q^{d_V - 1} V^* \otimes \Lambda_q^1 V^* \to \Lambda_q^{d_V} V^*,$$
$$\Lambda_q^1 V \otimes \Lambda_q^{d_V - 1} V \to \Lambda_q^{d_V} V,$$

it can be shown that the cofactor matrix $\tilde{\mathbf{u}}$ with entries $\tilde{u}_k^i := (-q)^{k-i} D_{\hat{i}}^{\hat{k}}$, where $\hat{k} := [d_V] \setminus \{k\}$, satisfies

$$\tilde{\mathbf{u}}\mathbf{u} = \mathbf{u}\tilde{\mathbf{u}} = D_q \mathbf{I}.$$

Then we can formally define $\mathbf{u}^{-1} = D_q^{-1} \tilde{\mathbf{u}}$. This gives the following Hopf structure on $\mathscr{O}_q(GL_q(V))$:

(1) $\Delta(\mathbf{u}) = \mathbf{u} \dot{\otimes} \mathbf{u}$, $\Delta(D_q^{-1}) = D_q^{-1} \otimes D_q^{-1}$.

(2) $\epsilon(\mathbf{u}) = \mathbf{I}$.
(3) $S(u^i_j) = \tilde{u}^i_j D_q^{-1}$, $S(D_q^{-1}) = D_q$, where u^i_j are the entries of $\mathbf{u}$ and $\tilde{u}^i_j$ are the entries of $\tilde{\mathbf{u}}$.

4.5. **A reduction system for $\mathscr{O}(M_q(V))$.** The Poincaré series of $\mathscr{O}(M_q(V))$ coincides with the Poincaré series of the commutative algebra $\mathbb{C}[u^j_i]$. Because, just as in the classical case, $\mathscr{O}(M_q(V))$ has a basis consisting of the standard monomials $(u^1_1)^{k_{11}}(u^1_2)^{k_{12}}\cdots(u^{d_V}_{d_V})^{k_{d_V d_V}}$, k_{ij} being nonnegative integers. To show this [49, 4], the monomials are ordered lexicographically, and the defining equations (38) of $\mathscr{O}(M_q(V))$ are recast in the form of a reduction system:

$$\begin{array}{lcl} u^j_k u^i_k & \to & q^{-1} u^i_k u^j_k \quad (i<j) \\ u^k_j u^k_i & \to & q^{-1} u^k_i u^k_j \quad (i<j) \\ u^j_k u^i_l & \to & u^i_l u^j_k \quad (i<j, k<l) \\ u^j_l u^i_k & \to & u^i_k u^j_l - (q-q^{-1}) u^i_l u^j_k \quad (i<j, k<l). \end{array} \tag{43}$$

Then, by the diamond lemma [30], it suffices to show that all ambiguities in this reduction system are resolvable. This means any term of the form $u^i_j u^k_l u^r_s$, when reduced in any way, leads to the same result. This has to be checked for 24 different types of configurations of the three indices $(i,j),(k,l),(r,s)$; see [4, 30, 49] for details.

4.6. **Compactness, unitary transformations.** What sets the standard quantum group apart from other known deformations [4, 37, 54, 55, 59] of $GL(V)$ is that it has a real form that is compact. To see what this means, we have to recall the notion of compactness due to Woronowicz in the quantum setting; see [62] or [30, Chapter 11] for details.

Let $\mathcal{A}$ be the coordinate Hopf algebra of a quantum group G_q. Suppose there is an involution $*$ on $\mathcal{A}$ so that it is a Hopf $*$-algebra [30]. We say that $*$ defines a real form of the quantum group G_q. A finite-dimensional corepresentation of $\mathcal{A}$ on a vector space V with a Hermitian form is called unitary if the matrix $m = (m^k_j)_{j,k}$ of this corepresentation with respect to an orthonormal basis $\{e_i\}$ of V satisfies $m^* m = m m^* = I$, where $m^* := \big((m^j_k)^*\big)_{j,k}$. The algebra $\mathcal{A}$ is called a compact matrix group algebra (CMQG) if (1) it is the linear span of all matrix elements of finite-dimensional corepresentations of $\mathcal{A}$, and (2) it is generated as an algebra by finitely many elements. Then

Theorem 4.1 (Woronowicz [62] (also see [30, Chapter 11]))**.** *(a) A Hopf $*$-algebra $\mathcal{A}$ is a CMQG algebra if and only if there is a finite-dimensional unitary corepresentation of $\mathcal{A}$ whose matrix elements generate $\mathcal{A}$ as an algebra.*

(b) If $\mathcal{A}$ is a CMQG algebra, then the quantum analog of the Peter-Weyl theorem holds and any finite-dimensional corepresentation of $\mathcal{A}$ is unitarizable, and hence, a direct sum of irreducible corepresentations.

Assume that objects are defined over $\mathbb{C}$ and q is a real number such that $q \neq 0, \pm 1$. There is a unique involution $*$ on the algebra $\mathscr{O}(GL_q(V))$ such that $(u^i_j)^* = S(u^j_i)$. This involution makes $\mathscr{O}(GL_q(V))$ into a Hopf $*$-algebra, denoted $\mathscr{O}(\mathtt{U}_q(V))$, and called the coordinate algebra of the quantum unitary group $\mathtt{U}_q(V)$—which is, again, a virtual object. Furthermore, $\mathscr{O}(\mathtt{U}_q(V))$ is a CMQG algebra.

Woronowicz [62] has shown that the usual results for real compact groups, such as Harmonic analysis, existence of orthonormal bases, and so on, generalize to CMQG algebras.

4.7. **Representations of $GL_q(V)$.** The *weight space* N^ζ of a $U_q(\mathfrak{g}_V)$-module N for the weight $\zeta \in \mathbf{X}(\mathfrak{g}_V)$ is the K-vector space $\{x \in N : q^h x = q^{\langle \zeta, h\rangle} x\}$ (we will only consider type 1 representations of $U_q(\mathfrak{g}_V)$ in this paper). Let $\mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V)$ be as in [24, Chapter 7], the category of finite-dimensional $U_q(\mathfrak{g}_V)$-modules such that the weight of any nonzero weight space belongs to $\mathbb{Z}_{\geq 0}^{d_V} \subseteq \mathbf{X}(\mathfrak{g}_V)$. It is semisimple, the irreducible objects being the highest weight modules V_λ for partitions λ.

Now by [30, Corollary 54, Chapter 11], there is a nondegenerate Hopf pairing between $U_q(\mathfrak{g}_V)$ and $\mathscr{O}(GL_q(V))$. So, as discussed in §2.6, any right $\mathscr{O}(GL_q(V))$-comodule is also a left $U_q(\mathfrak{g}_V)$-module. All of the objects of $\mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V)$ in fact come from $\mathscr{O}(GL_q(V))$-comodules; from now on, V_λ is understood to be both a $U_q(\mathfrak{g}_V)$-module and the corresponding $\mathscr{O}(GL_q(V))$-module. By the Peter-Weyl theorem for $\mathscr{O}(M_q(V))$ [30, Theorem 21, Chapter 11], the objects of $\mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V)$ are exactly the $\mathscr{O}(M_q(V))$-comodules.

For any object N of $\mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V)$ and partition λ, let $N[\lambda]$ be the V_λ-isotypic component of N. Set $N[\trianglelefteq \lambda] = \bigoplus_{\mu \trianglelefteq \lambda} N[\mu]$, $N[\triangleleft \lambda] = \bigoplus_{\mu \triangleleft \lambda} N[\mu]$. Let $\varsigma_\lambda^N : N \twoheadrightarrow N[\lambda]$ be the canonical surjection and $\iota_\lambda^N : N[\lambda] \hookrightarrow N$ the canonical inclusion. Define the projector $\pi_\lambda^N : N \to N$ by $\pi_\lambda^N = \iota_\lambda^N \circ \varsigma_\lambda^N$.

5. Bases for $GL_q(V)$ modules

We recall some facts we will need about the Gelfand-Tsetlin basis and canonical basis of V_λ. We then recall the construction of global crystal bases in the sense of [26, 27] and of the similar notion of based modules of [35]. We will also make use of the projected canonical basis defined in [9].

5.1. **Gelfand-Tsetlin bases and Clebsch-Gordon coefficients.** Standard results for the unitary group $\mathtt{U}(V)$ have their analogs for $\mathtt{U}_q(V)$. In this section, we describe results of this kind that we need; see [30, 60] for their detailed description. As in §4.6, we work over the field $\mathbb{C}$ and q is assumed to be a real number such that $q \neq 0, \pm 1$.

Recall that V_λ denotes the irreducible left $U_q(\mathfrak{g}_V)$-module of highest weight λ; this also corresponds to a right $\mathscr{O}(GL_q(V))$-comodule. Let $\{|M\rangle\}$ denote the orthonormal Gelfand-Tsetlin basis for V_λ, where M ranges over Gelfand-Tsetlin tableaux of shape λ. Gelfand-Tsetlin tableaux are equivalent to semistandard Young tableaux (SSYT) and in examples, as below, we will use SSYT to label the elements of this basis.

Example 5.1. The orthonormal Gelfand-Tsetlin of $V^{\otimes r}$ is orthonormal with respect to the Hermitian form on $V^{\otimes r}$ in which the standard monomial basis is orthonormal. The orthonormal Gelfand-Tsetlin basis of $V^{\otimes 2}$ when $d_V = 2$ is given by

$$\begin{aligned}
|\boxed{1}\boxed{1}\rangle &= v_{11}, \\
|\boxed{1}\boxed{2}\rangle &= \left(\tfrac{q}{[2]}\right)^{1/2}(qv_{12} + v_{21}), \\
|\boxed{2}\boxed{2}\rangle &= v_{22}, \\
\left|\begin{smallmatrix}\boxed{1}\\ \boxed{2}\end{smallmatrix}\right\rangle &= \left(\tfrac{q}{[2]}\right)^{1/2}(v_{21} - q^{-1}v_{12}),
\end{aligned} \tag{44}$$

where $v_{ij} := v_i \otimes v_j$. Note that in the $d_V = 2$ case, the orthonormal Gelfand-Tsetlin basis of $V^{\otimes r}$ is proportional to the projected upper canonical basis $\tilde{B}_V^r$ of §6.2.

Remark 5.2. The orthonormal Gelfand-Tsetlin basis described in [30, §7.3] is for a slightly larger quantized enveloping algebra $\breve{U}_q(\mathfrak{g}_V)$, with a different coproduct than that used here. The normalization required to make the Gelfand-Tsetlin basis orthonormal is therefore slightly different here than in this reference.

The tensor product of two irreducible $\mathscr{O}(GL_q(V))$-comodules decomposes as

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu,i} (V_\nu)_i, \tag{45}$$

where i labels different copies of V_ν—the number of these copies is the Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$.

The Clebsch-Gordon (Wigner) coefficients (CGCs) of this tensor product are defined by the formula

$$|M\rangle_i = \sum_{N,K} C_{NKM,i}^{\lambda,\mu,\nu} |N\rangle \otimes |K\rangle, \tag{46}$$

where N and K range over Gelfand-Tsetlin tableaux of shapes λ and μ, respectively, and $|M\rangle_i$ denotes the Gelfand-Tsetlin basis element of $(V_\nu)_i$ in (45) labeled by the Gelfand-Tsetlin tableau M of shape ν. By orthonormality, (46) can be inverted to obtain

$$|N\rangle \otimes |K\rangle = \sum_{\nu,i,M} \overline{C_{NKM,i}^{\lambda,\mu,\nu}} |M\rangle_i, \tag{47}$$

where the bar denotes complex conjugation, ν and i range as in the right-hand side of (45), and M ranges over Gelfand-Tsetlin tableaux of shape ν.

We denote $C_{NKM,i}^{\lambda,\mu,\nu}$ by simply $C_{NKM,i}$ if the shapes are understood. These coefficients have been intensively studied in the literature; see [30, 60] and the references therein. An explicit formula for them is known when either V_λ or V_μ is a fundamental vector representation, or more generally, a symmetric representation. In the presence of multiplicities, the Clebsch-Gordon coefficients are not uniquely determined, and do not have explicit formulae in general.

5.2. **Crystal bases.** An *upper crystal basis at* $q = \infty$ of $N \in \mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V)$ is a pair $(\mathscr{L}(N), \mathscr{B})$, where $\mathscr{L}(N)$ is a K_∞-submodule of N and $\mathscr{B}$ is a $\mathbb{Q}$-basis of $\mathscr{L}(N)/q^{-1}\mathscr{L}(N)$ which satisfy a certain compatibility with the Kashiwara operators $\tilde{E}_i^{\text{up}}, \tilde{F}_i^{\text{up}}$ (see [27, §3.1]). The lattice $\mathscr{L}(N)$ of the pair is called an *upper crystal lattice at* $q = \infty$ of N.

Kashiwara [27] gives a fairly explicit construction of an upper crystal basis of V_λ, which we denote by $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$. The basis $\mathscr{B}(\lambda)$ is naturally labeled by $\mathrm{SSYT}_{d_V}(\lambda)$ and we let b_P denote the basis element corresponding to $P \in \mathrm{SSYT}_{d_V}(\lambda)$ (see, for instance, [24, Chapter 7]). A fundamental result of [26, 27] is that an upper crystal basis is always isomorphic to a direct sum $\bigoplus_j(\mathscr{L}(\lambda^j), \mathscr{B}(\lambda^j))$, i.e., each $N \in \mathscr{O}^{\geq 0}_{\mathrm{int}}(\mathfrak{g}_V)$ has a unique upper crystal basis.

The *crystal graph* of an upper crystal basis $(\mathscr{L}, \mathscr{B})$ is the colored directed graph with vertex set $\mathscr{B}$, and, for each $\flat \in \mathscr{B}$ such that $\tilde{F}_i^{\mathrm{up}}(\flat) \neq 0$, a directed edge from $\flat$ to $\tilde{F}_i^{\mathrm{up}}(\flat)$ with color i. A *crystal component* of a crystal graph is a connected component of the underlying undirected colorless graph. By the uniqueness result for upper crystal bases, a crystal graph is always the disjoint union of crystal graphs of some $(\mathscr{L}(\lambda^j), \mathscr{B}(\lambda^j))$; also, it is well known that the crystal graphs of irreducibles are connected, so the decomposition of N into irreducibles is given by the decomposition of the crystal graph of $(\mathscr{L}(N), \mathscr{B})$ into connected components.

5.3. **Global crystal bases.**

We next define upper based $U_q(\mathfrak{g}_V)$-modules, which is similar to the based modules of [35, Chapter 27] (see [9, §4.2]).

The $\mathbf{A}$-*form* $U_q(\mathfrak{g}_V)_{\mathbf{A}}$ *of* $U_q(\mathfrak{g}_V)$ is the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g}_V)$ generated by $E_i^{(m)} := \frac{E_i^m}{[m]!}, F_i^{(m)} := \frac{F_i^m}{[m]!}, q^h, \left\{ {q^h \atop m} \right\}$ for $i \in [d_V - 1]$, $m \in \mathbb{Z}_{\geq 0}$, and $h \in \mathbf{X}(\mathfrak{g}_V)^*$, where

$$\left\{ {x \atop m} \right\} := \prod_{k=1}^{m} \frac{q^{1-k}x - q^{k-1}x^{-1}}{q^k - q^{-k}}.$$

We also define the $\mathbb{Q}[q, q^{-1}]$-*form* $U_q(\mathfrak{g}_V)_{\mathbb{Q}}$ *of* $U_q(\mathfrak{g}_V)$ to be $\mathbb{Q} \otimes_{\mathbb{Z}} U_q(\mathfrak{g}_V)_{\mathbf{A}}$.

Definition 5.3. An *upper based* $U_q(\mathfrak{g}_V)$-*module* is a pair (N, B), where N is an object of $\mathscr{O}^{\geq 0}_{\mathrm{int}}(\mathfrak{g}_V)$ and B is a K-basis of N such that

- *(a)* $B \cap N^\zeta$ is a basis of N^ζ, for any $\zeta \in \mathbf{X}(\mathfrak{g}_V)$;
- *(b)* Define $N_{\mathbf{A}} := \mathbf{A}B$. The $\mathbb{Q}[q, q^{-1}]$-submodule $\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}}$ of N is stable under $U_q(\mathfrak{g}_V)_{\mathbb{Q}}$;
- *(c)* the $\mathbb{Q}$-linear involution $\bar{\cdot} : N \to N$ defined by $\overline{ab} = \bar{a}b$ for all $a \in K$ and all $b \in B$ intertwines the $\bar{\cdot}$-involution of $U_q(\mathfrak{g}_V)$, i.e. $\overline{fn} = \bar{f}\bar{n}$ for all $f \in U_q(\mathfrak{g}_V), n \in N$;
- *(d)* Set $\mathscr{L}(N) = K_\infty B$ and let $\mathscr{B}$ denote the image of B in $\mathscr{L}(N)/q^{-1}\mathscr{L}(N)$. Then $(\mathscr{L}(N), \mathscr{B})$ is an upper crystal basis of N at $q = \infty$.

The $\bar{\cdot}$-*involution* of an upper based $U_q(\mathfrak{g}_V)$-module (N, B) is the involution on N defined in (c), its *balanced triple* is $(\mathbb{Q}[q, q^{-1}]B, K_0B, K_\infty B)$, and its *upper crystal basis* is that of (d). The *crystal graph* $\mathscr{G}$ of (N, B) is the crystal graph of its upper crystal basis, and, as a slight abuse notation, we identify the vertex set of $\mathscr{G}$ with B; a *crystal component* of (N, B) is a crystal component of $\mathscr{G}$, and is identified with a subset of B. Since in this paper based modules are emphasized over crystal bases, it is convenient to define the following global versions of the Kashiwara operators:

$$\begin{aligned} G\tilde{F}_i^{\text{up}} &: B \to B \sqcup \{0\}, \quad G\tilde{F}_i^{\text{up}} := G \circ \tilde{F}_i^{\text{up}} \circ G^{-1}, \\ G\tilde{E}_i^{\text{up}} &: B \to B \sqcup \{0\}, \quad G\tilde{E}_i^{\text{up}} := G \circ \tilde{E}_i^{\text{up}} \circ G^{-1}, \end{aligned} \tag{48}$$

for any $i \in [d_V - 1]$, where G^{-1} is the canonical isomorphism $B \xrightarrow{\cong} \mathscr{B}$. An element $b \in B$ is *highest weight* if $G\tilde{E}_i^{\text{up}} b = 0$ for all $i \in [d_V - 1]$.

Remark 5.4. In the language of Kashiwara [27], the basis B in the definition above is an upper global crystal basis with respect to its balanced triple. To define global upper crystal bases, Kashiwara first defines a balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}}, \overline{\mathscr{L}(N)}, \mathscr{L}(N))$ and a basis $\mathscr{B} \subseteq \mathscr{L}/q^{-1}\mathscr{L}$ and then defines B to be the inverse image of $\mathscr{B}$ under the isomorphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}} \cap \overline{\mathscr{L}(N)} \cap \mathscr{L}(N) \xrightarrow{\cong} \mathscr{L}/q^{-1}\mathscr{L}.$$

Let η_λ be a highest weight vector of V_λ. The $\bar{\cdot}$-involution on V_λ is defined by setting $\overline{\eta_\lambda} = \eta_\lambda$ and requiring that it intertwines the $\bar{\cdot}$-involution of $U_q(\mathfrak{g}_V)$. The upper $\mathbb{Q}[q, q^{-1}]$-form of V_λ of [27] is denoted $V_\lambda^{\mathbb{Q}\text{ up}}$, which is a $U_q(\mathfrak{g}_V)_{\mathbb{Q}}$-submodule of V_λ. We can now state the fundamental result about the existence of global crystal bases and based modules for V_λ.

Theorem 5.5 (Kashiwara [27])**.** *The triple $(V_\lambda^{\mathbb{Q}\text{ up}}, \overline{\mathscr{L}(\lambda)}, \mathscr{L}(\lambda))$ is balanced. Then, letting G_λ be the inverse of the canonical isomorphism*

$$V_\lambda^{\mathbb{Q}\text{ up}} \cap \overline{\mathscr{L}(\lambda)} \cap \mathscr{L}(\lambda) \xrightarrow{\cong} \mathscr{L}(\lambda)/q^{-1}\mathscr{L}(\lambda),$$

$B(\lambda) := G_\lambda(\mathscr{B}(\lambda))$ is the upper global crystal basis of V_λ *and* $(V_\lambda, B(\lambda))$ *is an upper based $U_q(\mathfrak{g}_V)$-module.*

Note that Kashiwara proves that the triples are balanced and the conclusions about based modules follow easily (see [35, 27.1.4] or [24, Theorem 6.2.2]). We may now define the upper integral form of V_λ to be $V_\lambda^{\mathbf{A}\text{ up}} := \mathbf{A}B(\lambda)$. We say that the element η_λ is the *canonical highest weight vector* of $(V_\lambda, B(\lambda))$.

We will need some facts about lower based modules from [35, Chapter 27], or rather, their corresponding statements for upper based $U_q(\mathfrak{g}_V)$-modules. It is shown in [27, §5.2] that (see §4.7 for notation)

> if (N, B) is an upper based $U_q(\mathfrak{g}_V)$-module, then so are $(N[\trianglelefteq \lambda], B[\trianglelefteq \lambda])$, $(N[\triangleleft\lambda], B[\triangleleft\lambda])$, and $(N[\lambda], \varsigma_\lambda^N(B[\lambda]))$, where $B[\trianglelefteq \lambda] = N[\trianglelefteq \lambda] \cap B$, $B[\triangleleft\lambda] = N[\triangleleft\lambda] \cap B$, and $B[\lambda] = B[\trianglelefteq \lambda] \backslash B[\triangleleft\lambda]$. Moreover, $(N[\lambda], \varsigma_\lambda^N(B[\lambda]))$ is isomorphic as an upper based $U_q(\mathfrak{g}_V)$-module to a direct sum of copies of $(V_\lambda, B(\lambda))$. (49)

As a consequence, the $U_q(\mathfrak{g}_V)$-cells of the module with basis (N, B) coincide with its crystal components. An additional consequence is that $b \in B$ being highest weight is equivalent to $E_i b = 0$ for all $i \in [d_V - 1]$ (warning: this is not true for lower based modules).

5.4. **Projected based modules.** We now define the projected based $U_q(\mathfrak{g}_V)$-module $(N, \tilde{B})$ of a based $U_q(\mathfrak{g}_V)$-module (N, B), following [9]. For this we need an integral form that is different from $N_{\mathbf{A}}$. Set $\mathscr{L} = \mathscr{L}(N)$. The upper based $U_q(\mathfrak{g}_V)$-module $(N[\trianglelefteq \lambda], B[\trianglelefteq \lambda])$ from the previous subsection has balanced triple

$$(N_{\mathbf{A}}[\trianglelefteq \lambda]/N_{\mathbf{A}}[\triangleleft\lambda], \overline{\mathscr{L}[\trianglelefteq \lambda]}/\overline{\mathscr{L}[\triangleleft\lambda]}, \mathscr{L}[\trianglelefteq \lambda]/\mathscr{L}[\triangleleft\lambda]), \tag{50}$$

where $N_{\mathbf{A}}[\trianglelefteq \lambda]$ and $\mathscr{L}[\trianglelefteq \lambda]$ (resp. $N_{\mathbf{A}}[\triangleleft\lambda]$ and $\mathscr{L}[\triangleleft\lambda]$) are the $\mathbf{A}$- and K_∞- span of $B[\trianglelefteq \lambda]$ (resp. $B[\triangleleft\lambda]$). Now define

$$\begin{array}{lll} N_{\mathbf{A},\lambda} & := \varsigma_\lambda^N(N_{\mathbf{A}}[\trianglelefteq \lambda]) & \subseteq N[\lambda], \\ \mathscr{L}_\lambda & := \varsigma_\lambda^N(\mathscr{L}[\trianglelefteq \lambda]) & \subseteq N[\lambda], \\ \tilde{N}_{\mathbf{A}} & := \bigoplus_\lambda N_{\mathbf{A},\lambda} & \subseteq N. \end{array} \tag{51}$$

We will make use of the following result giving several descriptions of projections of upper based $U_q(\mathfrak{g}_V)$-modules. This is slightly more general than the similar result [9, Theorem 6.1], which is proved in the context of Schur-Weyl duality in type A.

Theorem 5.6. *Maintain the notation above and that of §5.3. Let $b \in B[\lambda]$ and $\flat$ its image in $\mathscr{L}/q^{-1}\mathscr{L}$. The element $\varsigma_\lambda^N(b)$ belongs to a copy of $(V_\lambda, B(\lambda))$ in $(N[\lambda], \varsigma_\lambda^N(B[\lambda]))$ with canonical highest weight vector $\varsigma_\lambda^N(b_{hw})$ for some $b_{hw} \in B[\lambda]$. Let $b_P \in \mathscr{B}(\lambda)$ be such that $\varsigma_\lambda^N(b) = G_\lambda(b_P)$ in this copy and let $V_{b_{hw}} = U_q(\mathfrak{g}_V)b_{hw} \subseteq N$. Then the triples in (b) and (c) are balanced and the projected upper canonical basis element $\tilde{b}$ has the following descriptions*

(a) *the unique $\bar{\cdot}$-invariant element of $\tilde{N}_{\mathbf{A}}$ congruent to $b \mod q^{-1}\mathscr{L}$,*
(b) *$\tilde{G}(\flat)$, where $\tilde{G}$ is the inverse of the canonical isomorphism*

$$\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{N}_{\mathbf{A}} \cap \overline{\mathscr{L}} \cap \mathscr{L} \xrightarrow{\cong} \mathscr{L}/q^{-1}\mathscr{L},$$

(c) *$\tilde{G}_\lambda(\flat_\lambda)$, where $\flat_\lambda$ is image of $\varsigma_\lambda^N(b)$ in $\mathscr{L}_\lambda/q^{-1}\mathscr{L}_\lambda$ and $\tilde{G}_\lambda$ is the inverse of the canonical isomorphism*

$$\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A},\lambda} \cap \overline{\mathscr{L}_\lambda} \cap \mathscr{L}_\lambda \xrightarrow{\cong} \mathscr{L}_\lambda/q^{-1}\mathscr{L}_\lambda,$$

(d) *the global crystal basis element $G_\lambda(b_P)$ of $V_{b_{hw}}$,*
(e) *$\pi_\lambda^N(b)$.*

Then $(N, \tilde{B})$, with $\tilde{B} := \{\tilde{b} : b \in B\}$, is an upper based $U_q(\mathfrak{g}_V)$-module, referred to as the projected upper based $U_q(\mathfrak{g}_V)$-module of (N, B).

Proof. The proof is similar to that of [9, Theorem 6.1], which follows in a straightforward way from results of [27, §5.2] and the uniqueness of upper crystal bases. The proof of [9, Theorem 6.1] goes by showing that the elements in (b)–(e) are the same and then showing that these are $\bar{\cdot}$-invariant, hence equal to the element in (a). We replace the proof of $\bar{\cdot}$-invariance in [9] by the following: the element in (d) is $\bar{\cdot}$-invariant because the $\bar{\cdot}$-involutions on N and $V_{b_{\rm hw}}$ intertwine the bar-involution on $U_q(\mathfrak{g}_V)$, and $\overline{b}_{\rm hw} = b_{\rm hw}$. □

Given (N, B) and $(N, \tilde{B})$ as in theorem, let $(m_{\tilde{b'}b})_{\{\tilde{b'} \in \tilde{B}, b \in B\}}$ be the transition matrix from B to $\tilde{B}$ (so that $b = \sum_{\tilde{b'} \in \tilde{B}} m_{\tilde{b'}b} \tilde{b'}$ for all $b \in B$). It follows that for any partition μ and $b \in B$,

$$\pi_\mu^N(b) = \sum_{\tilde{b'} \in \tilde{B}[\mu]} m_{\tilde{b'}b} \tilde{b'}. \tag{52}$$

For later use, we record the following easy corollary.

Corollary 5.7. *Maintain the notation of the previous paragraph. For any $b \in B[\lambda]$,*

$$\pi_\mu^N(b) \begin{cases} = \tilde{b} & \text{if } \mu = \lambda, \\ = 0 & \text{if } \mu \not\trianglelefteq \lambda, \\ \in q^{-1}\mathscr{L} \cap q\overline{\mathscr{L}} & \text{if } \mu \triangleleft \lambda. \end{cases} \tag{53}$$

Proof. Theorem 5.6 (e) and (49) yield the top and middle case of (53), respectively. Next, note that Theorem 5.6 (b) implies $\sum_{\tilde{b'} \in \tilde{B}, b' \neq b} m_{\tilde{b'}b} \tilde{b'} = b - \tilde{b} \in q^{-1}\mathscr{L} \cap q\overline{\mathscr{L}}$. The bottom case then follows from (52). $\square$

5.5. **Tensor products of based modules.** Let $(N, B), (N', B')$ be upper based $U_q(\mathfrak{g}_V)$-modules. There is a basis $B \heartsuit B'$ which makes $N \otimes N'$ into an upper based $U_q(\mathfrak{g}_V)$-module. However, first, we need an involution on $N \otimes N'$ that intertwines the $\bar{\cdot}$-involution on $U_q(\mathfrak{g}_V)$. This definition is not obvious and requires Lusztig's quasi-$\mathscr{R}$-matrix, but adapted to our coproduct as in [16]: let $\Theta = (\varphi \otimes \varphi)(\tilde{\Theta}^{-1})$ where $\tilde{\Theta}$ is exactly Lusztig's quasi-$\mathscr{R}$-matrix from [35, §4.1.2]. It is an element of a certain completion $(U_q(\mathfrak{g}_V) \otimes U_q(\mathfrak{g}_V))^\wedge$ of the algebra $U_q(\mathfrak{g}_V) \otimes U_q(\mathfrak{g}_V)$. Then the involution $\bar{\cdot} : N \otimes N' \to N \otimes N'$ is defined by $\overline{n \otimes n'} = \Theta(\overline{n} \otimes \overline{n'})$. (This involution is denoted Ψ in [35].)

As discussed in [9, §4.4], the corresponding result for the based modules of Lusztig ([35, Theorem 27.3.2]) adapts to this setting:

Theorem 5.8. *Maintain the notation above with $(N, B), (N', B')$ upper based $U_q(\mathfrak{g}_V)$-modules and set $(N \otimes N')_{\mathbb{Z}[q^{-1}]} = \mathbb{Z}[q^{-1}] B \otimes B'$. For any $(b, b') \in B \times B'$, there is a unique element $b \heartsuit b' \in (N \otimes N')_{\mathbb{Z}[q^{-1}]}$ such that $\overline{b \heartsuit b'} = b \heartsuit b'$ and $b \heartsuit b' - b \otimes b' \in q^{-1}(N \otimes N')_{\mathbb{Z}[q^{-1}]}$.*

Set $B \heartsuit B' = \{b \heartsuit b' : b \in B, b' \in B'\}$. Then the pair $(N \otimes N', B \heartsuit B')$ is an upper based $U_q(\mathfrak{g}_V)$-module. Moreover, the product $\heartsuit$ is associative.

We define the $\heartsuit$ product on all of $N \times N'$ by extending the product just defined K-bilinearly.

We will come across the following situation in our application to the Kronecker problem.

Proposition 5.9. *Maintain the notation of this and the previous two subsections. Let $(N_1, B_1), \ldots, (N_l, B_l)$ be upper based $U_q(\mathfrak{g}_V)$-modules. Let $b_i \in B_i$, $i \in [l]$, be given and define λ^i such that $b_i \in B_i[\lambda^i]$. For each $i \in [l]$, let π_i be either $\pi_{\lambda^i}^{N_i}$ or the identity map on N_i; set $B_i' = \tilde{B}_i$ (resp. $B_i' = B_i$) if $\pi_i = \pi_{\lambda^i}^{N_i}$ (resp. $\pi_i = \mathrm{Id}^{N_i}$), where $(N_i, \tilde{B}_i)$ is the projected upper based $U_q(\mathfrak{g}_V)$-module of (N_i, B_i). Then*

$$\pi_1(b_1) \tilde{\heartsuit} \cdots \tilde{\heartsuit} \pi_l(b_l) = \pi_1 \otimes \cdots \otimes \pi_l(b_1 \heartsuit \cdots \heartsuit b_l),$$

where $\tilde{\heartsuit}$ (resp. $\heartsuit$) denotes the construction of Theorem 5.8 for the upper based $U_q(\mathfrak{g}_V)$-modules $(N_1, B'_1), \ldots, (N_l, B'_l)$ (resp. $(N_1, B_1), \ldots, (N_l, B_l)$).

Proof. Set $b := b_1 \heartsuit \cdots \heartsuit b_l$ and $\pi := \pi_1 \otimes \cdots \otimes \pi_l$. It suffices to show that $\pi(b)$ satisfies the defining properties of $\pi_1(b_1) \tilde{\heartsuit} \cdots \tilde{\heartsuit} \pi_l(b_l)$. Since the elements of $\tilde{B}_i$ are $\bar{\cdot}$-invariant, the transition matrix from B_i to $\tilde{B}_i$ consists of $\bar{\cdot}$-invariant elements of K. It follows from (52) that the projector π_i intertwines the $\bar{\cdot}$-involution on N_i. Hence $\overline{\pi(b)} = \pi(b)$.

It is evident from description (b) of Theorem 5.6 that the lattice $\mathscr{L}(N_i)$ is the same for the based modules (N_i, B_i) and $(N_i, \tilde{B}_i)$, and that $\pi_i(\mathscr{L}(N_i)) = \mathscr{L}(N_i)$. Set $\mathscr{L} := \mathscr{L}(N_1) \otimes_{K_\infty} \cdots \otimes_{K_\infty} \mathscr{L}(N_l)$. Hence applying π to $b - (b_1 \otimes \cdots \otimes b_l) \in q^{-1}\mathscr{L}$ implies

$$\pi(b) - \pi_1(b_1) \otimes \cdots \otimes \pi_l(b_l) \in q^{-1}\mathscr{L},$$

so $\pi(b)$ satisfies the defining properties of $\pi_1(b_1) \tilde{\heartsuit} \cdots \tilde{\heartsuit} \pi_l(b_l)$. □

6. Quantum Schur-Weyl duality and canonical bases

Write $V = V_{\epsilon_1}$ for the natural representation of $U_q(\mathfrak{g}_V)$. The action of $U_q(\mathfrak{g}_V)$ on the weight basis $v_1, \ldots, v_{d_V}$ of V is given by $q^{\epsilon^i} v_j = q^{\delta_{ij}} v_j$, $F_i v_i = v_{i+1}$, $F_i v_j = 0$ for $i \neq j$, and $E_i v_{i+1} = v_i$, $E_i v_j = 0$ for $j \neq i+1$.

In this section we describe the commuting actions of $U_q(\mathfrak{g}_V)$ and $\mathscr{H}_r$ on $\mathbf{T} := V^{\otimes r}$ as in [25, 22, 50, 16] and give several characterizations of the upper canonical basis and projected upper canonical basis of $\mathbf{T}$; we closely follow [16, 9] and are consistent with their conventions. This background will be needed to construct an upper canonical basis for $\check{\Lambda}^r \check{X}$ in §14 and motivates the hypothesized basis for $\check{X}^{\otimes r}$ detailed in Conjecture 19.1.

6.1. **Commuting actions on $\mathbf{T} = V^{\otimes r}$.** The action of $U_q(\mathfrak{g}_V)$ on $\mathbf{T}$ is determined by the coproduct Δ (29). The commuting action of $\mathscr{H}_r$ on $\mathbf{T}$ is defined by sending T_i to $\hat{\mathscr{R}}_i$, where $\hat{\mathscr{R}}_i$ denotes the $U_q(\mathfrak{g}_V)$-isomorphism of $V^{\otimes r}$ equal to $\hat{\mathscr{R}}_{V,V}$ on the i and $i+1$-st tensor factors and the identity elsewhere. Here $\hat{\mathscr{R}}_{V,V}$ denotes the $\hat{\mathscr{R}}$-matrix defined in §4.2. Equation (32) for $\hat{\mathscr{R}}_{V,V}$ gives an explicit form for the $\mathscr{H}_r$ action, which we reformulate as follows: for a word $\mathbf{k} = k_1 \ldots k_r \in [d_V]^r$, let $v_\mathbf{k} = v_{k_1} \otimes v_{k_2} \otimes \ldots \otimes v_{k_r}$ be the corresponding tensor monomial. Recall from §2.3 the right action of $\mathcal{S}_r$ on words of length r. Then

$$v_\mathbf{k} T_i^{-1} = \begin{cases} v_{\mathbf{k}\, s_i} & \text{if } k_i < k_{i+1}, \\ q^{-1} v_\mathbf{k} & \text{if } k_i = k_{i+1}, \\ (q^{-1} - q) v_\mathbf{k} + v_{\mathbf{k}\, s_i} & \text{if } k_i > k_{i+1}. \end{cases} \tag{54}$$

Remark 6.1. This convention for the action of $\mathscr{H}_r$ on $\mathbf{T}$ is consistent with that in [16, 50], but not with that in [22]. Note that $v_\mathbf{k}, T_i^{-1}$ are denoted M_α, H_i respectively in [16].

Schur-Weyl duality generalizes nicely to the quantum setting:

Theorem 6.2 (Jimbo [25])**.** *As a $(U_q(\mathfrak{g}_V), K\mathscr{H}_r)$-bimodule, $\mathbf{T}$ decomposes into irreducibles as*

$$\mathbf{T} \cong \bigoplus_{\lambda \vdash_{d_V} r} V_\lambda \otimes M_\lambda.$$

As an $\mathscr{H}_r$-module, $\mathbf{T}$ decomposes into a direct sum of weight spaces: $\mathbf{T} \cong \bigoplus_{\zeta \in \mathbf{X}(\mathfrak{g}_V)} \mathbf{T}^\zeta$. The weight space $\mathbf{T}^\zeta$ is the K-vector space spanned by $v_\mathbf{k}$ such that $\mathbf{k}$ has content ζ. Let $\epsilon_+ := M^{\mathbf{A}}_{(r)}$ be the trivial $\mathscr{H}_r$-module, i.e. the one-dimensional module identified with the map $\mathscr{H}_r \to \mathbf{A}$, $T_i \mapsto q$. It is not difficult to prove using (54) (see [16, §4])

Proposition 6.3. *The map $\mathbf{T}^\zeta_\mathbf{A} \to \epsilon_+ \otimes_{\mathscr{H}_{J_\zeta}} \mathscr{H}_r$ given by $v_\mathbf{k} \mapsto \epsilon_+ \otimes_{\mathscr{H}_{J_\zeta}} \overline{T}_{d(\mathbf{k})}$ is an isomorphism of right $\mathscr{H}_r$-modules.*

Here $d(\mathbf{k})$ is as in §2.3 and $\mathbf{T_A}$ is the integral form of $\mathbf{T}$, defined below.

6.2. **Upper canonical basis of T.** We now apply the general theory of §5.3, §5.5 to construct a global crystal basis of $\mathbf{T}$. Recall from §5.5 that there is a $\bar{\cdot}$-involution on $\mathbf{T}$ defined using the quasi-$\mathscr{R}$-matrix. The $\bar{\cdot}$-involution on $\mathscr{H}_r$ intertwines that of $\mathbf{T}$, i.e., $\overline{vh} = \overline{v}\ \overline{h}$, for any $v \in \mathbf{T}$, $h \in \mathscr{H}_r$ [35, 16].

Let $V_\mathbf{A} = \mathbf{A}\{v_i : i \in [d_V]\}$, which is the same as the integral form $V^{\mathbf{A}\ \mathrm{up}}_{\epsilon_1}$ from §5.3. By Theorem 5.8 and associativity of the $\heartsuit$ product, $(\mathbf{T}, B^r)$ is an upper based $U_q(\mathfrak{g}_V)$-module with balanced triple $(\mathbb{Q} \otimes_\mathbb{Z} \mathbf{T_A}, \overline{\mathscr{L}}, \mathscr{L})$, where

$$\begin{array}{ll} \mathscr{L} & := \mathscr{L}(\epsilon_1) \otimes_{K_\infty} \ldots \otimes_{K_\infty} \mathscr{L}(\epsilon_1), \\ \mathbf{T}_{\mathbb{Z}[q^{-1}]} & := \mathbb{Z}[q^{-1}]\{v_\mathbf{k} : \mathbf{k} \in [d_V]^r\}, \\ \mathbf{T_A} & := V_\mathbf{A} \otimes_\mathbf{A} \ldots \otimes_\mathbf{A} V_\mathbf{A} = \mathbf{A} \otimes_\mathbb{Z} \mathbf{T}_{\mathbb{Z}[q^{-1}]}, \\ B^r & := B(\epsilon_1) \heartsuit \ldots \heartsuit B(\epsilon_1). \end{array} \tag{55}$$

We call B^r the *upper canonical basis* of $\mathbf{T}$ and, for each $\mathbf{k} \in [d_V]^r$, write $c_\mathbf{k}$ for the element $v_{k_1} \heartsuit \ldots \heartsuit v_{k_r} \in B^r$. Figure 1 from the introduction gives the upper canonical basis in terms of the monomial basis for $r = 3,\ d_V = 2$.

Theorem 6.4 ([22, 16] (see [9, Theorem 5.6]))**.** *The upper canonical basis element $c_\mathbf{k}$, $\mathbf{k} \in [d_V]^r$, has the following equivalent descriptions*

(i) *the unique $\bar{\cdot}$-invariant element of $\mathbf{T}_{\mathbb{Z}[q^{-1}]}$, congruent to $v_\mathbf{k} \mod q^{-1}\mathbf{T}_{\mathbb{Z}[q^{-1}]}$;*
(ii) *$v_{k_1} \heartsuit \ldots \heartsuit v_{k_r}$;*
(iii) *The image of $C_{d(\mathbf{k})}$ under the isomorphism in Proposition 6.3.*

The next result is a slightly more precise version of Theorem 1.3. As explained in the introduction, it connects quantum Schur-Weyl duality with the RSK correspondence and is our model for constructing a basis of $\check{X}^{\otimes r}$ that solves the Kronecker problem.

Theorem 6.5 ([22] (see [9, Corollary 5.7]))**.**

(i) *The $\mathscr{H}_r$-module with basis $(\mathbf{T}, B^r)$ decomposes into $\mathscr{H}_r$-cells as*
$$B^r = \bigsqcup_{T \in SSYT^r_{d_V}} \Gamma_T, \quad \textit{where} \ \ \Gamma_T := \{c_\mathbf{k} : P(\mathbf{k}) = T\}.$$
(ii) *The $\mathscr{H}_r$-cell Γ_T of $\mathbf{T}$ is isomorphic to $(M_{\mathrm{sh}(T)}, \Gamma_{\mathrm{sh}(T)})$ of §3.2.*
(iii) *The $U_q(\mathfrak{g}_V)$-module with basis $(\mathbf{T}, B^r)$ decomposes into $U_q(\mathfrak{g}_V)$-cells as*
$$B^r = \bigsqcup_{\lambda \vdash_{d_V} r,\ T \in SYT(\lambda)} \Lambda_T, \quad \textit{where} \ \ \Lambda_T = \{c_\mathbf{k} : Q(\mathbf{k}) = T\}.$$

(iv) *The $U_q(\mathfrak{g}_V)$-cell Λ_T is isomorphic to $(V_{\mathrm{sh}(T)}, B(\mathrm{sh}(T)))$ of Theorem 5.5.*

We conclude this subsection with an explicit description of the projected upper canonical basis $(\mathbf{T}, \tilde{B}^r)$ of $(\mathbf{T}, B^r)$.

Theorem 6.6 ([9, Theorem 6.1])**.** *Let $(\mathbf{T}, \tilde{B}^r = \{\tilde{c}_{\mathbf{k}} : \mathbf{k} \in [d_V]^r\})$ be the projected upper canonical basis of $(\mathbf{T}, B^r)$ and $\tilde{\mathbf{T}}_{\mathbf{A}} = \bigoplus_\lambda \pi_\lambda^{\mathbf{T}}(\mathbf{T}_{\mathbf{A}}[\trianglelefteq \lambda])$ the integral form as in Theorem 5.6. Let $\mathbf{l} \in [d_V]^r$ and $\lambda = \mathrm{sh}(\mathbf{l})$. Set $\mathbf{j} = RSK^{-1}(Z_\lambda, Q(\mathbf{l}))$, where Z_λ is the superstandard tableau of shape λ (see §2.3). Let $V_{Q(\mathbf{l})} = U_q(\mathfrak{g}_V)c_{\mathbf{j}}$. Then the triple in (b) is balanced and the projected upper canonical basis element $\tilde{c}_{\mathbf{l}}$ has the following descriptions*

(a) *the unique $\bar{\cdot}$-invariant element of $\tilde{\mathbf{T}}_{\mathbf{A}}$ congruent to $v_{\mathbf{l}} \mod q^{-1}\mathscr{L}$,*
(b) *$\tilde{G}(b_{\mathbf{l}})$, where $b_{\mathbf{l}}$ is the image of $c_{\mathbf{l}}$ in $\mathscr{L}/q^{-1}\mathscr{L}$ and $\tilde{G}$ is the inverse of the canonical isomorphism*

$$\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{\mathbf{T}}_{\mathbf{A}} \cap \overline{\mathscr{L}} \cap \mathscr{L} \xrightarrow{\cong} \mathscr{L}/q^{-1}\mathscr{L},$$

(c) *the global crystal basis element $G_\lambda(b_{P(\mathbf{l})})$ of $V_{Q(\mathbf{l})}$,*
(d) *$\pi_\lambda^{\mathbf{T}}(c_{\mathbf{l}})$.*

The $U_q(\mathfrak{g}_V)$- and $\mathscr{H}_r$-cells of $(\mathbf{T}, \tilde{B}^r)$ are given by Theorem 6.5 with $\tilde{c}$ in place of c.

See [9, Figure 3] for the example of the projected upper canonical basis corresponding to the upper canonical basis of Figure 1.

6.3. **Graphical calculus for $U_q(\mathfrak{gl}_2)$-modules.** Our study of upper based $U_q(\mathfrak{g}_V)$-modules for two-row Kronecker in §15–17 depends heavily on the graphical calculus for $U_q(\mathfrak{gl}_2)$-modules, which we now describe. Our main reference for this is [19], though our notation differs slightly from theirs. In this subsection, fix $d_V = 2$ and let $F = F_1,\ E = E_1$.

Let $\lambda \vdash_{d_V} r$, $Q \in \mathrm{SYT}(\lambda)$, and Λ_Q be the $U_q(\mathfrak{g}_V)$-cell of the upper based $U_q(\mathfrak{g}_V)$-module $(V^{\otimes r}, B^r)$ of §6.2. Consider the quotient map from the minimal cellular submodule $K\Lambda_Q'$ containing $K\Lambda_Q$ onto $K\Lambda_Q$, and let $D(Q) \subseteq [d_V]^r$ be the set of $\mathbf{k}$ such that $c_{\mathbf{k}} \in K\Lambda_Q'$. Define $e_{\mathbf{k}} \in K\Lambda_Q$ to be the image of $c_{\mathbf{k}}$, $\mathbf{k} \in D(Q)$, under this map. Then

$$e_{\mathbf{k}} = \begin{cases} G_\lambda(b_{P(\mathbf{k})}) = \frac{[r-j]!}{[r]!} F^j \eta_\lambda & \text{if } \mathrm{sh}(\mathbf{k}) = \lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{56}$$

where j is the number of 2's in the first row of $P(\mathbf{k})$ and η_λ is the canonical highest weight vector of $(V_\lambda, B(\lambda))$. This formula follows from Theorem 6.5. Note that $(K\Lambda_Q, \Lambda_Q)$ is isomorphic to $(V_{\mathrm{sh}(Q)}, B(\mathrm{sh}(Q)))$, but it is convenient to keep the extra data of the SYT Q in what follows.

Definition 6.7. Let $\lambda^{(1)} \vdash_{d_V} i_1, \ldots, \lambda^{(l)} \vdash_{d_V} i_l$, $Q_j \in \mathrm{SYT}(\lambda^{(j)})$, and $\mathbf{k} = \mathbf{k}^{(\mathbf{1})} \cdots \mathbf{k}^{(\mathbf{l})}$ such that $\mathbf{k}^{(\mathbf{j})} \in D(Q_j)$. The canonical basis element $e_{\mathbf{k}^{(\mathbf{1})}} \heartsuit \ldots \heartsuit e_{\mathbf{k}^{(\mathbf{l})}} \in K\Lambda_{Q_1} \otimes \ldots \otimes K\Lambda_{Q_l}$ is described by the *diagram* of $\mathbf{k}$, denoted diagram($\mathbf{k}$), which is the picture obtained from $\mathbf{k}$ by pairing 2's and 1's as left and right parentheses and then drawing an arc between matching pairs as shown in Figure 2.

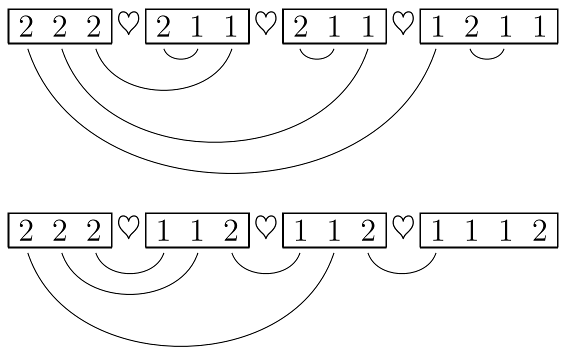


Figure 2: The diagram corresponding to two elements of $K(\Lambda_{Z^*_{(3)}} \otimes \Lambda_{Z^*_{(3)}} \otimes \Lambda_{Z^*_{(3)}} \otimes \Lambda_{Z^*_{(4)}})$. The top element evaluates to zero because it contains extra internal arcs. The bottom element is the canonical basis element $d_{2221121121112}$ in the notation of Theorem 6.9

We also record in the diagram the partitions $\lambda^{(1)}, \dots, \lambda^{(l)}$. An arc is *internal* if its ends belong to the same $\mathbf{k^{(j)}}$, and is *external* otherwise. An *extra internal arc* is an internal arc with ends in $\mathbf{k^{(j)}}$ that does not occur in the diagram of $\mathbf{l^{(j)}}$ for those $\mathbf{l^{(j)}}$ satisfying $Q(\mathbf{l^{(j)}}) = Q_j$ (all such diagrams have the same internal arcs).

Equation (56) implies the following important fact:

A diagram contains an extra internal arc if and only if the corresponding basis element evaluates to zero. (57)

Remark 6.8. Strictly speaking, determining the extra internal arcs requires the data Q_j, but deciding whether there are extra internal arcs, which is what we really care about, only requires knowing the $\lambda^{(j)}$.

For any upper based $U_q(\mathfrak{g}_V)$-module (N, B), define the functions $\varphi, \varepsilon : B \to \mathbb{Z}_{\geq 0}$ by

$$\varphi(b) := \max\{m : (G\tilde{F}^{\mathrm{up}})^m(b) \neq 0\}, \qquad \varepsilon(b) := \max\{m : (G\tilde{E}^{\mathrm{up}})^m(b) \neq 0\}. \tag{58}$$

These are the standard functions from crystal basis theory, but are usually defined for local rather than global crystal basis elements. In the case $(N, B) = (V^{\otimes r}, B^r)$, the statistic $\varphi(c_{\mathbf{k}})$ (resp. $\varepsilon(c_{\mathbf{k}})$) is the number of unpaired 1's (resp. 2's) in the diagram of $\mathbf{k}$. We also write $\varphi(\mathbf{k})$ (resp. $\varepsilon(\mathbf{k})$) in place of $\varphi(c_{\mathbf{k}})$ (resp. $\varepsilon(c_{\mathbf{k}})$).

Theorem 6.9 ([19, §2.3])**.** *Maintain the notation of Definition 6.7. The action of F and E on the upper canonical basis $\Lambda_{Q_1} \heartsuit \dots \heartsuit \Lambda_{Q_l}$ is given as follows. Let $d_{\mathbf{k}} = e_{\mathbf{k^{(1)}}} \heartsuit \cdots \heartsuit e_{\mathbf{k^{(l)}}} \in K(\Lambda_{Q_1} \otimes \dots \otimes \Lambda_{Q_l})$. Then*

$$F(d_{\mathbf{k}}) = \sum_{j=1}^{\varphi(\mathbf{k})} [j] d_{\mathscr{F}_{(j)}(\mathbf{k})},$$

where $\mathscr{F}_{(j)}(\mathbf{k})$ is the word obtained by replacing the j-th unpaired 1 in $\mathbf{k}$ with a 2 (if the diagram of $\mathscr{F}_{(j)}(\mathbf{k})$ has an extra internal arc, then $d_{\mathscr{F}_{(j)}(\mathbf{k})} = 0$; see [19]*). Similarly,*

$$E(d_{\mathbf{k}}) = \sum_{j=1}^{\varepsilon(\mathbf{k})} [j] d_{\mathscr{E}_{(j)}(\mathbf{k})},$$

where $\mathscr{E}_{(j)}(\mathbf{k})$ is the word obtained by replacing the $\varepsilon(\mathbf{k})-j+1$-th unpaired 2 in $\mathbf{k}$ with a 1 (so that $\mathscr{E}_{(1)}(\mathbf{k})$ changes the rightmost unpaired 2).

Remark 6.10. Throughout the paper we will usually only state results for F and omit the analogous statements for E.

In preparation for the application to the two-row Kronecker problem, we record the following corollary of Proposition 5.9. Let $(V^{\otimes 2}, B^2)$ be the upper based $U_q(\mathfrak{g}_V)$-module from §6.2. The corresponding projected basis is (where edges indicate the action of F)

$$\begin{tikzcd}
 & \tilde{c}_{21}=c_{21} & \\
\tilde{c}_{22}=c_{22} & & \tilde{c}_{11}=c_{11} \arrow[ul] \arrow[dl, "{[2]}"] \\
 & \tilde{c}_{12}=c_{12}+\frac{1}{[2]}c_{21} \arrow[ul] &
\end{tikzcd}.$$

Corollary 6.11. *Maintain the notation of Definition 6.7 and specialize the setup of Proposition 5.9 as follows: fix $t \in [l]$ and set $(N_t, B_t) = (V^{\otimes 2}, B^2)$ and $(N_j, B_j) = (K\Lambda_{Q_j}, \Lambda_{Q_j})$ for $j \in [l] \setminus \{t\}$. Set $\pi_t = \pi^{N_t}_{(2,0)}$ and $\pi_j = \mathrm{Id}$ for $j \neq t$. For convenience set $i_t = 2$ and $r = \sum_{j=1}^{l} i_j$. Then*

$$e_{\mathbf{k}^{(1)}} \tilde{\heartsuit} \cdots \tilde{\heartsuit} e_{\mathbf{k}^{(t-1)}} \tilde{\heartsuit} \tilde{c}_{\mathbf{k}^{(t)}} \tilde{\heartsuit} e_{\mathbf{k}^{(t+1)}} \tilde{\heartsuit} \cdots \tilde{\heartsuit} e_{\mathbf{k}^{(l)}} = d_{\mathbf{k}} + \frac{1}{[2]} d_{\mathbf{k}} C_m,$$

where $\mathbf{k}^{(t)} \in \{11, 12, 22\}$, $\mathbf{k} = \mathbf{k}^{(1)} \cdots \mathbf{k}^{(l)}$, $d_{\mathbf{k}}$ is the image of $c_{\mathbf{k}}$ under the projection

$$K\Lambda'_{Q_1} \otimes \ldots K\Lambda'_{Q_{t-1}} \otimes V^{\otimes 2} \otimes K\Lambda'_{Q_{t+1}} \otimes \ldots \otimes K\Lambda'_{Q_l} \twoheadrightarrow K\Lambda_{Q_1} \otimes \ldots K\Lambda_{Q_{t-1}} \otimes V^{\otimes 2} \otimes K\Lambda_{Q_{t+1}} \otimes \ldots \otimes K\Lambda_{Q_l},$$

and $m = 1 + \sum_{j=1}^{t-1} i_j$. Moreover, $d_{\mathbf{k}} C_m = d_{\mathbf{k}'}$, where $\mathbf{k}'$ is determined by the graphical calculus (the diagram $\asymp$ is attached below that of $\mathbf{k}$ in position m—see [19, §2.1]), and $d_{\mathbf{k}'} = 0$ if the diagram of $\mathbf{k}'$ has an extra internal arc with ends in the j-th projector for any $j \neq t$.

Proof. The projector π_t is just $\frac{1}{[2]} C'_m = 1 + \frac{1}{[2]} C_m$. □

7. Notation for $GL_q(V) \times GL_q(W)$

Let V, W, and $X = V \star W$ be vector spaces of dimensions d_V, d_W, d_X, where $\star$ is our notation for tensor product between objects associated to V and objects associated to W. As in the previous sections, V is the defining $\mathscr{O}(GL_q(V))$-comodule and $U_q(\mathfrak{g}_V)$-module with weight basis $v_1, \cdots, v_{d_V}$. Similarly, W is the defining $\mathscr{O}(GL_q(W))$-comodule and $U_q(\mathfrak{g}_W)$-module with weight basis $w_1, \cdots, w_{d_W}$. In general, notation from the previous sections for objects associated to V will be used for W as well, often with subscripts or superscripts to indicate whether they correspond to V or W.

For a word $\mathbf{k} = k_1 \ldots k_r \in [d_V]^r$ (resp. $\mathbf{l} = l_1 \ldots l_r \in [d_W]^r$), let $v_{\mathbf{k}} = v_{k_1} \otimes v_{k_2} \otimes \ldots \otimes v_{k_r} \in V^{\otimes r}$ (resp. $w_{\mathbf{k}} = w_{l_1} \otimes w_{l_2} \otimes \ldots \otimes w_{l_r} \in W^{\otimes r}$) denote the corresponding tensor monomial. Let $x_{\substack{i\\j}} = v_i \star w_j \in X$ and $x_{\substack{\mathbf{k}\\\mathbf{l}}} = x_{\substack{k_1\\l_1}} \otimes \cdots \otimes x_{\substack{k_r\\l_r}} = v_{\mathbf{k}} \star w_{\mathbf{l}} \in X^{\otimes r}$, for $\mathbf{k} \in [d_V]^r,\ \mathbf{l} \in [d_W]^r$.

We sometimes identify $[d_V] \times [d_W]$ with $[d_X]$ via the bijection $\rho : (a, b) \mapsto (a-1)d_W + b$. We will use the notation $y_{\rho(a,b)} = x_{\frac{a}{b}}$ and $y_{\mathbf{j}} = y_{j_1} \otimes \cdots \otimes y_{j_r}$, $\mathbf{j} \in [d_X]^r$.

The weight lattice $\mathbf{X}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ of $\mathfrak{g}_V \oplus \mathfrak{g}_W$ is equal to $\mathbf{X}(\mathfrak{g}_V) \oplus \mathbf{X}(\mathfrak{g}_W)$. The partial order $\trianglelefteq, \triangleleft$ on weights is defined the same way as for $\mathbf{X}(\mathfrak{g}_V)$, thus $(\alpha, \beta) \trianglelefteq (\gamma, \delta)$ if and only if $\alpha \trianglelefteq \gamma$ and $\beta \trianglelefteq \delta$. A pair of partitions (λ, μ) is identified with the weight $(\lambda, \mu) \in \mathbf{X}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$.

Define $\mathscr{O}^{\geq 0}_{\text{int}}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ to be the category of finite-dimensional $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules such that the weight of any nonzero weight space belongs to $\mathbb{Z}^{d_V}_{\geq 0} \oplus \mathbb{Z}^{d_W}_{\geq 0} \subseteq \mathbf{X}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$. For any object N in $\mathscr{O}^{\geq 0}_{\text{int}}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ and partitions λ, μ, define $\pi^N_{\lambda,\mu} : N \to N$ to be the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-projector with image the $V_\lambda \star W_\mu$-isotypic component of N.

The definitions and results for based modules from §5 carry over in the obvious way to objects of $\mathscr{O}^{\geq 0}_{\text{int}}(\mathfrak{g}_V \oplus \mathfrak{g}_W)$. From now on, $B^r_V := \{c^V_{\mathbf{k}} : \mathbf{k} \in [d_V]^r\}$ denotes the upper canonical basis of $V^{\otimes r}$ constructed in §6.2 and $B^r_W := \{c^W_{\mathbf{k}} : \mathbf{k} \in [d_W]^r\}$ denotes the upper canonical basis of $W^{\otimes r}$. The basis $B^r_V \star B^r_W$ of $X^{\otimes r}$ is the *upper canonical basis of* $X^{\otimes r}$ and its elements are denoted $c_{\substack{\mathbf{k}\\ \mathbf{l}}} := c^V_{\mathbf{k}} \star c^W_{\mathbf{l}}$. This makes $(X^{\otimes r}, B^r_V \star B^r_W)$ an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module with balanced triple $(X^{\otimes r}_{\mathbf{A}}, \overline{\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W}, \mathscr{L}_V \star_{K_\infty} \mathscr{L}_W)$, where $X_{\mathbf{A}} = V_{\mathbf{A}} \star W_{\mathbf{A}}$.

8. The nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$

Here we give the definition of the nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ as an FRT-algebra. The theory of this object and the corresponding Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ is developed in the next three sections, following the treatment of their standard counterparts in §4.2–4.6. In §8.2, we construct the nonstandard symmetric and exterior $\mathscr{O}(M_q(\check{X}))$-comodule algebras, which are the nonstandard analogs of the quantum symmetric and exterior $\mathscr{O}(M_q(V))$-comodule algebras. Subsections 8.3 and 8.4 address explicit computations for nonstandard objects, illustrating that these are significantly more complicated than their standard counterparts.

8.1. **Definition of $\mathscr{O}(M_q(\check{X}))$.** Let V and W be K-vector spaces of dimensions d_V and d_W, respectively, and let $X = V \star W$ be their tensor product (this notation for tensor product is explained in §2.2); we write $\check{X}$ in place of X when this space is associated with the nonstandard objects we are about to define. Let $\hat{\mathscr{R}}_{V,V}$ be as in §4.2 and $\hat{\mathscr{R}}_{W,W}$ be defined in the same way with W in place of V and standard basis $w_1, \ldots, w_{d_W}$ in place of $v_1, \ldots, v_{d_V}$. Define $\hat{\mathscr{R}}_{\check{X},\check{X}} := \hat{\mathscr{R}}_{V,V} \star \hat{\mathscr{R}}_{W,W} \in M_{d_X{}^2}(K)$. This is different from the $\hat{\mathscr{R}}$-matrix $\hat{\mathscr{R}}_{X,X}$ obtained by thinking of $X = V \star W$ as a corepresentation of the quantum coordinate algebra $\mathscr{O}(GL_q(X))$.

Both $\hat{\mathscr{R}}_{V,V}$ and $\hat{\mathscr{R}}_{W,W}$ are diagonalizable with eigenvalues q and $-q^{-1}$. Hence, $\hat{\mathscr{R}}_{\check{X},\check{X}}$ is diagonalizable with eigenvalues $q^2, -1, q^{-2}$. The *nonstandard symmetric square* $\check{S}^2\check{X} \subseteq \check{X} \otimes \check{X}$ is defined to be the sum of the eigenspaces of $\hat{\mathscr{R}}_{\check{X},\check{X}}$ corresponding to the eigenvalues q^2 and q^{-2}. The *nonstandard exterior square* $\check{\Lambda}^2\check{X} \subseteq \check{X} \otimes \check{X}$ is defined to be the eigenspace of $\hat{\mathscr{R}}_{\check{X},\check{X}}$ for the eigenvalue -1. Let $P^{\check{X}}_+ : \check{X}^{\otimes 2} \to \check{X}^{\otimes 2}$ (resp. $P^{\check{X}}_-$) be the projector with image $\check{S}^2\check{X}$ (resp. $\check{\Lambda}^2\check{X}$). These spaces and projectors are expressed in terms of V and W as

$$\begin{aligned} \check{S}^2\check{X} &= S_q^2V \star S_q^2W \oplus \Lambda_q^2V \star \Lambda_q^2W, \\ \check{\Lambda}^2\check{X} &= S_q^2V \star \Lambda_q^2W \oplus \Lambda_q^2V \star S_q^2W, \end{aligned} \tag{59}$$

and,

$$\begin{aligned} P_+^{\check{X}} &= P_+^V \star P_+^W + P_-^V \star P_-^W, \\ P_-^{\check{X}} &= P_-^V \star P_+^W + P_+^V \star P_-^W. \end{aligned} \tag{60}$$

Let $\check{Z} = \check{X} * \check{X}^* \cong U^V \star U^W$ with standard basis $\{z_i^j : i, j \in [d_X]\}$. Let $\check{\mathbf{z}} = \mathbf{u}^V \star \mathbf{u}^W$ be the variable matrix (z_i^j), specifying the linear functions on an endomorphism of $\check{X}$. Let $K\langle z_i^j \rangle = T(\check{Z})$ denote the free algebra over the variable entries of $\check{\mathbf{z}}$.

Definition 8.1. The *nonstandard coordinate algebra* $\mathscr{O}(M_q(\check{X}))$ of the virtual *nonstandard matrix space* $M_q(\check{X})$ is the quotient of $K\langle z_i^j \rangle$ by the relations

$$P_+^{\check{X}}(\check{\mathbf{z}} \otimes \check{\mathbf{z}}) = (\check{\mathbf{z}} \otimes \check{\mathbf{z}})P_+^{\check{X}}. \tag{61}$$

We now establish some basic facts and make some remarks about the nonstandard coordinate algebra. Since $I = P_-^{\check{X}} + P_+^{\check{X}}$, (61) is equivalent to

$$P_-^{\check{X}}(\check{\mathbf{z}} \otimes \check{\mathbf{z}}) = (\check{\mathbf{z}} \otimes \check{\mathbf{z}})P_-^{\check{X}}. \tag{62}$$

Similar to the description (40) of the quantum coordinate algebra $\mathscr{O}(M_q(V))$, we have:

> the nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ is the quotient of $T(\check{Z})$ by the two-sided ideal $\check{\mathcal{I}}$ generated by $\check{\mathcal{I}}_2 := \check{S}^2\check{X} * \check{\Lambda}^2\check{X}^* \oplus \check{\Lambda}^2\check{X} * \check{S}^2\check{X}^*$. (63)

It is easy to see that $\check{S}^2\check{X}$ (resp. $\check{\Lambda}^2\check{X}$) specializes to S^2X (resp. Λ^2X) at $q = 1$. Thus the degree 2 part of $\mathscr{O}(M_q(\check{X}))$ coincides with the degree 2 part of $\mathscr{O}(M(X))$ at $q = 1$. This means that the z_j^i's commute at $q = 1$. These specializations are made precise and checked carefully in Appendix A.

Remark 8.2. The definition of $A(\mathscr{R})$ in [54] requires $\mathscr{R}$ to be nonsingular. The relations (61) or (62) are like the defining relations for an FRT-algebra except with a singular $\mathscr{R}$, however (61) or (62) is equivalent to

$$\hat{\mathcal{R}}_{\check{X},\check{X}}(a,b)(\check{\mathbf{z}} \otimes \check{\mathbf{z}}) = (\check{\mathbf{z}} \otimes \check{\mathbf{z}})\hat{\mathcal{R}}_{\check{X},\check{X}}(a,b), \tag{64}$$

where

$$\hat{\mathcal{R}}_{\check{X},\check{X}}(a,b) = aP_+^{\check{X}} + bP_-^{\check{X}}$$

for any distinct constants a, b. Thus if a, b are distinct and nonzero, then $\mathscr{O}(M_q(\check{X}))$ is an FRT-algebra with $\mathscr{R}$-matrix $\mathcal{R}_{\check{X},\check{X}}(a,b) = \tau \circ \hat{\mathcal{R}}_{\check{X},\check{X}}(a,b)$.

As explained in §4.2, any FRT-algebra is a bialgebra, hence $\mathscr{O}(M_q(\check{X}))$ is a bialgebra with coproduct and counit given by $\Delta(\check{\mathbf{z}}) = \check{\mathbf{z}}\dot{\otimes}\check{\mathbf{z}}$, and $\epsilon(\check{\mathbf{z}}) = \mathbf{I}$.

Proposition 8.3. *Let $\mathscr{O}(M_q(V))$ and $\mathscr{O}(M_q(W))$ be the quantum coordinate algebras defined in §4.3. There is a bialgebra homomorphism*

$$\psi : \mathscr{O}(M_q(\check{X})) \to \mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W)),$$

determined by $\check{\mathbf{z}} \mapsto \mathbf{u}^V \star \mathbf{u}^W$.

