CONNECTIONS, CURVATURE, AND COHOMOLOGY

Volume II

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Connections, Curvature, and Cohomology

Volume II

Lie Groups, Principal Bundles, and Characteristic Classes

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VOLUME II

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Respectfully dedicated to the memory of

HEINZ HOPF

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Preface

This monograph developed out of the Abendseminar of 1958–1959 at the University of Zürich. It was originally a joint enterprise of the first author and H. H. Keller, who planned a brief treatise on connections in smooth fibre bundles. Then, in 1960, the first author took a position in the United States and geographic considerations forced the cancellation of this arrangement.

The collaboration between the first and third authors began with the former's move to Toronto in 1962; they were joined by the second author in 1965. During this time the purpose and scope of the book grew to its present form: a three-volume study, ab initio, of the de Rham cohomology of smooth bundles. In particular, the material in volume I has been used at the University of Toronto as the syllabus for an introductory graduate course on differentiable manifolds.

During the long history of this book we have had numerous valuable suggestions from many mathematicians. We are especially grateful to the faculty and graduate students of the institutions below. Our exposition of Poincaré duality is based on the master's thesis of C. Auderset, while particular thanks are due to D. Toledo for his frequent and helpful contributions. Our thanks also go to E. Stamm and the Academic Press reviewer for their criticisms of the manuscript, to which we paid serious attention. A. E. Fekete, who prepared the subject index, has our special gratitude.

We are indebted to the institutions whose facilities were used by one or more of us during the writing. These include the Departments of Mathematics of Cornell University, Flinders University, the University of Fribourg, and the University of Toronto, as well as the Institut für theoretische Kernphysik at Bonn and the Forschungsinstitut für Mathematik der Eidgenössischen Technischen Hochschule, Zürich.

The entire manuscript was typed with unstinting devotion by Frances Mitchell, to whom we express our deep gratitude.

A first class job of typesetting was done by the compositors.

A. E. Fekete and D. Johnson assisted us with the proof reading; however, any mistakes in the text are entirely our own responsibility.

Finally, we would like to thank the production and editorial staff at Academic Press for their unfailing helpfulness and cooperation. Their universal patience, while we rewrote the manuscript (ad infinitum), oscillated amongst titles, and ruined production schedules, was in large measure, responsible for the completion of this work.

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Introduction

The purpose of this monograph is to develop the theory of differential forms and de Rham cohomology for smooth manifolds and fibre bundles. The present volume deals with Lie groups and with fibre bundles whose structure group is a Lie group. In particular, the last half of the book is devoted to a detailed exposition of the Chern–Weil theory of characteristic classes.

The characteristic classes of a bundle are the cohomology classes in the image of a canonical homomorphism (the *Weil homomorphism*),

$$h : (\vee E^*)_I \to H(B),$$

where B is the base manifold and $(\vee E^*)_I$ is the algebra of invariant symmetric multilinear functions in the Lie algebra, E, of the structure group. In Chern–Weil theory (for the case of principal bundles, where the structure group coincides with the fibre) they are constructed as follows:

First, a *principal connection* is introduced in the bundle. This is essentially the choice of a suitable "horizontal" subbundle of the tangent bundle of the total space, and should be regarded as an additional, geometric structure.

Second, the *curvature* is obtained from the connection. The curvature is a 2-form with values in E; it measures the geometric "twist" of the bundle.

Finally, if Γ is an invariant symmetric function in E then a closed differential form in B representing the class, $h(\Gamma)$, is obtained by replacing each of the arguments of Γ by the curvature form.

The resulting homomorphism, h, is independent of the choice of the connection and thus is a bundle invariant. It is essentially the only de Rham cohomology invariant of the bundle, in the sense that the cohomology algebra of the total space is completely determined by $H(G)$, the differential algebra $(A(B), \delta)$, and the Weil homomorphism (as will be shown in volume III).

The material in this volume is organized as follows (a more detailed description appears below): The first three chapters contain the basic results (developed *ab initio*) on Lie groups and transformation groups. The reader may then proceed immediately to article 1 of Chapter V, in which principal bundles are defined, and then to the first six articles of Chapter VI, where the Weil homomorphism is constructed. The rest of Chapters V and VI is devoted to a wide variety of examples and special cases.

Chapter IV deals with the cohomology of Lie groups. The following chapters do not depend on it (except for article 8, Chapter VI), so that it may be omitted without loss of continuity.

Chapters VII to X describe the characteristic classes of vector bundles —an alternate approach to the Chern–Weil theory. Much of the material in these chapters is independent of the preceding work. It culminates in the Gauss–Bonnet–Chern theorem in Chapter X, which identifies the Euler class, as defined in volume I, with a characteristic class.

This volume contains about 250 problems in which a great deal of additional material is developed. For instance, the last problem of Chapter II leads the reader to a simple, elementary proof that $\pi_2(G) = 0$ (G, a compact Lie group), while the problems of Chapter VII constitute a classical course in differential geometry. However, as in volume I, the text is self-contained and does not rely on the problems.

Although most of the material in this volume is either in the literature, or is well-established folklore, we have not attempted to associate with the theorems the names of their discoverers, except where this is already common usage.

This monograph is intended for graduate students, especially those interested in global analysis or differential geometry. The present volume relies heavily on Chapters 0–V of volume I, and, to a much lesser extent, on the rest of that volume. Aside from these prerequisites, however, it is completely self-contained.

This volume will be followed by volume III which deals with the cohomology of principal bundles and homogeneous spaces.

Chapter 0. In this chapter the algebra, analysis, and topology prerequisites given in volume I are reproduced. In addition the chapter contains a summary of the definitions, notation, and principal results of that volume.

Chapter I. Lie Groups. A Lie group, G, is a group which is also a smooth manifold and for which multiplication and inversion are smooth maps. The vector fields on G, invariant under left translation, form a Lie algebra, linearly isomorphic to the tangent space, E, at the unit element, e. Thus E becomes a Lie algebra, the Lie algebra of G.

This chapter contains the standard elementary material on Lie groups (exponential map, general representations, adjoint representation, classification of abelian Lie groups) in which the relations between a group and its Lie algebra are stressed.

In the last article the invariant (Haar) integral of a smooth function on a compact Lie group is defined from the point of view of differential forms.

Chapter II. Subgroups and Homogeneous Spaces. The first two main results of this chapter (Theorem I, sec. 2.1, and Theorem II, sec. 2.9) assert that if K is a closed subgroup of a Lie group, G, then (1) K, itself, is a Lie group and (2) the homogeneous space, G/K, of cosets has a natural manifold structure. Theorem I is applied in article 2 to obtain Lie group structures in various subgroups of $GL(F)$, and to determine their Lie algebras.

Finally, in article 5, these results are applied to compact connected Lie groups. It is shown (Theorem III, sec. 2.18) that every element of such a Lie group is in a maximal torus and that any two maximal tori are conjugate. The same machinery yields the *Weyl integration formula* (Theorem IV, sec. 2.19) which asserts that if f is a smooth function on a compact Lie group, G, such that $f(xy) = f(yx)$, $x, y \in G$, then

$$\int_G f(x)\, dx = |\, W_G\,|^{-1} \int_T f(y) \det(\iota - \mathrm{Ad}^{\perp} y)\, dy,$$

where T is a maximal torus. The results of article 5 are rarely quoted, except in Chapter IV.

Chapter III. Transformation Groups. A smooth right action of a Lie group, G, on a manifold, M, is a smooth map, $M \times G \to M$, written $(z, a) \mapsto z \cdot a$, such that $z \cdot ab = (z \cdot a) \cdot b$ and $z \cdot e = z$. Such actions are the subject of Chapter III.

Article 1 contains definitions and elementary results. In article 2 we consider the subsets, $z \cdot G$, of M as embedded homogeneous spaces and prove that a smooth map $\sigma : N \to M$ satisfying $\sigma(N) \subset z \cdot G$ determines a smooth map into the corresponding homogeneous space (Theorem I, sec. 3.7). This theorem is quoted once, in article 1, Chapter VIII.

An action of G on M determines a Lie algebra homomorphism: $E \to \mathscr{X}(M)$ (E, the Lie algebra of G). In this way E acts on the algebra of differential forms on M via the substitution operator and the Lie derivative. This material is developed in articles 3 and 4 and is chiefly applied in Chapters IV and VI.

Chapter IV. Invariant Cohomology. If a Lie group, G, acts on a manifold, M, then the subalgebra, $A_I(M)$, of differential forms on M, invariant under the action, is stable under δ. The main result of article 1 (Theorem I, sec. 4.3) asserts that if G is compact and connected then the natural homomorphism, $H(A_I(M)) \to H(M)$, is an isomorphism.

In article 2 we consider the case where G acts on itself by left multiplication. Then the algebra of invariant differential forms on G is isomorphic to $\wedge E^*$ (E, the Lie algebra of G) and so δ induces an operator, δ_E, in $\wedge E^*$ (this operator is carefully studied in volume III). The results of article I are applied in Theorem III, sec. 4.10, to obtain isomorphisms,

$$H(G) \cong H(\wedge E^*, \delta_E) \cong (\wedge E^*)_I ,$$

if G is compact and connected.

Article 4 is devoted to a theorem of Hopf (Theorem IV, sec. 4.12) which states that the cohomology algebra of a compact connected Lie group is an exterior algebra over a graded space whose dimension coincides with the dimension of a maximal torus.

Finally, in article 5 we consider a homogeneous space, G/K, and discuss the algebra of differential forms on G/K, invariant under the left action of G. This algebra is identified with a subalgebra of $\wedge E^*$. Thus (again by article 1), if G is compact and connected, $H(G/K)$ coincides with the cohomology of an explicit, finite dimensional, graded differential algebra.

Chapter V. Bundles with Structure Group. In article 1 principal bundles are defined. These are fibre bundles, $\mathscr{P} = (P, \pi, B, G)$, ($G$, a Lie group) together with an action of G on P such that the orbits, $z \cdot G$ ($z \in P$) coincide with the fibres, $G_{\pi z}$.

Such a bundle, together with an action of G on a manifold, F, determines (article 2) an associated bundle, $\xi = (M, \rho, B, F)$. If F is a vector space and G acts by linear transformations then ξ is a vector bundle.

In article 3 we consider bundles whose fibre or base is a homogeneous space, G/K, and (for example) describe the tangent bundle of G/K in terms of a certain representation of K.

The rest of the chapter deals with the Grassmann manifolds (k-planes in a real, complex, or quaternionic n-space) and the Stiefel manifolds (k-frames in n-space) and various associated bundles. These manifolds are identified with homogeneous spaces, so that the results of article 3 can be applied. The chapter closes with the computation of the cohomology algebra of the Stiefel manifolds (article 6).

Chapter VI. Principal Connections and the Weil Homomorphism. The first six articles of this chapter are devoted to the construction of the Weil homomorphism for a principal bundle, as outlined above. It is shown (Theorems I and II, sec. 6.19) that this homomorphism is a bundle invariant, natural with respect to bundle maps.

Article 7 deals with three special cases: (1) abelian structure groups, (2) reduction of structure group, (3) connections invariant under a group action. For instance, the Euler class of a principal circle bundle is identified with a characteristic class. This permits the computation of the cohomology algebra of complex projective space.

Finally, in article 8, the results of the chapter are applied to the principal bundle, $(G, \pi, G/K, K)$, (K, a closed subgroup of a Lie group, G). In particular, all the G-invariant principal connections are determined.

Chapter VII. Linear Connections. Let $\xi = (M, \pi, B, F)$ be a vector bundle. A p-form, Φ, on B with values in ξ is a smoothly varying family of skew-symmetric p-linear maps,

$$\Phi_x : T_x(B) \times \cdots \times T_x(B) \to F_x .$$

Articles 1 and 2 establish the basic properties of bundle-valued forms and develop an "index and argument-free" notation for operations with these forms.

In article 3 a linear connection in ξ is defined as an $\mathbb{R}$-linear map, $\nabla : \operatorname{Sec} \xi \to A^1(B; \xi)$, satisfying

$$\nabla(f\sigma) = \delta f \wedge \sigma + f \nabla\sigma, \qquad f \in \mathscr{S}(B), \quad \sigma \in \operatorname{Sec} \xi.$$

Every vector bundle, ξ, admits a linear connection, ∇, and a linear connection in ξ induces connections in the dual bundle, the associated tensor bundles, and any pullback of ξ. The curvature of a linear connection is a 2-form in B with values in L_ξ and is defined in article 4.

Fix a linear connection, ∇, in the vector bundle $\xi = (M, \pi, B, F)$ and let ψ be a smooth path in B. Then ∇ determines a bundle map, $\mathbb{R} \times F \to \xi$, which induces $\psi : \mathbb{R} \to B$ and restricts to isomorphisms in the fibres (article 5). It follows (Theorem I, sec. 7.18) that the pullbacks of ξ under homotopic maps are strongly isomorphic.

In article 6 it is shown that a linear connection in $\xi = (M, \pi, B, F)$ determines a horizontal subbundle of the tangent bundle, τ_M, and those horizontal bundles which correspond to linear connections are characterized.

Riemannian connections (article 7) in a Riemannian vector bundle, (ξ, g), are linear connections, ∇, which satisfy $\nabla g = 0$. Their curvatures take values in Sk_ξ. Article 8 considers smooth maps $\psi : B \to S^n$ (B, an oriented compact n-manifold). The degree of ψ is represented as the integral of an n-form constructed from the curvature of a Riemannian connection in the bundle $\psi^*(\tau_{S^n})$ (*Hopf index formula*).

Chapter VIII. Characteristic Homomorphism for Σ-bundles. A Σ-bundle is a vector bundle, ξ, together with a finite ordered set, $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$, of cross-sections in the associated tensor bundles, subject to the following condition: There is a coordinate representation, $\{(U_\alpha, \psi_\alpha)\}$, for ξ such that the σ_i correspond under ψ_α to constant functions, $U_\alpha \mapsto v_i$ $(\in \bigotimes F^* \otimes \bigotimes F)$. The subgroup, $G \subset GL(F)$, of isomorphisms leaving the v_i fixed is called the structure group of the Σ-bundle.

According to Theorem I, sec. 8.2, the condition above is equivalent to the following condition on (ξ, Σ_ξ): For each pair of points, x, y, in the base there is an isomorphism, $F_x \xrightarrow{\cong} F_y$, carrying $\sigma_i(x)$ to $\sigma_i(y)$ $(i = 1, ..., m)$.

A Σ-connection in a Σ-bundle is a linear connection, ∇, such that $\nabla\sigma_i = 0$ $(i = 1, ..., m)$ (article 2). With the aid of a Σ-connection we construct, in articles 3 and 4, a characteristic homomorphism,

$$h_\xi : (\vee E^*)_I \to H(B),$$

(B the base, E the Lie algebra of G). This is a bundle invariant and is natural with respect to homomorphisms of Σ-bundles (Theorems III and IV, sec. 8.13).

The analogy between the Weil and the characteristic homomorphism is made precise in article 7: To each Σ-bundle corresponds an associated principal bundle with G as fibre. Moreover, there is a one-to-one correspondence between Σ-connections and principal connections; and the characteristic homomorphism of a Σ-connection coincides with the Weil homomorphism of the associated principal connection.

Finally, a Σ-bundle with compact carrier (article 6) is a Σ-bundle, (ξ, Σ_ξ), together with an explicit trivialization, α, of ξ outside a compact subset of the base, such that under α the cross-sections σ_i become constant functions. Such a bundle determines a characteristic homomorphism,

$$h_\xi^c : (\vee^+ E^*)_I \to H_c(B),$$

which is an invariant of the triple $(\xi, \Sigma_\xi, \alpha)$.

Chapter IX. Pontrjagin, Pfaffian, and Chern Classes. For a real vector bundle, $\xi = (M, \pi, B, F)$, the Lie algebra of the structure group is simply L_F. Canonical elements, $C_p^F \in (\vee^p L_F^*)_I$, (corresponding to the coefficients of the characteristic polynomial of a linear transformation of F) are defined in Appendix A. These give rise, via the characteristic homomorphism, to cohomology classes in $H(B)$, called the Pontrjagin classes of ξ. Their properties are established in article 2.

In article 3 we consider pseudo-Riemannian bundles, and use the Riemannian metric to obtain simplified representatives for the Pontrjagin classes (Proposition VIII, sec. 9.11).

The Pfaffian of a skew transformation of an oriented pseudo-Euclidean space, F, (cf. Appendix A) determines an element of $(\vee \mathrm{Sk}_F{}^*)_I$. This yields a characteristic class, the Pfaffian class, for an oriented pseudo-Riemannian vector bundle. Its properties are established in secs. 9.12 and 9.13.

In articles 4 and 5 we consider complex vector bundles (i.e., vector bundles whose fibre is a complex space, F). The characteristic coefficients of a complex linear transformation determine elements of $(\vee L_F{}^*)_I$ and so we again obtain characteristic classes—the Chern classes of the complex vector bundle. If ξ is a real vector bundle, then the Pontrjagin classes of ξ coincide with the Chern classes of $\mathbb{C} \otimes \xi$.

According to Theorem I, sec. 9.21 (which ends the chapter), the Chern classes satisfy four basic axioms, and are uniquely determined by them.

Chapter X. Gauss–Bonnet–Chern Theorem. This chapter consists of an exposition of Chern's proof of the Gauss–Bonnet theorem, which asserts that the Pfaffian class of an oriented Riemannian vector bundle (of even rank) coincides with the Euler class of the associated sphere bundle.

This theorem, combined with the theorems in Chapters VIII and X, volume I, implies that the integral of the Pfaffian class of the tangent bundle of a compact oriented manifold of even dimension is equal to the Euler–Poincaré characteristic of the manifold.

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Chapter 0

Algebraic and Analytic Preliminaries

§1. Linear algebra

0.0. Notation. Throughout this book ι_X denotes the identity map of a set X. When it is clear which set we mean, we write simply ι. If U_{α_i} $(i = 1, \ldots, r)$ are subsets of X, then $U_{\alpha_1\alpha_2\cdots\alpha_r}$ denotes their intersection. The empty set is denoted by $\varnothing$.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the natural numbers, integers, rationals, reals, and complexes.

0.1. We shall assume the fundamentals of linear and multilinear algebra. We will consider only real vector spaces (except for the occasional complex space).

A pair of dual vector spaces is denoted by E^*, E and the scalar product between E^* and E is denoted by $\langle \ , \ \rangle$. If $F \subset E$, then

$$F^\perp = \{y^* \in E^* \mid \langle y^*, F \rangle = 0\}.$$

The dual of a linear map $\varphi: E \to F$ is denoted by φ^*. A direct sum of spaces E^p is denoted

$$\sum_p E^p \qquad \text{or} \qquad \bigoplus\nolimits_p E^p.$$

The determinant and the trace of a linear transformation $\varphi: E \to E$ are denoted respectively by $\det \varphi$, $\operatorname{tr} \varphi$.

A *determinant function* in an n-dimensional vector space is a nonzero skew-symmetric n-linear function. Every nonzero determinant function Δ_E in a real vector space defines an *orientation*.

Given two vector spaces E and F, we shall denote by $L(E; F)$ the space of linear maps $E \to F$. $L(E; E)$ will also be denoted by L_E. Finally if $E_1, \ldots, E_p$, and F are vector spaces, $L(E_1, \ldots, E_p; F)$ denotes the space of p-linear maps $E_1 \times \cdots \times E_p \to F$.

The group of linear automorphisms of a vector space E will be denoted by $GL(E)$.

A *Euclidean space* is a finite-dimensional real space, together with a positive definite inner product (also denoted by $\langle\ ,\ \rangle$). A *Hermitian space* is a finite-dimensional complex space together with a positive definite Hermitian inner product (also denoted by $\langle\ ,\ \rangle$).

If F is a real vector space, make $F^{\mathbb{C}} = \mathbb{C} \otimes F$ into a complex space by setting

$$\beta(\alpha \otimes x) = \beta\alpha \otimes x, \qquad \beta, \alpha \in \mathbb{C}, \quad x \in F.$$

$F^{\mathbb{C}}$ is called the *complexification* of F.

If $\langle\ ,\ \rangle$ is a positive definite inner product in F, then

$$\langle \alpha \otimes x, \beta \otimes y\rangle_{\mathbb{C}} = \alpha\bar{\beta}\langle x, y\rangle, \qquad \alpha, \beta \in \mathbb{C}, \quad x, y \in F$$

defines a Hermitian metric in $F^{\mathbb{C}}$.

An *indefinite inner product* in a finite-dimensional real vector space E is a non degenerate symmetric bilinear function $\langle\ ,\ \rangle$. If E_+ is a maximal subspace in which $\langle\ ,\ \rangle$ is positive definite, then $E = E_+ \oplus E_+^{\perp}$. The integer

$$\dim E_+ - \dim E_+^{\perp}$$

is independent of the choice of E_+, and is called the *signature* of $\langle\ ,\ \rangle$.

The symbol $\otimes$ denotes tensor over $\mathbb{R}$ (unless otherwise stated); for other rings R we write $\otimes_R$.

0.2. Quaternions and quaternionic vector spaces. Let H be an oriented four-dimensional Euclidean space. Choose a unit vector $e \in H$, and let $K = e^{\perp}$; it is a three-dimensional Euclidean space. Orient K so that, if e_1, e_2, e_3 is a positive basis of K, then e, e_1, e_2, e_3 is a positive basis of H.

Now define a bilinear map $H \times H \to H$ by

$$pq = -\langle p, q\rangle\, e + p \times q, \qquad p, q \in K$$

$$pe = p = ep, \qquad p \in H,$$

where $\times$ denotes the cross product in the oriented Euclidean space K. In this way H becomes an associative division algebra with unit element e. It is called the *algebra of quaternions* and is denoted by $\mathbb{H}$. The vectors of $\mathbb{H}$ are called *quaternions* and the vectors of K are called *pure quaternions.*

Every quaternion can be uniquely written in the form

$$p = \lambda e + q = \lambda + q, \qquad \lambda \in \mathbb{R}, \qquad q \in K.$$

λ and q are called the *real part* and the *pure quaternionic part of* p. *The conjugate* $\bar{p}$ *of a quaternion* $p = \lambda e + q$ is defined by $\bar{p} = \lambda e - q$. The map $p \mapsto \bar{p}$ defines an anti-automorphism of the algebra $\mathbb{H}$ called *conjugation.* The product of p and $\bar{p}$ is given by $p\bar{p} = |p|^2 e = |p|^2$.

Multiplication and the inner product in $\mathbb{H}$ are connected by the relation

$$\langle pr, qr\rangle = \langle p, q\rangle\langle r, r\rangle, \qquad p, q, r \in \mathbb{H}.$$

In particular,

$$|pr| = |p|\,|r|, \qquad p, r \in \mathbb{H}.$$

A *unit quaternion* is a quaternion of norm one. A pure unit quaternion q satisfies the relation $q^2 = -e$. If (e_1, e_2, e_3) is a positive orthonormal basis in K, then

$$e_1e_2 = e_3, \qquad e_2e_3 = e_1, \qquad e_3e_1 = e_2.$$

0.3. Algebras. An *algebra* A over $\mathbb{R}$ is a real vector space together with a real bilinear map $A \times A \to A$ (called *product*). A *system of generators* of an algebra A is a subset $S \subset A$ such that every element of A can be written as a finite sum of products of the elements of S.

A *homomorphism* between two algebras A and B is a linear map $\varphi: A \to B$ such that

$$\varphi(xy) = \varphi(x)\,\varphi(y), \qquad x, y \in A.$$

A *derivation* in an algebra A is a linear map $\theta: A \to A$ satisfying

$$\theta(xy) = \theta(x)y + x\theta(y).$$

A derivation which is zero on a system of generators is identically zero. If θ_1 and θ_2 are derivations in A, then so is $\theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$.

More generally, let $\varphi: A \to B$ be a homomorphism of algebras. Then a *φ-derivation* is a linear map $\theta: A \to B$ which satisfies

$$\theta(xy) = \theta(x)\,\varphi(y) + \varphi(x)\,\theta(y).$$

A *graded algebra* A over $\mathbb{R}$ is a graded vector space $A = \sum_{p \geqslant 0} A^p$, together with an algebra structure, such that

$$A^p \cdot A^q \subset A^{p+q}.$$

If

$$xy = (-1)^{pq}\,yx, \qquad x \in A^p, \quad y \in A^q,$$

then A is called *anticommutative.* If A has an identity, and $\dim A^0 = 1$, then A is called *connected.*

If A and B are graded algebras, then $A \otimes B$ can be made into a graded algebra in two ways:

(1) $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2$
(2) $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{q_1p_2}\, x_1x_2 \otimes y_1y_2$

where $x_1, x_2 \in A$, $y_1, y_2 \in B$, $\deg y_1 = q_1$, $\deg x_2 = p_2$. The first algebra is called the *canonical tensor product* of A and B, while the second one is called the *anticommutative* or *skew tensor product* of A and B. If A and B are anticommutative, then so is the skew tensor product.

An *antiderivation* in a graded algebra A is a linear map $\alpha: A \to A$, homogeneous of *odd* degree, such that

$$\alpha(xy) = \alpha(x)y + (-1)^p\, x\alpha(y), \qquad x \in A^p, \qquad y \in A.$$

If α_1 and α_2 are antiderivations, then $\alpha_2 \circ \alpha_1 + \alpha_1 \circ \alpha_2$ is a derivation. If α is an antiderivation and θ is a derivation, then $\alpha \circ \theta - \theta \circ \alpha$ is an antiderivation.

The *direct product* $\prod_\alpha A_\alpha$ of algebras A_α is the set of infinite sequences $\{(x_\alpha) \mid x_\alpha \in A_\alpha\}$; multiplication and addition is defined component by component. The *direct sum* $\sum_\alpha A_\alpha$ is the subalgebra of sequences with finitely many nonzero terms.

0.4. Lie algebras. A *Lie algebra* E is a vector space (not necessarily of finite dimension) together with a bilinear map $E \times E \to E$, denoted by $[\ ,\ ]$, subject to the conditions

$$[x, x] = 0$$

and

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0, \quad x, y, z \in E \qquad \text{(Jacobi identity)}.$$

A *homomorphism of Lie algebras* is a linear map $\varphi: E \to F$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)], \qquad x, y \in E.$$

0.5. Multilinear algebra. The tensor, exterior, and symmetric algebras over a vector space E are denoted by

$$\otimes E = \sum_{p \geq 0} \otimes^p E, \qquad \wedge E = \sum_{p \geq 0} \wedge^p E, \qquad \vee E = \sum_{p \geq 0} \vee^p E.$$

(If $\dim E = n$, $\wedge E = \sum_{p=0}^n \wedge^p E$.)

If F is a second space, a nondegenerate pairing between $E^* \otimes F^*$ and $E \otimes F$ is given by

$$\langle x^* \otimes y^*, x \otimes y \rangle = \langle x^*, x \rangle \langle y^*, y \rangle, \qquad x^* \in E^*,\ y^* \in F^*, \quad x \in E,\ y \in F.$$

If E or F has finite dimension, this yields an isomorphism $E^* \otimes F^* \cong (E \otimes F)^*$. In particular, in this case $(\otimes^p E)^* \cong \otimes^p E^*$.

Similarly, if $\dim E < \infty$, we may write $(\wedge^p E)^* = \wedge^p E^*$, $(\vee^q E)^* = \vee^q E^*$ by setting

$$\langle x^{*1} \wedge \cdots \wedge x^{*p}, x_1 \wedge \cdots \wedge x_p \rangle = \det(\langle x^{*i}, x_j \rangle)$$

and

$$\langle y^{*1} \vee \cdots \vee y^{*p}, y_1 \vee \cdots \vee y_p \rangle = \operatorname{perm}(\langle y^{*i}, y_j \rangle),$$

where "perm" denotes the permanent of a matrix.

The algebras of multilinear (resp. skew multilinear, symmetric multilinear) functions in a space E are denoted by

$$T(E) = \sum_{p \geqslant 0} T^p(E), \qquad A(E) = \sum_{p \geqslant 0} A^p(E)$$

and

$$S(E) = \sum_{p \geqslant 0} S^p(E).$$

The multiplications are given respectively by

$$(\Phi \otimes \Psi)(x_1, \ldots, x_{p+q}) = \Phi(x_1, \ldots, x_p)\, \Psi(x_{p+1}, \ldots, x_{p+q})$$

$$(\Phi \wedge \Psi)(x_1, \ldots, x_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \Phi(x_{\sigma(1)}, \ldots, x_{\sigma(p)})\, \Psi(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})$$

and

$$(\Phi \vee \Psi)(x_1, \ldots, x_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S^{p+q}} \Phi(x_{\sigma(1)}, \ldots, x_{\sigma(p)})\, \Psi(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}).$$

Here S^p denotes the symmetric group on p objects, while $\epsilon_\sigma = \pm 1$ according as the permutation σ is even or odd.

If $\dim E < \infty$, we identify the graded algebras $T(E)$ and $\otimes E^*$ (resp. $A(E)$ and $\wedge E^*$, $S(E)$ and $\vee E^*$) by setting

$$\Phi(x_1, \ldots, x_p) = \langle \Phi, x_1 \otimes \cdots \otimes x_p \rangle, \qquad \Phi \in \otimes^p E^*$$

$$\Psi(x_1, \ldots, x_p) = \langle \Psi, x_1 \wedge \cdots \wedge x_p \rangle, \qquad \Psi \in \wedge^p E^*$$

and

$$X(x_1, \ldots, x_p) = \langle X, x_1 \vee \cdots \vee x_p \rangle, \qquad X \in \vee^p E^*.$$

A linear map $\varphi: E \to F$ extends uniquely to homomorphisms

$$\bigotimes\varphi: \bigotimes E \to \bigotimes F, \qquad \wedge\varphi: \wedge E \to \wedge F, \qquad \vee\varphi: \vee E \to \vee F.$$

These are sometimes denoted by $\varphi_\otimes$, $\varphi_\wedge$, and $\varphi_\vee$.

To each $x \in E$ we associate the *substitution* operator $i(x): A(E) \to A(E)$, given by

$$(i(x)\Phi)(x_1, \ldots, x_{p-1}) = \Phi(x, x_1, \ldots, x_{p-1}), \qquad \Phi \in A^p(E), \quad p \geqslant 1,$$
$$i(x)\Phi = 0, \qquad \Phi \in A^0(E),$$

and the multiplication operator $\mu(x): \wedge E \to \wedge E$ given by

$$\mu(x)(a) = x \wedge a, \qquad a \in \wedge E,$$

$i(x)$ is an antiderivation in $A(E)$ and is dual to $\mu(x)$.

§2. Homological algebra

0.6. Rings and modules. Let R be a commutative ring. If M, N are R-modules, then the tensor product $M \otimes_R N$ is again an R-module (cf. [4, p. AII–56] or [5, §8, Chap. 3]). If Q is a third R-module and if $\varphi\colon M \times N \to Q$ is a map satisfying the conditions

$$\begin{aligned} &(1) \quad \varphi(x + y, u) = \varphi(x, u) + \varphi(y, u) \\ &(2) \quad \varphi(x, u + v) = \varphi(x, u) + \varphi(x, v) \end{aligned}$$

and

$$(3) \quad \varphi(\lambda x, u) = \varphi(x, \lambda u)$$

for $x, y \in M$, $u, v \in N$, $\lambda \in R$, then there is a unique additive map $\psi\colon M \otimes_R N \to Q$ such that

$$\varphi(x, u) = \psi(x \otimes u), \qquad x \in M, \quad u \in N$$

(cf. [4, Prop. I(b), p. AII–51] or [5, §8, Chap. 3]). If (iii) is replaced by the stronger

$$\varphi(\lambda x, u) = \lambda\varphi(x, u) = \varphi(x, \lambda u), \qquad x \in M, \quad u \in N, \quad \lambda \in R,$$

then ψ is R-linear.

The R-module of R-linear maps $M \to N$ is denoted by $\mathrm{Hom}_R(M; N)$. $\mathrm{Hom}_R(M; R)$ is denoted by M^*. A canonical R-linear map

$$\alpha\colon M^* \otimes_R N \to \mathrm{Hom}_R(M; N)$$

is given by

$$\alpha(f \otimes u)(x) = f(x)u, \qquad x \in M, \quad u \in N, \quad f \in M^*.$$

A module M is called *free* if it has a basis; M is called *projective* if there exists another R-module N such that $M \oplus N$ is free. If M is projective and finitely generated, then N can be chosen so that $M \oplus N$ has a finite basis.

If M is finitely generated and projective, then so is M^*, and for all R-modules N, the homomorphism α given just above is an isomorphism. In particular, the isomorphism

$$M^* \otimes_R M \xrightarrow{\cong} \mathrm{Hom}_R(M; M)$$

specifies a unique tensor $t_M \in M^* \otimes_R M$ such that

$$\alpha(t_M) = \iota_M .$$

It is called the *unit tensor for M*.

A *graded module* is a module M in which submodules M^p have been distinguished such that

$$M = \sum_{p \geqslant 0} M^p.$$

The elements of M^p are called *homogeneous of degree p*. If $x \in M^p$, then p is called the *degree of x* and we shall write $\deg x = p$.

If M and N are graded modules, then a gradation in the module $M \otimes_R N$ is given by

$$(M \otimes_R N)^r = \sum_{p+q=r} M^p \otimes_R N^q.$$

An R-linear map between graded modules, $\varphi: M \to N$, is called *homogeneous of degree k*, if

$$\varphi(M^p) \subset N^{p+k}, \qquad p \geqslant 0$$

An R-linear map which is homogeneous of degree zero is called a *homomorphism of graded modules*.

A *bigraded module* is a module which is the direct sum of submodules $M^{p,q} (p \geqslant 0,\ q \geqslant 0)$.

An *exact sequence of modules* is a sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \longrightarrow \cdots,$$

where the φ_i are R-linear maps satisfying

$$\ker \varphi_i = \operatorname{Im} \varphi_{i-1} .$$

Suppose

$$\begin{tikzcd}
M_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"', "\cong"] & M_2 \arrow[r, "\varphi_2"] \arrow[d, "\alpha_2"', "\cong"] & M_3 \arrow[r, "\varphi_3"] \arrow[d, "\alpha_3"'] & M_4 \arrow[r, "\varphi_4"] \arrow[d, "\alpha_4"', "\cong"] & M_5 \arrow[d, "\alpha_5"', "\cong"] \\
N_1 \arrow[r, "\psi_1"'] & N_2 \arrow[r, "\psi_2"'] & N_3 \arrow[r, "\psi_3"'] & N_4 \arrow[r, "\psi_4"'] & N_5
\end{tikzcd}$$

is a commutative row-exact diagram of R-linear maps. Assume that the maps α_1, α_2, α_4, α_5 are isomorphisms. Then the *five-lemma* states that α_3 is also an isomorphism.

On the other hand, if

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{11} & \longrightarrow & M_{12} & \longrightarrow & M_{13} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{21} & \longrightarrow & M_{22} & \longrightarrow & M_{23} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{31} & \longrightarrow & M_{32} & \longrightarrow & M_{33} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is a commutative diagram of R-linear maps with exact columns, and if the middle and bottom rows are exact, then the *nine-lemma* states that the top row is exact.

An algebra over R is an R-module A together with an R-linear map $A \otimes_R A \to A$. In particular if M is any R-module, the tensor, exterior, and symmetric algebras over M are written $\otimes_R M$, $\wedge_R M$ and, $\vee_R M$. If M is finitely generated and projective, there are isomorphisms, $(\otimes_R^p M)^* \cong \otimes_R^p M^*$, $(\wedge_R^p M)^* \cong \wedge_R^p M^*$, $(\vee_R^p M)^* \cong \vee_R^p M^*$, defined in exactly the same way as in sec. 0.5.

0.7. Differential spaces. A *differential space* is a vector space X together with a linear map $\delta: X \to X$ satisfying $\delta^2 = 0$. δ is called the *differential operator in* X. The elements of the subspaces

$$Z(X) = \ker \delta \qquad \text{and} \qquad B(X) = \operatorname{Im} \delta.$$

are called, respectively, *cocycles* and *coboundaries*. The space $H(X) = Z(X)/B(X)$ is called the *cohomology space* of X.

A *homomorphism* of differential spaces $\varphi: (X, \delta_X) \to (Y, \delta_Y)$ is a linear map for which $\varphi \circ \delta_X = \delta_Y \circ \varphi$. It restricts to maps between the cocycle and coboundary spaces, and so induces a linear map

$$\varphi_\# : H(X) \to H(Y).$$

A *homotopy operator* for two such homomorphisms, φ, ψ, is a linear map $h: X \to Y$ such that

$$\varphi - \psi = h \circ \delta + \delta \circ h.$$

If h exists then $\varphi_\# = \psi_\#$.

Suppose

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact sequence of homomorphisms of differential spaces. Every cocycle $z \in Z$ has a preimage $y \in Y$. In particular,

$$g(\delta y) = \delta z = 0$$

and so there is a cocycle $x \in X$ for which $f(x) = \delta y$. The class $\xi \in H(X)$ represented by x depends only on the class $\zeta \in H(Z)$ represented by z. The correspondence $\zeta \mapsto \xi$ defines a linear map

$$\partial \colon H(Z) \to H(X)$$

called the *connecting homomorphism* for the exact sequence. The triangle

$$\begin{tikzcd} H(X) \arrow[rr, "f_*"] & & H(Y) \arrow[dl, "g_*"] \\ & H(Z) \arrow[ul, "\partial"] & \end{tikzcd}$$

is exact.

If

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\varphi"] & Y \arrow[r] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, "\chi"] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0 \end{tikzcd}$$

is a row-exact diagram of differential spaces, then

$$\partial' \circ \chi_* = \varphi_* \circ \partial$$

(∂, ∂' the connecting homomorphisms).

0.8. Graded differential spaces and algebras. A graded space $X = \sum_{p \geqslant 0} X^p$ together with a differential operator δ homogeneous of degree $+1$ is called a *graded differential space*. In such a case the cocycle, coboundary, and cohomology spaces are graded:

$$Z^p(X) = Z(X) \cap X^p, \qquad B^p(X) = B(X) \cap X^p$$

and

$$H^p(X) = Z^p(X)/B^p(X).$$

A *homomorphism of graded differential spaces* is a homomorphism of differential spaces, homogeneous of degree zero.

Now assume that X has finite dimension and let $\varphi: X \to X$ be a homomorphism of graded differential spaces. Let

$$\varphi^p: X^p \to X^p \qquad \text{and} \qquad (\varphi_\#)^p: H^p(X) \to H^p(X)$$

be the restrictions of φ and $\varphi_\#$ to X^p and $H^p(X)$. The *algebraic Lefschetz formula* states that

$$\sum_{p \geq 0} (-1)^p \operatorname{tr} \varphi^p = \sum_{p \geq 0} (-1)^p \operatorname{tr} (\varphi_\#)^p.$$

In particular, if $\varphi = \iota$, we obtain the *Euler-Poincaré formula*

$$\sum_{p \geq 0} (-1)^p \dim X^p = \sum_{p \geq 0} (-1)^p \dim H^p(X).$$

A *graded differential algebra* A is a graded algebra together with an antiderivation, δ, homogeneous of degree one such that $\delta^2 = 0$. In this case $Z(A)$ is a graded subalgebra and $B(A)$ is a graded ideal in $Z(A)$. Thus $H(A)$ becomes a graded algebra. It is called the *cohomology algebra* of A. If A is anticommutative, then so is $H(A)$.

A *homomorphism of graded differential algebras* $\varphi: A \to B$ is a map which is a homomorphism of graded differential spaces and a homomorphism of algebras. It induces a homomorphism between the cohomology algebras,

$$\varphi_\# : H(A) \to H(B).$$

Next let A and B be graded differential algebras and consider the skew tensor product $A \otimes B$. Then the antiderivation in $A \otimes B$, given by

$$\delta(x \otimes y) = \delta x \otimes y + (-1)^p x \otimes \delta y, \qquad x \in A^p, \quad y \in B,$$

satisfies $\delta^2 = 0$. Thus $A \otimes B$ becomes a graded differential algebra. The tensor multiplication between A and B induces an isomorphism

$$H(A) \otimes H(B) \xrightarrow{\cong} H(A \otimes B)$$

of graded algebras. It is called the *Künneth isomorphism*.

§3. Analysis and topology

0.9. Smooth maps. Let E, F be real, finite dimensional vector spaces with the standard topology. Let $U \subset E$ be an open subset. A map $\varphi: U \to F$ is called *differentiable* at a point $a \in U$ if for some $\psi_a \in L(E; F)$

$$\lim_{t \to 0} \frac{\varphi(a + th) - \varphi(a)}{t} = \psi_a(h), \qquad h \in E.$$

In this case ψ_a is called the *derivative of* φ *at* a and is denoted by $\varphi'(a)$. We shall write

$$\varphi'(a; h) = \varphi'(a)h = \psi_a(h), \qquad h \in E.$$

If φ is differentiable at every point $a \in U$, it is called a *differentiable map* and the map

$$\varphi': U \to L(E; F)$$

given by $a \mapsto \varphi'(a)$ is called the *derivative of* φ. Since $L(E; F)$ is again a finite dimensional vector space, it makes sense for φ' to be differentiable. In this case the derivative of φ' is denoted by φ''; it is a map

$$\varphi'': U \to L(E; L(E; F)) = L(E, E; F).$$

More generally, the kth derivative of φ (if it exists) is denoted by $\varphi^{(k)}$,

$$\varphi^{(k)}: U \to L(\underbrace{E, \dots, E}_{k \text{ terms}}; F).$$

For each $a \in U$, $\varphi^{(k)}(a)$ is a symmetric k-linear map of $E \times \cdots \times E$ into F. If all derivatives of φ exist, φ is called *infinitely differentiable*, or *smooth*.

A smooth map $\varphi: U \to V$ between open subsets $U \subset E$ and $V \subset F$ is called a *diffeomorphism* if it has a smooth inverse.

Assume now that $\varphi: U \to F$ is a map with a continuous derivative such that for some point $a \in U$

$$\varphi'(a): E \xrightarrow{\cong} F$$

is a linear isomorphism. Then the *inverse function theorem* states that there are neighbourhoods U of a and V of $\varphi(a)$ such that φ restricts to a diffeomorphism $U \xrightarrow{\cong} V$.

We shall also need the basic properties of the Riemannian integral of a compactly supported function in $\mathbb{R}^n$ (linearity, transformation of coordinates, differentiation with respect to a parameter). The theory extends to vector-valued functions (integrate component by component).

Finally, we shall use the Picard existence and uniqueness theorem for ordinary differential equations as given in [6, p. 22].

0.10. The exponential map. Let E be an n-dimensional real or complex vector space and let $\sigma\colon E \to E$ be a linear transformation. It follows from the standard existence theorems of differential equations that there is a unique smooth map $\tau\colon \mathbb{R} \to L_E$ satisfying the linear differential equation

$$\dot{\tau} = \sigma \circ \tau$$

and the initial condition $\tau(0) = \iota$. The linear transformation $\tau(1)$ is called the *exponential of* σ and is denoted by $\exp \sigma$.

In this way we obtain a (nonlinear) map $\exp\colon L_E \to L_E$. It has the following properties:

(0) $\exp 0 = \iota$.

(1) If $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$, then $\exp(\sigma_1 + \sigma_2) = \exp \sigma_1 \circ \exp \sigma_2$.

(2) $\exp(k\sigma) = (\exp \sigma)^k$, $k \in \mathbb{Z}$.

(3) $\det \exp \sigma = \exp \operatorname{tr} \sigma$.

(4) If a Euclidean (Hermitian) inner product is defined in the real (complex) vector space E and if σ^* denotes the adjoint linear transformation, then

$$\exp \sigma^* = (\exp \sigma)^*.$$

(All these properties are easy consequences of the uniqueness theorem for solutions of differential equations.)

Relations (0) and (1) imply that $\exp \sigma$ is an automorphism with $(\exp \sigma)^{-1} = \exp(-\sigma)$. In particular, if σ is self-adjoint, then so is $\exp \sigma$ and if σ is skew (resp. Hermitian skew), then $\exp \sigma$ is a proper rotation (resp. unitary transformation) of E.

In terms of an infinite series we can write

$$\exp \sigma = \sum_{p=0}^{\infty} \frac{1}{p!} \sigma^p.$$

0.11. General topology. We shall assume the basics of point set topology: manipulation with open sets and closed sets, compactness, Hausdorff spaces, locally compact spaces, second countable spaces, connectedness, paracompact spaces, normality, open coverings, shrinking of an open covering, etc.

The closure of a subset A of a topological space X will be denoted by $\bar{A}$. If A and B are any two subsets of X, we shall write

$$A - B = \{x \in A \mid x \notin B\}.$$

A neighbourhood of A in X will always mean an *open* subset U of X such that $U \supset A$.

An *open cover* of X is a family $\mathcal{O}$ of open sets whose union is X. It is called *locally finite* if every point has a neighbourhood which meets only finitely many elements of $\mathcal{O}$. $\mathcal{O}$ is called a *refinement* of an open cover $\mathcal{U}$ if each $O \in \mathcal{O}$ is a subset of some $U \in \mathcal{U}$. X is called *paracompact* if every open cover of X has a locally finite refinement.

A *basis* for the topology of X is a family $\mathcal{O}$ of open sets such that each open subset of X is the union of elements of $\mathcal{O}$. If $\mathcal{O}$ is closed under finite intersections, it is called an *i*-basis. If X has a countable basis, it is called *second countable.*

§4. Summary of volume I

0.12. Manifolds and vector bundles. All manifolds are smooth (i.e., infinitely differentiable), second countable, Hausdorff, and finite dimensional. The set of smooth maps between manifolds M and N is written $\mathscr{S}(M; N)$. If $\varphi\colon M \to N$ has a smooth inverse, it is called a *diffeomorphism*. $\mathscr{S}(M)$ denotes the algebra of smooth real-valued functions on M.

If $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{S}(M)$-modules, then $\mathscr{M} \otimes_M \mathscr{N}$, $\mathrm{Hom}_M(\mathscr{M}; \mathscr{N})$, and $\wedge_M \mathscr{M}$ denote the obvious linear and multilinear constructions, taken over $\mathscr{S}(M)$.

A *vector bundle* is a quadruple $\xi = (E, \pi, B, F)$ where: (1) $\pi\colon E \to B$ is smooth; (2) F and each set F_x $(= \pi^{-1}(x))$ is a finite-dimensional vector space; and (3) there is an open cover $\{U_\alpha\}$ of B and a system of diffeomorphisms $\psi_\alpha\colon U_\alpha \times F \xrightarrow{\cong} \pi^{-1}U_\alpha$ such that ψ_α restricts to linear isomorphisms $\psi_{\alpha,x}\colon F \xrightarrow{\cong} F_x$ $(x \in U_\alpha)$. E, B, and F are called the *total space*, *base space*, and *typical fibre* of ξ; π is called the *projection*. F_x is called the *fibre at* x. The dimension of F is called the *rank* of ξ. The collection $\{(U_\alpha, \psi_\alpha)\}$ is called a *coordinate representation* for ξ. If $E = B \times F$ and π is the obvious projection, ξ is called *trivial*.

Let $\xi' = (E', \pi', B', F')$ be a second vector bundle. A *bundle map* or *homomorphism* $\xi \to \xi'$ is a smooth map $\varphi\colon E \to E'$ that restricts to linear maps $\varphi_x\colon F_x \to F_{\psi(x)}$, $x \in B$. The correspondence $x \mapsto \psi(x)$ defines a smooth map $\psi\colon B \to B'$.

If $\psi = \iota$, then φ is called a *strong* bundle map.

The *Cartesian product* of ξ and ξ' is the vector bundle $\xi \times \xi' = (E \times E', \pi \times \pi', B \times B', F \oplus F')$. If $F' = 0$ (so that $E' = B'$, $\pi' = \iota$), we write simply $\xi \times B'$.

A vector bundle ξ determines vector bundles ξ^*, $\otimes^p \xi$, $\wedge \xi$, $\vee^q \xi$, whose fibres at x are the spaces F_x^*, $\otimes^p F_x$, $\wedge F_x$ and $\vee^q F_x$. If η is a second vector bundle with the same base and with typical fibre H, then $\xi \oplus \eta$, $\xi \otimes \eta$, and $L(\xi; \eta)$ denote the vector bundles with fibres $F_x \oplus H_x$, $F_x \otimes H_x$, and $L(F_x; H_x)$. $\xi \oplus \eta$ is called the *Whitney sum* of ξ and η. The bundle $L(\xi; \xi)$ is written L_ξ.

The *cross-sections* in ξ are the smooth maps $\sigma\colon B \to E$ which satisfy $\pi \circ \sigma = \iota$. The *carrier* of σ, carr σ, is the closure of the set of $x \in B$ such that $\sigma(x) \neq 0$. The operations

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x), \qquad (f \cdot \sigma)(x) = f(x)\,\sigma(x)$$

make the cross-sections into an $\mathscr{S}(B)$-module; it is denoted by Sec ξ.

A *pseudo-Riemannian metric* in ξ is a smooth assignment to the fibres F_x of inner products $g(x)$ (also written $\langle\ ,\ \rangle_x$, or simply $\langle\ ,\ \rangle$). Thus $g \in \mathrm{Sec}\, \vee^2 \xi^*$. If each $g(x)$ is positive definite, g is called a *Riemannian metric*.

Suppose rank $\xi = r$. An *orientation* in ξ is an equivalence class of nowhere vanishing cross-sections in $\wedge^r \xi^*$ under the equivalence relation: $\Delta_1 \sim \Delta_2$ if $\Delta_1 = f \cdot \Delta_2$ for some $f \in \mathscr{S}(B)$, with $f(x) > 0$, $x \in B$. A cross-section in one of these classes is called a *determinant function* in ξ which *represents that orientation.*

0.13. Tangent bundle and differential forms. Let M be an n-manifold. The *tangent space*, $T_x(M)$, at $x \in M$ is the space of linear maps $\xi: \mathscr{S}(M) \to \mathbb{R}$, which satisfy $\xi(f \cdot g) = \xi(f) \cdot g(x) + f(x) \cdot \xi(g)$. The *tangent bundle* of M, written $\tau_M = (T_M, \pi, M, \mathbb{R}^n)$, is the vector bundle whose fibre at x is $T_x(M)$. The *derivative* of a smooth map $\varphi: M \to N$ is the bundle map $d\varphi: T_M \to T_N$ whose restriction to $T_x(M)$ is given by

$$((d\varphi)_x \xi)(f) = \xi(f \circ \varphi), \qquad f \in \mathscr{S}(M)$$

If each $(d\varphi)_x$ is surjective, φ is called a *submersion*; if also φ is surjective, then N is called a quotient manifold of M. If $d\varphi$ is injective, (M, φ) is called an *embedded manifold*; if in addition φ is a homeomorphism onto $\varphi(M)$, then M is called a *submanifold* of N.

Let $\alpha(t)$ $(t_0 < t < t_1)$ be a smooth path in M. Then $\dot{\alpha}(t) \in T_{\alpha(t)}(M)$ is defined by

$$\dot{\alpha}(t)(f) = \frac{d}{dt} f(\alpha(t)).$$

A *vector field* on M is a cross-section X in τ_M; the module of vector fields is denoted by $\mathscr{X}(M)$. An *orbit* of X is a smooth path $\alpha(t)$ such that $\dot{\alpha}(t) = X(\alpha(t))$. The Picard theorem asserts that for each $x \in M$ there is a unique orbit of X through x. If $X \in \mathscr{X}(M)$ and $f \in \mathscr{S}(M)$, then $X(f) \in \mathscr{S}(M)$ is defined by $(X(f))(x) = X(x)(f)$. The *Lie product*, $[X, Y]$, of $X, Y \in \mathscr{X}(M)$ is the unique vector field satisfying

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Vector fields $X \in \mathscr{X}(M)$, $Y \in \mathscr{X}(N)$ are called *φ-related* with respect to a smooth map $\varphi: M \to N$ if $(d\varphi)_x X(x) = Y(\varphi(x))$, $x \in M$. In this case we write $X \underset{\varphi}{\sim} Y$. If φ is a diffeomorphism, $\varphi_*(X)$ denotes the unique vector field on N which is φ-related to X.

A *differential form* on M is a cross-section, Φ, in $\wedge \tau_M^*$. If each $\Phi(x) \in \wedge^p T_x(M)^*$, then Φ has *degree p*. The differential forms are a graded algebra, $A(M) = \sum_p A^p(M)$, with multiplication given by $(\Phi \wedge \Psi)(x) = \Phi(x) \wedge \Psi(x)$. $A^p(M)$ can be regarded as the space of p-linear (over $\mathscr{S}(M)$) skew-symmetric maps $\mathscr{X}(M) \times \cdots \times \mathscr{X}(M) \to \mathscr{S}(M)$, via the equations

$$(\Phi(X_1, \ldots, X_p))(x) = \Phi(x; X_1(x), \ldots, X_p(x)).$$

Then

$$(\Phi \wedge \Psi)(X_1, \ldots, X_{p+q}) = \frac{1}{p!\, q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \Phi(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\, \Psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).$$

A smooth map $\varphi\colon M \to N$ determines the homomorphism $\varphi^*\colon A(M) \leftarrow A(N)$ defined by

$$(\varphi^*\Phi)(x; \xi_1, \ldots, \xi_p) = \Phi(\varphi(x); (d\varphi)\, \xi_1, \ldots, (d\varphi)\, \xi_p), \qquad p \geqslant 1$$

$$(\varphi^* f)(x) = f(\varphi(x)).$$

The maps $M, N \to M \times N$ given by $x \mapsto (x, b)$ and $y \mapsto (a, y)$ are called the *inclusions opposite* b and a. Their derivatives define an isomorphism $T_a(M) \oplus T_b(N) \xrightarrow{\cong} T_{(a,b)}(M \times N)$; these isomorphisms in turn identify $\tau_M \times \tau_N$ with $\tau_{M \times N}$. In particular, $X \in \mathscr{X}(M)$ determines the vector field, $i_M X$, in $\mathscr{X}(M \times N)$ given by $(x, y) \mapsto (X(x), 0)$; it is also denoted by $i_L X$. Moreover, the induced isomorphisms,

$$\wedge T_{(a,b)}(M \times N)^* \cong \sum_{p,q} \wedge^p T_a(M)^* \otimes \wedge^q T_b(N)^*,$$

define a *bigradation* in $A(M \times N)$: $A(M \times N) = \sum_{p,q} A^{p,q}(M \times N)$. If $\Phi \in A^p(M)$ and $\Psi \in A^q(N)$, then $\Phi \times \Psi \in A^{p,q}(M \times N)$ denotes the $(p + q)$-form given by $\pi_M^* \Phi \wedge \pi_N^* \Psi$ ($\pi_M\colon M \times N \to M$, $\pi_N\colon M \times N \to N$ are the obvious projections). Thus $(\Phi \times \Psi)(a, b) = \Phi(a) \otimes \Psi(b)$.

The *substitution operator* $i(X)$, the *Lie derivative* $\theta(X)$, and the *exterior derivative* δ are the operators in $A(M)$, homogeneous of degrees -1, 0, and 1, defined, respectively, by

$$(i(X)\Phi)(X_2, \ldots, X_p) = \Phi(X, X_2, \ldots, X_p),$$

$$(\theta(X)\Phi)(X_1, \ldots, X_p) = X(\Phi(X_1, \ldots, X_p)) - \sum_{j=1}^{p} \Phi(X_1, \ldots, [X, X_j], \ldots, X_p),$$

$$(\delta\Phi)(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j X_j(\Phi(X_0, \ldots, \hat{X}_j, \ldots, X_p)) + \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} \Phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),$$

$$\Phi \in A^p(M), \quad p \geqslant 1,$$

and

$$i(X)f = 0, \qquad \theta(X)f = X(f), \qquad (\delta f)(X) = X(f), \qquad f \in \mathscr{S}(M).$$

They are respectively an antiderivation, a derivation, and an antiderivation.

These operators satisfy the relations

$$i([X, Y]) = \theta(X) \circ i(Y) - i(Y) \circ \theta(X),$$
$$\theta([X, Y]) = \theta(X) \circ \theta(Y) - \theta(Y) \circ \theta(X),$$
$$\theta(X) = i(X) \circ \delta + \delta \circ i(X)$$

and

$$\delta^2 = 0.$$

Moreover, if φ: $M \to N$ is smooth and $X \underset{\varphi}{\sim} Y$, then

$$i(X) \circ \varphi^* = \varphi^* \circ i(Y) \qquad \text{and} \qquad \theta(X) \circ \varphi^* = \varphi^* \circ \theta(Y).$$

In any case $\varphi^* \circ \delta = \delta \circ \varphi^*$.

Let F be any finite-dimensional vector space. The cross-sections in the bundle $L(\wedge\tau_M^*\,;\, M \times F)$ (respectively, $L(\wedge^p\tau_M^*\,;\, M \times F)$) are called *differential forms with values in* F (respectively, *p-forms with values in* F); these modules are denoted by $A(M; F)$ and $A^p(M; F)$. If $\Omega \in A^p(M; F)$, then $\Omega(x)$ is a skew-symmetric, p-linear, F-valued function in $T_x(M)$. An isomorphism $A(M) \otimes F \to A(M; F)$ is given by

$$(\Phi \otimes a)(x; \xi_1, ..., \xi_p) = \Phi(x; \xi_1, ..., \xi_p) \cdot a.$$

The operators $i(X) \otimes \iota$, $\theta(X) \otimes \iota$, and $\delta \otimes \iota$ in $A(M; F)$ are denoted simply by $i(X)$, $\theta(X)$, and δ; they satisfy the relations given above in the case $F = \mathbb{R}$. A smooth map φ: $M \to N$ induces a map

$$\varphi^* = (\varphi^* \otimes \iota)\colon A(M; F) \leftarrow A(N; F).$$

If $\alpha\colon F \to H$ is linear, we define α_*: $A(M; H) \to A(M; H)$ by $(\alpha_*\Phi)(x; \xi_1, ..., \xi_p) = \alpha(\Phi(x; \xi_1, ..., \xi_p))$. α_* commutes with the operators $i(X)$, $\theta(X)$, δ, and φ^*.

An *orientation* of M is an orientation of τ_M; thus it is an equivalence class of nowhere vanishing n-forms. A smooth map φ: $M \to N$ (dim M = dim N) is called *orientation preserving* (respectively, *orientation reversing*) if $\varphi^*\Delta$ (respectively, $-\varphi^*\Delta$) represents the orientation of M when Δ represents that of N.

The space $A_c(M)$ of differential forms with compact carrier is an ideal in $A(M)$. Assume M oriented and of dimension n. Then the *integral* is defined; it is a linear map $\int_M: A_c^n(M) \to \mathbb{R}$, natural with respect to orientation preserving diffeomorphisms, and satisfying

$$\int_E f \cdot \Delta_E = \int_E f(x)\, dx^1 \cdots dx^n, \qquad f \in \mathscr{S}_c(E),$$

where Δ_E is the positive normed determinant function of an oriented Euclidean space E.

The integral extends to the linear map $\int_M = \int_M \otimes \iota: A_c(M; F) \to F$. If $\alpha: F \to H$ is linear, then $\alpha \circ \int_M = \int_M \circ \alpha_*$.

In particular, suppose $\dim E = n + 1$, and S^n is the unit sphere in E. Then $T_x(S^n) = x^\perp$, $x \in S^n$, and the n-form, Ω, on S^n given by $\Omega(x; h_1, \dots, h_n) = \Delta_E(x, h_1, \dots, h_n)$ orients S^n. Its integral is called the *volume* of S^n and is given by

$$\int_{S^n} \Omega = \begin{cases} \dfrac{2^{m+1}}{1 \cdot 3 \cdots (2m-1)} \pi^m, & n = 2m, \quad m \geqslant 1 \\[2ex] \dfrac{2}{m!} \pi^{m+1}, & n = 2m+1, \quad m \geqslant 0. \end{cases}$$

0.14. De Rham cohomology. Let M be an n-manifold. Then $(A(M), \delta)$ is a graded differential algebra; its cohomology is denoted by $H(M) = \sum_{p=0}^n H^p(M)$ and is called the *de Rham cohomology algebra* of M. The homomorphism $\varphi^*: A(M) \leftarrow A(N)$ determined by a smooth map induces a homomorphism $\varphi^\#: H(M) \leftarrow H(N)$.

If $\dim H(M) < \infty$, then the pth *Betti number*, b_p, of M is $\dim H^p(M)$. The polynomial $f(t) = \sum_p b_p t^p$ is called the *Poincaré polynomial* and the number

$$\chi_M = \sum_{p=0}^{n} (-1)^p\, b_p$$

is called the *Euler–Poincaré characteristic of* M. If M is compact, then $\dim H(M) < \infty$.

Smooth maps $\varphi, \psi: M \to N$ are *homotopic* if there is a smooth map $H: \mathbb{R} \times M \to N$ such that $H(0, x) = \varphi(x)$ and $H(1, x) = \psi(x)$. H is called a *connecting homotopy*. The operator $h: A^p(N) \to A^{p-1}(M)$ given by

$$(h\Phi)(x; \xi_1, \dots, \xi_{p-1}) = \int_0^1 (H^*\Phi)(t, x; d/dt, \xi_1, \dots, \xi_{p-1})\, dt$$

is called the *homotopy operator induced from H*; it satisfies

$$\psi^* - \varphi^* = h \circ \delta + \delta \circ h.$$

In particular, if φ and ψ are homotopic, then $\varphi^\# = \psi^\#$.

The ideal $A_c(M)$ is stable under δ and the corresponding cohomology algebra is denoted by $H_c(M)$. Multiplication of differential forms makes $H_c(M)$ into a left and right graded $H(M)$-module, and these structures are denoted by

$$(\alpha, \beta) \mapsto \alpha * \beta \qquad \text{and} \qquad (\beta, \alpha) \mapsto \beta * \alpha, \qquad \alpha \in H(M), \quad \beta \in H_c(M).$$

Assume M is oriented. Then $\int_M \circ \delta = 0$ and so $\int_M$ induces a linear map $\int_M^\#: H_c^n(M) \to \mathbb{R}$. The *Poincaré scalar product* is the bilinear map $\mathscr{P}_M: H(M) \times H_c(M) \to \mathbb{R}$, given by

$$\mathscr{P}_M(\alpha, \beta) = \int_M^\# \alpha * \beta, \qquad \alpha \in H^p(M), \quad \beta \in H_c^{n-p}(M),$$

and

$$\mathscr{P}_M(\alpha, \beta) = 0, \qquad \deg \alpha + \deg \beta \neq n.$$

It induces an *isomorphism* $D_M: H(M) \xrightarrow{\cong} H_c(M)^*$, called the *Poincaré isomorphism*. In particular, if M is connected, $\int_M^\#$ is an isomorphism; i.e.,

$$\ker \int_M = \operatorname{Im} \delta.$$

The unique cohomology class $\omega_M \in H_c^n(M)$ such that $\int_M^\# \omega_M = 1$ is called the *orientation class*. If M is compact, then $H_c(M) = H(M)$ and so $b_p = b_{n-p}$.

The map $\Phi \otimes \Psi \mapsto \Phi \times \Psi$ (cf. sec. 0.13) defines homomorphisms

$$\kappa: A(M) \otimes A(N) \to A(M \times N) \qquad \text{and} \qquad \kappa_c: A_c(M) \otimes A_c(N) \to A_c(M \times N).$$

These induce the *Künneth homomorphisms*

$$\kappa_\#: H(M) \otimes H(N) \to H(M \times N) \text{ and } (\kappa_c)_\#: H_c(M) \otimes H_c(N) \to H_c(M \times N).$$

$(\kappa_c)_\#$ is always an isomorphism, while $\kappa_\#$ is an isomorphism if either $H(M)$ or $H(N)$ has finite dimension.

Suppose $\varphi, \psi: M \to N$ are smooth maps between compact connected oriented n-manifolds. The *degree* of φ is the integer, $\deg \varphi$, defined by

$$\int_M \varphi^* \Phi = \deg \varphi \int_N \Phi, \qquad \Phi \in A^n(N).$$

Let $\varphi^{\#}$ restrict to $\varphi^{(p)}$ in $H^p(N)$ and let $\tilde{\psi}^{(p)}\colon H^p(M) \to H^p(N)$ be the dual of $\psi^{(n-p)}$ with respect to the Poincaré scalar products. Then the *coincidence number* of φ and ψ is the alternating sum

$$L(\varphi, \psi) = \sum_{p=0}^{n} (-1)^p \operatorname{tr}(\varphi^{(p)} \circ \tilde{\psi}^{(p)}).$$

If $M = N$, then the *Lefschetz number* of φ is the alternating sum

$$L(\varphi) = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \varphi^{(p)}.$$

0.15. Smooth fibre bundles. A *smooth fibre bundle* is a quadruple $\mathscr{B} = (E, \pi, B, F)$ where (1) E, B, F are manifolds (*total space*, *base space*, *typical fibre*) and $\pi\colon E \to B$ is smooth, and (2) there is an open cover $\{U_\alpha\}$ of B and a family of commutative diagrams

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\psi_\alpha", "\cong"'] \arrow[dr] & & \pi^{-1}U_\alpha \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

(ψ_α, a diffeomorphism).

For $x \in B$, $\pi^{-1}(x)$ is a closed submanifold of E; it is denoted by F_x and is called the *fibre at* x. Thus ψ_α restricts to diffeomorphisms $\psi_{\alpha,x}\colon F \xrightarrow{\cong} F_x$. The family $\{(U_\alpha, \psi_\alpha)\}$ is called a *coordinate representation* for $\mathscr{B}$.

If $\mathscr{B}' = (E', \pi', B', F')$ is a second bundle, a smooth map $\varphi\colon E \to E'$ is called *fibre preserving* if it restricts to smooth maps $\varphi_x\colon F_x \to F'_{\psi(x)}$ $(x \in B)$. The induced map $\psi\colon B \to B'$ is smooth.

Fix $\mathscr{B} = (E, \pi, B, F)$. The spaces $T_z(F_x)$ $(z \in F_x, x \in B)$ are the fibres of a subbundle of τ_E; it is called the *vertical subbundle* and is denoted by $(V_E, p, E, \mathbb{R}^r)$ $(r = \dim F)$. The fibre at $z \in E$ is written $V_z(E)$ and called the *vertical subspace* at z; thus $V_z(E) = \ker(d\pi)_z$. A *horizontal subbundle* is a subbundle H_E of τ_E such that $\tau_E = H_E \oplus V_E$. Its fibre at z is called the *horizontal subspace* (with respect to the choice of H_E) and is written $H_z(E)$.

If $\mathscr{B}$ is a vector bundle, then $T_z(F_x) = F_x$. These identifications define a bundle map $\alpha\colon V_E \to E$ inducing $\pi\colon E \to B$, and restricting to isomorphisms in the fibres.

An orientation in V_E is called an *orientation in* $\mathscr{B}$; thus an orientation of $\mathscr{B}$ is a smoothly varying orientation of the fibres F_x. If $\Psi \in A^r(E)$ and its restriction to each F_x represents the orientation of F_x, then Ψ is said to *represent the orientation of* $\mathscr{B}$. If Φ orients the manifold B, then $\pi^*\Phi \wedge \Psi$ orients E; this orientation depends only on the orientations of $\mathscr{B}$ and B, and is called the *local product orientation.*

If $\mathscr{B}$ is a vector bundle, the definition above is a second definition of an orientation in $\mathscr{B}$; in this case, we use the map α, above, to identify orientations in $\mathscr{B}$ as defined in sec. 0.12 with orientations in the vertical bundle.

The space $A_F(E)$ of *differential forms with fibre compact carrier* consists of those Φ such that $\operatorname{carr}\Phi \cap \pi^{-1}(K)$ is compact whenever K is a compact subset of B. If F is compact, then $A_F(E) = A(E)$; while if B is compact, then $A_F(E) = A_c(E)$.

Let $\Omega \in A^{p+r}(E)$ $(r = \dim F)$. Fix $\eta_1, \dots, \eta_r \in T_z(F_x)$; then, for $\zeta_i \in T_z(E)$, $\Omega(z; \zeta_1, \dots, \zeta_p, \eta_1, \dots, \eta_r)$ depends only on the vectors $\xi_i = (d\pi)_z\zeta_i$ $(\in T_x(B))$. Thus a $\wedge^p T_x(B)^*$-valued r-form, Ω_x, on F_x is given by

$$\langle \Omega_x(z; \eta_1, \dots, \eta_r), \xi_1 \wedge \cdots \wedge \xi_p \rangle = \Omega(z; \zeta_1, \dots, \zeta_p, \eta_1, \dots, \eta_r), \qquad (d\pi)\zeta_i = \xi_i .$$

Ω_x is called the *retrenchment* of Ω to F_x. If $\Omega \in A_F(E)$, then each Ω_x has compact carrier.

Suppose $\mathscr{B}$ is oriented; then an orientation is determined in each manifold F_x. The *fibre integral* is the linear map $\oint_F : A_F(E) \to A(B)$, homogeneous of degree $-r$, given by

$$\left(\oint_F \Omega\right)(x) = \int_{F_x} \Omega_x .$$

It is surjective and satisfies

$$\oint_F \pi^*\Phi \wedge \Omega = \Phi \wedge \oint_F \Omega \qquad \text{and} \qquad \oint_F \circ\, \delta = \delta \circ \oint_F .$$

If B is oriented and E is given the local product orientation, then the *Fubini theorem* asserts that

$$\int_E \Omega = \int_B \oint_F \Omega, \qquad \Omega \in A_c^m(E), \quad m = \dim E.$$

0.16. Sphere bundles. An *r-sphere bundle* is a smooth bundle with fibre the r-sphere. If $\xi = (E, \pi, B, F)$ is a vector bundle with a Riemannian metric, then the unit spheres $S_x \subset F_x$ are the fibres of a sphere bundle $\xi_S = (E_S, \pi_S, B, S)$ called the *associated sphere bundle.* An orientation in ξ defines an orientation in the fibres F_x; the induced orientations in the spheres S_x (cf. sec. 0.13) define an orientation in ξ_S.