Note that the $\star$ in $\mathbf{u}^V \star \mathbf{u}^W$ is serving two purposes: one is as the tensor product of a $d_V \times d_V$ matrix and a $d_W \times d_W$ matrix and the other is as the $\star$ product inside the ring $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$.

Proof. One has to check that the relations obtained by substituting $\check{\mathbf{z}} = \mathbf{u}^V \star \mathbf{u}^W$ in (61) defining $\mathscr{O}(M_q(\check{X}))$ are implied by the relations defining $\mathscr{O}(M_q(V))$ and $\mathscr{O}(M_q(W))$.

The defining relations (38) of $\mathscr{O}(M_q(V))$ are

$$P_+^V(\mathbf{u}^V \otimes \mathbf{u}^V) = (\mathbf{u}^V \otimes \mathbf{u}^V)P_+^V,$$

which are equivalent to (39):

$$P_-^V(\mathbf{u}^V \otimes \mathbf{u}^V) = (\mathbf{u}^V \otimes \mathbf{u}^V)P_-^V.$$

Similarly, the defining relations of $\mathscr{O}(M_q(W))$ are either of

$$P_+^W(\mathbf{u}^W \otimes \mathbf{u}^W) = (\mathbf{u}^W \otimes \mathbf{u}^W)P_+^W,$$
$$P_-^W(\mathbf{u}^W \otimes \mathbf{u}^W) = (\mathbf{u}^W \otimes \mathbf{u}^W)P_-^W.$$

Since $P_-^{\check{X}} = P_-^V \star P_+^W + P_+^V \star P_-^W$ (see (60)), these relations imply (61) when $\check{\mathbf{z}} = \mathbf{u}^V \star \mathbf{u}^W$.

To show that ψ is a bialgebra homomorphism, one has to additionally verify that

$$\Delta \circ \psi = (\psi \otimes \psi) \circ \Delta \quad \text{and} \quad \epsilon = \epsilon \circ \psi,$$

which is easy. □

Remark 8.4. Fix distinct a, b, and let $\mathcal{R} = \mathcal{R}_{\check{X},\check{X}}(a, b)$, as in Remark 8.2. Given a tensor product $\check{X}^{\otimes r}$, let $\mathcal{R}_i$ denote the transformation which acts like $\mathcal{R}$ on the i-th and $(i+1)$-st factors, the other factors remaining unaffected. Then, as is shown in §11.6 (see Remark 11.17), the pairs $\hat{\mathcal{R}}_i, \hat{\mathcal{R}}_{i+1}$ do not satisfy the braid relation—equivalently, the pairs $\mathcal{R}_i, \mathcal{R}_{i+1}$ do not satisfy the quantum Yang-Baxter equation. Thus although $\mathscr{O}(M_q(\check{X}))$ is an FRT-algebra, it is not coquasitriangular, hence the main theory of FRT-algebras [54] does not apply.

Remark 8.5. The nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ is much smaller than $\mathscr{O}(M_q(X))$ as will be seen in Proposition 8.11 and §13. However, it is the only FRT-algebra with a coalgebra homomorphism to $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$ such that its degree 2 corepresentations coincide with those of $\mathscr{O}(M(X))$ at $q = 1$. For example, another quantization of $\mathscr{O}(M(X))$ we considered is the FRT-algebra $A(\mathscr{R}_{\check{X},\check{X}})$, where $\mathscr{R}_{\check{X},\check{X}} = \tau \circ \hat{\mathscr{R}}_{\check{X},\check{X}}$. However, it is smaller than $\mathscr{O}(M_q(\check{X}))$ and even its degree 2 corepresentations are smaller than those of $\mathscr{O}(M(X))$, so it is not a good candidate for a quantization of $\mathscr{O}(M(X))$ for the Kronecker problem. See Remark 11.4 for a similar argument claiming that the nonstandard Hecke algebra is in some sense the only choice for a quantization of the symmetric group for the Kronecker problem.

8.2. **Nonstandard symmetric and exterior algebras.** Here we define the nonstandard symmetric and exterior algebras of $\check{X}$; these play an analogous for $\mathscr{O}(M_q(\check{X}))$ to the role played by the quantum symmetric and exterior algebras of V for $\mathscr{O}(M_q(V))$, as was described in §4.3.

Maintain the notation from §7 so that $x_{\substack{i\\j}} = v_i \star w_j \in \check{X}$, $y_{\rho(a,b)} = x_{\substack{a\\b}}$, where $\rho(a,b) = (a-1)d_W + b$, $y_{\mathbf{j}} = y_{j_1} \otimes \cdots \otimes y_{j_r}$, $\mathbf{j} \in [d_X]^r$, etc. The *standard monomial basis* of $\check{X}^{\otimes r}$ is $\{y_{\mathbf{j}} : \mathbf{j} \in [d_X]^r\}$ and its dual basis is the *standard monomial basis* of $(\check{X}^*)^{\otimes r}$, denoted $\{y^{\mathbf{j}} : \mathbf{j} \in [d_X]^r\}$.

Define the *standard bilinear form* $\langle \cdot, \cdot \rangle$ on $\check{X}^{\otimes r}$ (resp. $(\check{X}^*)^{\otimes r}$) to be the symmetric bilinear form for which the standard monomial basis is orthonormal (we do not want the Hermitian form here). Both the standard bilinear form on $\check{X}^{\otimes r}$ and that on $(\check{X}^*)^{\otimes r}$ induce the isomorphism $\alpha_r : \check{X}^{\otimes r} \xrightarrow{\cong} (\check{X}^*)^{\otimes r}$, $y_{\mathbf{j}} \mapsto y^{\mathbf{j}}$, $\mathbf{j} \in [d_X]^r$. (65)

The *nonstandard symmetric algebra* $\check{S}(\check{X})$ of $\check{X}$ is the free K-algebra in the $x_{\substack{i\\j}}$'s subject to the relations

$$P_-^{\check{X}}(\mathbf{x} \otimes \mathbf{x}) = 0, \tag{66}$$

where $\mathbf{x}$ is the column vector with entries $x_{\substack{i\\j}}$. Equivalently, $\check{S}(\check{X})$ is the quotient of the tensor algebra $T(\check{X}) = \bigoplus_{r \geq 0} \check{X}^{\otimes r}$ by the two-sided ideal generated by $\check{\Lambda}^2\check{X}$. It can be thought of as the coordinate ring of the virtual *nonstandard symmetric space* $\check{X}_{\text{sym}}$.

Similarly, the *nonstandard exterior algebra* $\check{\Lambda}(\check{X})$ of $\check{X}$ is the free K-algebra in the $x_{\substack{i\\j}}$'s subject to the relations

$$P_+^{\check{X}}(\mathbf{x} \otimes \mathbf{x}) = 0. \tag{67}$$

Equivalently, $\check{\Lambda}(\check{X})$ is the quotient of $T(\check{X})$ by the two-sided ideal generated by $\check{S}^2\check{X}$. It can be thought of as the coordinate ring of the virtual *nonstandard antisymmetric space* $\check{X}_\wedge$. Let $\check{S}^r\check{X}$ and $\check{\Lambda}^r\check{X}$ be the degree r components of $\check{S}(\check{X})$ and $\check{\Lambda}(\check{X})$, respectively.

Using the standard bilinear form (65) to identify $\check{X}$ and $\check{X}^*$ defines the right-hand versions $\check{S}^r\check{X}^*$ and $\check{\Lambda}^r\check{X}^*$ of the nonstandard symmetric and exterior algebras, i.e., $\check{S}^r\check{X}^*$ is the quotient of $T(\check{X}^*)$ by the two-sided ideal generated by $\check{\Lambda}^2\check{X}^*$, where $\check{\Lambda}^2\check{X}^* := \alpha_2(\check{\Lambda}^2\check{X})$.

Proposition 8.6. (1) *The nonstandard symmetric algebra $\check{S}(\check{X})$ (resp. $\check{S}(\check{X}^*)$) is a left (resp. right) $\mathscr{O}(M_q(\check{X}))$-comodule algebra via $y_i \mapsto \sum_j z_i^j \otimes y_j$ (resp. $y^j \mapsto \sum_i y^i \otimes z_i^j$). The nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ is the largest bialgebra quotient of $T(\check{Z})$ such that $\check{S}^2\check{X}$ is a left $\mathscr{O}(M_q(\check{X}))$-comodule and $\check{S}^2\check{X}^*$ is a right $\mathscr{O}(M_q(\check{X}))$-comodule.*

(2) *Similarly, the nonstandard exterior algebra $\check{\Lambda}(\check{X})$ (resp. $\check{\Lambda}(\check{X}^*)$) is a left (resp. right) $\mathscr{O}(M_q(\check{X}))$-comodule algebra. The nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$ is the largest bialgebra quotient of $T(\check{Z})$ such that $\check{\Lambda}^2\check{X}$ is a left $\mathscr{O}(M_q(\check{X}))$-comodule and $\check{\Lambda}^2\check{X}^*$ is a right $\mathscr{O}(M_q(\check{X}))$-comodule.*

Proof. The proof is similar to the standard case. This uses the fact that the matrices $P_\pm^{\check{X}}$ are symmetric, which follows from the fact that $P_\pm^V$, $P_\pm^W$ are symmetric. □

Let $\tilde{B}_+^V$ (resp. $\tilde{B}_-^V$) be a basis of $S_q^2 V$ (resp. $\Lambda_q^2 V$). For instance, we could let $\tilde{B}_V^2$ be the projected upper canonical basis of $V^{\otimes 2}$ and $\tilde{B}_+^V \subseteq \tilde{B}_V^2$ (resp. $\tilde{B}_-^V \subseteq \tilde{B}_V^2$) the subset $\tilde{\Lambda}_{\boxed{1}\boxed{2}} = \{\tilde{c}_{ij}^V : 1 \leq i \leq j \leq d_V\}$ (resp. $\tilde{\Lambda}_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} = \{\tilde{c}_{ij}^V : 1 \leq j < i \leq d_V\}$); see §6.2, particularly Theorems 6.6 and 6.5. The $d_V = 2$ case is described explicitly in the example below. Define $\tilde{B}_+^W$ and $\tilde{B}_-^W$ similarly. Then by (59), the following are bases of $\check{S}^2\check{X}$ and $\check{\Lambda}^2\check{X}$:

$$\begin{aligned} \check{S}^2\check{X} :&\quad \tilde{B}_+^V \star \tilde{B}_+^W \sqcup \tilde{B}_-^V \star \tilde{B}_-^W, \\ \check{\Lambda}^2\check{X} :&\quad \tilde{B}_+^V \star \tilde{B}_-^W \sqcup \tilde{B}_-^V \star \tilde{B}_+^W. \end{aligned} \tag{68}$$

Example 8.7. Let $d_V = d_W = 2$ and $\{v_1, v_2\}$, $\{w_1, w_2\}$ be the standard bases of V and W. Then $\tilde{B}_+^V = \{\tilde{c}_{11}^V, \tilde{c}_{21}^V, \tilde{c}_{22}^V\}$ and $\tilde{B}_-^V = \{\tilde{c}_{21}^V\}$, where

$$\begin{aligned} \tilde{c}_{11}^V &= v_{11}, \\ \tilde{c}_{12}^V &= \tfrac{1}{[2]}(qv_{12} + v_{21}), \\ \tilde{c}_{22}^V &= v_{22}, \\ \tilde{c}_{21}^V &= v_{21} - q^{-1}v_{12}. \end{aligned} \tag{69}$$

The elements $\tilde{c}_{\mathbf{k}}^W$ of $\tilde{B}_+^W$ and $\tilde{B}_-^W$ are similar with $w_{\mathbf{l}}$ in place of $v_{\mathbf{l}}$. Set $\tilde{c}_{\substack{\mathbf{k}\\\mathbf{l}}} = \tilde{c}_{\mathbf{k}}^V \star \tilde{c}_{\mathbf{l}}^W$. The bijection ρ in the $d_V = d_W = 2$ case is

$$\substack{1\\1} \leftrightarrow 1,\ \substack{1\\2} \leftrightarrow 2,\ \substack{2\\1} \leftrightarrow 3,\ \substack{2\\2} \leftrightarrow 4.$$

Then the basis (68) of $\check{\Lambda}^2\check{X}$ is expressed in terms of the monomial basis of $\check{X}^{\otimes 2}$ as follows. This basis is labeled by NST((2)), the set of nonstandard tabloids of shape (2) (this will be explained in full generality in §14, but for now we can take this as the definition in the two-row case).

$$\begin{aligned}
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} &:= \tilde{c}_{\substack{11\\21}} = x_{\substack{11\\21}} - q^{-1}x_{\substack{11\\12}} && = y_{21} - q^{-1}y_{12}, \\
\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} &:= \tilde{c}_{\substack{22\\21}} = x_{\substack{22\\21}} - q^{-1}x_{\substack{22\\12}} && = y_{43} - q^{-1}y_{34}, \\
\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} &:= \tilde{c}_{\substack{21\\11}} = x_{\substack{21\\11}} - q^{-1}x_{\substack{12\\11}} && = y_{31} - q^{-1}y_{13}, \\
\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} &:= \tilde{c}_{\substack{21\\22}} = x_{\substack{21\\22}} - q^{-1}x_{\substack{12\\22}} && = y_{42} - q^{-1}y_{24}, \\
\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} &:= \tilde{c}_{\substack{12\\21}} = \tfrac{1}{[2]}(qx_{\substack{12\\21}} - x_{\substack{12\\12}} + x_{\substack{21\\21}} - q^{-1}x_{\substack{21\\12}}) && = \tfrac{1}{[2]}(qy_{23} - y_{14} + y_{41} - q^{-1}y_{32}), \\
\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} &:= \tilde{c}_{\substack{21\\12}} = \tfrac{1}{[2]}(qx_{\substack{21\\12}} - x_{\substack{12\\12}} + x_{\substack{21\\21}} - q^{-1}x_{\substack{12\\21}}) && = \tfrac{1}{[2]}(qy_{32} - y_{14} + y_{41} - q^{-1}y_{23}).
\end{aligned} \tag{70}$$

The basis (68) of $\check{S}^2\check{X}$ is expressed in terms of the monomial basis of $\check{X}^{\otimes 2}$ as follows. This basis is denoted $\widetilde{\mathrm{NST}}((1,1))$, which is defined to be the projection of NST((1,1)) onto $\check{S}^2\check{X}$ (the projection is needed as $\mathrm{NST}((2)) \sqcup \mathrm{NST}((1,1))$ is a basis for $\check{X}^{\otimes 2}$, but

NST((1, 1)) is not a subset of $\check{S}^2\check{X}$).

$$
\begin{aligned}
\widetilde{\boxed{1}\boxed{1}} &:= \tilde{c}_{\substack{11\\11}} = x_{\substack{11\\11}} &&= y_{11},\\
\widetilde{\boxed{2}\boxed{2}} &:= \tilde{c}_{\substack{11\\22}} = x_{\substack{11\\22}} &&= y_{22},\\
\widetilde{\boxed{1}\boxed{2}} &:= \tilde{c}_{\substack{11\\12}} = \tfrac{1}{[2]}(qx_{\substack{11\\12}} + x_{\substack{11\\21}}) &&= \tfrac{1}{[2]}(qy_{12} + y_{21}),\\
\widetilde{\boxed{3}\boxed{3}} &:= \tilde{c}_{\substack{22\\11}} = x_{\substack{22\\11}} &&= y_{33},\\
\widetilde{\boxed{4}\boxed{4}} &:= \tilde{c}_{\substack{22\\22}} = x_{\substack{22\\22}} &&= y_{44},\\
\widetilde{\boxed{3}\boxed{4}} &:= \tilde{c}_{\substack{22\\12}} = \tfrac{1}{[2]}(qx_{\substack{22\\12}} + x_{\substack{22\\21}}) &&= \tfrac{1}{[2]}(qy_{34} + y_{43}),\\
\widetilde{\boxed{1}\boxed{3}} &:= \tilde{c}_{\substack{12\\11}} = \tfrac{1}{[2]}(qx_{\substack{12\\11}} + x_{\substack{21\\11}}) &&= \tfrac{1}{[2]}(qy_{13} + y_{31}),\\
\widetilde{\boxed{2}\boxed{4}} &:= \tilde{c}_{\substack{12\\22}} = \tfrac{1}{[2]}(qx_{\substack{12\\22}} + x_{\substack{21\\22}}) &&= \tfrac{1}{[2]}(qy_{24} + y_{42}),\\
\widetilde{\boxed{1}\boxed{4}} &:= \tilde{c}_{\substack{12\\12}} = \tfrac{1}{[2]^2}(q^2x_{\substack{12\\12}} + qx_{\substack{12\\21}} + qx_{\substack{21\\12}} + x_{\substack{21\\21}}) &&= \tfrac{1}{[2]^2}(q^2y_{14} + qy_{23} + qy_{32} + y_{41}),\\
\widetilde{\boxed{4}\boxed{1}} &:= \tilde{c}_{\substack{21\\21}} = x_{\substack{21\\21}} - q^{-1}x_{\substack{21\\12}} - q^{-1}x_{\substack{12\\21}} + q^{-2}x_{\substack{12\\12}} &&= y_{41} - q^{-1}y_{32} - q^{-1}y_{23} + q^{-2}y_{14}.
\end{aligned} \tag{71}
$$

Proposition 8.8.

(1) A basis of $\check{S}(\check{X})$ is $\{y_{j_1}y_{j_2}\cdots y_{j_r} : 1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq d_X,\ r \geq 0\}$.
(2) A basis of $\check{\Lambda}(\check{X})$ is $\{y_{j_1}y_{j_2}\cdots y_{j_r} : 1 \leq j_1 < j_2 < \cdots < j_r \leq d_X,\ r \geq 0\}$.

Here $y_{\rho(a,b)} = x_{\substack{a\\b}}$ as above, and $y_{j_1}\cdots y_{j_r}$ denotes the image of $y_{j_1} \otimes \cdots \otimes y_{j_r}$ in $\check{S}(\check{X})$ or $\check{\Lambda}(\check{X})$.

These bases will be called *standard monomial bases* of $\check{S}(\check{X})$ and $\check{\Lambda}(\check{X})$.

Proof. (1) The relations (66) (in the two-row case this means setting the elements in (70) to 0) can be reformulated in the form of the following reduction system:

$$
\begin{aligned}
x_{\substack{j\\k}}x_{\substack{i\\k}} &\to q^{-1}x_{\substack{i\\k}}x_{\substack{j\\k}} \quad (i<j)\\
x_{\substack{k\\j}}x_{\substack{k\\i}} &\to q^{-1}x_{\substack{k\\i}}x_{\substack{k\\j}} \quad (i<j)\\
x_{\substack{j\\k}}x_{\substack{i\\l}} &\to x_{\substack{i\\l}}x_{\substack{j\\k}} \quad (i<j, k<l)\\
x_{\substack{j\\l}}x_{\substack{i\\k}} &\to x_{\substack{i\\k}}x_{\substack{j\\l}} - (q-q^{-1})x_{\substack{i\\l}}x_{\substack{j\\k}} \quad (i<j, k<l).
\end{aligned} \tag{72}
$$

When $d_V = d_W$, these coincide with the defining relations (43) for the standard quantum matrix space $M_q(V)$ after the change of variables $x_{\substack{i\\j}} \mapsto u_i^j$. In this case, the ambiguities in this reduction system can be resolved just as in the case of the reduction system for $\mathscr{O}(M_q(V))$ [49, 4] (see §4.5). This is also so when $d_V \neq d_W$. Hence the result follows from the diamond lemma [30].

(2) The relations (67) (in the two-row case this means setting the elements in (71) to 0) can be reformulated in the form of the following reduction system:

$$\begin{array}{rcl} x_{\substack{i\\j}}^2 & \to & 0 \\ x_{\substack{j\\k}} x_{\substack{i\\k}} & \to & -q x_{\substack{i\\k}} x_{\substack{j\\k}} \quad (i<j) \\ x_{\substack{k\\j}} x_{\substack{k\\i}} & \to & -q x_{\substack{k\\i}} x_{\substack{k\\j}} \quad (i<j) \\ x_{\substack{j\\l}} x_{\substack{i\\k}} & \to & -x_{\substack{i\\k}} x_{\substack{j\\l}} \quad (i<j, k<l) \\ x_{\substack{j\\k}} x_{\substack{i\\l}} & \to & -x_{\substack{i\\l}} x_{\substack{j\\k}} + (q^{-1}-q) x_{\substack{i\\k}} x_{\substack{j\\l}} \quad (i<j, k<l). \end{array} \tag{73}$$

Ambiguities in this reduction system can also be resolved just as in (1); we omit the details. So the result again follows from the diamond lemma [30]. $\square$

The nonstandard symmetric and exterior algebras $\check{S}^r\check{X}$ and $\check{\Lambda}^r\check{X}$ become $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodules via the coalgebra homomorphism ψ of Proposition 8.3.

Proposition 8.9. *(1) As an $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodule,*

$$\check{S}^r\check{X} \cong \bigoplus_{\substack{\lambda\vdash r\\ \ell(\lambda)\le d_V,\ \ell(\lambda)\le d_W}} V_\lambda \star W_\lambda.$$

(2) Similarly, letting λ' be the conjugate of λ as in §2.3,

$$\check{\Lambda}^r\check{X} \cong \bigoplus_{\substack{\lambda\vdash r\\ \ell(\lambda)\le d_V,\ \ell(\lambda')\le d_W}} V_\lambda \star W_{\lambda'}.$$

Proof. By the proof of Proposition 8.8 (1), the $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodule action on $\check{S}^r\check{X}$ coincides with the two-sided coaction of $\mathscr{O}(M_q(V))$ on $\mathscr{O}(M_q(V))$ in the case $W = V^*$. Thus (1) is the q-analog of the Peter-Weyl theorem for the standard quantum coordinate algebra $O(M_q(V))$ [30, Theorem 21, Chapter 11].

Similarly, (2) is an antisymmetric version of the q-analog of the Peter-Weyl theorem. A more careful proof of (2) can be given using nonstandard Schur-Weyl duality (Theorem 12.1) and Proposition 11.13. $\square$

8.3. **Explicit product formulae.** We wish to give explicit formulae for products in the nonstandard symmetric and exterior algebras $\check{S}(\check{X})$ and $\check{\Lambda}(\check{X})$.

Recall the Gelfand-Tsetlin basis and its notation from §5.1 and, as there, assume that objects are over $\mathbb{C}$ and q is a real number such that $q \neq 0, \pm 1$. Let

$$B^{\mathrm{GT}}(\check{S}^r\check{X}) = \bigsqcup_\lambda \{|M_\lambda\rangle \star |N_\lambda\rangle : M_\lambda \in \mathrm{SSYT}_{d_V}(\lambda), N_\lambda \in \mathrm{SSYT}_{d_W}(\lambda)\} \tag{74}$$

be the orthonormal Gelfand-Tsetlin basis for $\check{S}^r\check{X}$ as per the decomposition in Proposition 8.9 (1) and

$$B^{\mathrm{GT}}(\check{\Lambda}^r\check{X}) = \bigsqcup_\lambda \{|M_\lambda\rangle \star |N_{\lambda'}\rangle : M_\lambda \in \mathrm{SSYT}_{d_V}(\lambda), N_{\lambda'} \in \mathrm{SSYT}_{d_W}(\lambda')\} \tag{75}$$

that for $\check{\Lambda}^r\check{X}$ as per Proposition 8.9 (2).

When $W = V^*$, the basis element $|M_\lambda\rangle \star |N_\lambda\rangle \in V_\lambda \star W_\lambda \subseteq \check{S}(\check{X})$ corresponds to the matrix coefficient $u_{M_\lambda N_\lambda}$ of the comodule V_λ of $\mathscr{O}(GL_q(V))$ under the isomorphism in the proof of Proposition 8.8 (1).

It is of interest to know explicit transformation matrices connecting the Gelfand-Tsetlin bases of $\check{S}^r\check{X}$ and $\check{\Lambda}^r\check{X}$ with their standard monomial bases in Proposition 8.8. In other words, we want to know the decompositions in Proposition 8.9 (1) and (2) in terms of the monomial bases. When $W = V^*$, this amounts to finding explicit formulae for the matrix coefficients of irreducible representations of $GL_q(V)$. This problem has been studied intensively in the literature. When $d_V = 2$, explicit formulae for matrix coefficients in terms little q-Jacobi polynomials are known. In general, the problem is not completely understood at present; see the survey [60] and the references therein.

The advantage of working with the Gelfand-Tsetlin bases of $\check{S}(\check{X}), \check{\Lambda}(\check{X})$, instead of the standard monomial bases in Proposition 8.8 is that multiplication is simpler in terms of the former, and has explicit formulae in terms of Clebsch-Gordon coefficients. We now state these formulae.

When $W = V^*$, the following multiplication formula for matrix coefficients can be deduced from (46), (47), and the bialgebra structure of $\mathscr{O}(M_q(V))$ (see [30, §7.2.2]):

$$u_{N_\lambda R_\lambda} u_{K_\mu S_\mu} = \sum_{\nu, M_\nu, L_\nu} \left(\sum_i C_{N_\lambda K_\mu M_\nu, i} \overline{C_{R_\lambda S_\mu L_\nu, i}} \right) u_{M_\nu L_\nu}, \tag{76}$$

where the bar denotes complex conjugation.

It follows that multiplication in the Gelfand-Tsetlin basis $\bigsqcup_{r\geq 0} B^{\mathrm{GT}}(\check{S}^r\check{X})$ of $\check{S}(\check{X})$ is given by

$$(|N_\lambda\rangle \star |R_\lambda\rangle)(|K_\mu\rangle \star |S_\mu\rangle) = \sum_{\nu, M_\nu, L_\nu} \left(\sum_i \overline{C_{N_\lambda K_\mu M_\nu, i}}\ \overline{C_{R_\lambda S_\mu L_\nu, i}} \right) |M_\nu\rangle \star |L_\nu\rangle, \tag{77}$$

where λ, μ are partitions with at most $\min(d_V, d_W)$ parts.

Similarly, multiplication in the basis $\bigsqcup_{r\geq 0} B^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ of $\check{\Lambda}(\check{X})$ is given by

$$(|N_\lambda\rangle \star |R_{\lambda'}\rangle)(|K_\mu\rangle \star |S_{\mu'}\rangle) = \sum_{\nu, M_\nu, L_\nu} \left(\sum_i \overline{C_{N_\lambda K_\mu M_\nu, i}}\ \overline{C_{R_{\lambda'} S_{\mu'} L_{\nu'}, i}} \right) |M_\nu\rangle \star |L_{\nu'}\rangle, \tag{78}$$

where λ, μ are partitions with $\leq d_V$ parts and largest part $\leq d_W$.

8.4. **Examples and computations for** $\mathscr{O}(M_q(\check{X}))$**.** Here we give some flavor of the defining relations of $\mathscr{O}(M_q(\check{X}))$ in terms of the bases defined in §8.2. We also show that the reduction system (43) for $\mathscr{O}(M_q(V))$ does not have an analog (in any way we can determine) in the nonstandard case. This means that computations in $\mathscr{O}(M_q(\check{X}))$ as well as the corepresentation theory of $\mathscr{O}(M_q(\check{X}))$ are significantly more difficult than in the standard case.

In this section and in examples later on, we use the notation

$$z_{\rho(i,j)}^{\rho(k,l)} = y_{\rho(i,j)} * y^{\rho(k,l)} = x_{\substack{i \\ j}} * x^{\substack{k \\ l}},$$

where y and ρ are as in §7, and $*$ is our symbol for tensor product in this setting as explained in §2.2. We also drop the $\otimes$ symbol for elements of $\check{Z}^{\otimes r}$ so that $z_a^{a'} z_b^{b'}$ means $z_a^{a'} \otimes z_b^{b'}$. For example,

$$z_2^1 z_3^4 = y_{23} * y^{14} = x_{\substack{12\\21}} * x^{\substack{12\\12}}.$$

Let $\check{B}_+^{\check{X}}, \check{B}_-^{\check{X}}$ be the bases of $\check{S}^2\check{X}$ and $\check{\Lambda}^2\check{X}$ from (68). Let $\check{B}_+^{\check{X}^*}$ and $\check{B}_-^{\check{X}^*}$ be defined similarly, with the elements y^{ij} in place of y_{ij}, i.e. $\check{B}_\pm^{\check{X}^*} := \alpha_2(\check{B}_\pm^{\check{X}})$ where α_2 is as in (65). With these bases, the defining relations of $\mathscr{O}(M_q(\check{X}))$ take the form:

$$\begin{aligned} b'_+ * b_- = 0, &\quad b'_+ \in \check{B}_+^{\check{X}}, b_- \in \check{B}_-^{\check{X}^*}, \\ b'_- * b_+ = 0, &\quad b'_- \in \check{B}_-^{\check{X}}, b_+ \in \check{B}_+^{\check{X}^*}. \end{aligned} \tag{79}$$

Example 8.10. Let $d_V = d_W = 2$. Let

$$\begin{aligned} \mathrm{NST}((2)) &= \left\{ \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}, \boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}, \boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}, \boxed{\begin{smallmatrix}3\\2\end{smallmatrix}}, \boxed{\begin{smallmatrix}2\\3\end{smallmatrix}} \right\}, \\ \widetilde{\mathrm{NST}}((1,1)) &= \left\{ \widetilde{\boxed{1\,|\,1}}, \widetilde{\boxed{2\,|\,2}}, \widetilde{\boxed{1\,|\,2}}, \widetilde{\boxed{3\,|\,3}}, \widetilde{\boxed{4\,|\,4}}, \widetilde{\boxed{3\,|\,4}}, \widetilde{\boxed{1\,|\,3}}, \widetilde{\boxed{2\,|\,4}}, \widetilde{\boxed{1\,|\,4}}, \widetilde{\boxed{4\,|\,1}} \right\}, \end{aligned}$$

be as in (70) and (71).

The defining relations (79) of $\mathscr{O}(M_q(\check{X}))$ are now 120 in number. We show one such typical relation below (to avoid extra notation, we use the same symbol for an element of $\check{B}_\pm^{\check{X}^*}$ as its corresponding element of $\check{B}_\pm^{\check{X}}$):

$$\begin{aligned} 0 = \widetilde{\boxed{1\,|\,4}} * \boxed{\begin{smallmatrix}3\\2\end{smallmatrix}} &= \tfrac{1}{[2]^2}(q^2 x_{\substack{12\\12}} + q x_{\substack{12\\21}} + q x_{\substack{21\\12}} + x_{\substack{21\\21}}) * \tfrac{1}{[2]}(q x^{\substack{12\\21}} - x^{\substack{12\\12}} + x^{\substack{21\\21}} - q^{-1} x^{\substack{21\\12}}) \\ &= \tfrac{1}{[2]^2}(q^2 y_{14} + q y_{23} + q y_{32} + y_{41}) * \tfrac{1}{[2]}(q y^{23} - y^{14} + y^{41} - q^{-1} y^{32}) \\ &= \tfrac{1}{[2]^3}\Big(+q^3 z_1^2 z_4^3 + q^2 z_2^2 z_3^3 + q^2 z_3^2 z_2^3 + q z_4^2 z_1^3 \\ &\qquad -q^2 z_1^1 z_4^4 - q z_2^1 z_3^4 - q z_3^1 z_2^4 - z_4^1 z_1^4 \\ &\qquad +q^2 z_1^4 z_4^1 + q z_2^4 z_3^1 + q z_3^4 z_2^1 + z_4^4 z_1^1 \\ &\qquad -q z_1^3 z_4^2 - z_2^3 z_3^2 - z_3^3 z_2^2 - q^{-1} z_4^3 z_1^2 \Big). \end{aligned} \tag{80}$$

The $2\binom{d_X+1}{2}\binom{d_X}{2}$ relations of (79), after taking appropriate linear combinations, can be recast in the form of a reduction system—just as the relations (38) were recast in the form of a reduction system (43)—where each reduction rule is of the form

$$z_a^{a'} z_b^{b'} = \sum_i \alpha_i z_{a_i}^{a'_i} z_{b_i}^{b'_i},$$

each $z_{a_i}^{a'_i} z_{b_i}^{b'_i}$ being *descending*, meaning that $(a_i, a'_i) \geq (b_i, b'_i)$ (say, lexicographically). The resulting reduction system is described in Appendix A. It turns out that this system does not satisfy the diamond property. For example, the monomial $z_1^1 z_1^2 z_2^3$, when reduced in two different ways, yields the following two distinct linear combinations of descending monomials:

$$\begin{aligned}
l_{121} &= (-1+q^2)\cdot z_2^1 z_1^3 z_1^2 + \frac{q^2-1}{[2]}\cdot z_2^1 z_1^4 z_1^1 + \frac{q-q^{-1}}{[2]}\cdot z_2^2 z_1^3 z_1^1 \\
&\quad + \frac{2q}{[2]}\cdot z_2^3 z_1^2 z_1^1 + \frac{1-q^2}{[2]}\cdot z_2^4 z_1^1 z_1^1, \\
l_{212} &= \frac{q^3+q-3q^{-1}+q^{-3}}{[2]}\cdot z_2^1 z_1^3 z_1^2 + \frac{2q-2q^{-1}}{[2]^2}\cdot z_2^1 z_1^4 z_1^1 + \frac{2-2q^{-2}}{[2]^2}\cdot z_2^2 z_1^3 z_1^1 \\
&\quad + \frac{q^2+4-q^{-2}}{[2]^2}\cdot z_2^3 z_1^2 z_1^1 - \frac{2q-2q^{-1}}{[2]^2}\cdot z_2^4 z_1^1 z_1^1.
\end{aligned}$$

This means we have the following nontrivial relation among descending monomials:

$$l_{121} - l_{212} = 0.$$

See Appendix A for details.

This failure of the diamond property has the following consequence:

Proposition 8.11. *The Poincaré series of $\mathscr{O}(M_q(\check{X}))$ does not, in general, coincide with the Poincaré series of the classical $\mathscr{O}(M(X))$.*

Here by Poincaré series of $\mathscr{O}(M_q(\check{X}))$ we mean the series

$$\sum_{r\geq 0} \dim(\mathscr{O}(M_q(\check{X}))_r)\, t^r,$$

where $\mathscr{O}(M_q(\check{X}))_r$ denotes the degree r component of $\mathscr{O}(M_q(\check{X}))$. As an example, when $d_V = d_W = 2$, $\dim(\mathscr{O}(M_q(\check{X}))_3) = 688$, whereas the classical $\dim(\mathscr{O}(M(X))_3) = 816$; this example will be explained thoroughly in §12.3.

Proposition 8.11 has important consequences. In the standard case, the Poincaré series of $\mathscr{O}(M_q(V))$ coincides with the Poincaré series of the classical $\mathscr{O}(M(V))$. By the Peter-Weyl theorem, this implies that the irreducible representations of $GL_q(V)$ are in one-to-one correspondence with those of $GL(V)$, and dimensions agree under this correspondence. Intuitively, this is why the irreducible representations of $GL_q(V)$ turn out to be deformations of the irreducible representations of $GL(V)$. Proposition 8.11 implies that this is no longer true for the nonstandard quantum group.

9. Nonstandard determinant and minors

Here we define the left and right nonstandard determinant and minors of $M_q(\check{X})$. After some examples in §9.2, we show (§9.3) that the left and right determinants and minors with respect to the orthonormal Gelfand-Tsetlin basis agree. Finally in §9.4, we give explicit formulae for certain nonstandard minors and present an intriguing conjecture about lengths of canonical basis elements related to these minors.

9.1. **Definitions.** Recall from Proposition 8.6 (2) that $\check{\Lambda}^r\check{X}^*$ (resp. $\check{\Lambda}^r\check{X}$) is a right (resp. left) $\mathscr{O}(M_q(\check{X}))$-comodule and let

$$\begin{aligned}\varphi_r^R &: \check{\Lambda}^r\check{X}^* \to \check{\Lambda}^r\check{X}^* \otimes \mathscr{O}(M_q(\check{X})),\\ \varphi_r^L &: \check{\Lambda}^r\check{X} \to \mathscr{O}(M_q(\check{X})) \otimes \check{\Lambda}^r\check{X}\end{aligned} \tag{81}$$

be the corresponding right and left corepresentations.

Recall that $\Omega_r^{d_X}$ is the set of subsets of $[d_X] = [d_V d_W]$ of size r. For a subset $I \in \Omega_r^{d_X}$, with $I = \{i_1, \dots, i_r\}$, $i_1 < i_2 < \cdots < i_r$, let y_I be the monomial $y_{i_1} \cdots y_{i_r}$ in the notation of Proposition 8.8. By this proposition, the set of standard monomials $\{y_I\}_{I \in \Omega_r^{d_X}}$ is a basis of $\check{\Lambda}^r\check{X}$.

We define the *right nonstandard determinant* $\check{D}^R$ to be the matrix coefficient of the right comodule $\check{\Lambda}^{d_X}\check{X}^*$, which is independent of the choice of basis as $\check{\Lambda}^{d_X}\check{X}^*$ is one-dimensional. The *left nonstandard determinant* $\check{D}^L$ is the matrix coefficient of the left comodule $\check{\Lambda}^{d_X}\check{X}$. The nonstandard determinants are nonzero since $\epsilon(\check{D}^R) = \epsilon(\check{D}^L) = 1$, where ϵ is the counit.

More generally, the *right nonstandard r-minors* of $M_q(\check{X})$ in the standard monomial basis are defined to be the matrix coefficients of the right corepresentation φ_r^R in the standard monomial basis of $\check{\Lambda}^r\check{X}^*$: for $I \in \Omega_r^{d_X}$, define the right nonstandard r-minors $\check{D}_J^{I,R}$ by

$$\varphi_r^R(y^I) = \sum_{J \in \Omega_r^{d_X}} y^J \otimes \check{D}_J^{I,R}.$$

The *left nonstandard r-minors* $\check{D}_J^{I,L}$ are defined by

$$\varphi_r^L(y_J) = \sum_{I \in \Omega_r^{d_X}} \check{D}_J^{I,L} \otimes y_I.$$

Let $M_\wedge^{r,R} = (\check{D}_J^{I,R})$ and $M_\wedge^{r,L} = (\check{D}_J^{I,L})$ denote the corresponding coefficient matrices.

Similarly, define the right and left nonstandard minors of $M_q(\check{X})$ in the orthonormal Gelfand-Tsetlin basis to be the matrix coefficients of the right and left corepresentations φ_r^R and φ_r^L in the orthonormal Gelfand-Tsetlin bases $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}^*)$ and $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ of $\check{\Lambda}^r\check{X}^*$ and $\check{\Lambda}^r\check{X}$. Here $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ is like the basis $B^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ of (75), except may differ from it by a diagonal transformation; orthonormal must be interpreted in a certain way here, which is explained in the proof of Proposition-Definition 9.3. Also, the bases $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}^*)$ and $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ are related by $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}^*) = \alpha_r(B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}))$, where α_r is as in (65); the same notation $|M_\lambda\rangle \star |N_{\lambda'}\rangle$ will be used for both bases.

These minors are defined explicitly as follows: for $|M_\lambda\rangle \star |N_{\lambda'}\rangle \in B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}^*)$, $\lambda \vdash r$, define the *right nonstandard r-minors* $\check{D}_{K_\mu, L_{\mu'}}^{M_\lambda, N_{\lambda'}, R}$ by

$$\varphi_r^R(|M_\lambda\rangle \star |N_{\lambda'}\rangle) = \sum_{|K_\mu\rangle \star |L_{\mu'}\rangle \in B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X}^*)} |K_\mu\rangle \star |L_{\mu'}\rangle \otimes \check{D}_{K_\mu, L_{\mu'}}^{M_\lambda, N_{\lambda'}, R}.$$

The *left nonstandard r-minors* $\check{D}^{M_\lambda,N_{\lambda'},L}_{K_\mu,L_{\mu'}}$ are defined similarly. Let $\tilde{M}^{r,R}_\wedge = \big(\check{D}^{M_\lambda,N_{\lambda'},R}_{K_\mu,L_{\mu'}}\big)$ and $\tilde{M}^{r,L}_\wedge = \big(\check{D}^{M_\lambda,N_{\lambda'},L}_{K_\mu,L_{\mu'}}\big)$ denote the corresponding coefficient matrices.

9.2. **Nonstandard minors in the two-row case.**

Example 9.1. Let us first give an explicit formula for $\check{D}^L$ and $\check{D}^R$ when $d_V = d_W = 2$. The nonstandard exterior algebra $\check{\Lambda}(\check{X})$ is the quotient of $T(\check{X}) = \bigoplus_{r\geq 0} \check{X}^{\otimes r}$ by the two-sided ideal $\check{\mathcal{I}}_\wedge$ generated by the elements

$$\widetilde{\boxed{1}\boxed{1}},\ \widetilde{\boxed{2}\boxed{2}},\ \widetilde{\boxed{1}\boxed{2}},\ \widetilde{\boxed{3}\boxed{3}},\ \widetilde{\boxed{4}\boxed{4}},\ \widetilde{\boxed{3}\boxed{4}},\ \widetilde{\boxed{1}\boxed{3}},\ \widetilde{\boxed{2}\boxed{4}},\ \widetilde{\boxed{1}\boxed{4}},\ \widetilde{\boxed{4}\boxed{1}}$$

from (71). The degree r component $\check{\Lambda}^r\check{X}$ has standard monomial basis $\{y_{i_1}\cdots y_{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq d_X\}$, where

$$y_1 = x_{\substack{1\\1}}, y_2 = x_{\substack{1\\2}}, y_3 = x_{\substack{2\\1}}, y_4 = x_{\substack{2\\2}}.$$

The relations $\widetilde{\boxed{1}\boxed{4}}, \widetilde{\boxed{4}\boxed{1}} = 0$ imply that $y_4y_1 = -y_1y_4$. Since y_1 and y_4 quasicommute with all the y_i's, and $y_i^2 = 0$ for all i, it is easy to show that $y_iy_jy_ky_l$ is zero modulo $\check{\mathcal{I}}_\wedge$, unless it is of the form $\prod y_{\sigma(i)}$, for some permutation σ, or is either $y_2y_3y_2y_3$ or $y_3y_2y_3y_2$. Furthermore, we have

$$\prod y_{\sigma(i)} = (-1)^{\ell(\sigma)} q^{\iota(\sigma)} y_1y_2y_3y_4,$$

where $\ell(\sigma)$ is the number of inversions in σ, and $\iota(\sigma)$ is the number of inversions in σ not involving $(2,3)$ or $(1,4)$. Also

$$\begin{aligned} y_2y_3y_2y_3 &= (q^{-1}-q)q^2y_1y_2y_3y_4 \\ y_3y_2y_3y_2 &= (q-q^{-1})q^2y_1y_2y_3y_4. \end{aligned}$$

The right (resp. left) determinant $\check{D}^R$ (resp. $\check{D}^L$) is the the matrix coefficient of the right comodule $\check{\Lambda}^4\check{X}^*$ (resp. left comodule $\check{\Lambda}^4\check{X}$). From the preceding remarks, it easily follows that (see §8.4 for notation)

$$\begin{aligned} \check{D}^R &= \Big(\sum_\sigma (-1)^{\ell(\sigma)} q^{\iota(\sigma)} z^i_{\sigma(i)}\Big) + (q^{-1}-q)q^2 z^1_2 z^2_3 z^3_2 z^4_3 + (q-q^{-1})q^2 z^1_3 z^2_2 z^3_3 z^4_2, \\ \check{D}^L &= \Big(\sum_\sigma (-1)^{\ell(\sigma)} q^{\iota(\sigma)} z^{\sigma(i)}_i\Big) + (q^{-1}-q)q^2 z^2_1 z^3_2 z^2_3 z^3_4 + (q-q^{-1})q^2 z^3_1 z^2_2 z^3_3 z^2_4. \end{aligned}$$

These expressions are equal in $\mathscr{O}(M_q(\check{X}))$, which can be checked from the relations (79), or we can appeal to the more abstract argument given in §9.3. Compare these with the formula (42) for the standard quantum determinant.

We will also give another formula for the nonstandard determinant in Proposition 9.5 in terms of the upper canonical basis $B^{d_X}_V \star B^{d_X}_W$. In this case it is

$$\check{D}^L = \check{D}^R = \Big(c_{\substack{2121\\2211}} - c_{\substack{2211\\2121}}\Big) * x^{\substack{2121\\2211}}.$$

Example 9.2. The nonstandard minors in the two-row, $r = 2$, case are as follows (see Example 8.7 for notation):

$$\begin{pmatrix} -q\boxed{\begin{matrix}1\\2\end{matrix}} \\ -q\boxed{\begin{matrix}1\\3\end{matrix}} \\ -q^{-1}\boxed{\begin{matrix}3\\2\end{matrix}} - q\boxed{\begin{matrix}2\\3\end{matrix}} \\ \boxed{\begin{matrix}3\\2\end{matrix}} - \boxed{\begin{matrix}2\\3\end{matrix}} \\ -q\boxed{\begin{matrix}2\\4\end{matrix}} \\ -q\boxed{\begin{matrix}3\\4\end{matrix}} \end{pmatrix} \dot{*} - \frac{1}{[2]}\begin{pmatrix} \boxed{\begin{matrix}1\\2\end{matrix}} & \boxed{\begin{matrix}1\\3\end{matrix}} & \boxed{\begin{matrix}3\\2\end{matrix}} + \boxed{\begin{matrix}2\\3\end{matrix}} & -q\boxed{\begin{matrix}3\\2\end{matrix}} + q^{-1}\boxed{\begin{matrix}2\\3\end{matrix}} & \boxed{\begin{matrix}2\\4\end{matrix}} & \boxed{\begin{matrix}3\\4\end{matrix}} \end{pmatrix},$$

The coefficient matrix $M_\wedge^{2,R}$

$$-\frac{1}{[2]}\begin{pmatrix} \boxed{\begin{matrix}1\\2\end{matrix}} \\ \boxed{\begin{matrix}1\\3\end{matrix}} \\ \boxed{\begin{matrix}3\\2\end{matrix}} + \boxed{\begin{matrix}2\\3\end{matrix}} \\ -q\boxed{\begin{matrix}3\\2\end{matrix}} + q^{-1}\boxed{\begin{matrix}2\\3\end{matrix}} \\ \boxed{\begin{matrix}2\\4\end{matrix}} \\ \boxed{\begin{matrix}3\\4\end{matrix}} \end{pmatrix} \dot{*} \begin{pmatrix} -q\boxed{\begin{matrix}1\\2\end{matrix}} & -q\boxed{\begin{matrix}1\\3\end{matrix}} & -q^{-1}\boxed{\begin{matrix}3\\2\end{matrix}} - q\boxed{\begin{matrix}2\\3\end{matrix}} & \boxed{\begin{matrix}3\\2\end{matrix}} - \boxed{\begin{matrix}2\\3\end{matrix}} & -q\boxed{\begin{matrix}2\\4\end{matrix}} & -q\boxed{\begin{matrix}3\\4\end{matrix}} \end{pmatrix},$$

The coefficient matrix $M_\wedge^{2,L}$

where $\dot{*}$ is like $\dot{\otimes}$ of (30) using the tensor product $*$. Here the matrix $M_\wedge^{2,R}$ is with respect to the ordered basis $(y^{12}, y^{13}, y^{14}, y^{23}, y^{24}, y^{34})$ of $\check{\Lambda}^2\check{X}^*$ so that, for instance, its third column gives the entries in the second tensor factor of $\varphi_2^R(y^{14}) = \sum_{J\in\Omega_2^4} y^J \otimes \check{D}_J^{14,R}$. The matrix $M_\wedge^{2,L}$ is with respect to the ordered basis $(y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34})$ of $\check{\Lambda}^2\check{X}$ so that, for instance, its fourth row gives the entries in the second tensor factor of $\varphi_2^L(y_{23}) = \sum_{I\in\Omega_2^4} \check{D}_{23}^{I,L} \otimes y_I$.

In the two-row case, the orthonormal Gelfand-Tsetlin basis $B'^{\mathrm{GT}}(\check{\Lambda}^r\check{X})$ and the NSC basis (as defined in §14) of $\check{\Lambda}^r\check{X}$ differ by a diagonal transformation; this follows from the fact that the Gelfand-Tsetlin basis and projected upper canonical basis of $V^{\otimes r}$ differ by a diagonal transformation—see (69) and Example 5.1. For the example at hand, $B'^{\mathrm{GT}}(\check{\Lambda}^2\check{X})$ is given by

$$\boxed{\begin{matrix}1\\2\end{matrix}}^{\mathrm{GT}} = a\boxed{\begin{matrix}1\\2\end{matrix}}, \quad \boxed{\begin{matrix}1\\3\end{matrix}}^{\mathrm{GT}} = a\boxed{\begin{matrix}1\\3\end{matrix}}, \quad \boxed{\begin{matrix}3\\2\end{matrix}}^{\mathrm{GT}} = \boxed{\begin{matrix}3\\2\end{matrix}}, \quad \boxed{\begin{matrix}2\\3\end{matrix}}^{\mathrm{GT}} = \boxed{\begin{matrix}2\\3\end{matrix}}, \quad \boxed{\begin{matrix}2\\4\end{matrix}}^{\mathrm{GT}} = a\boxed{\begin{matrix}2\\4\end{matrix}}, \quad \boxed{\begin{matrix}3\\4\end{matrix}}^{\mathrm{GT}} = a\boxed{\begin{matrix}3\\4\end{matrix}},$$

where $a = \left(\frac{q}{[2]}\right)^{1/2}$. The coefficient matrices $\tilde{M}^{2,R}_{\wedge}$ and $\tilde{M}^{2,L}_{\wedge}$ with respect to the Gelfand-Tsetlin basis are

$$\begin{pmatrix} \begin{array}{|c|}\hline 1\\ 2\\ \hline\end{array}^{\mathrm{GT}} \\ \begin{array}{|c|}\hline 1\\ 3\\ \hline\end{array}^{\mathrm{GT}} \\ \begin{array}{|c|}\hline 3\\ 2\\ \hline\end{array}^{\mathrm{GT}} \\ \begin{array}{|c|}\hline 2\\ 3\\ \hline\end{array}^{\mathrm{GT}} \\ \begin{array}{|c|}\hline 2\\ 4\\ \hline\end{array}^{\mathrm{GT}} \\ \begin{array}{|c|}\hline 3\\ 4\\ \hline\end{array}^{\mathrm{GT}} \end{pmatrix} \dot{*} \begin{pmatrix} \begin{array}{|c|}\hline 1\\ 2\\ \hline\end{array}^{\mathrm{GT}} & \begin{array}{|c|}\hline 1\\ 3\\ \hline\end{array}^{\mathrm{GT}} & \begin{array}{|c|}\hline 3\\ 2\\ \hline\end{array}^{\mathrm{GT}} & \begin{array}{|c|}\hline 2\\ 3\\ \hline\end{array}^{\mathrm{GT}} & \begin{array}{|c|}\hline 2\\ 4\\ \hline\end{array}^{\mathrm{GT}} & \begin{array}{|c|}\hline 3\\ 4\\ \hline\end{array}^{\mathrm{GT}} \end{pmatrix}.$$

The coefficient matrix $\tilde{M}^{2,R}_{\wedge} = \tilde{M}^{2,L}_{\wedge}$

It will be shown in Proposition 9.5 that for any $r \in [d_X]$, the coefficient matrix $\tilde{M}^{r,R}_{\wedge} = \tilde{M}^{r,L}_{\wedge}$ has a similar form as an outer product.

9.3. **Symmetry of the determinants and minors.** A basic property of the standard quantum minors is the agreement of the left and right-handed versions (41). We now show that the same holds in the nonstandard case. This will be important for defining the Hopf algebra $\mathscr{O}(GL_q(\check{X}))$.

Proposition-Definition 9.3. *The left and right nonstandard determinants agree, so we can define*

$$\check{D} := \check{D}^L = \check{D}^R.$$

More generally, assuming that all objects are over $\mathbb{C}$ *and* q *is transcendental, the left and right nonstandard minors in the bases* $B'^{GT}(\check{\Lambda}^r\check{X})$ *and* $B'^{GT}(\check{\Lambda}^r\check{X}^*)$ *agree, so we can define*

$$\check{D}^{M_\lambda,N_{\lambda'}}_{K_\mu,L_{\mu'}} := \check{D}^{M_\lambda,N_{\lambda'},L}_{K_\mu,L_{\mu'}} = \check{D}^{M_\lambda,N_{\lambda'},R}_{K_\mu,L_{\mu'}}.$$

Equivalently, $\tilde{M}^{r,L}_{\wedge} = \tilde{M}^{r,R}_{\wedge}$.

In what follows, the tensor symbol $*$ will be treated as an "outer tensor" in the sense that we are only using the coalgebra structure of $\mathscr{O}(M_q(\check{X}))$ for this tensor, not its bialgebra structure: if N' is a left $\mathscr{O}(M_q(\check{X}))$-comodule and N is a right $\mathscr{O}(M_q(\check{X}))$-comodule, then we consider $N' * N$ as an $\mathscr{O}(M_q(\check{X}))$-bicomodule.

To prove Proposition-Definition 9.3, we will first show that $\check{\Lambda}^r\check{X}^*$ and $(\check{\Lambda}^r\check{X})_R$ are isomorphic as right $\mathscr{O}(M_q(\check{X}))$-comodules, where $(\cdot)_R$ is the notation for dualizing comodules explained in §2.5. Recall that $\check{Z} = \check{X} \otimes \check{X}^*$ and by (63), $\mathscr{O}(M_q(\check{X}))$ is the quotient of the tensor algebra $T(\check{Z})$ by the two-sided ideal $\check{\mathcal{I}}$ generated by $\check{\mathcal{I}}_2 = \check{S}^2\check{X} * \check{\Lambda}^2\check{X}^* \oplus \check{\Lambda}^2\check{X} * \check{S}^2\check{X}^*$. Let $\check{\mathcal{J}}$ be the two-sided ideal of $T(\check{Z})$ generated by $\check{S}^2\check{X} * \check{S}^2\check{X}^*$ and set

$$\mathcal{R} := T(\check{Z})/(\check{\mathcal{I}} + \check{\mathcal{J}}) \cong \mathscr{O}(M_q(\check{X}))/\overline{\check{\mathcal{J}}},$$

where $\overline{\check{\mathcal{J}}}$ denotes the image of $\check{\mathcal{J}}$ in $\mathscr{O}(M_q(\check{X}))$. The quotient coalgebra $\mathcal{R}$ turns out to be cosimple, as we now show, and will help us understand the nonstandard exterior algebra.

Lemma 9.4. *Let $\mathcal{R}$, $\check{\mathcal{I}}$, $\check{\mathcal{J}}$ be as above and $\mathcal{R}_r, \check{\mathcal{I}}_r, \check{\mathcal{J}}_r$ denote their degree r parts.*

1. *There is an isomorphism $\mathcal{R}_r \cong \check{\Lambda}^r \check{X} * \check{\Lambda}^r \check{X}^*$ of $\mathscr{O}(M_q(\check{X}))$-bicomodules.*
2. *The coalgebra $\mathcal{R}_r$ is cosimple.*
3. *There is an isomorphism $\check{\Lambda}^r \check{X}^* \cong (\check{\Lambda}^r \check{X})_R$ (resp. $\check{\Lambda}^r \check{X} \cong (\check{\Lambda}^r \check{X}^*)_L$) of right (resp. left) $\mathscr{O}(M_q(\check{X}))$-comodules.*

Proof. To prove (1), define

$$Y^2 := \check{S}^2 \check{X} * (\check{X}^*)^{\otimes 2},$$
$$Y'^2 := X^{\otimes 2} * \check{S}^2 \check{X}^*,$$

$$M^r := \sum_{i=1}^{r-1} \check{X}^{\otimes i-1} \otimes \check{S}^2 \check{X} \otimes (\check{X})^{\otimes r-i-1}, \ r > 2,$$
$$M'^r := \sum_{i=1}^{r-1} (\check{X}^*)^{\otimes i-1} \otimes \check{S}^2 \check{X}^* \otimes (\check{X}^*)^{\otimes r-i-1}, \ r > 2,$$
$$Y^r := \sum_{i=1}^{r-1} \check{Z}^{\otimes i-1} \otimes Y^2 \otimes \check{Z}^{\otimes r-i-1} = M^r * (\check{X}^*)^{\otimes r}, \ r > 2,$$
$$Y'^r := \sum_{i=1}^{r-1} \check{Z}^{\otimes i-1} \otimes Y'^2 \otimes \check{Z}^{\otimes r-i-1} = \check{X}^{\otimes r} * M'^r, \ r > 2.$$

All of the Y's are $\mathscr{O}(M_q(\check{X}))$-bicomodules, and M^r and M'^r are left and right $\mathscr{O}(M_q(\check{X}))$-comodules, respectively. The bicomodule $\mathcal{R}$ is the quotient of $T(\check{Z})$ by the two-sided ideal generated by $\check{\mathcal{I}}_2 + \check{\mathcal{J}}_2 = Y^2 + Y'^2$, hence we have the following isomorphisms of $\mathscr{O}(M_q(\check{X}))$-bicomodules

$$\mathcal{R}_r \cong \check{Z}^{\otimes r}/(Y^r + Y'^r) \tag{82}$$
$$\cong \check{X}^{\otimes r} * (\check{X}^*)^{\otimes r} / \big(M^r * (\check{X}^*)^{\otimes r} + \check{X}^{\otimes r} * M'^r\big) \tag{83}$$
$$\cong \big(\check{X}^{\otimes r}/M^r\big) * \big((\check{X}^*)^{\otimes r}/M'^r\big) \tag{84}$$
$$\cong \check{\Lambda}^r \check{X} * \check{\Lambda}^r \check{X}^*, \tag{85}$$

where the last isomorphism is by the definition of $\check{\Lambda}^r \check{X}$, and the second to last is just an application of the general fact that

for vector spaces $A \subseteq B$ and $A' \subseteq B'$, $B \otimes B'/(A \otimes B' + B \otimes A') \cong B/A \otimes B'/A'$.

Now statement (2) follows from (1) and by applying the following claim to the algebra $\mathcal{R}_r^*$ dual to the coalgebra $\mathcal{R}_r$.

Suppose H is a finite-dimensional algebra over a field and M (resp. N) is a left (resp. right) H-module. If $H \cong M \otimes N$ as H-bimodules and $\dim(M) = \dim(N)$, then H is split simple.

The claim holds because $H \cong M \otimes N$ implies that M is a faithful left H-module, hence $H \hookrightarrow \mathrm{End}(M)$ is an inclusion of algebras. Counting dimensions shows that this inclusion is an isomorphism.

Statement (3) follows from (2). $\square$

Proof of Proposition-Definition 9.3. Lemma 9.4 (3) implies that the coefficient matrices of $\check{\Lambda}^r \check{X}^*$ and $(\check{\Lambda}^r \check{X})_R$ are similar, i.e., there exists a nonsingular similarity matrix Q such that $\tilde{M}_\wedge^{r,L} = Q^{-1} \tilde{M}_\wedge^{r,R} Q$. We want to show that Q is the identity matrix. Since the decomposition of $\check{\Lambda}^r \check{X}^*$ as an $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodule is multiplicity-free (Proposition 8.9 (2)), a Gelfand-Tsetlin basis for $\check{\Lambda}^r \check{X}^*$ is uniquely determined up to a diagonal transformation. This means that the $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodule isomorphism $\check{\Lambda}^r \check{X}^* \xrightarrow{\cong} (\check{\Lambda}^r \check{X})_R$ must take $B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X}^*)$ to a basis of $(\check{\Lambda}^r \check{X})_R$ that differs from the basis dual to $B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X})$ by a diagonal transformation, i.e., Q is diagonal.

If Q were not the identity, the basis elements could be normalized by square roots of the entries of Q to fix this (we are assuming all objects are defined over $\mathbb{C}$ and q is transcendental). See the discussion below for a better explanation of this normalization. $\square$

A good way to say how the Gelfand-Tsetlin bases must be normalized uses the realization of $\check{\Lambda}^r \check{X}$ as a subset of $\check{X}^{\otimes r}$ described in §14. Recall from (65) that $\alpha_r : \check{X}^{\otimes r} \xrightarrow{\cong} (\check{X}^*)^{\otimes r}$ is the isomorphism induced by the standard bilinear form on $\check{X}^{\otimes r}$. Restricting α_r to $\check{\Lambda}^r \check{X}$ yields an isomorphism $\check{\Lambda}^r \check{X} \xrightarrow{\alpha_r} \check{\Lambda}^r \check{X}^*$ of vector spaces. Also, the composition $(\check{\Lambda}^r \check{X})_R \hookrightarrow (\check{X}^{\otimes r})_R \xrightarrow{\cong} (\check{X}^*)^{\otimes r}$ has image $\check{\Lambda}^r \check{X}^*$, so restricting the codomain yields an $\mathscr{O}(M_q(\check{X}))$-comodule isomorphism $\beta : (\check{\Lambda}^r \check{X})_R \xrightarrow{\cong} \check{\Lambda}^r \check{X}^*$. This follows, for instance, from Lemma 9.4 (3) and Theorem 12.1.

Now, given a basis B of $\check{\Lambda}^r \check{X}$, there are two ways to obtain a basis of $\check{\Lambda}^r \check{X}^*$: one is to take the image $\alpha_r(B)$, and the other is to take the basis B^* dual to B and apply β to obtain $\beta(B^*)$. We want to know when these bases agree. This is equivalent to $\beta^{-1}(\alpha_r(B))$ and B being dual bases, which exactly means that B is orthonormal with respect to the standard bilinear form restricted to $\check{\Lambda}^r \check{X}$. In the present situation, $B = B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X})$ and, by definition, $B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X}^*) = \alpha_r(B)$. The matrix Q is the transition matrix between $\beta(B^*)$ and $\alpha_r(B)$, so Q being the identity matrix is equivalent to $B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X})$ being orthonormal.

9.4. **Formulae for nonstandard minors.** Here we give as explicit as possible formulae for the nonstandard determinant $\check{D}$ and for certain nonstandard minors in the orthonormal Gelfand-Tsetlin basis.

Proposition 9.5. *In terms of the bases $B'^{GT}(\check{\Lambda}^r \check{X}) \subseteq \check{\Lambda}^r \check{X} \subseteq \check{X}^{\otimes r}$ and $B'^{GT}(\check{\Lambda}^r \check{X}^*) \subseteq \check{\Lambda}^r \check{X}^* \subseteq (\check{X}^*)^{\otimes r}$, there holds*

$$\check{D}_{K_\mu, L_{\mu'}}^{M_\lambda, N_{\lambda'}} = |K_\mu\rangle \star |L_{\mu'}\rangle * |M_\lambda\rangle \star |N_{\lambda'}\rangle. \tag{86}$$

In terms of the NSC basis of §14, the highest weight nonstandard minors are given by

$$\check{D}^{Z_\lambda, Z_{\lambda'}}_{Z_\lambda, Z_{\lambda'}} = NSC_{Z_\lambda, Z_{\lambda'}} * x^{\mathbf{k}}_{\mathbf{l}} := \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} c_{\substack{RSK^{-1}(Z_\lambda, Q) \\ RSK^{-1}(Z_{\lambda'}, Q^T)}} * x^{\mathbf{k}}_{\mathbf{l}}, \tag{87}$$

where $\mathbf{k} = RSK^{-1}(Z_\lambda, (Z^*_{\lambda'})^T)$, $\mathbf{l} = RSK^{-1}(Z_{\lambda'}, Z^*_{\lambda'})$, *and* Z_λ, $Z^*_{\lambda'}$, *and* $\ell(Q^T)$ *are as in §2.3.*

Proof. The first formula (86) follows from the proof of Proposition-Definition 9.3 and the discussion following it. Maintain the notation of this discussion. The coefficient matrix of $\check{X}^{\otimes r}$ in the standard monomial basis is $\left(y_{\mathbf{j}} * y^{\mathbf{j}'}\right)_{\mathbf{j},\mathbf{j}' \in [d_X]^r}$. In general, for any orthonormal basis B of $\check{X}^{\otimes r}$ with respect to the standard bilinear form, the coefficient matrix with respect to B has the similar form $\left(b * \alpha_r(b')\right)_{b,b' \in B}$. The coefficient matrix of any $\mathscr{O}(M_q(\check{X}))$-subcomodule N of $\check{X}^{\otimes r}$ with respect to an orthonormal basis of N has a similar form. In the present situation, the subcomodule is $\check{\Lambda}^r \check{X}$ and the orthonormal basis is $B'^{\mathrm{GT}}(\check{\Lambda}^r \check{X})$.

The right-hand side of (87) lies in $\check{\Lambda}^r \check{X} * (\check{X}^*)^{\otimes r}$. By Lemma 9.4, $\check{\Lambda}^r \check{X}$ is irreducible, and thus by Theorem 1.4(d), the image of $\check{\Lambda}^r \check{X} * (\check{X}^*)^{\otimes r}$ in $\mathscr{O}(M_q(\check{X}))$ is equal to the coefficient coalgebra $\check{\Lambda}^r \check{X} * \check{\Lambda}^r \check{X}^* \subseteq \mathscr{O}(M_q(\check{X}))$. Considering $\check{\Lambda}^r \check{X} * \check{\Lambda}^r \check{X}^*$ as an $\mathscr{O}(M_q(V)) \otimes \mathscr{O}(M_q(W))$-bicomodule, the highest weight space of left and right weight (λ, λ') is one-dimensional, so must be spanned by the right-hand side of (87) provided it is nonzero.

Since the nonstandard minors are matrix coefficients, we have $\epsilon(\check{D}^{Z_\lambda, Z_{\lambda'}}_{Z_\lambda, Z_{\lambda'}}) = 1$. Then to prove (87), it remains to check that the counit evaluates to 1 on this quantity. This amounts to checking that the coefficient of $x_{\substack{\mathbf{k} \\ \mathbf{l}}}$ in $c_{\substack{\mathbf{k}' \\ \mathbf{l}'}}$, for $\mathbf{k}', \mathbf{l}'$ such that $P(\mathbf{k}') = Z_\lambda, P(\mathbf{l}') = Z_{\lambda'}$, is 1 if $\mathbf{k} = \mathbf{k}', \mathbf{l} = \mathbf{l}'$ and 0 otherwise. By theorem 6.4 (iii), the coefficient of $v_{\mathbf{l}}$ in $c_{\mathbf{l}'}$ is just an adjusted Kazhdan-Lusztig polynomial $\overline{P^-_{d(\mathbf{l}),d(\mathbf{l}')}}$, where $P^-_{d(\mathbf{l}),d(\mathbf{l}')}$ is as in (24). Noting that the $\mathbf{l}$ defined in the proposition is the maximal in Bruhat order element of the minimal coset representatives ${}^{J_{\lambda'}}W$, the desired result follows from the fact that $P'_{x,w}$ is 0 unless $x \leq w$. □

The *length squared* $\|x\|^2$ of an element $x \in \check{X}^{\otimes r}$ is defined to be $\langle x, x \rangle$ using the standard bilinear form of (65). Let $[k]_-$ be the q-analog $q^{-k+1}[k] = 1 + q^{-2} + \cdots + q^{-2k+2}$ of k. The following computation for $\lambda = (r)$ should be mostly familiar:

$$\|\mathrm{NSC}_{Z_\lambda, Z_{\lambda'}}\|^2 = \|c_{\substack{1\,2\,\cdots\,r \\ r\,r-1\cdots 1}}\|^2 = \Big\| \sum_{\sigma \in \mathcal{S}_r} (-q)^{-\ell(\sigma)} x_{\substack{1\,2\,\cdots\,r \\ (r\,r-1\cdots 1)\sigma}} \Big\|^2 = \sum_{\sigma \in \mathcal{S}_r} q^{-2\ell(\sigma)} = [r]_-!. \tag{88}$$

We conjecture the following generalization:

Conjecture 9.6. *The length squares of the highest weight nonstandard columns are given by the following q-analogs of $r!$:*

$$\|NSC_{Z_\lambda, Z_{\lambda'}}\|^2 := \Big\| \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} c_{\substack{RSK^{-1}(Z_\lambda, Q) \\ RSK^{-1}(Z_{\lambda'}, Q^T)}} \Big\|^2 = |SYT(\lambda)| \prod_{b \in \lambda} [h(b)]_-,$$

where $\lambda \vdash_{d_V} r$, $\ell(\lambda') \leq d_W$, the product ranges over the squares b of the diagram of λ, and $h(b)$ denotes the hook length of b. Here Z_λ is the superstandard tableau of shape and content λ and $c_{\substack{\mathbf{k}\\ \mathbf{l}}}$ is the product $c^V_{\mathbf{k}} \star c^W_{\mathbf{l}}$ of upper canonical basis elements (see §7).

This conjecture has been checked for all λ of size less than or equal to 6. We suspect that these coefficients reflect something inherent in the integral form $\check{\Lambda}^r \check{X}^{\mathbf{A}}$ of $\check{\Lambda}^r \check{X}$ (see §14.1), rather than something specifically about canonical bases.