Suppose $\mathscr{B} = (M, \pi, B, S)$ is an oriented r-sphere bundle. Then there are differential forms $\Omega \in A^r(M)$, $\Phi \in A^{r+1}(B)$ such that $\oint_S \Omega = -1$ and $\pi^*\Phi = \delta\Omega$ (thus $\delta\Phi = 0$). The cohomology class represented by Φ (in $H^{r+1}(B)$) depends only on the oriented bundle $\mathscr{B}$. It is called the *Euler class* of $\mathscr{B}$ and is written $\chi_{\mathscr{B}}$.

Let $\mathscr{B} = (M, \pi, B, S)$ be an oriented sphere bundle with $\dim B = n = \dim S + 1$. Assume B is oriented. A *cross-section in $\mathscr{B}$ with finitely many singularities* $a_1, \ldots, a_k$ is a smooth map $\sigma: B - \{a_1, \ldots, a_k\} \to M$ such that $\pi \circ \sigma = \iota$. (Such cross-sections always exist, if $k \geqslant 1$.) Using the local product structure we obtain, from σ, smooth maps

$$\sigma_i: U_i - \{a_i\} \to S,$$

where U_i is a neighbourhood of a_i. The orientation of U_i determines an orientation in a 'sphere' S_i about a_i. Let τ_i be the restriction of σ_i to S_i; then the degree of τ_i is independent of the various choices. It is called the *index* of σ at a_i and is written $j_{a_i}(\sigma)$ or simply $j_i(\sigma)$.

The sum $j(\sigma) = \sum_i j_i(\sigma)$ is called the *index sum* of σ. It satisfies the relation

$$\int_B^{\#} \chi_{\mathscr{B}} = j(\sigma).$$

Moreover, if $\mathscr{B}$ is the associated sphere bundle of the tangent bundle of a compact oriented n-manifold B, then $\chi_{\mathscr{B}} \in H^n(B)$ and

$$\int_B^{\#} \chi_{\mathscr{B}} = \chi_B .$$

Chapter I

Lie Groups

§1. Lie algebra of a Lie group

1.1. Definition: A *Lie group* is a set G which is both a group and a smooth manifold, and for which the following maps are smooth:

(i) The *multiplication map* $\mu: G \times G \to G$ given by

$$(x, y) \mapsto xy.$$

(ii) The *inversion map* $\nu: G \to G$ given by

$$x \mapsto x^{-1}.$$

The unit element of a Lie group is denoted by e.

A *homomorphism of Lie groups* $\varphi: G \to H$ is a smooth homomorphism of groups. An *isomorphism of Lie groups* is a map that is both a homomorphism and a diffeomorphism.

Let G be a Lie group. Each $a \in G$ determines smooth maps λ_a, $\rho_a: G \to G$, given by

$$\lambda_a(x) = ax \qquad \text{and} \qquad \rho_a(x) = xa.$$

They are called *left* and *right translation* by a. The group axioms yield the relations

$$\lambda_a \circ \lambda_b = \lambda_{ab}, \qquad \rho_a \circ \rho_b = \rho_{ba},$$
$$\lambda_e = \rho_e = \iota, \qquad \text{and} \qquad \lambda_a \circ \rho_b = \rho_b \circ \lambda_a.$$

In particular, λ_a and ρ_b are diffeomorphisms, with inverses $\lambda_{a^{-1}}$ and $\rho_{b^{-1}}$.

We shall denote the derivatives of λ_a, ρ_b by

$$L_a = d\lambda_a: T_G \to T_G \qquad \text{and} \qquad R_b = d\rho_b: T_G \to T_G.$$

The relations above yield the relations

$$L_a \circ L_b = L_{ab}, \qquad R_a \circ R_b = R_{ba},$$
$$R_e = L_e = \iota_{T_G}, \qquad \text{and} \qquad L_a \circ R_b = R_b \circ L_a.$$

If $\varphi: G \to H$ is a homomorphism of Lie groups, then

$$\varphi \circ \lambda_a = \lambda_{\varphi(a)} \circ \varphi \qquad \text{and} \qquad \varphi \circ \rho_b = \rho_{\varphi(b)} \circ \varphi.$$

Hence

$$d\varphi \circ L_a = L_{\varphi(a)} \circ d\varphi \qquad \text{and} \qquad d\varphi \circ R_b = R_{\varphi(b)} \circ d\varphi.$$

In particular, each $(d\varphi)_x: T_x(G) \to T_{\varphi(x)}(H)$ $(x \in G)$ is injective (respectively, surjective) if and only if $(d\varphi)_e$ is.

Now consider the multiplication and inversion maps. Their derivatives are bundle maps

$$d\mu: T_G \times T_G \to T_G \qquad \text{and} \qquad d\nu: T_G \to T_G\,.$$

Lemma I: Let $\xi \in T_a(G)$, $\eta \in T_b(G)$. Then

(1) $d\mu(\xi, \eta) = R_b\xi + L_a\eta$

and

(2) $d\nu(\xi) = -(L_a^{-1} \circ R_a^{-1})(\xi)$.

Proof: (1) Let $j_a: G \to \{a\} \times G$ and $j_b: G \to G \times \{b\}$ denote the inclusions opposite a and b respectively. Then

$$d\mu(\xi, \eta) = (d\mu \circ dj_b)(\xi) + (d\mu \circ dj_a)(\eta) = R_b(\xi) + L_a(\eta).$$

(2) Since $x \mapsto \mu(x, \nu(x))$ is the constant map $G \to e$, we have

$$d\mu(\xi, d\nu(\xi)) = 0.$$

Now (2) follows from (1).

Q.E.D.

1.2. Invariant vector fields. The left and right translations of a Lie group G induce automorphisms $(\lambda_a)_*$ and $(\rho_a)_*$ of the real Lie algebra, $\mathscr{X}(G)$, of vector fields on G (cf. sec. 0.13). A vector field X on G is called *left invariant* if $L_a(X(x)) = X(ax)$, $a, x \in G$; i.e., if $(\lambda_a)_*X = X$, $a \in G$. In view of Lemma I, sec. 1.1, this is equivalent to

$$i_R X \underset{\mu}{\sim} X$$

$(i_R X(x, y) = (0, X(y))$.

Since each $(\lambda_a)_*$ preserves Lie products, the left invariant vector fields form a subalgebra, $\mathscr{X}_L(G)$, of $\mathscr{X}(G)$.

Proposition I: A strong bundle isomorphism $\alpha: G \times T_e(G) \xrightarrow{\cong} T_G$ is given by

$$(a, h) \mapsto L_a(h).$$

Proof: α restricts to isomorphisms in the fibres. Moreover it is given by

$$\alpha(a, h) = d\mu(0_a, h)$$

(cf. Lemma I, sec. 1.1) and hence it is smooth.

Q.E.D.

Corollary I: An isomorphism $\mathscr{X}_L(G) \xrightarrow{\cong} T_e(G)$ is given by

$$X \mapsto X(e).$$

In particular $\dim \mathscr{X}_L(G) = \dim G$.

Corollary II: An isomorphism of $\mathscr{S}(G)$-modules

$$\mathscr{X}_L(G) \otimes \mathscr{S}(G) \xrightarrow{\cong} \mathscr{X}(G)$$

is given by $X \otimes f \mapsto f \cdot X$.

Definition: Let $h \in T_e(G)$. The unique left invariant vector field X such that $X(e) = h$ is denoted by X_h, and is called the *left invariant vector field generated by h.*

Similarly, a vector field Y is called *right invariant* if $(\rho_b)_* Y = Y$, $b \in G$. The Lie algebra of right invariant vector fields is denoted by $\mathscr{X}_R(G)$. The same proof as given in Proposition I shows that

$$Y \mapsto Y(e)$$

defines an isomorphism $\mathscr{X}_R(G) \xrightarrow{\cong} T_e(G)$. The right invariant vector field corresponding to $h \in T_e(G)$ under this isomorphism is called the *right invariant vector field generated by h*, and is denoted by Y_h.

Proposition II: If $X \in \mathscr{X}_L(G)$ and $Y \in \mathscr{X}_R(G)$, then

$$[X, Y] = 0.$$

Proof: Define $i_L Y \in \mathscr{X}(G \times G)$ by $i_L Y(x, y) = (Y(x), 0)$. Then

$$i_R X \underset{\mu}{\sim} X \qquad \text{and} \qquad i_L Y \underset{\mu}{\sim} Y,$$

and it follows from Proposition IX, sec. 3.14, volume I, and Proposition VIII, sec. 3.13, volume I, that

$$0 = [i_R X, i_L Y] \underset{\mu}{\sim} [X, Y].$$

Since μ is surjective, $[X, Y] = 0$.

Q.E.D.

Finally, consider the inversion map $\nu: x \mapsto x^{-1}$ of G. Since $\nu^2 = \iota$, ν is a diffeomorphism. Clearly,

$$\nu \circ \lambda_a = \rho_{a^{-1}} \circ \nu, \qquad d\nu \circ L_a = R_{a^{-1}} \circ d\nu, \qquad \text{and} \qquad \nu_* \circ (\lambda_a)_* = (\rho_{a^{-1}})_* \circ \nu_*.$$

In particular, ν_* restricts to an isomorphism

$$\mathscr{X}_L(G) \xrightarrow{\cong} \mathscr{X}_R(G)$$

of Lie algebras. In view of Lemma I (2) sec. 1.1, we have

$$\nu_* X_h = -Y_h, \qquad h \in T_e(G),$$

and hence, for $h, k \in T_e(G)$,

$$[X_h, X_k](e) = -[Y_h, Y_k](e).$$

1.3. Lie algebra of a Lie group. The *Lie algebra* of a Lie group G is the vector space, $T_e(G)$, together with the Lie algebra structure induced from $\mathscr{X}_L(G)$ by the isomorphism of Corollary I to Proposition I, sec. 1.2. Thus, for $h, k \in T_e(G)$,

$$[h, k] = [X_h, X_k](e).$$

(Note that the isomorphism $\mathscr{X}_R(G) \xrightarrow{\cong} T_e(G)$ determines a second Lie product $[\ ,\ ]^\sim$ in $T_e(G)$. In view of sec. 1.2, we have

$$[h, k] = -[h, k]^\sim, \qquad h, k \in T_e(G).$$

Thus the map $h \mapsto -h$ defines an isomorphism between these Lie algebra structures.)

Now consider a homomorphism of Lie groups, $\varphi: G \to H$. Since $\varphi(e) = e$ (e denotes the unit of both groups), the derivative $d\varphi$ restricts to a linear map

$$(d\varphi)_e: T_e(G) \to T_e(H).$$

This map will be denoted by φ'.

Proposition III: φ' is a homomorphism of Lie algebras.

Proof: It follows from sec. 1.1 that

$$X_h \underset{\varphi}{\sim} X_{\varphi' h}, \qquad h \in T_e(G).$$

Hence $[X_h, X_k] \underset{\varphi}{\sim} [X_{\varphi' h}, X_{\varphi' k}]$. Evaluate this relation at e to obtain $\varphi'[h, k] = [\varphi' h, \varphi' k]$.

Q.E.D.

If $\psi: H \to K$ is a second homomorphism of Lie groups, then

$$(\psi \circ \varphi)' = \psi' \circ \varphi'.$$

1.4. Examples: **1.** *The vector group*: If V is a finite-dimensional real or complex vector space, vector addition makes V into a Lie group.

2. *The group $GL(V)$*: Consider the group $GL(V)$ of linear automorphisms of an n-dimensional vector space V (real or complex). It is an open subset of the vector space $L_V = L(V; V)$, and hence a manifold; moreover, multiplication and inversion are smooth and so $GL(V)$ is a Lie group.

Since $GL(V)$ is an open subset of L_V, its tangent bundle is the restriction of the tangent bundle of L_V,

$$T_{GL(V)} = GL(V) \times L_V.$$

In particular, the underlying vector space of the corresponding Lie algebra is L_V.

Next, observe that the left translations λ_τ, $\tau \in GL(V)$, are given by

$$\lambda_\tau(\sigma) = \tau \circ \sigma, \qquad \tau, \sigma \in GL(V).$$

It follows that

$$L_\tau(\sigma, \alpha) = (\tau \circ \sigma, \tau \circ \alpha), \qquad \sigma \in GL(V), \quad \alpha \in L_V.$$

Hence the left invariant vector field generated by $\alpha \in L_V$ is given by

$$X_\alpha(\tau) = (\tau, \tau \circ \alpha), \qquad \tau \in GL(V).$$

To determine the Lie product, let f be a linear function in L_V and denote its restriction to $GL(V)$ also by f. Then

$$(X_\alpha f)(\tau) = f(\tau \circ \alpha),$$

and so

$$([X_\alpha, X_\beta]f)(\tau) = f(\tau \circ (\alpha \circ \beta - \beta \circ \alpha)).$$

Since $\tau \in GL(V)$ and $f \in L_V^*$ were arbitrary, we obtain

$$[X_\alpha, X_\beta] = X_{\alpha\circ\beta-\beta\circ\alpha}.$$

In particular, the Lie algebra structure of L_V induced from the Lie group structure of $GL(V)$ is given by

$$[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha.$$

3. *The group of invertibles of an associative algebra*: Let A be an associative finite-dimensional algebra over $\mathbb{R}$, with unit element. For $a \in A$, define $\mu(a)$: $A \to A$ to be left multiplication by a. Then a has an inverse in A if and only if $\mu(a)$ is a linear isomorphism; i.e., if and only if

$$\det \mu(a) \neq 0.$$

The invertible elements of A form a group $G(A)$ under composition; the condition above shows that $G(A)$ is open in A. Hence $G(A)$ is a Lie group. The same argument as given for $GL(V)$ in L_V shows that the Lie algebra of $G(A)$ is A, with Lie bracket given by

$$[\alpha, \beta] = \alpha\beta - \beta\alpha, \qquad \alpha, \beta \in A.$$

4. *Direct products*: Let G, H be Lie groups. The product manifold $G \times H$ can be made into a Lie group by setting

$$(x, y) \cdot (x', y') = (x \cdot x', y \cdot y'), \qquad x, x' \in G \quad y, y' \in H.$$

This Lie group is called the *direct product* of G and H.

The projections π_G: $G \times H \to G$ and π_H: $G \times H \to H$, and the inclusions G, $H \to G \times H$, opposite e, are all homomorphisms of Lie groups. The Lie algebra homomorphisms π_G', π_H' are given by

$$\pi_G'(h, k) = h \qquad \text{and} \qquad \pi_H'(h, k) = k.$$

It follows that the Lie product in $T_e(G \times H)$ is given by

$$[(h, k), (h', k')] = ([h, h'], [k, k']), \qquad h, h' \in T_e(G), \quad k, k' \in T_e(H).$$

5. *Tangent bundle*: If G is a Lie group, then the map

$$d\mu\colon T_G \times T_G \to T_G$$

makes T_G into a Lie group, with inversion map $d\nu$. (The associative law is obtained by differentiating the relation $\mu \circ (\mu \times \iota) = \mu \circ (\iota \times \mu)$.) The zero cross-section o: $G \to T_G$ is a homomorphism of Lie groups.

6. *The* 1-*component*: Let G be a Lie group, and let G^0 denote that connected component of the manifold G which contains e; it is an open submanifold. Since μ, ν are continuous and $G^0 \times G^0$, G^0 are connected it follows that

$$\mu(G^0 \times G^0) \subset G^0 \qquad \text{and} \qquad \nu(G^0) \subset G^0.$$

Similarly, $aG^0a^{-1} \subset G^0$, $a \in G$. Thus G^0 is a normal subgroup of G. It is clearly a Lie group and is called the 1-*component of* G. The quotient group G/G^0 is called the *component group* of G.

7. The nonzero reals $\mathbb{R}_{\bullet} = \mathbb{R} - \{0\}$ and the nonzero complex numbers $\mathbb{C}_{\bullet} = \mathbb{C} - \{0\}$ are each a Lie group under multiplication. If V (respectively, W) is a real (respectively, complex) vector space, then the maps

$$\det: GL(V) \to \mathbb{R}_{\bullet} \qquad \text{and} \qquad \det: GL(W) \to \mathbb{C}_{\bullet}$$

are Lie group homomorphisms. Their derivatives are given, respectively, by

$$\operatorname{tr}: L_V \to \mathbb{R} \qquad \text{and} \qquad \operatorname{tr}: L_W \to \mathbb{C};$$

i.e., $\det' = \operatorname{tr}$ (cf. sec. 1.3).

§2. The exponential map

1.5. One-parameter subgroups. A *1-parameter subgroup* of a Lie group G is a homomorphism, α, of the additive group of real numbers into G,

$$\alpha\colon \mathbb{R} \to G.$$

In other words, a 1-parameter subgroup is a smooth map $\alpha\colon \mathbb{R} \to G$ such that

$$\alpha(s + t) = \alpha(s)\, \alpha(t), \qquad s, t \in \mathbb{R}.$$

In particular, $\alpha(0) = e$ and $\alpha(-t) = \alpha(t)^{-1}$.

Suppose $\alpha\colon \mathbb{R} \to G$ is a 1-parameter subgroup. Then (cf. sec. 0.13) α determines a path $\dot{\alpha}\colon \mathbb{R} \to T_G$:

$$\dot{\alpha}(t) = (d\alpha)_t \left(\frac{d}{dt}\right) \in T_{\alpha(t)}(G).$$

In particular, $\dot{\alpha}(0) \in T_e(G)$.

Proposition IV: Let $\alpha\colon \mathbb{R} \to G$ be a smooth map such that $\alpha(0) = e$ and let $\dot{\alpha}(0) = h$. Then the following are equivalent:

(1) α is a 1-parameter subgroup.
(2) α is an orbit of X_h .
(3) α is an orbit of Y_h .

Proof: (1) $\Rightarrow$ (2): Denote the vector field $t \mapsto d/dt$ on $\mathbb{R}$ by T; it is the left and right invariant vector field generated by $T(0)$. Hence if α is a 1-parameter subgroup,

$$T \underset{\alpha}{\sim} X_h;$$

i.e., α is an orbit of X_h .

(2) $\Rightarrow$ (1): Assume α is an orbit of X_h and fix $s \in \mathbb{R}$. Then

$$t \mapsto \alpha(s + t) \qquad \text{and} \qquad t \mapsto \alpha(s)\, \alpha(t)$$

are both orbits of X_h (use the left invariance of X_h), and agree at $t = 0$. Hence (cf. Proposition X, sec. 3.15, volume I)

$$\alpha(s + t) = \alpha(s)\,\alpha(t).$$

(3) $\Leftrightarrow$ (1): Same proof as (2) $\Leftrightarrow$ (1).

Q.E.D.

Proposition V: To every vector $h \in T_e(G)$ corresponds a unique 1-parameter subgroup, α, such that

$$\dot{\alpha}(0) = h.$$

Proof: The uniqueness is immediate from Proposition IV. Now we prove existence. According to Proposition X, sec. 3.15, volume I, for some $\epsilon > 0$ there is an orbit

$$\alpha_0 : (-\epsilon, \epsilon) \to G,$$

for X_h, satisfying $\alpha_0(0) = e$.

Now fix $t_0 \in (0, \epsilon)$. Define smooth maps

$$\alpha_p: (pt_0 - \epsilon, pt_0 + \epsilon) \to G, \qquad p \in \mathbb{Z},$$

by

$$\alpha_p(t) = \alpha_0(t_0)^p \alpha_0(t - pt_0).$$

Since X_h is left invariant, these maps are all orbits for X_h. Moreover,

$$\alpha_{p-1}(pt_0) = \alpha_0(t_0)^p = \alpha_p(pt_0).$$

Hence α_{p-1} and α_p agree in the intersection of their domains.

It follows that a smooth map $\alpha: \mathbb{R} \to G$ is given by

$$\alpha(t) = \alpha_p(t), \qquad t \in (pt_0 - \epsilon, pt_0 + \epsilon);$$

α is an orbit for X_h satisfying $\alpha(0) = e$; thus by Proposition IV it is a 1-parameter subgroup.

Q.E.D.

The 1-parameter subgroup, α, that satisfies $\dot{\alpha}(0) = h$ is called the *1-parameter subgroup generated by h*, and is denoted by α_h. In particular, the 1-parameter subgroup generated by 0 is the constant map $t \mapsto e$.

Example: Let $\mathbb{C}_{\bullet}$ be the multiplicative group of nonzero complex numbers: $\mathbb{C}_\bullet = \{z \in \mathbb{C} \mid z \neq 0\}$. Then the corresponding Lie algebra is $\mathbb{C}$, considered as a real vector space.

The 1-parameter subgroup generated by a vector $h \in \mathbb{C}$ is given by

$$\alpha_h(t) = \exp th.$$

1.6. The exponential map. Let G be a Lie group with Lie algebra E $(= T_e(G))$. Define a set map

$$\psi\colon \mathbb{R} \times E \to G$$

by

$$\psi(t, h) = \alpha_h(t), \qquad t \in \mathbb{R}, \quad h \in E.$$

Lemma II: ψ is a smooth map. It satisfies

$$\psi(st, h) = \psi(t, sh), \qquad s, t \in \mathbb{R}, \quad h \in E.$$

Proof: The equation holds because both sides define the 1-parameter subgroup generated by sh (cf. sec. 1.5).

To show that ψ is smooth, define a vector field Z on the manifold $E \times G$ by

$$Z(h, a) = (0, X_h(a)).$$

In view of Theorem II, sec. 3.15, volume I, there are neighbourhoods I of 0 in $\mathbb{R}$, V of 0 in E and U of e in G, and there is a smooth map

$$\varphi\colon I \times (V \times U) \to E \times G$$

such that

$$\dot{\varphi}(t, h, a) = Z(\varphi(t, h, a)), \qquad t \in I, \quad h \in V, \quad a \in U,$$

and

$$\varphi(0, h, a) = (h, a).$$

Now write

$$\varphi(t, h, e) = (\varphi_E(t, h), \varphi_G(t, h)), \qquad t \in I, \quad h \in V.$$

Then $\dot{\varphi}_E(t, h) = 0$, $\varphi_E(0, h) = h$, and so

$$\varphi_E(t, h) = h, \qquad t \in I, \quad h \in V.$$

It follows that

$$\dot{\varphi}_G(t, h) = X_{\varphi_E(t,h)}(\varphi_G(t, h)) = X_h(\varphi_G(t, h)).$$

Hence $\varphi_G(t, h) = \alpha_h(t) = \psi(t, h)$ and so ψ is smooth in $I \times V$.

Now the functional equation

$$\psi(t + \tau, h) = \psi(t, h)\,\psi(\tau, h), \qquad t, \tau \in \mathbb{R}, \quad h \in E$$

implies that ψ is smooth in $\mathbb{R} \times V$. Finally, applying the equation $\psi s(t, h) = \psi(t, sh)$, we see that ψ is smooth in $\mathbb{R} \times E$.

Q.E.D.

Definition: The *exponential map* for G is the smooth map $\exp\colon E \to G$ given by

$$\exp h = \psi(1, h) = \alpha_h(1).$$

It follows from Lemma II that the 1-parameter group generated by $h \in E$ can be written as

$$\alpha_h(t) = \exp th, \qquad t \in \mathbb{R}.$$

In particular $\exp ph = (\exp h)^p$, $p \in \mathbb{Z}$, $h \in E$.

Proposition VI: The exponential map satisfies

$$\exp 0 = e \qquad \text{and} \qquad (d \exp)_0 = \iota.$$

Proof: Fix $h \in E$. Then

$$h = \dot{\alpha}_h(0) = (\exp th)^{\dot{}}(0) = (d \exp)_0(h).$$

Q.E.D.

Corollary I: There are neighbourhoods V of 0 in E and U of e in G such that the exponential map restricts to a diffeomorphism

$$\exp\colon V \xrightarrow{\cong} U.$$

Corollary II: Let $E = E_1 \oplus \cdots \oplus E_r$ be a decomposition of E as a direct sum of subspaces. Define $\varphi\colon E \to G$ by

$$\varphi(h_1 \oplus \cdots \oplus h_r) = \exp h_1 \cdot \cdots \cdot \exp h_r\,, \qquad h_i \in E_i\,.$$

Then $(d\varphi)_0 = \iota$, and so φ maps a neighbourhood of 0 diffeomorphically onto a neighbourhood of e.

Proof: Clearly $(d\varphi)_0$ restricts to the identity in each E_i ; hence it is the identity in E.

Q.E.D.

Corollary III: If G is connected, then $\exp(E)$ generates G.

Proof: By Corollary I, $\exp(E)$ contains a neighbourhood of e. Thus the corollary follows from Lemma III below.

Q.E.D.

Lemma III: If G is connected, and $U \subset G$ is a neighbourhood of e, then U generates G.

Proof: U generates an open subgroup H of G. Thus each coset Ha $(a \in G)$ is open and

$$G = H \cup \bigcup_{a \notin H} Ha$$

partitions G into two disjoint open sets. Since G is connected, $G = H$.

Q.E.D.

Examples: **1.** Consider the case $G = GL(V)$, $E = L_V$ (cf. Example 2, sec. 1.4). Then exp is the map given in sec. 0.10.

2. Let H be a second Lie group with Lie algebra F. Then the exponential map for $G \times H$ is given by

$$\exp(h, k) = (\exp_G(h), \exp_H(k)), \qquad h \in E, \quad k \in F.$$

1.7. Homomorphisms. Proposition VII: Let $\varphi: G \to H$ be a homomorphism of Lie groups. Then the induced homomorphism, φ', of Lie algebras satisfies

$$\varphi \circ \exp_G = \exp_H \circ \varphi'.$$

Proof: Fix $h \in T_e(G)$. Then

$$\alpha: t \mapsto \varphi(\exp_G(th)) \qquad \text{and} \qquad \beta: t \mapsto \exp_H(t\varphi'(h))$$

are 1-parameter subgroups of H. Moreover,

$$\dot{\alpha}(0) = \varphi'(h) = \dot{\beta}(0),$$

and hence (Proposition V, sec. 1.5) $\alpha = \beta$. In particular

$$\varphi(\exp_G(h)) = \exp_H(\varphi'(h)), \qquad h \in T_e(G).$$

Q.E.D.

Corollary I: Assume $\psi: G \to H$ is a second homomorphism of Lie groups and that $\varphi' = \psi'$. If G is connected, then $\varphi = \psi$.

Proof: Proposition VII implies that φ and ψ agree in $\exp_G(T_e(G))$. By Corollary III to Proposition VI, sec. 1.6, this set generates G. Since φ and ψ are group homomorphisms, it follows that $\varphi = \psi$.

Q.E.D.

Corollary II: The homomorphism φ is injective if and only if

$$d\varphi: T_G \to T_H$$

is injective. In this case φ embeds G into H.

Proof: If $d\varphi$ is injective, then certainly φ is injective. Conversely, assume φ is injective. Let V be a neighbourhood of 0 in $T_e(G)$ such that the restriction of $\exp_G$ to V is injective. Then since $\exp_H \circ \varphi' = \varphi \circ \exp_G$, the restriction of $\exp_H \circ \varphi'$ to V is injective. In particular, the restriction of φ' to V is injective.

Since φ' is linear and V is an open subset of $T_e(G)$, it follows that φ' is injective. Since

$$(d\varphi)_a = L_{\varphi(a)} \circ \varphi' \circ L_{a^{-1}}, \qquad a \in G,$$

each $(d\varphi)_a$ is injective. Hence so is $d\varphi$.

Q.E.D.

Corollary III: If φ is bijective, then it is a diffeomorphism and hence an isomorphism between Lie groups.

Proof: Since φ is injective, Corollary II implies that $d\varphi: T_G \to T_H$ is injective. Now Proposition IV, sec. 3.8, volume I, implies that φ is a diffeomorphism.

Q.E.D.

Proposition VIII: A continuous group homomorphism $\varphi: G \to H$ between Lie groups is smooth.

Proof: Consider first the case that $G = \mathbb{R}$. It has to be shown that a continuous map $\alpha: \mathbb{R} \to H$ which satisfies

$$\alpha(s + t) = \alpha(s)\,\alpha(t), \qquad s, t \in \mathbb{R},$$

is smooth. In view of Corollary I to Proposition VI, sec. 1.6, there is a neighbourhood V of 0 in $T_e(H)$ which $\exp_H$ maps diffeomorphically onto a neighbourhood U of e in H.

Without loss of generality we may assume that

$$\alpha(t) \in U, \qquad |t| \leqslant 1.$$

Define a continuous map $\beta: I \to V$ ($I = \{t \in \mathbb{R} \mid |t| \leqslant 1\}$) by

$$\beta(t) = \exp_H^{-1} \alpha(t).$$

Since α is a homomorphism,

$$\exp_H(q \cdot \beta(t)) = \alpha(qt) = \exp_H(\beta(qt)) \qquad q \in \mathbb{Z}, \quad |qt| \leqslant 1, \quad |t| \leqslant 1.$$

Hence $q \cdot \beta(t) \in V$ if and only if

$$q \cdot \beta(t) = \beta(qt).$$

Fix $q \neq 0$. Consider the set

$$\{t \in (1/q)I \mid q \cdot \beta(t) \in V\}.$$

The above relation shows that this set is both closed and open in $(1/q)I$, and hence equal to $(1/q)I$. Thus

$$q \cdot \beta(t) = \beta(qt), \qquad |qt| \leqslant 1.$$

It follows that for $p/q \in \mathbb{Q}$ and $|p/q| \leqslant 1$,

$$\beta(p/q) = (p/q)\beta(1).$$

Since β is continuous,

$$\beta(t) = t\beta(1), \qquad |t| \leqslant 1.$$

Now we have

$$\alpha(t) = \exp_H(t\beta(1)), \qquad |t| \leqslant 1.$$

Since α is a homomorphism (as is $t \mapsto \exp_H(t\beta(1))$) and the interval $(-1, 1)$ generates the additive group $\mathbb{R}$, it follows that

$$\alpha(t) = \exp_H(t\beta(1)), \qquad t \in \mathbb{R},$$

and so α is smooth.

Finally, consider the general case, $\varphi\colon G \to H$. Choose a basis $e_1, \ldots, e_n$ of $T_e(G)$ and consider the smooth map $\psi\colon \mathbb{R}^n \to G$ given by

$$\psi(t_1, \ldots, t_n) = \exp_G(t_1 e_1) \cdot \cdots \cdot \exp_G(t_n e_n).$$

By Corollary II to Proposition VI, sec. 1.6, ψ maps a neighbourhood V of 0 diffeomorphically onto a neighbourhood U of e.

On the other hand, the maps $t \mapsto \varphi(\exp_G(te_i))$ $(i = 1, \ldots, n)$ are continuous homomorphisms $\mathbb{R} \to H$; thus they are smooth by the argument above. Since φ is a homomorphism, we have

$$(\varphi \circ \psi)(t_1, \ldots, t_n) = \varphi(\exp_G(t_1 e_1)) \cdots \varphi(\exp_G(t_n e_n))$$

and so $\varphi \circ \psi$ is smooth. In particular, φ is smooth in U. But for any $a \in G$,

$$\varphi(ax) = \varphi(a)\,\varphi(x).$$

Thus φ is smooth in a neighbourhood of a and hence in G.

Q.E.D.

§3. Representations

In this article G denotes a fixed Lie group with Lie algebra E.

1.8. The derivative of a representation. A *representation* of G in a finite-dimensional vector space W (real or complex) is a homomorphism of Lie groups

$$P: G \to GL(W).$$

Since the Lie algebra of $GL(W)$ is the space L_W of linear transformations of W (cf. Example 2, sec. 1.4), the derivative of the homomorphism P is a homomorphism of Lie algebras,

$$P': E \to L_W$$

(cf. Proposition III, sec. 1.3). P' will be called the *derivative of the representation* P.

A Lie algebra homomorphism $\theta: E \to L_W$ is called a *representation of* E *in* W. Thus P' is a representation of E in W.

A representation, P, of G (respectively, θ of E) is called *faithful* if $\ker P = e$ (respectively, if $\ker \theta = 0$).

If P is a representation of G in W, then the *invariant subspace of* P is the subspace $W_{P=I}$ (or simply W_I) given by

$$W_I = \{w \in W \mid P(x)w = w,\ x \in G\}.$$

Similarly, if θ is a representation of E in W, then the *invariant subspace for* θ is the subspace $W_{\theta=0}$ (or W_0) given by

$$W_{\theta=0} = \{w \in W \mid \theta(h)w = 0,\ h \in E\}.$$

A subspace $V \subset W$ is called *stable for* P (respectively, *stable for* θ) if each of the operators $P(x)$, $x \in G$ (respectively $\theta(h)$, $h \in E$) maps V to itself.

Now fix $h \in E$. Then $P(\exp th)$, and $P'(h)$ are linear transformations of W. In particular, we regard the 1-parameter group

$$P_h: t \mapsto P(\exp th)$$

as a path in the vector space L_W. Thus differentiation yields a path $\dot{P}_h(t)$ in L_W.

On the other hand recall from Example 2, sec. 1.4, that $T_{GL(W)} = GL(W) \times L_W$. Moreover,

$$X_{P'(h)}(\tau) = (\tau, \tau \circ P'(h)), \qquad \tau \in GL(W), \quad h \in E.$$

Applying this formula with $\tau = P_h(t)$ gives

Lemma IV: $\dot{P}_h(t) = P_h(t) \circ P'(h)$.

Proposition IX: (1) The invariant subspaces W_I and W_0 for P and P' are related by

$$W_I \subset W_0 .$$

If G is connected, then $W_I = W_0$.

(2) If $V \subset W$ is stable for P, then it is stable for P'. If V is stable for P' and G is connected, then V is stable for P.

Proof: (1) Suppose $h \in E$ and $w \in W_I$. Then $P_h(t)w = w$, and so

$$\dot{P}_h(t)w = (P_h(t)w)^{\cdot} = 0.$$

Now Lemma IV yields $P'(h)w = 0$. Thus $W_I \subset W_0$.

Conversely, let $h \in E$ and assume $w \in W_0$. Then Lemma IV implies that $P_h(t)w = w$, $t \in \mathbb{R}$. It follows that

$$P(\exp h)\, w = w, \qquad h \in E.$$

Now if G is connected we can apply Corollary III to Proposition VI, sec. 1.6, to obtain $P(x)w = w$, $x \in G$.

(2) is proved in the same way.

Q.E.D.

1.9. Examples: In this section P (respectively, θ) denotes a fixed representation of G (respectively, E) in W.

1. *Contragredient representation*: The representation, $P^\natural$, of G in W^* *contragredient to* P is defined by

$$P^\natural(x) = (P(x)^{-1})^*, \qquad x \in G.$$

The representation $\theta^\natural$ of E in W^* *contragredient to* θ is defined by

$$\theta^\natural(h) = -\theta(h)^*, \qquad h \in E.$$

Evidently

$$(P^{\natural})' = (P')^{\natural}.$$

2. *Multilinear representations*: Representations $\otimes P$, $\wedge P$ and $\vee P$ of G in $\otimes W$, $\wedge W$, $\vee W$ are given by

$$(\otimes P)(x) = \otimes P(x), \qquad (\wedge P)(x) = \wedge P(x)$$

and

$$(\vee P)(x) = \vee P(x), \qquad x \in G,$$

(cf. sec. 0.5).

Representations $\theta_\otimes$, $\theta_\wedge$, $\theta_\vee$ of E in $\otimes W$, $\wedge W$, and $\vee W$ are given by

$$\theta_\otimes(h)(w_1 \otimes \cdots \otimes w_p) = \sum_{i=1}^{p} w_1 \otimes \cdots \theta(h)w_i \cdots \otimes w_p,$$

$$\theta_\wedge(h)(w_1 \wedge \cdots \wedge w_p) = \sum_{i=1}^{p} w_1 \wedge \cdots \theta(h)w_i \cdots \wedge w_p,$$

$$\theta_\vee(h)(w_1 \vee \cdots \vee w_p) = \sum_{i=1}^{p} w_1 \vee \cdots \theta(h)w_i \cdots \vee w_p, \qquad p \geqslant 1,$$

and

$$\theta_\otimes(h)\lambda = 0, \qquad \theta_\wedge(h)\lambda = 0, \qquad \theta_\vee(h)\lambda = 0, \qquad \lambda \in \mathbb{R}.$$

Evidently,

$$(\otimes P)' = (P')_\otimes, \qquad (\wedge P)' = (P')_\wedge \qquad \text{and} \qquad (\vee P)' = (P')_\vee.$$

3. Recall that $T^p(W)$ denotes the space of p-linear functions in W. Define a representation, P^p, of G in $T^p(W)$ by setting

$$(P^p(x)\Phi)(w_1, ..., w_p) = \Phi(P(x^{-1})w_1, ..., P(x^{-1})w_p),$$
$$w_i \in W, \quad x \in G, \quad \Phi \in T^p(W).$$

Then the derivative of P^p is given by

$$[(P^p)'(h)](\Phi)(w_1, ..., w_p) = -\sum_{i=1}^{p} \Phi(w_1, ..., P'(h)w_i, ..., w_p), \qquad h \in E.$$

4. *Differential spaces*: Let (W, d) be a differential space (cf. sec. 0.7) and denote its homology by $H(W)$. Assume that P is a representation of G in W such that

$$P(x) \circ d = d \circ P(x), \qquad x \in G.$$

Then $P(x)$ determines a linear map

$$P(x)_\#: H(W) \to H(W)$$

and $P_\#: x \mapsto P(x)_\#$ is a representation of G in $H(W)$.

On the other hand, the representation, P', of E satisfies

$$P'(h) \circ d = d \circ P'(h), \qquad h \in E$$

(differentiate the relation above). Hence $P'(h)$ determines an operator $P'(h)_\#$ in $H(W)$ and

$$(P')_\#: h \mapsto P'(h)_\#$$

is a representation of E in $H(W)$.

It follows immediately from the definitions that $(P')_\#$ is the derivative of $P_\#$,

$$(P_\#)' = (P')_\# .$$

1.10. The adjoint representation. Each $a \in G$ determines the *inner automorphism*, τ_a, of G given by

$$\tau_a(x) = axa^{-1}, \qquad x \in G.$$

Hence the derivative, τ_a', of τ_a is an automorphism of the Lie algebra E. It is denoted by Ad a. Since $\tau_a = \lambda_a \circ \rho_a^{-1}$,

$$\operatorname{Ad} a = L_a \circ R_a^{-1}, \qquad a \in G.$$

Proposition X: The correspondence Ad: $a \mapsto \operatorname{Ad} a$ defines a representation of G in E.

Proof: Evidently $\tau_a \circ \tau_b = \tau_{ab}$, and so

$$\operatorname{Ad} a \circ \operatorname{Ad} b = \operatorname{Ad} ab.$$

Thus Ad is a group homomorphism. It remains to show that Ad is smooth.

Define a smooth map $T: G \times G \to G$ by setting

$$T(y, x) = \tau_y(x), \qquad y, x \in G.$$

Its derivative, dT, is smooth. But

$$(dT)_{(y,e)}(0, h) = \operatorname{Ad} y\,(h).$$

Hence, for each $h \in E$, the maps $y \mapsto \operatorname{Ad} y\,(h)$ are smooth. It follows that Ad is smooth.

Q.E.D.

The representation Ad is called the *adjoint representation of G*.

On the other hand, a representation, ad, of the Lie algebra E in the vector space E is given by

$$(\mathrm{ad}\, h)(k) = [h, k], \qquad h, k \in E.$$

It is called the *adjoint representation of E*.

Proposition XI: ad is the derivative of Ad.

Lemma V: Fix $a \in G$, $h \in E$. Then

$$X_h(a) = Y_{\mathrm{Ad}\, a(h)}(a).$$

Proof: Recall that $\mathrm{Ad}\, a = R_a^{-1} \circ L_a$. Hence

$$Y_{\mathrm{Ad}\, a(h)}(a) = (R_a \circ \mathrm{Ad}\, a)(h) = L_a(h) = X_h(a).$$

Q.E.D.

Proof of the proposition: Fix $h \in E$ and let $e_1, \ldots, e_n$ be a basis for E. Then functions f_i on G are defined by

$$\mathrm{Ad} x(h) = \sum_{i=1}^{n} f_i(x)\, e_i, \qquad x \in G.$$

They satisfy

$$\mathrm{Ad}' k(h) = \sum_{i=1}^{n} (X_k(f_i))(e)\, e_i, \qquad k \in E.$$

On the other hand, we can apply Lemma V to obtain

$$X_h = \sum_{i=1}^{n} f_i Y_{e_i}.$$

Since $[X_k, Y_{e_i}] = 0$ (cf. Proposition II, sec. 1.2), it follows that

$$[X_k, X_h] = \sum_{i=1}^{n} X_k(f_i) Y_{e_i}.$$

Evaluate this at e to obtain $[k, h] = \mathrm{Ad}' k(h)$.

Q.E.D.

Corollary: $\mathrm{Ad}(\exp h) = \exp(\mathrm{ad}\, h)$, $h \in E$.

Proof: Apply Proposition VII, sec. 1.7.

Q.E.D.

§4. Abelian Lie groups

1.11. An *abelian Lie group* is a Lie group G satisfying $xy = yx$ ($x, y \in G$). An *abelian Lie algebra* is a Lie algebra E such that $[h, k] = 0$ ($k, h \in E$). Let G be a Lie group with Lie algebra E and consider the following conditions:

(1) G is abelian.

(2) The adjoint representation of G is trivial: $\operatorname{Ad} a = \iota$ ($a \in G$).

(3) The left and right invariant vector fields coincide,

$$X_h = Y_h, \qquad h \in E.$$

(4) The adjoint representation of E is trivial: $\operatorname{ad} h = 0$ ($h \in E$).

(5) E is abelian.

Proposition XII: The conditions above satisfy

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5).$$

If G is connected, they are all equivalent.

Proof: The sequence of implications is an immediate consequence of the relations $\tau_a' = \operatorname{Ad} a$ and $\operatorname{Ad}' = \operatorname{ad}$, together with Lemma V, sec. 1.10. If G is connected and E is abelian, then Corollary I to Proposition VII, sec. 1.7, shows that

$$\operatorname{Ad} = \gamma \qquad \text{and} \qquad \tau_a = \iota \quad (a \in G),$$

where $\gamma\colon G \to \iota$ is the constant homomorphism.

Q.E.D.

Examples: 1. Vector spaces under addition (cf. Example 1, sec. 1.4) are abelian.

2. Consider the unit circle of the complex plane

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

It is an abelian Lie group under multiplication. The tangent space, $T_e(S^1)$, is given by $T_e(S^1) = (1)^{\perp}$; i.e., it is the pure imaginary axis. We identify $\mathbb{R}$ with $T_e(S^1)$ by the correspondence $t \mapsto 2\pi i t$. With this identification the exponential map $\exp\colon \mathbb{R} \to S^1$ is given by

$$\exp h = e^{2\pi i h}, \qquad h \in \mathbb{R}.$$

In particular, $\exp^{-1}(1) = \mathbb{Z}$.

3. *Tori*: Recall that $\mathbb{R}^n$ is an abelian Lie group under addition. Consider the closed subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$ consisting of n-tuples of integers. In Example 3, sec. 1.4, volume I, the factor group $T^n = \mathbb{R}^n/\mathbb{Z}^n$ was made into a smooth manifold in such a way that the projection

$$\pi^n\colon \mathbb{R}^n \to T^n$$

was a local diffeomorphism. With this smooth structure T^n becomes a connected abelian Lie group. It is called the *n-torus*.

If $n = 1$, then T^1 is the circle S^1 and π is the exponential map (Example 2 above). Since

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}, \qquad \mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$$

(as Lie groups), it follows that

$$T^n \cong S^1 \times \cdots \times S^1$$

(as Lie groups). In particular, T^n is compact. Moreover,

$$\pi^n = \pi \times \cdots \times \pi = \exp \times \cdots \times \exp = \exp_{T^n}.$$

Thus we may identify π^n with the exponential map for T^n.

1.12. Proposition XIII: Every connected abelian Lie group G is isomorphic to the direct product $T^p \times \mathbb{R}^q$ (for some $p, q \in \mathbb{N}$). In particular, a compact connected abelian Lie group is a torus.

The proposition follows at once from Lemmas VI and VII below.

Lemma VI: Let E be the Lie algebra of G. Then

$$\exp(h + k) = \exp h \cdot \exp k, \qquad h, k \in E;$$

i.e., exp is a Lie group homomorphism.

Proof: Since G is abelian,

$$\alpha\colon t \mapsto \exp th \cdot \exp tk$$

is a 1-parameter subgroup. But $\dot{\alpha}(0) = h + k$; hence

$$\exp th \cdot \exp tk = \alpha(t) = \exp t(h + k).$$

Now set $t = 1$.

Q.E.D.

Corollary: exp is a surjective local diffeomorphism. $\exp^{-1}(e)$ is a closed discrete subgroup of E.

Proof: Apply Proposition VI, sec. 1.6, and its third corollary.

Q.E.D.

Lemma VII: Let K be a closed discrete subgroup of $\mathbb{R}^n$. Then there are linearly independent vectors $e_1, \ldots, e_p \in \mathbb{R}^n$ such that K consists of the integral combinations of the e_i:

$$K = \left\{ \sum_{i=1}^{p} q_i e_i \mid q_i \in \mathbb{Z} \right\}.$$

Proof: Clearly, we may assume that K contains a basis of $\mathbb{R}^n$, and we argue by induction on n. Fix a positive inner product in $\mathbb{R}^n$. Choose $e_1 \in K$ so that $e_1 \neq 0$ and $|e_1| \leqslant |x|$ for $x \in K$. Then $(\mathbb{R} \cdot e_1) \cap K$ consists of the integer multiples of e_1.

Now consider the projection

$$\pi: \mathbb{R}^n \to \mathbb{R}^n/(\mathbb{R} \cdot e_1).$$

It is easy to see that $\pi(K)$ is a closed discrete subgroup of $\mathbb{R}^n/(\mathbb{R} \cdot e_1)$. Hence, by the induction hypothesis, there are linearly independent vectors $\pi(e_2), \ldots, \pi(e_n)$ in $\mathbb{R}^n/(\mathbb{R} \cdot e_1)$ such that $e_i \in K$, and every element of $\pi(K)$ is an integral linear combination of the $\pi(e_i)$, $(i \geqslant 2)$.

Now an easy argument shows that the vectors $e_1, \ldots, e_n$ satisfy the conditions of the lemma. This closes the induction and completes the proof.

Q.E.D.

An element a of a Lie group G is called a *generator of* G if the set a^k $(k \in \mathbb{Z})$ is dense in G. A Lie group that has a generator is clearly abelian. Now we show that a torus T^n has a generator. In fact, let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$ and set $\mathbb{Z}^n = \mathbb{Z}e_1 \times \cdots \times \mathbb{Z}e_n$.

Choose numbers $\theta i \in \mathbb{R}$ so that the conditions λ_0, $\lambda_\nu \in \mathbb{Q}$ and

$$\lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu \theta^\nu = 0$$

imply $\lambda_\nu = 0$, $\nu = 0, \ldots, n$. Set $a = \sum_{\nu=1}^{n} \theta^\nu e_\nu$. Then $\exp a$ is a generator for T^n.

§5. Integration on compact Lie groups

1.13. Invariant n-forms. Let G be an n-dimensional Lie group with Lie algebra E. An n-form Ω on G is called *left* (respectively, *right*) *invariant* if

$$\lambda_a^*\Omega = \Omega, \qquad a \in G$$

(respectively, $\rho_a^*\Omega = \Omega$, $a \in G$). If Ω is both left and right invariant, it is called *biinvariant* or simply *invariant*.

To every determinant function $\Delta_e \in \wedge^n E^*$ corresponds a unique left invariant form Δ_L such that $\Delta_L(e) = \Delta_e$ and conversely. It is given by

$$\Delta_L(x; \xi_1, \ldots, \xi_n) = \Delta_e(L_x^{-1}(\xi_1), \ldots, L_x^{-1}(\xi_n)), \qquad x \in G, \quad \xi_i \in T_x(G).$$

Similarly, the unique right invariant form Δ_R which satisfies $\Delta_R(e) = \Delta_e$ is given by

$$\Delta_R(x; \xi_1, \ldots, \xi_n) = \Delta_e(R_x^{-1}(\xi_1), \ldots, R_x^{-1}(\xi_n)).$$

These relations yield

$$\rho_a^* \Delta_L = \det(\mathrm{Ad}\, a^{-1})\, \Delta_L \qquad \text{and} \qquad \Delta_R(a) = \det(\mathrm{Ad}\, a)\, \Delta_L(a), \qquad a \in G. \tag{1.1}$$

Hence G admits a nonzero invariant n-form if and only if

$$\det(\mathrm{Ad}\, x) = 1, \qquad x \in G.$$

Let Δ_e orient E; then the corresponding left invariant n-form Δ_L orients G. Similarly, Δ_R orients G. These orientations depend only on the orientation of E represented by Δ_e, and are called (respectively) the *left* and *right orientations of* G corresponding to the given orientation of E. They coincide if and only if

$$\det(\mathrm{Ad}\, x) > 0, \qquad x \in G$$

(cf. formula (1.1)). Observe that each left translation preserves the left orientation and each right translation preserves the right orientation.

Example: *Unimodular Lie groups and Lie algebras*: A Lie group G (respectively, a Lie algebra E) is called *unimodular* if $|\det(\mathrm{Ad}\, x)| = 1$,

$x \in G$ (respectively, if $\operatorname{tr}(\operatorname{ad} h) = 0$, $h \in E$). In particular, if G is a connected unimodular Lie group then $\det(\operatorname{Ad} x) = 1$, $x \in G$, and so G admits a nonzero invariant n-form.

Let G be any Lie group with Lie algebra E. It follows from Proposition XI, sec. 1.10, that in E

$$(\det \circ \operatorname{Ad})' = \operatorname{tr} \circ \operatorname{ad}.$$

Hence if G is unimodular, so is E; and these conditions are equivalent if G is connected.

Finally, observe that a compact Lie group G is unimodular. Indeed, in this case the image of $\det \circ \operatorname{Ad}$ is a compact subgroup of the multiplicative group $\mathbb{R}_{\bullet}$; this can only be $\{1\}$ or $\{\pm 1\}$. In particular, *every compact connected Lie group admits a nonzero invariant n-form.*

1.14. Integration of functions. Let G be an n-dimensional Lie group with Lie algebra E. Orient G by a left invariant n-form Δ_L. Let W be a finite-dimensional vector space (real or complex).

For each smooth function $f\colon G \to W$ of compact support, we can form the integral

$$\int_G f \cdot \Delta_L$$

(cf. sec. 0.13). The left invariance of Δ_L and the fact that left translations preserve the orientation give

$$\int_G f \cdot \Delta_L = \int_G \lambda_a^* f \cdot \Delta_L, \qquad a \in G.$$

On the other hand we have, in view of (1.1),

$$\int_G \rho_a^* f \cdot \Delta_L = \det(\operatorname{Ad} a) \int_G \rho_a^* f \cdot \rho_a^* \Delta_L = \det(\operatorname{Ad} a) \int_G \rho_a^*(f \cdot \Delta_L).$$

Since ρ_a preserves the left orientation precisely if $\det(\operatorname{Ad} a) > 0$, it follows that (cf. Proposition XII, sec. 4.13, volume I),

$$\int_G \rho_a^* f \cdot \Delta_L = |\det(\operatorname{Ad} a)| \cdot \int_G f \cdot \Delta_L, \qquad a \in G.$$

If G is unimodular (in particular, if G is compact), this formula reduces to

$$\int_G \rho_a^* f \cdot \Delta_L = \int_G f \cdot \Delta_L, \qquad a \in G.$$

1.15. Integration over compact groups. Let G be a *compact* n-dimensional Lie group with Lie algebra E. Give G the left orientation induced by an orientation of E. Let Δ be the unique left invariant n-form such that

$$\int_G \Delta = 1.$$

Let $f \in \mathscr{S}(G; W)$ (W, a vector space). Then the vector, $\int_G f \cdot \Delta$, is independent of the orientation. It is called the *integral of* f, and we write

$$\int_G f \cdot \Delta = \int_G f(a)\, da.$$

In particular, $\int_G da = 1$.

Since G is unimodular, the relations in sec. 1.14 give

$$\int_G f(ab)\, da = \int_G f(a)\, da = \int_G f(ba)\, da, \qquad b \in G. \tag{1.2}$$

More generally, assume φ is a diffeomorphism of G such that

$$\varphi^* \Delta = \epsilon \cdot \Delta,$$

where ϵ: $G \to \mathbb{R}$ is smooth and $|\epsilon| = 1$. Then for $f \in \mathscr{S}(G; W)$

$$\int_G f(\varphi(a))\, da = \int_G f(a)\, da.$$

In particular (apply Lemma I, sec. 1.1) this condition holds for $\varphi = \nu$, the inversion map. Thus

$$\int_G f(a^{-1})\, da = \int_G f(a)\, da. \tag{1.3}$$

Finally, if α: $W \to V$ is a linear map, then (cf. sec. 0.13)

$$\alpha\left(\int_G f(a)\, da\right) = \int_G (\alpha \circ f)(a)\, da. \tag{1.4}$$

Next let φ: $G \to G$ be a smooth map. It induces the smooth map ψ: $G \to L_E$ given by

$$\psi(x) = (L_{\varphi(x)}^{-1})_{\varphi(x)} \circ (d\varphi)_x \circ (L_x)_e\,, \qquad x \in G.$$

Proposition XIV: If G is compact and connected, then

$$\deg \varphi \cdot \int_G f(x)\, dx = \int_G f(\varphi(x)) \cdot \det \psi(x)\, dx, \qquad f \in \mathscr{S}(G; W).$$

Proof: Evidently, $\varphi^*\Delta = \det\psi \cdot \Delta$. Hence

$$\deg\varphi \cdot \int_G f(x)\,dx = \int_G \varphi^*(f\Delta) = \int_G \varphi^*f \cdot \det\psi \cdot \Delta$$
$$= \int_G f(\varphi(x))\det\psi(x)\,dx.$$

Q.E.D.

Corollary: $\deg\varphi = \int_G \det\psi(x)\,dx$.

Examples: 1. If $\varphi = \lambda_a$, ρ_a or ν, then ψ is given by

$$\psi(x) = \iota_E, \qquad \psi(x) = \operatorname{Ad} a^{-1}, \qquad \text{or} \qquad \psi(x) = -\operatorname{Ad} x,$$

respectively. In this case the Proposition yields formulae (1.2) and (1.3) above, in turn.

2. $\varphi(x) = x^2$. Then $\psi(x) = \iota + \operatorname{Ad} x^{-1}$. Now the Corollary to Proposition XIV, together with formula 1.3, yields

$$\deg\varphi = \int_G \det(\iota + \operatorname{Ad} x)\,dx.$$

3. φ is a homomorphism. Then $\psi(x) = \varphi'$, $x \in G$, whence

$$\deg\varphi \cdot \int_G f(x)\,dx = \det\varphi' \cdot \int_G f(\varphi(x))\,dx, \qquad f \in \mathscr{S}(G; W).$$

Setting $f = 1$, we obtain $\deg\varphi = \det\varphi'$. In particular, $\det\varphi'$ is an integer.

Now assume that $\deg\varphi \neq 0$. Then the relations above yield

$$\int_G f(x)\,dx = \int_G f(\varphi(x))\,dx.$$

Moreover, in this case φ' is a linear isomorphism. Hence each map $(d\varphi)_x$ ($x \in G$) is an isomorphism. Thus φ is a local diffeomorphism, and so the set $\varphi^{-1}(e)$ is finite. Now Theorem I, sec. 6.3, volume I, implies that the integer $|\det\varphi'|$ is equal to the number of points in $\varphi^{-1}(e)$.

1.16. Invariant subspace of a representation. Let P be a representation of a compact Lie group in a finite-dimensional vector space W. Since P is a smooth, map $G \to L_W$, we can form the integral,

$$P_0 = \int_G P(x)\,dx,$$

to obtain a linear transformation of W. (Note that P_0 is not in general a linear automorphism of W.)

Proposition XV: With the notation and hypotheses above

(1) $P_0 \circ P(x) = P_0 = P(x) \circ P_0$, $x \in G$.

(2) $P_0^2 = P_0$

(3) If $P^\natural$ denotes the contragradient representation, then $(P^\natural)_0 = P_0^*$.

(4) A vector w is invariant (i.e. $w \in W_I$) if and only if $P_0 w = w$.

Proof: We rely throughout on formulae (1.2), (1.3), and (1.4) of sec. 1.15. To prove (1) observe that for $x \in G$,

$$P_0 \circ P(x) = \left(\int_G P(y)\, dy\right) \circ P(x) = \int_G P(y) \circ P(x)\, dy$$

$$= \int_G P(yx)\, dy = \int_G P(y)\, dy = P_0\,.$$

Similarly, $P(x) \circ P_0 = P_0$ and so (1) follows. This relation yields

$$P_0^2 = P_0 \circ \int_G P(x)\, dx = \int_G P_0 \circ P(x)\, dx = \int_G P_0\, dx = P_0\,.$$

Next note that

$$(P^\natural)_0 = \int_G P(x^{-1})^*\, dx = \left(\int_G P(x)\, dx\right)^* = P_0^*.$$

To prove (4) let $w \in W_I$. Then

$$P_0 w = \int_G (P(x)w)\, dx = \int_G w\, dx = w.$$

On the other hand, if $P_0 w = w$, then (1) yields

$$P(x)w = (P(x) \circ P_0)w = P_0 w = w, \qquad x \in G,$$

and so $w \in W_I$.

Q.E.D.

Corollary I: The dimension of W_I is given by

$$\dim W_I = \int_G \operatorname{tr} P(x)\, dx.$$

In particular, $W_I = 0$ if and only if

$$\int_G \operatorname{tr} P(x)\, dx = 0.$$

Proof: Since $P_0^2 = P_0$ and $\operatorname{Im} P_0 = W_I$,

$$\dim W_I = \operatorname{tr} P_0 = \int_G \operatorname{tr} P(x)\, dx.$$

Q.E.D.

Corollary II: If W_I^* is the invariant subspace for $P^\natural$, then

$$\dim W_I^* = \dim W_I.$$

Corollary III: Consider the induced representations, $\wedge^k P$, in $\wedge^k W$ for $k = 0, \dots, r$ ($r = \dim W$), and let

$$c_k = \dim(\wedge^k W)_I, \qquad k = 0, 1, \dots, r.$$

Suppose G is connected. Then

$$\int_G \det(P(x) + \lambda\iota)\, dx = \sum_{k=0}^{r} c_k \lambda^{r-k} = \sum_{k=0}^{r} c_k \lambda^k.$$

Proof: Corollary I gives

$$c_k = \int_G \operatorname{tr} \wedge^k P(x)\, dx.$$

But $\operatorname{tr} \wedge^k P(x)$ is the coefficient of λ^{r-k} in the polynomial $\det(P(x) + \lambda\iota)$ (cf. sec. A.2). Thus

$$\int_G \det(P(x) + \lambda\iota)\, dx = \sum_{k=0}^{r} \left[\int_G \operatorname{tr} \wedge^k P(x)\, dx\right] \lambda^{r-k}$$
$$= \sum_{k=0}^{r} c_k \lambda^{r-k}.$$

To establish the other equality, note that because G is compact and connected, the homomorphism $\det \circ P : G \to \mathbb{R}_{\bullet}$ has a compact connected image; i.e.,

$$\det P(x) = 1, \qquad x \in G.$$

It follows that for $\lambda \neq 0$,

$$\det(P(x) + \lambda\iota) = \lambda^r \det(\lambda^{-1}\iota + P(x^{-1})).$$

Integrating over G, we obtain

$$\sum_{k=0}^{r} c_k\lambda^{r-k} = \lambda^r \sum_{k=0}^{r} c_k\lambda^{k-r} = \sum_{k=0}^{r} c_k\lambda^k.$$

Q.E.D.

1.17. Invariant inner products. Let P be a representation of a Lie group in a real (respectively, complex) vector space W. A Euclidean (respectively, Hermitian) inner product $\langle\,,\rangle$ in W is called *invariant with respect to* P, if it satisfies

$$\langle P(x)u, P(x)v\rangle = \langle u, v\rangle, \qquad x \in G, \quad u, v \in W.$$

If $\langle\,,\rangle$ is such an inner product, it follows that for each $h \in T_e(G)$, the map $P'(h)\colon W \to W$ is skew.

Proposition XVI: Every representation of a compact Lie group admits an invariant inner product.

Proof: Let $(\,,)$ be any Euclidean (respectively, Hermitian) inner product. Define $\langle\,,\rangle$ by setting

$$\langle u, v\rangle = \int_G (P(a)u, P(a)v)\, da.$$

Then $\langle\,,\rangle$ has the desired properties.

Q.E.D.

Corollary: Let G be a compact connected Lie group. Then the map $\varphi\colon x \mapsto x^2$ is surjective.

Proof: Recall from Example 2 sec. 1.15, that

$$\deg\varphi = \int_G \det(\iota + \operatorname{Ad} x)\, dx.$$

Now choose an inner product in $T_e(G)$ which is invariant under the adjoint representation. Thus each $\operatorname{Ad} x$ is a proper rotation, and it follows from elementary linear algebra that

$$\det(\iota + \operatorname{Ad} x) \geqslant 0, \qquad x \in G.$$

Since

$$\det(\iota + \operatorname{Ad} e) = \det(2\iota) = 2^n, \qquad (n = \dim G),$$

we obtain $\deg \varphi > 0$. Hence φ is surjective.

Q.E.D.

Remark: In sec. 2.18 it will be shown that, for a compact connected Lie group, the map $x \mapsto x^p$ is surjective for every integer $p \neq 0$.

The following example of Hopf shows that the map $x \mapsto x^2$ is not necessarily surjective if G is not compact. Let G be the group $SL(2; \mathbb{R})$ consisting of linear transformations $\alpha: \mathbb{R}^2 \to \mathbb{R}^2$ with $\det \alpha = 1$. (It follows from Theorem I, sec. 2.1, of the next chapter that G is a Lie group.)

The Cayley–Hamilton theorem yields

$$\alpha^2 - (\operatorname{tr} \alpha)\, \alpha + \iota = 0, \qquad \alpha \in G,$$

whence

$$\operatorname{tr} \alpha^2 - (\operatorname{tr} \alpha)^2 + 2 = 0.$$

Hence $\operatorname{tr} \alpha^2 \geqslant -2$ if $\alpha \in G$. In particular, the transformation $\beta \in G$ given by

$$\beta(e_1) = -2e_1, \qquad \beta(e_2) = -\tfrac{1}{2}e_2$$

has trace < -2, and so is *not* the square of any α in G.

A representation of a Lie group in a vector space W is called *semisimple*, if every stable subspace $W_1 \subset W$ has a stable complement; i.e., if $W_1 \subset W$ is stable, then there is a stable subspace W_2 such that $W = W_1 \oplus W_2$.

Proposition XVII: Every representation of a compact Lie group in a finite-dimensional vector space is semisimple.

Proof: In view of Proposition XVI there exists an invariant inner product in W. Now let $W_1 \subset W$ be stable. Then

$$W = W_1 \oplus W_1^\perp$$

and $W_1^\perp$ is also a stable subspace.

Q.E.D.

Problems

G is a Lie group with Lie algebra E.

1. Show that a 1-parameter subgroup is either $\mathbb{R}$ or S^1.

2. Construct a nonabelian Lie group with trivial adjoint representation and abelian component group. What is the minimum number of components of such a group?

3. Let $h, k \in E$ and $f \in \mathscr{S}(G)$.

(i) Show that

$$X_h(f)(x) = \left[\frac{d}{dt} f(x \cdot \exp th)\right]_{t=0} \quad \text{and} \quad Y_k(f)(x) = \left[\frac{d}{dt} f(\exp tk \cdot x)\right]_{t=0}.$$

(ii) Use the fact that

$$\frac{\partial^2}{\partial t\, \partial \tau} f(\exp \tau k \cdot x \cdot \exp th) = \frac{\partial^2}{\partial \tau\, \partial t} f(\exp \tau k \cdot x \cdot \exp th)$$

to conclude that $[X_h, Y_k] = 0$.

(iii) Show that

$$\frac{\partial^2}{\partial t\, \partial \tau} [f(\exp th \cdot \exp \tau k \cdot \exp -th)]_{t=0,\tau=0} = [X_h, X_k](f)(e)$$

and

$$\left[\frac{\partial^2}{\partial t\, \partial \tau} f(\exp th \cdot \exp \tau k \cdot \exp -th \cdot \exp -\tau k)\right]_{t=0,\tau=0} = X_{[h,k]}(f)(e).$$

4. Let $q: M \to N$ be a smooth map such that $(dq)_a = 0$.

(i) Show that a bilinear map $\beta: T_a(M) \times T_a(M) \to T_{q(a)}(N)$ is defined by

$$\beta(\xi, \eta)(f) = X(Y(q^*f))(a), \qquad f \in \mathscr{S}(N),$$

where $X(a) = \xi$ and $Y(a) = \eta$.

(ii) Show that β is symmetric.

(iii) Determine β in the case $M = G \times G$, $N = G$, $q(x, y) = xyx^{-1}y^{-1}$ and $a = e \times e$.

5. Show that the upper-triangular real $(n \times n)$-matrices with 1's on the main diagonal form a Lie group G. Show that G is nonabelian if $n > 2$. Find the Lie algebra of G, and prove that the exponential map is a global diffeomorphism.

6. Use the Cayley map (cf. Example 9, sec. 1.5, volume I) to make the group of proper rotations of Euclidean space into a Lie group.

7. Tori. Let T be an n-torus with Lie algebra L_T. The subset Γ_T of L_T given by $\Gamma_T = \exp^{-1}(e)$ is called the *integer lattice of* L_T.

(i) Show that $\Gamma_T \cong \mathbb{Z}^n$ ($\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, n terms, cf. sec. 1.12).

(ii) If $\varphi\colon T \to S$ is a homomorphism into another torus, show that $\varphi'\colon L_T \to L_S$ restricts to a group homomorphism $\varphi_\Gamma\colon \Gamma_T \to \Gamma_S$. Show that this defines a bijection between the set of homomorphisms $T \to S$ and the set $\mathrm{Hom}(\mathbb{Z}^n; \mathbb{Z}^m)$ ($n = \dim T$, $m = \dim S$).

(iii) Show that a subspace $L \subset L_T$ is the Lie algebra of a subtorus if and only if L is generated (over $\mathbb{R}$) by vectors in Γ_T.

(iv) Given a subtorus S_1 of T, find a second subtorus S_2 such that the map $S_1 \times S_2 \to T$ given by $(x_1, x_2) \to x_1x_2$ is an isomorphism of Lie groups.

8. Power maps. Define $P_k\colon G \to G$ by $P_k(x) = x^k$, ($k \in \mathbb{Z}$).

(i) Show that

$$(dP_k)_x = (L_x)^k \circ \Phi_k(x) \circ (L_x)^{-1}$$

where, if $k \geqslant 1$,

$$\Phi_k(x) = \sum_{j=0}^{k-1} (\mathrm{Ad}\, x^{-1})^j.$$

Find Φ_k if $k < 1$.

(ii) Fix $x \in G$. Show that $\det \Phi_k(x) = 0$ if and only if there exists an $h \in E$ such that

$$(\mathrm{Ad}\, x)^k h = h \qquad \text{and} \qquad (\mathrm{Ad}\, x)h \neq h.$$

(iii) If G is compact and connected, show that $\det \Phi_k(x) \geqslant 0$, $x \in G$. Conclude that the maps P_k are all surjective. Use this to show that the exponential map is surjective.

(iv) If G is compact and connected, show that

$$\deg P_2 = \dim(\wedge E^*)_I,$$

where $(\wedge E^*)_I$ denotes the subalgebra of $\wedge E^*$ invariant under the representation $\wedge \operatorname{Ad}^\natural$.

9. **The group $\mathbb{R}P^3$.** Fix a Euclidean inner product, and an orientation in $\mathbb{R}^3$.

(i) Show that the cross product makes $\mathbb{R}^3$ into a Lie algebra. Let $\psi(h)(x) = h \times x$ and show that ψ is an isomorphism from $\mathbb{R}^3$ to the Lie algebra of skew transformations of $\mathbb{R}^3$.

(ii) Show that

$$\exp \psi(h)(x) = (\cos |h|)x + \left(2\frac{\langle h, x\rangle}{\langle h, h\rangle}\sin^2\frac{|h|}{2}\right)h + \frac{\sin |h|}{|h|}(h \times x),$$

$$h, x \in \mathbb{R}^3.$$

(iii) Let B be the closed ball of radius π in $\mathbb{R}^3$. Regard $\mathbb{R}P^3$ as the quotient space of B under the equivalence relation $x \sim y$ if and only if either $x = y$ or $|x| = \pi$ and $x = -y$. Use (ii) to obtain an embedding $\mathbb{R}P^3 \to GL(\mathbb{R}^3)$ whose image is the set of proper isometries of $\mathbb{R}^3$.

(iv) Conclude that $\mathbb{R}P^3$ is a Lie group with Lie algebra $\mathbb{R}^3$. Write down the exponential map explicitly. Obtain expressions for the left and right invariant vector fields.

10. Let $(x_1, x_2) = x$ and $(y_1, y_2) = y$ belong to $\mathbb{R}^2$. Set $xy = (x_1 + y_1 e^{-x_2}, x_2 + y_2)$ and show that this makes $\mathbb{R}^2$ into a Lie group. Find the 1-parameter subgroups, the left and right invariant vector fields, and the Lie algebra.

11. **1-parameter subgroups.** (i) Show that quaternionic multiplication makes S^3 into a Lie group. Show that the 1-parameter subgroups are the great circles through e.

(ii) Let $x(t)$ be the 1-parameter subgroup of $GL(\mathbb{R}^3)$ generated by a skew transformation σ. Show that x is periodic with period $2\pi/(-\frac{1}{2}\operatorname{tr}\sigma^2)^{1/2}$.

12. Representations. Let V, W be complex vector spaces. Two representations $P\colon G \to GL(V)$ and $Q\colon G \to GL(W)$ are called *equivalent* if there exists a linear isomorphism $\varphi\colon V \xrightarrow{\cong} W$ such that

$$\varphi \circ P(x) = Q(x) \circ \varphi, \qquad x \in G.$$

A representation P in V is called *irreducible*, if V is not the direct sum of nontrivial stable subspaces. The *character* of P is the complex-valued function χ_P on G given by $\chi_P(x) = \operatorname{tr} P(x)$.