Remark 9.7. An interesting and, as far as we know, unstudied problem is to compute the lengths of canonical basis elements $c_{\mathbf{k}}$ (here the symmetric bilinear form is that for which the monomial basis of $V^{\otimes r}$ is orthonormal). These lengths are polynomials in q^{-1}, and, by the nonnegativity of type A Kazhdan-Lusztig polynomials, have nonnegative integer coefficients. Finding a combinatorial interpretation for these lengths seems like a difficult but tractable problem, or at least much easier than understanding Kazhdan-Lusztig polynomials combinatorially.

10. The nonstandard quantum groups $GL_q(\check{X})$ and $\mathtt{U}_q(\check{X})$

Here we define the nonstandard coordinate Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ of the (virtual) *nonstandard quantum group* $GL_q(\check{X})$ by inverting the determinant of $\mathscr{O}(M_q(\check{X}))$ and using the results of the previous section to put a Hopf structure on the resulting bialgebra. A natural $*$-structure is put on $\mathscr{O}(GL_q(\check{X}))$, and $\mathtt{U}_q(\check{X})$ is defined to be the virtual object corresponding to this $*$-Hopf algebra—this is the analog of the unitary group in this setting. Finally, we restate our main theorem about $GL_q(\check{X})$ and $\mathtt{U}_q(\check{X})$ (Theorem 1.4, restated as Theorem 10.7) and assemble its proof.

10.1. **Hopf structure.** To define a cofactor matrix of $\check{\mathbf{z}}$ we need the following.

Proposition 10.1. *The left $\mathscr{O}(M_q(\check{X}))$-comodule homomorphism*
$$\check{\Lambda}^r \check{X} \otimes \check{\Lambda}^{d_X - r} \check{X} \to \check{\Lambda}^{d_X} \check{X}, \;\; y_I \otimes y_J \mapsto y_I y_J$$
is a nondegenerate pairing. A similar statement holds for the corresponding right comodules.

Proof. Note that the given map is a left $\mathscr{O}(M_q(\check{X}))$-comodule homomorphism because $\check{\Lambda}(\check{X})$ is a left $\mathscr{O}(M_q(\check{X}))$-comodule algebra. The nondegeneracy follows from Proposition 8.8, the reduction system (73), and the nondegeneracy of the pairing in the $q = 1$ case. □

Proposition 10.2. *There exists a cofactor matrix $\tilde{\check{\mathbf{z}}}$ so that*
$$\tilde{\check{\mathbf{z}}}\check{\mathbf{z}} = \check{\mathbf{z}}\tilde{\check{\mathbf{z}}} = \check{D}\mathbf{I}.$$

Proof. The matrix form of the nondegenerate pairing in Proposition 10.1 yields a q-analog of Laplace expansion for $\mathscr{O}(M_q(\check{X}))$ in the present context. In particular, we have nondegenerate pairings
$$\check{\Lambda}^{d_X - 1} \check{X}^* \otimes \check{\Lambda}^1 \check{X}^* \xrightarrow{m^*_{d_X-1,1}} \check{\Lambda}^{d_X} \check{X}^*, \tag{89}$$
$$\check{\Lambda}^1 \check{X} \otimes \check{\Lambda}^{d_X - 1} \check{X} \xrightarrow{m_{1,d_X-1}} \check{\Lambda}^{d_X} \check{X}. \tag{90}$$

The homomorphism in (89) (resp. (90)) is a right (resp. left) $\mathscr{O}(M_q(\check{X}))$-comodule homomorphism. Note that $\check{\Lambda}^1\check{X} = \check{X}$ is the fundamental vector representation.

Let $B'^{\mathrm{GT}}(\check{\Lambda}^1\check{X}) = \{x_{\substack{i\\j}}\}$ and $B'^{\mathrm{GT}}(\check{\Lambda}^{d_X-1}\check{X}) = \{x^\vee_{\substack{i\\j}}\}$ be the Gelfand-Tsetlin bases of $\check{X}$ and $\check{\Lambda}^{d_X-1}\check{X}$ as in the proof of Proposition-Definition 9.3; in terms of our previous notation for Gelfand-Tsetlin basis elements, $x^\vee_{\substack{i\\j}} := |M_\lambda\rangle \star |N_{\lambda'}\rangle$, where $\lambda = ({d_W}^{d_V-1}, d_W-1)$ and M_λ (resp. $N_{\lambda'}$) is superstandard in the first d_W-1 (resp. d_V-1) columns and its last column has entries $[d_V]\setminus\{i\}$ (resp. $[d_W]\setminus\{j\}$). Let $\check{d} = x_{\substack{1\\1}}x^\vee_{\substack{1\\1}}$ be our chosen basis element for $\check{\Lambda}^{d_X}\check{X}$. Then since $\check{\Lambda}^{d_X-1}\check{X} \cong \Lambda_q^{d_V-1}V\otimes(\Lambda_q^{d_V}V)^{\otimes d_W-1}\star\Lambda_q^{d_W-1}W\otimes(\Lambda_q^{d_W}W)^{\otimes d_V-1}$ as an $\mathscr{O}(M_q(V))\star\mathscr{O}(M_q(W))$-comodule, just as in the standard case (see the proof of [30, Proposition 8, Chapter 9]), there holds

$$x_{\substack{i\\j}}\otimes x^\vee_{\substack{k\\l}} \xrightarrow{m_{1,d_X-1}} \delta_{ik}\delta_{jl}(-q)^{i-1+j-1}\check{d}, \tag{91}$$

Applying the left corepresentation maps corresponding to the comodules in (90) to both sides of (91) implies that

$$\sum_{k,l}(-q)^{k+l-i-j}\, z^{\rho(k,l)}_{\rho(r,s)}\,\check{D}^{\hat{k}\hat{l},L}_{\hat{i}\hat{j}} = \delta_{ri}\delta_{sj}\check{D}^L,$$

where the $\check{D}^{\hat{k}\hat{l},L}_{\hat{i}\hat{j}}$ is an abbreviated notation for the entries of $\tilde{M}^{d_X-1,L}_{\wedge}$. Thus the cofactor matrix $\tilde{\check{\mathbf{z}}}$ with entries $\tilde{z}^{\rho(i,j)}_{\rho(k,l)} := (-q)^{k+l-i-j}\check{D}^{\hat{k}\hat{l},L}_{\hat{i}\hat{j}}$ satisfies

$$\check{\mathbf{z}}\tilde{\check{\mathbf{z}}} = \check{D}\mathbf{I}.$$

(Recall our convention that lower indices correspond to rows and upper indices to columns.) By Proposition-Definition 9.3, $\check{D}^{\hat{k}\hat{l},L}_{\hat{i}\hat{j}} = \check{D}^{\hat{k}\hat{l},R}_{\hat{i}\hat{j}}$ and $\check{D}^L = \check{D}^R$. A similar computation using (89) then yields

$$\tilde{\check{\mathbf{z}}}\check{\mathbf{z}} = \check{D}\mathbf{I}.$$

□

This result implies, just as in the standard case [30, §9.2.2], that $\check{D}$ belongs to the center of $\mathscr{O}(M_q(\check{X}))$. The coordinate algebra $\mathscr{O}(GL_q(\check{X}))$ of the sought quantum group $GL_q(\check{X})$ is obtained by adjoining the inverse $\check{D}^{-1}$ to $\mathscr{O}(M_q(\check{X}))$. We formally define $\check{\mathbf{z}}^{-1} = \check{D}^{-1}\tilde{\check{\mathbf{z}}}$. This allows us to define a Hopf structure on $\mathscr{O}(GL_q(\check{X}))$ just as in the standard case (§4.4).

Proposition 10.3. *There is a unique Hopf algebra structure on $\mathscr{O}(GL_q(\check{X}))$ so that*

(1) $\Delta(\check{\mathbf{z}}) = \check{\mathbf{z}}\dot{\otimes}\check{\mathbf{z}}$, $\Delta(\check{D}^{-1}) = \check{D}^{-1}\otimes\check{D}^{-1}$.
(2) $\epsilon(\check{\mathbf{z}}) = \mathbf{I}$.
(3) $S(z^i_j) = \tilde{z}^i_j\check{D}^{-1}$, $S(\check{D}^{-1}) = \check{D}$, *where* z^i_j *are the entries of* $\check{\mathbf{z}}$ *and* $\tilde{z}^i_j$ *are the entries of* $\tilde{\check{\mathbf{z}}}$.

Proof. Most of the work has been done in Proposition 10.2. The remaining details are similar to the standard case [30, Proposition 10, Chapter 9]. □

10.2. **Compact real form.**

Proposition 10.4. *The algebra $\mathscr{O}(GL_q(\check{X}))$ is a Hopf $*$-algebra with the involution $*$ determined by $(z^i_j)^* = S(z^j_i)$.*

Proof. This follows from [30, Proposition 3, Chapter 9]. This requires that the rule $\check{\mathbf{z}} \mapsto \check{\mathbf{z}}^T$ determines an algebra automorphism of $\mathscr{O}(GL_q(\check{X}))$, which follows from the fact that $P_+^{\check{X}}$ is symmetric. □

Proposition 10.5. *Let $\psi : \mathscr{O}(M_q(\check{X})) \to \mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$ be as in Proposition 8.3. There holds*

$$\psi(\check{D}) = D_q(V)^{d_W} D_q(W)^{d_V}, \tag{92}$$

where $D_q(V), D_q(W)$ are the quantum determinants of $\mathscr{O}(M_q(V))$ and $\mathscr{O}(M_q(W))$, respectively (see §4.4). There is a unique Hopf $*$-algebra homomorphism

$$\tilde{\psi} : \mathscr{O}(GL_q(\check{X})) \to \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$$

extending ψ.

Proof. The composition

$$\check{\Lambda}^{d_X}\check{X} \xrightarrow{\varphi^L_{d_X}} \mathscr{O}(M_q(\check{X})) \otimes \check{\Lambda}^{d_X}\check{X} \xrightarrow{\psi\otimes id} \mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W)) \otimes \check{\Lambda}^{d_X}\check{X}$$

takes $y_1 \cdots y_{d_X}$ to $\psi(\check{D}) \otimes y_1 \cdots y_{d_X}$. Thus $\psi(\check{D})$ is the matrix coefficient of $\check{\Lambda}^{d_X}\check{X}$ considered as a left $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodule. By Proposition 8.9 (2), $\check{\Lambda}^{d_X}\check{X} \cong V_{({d_W}^{d_V})} \star W_{({d_V}^{d_W})}$, where $({d_W}^{d_V})$ (resp. $({d_V}^{d_W})$) is the rectangular partition with d_V rows and d_W columns (resp. d_W rows and d_V columns). Since $V_{({d_W}^{d_V})} \cong (\Lambda_q^{d_V} V)^{\otimes d_W}$ (as $\mathscr{O}(M_q(V))$-comodules), the identity (92) follows.

Any algebra homomorphism extending ψ must satisfy $\check{D}^{-1} \mapsto D_q(V)^{-d_W} D_q(W)^{-d_V}$, hence the uniqueness of $\tilde{\psi}$. It is easy to check that $\tilde{\psi}$ is a bialgebra homomorphism and a bialgebra homomorphism of Hopf algebras is always a Hopf algebra homomorphism [30, §1.2.4]. That $\tilde{\psi}$ intertwines the $*$-involutions, i.e. $\tilde{\psi}(z^*) = \tilde{\psi}(z)^*$, follows from the fact that it intertwines the antipodes and it intertwines the algebra automorphisms of $\mathscr{O}(GL_q(\check{X}))$ and $\mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$ determined by $\check{\mathbf{z}} \mapsto \check{\mathbf{z}}^T$ and $\mathbf{u}^V \mapsto (\mathbf{u}^V)^T$, $\mathbf{u}^W \mapsto (\mathbf{u}^W)^T$, respectively. □

Proposition 10.6. *The Hopf $*$-algebra $\mathscr{O}(GL_q(\check{X}))$ is a CMQG algebra (see §4.6).*

Proof. The fundamental corepresentation $\check{X}$ of $\mathscr{O}(GL_q(\check{X}))$ is unitary by Proposition 10.2, and $\check{D}^{-1}$ is a unitary element of $\mathscr{O}(GL_q(\check{X}))$. Furthermore, $\mathscr{O}(GL_q(\check{X}))$ is generated by the matrix elements of the unitary corepresentation $\check{\mathbf{z}} \oplus (\check{D}^{-1})$. Hence, the result follows from Theorem 4.1 (a). □

10.3. **Complete reducibility.** We conclude this section by restating our main theorem about $GL_q(\check{X})$ and $\mathtt{U}_q(\check{X})$ and collecting its proof.

Theorem 10.7. *Assume that all objects are over the field $\mathbb{C}$ and q is real and transcendental. Then*

(a) The Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ can be made into a Hopf $*$-algebra. This is considered to be the coordinate ring of the compact real form of the nonstandard quantum group $GL_q(\check{X})$. This virtual compact real form is denoted $\mathtt{U}_q(\check{X})$, which is a compact quantum group in the sense of Woronowicz [62].

(b) There is a Hopf $*$-algebra homomorphism

$$\tilde{\psi} : \mathscr{O}(GL_q(\check{X})) \to \mathscr{O}(GL_q(V)) \otimes \mathscr{O}(GL_q(W)),$$

(c) Every finite-dimensional representation of $\mathtt{U}_q(\check{X})$ (meaning a corepresentation of $\mathscr{O}(GL_q(\check{X}))$) is unitarizable, and hence, is a direct sum of irreducible representations.

(d) An analog of the Peter-Weyl theorem holds:

$$\mathscr{O}(GL_q(\check{X})) = \bigoplus_{\alpha \in \check{\mathscr{P}}} (\check{\mathcal{X}}_\alpha)_L \otimes \check{\mathcal{X}}_\alpha,$$

where $\check{\mathscr{P}}$ is an index set for the irreducible right comodules of $\mathscr{O}(GL_q(\check{X}))$ and $\check{\mathcal{X}}_\alpha$ is the comodule labeled by α.

Proof. Part (a) is Proposition 10.4 and Proposition 10.6, part (b) is contained in Proposition 10.5, and parts (c) and (d) follow from Proposition 10.6 and Theorem 4.1. $\square$

11. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$

We now turn to the nonstandard Hecke algebra, which plays the role of the symmetric group in the nonstandard setting. The group algebra $\mathbb{Q}\mathcal{S}_r$ is a Hopf algebra with coproduct $\Delta_{\mathbb{Q}\mathcal{S}_r} : \mathbb{Q}\mathcal{S}_r \to \mathbb{Q}\mathcal{S}_r \star \mathbb{Q}\mathcal{S}_r$, $w \mapsto w \star w$. This makes the tensor product of $\mathbb{Q}\mathcal{S}_r$-modules into a $\mathbb{Q}\mathcal{S}_r$-module, and allows us to define Kronecker coefficients. The Hecke algebra $\mathscr{H}_r$ is not a Hopf algebra in a natural way. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$, which we soon define, approximates the Hopf algebra $\mathbb{Q}\mathcal{S}_r$ in the smallest possible way in a certain sense. Despite its being as small as possible, $\check{\mathscr{H}}_r$ has dimension much larger than that of $\mathcal{S}_r$.

In this section we show that $\check{\mathscr{H}}_r$ is semisimple and describe some of its basic properties and representation theory. This section is mostly a summary of results from the older unpublished version of this paper [45] and [8, 11].

11.1. **Definition of $\check{\mathscr{H}}_r$ and basic properties.** Let $S = \{s_1, \ldots, s_{r-1}\}$ be the set of simple reflections of the Coxeter group $\mathcal{S}_r$. Let $\mathscr{H}_r = \mathscr{H}(\mathcal{S}_r)$ be the type A Hecke algebra as introduced in §3.2. Let $C'_s = T_s + q$ and $C_s = T_s - q^{-1}$ for each $s \in S$. These are the simplest lower and upper Kazhdan-Lusztig basis elements (see §3.1 and [9]). They are also proportional to the primitive central idempotents of $\mathbf{A}'(\mathscr{H}_r)_{\{s\}} \cong \mathbf{A}'\mathscr{H}_2$, where

$\mathbf{A}' := \mathbf{A}[\frac{1}{[2]}]$. Specifically, $\frac{1}{[2]}C'_{s_1}$ (resp. $-\frac{1}{[2]}C_{s_1}$) is the idempotent corresponding to the trivial (resp. sign) representation of $\mathbf{A}'\mathscr{H}_2$.

Definition 11.1. The *type A nonstandard Hecke algebra* $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \star \mathscr{H}_r$ generated by the elements

$$\mathcal{P}_s := C'_s \star C'_s + C_s \star C_s,\ s \in S. \tag{93}$$

We let $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \star \mathscr{H}_r$ denote the canonical inclusion, which we think of as a deformation of the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r} : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \star \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \star w$.

The nonstandard Hecke algebra is also the subalgebra of $\mathscr{H}_r \star \mathscr{H}_r$ generated by

$$\mathcal{Q}_s := [2]^2 - \mathcal{P}_s = -C'_s \star C_s - C_s \star C'_s,\ s \in S.$$

We will write $\mathcal{P}_{i_1 i_2 \dots i_l}$ as shorthand for $\mathcal{P}_{s_{i_1}}\mathcal{P}_{s_{i_2}}\cdots\mathcal{P}_{s_{i_l}}$. For a ring homomorphism $K \to \mathbf{A}$, we have the specialization $K\check{\mathscr{H}}_r := K \otimes_{\mathbf{A}} \check{\mathscr{H}}_r$ of the nonstandard Hecke algebra.

Remark 11.2. The notation $\star$ is our notation for tensor product when tensoring objects associated to V with objects associated to W. We use this notation in the present context because of the natural action of $\mathscr{H}_r \star \mathscr{H}_r$ on $V^{\otimes r} \star W^{\otimes r}$ that will be discussed in §12.

Write $\epsilon_+ = M^{\mathbf{A}}_{(r)}, \epsilon_- = M^{\mathbf{A}}_{(1^r)}$ for the one-dimensional trivial and sign representations of $\mathscr{H}_r$, which are defined by

$$\epsilon_+ : C'_s \mapsto [2], \qquad \epsilon_- : C'_s \mapsto 0, \quad s \in S.$$

We identify these algebra homomorphisms $\epsilon_+, \epsilon_- : \mathscr{H}_r \to \mathbf{A}$ with right $\mathscr{H}_r$-modules in the usual way.

There are also one-dimensional trivial and sign representations of $\check{\mathscr{H}}_r$, which we denote by $\check{\epsilon}_+$ and $\check{\epsilon}_-$:

$$\check{\epsilon}_+ : \mathcal{P}_s \mapsto [2]^2, \qquad \check{\epsilon}_- : \mathcal{P}_s \mapsto 0, \quad s \in S.$$

There is an $\mathbf{A}$-algebra automorphism $\theta : \mathscr{H}_r \to \mathscr{H}_r$ defined by $\theta(T_s) = -T_s^{-1},\ s \in S$. Note that $\theta(C'_s) = -C_s, \theta(C_s) = -C'_s$. Let 1^{op} be the $\mathbf{A}$-anti-automorphism of $\mathscr{H}_r$ given by $1^{\mathrm{op}}(T_w) = T_{w^{-1}}$. Let θ^{op} be the $\mathbf{A}$-anti-automorphism of $\mathscr{H}_r$ given by $\theta^{\mathrm{op}} = \theta \circ 1^{\mathrm{op}} = 1^{\mathrm{op}} \circ \theta$. Let η be the unique $\mathbf{A}$-algebra homomorphism from $\mathbf{A}$ to $\mathscr{H}_r$. At $q = 1$, the maps $\eta, \epsilon_+, 1^{\mathrm{op}}$ specialize to the unit, counit, and antipode of the Hopf algebra $\mathbb{Z}\mathcal{S}_r$.

Proposition 11.3 ([8])**.** *Set $\mathbf{A}' = \mathbf{A}[\frac{1}{[2]}]$ and $A'_1 = \mathbb{Z}[\frac{1}{2}]$. We have $\mathbf{A}'\check{\mathscr{H}}_2 \cong \mathbf{A}'\mathscr{H}_2$ by $\mathcal{P}_1 \mapsto [2]C'_{s_1}$. Then*

(i) *$\mathbf{A}'\mathscr{H}_2$ is a Hopf algebra with coproduct $\Delta = \check{\Delta}$, antipode 1^{op}, counit ϵ_+, and unit η.*

(ii) *the Hopf algebra $\mathbf{A}'\mathscr{H}_2|_{q=1}$, with Hopf algebra structure coming from (i), is isomorphic to the group algebra $A'_1\mathcal{S}_2$ with its usual Hopf algebra structure.*

Moreover, the Hopf algebra structure of (i) is the unique way to make the algebra $\mathbf{A}'\mathscr{H}_2$ into a Hopf algebra so that (ii) is satisfied.

Remark 11.4. If we want to construct a subalgebra H of $K(\mathscr{H}_r \star \mathscr{H}_r)$ that is compatible with the coproducts $K(\mathscr{H}_r)_{\{s_i\}} \hookrightarrow K((\mathscr{H}_r)_{\{s_i\}} \star (\mathscr{H}_r)_{\{s_i\}})$, then the nonstandard Hecke algebra $K\check{\mathscr{H}}_r$ is the smallest possible choice and the choice making H and $H \hookrightarrow K(\mathscr{H}_r \star \mathscr{H}_r)$ as close as possible to $\mathbb{Q}\mathcal{S}_r$ and $\Delta_{\mathbb{Q}\mathcal{S}_r}$ at $q = 1$.

The next proposition gives another way that $\check{\mathscr{H}}_r$ is like a Hopf algebra.

Proposition 11.5 ([8])**.** *The involutions 1^{op} and θ^{op} are antipodes in the following sense:*

$$\mu \circ (1^{op} \otimes 1) \circ \check{\Delta} = \eta \circ \check{\epsilon}_+, \tag{94}$$

$$\mu \circ (\theta^{op} \otimes 1) \circ \check{\Delta} = \eta \circ \check{\epsilon}_-, \tag{95}$$

where these are equalities of maps from $\check{\mathscr{H}}_r$ to $\mathscr{H}_r$ and μ is the multiplication map for $\mathscr{H}_r$.

We next study some algebra involutions of $\mathscr{H}_r \star \mathscr{H}_r$ and their restrictions to $\check{\mathscr{H}}_r$. This will help us understand the representation theory of $\check{\mathscr{H}}_r$. Let $\tau : \mathscr{H}_r \star \mathscr{H}_r \to \mathscr{H}_r \star \mathscr{H}_r$, be the algebra involution given by $\tau(a \star b) \mapsto b \star a$. Let $\mathscr{A}$ be the subgroup of the group of algebra automorphisms of $\mathscr{H}_r \star \mathscr{H}_r$ generated by $\theta \star 1$, $1 \star \theta$, and τ. Let $\mathscr{A}_{\theta,\tau}$ be the subgroup of $\mathscr{A}$ generated by $\theta \star \theta$ and τ. Note that $\mathscr{A} \cong \mathscr{A}_{\theta,\tau} \rtimes \mathcal{S}_2$ is isomorphic to the dihedral group of order eight.

Proposition 11.6. *The elements of $\mathscr{A}$ restrict nicely to $\check{\mathscr{H}}_r$:*

(i) $\alpha(\mathcal{P}_s) = \begin{cases} \mathcal{P}_s & \text{if } \alpha \in \mathscr{A}_{\theta,\tau} \\ \mathcal{Q}_s & \text{if } \alpha \in (\theta \star 1)\mathscr{A}_{\theta,\tau} \end{cases}$ *for all $s \in S$.*

(ii) *$\check{\mathscr{H}}_r$ is left stable by the elements of $\mathscr{A}$.*

(iii) *There is an $\mathbf{A}$-algebra involution $\Theta : \check{\mathscr{H}}_r \to \check{\mathscr{H}}_r$ determined by $\Theta(\mathcal{P}_s) = \mathcal{Q}_s$, $s \in S$.*

(iv) *The restriction of an element of $\mathscr{A}$ to $\check{\mathscr{H}}_r$ corresponds to the map $\mathscr{A} \to \mathrm{Aut}\,(\check{\mathscr{H}}_r)$ given by $\theta \star 1 \mapsto \Theta$, $1 \star \theta \mapsto \Theta$, $\tau \mapsto 1$.*

(v) *The nonstandard Hecke algebra is at the beginning of the chain of subalgebras $\check{\mathscr{H}}_r \subseteq (S^2\mathscr{H}_r)^\theta \subseteq S^2\mathscr{H}_r \subseteq \mathscr{H}_r \star \mathscr{H}_r$, where $(S^2\mathscr{H}_r)^\theta$ is the subalgebra of $\mathscr{H}_r \star \mathscr{H}_r$ fixed by the elements of $\mathscr{A}_{\theta,\tau}$.*

Proof. Statement (i) follows from the definition of $\mathcal{P}_s$ (93). The remaining statements follow easily from (i). □

11.2. **Semisimplicity of $K\check{\mathscr{H}}_r$.**

Lemma 11.7. *Let $K = \mathbb{Q}(q)$ and let $H \subseteq M_m(K)$ be a K-subalgebra of the matrix algebra $M_m(K)$. If for every $M \in H$, the transpose M^T is also in H, then H is semisimple.*

Proof. Since H is a finite-dimensional algebra over a field, its Jacobson radical $J(H)$ is nilpotent, i.e., $J(H)^k = 0$ for some k. Thus H is semisimple (equivalently, $J(H) = 0$) if and only if there does not exist a nonzero two-sided ideal I of H such that $I^2 = 0$. Now suppose I is a two-sided ideal of H such that $I^2 = 0$ and let $M \in I$. Then $N := M^T M \in I$ implies $N^T N = N^2 = 0$. Taking the trace of both sides of this equation and letting N_{ij}

denote the entries of N yields $\sum N_{ij}^2 = 0$. It follows that $N = 0$. The same argument then shows $M = 0$, thus $I = 0$. □

Proposition 11.8. *The nonstandard Hecke algebra $K\check{\mathscr{H}}_r$ is semisimple, where $K = \mathbb{Q}(q)$. There is a finite extension K' of K such that $K'\check{\mathscr{H}}_r$ is split semisimple.*

Proof. We know that there is a right action of $K\mathscr{H}_r$ on $V^{\otimes r}$, which is faithful when $d_V = \dim_K V \geq r$. It is easy to see from (54) that the action of each C'_s, when expressed in the standard monomials basis of $V^{\otimes r}$, is a symmetric matrix. Consequently there is a faithful representation of $K(\mathscr{H}_r \star \mathscr{H}_r)$ on $(V \star W)^{\otimes r}$ such that the matrix corresponding to each $\mathcal{P}_s = C'_s \star C'_s + C_s \star C_s$ is symmetric.

Next, let us check that $K\check{\mathscr{H}}_r \xrightarrow{K\check{\Delta}} K(\mathscr{H}_r \star \mathscr{H}_r)$ is an inclusion (this deserves some care since it fails for the specialization $\mathbf{A} \to \mathbb{Z}$, $q \mapsto 1$). We have the inclusion $\check{\mathscr{H}}_r \xrightarrow{\check{\Delta}} \mathscr{H}_r \star \mathscr{H}_r$ of $\mathbf{A}$-modules. Since localizations are flat, K is a flat $\mathbf{A}$-module, and thus $K\check{\Delta}$ is also an inclusion. Then $K\check{\mathscr{H}}_r$ is a subalgebra of $M_{(r^2)^r}(K)$ generated by symmetric matrices, so by Lemma 11.7 $K\check{\mathscr{H}}_r$ is semisimple.

The second fact follows from the general fact that any finite-dimensional associative algebra over a field becomes split after a finite field extension [18, Proposition 7.13]. □

Remark 11.9. The specialization $\check{\mathscr{H}}_r|_{q=1} := \mathbb{Q} \otimes_{\mathbf{A}} \check{\mathscr{H}}_r$, the map $\mathbf{A} \to \mathbb{Q}$ given by $q \mapsto 1$, has $\mathbb{Q}$ dimension equal to $\dim_K K\check{\mathscr{H}}_r$. This is because $\mathbb{Q} \otimes_{\mathbb{Z}} \check{\mathscr{H}}_r$ is a free $\mathbb{Q}[q, q^{-1}]$-module since it is a submodule of a free $\mathbb{Q}[q, q^{-1}]$-module and $\check{\mathscr{H}}_r|_{q=1} = \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} \mathbb{Q} \otimes_{\mathbb{Z}} \check{\mathscr{H}}_r$. It can then be shown that the algebra $\check{\mathscr{H}}_r|_{q=1}$ is not semisimple for $r > 2$. It has Jacobson radical $J = \ker(\check{\Delta}|_{q=1})$ and the quotient $\check{\mathscr{H}}_r|_{q=1}/J \cong \mathrm{im}\,(\check{\Delta}|_{q=1})$ is equal to $\mathrm{im}\,(\Delta) \cong \mathbb{Q}\mathcal{S}_r$, where $\Delta : \mathbb{Q}\mathcal{S}_r \to \mathbb{Q}\mathcal{S}_r \star \mathbb{Q}S_r$ is the usual coproduct.

11.3. **Representation theory of $S^2\mathscr{H}_r$.** We briefly discuss the representation theory of $S^2\mathscr{H}_r$ because this is close to the representation theory of $\check{\mathscr{H}}_r$, especially in the two-row case. We will return to this again in §13.

First note that we have the following commutativity property for any $\mathscr{H}_r$-modules M and M':

$$\mathrm{Res}_{S^2\mathscr{H}_r} M \star M' \cong \mathrm{Res}_{S^2\mathscr{H}_r} M' \star M, \tag{96}$$

where the isomorphism is given by τ.

Recall from §2.3 that $\mathscr{P}_r$ denotes the set of partitions of size r and $\mathscr{P}'_r$ the set of partitions of r that are not a single row or column shape. Recall from §3.2 that $M_\lambda^{\mathbf{A}}$ denotes the Specht module of $\mathscr{H}_r$ so that $M_\lambda \cong K \otimes_{\mathbf{A}} M_\lambda^{\mathbf{A}}$.

Proposition-Definition 11.10. *Define the following $S^2\mathscr{H}_r$-modules. After tensoring these with K, this is the list of distinct $KS^2\mathscr{H}_r$-irreducibles*

(1) $M_{\{\lambda,\mu\}}^{\mathbf{A}} := \mathrm{Res}_{S^2\mathscr{H}_r} M_\lambda^{\mathbf{A}} \star M_\mu^{\mathbf{A}}$, $\{\lambda, \mu\} \subseteq \mathscr{P}_r$, $\lambda \neq \mu$,
(2) $S^2 M_\lambda^{\mathbf{A}} := \mathrm{Res}_{S^2\mathscr{H}_r} S^2 M_\lambda^{\mathbf{A}}$, $\lambda \in \mathscr{P}_r$,
(3) $\Lambda^2 M_\lambda^{\mathbf{A}} := \mathrm{Res}_{S^2\mathscr{H}_r} \Lambda^2 M_\lambda^{\mathbf{A}}$, $\lambda \in \mathscr{P}'_r$.

Let $M_{\{\lambda,\mu\}}$, S^2M_λ, $\Lambda^2 M_\lambda$ denote the corresponding $KS^2\mathscr{H}_r$-modules.

Proof. This follows from a general result about the structure of S^2H for H any split semisimple algebra over a field R of characteristic zero. Such a result is proved in [33, §5] in the case $R = \mathbb{C}$. The proof goes by using the result for the case $H = \mathrm{End}(V)$, $V = R^{\oplus k}$ to deduce the general case. And this special case follows from

$$S^2(\mathrm{End}(V)) \cong S^2(V^* \otimes V) \cong S^2V^* \otimes S^2V \oplus \Lambda^2V^* \otimes \Lambda^2 V \cong \mathrm{End}(S^2V) \oplus \mathrm{End}(\Lambda^2V).$$

On the level of vector spaces, this is just the degree 2 part of Proposition 8.9 (1) in the case $V = V^*$, $W = V$, $q = 1$. One checks that this is in fact an isomorphism of algebras that holds over any field R of characteristic zero. $\square$

11.4. **Some representation theory of $\check{\mathscr{H}}_r$.** In this subsection, we give some flavor for the representation theory of $\check{\mathscr{H}}_r$. This is far from being a thorough treatment as our knowledge is limited outside the two-row case (see Theorem 13.4 for a complete answer in this case). From computations we have done up to $r = 6$, we suspect that most of the $\check{\mathscr{H}}_r$-irreducibles are restrictions of $(S^2\mathscr{H}_r)^\theta$-irreducibles (see Proposition 11.6), except for the trivial and sign representations.

We have already defined the trivial and sign representations $\check{\epsilon}_+$, $\check{\epsilon}_-$ of $\check{\mathscr{H}}_r$. In addition, there are the following six types of $\check{\mathscr{H}}_r$-modules:

$$\begin{aligned}
&(1)\ \check{M}^{\mathbf{A}}_{\{\lambda,\mu\}} := \mathrm{Res}_{\check{\mathscr{H}}_r} M^{\mathbf{A}}_{\{\lambda,\mu\}},\\
&(2)\ S^2\check{M}^{\mathbf{A}}_{\lambda} := \mathrm{Res}_{\check{\mathscr{H}}_r} S^2M^{\mathbf{A}}_{\lambda},\\
&(3)\ \Lambda^2\check{M}^{\mathbf{A}}_{\lambda} := \mathrm{Res}_{\check{\mathscr{H}}_r} \Lambda^2M^{\mathbf{A}}_{\lambda},\\
&(4)\ S'\check{M}^{\mathbf{A}}_{\lambda} := \ker(\mathrm{tr}) \cap S^2\check{M}^{\mathbf{A}}_{\lambda},\\
&(5)\ (S'\check{M}^{\mathbf{A}}_{\lambda})^{\#},\\
&(6)\ (\Lambda^2\check{M}^{\mathbf{A}}_{\lambda})^{\#},
\end{aligned} \tag{97}$$

where the right-hand sides of the first three lines are restrictions of the $S^2\mathscr{H}_r$-modules of Proposition 11.10. The last three modules will be explained below, after we discuss contragradients of $\check{\mathscr{H}}_r$-modules. Let $\check{M}_{\{\lambda,\mu\}}$, $S^2\check{M}_\lambda$, etc. denote the corresponding $K\check{\mathscr{H}}_r$-modules.

Any anti-automorphism S of an $\mathbf{A}$-algebra H allows us to define contragradients of H-modules: let $\langle \cdot,\cdot\rangle : M \otimes M^* \to \mathbf{A}$ be the canonical pairing, where M^* is the $\mathbf{A}$-module $\mathrm{Hom}_{\mathbf{A}}(M, \mathbf{A})$. Then the H-module structure on M^* is defined by

$$\langle m, hm'\rangle = \langle S(h)m, m'\rangle \text{ for any } h \in H,\ m \in M, m' \in M^*.$$

Recall the anti-automorphisms 1^{op}, θ^{op} of $\mathscr{H}_r$ defined in §11.1. There are also anti-automorphisms $1^{\mathrm{op}} := 1^{\mathrm{op}} \star 1^{\mathrm{op}}$ and $(\Theta)^{\mathrm{op}} := 1^{\mathrm{op}} \circ \Theta$ of $\check{\mathscr{H}}_r$, where Θ is defined in Proposition 11.6 (iii). For an $\mathscr{H}_r$-module M, we write $M^\diamond$ (resp. $M^{\#}$) for the contragradient of M corresponding to the anti-automorphism 1^{op} (resp. θ^{op}). For an $\check{\mathscr{H}}_r$-module $\check{M}$, we also write $\check{M}^\diamond$ (resp. $\check{M}^{\#}$) for the contragradient of $\check{M}$ corresponding to the anti-automorphism 1^{op} (resp. Θ^{op}).

Proposition 11.11 ([11] (see also [38, Exercises 2.7, 3.14])). *The contragradients of the Specht module $M_\lambda^{\mathbf{A}}$ of $\mathscr{H}_r$ are given by*

$$(M_\lambda^{\mathbf{A}})^\diamond \cong M_\lambda^{\mathbf{A}} \quad \textit{and} \quad (M_\lambda^{\mathbf{A}})^\# \cong M_{\lambda'}^{\mathbf{A}}.$$

We now explain (4)–(6) of (97). Let $\mathbf{A} \xrightarrow{I} (M_\lambda^{\mathbf{A}})^\diamond \star M_\lambda^{\mathbf{A}}$ be the canonical inclusion given by sending $1 \in \mathbf{A}$ to $I \in \mathrm{End}(M_\lambda^{\mathbf{A}}) \cong (M_\lambda^{\mathbf{A}})^\diamond \star M_\lambda^{\mathbf{A}}$. Let $M_\lambda^{\mathbf{A}} \star (M_\lambda^{\mathbf{A}})^\diamond \xrightarrow{\mathrm{tr}} \mathbf{A}$ be the canonical surjection. It follows from Proposition 11.5 (see, e.g., [11]) and the $\mathscr{H}_r$-module isomorphism $(M_\lambda^{\mathbf{A}})^\diamond \cong M_\lambda^{\mathbf{A}}$ that there are the following $\check{\mathscr{H}}_r$-module homomorphisms

$$\begin{gathered}\check{\epsilon}_+ \xrightarrow{I} \check{M}_{\{\lambda,\lambda\}}^{\mathbf{A}}, \\ \ker(\mathrm{tr}) \hookrightarrow \check{M}_{\{\lambda,\lambda\}}^{\mathbf{A}} \xrightarrow{\mathrm{tr}} \check{\epsilon}_+.\end{gathered}$$

Since $\frac{1}{|\mathrm{SYT}(\lambda)|} I$ is a splitting of tr, we obtain the decomposition

$$\ker(\mathrm{tr}) \oplus \check{\epsilon}_+ \cong \check{M}_{\{\lambda,\lambda\}}^{\mathbf{A}} \tag{98}$$

of $\check{\mathscr{H}}_r$-modules.

Define $S'\check{M}_\lambda^{\mathbf{A}} := \ker(\mathrm{tr}) \cap S^2\check{M}_\lambda^{\mathbf{A}}$. The decomposition (98) yields the decomposition

$$S^2\check{M}_\lambda^{\mathbf{A}} \cong S'\check{M}_\lambda^{\mathbf{A}} \oplus \check{\epsilon}_+. \tag{99}$$

Applying # to (98) and (99) yields

$$\begin{aligned}(\check{M}_{\{\lambda,\lambda\}}^{\mathbf{A}})^\# &\cong (\ker(\mathrm{tr}))^\# \oplus \check{\epsilon}_-, \\ (S^2\check{M}_\lambda^{\mathbf{A}})^\# &\cong (S'\check{M}_\lambda^{\mathbf{A}})^\# \oplus \check{\epsilon}_-.\end{aligned} \tag{100}$$

Proposition 11.12. *There hold the following isomorphisms of $\check{\mathscr{H}}_r$-modules*

$$\begin{array}{ccccc}(\mathrm{Res}_{\check{\mathscr{H}}_r} M_l^{\mathbf{A}} \otimes M_r^{\mathbf{A}})^\diamond & \cong & \mathrm{Res}_{\check{\mathscr{H}}_r} (M_l^{\mathbf{A}})^\diamond \otimes (M_r^{\mathbf{A}})^\diamond & \cong & \mathrm{Res}_{\check{\mathscr{H}}_r} (M_l^{\mathbf{A}})^\# \otimes (M_r^{\mathbf{A}})^\#, \\ (\mathrm{Res}_{\check{\mathscr{H}}_r} M_l^{\mathbf{A}} \otimes M_r^{\mathbf{A}})^\# & \cong & \mathrm{Res}_{\check{\mathscr{H}}_r} (M_l^{\mathbf{A}})^\diamond \otimes (M_r^{\mathbf{A}})^\# & \cong & \mathrm{Res}_{\check{\mathscr{H}}_r} (M_l^{\mathbf{A}})^\# \otimes (M_r^{\mathbf{A}})^\diamond,\end{array} \tag{101}$$

for any $\mathscr{H}_r$-modules $M_l^{\mathbf{A}}, M_r^{\mathbf{A}}$. Hence the following $\check{\mathscr{H}}_r$-modules are isomorphic

$$\check{\epsilon}_\pm^\diamond \cong \check{\epsilon}_\pm, \tag{102}$$

$$\check{\epsilon}_\pm^\# \cong \check{\epsilon}_\mp, \tag{103}$$

$$(\check{M}_{\{\lambda,\mu\}}^{\mathbf{A}})^\diamond \cong \check{M}_{\{\lambda,\mu\}}^{\mathbf{A}} \cong \check{M}_{\{\lambda',\mu'\}}^{\mathbf{A}}, \tag{104}$$

$$(\check{M}_{\{\lambda,\mu\}}^{\mathbf{A}})^\# \cong \check{M}_{\{\lambda,\mu'\}}^{\mathbf{A}} \cong \check{M}_{\{\lambda',\mu\}}^{\mathbf{A}}, \tag{105}$$

$$(S^2\check{M}_\lambda^{\mathbf{A}})^\diamond \cong S^2\check{M}_\lambda^{\mathbf{A}} \cong S^2\check{M}_{\lambda'}^{\mathbf{A}}, \tag{106}$$

$$(\Lambda^2\check{M}_\lambda^{\mathbf{A}})^\diamond \cong \Lambda^2\check{M}_\lambda^{\mathbf{A}} \cong \Lambda^2\check{M}_{\lambda'}^{\mathbf{A}}, \tag{107}$$

$$(S'\check{M}_\lambda^{\mathbf{A}})^\diamond \cong S'\check{M}_\lambda^{\mathbf{A}} \cong S'\check{M}_{\lambda'}^{\mathbf{A}}. \tag{108}$$

Proof. The isomorphisms in (101) follow from Proposition 11.6 (iv) (see also [8, Proposition 2.9]). The isomorphisms (102)–(107) are straightforward from (101) and Proposition 11.11. The isomorphisms in (108) follow from (106), (102), and (99). ☐

11.5. **The sign representation in the canonical basis.** For future reference, we record the right action of $\frac{\mathcal{Q}_s}{[2]^2}$ on $\mathscr{H}_r \star \mathscr{H}_r$ in terms of the basis $\Gamma_{\mathcal{S}_r} \star \Gamma_{\mathcal{S}_r}$.

$$C_v \star C_w \frac{\mathcal{Q}_s}{[2]^2} = \tag{109}$$
$$\begin{cases} 0 & \text{if } s \in R(v), s \in R(w), \\ C_v \star C_w + \frac{1}{[2]} \sum\limits_{s \in R(w')} \mu(w', w) C_v \star C_{w'} & \text{if } s \in R(v) \text{ and } s \notin R(w), \\ C_v \star C_w + \frac{1}{[2]} \sum\limits_{s \in R(v')} \mu(v', v) C_{v'} \star C_w & \text{if } s \notin R(v) \text{ and } s \in R(w), \\ -\frac{1}{[2]} \Big(\sum\limits_{s \in R(v')} \mu(v', v) C_{v'} \star C_w + \sum\limits_{s \in R(w')} \mu(w', w) C_v \star C_{w'} \Big) + & \\ -\frac{2}{[2]^2} \sum\limits_{s \in R(v'), s \in R(w')} \mu(v', v) \mu(w', w) C_{v'} \star C_{w'} & \text{if } s \notin R(w) \text{ and } s \notin R(v). \end{cases}$$

This is immediate from (25). This also gives the action on any cellular subquotient $\mathbf{A}\Gamma$ of $\mathbf{A}\Gamma_{\mathcal{S}_r} \star \Gamma_{\mathcal{S}_r}$ by restricting μ to Γ.

The next proposition expresses the decomposition (100) in terms of canonical bases. This will allow us to construct a canonical basis for $\check{\Lambda}(\check{X})$ explicitly as a subset of $T(\check{X})$ in §14.1.

Proposition 11.13 ([11])**.** *The sign representation $K\check{\epsilon}_-$ of $K\check{\mathscr{H}}_r$ occurs with multiplicity one in $M_\lambda \star M_{\lambda'}$ and not at all in $M_\lambda \star M_\mu$, $\mu \neq \lambda'$. Moreover, the inclusion $\check{\imath}_- : K\check{\epsilon}_- \hookrightarrow M_\lambda \star M_{\lambda'}$ can be expressed in terms of the basis $\Gamma_\lambda \star \Gamma_{\lambda'}$ as*

$$1 \mapsto \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} C_Q \star C_{Q^T},$$

and the surjection $\check{s}_- : M_\lambda \star M_{\lambda'} \twoheadrightarrow K\check{\epsilon}_-$ by

$$\sum_{Q', Q \in SYT(\lambda)} a^{Q'Q^T} C_{Q'} \star C_{Q^T} \mapsto \frac{1}{|SYT(\lambda)|} \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} a^{QQ^T} \quad (a^{Q'Q^T} \in K),$$

where $\ell(Q^T)$ is as in §2.3.

11.6. **The algebra $\check{\mathscr{H}}_3$.** In [45], the example $\check{\mathscr{H}}_3$ is described in detail. This example is generalized in [8], and we follow this reference to recall some of the main results.

Let us first describe the irreducible representations of $K\check{\mathscr{H}}_3$.

Proposition 11.14. *For $K = \mathbb{Q}(q)$, the irreducible representations of $K\check{\mathscr{H}}_3$ consist of the trivial and sign representations*

$$\check{\epsilon}_+, \ \check{\epsilon}_-,$$

and the two two-dimensional representations

$$\check{M}_{\{(3),(2,1)\}}, \quad S'\check{M}_{(2,1)}.$$

This is shown in [45] and is a special case of [8, Theorem 3.4]. Note that the two-dimensional representations both specialize to the representation $M_{(2,1)}|_{q=1}$ of $\mathbb{Q}\mathcal{S}_3$ at $q = 1$. Thus the algebra $K\check{\mathscr{H}}_3$ is similar to $K\mathscr{H}_3$ except with two two-dimensional representations instead of one. Let us now see how these two-dimensional representations differ.

These two-dimensional representations are both of the form $\check{N}(a)$, where $\check{N}(a) \cong K^{\oplus 2}$, $a \in K$, is the representation of $\check{\mathscr{H}}_3$ determined by the following matrices giving the action of $\mathcal{P}_i$ on $K^{\oplus 2}$:

$$\mathcal{P}_1 \mapsto \begin{pmatrix} [2]^2 & 0 \\ a & 0 \end{pmatrix}, \ \mathcal{P}_2 \mapsto \begin{pmatrix} 0 & a \\ 0 & [2]^2 \end{pmatrix}. \tag{110}$$

Here we have specified a basis (e_1, e_2) for $\check{N}(a)$ and are thinking of matrices as acting on the right on row vectors, so that the j-th row of these matrices gives the coefficients of $e_j\mathcal{P}_i$ in the basis (e_1, e_2). Those a for which $\check{N}(a)$ defines a representation of $K\check{\mathscr{H}}_3$ that is irreducible are

$$a_1 = [2],\ a_2 = [2]^2 - 2,$$

and

$$\check{M}_{\{(3),(2,1)\}} = \check{N}(a_1), \quad S'\check{M}_{(2,1)} = \check{N}(a_2).$$

Remark 11.15. In view of the generalization of the nonstandard Hecke algebra in [8], the constants a_1, a_2 have a nice explanation: a_1, a_2 is the beginning of the sequence $T_1(\frac{1}{[2]}), T_2(\frac{1}{[2]}), \ldots$ (up to a certain normalization by factors of $\pm[2]^j$), where the $T_k(x)$ are the Chebyshev polynomials of the first kind.

The algebra $\check{\mathscr{H}}_3$ has a nice presentation using these coefficients.

Theorem 11.16 ([45],[8])**.** *The algebra $\check{\mathscr{H}}_3$ is the associative* $\mathbf{A}$*-algebra generated by $\mathcal{P}_s, s \in S = \{s_1, s_2\}$, with quadratic relations*

$$(\mathcal{P}_s)^2 = [2]^2\mathcal{P}_s, \quad s \in S, \tag{111}$$

and nonstandard braid relation

$$\mathcal{P}_1(\mathcal{P}_{21} - a_1^2)(\mathcal{P}_{21} - a_2^2) = \mathcal{P}_2(\mathcal{P}_{12} - a_1^2)(\mathcal{P}_{12} - a_2^2). \tag{112}$$

Moreover, $\check{\mathscr{H}}_3$ is free as an $\mathbf{A}$*-module.*

Remark 11.17. Since $\mathcal{P}_1, \mathcal{P}_2$ do not satisfy the relation

$$C_1'C_2'C_1' - C_1' = C_2'C_1'C_2' - C_2'$$

satisfied by C_1', C_2' (which is equivalent to the braid relation for T_1, T_2), the $\mathscr{R}$-matrices $\mathcal{R}_1, \mathcal{R}_2$ (in the notation of Remark 8.4) do not satisfy the quantum Yang-Baxter equation. Hence the Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ is not coquasitriangular.

11.7. **A canonical basis of $\check{\mathscr{H}}_3$.** We now ask if $\check{\mathscr{H}}_r$ has a canonical basis $\check{\mathcal{C}}^r$ akin to the Kazhdan-Lusztig basis of $\mathscr{H}_r$. There are two properties we would like such a basis to satisfy.

The first is that for each $J \subseteq S$, the nonstandard right (resp. left) J-descent space is spanned by a subset of $\check{\mathcal{C}}^r$, where the *nonstandard right (resp. left) J-descent space* of $\check{\mathscr{H}}_r$ is defined to be

$$\{h \in \check{\mathscr{H}}_r : h\mathcal{Q}_s = [2]^2 h \text{ for all } s \in J\},$$
$$(\text{resp. } \{h \in \check{\mathscr{H}}_r : \mathcal{Q}_s h = [2]^2 h \text{ for all } s \in J\}).$$

The second property is that $\check{\mathcal{C}}^r$ be a cellular basis in the sense of Graham-Lehrer [21]. Graham and Lehrer's theory of cellular algebras [21] was in fact made to abstract and generalize some of the nice properties satisfied by the Kazhdan-Lusztig basis of $\mathscr{H}_r$. We now review the definition of a cellular basis and recall the cellular basis of $R\check{\mathscr{H}}_3$ (for suitable R) from [45, 8].

Let H be an algebra over a commutative ring R.

Definition 11.18. Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $\mathcal{T}(\lambda)$ and distinct elements $C^\lambda_{ST} \in H$ for all $S, T \in \mathcal{T}(\lambda)$ such that

$$\mathcal{C} = \{C^\lambda_{ST} : \lambda \in \Lambda \text{ and } S, T \in \mathcal{T}(\lambda)\}$$

is a (free) R-basis of H. For $\lambda \in \Lambda$, let $H_{<\lambda}$ be the R-submodule of H with basis $\{C^\mu_{ST} : \mu < \lambda \text{ and } S, T \in \mathcal{T}(\mu)\}$.

The triple $(\mathcal{C}, \Lambda, \mathcal{T})$ is a *cellular basis* of H if

(i) the R-linear map $* : H \to H$ determined by $(C^\lambda_{ST})^* = C^\lambda_{TS}$, for all $\lambda \in \Lambda$ and all S and T in $\mathcal{T}(\lambda)$, is an algebra anti-isomorphism of H,

(ii) for any $\lambda \in \Lambda$ and $h \in H$ there exist $r_{S',S} \in R$, for $S', S \in \mathcal{T}(\lambda)$, such that for all $T \in \mathcal{T}(\lambda)$

$$hC^\lambda_{ST} \equiv \sum_{S' \in \mathcal{T}(\lambda)} r_{S',S} C^\lambda_{S'T} \mod H_{<\lambda}.$$

For each $\lambda \in \Lambda$, the *cell representation* corresponding to λ is the left H-module that is the submodule of $H_{\leq\lambda}/H_{<\lambda}$ with R-basis $\{C^\lambda_{ST} : S \in \mathcal{T}(\lambda)\}$ for some $T \in \mathcal{T}(\lambda)$; this basis is independent of T.

Let R be a commutative ring with a map $\mathbf{A} \to R$ such that the images of a_1 and a_2 are invertible. There are several cellular bases for $R\check{\mathscr{H}}_3$. We define one such cellular basis $(\check{\mathcal{C}}^3, \check{\Lambda}, \check{\mathcal{T}})$ for the poset $\check{\Lambda}$ given by

$$\begin{array}{c} \check{\epsilon}_+ \\ | \\ \check{N}(a_1) \\ | \\ \check{N}(a_2) \\ | \\ \check{\epsilon}_- \end{array} \tag{113}$$

Set $\check{\mathcal{T}}(\check{\epsilon}_\pm) = \{\pm\}$, $\check{\mathcal{T}}(\check{N}(a_i)) = \{1,2\}$. The basis $\check{\mathcal{C}}^3$ consists of

$$\begin{array}{lcl}
C^{\check{\epsilon}_+} & := & 1, \\
C^{\check{N}(a_1)}_{11} & := & a_1\mathcal{Q}_1, \\
C^{\check{N}(a_1)}_{21} & := & \mathcal{Q}_{21}, \\
C^{\check{N}(a_1)}_{22} & := & a_1\mathcal{Q}_2, \\
C^{\check{N}(a_1)}_{12} & := & \mathcal{Q}_{12}, \\
C^{\check{N}(a_2)}_{11} & := & a_2\mathcal{Q}_1(\mathcal{Q}_{21}-a_1^2), \\
C^{\check{N}(a_2)}_{21} & := & \mathcal{Q}_{21}(\mathcal{Q}_{21}-a_1^2), \\
C^{\check{N}(a_2)}_{22} & := & a_2\mathcal{Q}_2(\mathcal{Q}_{12}-a_1^2), \\
C^{\check{N}(a_2)}_{12} & := & \mathcal{Q}_{12}(\mathcal{Q}_{12}-a_1^2), \\
C^{\check{\epsilon}_-} & := & \mathcal{Q}_1(\mathcal{Q}_{21}-a_1^2)(\mathcal{Q}_{21}-a_2^2).
\end{array} \tag{114}$$

It is is not hard to check, given Theorem 11.16, that $(\check{\mathcal{C}}^3, \check{\Lambda}, \check{\mathcal{T}})$ is a cellular basis; see [8, Proposition 5.3] for a careful proof. In the case $R = K$, the radical [21, Definition 3.1] of each cell representation is 0. Thus we recover the four absolutely irreducible $K\check{\mathscr{H}}_3$-modules of Proposition 11.14 as cell representations.

Remark 11.19. It is shown in the older version of this paper [45] that a similar basis B of $\check{\mathscr{H}}_3$ has the following positivity property: the coefficients of the expansion of any $b \in B$ in terms of the basis $\{C_v \star C_w : v, w \in \mathcal{S}_3\}$ are $\bar{\cdot}$-invariant Laurent polynomials in q with nonnegative integer coefficients. We are uncertain if a similar, perhaps slightly weaker, form of positivity holds for $\check{\mathscr{H}}_4$ and beyond. In Example 19.3, we construct a nice basis of each $\check{\mathscr{H}}_4$-irreducible. The coefficients for the action of $\mathcal{Q}_i$ on these bases are $\bar{\cdot}$-invariant Laurent polynomials in q and, after factoring out powers of $q - q^{-1}$, have all nonnegative or all nonpositive integer coefficients. Similarly, Theorem 17.7 and Example 17.8 show that a weak form of positivity holds for the basis +HNSTC(ν) of $\check{X}_\nu$, but that this cannot be strengthened.

11.8. **The algebra $\check{\mathscr{H}}_4$.** The algebra $\check{\mathscr{H}}_4$ turns out to be considerably more complicated and is of dimension 114. The irreducible representations of $K\check{\mathscr{H}}_4$ are

$$K\check{\epsilon}_-,\ S'\check{M}_{(2,2)},\ \check{\Lambda}^2\check{M}_{(3,1)},\ \check{M}_{\{(3,1),(2,2)\}},\ S'\check{M}_{(3,1)},\ (S'\check{M}_{(3,1)})^\#,\ \check{M}_{\{(4),(2,2)\}},\ \check{M}_{\{(4),(3,1)\}},\ K\check{\epsilon}_+.$$

The dimension count corresponding to expressing $K\check{\mathscr{H}}_4$ as the sum of its minimal two-sided ideals is

$$1^2 + 2^2 + 3^2 + 6^2 + 5^2 + 5^2 + 2^2 + 3^2 + 1^2 = 114.$$

We have not been able to determine a presentation for $\check{\mathscr{H}}_4$ akin to the presentation (22) for the Hecke algebra or the presentation of $\check{\mathscr{H}}_3$ from Theorem 11.16.

The ideal of relations expressing $\check{\mathscr{H}}_4$ as a quotient of the free $\mathbf{A}$-algebra in the $\mathcal{P}_i$'s is not generated by the quadratic relations (111) and the nonstandard braid relations (112) for the parabolic subalgebras $\langle \mathcal{P}_i, \mathcal{P}_{i+1}\rangle \subseteq \check{\mathscr{H}}_4$. We determined by computer the ideal of relations by a simple procedure of generating monomials in the $\mathcal{P}_i$'s systematically and of increasing degree while retaining only those which were not linear combinations of earlier monomials. The top degree obtained thus was 9. In other words, every monomial of degree 10 and above is a linear combination of some smaller monomials. However, these linear combinations seem fairly complicated. To give an idea of the difficulties involved, the simplest relation among the generators of $\check{\mathscr{H}}_4$ that cannot be deduced from the quadratic and nonstandard braid relations is a linear combination of 74 monomials of degrees ≤ 7; it is reported in the older version of this paper [45].

12. Nonstandard Schur-Weyl duality

We prove a nonstandard analog of quantum Schur-Weyl duality for the coaction of $\mathscr{O}(M_q(\check{X}))$ and action of $\check{\mathscr{H}}_r$ on $\check{X}^{\otimes r}$. This allows us to relate the representation theory of $\check{\mathscr{H}}_r$ to the corepresentation theory of $\mathscr{O}(M_q(\check{X}))$. We illustrate this for the two-row, $r = 3$ case.

12.1. **Nonstandard Schur-Weyl duality.** As explained in §6.1, $\mathscr{H}_r$ acts on $V^{\otimes r}$ on the right by sending T_i to the endomorphism of $V^{\otimes r}$ given by $\hat{\mathscr{R}}_{V,V}$ acting on the i and $i+1$-st tensor factors. Let $K\mathscr{S}(V,r)$ be the *q-Schur algebra* over K, which is defined to be the endomorphism algebra $\mathrm{End}_{K\mathscr{H}_r}(V^{\otimes r})$. It is known that this is also equal to the algebra dual to the coalgebra $\mathscr{O}(M_q(V))_r$. By definition, the right action of $\mathscr{H}_r$ on $V^{\otimes r}$ commutes with the action of $K\mathscr{S}(V,r)$ on $V^{\otimes r}$, hence $V^{\otimes r}$ is a $(K\mathscr{S}(V,r), K\mathscr{H}_r)$-bimodule. Quantum Schur-Weyl duality now takes the same form $V^{\otimes r} \cong \bigoplus_\lambda V_\lambda \otimes M_\lambda$ as Theorem 6.2, except this is considered as an isomorphism of $(K\mathscr{S}(V,r), K\mathscr{H}_r)$-bimodules rather of $(U_q(\mathfrak{g}_V), K\mathscr{H}_r)$-bimodules.

Next, consider the commuting actions of $\mathscr{H}_r \star \mathscr{H}_r$ and $K\mathscr{S}(V,r) \star K\mathscr{S}(W,r)$ on $X^{\otimes r}$. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$ acts on $\check{X}^{\otimes r}$ by sending $\mathcal{P}_i$ to the endomorphism of $\check{X}^{\otimes r}$ given by $[2]^2 P_+^{\check{X}}$ acting on the i and $i+1$-st tensor factors. This is a well-defined action of $\check{\mathscr{H}}_r$ on $\check{X}^{\otimes r}$ by the definition of the action of $\mathscr{H}_r$ on $V^{\otimes r}$ and the similar forms of $\mathcal{P}_i$ (93) and $P_+^{\check{X}}$ (60). Define the *nonstandard Schur algebra*, denoted $K\check{\mathscr{S}}(\check{X}, r)$, to be the algebra dual to the coalgebra $\mathscr{O}(M_q(\check{X}))_r$. By the definition of $\mathscr{O}(M_q(\check{X}))$, the action of $\check{\mathscr{H}}_r$ on $\check{X}^{\otimes r}$ commutes with the action of $K\check{\mathscr{S}}(\check{X}, r)$—this will be shown carefully in the proof below.

We have the following nonstandard analog of quantum Schur-Weyl duality:

Theorem 12.1. *As a $(K\check{\mathscr{S}}(\check{X}, r), K\check{\mathscr{H}}_r)$-bimodule, $\check{X}^{\otimes r}$ decomposes into irreducibles as*

$$\check{X}^{\otimes r} \cong \bigoplus_{\alpha \in \check{\mathscr{P}}_r} \check{\mathcal{X}}_\alpha \otimes \check{M}_\alpha, \tag{115}$$

where $\check{\mathscr{P}}_r$ is an index set so that $\check{\mathcal{X}}_\alpha$ ranges over irreducible $\mathscr{O}(M_q(\check{X}))_r$-comodules (which are the same as irreducible $K\check{\mathscr{S}}(\check{X}, r)$-modules) and $\check{M}_\alpha$ ranges over $K\check{\mathscr{H}}_r$-irreducibles.

Remark 12.2. This theorem should use $\check{X}^*$ in place of $\check{X}$ to be consistent with our conventions in §8–10 (see Remark 2.1). The proof uses $\check{X}^*$ in place of $\check{X}$ and pays careful attention to duals. The left $K\check{\mathscr{S}}(\check{X}, r)$-module $(\check{X}^*)^{\otimes r}$ gives rise to an injection $K\check{\mathscr{S}}(\check{X}, r) \hookrightarrow \mathrm{End}(\check{X}^{\otimes r})$ and this is our starting point in the proof below.

Proof. Let $\mathrm{End}_{K\check{\mathscr{H}}_r}(\check{X}^{\otimes r})$ be the algebra of endomorphisms of $\check{X}^{\otimes r}$ intertwining the action of $K\check{\mathscr{H}}_r$. By well-known algebraic generalities used to prove classical Schur-Weyl duality (see, e.g., [20, Lemma 6.22]; the algebras in this lemma are over $\mathbb{C}$, but it is not hard to check that it extends to the present setting) and the semisimplicity of $K\check{\mathscr{H}}_r$, it suffices to show that

$$K\check{\mathscr{S}}(\check{X}, r) = \mathrm{End}_{K\check{\mathscr{H}}_r}(\check{X}^{\otimes r}).$$

Clearly, $f \in \mathrm{End}(\check{X}^{\otimes r})$ lies in $\mathrm{End}_{K\check{\mathscr{H}}_r}(\check{X}^{\otimes r})$ if and only if f commutes with each $\frac{\mathcal{P}_i}{[2]^2}$. Note that, in general, an endomorphism g commutes with a projector p if and only if $g(\mathrm{im}\,(p)) \subseteq \mathrm{im}\,(p)$ and $g(\ker(p)) \subseteq \ker(p)$. Hence f commutes with $\frac{\mathcal{P}_i}{[2]^2}$ if and only if

$$\begin{aligned} &f\big(\check{X}^{\otimes i-1} \otimes \check{S}^2\check{X} \otimes \check{X}^{\otimes r-i-1}\big) \subseteq \check{X}^{\otimes i-1} \otimes \check{S}^2\check{X} \otimes \check{X}^{\otimes r-i-1}, \text{ and}\\ &f\big(\check{X}^{\otimes i-1} \otimes \check{\Lambda}^2\check{X} \otimes \check{X}^{\otimes r-i-1}\big) \subseteq \check{X}^{\otimes i-1} \otimes \check{\Lambda}^2\check{X} \otimes \check{X}^{\otimes r-i-1}. \end{aligned}$$

This is equivalent to

$$f \in (\check{Z}^*)^{\otimes i-1} \otimes \big(\check{S}^2\check{X}^* * \check{S}^2\check{X} \oplus \check{\Lambda}^2\check{X}^* * \check{\Lambda}^2\check{X}\big) \otimes (\check{Z}^*)^{\otimes r-i-1},$$

where $\check{Z} = \check{X} \otimes \check{X}^*$. Regarding f as an element of $\mathrm{Hom}(\check{Z}^{\otimes r}, K)$, this is equivalent to f vanishing on

$$\check{\mathcal{I}}_{r,i} := \check{Z}^{\otimes i-1} \otimes \big(\check{S}^2\check{X} * \check{\Lambda}^2\check{X}^* \oplus \check{\Lambda}^2\check{X} * \check{S}^2\check{X}^*\big) \otimes \check{Z}^{\otimes r-i-1}.$$

Since by (63), $\mathscr{O}(M_q(\check{X}))_r$ is the quotient of $\check{Z}^{\otimes r}$ by $\sum_{i=1}^{r-1} \check{\mathcal{I}}_{r,i}$, $f \in K\check{\mathscr{S}}(\check{X}, r)$ if and only if f vanishes on $\sum_{i=1}^{r-1} \check{\mathcal{I}}_{r,i}$ if and only if f commutes with each $\frac{\mathcal{P}_i}{[2]^2}$. □

12.2. **Consequences for the corepresentation theory of $\mathscr{O}(M_q(\check{X}))$.** As a corollary to nonstandard Schur-Weyl duality, we obtain a result similar to Theorem 10.7 (d) (but without the assumption that the field is $\mathbb{C}$).

Corollary 12.3. *As a coalgebra over $K = \mathbb{Q}(q)$, $\mathscr{O}(M_q(\check{X}))$ is cosemisimple.*

Note that, just as in the classical case, nonstandard Schur-Weyl duality allows us to describe the irreducible $\mathscr{O}(M_q(\check{X}))$-comodule $\check{\mathcal{X}}_\alpha$ if a primitive idempotent $\check{e}_\alpha$ corresponding to $\check{M}_\alpha$ is known: $\check{\mathcal{X}}_\alpha$ is equal to $\check{X}^{\otimes r}\check{e}_\alpha$. Computing such idempotents explicitly is difficult, but there is a similar and easier way to relate the corepresentation theory of $\mathscr{O}(M_q(\check{X}))$ to the representation theory of $\check{\mathscr{H}}_r$: combining standard quantum Schur-Weyl duality applied to $V^{\otimes r}$ and $W^{\otimes r}$ with (115) yields

$$\bigoplus_{\alpha \in \check{\mathscr{P}}_r} \check{\mathcal{X}}_\alpha \otimes \check{M}_\alpha \cong \bigoplus_{\lambda \vdash_{d_V} r,\ \mu \vdash_{d_W} r} V_\lambda \star W_\mu \otimes M_\lambda \star M_\mu. \tag{116}$$

Viewing this as an isomorphism of $(K\mathscr{S}(V, r) \star K\mathscr{S}(W, r), \check{\mathscr{H}}_r)$-bimodules shows that there are nonnegative integers $n_\alpha^{\lambda,\mu} = n_{\lambda,\mu}^\alpha$ that correspond to the multiplicities in the following two decomposition problems:

$$\check{\mathcal{X}}_\alpha \cong \bigoplus_{\lambda,\mu} (V_\lambda \star W_\mu)^{\oplus n_\alpha^{\lambda,\mu}}, \quad \mathrm{Res}_{\check{\mathscr{H}}_r} M_\lambda \star M_\mu \cong \bigoplus_\alpha \check{M}_\alpha^{\oplus n_{\lambda,\mu}^\alpha}. \tag{117}$$

Applying (117) to α corresponding to the trivial or sign representation of $\check{\mathscr{H}}_r$ together with Proposition 11.13 (and the analogous statement for $\check{\epsilon}_+$) yields

Corollary 12.4. *The $\mathscr{O}(M_q(\check{X}))$-comodules $\check{\Lambda}^r \check{X}$ and $\check{S}^r \check{X}$ are irreducible and decompose into irreducible $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodules as in Proposition 8.9.*

This was already known from Lemma 9.4 and Proposition 8.9, but this gives another proof.

12.3. **The two-row, $r = 3$ case.** In the next section, we will give a complete description of nonstandard Schur-Weyl duality in the two-row case, but for now let us give some feel for the result by working out the two-row ($d_V = d_W = 2$), $r = 3$ case. This will make use of our understanding of $\check{\mathscr{H}}_3$ from §11.6.