(i) Let P, Q be representations of G in V and W, respectively. Show that a representation R of G in the space $L(V; W)$ is given by

$$R(x)\psi = Q(x) \circ \psi \circ P(x)^{-1}, \qquad \psi \in L(V; W).$$

Show that R is equivalent to the representation $P^\natural \otimes Q$ in $V^* \otimes W$. Show that P and Q are equivalent if and only if the space $L(V; W)_I$ contains a linear isomorphism. If P and Q are irreducible, show that they are equivalent if and only if $L(V; W)_I \neq 0$.

(ii) Show that equivalent representations have the same character. Prove the relations (the last only if G is compact)

$$\chi_{P\oplus Q} = \chi_P + \chi_Q, \qquad \chi_{P\otimes Q} = \chi_P \cdot \chi_Q, \qquad \chi_{P^\natural} = \bar{\chi}_P$$

(where $\bar{\chi}_P$ is the complex conjugate of χ_P).

13. Representations of compact Lie groups. Let G be compact.

(i) Show that each representation of G is the direct sum of irreducible representations.

(ii) Let P and Q be irreducible representations of G in complex vector spaces. Show that

$$\int_G \chi_P \bar{\chi}_Q = \begin{cases} 1 & \text{if } P \text{ and } Q \text{ are equivalent} \\ 0 & \text{otherwise.} \end{cases}$$

Conclude that P and Q are equivalent if and only if $\chi_P = \chi_Q$.

(iii) Assume that $\{(P_\lambda, V_\lambda)\}$ is a collection of inequivalent irreducible complex representations such that every irreducible complex representation is equivalent to some P_λ. Define a canonical G-linear isomorphism

$$\oplus_\lambda (V_\lambda \otimes L(V_\lambda; V)_I) \xrightarrow{\cong} V,$$

where the representations are $\oplus_\lambda (P_\lambda \otimes \iota)$ and P, respectively.

14. Finite groups. Let Γ be a finite group and let $|\Gamma|$ denote the order of Γ. Let $\mathbb{C}(\Gamma)$ be the complex vector space with the elements of Γ as basis.

(i) Show that the multiplication of Γ makes $\mathbb{C}(\Gamma)$ into an algebra.

(ii) If f is a complex-valued function on Γ show that

$$\int_\Gamma f(x)\, dx = |\Gamma|^{-1} \sum_{x\in\Gamma} f(x).$$

(iii) Show that left and right multiplications determine equivalent representations L and R of Γ in the space $\mathbb{C}(\Gamma)$. They are called the *left* (respectively, *right*) *regular representations of* Γ. Show that

$$R(x) \circ L(y) = L(y) \circ R(x), \qquad x, y \in \Gamma.$$

(iv) If P is a representation of Γ in a complex vector space V, show that

$$\operatorname{tr}(R(x) \otimes P(x)) = 0, \qquad x \neq e.$$

Conclude that

$$\int_\Gamma \operatorname{tr} R(x) \otimes P(x)\, dx = \dim V.$$

(v) Show that L determines a representation L_I of Γ in the invariant subspace $[\mathbb{C}(\Gamma) \otimes V]_I$ (with respect to $R \otimes P$). Show that a linear map $\varphi\colon \mathbb{C}(\Gamma) \otimes V \to V$ is given by

$$\varphi(x \otimes v) = P(x)v, \qquad x \in \Gamma, \quad v \in V.$$

Show that φ restricts to an isomorphism

$$\bar{\varphi}\colon [\mathbb{C}(\Gamma) \otimes V]_I \xrightarrow{\cong} V$$

and that $\bar{\varphi}$ is an equivalence between the representations $L \otimes \iota$ and P.

Conclude that the right regular representation is a direct sum of irreducible representations, and that each irreducible representation occurs p times, where p is the dimension of its representation space.

15. Let A be a real finite-dimensional associative algebra.

(i) Show that the group of units, G_A, of A is dense in A.

(ii) Show that left multiplication defines a representation of G_A in A. What is its derivative?

(iii) Define the adjoint representation of G_A in terms of the multiplication in A.

16. Local homeomorphisms. Let Q be a second countable Hausdorff space and let $\pi: Q \to M$ be a local homeomorphism into a smooth manifold. Show that there is a unique smooth structure on Q which makes π into a local diffeomorphism.

17. Covering spaces. Let (Q, π, M, F) be a smooth bundle and assume that π is a local diffeomorphism. Then Q is called a *covering manifold of* M and π is called a *covering projection.*

(i) If $\pi: Q \to M$ is a covering projection, show that the fibre consists of finitely or countably many points.

(ii) Show that the composite of two covering projections is a covering projection.

18. Universal covering manifold. Let M be a connected manifold and fix a point $x_0 \in M$. Let X denote the set of continuous maps φ: $[0, 1] \to M$ satisfying $\varphi(0) = x_0$. For each open subset U of M and each compact subset C of $[0, 1]$, set $X_{C,U} = \{\varphi \in X \mid \varphi(C) \subset U\}$. Give X the weakest topology such that each $X_{C,U}$ is open. Define an equivalence relation, $\sim$, in X as follows: $\varphi \sim \psi$ if $\varphi(1) = \psi(1)$ and if there is a continuous homotopy φ_t connecting φ and ψ such that $\varphi_t(0) = x_0$, $\varphi_t(1) = \varphi(1)$ $(0 \leqslant t \leqslant 1)$. Let $\tilde{M}$ be the set of equivalence classes with the quotient topology.

(i) Show that the map $X \to M$ given by $\varphi \to \varphi(1)$ induces a continuous map π: $\tilde{M} \to M$.

(ii) Show that $\tilde{M}$ is a connected manifold and that π is a covering projection.

(iii) Let ρ: $P \to N$ be a covering projection and let φ: $M \to N$ be a smooth map. Fix $y_0 \in \pi^{-1}(x_0)$ and $z_0 \in \rho^{-1}(\varphi(x_0))$. Show that there is a unique smooth map $\tilde{\varphi}$: $\tilde{M} \to P$ such that $\rho \circ \tilde{\varphi} = \varphi \circ \pi$ and $\tilde{\varphi}(y_0) = z_0$. Conclude that $\tilde{M}$ has a universal property: It is called the *universal covering manifold of* M.

(iv) Show that the universal covering of $\tilde{M}$ is $\tilde{M}$, with the identity map as projection.

19. Covering groups. Let G be a connected Lie group. Let $\pi: \tilde{G} \to G$ be a covering projection and fix $\tilde{e} \in \pi^{-1}(e)$.

(i) Show that there is a unique Lie group structure on $\tilde{G}$ such that π is a homomorphism of Lie groups and $\tilde{e}$ is the identity element. $\tilde{G}$ is called a *covering group* of G. If $\tilde{G}$ is the universal covering manifold, it is called the *universal covering group of* G.

(ii) Show that $\tilde{G}$ has the same Lie algebra as G.

(iii) Show that $\pi^{-1}(e)$ is a countable, closed, discrete normal subgroup of $\tilde{G}$. Conclude that $\pi^{-1}(e)$ is contained in the center of $\tilde{G}$.

(iv) Show that the universal covering group of a connected abelian group is its Lie algebra and that the exponential map is the covering projection.

20. Local homorphisms. A *local homomorphism* from a Lie group G into an abstract group Γ is a set map $\varphi: U \to \Gamma$ (U, a neighbourhood of e) such that $\varphi(e) = e_\Gamma$ and, for some neighbourhood V of e,

$$\varphi(xy) = \varphi(x) \cdot \varphi(y), \qquad \varphi(x^{-1}) = \varphi(x)^{-1}, \qquad x, y \in V.$$

(i) Given such a φ find a minimal covering group $\rho: \hat{G} \to G$ and a group homomorphism $\psi: \hat{G} \to \Gamma$ such that $\varphi \circ \rho = \psi$ in some neighbourhood of the identity of $\hat{G}$.

(ii) Construct a group homomorphism $\tilde{\varphi}: \tilde{G} \to \Gamma$ such that $\varphi \circ \pi = \tilde{\varphi}$ in a neighbourhood of $\tilde{e}$ ($\pi: \tilde{G} \to G$ is the universal covering projection).

(iii) Show that a homomorphism between Lie groups lifts to a homomorphism between the universal covering groups.

21. Let G be a Lie group with Lie algebra E. Let $t \mapsto h(t)$ be a smooth curve in E. Show that there is a unique smooth curve $t \mapsto x(t)$ $(0 \leqslant t \leqslant 1)$ in G such that $x(0) = e$ and

$$\dot{x}(t) = R_{x(t)}h(t), \qquad 0 \leqslant t \leqslant 1.$$

(*Hint*: Fix $s \in [0,1]$ and show that the equations $\dot{x}_s(t) = R_{x_s(t)}h(t+s)$, $x_s(0) = e$ have a solution in $-\epsilon < t < \epsilon$, where ϵ is independent of s.)

Chapter II

Subgroups and Homogeneous Spaces

§1. Lie subgroups

2.1. Definition: A *Lie subgroup* of a Lie group G is a pair (φ, K), where

(1) K is a Lie group

and

(2) $\varphi: K \to G$ is an injective homomorphism of Lie groups.

If (φ, K) is a Lie subgroup of G, then φ embeds the abstract group K as a subgroup of the abstract group G. Moreover, it follows from Corollary II to Proposition VII, sec. 1.7, that

$$d\varphi: T_K \to T_G$$

is injective. Hence φ embeds the manifold K in the manifold G. According to sec. 3.10, volume I, K is not necessarily a submanifold of G.

Let K be a subgroup of a Lie group G (in the sense of abstract groups). The topology of G induces a topology in K; K is called a *closed subgroup* of G if it is a closed subset of G.

Theorem I: Let K be a closed subgroup of a Lie group G. Then the topological space, K, (topology induced from G) admits a unique smooth structure with respect to which it is a Lie group.

With this smooth structure the inclusion map, $i: K \to G$, is a homomorphism of Lie groups, and the pair (i, K) is a Lie subgroup of G. Moreover, K is a submanifold of G.

Proof: We prove the uniqueness and embedding part of the theorem first. In fact, assume K has been given a smooth structure so that it is a Lie group. Since the inclusion map $i: K \to G$ is continuous, Proposition VIII, sec. 1.7, implies that it is smooth.

Since i is injective (i, K) is a Lie subgroup of G. Hence i embeds the manifold K into the manifold G. Since i is a homeomorphism onto its image, K is a submanifold of G. Thus the corollary to Proposition VI, sec. 3.10, volume I, implies that the smooth structure of K is uniquely determined, if it exists.

To construct the smooth structure of K, we first establish four lemmas.

2.2. Lemma I: Let $K \subset G$ be a closed subgroup. Assume that $t_i (i = 1, 2, \ldots)$ is a sequence of real numbers and h_i $(i = 1, 2, \ldots)$ is a sequence of vectors in $T_e(G)$ such that

(1) $t_i \to 0 \quad$ but $\quad t_i \neq 0, \quad i = 1, 2, \ldots,$

(2) $h_i \to h \; (\in T_e(G))$

and

(3) $\exp t_i h_i \in K, \qquad i = 1, 2, \ldots .$

Then $\exp th \in K$, $t \in \mathbb{R}$.

Proof: (cf. [11, Lemma 4.2, p. 228].) Since

$$\exp(-t_i h_i) = (\exp t_i h_i)^{-1},$$

we may assume that each $t_i > 0$. Fix $t > 0$ and let n_i be the unique integer such that

$$(t/t_i) - 1 < n_i \leqslant t/t_i .$$

Then $t - t_i < n_i t_i \leqslant t$.

Since $t_i \to 0$, we obtain

$$\lim_{i\to\infty} n_i t_i = t, \qquad \lim_{i\to\infty} n_i t_i h_i = th.$$

Now

$$\exp th = \lim_{i\to\infty} \exp(n_i t_i h_i) = \lim_{i\to\infty} (\exp t_i h_i)^{n_i}.$$

Since $\exp t_i h_i \in K$ $(i = 1, 2, \ldots)$, so does its n_ith power. Since K is closed, it follows that $\exp th \in K$, $t \in \mathbb{R}$.

Q.E.D.

Lemma II: Let $H \subset G$ be any subgroup. Let $S(H)$ denote the set of smooth maps $\alpha \colon \mathbb{R} \to G$ such that $\operatorname{Im} \alpha \subset H$ and $\alpha(0) = e$ ($S(H)$ includes the constant path $\mathbb{R} \to e$). Let F be the subset of $T_e(G)$ given by

$$F = \{h \in T_e(G) \mid h = \dot{\alpha}(0) \text{ for some } \alpha \in S(H)\}.$$

Then F is a subspace of $T_e(G)$ and, if H is closed,

$$\exp th \in H, \qquad h \in F, \quad t \in \mathbb{R}.$$

Proof: Let $h, k \in F$ and $\lambda, \mu \in \mathbb{R}$. Choose smooth maps $\alpha \in S(H)$ and $\beta \in S(H)$ such that

$$\dot{\alpha}(0) = h \qquad \text{and} \qquad \dot{\beta}(0) = k.$$

Define $\gamma \in S(H)$ by $\gamma(t) = \alpha(\lambda t)\beta(\mu t)$, $t \in \mathbb{R}$. Then $\dot{\gamma}(0) = \lambda h + \mu k$. Hence $\lambda h + \mu k \in F$, and so F is a subspace of $T_e(G)$.

Now assume H is a closed subgroup. Let $\alpha \in S(H)$ and write $\dot{\alpha}(0) = h$. Since exp is a local diffeomorphism at 0, it follows that, for some $\epsilon > 0$,

$$\alpha(t) = \exp h(t), \qquad |t| < \epsilon,$$

where $h(t)$ is a smooth path in $T_e(G)$ and $h(0) = 0$. Hence

$$h = \dot{\alpha}(0) = \dot{h}(0) = \lim_{t \to 0} h(t)/t = \lim_{i \to \infty} ih(1/i).$$

Set $t_i = 1/i$ and $h_i = ih(1/i)$ $(i = 1, \ldots)$. Since $t_i h_i = h(1/i)$, we have

$$\exp t_i h_i = \exp h(1/i) = \alpha(1/i) \in H, \qquad i = 1, 2, \ldots .$$

Hence we can apply Lemma I to t_i, h_i to obtain $\exp th \in H$, $t \in \mathbb{R}$.

Q.E.D.

Lemma III: Let K be a closed subgroup of G. Let $F \subset T_e(G)$ be the corresponding subspace as defined in Lemma II and let $L \subset T_e(G)$ be a complementary subspace, $T_e(G) = L \oplus F$. Then there is a neighbourhood U_L of 0 in L such that

$$\exp h \notin K, \qquad h \in U_L, \quad h \neq 0.$$

Proof: Otherwise there would exist a sequence, h_i, of nonzero vectors in L such that $h_i \to 0$ and $\exp h_i \in K$. Choose real numbers t_i so that

$$1 \leqslant |t_i h_i| \leqslant 2, \qquad i = 1, 2, \ldots ,$$

with respect to some norm in $T_e(G)$. Then by choosing a subsequence, if necessary, we can assume that

$$\lim_{i \to \infty} t_i h_i = h \neq 0.$$

Since L is closed in $T_e(G)$, it follows that $h \in L$.

Finally, applying Lemma I to the vectors $t_i h_i$, and the numbers $1/t_i$, we obtain

$$\exp th \in K, \qquad t \in \mathbb{R}$$

and hence $h \in F$. But $F \cap L = 0$. This contradiction proves the lemma.

Q.E.D.

Lemma IV: Let K be a closed subgroup of G and define the subspace $F \subset T_e(G)$ as in Lemma II. Choose a second subspace $L \subset T_e(G)$ so that $T_e(G) = L \oplus F$. Consider the map $\varphi: L \times F \to G$ given by

$$\varphi(k, h) = \exp k \cdot \exp h, \qquad k \in L, \quad h \in F.$$

Then there are neighbourhoods V_L and V_F of 0 in L and F and there is a neighbourhood U of e in G such that

(i) φ restricts to a diffeomorphism $\varphi: V_L \times V_F \xrightarrow{\cong} U$

(ii) $U \cap K = \exp(V_F)$.

Proof: By Corollary II to Proposition VI, sec. 1.6, there exist V_L, V_F, and U such that φ restricts to a diffeomorphism

$$\varphi: V_L \times V_F \xrightarrow{\cong} U.$$

In view of Lemma III we can choose V_L so that

$$\exp k \notin K, \qquad k \in V_L - \{0\}.$$

Now we show that $U \cap K = \exp(V_F)$. Lemma II implies that $\exp(V_F) \subset K$. Moreover,

$$\exp(V_F) = \varphi(0 \times V_F) \subset U$$

and so $\exp(V_F) \subset U \cap K$.

On the other hand, let $x \in U \cap K$ be arbitrary. Since $x \in U$, we can write

$$x = \exp k \cdot \exp h, \qquad k \in V_L, \quad h \in V_F,$$

whence $\exp k = x \cdot (\exp h)^{-1} \in K$. Now Lemma III implies that $k = 0$ and so we have

$$x = \exp h \in \exp(V_F).$$

This shows that $U \cap K \subset \exp(V_F)$.

Q.E.D.

2.3. Proof of Theorem I continued. We retain the notation of Lemma IV, sec. 2.2. Observe that since the topology of K is induced from that of G, it is second countable and Hausdorff. Next we construct a smooth atlas for K.

In fact, for each $a \in K$, set $V_a = a \cdot (U \cap K)$; it is an open subset of K. In particular, Lemma IV of sec. 2.2 shows that exp restricts to a homeomorphism

$$V_F \xrightarrow{\cong} V_e .$$

Give V_e the manifold structure that makes exp into a diffeomorphism; it is then a submanifold of U and hence a submanifold of G. Thus every open subset of V_e is a submanifold of G.

Now consider the collection $\{V_a, v_a, V_e\}$, where $v_a\colon V_a \to V_e$ is the homeomorphism given by $v_a(x) = a^{-1}x$. The transition map

$$v_b \circ v_a^{-1}\colon v_a(V_a \cap V_b) \to v_b(V_a \cap V_b)$$

is the restriction to $v_a(V_a \cap V_b)$ of the smooth map $\lambda_b^{-1} \circ \lambda_a$ in G. Since $v_a(V_a)$, $v_b(V_b)$ are submanifolds of G, $v_b \circ v_a^{-1}$ is smooth (cf. sec. 3.10, volume I). It follows that $\{V_a, v_a, V_e\}$ is a smooth atlas for K.

The smooth structure on K defined by this atlas makes K into a submanifold of G. In fact, the restriction of the inclusion i to the open set $V_e \subset K$ is a smooth embedding, as we have seen above. Since

$$i = \lambda_a \circ i \circ v_a\colon V_a \to G,$$

the restriction of i to each V_a is also a smooth embedding.

Thus i is smooth and each $(di)_a$ is injective. Since i is a homeomorphism onto its image, K is a submanifold of G.

Finally, the multiplication and inversion maps

$$\mu\colon K \times K \to K \qquad \text{and} \qquad \nu\colon K \to K$$

are the restrictions of the (smooth) multiplication and inversion maps for G. Since K is a submanifold these maps are smooth. Hence K is a Lie group.

Q.E.D.

Remark: It follows from the construction of the smooth structure of K that $T_e(K) = F$ and hence F is the Lie algebra of K. Thus $h \in F$ if and only if $\exp th \in K$, $t \in \mathbb{R}$.

2.4. Examples: 1. Any compact subgroup of G is a Lie subgroup.

2. The kernel of a homomorphism $\varphi: G \to H$ of Lie groups is a closed (and hence Lie) normal subgroup. Its Lie algebra F is the kernel of φ'.

3. Let K_1 and K_2 be closed subgroups of G. Then $K_1 \cap K_2$ is again a closed subgroup. Give K_1, K_2, and $K_1 \cap K_2$ the Lie group structures defined in Theorem I and let F_1, F_2, and F be the corresponding Lie algebras. Then

$$F = F_1 \cap F_2 .$$

In fact, let V be a neighbourhood of 0 in $E = T_e(G)$ such that

$$\exp: V \xrightarrow{\cong} U$$

is a diffeomorphism, where U is some neighbourhood of e in G. Lemma IV above shows that V can be chosen so that

$$\exp: V \cap F_1 \xrightarrow{\cong} U \cap K_1 , \qquad \exp: V \cap F_2 \xrightarrow{\cong} U \cap K_2$$

and

$$\exp: V \cap F \xrightarrow{\cong} U \cap K_1 \cap K_2$$

are diffeomorphisms. The first two equations imply that

$$\exp: V \cap F_1 \cap F_2 \to U \cap K_1 \cap K_2$$

is a diffeomorphism. Comparing this with the third equation, we obtain $V \cap F = V \cap F_1 \cap F_2$. It follows that $F = F_1 \cap F_2$.

4. *Centralizer and normalizer*: Let G be a Lie group with Lie algebra E. Let $A \subset G$ be a fixed subset. The closed subgroup, Z_A, (or $Z(A)$) of G given by

$$Z_A = \{x \in G \mid xa = ax, \ a \in A\}$$

is called the *centralizer of* A. The Lie algebra, F, of Z_A consists of those vectors $h \in E$ which satisfy

$$a \cdot \exp th \cdot a^{-1} = \exp th, \qquad t \in \mathbb{R}, \quad a \in A.$$

Differentiating this relation we obtain

$$F = \{h \in E \mid (\operatorname{Ad} a)h = h, \ a \in A\}.$$

Suppose A itself is a Lie subgroup of G. Then the adjoint representation of G restricts to a representation $\operatorname{Ad}_{G,A}$ of A in E and F is the invariant subspace for this representation. Moreover, the derivative, ad_E, of Ad_G is given by

$$(\mathrm{ad}_E\, h)(k) = [h, k], \qquad h \in T_e(A), \quad k \in E.$$

Thus if A is connected, Proposition IX, sec. 1.8, gives

$$F = \{k \in E \mid [h, k] = 0,\ h \in T_e(A)\}.$$

In particular, consider the case $A = G$. In this case Z_G is called the *centre of the group* G. The *centre of* E is defined to be the Lie algebra

$$Z_E = \{k \in E \mid [h, k] = 0,\ h \in E\}$$

and the above discussion shows that the centre of E is the Lie algebra of the centre of G if G is connected.

The *normalizer*, N_A, (or $N(A)$) of a subset $A \subset G$ is defined to be the subgroup of G given by

$$N_A = \{x \in G \mid xA = Ax\}.$$

If A is closed in G, then so is N_A and hence it is a Lie subgroup of G. Clearly $Z_A \subset N_A$. If A consists of a single element a, then $N_a = Z_a$ and its Lie algebra is given by

$$T_e(N_a) = \{h \in E \mid (\mathrm{Ad}\, a)h = h\}.$$

5. *Representations*: Let P represent G in a vector space W. Let V be a subspace of W. Then closed (and hence Lie) subgroups $K_1, K_2 \subset G$ are defined by

$$K_1 = \{x \in G \mid P(x)V = V\} \qquad \text{and} \qquad K_2 = \{x \in G \mid P(x)w = w,\ w \in V\}.$$

In view of Proposition IX, sec. 1.8, the Lie algebras F_1 of K_1 and F_2 of K_2 are given by

$$F_1 = \{h \in E \mid P'(h)V \subset V\} \qquad \text{and} \qquad F_2 = \{h \in E \mid P'(h)w = 0,\ w \in V\}.$$

§2. Linear groups

2.5. Closed subgroups of $GL(n; \mathbb{R})$. Fix an n-dimensional real vector space, F. Recall from Example 2, sec. 1.4, that $GL(F)$ is the Lie group of linear automorphisms of F, and that its Lie algebra is L_F. This group is often denoted by $GL(n; \mathbb{R})$.

We shall consider several closed subgroups of $GL(n; \mathbb{R})$.

1. *The special linear group* $SL(n; \mathbb{R})$ (or $SL(F)$): This is the closed subgroup of $GL(n; \mathbb{R})$ consisting of those automorphisms α satisfying $\det \alpha = 1$. Its Lie algebra (Example 5, sec. 2.4, with $P = \det$) is the subalgebra, L_F^0, of L_F of transformations of zero trace. In particular,

$$\dim SL(n; \mathbb{R}) = \dim L_F^0 = n^2 - 1.$$

2. *The orthogonal group* $O(n)$: Assume $\langle\ ,\ \rangle$ is a Euclidean inner product in F. The orthogonal group $O(n)$ (or $O(F)$) is the group of rotations in F; thus it is the subgroup of $GL(F)$ whose elements, α, satisfy

$$\langle \alpha(x), \alpha(y)\rangle = \langle x, y\rangle, \qquad x, y \in F.$$

$O(n)$ is a bounded subset of L_F (sup norm) and hence is compact. It will be shown in sec. 3.6 that $O(n)$ has two components. These are given by

$$O^+(n) = \{\tau \in O(n) \mid \det \tau = 1\} \qquad \text{and} \qquad O^-(n) = \{\tau \in O(n) \mid \det \tau = -1\}.$$

Now we show that the Lie algebra of $O(n)$ is the Lie algebra $\mathrm{Sk}(n)$ (or $\mathrm{Sk}(F)$) of skew linear transformations of F. Define a representation P of $GL(F)$ in $\vee^2 F^*$ by setting

$$P(\alpha)(x^* \vee y^*) = (\alpha^{-1})^* x^* \vee (\alpha^{-1})^* y^*.$$

The derivative of P is given by

$$P'(\varphi)(x^* \vee y^*) = -\varphi^*(x^*) \vee y^* - x^* \vee \varphi^*(y^*), \qquad \varphi \in L_F.$$

Regard the inner product in F as an element $g \in \vee^2 F^*$. Then $O(n)$ is the subgroup which fixes g. Hence (Example 5, sec. 2.4) the Lie algebra of $O(n)$ consists of those transformations φ such that $P'(\varphi)g = 0$. But this is $\mathrm{Sk}(n)$. In particular, it follows that

$$\dim O(n) = \dim \mathrm{Sk}(n) = \tfrac{1}{2}n(n-1) = \binom{n}{2}.$$

3. *The special orthogonal group* $SO(n)$: $SO(n)$ is the compact subgroup of $O(n)$ whose elements have determinant $+1$. It is the 1-component of $O(n)$ (cf. sec. 1.4), and so has the same Lie algebra. The elements of $SO(n)$ are called *proper* rotations.

2.6. Closed subgroups of $GL(n; \mathbb{C})$. 1. *The complex general linear group*, $GL(n; \mathbb{C})$: Let F be an n-dimensional complex space. Denote the underlying $2n$-dimensional real vector space by $F_{\mathbb{R}}$. Then the group $GL(n; \mathbb{C})$ (or $GL(F)$) of complex linear automorphisms of F is the subgroup in $GL(F_{\mathbb{R}})$ of the automorphisms that commute with multiplication by i. In particular it is closed in $GL(F_{\mathbb{R}})$ and hence a Lie subgroup.

$GL(n; \mathbb{C})$ is connected (cf. [7, p. 136]). Its Lie algebra is the real space $L(n; \mathbb{C})$ (or L_F) of complex linear transformations of F. Thus

$$\dim GL(n; \mathbb{C}) = \dim L(n; \mathbb{C}) = 2n^2.$$

2. *The special complex linear group* $SL(n; \mathbb{C})$: This is the closed subgroup in $GL(n; \mathbb{C})$ of the automorphisms of determinant 1. As in the case of $SL(n; \mathbb{R})$, it follows from Example 5, sec. 2.4, that the Lie algebra of $SL(n; \mathbb{C})$ consists of the complex linear transformations with trace zero. Thus

$$\dim SL(n; \mathbb{C}) = 2n^2 - 2.$$

3. *The unitary group* $U(n)$: Suppose now that the complex space F has been equipped with a Hermitian inner product $\langle\ ,\ \rangle$. Then the unitary group $U(n)$ (or $U(F)$) is defined to be the (closed) subgroup of $GL(n; \mathbb{C})$ consisting of the Hermitian isometries. Regard $\mathrm{Re}\langle\ ,\ \rangle$ as a Euclidean inner product in the underlying $2n$-dimensional real space $F_{\mathbb{R}}$. Then

$$U(n) = GL(F) \cap O(F_{\mathbb{R}}).$$

In particular, $U(n)$ is compact. It will be shown in sec. 3.6 that $U(n)$ is connected.

Example 3 of sec. 2.4 shows that the Lie algebra of $U(n)$ is the space

$$L_F \cap \mathrm{Sk}(F_{\mathbb{R}});$$

i.e., it is the *real* vector space consisting of those complex linear transformations of F which are skew with respect to $\langle\ ,\ \rangle$. Thus

$$\dim U(n) = n^2.$$

In particular, consider the case $F = \mathbb{C}$. We may take $\langle\ ,\ \rangle$ to be the Hermitian product

$$\langle z, w \rangle = z\bar{w}, \qquad z, w \in \mathbb{C}.$$

A complex linear map $\varphi: \mathbb{C} \to \mathbb{C}$ is multiplication by some complex number z, and φ is an isometry if and only if $|z| = 1$. In this way the group $U(1)$ is identified with the unit circle S^1 (cf. Example 2, sec. 1.11).

4. *The special unitary group* $SU(n)$ (or $SU(F)$): $SU(n)$ is that subgroup of $U(n)$ whose elements have determinant 1. Its Lie algebra consists of the skew Hermitian transformations with trace zero. Hence (note that the trace of a skew Hermitian transformation has real part zero)

$$\dim SU(n) = n^2 - 1.$$

$SU(n)$ is clearly compact. In sec. 3.6 we show that it is connected.

2.7. The symplectic and quaternionic groups. 1. *The group* $\mathrm{Sp}(n; \mathbb{R})$. Let F be a real vector space of dimension $2n$ and assume that a nondegenerate *skew-symmetric* scalar product $\langle\ ,\ \rangle$ is defined in F. Let $\mathrm{Sp}(n; \mathbb{R})$ be the closed Lie subgroup of $GL(F)$ that consists of the linear automorphisms τ satisfying

$$\langle \tau(u), \tau(v) \rangle = \langle u, v \rangle, \qquad u, v \in F.$$

The Lie algebra of $\mathrm{Sp}(n; \mathbb{R})$ is the subalgebra, $\mathrm{Sy}(n; \mathbb{R})$, of $L(2n; \mathbb{R})$ consisting of those transformations, φ, satisfying

$$\langle \varphi(u), v \rangle + \langle u, \varphi(v) \rangle = 0, \qquad u, v \in F.$$

It has dimension $n(2n + 1)$. Hence

$$\dim \mathrm{Sp}(n; \mathbb{R}) = n(2n + 1).$$

$\mathrm{Sp}(n; \mathbb{R})$ is called the *real symplectic group*. *Note*: In the literature the terminology "symplectic group" and notation $\mathrm{Sp}(n)$ is frequently used for the compact group defined in Example 4 below.

2. *The group* $\mathrm{Sp}(n; \mathbb{C})$: Let F be a complex vector space of dimension $2n$ and let $\langle\ ,\ \rangle$ be a nondegenerate, skew-symmetric, complex bilinear function in F. Then the complex linear automorphisms, τ, of F which satisfy

$$\langle \tau(u), \tau(v) \rangle = \langle u, v \rangle, \qquad u, v \in F$$

form a closed subgroup of $GL(F)$. It is denoted by $\mathrm{Sp}(n; \mathbb{C})$ and is called the *complex symplectic group*. The corresponding Lie algebra consists of the complex linear transformations, α, of F which satisfy

$$\langle \alpha(u), v \rangle + \langle u, \alpha(v) \rangle = 0.$$

It is a complex Lie algebra of complex dimension $n(2n + 1)$. Hence

$$\dim \mathrm{Sp}(n; \mathbb{C}) = 2n(2n + 1).$$

3. *The group of unit quaternions*: Let $\mathbb{H}$ be the algebra of quaternions with quaternionic norm. Then

$$| \alpha\beta | = | \alpha | \, | \beta |, \qquad \alpha, \beta \in \mathbb{H},$$

and so the unit quaternions form a compact Lie group (nonabelian) whose underlying manifold is the three-sphere S^3 (cf. sec. 0.2).

S^3 is a closed subgroup of the Lie group of nonzero quaternions (under multiplication). Since this latter group is the group $G(\mathbb{H})$ of units of $\mathbb{H}$, it follows from Example 3, sec. 1.4, that its Lie algebra is $\mathbb{H}$, with Lie bracket given by

$$[\alpha, \beta] = \alpha\beta - \beta\alpha, \qquad \alpha, \beta \in \mathbb{H}.$$

Now S^3 is a subgroup of $G(\mathbb{H})$. Thus its Lie algebra E is the subalgebra given by

$$E = T_e(S^3) = (e)^{\perp};$$

i.e., E is the Lie algebra of pure quaternions.

4. *The quaternionic group $Q(n)$*: Let V be an n-dimensional Euclidean space, and consider the *quaternionic space*

$$F = \mathbb{H} \otimes_{\mathbb{R}} V.$$

Define a real bilinear map $F \times F \to \mathbb{H}$ by

$$\langle q \otimes u, q' \otimes u' \rangle = q\bar{q}' \langle u, u' \rangle, \qquad q, q' \in \mathbb{H}, \quad u, u' \in V$$

($\bar{q}'$ is the conjugate of q', cf. sec. 0.2).

F is a left vector space over $\mathbb{H}$, with scalar multiplication given by

$$q \cdot (p \otimes u) = qp \otimes u, \qquad q, p \in \mathbb{H}, \quad u \in V.$$

The underlying real space $F_{\mathbb{R}}$ has dimension $4n$, and $\mathrm{Re}\langle \;,\; \rangle$ may be regarded as a Euclidean inner product in $F_{\mathbb{R}}$.

A real linear map $\alpha: F \to F$ is called *quaternionic linear* if

$$\alpha(qx) = q\alpha(x), \qquad q \in \mathbb{H}, \quad x \in F.$$

If, further,

$$\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle, \qquad x, y \in F,$$

then α is called a *quaternionic linear isometry*. These isometries form a closed subgroup of $O(F_{\mathbb{R}})$; it is denoted by $Q(F)$ or $Q(n)$ and is called the *quaternionic group*. $Q(n)$ is a compact Lie group; in sec. 3.6 it will be shown that $Q(n)$ is connected.

The (real) Lie algebra of $Q(n)$ consists of the $\mathbb{H}$-linear transformations, α, of F satisfying

$$\langle \alpha(u), v \rangle + \langle u, \alpha(v) \rangle = 0, \qquad u, v \in F.$$

It is denoted by $\mathrm{Sk}(n; \mathbb{H})$ and has dimension $n(2n + 1)$. Thus

$$\dim Q(n) = n(2n + 1).$$

If $n = 1$, then $F = \mathbb{H}$ and $\langle\ ,\ \rangle$ is given by

$$\langle p, q \rangle = p\bar{q}, \qquad p, q \in \mathbb{H}.$$

A quaternionic linear map φ of $\mathbb{H}$ is simply multiplication on the right by some $p \in \mathbb{H}$. Since

$$\langle \varphi(q_1), \varphi(q_2) \rangle = q_1 p\bar{p}\bar{q}_2 ,$$

it follows that φ is an isometry if and only if p has norm 1; i.e., $Q(1)$ is the group of unit quaternions.