With the given assumptions, (115) takes the form

$$\check{X}^{\otimes 3} \cong \check{S}^3 \check{X} \otimes K\check{\epsilon}_+ \oplus \check{\mathcal{X}}_{\{(3),(2,1)\}} \otimes \check{M}_{\{(3),(2,1)\}} \oplus \check{\mathcal{X}}_{+(2,1)} \otimes S'\check{M}_{(2,1)} \oplus \check{\Lambda}^3 \check{X} \otimes K\check{\epsilon}_-. \tag{118}$$

The $\mathscr{O}(M_q(\check{X}))$-comodules $\check{\mathcal{X}}_{\{(3),(2,1)\}}$, $\check{\mathcal{X}}_{+(2,1)}$ have not yet been defined, but we know from (117), and the fact that $\mathrm{Res}_{\check{\mathscr{H}}_3} M_{(2,1)} \star M_{(2,1)} \cong \check{\epsilon}_+ \oplus S'\check{M}_{(2,1)} \oplus \check{\epsilon}_-$, that they decompose into irreducible $\mathscr{O}(M_q(V)) \star \mathscr{O}(M_q(W))$-comodules as

$$\begin{array}{lcl} \check{S}^3 \check{X} & \cong & V_{(3)} \star W_{(3)} \oplus V_{(2,1)} \star W_{(2,1)}, \\ \check{\mathcal{X}}_{\{(3),(2,1)\}} & \cong & V_{(3)} \star W_{(2,1)} \oplus V_{(2,1)} \star W_{(3)} \\ \check{\mathcal{X}}_{+(2,1)} & \cong & V_{(2,1)} \star W_{(2,1)}, \\ \check{\Lambda}^3 \check{X} & \cong & V_{(2,1)} \star W_{(2,1)}. \end{array} \tag{119}$$

(The first and last lines are already known from Proposition 8.9.)

As a check, the dimension count for (118) is

$$4^3 = 20 \cdot 1 + 16 \cdot 2 + 4 \cdot 2 + 4 \cdot 1.$$

See Example 19.2 for a nice basis of $\check{X}^{\otimes 3}$ that realizes nonstandard Schur-Weyl duality in this case.

We can also compare this to standard quantum Schur-Weyl duality for $X^{\otimes 3}$ as a $\mathscr{O}(GL_q(X))$-comodule (this has the same form as Schur-Weyl duality between $GL(X)$ and $\mathcal{S}_3$):

$$X^{\otimes 3} \cong X_{(3)} \otimes M_{(3)} \oplus X_{(2,1)} \otimes M_{(2,1)} \oplus X_{(1,1,1)} \otimes M_{(1,1,1)},$$

where X_λ denotes the $\mathscr{O}(GL_q(X))$-comodule of highest weight λ. The dimension count here is

$$4^3 = 20 \cdot 1 + 20 \cdot 2 + 4 \cdot 1.$$

By comparing this to Schur-Weyl duality for $V^{\otimes 3}$ and $W^{\otimes 3}$ and setting $q = 1$ (we are just computing Kronecker coefficients for partitions of size 3) we obtain the left-hand isomorphisms below.

$$\begin{aligned} X_{(3)}|_{q=1} &\cong \big(V_{(3)} \star W_{(3)} \oplus V_{(2,1)} \star W_{(2,1)}\big)\,|_{q=1} && \cong \check{S}^3\check{X}|_{q=1}, \\ X_{(2,1)}|_{q=1} &\cong \big(V_{(3)} \star W_{(2,1)} \oplus V_{(2,1)} \star W_{(3)} \oplus V_{(2,1)} \star W_{(2,1)}\big)\,|_{q=1} && \cong \check{\mathscr{X}}_{\{(3),(2,1)\}}|_{q=1} \oplus \check{\mathscr{X}}_{+(2,1)}|_{q=1}, \\ X_{(1,1,1)}|_{q=1} &\cong \big(V_{(2,1)} \star W_{(2,1)}\big)\,|_{q=1} && \cong \check{\Lambda}^3\check{X}|_{q=1}. \end{aligned}$$

The right-hand isomorphisms are by (119), all objects here being thought of as $\mathscr{O}(GL(V))\star\mathscr{O}(GL(W))$-comodules. Thus $\check{\mathscr{X}}_{\{(3),(2,1)\}} \oplus \check{\mathscr{X}}_{+(2,1)}$ is some quantization of $X_{(2,1)}|_{q=1}$, which we believe to be a better quantization than $X_{(2,1)}$ for the Kronecker problem.

We can also use the knowledge just gained about $\mathscr{O}(M_q(\check{X}))_3$-comodules to understand the Peter-Weyl theorem for $\mathscr{O}(M_q(\check{X}))$ (Theorem 10.7 (d)) and explain the dimension count $\dim(\mathscr{O}(M_q(\check{X}))_3) = 688$. By (119) and Theorem 10.7 (d),

$$\mathscr{O}(M_q(\check{X}))_3 \cong \check{S}^3\check{X}\otimes\check{S}^3\check{X}^*\oplus(\check{\mathscr{X}}_{\{(3),(2,1)\}})_L\otimes\check{\mathscr{X}}_{\{(3),(2,1)\}}\oplus(\check{\mathscr{X}}_{+(2,1)})_L\otimes\check{\mathscr{X}}_{+(2,1)}\oplus\check{\Lambda}^3\check{X}\otimes\check{\Lambda}^3\check{X}^*,$$

with corresponding dimensions $688 = 20^2 + 16^2 + 4^2 + 4^2$. Compare this to the dimension count $\dim(\mathscr{O}(M_q(X))_3) = 816 = 20^2 + 20^2 + 4^2$ for the standard Peter-Weyl theorem.

13. Nonstandard representation theory in the two-row case

It turns out that in the two-row ($d_V = d_W = 2$) case, the nonstandard Hecke algebra $\check{\mathscr{H}}_r$ is quite close to $S^2\mathscr{H}_r$ (by the "two-row case" of $\check{\mathscr{H}}_r$, we mean the nonstandard Temperley-Lieb quotient $\check{\mathscr{H}}_{r,2}$, defined in §13.6) and the nonstandard coordinate algebra $\mathscr{O}(GL_q(\check{X}))$ is close to the smash coproduct $\mathscr{O}_q^\tau := \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \rtimes \mathscr{F}(\mathcal{S}_2)$ (as defined in Appendix B). To develop a theory of crystal bases for $\mathscr{O}(GL_q(\check{X}))$ and its comodules, we would prefer to work with a "nonstandard enveloping algebra" that is Hopf dual to $\mathscr{O}(GL_q(\check{X}))$. However, we have not been able to construct such a nonstandard enveloping algebra explicitly. For the two-row case however, the approximation $U_q^\tau := U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W) \rtimes \mathcal{S}_2$ has proved to be close enough, and this is the object we work with in the detailed study of two-row Kronecker in §15–17.

Throughout this section, assume $d_V = d_W$. We discuss the (co)representation theory of U_q^τ and $\mathscr{O}_q^\tau$ and a Schur-Weyl duality between these and $S^2\mathscr{H}_r$. A complete description of nonstandard Schur-Weyl duality between $\check{\mathscr{H}}_{r,2}$ and $\mathscr{O}(M_q(\check{X}))$ is then given in §13.6. We also extend the theory of upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules to U_q^τ-modules (§13.5).

13.1. **The Hopf algebra U_q^τ.** Define U_q^τ to be the wreath product $U_q(\mathfrak{gl}_{d_V}) \wr \mathcal{S}_2$, also equal to $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W) \rtimes \mathcal{S}_2$ since we are assuming $d_V = d_W$. Explicitly, U_q^τ is the algebra containing $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ and an element τ such that $\tau g_V \tau = g_W$, for any $g \in U_q(\mathfrak{gl}_{d_V})$, where g_V, g_W denote the corresponding elements of $U_q(\mathfrak{g}_V)$ and $U_q(\mathfrak{g}_W)$, respectively.

The algebra U_q^τ is a Hopf algebra containing $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ as a Hopf subalgebra. Its coproduct $\Delta : U_q^\tau \to U_q^\tau \otimes U_q^\tau$ is the unique algebra homomorphism extending the coproduct of $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ (see (29)) and satisfying

$$\Delta(\tau) = \tau \otimes \tau. \tag{120}$$

The counit ε of U_q^τ is the algebra homomorphism determined by $\varepsilon(q^h) = \varepsilon(\tau) = 1$, $\varepsilon(E_i) = \varepsilon(F_i) = 0$, and the antipode S is the algebra anti-homomorphism determined by

$$S(q^h) = q^{-h}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i, \quad S(\tau) = \tau. \tag{121}$$

The bar-involution on $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ extends to a $\mathbb{Q}$-linear automorphism of U_q^τ by $\overline{\tau} = \tau$, also called the *bar-involution* and denoted $\bar{\cdot}$.

13.2. **The Hopf algebra $\mathscr{O}_q^\tau$.** Let $\mathscr{F}(\mathcal{S}_2)$ denote the Hopf algebra of functions on the group $\mathcal{S}_2$ taking values in K. Dual to U_q^τ, there is an object

$$\mathscr{O}_q^\tau := \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \rtimes \mathscr{F}(\mathcal{S}_2).$$

Here, $\rtimes$ is a "semidirect coproduct," often called a smash coproduct in this setting. Since this is a less familiar operation than the semidirect product, we give it a careful treatment, but leave this to Appendix B since the details are somewhat technical. The object $\mathscr{O}_q^\tau$ can be given the structure of a Hopf algebra (Proposition B.2) such that the pairing between U_q^τ and $\mathscr{O}_q^\tau$ coming from the pairing between $U_q(\mathfrak{g}_V)$ and $\mathscr{O}(GL_q(V))$ is a nondegenerate Hopf-pairing (Corollary B.3). Moreover, we show (Proposition B.4) that there is a Hopf algebra homomorphism $\tilde{\psi}^\tau$ such that the composition

$$\mathscr{O}(GL_q(\check{X})) \xrightarrow{\tilde{\psi}^\tau} \mathscr{O}_q^\tau \xrightarrow{\pi} \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$$

is equal to $\tilde{\psi}$ of Proposition 10.5; here π is the canonical surjection.

13.3. **Representation theory of U_q^τ and $\mathscr{O}_q^\tau$.** Let $\mathscr{O}_{\text{int}}^{\geq 0}(U_q^\tau)$ be the full subcategory of U_q^τ-modules with objects

$$\{\mathcal{X} \in U_q^\tau\textbf{-Mod} : \operatorname{Res}_{U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)} \mathcal{X} \in \mathscr{O}_{\text{int}}^{\geq 0}(\mathfrak{g}_V \oplus \mathfrak{g}_W)\}.$$

All such modules are completely reducible and the irreducibles are described below. This follows from general results about the representation theory of the wreath product of a universal enveloping algebra (or an algebra with similar properties) and a finite group, as treated in [29].

Let $\eta_{\lambda,\mu}$ denote the canonical highest weight vector of $V_\lambda \star W_\mu$. Recall that $\mathscr{P}_{r,l}$ denotes the set of partitions of size r with at most l parts. The irreducible objects of $\mathscr{O}_{\text{int}}^{\geq 0}(U_q^\tau)$ are

(1) $\mathcal{X}_{\{\lambda,\mu\}} := V_\lambda \star W_\mu \oplus V_\mu \star W_\lambda$, $\lambda \in \mathscr{P}_{r_1,d_V}$, $\mu \in \mathscr{P}_{r_2,d_V}$, $r_1, r_2 \geq 0$, $\lambda \neq \mu$, with the action of τ determined by $\tau(\eta_{\lambda,\mu}) = \eta_{\mu,\lambda}$,
(2) $\mathcal{X}_{+\lambda} := V_\lambda \star W_\lambda$, $\lambda \in \mathscr{P}_{r,d_V}$, $r \geq 0$, with the action of τ determined by $\tau(\eta_{\lambda,\lambda}) = \eta_{\lambda,\lambda}$,
(3) $\mathcal{X}_{-\lambda} := V_\lambda \star W_\lambda$, $\lambda \in \mathscr{P}_{r,d_V}$, $r \geq 0$, with the action of τ determined by $\tau(\eta_{\lambda,\lambda}) = -\eta_{\lambda,\lambda}$.

Since U_q^τ and $\mathscr{O}_q^\tau$ are dually paired Hopf algebras, any right $\mathscr{O}_q^\tau$-comodule is also a left U_q^τ-module (see §2.6). All of the objects of $\mathscr{O}_{\text{int}}^{\geq 0}(U_q^\tau)$ in fact come from $\mathscr{O}_q^\tau$-comodules and the irreducibles are the same: let $\mathcal{X}$ be any of the U_q^τ-modules (1)–(3). First of all, given the forms of (1)–(3), it follows from §4.7 that there is a corepresentation $\varphi : \mathcal{X} \to$

$\mathscr{X} \otimes \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$, $x \mapsto \sum x_{(0)} \otimes x_{(1)}$ corresponding to $\mathrm{Res}_{U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)} \mathscr{X}$. The corepresentation $\varphi^\tau : \mathscr{X} \to \mathscr{X} \otimes \mathscr{O}_q^\tau$ giving rise to the U_q^τ-module $\mathscr{X}$ is then given by

$$\begin{cases} \varphi^\tau(x) = \sum x_{(0)} \otimes x_{(1)} \sharp e^\vee + \tau(x_{(0)}) \otimes \tau(x_{(1)}) \sharp \tau^\vee, & \text{if } \mathscr{X} = \mathscr{X}_{\{\lambda,\mu\}} \text{ or } \mathscr{X} = \mathscr{X}_{+\lambda}, \\ \varphi^\tau(x) = \sum x_{(0)} \otimes x_{(1)} \sharp e^\vee - \tau(x_{(0)}) \otimes \tau(x_{(1)}) \sharp \tau^\vee, & \text{if } \mathscr{X} = \mathscr{X}_{-\lambda}, \end{cases}$$

where $\tau(\cdot)$ denotes both the involution of $\mathscr{X}$ and of $\mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$ given by $\tau(f \star g) = g \star f$; see Appendix B for notation. Moreover, by the Peter-Weyl theorem for $\mathscr{O}(M_q(V))$, the comodules corresponding to (1)–(3) are all the irreducible comodules of $\mathscr{O}_q^\tau$ up to tensoring with powers of the determinant.

13.4. **Schur-Weyl duality between U_q^τ and $S^2\mathscr{H}_r$.** Recall from §11.3 the irreducible $KS^2\mathscr{H}_r$-modules $M_{\{\lambda,\mu\}}$, S^2M_λ, $\Lambda^2 M_\lambda$. Recall that $\mathscr{P}'_{r,l}$ is the subset of $\mathscr{P}_{r,l}$ consisting of those partitions that are not a single row or column shape.

Proposition 13.1. *As a $(U_q^\tau, S^2\mathscr{H}_r)$-bimodule, $X^{\otimes r}$ decomposes into irreducibles as*

$$X^{\otimes r} \cong \bigoplus_{\substack{\{\lambda,\mu\} \subseteq \mathscr{P}_{r,d_V} \\ \lambda \neq \mu}} \mathscr{X}_{\{\lambda,\mu\}} \otimes M_{\{\lambda,\mu\}} \oplus \bigoplus_{\lambda \in \mathscr{P}_{r,d_V}} \mathscr{X}_{+\lambda} \otimes S^2 M_\lambda \oplus \bigoplus_{\lambda \in \mathscr{P}'_{r,d_V}} \mathscr{X}_{-\lambda} \otimes \Lambda^2 M_\lambda.$$

This follows easily from Proposition 11.10 and the decomposition of $V^{\otimes r} \star W^{\otimes r}$ as a $(U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathscr{H}_r \star \mathscr{H}_r)$-bimodule.

Now we can define the symmetric (resp. exterior) Kronecker coefficient $g_{+\lambda\nu}$ (resp. $g_{-\lambda\nu}$) to be either of the following quantities

- the multiplicity of $M_\nu|_{q=1}$ in $S^2 M_\lambda|_{q=1}$ (resp. $\Lambda^2 M_\lambda|_{q=1}$),
- the multiplicity of $\mathscr{X}_{+\lambda}|_{q=1}$ (resp. $\mathscr{X}_{-\lambda}|_{q=1}$) in $\mathrm{Res}_{U^\tau}(X_\nu|_{q=1})$,

where $U^\tau := U(\mathfrak{g}_V \oplus \mathfrak{g}_W) \rtimes \mathcal{S}_2$. These multiplicities are the same by Proposition 13.1 and standard Schur-Weyl duality for $X^{\otimes r}$ $\big(X^{\otimes r} \cong \bigoplus_{\nu \vdash_{d_X} r} X_\nu \otimes M_\nu\big)$.

13.5. **Upper based U_q^τ-modules.** For the detailed study of two-row Kronecker in §15–17, we need some theory of canonical bases for U_q^τ-modules.

Definition 13.2. A *weak upper based U_q^τ-module* is a pair (N, B), where N is an object of $\mathscr{O}_{\mathrm{int}}^{\geq 0}(U_q^\tau)$ and B is a K-basis of N such that $(\mathrm{Res}_{U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)} N, B)$ is an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module and $\tau(b) \in \pm B$ for all $b \in B$.

A weak upper based U_q^τ-module (N, B) is a *upper based U_q^τ-module* if for any highest weight $b \in B$ of weight (λ, λ), $\tau(b) = \pm b$.

In order to check that the tensor product of weak upper based U_q^τ-modules is a weak upper based U_q^τ-module, we must first check that

$$\text{if the } \bar{\cdot}\text{-involution on } U_q^\tau \text{ intertwines that of } N_1 \text{ and } N_2 \text{, then it intertwines the } \bar{\cdot}\text{-involution on } N_1 \otimes N_2. \tag{122}$$

This amounts to checking that $\Theta\ \bar{\cdot} \otimes \bar{\cdot}(\Delta(\tau)) = \Delta(\overline{\tau})\Theta$, which follows from $\overline{\tau} = \tau$, $\Delta(\tau) = \tau \otimes \tau$, and $\Theta = \Theta_V \Theta_W$, where Θ_V (resp. Θ_W) denotes the quasi-$\mathscr{R}$-matrix for $U_q(\mathfrak{g}_V)$ (resp. $U_q(\mathfrak{g}_W)$). Here $\bar{\cdot} \otimes \bar{\cdot}$ denotes the map $U_q^\tau \otimes U_q^\tau \to U_q^\tau \otimes U_q^\tau$, $x \otimes x' \mapsto \overline{x} \otimes \overline{x'}$.

Proposition 13.3. *If $(N, B), (N', B')$ are weak upper based U_q^τ-modules, then $(N \otimes N', B \heartsuit B')$ is a weak upper based U_q^τ-module with $\tau(b \heartsuit b') = \tau(b) \heartsuit \tau(b')$.*

Proof. Note that by the definition of weak upper based U_q^τ-module, the $\bar{\cdot}$-involution on U_q^τ intertwines that of N and N'. Then by (122), there holds $\overline{\tau(b \heartsuit b')} = \overline{\tau}(\overline{b \heartsuit b'}) = \tau(b \heartsuit b')$. Moreover, $\tau(b \heartsuit b') - \tau(b) \otimes \tau(b') \in q^{-1}(N \otimes N')_{\mathbb{Z}[q^{-1}]}$. Hence $\tau(b \heartsuit b')$ satisfies the defining properties of $\tau(b) \heartsuit \tau(b')$. The hypotheses then imply $\tau(b \heartsuit b') = \tau(b) \heartsuit \tau(b') \in \pm B \heartsuit B'$, hence $(N \otimes N', B \heartsuit B')$ is a weak upper based U_q^τ-module. $\square$

An upper based U_q^τ-module, unlike a weak upper based U_q^τ-module, has irreducible U_q^τ-cells. The are four types of isomorphism classes of upper based U_q^τ-modules (N, B) for which N is irreducible (numbered to match Proposition-Definition 11.10):

$$\begin{array}{ll} (1_+) & (\mathscr{X}_{\{\lambda,\mu\}}, B_V(\lambda) \star B_W(\mu) \sqcup B_V(\mu) \star B_W(\lambda))_+, \\ (1_-) & (\mathscr{X}_{\{\lambda,\mu\}}, B_V(\lambda) \star B_W(\mu) \sqcup B_V(\mu) \star B_W(\lambda))_-, \\ (2) & (\mathscr{X}_{+\lambda}, B_V(\lambda) \star B_W(\lambda)), \\ (3) & (\mathscr{X}_{-\lambda}, B_V(\lambda) \star B_W(\lambda)), \end{array} \tag{123}$$

where the action of τ is given by

$$\begin{aligned} \tau(G_\lambda(b^V_{P_V}) \star G_\mu(b^W_{P_W})) &= G_\mu(b^V_{P_W}) \star G_\lambda(b^W_{P_V}), \\ \tau(G_\lambda(b^V_{P_V}) \star G_\mu(b^W_{P_W})) &= -G_\mu(b^V_{P_W}) \star G_\lambda(b^W_{P_V}), \\ \tau(G_\lambda(b^V_{P_V}) \star G_\lambda(b^W_{P_W})) &= G_\lambda(b^V_{P_W}) \star G_\lambda(b^W_{P_V}), \\ \tau(G_\lambda(b^V_{P_V}) \star G_\lambda(b^W_{P_W})) &= -G_\lambda(b^V_{P_W}) \star G_\lambda(b^W_{P_V}). \end{aligned}$$

13.6. **The nonstandard two-row case.** We know that $V_{\mathbf{A}}^{\otimes r}$ is a right $\mathscr{H}_r$-module, where $V_{\mathbf{A}} = \mathbf{A}\{v_i : i \in [d_V]\}$ is the integral form of V. This defines an $\mathbf{A}$-algebra homomorphism $\mathscr{H}_r \to \mathrm{End}_{\mathbf{A}}(V_{\mathbf{A}}^{\otimes r})$. Define the *Temperley-Lieb* quotient $\mathscr{H}_{r,d_V}$ of $\mathscr{H}_r$ to be the image of this homomorphism for d_V equal to the dimension of V. Equivalently, this is the quotient of $\mathscr{H}_r$ by the two-sided ideal

$$\bigoplus_{\substack{\lambda \vdash r,\ \ell(\lambda) > d, \\ P \in \mathrm{SYT}(\lambda)}} \mathbf{A}\Gamma_P = \mathbf{A}\{C_w : \ell(\mathrm{sh}(w)) > d\}.$$

Here Γ_P denotes a right Kazhdan-Lusztig cell of $\mathcal{S}_r$ (see §3.2).

Next, define the *nonstandard Temperley-Lieb* quotient $\check{\mathscr{H}}_{r,d}$ of $\check{\mathscr{H}}_r$ to be the subalgebra of $\mathscr{H}_{r,d} \star \mathscr{H}_{r,d}$ generated by the elements

$$\mathcal{P}_s := C'_s \star C'_s + C_s \star C_s,\ s \in S.$$

Recall that $\mathscr{P}_{r,2}$ is the set of partitions of size r with at most two parts and $\mathscr{P}'_{r,2}$ is the subset of $\mathscr{P}_{r,2}$ consisting of those partitions that are not a single row or column shape. Define the index set $\check{\mathscr{P}}_{r,2}$ for the $K\check{\mathscr{H}}_{r,2}$-irreducibles as follows:

$$\check{\mathscr{P}}_{r,2} = \{\{\lambda, \mu\} : \lambda, \mu \in \mathscr{P}_{r,2},\ \lambda \neq \mu\} \sqcup \{+\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{-\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{\check{\epsilon}_+\}. \tag{124}$$

Theorem 13.4 ([11])**.** *The algebra $K\check{\mathscr{H}}_{r,2}$ is split semisimple and the list of distinct irreducibles is*

(1) $\check{M}_\alpha := \mathrm{Res}_{\check{\mathscr{H}}_{r,2}} M_\lambda \otimes M_\mu$, *for* $\alpha = \{\lambda, \mu\} \in \check{\mathscr{P}}_{r,2}$,
(2) $\check{M}_\alpha := S'\check{M}_\lambda$, *for* $\alpha = +\lambda \in \check{\mathscr{P}}_{r,2}$,
(3) $\check{M}_\alpha := \Lambda^2\check{M}_\lambda$, *for* $\alpha = -\lambda \in \check{\mathscr{P}}_{r,2}$,
(4) $\check{M}_\alpha := K\check{\epsilon}_+$, *for* $\alpha = \check{\epsilon}_+ \in \check{\mathscr{P}}_{r,2}$.

Nonstandard Schur-Weyl duality in the two-row case thus takes the form

$$\check{X}^{\otimes r} \cong \check{S}^r\check{X} \otimes K\check{\epsilon}_+ \oplus \bigoplus_{\substack{\{\lambda,\mu\}\subseteq \mathscr{P}_{r,2}\\ \lambda\neq\mu}} \check{\mathcal{X}}_{\{\lambda,\mu\}} \otimes \check{M}_{\{\lambda,\mu\}} \oplus \bigoplus_{\lambda\in\mathscr{P}'_{r,2}} \check{\mathcal{X}}_{+\lambda} \otimes S'\check{M}_\lambda \oplus \bigoplus_{\lambda\in\mathscr{P}'_{r,2}} \check{\mathcal{X}}_{-\lambda} \otimes \Lambda^2\check{M}_\lambda. \tag{125}$$

Here $\check{\mathcal{X}}_{\{\lambda,\mu\}}, \check{\mathcal{X}}_{\pm\lambda}$ are irreducible $\mathscr{O}(M_q(\check{X}))$-comodules (for $d_V = d_W = 2$) such that the corresponding $\mathscr{O}_q^\tau$-modules obtained via $\tilde{\psi}^\tau$ are isomorphic to $\mathcal{X}_{\{\lambda,\mu\}}, \mathcal{X}_{\pm\lambda}$, respectively. This follows from Proposition 13.1, §13.3, and arguments similar to those producing (117). Then, setting $\check{\mathcal{X}}_{\check{\epsilon}_+} := \check{S}^r\check{X}$, we have

Corollary 13.5. *The distinct irreducible $\mathscr{O}(M_q(\check{X}))$-comodules in the two-row case are the $\check{\mathcal{X}}_\alpha$, as r ranges over nonnegative integers and α ranges over $\check{\mathscr{P}}_{r,2}$.*

14. A canonical basis for $\check{Y}_\alpha$

Throughout this section, let $\alpha \vDash_l^{d_X} r$, set $\check{\mathbf{T}} = \check{X}^{\otimes r}$, and define

$$\check{Y}_\alpha := \check{\Lambda}^{\alpha_1}\check{X} \otimes \check{\Lambda}^{\alpha_2}\check{X} \otimes \ldots \otimes \check{\Lambda}^{\alpha_l}\check{X} \subseteq \check{\mathbf{T}}. \tag{126}$$

We define a canonical basis of $\check{Y}_\alpha$ by first defining a canonical basis of $\check{\Lambda}^r\check{X}$ and then putting these together with Lusztig's construction for tensoring based modules (§5.5). The bases of $\check{\Lambda}^r\check{X}$ and $\check{Y}_\alpha$ are labeled by what we call nonstandard columns and nonstandard tabloids, respectively. Almost all the results in this section hold for general d_V, d_W, though a few things are only made explicit in the two-row ($d_V = d_W = 2$) case.

Remark 14.1. For convenience, we require $\alpha_i > 0$, but this is not essential as $\check{\Lambda}^0\check{X} = K$.

14.1. Nonstandard columns label a canonical basis for $\check{\Lambda}^r\check{X}$.

Here we define a basis NSC^r of $\check{\Lambda}^r\check{X}$ making $(\check{\Lambda}^r\check{X}, \mathrm{NSC}^r)$ into an upper based U_q^τ-module. Since $\mathrm{Res}_{U_q(\mathfrak{g}_V\oplus\mathfrak{g}_W)}\check{\Lambda}^r\check{X} \cong \bigoplus_{\lambda\vdash r} V_\lambda \star W_{\lambda'}$ and the weights $\{(\lambda,\lambda') : \lambda\vdash r\}$ are pairwise incomparable, it follows from (49) that $\mathrm{Res}_{U_q(\mathfrak{g}_V\oplus\mathfrak{g}_W)}\check{\Lambda}^r\check{X}$ can be made into an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module in a unique way (this is not so for $\check{S}^r\check{X}$). The main content of this subsection is to realize the basis NSC^r explicitly as a subset of $\check{\mathbf{T}}$, which will turn out to be important later on.

We define the elements NSC_{P_V,P_W} of the basis NSC^r by the following formula; Proposition 14.6 will explain this formula and establish some important facts about this basis.

Definition 14.2. Let λ be a partition and $P_V \in \mathrm{SSYT}_{d_V}(\lambda)$, $P_W \in \mathrm{SSYT}_{d_W}(\lambda')$. Define the element NSC_{P_V,P_W} of $\check{\mathbf{T}}$ by

$$NSC_{P_V,P_W} := \sum_{Q\in SYT(\lambda)} (-1)^{\ell(Q^T)}\tilde{c}_{\substack{RSK^{-1}(P_V,Q)\\ RSK^{-1}(P_W,Q^T)}}, \tag{127}$$

$$\boxed{1} = c_{\substack{1\\1}} \qquad \boxed{2} = c_{\substack{1\\2}} \qquad \boxed{3} = c_{\substack{2\\1}} \qquad \boxed{4} = c_{\substack{2\\2}}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = c_{\substack{11\\21}} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} = c_{\substack{21\\11}} \qquad \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} = c_{\substack{12\\21}} + \tfrac{1}{[2]}c_{\substack{21\\21}} \quad \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = c_{\substack{21\\12}} + \tfrac{1}{[2]}c_{\substack{21\\21}} \quad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} = c_{\substack{22\\21}} \quad \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} = c_{\substack{21\\22}}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} = c_{\substack{211\\121}} - c_{\substack{121\\211}} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} = c_{\substack{211\\221}} - c_{\substack{121\\212}} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} = c_{\substack{212\\121}} - c_{\substack{221\\211}} \qquad \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} = c_{\substack{212\\221}} - c_{\substack{221\\212}}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array} = c_{\substack{2121\\2211}} - c_{\substack{2211\\2121}}$$

Figure 3: Nonstandard columns of height r are identified with NSC^r. These are a basis of $\check{\Lambda}^r\check{X}$.

where $\tilde{c}_{\substack{\mathbf{k}\\\mathbf{l}}} := \tilde{c}^V_{\mathbf{k}} \star \tilde{c}^W_{\mathbf{l}}$ and $\tilde{c}^V_{\mathbf{k}}$ (resp. $\tilde{c}^W_{\mathbf{k}}$) is the projected upper canonical basis element of $V^{\otimes r}$ (resp. $W^{\otimes r}$) from Theorem 6.6, and $\ell(Q^T)$ is as in §2.3. Also define the sets

$$\begin{aligned} \mathrm{NSC}(\lambda) :=\ & \{\mathrm{NSC}_{P_V,P_W} : P_V \in \mathrm{SSYT}_{d_V}(\lambda), P_W \in \mathrm{SSYT}_{d_W}(\lambda')\}, \\ \mathrm{NSC}^r :=\ & \textstyle\bigsqcup_{\lambda \vdash r} \mathrm{NSC}(\lambda). \end{aligned} \tag{128}$$

Remark 14.3. Recall that Z_λ is the superstandard tableau of shape and content λ (§2.3). For the highest weight NSC, i.e. those of the form $\mathrm{NSC}_{Z_\lambda, Z_{\lambda'}}$, we do not need the projected basis:

$$NSC_{Z_\lambda,Z_{\lambda'}} = \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} c_{\substack{RSK^{-1}(Z_\lambda,Q)\\ RSK^{-1}(Z_{\lambda'},Q^T)}}.$$

This is immediate from Theorem 6.6.

For each $\mathrm{NSC}_{P_V,P_W} \in \mathrm{NSC}(\lambda)$, choose some $Q \in \mathrm{SYT}(\lambda)$ and set $\mathbf{k} = RSK^{-1}(P_V, Q), \mathbf{l} = RSK^{-1}(P_W, Q^T)$ and label this element by the column $\begin{array}{|c|}\hline j_r\\ \vdots \\ j_2\\\hline j_1\\\hline\end{array}$, where $j_i := \rho(k_i, l_i)$ and $\rho(a, b) = (a-1)d_W + b$ as in §7. The set of columns obtained in this way from NSC^r is the set of *nonstandard columns* of *height* r. In the case $d_V = d_W = 2$, the nonstandard columns of height r and their identifications with NSC^r, $r \in [d_X]$, are shown in Figure 3. The choices for the SYT Q above are implicit in these identifications, as the following example clarifies.

Example 14.4. For $P_V = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$, $P_W = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$, let us explain the identification of $\mathrm{NSC}_{P_V,P_W} = c_{\substack{211\\221}} - c_{\substack{121\\212}}$ with the nonstandard column $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}$. This corresponds to the choice $Q = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ so that $\mathbf{k} = 211, \mathbf{l} = 221$; hence $\rho(\mathbf{k}, \mathbf{l}) = 421$.

Remark 14.5. In [1], it is shown how to put a $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal structure on the set $\mathrm{SSYT}_{d_X}((1^r))$. This gives a natural way to identify NSC^r with $\mathrm{SSYT}_{d_X}((1^r))$ for general d_V, d_W.

Define the integral form $\check{\Lambda}^r \check{X}^{\mathbf{A}}$ and the K_∞-lattice $\mathscr{L}_{(r)}$ of $\check{\Lambda}^r \check{X}$ by

$$\begin{aligned} \check{\Lambda}^r \check{X}^{\mathbf{A}} &= \mathbf{A}\mathrm{NSC}^r \\ \mathscr{L}_{(r)} &= K_\infty \mathrm{NSC}^r. \end{aligned} \tag{129}$$

Let $\check{p}_{(r)} : \check{\mathbf{T}} \to \check{\mathbf{T}}$ be the $K\check{\mathscr{H}}_r$-module projector with image the $K\check{\epsilon}_-$-isotypic component of $\check{\mathbf{T}}$, which is equal to $\check{\Lambda}^r \check{X}$. Define the projector $\check{\rho}_-^{\lambda,\lambda'} : M_\lambda \star M_{\lambda'} \to M_\lambda \star M_{\lambda'}$ by $\check{\rho}_-^{\lambda,\lambda'} = \check{\imath}_- \circ \check{s}_-$, where $\check{\imath}_-, \check{s}_-$ are as in Proposition 11.13. Extend this to a projector $\check{p}_-^{\lambda,\lambda'} : \check{\mathbf{T}} \to \check{\mathbf{T}}$ that acts as $\check{\rho}_-^{\lambda,\lambda'}$ on each $M_\lambda \star M_{\lambda'}$-isotypic component for the $\mathscr{H}_r \star \mathscr{H}_r$-action and 0 on the other isotypic components. Recall from §7 that $\pi^{\check{\mathbf{T}}}_{\lambda,\lambda'} : \check{\mathbf{T}} \to \check{\mathbf{T}}$ is the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-projector with image the $V_\lambda \star W_{\lambda'}$-isotypic component of $\check{\mathbf{T}}$. Then by Proposition 8.9 (2),

$$\check{p}_{(r)} = \sum_{\lambda} \check{p}_{(r)} \pi^{\check{\mathbf{T}}}_{\lambda,\lambda'} = \sum_{\lambda} \check{p}_-^{\lambda,\lambda'} \pi^{\check{\mathbf{T}}}_{\lambda,\lambda'}. \tag{130}$$

Proposition 14.6. *Maintain the notation above and that of Proposition 11.13.*

(a) *The set NSC(λ) is an* $\mathbf{A}$*-basis of* $V_\lambda^{\mathbf{A}} \star W_{\lambda'}^{\mathbf{A}} \subseteq \check{\Lambda}^r \check{X}^{\mathbf{A}}$*, and*

$$(V_\lambda \star W_{\lambda'}, B_V(\lambda) \star B_W(\lambda')) \to (\check{\Lambda}^r \check{X}, NSC^r), \qquad G_\lambda(b^V_{P_V}) \star G_{\lambda'}(b^W_{P_W}) \mapsto NSC_{P_V,P_W}$$

is an inclusion of upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$*-modules (where* G_λ, b^V, b^W *are as in §5.2–5.3). These inclusions combine to give an isomorphism of upper based* U_q^τ*-modules*

$$\bigoplus_{\substack{\lambda \vdash_{d_V} r, \\ \lambda' \vdash_{d_W} r}} (V_\lambda \star W_{\lambda'}, B_V(\lambda) \star B_W(\lambda')) \xrightarrow{\cong} (\check{\Lambda}^r \check{X}, NSC^r).$$

(b)

$$\check{p}_-^{\lambda,\lambda'} \pi^{\check{\mathbf{T}}}_{\lambda,\lambda'}(c_{\substack{\mathbf{k}\\ \mathbf{l}}}) \begin{cases} = (-1)^{\ell(Q(\mathbf{l})^T)} NSC_{P(\mathbf{k}),P(\mathbf{l})} & \text{if } Q(\mathbf{k}) = Q(\mathbf{l})^T \text{ has shape } \lambda, \\ = 0 & \text{if } \mathrm{sh}(\mathbf{k}) = \mathrm{sh}(\mathbf{l})' = \lambda \text{ and } Q(\mathbf{k}) \neq Q(\mathbf{l})^T, \\ = 0 & \text{if } \mathrm{sh}(\mathbf{k}) \not\trianglerighteq \lambda \text{ or } \mathrm{sh}(\mathbf{l}) \not\trianglerighteq \lambda', \\ \in q^{-1}\mathscr{L}_{(r)} \cap q\overline{\mathscr{L}}_{(r)} & \text{otherwise,} \end{cases}$$

(c)

$$\check{p}_{(r)}(c_{\substack{\mathbf{k}\\ \mathbf{l}}}) \begin{cases} = (-1)^{\ell(Q(\mathbf{l})^T)} NSC_{P(\mathbf{k}),P(\mathbf{l})} & \text{if } Q(\mathbf{k}) = Q(\mathbf{l})^T, \\ = 0 & \text{if } \mathrm{sh}(\mathbf{k}) = \mathrm{sh}(\mathbf{l})' \text{ and } Q(\mathbf{k}) \neq Q(\mathbf{l})^T, \\ \in q^{-1}\mathscr{L}_{(r)} \cap q\overline{\mathscr{L}}_{(r)} & \text{otherwise,} \end{cases}$$

(d) $\check{p}_{(r)}(\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W) = \mathscr{L}_{(r)}$.

Proof. We prove (b) first. Assume we are in one of the top two cases of (b). By Theorem 6.6 (d), $\pi^{\check{\mathbf{T}}}_{\lambda,\lambda'}(c_{\substack{\mathbf{k}\\ \mathbf{l}}}) = \tilde{c}_{\substack{\mathbf{k}\\ \mathbf{l}}}$. Theorem 6.5 (ii) for $\tilde{B}$ shows that

$$\{\tilde{c}_{\substack{\mathbf{k}'\\ \mathbf{l}'}} : P(\mathbf{k}') = P(\mathbf{k}), P(\mathbf{l}') = P(\mathbf{l})\} \xrightarrow{\cong} \Gamma_\lambda \otimes \Gamma_{\lambda'}, \quad \tilde{c}_{\substack{\mathbf{k}'\\ \mathbf{l}'}} \mapsto C_{Q(\mathbf{k}')} \otimes C_{Q(\mathbf{l}')}$$

is an isomorphism of $\mathscr{H}_r \otimes \mathscr{H}_r$-cells. Now applying $\check{p}_-^{\lambda,\lambda'}$ to $\tilde{c}_{\substack{\mathbf{k}\\ \mathbf{l}}}$, using Proposition 11.13, yields the top two cases of (b). For the bottom two cases of (b), we use Corollary 5.7 to apply the projector $\pi^{\check{\mathbf{T}}}_{\lambda,\lambda'}$ and then the easy fact that $\check{p}_-^{\lambda,\lambda'}(K_\infty(\Gamma_\lambda \otimes \Gamma_{\lambda'})) \subseteq K_\infty(\Gamma_\lambda \otimes \Gamma_{\lambda'})$.

Next, we use (130) to show (b) implies (c). This is straightforward, noting that the third case of (b) applies if $\mathrm{sh}(\mathbf{k}) = \mathrm{sh}(\mathbf{l})' \neq \lambda$. Statement (d) is immediate from (c) and definitions.

Finally, we prove (a). By Theorem 6.6, the following bijection of the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cells of $(\check{\mathbf{T}}, \tilde{B}_V \star \tilde{B}_W)$ on the left-hand side with the right-hand side gives rise to an isomorphism of upper based U_q^τ-modules:

$$\begin{aligned} \bigsqcup_{\substack{\lambda \vdash_{d_V} r,\\ \lambda' \vdash_{d_W} r}} \{\tilde{c}_{\substack{\mathbf{k}'\\ \mathbf{l}'}} : Q(\mathbf{k}') = (Z^*_{\lambda'})^T, Q(\mathbf{l}') = Z^*_{\lambda'}\} &\xrightarrow{\cong} \bigsqcup_{\substack{\lambda \vdash_{d_V} r,\\ \lambda' \vdash_{d_W} r}} B_V(\lambda) \star B_W(\lambda'), \\ \tilde{c}_{\substack{\mathbf{k}'\\ \mathbf{l}'}} &\mapsto G_\lambda(b^V_{P(\mathbf{k}')}) \otimes G_{\lambda'}(b^W_{P(\mathbf{l}')}). \end{aligned}$$

Composing the inverse of this with $\check{p}_{(r)}$, using (c), yields (a). $\square$

Remark 14.7. In this section and onward, we mostly work with U_q^τ-modules instead of $\mathscr{O}(GL_q(\check{X}))$-comodules because we have a theory of based modules for the former. Regarding the previous proposition, we know that $\check{\Lambda}^r \check{X}$ is an irreducible $\mathscr{O}(GL_q(\check{X}))$-comodule, but it would be desirable to see this explicitly by describing the action of generators of the hypothetical nonstandard enveloping algebra on the basis NSC^r.

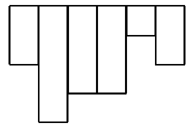
14.2. **Nonstandard tabloids label a canonical basis of $\check{Y}_\alpha$.** We define two products $\heartsuit_{\check{p}_-}$ and $\check{\heartsuit}$ that we use to construct a canonical basis of $\check{Y}_\alpha$ from the canonical bases NSC^{α_j} of $\check{\Lambda}^{\alpha_j}\check{X}$. These products turn out to agree, so we believe the resulting canonical basis to be the "correct" basis for $\check{Y}_\alpha$. Before defining these products, we first introduce nonstandard tabloids, which will label this basis of $\check{Y}_\alpha$.

Definition 14.8. A *column-diagram* of shape $\alpha \vDash_l^{d_X} r$ is a sequence of columns of heights $\alpha_1, \ldots, \alpha_l$ with their tops aligned as in Example 14.9. A *nonstandard tabloid* (NST) of shape α is a column-diagram whose columns are filled from the set of nonstandard columns (see Figure 3 for the two-row case). The set of all NST of shape α is denoted $\mathrm{NST}(\alpha)$.

The *column reading word* of an NST T is the word $\mathbf{j}$ obtained by reading its columns from bottom to top and then left to right. The V*-word* (resp. W*-word*) of T is $\mathbf{k}$ (resp. $\mathbf{l}$), where $(\mathbf{k}, \mathbf{l}) = \rho^{-1}(\mathbf{j})$ (see the beginning of §14.1).

For $T \in \mathrm{NST}(\alpha)$, let $T|_c$ denote the c-th column of T ($T|_1$ is the leftmost column). For a sequence of integers $1 \leq c_1 < c_2 < \cdots < c_k \leq l$, let $T|_{\{c_1,\ldots,c_k\}}$ be the NST $T|_{c_1}T|_{c_2}\cdots T|_{c_k}$, i.e., the NST consisting of the specified columns of T, in the same order as they occur in T. Such an NST is a *subtabloid* of T and, if $\{c_1, \ldots, c_k\}$ is equal to the interval $[c_1, c_k]$, it is *contiguous*.

Example 14.9. A column-diagram and nonstandard tabloid of shape $(2,4,3,3,1,2)$:



$$\begin{array}{|c|c|c|c|c|c|} \hline 2 & 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 3 & & 2 \\ \hline & 3 & 4 & 4 \\ \cline{3-4} & 4 \\ \cline{2-2} \end{array}$$

The column reading word (top), V-word (middle), and W-word (bottom) of the NST are

$$\begin{matrix} 32\,4321\,421\,432\,2\,23 \\ 21\,2211\,211\,221\,1\,12 \\ 12\,2121\,221\,212\,2\,21 \end{matrix}\ .$$

If T is the NST above, then $T|_{[3,5]} = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \cline{3-3} 2 & 3 \\ 4 & 4 \\ \cline{1-2} \end{array}$ is a contiguous subtabloid of T.

Let $\check{p}_\alpha : \check{\mathbf{T}} \to \check{\mathbf{T}}$ be the $K(\check{\mathscr{H}}_{\alpha_1} \otimes \cdots \otimes \check{\mathscr{H}}_{\alpha_l})$ -module projector with image $\check{Y}_\alpha$. There holds $\check{p}_\alpha = \check{p}_{\alpha_1} \otimes \ldots \otimes \check{p}_{\alpha_l}$.

If $x_i \in V^{\otimes r_i} \star W^{\otimes r_i}, i = 1,2$, then let $x_1 \heartsuit x_2$ be the element of $V^{\otimes r} \star W^{\otimes r}$, $r = r_1 + r_2$, defined using the $\heartsuit$ product from §6.2 for V and W; equivalently, we can define $\heartsuit$ by setting $c_{\substack{\mathbf{k^1}\\ \mathbf{l^1}}} \heartsuit c_{\substack{\mathbf{k^2}\\ \mathbf{l^2}}} = c_{\substack{\mathbf{k^1}\mathbf{k^2}\\ \mathbf{l^1}\mathbf{l^2}}}, \mathbf{k^i} \in [d_V]^{r_i}, \mathbf{l^i} \in [d_W]^{r_i}$, and extending bilinearly. Now for T an NST of shape α, define the element $T_\heartsuit \in \check{\mathbf{T}}$ to be $T|_1 \heartsuit \cdots \heartsuit T|_l$.

$$\text{We identify } T \text{ with the element } T|_1 \heartsuit_{\check{p}_-} \cdots \heartsuit_{\check{p}_-} T|_l := \check{p}_\alpha(T_\heartsuit) \text{ of } \check{Y}_\alpha. \tag{131}$$

Remark 14.10. In the two-row case, the $\heartsuit_{\check{p}_-}$ product can be computed as follows: let J_α be as in §2.3 and let $J_2 \subseteq J_\alpha$ be the subset of J_α corresponding to the parts of α equal to 2, i.e. $J_2 = \{j \in J_\alpha : j-1, j+1 \notin J_\alpha\}$. Then for any $T \in$ NST(α) there holds

$$T = \check{p}_{J_\alpha}(T_\heartsuit) = \check{p}_{J_2}(T_\heartsuit) = T_\heartsuit\Big(\prod_{j \in J_2} \frac{\mathcal{Q}_j}{[2]^2}\Big), \tag{132}$$

where the second equality holds because $\mathrm{Res}_{U_q(\mathfrak{sl}(V) \oplus \mathfrak{sl}(W))} \check{\Lambda}^3 \check{X} \cong \mathrm{Res}_{U_q(\mathfrak{sl}(V) \oplus \mathfrak{sl}(W))} \check{\Lambda}^1 \check{X}$.

Example 14.11. For the NST T shown below, we compute $T_\heartsuit$ and T in terms of the upper canonical basis of $X^{\otimes r}$:

$$\boxed{\begin{matrix}2\\3\end{matrix}} \heartsuit \boxed{\begin{matrix}1\\3\end{matrix}} = \big(c_{\substack{21\\12}} + \frac{1}{[2]} c_{\substack{21\\21}}\big) \heartsuit c_{\substack{21\\11}} = c_{\substack{2121\\1211}} + \frac{1}{[2]} c_{\substack{2121\\2111}},$$

$$\begin{array}{|c|c|} \hline 2 & 1 \\ 3 & 3 \\ \hline \end{array} = \boxed{\begin{matrix}2\\3\end{matrix}} \heartsuit_{\check{p}_-} \boxed{\begin{matrix}1\\3\end{matrix}} = \big(c_{\substack{21\\12}} + \frac{1}{[2]} c_{\substack{21\\21}}\big) \heartsuit_{\check{p}_-} c_{\substack{21\\11}} = \big(c_{\substack{2121\\1211}} + \frac{1}{[2]} c_{\substack{2121\\2111}}\big) \frac{\mathcal{Q}_1}{[2]^2} \frac{\mathcal{Q}_3}{[2]^2} = c_{\substack{2121\\1211}} + \frac{1}{[2]} c_{\substack{2121\\1121}} + \frac{1}{[2]} c_{\substack{2121\\2111}}.$$

The last equality can be computed using (109).

The $\heartsuit_{\check{p}_-}$ product was the first way we computed a nice basis for $\check{Y}_\alpha$ and is well-suited for explicit computation in the two-row case. Later, we realized that this is a special case of Lusztig's construction (Theorem 5.8), which is theoretically cleaner. We now explain Lusztig's construction in detail in this context and denote the product by $\check{\heartsuit}$.

Define the following K_∞-lattice of $\check{Y}_\alpha$

$$\mathscr{L}_\alpha := \mathscr{L}_{(\alpha_1)} \otimes_{K_\infty} \ldots \otimes_{K_\infty} \mathscr{L}_{(\alpha_l)} = \check{p}_\alpha(\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W), \tag{133}$$

where the equality is by Proposition 14.6 (d). Define the integral form

$$\check{Y}^{\mathbf{A}}_{\alpha} := \check{\Lambda}^{\alpha_1}\check{X}^{\mathbf{A}} \otimes_{\mathbf{A}} \ldots \otimes_{\mathbf{A}} \check{\Lambda}^{\alpha_l}\check{X}^{\mathbf{A}}. \tag{134}$$

It follows from Theorem 5.8 that

> there is a unique $\bar{\cdot}$-invariant element $T|_1 \check{\heartsuit} T|_2 \check{\heartsuit} \cdots \check{\heartsuit} T|_l$ of $\check{Y}^{\mathbf{A}}_{\alpha}$ congruent to $T|_1 \otimes T|_2 \otimes \cdots \otimes T|_l \mod q^{-1}\mathscr{L}_\alpha$, for any $T \in \mathrm{NST}(\alpha)$.

As will be justified by the next proposition, we may identify the element $T|_1 \check{\heartsuit} T|_2 \check{\heartsuit} \cdots \check{\heartsuit} T|_l$ with T. Then, with this identification and by Proposition 13.3, $(\check{Y}_\alpha, \mathrm{NST}(\alpha))$ is a weak upper based U^τ_q-module with balanced triple $(\check{Y}^{\mathbf{A}}_\alpha, \overline{\mathscr{L}}_\alpha, \mathscr{L}_\alpha)$.

Proposition 14.12. *The products $\heartsuit_{\check{p}_-}$ and $\check{\heartsuit}$ agree.*

Proof. It suffices to show that the $\heartsuit_{\check{p}_-}$ product satisfies the characterizing conditions of the $\check{\heartsuit}$ product. Let $\alpha = (\beta, \gamma) \vDash^{d_X}_{l} r$ and $T \in \check{Y}^{\mathbf{A}}_\beta, T' \in \check{Y}^{\mathbf{A}}_\gamma$ and assume by induction that we have shown that T and T' are equal to the $\heartsuit_{\check{p}_-}$ and the $\check{\heartsuit}$ products of their columns. It follows from the $\bar{\cdot}$-invariance of the $\tilde{c}_{\substack{\mathbf{k}\\ \mathbf{l}}}$ and the proof of Proposition 14.6 (c) that $\check{p}_{(r)}$ intertwines the $\bar{\cdot}$-involution on $\check{\mathbf{T}}$. Since $\check{\mathscr{H}}_r$ acts faithfully on $\check{\mathbf{T}}$ when $d_V, d_W \geq r$, this implies that the minimal central idempotent of $K\check{\mathscr{H}}_r$ corresponding to $\check{\epsilon}_-$ is $\bar{\cdot}$-invariant. Thus $\check{p}_\alpha$ intertwines the $\bar{\cdot}$-involution, hence $\check{p}_\alpha(T\heartsuit T')$ is $\bar{\cdot}$-invariant.

Next, we can write $T\heartsuit T' \in T \otimes T' + q^{-1}\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W$, which implies

$$\check{p}_\alpha(T\heartsuit T') \in \check{p}_\alpha(T\otimes T') + q^{-1}\check{p}_\alpha(\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W) = \check{p}_\alpha(T\otimes T') + q^{-1}\mathscr{L}_\alpha = T\otimes T' + q^{-1}\mathscr{L}_\alpha,$$

where the first equality is by (133) and the last equality is simply because $T \in \check{Y}_\beta, T' \in \check{Y}_\gamma$, so $\check{p}_\alpha(T\otimes T') = \check{p}_\beta(T) \otimes \check{p}_\gamma(T') = T \otimes T'$. Thus $T\heartsuit_{\check{p}_-}T' = \check{p}_\alpha(T\heartsuit T') = T\check{\heartsuit}T'$. □

Remark 14.13. This proposition is very similar to Proposition 5.9, the difference being that the projector $\check{p}_\alpha = \check{p}_{\alpha_1} \otimes \ldots \otimes \check{p}_{\alpha_l}$ is more complicated than $\pi = \pi_1 \otimes \cdots \otimes \pi_l$. A little extra care is needed to check that $\check{p}_\alpha$ intertwines the $\bar{\cdot}$-involution, but otherwise the proofs are essentially the same.

Remark 14.14. We believe that the $\mathbf{A}$-module $\check{p}_\alpha(\check{\mathbf{T}}_{\mathbf{A}})$ is not a good choice for an integral form of $\check{Y}_\alpha$. It can be strictly larger than the integral form $\check{Y}^{\mathbf{A}}_\alpha$. For example,

$$\check{p}_{(2)}(c_{\substack{12\\12}}) = c_{\substack{12\\12}}\frac{\mathcal{Q}_1}{[2]^2} = -\frac{1}{[2]}c_{\substack{21\\12}} - \frac{1}{[2]}c_{\substack{12\\21}} - \frac{2}{[2]^2}c_{\substack{21\\21}} = -\frac{1}{[2]}\left(\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}} + \boxed{\begin{smallmatrix}3\\2\end{smallmatrix}}\right) \notin \check{Y}^{\mathbf{A}}_{(2)}$$

(the second equality can be computed using (109)).

Example 14.15. Continuing Example 14.11, we compute the corresponding $\check{\heartsuit}$ product of nonstandard columns: from $c_{12} \otimes c_{11} = c_{1211} + q^{-1}c_{1121} + q^{-2}c_{1112}$ we deduce

$$\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} := \left(c_{\substack{21\\12}} + \frac{1}{[2]}c_{\substack{21\\21}}\right) \otimes c_{\substack{21\\11}} = c_{\substack{2121\\1211}} + q^{-1}c_{\substack{2121\\1121}} + q^{-2}c_{\substack{2121\\1112}} + \frac{1}{[2]}c_{\substack{2121\\2111}} =$$

$$c_{\substack{2121\\1211}} + \frac{1}{[2]}c_{\substack{2121\\1121}} + \frac{1}{[2]}c_{\substack{2121\\2111}} + q^{-2}\left(c_{\substack{2121\\1112}} + \frac{1}{[2]}c_{\substack{2121\\1121}}\right) \equiv c_{\substack{2121\\1211}} + \frac{1}{[2]}c_{\substack{2121\\1121}} + \frac{1}{[2]}c_{\substack{2121\\2111}} \mod q^{-1}\mathscr{L}_{(2,2)}.$$

This last equivalence follows from

$$c_{\substack{2121\\1112}} + \frac{1}{[2]}c_{\substack{2121\\1121}} = c_{\substack{21\\11}}\heartsuit\big(c_{\substack{21\\12}} + \frac{1}{[2]}c_{\substack{21\\21}}\big) = c_{\substack{21\\11}} \otimes \big(c_{\substack{21\\12}} + \frac{1}{[2]}c_{\substack{21\\21}}\big) = \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,\check{\heartsuit}\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}.$$

To summarize,

$$c_{\substack{2121\\1211}} + \frac{1}{[2]}c_{\substack{2121\\1121}} + \frac{1}{[2]}c_{\substack{2121\\2111}} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} - q^{-2}\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array} \equiv \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \mod q^{-1}\mathscr{L}_{(2,2)}. \tag{135}$$

The left-hand quantity of (135) is $\bar{\ }$-invariant, so it must be $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\check{\heartsuit}\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$, in agreement with the computation of the $\heartsuit_{\check{p}_-}$ product in Example 14.11.

Example 14.16. Here is the result of a similar computation for $\alpha = (2,1)$:

$$\begin{array}{|c|c|}\hline 2&1\\\hline 4\\\cline{1-1}\end{array} = \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline\end{array} - q^{-2}\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2&1\\\hline 4\\\cline{1-1}\end{array} = c_{\substack{211\\221}} + \frac{1}{[2]}c_{\substack{211\\212}}, \text{ and } \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} = c_{\substack{211\\122}} + \frac{1}{[2]}c_{\substack{211\\212}}.$$

Remark 14.17. By §5.5, $\check{\heartsuit}$ is associative: if $(\beta,\gamma,\delta) \vDash_l^{dx} r$ and if A, B, C are NST of shape β,γ,δ respectively, then $A\check{\heartsuit}(B\check{\heartsuit}C) = (A\check{\heartsuit}B)\check{\heartsuit}C$ as elements of $\check{Y}_{(\beta,\gamma,\delta)}$. It is important to keep in mind that the product $\check{\heartsuit}$ depends implicitly on (β,γ,δ), even though this is not included in the notation.

14.3. **The action of the Kashiwara operators and τ on NST.** We have shown that $(\check{Y}_\alpha, \mathrm{NST}(\alpha))$ is a weak upper based U_q^τ-module; let $(\mathscr{L}_\alpha, \mathcal{NST}(\alpha))$ be its upper crystal basis. The action of τ on NST is deduced easily and the action of the Kashiwara operators $(\tilde{F}_i)_V^{\mathrm{up}}, (\tilde{F}_j)_W^{\mathrm{up}}$, $i \in [d_V - 1], j \in [d_W - 1]$ on $\mathcal{NST}$ is given by the well-known rule for tensoring $U_q(\mathfrak{g}_V)$ crystal bases (see e.g. [24, Chapter 7]). We now describe these actions explicitly in the two-row case.

Proposition 14.18. *The action of τ on NSC^r is given by*

$$\tau(NSC_{P_V,P_W}) = (-1)^{\ell((Z_\lambda^*)^T)} NSC_{P_W,P_V}, \text{ where } \lambda = \mathrm{sh}(P_V). \tag{136}$$

This can be made more explicit using

$$\ell((Z_\lambda^*)^T) \equiv \binom{n}{2} + \sum_i \binom{\lambda_i}{2} + i\binom{\lambda_i'}{2} \mod 2. \tag{137}$$

Moreover, for an NST T with l columns, $\tau(T) = \tau(T|_1)\check{\heartsuit}\cdots\check{\heartsuit}\,\tau(T|_l)$.

Proof. Formula (136) is straightforward from definitions and the fact $\tau(\tilde{c}_{\substack{\mathbf{k}\\\mathbf{l}}}) = \tilde{c}_{\substack{\mathbf{l}\\\mathbf{k}}}$. The length computation (137) is slightly involved and we omit the proof. The last statement is exactly Proposition 13.3 in this setting. □

For the remainder of this section set $d_V = d_W = 2$.

Definition 14.19. The V-diagram (resp. W-diagram) of an NST T of shape α is the diagram obtained from its V-word (resp. W-word) according to the rule in §6.3. The *V-arcs* (resp. *W-arcs*) of T are the arcs of the V-diagram (resp. W-diagram) of T. Internal and external V-arcs (resp. W-arcs) are defined as in Definition 6.7 with $i_j = \alpha_j$

and $\lambda^{(j)} = \mathrm{sh}(P_V)$ (resp. $\lambda^{(j)} = \mathrm{sh}(P_W)$) for all $j \in [l]$, where P_V, P_W are defined by $T|_j = \mathrm{NSC}_{P_V,P_W}$. An *arc* of T is either a V-arc or a W-arc of T.

Let φ_V, φ_W be as in (58), defined using $G\tilde{F}_V^{\mathrm{up}}$ and $G\tilde{F}_W^{\mathrm{up}}$, respectively ($G\tilde{F}^{\mathrm{up}}$ is the global Kashiwara operator defined in (48)). Then for any NST T, the statistic $\varphi_V(T)$ (resp. $\varphi_W(T)$) is the number of unpaired 1's in the V-diagram (resp. W-diagram) of T.

For an NST T with V-word $\mathbf{k}$, let $\mathscr{F}_{(j)V}(T)$ be the NST corresponding to $\mathscr{F}_{(j)}(\mathbf{k})$ (as defined in Theorem 6.9), defined precisely as follows: if $\mathscr{F}_{(j)}(\mathbf{k})$ has an extra internal arc, then $\mathscr{F}_{(j)V}(T) = 0$, and otherwise $\mathscr{F}_{(j)V}(T)$ is the NST obtained from T by replacing the column $T|_c$ containing the j-th unpaired 1 in $\mathbf{k}$ with $G\tilde{F}_V^{\mathrm{up}}(T|_c)$ if the j-th unpaired 1 lies in the c-th column of V-diagram(T). The NST $\mathscr{F}_{(j)W}(T)$, $\mathscr{E}_{(j)V}(T)$, and $\mathscr{E}_{(j)W}(T)$ are defined in a similar way using W-word in place of V-word and $\mathscr{E}_{(j)}(\mathbf{k})$ in place of $\mathscr{F}_{(j)}(\mathbf{k})$ as appropriate.

Proposition 14.20. *The pair $(\check{Y}_\alpha, NST(\alpha))$ is a weak upper based U_q^τ-module with global Kashiwara operators given by*

$$\begin{aligned} G\tilde{F}_V^{up}(T) &= \mathscr{F}_{(\varphi_V(T))V}(T), \\ G\tilde{F}_W^{up}(T) &= \mathscr{F}_{(\varphi_W(T))W}(T). \end{aligned} \tag{138}$$

Thus the highest weight $NST(\alpha)$ are those whose V-diagram and W-diagram have no unpaired 2's. The action of τ on $NST(\alpha)$ is given by $\tau(T) = (-1)^j T'$, where T' is obtained from T by changing 2's to 3's and 3's to 2's with the exception that columns $\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$, $\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}$ do not change, and j is the number of parts of α equal to 3 or 4.

The following stronger result for the action of F_V and F_W is also useful.

Proposition 14.21. *For an NST T there holds*

$$\begin{aligned} F_V T &= \textstyle\sum_{j=1}^{\varphi_V(T)} [j] \mathscr{F}_{(j)V}(T), \\ F_W T &= \textstyle\sum_{j=1}^{\varphi_W(T)} [j] \mathscr{F}_{(j)W}(T). \end{aligned}$$

Proof. This is a special case of Proposition 6.9 since $\check{\Lambda}^r \check{X}$, considered as an upper based $U_q(\mathfrak{g}_V)$-module, is isomorphic to a direct sum of $(V_{(r-i,i)}, B((r-i,i)))$. □

15. A global crystal basis for two-row Kronecker coefficients

For the remainder of this paper, set $d_V = d_W = 2$ (the two-row case). We now come to our main result on the two-row Kronecker problem and the deepest canonical basis theory of this paper.

We show that for any $\nu \vdash r$, the $\mathscr{O}(GL_q(\check{X}))$-comodule quotient $\check{X}_\nu$ of $\check{Y}_{\nu'}$ defined in §1.6 satisfies $\mathrm{Res}_{U^\tau}(\check{X}_\nu|_{q=1}) \cong \mathrm{Res}_{U^\tau}(X_\nu|_{q=1})$, where $X_\nu|_{q=1}$ is the $U(\mathfrak{g}_X)$-module of highest weight ν and $U^\tau = U(\mathfrak{gl}_2) \wr \mathcal{S}_2$. Recall from §1.6 that $\check{X}_\nu$ is defined to be $\check{Y}_{\nu'}/\check{Y}_{\triangleright\nu'}$, where the submodule $\check{Y}_{\triangleright\alpha}$ of $\check{Y}_\alpha$ (for simplicity, $\check{Y}_{\triangleright\alpha}$ was discussed only for partitions α in §1.6, but it can be defined for any composition $\alpha \vDash^{d_X} r$) is defined "by hand" for $\ell(\alpha) = 2$ and for $\ell(\alpha) > 2$ is defined to be the (generally, not direct) sum over all $i \in [l-1]$ of

$$\check{Y}_{\triangleright^i \alpha} := \check{Y}_{(\alpha_1,\ldots,\alpha_{i-1})} \otimes \check{Y}_{\triangleright(\alpha_i,\alpha_{i+1})} \otimes \check{Y}_{(\alpha_{i+2},\ldots,\alpha_l)}.$$

We define a subset $+\mathrm{HNSTC}(\nu) \subseteq \check{X}_\nu$ to be the image of (a rescaled version of) a certain subset of $\mathrm{NST}(\nu')$, and we show that $+\mathrm{HNSTC}(\nu)$ is a global crystal basis of $\check{X}_\nu$. This gives an elegant solution to the two-row Kronecker problem: the Kronecker coefficient $g_{\lambda\mu\nu}$ is equal to the number of highest weight $+\mathrm{HNSTC}(\nu)$ of weight (λ,μ). This section is devoted to the algebraic portion of the proof of this as well as the verification that $\check{X}_\nu$ behaves correctly at $q = 1$. One of the main difficulties is that $\check{Y}_{\triangleright^i \alpha}$ is not easily expressed in terms of the basis $\mathrm{NST}(\alpha)$. We develop some tools to remedy this: a grading on $\check{Y}_\alpha$ (§15.1) and a canonical basis for $\check{Y}_{\triangleright^i \alpha}$ (§15.5).

15.1. **Invariants.** As defined explicitly below, an invariant is a minimal NST that is killed by F_V, F_W, E_V, and E_W. This allows us to define a grading on $\check{Y}_\alpha$ corresponding to how many invariants an NST contains. This will help us organize the relations satisfied by the image of $\mathrm{NST}(\nu')$ in $\check{X}_\nu$.

An *invariant* is an NST equal to one of:

$$\begin{ytableau} 4 & 1 \end{ytableau}\,,\ \begin{ytableau} 2 & 1 \\ 4 & 3 \end{ytableau}\,,\ \begin{ytableau} 3 & 1 \\ 4 & 2 \end{ytableau}\,,\ \begin{ytableau} 2 & 1 \\ 3 & 2 \\ 4 & 3 \end{ytableau}\,,\ \begin{ytableau} 2 & 1 \\ 3 \\ 4 \end{ytableau}\,,\ \begin{ytableau} 4 & 1 \\ \none & 2 \\ \none & 3 \end{ytableau} \text{ or } \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}.$$

If the columns of an invariant have the same height j, then j is the *height* of the invariant. We will not be too interested in the invariants $\begin{ytableau} 2 & 1 \\ 3 \\ 4 \end{ytableau}$ and $\begin{ytableau} 4 & 1 \\ \none & 2 \\ \none & 3 \end{ytableau}$ because they belong to $\check{Y}_{\triangleright(3,1)}$ and $\check{Y}_{\triangleright(1,3)}$, respectively.

Definition 15.1. An *invariant column pair* of an NST T is a pair of columns of T that are paired by two arcs (see Definition 14.19).

The *invariant record* of an NST T is the tuple (i_4, i_3, i_2, i_1), where

i_4 is the number of height-4 columns,
i_j is the number of invariant column pairs of height-j of T, $(j = 1, 2, 3)$.

The *degree* of T, denoted $\deg(T)$, is i_4 plus the number of invariant column pairs. The *invariant-free part* of an NST T is the (possibly empty) NST obtained by removing all invariant column pairs and all height-4 columns.

The columns of an invariant column pair have no V- or W-arcs with an end outside the pair, so the definition of the invariant-free part is sound in that after all invariant column pairs are removed, the resulting NST has no invariant column pairs. It is easy to check that the invariants listed above, except the height-4 column, are invariant column pairs, and all invariant column pairs are of this form (after removing the columns not in the pair).

Before introducing an associated graded of $\check{Y}_\alpha$, we recall some algebraic generalities. Let $0 \subseteq X_r \subseteq X_{r-1} \subseteq \cdots \subseteq X_0 = X$ be a filtered R-module. The associated graded of

X is the graded R-module $\operatorname{gr}(X) := \bigoplus_{i=0}^{r-1} X_i/X_{i+1}$. The rule $X \mapsto \operatorname{gr}(X)$ is a functor from filtered R-**Mod** to graded R-**Mod**. A submodule (resp. quotient module) M of X inherits a filtration from that of X, so we write $\operatorname{gr}(M)$ for the associated graded module; it is a submodule (resp. quotient module) of $\operatorname{gr}(X)$.

For $x \in X$, the *degree of* x, denoted $\deg(x)$, is the largest integer h such that $x \in X_h$. For $x \in X_h$, let $\operatorname{in}_h(x)$ denote the image of x under the composition $X_h \twoheadrightarrow X_h/X_{h+1} \hookrightarrow \operatorname{gr}(X)$. And for $x \in X$, set $\operatorname{in}(x) = \operatorname{in}_{\deg(x)}(x)$.

Proposition-Definition 15.2. *Let $\alpha \vDash_l^{d_X} r$, as usual. Set*

$$\begin{array}{rcl} NST(\alpha)_{\geq h} & := & \{T \in NST(\alpha) : \deg(T) \geq h\}, \\ NST(\alpha)_h & := & \{T \in NST(\alpha) : \deg(T) = h\}, \end{array}$$

for $h \geq 0$, and let $(\check{Y}_\alpha)_h$, $(\mathscr{L}_\alpha)_h$, $(\check{Y}_\alpha^{\mathbf{A}})_h$ be the K, K_∞, and $\mathbf{A}$ span of $NST(\alpha)_{\geq h}$, respectively. The pair $((\check{Y}_\alpha)_h, NST(\alpha)_{\geq h})$ is a weak upper based U_q^τ-module with balanced triple $((\check{Y}_\alpha^{\mathbf{A}})_h, \overline{(\mathscr{L}_\alpha)}_h, (\mathscr{L}_\alpha)_h)$. The filtration

$$0 \subseteq ((\check{Y}_\alpha)_l, NST(\alpha)_{\geq l}) \subseteq \ldots \subseteq ((\check{Y}_\alpha)_1, NST(\alpha)_{\geq 1}) \subseteq ((\check{Y}_\alpha)_0, NST(\alpha)_{\geq 0}) = (\check{Y}_\alpha, NST(\alpha))$$

is a filtration of weak upper based U_q^τ-modules, and hence $(\operatorname{gr}(\check{Y}_\alpha), \operatorname{in}(NST(\alpha)))$ is a weak upper based U_q^τ-module.

Proof. The inclusion $(\check{Y}_\alpha)_h \hookrightarrow \check{Y}_\alpha$ is a U_q^τ-module homomorphism because applying a Chevalley generator or τ to an NST T yields a linear combination $\sum_i c_i T^i$ of NST such that every invariant column pair of T^i is an invariant column pair of T; see Proposition 14.21. □

Note that $(\operatorname{gr}(\check{Y}_\alpha), \operatorname{in}(\mathrm{NST}(\alpha)))$ and $(\check{Y}_\alpha, \mathrm{NST}(\alpha))$ are not isomorphic as weak upper based U_q^τ-modules.

15.2. **Two-column moves.** We now define U_q^τ-submodules $\check{Y}_{\rhd\gamma} \subseteq \check{Y}_\gamma$ for γ a composition of length 2. Let γ' be the conjugate of the partition obtained by sorting the parts of γ in weakly decreasing order. We are in luck: it turns out that $\operatorname{Res}_{U^\tau}(X_{\gamma'}|_{q=1})$ is multiplicity-free, so in order for $\operatorname{Res}_{U_q^\tau} \check{Y}_\gamma/\check{Y}_{\rhd\gamma}$ to be a q-analog of $\operatorname{Res}_{U^\tau}(X_{\gamma'}|_{q=1})$, the submodule $\check{Y}_{\rhd\gamma}$ must be a direct sum of certain U_q^τ-irreducibles of $\check{Y}_\gamma$, which are easily computed. In other words, there is only one way to define $\check{Y}_{\rhd\gamma}$ so that $\check{Y}_\gamma/\check{Y}_{\rhd\gamma}$ is what it is supposed to be at $q = 1$.

The Figures 4–11 below serve several purposes: they give, for each partition γ of length 2 such that $\gamma_1 \leq 3$, an explicit description of $\check{Y}_{\rhd\gamma}$ by depicting a basis $\mathrm{NST}(\rhd\gamma)$ for this space, and these bases will play an important role in subsequent arguments; they give examples of the action of F_V, F_W, and τ on NST as determined by Propositions 14.20 and 14.21 (the horizontal (resp. vertical) arrows give the action of F_V (resp. F_W) on the basis elements and the labels on the arrows indicate the coefficient; the action of τ is only given for highest weight basis elements to avoid cluttering the diagrams); in some cases, the basis elements agree with NST of a different shape by thinking of all the $\check{Y}_\gamma$ with $\gamma \vDash_2^{d_X} r$ as subspaces of $\check{X}^{\otimes r}$, and we indicate this in the figures.

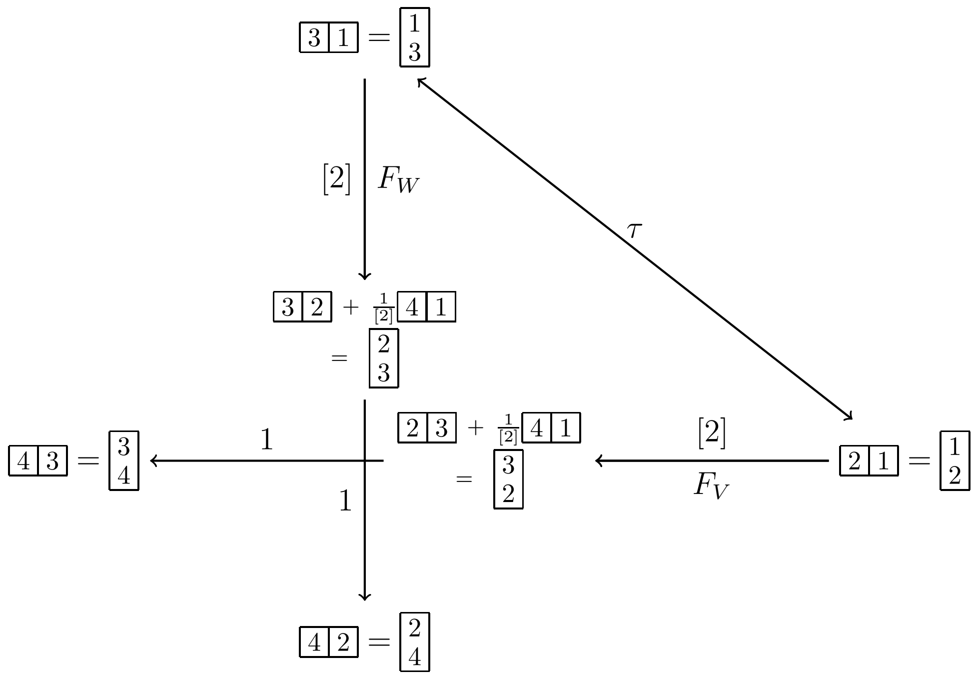


Figure 4: The basis NST($\triangleright(1,1)$) of $\check{Y}_{\triangleright(1,1)}$, which consists of graded and nonintegral $\triangleright$NST.

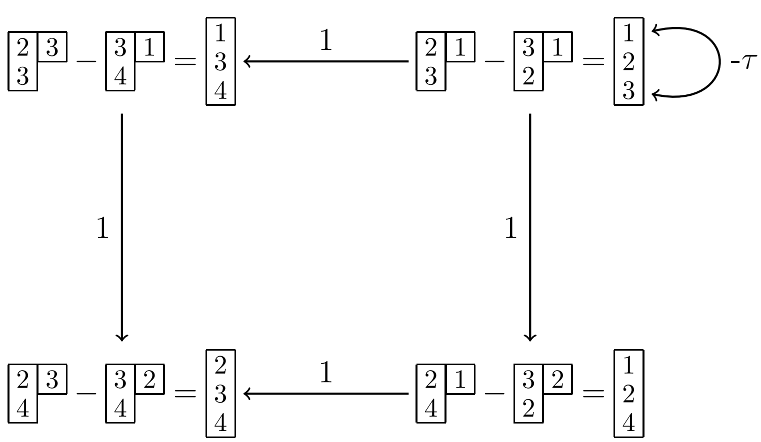


Figure 5: The basis NST($\triangleright(2,1)$) of $\check{Y}_{\triangleright(2,1)}$, which consists of degree-preserving and integral $\triangleright$NST.

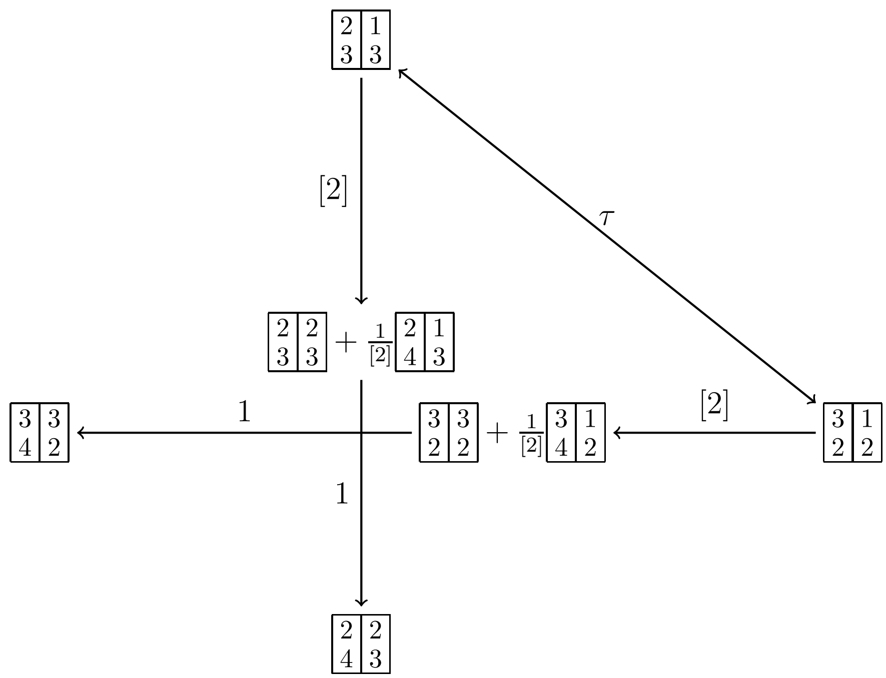


Figure 6: The graded elements of the basis $\mathrm{NST}(\triangleright(2,2))$ of $\check{Y}_{\triangleright(2,2)}$, which are all nonintegral.

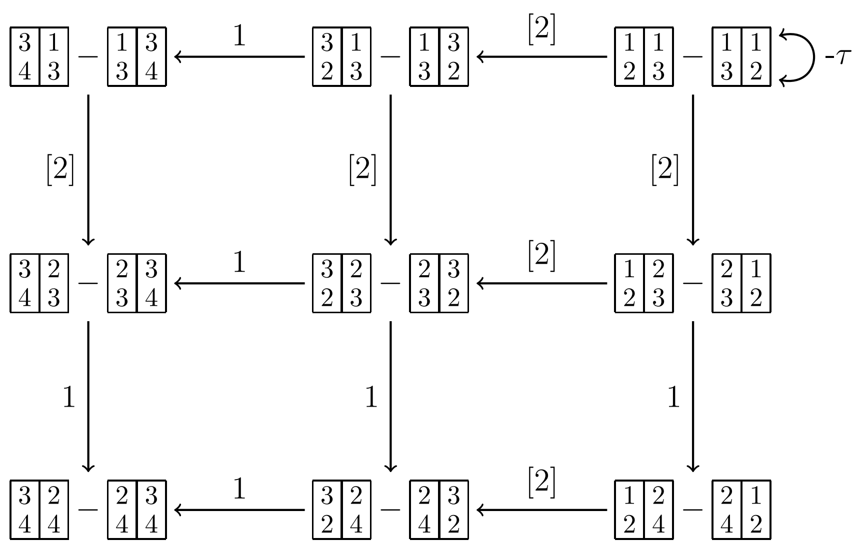


Figure 7: Some of the degree-preserving elements of the basis $\mathrm{NST}(\triangleright(2,2))$ of $\check{Y}_{\triangleright(2,2)}$, which are all integral.

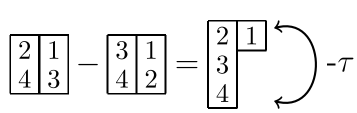


Figure 8: A degree-preserving and integral element of the basis $\mathrm{NST}(\triangleright(2,2))$ of $\check{Y}_{\triangleright(2,2)}$.

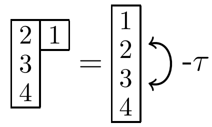


Figure 9: The element of the basis NST($\triangleright(3,1)$) of $\check{Y}_{\triangleright(3,1)}$, which is graded and integral.

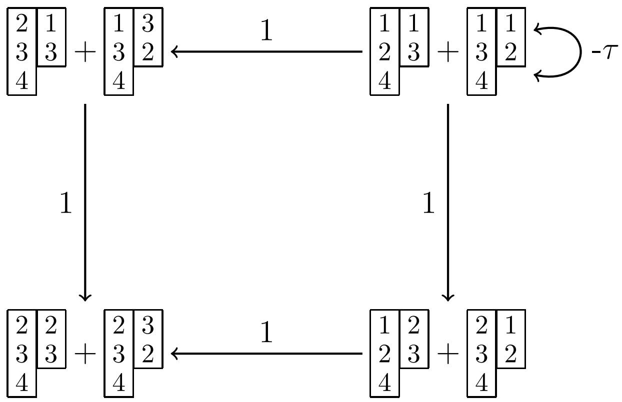


Figure 10: The basis NST($\triangleright(3,2)$) of $\check{Y}_{\triangleright(3,2)}$, which consists of degree-preserving and integral $\triangleright$NST.

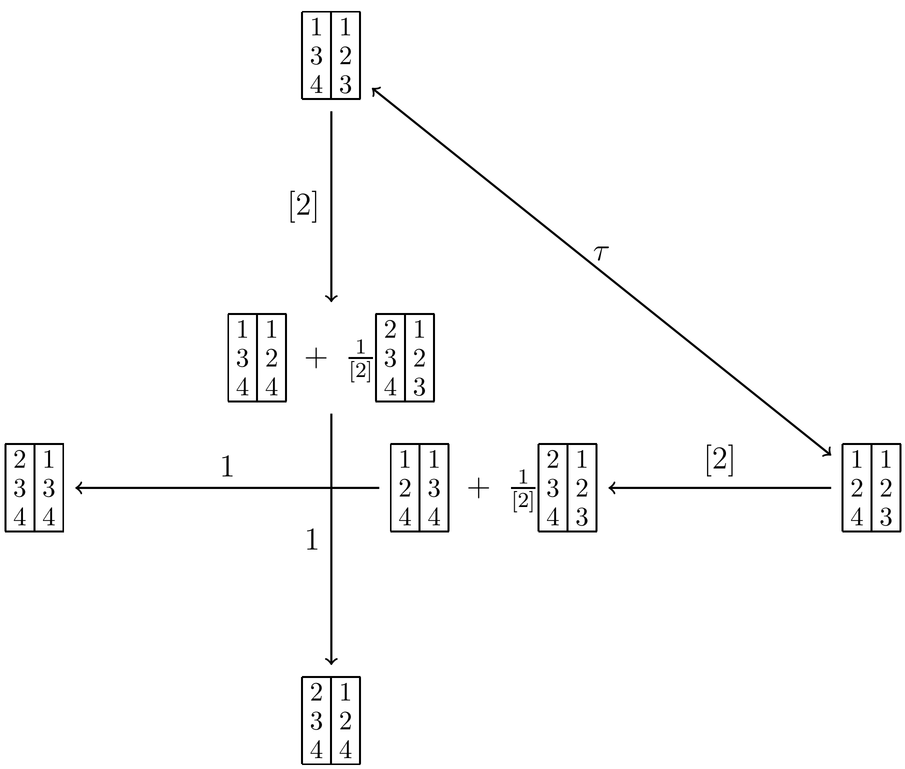


Figure 11: The basis NST($\triangleright(3,3)$) of $\check{Y}_{\triangleright(3,3)}$, which consists of graded and nonintegral $\triangleright$NST.

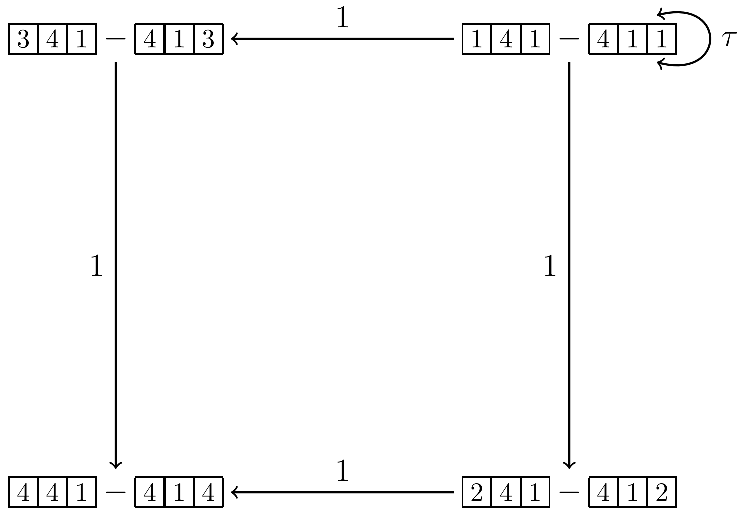


Figure 12: The elements of $\mathrm{NST}(\triangleright(1,1,1))$, which span a U_q^τ-submodule of $\check{Y}_{\triangleright(1,1,1)}$ and are all degree-preserving and integral. This shows that height-1 invariants commute with height-1 columns in $\check{X}_{(1,1,1)}$.

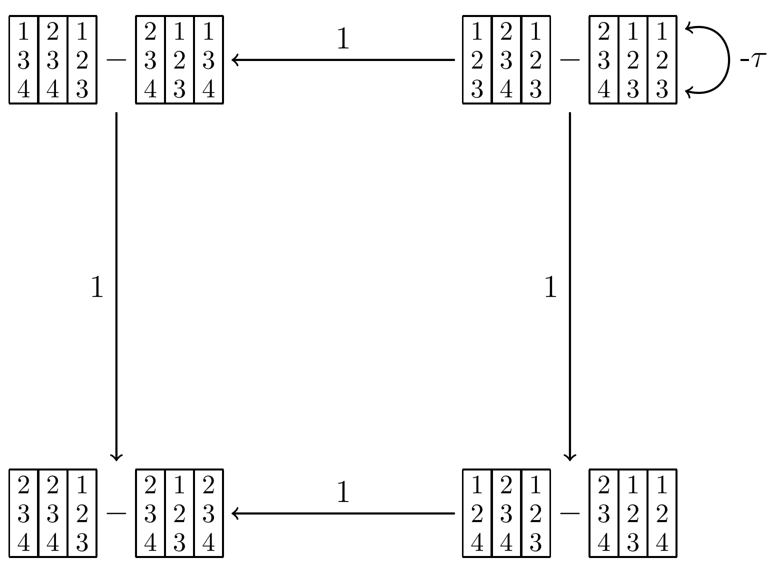


Figure 13: The elements of $\mathrm{NST}(\triangleright(3,3,3))$, which span a U_q^τ-submodule of $\check{Y}_{\triangleright(3,3,3)}$ and are all degree-preserving and integral. This shows that height-3 invariants commute with height-3 columns in $\check{X}_{(3,3,3)}$.

$$\begin{array}{|c|c|c|}\hline 2&2&1\\ \hline 3&4&3\\ \hline 4 \\ \cline{1-1}\end{array}\cdots\boxed{1} \quad \overset{-\tau}{\longleftrightarrow} \quad \begin{array}{|c|c|c|}\hline 2&3&1\\ \hline 3&4&2\\ \hline 4 \\ \cline{1-1}\end{array}\cdots\boxed{1}$$

Figure 14: The elements of $\mathrm{NST}(\triangleright(3,2^{2t},1))$, $t \geq 1$, which span a U_q^τ-submodule of $\check{Y}_{\triangleright(3,2^{2t},1)}$ and are graded and integral; the dots represent $t-1$ height-2 invariants (so that this picture represents 2^t NST).

15.3. **Invariant moves.** For γ equal to $(1,1,1)$, $(3,3,3)$, or $(3,2^{2t},1)$ $(t \geq 1)$, let $\mathrm{NST}(\triangleright\gamma)$ be the elements of $\check{Y}_{\triangleright\gamma}$ shown in Figure 12, 13, or 14, respectively. The notation $\mathrm{NST}(\triangleright\gamma)$ is somewhat misleading in this case as this set is not a basis for $\check{Y}_{\triangleright\gamma}$, but it allows many of the definitions and results below to be stated uniformly.

Let $\triangleright$NST be the union of $\mathrm{NST}(\triangleright\gamma)$ over all γ for which this is defined. A $\triangleright$NST corresponding to Figure 4, 6, or 11 is a *nonintegral* $\triangleright$*NST* and an *integral* $\triangleright$*NST* otherwise. A $\triangleright$NST corresponding to Figure 4, 6, 11, 9, or 14 is a *graded* $\triangleright$*NST* and a *degree-preserving* $\triangleright$*NST* otherwise. A $\triangleright$NST corresponding to Figure 12, 13, or 14 is an *invariant* $\triangleright$*NST* and a *two-column* $\triangleright$*NST* otherwise. The reasons for this terminology will be explained shortly.

Proposition 15.3. *The elements shown in Figures 12, 13, and 14 belong to* $\check{Y}_{\triangleright(1,1,1)}$, $\check{Y}_{\triangleright(3,3,3)}$, *and* $\check{Y}_{\triangleright(3,2^{2t},1)}$, *respectively.*

Part of this proposition can be rephrased as saying that for $j = 1, 3$, an invariant of height j commutes with columns of height j in $\check{X}_{(j,j,j)'}$. This can be proved by directly calculating the elements in the figures in terms of tensor products of NSC, however we postpone the proof to §15.6, where we establish a result that makes the proof easy.

The importance of the invariant $\triangleright$NST will become more clear in §16. Essentially, they are needed because, though the corresponding relations in $\check{X}_{\gamma'}$ are consequences of the relations corresponding to two-column $\triangleright$NST, they are not consequences in an easy, combinatorial way.

15.4. **Nonstandard tabloid classes.** Here we introduce nonstandard tabloid classes (NSTC) and the subset +HNSTC of NSTC. The +HNSTC of shape ν are the combinatorial objects that will eventually be identified with a basis of $\check{X}_\nu$. We begin by defining directed graphs $\mathcal{TG}(\nu)$ on (a rescaled version of) $\mathrm{NST}(\nu)$.

Definition 15.4. A *scaled nonstandard tabloid* (SNST) is an element of

$$\bigsqcup_{T\in\mathrm{NST}} \left\{(-\tfrac{1}{[2]})^{\deg(T)}T, -(-\tfrac{1}{[2]})^{\deg(T)}T\right\} \subseteq K\mathrm{NST},$$

where KNST denotes the K-vector space with basis NST. The shape of an SNST T is the shape of the NST in KT, and $\mathrm{SNST}(\alpha)$ denotes the set of SNST of shape α.

The notions of subtabloid, invariant record, degree, etc. for nonstandard tabloids extend in the obvious way to scaled nonstandard tabloids. For instance, the invariant-free part of aT, for an NST T and $a \in K$, is aT', where T' is the invariant-free part of T. Also, $(aT)|_{[i,j]} := a(T|_{[i,j]})$, for any NST T and $a \in K$.

Definition 15.5. For each $\nu \vdash r$, let $\mathcal{TG}(\nu)$ be the directed graph with vertex set $\mathrm{SNST}(\nu') \sqcup \{0\}$ and edge set given by $T \to T'$ if the following conditions are satisfied for some t and $i \in [\nu_1 - t + 1]$:

- $cT|_{[i,i+t-1]} - c'T'|_{[i,i+t-1]} \in \mathrm{NST}(\triangleright(\nu'_i, \dots, \nu'_{i+t-1}))$, for some c, c' both in $\{(-[2])^j \ : \ j \in \mathbb{Z}_{\geq 0}\}$ or both in $\{-(-[2])^j \ : \ j \in \mathbb{Z}_{\geq 0}\}$;
- T and T' agree outside columns $i, \dots, i+t-1$, i.e. $KT|_{[i-1]} = KT'|_{[i-1]}$ and $KT|_{[i+t,l]} = KT'|_{[i+t,l]}$;
- $\deg(T) \leq \deg(T')$

(we also allow $T' = 0$ and define $\deg(0) = \infty$, i.e. $T \to 0$ if $cT|_{[i,i+t-1]} \in \mathrm{NST}(\triangleright(\nu'_i, \dots, \nu'_{i+t-1}))$ for some $c \in \{[2]^{\deg(T)}, -[2]^{\deg(T)}\}$, $i \in [\nu_1 - t + 1]$).

See Figure 15 for the example $\mathcal{TG}((3,2,2,1)')$.

A directed edge is a *graded move* (resp. *degree-preserving move, integral, nonintegral, invariant move, two-column move*) if the corresponding $\triangleright$NST is graded (resp. degree-preserving, integral, nonintegral, invariant, two-column); a directed edge is a *move defined by Figure i* if the corresponding $\triangleright$NST appear in this figure. Also, if the directed edge corresponds to t, i in the definition above, then we say that it is a (graded, degree-preserving, etc.) *move at* $[i, i+t-1]$.

Definition 15.6. A *nonstandard tabloid class* (NSTC) $\mathbf{T}$ of shape ν is a strong component of $\mathcal{TG}(\nu)$ (we will often identify a strong component with its vertex set). The set of NSTC of shape ν is denoted $\mathrm{NSTC}(\nu)$. An NSTC is *nonorientable* if it contains T and $-T$ for some SNST T. An NSTC $\mathbf{T}$ is *dishonest* if it is the vertex 0 itself, it is nonorientable, or it has a directed edge to some other NSTC. Otherwise, we say $\mathbf{T}$ is *honest*.

A SNST is *honest* (resp. *dishonest*) if it belongs to an honest (resp. dishonest) NSTC. If two SNST T, T' lie the same NSTC, then we say that T and T' are *equivalent* and also denote this by $T \equiv T'$.

It is easy to check directly (and is done in the proof of Theorem 15.21 (iv)) that if $T \equiv T'$, then T is highest weight if and only if T' is. We then define an NSTC to be *highest weight* if every SNST in its class is highest weight.

The next proposition shows that the strong components of $\mathcal{TG}(\nu)$ are quite easy to describe and justifies our terminology degree-preserving and graded.

Proposition 15.7.

(a) *If $T \to T'$ is a degree-preserving move, then the invariant records of T and T' agree.*
(b) *If $T \to T'$ is a graded move, then* $\deg(T) < \deg(T')$.
(c) *An NSTC is a connected component in the undirected graph on SNST consisting of degree-preserving moves.*

Proof. The key point here is that modifying part of an NST only affects arcs having one or both ends in the modified part. With this in mind, one can check (a) directly for each

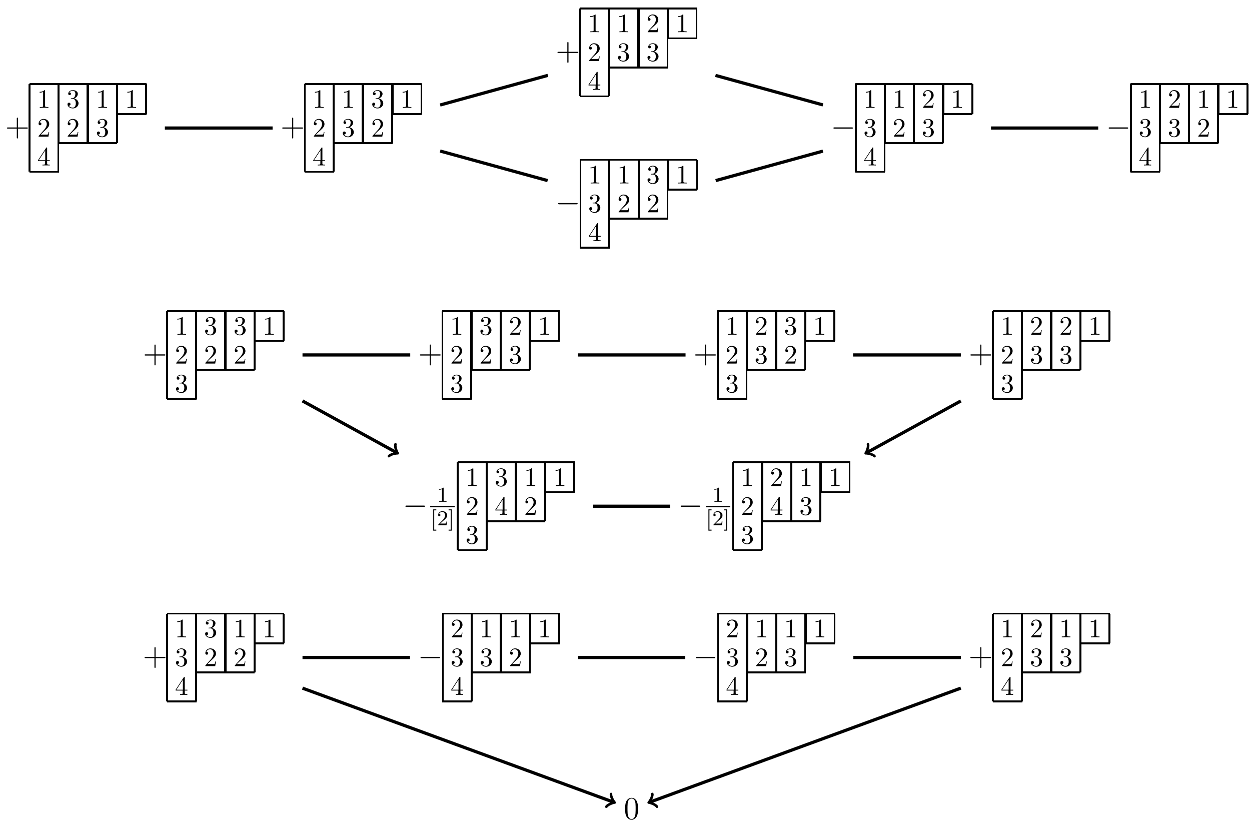


Figure 15: The graph $\mathcal{TG}((3,2,2,1)')$ restricted to highest weight SNST of weight $((5,3),(5,3))$, and, for each pair $\{T,-T\}$ of SNST, we have only drawn one of the pair. Edges without arrows indicate a directed edge in both directions and are degree-preserving moves; edges with arrows are graded moves. There are two strong components that are honest NSTC (the one of size 6 and the one of size 2) corresponding to the fact that the Kronecker coefficient $g_{(5,3),(5,3),(3,2,2,1)'} = 2$.

degree-preserving move. For instance, if T' is obtained from T by replacing a contiguous subtabloid of T equal to $\boxed{\begin{smallmatrix}1&1\\2&3\end{smallmatrix}}$ with $\boxed{\begin{smallmatrix}1&1\\3&2\end{smallmatrix}}$, then the V- and W-words of T and T' look like

$$\underset{T}{\cdots\begin{smallmatrix}1121\\2111\end{smallmatrix}\cdots} \quad\longleftrightarrow\quad \underset{T'}{\cdots\begin{smallmatrix}2111\\1121\end{smallmatrix}\cdots}$$

The column $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$ of this contiguous subtabloid of T is paired by two arcs to another column of T if and only if the column $\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}$ of T' is paired by two arcs to another column of T'. A similar statement holds for $\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}$.

Statement (b) holds by the definition $\deg(0) = \infty$ except in the following case: $T \to T'$ is a graded move at $[i, i+1]$ and $T|_{[i,i+1]} = C, T'|_{[i,i+1]} = C'$, for $C + \frac{1}{[2]}C'$ a graded $\triangleright$NST in the center of Figure 4, 6, or 11. To see that $\deg(T) < \deg(T')$ in this case, note that T' contains an invariant column pair at columns i and $i+1$, while an invariant column pair of T cannot contain column i or $i+1$. Since the arcs of T and T' not involving columns i and $i+1$ are the same, any invariant column pair of T is an invariant column pair of T'.

Statement (c) follows from (a) and (b). □

The *invariant record* (resp. *degree*) of an honest NSTC is the invariant record (resp. *degree*) of any SNST in its class. An honest NSTC is *invariant-free* if its invariant record is $(0,0,0,0)$.

Define the set of *positive honest nonstandard tabloid classes* (+HNSTC) as follows: for each honest NSTC $\mathbf{T}$, declare either $+\mathbf{T}$ or $-\mathbf{T}$ to be positive. It does not really matter how these choices are made, but for computations in §17 we have found it convenient to adopt the following convention. A 3-2 arc of an SNST is an arc between a height-3 column and a height-2 column (Definition 17.1).

If every (equivalently, any) $T \in \mathbf{T}$ has no 3-2 arc, then declare $\mathbf{T}$ to be positive if it contains $(-\frac{1}{[2]})^{\deg(T)}T$ for some NST T. Otherwise, declare $\mathbf{T}$ to be positive if it contains $(-\frac{1}{[2]})^{\deg(T)}T$ for some NST T whose 3-2 arc is a 3-2 W-arc. (139)

Let +HSNST be the set of SNST that belong to some +HNSTC. The notions of invariant-free, NSTC(ν), SNST(ν), etc. carry over in the obvious way to +HNSTC and +HSNST. Let +HNSTC$(\nu)_h$ (resp. +HNSTC$(\nu)_{\geq h}$) be the subset of +HNSTC of shape ν and degree h (resp. at least h).

In the next subsection, we identify +HNSTC(ν) with a subset of $\check{X}_\nu$. We will show (as part of Theorem 15.21) that +HNSTC(ν) is a basis for $\check{X}_\nu$ and the number of highest weight elements of +HNSTC(ν) of weight (λ, μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$.

A U_q^τ-cell of $(\check{X}_{(3,2,1)}, +\text{HNSTC}((3,2,1)))$ is shown in Figure 16. An NST representative of each highest weight +HNSTC of shape $(3,2,2,2,2,1)'$ is shown in Figure 17.

15.5. **Justifying the combinatorics.** Here we establish the precise relationship between the combinatorial NSTC(ν) of the previous subsection and the relations satisfied by the image of SNST(ν') in $\check{X}_\nu$.

Let us recall some of the notation introduced at the beginning of the section and introduce some new notation: let $\alpha \vDash_l^{dx} r$ and suppose $\alpha = (\beta, \gamma, \delta)$ with $\beta = (\alpha_1, \dots, \alpha_{i-1})$, $\gamma = (\alpha_i, \dots, \alpha_{i+t-1})$, $\delta = (\alpha_{i+t}, \dots, \alpha_l)$, and γ is such that NST($\triangleright\gamma$) is defined. Set

$$\begin{array}{lcl} \check{Y}_{\triangleright\gamma} & := & K\,\mathrm{NST}(\triangleright\gamma), \\ \check{Y}_{(\beta,\triangleright\gamma,\delta)} & := & \check{Y}_\beta \otimes \check{Y}_{\triangleright\gamma} \otimes \check{Y}_\delta \subseteq \check{Y}_\alpha, \\ \check{Y}_{\triangleright^i\alpha} & := & \check{Y}_{(\beta,\triangleright\gamma,\delta)}, \ \text{ if } \ell(\gamma)=2, \\ \check{Y}_{\triangleright\alpha} & := & \sum_{i=1}^{l-1} \check{Y}_{\triangleright^i\alpha}, \\ \check{X}_\nu & := & \check{Y}_{\nu'}/\check{Y}_{\triangleright\nu'}, \ \text{ for } \nu \vdash r, \\ \mathrm{gr}(\check{X}_\nu) & := & \mathrm{gr}(\check{Y}_{\nu'})/\mathrm{gr}(\check{Y}_{\triangleright\nu'}). \end{array} \tag{140}$$

Let ϖ_ν denote the projection $\check{Y}_{\nu'} \twoheadrightarrow \check{X}_\nu$. It is sometimes convenient to use

$$\mathrm{gr}(\check{X}_\nu) = \bigoplus_{h\geq 0} (\check{Y}_{\nu'})_h / ((\check{Y}_{\triangleright\nu'})_h + (\check{Y}_{\nu'})_{h+1}). \tag{141}$$

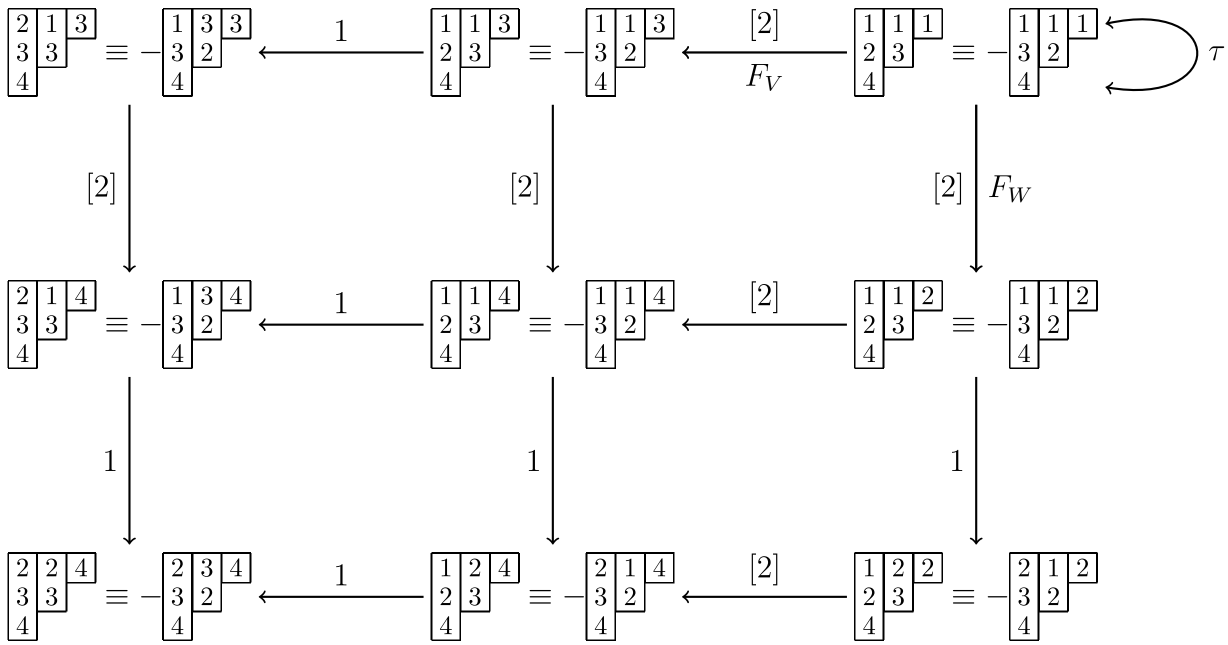


Figure 16: A U_q^τ-cell of +HNSTC((3, 2, 1)); all SNST belonging to each +HNSTC in this cell are shown.

The difficulty in understanding the image of SNST(α) in $\check{Y}_\alpha/\check{Y}_{\triangleright\alpha}$ is that $\check{Y}_{(\beta,\triangleright\gamma,\delta)}$ is not easily expressed in terms of the basis SNST(α). This is mostly because of the nonintegral $\triangleright$NST. To remedy this, we define a basis NST($\beta, \triangleright\gamma, \delta$) of $\check{Y}_{(\beta,\triangleright\gamma,\delta)}$, which is analogous to the basis $B_1' \tilde{\heartsuit} \cdots \tilde{\heartsuit} B_l'$ of Proposition 5.9. Let $(\check{Y}_{(\beta,\triangleright\gamma,\delta)}, \mathrm{NST}(\beta, \triangleright\gamma, \delta))$ be the weak upper based U_q^τ-module obtained by tensoring the weak upper based U_q^τ-modules $(\check{Y}_\beta, \mathrm{NST}(\beta)), (\check{Y}_{\triangleright\gamma}, \mathrm{NST}(\triangleright\gamma)), (\check{Y}_\delta, \mathrm{NST}(\delta))$ (see §5.5 and §13.5). Let $\check{\heartsuit}_\triangleright$ denote the $\heartsuit$ product in this setting, so that

$$\begin{aligned}\mathrm{NST}(\beta, \triangleright\gamma, \delta) &:= \mathrm{NST}(\beta)\check{\heartsuit}_\triangleright \mathrm{NST}(\triangleright\gamma)\check{\heartsuit}_\triangleright \mathrm{NST}(\delta) \\ &= \Big\{ B\check{\heartsuit}_\triangleright \tilde{C}\check{\heartsuit}_\triangleright D : B \in \mathrm{NST}(\beta), \tilde{C} \in \mathrm{NST}(\triangleright\gamma), D \in \mathrm{NST}(\delta)\Big\}.\end{aligned}$$

The main result of this subsection will follow from the following two general propositions about based modules.

Proposition 15.8. *Let $(N_1', B_1'), (N_1, B_1), (N_2, B_2)$ be upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules and N_1' a submodule of N_1. Suppose that the canonical inclusion $\iota : N_1' \hookrightarrow N_1$ induces a morphism of balanced triples, i.e., ι restricts to maps*

$$\mathbb{Q}[q, q^{-1}]B_1' \to \mathbb{Q}[q, q^{-1}]B_1, \quad K_0 B_1' \to K_0 B_1, \quad K_\infty B_1' \to K_\infty B_1.$$

Then $b_1' \heartsuit' b_2 = b_1' \heartsuit b_2$ for all $b_1' \in B_1',\ b_2 \in B_2$, where $\heartsuit'$ (resp. $\heartsuit$) denotes the product of §5.5 for tensoring (N_1', B_1') and (N_2, B_2) (resp. (N_1, B_1) and (N_2, B_2)).

Proof. Let $\mathscr{L}' = K_\infty(B_1' \otimes B_2)$ (resp. $\mathscr{L} = K_\infty(B_1 \otimes B_2)$) be the crystal lattice of $(N_1' \otimes N_2, B_1' \heartsuit' B_2)$ (resp. $(N_1 \otimes N_2, B_1 \heartsuit B_2)$). This follows simply from the fact that the

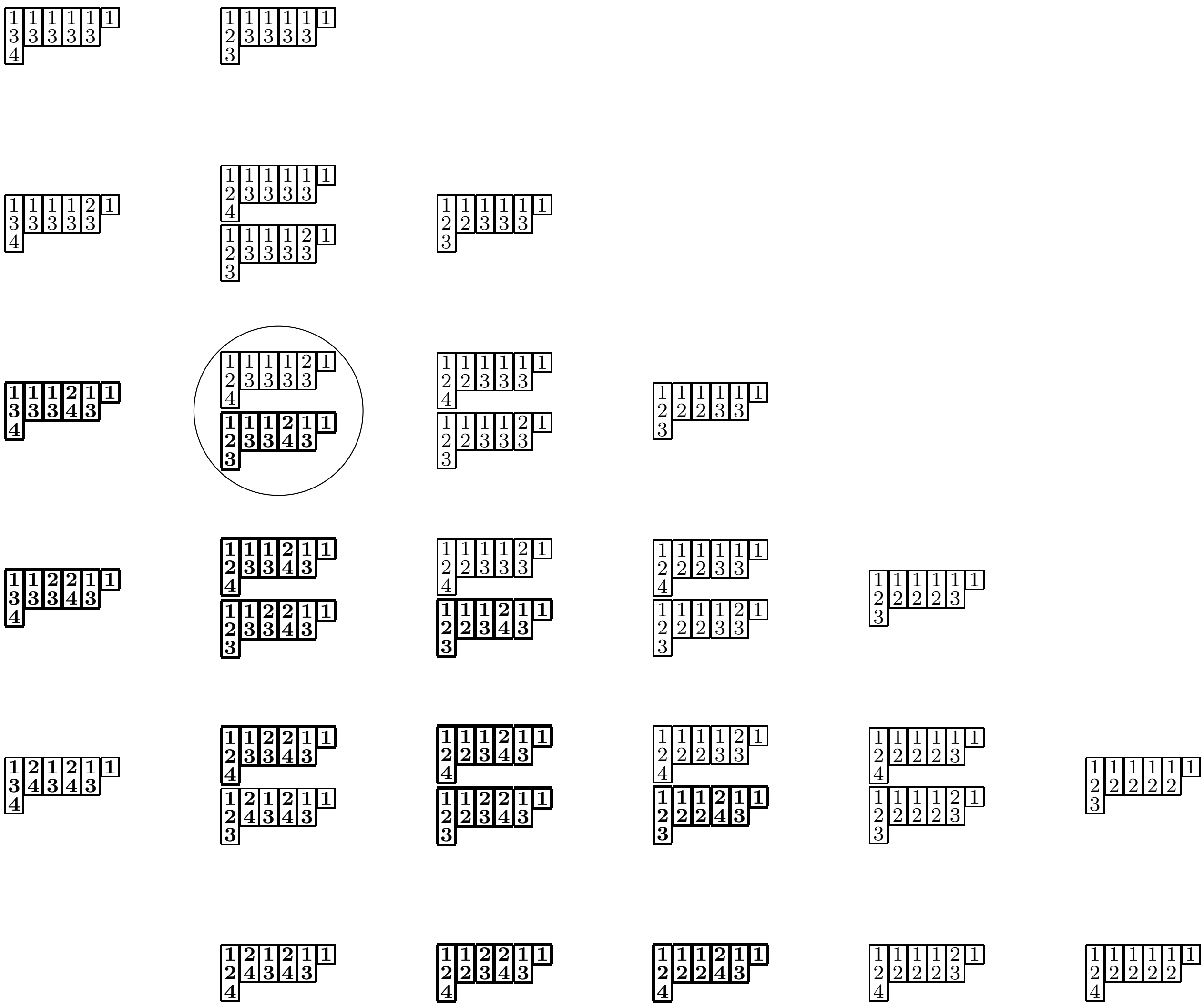


Figure 17: An NST representative of each highest weight +HNSTC of shape $\nu = (3,2,2,2,2,1)'$ (these are the straightened representatives, defined in §16). The NST of weight $(\lambda,\mu) = ([l_2,l_1],[m_2,m_1])$ are drawn at position $(\frac{l_1}{2},\frac{m_1}{2})$ so, for instance, the two NST circled at position (1,3) corresponds to $g_{(7,5)\,(9,3)\,\nu} = 2$. The bold borders and numbers make it easier to read off the NST of fixed degree.

canonical isomorphism

$$\mathbb{Q}[q,q^{-1}](B_1 \otimes B_2) \cap \overline{\mathscr{L}} \cap \mathscr{L} \xrightarrow{\cong} \mathscr{L}/q^{-1}\mathscr{L}$$

restricts to the canonical isomorphism

$$\mathbb{Q}[q,q^{-1}](B_1' \otimes B_2) \cap \overline{\mathscr{L}'} \cap \mathscr{L}' \xrightarrow{\cong} \mathscr{L}'/q^{-1}\mathscr{L}'.$$

$\square$

Recall that $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-irreducibles are parameterized by pairs of partitions and that $\lhd$ denotes dominance order on pairs of partitions as well as dominance order on partitions, as discussed in §7.

Proposition 15.9. *Let $(N_1', B_1'), (N_1, B_1), (N_2, B_2)$ be upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules and N_1' a submodule of N_1. Let $\heartsuit'$ and $\heartsuit$ be as in the previous proposition. Then for all pairs of partitions θ, $b_1' \in B_1'[\theta]$, and $b_2 \in B_2$,*

$$b_1' \heartsuit' b_2 - b_1' \heartsuit b_2 \in N_1[\lhd\theta] \otimes N_2.$$

Proof. First assume that $N_1 = N_1[\theta]$ for some pair of partitions $\theta = (\lambda, \mu)$. The result in this case follows from the fact that (N_1', B_1') and (N_1, B_1) are both isomorphic to a direct sum of $(V_\lambda \star W_\mu, B(\lambda) \star B(\mu))$ (the result follows despite the fact that the inclusion $N_1' \hookrightarrow N_1$ need not induce a morphism of upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules or even a map $K_\infty B_1' \to K_\infty B_1$). Now for the general case, let θ, b_1', b_2 be as in the statement. Recall that $\varsigma_\theta^{N_1}(b_1')$ is the image of b_1' in $N_1[\theta]$. Then

$$b_1' \heartsuit' b_2 \equiv \varsigma_\theta^{N_1'}(b_1') \tilde{\heartsuit}' b_2 = \varsigma_\theta^{N_1}(b_1') \tilde{\heartsuit} b_2 \equiv b_1' \heartsuit b_2,$$

where the equivalences are mod $N_1[\lhd\theta] \otimes N_2$ and the equality is by the result for the $N_1 = N_1[\theta]$ case; the product $\tilde{\heartsuit}'$ (resp. $\tilde{\heartsuit}$) is for the tensor product of $(N_1'[\theta], \varsigma_\theta^{N_1'}(B_1'[\theta]))$ (resp. $(N_1[\theta], \varsigma_\theta^{N_1}(B_1[\theta]))$) and (N_2, B_2). □

Theorem 15.10. *Let $\alpha = (\beta, \gamma, \delta)$ as above. For any $B \in NST(\beta), \tilde{C} \in NST(\rhd\gamma), D \in NST(\delta)$, there holds*

$$B \check{\heartsuit} \tilde{C} \check{\heartsuit} D - B \check{\heartsuit}_\rhd \tilde{C} \check{\heartsuit}_\rhd D \begin{cases} = 0 & \text{if } \tilde{C} \text{ is integral,} \\ \in (\check{Y}_\alpha)_{h+1} & \text{if } \tilde{C} \text{ is nonintegral,} \end{cases} \tag{142}$$

where $h = \deg(B \check{\heartsuit} \tilde{C} \check{\heartsuit} D)$ ($\deg(x)$ is defined for any $x \in \check{Y}_\alpha$ in §15.1). Hence

$$\operatorname{gr}(\check{Y}_{(\beta,\rhd\gamma,\delta)}) = \operatorname{gr}(\check{Y}_\beta \check{\heartsuit} \check{Y}_{\rhd\gamma} \check{\heartsuit} \check{Y}_\delta). \tag{143}$$

Proof. The top case of (142) follows from Proposition 15.8 and the bottom from Proposition 15.9. The propositions are applied with $(N_1', B_1') = (\check{Y}_{\rhd\gamma}, \mathrm{NST}(\rhd\gamma))$ and $(N_1, B_1) = (\check{Y}_\gamma, \mathrm{NST}(\gamma))$, except the application of Proposition 15.8 in the case $\gamma = (2,2)$. For this case, since nonintegral $\mathrm{NST}(\rhd(2,2))$ span a submodule of $\check{Y}_{\rhd(2,2)}$, we instead apply Proposition 15.8 with (N_1', B_1') equal to the based module corresponding to this submodule.

For the bottom case, the necessary connection between the $\lhd$ of Proposition 15.9 and degree is made as follows: put $\gamma = (j, j)$, $j = 1, 2$, or 3. The θ of the proposition is equal $((j+1, j-1), (j,j))$ or $((j,j),(j+1,j-1))$, hence $N_1[\lhd\theta]$ is spanned by the height-j invariant(s). Further, by Proposition 15.7 (b) and the fact that the action of the global Kashiwara operators $G\tilde{F}_V^{\mathrm{up}}, G\tilde{F}_W^{\mathrm{up}}$ on NST preserve degree, there holds $\check{Y}_\beta \otimes N_1[\lhd\theta] \otimes \check{Y}_\delta \subseteq (\check{Y}_\alpha)_{h+1}$. □

Now we can show that combinatorics of the previous subsection is actually relevant to the algebraic objects $\check{X}_\nu$ and $\operatorname{gr}(\check{X}_\nu)$.

Proposition 15.11. *Let T, T' be SNST of shape ν' for some $\nu \vdash_{d_X} r$, and let $\mathbf{T}, \mathbf{T}'$ be the NSTC containing T and T', respectively.*

(i) *If $T \equiv T'$ (equivalently, $\mathbf{T} = \mathbf{T}'$), then $\varpi_\nu(T) = \varpi_\nu(T')$ and we identify this element of $\check{X}_\nu$ with the NSTC $\mathbf{T}$.*
(ii) *If $T \to 0$ is an integral move, then $\mathbf{T} = 0$ as an element of $\check{X}_\nu$.*
(iii) *If T is dishonest, then $\mathrm{in}_h(\mathbf{T}) = 0$ in $(\check{X}_\nu)_h/(\check{X}_\nu)_{h+1}$, where $h = \deg(T)$.*
(iv) *The set $\mathrm{in}_h(+HNSTC(\nu)_h)$ spans $\mathrm{gr}(\check{X}_\nu)_h$. In particular, $\mathrm{in}(+HNSTC(\nu))$ spans $\mathrm{gr}(\check{X}_\nu)$.*
(v) *The set $+HNSTC(\nu)_{\geq h}$ spans $(\check{X}_\nu)_h$. In particular, $+HNSTC(\nu)$ spans $\check{X}_\nu$.*

Note that part (i) of the proposition implies that if $T \equiv T'$ $(T, T' \in \mathrm{SNST}(\nu'))$ in the notation of the previous subsection, then $T \equiv T' \mod \check{Y}_{\triangleright\nu'}$.

Proof. By the top case of (142) of Theorem 15.10, if $T \to T'$ is an integral move, then $\varpi_\nu(T) = \varpi_\nu(T')$. Since an NSTC is just a connected component in the undirected graph consisting of degree-preserving moves (Proposition 15.7(c)) and degree-preserving moves are integral, (i) follows. This implies (ii) as well.

Next suppose $T \to T'$ is a graded move and $\deg(T) = h$ (and, since the move is graded, $\deg(T') > h$). Then $cT - c'T'$ is of the form $B\check{\heartsuit}\tilde{C}\check{\heartsuit}D$ in the notation of Theorem 15.10 and Definition 15.5, so $B\check{\heartsuit}\tilde{C}\check{\heartsuit}D \in \check{Y}_{\triangleright\nu'} + (\check{Y}_{\nu'})_{h+1}$ by (142), hence $T \in (\check{Y}_{\triangleright\nu'})_h + (\check{Y}_{\nu'})_{h+1}$, which proves (iii).

By (i), $+\mathrm{HNSTC}(\nu)$ is a well-defined subset of $\check{X}_\nu$ and, moreover, by Proposition 15.7, $+\mathrm{HNSTC}(\nu)_{\geq h}$ is a well-defined subset of $(\check{X}_\nu)_h$. Then (iii) and the fact that $\mathrm{NST}(\nu')_{\geq h}$ is a basis of $(\check{Y}_{\nu'})_h$ prove (iv). Statement (v) follows easily from (iv). □

Example 15.12. The theorem above does not hold without the gr: suppose $\alpha = (1,1,1)$, $\beta = (1)$, $\gamma = (1,1)$, $\delta = ()$, $B = \boxed{3}$, $\tilde{C} = \boxed{2}\boxed{1}$, and consider the NST $B\check{\heartsuit}\tilde{C} = \boxed{3}\boxed{2}\boxed{1}$. Since $\boxed{2}\boxed{1} \in \check{Y}_{\triangleright\gamma}$, we would like to conclude that $\varpi_{\alpha'}(\boxed{3}\boxed{2}\boxed{1}) = 0$ in $\check{X}_{\alpha'}$, however this is only true in $\mathrm{gr}(\check{X}_{\alpha'})$. This can be seen by expressing $\boxed{3}\check{\heartsuit}_{\triangleright}\boxed{2}\boxed{1} \in \mathrm{NST}(\beta, \triangleright\gamma)$ in terms of $\mathrm{NST}(\alpha)$ as

$$\boxed{3}\check{\heartsuit}_{\triangleright}\boxed{2}\boxed{1} = \boxed{3}\boxed{2}\boxed{1} + \frac{1}{[2]}\boxed{1} \otimes \boxed{4}\boxed{1} = \boxed{3}\boxed{2}\boxed{1} + \frac{1}{[2]}\boxed{1}\boxed{4}\boxed{1}.$$

The next subsection will give a nice way of doing such computations, but for now we can verify it by checking that the right-hand side is $\bar{\cdot}$-invariant, which is clear, and that

$$\begin{aligned}
\boxed{3}\boxed{2}\boxed{1} + \tfrac{1}{[2]}\boxed{1} \otimes \boxed{4}\boxed{1} &= c_{\substack{211\\121}} + \tfrac{1}{[2]} c_{\substack{121\\121}} \\
&= c_{\substack{2\\1}} \otimes c_{\substack{11\\21}} - q^{-1} c_{\substack{1\\1}} \otimes c_{\substack{21\\21}} - q^{-2} c_{\substack{1\\1}} \otimes c_{\substack{12\\21}} + \tfrac{1}{[2]} c_{\substack{1\\1}} \otimes c_{\substack{21\\21}} \\
&= \boxed{3} \otimes \boxed{2}\boxed{1} - q^{-2}\boxed{1} \otimes \left(\boxed{2}\boxed{3} + \tfrac{1}{[2]}\boxed{4}\boxed{1}\right),
\end{aligned}$$

which lies in $\check{Y}_{(\beta,\triangleright\gamma)}$ and is congruent to $\boxed{3} \otimes \boxed{2}\boxed{1} \mod q^{-1}K_\infty\mathrm{NST}(\beta, \triangleright\gamma)$.

Therefore $\boxed{3}\boxed{2}\boxed{1} = \boxed{3}\check{\heartsuit}_{\triangleright}\boxed{2}\boxed{1} - \frac{1}{[2]}\boxed{1} \otimes \boxed{4}\boxed{1}$ does not belong to $\check{Y}_{(\beta,\triangleright\gamma)}$, but it does belong to $\check{Y}_{(\beta,\triangleright\gamma)} + (\check{Y}_\alpha)_1$. Hence its image under $\check{Y}_\alpha \xrightarrow{\mathrm{in}_0} \mathrm{gr}(\check{Y}_\alpha) \twoheadrightarrow \mathrm{gr}(\check{X}_{\alpha'})$ is 0.

Remark 15.13. We can also construct a basis for $\check{Y}_{\triangleright\gamma}$ in the case $\ell(\gamma) = 2$ and $\gamma_1 < \gamma_2$. This has a similar form to the basis $\mathrm{NST}(\triangleright(\gamma_2, \gamma_1))$. Theorem 15.10 above can be extended to this case and many of the results of this section, including Theorem 15.21 (i) and (ii), can be extended to the case ν' is not necessarily a partition (after introducing some more invariant $\triangleright$NST). We believe that all of Theorem 15.21 can be extended to this case, but we have not worked out the necessary combinatorics. This is a more precise version of a special case of Conjecture 19.1, as explained in the discussion following the conjecture.

15.6. **Explicit formulae for nonintegral NST**$(\beta, \triangleright\gamma, \delta)$**.** Using Corollary 6.11, we now make explicit how to express $\mathrm{NST}(\beta, \triangleright\gamma, \delta)$ in terms of $\mathrm{NST}(\alpha)$ for those elements of $\mathrm{NST}(\beta, \triangleright\gamma, \delta)$ corresponding to nonintegral $\mathrm{NST}(\triangleright\gamma)$.

Denote some of the invariants as follows

$$I_1^V = I_1^W = \boxed{4\,|\,1},\ I_2^V = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array},\ I_2^W = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array},\ I_3^V = I_3^W = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 2 \\ \hline 4 & 3 \\ \hline\end{array}.$$

For $\gamma = (j, j)$, $j \in [3]$, define

$$\mathrm{NST}_V(\gamma) := \{T \in \mathrm{NST}(\gamma) : \varphi_V(T) + \varepsilon_V(T) = 2, \varphi_W(T) + \varepsilon_W(T) = 0\} \sqcup \{I_j^V\}.$$

Note that $\mathrm{NST}_V(\gamma)$ spans an upper based $U_q(\mathfrak{sl}(V))$-submodule of $(\check{Y}_\gamma, \mathrm{NST}(\gamma))$ that is isomorphic to $(\mathrm{Res}_{U_q(\mathfrak{sl}(V))} V^{\otimes 2}, B_V^2)$ (where $(V^{\otimes 2}, B_V^2)$ is as in §6.2). For each $C \in \mathrm{NST}_V(\gamma)$, let $\tilde{C} := \pi^{\check{Y}_\gamma}_{((j+1,j-1),(j,j))}(C)$ be the corresponding element of $\mathrm{NST}(\triangleright\gamma)$ (see the discussion before Corollary 6.11), which is a nonintegral $\triangleright$NST. Denote the set of such $\tilde{C}$ by $\mathrm{NST}_V(\triangleright\gamma)$. This is just the horizontal $\mathfrak{g}_V$-string in Figure 4, 6, or 11 for $j = 1, 2, 3$, respectively. Define $\mathrm{NST}_W(\gamma)$ and $\mathrm{NST}_W(\triangleright\gamma)$ similarly, with the roles of W-word and V-word interchanged and I_j^W in place of I_j^V.

Since $\{T \in \mathrm{NST}(\alpha) : T|_{[i,i+1]} \in \mathrm{NST}_V(\gamma))\}$ spans an upper based $U_q(\mathfrak{sl}(V))$-submodule of $(\check{Y}_\alpha, \mathrm{NST}(\alpha))$, and this is a direct sum of upper based $U_q(\mathfrak{sl}(V))$-modules of the form in Corollary 6.11 (restrict the $U_q(\mathfrak{g}_V)$-modules of the corollary to $U_q(\mathfrak{sl}(V))$), with B_t of the corollary equal to $\mathrm{NST}_V(\gamma)$, we obtain

Corollary 15.14. *Maintain the notation above, with $\gamma = (j, j)$, $j \in [3]$, and let $B \in NST(\beta), C \in NST_V(\gamma), D \in NST(\delta)$. Let $T = B\check{\heartsuit}C\check{\heartsuit}D$ and $\mathbf{k}'$ be the unpaired V-word of C. Then*

$$B\check{\heartsuit}_{\triangleright}\tilde{C}\check{\heartsuit}_{\triangleright}D = \begin{cases} B\check{\heartsuit}C\check{\heartsuit}D + \frac{1}{[2]}B'\check{\heartsuit}I_j^V\check{\heartsuit}D & \text{if } \mathbf{k}' = 11, \\ B\check{\heartsuit}C\check{\heartsuit}D + \frac{1}{[2]}B\check{\heartsuit}I_j^V\check{\heartsuit}D & \text{if } \mathbf{k}' = 12, \\ B\check{\heartsuit}C\check{\heartsuit}D + \frac{1}{[2]}B\check{\heartsuit}I_j^V\check{\heartsuit}D' & \text{if } \mathbf{k}' = 22, \end{cases}$$

where $B' \in NST(\beta) \sqcup \{0\}, D' \in NST(\delta) \sqcup \{0\}$ are determined by the graphical calculus as in Corollary 6.11, and depend on which (if any) V-arcs of T are paired with k_1', k_2' in the V-diagram of T.

A similar statement holds for $C \in NST_W(\gamma)$ by considering W-words instead of V-words.

Example 15.15. The following example corresponds to Corollary 15.14 for $\beta = (), \gamma = (2,2), \delta = (1,1)$, and to the case $\mathbf{k}' = 22$:

$$\begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\,\check{\heartsuit}_{\triangleright}\begin{array}{|c|c|}\hline 1&1\\\hline\end{array} = \begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 1&1\\\hline\end{array} + \frac{1}{[2]}\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 3&1\\\hline\end{array} \text{ (in } \check{Y}_{(2,2,1,1)}\text{)}.$$

Hence

$$\begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 1&1\\\hline\end{array} = -\frac{1}{[2]}\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 3&1\\\hline\end{array} = 0 \text{ in } \check{X}_{(2,2,1,1)'}.$$

Compare this to

$$\begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\,\check{\heartsuit}_{\triangleright}\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} = \begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} + \frac{1}{[2]}\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 3&2\\\hline\end{array} \text{ (in } \check{Y}_{(2,2,1,1)}\text{)}.$$

Combining this with

$$\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\,\check{\heartsuit}_{\triangleright}\Big(\begin{array}{|c|c|}\hline 3&2\\\hline\end{array} + \tfrac{1}{[2]}\begin{array}{|c|c|}\hline 4&1\\\hline\end{array}\Big) = \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 3&2\\\hline\end{array} + \frac{1}{[2]}\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 4&1\\\hline\end{array} \text{ (in } \check{Y}_{(2,2,1,1)}\text{)}$$

yields

$$\begin{array}{|c|c|}\hline 3&3\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} = \frac{1}{[2]^2}\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\begin{array}{|c|c|}\hline 4&1\\\hline\end{array} \text{ in } \check{X}_{(2,2,1,1)'}.$$

Remark 15.16. Corollary 15.14 is a more precise result than the bottom case of (142) of Theorem 15.10, however we believe the method of proof of the theorem to be valuable for its potential use outside the two-row case.

We now have the tools to easily prove Proposition 15.3. Recall that this states that the invariant $\mathrm{NST}(\triangleright\gamma)$ actually belong to $\check{Y}_{\triangleright\gamma}$.

Proof of Proposition 15.3. The statement for Figure 12 follows from

$$-\frac{1}{[2]}\begin{array}{|c|c|c|}\hline 4&1&2\\\hline\end{array} \equiv \begin{array}{|c|c|c|}\hline 2&3&2\\\hline\end{array} \equiv -\frac{1}{[2]}\begin{array}{|c|c|c|}\hline 2&4&1\\\hline\end{array}, \tag{144}$$

where the equivalences are mod $\check{Y}_{\triangleright(1,1,1)}$ and are by Corollary 15.14. Next, U_q^τ applied to $\begin{array}{|c|c|c|}\hline 4&1&2\\\hline\end{array} - \begin{array}{|c|c|c|}\hline 2&4&1\\\hline\end{array} \in \check{Y}_{\triangleright(1,1,1)}$ yields the U_q^τ-module in Figure 12, hence this is a U_q^τ-submodule of $\check{Y}_{\triangleright(1,1,1)}$.

The statement for Figure 13 has a similar proof to that for Figure 12 because $\mathrm{Res}_{U_q(\mathfrak{sl}(V)\oplus\mathfrak{sl}(W))}\check{\Lambda}^3\check{X} \cong \mathrm{Res}_{U_q(\mathfrak{sl}(V)\oplus\mathfrak{sl}(W))}\check{\Lambda}^1\check{X}$.

Let us show the statement for Figure 14 for $t = 1$, the general case being similar. The $t = 1$ case follows from

$$-\frac{1}{[2]}\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 3&4&3\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\begin{array}{|c|}\hline 1\\\hline\end{array} \equiv \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 3&3&3\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\begin{array}{|c|}\hline 1\\\hline\end{array} \equiv -\begin{array}{|c|c|c|}\hline 2&3&3\\\hline 3&2&2\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\begin{array}{|c|}\hline 1\\\hline\end{array} \equiv \frac{1}{[2]}\begin{array}{|c|c|c|}\hline 2&3&1\\\hline 3&4&2\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\begin{array}{|c|}\hline 1\\\hline\end{array} \equiv \frac{1}{[2]}\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 3&4&3\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\begin{array}{|c|}\hline 1\\\hline\end{array}, \tag{145}$$

where the equivalences are mod $\check{Y}_{\triangleright(3,2,2,1)}$, the first and third equivalence are by Corollary 15.14, the second is by moves defined by Figures 5 and 10, and the fourth by a move defined by Figure 8. Here we are implicitly using Proposition 15.11 (i). □

15.7. **A basis for the two-row Kronecker problem.** After two preliminary lemmas, we state the technical version of our main theorem about the two-row Kronecker problem. We give the main body of the proof in this subsection, although it depends on combinatorics and a detailed case-by-case analysis given in future sections. The reader may want to postpone a careful reading of the lemmas until seeing how they are applied in the proof.

Lemma 15.17. *Let B be a basis of a module N in $\mathscr{O}_{int}^{\geq 0}(\mathfrak{gl}_2)$ and G a digraph with vertex set B that is a disjoint union of directed paths. Let $\tilde{F}_*$, (resp. $\tilde{E}_*$) be the function from B to $B \sqcup \{0\}$ that takes $b \in B$ to the vertex obtained by following the edge leaving (resp. going to) b if it exists and to 0 otherwise. Define functions $\varphi, \varepsilon : B \to \mathbb{Z}_{\geq 0}$ by*

$$\varphi(b) := \max\{i : \tilde{F}_*^i(b) \neq 0\}, \qquad \varepsilon(b) := \max\{i : \tilde{E}_*^i(b) \neq 0\}.$$

Suppose that

$$\begin{aligned} F(b) &= [\varphi(b)]\tilde{F}_*(b) + \sum_{\substack{b' \in B, \\ \varphi(b')+\varepsilon(b')<\varphi(b)+\varepsilon(b)}} a^-_{b'b} b', \quad a^-_{b'b} \in \mathbb{Q}[q,q^{-1}],\ \overline{a^-_{b'b}} = a^-_{b'b},\ \deg(a^-_{b'b}) < \varphi(b), \\ E(b) &= [\varepsilon(b)]\tilde{E}_*(b) + \sum_{\substack{b' \in B, \\ \varphi(b')+\varepsilon(b')<\varphi(b)+\varepsilon(b)}} a^+_{b'b} b', \quad a^+_{b'b} \in \mathbb{Q}[q,q^{-1}],\ \overline{a^+_{b'b}} = a^+_{b'b},\ \deg(a^+_{b'b}) < \varepsilon(b), \end{aligned} \tag{146}$$

for any $b \in B$. Then the pair (N, B) satisfies (c) and (d) of Definition 5.3.