2.8. The groups. $SO(2)$, $SO(3)$, $SO(4)$. **1.** $SO(2)$: Regard $\mathbb{C}$ as a Euclidean plane. A proper rotation of the plane is then multiplication by some $e^{i\theta}$. Thus (cf. Example 3, sec. 2.6)

$$SO(2) = U(1) = S^1.$$

2. $SO(3)$: Consider a three-dimensional Euclidean space F. Recall that $SO(F)$ has Lie algebra $\mathrm{Sk}(F)$ (Example 3, sec. 2.5). Orient F, and make F into a Lie algebra by setting

$$[x, y] = x \times y, \qquad x, y \in F$$

(cross-product). Then an isomorphism $\alpha: F \xrightarrow{\cong} \mathrm{Sk}(F)$ of Lie algebras is defined by $\alpha(a)(x) = a \times x$.

Next we shall establish a diffeomorphism $SO(3) \cong \mathbb{R}P^3$, where $\mathbb{R}P^3$ denotes the three-dimensional projective space. Let $\mathbb{H}$ be the space of quaternions, and identify F with the orthogonal complement of the unit element $e \in \mathbb{H}$. Let S^3 be the unit sphere in $\mathbb{H}$. Then every unit vector $p \in S^3$ determines the proper rotation τ_p of F given by

$$\tau_p(x) = pxp^{-1}, \qquad x \in F.$$

In this way we obtain a homomorphism of Lie groups

$$\tau: S^3 \to SO(3).$$

(S^3 is given the Lie group structure defined in Example 3, sec. 2.7.)

τ is surjective, and its kernel consists of the vectors e and $-e$ (cf. [7, p. 327)]. Since

$$\tau_p = \tau_{-p}, \qquad p \in S^3,$$

the map τ factors over the canonical projection $\pi: S^3 \to \mathbb{R}P^3$ to yield a commutative diagram

$$\begin{tikzcd} S^3 \arrow[r, "\tau"] \arrow[d, "\pi"'] & SO(3). \\ \mathbb{R}P^3 \arrow[ur, "\sigma"'] & \end{tikzcd}$$

The induced map σ is a diffeomorphism.

3. $SO(4)$: Let S^3 be the unit sphere in the space of quaternions. Define $\tau: S^3 \times S^3 \to SO(4)$ by

$$\tau(p, q)(x) = pxq^{-1} \qquad p, q \in S^3, \quad x \in \mathbb{H}.$$

Then τ is a surjective homomorphism of Lie groups, and the kernel of τ consists of the pairs (e, e) and $(-e, -e)$ (cf. [7, p. 329]). It follows that τ induces an isomorphism of groups

$$\varphi: (S^3 \times S^3)/\mathbb{Z}_2 \to SO(4),$$

where $\mathbb{Z}_2$ denotes the normal subgroup of $S^3 \times S^3$ consisting of (e, e) and $(-e, -e)$.

Restricting τ to the normal subgroup $S^3 \times e$ of $S^3 \times S^3$, we obtain a Lie group isomorphism of $S^3 \times e$ onto a normal subgroup H_1 of $SO(4)$. Similarly, the restriction of τ to $e \times S^3$ determines a second normal subgroup H_2 of $SO(4)$ that is isomorphic to S^3.

We finally note that $SO(4)$, as a manifold, is diffeomorphic to the product $\mathbb{R}P^3 \times S^3$. In fact, let ψ: $S^3 \times S^3 \to SO(4)$ be given by

$$\psi(p, q)(x) = pxp^{-1}q \qquad p, q \in S^3, \quad x \in \mathbb{H}.$$

Then ψ factors over the projection $\pi \times \iota$: $S^3 \times S^3 \to \mathbb{R}P^3 \times S^3$ to yield a diffeomorphism

$$\mathbb{R}P^3 \times S^3 \xrightarrow{\cong} SO(4).$$

§3. Homogeneous spaces

In this article G denotes a fixed Lie group with Lie algebra E.

2.9. Definitions: Let $K \subset G$ be a closed subgroup. Consider the set G/K of left cosets; i.e., an element of G/K is a subset of G of the form aK, $(a \in G)$. The projection $a \mapsto aK$ defines a surjective map

$$\pi\colon G \to G/K.$$

It will be convenient to write $\pi(a) = \bar{a}$.

We make G/K into a topological space by calling $O \subset G/K$ open if and only if $\pi^{-1}(O)$ is open. If $U \subset G$ is open, then

$$\pi^{-1}(\pi(U)) = \bigcup_{a \in K} Ua.$$

Hence it is open in G. Thus $\pi(U)$ is open and π is an open map. It follows that the topology of G/K is second countable. Moreover, since K is closed, G/K is a Hausdorff space.

In this article we make G/K into a smooth manifold. In fact we prove

Theorem II: There is a unique smooth structure on G/K such that π is smooth and G/K is a quotient manifold of G. The dimension of G/K is given by

$$\dim G/K = \dim G - \dim K.$$

The manifold G/K so obtained is called a (smooth) *homogeneous space.*

The uniqueness of the smooth structure on G/K follows immediately from the Corollary to Proposition V, sec. 3.9, volume I. To construct the smooth structure on G/K, denote the Lie algebra of K by F, choose a subspace $L \subset E$ so that $E = L \oplus F$ and define $\tau\colon L \times K \to G$ by

$$\tau(k, y) = \exp k \cdot y \qquad k \in L, \quad y \in K.$$

Lemma V: There is a neighbourhood W_L of 0 in L such that τ restricts to a diffeomorphism

$$\tau\colon W_L \times K \xrightarrow{\cong} O$$

onto an open set $O \subset G$.

Proof: According to Lemma IV, sec. 2.2, there are neighbourhoods V_L of 0 in L, V_F of 0 in F and U of e in G with the following properties:

(i) $(k, h) \mapsto \exp k \cdot \exp h$ defines a diffeomorphism

$$\varphi\colon V_L \times V_F \xrightarrow{\cong} U$$

and

(ii) $U \cap K = \exp(V_F)$.

For any subsets $A, B \subset G$ define $A^{-1}B \subset G$ by

$$A^{-1}B = \{a^{-1}b \mid a \in A,\ b \in B\}.$$

Because the map $(x, y) \mapsto x^{-1}y$ is continuous we can find a neighbourhood W_L of 0 in L such that

$$W_L \subset V_L \qquad \text{and} \qquad (\exp W_L)^{-1} \cdot (\exp W_L) \subset U.$$

It will be shown W_L satisfies the conditions of the lemma. We prove first that the restriction of τ to $W_L \times K$ is injective.

In fact, assume that $\tau(k_1, y_1) = \tau(k_2, y_2)$ for some $k_1, k_2 \in W_L$ and $y_1, y_2 \in K$. Then

$$(\exp k_1)y_1 = (\exp k_2)y_2$$

and so

$$y_2y_1^{-1} \in [(\exp W_L)^{-1} \cdot \exp W_L] \cap K \subset U \cap K = \exp V_F.$$

Thus, for some $h \in V_F$, $y_2y_1^{-1} = \exp h$.

It follows that

$$\varphi(k_1, 0) = \exp k_1 = \exp k_2 \cdot \exp h = \varphi(k_2, h).$$

Since φ is injective,

$$k_1 = k_2 \qquad \text{and} \qquad h = 0.$$

Hence $y_1 = y_2$, and so τ is injective in $W_L \times K$.

It remains to be shown that τ is a local diffeomorphism at all points of $W_L \times K$.

The commutative diagram

$$\begin{tikzcd} W_L \times V_F \arrow[r, "\iota\times\exp", "\cong"'] \arrow[d, "\varphi"', "\cong"] & W_L \times (U \cap K) \arrow[dl, "\tau"] \\ \varphi(W_L \times V_F) & \end{tikzcd}$$

implies that τ is a diffeomorphism in $W_L \times (U \cap K)$. But $\tau(k, b) = \tau(k, e) \cdot b$ for each $b \in K$. It follows that τ is a local diffeomorphism at all points of $W_L \times K$.

Q.E.D.

2.10. Proof of Theorem II. Choose W_L as in Lemma V. Set

$$\alpha = \pi \circ \exp: W_L \to G/K.$$

Then the diagram

$$\begin{tikzcd} W_L \times K \arrow[r, "\tau", "\cong"'] \arrow[d, "\pi_1"'] & O \arrow[d, "\pi"] \\ W_L \arrow[r, "\alpha"] & G/K \end{tikzcd}$$

commutes. We shall show that α is a homeomorphism onto the open set $\pi(O)$. Denote $\pi(O)$ by $\tilde{O}$.

α is obviously continuous, and it follows from the diagram that α is open (because π is). The diagram also shows that $\alpha(W_L) = \tilde{O}$. To prove that α is injective, assume

$$\alpha(k_1) = \alpha(k_2), \qquad k_1, k_2 \in W_L .$$

Then, for some $b \in K$,

$$\exp k_1 = \exp k_2 \cdot b; \qquad \text{i.e.,} \qquad \tau(k_1, e) = \tau(k_2, b).$$

Since τ is injective, $k_1 = k_2$.

Note also that $O = OK$ and so $\pi^{-1}(\tilde{O}) = O$.

We shall now construct a smooth atlas on G/K indexed by the points of G. First define a smooth structure on $\tilde{O}$ via the homeomorphism α. Then α becomes a diffeomorphism, and the diagram implies that the restriction of π to O is smooth.

Next, define maps $T_a: G/K \to G/K$ $(a \in G)$ by

$$T_a(bK) = abK.$$

It will be convenient to write this as $\bar{b} \mapsto a \cdot \bar{b}$. Since

$$T_a \circ \pi = \pi \circ \lambda_a \qquad \text{and} \qquad T_a \circ T_{a^{-1}} = \iota,$$

it follows that each T_a is a homeomorphism.

Now set

$$\tilde{O}_a = a \cdot \tilde{O} \qquad \text{and} \qquad \alpha_a = T_a \circ \alpha, \qquad a \in G.$$

We shall show that $\{\tilde{O}_a, \alpha_a^{-1}, W_L\}$ is a smooth atlas for G/K.

In fact, let $a \in G$ and $b \in G$ be two points such that $\tilde{O}_a \cap \tilde{O}_b \neq \varnothing$. Then the identification map,

$$u_{ba}\colon \alpha_a^{-1}(\tilde{O}_a \cap \tilde{O}_b) \to \alpha_b^{-1}(\tilde{O}_a \cap \tilde{O}_b),$$

is given by

$$\begin{aligned} u_{ba} &= \alpha_b^{-1} \circ \alpha_a = \alpha^{-1} \circ T_b^{-1} \circ T_a \circ \alpha \\ &= \alpha^{-1} \circ T_{b^{-1}a} \circ \pi \circ \exp = \alpha^{-1} \circ \pi \circ \lambda_{b^{-1}a} \circ \exp. \end{aligned}$$

Since π is smooth in $O = \pi^{-1}\tilde{O}$, it follows that the map u_{ba} is smooth. Hence this atlas makes G/K into a smooth manifold.

Finally, in the commutative diagrams

$$\begin{tikzcd} W_L \times K \arrow[r, "\tau", "\cong"'] \arrow[d, "\pi_1"] & O \arrow[r, "\lambda_a", "\cong"'] \arrow[d, "\pi"] & a \cdot O \arrow[d, "\pi"] \\ W_L \arrow[r, "\alpha", "\cong"'] & \tilde{O} \arrow[r, "T_a", "\cong"'] & \tilde{O}_a, \end{tikzcd} \qquad a \in G,$$

all the horizontal maps are diffeomorphisms. It follows that $\pi\colon G \to G/K$ is smooth and that each $(d\pi)_a\colon T_a(G) \to T_{\bar{a}}(G/K)$ is surjective. Therefore π makes G/K into a quotient manifold of G.

As regards dimension, observe that

$$\begin{aligned} \dim G/K &= \dim W_L = \dim L = \dim E - \dim F \\ &= \dim G - \dim K. \end{aligned}$$

Q.E.D.

2.11. Consequences of Theorem II. Corollary I: The derivative $(d\pi)_e\colon E \to T_{\bar{e}}(G/K)$ induces a linear isomorphism

$$E/F \xrightarrow{\cong} T_{\bar{e}}(G/K).$$

Proof: Observe that $\pi(K) = \bar{e}$; since $F = T_e(K)$, it follows that $F \subset \ker(d\pi)_e$.

On the other hand,

$$\dim \operatorname{Im}(d\pi)_e = \dim T_{\bar{e}}(G/K) = \dim E - \dim F$$

and so $\ker(d\pi)_e = F$.

Q.E.D.

Corollary II: For each $a \in G$, there is a smooth map $\sigma_a\colon \tilde{O}_a \to G$ such that $\pi \circ \sigma_a = \iota$.

Next, define a map $T\colon G \times G/K \to G/K$ by setting

$$T(a, bK) = T_a(bK) = abK.$$

We also write $T(a, \bar{b}) = a \cdot \bar{b}$.

The diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "\iota\times\pi"'] & G \arrow[d, "\pi"] \\ G \times G/K \arrow[r, "T"] & G/K \end{tikzcd}$$

commutes. Since G/K is a quotient manifold of G, the smoothness of μ implies that T is smooth (cf. sec. 3.9, volume I). T satisfies the relations

$$(ab) \cdot \bar{x} = a \cdot (b \cdot \bar{x}) \qquad \text{and} \qquad e \cdot \bar{x} = \bar{x} \qquad a, b \in G, \quad \bar{x} \in G/K.$$

T will be called the *action of G on G/K* (cf. sec. 3.1 for general actions of Lie groups).

Finally, note that if we consider the quotient space of *right* cosets Ka ($a \in G$), then Theorem II remains true. The proof differs from that given above only insofar as elements of K must be written on the left instead of on the right in all formulae. The space of *right* cosets will be denoted by $K\backslash G$.

2.12. Factor groups. Suppose K is a closed normal subgroup of a Lie group G. Then the coset space G/K becomes a group, with multiplication, $\bar{\mu}$, given by

$$\bar{\mu}(\pi(a), \pi(b)) = \pi(ab), \qquad a, b \in G.$$

Since G/K is a quotient manifold of G, it follows (as above for T) that $\bar{\mu}$ is smooth. Similarly, inversion in G/K is smooth. Thus G/K is a Lie group. It is called the *factor group* of G with respect to K.

Proposition I: Let $\varphi\colon G \to H$ be a surjective homomorphism of Lie groups and let K be the kernel of φ. Then K is a closed normal subgroup of G and the map $\psi\colon G/K \to H$ defined by the commutative diagram,

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ G/K \arrow[ur, "\psi"'] & \end{tikzcd}$$

is an isomorphism of Lie groups.

Proof: Since G/K is a quotient manifold of G the map $\psi\colon G/K \to H$ is smooth. On the other hand, ψ is a bijective homomorphism. Hence, Corollary III of Proposition VII, sec. 1.7, implies that ψ is an isomorphism of Lie groups.

Q.E.D.

Example: The group $SU(n)$ is a normal subgroup of $U(n)$ and the factor group is isomorphic to S^1:

$$U(n)/SU(n) \cong S^1.$$

In fact, regard the determinant as a surjective homomorphism

$$\det\colon U(n) \to S^1.$$

Then $\ker \det = SU(n)$, and the result follows from Proposition I.

§4. The bundle structure of a homogeneous space

In this article G denotes a fixed Lie group of dimension n with Lie algebra E. K is a closed r-dimensional subgroup with Lie algebra F and $i\colon K \to G$ denotes the inclusion map.

2.13. The bundle $(G, \pi, G/K, K)$. By Theorem II (sec. 2.9) the projection $\pi\colon G \to G/K$ makes G/K into a quotient manifold of G. In this section we shall show that $(G, \pi, G/K, K)$ is a smooth fibre bundle.

In view of Corollary II to Theorem II, sec. 2.11, there exists a covering of G/K by open sets V_α and a family of smooth maps $\sigma_\alpha\colon V_\alpha \to G$ such that $\pi \circ \sigma_\alpha = \iota$. Define smooth maps $\psi_\alpha\colon V_\alpha \times K \to G$ by

$$\psi_\alpha(x, y) = \sigma_\alpha(x)y, \qquad x \in V_\alpha, \quad y \in K.$$

Then

$$\pi\psi_\alpha(x, y) = x, \qquad x \in V_\alpha, \quad y \in K.$$

Moreover, each ψ_α is a diffeomorphism onto $\pi^{-1}(V_\alpha)$ with smooth inverse $\varphi_\alpha\colon \pi^{-1}(V_\alpha) \to V_\alpha \times K$ given by

$$\varphi_\alpha(z) = (x, \sigma_\alpha(x)^{-1}z), \qquad z \in \pi^{-1}(V_\alpha), \quad x = \pi z.$$

It follows that $(G, \pi, G/K, K)$ is a smooth fibre bundle with coordinate representation $\{(V_\alpha, \psi_\alpha)\}$.

Remark: The coordinate representation $\{(V_\alpha, \psi_\alpha)\}$ satisfies

$$\psi_\alpha(x, y_1 y_2) = \psi_\alpha(x, y_1) \cdot y_2, \qquad x \in V_\alpha, y_1, y_2 \in K.$$

2.14. Orientations and fibre integration. Choose elements $\Delta_E \in \wedge^n E^*$ and $\Delta_F \in \wedge^r E^*$ such that $\Delta_E \neq 0$ and Δ_F restricts to a nonzero determinant function in F. Recall that a strong bundle isomorphism $G \times E \xrightarrow{\cong} \tau_G$ is given by

$$(a, h) \mapsto L_a(h)$$

(cf. Proposition I, sec. 1.2). Hence differential forms $\Delta_G \in A^n(G)$ and $\Delta_K \in A^r(G)$ are defined by

$$\Delta_G(a; L_a h_1, \dots, L_a h_n) = \Delta_E(h_1, \dots, h_n)$$

and

$$\Delta_K(a; L_a h_1, \ldots, L_a h_r) = \Delta_F(h_1, \ldots, h_r), \qquad a \in G, \quad h_i \in E.$$

Evidently,

$$\lambda_a^* \Delta_G = \Delta_G \qquad \text{and} \qquad \lambda_a^* \Delta_K = \Delta_K, \qquad a \in G.$$

Lemma VI: With the notation above

(1) Δ_G orients the manifold G.

(2) Δe orients the bundle $(G, \pi, G/K, K)$ (cf. sec. 0.15).

(3) If K is connected, then G/K is orientable. Moreover, an orientation in $T_{\bar{e}}(G/K)$ extends to a unique orientation in G/K such that the diffeomorphisms T_a ($a \in G$) are all orientation preserving.

Proof: (1) This is clear (cf. sec. 1.13).

(2) First note that $i^*\Delta_K$ is a left invariant r-form on K whose restriction to the Lie algebra F orients F. Hence $i^*\Delta_K$ orients K.

Now fix $a \in G$ and let $j_{\bar{a}}: K_{\bar{a}} \to G$ be the inclusion map ($K_{\bar{a}}$ denotes the fibre over $\bar{a}$). There is a unique diffeomorphism $\varphi_a: K \xrightarrow{\cong} K_{\bar{a}}$, such that

$$j_{\bar{a}} \circ \varphi_a = \lambda_a \circ i.$$

Thus

$$j_{\bar{a}}^* \Delta_K = (\varphi_a^{-1})^* i^* \lambda_a^*(\Delta_K) = (\varphi_a^{-1})^* i^*(\Delta_K)$$

and so $j_{\bar{a}}^* \Delta_K$ orients $K_{\bar{a}}$.

(3) Fix an orientation in $T_{\bar{e}}(G/K)$. For each $y \in K$, the bundle map dT_y restricts to a linear automorphism of $T_{\bar{e}}(G/K)$; since K is connected, these automorphisms all preserve the orientation. Thus, for each $\bar{a} \in G/K$, the orientation of $T_{\bar{a}}(G/K)$ induced by the isomorphism,

$$dT_a: T_{\bar{e}}(G/K) \xrightarrow{\cong} T_{\bar{a}}(G/K),$$

is independent of the choice of a in $K_{\bar{a}}$. These orientations determine the desired orientation of G/K. It is obviously unique.

Q.E.D.

In view of Lemma VI, the fibre integral

$$\oint_K : A_K(G) \to A(G/K)$$

is defined (cf. sec. 0.15). Since, by construction, the left translations λ_a all preserve the bundle orientation, we have (cf. Proposition VIII, sec. 7.12, volume I)

$$\oint_K \circ \lambda_a^* = T_a^* \circ \oint_K . \tag{2.1}$$

Now assume that both G and K are compact and connected. Then $A_K(G) = A(G)$. Choose Δ_E and Δ_F so that the corresponding forms Δ_G, Δ_K satisfy

$$\int_G \Delta_G = 1 \qquad \text{and} \qquad \int_K i^* \Delta_K = 1.$$

Set $\Delta_{G/K} = \oint_K \Delta_G$.

Proposition II: With the notation and hypotheses above:

(1) $T_a^* \Delta_{G/K} = \Delta_{G/K}$, $a \in G$.

(2) $\Delta_{G/K}$ orients G/K, and $\int_{G/K} \Delta_{G/K} = 1$.

(3) $\Delta_G = \pi^* \Delta_{G/K} \wedge \Delta_K$.

Proof: (1) This follows from the relation (cf. formula (2.1))

$$T_a^* \oint_K \Delta_G = \oint_K \lambda_a^* \Delta_G = \oint_K \Delta_G , \qquad a \in G.$$

(2) In view of Lemma VI, (3), G/K is orientable. Orient it so that Δ_G represents the local product orientation in G (cf. sec. 0.15). Then

$$\int_{G/K} \Delta_{G/K} = \int_{G/K} \oint_K \Delta_G = \int_G \Delta_G = 1$$

(cf. sec. 0.15, or Theorem I, sec. 7.14, volume I). Thus, for some $\bar{a} \in G/K$, $\Delta_{G/K}(\bar{a})$ is strictly positive. Since the T_a are all orientation preserving, it now follows from (1) that $\Delta_{G/K}$ represents the orientation of G/K.

(3) Observe that $\oint_K \Delta_K$ is a function on G/K, and that for $\bar{a} \in G/K$

$$\left(\oint_K \Delta_K\right)(\bar{a}) = \left(T_a^* \oint_K \Delta_K\right)(\bar{e}) = \left(\oint_K \lambda_a^* \Delta_K\right)(\bar{e}) = \int_K i^* \Delta_K = 1;$$

i.e., $\oint_K \Delta_K = 1$.

Now set

$$\pi^* \Delta_{G/K} \wedge \Delta_K = \Phi.$$

Then in view of sec. 0.15 (cf. also Proposition IX, sec. 7.13, volume I)

$$\int_G \Phi = \int_{G/K} \Delta_{G/K} \oint_K \Delta_K = 1.$$

On the other hand,

$$\lambda_a^* \Phi = \pi^* T_a^* \Delta_{G/K} \wedge \Delta_K = \pi^* \Delta_{G/K} \wedge \Delta_K = \Phi, \qquad a \in G.$$

These relations imply that $\Phi = \Delta_G$.

Q.E.D.

§5. Maximal tori

In this article G denotes a compact connected n-dimensional Lie group. Its Lie algebra, E, is equipped with a fixed Euclidean inner product, $\langle\ ,\ \rangle$, with respect to which the transformations $\mathrm{Ad}\ x$ $(x \in G)$ are isometries (cf. Proposition XVI, sec. 1.17).

2.15. Maximal tori. Let T be a closed connected abelian subgroup of G. Then T is compact and (cf. Theorem I, sec. 2.1) a Lie subgroup. It follows from Proposition XIII, sec. 1.12, that T is a torus.

In particular, the closure of a 1-parameter subgroup is a torus. A *maximal torus* in G is a torus that is not properly contained in another torus. Clearly, an automorphism of G carries a maximal torus onto a maximal torus.

Let T be a torus in G and denote by F the Lie algebra of T. Then $F \subset E$ and hence we can write

$$E = F^{\perp} \oplus F,$$

where $F^{\perp}$ denotes the orthogonal complement of F in E. Proposition XII, sec. 1.11, shows that the adjoint representation of T in F is trivial. Moreover, $F^{\perp}$ is stable under $\mathrm{Ad}\ y$ $(y \in T)$. Thus we can write

$$\mathrm{Ad}\ y = \mathrm{Ad}^{\perp}\ y \oplus \iota, \qquad y \in T, \tag{2.2}$$

where $\mathrm{Ad}^{\perp}\ y$ denotes the restriction of $\mathrm{Ad}\ y$ to $F^{\perp}$.

Lemma VII: Let T be a torus with Lie algebra F and let $a \in T$ be a generator (cf. sec. 1.12). Then

$$F \subset \ker(\iota - \mathrm{Ad}\ a)$$

and equality holds if and only if T is maximal.

Proof: Let S be any torus in G such that $T \subset S$ and denote the Lie algebra of S by H. Then

$$F \subset H \subset \ker(\iota - \mathrm{Ad}\ a).$$

Thus $F \subset \ker(\iota - \mathrm{Ad}\ a)$ and if equality holds, then T is maximal.

Conversely, assume that T is maximal. Let $L = \ker(\iota - \operatorname{Ad} a)$. Formula (2.2) shows that

$$L = F^{\perp} \cap L \oplus F.$$

Now, if $h \in F^{\perp} \cap L$, then the 1-parameter subgroup H generated by h centralizes a and so it centralizes T. It follows that the closure S of the group $H \cdot T$ is compact, connected, and abelian; i.e., it is a torus in G with h in its Lie algebra.

Since T is maximal, and $S \supset T$, we have $S = T$. Hence

$$h \in F \cap F^{\perp} = \{0\};$$

i.e., $\ker(\iota - \operatorname{Ad} a) = F$.

Q.E.D.

2.16. The Weyl group. Let T be a maximal torus in G and consider its normalizer, $N(T)$. $N(T)$ is a compact Lie group (cf. Example 4, sec. 2.4) and T is a closed normal subgroup of $N(T)$. The factor group $W_G = N(T)/T$ is called the *Weyl group* of G (with respect to the maximal torus T).

Remark: It follows from Theorem III, sec. 2.18, that the Weyl groups of G with respect to any two maximal tori are isomorphic.

Example: Let $U(n)$ be the unitary group. To find a maximal torus, let e_j ($j = 1, \ldots, n$) be an orthonormal basis of $\mathbb{C}^n$ and consider the unitary transformations given by

$$\tau e_j = \epsilon_j e_j, \qquad j = 1, \ldots, n, \quad \epsilon_j \in \mathbb{C}, \quad |\epsilon_j| = 1.$$

Clearly these transformations form a torus subgroup, T, of $U(n)$.

To show that T is maximal assume that S is an abelian subgroup of $U(n)$ such that $T \subset S$. Then every $\sigma \in S$ and $\tau \in T$ commute. Choose $\tau \in T$ such that the ϵ_j are distinct. Then the transformations that commute with τ preserve the one-dimensional spaces (e_j):

$$\sigma e_j = \lambda_j e_j, \qquad \sigma \in S.$$

Thus $S \subset T$.

To determine $N(T)$, let $\alpha \in N(T)$. Then $\alpha^{-1} T \alpha \subset T$ and so we have, for $\tau \in T$,

$$(\alpha^{-1} \tau \alpha)\, e_i = \epsilon_i e_i, \qquad i = 1, 2, \ldots, n;$$

i.e., the vectors $\alpha(e_i)$ are eigenvectors for τ. Choosing $\tau \in T$ such that the ϵ_i are distinct we see that each $\alpha(e_i)$ must be a scalar multiple of one of the vectors $e_1, \ldots, e_n$. Since α is injective, it follows that

$$\alpha(e_i) = \lambda_i e_{\omega(i)}, \qquad i = 1, \ldots, n,$$

where ω is a permutation of $(1, \ldots, n)$. Conversely, every unitary map of this form normalizes T.

In particular, the correspondence $\alpha \mapsto \omega$ defines a surjective group homomorphism

$$\rho: N(T) \to S^n$$

(S^n the group of all permutations of the set $\{1, \ldots, n\}$). Evidently $\ker \rho = T$, and so ρ induces an isomorphism

$$W_{U(n)} \xrightarrow{\cong} S^n.$$

Proposition III: Let T be a maximal torus in G. Then T is the 1-component of $N(T)$. In particular, the Weyl group W_G is finite.

Proof: Let H denote the Lie algebra of $N(T)$. We show that $H = F$. Since $F \subset H$, we have

$$H = (F^\perp \cap H) \oplus F.$$

Since T is a subgroup of $N(T)$, H is stable under the transformations $\operatorname{Ad} y$ $(y \in T)$ and so $F^\perp \cap H$ is stable under these transformations. Since $\operatorname{ad} = \operatorname{Ad}'$ (cf. Proposition XI, sec. 1.10), this implies that $F^\perp \cap H$ is stable under $\operatorname{ad} h$,

$$[h, k] \in F^\perp \cap H, \qquad h \in F, \quad k \in F^\perp \cap H.$$

On the other hand, since T is normal in $N(T)$, F is stable under the maps $\operatorname{Ad} x$, $x \in N(T)$. Hence it is stable under the transformations $\operatorname{ad} k$ $(k \in H)$. In particular, for $h \in F$ and $k \in F^\perp \cap H$, we have

$$[h, k] \in F \cap (F^\perp \cap H) = 0,$$

whence $(\operatorname{Ad} y)k = k$, $y \in T$, $k \in F^\perp \cap H$.

Since T is maximal, it follows from this relation and Lemma VII, sec. 2.15 that $F^\perp \cap H = 0$. Thus $F = H$. In particular, T and $N(T)$ have the same dimension, and so T is a connected, open subgroup of $N(T)$; i.e., T is the 1-component of $N(T)$.

It follows that the compact group W_G is discrete, and hence finite.

Q.E.D.

Since T is normal in $N(T)$, a left action, Φ, of $N(T)$ in T is given by

$$\Phi(x, y) = xyx^{-1}, \qquad y \in T, \quad x \in N(T).$$

Because T is abelian, $\Phi(x, y)$ depends only on y and the coset $\bar{x}$ of x in W_G. Thus a left action, Ψ, of the finite group W_G on T is given by

$$\Psi(\bar{x}, y) = \Phi(x, y), \qquad x \in N(T), \quad y \in T.$$

2.17. The map ψ. Let T be a maximal torus in G with Lie algebra F. Define a map,

$$\varphi: G \times T \to G,$$

by

$$\varphi(x, y) = xyx^{-1}, \qquad x \in G, \quad y \in T.$$

Since T is abelian, it follows that

$$\varphi(xz, y) = \varphi(x, y), \qquad z \in T.$$

Hence φ factors over the projection,

$$\pi \times \iota: G \times T \to G/T \times T,$$

to yield a smooth commutative diagram

$$\begin{tikzcd} G \times T \arrow[r, "\varphi"] \arrow[d, "\pi \times \iota"'] & G \\ G/T \times T \arrow[ur, "\psi"'] & \end{tikzcd}.$$

It follows from the definitions that the diagram,

$$\begin{tikzcd} W_G \times T \arrow[r, "\Psi"] \arrow[d, "j \times \iota"'] & T \arrow[d, "i"] \\ G/T \times T \arrow[r, "\psi"'] & G \end{tikzcd},$$

commutes, where i and j are the inclusions (j is induced by the inclusion map $N(T) \to G$). In other words, Ψ is the restriction of ψ to $N(T)/T \times T$.

2.18. Degree of ψ. Orient $F^{\perp}$ and F and give E the orientation induced by the decomposition $E = F^{\perp} \oplus F$. Orient $T_{\bar{e}}(G/T)$ so that

$$(d\pi)_e: F^{\perp} \xrightarrow{\cong} T_{\bar{e}}(G/T)$$

preserves the orientations. The orientations of F and E determine left orientations in T and G; because T and G are connected, these orientations are also invariant under right translations. Finally (cf. Lemma VI, sec. 2.14) the orientation of $T_{\bar{e}}(G/T)$ determines an orientation of G/T.

Since ψ is a map between oriented, compact manifolds of the same dimension, the degree of ψ is defined (cf. sec. 0.14).

Proposition IV: With orientations as described above, the degree of ψ is equal to the order of the Weyl group,

$$\deg \psi = | W_G |.$$

In particular, ψ is surjective.

Lemma VIII: The derivative of ψ at $(\bar{e}, y)$ $(y \in T)$ is given by the commutative diagram,

$$\begin{tikzcd} F^{\perp} \oplus T_y(T) \arrow[r, "\alpha_y"] \arrow[d, "(d\pi)_e \oplus \iota"', "\cong"] & T_y(G) \\ T_{\bar{e}}(G/T) \oplus T_y(T) \arrow[ur, "d\psi"'] & \end{tikzcd},$$

where $\alpha_y(h, R_y k) = R_y((\iota - \mathrm{Ad}^{\perp} y)h + k), \quad h \in F^{\perp}, k \in F.$

Proof: Observe (via Lemma I, sec. 1.1) that α_y is the restriction of $(d\varphi)_{(e,y)}$.

Q.E.D.

Lemma IX: Let a be a generator of T and let $\bar{x} \in G/T$. Then

$$(d\psi)_{(\bar{x},a)}: T_{\bar{x}}(G/T) \oplus T_a(T) \to T_{xax^{-1}}(G)$$

is an orientation preserving isomorphism.

Proof: In fact, the diagram

$$\begin{tikzcd} G/T \times T \arrow[r, "\psi"] \arrow[d, "\tau_x \times \iota"'] & G \arrow[d, "\tau_x"] \\ G/T \times T \arrow[r, "\psi"'] & G \end{tikzcd}$$

commutes, where T denotes the left action of G on G/T and

$$\tau_x(y) = xyx^{-1}.$$

The vertical arrows are orientation preserving diffeomorphisms. Thus we can reduce to the case $\bar{x} = \bar{e}$.

Since a generates T, Lemma VII of sec. 2.15 implies that the map

$$\iota - \mathrm{Ad}^{\perp} a : F^{\perp} \to F^{\perp}$$

is a linear isomorphism. Hence α_a is an isomorphism, and so Lemma VIII shows that $(d\psi)_{(\bar{e},a)}$ is an isomorphism. Moreover, since $\mathrm{Ad}^{\perp} a$ is a proper rotation,

$$\det(\iota - \mathrm{Ad}^{\perp} a) > 0,$$

and so α_a preserves orientations. Thus so does $d\psi_{(\bar{e},a)}$.

Q.E.D.

Proof of the proposition: Regard W_G as a subset of G/T. First we show that, for a generator $a \in T$,

$$\psi^{-1}(a) = \{(\bar{x}, x^{-1}ax) \mid \bar{x} \in W_G\}.$$

In fact, if $x \in G$, $y \in T$, then

$$a = \varphi(x, y) = xyx^{-1}$$

holds if and only if $x^{-1}ax = y$.

Since a generates T and $y \in T$, this implies that $x \in N(T)$. Thus

$$\varphi^{-1}(a) = \{(x, x^{-1}ax) \mid x \in N(T)\},$$

whence

$$\psi^{-1}(a) = \{(\bar{x}, x^{-1}ax) \mid \bar{x} \in W_G\}.$$

According to Proposition III, sec. 2.16, W_G is finite. Moreover, for $\bar{x} \in W_G$, $x^{-1}ax$ is again a generator of T. Thus Lemma IX shows that $d\psi$ is an orientation preserving isomorphism at each point $(\bar{x}, x^{-1}ax)$. Now Theorem I, sec. 6.3, volume I, implies that

$$\deg \psi = \text{cardinality of } \psi^{-1}(a) = |\, W_G \,|.$$

Q.E.D.