Proof. The $\bar{\cdot}$-invariance of $a^-_{b'b}, a^+_{b'b}$ easily implies (c).

Next, note that it follows from the form of (146) that N is filtered by the submodules $N_{\leq k} := K\{b \in B : \varphi(b) + \varepsilon(b) \leq k\}$ and the subquotient $N_{\leq k}/N_{\leq k-1}$ is isomorphic to $\bigoplus_{\lambda_1 - \lambda_2 = k} N[\lambda]$. Set $\pi_k^N = \sum_{\lambda_1-\lambda_2=k} \pi_\lambda^N$. Recall that $N[\lambda]$ is the V_λ-isotypic component of N and $\pi_\lambda^N : N \to N$ is the projector with image $N[\lambda]$.

To prove that (N, B) satisfies (d), let $\mathscr{L}(N) = K_\infty B$ as in Definition 5.3. To show that $\mathscr{L}(N)$ is an upper crystal lattice at $q = \infty$, we first show that for any $b \in B$, $\tilde{F}^{\mathrm{up}}(\pi_k^N(b)) \in \mathscr{L}(N)$ with $k = \varphi(b) + \varepsilon(b)$. Using the filtration of N just mentioned, we see that

$$\pi_k^N(b) = \frac{[k-j]!F^j}{[k]!}\frac{E^j}{[j]!}\, b \equiv b, \tag{147}$$

where $j := \varepsilon(b)$, and the equivalence is mod $q^{-1}\mathscr{L}(N)$ and is proved using (146) to evaluate the middle expression. Hence

$$\tilde{F}^{\mathrm{up}}(\pi_k^N(b)) = \pi_k^N(\tilde{F}_*(b)) \equiv \tilde{F}_*(b) \in \mathscr{L}(N), \tag{148}$$

where the equivalence is mod $q^{-1}\mathscr{L}(N)$. To show that $\tilde{F}^{\mathrm{up}}(\mathscr{L}(N)) \subseteq \mathscr{L}(N)$ we must also show that $\tilde{F}^{\mathrm{up}}(b - \pi_k^N(b)) \in \mathscr{L}(N)$. But this follows from what was just shown applied inductively, as $b - \pi_k^N(b) \in q^{-1}\mathscr{L}(N) \cap N_{\leq k-1}$. The proof that $\tilde{E}^{\mathrm{up}}(\mathscr{L}(N)) \subseteq \mathscr{L}(N)$ is similar. Finally, it follows from (147) and (148) and similar statements for E in place of F that $(\mathscr{L}(N), \mathscr{B})$ is an upper crystal basis at $q = \infty$ (where $\mathscr{B}$ is the image of B in $\mathscr{L}(N)/q^{-1}\mathscr{L}(N)$). □

For the statement of Theorem 15.21, the integral forms needed to define specializations at $q = 1$ must be chosen carefully (recall that if N is a K-module and $N_{\mathbf{A}}$ is an $\mathbf{A}$-submodule of N that is understood from context, then $N|_{q=1}$ is defined to be $\mathbb{Q} \otimes_{\mathbf{A}} N_{\mathbf{A}}$, the map $\mathbf{A} \to \mathbb{Q}$ given by $q \mapsto 1$). Define the following integral forms and basis of $\check{Y}_{\triangleright^i \nu'}$:

$$\begin{array}{rcl}
\check{Y}^{S\mathbf{A}}_{\nu'} & := & \mathbf{A}\mathrm{SNST}(\nu'), \\
\mathrm{SNST}(\triangleright^i \nu') & := & \mathbf{A}\Big\{(-\tfrac{1}{[2]})^{\deg(\tilde{T})}\tilde{T} : \tilde{T} \in \mathrm{NST}(\beta, \triangleright\gamma, \delta)\Big\}, \text{ where } \nu' = (\beta, \gamma, \delta) \\
 & & \text{as in §15.5 and } \gamma = (\nu'_i, \nu'_{i+1}), \\
\check{Y}^{S\mathbf{A}}_{\triangleright^i \nu'} & := & \mathbf{A}\mathrm{SNST}(\triangleright^i \nu'), \\
\check{Y}^{S\mathbf{A}}_{\triangleright \nu'} & := & \sum_{i=1}^{l-1} \check{Y}^{S\mathbf{A}}_{\triangleright^i \nu'}, \\
\check{X}'^{\mathbf{A}}_{\nu} & := & \check{Y}^{S\mathbf{A}}_{\nu'} / \check{Y}^{S\mathbf{A}}_{\triangleright \nu'}, \\
\check{X}^{\mathbf{A}}_{\nu} & := & \check{Y}^{S\mathbf{A}}_{\nu'} / (\check{Y}_{\triangleright \nu'} \cap \check{Y}^{S\mathbf{A}}_{\nu'}) = \varpi(\check{Y}^{S\mathbf{A}}_{\nu'}) = \mathbf{A}\mathrm{NSTC}(\nu) = \mathbf{A}{+}\mathrm{HNSTC}(\nu).
\end{array} \tag{149}$$

For the second line, note that $\deg(x)$ is defined for any $x \in \check{Y}_{\nu'}$ in §15.1. On the last line, the second equality follows from Proposition 15.11 (i); the third equality holds because those $\mathrm{NSTC}(\nu)$ that are dishonest because of an integral move are 0 in $\check{X}_\nu$ by Proposition 15.11 (ii), and those that are dishonest because of a nonintegral move lie in $\mathbf{A}{+}\mathrm{HNSTC}(\nu)$ by Corollary 15.14 and Proposition 15.7 (b).

Remark 15.18. It is true that $\check{X}'^{\mathbf{A}}_{\nu} = \check{X}^{\mathbf{A}}_{\nu}$, however this is not yet clear, and in fact, this can fail outside the two-row case. We have that $\check{Y}^{S\mathbf{A}}_{\triangleright \nu'} \subseteq (\check{Y}_{\triangleright \nu'} \cap \check{Y}^{S\mathbf{A}}_{\nu'})$. Hence there is a canonical surjection $s_X : \check{X}'^{\mathbf{A}}_{\nu} \twoheadrightarrow \check{X}^{\mathbf{A}}_{\nu}$. This is all we can say in general, without the $d_V = d_W = 2$ assumption.

It will be shown in Theorem 15.21 that $\check{X}^{\mathbf{A}}_{\nu}$ has $\mathbf{A}$-basis $+\mathrm{HNSTC}(\nu)$ and that this basis has the correct size $|\mathrm{SSYT}_{d_X}(\nu)|$. Thus there are no relations amongst $+\mathrm{HSNST}(\nu')$ not already accounted for by Proposition 15.11. Also, from the proof of Proposition 15.3, we see that all the identifications of $+\mathrm{HSNST}(\nu')$ made to get $+\mathrm{HNSTC}(\nu)$ actually belong to $\check{Y}^{S\mathbf{A}}_{\triangleright \nu'}$. Thus the surjection s_X is actually an equality. However, we do not need this since once we know $+\mathrm{HNSTC}(\nu)$ is an $\mathbf{A}$-basis of $\check{X}^{\mathbf{A}}_{\nu}$, we can just use this integral form of $\check{X}_\nu$ and forget about $\check{X}'^{\mathbf{A}}_{\nu}$.

The next lemma roughly says that the quotient of a vector space with basis by relations that only involve two basis elements is easily understood in terms of the components of a graph.

Lemma 15.19. *Let Y be a K-vector space with basis $B = \{b_1, \dots, b_s\}$ and set $Y_{\mathbf{A}} = \mathbf{A}B$. Let $r_i = c_i b_{j_i} + c'_i b_{j'_i}$, $i \in [t]$, $j_i, j'_i \in [s]$ be elements of $Y_{\mathbf{A}}$ with $c_i, c'_i \in \mathbf{A}$ and $c_i \neq 0$; let M (resp. $M_{\mathbf{A}}$) be the K-submodule (resp. $\mathbf{A}$-submodule) of Y spanned by $r_1, \dots, r_t$. Let X (resp. $X_{\mathbf{A}}$) be the quotient Y/M (resp. $Y_{\mathbf{A}}/M_{\mathbf{A}}$). Suppose the c_i, c'_i have no poles or zeros at $q = 1$ except for those c'_i that are 0, and, for any well-defined product*

$$p = \prod_{j \in S} \frac{c_j}{c'_j} \prod_{j \in S'} \frac{c'_j}{c_j}, \qquad S \sqcup S' \subseteq [t],$$

$p = 1$ if and only if $p|_{q=1} = 1$. Then $\dim_K X = \dim_{\mathbb{Q}} X|_{q=1}$ and the exact sequence

$$0 \to M_{\mathbf{A}} \to Y_{\mathbf{A}} \to X_{\mathbf{A}} \to 0$$

remains exact after tensoring with $\mathbb{Q}$, the map $\mathbf{A} \to \mathbb{Q}$ given by $q \mapsto 1$.

Proof. Let G be the weighted digraph with vertex set $B \sqcup \{0\}$ and, for each $i \in [t]$,

$$\begin{cases} \text{a directed edge } b_{j_i} \to b_{j'_i} \text{ with weight } \frac{c'_i}{c_i} \text{ and} & \\ \text{a directed edge } b_{j'_i} \to b_{j_i} \text{ with weight } \frac{c_i}{c'_i} & \text{if } c'_i \neq 0, \\ \text{a directed edge } b_{j_i} \to 0 \text{ with weight } c_i & \text{if } c'_i = 0. \end{cases}$$

Let $G|_{q=1}$ be the same digraph with edge weights evaluated at $q = 1$ (by the hypotheses of the lemma, these evaluations are well-defined elements of $\mathbb{Q} \setminus \{0\}$).

We say that a component C of the underlying undirected graph of G (or of $G|_{q=1}$) is *honest* if $0 \notin V(C)$, and, for every directed cycle in G with vertex contained in $V(C)$, the product of its edge weights is 1. Then $\dim_K X$ (resp. $\dim_{\mathbb{Q}} X|_{q=1}$) is the number of honest components of G (resp. $G|_{q=1}$). The hypotheses of the lemma certainly imply that G and $G|_{q=1}$ have the same number of honest components.

To prove the statement about exactness, first observe that applying the functor $\mathbb{Q} \otimes_{\mathbf{A}} \cdot$ to the exact sequence above is the same as first applying $\mathbb{Q} \otimes_{\mathbb{Z}} \cdot$ and then applying $\mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} \cdot$. Applying $\mathbb{Q} \otimes_{\mathbb{Z}} \cdot$ yields the sequence

$$0 \to \mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathbf{A}} \to \mathbb{Q} \otimes_{\mathbb{Z}} Y_{\mathbf{A}} \to \mathbb{Q} \otimes_{\mathbb{Z}} X_{\mathbf{A}} \to 0,$$

which is exact because localizations are flat. Since torsion-free $\mathbb{Q}[q, q^{-1}]$-modules are free, $\mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathbf{A}}$ is a free $\mathbb{Q}[q, q^{-1}]$-module, hence $\dim_{\mathbb{Q}} M|_{q=1} = \operatorname{rank}_{\mathbb{Q}[q,q^{-1}]} \mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathbf{A}} = \dim_K M$. We also have $\dim_{\mathbb{Q}} Y|_{q=1} = \dim_K Y$ and we just proved $\dim_K X = \dim_{\mathbb{Q}} X|_{q=1}$. Since

$$0 \to \mathbb{Q} \otimes_{\mathbf{A}} M_{\mathbf{A}} \to \mathbb{Q} \otimes_{\mathbf{A}} Y_{\mathbf{A}} \to \mathbb{Q} \otimes_{\mathbf{A}} X_{\mathbf{A}} \to 0$$

is right exact, it is exact by the dimension count

$$\dim_{\mathbb{Q}} M|_{q=1} = \dim_K M = \dim_K Y - \dim_K X = \dim_{\mathbb{Q}} Y|_{q=1} - \dim_{\mathbb{Q}} X|_{q=1}.$$

□

Definition 15.20. A $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal component (resp. -cell) $\mathscr{G}$ of $(\check{X}_\nu, +\text{HNSTC}(\nu))$ is a (λ, μ)*-crystal component (resp. cell) of +HNSTC(ν)* if its highest weight +HNSTC has weight (λ, μ). This means that if $\mathscr{G}$ is drawn as in Figure 16, then it occupies a grid of width $\lambda_1 - \lambda_2 + 1$ and height $\mu_1 - \mu_2 + 1$.

A U_q^τ-cell $\mathscr{G}$ of $(\check{X}_\nu, +\text{HNSTC}(\nu))$ is a $\{\lambda, \mu\}$*-cell (resp. $\varepsilon\lambda$-cell) of +HNSTC(ν)* if the corresponding cellular subquotient is isomorphic to $\mathcal{X}_{\{\lambda,\mu\}}$ (resp. $\mathcal{X}_{\varepsilon\lambda}$) (see §13.3 for notation).

A crystal component or cell of +HNSTC(ν) is *invariant-free* if every (equivalently, any) +HNSTC it contains is invariant-free.

We now state our main result on the two-row Kronecker problem. This is a stronger and more technical version of Theorem 1.6. We give most of the proof now, although it depends on several results that we prove later. One is Corollary 16.5 (see §16.2), in which we use a counting argument to show that $|{+}\mathrm{HNSTC}(\nu)| = |\mathrm{SSYT}_{d_X}(\nu)|$. This, together with what we have proved so far, is enough to obtain a combinatorial formula for two-row Kronecker coefficients. The stronger statements (v) and (vi) require a more detailed combinatorial understanding of the relations satisfied by $\varpi_\nu(\mathrm{NST}(\nu'))$ and the rather involved case-by-case analysis of the $\mathscr{F}_{(j)V}(T)$ given in §17.2.

Theorem 15.21. *The pair $(+HNSTC(\nu), \check{X}_\nu)$ yields a "crystal basis-theoretic" solution to the two-row Kronecker problem. Precisely, we prove the following.*

(i) $\dim_K \check{X}_\nu = \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbf{A}} \check{X}'^{\mathbf{A}}_\nu$.

(ii) $\mathbb{Q} \otimes_{\mathbf{A}} \check{X}'^{\mathbf{A}}_\nu \cong \mathrm{Res}_{U^\tau}(X_\nu|_{q=1})$ *as U^τ-modules (where $U^\tau := U(\mathfrak{gl}_2) \wr \mathcal{S}_2$).*

(iii) *The set $+HNSTC(\nu)_{\geq h}$ is an $\mathbf{A}$-basis of $(\check{X}^{\mathbf{A}}_\nu)_h$. In particular, $+HNSTC(\nu)$ is an $\mathbf{A}$-basis of $\check{X}^{\mathbf{A}}_\nu$.*

(iv) *The pair $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+HNSTC(\nu)))$ is an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module.*

(v) *The pair $(\check{X}_\nu, +HNSTC(\nu))$ is an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module.*

(vi) *The upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+HNSTC(\nu)))$ and $(\check{X}_\nu, +HNSTC(\nu))$ are upper based U_q^τ-modules.*

(vii) *The Kronecker coefficient $g_{\lambda\mu\nu}$ is the number of highest weight $+HNSTC$ of shape ν and weight (λ, μ), or equivalently, the number of (λ, μ)-crystal components of $+HNSTC(\nu)$.*

(viii) *The symmetric or exterior Kronecker coefficient $g_{\varepsilon\,\lambda\nu}$ is the number of highest weight $+HNSTC$ $\mathbf{T}$ of shape ν and weight (λ, λ) such that $\tau\mathbf{T} = \varepsilon\mathbf{T}$, or equivalently, the number of $\varepsilon\lambda$-cells of $+HNSTC(\nu)$. Moreover, the condition $\tau\mathbf{T} = \varepsilon\mathbf{T}$ is equivalent to $(-1)^{\nu_3 + \cup^{3\text{-}2}(\mathbf{T})} = \varepsilon$ (the notation $\cup^{3\text{-}2}$ is introduced after Definition 17.1).*

Remark 15.22.

(1) It follows from (iii) that the degree of a +HNSTC as an element of the filtered module $\check{X}_\nu$ is the same as its degree defined after Proposition 15.7 (the degree of a dishonest NSTC as an element of the filtered $\check{X}_\nu$ is more difficult to determine). Thus $\bigsqcup_{h\geq 0} \mathrm{in}_h(+\mathrm{HNSTC}(\nu)_h)$ and $\mathrm{in}(+\mathrm{HNSTC}(\nu))$ are identical, so we can safely use the latter as a shorthand for the former.

(2) The $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cells of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+\mathrm{HNSTC}(\nu)))$ and $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$ coincide under the bijection $\mathrm{in}(+\mathrm{HNSTC}(\nu)) \cong +\mathrm{HNSTC}(\nu)$. The same goes for U_q^τ-cells. The $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cells of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+\mathrm{HNSTC}(\nu)))$ and $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$ also coincide with their crystal components (this is always true for upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules—see §5.3). We give a fairly explicit description of these cells in Corollaries 17.6 and 17.10 after we have a better combinatorial understanding of +HNSTC.

(3) It follows from §13.6 that the $\check{Y}_{\triangleright\gamma}$ for $\ell(\gamma) = 2$ are $\mathscr{O}(GL_q(\check{X}))$-comodules, hence $\check{X}_\nu$ is a $\mathscr{O}(GL_q(\check{X}))$-comodule. It also follows from (vi) and §13.6 that the U_q^τ-cells and $K\check{\mathscr{S}}(\check{X}, r)$-cells of $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$ are the same except that $(\check{X}_{(r)}, +\mathrm{HNSTC}((r)))$ is a $K\check{\mathscr{S}}(\check{X}, r)$-cell and the union of $\lfloor \frac{r}{2} \rfloor + 1$ U_q^τ-cells (recall

that $K\check{\mathscr{S}}(\check{X}, r)$ is the algebra dual to the coalgebra $\mathscr{O}(M_q(\check{X}))_r)$. See the comment after Corollary 17.10 for more details.

(4) The pairs $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+\mathrm{HNSTC}(\nu)))$ and $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$ are not isomorphic as upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules.

(5) The proof of (iv) and (v) uses Lemma 15.17, which uses information about a basis to deduce the existence of a (local) crystal basis. This may seem backward since one usually first proves the existence of a (local) crystal basis to deduce the existence of a global crystal basis. However, because we have explicit formulae for the action of the Chevalley generators on NST in the two-row case (Proposition 14.21), our proof is essentially the same as, and perhaps somewhat more convenient than, one that first proves the existence of a (local) crystal basis.

(6) The proof of (v) might be easier if we could construct a basis for all of $\check{Y}_{\rhd\nu'}$ in which $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$ occurs as a cellular quotient (see Conjecture 19.1).

Most of the proof of Theorem 15.21.

Statement (i): by the definitions (149), we have the exact sequence of $\mathbf{A}$-modules

$$0 \to \check{Y}^{S\mathbf{A}}_{\rhd\nu'} \to \check{Y}^{S\mathbf{A}}_{\nu'} \to \check{X}'^{\mathbf{A}}_{\nu} \to 0. \tag{150}$$

Since localizations are flat, tensoring with K yields the exact sequence

$$0 \to \check{Y}_{\rhd\nu'} \to \check{Y}_{\nu'} \to \check{X}_{\nu} \to 0.$$

Then Lemma 15.19 applied with $Y = \check{Y}_{\nu'}$, $B = \mathrm{SNST}(\nu')$, $M = \check{Y}_{\rhd\nu'}$, and $\{r_i : i \in [t]\} = \bigsqcup_{i\in[l-1]} \mathrm{SNST}(\rhd^i\nu')$ yields the desired $\dim_K \check{X}_\nu = \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbf{A}} \check{X}'^{\mathbf{A}}_{\nu}$. Here we have used the top case of (142) of Theorem 15.10 and Corollary 15.14 to show that any product p of Lemma 15.19 is $\pm[2]^k$, $k \in \mathbb{Z}$, so the hypotheses of the lemma are satisfied.

Statement (ii): the above application of Lemma 15.19 also implies that (150) remains exact after tensoring with $\mathbb{Q}$ (the map $\mathbf{A} \to \mathbb{Q}$ given by $q \mapsto 1$), i.e., $\mathbb{Q} \otimes_{\mathbf{A}} \check{X}'^{\mathbf{A}}_{\nu} \cong \check{Y}_{\nu'}|_{q=1}/\check{Y}_{\rhd\nu'}|_{q=1}$. This quotient is isomorphic to $\mathrm{Res}_{U^\tau}(X_\nu|_{q=1})$: this is true in the $\ell(\nu') \le 2$ case because the $\check{Y}_{\rhd\nu'}$ were defined to make this true; also note that the decomposition of $\check{Y}_{\rhd\nu'}|_{q=1}$ into U^τ-modules is multiplicity free for $\ell(\nu') = 2$; the case $\ell(\nu') > 2$ then follows because the fact $\check{Y}_{\rhd\nu'}|_{q=1} = \sum_{i=1}^{l-1} \check{Y}_{\rhd^i\nu'}|_{q=1}$ implies that $\check{Y}_{\nu'}|_{q=1}/\check{Y}_{\rhd\nu'}|_{q=1}$ matches the definition of the Schur functor $L_{\nu'}X$ given in [61, Chapter 2] (see §1.6).

Statement (iii): Proposition 15.11 (v) implies the left-hand inequality, (i) and (ii) imply the middle equality, and Proposition 16.4 implies the right-hand inequality of

$$|+\mathrm{HNSTC}(\nu)| \ge \dim_K \check{X}_\nu = |\mathrm{SSYT}_{d_X}(\nu)| \ge |+\mathrm{HNSTC}(\nu)|,$$

hence equality must hold throughout. Thus $+\mathrm{HNSTC}(\nu)$ is a basis of $\check{X}_\nu$, and, since any nontrivial relation with coefficients in $\mathbf{A}$ satisfied by the $+\mathrm{HNSTC}(\nu)$ is also a relation over K, we conclude that $+\mathrm{HNSTC}(\nu)$ is an $\mathbf{A}$-basis of $\check{X}^{\mathbf{A}}_\nu$. Now this further implies by Proposition 15.11 (iv) and (v) that $+\mathrm{HNSTC}(\nu)_{\ge h}$ is an $\mathbf{A}$-basis for $(\check{X}^{\mathbf{A}}_\nu)_h$ (where $(\check{X}^{\mathbf{A}}_\nu)_h := \check{X}^{\mathbf{A}}_\nu \cap (\check{X}_\nu)_h)$ and $\mathrm{in}_h(+\mathrm{HNSTC}(\nu)_h)$ is an $\mathbf{A}$-basis for $\mathrm{gr}(\check{X}^{\mathbf{A}}_\nu)_h$.

Statement (iv): we need to check conditions (a)–(d) of Definition 5.3 for each pair $P_h := (\mathrm{gr}(\check{X}_\nu)_h, \mathrm{in}_h(+\mathrm{HNSTC}(\nu)_h))$. First of all, we have seen in Proposition-Definition 15.2 that $(\mathrm{gr}(\check{Y}_{\nu'})_h, \mathrm{in}_h(\mathrm{NST}(\nu')_h))$ is a weak upper based U_q^τ-module. Scaling this basis by the factor $(-\frac{1}{[2]})^h$ yields the weak upper based U_q^τ-module $(\mathrm{gr}(\check{Y}_{\nu'})_h, \mathrm{in}_h(+\mathrm{HSNST}(\nu')_h))$. By Proposition 15.11 (i) and (iii), the pair $(\mathrm{gr}(\check{X}_\nu)_h, \mathrm{in}_h(+\mathrm{HNSTC}(\nu)_h))$ is then obtained from this one by quotienting by $\mathrm{gr}(\check{Y}_{\triangleright\nu'})_h$, which amounts to identifying $+\mathrm{HSNST}(\nu')$ in the same strong component of $\mathcal{TG}(\nu)$ and getting rid of those $+\mathrm{HSNST}(\nu')$ that become 0, which are exactly the dishonest ones. Given this, condition (a) is clear and condition (b) for P_h is easy to check from condition (b) for $(\mathrm{gr}(\check{Y}_{\nu'})_h, \mathrm{in}_h(\mathrm{NST}(\nu')_h))$.

To prove that (c) and (d) hold, we apply Lemma 15.17 (we apply the lemma to show that (d) holds for P_h as a $U_q(\mathfrak{g}_V)$-module with basis; that (d) holds for P_h as a $U_q(\mathfrak{g}_W)$-module with basis is similar): first note that

(151) if $T \to T'$ is a degree-preserving move at $[i, i+t-1]$, then the unpaired V-diagram of T is the same as that of T'. Moreover, the V-diagram of T and T' are identical outside columns i through $i+t-1$.

Then, for $\mathbf{T} \in +\mathrm{HNSTC}(\nu)_h$, define $\tilde{F}_*(\mathbf{T})$ to be the $+\mathrm{HNSTC}$ containing $\mathscr{F}_{(\varphi_V(T))V}(T)$ for any $T \in \mathbf{T}$. To show that this is well-defined we must show that if $T \to T'$ is a degree-preserving move at $[i, i+t-1]$, then $\mathscr{F}_{(\varphi_V(T))V}(T) \to \mathscr{F}_{(\varphi_V(T))V}(T')$ is a degree-preserving move. Given (151), this is evident from Figures 5, 7, 8, 10, 12, and 13 if T and $\mathscr{F}_{(\varphi_V(T))V}(T)$ differ in the j-th column and $j \in [i, i+t-1]$, and is clear if $j \notin [i, i+t-1]$. The graph G and $\tilde{E}_*$ of the lemma are then determined uniquely so that the conditions of the lemma are satisfied. The conditions in (146) on the coefficients $a^-_{b'b}$ and $a^+_{b'b}$ hold by Proposition 14.21 and Proposition 15.11 (i), (iii).

Statement (v): the proof is similar to that of (iv). The following modifications will suffice: to show condition (b) for the pair $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$, we need check that the structure coefficients for the action of $F_V^{(m)}, F_W^{(m)}, E_V^{(m)}, E_W^{(m)}$ in the basis $+\mathrm{HNSTC}(\nu)$ lie in $\mathbf{A}$. This follows from condition (b) for $(\check{Y}_{\nu'}, \mathrm{NST}(\nu'))$, Corollary 15.14 together with Proposition 15.7 (b), and Proposition 15.11 (i), (ii). This is easy because scaling the elements of $\mathrm{NST}(\nu')_h$ by the factor $(-\frac{1}{[2]})^h$ only makes it easier to have the structure coefficients lie in $\mathbf{A}$. The main difficulty is showing that the degree bounds on $a^-_{b'b}$ and $a^+_{b'b}$ given in (146) hold. These follow from Theorem 17.7 (see §17.2).

Statement (vi): the given upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules are weak upper based U_q^τ-modules since $(\check{Y}_{\nu'}, \mathrm{NST}(\nu'))$ is a weak upper based U_q^τ-module. That they are in fact upper based U_q^τ-modules is the contents of Proposition 17.9.

Statement (vii): this follows from (ii), (v), and the general facts about upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules in (49).

Statement (viii): this follow from (ii), (vi), and the general facts about upper based U_q^τ-modules in §13.5. □

16. Straightened NST and semistandard tableaux

In this section we study the equivalence classes +HNSTC of +HSNST combinatorially. We define a lexicographic order on +HSNST and give explicit necessary conditions on the columns of a +HSNST for it to be the smallest +HSNST in its class—we call a

+HSNST and its corresponding NST satisfying these conditions *straightened*. We then exhibit a bijection between straightened NST of shape ν' and $\mathrm{SSYT}_{d_X}(\nu)$, completing the proof of Theorem 15.21 (iii) and (iv) and showing that these necessary conditions are also sufficient.

16.1. **Lexicographic order on NST.** We define a total order $<$ on nonstandard columns, which is very similar to the order that columns in a semistandard tableau must satisfy:

$$\boxed{\begin{matrix}1\\2\\3\\4\end{matrix}} < \boxed{\begin{matrix}1\\2\\3\end{matrix}} < \boxed{\begin{matrix}1\\2\\4\end{matrix}} < \boxed{\begin{matrix}1\\3\\4\end{matrix}} < \boxed{\begin{matrix}2\\3\\4\end{matrix}} < \boxed{\begin{matrix}1\\2\end{matrix}} < \boxed{\begin{matrix}1\\3\end{matrix}} < \boxed{\begin{matrix}2\\3\end{matrix}} < \boxed{\begin{matrix}3\\2\end{matrix}} < \boxed{\begin{matrix}2\\4\end{matrix}} < \boxed{\begin{matrix}3\\4\end{matrix}} < \boxed{1} < \boxed{2} < \boxed{3} < \boxed{4}. \qquad (152)$$

We consider the lexicographic order on $+\mathrm{HSNST}(\nu')$ induced from this order: for $T, T' \in +\mathrm{HSNST}(\nu')$, $T < T'$ if $T|_j < T'|_j$ and $T|_{[j-1]} = T'|_{[j-1]}$, $j \in [l]$. We also consider this order on $\mathrm{NST}(\nu')$.

A nonstandard column of height 2 is of type V (resp. W) if it does not contain an internal V-arc (resp. W-arc):

$$\underbrace{\boxed{\begin{matrix}1\\2\end{matrix}}\quad\boxed{\begin{matrix}3\\2\end{matrix}}\quad\boxed{\begin{matrix}3\\4\end{matrix}}}_{\text{type } V}\qquad\underbrace{\boxed{\begin{matrix}1\\3\end{matrix}}\quad\boxed{\begin{matrix}2\\3\end{matrix}}\quad\boxed{\begin{matrix}2\\4\end{matrix}}}_{\text{type } W}.$$

We will occasionally write height-3 columns in a more compact form as

$$\boxed{1^\vee} := \boxed{\begin{matrix}1\\2\\3\end{matrix}},\ \boxed{2^\vee} := \boxed{\begin{matrix}1\\2\\4\end{matrix}},\ \boxed{3^\vee} := \boxed{\begin{matrix}1\\3\\4\end{matrix}},\ \boxed{4^\vee} := \boxed{\begin{matrix}2\\3\\4\end{matrix}}.$$

For an NST or SNST T, we sometimes use m with various subscripts to denote the number of columns of each type comprising T, i.e. m_1 is the number of columns of T equal to $\boxed{1}$, $m_{\substack{2\\3}}$ is the number of columns of T equal to $\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}$, etc.

We begin with a basic result about the order and number of columns in an honest SNST.

Proposition 16.1. *Let T be an honest SNST and let m_1, m_2, etc. denote the number of columns of T of each type, as above.*

(i) *If two honest SNST differ by moving some type V columns past type W columns, then they are equivalent.*

(ii) *If two honest SNST differ by moving a height-2 invariant past some columns of height 2, then they are equivalent.*

(iii) *The type V columns of the invariant-free part of T are weakly increasing; $m_{\substack{3\\2}} \leq 1$.*

(iv) *The type W columns of the invariant-free part of T are weakly increasing; $m_{\substack{2\\3}} \leq 1$.*

(v) *The height-1 columns of the invariant-free part of T are weakly increasing; at least one of m_2 and m_3 is 0.*

(vi) *The height-3 columns of the invariant-free part of T are weakly increasing; at least one of $m_{2^\vee}$ and $m_{3^\vee}$ is 0.*

Proof. Statement (i) follows from the moves defined by Figure 7. Statement (ii) follows from (i) and the moves defined by Figure 8 (which simply correspond to replacing a contiguous subtabloid equal to a height-2 invariant with the other height-2 invariant).

Given (i) and (ii), T is equivalent to a scaled nonstandard tabloid T' satisfying: (1) the type V columns of T' that remain in its invariant-free part form a contiguous subtabloid, (2) all of its height-2 invariant column pairs are contiguous and occur at the end of its subtabloid of height-2 columns, and (3) the type V columns of its invariant-free part are in the same order as the type V columns of the invariant-free part of T. Since T is honest, so is T' and then (iii) is immediate from the moves defined by Figure 6. The proof of (iv) is similar.

By the moves defined by Figure 4, T is honest implies T does not contain a contiguous subtabloid $\boxed{i}\boxed{j}$ with $i > j$ unless $\boxed{i}\boxed{j} = \boxed{4}\boxed{1}$. By the invariant moves defined by Figure 12, T is equivalent to a +HSNST whose height-1 columns are of the form

$$\boxed{1}\cdots\boxed{1}\boxed{2}\cdots\boxed{2}\boxed{3}\cdots\boxed{3}\boxed{4}\cdots\boxed{4}\boxed{4}\boxed{1}\cdots\boxed{4}\boxed{1}.$$

Also, by the moves defined by Figure 4, T does not contain $\boxed{2}\boxed{3}$ or $\boxed{3}\boxed{2}$ as a contiguous subtabloid, so at least one of m_2, m_3 is zero. This proves (v). The proof of (vi) is similar using the moves defined by Figure 11. □

Proposition 16.2. *Let $\mathbf{T}$ be an invariant-free honest NSTC of shape $\nu' \vdash r$ and T the lexicographically minimum SNST in this class. Then*

(0.1) *the height-j columns of T are weakly increasing for all $j \in [d_X]$,*
(1.1) *at least one of m_2 and m_3 is zero,*
(1.2) $m_1 > 0$ *implies* $m_{\substack{3\\4}} = m_{\substack{2\\4}} = 0$,
(1.3) $m_2 > 0$ *implies* $m_{\substack{3\\4}} = 0$,
(1.4) $m_3 > 1$ *implies* $m_{\substack{2\\4}} = 0$,
(1.5) $m_1 > 0$ *implies* $m_{\substack{3\\2}} = 0$,
(2.1) $m_{\substack{2\\3}} \le 1$ *and* $m_{\substack{3\\2}} \le 1$,
(3.1) *at least one of $m_{2^\vee}$ and $m_{3^\vee}$ is zero,*
(3.2) $m_{4^\vee} > 0$ *implies* $m_{\substack{1\\2}} = m_{\substack{1\\3}} = 0$,
(3.3) $m_{3^\vee} > 0$ *implies* $m_{\substack{1\\2}} = 0$,
(3.4) $m_{2^\vee} > 1$ *implies* $m_{\substack{1\\3}} = 0$,
(3.5) $m_{4^\vee} > 0$ *implies* $m_{\substack{3\\2}} = 0$,
(4.1) *at least one of $m_{4^\vee}$ and m_1 is zero.*

Proof. Statements (0.1), (1.1), and (3.1) follow from Proposition 16.1 and the invariant-free assumption. Statement (2.1) is a restatement of parts of Proposition 16.1 (iii) and (iv). Given (0.1), statements (3.2) and (3.3) are by the moves defined by Figure 10.

To prove (3.4), suppose the contrary, that $m_{2^\vee} > 1$ and $m_{\substack{1\\3}} > 0$. Given (0.1), (3.1), and (3.2), we can apply the move

$$\cdots\begin{array}{|c|c|c|}\hline 1&1&1\\ 2&2&3\\ \cline{3-3} 4&4\\ \cline{1-2}\end{array}\cdots \longleftrightarrow - \cdots\begin{array}{|c|c|c|}\hline 1&1&1\\ 2&3&2\\ \cline{3-3} 4&4\\ \cline{1-2}\end{array}\cdots$$

to T, which, by Proposition 16.1 (vi), contradicts that T is honest.

If $m_{4^\vee} > 0$ and $m_{\substack{2\\3}} > 0$ and $m_{\substack{3\\2}} > 0$, then by (3.2) and (0.1) we can apply the sequence of moves

$$\cdots\begin{array}{|c|c|c|}\hline 2&2&3\\ 3&3&2\\ \cline{2-3} 4\\ \cline{1-1}\end{array}\cdots \longleftrightarrow - \cdots\begin{array}{|c|c|c|}\hline 2&3&3\\ 3&2&2\\ \cline{2-3} 4\\ \cline{1-1}\end{array}\cdots \longrightarrow 0,$$

which contradicts that $\mathbf{T}$ is honest. This proves (3.5).

Statements (1.1)–(1.5) have proofs similar to (3.1)–(3.5), using Figure 5 instead of Figure 10 and Proposition 16.1 (v) instead of (vi).

Finally, for statement (4.1), assume for a contradiction that $m_{4^\vee}$ and m_1 are positive. By (0.1), (1.2), (1.5), (3.2), and (3.5) $m_{\substack{1\\2}} = m_{\substack{1\\3}} = m_{\substack{3\\2}} = m_{\substack{2\\4}} = m_{\substack{3\\4}} = 0$. If $m_{\substack{2\\3}} = 0$ then T is dishonest by the moves defined by Figure 9. If $m_{\substack{2\\3}} = 1$ then the moves

$$\cdots\begin{array}{|c|c|c|}\hline 2&2&1\\ \cline{3-3} 3&3\\ \cline{2-2} 4\\ \cline{1-1}\end{array}\cdots \longleftrightarrow - \cdots\begin{array}{|c|c|c|}\hline 2&3&1\\ \cline{3-3} 3&2\\ \cline{2-2} 4\\ \cline{1-1}\end{array}\cdots \longleftrightarrow - \cdots\begin{array}{|c|c|c|}\hline 2&2&1\\ \cline{3-3} 3&3\\ \cline{2-2} 4\\ \cline{1-1}\end{array}\cdots,$$

show that $\mathbf{T}$ is nonorientable, contradicting the assumption that it is honest. □

The invariant moves and Proposition 16.1 (ii) reduce the analysis of $\mathcal{TG}(\nu)$ to the invariant-free case, as we now verify.

Proposition 16.3. *Let $\check{\mathbf{T}}$ be an honest NSTC of shape ν and $\check{T}$ a SNST in this class that is lexicographically minimum. Then the invariant-free part T of $\check{T}$ satisfies the conditions of Proposition 16.2.*

Proof. By Proposition 16.1 (ii) and invariant moves, if $T \to T'$ is any move, then there is a path in $\mathcal{TG}(\nu)$ from $\check{T}$ to the SNST(ν') obtained from $\check{T}$ by keeping its invariant column pairs fixed and replacing its invariant-free part with T'. Note that if $T \to T' = 0$ is a move defined by Figure 9, then a move defined by Figure 14 is needed to conclude that $\check{T}$ has a path to 0. It follows that T satisfies the conditions of Proposition 16.2. □

An NST or SNST $\check{T}$ is *straightened* if its invariant-free part T satisfies the conditions of Proposition 16.2 and its invariant column pairs are positioned to make $\check{T}$ as small as possible in lexicographic order, i.e., height-3 invariant column pairs are contiguous and lie between the last $\boxed{3^\vee}$ column of T and the first $\boxed{4^\vee}$ column of T, height-2 invariant column pairs are contiguous, are all of the form $\begin{array}{|c|c|}\hline 2&1\\ 4&3\\ \hline\end{array}$, and lie between the last $\begin{array}{|c|}\hline 3\\ 2\\ \hline\end{array}$ and the first $\begin{array}{|c|}\hline 2\\ 4\\ \hline\end{array}$ of T, and height-1 invariant column pairs are contiguous and lie between the last $\boxed{3}$ and the first $\boxed{4}$ of T. *Straightening* an NST T is the process of following moves from T to the straightened representative of its class, or following moves that show it to be dishonest.

We have shown in this subsection that, as sets,

$$\begin{aligned}&+\mathrm{HNSTC}(\nu) \cong \text{lexicographically minimum representatives of } +\mathrm{HNSTC}(\nu)\\ &\subseteq \text{straightened elements of } +\mathrm{HSNST}(\nu').\end{aligned}$$

In the next subsection, we will show that this containment is actually an equality.

16.2. **Bijection with semistandard tableaux.** In this subsection we prove the following combinatorial result, completing the proof of Theorem 15.21 (iii) and (iv).

Proposition 16.4. *There is a bijection between straightened NST of shape ν' and $SSYT_{d_X}(\nu)$.*

Proof. We define a map f from straightened NST of shape ν' to $\mathrm{SSYT}_{d_X}(\nu)$ and check that it has an inverse (we omit a fully explicit description of the inverse and check this somewhat informally). To define f, let $\check{T}$ be an arbitrary straightened NST of shape ν' with invariant record (i_4, i_3, i_2, i_1) and invariant-free part T. As above, let $m_{1^\vee}, m_{2^\vee}, \dots$ denote the number of columns of each type occurring in T. The semistandard tableau $f(\check{T})$ will be defined through pictures and by specifying the number of columns $m'_{1^\vee}, m'_{2^\vee}, \dots$ of each type it is to contain. Regardless of the values of (i_4, i_3, i_2, i_1) and $m_{1^\vee}, m_{2^\vee}, \dots,$ we always set $m'_{1^\vee} = m_{1^\vee}$, $m'_{4^\vee} = m_{4^\vee}$, $m'_1 = m_1$, $m'_4 = m_4$, and there is no choice for the height-4 columns of $f(\check{T})$. The remainder of $f(\check{T})$ is defined as follows: a right, middle, and left subtableaux of this remainder (as shown in the pictures below, to the right of the $\mapsto$) are defined from i_1 and a right subtabloid of T, i_2 and a middle subtabloid of T, and i_3 and a left subtabloid of T (as shown in the pictures below, to the left of the $\mapsto$), respectively. We freely use Proposition 16.2 to break up the definition of f into cases—the parenthetical comments in the cases are consequences of this proposition.

If $m_1 > 0$, then the right subtableau of $f(\check{T})$ is defined from i_1 and the right subtabloid of T as follows:

$$\underbrace{\boxed{1}\cdots\cdots\boxed{1}}_{m_1}\underbrace{\boxed{2}\cdots\boxed{2}}_{m_2}\underbrace{\boxed{3}\cdots\boxed{3}}_{m_3} \mapsto \underbrace{\boxed{1}\cdots\cdots\boxed{1}}_{m_1}\underbrace{\boxed{2}\cdots\boxed{2}}_{i_1+m_2}\underbrace{\boxed{3}\cdots\boxed{3}}_{i_1+m_3}.$$

Here, and elsewhere in the paper, the two sets of dots is an added visual aid to indicate that this type of column appears at least once.

If $m_1 = 0$, then the right subtableau of $f(\check{T})$ is defined from i_1 and the right subtabloid of T as follows:

$$\underbrace{\begin{array}{|c|}\hline 2\\ 4\\ \hline\end{array}\cdots\begin{array}{|c|}\hline 2\\ 4\\ \hline\end{array}}_{m_{\substack{2\\4}}}\underbrace{\begin{array}{|c|}\hline 3\\ 4\\ \hline\end{array}\cdots\begin{array}{|c|}\hline 3\\ 4\\ \hline\end{array}}_{m_{\substack{3\\4}}}\underbrace{\boxed{2}\cdots\boxed{2}}_{m_2}\underbrace{\boxed{3}\cdots\boxed{3}}_{m_3} \mapsto$$

$$\begin{cases}
\underbrace{\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}}_{m_{\substack{2\\4}}}\underbrace{\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}}_{m_{\substack{3\\4}}}\underbrace{\boxed{3}\cdots\boxed{3}}_{2i_1+m_3} & \text{if } m_{\substack{2\\4}}>0 \text{ and } m_2=0 \text{ (and } m_3\le 1\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}}_{m_{\substack{2\\4}}}\underbrace{\boxed{2}\cdots\cdots\boxed{2}}_{i_1+m_2}\underbrace{\boxed{3}\cdots\boxed{3}}_{i_1} & \text{if } m_{\substack{2\\4}}>0 \text{ and } m_2>0 \text{ (and } m_{\substack{3\\4}}=m_3=0\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}}_{m_{\substack{3\\4}}}\underbrace{\boxed{2}\cdots\cdots\boxed{2}}_{i_1+m_2}\underbrace{\boxed{3}\cdots\cdots\boxed{3}}_{i_1+m_3} & \text{if } m_{\substack{2\\4}}=0 \text{ and } i_1>0,\\
\underbrace{\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}}_{m_{\substack{3\\4}}}\underbrace{\boxed{2}\cdots\boxed{2}}_{m_2}\underbrace{\boxed{3}\cdots\boxed{3}}_{m_3} & \text{if } m_{\substack{2\\4}}=0 \text{ and } i_1=0.
\end{cases}$$

Note that the last case actually is a semistandard tableau because $m_{\substack{3\\4}}>0$ implies $m_2=0$.

Let us check that i_1 and the right subtabloid of T can be recovered from the right subtableau of $f(\check{T})$. First observe that we can determine from $f(\check{T})$ which case applies: the first case applies if $m'_{\substack{2\\4}}>0$ and $m'_2=0$, the second if $m'_{\substack{2\\4}}>0$ and $m'_2>m'_3$, the third if ($m'_{\substack{2\\4}}>0$ and $0<m'_2\le m'_3$) or ($m'_{\substack{2\\4}}=0$ and $\min(m'_2,m'_3)>0$), and the fourth if $m'_{\substack{2\\4}}=0$ and $\min(m'_2,m'_3)=0$. Next, we can recover i_1 by $i_1=\lfloor\frac{m'_3}{2}\rfloor$ if the first case applies, and $i_1=\min(m'_2,m'_3)$ otherwise. Also, in the first case $m_3\le 1$, so m_3 is determined by $m_3\equiv m'_3 \mod 2$. In the remaining cases, the right subtabloid of T is easily recovered from the right subtableau of $f(\check{T})$.

Next, the middle subtableau of $f(\check{T})$ is defined from i_2 and the middle subtabloid of T as follows:

$$\underbrace{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}}_{m_{\substack{2\\3}}}\underbrace{\boxed{\begin{smallmatrix}3\\2\end{smallmatrix}}}_{m_{\substack{3\\2}}}\mapsto\begin{cases}
\underbrace{\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}}_{2i_2+m_{\substack{2\\3}}} & \text{if } m_{4^\vee}=0 \text{ and } m_1>0 \text{ (and } m_{\substack{3\\2}}=m_{\substack{2\\4}}=m_{\substack{3\\4}}=0\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}}_{2i_2+m_{\substack{2\\3}}} & \text{if } m_{4^\vee}>0 \text{ and } m_1=0 \text{ (and } m_{\substack{1\\2}}=m_{\substack{1\\3}}=m_{\substack{3\\2}}=0\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}}_{2i_2+m_{\substack{3\\2}}} & \text{if } m_{4^\vee}=0, m_1=0, \text{ and } m_{\substack{2\\3}}=0,\\
\underbrace{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}}_{2i_2+m_{\substack{3\\2}}+m_{\substack{2\\3}}} & \text{if } m_{4^\vee}=0, m_1=0, \text{ and } m_{\substack{2\\3}}>0.
\end{cases}$$

Note that by Proposition 16.2, exactly one of these conditions is satisfied. The integer i_2 and the middle subtabloid of T can be recovered from the middle subtableau of $f(\check{T})$ as follows: since $m_{4^\vee} = m'_{4^\vee}$, $m_1 = m'_1$, which of $m'_1, m'_{4^\vee}, m'_{\substack{2\\3}}$ are 0 determines which case applies. Then $i_2 = \lfloor \frac{1}{2}(m'_{\substack{1\\4}} + m'_{\substack{2\\3}}) \rfloor$ in the first three cases and $i_2 = \lfloor \frac{1}{2}(m'_{\substack{2\\3}} - 1) \rfloor$ in the last case. Also, $m_{\substack{2\\3}} \leq 1$ and $m_{\substack{3\\2}} \leq 1$, so $m_{\substack{2\\3}}$ and $m_{\substack{3\\2}}$ are determined by $m'_{\substack{1\\4}} \equiv m_{\substack{2\\3}} \mod 2$ in the first case, $m'_{\substack{2\\3}} \equiv m_{\substack{2\\3}} \mod 2$ in the second, $m'_{\substack{1\\4}} \equiv m_{\substack{3\\2}} \mod 2$ in the third, and $m'_{\substack{2\\3}} - 1 \equiv m_{\substack{3\\2}} \mod 2$ and $m_{\substack{2\\3}} = 1$ in the fourth.

The columns of height-3 are handled similarly to the height-1 columns. If $m_{4^\vee} > 0$, then the left subtableau of $f(\check{T})$ is defined from i_3 and the left subtabloid of T as follows:

$$\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}_{m_{4^\vee}} \mapsto \underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{i_3+m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{i_3+m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}_{m_{4^\vee}}.$$

If $m_{4^\vee} = 0$, then the left subtableau of $f(\check{T})$ is defined from i_3 and the left subtabloid of T as follows:

$$\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}}_{m_{\substack{1\\2}}}\underbrace{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}}_{m_{\substack{1\\3}}} \mapsto$$

$$\begin{cases}
\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{2i_3+m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}}_{m_{\substack{1\\2}}}\underbrace{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}}_{m_{\substack{1\\3}}} & \text{if } m_{\substack{1\\3}} > 0 \text{ and } m_{3^\vee} = 0 \text{ (and } m_{2^\vee} \leq 1\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{i_3}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{i_3+m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}}_{m_{\substack{1\\3}}} & \text{if } m_{\substack{1\\3}} > 0 \text{ and } m_{3^\vee} > 0 \text{ (and } m_{\substack{1\\2}} = m_{2^\vee} = 0\text{)},\\
\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{i_3+m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{i_3+m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}}_{m_{\substack{1\\2}}} & \text{if } m_{\substack{1\\3}} = 0 \text{ and } i_3 > 0,\\
\underbrace{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}}_{m_{2^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}_{m_{3^\vee}}\underbrace{\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\cdots\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}}_{m_{\substack{1\\2}}} & \text{if } m_{\substack{1\\3}} = 0 \text{ and } i_3 = 0.
\end{cases}$$

The integer i_3 and the left subtabloid of T can be recovered from the left subtableau of $f(\check{T})$ as follows: the first case applies if $m'_{\substack{1\\3}} > 0$ and $m'_{3^\vee} = 0$, the second if $m'_{\substack{1\\3}} > 0$ and $m'_{3^\vee} > m'_{2^\vee}$, the third if ($m'_{\substack{1\\3}} > 0$ and $0 < m'_{3^\vee} \leq m'_{2^\vee}$) or ($m'_{\substack{1\\3}} = 0$ and $\min(m'_{3^\vee}, m'_{2^\vee}) > 0$), and the fourth if $m'_{\substack{1\\3}} = 0$ and $\min(m'_{3^\vee}, m'_{2^\vee}) = 0$. In the first case, $i_3 = \lfloor \frac{m'_{2^\vee}}{2} \rfloor$, and otherwise $i_3 = \min(m'_{3^\vee}, m'_{2^\vee})$. In the first case, $m_{2^\vee}$ is the element of $\{0, 1\}$ with the same parity as $m'_{2^\vee}$. Otherwise, $m_{2^\vee}, m_{3^\vee}, m_{\substack{1\\2}}$, and $m_{\substack{1\\3}}$ are easily recovered from $m'_{2^\vee}, m'_{3^\vee}, m'_{\substack{1\\2}}$, and $m'_{\substack{1\\3}}$ once the correct case is known. $\square$

The proof of Theorem 15.21 (iii) and Propositions 16.3 and 16.4 establish the following.

Corollary 16.5. *As sets,*

$$\begin{aligned} &+HNSTC(\nu) \cong \textit{lexicographically minimum representatives of } +HNSTC(\nu) \\ &= \textit{straightened elements of } +HSNST(\nu') \cong SSYT_{d_X}(\nu). \end{aligned}$$

16.3. **Invariant-free straightened highest weight NST.** It is not difficult to analyze the straightening conditions of Proposition 16.2 to determine exactly the form of the invariant-free straightened highest weight NST. This will be used to obtain explicit formulae for Kronecker coefficients in §18.

Proposition 16.6. *Let T be an invariant-free NST. Then T is straightened and highest weight if and only if*

$$T = \underbrace{\boxed{\begin{matrix}1\\2\\3\end{matrix}}\cdots\boxed{\begin{matrix}1\\2\\3\end{matrix}}}_{m_{1^\vee}}\underbrace{\boxed{\begin{matrix}1\\2\\4\end{matrix}}\cdots\boxed{\begin{matrix}1\\2\\4\end{matrix}}}_{m_{2^\vee}}\underbrace{\boxed{\begin{matrix}1\\3\\4\end{matrix}}\cdots\boxed{\begin{matrix}1\\3\\4\end{matrix}}}_{m_{3^\vee}}\underbrace{\boxed{\begin{matrix}1\\2\end{matrix}}\cdots\boxed{\begin{matrix}1\\2\end{matrix}}}_{m_{\substack{1\\2}}}\underbrace{\boxed{\begin{matrix}1\\3\end{matrix}}\cdots\boxed{\begin{matrix}1\\3\end{matrix}}}_{m_{\substack{1\\3}}}\underbrace{\boxed{\begin{matrix}2\\3\end{matrix}}}_{m_{\substack{2\\3}}}\ \underbrace{\boxed{1}\cdots\boxed{1}}_{m_1},$$

where

(a) *at least one of* $m_{2^\vee}, m_{3^\vee}$ *is zero,*
(b) $m_{\substack{1\\2}} > 0$ *implies* $m_{3^\vee} = 0$,
(c) $m_{\substack{1\\3}} > 0$ *implies* $m_{2^\vee} \le 1$,
(d) $m_{\substack{2\\3}} \le 1$,
(e) $m_{\substack{2\\3}} = 1$ *implies* $m_1 > 0$,
(f) $m_{2^\vee} \le m_1 + m_{\substack{1\\3}}$,
(g) $m_{3^\vee} \le m_1 + m_{\substack{1\\2}}$.

Proof. If T is straightened, then it satisfies (a)–(d). If it is also highest weight, then $m_4 = m_3 = m_2 = 0$ as shown in the picture of T above, and (e)–(g) hold. Next, $m_{\substack{3\\2}} = m_{\substack{2\\4}} = m_{\substack{3\\4}} = 0$ because if $m_1 > 0$, this is by the straightening conditions (1.2) and (1.5) and if $m_1 = 0$, this is by the highest weight assumption. Finally, $m_{4^\vee} = 0$ by (4.1) if $m_1 > 0$, by (3.2) if $m_{\substack{1\\2}} > 0$ or $m_{\substack{1\\3}} > 0$, and by the highest weight assumption otherwise. This proves the "only if" direction. For the "if" direction, one checks directly and easily that if T has the form above, then T is straightened and highest weight. $\square$

17. A Kronecker graphical calculus and applications

In §17.1 we give a description of the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal components of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+\mathrm{HNSTC}(\nu)))$ in terms of arcs. We hope this to be the beginnings of a Kronecker version of the $U_q(\mathfrak{sl}_2)$ graphical calculus of [19]. This graphical description of crystal components will make it easier to obtain explicit formulae for Kronecker coefficients in §18. We also use it to write down the action of the Chevalley generators on +HNSTC (§17.2) and the action of τ on +HNSTC (§17.3).

17.1. **Kronecker graphical calculus.** Though it is not strictly necessary for the results in the next two sections, we believe it to be useful to give a description of crystal components that is independent of any +HNSTC in the component and any representative +HSNST in its class. We hope this to be the beginnings of a graphical calculus that describes $\mathscr{O}(M_q(\check{X}))$-comodules and morphisms between them in terms of their +HNSTC bases or some generalization thereof. However, to more fully develop such a theory it seems that we need a canonical basis for all of $\check{X}^{\otimes r}$ as detailed in Conjecture 19.1.

Definition 17.1. A *k-l V-arc* of an NST or SNST is an external V-arc between a height-k column and a height-l column. We define *k-l W-arcs* similarly, and a *k-l arc* is either a k-l V-arc or a k-l W-arc. We also define a 3-2-1 arc to be a k-l arc and a k-l' arc that share a height-k column and such that $\{k, l, l'\} = \{1, 2, 3\}$. A 3-2-1 arc is a 3-2-1 V-arc (resp. W-arc) if both of its arcs are V-arcs (resp. W-arcs) or if the longer of its two arcs is a V-arc (resp. W-arc) (see Example 17.2).

A *pure* k-l V-arc is a k-l V-arc that is not part of a 3-2-1 arc. Pure k-l W-arcs and pure k-l arcs are defined similarly.

We write

$$\cup^{k\text{-}l\ V}(T), \cup^{k\text{-}l\ W}(T), \cup^{k\text{-}l}(T), \cup^{3\text{-}2\text{-}1\ V}(T), \cup^{3\text{-}2\text{-}1\ W}(T), \cup^{3\text{-}2\text{-}1}(T)$$

for the number of pure k-l V-arcs, pure k-l W-arcs, pure k-l arcs, 3-2-1 V-arcs, 3-2-1 W-arcs, and 3-2-1 arcs of T, respectively. Also let

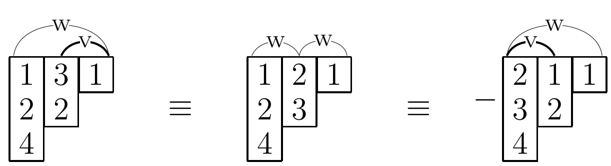
$$\cup^{\text{ext}}(T)$$

denote the total number of external arcs of T. Finally, let

$$\text{ext-free}_V(T), \text{ext-free}_W(T)$$

be the number of type V columns of T not at the end of an external arc, type W columns of T not at the end of an external arc, respectively.

Example 17.2. Here are three equivalent SNST with their external V- and W-arcs drawn.



T^1 T^2 T^3

There is one 3-2-1 W-arc in each T^i, and there are no pure k-l V- or W-arcs in any of the T^i.

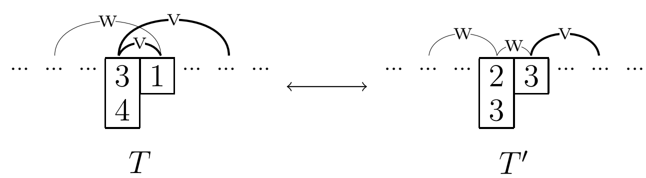
Proposition 17.3. *From the diagram of a +HSNST(ν') T we extract the following information:*

(A) *(the unpaired V-diagram, the unpaired W-diagram)*
(B) *the invariant record (i_4, i_3, i_2, i_1) of T,*
(C) $(\cup^{2\text{-}1}(T), \cup^{3\text{-}2}(T))$,
(D) $(\cup^{3\text{-}1}(T), \cup^{3\text{-}2\text{-}1}(T), |\textit{ext-free}_V(T) - \textit{ext-free}_W(T)|)$,
(E) $(\cup^{3\text{-}1\ V}(T), \cup^{3\text{-}1\ W}(T), \cup^{3\text{-}2\text{-}1\ V}(T), \cup^{3\text{-}2\text{-}1\ W}(T), \textit{ext-free}_V(T), \textit{ext-free}_W(T))$,

The data above is constant on +HNSTC, (B)–(E) are constant on $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal components of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+HNSTC(\nu)))$, and (B)–(D) are constant on U_q^τ-cells of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+HNSTC(\nu)))$.

Note that the crystal components and cells of $(\mathrm{gr}(\check{X}_\nu), \mathrm{in}(+\mathrm{HNSTC}(\nu)))$ are the same and coincide with those of $(\check{X}_\nu, +\mathrm{HNSTC}(\nu))$, but we do not yet know that the latter is an upper based $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$- or U_q^τ-module.

Proof. Proposition 15.7 (a) says exactly that (B) is constant on NSTC. Next, one checks that degree-preserving moves preserve the number of internal and external V- and W-arcs. Thus (A) is constant on NSTC. Also, observe that an honest NST does not contain any k-k arcs, $k \in [d_X]$, except for those that are part of an invariant column pair. With these facts in mind, it is easy to check that for each degree-preserving move, (C)–(E) are constant on +HNSTC. For instance, for a degree-preserving move of the form below, we have drawn the V- and W-arcs (the arcs that have one end on the dots may not exist–call them potential arcs).



This turns the shown 2-1 V-arc of T into the shown 2-1 W-arc of T', but (C)–(E) remain constant: the potential V-arc does not exist because, as just mentioned, T' honest implies that it does not contain a 1-1 V-arc not part of an invariant column pair; the potential W-arc cannot have its undetermined end on a height-2 column because T' honest implies that it does not contain a 2-2 W-arc not part of an invariant column pair; if the potential W-arc has its undetermined end on a height-3 column, then it is part of a 3-2-1 W-arc in T and T'.

Since the action of $G\tilde{F}_V^{\mathrm{up}}$ and $G\tilde{F}_W^{\mathrm{up}}$ on NST does not modify external V- and W-arcs, (B)–(E) are constant on $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal components. This, together with the fact that the action of τ on +HSNST interchanges V- and W-arcs (and sometimes multiplies by -1), proves that (B)–(D) are constant on U_q^τ-cells. □

Proposition 16.6 implies that any invariant-free straightened highest weight NST belongs to exactly one of the eight cases of Figure 18, which are drawn with the following conventions:

$$\text{two sets of dots indicates that that type of column appears at least once; one set of dots indicates that that type of column can appear any number of times; a column appearing once (and with no dots) indicates that that type of column appears exactly once. All external arcs are drawn (pure 2-1 and 3-2 arcs are not labeled as } V\text{-arcs or } W\text{-arcs, as indicated by the funny dashed style, because these are not constant on +HNSTC).} \tag{153}$$

These eight cases are grouped into five cases corresponding to the different types of invariant-free Kronecker coefficients $\hat{g}^*_{\lambda\mu\nu}$, as described in the next section (these coefficients give a convenient way of decomposing Kronecker coefficients into a sum of smaller nonnegative coefficients).

Propositions 16.4, 16.6, and 17.3 have the following corollary, which partially realizes our original goal of obtaining a bijection (5) that is some kind of "Kronecker analog" of the RSK correspondence.

Corollary 17.4. *The map from $+HSNST(\nu')$ to (A),(B),(C),(E) of Proposition 17.3 has fibers given by $+HNSTC(\nu)$. Pre-composing this map with the bijection of Proposition 16.4 yields a bijection*

$$\begin{array}{ccccc} SSYT_{d_X}(\nu) & \cong & +HNSTC(\nu) & \xrightarrow{\cong} & \bigsqcup_{\lambda,\mu} SSYT_{d_V}(\lambda) \times SSYT_{d_W}(\mu) \times \mathbf{g}_{\lambda\mu\nu} \\ f(\mathbf{T}) & \longleftrightarrow & \mathbf{T} & \mapsto & P(\mathbf{k}), \qquad P(\mathbf{l}), \qquad (B),(C),(E), \end{array} \tag{154}$$

where $\mathbf{k}, \mathbf{l}$ are the V- and W-word of any $T \in \mathbf{T}$ and $\mathbf{g}_{\lambda\mu\nu}$ is a subset of $\mathbb{Z}^{12}_{\geq 0}$ that depends on λ, μ, ν, but not on $\mathbf{T}$, and has cardinality $g_{\lambda\mu\nu}$ (here, (B),(C), and (E) are encoded as a 12-tuple of nonnegative integers).

Moreover, the set $\mathbf{g}_{\lambda\mu\nu}$ is not hard to read off from Figure 18, and we make this even more explicit in the next section.

Remark 17.5. We have only partially realized the goal of obtaining a bijection as in (5) because we really want SYT(ν)-many bijections that are all slightly different, but similar to (154). We expect these bijections to be realized algebraically by finding a basis for $\check{X}^{\otimes r}$ whose U_q^τ-cells can be partitioned into SYT(ν)-many cellular subquotients, called fat cells in Conjecture 19.1, each similar to (but not necessarily isomorphic as a based module to) $(\check{X}_\nu, +\text{HNSTC}(\nu))$. See Conjecture 19.1 for more details.

By projecting the right-hand side of (154) onto $\bigsqcup_{\lambda,\mu} \mathbf{g}_{\lambda\mu\nu}$, we also obtain a nice description of $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal components in terms of the Kronecker graphical calculus. Let us assume Theorem 15.21 for this corollary so that we can state it in terms of the basis $+\text{HNSTC}(\nu)$ rather than in$(+\text{HNSTC}(\nu))$.

Corollary 17.6. *The $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module with basis $(\check{X}_\nu, +HNSTC(\nu))$ decomposes into $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal components (or $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cells) as*

$$+HNSTC(\nu) = \bigsqcup_{\lambda,\mu \vdash_2 r,\ \zeta \in \mathbf{g}_{\lambda\mu\nu}} \check{\Lambda}_{\nu,\zeta},$$

where $\check{\Lambda}_{\nu,\zeta}$ consists of those $\mathbf{T} \in +HNSTC(\nu)$ such that the 12-tuple of $\mathbf{T}$ is equal to ζ.

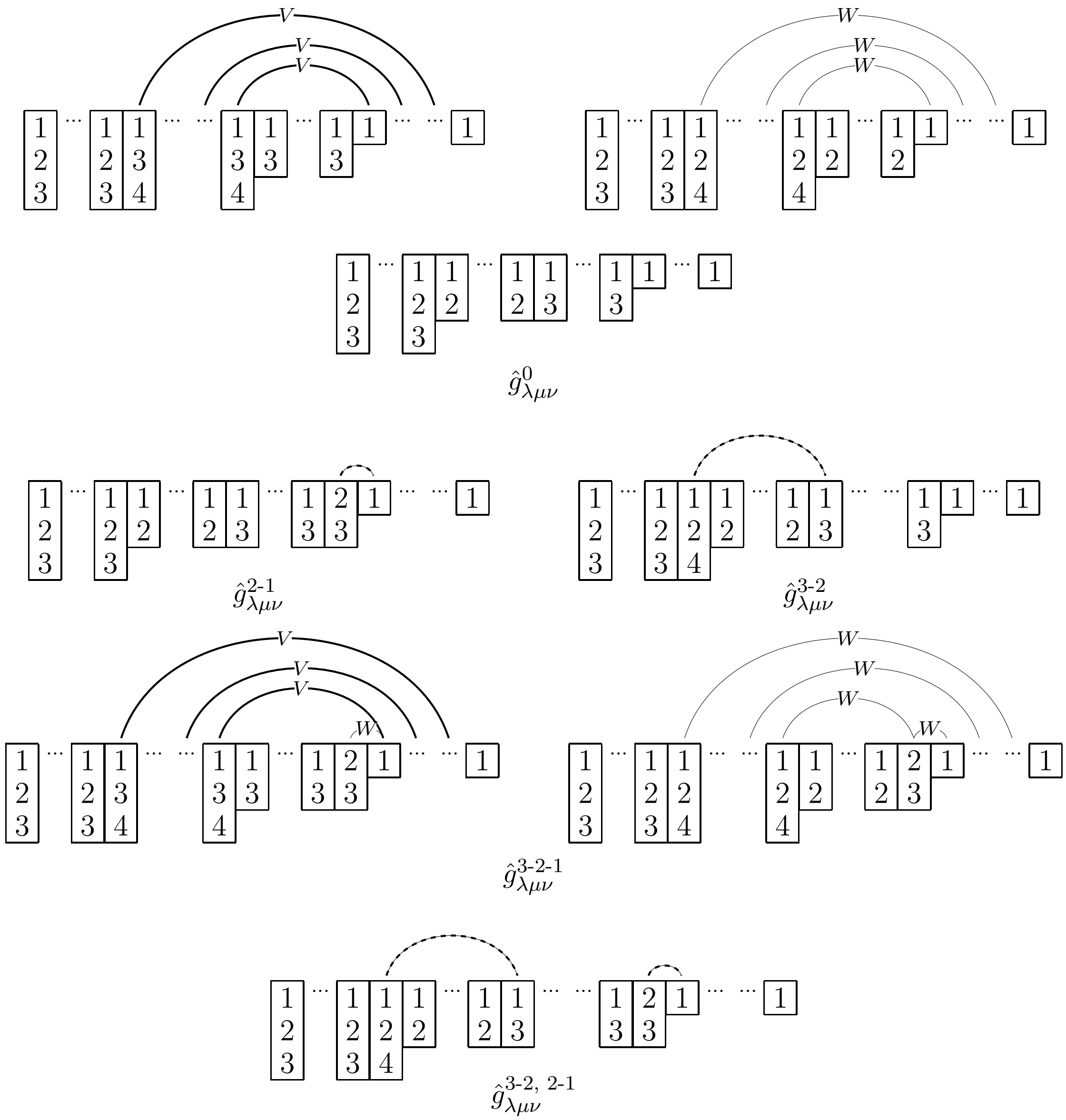


Figure 18: The Kronecker graphical calculus for straightened highest weight invariant-free NST, drawn with the conventions of (153). The eight cases are grouped into five cases corresponding to the different types of invariant-free Kronecker coefficients, as described in §18.