Theorem III: Every element of a compact, connected Lie group G is contained in a maximal torus, and any two maximal tori are conjugate.

Proof: Let T be a maximal torus and let $a \in G$. Since ψ is surjective, there are elements $b \in G$ and $y \in T$ such that $a = byb^{-1}$. Hence a is in the maximal torus bTb^{-1}.

If S is any maximal torus, let a be a generator of S. Then, for some $b \in G$, $a \in bTb^{-1}$. This implies that

$$S \subset bTb^{-1}.$$

Since S is maximal, it follows that $S = bTb^{-1}$; i.e., S is conjugate to T.

Q.E.D.

Corollary I: For every compact connected Lie group G the map $\exp: E \to G$ is surjective.

Proof: Given $a \in G$ choose a torus T such that $a \in T$, and observe that the map $\exp: F \to T$ (F is the Lie algebra of T) is surjective (cf. the corollary to Lemma VI, sec. 1.12).

Q.E.D.

Corollary II: For every compact connected Lie group G, the maps $x \mapsto x^p$, $p = \pm 1, \pm 2, \ldots$ are surjective.

Proof: Note that $\exp ph = (\exp h)^p$ and apply Corollary I.

Q.E.D.

2.19. The Weyl integration formula. We retain the notation of the preceding sections. In particular, G, T, and G/T are oriented as described in sec. 2.18; E and F are the Lie algebras of G and T; and if $y \in T$, $\mathrm{Ad}^{\perp} y$ denotes the restriction of $\mathrm{Ad}\, y$ to $F^{\perp}$.

A *central function* f on G is a smooth function such that

$$f(xyx^{-1}) = f(y), \qquad x, y \in G,$$

or, equivalently,

$$f(xy) = f(yx), \qquad x, y \in G.$$

In this and the next section we establish

Theorem IV: Let f be a central function on a compact connected Lie group G. Then

$$\int_G f(x)\, dx = |\, W_G\, |^{-1} \int_T f(y) \det(\iota - \mathrm{Ad}^{\perp} y)\, dy.$$

Before proving the theorem we establish some notation. Let Δ_G, Δ_T, $\Delta_{G/T}$ be the differential forms as constructed in sec. 2.14; thus in particular Δ_G orients G, $i^*\Delta_T$ orients T, $\Delta_{G/T}$ orients G/T, and

$$\int_G \Delta_G = 1, \qquad \int_T i^* \Delta_T = 1, \qquad \int_{G/T} \Delta_{G/T} = 1.$$

Proposition V: The map ψ of sec. 2.17 satisfies

$$\psi^* \Delta_G = \Delta_{G/T} \times g \cdot i^* \Delta_T,$$

where $g \in \mathscr{S}(T)$ is given by

$$g(y) = \det(\iota - \mathrm{Ad}^{\perp}\, y).$$

Proof: Since $\Delta_{G/T} \times i^*\Delta_T$ orients $G/T \times T$, we can write

$$\psi^* \Delta_G = g_1 \cdot (\Delta_{G/T} \times i^* \Delta_T),$$

for some $g_1 \in \mathscr{S}(G/T \times T)$. Combining the relations

$$\lambda_a^* \Delta_G = \Delta_G = \rho_a^* \Delta_G, \qquad T_a^* \Delta_{G/T} = \Delta_{G/T}$$

and

$$\psi \circ (T_a \times \iota) = \lambda_a \circ \rho_a^{-1} \circ \psi, \qquad a \in G,$$

we find that $(T_a \times \iota)^* g_1 = g_1$, $a \in G$. It follows that

$$\psi^* \Delta_G = \Delta_{G/T} \times g \cdot i^* \Delta_T,$$

where $g(y) = g_1(\bar{e}, y)$.

On the other hand, using Lemma VIII, sec. 2.18, we find that

$$\begin{aligned}(\psi^*\Delta_G)(\bar{e}, y; (d\pi)\, k_1, \ldots, (d\pi)\, k_{n-r}, R_y h_1, \ldots, R_y h_r)\\ = \det(\iota - \mathrm{Ad}^{\perp}\, y)\, \Delta_G(e, k_1, \ldots, k_{n-r}, h_1, \ldots, h_r),\end{aligned}$$

where $k_i \in F^{\perp}$, $h_j \in F$. Since $\Delta_G = \pi^*\Delta_{G/T} \wedge \Delta_T$ (cf. Proposition II, sec. 2.14) and since $\rho_y^* i^*\Delta_T = i^*\Delta_T$ ($y \in T$), we obtain

$$\begin{aligned}&\Delta_G(e; k_1, \ldots, k_{n-r}, h_1, \ldots, h_r)\\ &\quad = \Delta_{G/T}(\bar{e}; (d\pi)\, k_1, \ldots, (d\pi)\, k_{n-r}) \cdot (i^* \Delta_T)(e; h_1, \ldots, h_r)\\ &\quad = (\Delta_{G/T} \times i^* \Delta_T)(\bar{e}, y; (d\pi)\, k_1, \ldots, (d\pi)\, k_{n-r}, R_y h_1, \ldots, R_y h_r).\end{aligned}$$

Combining these two relations shows that

$$g(y) = \det(\iota - \mathrm{Ad}^{\perp} y).$$

Q.E.D.

2.20. Proof of Theorem IV: Since f is a central function, we have $\psi^* f = 1 \times i^* f$ and so Proposition V yields

$$\psi^*(f \cdot \Delta_G) = \psi^* f \cdot \psi^* \Delta_G = \Delta_{G/T} \times g \cdot i^*(f \Delta_T).$$

Applying the Fubini theorem (cf. Proposition XIII, sec. 4.13, volume I), we obtain

$$\int_{G/T \times T} \psi^*(f \cdot \Delta_G) = \int_{G/T} \Delta_{G/T} \cdot \int_T g \cdot i^*(f \Delta_T)$$

$$= \int_T f(y) \cdot \det(\iota - \mathrm{Ad}^{\perp} y)\, dy.$$

On the other hand

$$\int_{G/T \times T} \psi^*(f \cdot \Delta_G) = \deg \psi \cdot \int_G f \cdot \Delta_G = | W_G | \cdot \int_G f(x)\, dx$$

(cf. Proposition IV, sec. 2.18) and thus it follows that

$$\int_G f(x)\, dx = | W_G |^{-1} \int_T f(y) \det(\iota - \mathrm{Ad}^{\perp} y)\, dy.$$

Q.E.D.

Corollary: $| W_G | = \int_T \det(\iota - \mathrm{Ad}^{\perp} y)\, dy.$

Problems

1. **Centre.** (i) Show that the centre of a Lie group is a closed normal subgroup and that the centre of a Lie algebra is an ideal.

(ii) Find the centres of the groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ and of their Lie algebras.

2. **The derived group.** Let G be a Lie group. A *commutator* in G is an element of the form $xyx^{-1}y^{-1}$; it is denoted by $[x, y]$. The *derived group* G' is the subgroup generated by the commutators.

Let $G^{(p)} = G \times \cdots \times G$ (p factors) and define $c_p\colon G^{(2p)} \to G$ by

$$c_p(x_1, y_1, \ldots, x_p, y_p) = [x_1, y_1] \cdot \cdots \cdot [x_p, y_p].$$

(i) Show that $G' = \bigcup_p \operatorname{Im} c_p$.

(ii) Construct a distribution ξ on G (i.e., a subbundle of τ_G) with fibre F_x at $x \in G$ and satisfying the following conditions: (a) $L_x(F_e) = F_x$; (b) $\operatorname{Im}(dc_p)_z \subset F_{c_p(z)}$; and (c) for each $x \in G'$, there is some p and some $z \in G^{(2p)}$ such that $\operatorname{Im}(dc_p)_z = F_x$.

Show that these conditions uniquely determine ξ.

(iii) Show that the distribution ξ is involutive, and that the integral manifold through e is G'. Conclude that G' is a Lie subgroup of G (cf. problem 8, Chap. III, volume I).

(iv) Show that if G is connected then so is G'. Construct an example where G^0 is abelian and equal to G', but $G \neq G^0$.

(v) Construct an example where G is compact and G' is a proper, dense subgroup of G.

3. **The Lie algebra of G'.** Let E be the Lie algebra of a Lie group G. Let F be the subspace spanned by vectors of the form $(\operatorname{Ad} x - \iota)(h)$ ($x \in G$, $h \in E$).

(i) Show that F is the Lie algebra of G'. (*Hint*: First show that $L_x(F) = R_x(F)$, $x \in G$. Then compute the derivative of c_p and conclude that $\operatorname{Im}(dc_p)_z \subset L_{c_p(z)}(F)$, $z \in G^{(2p)}$. Finally, by considering the paths $x \cdot \exp th \cdot x^{-1} \cdot \exp(-th)$, show that $T_e(G') \subset F$).

(ii) Let L be any Lie algebra. The *derived algebra* L' is the space spanned by vectors of the form $[h, k]$, $h, k \in L$. Show that $E' \subset T_e(G')$.

(iii) If G is connected, show that E' is the Lie algebra of G'.

4. Homomorphisms. If $\varphi: G \to H$ is a homomorphism of Lie groups, show that $\ker \varphi$ is a closed normal subgroup of G. Show that $G/\ker \varphi$ is a Lie subgroup of H and conclude that $\operatorname{Im} \varphi$ is a Lie subgroup of H.

5. The Lie subgroup associated with a subalgebra. Let E be the Lie algebra of a Lie group G and let F be a subalgebra of E.

(i) Show that there is a unique connected Lie subgroup K of G with Lie algebra F. (*Hint*: Show that the spaces $L_x(F)$ $(x \in G)$ define an involutive distribution ξ on G and take K to be the maximal connected integral manifold of ξ through e (cf. problem 8, Chap. III, volume I).)

(ii) Show that the closure $\bar{K}$ of K is a Lie subgroup containing K as a normal subgroup.

(iii) Show that the derived group $(\bar{K})'$ is contained in K.

6. Connected subgroups. Let G be a Lie group and let K be a subgroup (i.e., a subset closed under multiplication and inversion). Assume that, for each $y \in K$, there is a smooth path y_t in G, joining e to y and such that each $y_t \in K$. Prove that K is a Lie subgroup of G.

Hint: Let E be the Lie algebra of G and let F be the subset of E whose elements are tangent vectors at e to smooth curves which are contained in K. Show that F is a subalgebra. Then apply problem 5.

7. Killing form. The *Killing form* of a Lie algebra E is the bilinear function given by

$$K(h, k) = \operatorname{tr}(\operatorname{ad} h \circ \operatorname{ad} k), \qquad h, k \in E.$$

(i) Show that K is symmetric and that the transformations $\operatorname{ad} h$ $(h \in E)$ are skew with respect to K.

(ii) If E is the Lie algebra of G, show that the transformations $\operatorname{Ad} x$ $(x \in G)$ are isometries with respect to K.

(iii) Suppose E is the Lie algebra of a compact group G. Show that $E = Z_E \oplus E'$. Show that Z_E is the null space of the Killing form, and that the restriction of K to E' is negative definite. Show that $(E')' = E'$.

8. The groups Aut E, Aut G. (i) Let E be a Lie algebra. Show that the group $\operatorname{Aut} E$ of automorphisms of E is a closed Lie subgroup of $GL(E)$. Show that the Lie algebra of $\operatorname{Aut} E$ is the Lie algebra of derivations of E.

(ii) Let E be the Lie algebra of a connected Lie group G. Denote the group of automorphisms of the Lie group G by Aut G. Show that $\sigma \mapsto \sigma'$ defines a group homomorphism φ_G: Aut $G \to$ Aut E.

(iii) Let $\tilde{G} \xrightarrow{p} G$ be the universal covering group (cf. problem 19, Chap. I). Show that $\varphi_{\tilde{G}}$ is an isomorphism of groups, and hence make Aut $\tilde{G}$ into a Lie group. Show that $\varphi_{\tilde{G}}^{-1} \circ \varphi_G$ maps Aut G injectively onto the subgroup of Aut $\tilde{G}$ consisting of those elements which normalize $\ker p$. (*Hint*: cf. problem 20, Chap. I).

(iv) Conclude that Aut G is a Lie group and that φ_G is an isomorphism of the Lie group Aut G onto a closed subgroup of Aut E. What is the Lie algebra of Aut G?

(v) Show that a homomorphism of Lie groups τ: $G \to$ Aut G is defined by $\tau(a)(x) = axa^{-1}$ $(a, x \in G)$. Show that $(\varphi_G \circ \tau)(a) = \operatorname{Ad} a$. Conclude that Im τ and Im Ad are normal Lie subgroups, and that

$$\operatorname{Im} \tau \cong \operatorname{Im} \operatorname{Ad} \cong G/Z_G .$$

What is the Lie algebra of Im Ad?

(vi) Consider a vector space V as a Lie group (under addition). Find Aut V.

9. **Semidirect products.** (i) Let H and K be Lie groups. Obtain a bijection between Lie group homomorphisms φ: $H \to$ Aut K and smooth maps ψ: $H \times K \to K$ that satisfy

$$\psi(a_1a_2 , b) = \psi(a_1 , \psi(a_2 , b)) \qquad \text{and} \qquad \psi(a, b_1b_2) = \psi(a, b_1)\, \psi(a, b_2).$$

(ii) Let φ: $H \to$ Aut K be a Lie group homomorphism. Define a multiplication in $H \times K$ by

$$(a, b) \cdot (a_1 , b_1) = (aa_1 , [\varphi(a_1^{-1})(b)] \cdot b_1).$$

Show that this makes $H \times K$ into a Lie group; it is called the *semidirect product* of H and K (via φ), and written $H \times_\varphi K$.

(iii) Let F and E be Lie algebras. An *action of F on E by derivations* is a homomorphism θ of F into the Lie algebra of derivations of E. Given such a homomorphism show that the multiplication in $F \oplus E$ defined by

$$[(h_1 , k_1), (h_2 , k_2)] = ([h_1 , h_2], [k_1 , k_2] + \theta(h_1)(k_2) - \theta(h_2)(k_1))$$

is a Lie product. This Lie algebra is called the *semidirect product* of F and E (via θ) and is written $F \oplus_\theta E$.

(iv) Show that the Lie algebra of a semidirect product of Lie groups is a semidirect product of the Lie algebras.

(v) Show that a Lie group G is a semidirect product of Lie groups H and K if and only if: (a) H and K are closed Lie subgroups of G, (b) K is normal in G, (c) $H \cap K = \{e\}$, and (d) every element in G is of the form ab, $a \in H$, $b \in K$.

(vi) *Tangent group.* Show that the Lie group T_G is the semidirect product $G \times_{\mathrm{Ad}} E$.

(vii) *Affine group.* A map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *affine* if it is of the form $\varphi(x) = \psi(x) + a$, where ψ is linear and $a \in \mathbb{R}^n$. Show that the affine bijections of $\mathbb{R}^n$ form a group under composition. Identify this with the group $GL(\mathbb{R}^n) \times_T \mathbb{R}^n$, where T is the standard representation of $GL(\mathbb{R}^n)$ in $\mathbb{R}^n$.

10. The group $SU(2)$. Make $\mathbb{C}^2$ into a Hermitian space.

(i) Show that the complex linear transformations φ of $\mathbb{C}^2$ that satisfy $\varphi = \lambda\tau$ ($\lambda \in \mathbb{R}$, $\tau \in SU(2)$) form a real four-dimensional subalgebra A of $L(2; \mathbb{C})$. Show that A is isomorphic to $\mathbb{H}$.

(ii) Obtain an isomorphism $SU(2) \cong Q(1)$ of Lie groups. Conclude that $SU(2)$ is diffeomorphic to S^3.

11. The group $SO(4)$. (i) Show that $SO(4)$ contains two normal subgroups H_1 and H_2 each isomorphic to $SO(3)$.

(ii) Show that $SO(4) = (H_1 \times H_2)/\mathbb{Z}_2$.

(iii) Let $\tau \in SO(4)$ and suppose $\tau \neq \pm\iota$. Write $\mathbb{R}^4 = F \oplus F^\perp$, where F and $F^\perp$ are planes, oriented so that $\mathbb{R}^4$ has the induced orientation, and stable under τ. Let τ_F and $\tau_{F^\perp}$ be the restrictions of τ to F and $F^\perp$, and let θ and $\theta^\perp$ ($-\pi < \theta$, $\theta^\perp \leqslant \pi$) be the corresponding rotation angles. Show that $\tau \in H_1$ if and only if $\theta = \theta^\perp$ and $\tau \in H_2$ if and only if $\theta = -\theta^\perp$.

12. General Lorentz groups. Give $\mathbb{R}^{p+q}$ an inner product of type (p, q) (i.e., for some orthogonal basis $x_1, \ldots, x_p, y_1, \ldots, y_q$, $\langle x_i, x_i\rangle = 1$, $\langle y_j, y_j\rangle = -1$). An *isometry* of $(\mathbb{R}^{p+q}, \langle\ ,\ \rangle)$ is a linear isomorphism φ which satisfies $\langle\varphi(x), \varphi(y)\rangle = \langle x, y\rangle$, $x, y \in \mathbb{R}^{p+q}$.

(i) Show that the isometries of $\mathbb{R}^{p+q}$ form a closed Lie subgroup of $GL(p + q; \mathbb{R})$; it is denoted by $O(p, q)$ and is called the *Lorentz group of type* (p, q).

(ii) Show that $\det \varphi = \pm 1$, $\varphi \in O(p, q)$. Show that, if p and q are > 0, then $O(p, q)$ is not compact.

(iii) Find the Lie algebra of $O(p, q)$.

13. The Lorentz group. The Lie group $O(3, 1)$ is called simply the *Lorentz group*.

(i) Show that $O(3, 1)$ has four components. Characterize its 1-component, $O^0(3, 1)$.

(ii) Show that an inner product of type $(3, 1)$ is defined in the space, S, of selfadjoint mappings of $\mathbb{C}^2$ by

$$\langle \sigma, \tau \rangle = \tfrac{1}{2}(\operatorname{tr} \sigma \circ \tau - \operatorname{tr} \sigma \cdot \operatorname{tr} \tau), \qquad \sigma, \tau \in S.$$

Conclude that $O(3, 1)$ is the group of isometries of S.

(iii) Show that $SL(2; \mathbb{C})$ is the universal covering group of $O^0(3, 1)$ and that the covering projection is given by

$$(\pi\alpha)(\sigma) = \alpha \circ \sigma \circ \tilde{\alpha}, \qquad \alpha \in SL(2; \mathbb{C}), \quad \sigma \in S.$$

Find the kernel of π.

14. The Möbius group. (i) Show that the fractional linear transformations

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1,$$

form a group M of smooth transformations of the Riemann sphere S^2.

(ii) Show that M is a Lie group with $SL(2; \mathbb{C})$ as universal covering group.

(iii) Show that M is isomorphic to the Lorentz group $O^0(3, 1)$.

(iv) Show that the map $M \times S^2 \to S^2$ given by $(T, z) \mapsto T(z)$ is smooth.

15. Elliptic isometries: Let M_E denote the subset of M consisting of the transformations of the form

$$T(z) = \frac{az + b}{-\bar{b}z + \bar{a}}, \qquad |a|^2 + |b|^2 = 1.$$

(i) Show that Me is a closed subgroup of M, diffeomorphic to $\mathbb{R}P^3$.

(ii) Define a Riemannian metric, g, in S^2 such that $g(z; \zeta_1, \zeta_2) = (1 + |z|^2)^{-2} \operatorname{Re}(\zeta_1 \bar{\zeta}_2)$; it is called the *elliptic metric* in S^2. Show that M_E acts via isometries on S^2 with respect to g.

(iii) Let σ be the stereographic projection of the 2-sphere of diameter 1 from the north pole to the tangent plane T_S at the south pole. Identify T_S with the complex plane with elliptic metric. Show that σ is an isometry. Conclude that $M_E \cong SO(3)$.

16. Hyperbolic isometries: Let M_H be the subset of M consisting of the transformations of the form

$$T(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \qquad |a|^2 - |b|^2 = 1.$$

(i) Show that M_H is a closed subgroup of M, diffeomorphic to the manifold obtained from the hyperboloid $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$ in $\mathbb{R}^4$ by identifying antipodal points.

(ii) Show that M_H is a group of isometries of the unit disc $\Omega(|z| < 1)$ with respect to the Riemannian metric given by

$$g(z; \zeta_1, \zeta_2) = \operatorname{Re} \frac{\zeta_1 \bar{\zeta}_2}{(1 - |z|^2)^2}, \qquad |z| < 1.$$

The unit disc with this Riemannian metric is called the *hyperbolic plane*. Show that a fractional linear transformation is in M_H if and only if it maps Ω onto Ω.

(iii) Give $\mathbb{R}^3$ an inner product of type (2,1). Consider the hyperboloid H of vectors x satisfying $\langle x, x \rangle = -\frac{1}{4}$. Show that the inner product induces a positive definite Riemannian metric in H. Show that hyperbolic stereographic projection is an isometry of the lower shell of H with Ω. Conclude that $M_H = O^0(2, 1)$.

17. (i) Show that every homomorphism φ: $U(n) \to S^1$ has the form $\varphi(\tau) = (\det \tau)^p (\overline{\det \tau})^q$ ($p, q \in \mathbb{Z}$).

(ii) Show that the Lie groups $S^1 \times SU(n)$ and $U(n)$ are diffeomorphic, but not isomorphic. Construct a covering projection $S^1 \times SU(n) \to U(n)$.

(iii) Show that the Lie groups $\mathbb{Z}_2 \times SO(n)$ and $O(n)$ are always diffeomorphic, but isomorphic if and only if n is odd.

(iv) Let φ: $U(n) \to S^1$ be the homomorphism given by $\varphi(\tau) = \det \tau$. Show that there is no homomorphism ψ: $S^1 \to U(n)$ such that $\varphi \circ \psi = \iota$.

(v) Let φ: $O(n) \to S^0$ be the homomorphism given by $\varphi(\tau) = \det \tau$. Show that there is a homomorphism ψ: $S^0 \to O(n)$ such that $\varphi \circ \psi = \iota$ if and only if n is odd.

(vi) Show that

$$U(n) = SO(2n) \cap \mathrm{Sp}(n; \mathbb{R}) = SO(2n) \cap GL(n; \mathbb{C}).$$

(vii) Show that $Q(n) = SO(4n) \cap GL(n; \mathbb{H})$.

18. Let P be a representation by isometries of a compact connected Lie group G in a Euclidean n-space W. Fix a vector $v_0 \in W$ and let $K = \{a \in G \mid P(a)\, v_0 = v_0\}$. Assume that K is connected and that $G \cdot v_0$ contains an orthonormal basis.

(i) Construct a smooth map α: $G/K \to W$, such that $\alpha(\bar{a}) = P(a)v_0$.

(ii) Show that there is an r-form Δ on G/K ($r = \dim G/K$) such that

$$T_a{}^* \Delta = \Delta, \quad a \in G, \qquad \text{and} \qquad \int_{G/K} \Delta = 1,$$

where T is the action of G on G/K.

(iii) If φ: $W \to W$ is a linear map show that

$$(1/n) \operatorname{tr} \varphi = \int_{G/K} f \cdot \Delta,$$

where $f(x) = \langle \varphi\alpha(x), \alpha x \rangle$, $x \in G/K$.

(iv) Apply this to the natural representation of $SO(n)$ in $\wedge W$ to obtain an integral formula for the characteristic coefficients of a linear transformation of W.

19. Clifford algebras. Let $(E, \langle\ ,\ \rangle)$ be an n-dimensional space with a symmetric bilinear function $\langle\ ,\ \rangle$. Let $\mathscr{I} \subset \otimes E$ be the ideal generated by elements of the form $x \otimes x - \langle x, x \rangle$, $x \in E$. The factor algebra $C_E = \otimes E/\mathscr{I}$ is called the *Clifford algebra* of $(E, \langle\ ,\ \rangle)$; the canonical projection is written π: $\otimes E \to C_E$.

(i) Obtain a $\mathbb{Z}_2$-gradation $C_E = C_E^0 \oplus C_E^1$ of C_E from the decomposition $\otimes E = \sum_{p \text{ even}} \otimes^p E \oplus \sum_{p \text{ odd}} \otimes^p E$.

(ii) Show that $\pi\mid_E$ is injective and identify E with $\pi(E)$.

(iii) Show that C_E satisfies a universal property that determines it uniquely.

(iv) If the bilinear function $\langle\ ,\ \rangle$ is zero, show that $C_E = \wedge E$.

(v) Assume a direct decomposition $E = F \oplus H$ such that $\langle y, z\rangle = 0$, $y \in F$, $z \in H$. Prove that $C_E \cong C_F \otimes C_H$ (as $\mathbb{Z}_2$-graded algebras), where the right-hand side is the anticommutative tensor product. Conclude that $\dim C_E = 2^n$.

(vi) Let C_k be the subspace of C_E spanned by the vectors 1, $x_{i_1} \bullet \cdots \bullet x_{i_j}$ ($x_\nu \in E, j \leqslant k$). Show that $C_k \cdot C_l \subset C_{k+l}$. Obtain an algebra structure in $\bigoplus_k C_k/C_{k-1}$, and show that this algebra is isomorphic to $\wedge E$.

(vii) Let C_n^+ (respectively, C_n^-) denote the Clifford algebra of an n-dimensional space with a positive (respectively, negative) definite inner product. Establish isomorphisms

$$C_1^- \cong \mathbb{C}, \qquad C_1^+ \cong \mathbb{R} \oplus \mathbb{R}, \qquad C_2^- \cong \mathbb{H}, \qquad C_2^+ \cong L_{\mathbb{R}^2},$$
$$C_3^- \cong \mathbb{H} \otimes \mathbb{C}, \qquad C_3^+ \cong L(\mathbb{C}^2), \qquad C_4^- \cong \mathbb{H} \otimes \mathbb{H}, \qquad C_4^+ \cong \mathbb{H} \otimes \mathbb{H}.$$

20. The groups Pin and Spin. Let C_E be the Clifford algebra of an n-dimensional Euclidean space and let ω be the involution of C_E given by $\omega(x) = x$, $x \in C_E^0$ and $\omega(x) = -x$, $x \in C_E^1$. Let C_E^* be the group of units of C_E. Let L be C_E, regarded as a Lie algebra.

(i) Show that C_E^* is a Lie group with Lie algebra L. What is the adjoint representation?

(ii) Show that a representation P of C_E^* in C_E is given by

$$P(u)(v) = \omega(u) \cdot v \cdot u^{-1}, \qquad u \in C_E^*, \quad v \in C_E.$$

Let CL_E be the subgroup of C_E^* consisting of those u such that $P(u)$ stabilizes E. Show that CL_E is a closed Lie subgroup of C_E^*; it is called the *Clifford group*.

(iii) Show that CL_E acts in E by isometries. Obtain a surjective homomorphism,

$$CL_E \to O(n),$$

with kernel $\{\lambda e \mid \lambda \in \mathbb{R}\}$.

(iv) Let $\mathrm{Pin}(n)$ be the subgroup of CL_E consisting of the elements a that satisfy $\det \mu(a) = 1$ ($\mu(a)$ is left multiplication of C_E by a). Obtain an exact sequence of Lie groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(n) \to O(n) \to 1.$$

(v) Let Spin (n) = Pin$(n) \cap C_E^0$. Obtain an exact sequence of Lie groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\pi} SO(n) \to 1, \qquad n \geqslant 3.$$

If $n \geqslant 3$ show that Spin(n) is a compact connected Lie group.

(vi) Show that π is the universal covering projection for $SO(n)$, $n \geqslant 3$.

21. The Lie algebra of Spin (*n*) Let L denote the Lie algebra obtained from the associative algebra C_E by setting

$$[a, b] = ab - ba, \qquad a, b \in C_E.$$

(i) Show that a representation θ of L in C_E is given by

$$\theta(a)u = au - ua, \qquad a \in L, \quad u \in C_E$$

and that each $\theta(a)$ is a derivation.

(ii) Let F be the subalgebra of L consisting of those vectors a for which E is stable under $\theta(a)$. Show that, for $a \in F$, the transformations $\theta(a) : E \to E$ are skew. Show that if vectors $a, b \in E$ satisfy $\langle a, b\rangle = 0$, then $ab \in F$. Conclude that

$$\dim F = \tfrac{1}{2}n(n-1).$$

(iii) Show that the map $\theta : F \to \mathrm{Sk}(E)$ is an isomorphism of Lie algebras. Identify F with the Lie algebra of Spin(n) and Sk(E) with the Lie algebra of $SO(n)$ and show that $\theta = \pi'$, where π: Spin$(n) \to SO(n)$ is the covering projection (cf. problem 26).

22. (i) Let $\pi: \tilde{G} \to G$ be a covering projection with $\tilde{G}$ compact and connected. Show that if T is a maximal torus of G, then $\pi^{-1}(T)$ is a maximal torus of $\tilde{G}$. Conclude that $\tilde{G}/\pi^{-1}(T)$ is diffeomorphic to G/T.

(ii) Obtain diffeomorphisms

$$SO(3)/SO(2) \cong S^2, \qquad SU(2)/U(1) \cong S^2$$

and

$$SO(4)/(SO(2) \times SO(2)) \cong S^2 \times S^2.$$

23. Power maps. Let G be a compact connected Lie group. Use the Weyl integration formula to conclude that the power map P_k has degree k^r, where r is the dimension of a maximal torus (cf. problem 8, Chap. I).

24. Conjugacy classes of subgroups. Two subgroups H and K of G are called *conjugate* if there exists an element $a \in G$ such that $H = aKa^{-1}$. The set of all subgroups conjugate to a given subgroup K is called the *conjugacy class of* K and is denoted by (K). If K is conjugate to a subgroup of H, then we write $(K) \leqslant (H)$.

(i) Suppose K_i $(i = 1, 2, 3)$ are Lie subgroups of G such that $(K_1) \leqslant (K_2)$ and $(K_2) \leqslant (K_3)$. Conclude that $(K_1) \leqslant (K_3)$.

(ii) Assume K and H are Lie subgroups of G and K has finitely many components. Show that then the relations $(K) \leqslant (H)$ and $(H) \leqslant (K)$ imply that $(H) = (K)$. Construct an example to show that the hypothesis that K has finitely many components is not superfluous.

25. Let T_1 and T_2 be maximal tori in G (G, compact and connected).

(i) Show that $aT_1a^{-1} = T_2$ for some element $a \in G$ which commutes with all elements of $T_1 \cap T_2$.

(ii) Let S be a subset of T_1 and assume that for some $b \in G$, $bSb^{-1} \subset T_1$. Show that there is an element $n \in N(T_1)$, such that $nxn^{-1} = bxb^{-1}$, $x \in S$.

26. Lie groups of rank 1. Suppose G is a compact, connected non-abelian Lie group with Lie algebra E. Assume that G contains a maximal torus, T, of dimension 1. Show that $\dim G = 3$.

Hint: Proceed as follows:

(i) Show that every 1-parameter subgroup of G is a circle.

(ii) Give E an invariant inner product and let S denote the unit sphere in E. Fix a suitable $h \in S$, and show that the map $x \mapsto (\mathrm{Ad}\, x)h$ induces a diffeomorphism $G/T \xrightarrow{\cong} S$.

(iii) Use (ii) to show that $W_G = \mathbb{Z}_2$. In particular, find an element $n \in N(T)$ such that $nxn^{-1} = x^{-1}$, $x \in T$. Conclude that the maps $T \to G$ given by $x \mapsto x$ and $x \mapsto x^{-1}$ are homotopic.

(iv) Let i: $T \to G$ be the inclusion. Show that $i^{\#}H^{+}(G) = 0$. Conclude that the circle bundle $(G, \pi, G/T, T)$ has nontrivial Euler class. Conclude that $\dim G/T = 2$ and $\dim G = 3$.

27. Assume that G is compact and connected. Show that every Lie subgroup of G which contains a maximal torus of G is compact. (*Hint*: cf. problem 5.)

28. Roots I. Let G be compact and connected Lie group with maximal torus T. Denote the corresponding Lie algebras by E and F. Equip E with an invariant inner product. Denote by $\mathrm{Ad}_{G,T}$ the restriction of the map, Ad: $G \to SO(E)$, to T.

(i) Construct a direct decomposition $E = F \oplus \sum_\mu E_\mu$ where the E_μ are mutually perpendicular two-dimensional subspaces each stable under the action of T.

(ii) Show that the induced homomorphisms $\mathrm{Ad}_\mu : T \to SO(E_\mu)$ are surjective.

(iii) Fix an orthonormal basis a_μ, b_μ of E_μ. Show that a linear function $\theta_\mu \in F^*$ is determined by the condition: for $h \in F$, the matrix of $\mathrm{Ad}_\mu(\exp h)$ with respect to (a_μ, b_μ) is given by

$$\begin{pmatrix} \cos 2\pi\theta_\mu(h) & -\sin 2\pi\theta_\mu(h) \\ \sin 2\pi\theta_\mu(h) & \cos 2\pi\theta_\mu(h) \end{pmatrix}.$$

(iv) Show that θ_μ restricts to a linear function $\Gamma_T \to \mathbb{Z}$ (cf. problem 7, Chap. I). It is called an *integral root* for E. Show that θ_μ depends only on the orientation of E_μ determined by (a_μ, b_μ), and that reversing the orientation changes θ_μ to $-\theta_\mu$. The set $\{\pm\theta_\mu\}$ is called the *set of integral roots for G.*

(v) If $E = E'$, show that the θt span F^* (cf. problem 3).

(vi) Generalize (iii) and (iv) to any representation of G.

29. Roots II. We adopt the notation of problem 28.

(i) If A and B are subspaces of E, then $[A, B]$ denotes the linear closure of the vectors $[h, k]$ $(h \in A, k \in B)$. Show that $[E_\mu, E_\mu]$ is a T-stable space of dimension $\leqslant 1$ and conclude that $[E_\mu, E_\mu] \subset F$.

(ii) Fix an integral root θ and let L be the sum of subspaces E_μ such that $\theta_\mu = \lambda_\mu \theta$ for some $\lambda_\mu \in \mathbb{R}$. Show that $F \oplus L$ is a subalgebra of E. Show that $\ker\theta$ is the centre of $F \oplus L$. Let F_1 be the one-dimensional subspace of F orthogonal to $\ker\theta$ and show that

$$(F_1 \oplus L)' = F_1 \oplus L.$$

(iii) Construct a compact Lie group G_1 with Lie algebra $F_1 \oplus L$. Show that F_1 is the Lie algebra of a maximal torus.

(iv) Conclude from (iii) (via problem 26) that $\dim L = 2$. Hence show that no θ_ν is a scalar multiple of another θ_μ. Thus prove that the decomposition in problem 28(i) is unique.

(v) Show that there are only finitely many Lie subgroups of G that contain T.