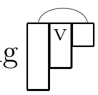
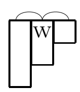
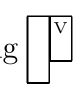
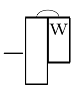
Note that the sets $\{\mathbf{g}_{\lambda\mu\nu}\}_{\lambda,\mu}$ for fixed ν are actually disjoint as subsets of $\mathbb{Z}^{12}_{\geq 0}$ (this will be seen in (166)), and this was used implicitly in the definition of $\check{\Lambda}_{\nu,\zeta}$.

In light of this corollary and Proposition 17.3, it makes sense to write $\cup^{2\text{-}1}(\mathscr{G}) = \cup^{2\text{-}1}(\mathbf{T}) = \cup^{2\text{-}1}(T), \cup^{3\text{-}2}(\mathscr{G}) = \cup^{3\text{-}2}(\mathbf{T}) = \cup^{3\text{-}2}(T)$, etc. for $\mathbf{T}$ the +HNSTC containing an +HSNST T and $\mathscr{G}$ the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-crystal component containing $\mathbf{T}$.

17.2. **Action of the Chevalley generators on +HNSTC.** For $g \in U_q^\tau$ and $\mathbf{T} \in$ +HNSTC(ν), define the structure coefficients $a^g_{\mathbf{T}'\mathbf{T}}$ by $g\mathbf{T} = \sum_{\mathbf{T}' \in +\mathrm{HNSTC}(\nu)} a^g_{\mathbf{T}'\mathbf{T}} \mathbf{T}'$. We now determine the structure coefficients $a^g_{\mathbf{T}'\mathbf{T}}$ when g is one of the Chevalley generators F_V, F_W, E_V, E_W. This requires a somewhat involved case-by-case analysis.

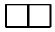
Let $\mathbf{T}$ be a +HNSTC. By Proposition 14.21, $F_V\mathbf{T} = \sum_{j=1}^{\varphi_V(\mathbf{T})} [j] \mathscr{F}_{(j)V}(\mathbf{T})$. Here, $\mathscr{F}_{(j)V}(\mathbf{T})$ is defined to be the NSTC containing $\mathscr{F}_{(j)V}(T)$ for any $T \in \mathbf{T}$ (this definition is sound by the same proof given for the case $j = \varphi_V(\mathbf{T})$ in the proof of Theorem 15.21 (iv)). The drawings in cases (A)–(G) below describe $\mathscr{F}_{(j)V}(\mathbf{T})$, for $j < \varphi_V(\mathbf{T})$, in terms of +HNSTC. Cases (A)–(G) correspond to the eight cases in Figure 18, except that case (B) combines the left case for $\hat{g}^0_{\lambda\mu\nu}$ and the left case for $\hat{g}^{3\text{-}2\text{-}1}_{\lambda\mu\nu}$.

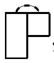
In the drawings below, the left-hand side represents part of some $T \in \mathbf{T}$ and the right-hand side represents part of $\mathscr{F}_{(j)V}(\mathbf{T})$, expressed in terms of +HNSTC (it turns out that $\mathscr{F}_{(j)V}(\mathbf{T})$ is always equal to an honest NSTC or to 0, though a priori we only know it to be some **A**-linear combination of +HNSTC). Note that $\mathscr{F}_{(j)V}(T)$, $j < \varphi_V(\mathbf{T})$, has one more external V-arc than T, for any $T \in \mathbf{T}$. Internal to case (*), subcases are labeled in the format (*.k-l) to indicate that the external V-arc created in the change from T to $\mathscr{F}_{(j)V}(T)$ is a k-l V-arc, and (*.k-l.1), (*.k-l.2), etc. is used if there is more than one way to add an external k-l V-arc. In general, k and l depend on the choice of $T \in \mathbf{T}$, but we specify as little information about T as possible. One way this is done is that only the columns involved in the change from $\mathbf{T}$ to $\mathscr{F}_{(j)V}(\mathbf{T})$ are drawn; so, for instance, unless specified otherwise, there may be some type V or type W columns that are not shown that lie between height-3 and height-1 columns that are shown. Other conventions for the drawings are that V-arcs are thick, W-arcs are thin, and arcs that we do not want to specify have the funny dashed style.

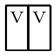
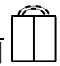
For example, the drawing  $\rightsquigarrow$  in subcase (C.2-1) indicates that if $T = \begin{array}{|c|c|c|}1&1&1\\2&2&\\4&&\end{array}$, then $\mathscr{F}_{(3)V}(T) = \begin{array}{|c|c|c|}1&3&1\\2&2&\\4&&\end{array} \equiv \begin{array}{|c|c|c|}1&2&1\\2&3&\\4&&\end{array}$; another specific example covered by this subcase is that $T = \begin{array}{|c|c|c|c|c|c|c|c|}1&1&1&1&1&1&1&3\\2&2&2&2&2&2&&\\3&4&4&4&&&&\end{array}$ implies $\mathscr{F}_{(8)V}(T) = \begin{array}{|c|c|c|c|c|c|c|c|}1&1&1&1&1&3&1&3\\2&2&2&2&2&2&&\\3&4&4&4&&&&\end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|c|}1&1&1&1&1&2&1&3\\2&2&2&2&2&3&&\\3&4&4&4&&&&\end{array}$. As another example of how to interpret the subcases, the drawing  $\rightsquigarrow -$  appears in (A.3-2) and (D.3-2); in (A.3-2) it means that if $\mathbf{T}$ has no external arcs and shape $[n_4, n_3, n_2, n_1]$ (see §2.3 for notation), then $\mathscr{F}_{(n_3)V}(\mathbf{T})$ has a pure 3-2 arc and no 2-1 arcs, and in (D.3-2) it means that if $\mathbf{T}$ has a pure 2-1 arc and no 3-2 arcs and shape $[n_4, n_3, n_2, n_1]$, then $\mathscr{F}_{(n_3)V}(\mathbf{T})$ has a pure 3-2 arc and a pure 2-1 arc. See the examples interspersed between the cases for more about how to interpret these drawings.

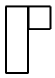
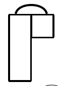
The subcases are fairly redundant, but we include them all for completeness. The subcases that are genuinely different from all previous ones are marked by bold labels.

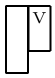
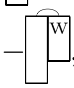
Case (A): $\mathbf{T}$ has no external arcs:

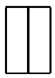
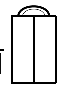
(A.1-1)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

(A.2-1)  $\rightsquigarrow$ ,

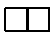
(A.2-2)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

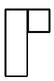
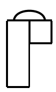
(A.3-1)  $\rightsquigarrow$  (if $\text{ext-free}_V(\mathbf{T}) = 0$),

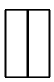
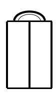
(A.3-2)  $\rightsquigarrow -$ ,

(A.3-3)  $\rightsquigarrow -\frac{1}{[2]}$  or 0.

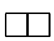
Case (B): $\mathbf{T}$ has a pure 3-1 V-arc or a 3-2-1 V-arc (combines two cases from Figure 18):

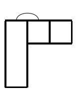
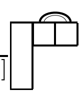
(B.1-1)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

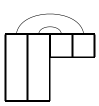
(B.3-1)  $\rightsquigarrow$ ,

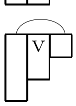
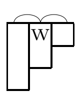
(B.3-3)  $\rightsquigarrow -\frac{1}{[2]}$  or 0.

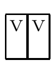
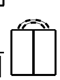
Case (C): $\mathbf{T}$ has at least one pure 3-1 W-arc and no 3-2 or 2-1 arcs:

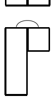
(C.1-1.1)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

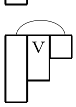
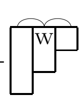
(C.1-1.2)  $\rightsquigarrow -\frac{1}{[2]}$ ,

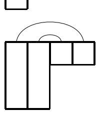
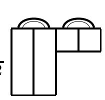
(C.1-1.3)  $\rightsquigarrow 0$,

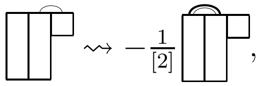
(C.2-1)  $\rightsquigarrow$ ,

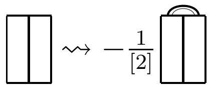
(C.2-2)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

(C.3-1)  $\rightsquigarrow 0$ (if $\text{ext-free}_V(\mathbf{T}) = 0$),

(C.3-2)  $\rightsquigarrow -$ ,

(C.3-3.1)  $\rightsquigarrow \frac{1}{[2]^2}$  or 0,

(C.3-3.2) 

(C.3-3.3)  or 0.

For example, if $T = \begin{array}{llllllll} 1&1&1&1&1&1&1&1\\ 2&2&2&2&2&&&\\ 4&4&4&&&&& \end{array}$, then subcase (C.2-1) indicates that

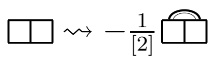
$$\mathscr{F}_{(7)V}(T) = \begin{array}{llllllll} 1&1&1&1&3&1&1&1\\ 2&2&2&2&2&&&\\ 4&4&4&&&&& \end{array} \equiv \begin{array}{llllllll} 1&1&1&1&2&1&1&1\\ 2&2&2&2&3&&&\\ 4&4&4&&&&& \end{array},$$

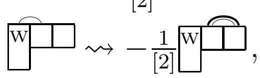
and subcase (C.3-2) indicates that

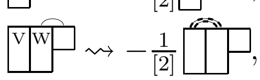
$$\mathscr{F}_{(3)V}(T) = \begin{array}{llllllll} 1&1&2&1&1&1&1&1\\ 2&2&3&2&2&&&\\ 4&4&4&&&&& \end{array} \equiv -\begin{array}{llllllll} 1&1&1&1&2&1&1&1\\ 2&2&2&2&3&&&\\ 4&4&4&&&&& \end{array}.$$

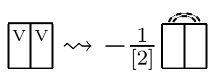
Set $T' = \begin{array}{llllllll} 1&1&1&1&2&1&1&1\\ 2&2&2&2&3&&&\\ 4&4&4&&&&& \end{array}$ and let $\mathbf{T}'$ (resp. $\mathbf{T}$) be the +HNSTC containing T' (resp. T). Examining all of case (C), we observe that these are the only contributions to the structure coefficient $a^{F_V}_{\mathbf{T}'\mathbf{T}}$, hence $a^{F_V}_{\mathbf{T}'\mathbf{T}} = [7] - [3]$.

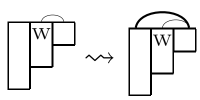
Case (D): $\mathbf{T}$ has a pure 2-1 arc and no 3-2 arcs:

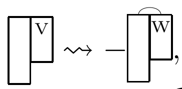
(D.1-1.1)  or 0,

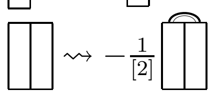
(D.1-1.2) 

(D.2-1) 

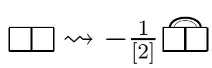
(D.2-2)  or 0,

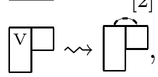
(D.3-1)  (if ext-free$_V(\mathbf{T}) = 0$),

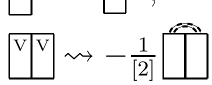
(D.3-2) 

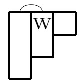
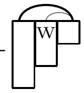
(D.3-3)  or 0.

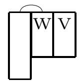
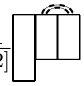
Case (E): $\mathbf{T}$ has a pure 3-2 arc and no 2-1 arcs:

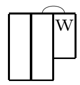
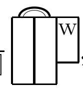
(E.1-1)  or 0,

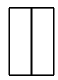
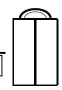
(E.2-1) 

(E.2-2)  or 0,

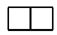
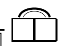
(E.3-1)  $\rightsquigarrow -$  (if ext-free$_V(\mathbf{T}) = 0$),

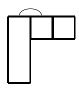
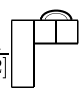
(E.3-2)  $\rightsquigarrow \frac{1}{[2]}$ ,

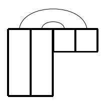
(E.3-3.1)  $\rightsquigarrow -\frac{1}{[2]}$ ,

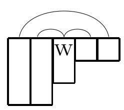
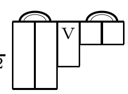
(E.3-3.2)  $\rightsquigarrow -\frac{1}{[2]}$  or 0.

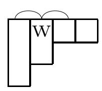
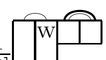
Case (F): $\mathbf{T}$ has a 3-2-1 W-arc:

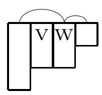
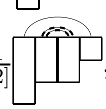
(F.1-1.1)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

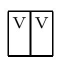
(F.1-1.2)  $\rightsquigarrow -\frac{1}{[2]}$ ,

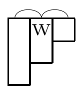
(F.1-1.3)  $\rightsquigarrow 0$,

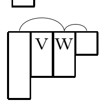
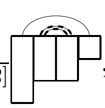
(F.1-1.4)  $\rightsquigarrow -\frac{1}{[2]^2}$ ,

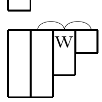
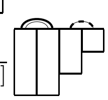
(F.1-1.5)  $\rightsquigarrow -\frac{1}{[2]}$  (if $\cup^{3\text{-}1\ W}(\mathbf{T}) = 0$),

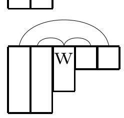
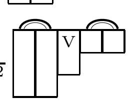
(F.2-1)  $\rightsquigarrow -\frac{1}{[2]}$ ,

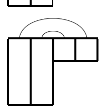
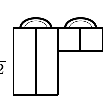
(F.2-2)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

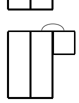
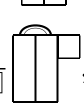
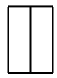
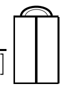
(F.3-1)  $\rightsquigarrow 0$ (if ext-free$_V(\mathbf{T}) = 0$),

(F.3-2)  $\rightsquigarrow \frac{1}{[2]}$ ,

(F.3-3.1)  $\rightsquigarrow -\frac{1}{[2]}$  (if $\cup^{3\text{-}1\ W}(\mathbf{T}) = 0$),

(F.3-3.2)  $\rightsquigarrow \frac{1}{[2]^2}$ ,

(F.3-3.3)  $\rightsquigarrow \frac{1}{[2]^2}$  or 0,

(F.3-3.4)  $\rightsquigarrow -\frac{1}{[2]}$ , (F.3-3.5)  $\rightsquigarrow -\frac{1}{[2]}$  or 0.

For example, if $T = \begin{ytableau} 1 & 1 & 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 4 & 4 \end{ytableau}$, then subcase (F.1-1.4) indicates that

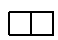
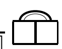
$$\mathscr{F}_{(4)V}(T) = \begin{ytableau} 1 & 1 & 1 & 2 & 3 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 4 & 4 \end{ytableau} \equiv -\frac{1}{[2]} \begin{ytableau} 1 & 1 & 1 & 1 & 4 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 4 & 4 \end{ytableau} \equiv \frac{1}{[2]} \begin{ytableau} 1 & 1 & 1 & 1 & 4 & 1 \\ 2 & 2 & 3 & 2 \\ 3 & 4 & 4 \end{ytableau} \equiv -\frac{1}{[2]^2} \begin{ytableau} 1 & 2 & 1 & 1 & 4 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 4 & 3 \end{ytableau},$$

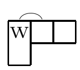
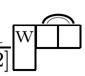
and subcase (F.3-3.2) indicates that

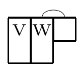
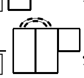
$$\mathscr{F}_{(2)V}(T) = \begin{ytableau} 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ 3 & 4 & 4 \end{ytableau} \equiv -\frac{1}{[2]} \begin{ytableau} 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 4 \\ 3 & 4 & 3 \end{ytableau} \equiv -\frac{1}{[2]} \begin{ytableau} 1 & 2 & 1 & 3 & 2 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 4 & 3 \end{ytableau} \equiv \frac{1}{[2]^2} \begin{ytableau} 1 & 2 & 1 & 1 & 4 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 4 & 3 \end{ytableau}.$$

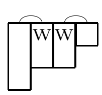
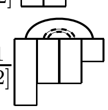
Set $T' = \frac{1}{[2]^2} \begin{ytableau} 1 & 2 & 1 & 1 & 4 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 4 & 3 \end{ytableau}$ and let $\mathbf{T}'$ (resp. $\mathbf{T}$) be the +HNSTC containing T' (resp. T). Examining all of case (F), we see that these are the only cases that contribute to $a^{F_V}_{\mathbf{T}'\mathbf{T}}$, hence $a^{F_V}_{\mathbf{T}'\mathbf{T}} = [2] - [4]$.

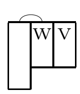
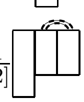
Case (G): $\mathbf{T}$ has a pure 3-2 arc and a pure 2-1 arc:

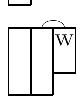
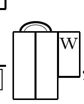
(G.1-1.1)  $\rightsquigarrow -\frac{1}{[2]}$  or 0,

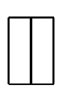
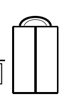
(G.1-1.2)  $\rightsquigarrow -\frac{1}{[2]}$ ,

(G.2-1)  $\rightsquigarrow -\frac{1}{[2]}$ ,

(G.3-1)  $\rightsquigarrow \frac{1}{[2]}$  (if ext-free$_V(\mathbf{T}) = 0$),

(G.3-2)  $\rightsquigarrow \frac{1}{[2]}$ ,

(G.3-3.1)  $\rightsquigarrow -\frac{1}{[2]}$ ,

(G.3-3.2)  $\rightsquigarrow -\frac{1}{[2]}$  or 0.

The next theorem summarizes the findings of this case-by-case analysis. We choose to state the theorem in the language of Corollary 17.4; recall that by this corollary, a +HNSTC $\mathbf{T}$ is determined by its unpaired V- and W-diagrams and a 12-tuple whose first four entries are the invariant-record of $\mathbf{T}$ and next eight entries are

$$\cup^{\text{2-1}}(\mathbf{T}), \cup^{\text{3-2}}(\mathbf{T}), \cup^{\text{3-1 }V}(\mathbf{T}), \cup^{\text{3-1 }W}(\mathbf{T}), \cup^{\text{3-2-1 }V}(\mathbf{T}), \cup^{\text{3-2-1 }W}(\mathbf{T}), \text{ext-free}_V(\mathbf{T}), \text{ext-free}_W(\mathbf{T}).$$

Theorem 17.7. *For +HNSTC $\mathbf{T}', \mathbf{T}$ of shape $\nu = [n_4, n_3, n_2, n_1]$, the structure coefficient* $a^{F_V}_{\mathbf{T}'\mathbf{T}}$

$$\begin{cases} = [\varphi_V(\mathbf{T})] & \text{if} \quad \mathbf{T}' = \mathscr{F}_{(\varphi_V(\mathbf{T}))V}(\mathbf{T}), & (155) \\ = [n_3 + 2n_2] - [n_3] & \begin{array}{ll} \mathbf{T} & (i_4, i_3, i_2, i_1, 0, 0, 0, k, 0, 0, n_2, 0) \\ \mathbf{T}' & (i_4, i_3, i_2, i_1, 0, 0, 0, k-1, 0, 1, n_2 - 1, 0) \end{array} & (156) \\ = [n_3 + 2n_2 - 2] - [n_3] & \begin{array}{ll} \mathbf{T} & (i_4, i_3, i_2, i_1, 0, 0, 0, k, 0, 1, n_2 - 1, 0) \\ \mathbf{T}' & (i_4, i_3, i_2 + 1, i_1, 0, 0, 0, k+1, 0, 0, n_2 - 1, 0) \end{array} & (157) \\ = [n_3 - 1] - [n_3 + 2n_2 - 1] & \begin{array}{ll} \mathbf{T} & (i_4, i_3, i_2, i_1, 0, 0, 0, k, 0, 1, n_2 - 1, 0) \\ \mathbf{T}' & (i_4, i_3 + 1, i_2, i_1 + 1, 0, 0, 0, k-1, 0, 0, n_2, 0) \end{array} & (158) \\ \in \bigcup_{j=0}^{\varphi_V(\mathbf{T})-1} \{-[j], [j]\} & \text{otherwise,} & (159) \end{cases}$$

where i_j and k are nonnegative integers, and the conditions for (156)–(158) *are that $\mathbf{T}$ and $\mathbf{T}'$ are determined by the 12-tuples shown and the unpaired V-diagrams (resp. W-diagrams) of $\mathbf{T}$ and $\mathbf{T}'$ have the same number of 2's.*

Similarly, the structure coefficient

$$a^{E_V}_{\mathbf{T}'\mathbf{T}} \begin{cases} = [\varepsilon_V(\mathbf{T})] & \text{if } \mathbf{T}' = \mathscr{E}_{(\varepsilon_V(\mathbf{T}))V}(\mathbf{T}), \\ = [n_1] - [n_1 + 2n_2] & \text{same 12-tuples as (156)}, \\ = [n_1] - [n_1 + 2n_2 - 2] & \text{same 12-tuples as (157)}, \\ = [n_1 + 2n_2 - 1] - [n_1 - 1] & \text{same 12-tuples as (158)}, \\ \in \bigcup_{j=0}^{\varepsilon_V(\mathbf{T})-1} \{-[j], [j]\} & \text{otherwise.} \end{cases}$$

For the middle three cases, we also require that the V-words (resp. W-words) of $\mathbf{T}$ and $\mathbf{T}'$ have the same number of 1's.

Similar statements hold with W in place of V—the coefficients remain the same, though the tuples in (156)–(158) *must have their entries in positions 7 and 8, 9 and 10, and 11 and 12 swapped to accommodate interchanging V and W.*

In particular, for $g = F_V, F_W, E_V$, or E_W, the structure coefficient $a^g_{\mathbf{T}'\mathbf{T}}$ is a $\bar{\cdot}$-invariant Laurent polynomial in q with all nonnegative or all nonpositive coefficients.

Proof. The case-by-case analysis shows that $\mathscr{F}_{(j)V}(\mathbf{T})$ is always equal (in $\check{X}_\nu$) to some honest NSTC or to 0. The theorem then follows by determining those $\mathscr{F}_{(j')V}(\mathbf{T})$ that are proportional to another $\mathscr{F}_{(j)V}(\mathbf{T})$. The only such $\mathscr{F}_{(j')V}(\mathbf{T})$ are those corresponding to subcases (C.2-1) and (C.3-2), (F.2-1) and (F.3-2), and (F.1-1.4) and (F.3-3.2), which yield (156), (157), and (158), respectively. □

The next example shows that the structure coefficients $a^g_{\mathbf{T}'\mathbf{T}}$, for g a canonical basis element of $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$, are not always polynomials in q with all nonnegative or all nonpositive coefficients. Note that this also implies that there is no way to readjust the signs of +HNSTC so that the $a^{F_V}_{\mathbf{T}'\mathbf{T}}$ have only nonnegative coefficients.

Example 17.8. Let $T = \begin{array}{|c|}\hline 1\\ 2\\ 3\\ \hline\end{array} \cdots \begin{array}{|c|c|c|}\hline 1&1&1\\ 2&2&2\\ \hline 3&4\\ \cline{1-2}\end{array} \cdots\cdots \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2\\ \cline{1-1}\end{array}$ and $T' = -\frac{1}{[2]} \begin{array}{|c|}\hline 1\\ 2\\ 3\\ \hline\end{array} \cdots \begin{array}{|c|c|c|c|}\hline 1&1&2&1\\ 2&2&3&2\\ \hline 3&4\\ \cline{1-2}\end{array} \cdots \begin{array}{|c|c|c|}\hline 1&4&1\\ \hline 2\\ \cline{1-1}\end{array}$ be +HSNST of shape $\nu = [0, n_3, n_2, 2]$, where the double dots indicate that there is at least one column of this type. Let $\mathbf{T}$ (resp. $\mathbf{T}'$) be the +HNSTC containing T (resp. T'). The coefficient $a^{F_V^{(2)}}_{\mathbf{T}'\mathbf{T}}$ is computed as follows. Let

$$\mathbf{T}^1 = \begin{array}{|c|}\hline 1\\ 2\\ 3\\ \hline\end{array} \cdots \begin{array}{|c|c|c|c|}\hline 1&1&2&1\\ 2&2&3&2\\ \hline 3&4\\ \cline{1-2}\end{array} \cdots \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2\\ \cline{1-1}\end{array}, \quad \mathbf{T}^2 = -\frac{1}{[2]} \begin{array}{|c|}\hline 1\\ 2\\ 3\\ \hline\end{array} \cdots \begin{array}{|c|c|c|}\hline 1&1&1\\ 2&2&2\\ \hline 3&3\\ \cline{1-2}\end{array} \cdots\ \cdots \begin{array}{|c|c|c|}\hline 1&4&1\\ \hline 2\\ \cline{1-1}\end{array}.$$

There holds

$$\begin{aligned} a^{F_V^{(2)}}_{\mathbf{T}'\mathbf{T}} &= \tfrac{1}{[2]} \textstyle\sum_{\mathbf{T}''} a^{F_V}_{\mathbf{T}'\mathbf{T}''} a^{F_V}_{\mathbf{T}''\mathbf{T}} \\ &= \tfrac{1}{[2]} \big(a^{F_V}_{\mathbf{T}'\mathbf{T}^1} a^{F_V}_{\mathbf{T}^1\mathbf{T}} + a^{F_V}_{\mathbf{T}'\mathbf{T}^2} a^{F_V}_{\mathbf{T}^2\mathbf{T}} \big) \\ &= \tfrac{1}{[2]} \Big([n_3 + 2n_2 - 1]([n_3 + 2n_2] - [n_3]) + (-[n_3])[n_3 + 2n_2 + 1] \Big), \end{aligned} \tag{160}$$

where the third equality uses (156). To justify the second equality, note that $a^{F_V}_{\mathbf{D}'\mathbf{D}}$ is nonzero only if $\mathbf{D}'$ has at least as many height-j invariants as $\mathbf{D}$. Given this, we can read off from case (C) that $\mathbf{T}^1$ and $\mathbf{T}^2$ are the only possibilities for $\mathbf{T}''$ that contribute to the sum.

There are shapes ν for which the coefficient $a^{F_V^{(2)}}_{\mathbf{T}'\mathbf{T}}$ is not a polynomial in q with all nonnegative or all nonpositive coefficients. For example, if $n_3 = 3$ and $n_2 = 2$, then $a^{F_V^{(2)}}_{\mathbf{T}'\mathbf{T}} = q^{10} - q^4 - q^{-4} + q^{-10}$.

17.3. **The action of τ on +HNSTC.** We have enough information now about straightened highest weight NST to prove Theorem 15.21 (vi), finally completing the proof of the theorem.

Proposition 17.9. *Let $\mathbf{T}$ be a highest weight +HNSTC of shape ν and weight (λ, λ). Then*

$$\tau(\mathbf{T}) = \begin{cases} (-1)^{\nu_3}\mathbf{T} & \text{if } \cup^{3\text{-}2}(\mathbf{T}) = 0, \\ (-1)^{\nu_3 - 1}\mathbf{T} & \text{if } \cup^{3\text{-}2}(\mathbf{T}) = 1. \end{cases} \tag{161}$$

Hence the weak upper based U_q^τ-module $(\check{X}_\nu, +HNSTC(\nu))$ is an upper based U_q^τ-module.

Proof. We compute $\tau(\mathbf{T})$ by computing the action of τ on the straightened representative T of $\mathbf{T}$ using Proposition 14.20 and then straightening the result. This is a straightforward computation as the invariant-free part $\hat{T}$ of T is represented by one of the eight cases in Figure 18. Note that $\lambda = \mu$ implies that the left and right cases for $\hat{g}^0_{\lambda\mu\nu}$ and the two cases for $\hat{g}^{3\text{-}2\text{-}1}_{\lambda\mu\nu}$ cannot occur, and, in the remaining cases, $\text{ext-free}_V(\hat{T}) = \text{ext-free}_W(\hat{T})$. Finally, the computation of $\tau(\hat{T})$ gives the desired result for $\tau(\mathbf{T})$ by observing that τ fixes height-j invariants for $j \in [3]$ and takes the height-4 invariant to its negative. $\square$

Define an index set $\mathscr{P}^{\tau}_{r,2}$ for the U_q^{τ}-irreducibles having nonzero multiplicity in $X^{\otimes r}$ (see Proposition 13.1):

$$\mathscr{P}^{\tau}_{r,2} := \{\{\lambda,\mu\} : \lambda,\mu \in \mathscr{P}_{r,2},\ \lambda \neq \mu\} \sqcup \{+\lambda : \lambda \in \mathscr{P}_{r,2}\} \sqcup \{-\lambda : \lambda \in \mathscr{P}'_{r,2}\}.$$

Let $\pi : \mathbb{Z}^{12}_{\geq 0} \twoheadrightarrow \mathbb{Z}^{9}_{\geq 0}$ be the projection from (B), (C), (E) to (B), (C), (D) of Proposition 17.3 obtained by projecting (E) onto (D) in the obvious way and leaving (B) and (C) fixed. For $\alpha \in \mathscr{P}^{\tau}_{r,2}$ and $\nu \vdash_{d_X} r$, define subsets $\mathbf{g}^{\tau}_{\alpha\nu}$ of $\mathbb{Z}^{9}_{\geq 0}$ by

$$\mathbf{g}^{\tau}_{\alpha\nu} := \begin{cases} \pi(\mathbf{g}_{\lambda\mu\nu}) = \pi(\mathbf{g}_{\mu\lambda\nu}) & \text{if } \alpha = \{\lambda,\mu\}, \\ \pi(\mathbf{g}_{\lambda\lambda\nu}) \cap \{\zeta \in \mathbb{Z}^{9}_{\geq 0} : \varepsilon = (-1)^{\nu_3+\zeta_{\text{3-2}}}\} & \text{if } \alpha = \varepsilon\lambda, \end{cases} \tag{162}$$

where, in the second case, $\zeta_{\text{3-2}}$ is the $\cup^{\text{3-2}}(\cdot)$ coordinate of ζ. Corollary 17.6 together with the previous proposition then yields

Corollary 17.10. *The U_q^{τ}-module with basis $(\check{X}_{\nu}, +HNSTC(\nu))$ decomposes into U_q^{τ}-cells as*

$$+HNSTC(\nu) = \bigsqcup_{\alpha \in \mathscr{P}^{\tau}_{r,2},\ \zeta \in \mathbf{g}^{\tau}_{\alpha\nu}} \check{\Lambda}^{\tau}_{\nu,\zeta},$$

where $\check{\Lambda}^{\tau}_{\nu,\zeta}$ is the set of $\mathbf{T} \in +HNSTC(\nu)$ such that the 9-tuple of $\mathbf{T}$ given by (B)–(D) of Proposition 17.3 is equal to ζ. Moreover, similar to the comment after Corollary 17.6, (166) *implies that given $\check{\Lambda}^{\tau}_{\nu,\zeta}$, α can be determined from ν and ζ and the U_q^{τ}-module $K\check{\Lambda}^{\tau}_{\nu,\zeta}$ is isomorphic to $\mathcal{X}_{\alpha}$.*

Recall that $K\check{\mathscr{S}}(\check{X}, r)$ is the algebra dual to the coalgebra $\mathscr{O}(M_q(\check{X}))_r$. It follows from §13.6 that the U_q^{τ}-cells and $K\check{\mathscr{S}}(\check{X}, r)$-cells of $(\check{X}_{\nu}, +\text{HNSTC}(\nu))$ are the same except that $(\check{X}_{(r)}, +\text{HNSTC}((r)))$ is a $K\check{\mathscr{S}}(\check{X}, r)$-cell and the union

$$\bigsqcup_{\zeta} (K\check{\Lambda}^{\tau}_{(r),\zeta}, \check{\Lambda}^{\tau}_{(r),\zeta}) = \bigsqcup_{\lambda \vdash_2 r} (\mathcal{X}_{+\lambda}, B_V(\lambda) \star B_W(\lambda))$$

of $\lfloor \frac{r}{2} \rfloor + 1$ U_q^{τ}-cells.

18. Explicit formulae for Kronecker coefficients

Here we deduce explicit formulae for Kronecker coefficients by counting the number of (λ,μ)-cells of $(\check{X}_{\nu}, +\text{HNSTC})$. This count can be simplified by writing it as a sum of what we call invariant-free Kronecker coefficients, which correspond to counting invariant-free $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cells. We then use the Kronecker graphical calculus to organize the invariant-free Kronecker coefficients into nonnegative sums of smaller coefficients, and we use this to give a fairly simple, explicit formula for two-row Kronecker coefficients. This is the first obviously positive formula for these coefficients. We also give an elegant formula for symmetric and exterior Kronecker coefficients (§18.2). In §18.3, we compare our formulae to ones in [15] and use them to determine exactly when two-row Kronecker coefficients vanish, reproducing a result of [13].

18.1. **Invariant-free Kronecker coefficients and explicit formulae.** We can now give a fairly simple and explicit description of two-row Kronecker coefficients.

Definition 18.1. The *invariant-free Kronecker coefficient* $\hat{g}_{\lambda\mu\nu}$ is the number of invariant-free (λ,μ)-cells of +HNSTC(ν). We write $\hat{g}_{\lambda\mu\nu}$ as the sum of five terms according to the Kronecker graphical calculus described above (see Figure 18).

Contribution to $\hat{g}_{\lambda\mu\nu}$	The number of invariant-free (λ,μ)-cells of +HNSTC(ν) containing
$\hat{g}^{0}_{\lambda\mu\nu}$	no 3-2 or 2-1 arcs.
$\hat{g}^{\text{2-1}}_{\lambda\mu\nu}$	a pure 2-1 arc but no 3-2 arcs.
$\hat{g}^{\text{3-2}}_{\lambda\mu\nu}$	a pure 3-2 arc but no 2-1 arcs.
$\hat{g}^{\text{3-2-1}}_{\lambda\mu\nu}$	a 3-2-1 arc.
$\hat{g}^{\text{3-2, 2-1}}_{\lambda\mu\nu}$	a pure 3-2 arc and a pure 2-1 arc.

We use $\hat{g}^{*}_{\lambda\mu\nu}$ to refer to any of these five types of invariant-free Kronecker coefficients.

Thus, letting $\lambda,\mu,\nu = [l_2,l_1],[m_2,m_1],[n_4,n_3,n_2,n_1]$, there holds

$$g_{\lambda\mu\nu} = \sum_{i_1,i_2,i_3} \hat{g}_{\hat{\lambda}\hat{\mu}\hat{\nu}} = \sum_{i_1,i_2,i_3} \hat{g}^{0}_{\hat{\lambda}\hat{\mu}\hat{\nu}} + \hat{g}^{\text{2-1}}_{\hat{\lambda}\hat{\mu}\hat{\nu}} + \hat{g}^{\text{3-2}}_{\hat{\lambda}\hat{\mu}\hat{\nu}} + \hat{g}^{\text{3-2-1}}_{\hat{\lambda}\hat{\mu}\hat{\nu}} + \hat{g}^{\text{3-2, 2-1}}_{\hat{\lambda}\hat{\mu}\hat{\nu}}, \tag{163}$$

where

$$\begin{aligned} \hat{\lambda} &= [l_2 - i_1 - 2i_2 - 3i_3 - 2n_4, l_1], \\ \hat{\mu} &= [m_2 - i_1 - 2i_2 - 3i_3 - 2n_4, m_1], \\ \hat{\nu} &= [0, n_3 - 2i_3, n_2 - 2i_2, n_1 - 2i_1], \end{aligned} \tag{164}$$

and the sum is over all i_1, i_2, i_3 such that $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ are partitions.

Remarkably, the invariant-free Kronecker coefficients are at most 2. Moreover, the coefficients $\hat{g}^{*}_{\lambda\mu\nu}$ are at most 1. We now determine exactly when each of these is 0 or 1.

Theorem 18.2. *Maintain the notation above. The two-row Kronecker coefficients are given by* (163) *and each type of invariant-free Kronecker coefficient is 0 or 1; it is 1 if and only if (l_1,m_1) lies in the corresponding (one-dimensional) polytope shown in Figure 19, $l_1 \equiv m_1 \equiv r \mod 2$, and*

$$\begin{array}{rl} \text{[no extra condition]} & \text{for type } \hat{g}^{0}_{\lambda\mu\nu}, \\ n_1 \geq 1 \text{ and } n_2 \geq 1 & \text{for type } \hat{g}^{\text{2-1}}_{\lambda\mu\nu}, \\ n_2 \geq 1 \text{ and } n_3 \geq 1 & \text{for type } \hat{g}^{\text{3-2}}_{\lambda\mu\nu}, \\ n_1 \geq 1,\ n_2 \geq 1, \text{ and } n_3 \geq 1 & \text{for type } \hat{g}^{\text{3-2-1}}_{\lambda\mu\nu}, \\ n_1 \geq 1,\ n_2 \geq 2, \text{ and } n_3 \geq 1 & \text{for type } \hat{g}^{\text{3-2, 2-1}}_{\lambda\mu\nu}. \end{array} \tag{165}$$

Proof. Let $\mathscr{G}$ be an invariant-free (λ,μ)-cell of +HNSTC(ν). The arcs of $\mathscr{G}$ are given by one of the eight cases in Figure 18. The partitions λ and μ are can be expressed in terms of ν and the arcs of $\mathscr{G}$:

$$\begin{aligned} \tfrac{l_1+m_1}{2} &= n_3 + n_2 + n_1 - \cup^{\text{ext}}(\mathscr{G}), \\ \tfrac{l_1-m_1}{2} &= \text{ext-free}_V(\mathscr{G}) - \cup^{\text{3-1 } V}(\mathscr{G}) - \cup^{\text{3-2-1 } V}(\mathscr{G}) - (\text{ext-free}_W(\mathscr{G}) - \cup^{\text{3-1 } W}(\mathscr{G}) - \cup^{\text{3-2-1 } W}(\mathscr{G})). \end{aligned}$$

Then for each case of Figure 18 we have

$$\begin{array}{lrcllrcl}
\hat{g}^{0}_{\lambda\mu\nu}(\text{middle}) & \frac{l_1+m_1}{2} & = & n_3+n_2+n_1 & \frac{l_1-m_1}{2} & = & \text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G}),\\
\hat{g}^{0}_{\lambda\mu\nu}(\text{left}) & m_1 & = & n_3+2n_2+n_1 & l_1 & = & n_1+n_3-2\cup^{3\text{-}1}(\mathscr{G}),\\
\hat{g}^{0}_{\lambda\mu\nu}(\text{right}) & l_1 & = & n_3+2n_2+n_1 & m_1 & = & n_1+n_3-2\cup^{3\text{-}1}(\mathscr{G}),\\
\hat{g}^{2\text{-}1}_{\lambda\mu\nu} & \frac{l_1+m_1}{2} & = & n_3+n_2+n_1-1 & \frac{l_1-m_1}{2} & = & \text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G}),\\
\hat{g}^{3\text{-}2}_{\lambda\mu\nu} & \frac{l_1+m_1}{2} & = & n_3+n_2+n_1-1 & \frac{l_1-m_1}{2} & = & \text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G}),\\
\hat{g}^{3\text{-}2\text{-}1}_{\lambda\mu\nu}(\text{left}) & m_1 & = & n_3+2n_2+n_1-2 & l_1 & = & n_1+n_3-2-2\cup^{3\text{-}1}(\mathscr{G}),\\
\hat{g}^{3\text{-}2\text{-}1}_{\lambda\mu\nu}(\text{right}) & l_1 & = & n_3+2n_2+n_1-2 & m_1 & = & n_1+n_3-2-2\cup^{3\text{-}1}(\mathscr{G}),\\
\hat{g}^{3\text{-}2,\ 2\text{-}1}_{\lambda\mu\nu} & \frac{l_1+m_1}{2} & = & n_3+n_2+n_1-2 & \frac{l_1-m_1}{2} & = & \text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G}).
\end{array} \tag{166}$$

The theorem follows by simply recording the contribution to $\hat{g}^{*}_{\lambda\mu\nu}$ for each of the eight cases. There is only one free parameter in each case: $\cup^{3\text{-}1}(\mathscr{G})$ in the $\hat{g}^{3\text{-}2\text{-}1}_{\lambda\mu\nu}$ cases and the left and right cases of $\hat{g}^{0}_{\lambda\mu\nu}$ and $\text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G})$ in the other cases. The eight line segments of Figure 19 are thus obtained from the eight cases.

The endpoints of each line segment are read off easily from (166) and the constraints

$$\begin{array}{rcl}
\cup^{3\text{-}1}(\mathscr{G})+\cup^{3\text{-}2\text{-}1}(\mathscr{G}) & \leq & \min(n_3,n_1),\\
|\text{ext-free}_V(\mathscr{G})-\text{ext-free}_W(\mathscr{G})| & \leq & n_2-\cup^{2\text{-}1}(\mathscr{G})-\cup^{3\text{-}2}(\mathscr{G})-\cup^{3\text{-}2\text{-}1}(\mathscr{G}).
\end{array}$$

The additional conditions in (165) are clearly necessary and it is not hard to see that they are the only additional conditions needed. □

18.2. **The symmetric and exterior Kronecker coefficients.** Maintain the notation from §13.4 and Definition 15.20. We now obtain an elegant formula for symmetric and exterior Kronecker coefficients. Let $\hat{g}^{*}_{\varepsilon\lambda\nu}$ be the number of invariant-free $\varepsilon\lambda$-cells of $+\mathrm{HNSTC}(\nu)$ containing arcs as specified by $*$.

Corollary 18.3. *Maintain the notation above and that of* (163) *and* (164). *Then*

$$g_{\varepsilon\lambda\nu}=\sum_{i_1,i_2,i_3}\hat{g}^{0}_{\varepsilon'\hat{\lambda}\hat{\nu}}+\hat{g}^{2\text{-}1}_{\varepsilon'\hat{\lambda}\hat{\nu}}+\hat{g}^{3\text{-}2}_{\varepsilon'\hat{\lambda}\hat{\nu}}+\hat{g}^{3\text{-}2,\ 2\text{-}1}_{\varepsilon'\hat{\lambda}\hat{\nu}}, \tag{167}$$

where $\varepsilon'=(-1)^{n_4}\varepsilon$ (we have identified the symbol $+$ with $+1$ and $-$ with -1). Moreover, the coefficients $\hat{g}^{*}_{\varepsilon\lambda\nu}$ are 0 or 1, and they are 1 if and only if the following conditions are met:

$$\begin{array}{ll}
\hat{g}^{0}_{\varepsilon\lambda\nu} & n_2 \textit{ is even, } \varepsilon=(-1)^{n_3}, \textit{ and}\\
& l_1=n_1+n_2+n_3,\\
\hat{g}^{2\text{-}1}_{\varepsilon\lambda\nu} & n_2 \textit{ is odd, } \varepsilon=(-1)^{n_3}, \textit{ and}\\
& l_1=n_1+n_2+n_3-1,\quad n_1\geq 1, n_2\geq 1,\\
\hat{g}^{3\text{-}2}_{\varepsilon\lambda\nu} & n_2 \textit{ is odd, } \varepsilon=(-1)^{n_3+1}, \textit{ and}\\
& l_1=n_1+n_2+n_3-1,\quad n_2\geq 1, n_3\geq 1,\\
\hat{g}^{3\text{-}2,\ 2\text{-}1}_{\varepsilon\lambda\nu} & n_2 \textit{ is even, } \varepsilon=(-1)^{n_3+1}, \textit{ and}\\
& l_1=n_1+n_2+n_3-2,\quad n_1\geq 1, n_2\geq 2, n_3\geq 1.
\end{array} \tag{168}$$

Proof. This follows from the discussion above, Theorem 18.2, and Proposition 17.9. The parity conditions on n_2 come from the condition $l_1\equiv m_1\equiv r \mod 2$ from Theorem 18.2, the fact that $n_1+n_3\equiv r \mod 2$, and the constraint on l_1 in each case. □

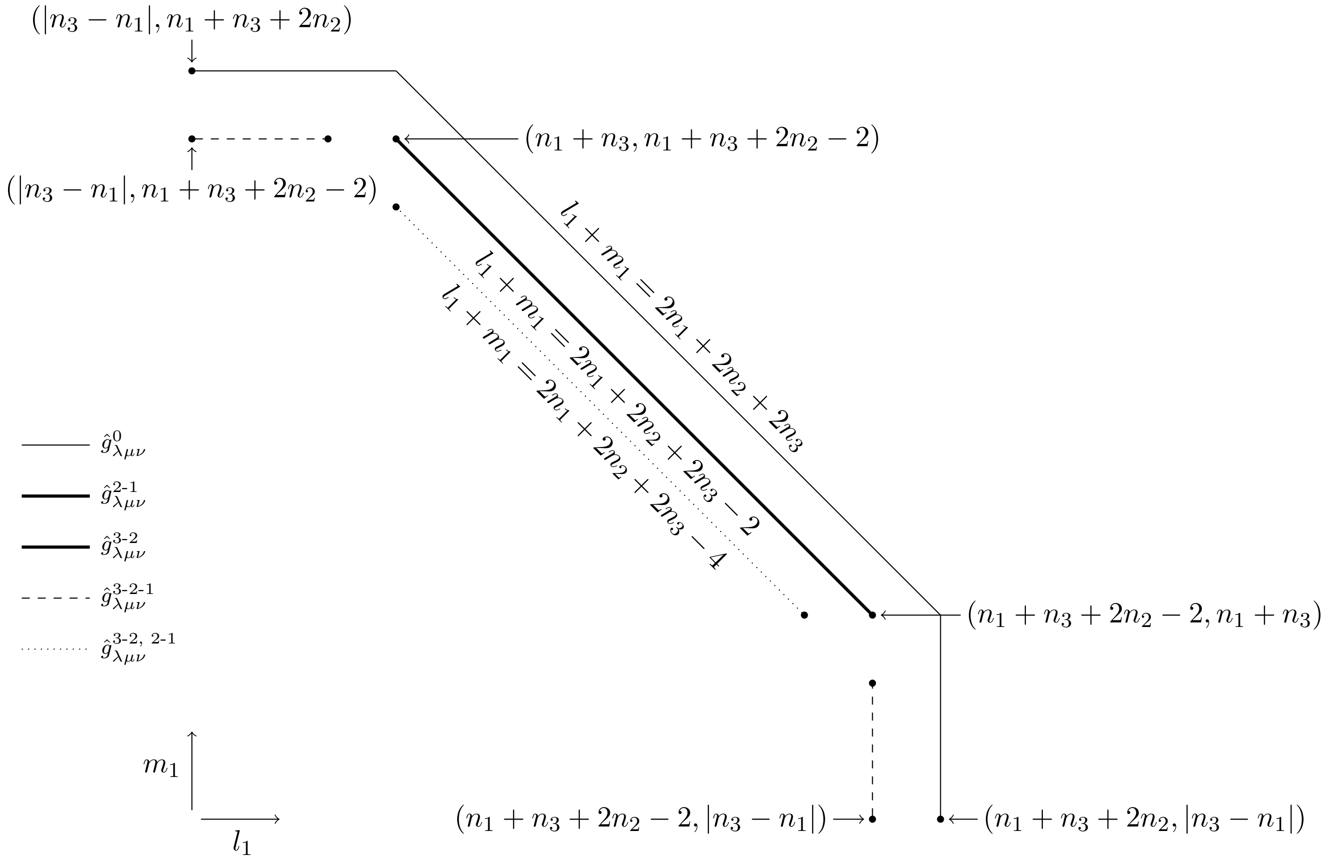


Figure 19: Polytopes for the five types of invariant-free Kronecker coefficients.

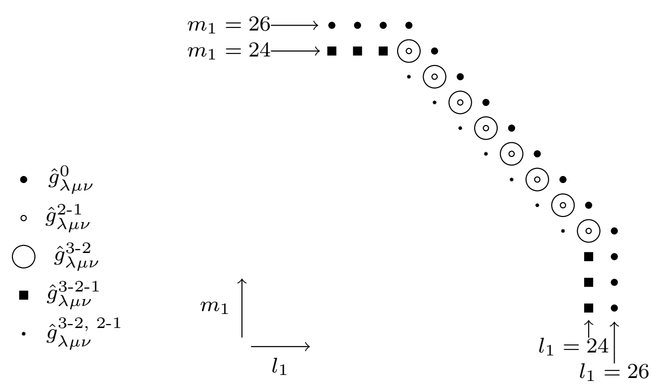


Figure 20: The contributions to the invariant-free Kronecker coefficient $\hat{g}_{\lambda\mu\nu}$ for $\nu = [0, 7, 8, 3]$. The vertex styles distinguish the five types of invariant-free Kronecker coefficients.

We can now obtain particularly nice formulae for symmetric and exterior Kronecker coefficients by assembling them into generating functions and explicitly evaluating (167).

Define the *symmetric* (resp. *exterior*) *Kronecker generating function*

$$g_{\varepsilon\nu}(x) := \sum_{\lambda\vdash_2 r} g_{\varepsilon\lambda\nu} x^{l_1}, \quad \varepsilon = + \text{ (resp. } \varepsilon = -). \tag{169}$$

For $k, l \in \mathbb{Z}$, $k \le l$, define $[\![k,l]\!] = x^l + x^{l-2} + \cdots + x^{k'}$, where k' is k if $k' \equiv l \mod 2$ and $k'+1$ otherwise. Also set $[\![^{\varepsilon}k,l]\!]$ to be $[\![k,l]\!]$ if $(-1)^{k-l} = \varepsilon$ and 0 otherwise. Lastly, set $[\![l]\!] := [\![0,l]\!]$ and $[\![^{\varepsilon}l]\!] := [\![^{\varepsilon}0,l]\!]$.

Corollary 18.4. *The symmetric and exterior Kronecker generating functions are given by*

$$g_{\varepsilon\nu}(x) = \begin{cases} [\![n_1]\!][\![n_2]\!][\![n_3]\!] & \text{if } \quad (-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-1]\!][\![n_3]\!]x & \text{if } -(-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1]\!][\![n_2-1]\!][\![n_3-1]\!]x & \text{if } -(-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-2]\!][\![n_3-1]\!]x^2 & \text{if } \quad (-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1. \end{cases}$$

Proof. We first prove that

$$\begin{aligned} g_{\varepsilon\nu}(x) \;=\; & [\![n_1]\!][\![^{+}n_2]\!][\![^{\varepsilon'}n_3]\!] + [\![n_1-1]\!][\![^{+}n_2-1]\!][\![^{\varepsilon'}n_3]\!]x + \\ & [\![n_1]\!][\![^{+}n_2-1]\!][\![^{\varepsilon'}n_3-1]\!]x + [\![n_1-1]\!][\![^{+}n_2-2]\!][\![^{\varepsilon'}n_3-1]\!]x^2, \end{aligned} \tag{170}$$

where $\varepsilon' = (-1)^{n_4}\varepsilon$. This is straightforward from Corollary 18.3. Each term of (170) corresponds to one of (167). For example, for the $\hat{g}^{\text{2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}}$ contribution we have

$$\begin{aligned} \sum_{\lambda\vdash_2 r}\sum_{i_1,i_2,i_3} \hat{g}^{\text{2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}} x^{l_1} &= \sum_{i_1,i_2,i_3} x^{\hat{n}_1+\hat{n}_2+\hat{n}_3-1} \sum_{\lambda\vdash_2 r} \hat{g}^{\text{2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}} = x^{-1}\prod_{j=1}^{3}\sum_{i_j} x^{\hat{n}_j}\Big(\sum_{\lambda\vdash_2 r}\hat{g}^{\text{2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}}\Big) \\ &= x^{-1}[\![1,n_1]\!][\![^{+}1,n_2]\!][\![^{\varepsilon'}n_3]\!] = [\![n_1-1]\!][\![^{+}n_2-1]\!][\![^{\varepsilon'}n_3]\!]x, \end{aligned}$$

which accounts for the second term of (170). Here we have set $\hat{\lambda} = [\hat{l}_2, \hat{l}_1]$, $\hat{\nu} = [0, \hat{n}_3, \hat{n}_2, \hat{n}_1]$, and the first equality uses $l_1 = \hat{l}_1$ and the second that $\sum_{\lambda\vdash_2 r}\hat{g}^{\text{2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}}$ is independent of $\hat{\nu}$ and is either 0 or 1, hence equal to its cube. (The sum $\sum_{\lambda\vdash_2 r}\hat{g}^{\text{3-2, 2-1}}_{\varepsilon'\hat{\lambda}\hat{\nu}}$ does depend on $\hat{\nu}$, but this can be dealt with by changing the summation bounds on i_2.)

The result then follows from (170) by noting that for each of the four possibilities for $(-1)^{n_2}, (-1)^{n_3+n_4}\varepsilon$, exactly one term on the right-hand side is nonzero. □

18.3. **Comparisons with other formulae.** The recent paper [15] gives a very nice explicit formula for the Kronecker coefficients $g_{[d,0]\mu\nu}$, where r is even and $d = r/2$. Maintain the notation of (163) and (164) and define the generating function

$$g_{[d,0]\nu}(x) := \sum_{\mu\vdash_2 r} g_{[d,0]\mu\nu} x^{m_1}. \tag{171}$$

Then the main result of [15], rephrased in our notation, is

$$\begin{aligned} g_{[d,0]\nu}(x^{1/2}) \;=\; & \sum_{k=0}^{d}\Big(\sum_{j=0}^{k}\chi\{n_3-j, n_2-(k-j), n_1-j \text{ are even and nonnegative}\} + \\ & \sum_{j=1}^{k}\chi\{n_3-j, n_2-(k+1-j), n_1-j \text{ are even and nonnegative}\}\Big)x^k, \end{aligned} \tag{172}$$

where $\chi\{P\}$ is equal to 1 if P is true and 0 otherwise.

This result can be reproduced from Theorem 18.2 as it follows from the theorem that

$$g_{[d,0]\nu}(x^{1/2}) = \sum_{\substack{m_1=0,\\ m_1 \text{ even}}}^{r} \sum_{i_1,i_2,i_3} \hat{g}_{\hat\lambda\hat\mu\hat\nu}(x^{1/2})^{m_1} = \sum_{k:=\frac{m_1}{2}=0}^{d}\Big(\sum_{i_1,i_2,i_3}\hat{g}^{0}_{\hat\lambda\hat\mu\hat\nu} + \hat{g}^{\text{3-2-1}}_{\hat\lambda\hat\mu\hat\nu}\Big)x^k. \tag{173}$$

Here we have set $\lambda=[d,0]$ and have used that only the reduced Kronecker coefficients $\hat{g}^{0}_{\hat\lambda\hat\mu\hat\nu}$ and $\hat{g}^{\text{3-2-1}}_{\hat\lambda\hat\mu\hat\nu}$ contribute to $g_{\lambda\mu\nu}$. The right-hand sides of (172) and (173) are readily seen to be equal by noting that

$$\begin{aligned} \hat{g}^{0}_{\hat\lambda\hat\mu\hat\nu} &= \chi\{\hat n_1=\hat n_3\}\chi\{m_1=2\hat n_1+2\hat n_2\}, \\ \hat{g}^{\text{3-2-1}}_{\hat\lambda\hat\mu\hat\nu} &= \chi\{\hat n_1\ge 1\}\chi\{\hat n_2\ge 1\}\chi\{\hat n_1=\hat n_3\}\chi\{m_1=2\hat n_1+2\hat n_2-2\} \end{aligned} \tag{174}$$

(where $\hat\lambda,\hat\mu,\hat\nu$ are as in the right-hand side of (173)) and identifying j with $\hat n_1=\hat n_3$ and $\hat n_2$ with $k-j$ (resp. $k+1-j$) for the top (resp. bottom) line of (172). Interestingly, not only do our formulae coincide in this case, but they also decompose Kronecker coefficients into a sum of smaller nonnegative quantities in a similar way.

Using similar arguments to those for Proposition 18.4 and with the notation of the proposition, (173) can be converted into the following compact expression.

Proposition 18.5. *The Kronecker coefficients for* $\lambda=[d,0]$ *are given by*

$$g_{[d,0]\nu}(x^{1/2}) = [\![\min(n_1,n_3)]\!][\![n_2]\!] + [\![\min(n_1,n_3)-1]\!][\![n_2-1]\!]x.$$

Proof. From (173) and (174), we obtain

$$\begin{aligned} g_{[d,0]\nu}(x^{1/2}) \;&=\; \sum_{\substack{i_1,i_2,i_3,\\ \hat n_3=\hat n_1}} \hat{g}^{0}_{\hat\lambda\hat\mu\hat\nu}x^{\hat n_1+\hat n_2} + \hat{g}^{\text{3-2-1}}_{\hat\lambda\hat\mu\hat\nu}x^{\hat n_1+\hat n_2-1} \\ &=\; [\![\min(n_1,n_3)]\!][\![n_2]\!] + [\![1,\min(n_1,n_3)]\!][\![1,n_2]\!]x^{-1}. \end{aligned}$$

□

In [13], the authors give an explicit description of which two-row Kronecker coefficients are zero. We can use the explicit formulae for Kronecker coefficients established in this section to reproduce this result. (Note that the conditions in (7) of [13] involve a description of a cone Δ from [12], which is roughly the cone generated by positive Kronecker coefficients; also, there is a mistake in the 2008 arXiv version of [13] that will be corrected in a later version.)

Proposition 18.6. *Let $\lambda, \mu, \nu = [l_2, l_1], [m_2, m_1], [n_4, n_3, n_2, n_1]$ as above. The Kronecker coefficient $g_{\lambda\mu\nu}$ is 0 if and only if at least one of the following conditions is satisfied*

$$\frac{l_1 + m_1}{2} > n_1 + n_2 + n_3; \tag{175}$$

$$\frac{|l_1 - m_1|}{2} > \min(n_1, n_3) + n_2; \tag{176}$$

$$n_1 = n_3 = 0 \text{ and } \frac{l_1 + m_1}{2} \not\equiv n_2 \mod 2; \tag{177}$$

$$\min(l_1, m_1) = 0,\ \min(n_1, n_2, n_3) = 0,\ \text{and } \frac{\max(l_1, m_1)}{2} \not\equiv n_1 + n_2 \mod 2; \tag{178}$$

$$\{l_1, m_1\} = \{0, 2\},\ \text{and } n_1, n_2, n_3 \text{ are even}; \tag{179}$$

$$l_1 = m_1 = 0,\ \text{and } n_1 \text{ or } n_2 \text{ is odd.} \tag{180}$$

Note that if $\min(l_1, m_1) = 0$, then l_1 and m_1 are even, and n_1 and n_3 have the same parity.

Proof. By Theorem 18.2, $g_{\lambda\mu\nu} = 0$ if (175) or (176) holds. For the remainder of the proof, assume that (175) and (176) do not hold.

Assume in addition that n_1, n_3 are not both 0, and l_1, m_1 are not 0. By reducing to the $\lambda = \mu$ case, we will show that $g_{\lambda\mu\nu} > 0$. If $\lambda = \mu$ (equivalently, $l_1 = m_1$), then we can see directly from Corollary 18.4 (keeping in mind the parity condition $r \equiv l_1 \equiv m_1 \equiv n_1 + n_3 \mod 2$) that $g_{\lambda\mu\nu} = g_{+\lambda\nu} + g_{-\lambda\nu} > 0$. Moreover, if n_2 is even, then there is a (λ, λ)-cell $\mathscr{G}$ of $+\mathrm{HNSTC}(\nu)$ such that $\cup^{\mathrm{ext}}(\mathscr{G}) = 0$.

Now suppose (without loss of generality) that $l_1 > m_1$. Set $\hat{\lambda} = [l_2, m_1]$ and

$$\hat{\nu} = [\hat{n}_4, \hat{n}_3, \hat{n}_2, \hat{n}_1] = \begin{cases} [n_4, n_3, n_2 - \frac{(l_1 - m_1)}{2}, n_1] & \text{if } n_2 \geq \frac{l_1 - m_1}{2}, \\ [n_4, n_3 - (\frac{l_1 - m_1}{2} - n_2), 0, n_1 - (\frac{l_1 - m_1}{2} - n_2)] & \text{if } n_2 < \frac{l_1 - m_1}{2}. \end{cases}$$

By the previous paragraph, there is a $(\hat{\lambda}, \hat{\mu})$-cell $\hat{\mathscr{G}}$ of $+\mathrm{HNSTC}(\hat{\nu})$ such that if $\hat{n}_2 = 0$, then $\cup^{\mathrm{ext}}(\hat{\mathscr{G}}) = 0$. Then by Corollary 17.4 and Figure 18, there is a (λ, μ)-cell $\mathscr{G}$ of $+\mathrm{HNSTC}(\nu)$ obtained from $\hat{\mathscr{G}}$ by setting $\text{ext-free}_V(\mathscr{G}) = \text{ext-free}_V(\hat{\mathscr{G}}) + \min(n_2, \frac{l_1 - m_1}{2})$, and, if $n_2 < \frac{l_1 - m_1}{2}$, $\cup^{3\text{-}1\ W}(\mathscr{G}) = \cup^{3\text{-}1\ W}(\hat{\mathscr{G}}) + (\frac{l_1 - m_1}{2} - n_2)$, and otherwise keeping the data (B),(C),(E) of Proposition 17.3 the same for $\mathscr{G}$ and $\hat{\mathscr{G}}$.

Finally, it is a direct check using Theorem 18.2 that if $n_1 = n_3 = 0$, then $g_{\lambda\mu\nu}$ is 0 if (177) holds and 1 otherwise. If $\min(l_1, m_1) = 0$, then we can read off when $g_{\lambda\mu\nu} = 0$ from Proposition 18.5, which accounts for (178), (179), and (180). □

19. Future work

19.1. **A canonical basis for $\check{X}^{\otimes r}$.** Recall that $K\check{\mathscr{S}}(\check{X}, r)$ is the algebra dual to the coalgebra $\mathscr{O}(M_q(\check{X}))_r$ (see Theorem 12.1) and the index set $\check{\mathscr{P}}_{r,2}$ from (124) is

$$\check{\mathscr{P}}_{r,2} = \{\{\lambda, \mu\} : \lambda, \mu \in \mathscr{P}_{r,2},\ \lambda \neq \mu\} \sqcup \{+\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{-\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{\check{\epsilon}_+\}.$$

Recall from the introduction that we seek a basis $\check{B}^r$ of $\check{X}^{\otimes r} = \check{\mathbf{T}}$ (assume here that $d_V = d_W = 2$) so that the $(K\check{\mathscr{S}}(\check{X}, r), \check{\mathscr{H}}_r)$-bimodule with basis $(\check{\mathbf{T}}, \check{B}^r)$ is compatible (in the precise sense of §2.4) with the decomposition

$$\check{X}^{\otimes r} \cong \bigoplus_{\alpha \in \check{\mathscr{P}}_{r,2}} \check{\mathscr{X}}_\alpha \otimes \check{M}_\alpha, \tag{181}$$

and so that the $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule with basis $(\check{\mathbf{T}}|_{q=1}, \check{B}^r|_{q=1})$ is compatible with the decomposition

$$\check{X}^{\otimes r}|_{q=1} \cong \bigoplus_{\nu \vdash_{d_X} r} X_\nu|_{q=1} \otimes M_\nu|_{q=1}. \tag{182}$$

We now state a more detailed version of this conjecture.

For any $\alpha \vDash_l^{d_X} r$, let $J_\alpha \subseteq S$ be as in §2.3. Define the *nonstandard J_α-descent space* to be $\check{Y}_\alpha \subseteq \check{X}^{\otimes r}$. For any $b \in \check{X}^{\otimes r}$, the *nonstandard descent set* of b is the maximal J such that b belongs to the nonstandard J-descent space.

For $\alpha \in \check{\mathscr{P}}_{r,2}$, we write $(\lambda, \mu) \in \alpha$ if

$$\begin{aligned}
&\alpha = \check{\epsilon}_+ \text{ and } \lambda = \mu \vdash_{d_V} r,\\
&\alpha = \pm\nu \text{ and } \nu = \lambda = \mu, \text{ or}\\
&\alpha = \{\lambda, \mu\}.
\end{aligned}$$

Define the partial order $\preceq$ on $\check{\mathscr{P}}_{r,2}$ by

$$\begin{aligned}
\alpha^1 \preceq \alpha^2 \text{ if } (\lambda^i, \mu^i) \in \alpha^i,\ i = 1, 2 \text{ for some } \lambda^i, \mu^i \text{ such that } \lambda^1 \trianglelefteq \lambda^2 \text{ and } \mu^1 \trianglelefteq \mu^2\\
\text{with the exceptions that } +\lambda \not\preceq -\lambda,\ -\lambda \not\preceq +\lambda, \text{ and } \check{\epsilon}_+ \not\preceq \alpha \text{ for all } \alpha \neq \check{\epsilon}_+.
\end{aligned}$$

Note that this implies $\alpha \preceq \check{\epsilon}_+$ for all $\alpha \in \check{\mathscr{P}}_{r,2}$.

The multiplicity $m_{\alpha\nu}$ ($\alpha \in \check{\mathscr{P}}_{r,2}$, $\nu \vdash_{d_X} r$) was defined in the introduction to be the multiplicity of the $\mathcal{S}_r$-irreducible $M_\nu|_{q=1}$ in $\check{M}_\alpha|_{q=1}$. By (11), (10), and Theorem 1.6, this is also equal to the multiplicity of $\check{\mathscr{X}}_\alpha$ in $\check{X}_\nu$ (the $\check{\mathscr{X}}_\alpha$ are defined before Corollary 13.5).

Conjecture 19.1. *There is a basis $\check{B}^r$ of $\check{\mathbf{T}}$ making $(\check{\mathbf{T}}, \check{B}^r)$ into an upper based U_q^τ-module satisfying (i)–(vi) below. To each $b \in \check{B}^r$ there is associated*

$$\begin{aligned}
&\lambda(b) \vdash_{d_V} r,\ \mu(b) \vdash_{d_W} r,\ \nu(b) \vdash_{d_X} r,\ \alpha(b) \in \check{\mathscr{P}}_{r,2},\\
&P_V(b) \in SSYT_{d_V}(\lambda(b)),\ P_W(b) \in SSYT_{d_W}(\mu(b)),\\
&Q^1(b) \in SYT(\nu(b)),\ \zeta(b) \in \mathbf{g}_{\alpha(b)\nu(b)},
\end{aligned}$$

such that $(\lambda(b), \mu(b)) \in \alpha(b)$. Here, $\mathbf{g}_{\alpha\nu}$ is a set of cardinality $m_{\alpha\nu}$. Define the following subsets of $\check{B}^r$:

$$\begin{aligned}
\check{\Gamma}_{\alpha,P_V,P_W} &:= \{b : \alpha(b) = \alpha,\ P_V(b) = P_V,\ P_W(b) = P_W\},\\
\check{\Gamma}^1_{\alpha,P_V,P_W,\nu,\zeta} &:= \{b : \alpha(b) = \alpha,\ P_V(b) = P_V,\ P_W(b) = P_W,\ \nu(b) = \nu,\ \zeta(b) = \zeta\},\\
\check{\Lambda}_{\alpha,Q,\zeta} &:= \{b : \alpha(b) = \alpha,\ Q^1(b) = Q,\ \zeta(b) = \zeta\},\\
\check{\Lambda}^1_Q &:= \{b : Q^1(b) = Q\}.
\end{aligned}$$

(i) *The nonstandard J-descent spaces of $\check{\mathbf{T}}$ are spanned by subsets of $\check{B}^r$, i.e. for each $\alpha \vDash_l^{d_X} r$, $\check{Y}_\alpha$ is a $K\check{\mathscr{S}}(\check{X}, r)$-cellular submodule of $(\check{\mathbf{T}}, \check{B}^r)$.*

(ii) *The decomposition of* $\check{B}^r$ *into* $\check{\mathscr{H}}_r$*-cells is*

$$\check{B}^r = \bigsqcup_{\substack{\alpha\in\check{\mathscr{P}}_{r,2} \\ (\mathrm{sh}(P_V),\mathrm{sh}(P_W))\in\alpha}} \check{\Gamma}_{\alpha,P_V,P_W},$$

and $\mathbf{A}\check{\Gamma}_{\alpha,P_V,P_W} \cong \check{M}_\alpha^{\mathbf{A}}$. *The partial order on cells is refined by* $\prec$, *i.e.* $\check{\Gamma}_{\alpha,P_V,P_W} <_{\check{B}^r} \check{\Gamma}_{\alpha',P'_V,P'_W}$ *implies* $\alpha \prec \alpha'$ *and* P_V *(resp.* P_W*) has the same content as* P'_V *(resp.* P_W*).*

(iii) *We say that a* $\mathbb{Q}\mathcal{S}_r$*-cell of* $(\check{\mathbf{T}}|_{q=1}, \check{B}^r|_{q=1})$ *is a* quasi-cell. *The decomposition into quasi-cells is*

$$\check{B}^r|_{q=1} = \bigsqcup_{\substack{\alpha\in\check{\mathscr{P}}_{r,2},\ (\mathrm{sh}(P_V),\mathrm{sh}(P_W))\in\alpha \\ \nu\vdash_{d_X} r,\ \zeta\in\mathbf{g}_{\alpha\nu}}} \check{\Gamma}^1_{\alpha,P_V,P_W,\nu,\zeta},$$

and $\mathbb{Q}\check{\Gamma}^1_{\alpha,P_V,P_W,\nu,\zeta}|_{q=1} \cong \mathbb{Q}M_\nu|_{q=1}$. *The partial order on quasi-cells is refined by dominance order in* ν.

(iv) *The decomposition of* $\check{B}^r$ *into* $K\check{\mathscr{S}}(\check{X}, r)$*-cells is*

$$\check{B}^r = \bigsqcup_{\substack{\alpha\in\check{\mathscr{P}}_{r,2},\ \nu\vdash_{d_X} r \\ Q\in SYT(\nu),\ \zeta\in\mathbf{g}_{\alpha\nu}}} \check{\Lambda}_{\alpha,Q,\zeta},$$

and $K\check{\Lambda}_{\alpha,Q,\zeta} \cong \check{\mathcal{X}}_\alpha$. *The partial order on* $K\check{\mathscr{S}}(\check{X}, r)$*-cells is refined by* $\prec$.

(v) *The set of* $K\check{\mathscr{S}}(\check{X}, r)$*-cells* $\{\check{\Lambda}_{\alpha,Q,\zeta}\}$ *of* $(\check{\mathbf{T}}, \check{B}^r)$ *can be partitioned into* $K\check{\mathscr{S}}(\check{X}, r)$*-cellular subquotients* $\check{\Lambda}^1_Q = \bigsqcup_{\alpha,\zeta} \check{\Lambda}_{\alpha,Q,\zeta}$ *called* fat cells. *The decomposition into fat cells is given by*

$$\check{B}^r = \bigsqcup_{\nu\vdash_{d_X} r,\ Q\in SYT(\nu)} \check{\Lambda}^1_Q,$$

and $\mathbb{Q}\check{\Lambda}^1_Q|_{q=1} \cong \mathrm{Res}_{U^\tau}(X_\nu|_{q=1})$. *Moreover, each* $\check{\Lambda}^1_Q$ *is a subset of the nonstandard* $R(Q)$*-descent space, where*

$$R(Q) = \{s_i : i+1 \textit{ is strictly to the south of } i \textit{ in } Q\}$$

(this is the same as the descent set in (26)*). The partial order on fat cells at* $q = 1$ *is refined by dominance order in* $\mathrm{sh}(Q)$, *i.e.* $\check{\Lambda}^1_Q|_{q=1} <_{\check{B}^r|_{q=1}} \check{\Lambda}^1_{Q'}|_{q=1}$ *implies* $\mathrm{sh}(Q) \lhd \mathrm{sh}(Q')$.

(vi) *In the case* $Q = (Z^*_{\nu'})^T$, *the fat cell* $(K\check{\Lambda}^1_Q, \check{\Lambda}^1_Q)$ *is equal to* $(\check{X}_\nu, +HNSTC(\nu))$ *(after perhaps modifying the sign convention* (139) *for* $+HNSTC(\nu)$*).*

Several remarks are now in order. One of the difficulties in constructing $\check{B}^r$ is that the integral form $\check{\mathbf{T}}_{\mathbf{A}} := \mathbf{A}\check{B}^r$ will not be equal to $X_{\mathbf{A}}^{\otimes r}$, and the lattice $\check{\mathscr{L}} := K_\infty \check{B}^r$ will not be equal to $\mathscr{L}_V \star_{K_\infty} \mathscr{L}_W$ (in the notation of §7). We expect that $\check{\mathbf{T}}_{\mathbf{A}}$ and $\check{\mathscr{L}}$ will be close to the $\mathbf{A}$ and K_∞-span of SNST((1^r)), or at least something of the same flavor; see Example 19.3 for what happens in the $r = 4$ case. Note that it follows from the general theory of crystal bases that once $\check{\mathbf{T}}_{\mathbf{A}}$ and $\check{\mathscr{L}}$ are specified, it only suffices to specify the image of the highest weight elements of $\check{B}^r$ in $\check{\mathscr{L}}/q^{-1}\check{\mathscr{L}}$.

The assumption that $(\check{\mathbf{T}}, \check{B}^r)$ is an upper based U_q^τ-module implies that each U_q^τ-cell of $(\check{\mathbf{T}}, \check{B}^r)$ is isomorphic to one of the irreducible upper based U_q^τ-modules from (123). It then follows from §13.6 that the U_q^τ-cells and $K\check{\mathscr{S}}(\check{X}, r)$-cells of $(\check{\mathbf{T}}, \check{B}^r)$ are the same except that $\check{\Lambda}_{\check{\epsilon}_+, Z^*_{(r)}, \zeta}$ is a union of $\lfloor \frac{r}{2} \rfloor + 1$ U_q^τ-cells, where $Z^*_{(r)}$ is the SYT of shape (r). The reason that U_q^τ is mentioned in the conjecture is that we have a theory of based modules for U_q^τ, but not for $K\check{\mathscr{S}}(\check{X}, r)$. If such a theory is developed for $K\check{\mathscr{S}}(\check{X}, r)$ or for the hypothetical nonstandard enveloping algebra, then this conjecture should be strengthened to accommodate it.

The set $\mathbf{g}_{\alpha\nu}$ above could be taken to be $\mathbf{g}^\tau_{\alpha\nu}$ from Corollary 18.3 in the case $\nu \neq (r)$. We may want to allow this set to depend on the SYT Q.

Regarding (ii), it can be shown that $(\check{\mathbf{T}}, \check{B}^r)$ an upper based U_q^τ-module implies $\check{\Gamma}_{\alpha, P_V, P_W} \cong \check{\Gamma}_{\alpha, P'_V, P'_W}$ for all $\alpha \in \check{\mathscr{P}}_{r,2}$ and SSYT P_V, P'_V, P_W, P'_W such that $(\mathrm{sh}(P_V), \mathrm{sh}(P_W)) \in \alpha$, $(\mathrm{sh}(P'_V), \mathrm{sh}(P'_W)) \in \alpha$.

Regarding (iv), the statement about partial order follows from $(\check{\mathbf{T}}, \check{B}^r)$ being an upper based U_q^τ-module, except in the case one of cells is $\check{\Lambda}_{\check{\epsilon}_+, Z^*_{(r)}, \zeta}$, However, requirement (i) implies that this cell is a maximal element for the partial order on $K\check{\mathscr{S}}(\check{X}, r)$-cells.

Requirement (vi) can be strengthened to say that if $\alpha \vDash_l^{d_X} r$ and ν' is the partition obtained from α by sorting its parts and Q is the unique $\mathrm{SYT}(\nu)$ with $R(C_Q) = J_\alpha$, then there is a straightforward generalization $+\mathrm{HNSTC}'(\alpha)$ of $+\mathrm{HNSTC}(\nu)$, as in Remark 15.13, such that $(K\check{\Lambda}^1_Q, \check{\Lambda}^1_Q)$ is equal to $(\check{Y}_\alpha/\check{Y}_{\triangleright\alpha}, +\mathrm{HNSTC}'(\alpha))$.

Regarding (v), Example 19.3 shows that we should not demand that all the fat cells of the same shape (the shape being the shape of Q) are isomorphic as $K\check{\mathscr{S}}(\check{X}, r)$-modules with basis. This example also has two quasi-cells of the same shape that are not isomorphic as $\mathbb{Q}\mathcal{S}_4$-modules with basis.

Although we have not done many computations outside of the two-row case, we are hopeful that Conjecture 19.1 holds for general $d = d_V = d_W$. For this generalization, $\check{\mathscr{P}}_{r,2}$ would need to be replaced by an index set $\check{\mathscr{P}}_{r,d}$ parameterizing irreducible representations of $K\check{\mathscr{H}}_{r,d}$.

Example 19.2. Recall nonstandard Schur-Weyl duality in the two-row case for $r = 3$:

$$\check{X}^{\otimes 3} \cong \check{S}^3\check{X} \otimes \check{\epsilon}_+ \;\oplus\; \check{\mathcal{X}}_{\{(3),(2,1)\}} \otimes \check{M}_{\{(3),(2,1)\}} \;\oplus\; \check{\mathcal{X}}_{+(2,1)} \otimes S'\check{M}_{(2,1)} \;\oplus\; \check{\Lambda}^3\check{X} \otimes \check{\epsilon}_-.$$

Define the basis $\check{B}^3$ of $\check{X}^{\otimes 3}$ to be the union of the following $K\check{\mathscr{S}}(\check{X}, r)$-cells.

$$\begin{array}{lll}
\check{\Gamma}_{-(2,1), \begin{smallmatrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{smallmatrix}} & := & +\mathrm{HSNST}((3)), \\
\check{\Gamma}_{+(2,1), \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}} & & \text{is defined below}, \\
\check{\Gamma}_{+(2,1), \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}} & & \text{is defined below}, \\
\check{\Gamma}_{\{(3),(2,1)\}, \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}} & := & \text{ext-free}\big(+\mathrm{HSNST}((2,1))\big), \\
\check{\Gamma}_{\{(3),(2,1)\}, \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}} & := & \text{ext-free}\big(+\mathrm{HSNST}((1,2))\big),
\end{array}$$

$$\check{\Gamma}_{\check{\epsilon}_+,\boxed{1}\boxed{2}\boxed{3}} \quad := \quad \mathrm{Lex}(+\mathrm{HSNST}((1,1,1))),$$

where Lex(+HSNST(ν')) denotes the set of straightened +HSNST of shape ν' and ext-free denotes those +HSNST with no external arcs. Here we have suppressed ζ from the notation $\check{\Gamma}_{\alpha,Q,\zeta}$ because all the sets $\mathbf{g}_{\alpha\nu}$ have size 0 or 1.

This basis satisfies the requirements of Conjecture 19.1. There are four fat cells—the two fat cells of shape $(2,1)$ are the union of two $K\check{\mathscr{S}}(\check{X},r)$-cells. In this case, the quasi-cells are the same as the $\check{\mathscr{H}}_3$-cells. Each $\check{\mathscr{H}}_3$-cell of $\check{X}^{\otimes 3}$ is isomorphic to a right $\check{\mathscr{H}}_3$-cell of the cellular basis $\check{\mathcal{C}}^3$ of $K\check{\mathscr{H}}_3$ defined in (114).

$$\begin{tikzcd}
\frac{1}{2}\left(\begin{smallmatrix}2&3\\3&\end{smallmatrix}+\begin{smallmatrix}3&1\\4&\end{smallmatrix}\right) \arrow[d, "1"'] & \frac{1}{2}\left(\begin{smallmatrix}2&1\\3&\end{smallmatrix}+\begin{smallmatrix}3&1\\2&\end{smallmatrix}\right) \arrow[l, "1"'] \arrow[d, "1"'] \\
\frac{1}{2}\left(\begin{smallmatrix}2&3\\4&\end{smallmatrix}+\begin{smallmatrix}3&2\\4&\end{smallmatrix}\right) & \frac{1}{2}\left(\begin{smallmatrix}2&1\\4&\end{smallmatrix}+\begin{smallmatrix}3&2\\2&\end{smallmatrix}\right) \arrow[l, "1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{1}{2}\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix}+\begin{smallmatrix}4&1\\&3\end{smallmatrix}\right) \arrow[d, "1"'] & \frac{1}{2}\left(\begin{smallmatrix}3&1\\&2\end{smallmatrix}+\begin{smallmatrix}2&3\\&1\end{smallmatrix}\right) \arrow[l, "1"'] \arrow[d, "1"'] \\
\frac{1}{2}\left(\begin{smallmatrix}4&3\\&2\end{smallmatrix}+\begin{smallmatrix}4&2\\&3\end{smallmatrix}\right) & \frac{1}{2}\left(\begin{smallmatrix}4&1\\&2\end{smallmatrix}+\begin{smallmatrix}2&2\\&3\end{smallmatrix}\right) \arrow[l, "1"']
\end{tikzcd}$$

$$\check{\Gamma}_{+(2,1),\begin{smallmatrix}1&3\\2&\end{smallmatrix}} \qquad\qquad \check{\Gamma}_{+(2,1),\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$$

Example 19.3. We describe a basis $\check{B}^4$ of $\check{X}^{\otimes 4}$ satisfying the requirements of Conjecture 19.1. As remarked above, it is enough to specify the integral form $\check{\mathbf{T}}_{\mathbf{A}}$, the lattice $\check{\mathscr{L}}$, and the highest weight elements of $\check{B}^4$. Define the integral form $\check{\mathbf{T}}_{\mathbf{A}}$ and the lattice $\check{\mathscr{L}}$ to be the $\mathbf{A}$ and K_∞-span of SNST$'((1^4))$, respectively, where

$$\mathrm{SNST}'((1^4)) := \bigsqcup_{T\in \mathrm{NST}((1^4))} \left\{(-\tfrac{1}{[2]})^{\deg'(T)}T, -(-\tfrac{1}{[2]})^{\deg'(T)}T\right\},$$

just as in Definition 15.4 and $\deg'$ is a slightly different definition of degree: it agrees with the earlier definition except that $\deg'(\boxed{4}\boxed{2}\boxed{3}\boxed{1}) = \deg'(\boxed{4}\boxed{3}\boxed{2}\boxed{1}) := 1$ (with the earlier definition, they have degree 0).

The highest weight elements of $\check{B}^4$ are partitioned into the following $\check{\mathscr{H}}_4$-cells:

$$\begin{aligned}
\check{\Gamma}_{-(2,2),\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\begin{smallmatrix}1&1\\2&2\end{smallmatrix}} &= \left\{-\tfrac{1}{[2]}\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}\right\},\\
\check{\Gamma}_{+(2,2),\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\begin{smallmatrix}1&1\\2&2\end{smallmatrix}} &= \left\{-\tfrac{1}{[2]}\begin{smallmatrix}3&1\\4&2\end{smallmatrix}, -\tfrac{1}{[2]}\begin{smallmatrix}2&1\\4&3\end{smallmatrix}\right\},\\
\check{\Gamma}_{-(3,1),\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix},\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}} &= \left\{\begin{smallmatrix}1&1\\2&\\3&\end{smallmatrix}, \begin{smallmatrix}1&1\\3&2\end{smallmatrix} - \begin{smallmatrix}1&1\\2&3\end{smallmatrix}, \begin{smallmatrix}1&1\\&2\\&3\end{smallmatrix}\right\},\\
\check{\Gamma}_{\{(3,1),(2,2)\},\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix},\begin{smallmatrix}1&1\\2&2\end{smallmatrix}} &= \text{the 6 elements in Figure 21},\\
\check{\Gamma}_{\{(3,1),(2,2)\},\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}} &= \{\tau(b) : b \text{ is an element in Figure 21}\},\\
\check{\Gamma}_{+(3,1),\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix},\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}} &= \text{the 5 elements in Figure 22},
\end{aligned}$$

$$
\begin{aligned}
\check{\Gamma}_{\{(4),(2,2)\},\ytableaushort{1111},\ytableaushort{11,22}} &= \left\{\ytableaushort{11,22}, \ytableaushort{211,\none 2}\right\}, \\
\check{\Gamma}_{\{(4),(2,2)\},\ytableaushort{11,22},\ytableaushort{1111}} &= \left\{\ytableaushort{11,33}, \ytableaushort{311,\none 3}\right\}, \\
\check{\Gamma}_{\{(4),(3,1)\},\ytableaushort{1111},\ytableaushort{111,2}} &= \left\{\ytableaushort{111,2}, \ytableaushort{111,\none 2}, \ytableaushort{111,\none\none 2}\right\}, \\
\check{\Gamma}_{\{(4),(3,1)\},\ytableaushort{111,2},\ytableaushort{1111}} &= \left\{\ytableaushort{111,3}, \ytableaushort{111,\none 3}, \ytableaushort{111,\none\none 3}\right\}, \\
\check{\Gamma}_{\check{\epsilon}_+,\ytableaushort{1111},\ytableaushort{1111}} &= \left\{\ytableaushort{1111}\right\}, \\
\check{\Gamma}_{\check{\epsilon}_+,\ytableaushort{111,2},\ytableaushort{111,2}} &= \left\{-\frac{1}{[2]}\ytableaushort{1141}\right\}, \\
\check{\Gamma}_{\check{\epsilon}_+,\ytableaushort{11,22},\ytableaushort{11,22}} &= \left\{\frac{1}{[2]^2}\ytableaushort{4141}\right\}.
\end{aligned}
$$

As evidence that this basis is nice, the coefficients that show up for the action of F_V, F_W, E_V, and E_W lie in

$$\frac{1}{2}\mathbb{Z}\{[2],[3],[2]^2,[4],a_2\},$$

and the coefficients for the action of $\mathcal{Q}_i$ lie in

$$\frac{1}{2}\mathbb{Z}\{[2],[2]^2,a_2,c_0\},$$

where $a_2 = [2]^2 - 2$ and $c_0 = [2]^2 - 4$.

Figures 21 and 22 are examples of $\check{\mathscr{H}}_4$-cells that are not quasi-cells. The top row of Figure 21 is the quasi-cell $\check{\Gamma}^1_{\{(3,1),(2,2)\},Z_{(3,1)},Z_{(2,2)},(2,1,1)}$, which spans a $\mathbb{Q}\mathcal{S}_4$-cellular submodule of $\mathbb{Q}\check{\Gamma}_{\{(3,1),(2,2)\},Z_{(3,1)},Z_{(2,2)}}|_{q=1}$ isomorphic to $M_{(2,1,1)}|_{q=1}$ (we are omitting $\zeta \in \mathbf{g}_{\alpha\nu}$ from the notation because all Kronecker coefficients for $r=4$ are 1 or 0). The bottom row is a quasi-cell, which spans a $\mathbb{Q}\mathcal{S}_4$-cellular quotient of $\mathbb{Q}\check{\Gamma}_{\{(3,1),(2,2)\},Z_{(3,1)},Z_{(2,2)}}|_{q=1}$ isomorphic to $M_{(3,1)}|_{q=1}$. The top row of Figure 22 is the quasi-cell $\check{\Gamma}^1_{+(3,1),Z_{(3,1)},Z_{(3,1)},(2,2)}$, and the bottom row is the quasi-cell $\check{\Gamma}^1_{+(3,1),Z_{(3,1)},Z_{(3,1)},(3,1)}$.

The following fat cells of $(\check{X}^{\otimes 4}, \check{B}^4)$ are not isomorphic (as $K\check{\mathscr{S}}(\check{X},r)$-modules with basis)

$$+\mathrm{HNSTC}((2,1,1)) = \check{\Lambda}^1_{\ytableaushort{14,2,3}} \neq \check{\Lambda}^1_{\ytableaushort{13,2,4}}.$$

The fat cell $\check{\Lambda}^1_{\ytableaushort{14,2,3}}$ is the union of the $K\check{\mathscr{S}}(\check{X},r)$-cell $\check{\Lambda}_{-(3,1),\ytableaushort{14,2,3}}$ (which has highest weight $\ytableaushort{11,2,3}$) and the $K\check{\mathscr{S}}(\check{X},r)$-cell $\check{\Lambda}_{\{(3,1),(2,2)\},\ytableaushort{14,2,3}}$ (which has highest weights $-\ytableaushort{11,2,4}, \ytableaushort{11,3,4}$); the fat cell $\check{\Lambda}^1_{\ytableaushort{13,2,4}}$ is the union of the $K\check{\mathscr{S}}(\check{X},r)$-cell $\check{\Lambda}_{-(3,1),\ytableaushort{13,2,4}}$ (which has highest weight $\ytableaushort{11,32} - \ytableaushort{11,23}$) and the $K\check{\mathscr{S}}(\check{X},r)$-cell $\check{\Lambda}_{\{(3,1),(2,2)\},\ytableaushort{13,2,4}}$ (which has highest weights $\ytableaushort{31,22}, \ytableaushort{21,33}$).

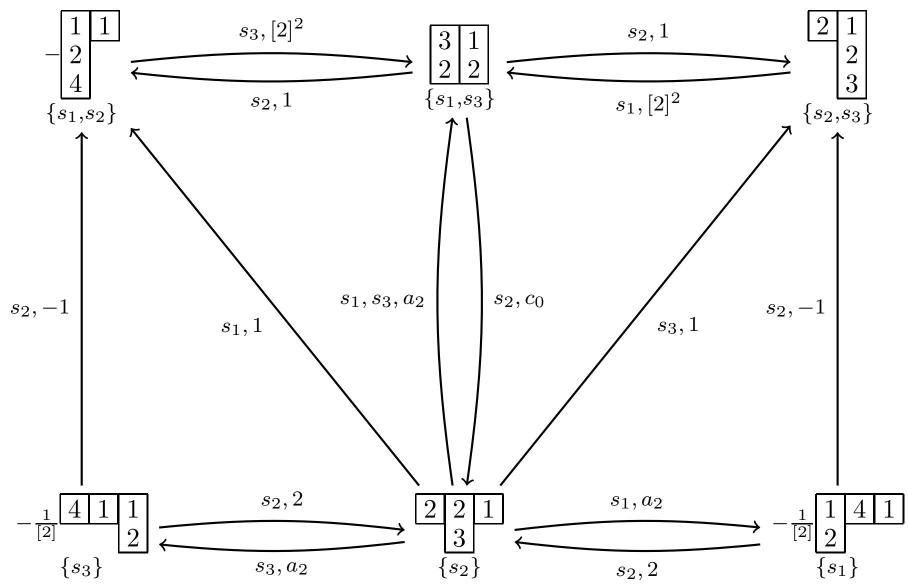


Figure 21: The $\check{\mathscr{H}}_4$-cell $\check{\Gamma}_{\{(3,1),(2,2)\},Z_{(3,1)},Z_{(2,2)}}$ of $\check{B}^4$. Below each basis element is its nonstandard descent set; if s_i is in the nonstandard descent set of b, then $b\mathcal{Q}_i = [2]^2 b$. The action of $\mathcal{Q}_i$ on the basis elements without a nonstandard descent at i is given by the edges and their labels, where $a_2 = [2]^2 - 2,\ c_0 = [2]^2 - 4$.

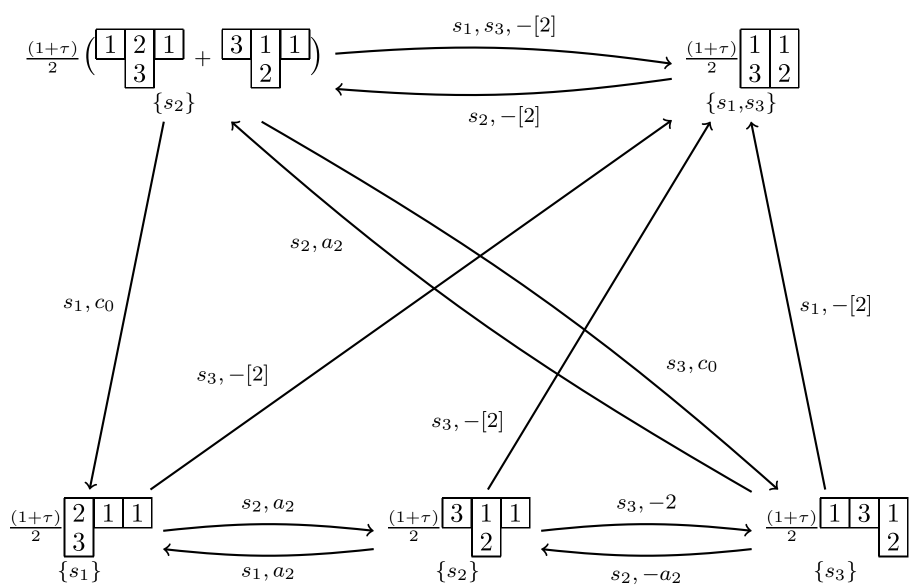


Figure 22: The $\check{\mathscr{H}}_4$-cell $\check{\Gamma}_{+(3,1),Z_{(3,1)},Z_{(3,1)}}$ of $\check{B}^4$. Conventions are the same as for Figure 21.

We have also been able to construct some $\check{\mathscr{H}}_r$-cells of a basis satisfying the requirements of Conjecture 19.1 for $r = 5$ and $r = 6$. In addition, we can construct some quasi-cells for $r = 7$ and $r = 8$.

We are also hopeful that the nonstandard Temperley-Lieb quotient $R\check{\mathscr{H}}_{r,d}$ of $R\check{\mathscr{H}}_r$ (see §13.6) for suitable R has a canonical basis $\check{\mathcal{C}}^{r,d}$ that is a cellular basis in the sense of [21] and is compatible with nonstandard descent spaces, as described in §11.7. We would also like this basis to have right cells isomorphic to the $\check{\mathscr{H}}_r$-cells of $\check{X}^{\otimes r}$ from Conjecture 19.1. We have only been able to construct such a basis for $r = 3$ (see (114)), and the next easiest case $r = 4$, $d = 2$ seems to be quite difficult. We would hope to obtain $\check{\mathcal{C}}^{r,d}$ by a globalization procedure similar to that used for the Hecke algebra in [28], or quantum group representations in [26, 27], but this may require having a presentation and monomial basis for $\check{\mathscr{H}}_{r,d}$, which we know to be difficult from §11.8. We would also like a relatively simple description of the lattice $K_\infty\check{\mathcal{C}}^{r,d}$, but even in the $r = 3$ case, the only description we have is as the K_∞-span of the canonical basis $\check{\mathcal{C}}^3$, rather than as the K_∞-span of certain monomials in the $\mathcal{Q}_i$, say.

19.2. **Defining $\check{X}_\nu$ outside the two-row case.** Let d_V, d_W be arbitrary and let $\nu \vdash_{d_X} r$, $\ell(\nu') = 2$. It is possible to define a version of $\check{X}_\nu$ in this case—this corresponds to the special case of the Kronecker problem in which one of the partitions has two columns and the other two are arbitrary. As before, for $\alpha \vDash_l^{d_X} r$, set

$$\check{Y}_\alpha = \check{\Lambda}^{\alpha_1}\check{X} \otimes \check{\Lambda}^{\alpha_2}\check{X} \otimes \ldots \otimes \check{\Lambda}^{\alpha_l}\check{X}.$$

Following the construction of Schur modules from [3, 61] mentioned in the introduction, one can define injective $\mathscr{O}(GL_q(\check{X}))$-comodule homomorphisms $\iota^L_{\nu'} : \check{Y}_{\nu'_1+1,\nu'_2-1} \hookrightarrow \check{Y}_{\nu'_1,\nu'_2}$, $\iota^R_{\nu'} : \check{Y}_{\nu'_2-1,\nu'_1+1} \hookrightarrow \check{Y}_{\nu'_1,\nu'_2}$. Then define

$$\begin{aligned} \check{Y}^L_{\rhd\nu'} &= \operatorname{im}(\iota^L_{\nu'}), \quad & \check{X}^L_\nu &= \check{Y}_{\nu'}/\check{Y}^L_{\rhd\nu'}, \\ \check{Y}^R_{\rhd\nu'} &= \operatorname{im}(\iota^R_{\nu'}), \quad & \check{X}^R_\nu &= \check{Y}_{\nu'}/\check{Y}^R_{\rhd\nu'}. \end{aligned}$$

This yields two $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules $\check{X}^L_\nu, \check{X}^R_\nu$ that specialize to $\operatorname{Res}_{U(\mathfrak{g}_V\oplus\mathfrak{g}_W)} X_\nu|_{q=1}$ at $q = 1$. The modules $\check{Y}_{\nu'}, \check{Y}^L_{\rhd\nu'}, \check{Y}^R_{\rhd\nu'}$ all come with canonical bases, but we do not yet understand the maps $\iota^L_{\nu'}$ and $\iota^R_{\nu'}$ combinatorially. This is not an easy task, but it would yield a combinatorial formula for Kronecker coefficients in the case that one of the partitions has two columns and the other two are arbitrary (actually, it would yield two different formulae, one from $\check{X}^L_\nu$ and one from $\check{X}^R_\nu$).

Extending this approach to $\ell(\nu') > 2$ meets with serious difficulties. The K-vector subspaces $\check{Y}^L_{\rhd\nu'}$ and $\check{Y}^R_{\rhd\nu'}$ of $\check{Y}_{\nu'}$ are not in general equal, even though they both have integral forms that specialize to the same thing at $q = 1$. As a consequence, if we define $\check{X}_\nu$ for $\ell(\nu') > 2$ as we have before (as in (140)), using either $\check{Y}^L_{\rhd\nu'}$ or $\check{Y}^R_{\rhd\nu'}$ to define $\check{Y}_{\rhd^i\nu'}$, then the resulting $\check{X}_\nu$ is in general too small, i.e. its K-dimension is less than the $\mathbb{Q}$-dimension of $X_\nu|_{q=1}$. We hope that understanding the quasi-cells of Conjecture 19.1 will help us better understand this difficulty and suggest a way around it.

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Appendix A. Reduction system for $\mathscr{O}(M_q(\check{X}))$

In this section we reformulate the relations (79) for $\mathscr{O}(M_q(\check{X}))$ in the form of a reduction system, as mentioned in §8.10, and show that this does not satisfy the diamond property.

To define the reduction system, define the following total order $\succeq$ on the variables $z_a^{a'} \in \check{Z}$, $a', a \in [d_X]$: $z_a^{a'} \succeq z_b^{b'}$ if $a > b$ or ($a = b$ and $a' \geq b'$), i.e. $(a, a') \geq (b, b')$ lexicographically. We say that a monomial $z_{a_1}^{a'_1} \cdots z_{a_r}^{a'_r} \in \mathscr{O}(M_q(\check{X}))_r$ is *descending* if $z_{a_1}^{a'_1} \succeq \cdots \succeq z_{a_r}^{a'_r}$ and *nondescending* otherwise. We shall now see that the set of degree 2 descending monomials

$$B^{\check{Z}} := \{z_a^{a'} z_b^{b'} : (a, a') \succeq (b, b')\} \subseteq \mathscr{O}(M_q(\check{X}))_2, \tag{183}$$

is a basis of $\mathscr{O}(M_q(\check{X}))_2$.

Set $\mathbf{A}' = \mathbf{A}[\frac{1}{[2]}]$. To define specializations at $q = 1$, define the following integral forms:

$$\begin{array}{rcl}
S_q^2 V_{\mathbf{A}'} & := & \mathbf{A}' \tilde{B}_+^V, \\
\Lambda_q^2 V_{\mathbf{A}'} & := & \mathbf{A}' \tilde{B}_-^V, \\
V_{\mathbf{A}'}^{\otimes r} & := & \mathbf{A}'\{v_{\mathbf{k}} : \mathbf{k} \in [d_V]^r\}, \\
\check{S}^2 \check{X}_{\mathbf{A}'} & := & S_q^2 V_{\mathbf{A}'} \star S_q^2 W_{\mathbf{A}'} \oplus \Lambda_q^2 V_{\mathbf{A}'} \star \Lambda_q^2 W_{\mathbf{A}'}, \\
\check{\Lambda}^2 \check{X}_{\mathbf{A}'} & := & S_q^2 V_{\mathbf{A}'} \star \Lambda_q^2 W_{\mathbf{A}'} \oplus \Lambda_q^2 V_{\mathbf{A}'} \star S_q^2 W_{\mathbf{A}'}, \\
(\check{\mathcal{I}}_2)_{\mathbf{A}'} & := & \check{S}^2 \check{X}_{\mathbf{A}'} * \check{\Lambda}^2 \check{X}^*_{\mathbf{A}'} \oplus \check{\Lambda}^2 \check{X}_{\mathbf{A}'} * \check{S}^2 \check{X}^*_{\mathbf{A}'}, \\
(\check{\mathcal{I}}_2^{\perp})_{\mathbf{A}'} & := & \check{S}^2 \check{X}_{\mathbf{A}'} * \check{S}^2 \check{X}^*_{\mathbf{A}'} \oplus \check{\Lambda}^2 \check{X}_{\mathbf{A}'} * \check{\Lambda}^2 \check{X}^*_{\mathbf{A}'}, \\
\check{Z}_{\mathbf{A}'}^{\otimes r} & := & \mathbf{A}'\{z_{a_1}^{a'_1} \otimes \cdots \otimes z_{a_r}^{a'_r} : \mathbf{a}, \mathbf{a}' \in [d_X]^r\},
\end{array} \tag{184}$$

where $\tilde{B}_+^V$, $\tilde{B}_-^V$ are as in §8.2, and $S_q^2 W_{\mathbf{A}'}$, $\Lambda_q^2 W_{\mathbf{A}'}$ are defined similarly to $S_q^2 V_{\mathbf{A}'}, \Lambda_q^2 V_{\mathbf{A}'}$ and $\check{S}^2 \check{X}^*_{\mathbf{A}'}, \check{\Lambda}^2 \check{X}^*_{\mathbf{A}'}$ are defined similarly to $\check{S}^2 \check{X}_{\mathbf{A}'}$, $\check{\Lambda}^2 \check{X}_{\mathbf{A}'}$. These are integral forms of the corresponding K vector spaces, meaning that $K \otimes_{\mathbf{A}'} S_q^2 V_{\mathbf{A}'} \cong S_q^2 V$, $K \otimes_{\mathbf{A}'} \check{S}^2 \check{X}_{\mathbf{A}'} \cong \check{S}^2 \check{X}$, etc. ($\check{S}^2 \check{X}$ is defined in (59)).

One checks that $V_{\mathbf{A}'}^{\otimes 2} = S_q^2 V_{\mathbf{A}'} \oplus \Lambda_q^2 V_{\mathbf{A}'}$. It follows that

$$\check{Z}_{\mathbf{A}'}^{\otimes 2} = (\check{\mathcal{I}}_2)_{\mathbf{A}'} \oplus (\check{\mathcal{I}}_2^{\perp})_{\mathbf{A}'},$$

and therefore

$$(\mathscr{O}(M_q(\check{X}))_2)_{\mathbf{A}'} := \check{Z}_{\mathbf{A}'}^{\otimes 2} / (\check{\mathcal{I}}_2)_{\mathbf{A}'} \cong (\check{\mathcal{I}}_2^{\perp})_{\mathbf{A}'} \tag{185}$$

is an integral form of $\mathscr{O}(M_q(\check{X}))_2$. Additionally, one checks from the explicit formulae for $\tilde{B}_+^V$ and $\tilde{B}_-^V$ in (69) that $S_q^2 V|_{q=1} = \mathbb{Q}\{v_{ij} + v_{ji} : 1 \leq i \leq j \leq d_V\}$ and $\Lambda_q^2 V|_{q=1} = \mathbb{Q}\{v_{ij} - v_{ji} : 1 \leq i < j \leq d_V\}$ (the formulae in (69) are only for the $d_V = d_W = 2$ case, but the general case is just as easy). Hence,

$$\check{\mathcal{I}}_2^{\perp}|_{q=1} = \mathbb{Q}\{z_a^{a'} z_b^{b'} + z_b^{b'} z_a^{a'} : (a,a') \succeq (b,b')\}, \tag{186}$$

$$\check{\mathcal{I}}_2|_{q=1} = \mathbb{Q}\{z_a^{a'} z_b^{b'} - z_b^{b'} z_a^{a'} : (a,a') \succ (b,b')\}. \tag{187}$$

Now since the degree 2 descending monomials lie in $\check{Z}_{\mathbf{A}'}^{\otimes 2}$, their images in $\mathscr{O}(M_q(\check{X}))_2$, i.e. $B^{\check{Z}}$, lie in $(\mathscr{O}(M_q(\check{X}))_2)_{\mathbf{A}'}$. Set $N_{\mathbf{A}'} := \mathbf{A}' B^{\check{Z}} \subseteq (\mathscr{O}(M_q(\check{X}))_2)_{\mathbf{A}'}$. By (186) and (187), $B^{\check{Z}}|_{q=1}$ is a basis of $\mathscr{O}(M_q(\check{X}))_2|_{q=1}$. Since any relation satisfied by $B^{\check{Z}}|_{q=1} \subseteq N|_{q=1}$ would yield one of $B^{\check{Z}}|_{q=1} \subseteq \mathscr{O}(M_q(\check{X}))_2|_{q=1}$, $B^{\check{Z}}|_{q=1}$ is also a basis of $N|_{q=1}$. Since a torsion-free $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{A}'$-module is free, $\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}'}$ is a free $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{A}'$-module. Thus $|B^{\check{Z}}| = \dim_{\mathbb{Q}} N|_{q=1} = \operatorname{rank}_{\mathbb{Q}[q,q^{-1}]} \mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}'}$. It follows that $\mathbb{Q} \otimes_{\mathbb{Z}} N_{\mathbf{A}'}$ is a free $\mathbb{Q}[q,q^{-1}]$-module with $\mathbb{Q}[q,q^{-1}]$-basis $B^{\check{Z}}$. We also have

$$|B^{\check{Z}}| = \binom{{d_X}^2 + 1}{2} = \operatorname{rank}_{\mathbf{A}'} (\mathscr{O}(M_q(\check{X}))_2)_{\mathbf{A}'} = \dim_K \mathscr{O}(M_q(\check{X}))_2,$$

where the second equality follows from (184). Hence $N_{\mathbf{A}'}$ is also an integral form of $\mathscr{O}(M_q(\check{X}))_2$ and, in particular, the set of degree 2 descending monomials $B^{\check{Z}}$ is a basis of $\mathscr{O}(M_q(\check{X}))_2$.

This yields a reduction system for $\check{Z}^{\otimes 2}/\check{\mathcal{I}}_2$ wherein we have

$$z_a^{a'} z_b^{b'} = \sum_i \alpha_i z_{a_i}^{a_i'} z_{b_i}^{b_i'},$$

where all $z_{a_i}^{a_i'} z_{b_i}^{b_i'}$ are descending.

By this reduction rule, there is a method of expanding any monomial $z_a^{a'} z_b^{b'} z_c^{c'}$ as

$$z_a^{a'} z_b^{b'} z_c^{c'} = \sum_i \alpha_i z_{a_i}^{a_i'} z_{b_i}^{b_i'} z_{c_i}^{c_i'}$$

wherein $(a_i, a_i') \succeq (b_i, b_i') \succeq (c_i, c_i')$. Thus every nondescending monomial may be expanded into a linear combination of descending monomials. Unfortunately, this reduction system does not obey the *diamond lemma*. In other words, there exist monomials $z_a^{a'} z_b^{b'} z_c^{c'}$ wherein two different simplifications using the above reduction rules yield two different expansions into descending monomials.

Consider the monomial $m = z_1^1 z_1^2 z_2^3$. For any monomial $z_a^{a'} z_b^{b'} z_c^{c'}$, if $(a,a') \not\succeq (b,b')$, then we may apply the reduction system above for the first two monomials and this is denoted as $(z_a^{a'} z_b^{b'}) z_c^{c'}$. We say that R_1 applies and display the result. Similar, we say that R_2 applies if $(b,b') \not\succeq (c,c')$ and denote this application by $z_a^{a'}(z_b^{b'} z_c^{c'})$. The monomial above has two expansions, viz., $R_1 R_2 R_1$ and $R_2 R_1 R_2$, reading both strings from *left to right*.

The first expansion yields

$$l_1 := (m) R_1 = q \cdot z_1^2 z_1^1 z_2^3,$$
$$l_{12} := (m) R_1 R_2 = (-1 + q^2) \cdot z_1^2 z_2^1 z_1^3 + q \cdot z_1^2 z_2^3 z_1^1.$$

The expression l_{12} has two monomials, m_1 and m_2. We have

$$(m_1) R_1 = z_2^1 z_1^2 z_1^3 = z_2^1 z_1^3 z_1^2;$$

the last equality follows since z_1^3 and z_1^2 commute, as is easy to show.

$$(m_2)R_1 = \frac{q-q^{-1}}{[2]} \cdot z_2^1 z_1^4 z_1^1 + \frac{1-q^{-2}}{[2]} \cdot z_2^2 z_1^3 z_1^1 + \frac{2}{[2]} \cdot z_2^3 z_1^2 z_1^1 + \frac{q^{-1}-q}{[2]} \cdot z_2^4 z_1^1 z_1^1,$$

Combining all this, we have $l_{121} = (m)R_1R_2R_1$:

$$\begin{aligned} l_{121} = (-1+q^2) \cdot z_2^1 z_1^3 z_1^2 + \frac{q^2-1}{[2]} \cdot z_2^1 z_1^4 z_1^1 + \frac{q-q^{-1}}{[2]} \cdot z_2^2 z_1^3 z_1^1 \\ + \frac{2q}{[2]} \cdot z_2^3 z_1^2 z_1^1 + \frac{1-q^2}{[2]} \cdot z_2^4 z_1^1 z_1^1 \end{aligned}$$

This is a linear combination of descending monomials so no further reductions are needed.

The second expansion is $(m)R_2R_1R_2$. First, we have

$$l_2 := (m)R_2 = \frac{1}{[2]}\left((q-q^{-1}) \cdot z_1^1 z_2^1 z_1^4 + (1-q^{-2}) \cdot z_1^1 z_2^2 z_1^3 + 2 \cdot z_1^1 z_2^3 z_1^2 + (q^{-1}-q) \cdot z_1^1 z_2^4 z_1^1\right)$$

This has 4 monomials, $m_1', \ldots, m_4'$, and applying R_1 to each monomial yields:

$$\begin{aligned} m_1' &= q \cdot z_2^1 z_1^1 z_1^4, \\ m_2' &= (q-q^{-1}) \cdot z_2^1 z_1^2 z_1^3 + z_2^2 z_1^1 z_1^3, \\ m_3' &= (q-q^{-1}) \cdot z_2^1 z_1^3 z_1^2 + \cdot z_2^3 z_1^1 z_1^2, \\ m_4' &= \frac{1}{[2]}\left((q^2-1) \cdot z_2^1 z_1^4 z_1^1 + (q-q^{-1}) \cdot z_2^2 z_1^3 z_1^1 + (q-q^{-1}) \cdot z_2^3 z_1^2 z_1^1 + 2 \cdot z_2^4 z_1^1 z_1^1\right). \end{aligned}$$

Whence, $l_{21} = (m)R_2R_1$ equals:

$$\begin{aligned} l_{21} \;&= \tfrac{1}{[2]}\big((q^2-1) \cdot z_2^1 z_1^1 z_1^4 &&+ \tfrac{[3]-3}{q} \cdot z_2^1 z_1^2 z_1^3 &&+ (2q-2q^{-1}) \cdot z_2^1 z_1^3 z_1^2 \\ &+ \tfrac{q(3-[3])}{[2]} \cdot z_2^1 z_1^4 z_1^1 &&+ (1-q^{-2}) \cdot z_2^2 z_1^1 z_1^3 &&+ \tfrac{3-[3]}{[2]} \cdot z_2^2 z_1^3 z_1^1 \\ &+ 2 \cdot z_2^3 z_1^1 z_1^2 &&+ \tfrac{3-[3]}{[2]} \cdot z_2^3 z_1^2 z_1^1 &&+ (2q^{-1}-2q) \cdot z_2^4 z_1^1 z_1^1\big). \end{aligned}$$

This has 9 monomials, $m_1'', \ldots, m_9''$, which on applying R_2 yields:

$$\begin{aligned} m_1'' &= (q-q^{-1}) \cdot z_2^1 z_1^3 z_1^2 + z_2^1 z_1^4 z_1^1, \\ m_2'' &= z_2^1 z_1^3 z_1^2, \\ m_3'' &= z_2^1 z_1^3 z_1^2, \\ m_4'' &= z_2^1 z_1^4 z_1^1, \\ m_5'' &= q \cdot z_2^2 z_1^3 z_1^1, \\ m_6'' &= z_2^2 z_1^3 z_1^1, \\ m_7'' &= q \cdot z_2^3 z_1^2 z_1^1, \\ m_8'' &= z_2^3 z_1^2 z_1^1, \\ m_9'' &= z_2^4 z_1^1 z_1^1. \end{aligned}$$

Finally, collating this, we get $l_{212} = (m)R_2R_1R_2$ as follows:

$$\begin{aligned} l_{212} = {} & \frac{q^3+q-3q^{-1}+q^{-3}}{[2]} \cdot z_2^1 z_1^3 z_1^2 + \frac{2q-2q^{-1}}{[2]^2} \cdot z_2^1 z_1^4 z_1^1 + \frac{2-2q^{-2}}{[2]^2} \cdot z_2^2 z_1^3 z_1^1 \\ & + \frac{q^2+4-q^{-2}}{[2]^2} \cdot z_2^3 z_1^2 z_1^1 - \frac{2q-2q^{-1}}{[2]^2} \cdot z_2^4 z_1^1 z_1^1, \end{aligned}$$

which is a linear combination of descending monomials. Observe that the expansions $(m)R_1R_2R_1$ and $(m)R_2R_1R_2$ do not coincide.

Appendix B. The Hopf algebra $\mathscr{O}_q^\tau$

Here we give the details of the construction of $\mathscr{O}_q^\tau$, a quantized coordinate algebra that is Hopf dual to U_q^τ. Let $\mathcal{H}$ be a bialgebra and $\mathcal{X}$ a left $\mathcal{H}$-comodule coalgebra via $\beta : \mathcal{X} \to \mathcal{H} \otimes \mathcal{X}$. As explained in [30, §10.2], the *left crossed coproduct coalgebra* $\mathcal{X} \rtimes \mathcal{H}$ *of* $\mathcal{H}$ *and* $\mathcal{X}$ is the coalgebra, equal to $\mathcal{X} \otimes \mathcal{H}$ as a vector space, with comultiplication and counit given by

$$\begin{aligned} \Delta(x\sharp h) &= \sum x_{(1)} \sharp (x_{(2)})^{(-1)} h_{(1)} \otimes (x_{(2)})^{(0)} \sharp h_{(2)}, \\ \epsilon(x\sharp h) &= \epsilon_{\mathcal{X}}(x)\epsilon_{\mathcal{H}}(h). \end{aligned} \tag{188}$$

Here $\Delta_{\mathcal{X}}, \Delta_{\mathcal{H}}$, and $\epsilon_{\mathcal{X}}, \epsilon_{\mathcal{H}}$ are the coproducts and counits of $\mathcal{X}$ and $\mathcal{H}$, and we have used the Sweedler notation $\Delta_{\mathcal{H}}(h) = \sum h_{(1)} \otimes h_{(2)}$, $\Delta_{\mathcal{X}}(x) = \sum x_{(1)} \otimes x_{(2)}$, $\beta(x) = \sum x^{(-1)} \otimes x^{(0)}$. The coalgebra $\mathcal{X} \rtimes \mathcal{H}$ is also called the smash coproduct and is denoted $\mathcal{X}\sharp\mathcal{H}$; we have chosen to use the $\sharp$ symbol to denote elements of $\mathcal{X} \rtimes \mathcal{H}$. With suitable assumptions, $\mathcal{X} \rtimes \mathcal{H}$ can be given the structure of a bialgebra. There is a general theorem due to Radford along these lines (see, e.g., [39, Theorem 10.6.5]); for our purposes, the following easy result will suffice.

Proposition B.1. *Maintain the notation above and further assume that* $\mathcal{H}$ *is commutative,* $\mathcal{X}$ *is a bialgebra, and* β *makes* $\mathcal{X}$ *into a left* $\mathcal{H}$*-comodule algebra. Then, giving* $\mathcal{X} \rtimes \mathcal{H}$ *the algebra structure of* $\mathcal{X} \otimes \mathcal{H}$ *makes it into a bialgebra.*

Proof. The assumption that β makes $\mathcal{X}$ into a left $\mathcal{H}$-comodule algebra means that

$$\sum x^{(-1)} x'^{(-1)} \otimes x^{(0)} x'^{(0)} = \sum (xx')^{(-1)} \otimes (xx')^{(0)}, \;\; x, x' \in \mathcal{X}. \tag{189}$$

We need to check that the coproduct Δ given in (188) is an algebra homomorphism. For $h, h' \in \mathcal{H},\ x, x' \in \mathcal{X}$,

$$\begin{aligned}
\Delta(x\sharp h)\Delta(x'\sharp h') &= \textstyle\sum x_{(1)}x'_{(1)}\sharp(x_{(2)})^{(-1)}h_{(1)}(x'_{(2)})^{(-1)}h'_{(1)} \otimes (x_{(2)})^{(0)}(x'_{(2)})^{(0)}\sharp h_{(2)}h'_{(2)} \\
&\quad \mathcal{H} \text{ is commutative} \\
&= \textstyle\sum x_{(1)}x'_{(1)}\sharp(x_{(2)})^{(-1)}(x'_{(2)})^{(-1)}h_{(1)}h'_{(1)} \otimes (x_{(2)})^{(0)}(x'_{(2)})^{(0)}\sharp h_{(2)}h'_{(2)} \\
&\quad \Delta_{\mathcal{H}} \text{ is an algebra homomorphism} \\
&= \textstyle\sum x_{(1)}x'_{(1)}\sharp(x_{(2)})^{(-1)}(x'_{(2)})^{(-1)}(hh')_{(1)} \otimes (x_{(2)})^{(0)}(x'_{(2)})^{(0)}\sharp(hh')_{(2)} \\
&\quad \text{by (189)} \\
&= \textstyle\sum x_{(1)}x'_{(1)}\sharp(x_{(2)}x'_{(2)})^{(-1)}(hh')_{(1)} \otimes (x_{(2)}x'_{(2)})^{(0)}\sharp(hh')_{(2)} \\
&\quad \Delta_{\mathcal{X}} \text{ is an algebra homomorphism} \\
&= \textstyle\sum (xx')_{(1)}\sharp((xx')_{(2)})^{(-1)}(hh')_{(1)} \otimes ((xx')_{(2)})^{(0)}\sharp(hh')_{(2)} \\
&= \Delta(hh'\sharp xx').
\end{aligned}$$

Additionally, observe that $\mathcal{X}$ a left $\mathcal{H}$-comodule algebra implies $\Delta(1\sharp 1) = 1\sharp 1 \otimes 1\sharp 1$. $\square$

In what follows we let $\tau : \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \to \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$ denote the involution given by $\tau(f \star g) = g \star f$. The next proposition follows easily from Proposition B.1 and the fact that τ is a Hopf algebra involution.

Proposition B.2. *Maintain the notation of Proposition B.1. Set $\mathcal{H} = \mathscr{F}(\mathcal{S}_2)$, the Hopf algebra of functions on the 2 element group with values in K; let e, τ be the elements of $\mathcal{S}_2$ (e the identity element) and let $e^\vee, \tau^\vee$ be the dual basis in $\mathscr{F}(\mathcal{S}_2)$. Set $\mathcal{X} = \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$, suppose $d_V = d_W$, and define β by*

$$\beta(x) = e^\vee \otimes x + \tau^\vee \otimes \tau(x),$$

for $x \in \mathcal{X}$. Then $\mathcal{H}, \mathcal{X}$, and β satisfy the hypotheses of Proposition B.1, hence

$$\mathscr{O}_q^\tau := \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \rtimes \mathscr{F}(\mathcal{S}_2)$$

is a bialgebra with multiplication equal to that of $\mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W)) \rtimes \mathscr{F}(\mathcal{S}_2)$ and coproduct given by

$$\Delta(x\sharp e^\vee) = \sum x_{(1)}\sharp e^\vee \otimes x_{(2)}\sharp e^\vee + x_{(1)}\sharp\tau^\vee \otimes \tau(x_{(2)})\sharp\tau^\vee,$$
$$\Delta(x\sharp\tau^\vee) = \sum x_{(1)}\sharp e^\vee \otimes x_{(2)}\sharp\tau^\vee + x_{(1)}\sharp\tau^\vee \otimes \tau(x_{(2)})\sharp e^\vee.$$

Here we have used the Sweedler notation $\Delta_{\mathcal{X}}(x) = \sum x_{(1)} \otimes x_{(2)}$. Moreover, $\mathscr{O}_q^\tau$ is a Hopf algebra with antipode S given by

$$S(x\sharp e^\vee) = S(x)\sharp e^\vee, \quad S(x\sharp\tau^\vee) = \tau(S(x))\sharp\tau^\vee.$$

Since there are nondegenerate Hopf pairings between $U_q(\mathfrak{g}_V)$ and $\mathscr{O}(GL_q(V))$ [30, Corollary 54, Chapter 11] and between $K\mathcal{S}_2$ and $\mathscr{F}(\mathcal{S}_2)$, it is straightforward to check that

Corollary B.3. *There is a nondegenerate pairing of Hopf algebras $\langle\cdot,\cdot\rangle : U_q^\tau \times \mathscr{O}_q^\tau \to K$ given by $\langle ya, x\sharp h\rangle = \langle y, x\rangle h(a)$, where $y \in U_q(\mathfrak{g}_V) \star U_q(\mathfrak{g}_W)$, $a \in K\mathcal{S}_2$, $x \in \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$, $h \in \mathscr{F}(\mathcal{S}_2)$.*

Let $\pi : \mathscr{O}_q^\tau \hookrightarrow \mathscr{O}(GL_q(V)) \star \mathscr{O}(GL_q(W))$, $x\sharp h \mapsto x\epsilon_{\mathcal{H}}(h)$ be the canonical surjection (in the notation of Proposition B.1). Recall the homomorphism $\tilde{\psi} : \mathscr{O}(GL_q(\check{X})) \to \mathscr{O}(GL_q(V))\star\mathscr{O}(GL_q(W)), \check{\mathbf{z}} \mapsto \mathbf{u}^V\star\mathbf{u}^W$ from Proposition 10.5. We now prove the following stronger result:

Proposition B.4. *Let $\mathscr{O}(GL_q(\check{X}))^{1,1}$ be the sub-Hopf algebra $\bigoplus_{r\in\mathbb{Z}} \mathscr{O}(GL_q(V))_r\star\mathscr{O}(GL_q(W))_r$ of $\mathscr{O}(GL_q(V))\star\mathscr{O}(GL_q(W))$ and let τ_1 (resp. τ_2) be the algebra involution of $\mathscr{O}(GL_q(\check{X}))^{1,1}$ determined by $\tau_1(u_i^j \star u_k^l) = u_k^j \star u_i^l$ (resp. $\tau_2(u_i^j \star u_k^l) = u_i^l \star u_k^j$). The map*

$$\tilde{\psi}^\tau : \mathscr{O}(GL_q(\check{X})) \to \mathscr{O}_q^\tau, \quad z \mapsto \tilde{\psi}(z)\sharp e^\vee + \tau_2(\tilde{\psi}(z))\sharp\tau^\vee \tag{190}$$

is a Hopf algebra homomorphism, and $\tilde{\psi}$ factors through $\tilde{\psi}^\tau$ via $\tilde{\psi} = \pi \circ \tilde{\psi}^\tau$.

Proof. In the proof of Proposition 8.3, it shown that

$$P_-^{\check{X}}(\mathbf{u}^V \star \mathbf{u}^W \otimes \mathbf{u}^V \star \mathbf{u}^W) = (\mathbf{u}^V \star \mathbf{u}^W \otimes \mathbf{u}^V \star \mathbf{u}^W)P_-^{\check{X}}.$$

Since $P_\pm^V$ is sent to $P_\pm^W$ under the isomorphism $\mathrm{End}(V^{\otimes 2}) \cong \mathrm{End}(W^{\otimes 2})$ induced by $V \cong W$, $P_-^{\check{X}} = P_-^V \star P_+^W + P_+^V \star P_-^W = P_-^W \star P_+^V + P_+^W \star P_-^V$. Hence, there also holds

$$P_-^{\check{X}}\tau_2(\mathbf{u}^V \star \mathbf{u}^W \otimes \mathbf{u}^V \star \mathbf{u}^W) = \tau_2(\mathbf{u}^V \star \mathbf{u}^W \otimes \mathbf{u}^V \star \mathbf{u}^W)P_-^{\check{X}}.$$

It follows that $\tilde{\psi}^\tau$ is a well-defined map. That it is an algebra homomorphism follows directly from the definition (190) and the fact that $e^\vee$ and $\tau^\vee$ are orthogonal idempotents.

To show that $\tilde{\psi}^\tau$ is a coalgebra homomorphism, one first checks directly from the definitions that

$$1 \otimes \tau_2 \circ \Delta = \tau_2 \otimes \tau_2 \circ \Delta, \quad \Delta \circ \tau_2 = 1 \otimes \tau_2 \circ \Delta, \quad 1 \otimes \tau \circ \Delta \circ \tau_2 = \tau_2 \otimes 1 \circ \Delta. \tag{191}$$

Here Δ is the coproduct of $\mathscr{O}(GL_q(\check{X}))^{1,1}$. Then, for any $z \in \mathscr{O}(GL_q(\check{X}))$, there holds

$$\begin{aligned}
\Delta \circ \tilde{\psi}^\tau(z) = {} & \Delta\big(\tilde{\psi}(z)\sharp e^\vee + \tau_2(\tilde{\psi}(z))\sharp\tau^\vee\big) \\
= {} & \textstyle\sum \tilde{\psi}(z)_{(1)}\sharp e^\vee \otimes \tilde{\psi}(z)_{(2)}\sharp e^\vee + \tilde{\psi}(z)_{(1)}\sharp\tau^\vee \otimes \tau(\tilde{\psi}(z)_{(2)})\sharp\tau^\vee \\
& + \tau_2(\tilde{\psi}(z))_{(1)}\sharp e^\vee \otimes \tau_2(\tilde{\psi}(z))_{(2)}\sharp\tau^\vee + \tau_2(\tilde{\psi}(z))_{(1)}\sharp\tau^\vee \otimes \tau(\tau_2(\tilde{\psi}(z))_{(2)})\sharp e^\vee \\
& \text{by (191)} \\
= {} & \textstyle\sum \tilde{\psi}(z)_{(1)}\sharp e^\vee \otimes \tilde{\psi}(z)_{(2)}\sharp e^\vee + \tau_2(\tilde{\psi}(z)_{(1)})\sharp\tau^\vee \otimes \tau_2(\tilde{\psi}(z)_{(2)})\sharp\tau^\vee \\
& + \tilde{\psi}(z)_{(1)}\sharp e^\vee \otimes \tau_2(\tilde{\psi}(z)_{(2)})\sharp\tau^\vee + \tau_2(\tilde{\psi}(z)_{(1)})\sharp\tau^\vee \otimes \tilde{\psi}(z)_{(2)}\sharp e^\vee \\
& \tilde{\psi} \text{ is a coalgebra homomorphism} \\
= {} & \textstyle\sum \tilde{\psi}(z_{(1)})\sharp e^\vee \otimes \tilde{\psi}(z_{(2)})\sharp e^\vee + \tau_2(\tilde{\psi}(z_{(1)}))\sharp\tau^\vee \otimes \tau_2(\tilde{\psi}(z_{(2)}))\sharp\tau^\vee \\
& + \tilde{\psi}(z_{(1)})\sharp e^\vee \otimes \tau_2(\tilde{\psi}(z_{(2)}))\sharp\tau^\vee + \tau_2(\tilde{\psi}(z_{(1)}))\sharp\tau^\vee \otimes \tilde{\psi}(z_{(2)})\sharp e^\vee \\
= {} & \textstyle\sum \big(\tilde{\psi}(z_{(1)})\sharp e^\vee + \tau_2(\tilde{\psi}(z_{(1)}))\sharp\tau^\vee\big) \otimes \big(\tilde{\psi}(z_{(2)})\sharp e^\vee + \tau_2(\tilde{\psi}(z_{(2)}))\sharp\tau^\vee\big) \\
= {} & \textstyle\sum \tilde{\psi}^\tau z_{(1)} \otimes \tilde{\psi}^\tau z_{(2)} \\
= {} & (\tilde{\psi}^\tau \otimes \tilde{\psi}^\tau) \circ \Delta(z).
\end{aligned}$$

This proves that $\tilde{\psi}^\tau$ is a coalgebra homomorphism. The compatibility of $\tilde{\psi}^\tau$ with the counits is clear. Thus $\tilde{\psi}^\tau$ is a bialgebra homomorphism. Since a bialgebra homomorphism of Hopf algebras is always a Hopf algebra homomorphism [30, §1.2.4], the result follows. □

References

[1] B. Adsul, M. Sohoni, and K. V. Subrahmanyam. Quantum deformations of the restriction of $GL_{mn}(\mathbb{C})$-modules to $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. *ArXiv e-prints*, October 2010. `arXiv:0905.0094v2`.

[2] Bharat Adsul and K. V. Subrahmanyam. A geometric approach to the Kronecker problem. I. The two row case. *Proc. Indian Acad. Sci. Math. Sci.*, 118(2):213–226, 2008.

[3] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman. Schur functors and Schur complexes. *Adv. in Math.*, 44(3):207–278, 1982.

[4] Michael Artin, William Schelter, and John Tate. Quantum deformations of GL_n. *Comm. Pure Appl. Math.*, 44(8-9):879–895, 1991.

[5] Sami H. Assaf. Dual Equivalence Graphs I: A combinatorial proof of LLT and Macdonald positivity. *ArXiv e-prints*, May 2010. `arXiv:1005.3759`.

[6] Cristina M. Ballantine and Rosa C. Orellana. On the Kronecker product $s_{(n-p,p)} * s_\lambda$. *Electron. J. Combin.*, 12:Research Paper 28, 26 pp. (electronic), 2005.

[7] Arkady Berenstein and Sebastian Zwicknagl. Braided symmetric and exterior algebras. *Trans. Amer. Math. Soc.*, 360(7):3429–3472, 2008.

[8] Jonah Blasiak. Nonstandard braid relations and Chebyshev polynomials. *ArXiv e-prints*, October 2010. `arXiv:1010.0421`.

[9] Jonah Blasiak. Quantum Schur-Weyl duality and projected canonical bases. *ArXiv e-prints*, February 2011. `arXiv:1102.1453`.

[10] Jonah Blasiak. W-graph versions of tensoring with the $\mathcal{S}_n$ defining representation. *J. Algebraic Combin.*, 34(4):545–585, 2011.

[11] Jonah Blasiak. Representation theory of the nonstandard Hecke algebra. *ArXiv e-prints*, February 2012. `arXiv:1201.2209v2`.

[12] S. Bravyi. Requirements for compatibility between local and multipartite quantum states. *ArXiv Quantum Physics e-prints*, January 2003. `arXiv:quant-ph/0301014`.

[13] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. Quasipolynomial formulas for the Kronecker coefficients indexed by two two-row shapes (extended abstract), 2008. `arXiv:0812.0861`.

[14] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. Reduced Kronecker coefficients and counter-examples to Mulmuley's strong saturation conjecture SH: with an appendix by Ketan Mulmuley. *Comput. Complex.*, 18(4):577–600, 2009.

[15] A. Brown, S. van Willigenburg, and M. Zabrocki. Expressions for Catalan Kronecker Products. *ArXiv e-prints*, September 2008. `arXiv:0809.3469`.

[16] Jonathan Brundan. Dual canonical bases and Kazhdan-Lusztig polynomials. *J. Algebra*, 306(1):17–46, 2006.

[17] Peter Buergisser and Christian Ikenmeyer. Geometric Complexity Theory and Tensor Rank. *ArXiv e-prints*, November 2010. `arXiv:1011.1350`.

[18] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I.* John Wiley & Sons Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.

[19] Igor B. Frenkel and Mikhail G. Khovanov. Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$. *Duke Math. J.*, 87(3):409–480, 1997.

[20] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[21] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[22] Ian Grojnowski and George Lusztig. On bases of irreducible representations of quantum GL_n. In *Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989)*, volume 139 of *Contemp. Math.*, pages 167–174. Amer. Math. Soc., Providence, RI, 1992.

[23] Takahiro Hayashi. Nonexistence of homomorphisms between quantum groups. *Tokyo J. Math.*, 15(2):431–435, 1992.

[24] Jin Hong and Seok-Jin Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[25] Michio Jimbo. A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.*, 11(3):247–252, 1986.

[26] Masaki Kashiwara. On crystal bases of the Q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[27] Masaki Kashiwara. Global crystal bases of quantum groups. *Duke Math. J.*, 69(2):455–485, 1993.

[28] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[29] Apoorva Khare. Functoriality of the BGG category $\mathscr{O}$. *Comm. Algebra*, 37(12):4431–4475, 2009.

[30] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[31] J. M. Landsberg. *Tensors: geometry and applications*, volume 128 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[32] Joseph M. Landsberg and Laurent Manivel. On the ideals of secant varieties of Segre varieties. *Found. Comput. Math.*, 4(4):397–422, 2004.

[33] Michael J. Larsen and Eric C. Rowell. An algebra-level version of a link-polynomial identity of Lickorish. *Math. Proc. Cambridge Philos. Soc.*, 144(3):623–638, 2008.

[34] Alain Lascoux. Produit de Kronecker des représentations du groupe symétrique. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979)*, volume 795 of *Lecture Notes in Math.*, pages 319–329. Springer, Berlin, 1980.

[35] George Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993.

[36] Ian G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[37] Yu. I. Manin. *Quantum groups and noncommutative geometry*. Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.

[38] Andrew Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[39] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[40] K. Mulmuley. Geometric complexity theory VIII: On canonical bases for the nonstandard quantum groups. Technical Report TR 2007-15, Computer Science Department, The University of Chicago, 2007.

[41] Ketan Mulmuley. Geometric complexity theory VII: Nonstandard quantum group for the plethysm problem. *CoRR*, abs/0709.0749, 2007.

[42] Ketan Mulmuley. Explicit Proofs and The Flip. *ArXiv e-prints*, September 2010. `arXiv:1009.0246`.

[43] Ketan Mulmuley. Geometric complexity theory VI: The flip via positivity. Technical report, The Computer Science Department, The University of Chicago, January 2011.

[44] Ketan Mulmuley and Milind A. Sohoni. Geometric complexity III: on deciding positivity of Littlewood-Richardson coefficients. *CoRR*, abs/cs/0501076, 2005.

[45] Ketan Mulmuley and Milind A. Sohoni. Geometric complexity theory IV: Quantum group for the Kronecker problem. *CoRR*, abs/cs/0703110, 2007.

[46] Ketan D. Mulmuley. On P vs. NP and geometric complexity theory: Dedicated to Sri Ramakrishna. *J. ACM*, 58:5:1–5:26, April 2011.

[47] Ketan D. Mulmuley and Milind Sohoni. Geometric complexity theory I: An approach to the P vs. NP and related problems. *SIAM J. Comput.*, 31(2):496–526 (electronic), 2001.

[48] Ketan D. Mulmuley and Milind Sohoni. Geometric complexity theory II: Towards explicit obstructions for embeddings among class varieties. *SIAM J. Comput.*, 38(3):1175–1206, 2008.

[49] Masatoshi Noumi, Hirofumi Yamada, and Katsuhisa Mimachi. Finite-dimensional representations of the quantum group $\mathrm{GL}_q(n;\mathbb{C})$ and the zonal spherical functions on $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$. *Japan. J. Math. (N.S.)*, 19(1):31–80, 1993.

[50] Arun Ram. A Frobenius formula for the characters of the Hecke algebras. *Invent. Math.*, 106(3):461–488, 1991.

[51] Arun Ram. Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc. (3)*, 75(1):99–133, 1997.

[52] Jeffrey B. Remmel. A formula for the Kronecker products of Schur functions of hook shapes. *J. Algebra*, 120(1):100–118, 1989.

[53] Jeffrey B. Remmel and Tamsen Whitehead. On the Kronecker product of Schur functions of two row shapes. *Bull. Belg. Math. Soc. Simon Stevin*, 1(5):649–683, 1994.

[54] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev. Quantization of Lie groups and Lie algebras. *Algebra i Analiz*, 1(1):178–206, 1989.

[55] Nicolai Reshetikhin. Multiparameter quantum groups and twisted quasitriangular Hopf algebras. *Lett. Math. Phys.*, 20(4):331–335, 1990.

[56] Mercedes H. Rosas. The Kronecker product of Schur functions indexed by two-row shapes or hook shapes. *J. Algebraic Comb.*, 14(2):153–173, 2001.

[57] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[58] Richard P. Stanley. Positivity problems and conjectures in algebraic combinatorics. In *Mathematics: frontiers and perspectives*, pages 295–319. Amer. Math. Soc., Providence, RI, 2000.

[59] Anthony Sudbery. Consistent multiparameter quantisation of $\mathrm{GL}(n)$. *J. Phys. A*, 23(15):L697–L704, 1990.

[60] Naum Ja. Vilenkin and Anatoliy U. Klimyk. *Representation of Lie groups and special functions. Vol. 3*, volume 75 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Classical and quantum groups and special functions, Translated from the Russian by V. A. Groza and A. A. Groza.

[61] Jerzy Weyman. *Cohomology of Vector Bundles and Syzygies (Cambridge Tracts in Mathematics)*. Cambridge University Press, 2003.

[62] Stanisław L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.

[63] Sebastian Zwicknagl. Braided Symmetric Algebras of Simple $U_q(\mathfrak{sl}_2)$-Modules and Their Geometry. *ArXiv e-prints*, September 2010. `arXiv:1009.2931`.

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