(vi) Show that there is an open dense subset O of G such that, for $a \in O$,

$$T \cap aTa^{-1} = Z_G .$$

30. Find the Weyl groups for $SO(2n)$, $SO(2n + 1)$ and $Q(n)$.

31. Let $\varphi: U(p_1) \times \cdots \times U(p_k) \to U(n)$ be a Lie group inclusion, where $p_1 + \cdots + p_k = n$. Find an automorphism of $U(n)$ that carries φ to the standard inclusion.

32. Let G be a compact Lie group and let $a \in G$.

(i) Show that a represents an element $\bar{a}$ of finite order in G/G^0.

(ii) If $\bar{a}$ has order p, show that a belongs to an abelian subgroup of G of the form $T \times \mathbb{Z}_p$, where T is a maximal torus in G.

33. Let G be a compact connected Lie group and let T be a maximal torus. Show that T is not properly contained in any abelian subgroup of G. Conclude that the centralizer of any torus S is connected.

34. **Regular points.** Let G be an n-dimensional compact connected Lie group. A point $a \in G$ is called *regular* if and only if it is contained in precisely one maximal torus. Otherwise a is called *singular*. Denote the set of regular (respectively, singular) points by R_G (respectively, S_G).

(i) Show that $a \in G$ is regular if and only if it is a regular value for the map ψ of sec. 2.17.

(ii) Show that R_G is an open dense subset of G, stable under conjugation by elements of G.

(iii) Show that ψ restricts to a covering projection,

$$\pi: G/T \times (T \cap R_G) \to R_G ,$$

with fibre the Weyl group.

(iv) Fix a maximal torus, T, and consider the representations Ad_μ of T in E_μ ($\mu = 1,\ldots, m$) (cf. problem 28). Denote the kernel of Ad_μ by U_μ . Let $a \in G$ and assume that

$$a \in U_1 \cap \cdots \cap U_p \qquad \text{and} \qquad a \notin U_{p+1} \cup \cdots \cup U_m .$$

Show that the Lie algebra of the normalizer, N_a, is given by

$$F \oplus E_1 \oplus \cdots \oplus E_p .$$

Conclude that

$$\dim N_a = r + 2p \qquad (r = \dim T).$$

Conclude that a is regular if and only if it is contained in no U_μ.

(v) Construct a compact $(n-3)$-manifold, M, and a smooth map $\varphi: M \to G$ such that $\operatorname{Im} \varphi = S_G$.

35. $\pi_2(G) = 0$. Let G be a compact connected Lie group. Show that any smooth map $\varphi: S^2 \to G$ is homotopic to the constant map.

Hint: Proceed as follows:

(i) Replace G by $S^3 \times S^3 \times G$ and reduce to the case that φ is an embedding with trivial normal bundle. Obtain a neighbourhood of $\varphi(S^2)$ which is diffeomorphic to $S^2 \times \mathbb{R}^{n-2}$.

(ii) Deform φ so that $\varphi(S^2)$ consists of regular points (cf. problem 34).

(iii) Lift φ to a map $\tilde{\varphi}: S^2 \to G/T \times T$.

(iv) Deform $\tilde{\varphi}$ into a map $S^2 \to G/T \times \{e\}$.

Chapter III

Transformation Groups

In this chapter G denotes a fixed Lie group with unit element e and Lie algebra E. M and N denote smooth manifolds.

§1. Action of a Lie group

3.1. Definition: A *right action* of G on a manifold M (or a set V) is a smooth map

$$T: M \times G \to M$$

(or a set map $V \times G \to V$), written $(z, a) \mapsto z \cdot a$, and satisfying

$$z \cdot (ab) = (z \cdot a) \cdot b \quad \text{and} \quad z \cdot e = z, \qquad a, b \in G, \quad z \in M.$$

The group G is said to act *transitively* on M if, for every two points $z_1, z_2 \in M$, there is an element $a \in G$ such that $z_1 \cdot a = z_2$.

An action T determines the diffeomorphisms T_a ($a \in G$) of M given by

$$T_a(z) = z \cdot a = T(z, a).$$

(Note that $T_a^{-1} = T_{a^{-1}}$.) T_a is called *right translation by a*.

On the other hand, to each $z \in M$, corresponds the smooth map $A_z: G \to M$ given by

$$A_z(a) = z \cdot a, \qquad a \in G.$$

It satisfies the relations

$$T_b \circ A_z = A_z \circ \rho_b \quad \text{and} \quad A_{z \cdot b} = A_z \circ \lambda_b = T_b \circ A_z \circ \tau_b, \qquad b \in G, \quad z \in M$$

(τ_b denotes conjugation in G by b).

Now assume $\hat{T}$ is a right action of G on N. Then a smooth map $\varphi: M \to N$ is called *equivariant* with respect to T and $\hat{T}$ if the diagram

$$\begin{tikzcd} M \times G \arrow[r, "T"] \arrow[d, "\varphi \times \iota"'] & M \arrow[d, "\varphi"] \\ N \times G \arrow[r, "\hat{T}"'] & N \end{tikzcd}$$

commutes. This is equivalent to each of the following three conditions

$$\varphi(z \cdot a) = \varphi(z) \cdot a, \qquad z \in M, \quad a \in G,$$

$$\varphi \circ T_a = \hat{T}_a \circ \varphi, \qquad a \in G,$$

and

$$\varphi \circ A_z = \hat{A}_{\varphi(z)}, \qquad z \in M.$$

(For $y \in N$, $\hat{A}_y: G \to N$ is the map $a \mapsto y \cdot a$.)

A *left action* of G on M is a smooth map

$$T: G \times M \to M,$$

written $T(a, z) = a \cdot z$, and such that

$$(ab) \cdot z = a \cdot (b \cdot z) \qquad \text{and} \qquad e \cdot z = z, \qquad a, b \in G, \quad z \in M.$$

The diffeomorphism $T_a: z \mapsto a \cdot z$ of M is called *left translation by a*. The smooth maps $A_z: G \to M$ ($z \in M$) given by

$$A_z(a) = a \cdot z$$

satisfy

$$T_b \circ A_z = A_z \circ \lambda_b \qquad \text{and} \qquad A_{b \cdot z} = A_z \circ \rho_b = T_b \circ A_z \circ \tau_b^{-1}.$$

Finally, if $\hat{T}$ is a left action of G on N, then $\varphi: M \to N$ is called *equivariant* if

$$\varphi(a \cdot z) = a \cdot \varphi(z), \qquad a \in G, \quad z \in M.$$

3.2. Examples: 1. The multiplication map $\mu: G \times G \to G$ of a Lie group G is both a left and right action of G on itself. The left and right translations by $a \in G$ are simply λ_a and ρ_a.

2. The group $G \times G$ acts from the left on G by

$$T((a, b), z) = azb^{-1}, \qquad (a, b) \in G \times G, \quad z \in G.$$

3. A right action, $\tilde{T}$, of G on $M \times G$ (M, any manifold) is given by

$$\tilde{T}((z, a), b) = (z, ab).$$

If T is *any* right action of G on M, then T is equivariant with respect to $\tilde{T}$ and T.

4. A left action of G on G is given by

$$a \cdot z = aza^{-1}.$$

5. A representation, P, of G in a vector space V defines a left action of G on V:

$$a \cdot v = P(a)v, \qquad a \in G, \quad v \in V.$$

6. Assume that a Lie group H acts from the left on a Lie group G. H is said to *act via homomorphisms*, if each map T_a: $G \to G$ ($a \in H$) is a homomorphism (and hence an automorphism) of G. Assuming that H acts on G via homomorphisms, define a multiplication on the product manifold $H \times G$ by

$$\mu((a, x), (b, y)) = (ab, T_b^{-1}(x)y), \qquad a, b \in H, \quad x, y \in G.$$

It is easy to verify that this multiplication makes $H \times G$ into a Lie group. It is called the *semidirect product of* H *and* G (with respect to the action T) and is denoted by $H \times_T G$. If the action, T, is *trivial*, ($T_a = \iota$, $a \in H$), the semidirect product is simply the direct product. In any case, $H \times e$ is a closed subgroup of $H \times_T G$, while $e \times G$ is a closed *normal* subgroup.

7. If T: $M \times G \to M$ is an action of G on M, then

$$dT: T_M \times T_G \to T_M$$

is an action of the tangent group T_G (cf. Example 5, sec. 1.4) on T_M. In particular, identify G with the zero vectors in T_G to obtain an action

$$T_M \times G \to T_M$$

of G on T_M. It is given explicitly by

$$\xi \cdot a = dT_a(\xi), \qquad \xi \in T_M, \quad a \in G.$$

8. If $M \times G \to M$ is an action of G on M, a subset $N \subset M$ is called *stable* if

$$z \cdot a \in N, \qquad z \in N, \quad a \in G.$$

If N is stable, the action restricts to a set map $N \times G \to N$. In particular, if N is a stable submanifold of M, this map is smooth (cf. Proposition VI, sec. 3.10, volume I) and hence it is a smooth action of G on N.

As an example, suppose $P: G \to O(V)$ represents G by isometries in a Euclidean space V. Then the unit sphere S of V is stable, and so the linear action of G in V restricts to an action $G \times S \to S$.

9. A right action, $T_R: M \times G \to M$, determines an *associated left action*, T_L, given by

$$T_L(a, z) = T_R(z, a^{-1}), \qquad z \in M, \quad a \in G.$$

3.3. Action on a homogeneous space. Let K be a closed subgroup of G and consider the homogeneous space G/K of left cosets. Then a left action T of G on G/K is given by

$$T(a, \bar{x}) = a \cdot \bar{x}, \qquad a \in G, \quad \bar{x} \in G/K$$

(cf. sec. 2.11). The projection $\pi: G \to G/K$ is equivariant with respect to the left action of G on itself, and T. The action of G on G/K is transitive. In fact, let $\bar{x}_1 = \pi x_1$ and $\bar{x}_2 = \pi x_2$ be arbitrary and set $a = x_2 x_1^{-1}$. Then $a \cdot \bar{x}_1 = \bar{x}_2$.

Similarly, a right action of G is defined on the space of right cosets.

Next consider the normalizer N_K of K (cf. Example 4, sec. 2.4). A right action

$$S: G/K \times N_K \to G/K$$

is given by

$$S(\bar{x}, a) = \overline{xa}, \qquad x \in G, \quad a \in N_K.$$

(Since $a \in N_K$, this map is well defined.)

To see that it is smooth, observe that the diagram

$$\begin{tikzcd} G \times N_K \arrow[r, "\mu"] \arrow[d, "\pi \times \iota"'] & G \arrow[d, "\pi"] \\ G/K \times N_K \arrow[r, "S"'] & G/K \end{tikzcd},$$

commutes and recall that π makes G/K into a quotient manifold of G. The diagram also shows that the projection π is equivariant with respect to the right actions of N_K on G and on G/K.

Finally, since K is a closed normal subgroup of N_K, we can form the factor group N_K/K. The action S factors over the projection

$$\rho: N_K \to N_K/K$$

to give a smooth commutative diagram

$$\begin{tikzcd}
G/K \times N_K \arrow[rr, "S"] \arrow[dr, "\iota \times \rho"'] & & G/K \\
& G/K \times N_K/K. \arrow[ur, "\bar{S}"'] &
\end{tikzcd}$$

Thus $\bar{S}$ is a right action of N_K/K on G/K.

§2. Orbits of an action

In this article, $T: M \times G \to M$ denotes a right action of G on M.

3.4. The isotropy subgroup. Every point $z \in M$ determines the closed subgroup $G_z \subset G$ given by

$$G_z = \{a \in G \mid z \cdot a = z\}.$$

Since G_z is closed, it is a Lie subgroup of G (cf. Theorem I, sec. 2.1). It is called the *isotropy subgroup at* z. If $G_z = \{e\}$ (respectively, G_z is discrete), for each $z \in M$, the action is called *free* (respectively, *almost free*).

Proposition I: The Lie algebra E_z of the isotropy group G_z is given by

$$E_z = \ker(dA_z)_e .$$

Proof: Since the restriction of A_z to G_z is constant, it follows that $E_z \subset \ker(dA_z)_e$. Conversely, assume that $h \in \ker(dA_z)_e$. To show that $h \in E_z$ we must prove that $\exp th \in G_z$, $t \in \mathbb{R}$.

But the path in M given by $\beta(t) = z \cdot \exp th$ satisfies

$$\dot{\beta}(t) = (dA_z \circ R_{\exp th})(h) = (dT_{\exp th} \circ dA_z)(h) = 0, \qquad t \in \mathbb{R},$$

(cf. sec. 3.1). It follows that $z \cdot \exp th = z$ and so $\exp th \in G_z$.

Q.E.D.

Corollary: The action is almost free if and only if each $(dA_z)_e$ is injective.

3.5. Orbits. For $z \in M$ the set $z \cdot G$ $(= \operatorname{Im} A_z)$ is called the *orbit of* G *through* z. M is the disjoint union of its orbits. Clearly, if G acts transitively on M, then M consists of a single orbit.

Let z, $z \cdot a$ be points in the same orbit. Then $G_{z \cdot a} = a^{-1}G_z a$. In particular, if the action is transitive, any two isotropy groups are conjugate.

Next observe that the relation $A_z(ab) = A_z(a) \cdot b$ shows that A_z

factors over the projection $\pi\colon\ G \to G_z\backslash G$ to yield a commutative diagram

$$\begin{tikzcd} G \arrow[r, "A_z"] \arrow[d, "\pi"'] & M \\ G_z\backslash G \arrow[ur, "\bar{A}_z"'] & \end{tikzcd}$$

Since $G_z\backslash G$ is a quotient manifold of G under π, the map $\bar{A}_z$ is smooth. Moreover, $\bar{A}_z$ is equivariant with respect to the right actions of G on $G_z\backslash G$ and M.

Proposition II: $\bar{A}_z$ embeds the homogeneous space $G_z\backslash G$ into M, with image the orbit $z \cdot G$.

Proof: $\bar{A}_z$ is obviously injective, and has image $z \cdot G$. Thus we need only show that the linear maps

$$(d\bar{A}_z)_{\bar{a}}\colon T_{\bar{a}}(G_z\backslash G) \to T_{z\cdot a}(M), \qquad \bar{a} \in G_z\backslash G,$$

are injective. In view of the equivariance of $\bar{A}_z$, it is sufficient to consider the case $\bar{a} = \bar{e}$. But it follows from Proposition I, sec. 3.4, and Corollary I of sec. 2.11, that

$$\ker(dA_z)_e = E_z = \ker(d\pi)_e .$$

Hence $(d\bar{A}_z)_{\bar{e}}$ is injective.

Q.E.D.

Corollary: If G acts transitively on M, then $\bar{A}_z$ is a diffeomorphism of $G_z\backslash G$ onto M.

Proof: Apply Proposition IV, sec. 3.8, volume I.

3.6. Examples. 1. Consider the right action T of G on itself by conjugation,

$$T(z, a) = a^{-1}za, \qquad z, a \in G.$$

The orbits of G under this action are called the *conjugacy classes of* G. Two elements z_1, z_2 are in the same orbit if and only if for some $a \in G$

$$a^{-1}z_1a = z_2 .$$

In this case they are called *conjugate*.

On the other hand, the isotropy subgroup at $a \in G$ is the normalizer N_a. Thus Proposition II, sec. 3.5, gives an embedding of $N_a \backslash G$ into G, with image the conjugacy class of a.

2. Let V be an n-dimensional Euclidean space. A left action T of $SO(n)$ on V is defined by

$$T(\sigma, z) = \sigma(z), \qquad \sigma \in SO(n), \quad z \in V.$$

The orbit of a point $a \in V$ $(a \neq 0)$ is the sphere $\{x \in V \mid |x| = |a|\}$, while the orbit of 0 consists only of 0.

The action T restricts to a transitive action of $SO(n)$ on the unit sphere S^{n-1}. The isotropy subgroup of a point $x \in S^{n-1}$ is the subgroup $SO(x^\perp)$, where $x^\perp$ denotes the orthogonal complement of x. Hence T induces an equivariant diffeomorphism (cf. the corollary to Proposition II, sec. 3.5):

$$SO(n)/SO(n-1) \xrightarrow{\cong} S^{n-1}, \qquad n \geqslant 2.$$

3. By replacing the Euclidean space, V, of Example 2 with a Hermitian space, W, we obtain an action of $U(n)$ with orbits the spheres of W. In particular, this yields an equivariant diffeomorphism

$$U(n)/U(n-1) \xrightarrow{\cong} S^{2n-1}, \qquad n \geqslant 1.$$

The action of $U(n)$ on W induces an action of the special unitary group $SU(n)$ which restricts to a transitive action on S^{2n-1} for $n \geqslant 2$.

Finally, the use of a quaternionic space leads to equivariant diffeomorphisms

$$Q(n)/Q(n-1) \xrightarrow{\cong} S^{4n-1}, \qquad n \geqslant 1.$$

Proposition III: The groups $SO(n)$, $U(n)$, $SU(n)$, and $Q(n)$ are connected.

Proof: $SO(1)$ $(= \iota)$ is connected. Assume by induction that $SO(n-1)$ is connected $(n \geqslant 2)$. Then, in view of Example 2, $SO(n)/SO(n-1)$ is also connected. Since (cf. sec. 2.13)

$$(SO(n), \pi, SO(n)/SO(n-1), SO(n-1))$$

is a fibre bundle, it follows that $SO(n)$ is connected and the induction is closed.

The same argument, using Example 3 above shows that $U(n)$, $SU(n)$, and $Q(n)$ are connected.

Q.E.D.

Corollary: $O(n)$ has two components (cf. Example 2, sec. 2.5).

3.7. Embedding of orbits. Consider the injective map of sec. 3.5,

$$\bar{A}_z\colon G_z\backslash G \to M.$$

In general, the pair $(G_z\backslash G, \bar{A}_z)$ is not a submanifold of M as the following example shows: Let $\mathbb{R}$ act on the 2-torus T^2 by setting

$$T_t\pi(x, y) = \pi(x + at, y + bt), \qquad t, x, y \in \mathbb{R},$$

where $\pi\colon \mathbb{R}^2 \to T^2$ denotes the projection and b/a is irrational. Then each orbit is dense in T^2 and so the orbits are not submanifolds of T^2.

Nonetheless we have

Theorem I: With the notation above, let

$$\begin{tikzcd} G_z\backslash G \arrow[rr, "\bar{A}_z"] & & M \\ & N \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

be a commutative diagram. Then σ is smooth if and only if τ is.

For the proof of this theorem we first establish four lemmas. In view of Corollary II to Theorem II, sec. 2.11, we can find a submanifold W_1 of G such that $e \in W_1$, and the projection $\pi\colon G \to G_z\backslash G$ restricts to a diffeomorphism of W_1 onto a neighbourhood of $\bar{e}$.

Lemma I: There is a submanifold V of M containing z and a connected neighbourhood W of e in W_1, and a neighbourhood U of z in M such that T restricts to a diffeomorphism

$$\psi\colon V \times W \xrightarrow{\cong} U.$$

Proof: Choose a submanifold V_1 of M such that $z \in V_1$ and

$$T_z(M) = T_z(V_1) \oplus \operatorname{Im}(dA_z)_e .$$

Write $T_{(z,e)}(V_1 \times W_1) = T_z(V_1) \oplus T_e(W_1)$ and note that

$$dT(\xi, \eta) = \xi + (d\bar{A}_z \circ d\pi)(\eta), \qquad \xi \in T_z(V_1), \quad \eta \in T_e(W_1).$$

In view of Proposition II, sec. 3.5,

$$T_e(W_1) \xrightarrow[\cong]{d\pi} T_{\bar{e}}(G_z\backslash G) \xrightarrow[\cong]{d\bar{A}_z} \operatorname{Im}(dA_z)_e\,.$$

It follows that dT maps $T_{(z,e)}(V_1 \times W_1)$ isomorphically onto $T_z(M)$. The lemma follows (cf. Theorem I, sec. 3.8, volume I) for suitably small neighbourhoods $V \subset V_1$ and $W \subset W_1$.

Q.E.D.

Lemma II: Suppose that, in the notation of Lemma I, $\psi(y, b) = z \cdot a$ for some $y \in V$, $b \in W$, $a \in G$. Then

$$(d\psi)_{(y,b)}(T_b(W)) = (dA_z)(T_a(G)).$$

Proof: Set $c = ab^{-1}$. Since $\psi(y, b) = y \cdot b$, we have

$$y = z \cdot ab^{-1} = z \cdot c.$$

Since the restriction of ψ to $\{y\} \times W$ is simply A_y $(= A_{z\cdot c})$, it follows that

$$\begin{aligned}(d\psi)_{(y,b)}(T_b(W)) &= (dA_{z\cdot c})(T_b(W)) \subset (dA_{z\cdot c})(T_b(G))\\ &= (dA_z)(T_a(G)).\end{aligned}$$

Moreover, combining Proposition II, sec. 3.5, with Lemma I, we obtain

$$\dim(d\psi)_{(y,b)}(T_b(W)) = \dim W = \dim(G_z\backslash G) = \dim dA_z(T_a(G)).$$

The lemma follows.

Q.E.D.

Lemma III: Let S denote the subset of V given by

$$S = \{y \in V \mid \psi(y, b) \in z \cdot G \text{ for some } b \in W\}.$$

Then S is countable.

Proof: Consider the open subset $O \subset G_z\backslash G$ given by (cf. Lemma I for U)

$$O = \bar{A}_z^{-1}(U).$$

Let φ: $O \to V$ be the composite given by

$$O \xrightarrow{\bar{A}_z} U \xrightarrow[\cong]{\psi^{-1}} V \times W \xrightarrow{\pi_V} V.$$

We show that $d\varphi = 0$.

In fact, let $\bar{a} \in O$ and let $\xi \in T_{\bar{a}}(O)$. Then we can write $z \cdot a = \psi(y, b)$ for some $y \in V$, $b \in W$. By Lemma II there exists an $\eta \in T_b(W)$ such that

$$(d\bar{A}_z)(\xi) = (d\psi)_{(y,b)}(\eta).$$

This yields

$$(d\varphi)\xi = (d\pi_V)(d\psi)^{-1}(d\bar{A}_z)\xi = (d\pi_V)\eta = 0,$$

whence $d\varphi = 0$.

Thus φ must be constant on each of the (countably many) components of O. Since $S = \operatorname{Im} \varphi$, S is a countable set.

Q.E.D.

Lemma IV: Give $(z \cdot G) \cap U$ the subspace topology induced from U. Then

$$\psi(\{z\} \times W) = z \cdot W$$

is a component of $(z \cdot G) \cap U$.

Proof: It is sufficient to show that $\{z\} \times W$ is a component of $\psi^{-1}((z \cdot G) \cap U)$. But

$$\psi^{-1}((z \cdot G) \cap U) = S \times W.$$

Moreover, in view of Lemma III,

$$S \times W = \bigcup_{i=0}^{\infty} (\{y_i\} \times W),$$

with $y_0 = z$. Since W is connected, the lemma follows.

Q.E.D.

3.8. Proof of Theorem I: If τ is smooth, then so is $\sigma = \bar{A}_z \circ \tau$. Conversely, assume that σ is smooth. Translating by elements of G allows us to restrict ourselves to proving that τ is smooth near those points $q \in N$ such that

$$\tau(q) = \bar{e} \qquad \text{and} \qquad \sigma(q) = z.$$

Choose U, V, W, and ψ as in sec. 3.7. Let Q be a connected neighbourhood of q such that

$$Q \subset \sigma^{-1}(U).$$

Restrict σ to a continuous map

$$\sigma_1 : Q \to (z \cdot G) \cap U,$$

where $(z \cdot G) \cap U$ is given the subspace topology. Since Q is connected, so is $\sigma_1(Q)$. Moreover,

$$\sigma_1(q) = z \in \psi(\{z\} \times W).$$

Thus Lemma IV yields

$$\operatorname{Im} \sigma_1 \subset \psi(\{z\} \times W).$$

In particular, the map $\psi^{-1} \circ \sigma : Q \to V \times W$ has the form

$$(\psi^{-1} \circ \sigma)(x) = (z, \chi(x)),$$

where $\chi: Q \to W$ is a (necessarily) smooth map. Moreover, the smooth map $\bar{\chi}: Q \to G_z \backslash G$ given by $\bar{\chi} = \pi \circ \chi$ satisfies

$$\begin{aligned}(\bar{A}_z \circ \bar{\chi})(x) &= (A_z \circ \chi)(x) = z \cdot \chi(x) \\ &= \psi(z, \chi(x)) = \sigma(x) = (\bar{A}_z \circ \tau)(x), \qquad x \in Q.\end{aligned}$$

Since $\bar{A}_z$ is injective, we obtain $\bar{\chi} = \tau$. It follows that τ is smooth in Q.

Q.E.D.

§3. Vector fields

In this article $T: M \times G \to M$ denotes a right action of G on M.

3.9. Fundamental vector fields. The action T determines the strong bundle map,

$$\alpha: M \times E \to T_M,$$

given by

$$\alpha(z, h) = (dT)_{(z,e)}(0_z, h) = dA_z(h).$$

Differentiating the relation $T_a \circ A_z = A_{z \cdot a} \circ \tau_a^{-1}$ (τ_a denotes conjugation by a) yields the commutative diagram

$$\begin{array}{ccc} M \times E & \xrightarrow{\alpha} & T_M \\ {\scriptstyle T_a \times \mathrm{Ad}\, a^{-1}} \downarrow & & \downarrow {\scriptstyle dT_a} \\ M \times E & \xrightarrow[\alpha]{} & T_M \end{array} \quad , \qquad a \in G. \tag{3.1}$$

Now fix $h \in E$. The constant map $M \to \{h\}$ corresponds, under α, to the vector field Z_h on M given by

$$Z_h(z) = dA_z(h), \qquad z \in M.$$

It is called the *fundamental vector field generated by h*. The orbits of Z_h are the paths in M given by

$$t \mapsto z \cdot \exp th.$$

More generally, α induces the homomorphism

$$\alpha_*: \mathscr{S}(M; E) \to \mathscr{X}(M),$$

given by

$$(\alpha_* f)(z) = \alpha(z, f(z)) = dA_z(f(z)), \qquad z \in M. \qquad f \in \mathscr{S}(M; E)$$

We denote $\alpha_* f$ by Z_f and call it the *vector field generated by the function f*. Thus

$$Z_f(z) = Z_{f(z)}(z), \qquad z \in M.$$

Now let $\hat{T}$: $N \times G \to N$ be a right action of G on N and let φ: $M \to N$ be a smooth equivariant map. Then the diagram,

$$\begin{array}{ccc} M \times E & \xrightarrow{\alpha} & T_M \\ {\scriptstyle \varphi\times\iota}\downarrow & & \downarrow{\scriptstyle d\varphi} \\ N \times E & \xrightarrow[\hat{\alpha}]{} & T_N \end{array} \quad ,$$

commutes. In particular, the fundamental fields on M and on N, generated by a vector $h \in E$, are φ-related.

Example: Consider the action of G on itself by *right* translations. The fundamental vector fields are precisely the *left* invariant vector fields (cf. sec. 1.2).

To see this, observe that in this case $A_z = \lambda_z$, $z \in G$. It follows that

$$dA_z(h) = L_z(h) = X_h(z), \qquad z \in G, \quad h \in E,$$

whence $Z_h = X_h$.

More generally, if G acts on $M \times G$ (M, any manifold) by right translations of G, then the fundamental fields are given by

$$Z_h(y, x) = X_h(x), \qquad h \in E, \quad y \in M, \quad x \in G.$$

Proposition IV: The map $E \to \mathscr{X}(M)$ given by $h \mapsto Z_h$ is a homomorphism of Lie algebras:

$$[Z_h , Z_k] = Z_{[h,k]} , \qquad h, k \in E.$$

Proof: Consider first the right action $\tilde{T}$ of G on $M \times G$ given by

$$\tilde{T}((z, a), b) = (z, ab).$$

In view of the example above, the fundamental vector fields for this action are given by

$$\tilde{Z}_h(y, x) = X_h(x).$$

It follows now from sec. 1.3, that

$$[\tilde{Z}_h , \tilde{Z}_k] = \tilde{Z}_{[h,k]} . \tag{3.2}$$

Next recall that T: $M \times G \to M$ is equivariant with respect to $\tilde{T}$ and T (Example 3, sec. 3.2). It follows that

$$\tilde{Z}_h \underset{T}{\sim} Z_h , \qquad \tilde{Z}_k \underset{T}{\sim} Z_k , \qquad \tilde{Z}_{[h,k]} \underset{T}{\sim} Z_{[h,k]} .$$

Thus formula (3.2) and Proposition VIII, sec. 3.13, volume I, yield

$$\tilde{Z}_{[h,k]} \underset{T}{\sim} [Z_h, Z_k]$$

and so, since T is surjective,

$$Z_{[h,k]} = [Z_h, Z_k].$$

Q.E.D.

3.10. Invariant vector fields. We saw in Example 7 of sec. 3.2 that a right action of G in M induces an action in T_M. Define an action of G in $\mathscr{X}(M)$ by setting

$$X \cdot a = (T_a)_* X, \qquad a \in G, \quad X \in \mathscr{X}(M).$$

Then

$$[X, Y] \cdot a = [X \cdot a, Y \cdot a], \qquad X, Y \in \mathscr{X}(M), \quad a \in G.$$

A vector field X on M is called *invariant* if $X \cdot a = X$ $(a \in G)$; i.e., if

$$X \underset{T_a}{\sim} X, \qquad a \in G.$$

The subalgebra of $\mathscr{X}(M)$ that consists of invariant vector fields is denoted by $\mathscr{X}^I(M)$.

Examples: **1.** If $M = G$ and if G acts on itself by right translations, then the algebra $\mathscr{X}^I(M)$ consists of the right invariant vector fields (sec. 1.2).

2. It follows from diagram (3.1), sec. 3.9, that the fundamental fields satisfy

$$Z_h \cdot a = Z_{(\mathrm{Ad}\, a^{-1})h}, \qquad h \in E, \quad a \in G.$$

Thus Z_h is invariant if $(\mathrm{Ad}\, a)h = h$, $a \in G$. If G is connected, this is equivalent to

$$[h, k] = 0, \qquad k \in E;$$

i.e., Z_h is invariant if h is in the centre of E (cf. Example 4, sec. 2.4).

3. Let $f \in \mathscr{S}(M; E)$ and $a \in G$. Define $a \cdot f \in \mathscr{S}(M; E)$ by

$$(a \cdot f)(z) = (\mathrm{Ad}\, a)(f(z \cdot a)), \qquad z \in M.$$

Then $Z_{a \cdot f} = Z_f \cdot a^{-1}$. Thus Z_f is invariant if

$$(\mathrm{Ad}\, a^{-1})(f(z)) = f(z \cdot a), \qquad z \in M, \quad a \in G.$$

Proposition V: The Lie bracket of a fundamental field Z_h and an invariant vector field X is zero.

Proof: Let $\tilde{X}$ be the vector field on $M \times G$ given by $\tilde{X}(z, a) = X(z)$. Then

$$dT(\tilde{X}(z, a)) = (X \cdot a)(z \cdot a)$$

and hence, since X is invariant, $\tilde{X} \underset{T}{\sim} X$.

On the other hand, as we saw in the proof of Proposition IV, sec. 3.9, the left invariant vector field X_h on G, regarded as a vector field $\tilde{Z}_h$ on $M \times G$, is T-related to Z_h. Thus

$$0 = [\tilde{Z}_h, \tilde{X}] \underset{T}{\sim} [Z_h, X].$$

Since T is surjective, it follows that $[Z_h, X] = 0$.

Q.E.D.

3.11. Fundamental subbundle. Recall from sec. 3.4 that T is called almost free if each isotropy subgroup G_z is discrete. In view of the corollary to Proposition I, sec. 3.4, this is equivalent to each of the following conditions:

(1) The Lie algebras E_z are zero.
(2) The fundamental vector fields Z_h $(h \neq 0)$ have no zeros.
(3) The bundle map $\alpha: M \times E \to T_M$ of sec. 3.9 restricts to linear injections in the fibres.

In this case $\operatorname{Im} \alpha$ is a subbundle of T_M, called the *fundamental subbundle* F_M. The rank of F_M is the dimension of G. Diagram (3.1), sec. 3.9, shows that is F_M stable under the action dT of G in T_M. Moreover, α is a strong isomorphism,

$$\alpha: M \times E \xrightarrow{\cong} F_M,$$

and so F_M is trivial. Thus the correspondence $f \to Z_f$ defines an isomorphism

$$\mathscr{S}(M; E) \xrightarrow{\cong} \operatorname{Sec} F_M.$$

§4. Differential forms

In this article $T: M \times G \to M$ denotes a right action of G on M.

3.12. Invariant differential forms. The right translations T_a of M $(a \in G)$ induce automorphisms T_a^* of the graded algebra $A(M)$ of differential forms on M. Evidently,

$$T_{ab}^* = T_a^* \circ T_b^* \qquad \text{and} \qquad T_e^* = \iota \qquad a, b \in G.$$

Since, for $X \in \mathscr{X}(M)$, $a \in G$ (cf. sec. 3.10),

$$(X \cdot a)(z) = dT_a(X(z \cdot a^{-1})),$$

it follows that (cf. sec. 0.13)

$$i(X) \circ T_a^* = T_a^* \circ i(X \cdot a) \qquad \text{and} \qquad \theta(X) \circ T_a^* = T_a^* \circ \theta(X \cdot a).$$

Moreover, clearly

$$T_a^* \circ \delta = \delta \circ T_a^*.$$

A differential form Φ on M is called *invariant under the action of* G if it satisfies

$$T_a^* \Phi = \Phi, \qquad a \in G.$$

The invariant differential forms are a graded subalgebra of $A(M)$, which will be denoted by $A_I(M)$. In particular, the invariant functions form a subalgebra of $\mathscr{S}(M)$ which we denote by $\mathscr{S}_I(M)$. (The invariant vector fields on M are a module over $\mathscr{S}_I(M)$.)

Since T_a^* commutes with δ, it follows that the subalgebra $A_I(M)$ is stable under δ. The other commutation relations above show that the subalgebra $A_I(M)$ is stable under $i(X)$ and $\theta(X)$ provided that X is an invariant vector field on M.

3.13. The operators $i(h)$ and $\theta(h)$. Consider the fundamental vector field Z_h generated by $h \in E$ (cf. sec. 3.9). The operators $i(Z_h)$ and $\theta(Z_h)$ in $A(M)$ will often be denoted simply by $i(h)$ and $\theta(h)$. Proposition I, sec. 4.2, and Proposition II, sec. 4.3, both of volume I, together with the relation $Z_{[h,k]} = [Z_h, Z_k]$ $(h, k \in E)$, imply that

$$i([h,k]) = \theta(h) \circ i(k) - i(k) \circ \theta(h),$$

$$\theta([h,k]) = \theta(h) \circ \theta(k) - \theta(k) \circ \theta(h),$$

and

$$\theta(h) = i(h) \circ \delta + \delta \circ i(h), \qquad h, k \in E.$$

A differential form $\Phi \in A(M)$ is called *horizontal with respect to the action of* G if it satisfies

$$i(h)\Phi = 0, \qquad h \in E.$$

Since each $i(h)$ is an antiderivation, the horizontal forms are a graded subalgebra of $A(M)$. This subalgebra will be denoted by $A(M)_{i=0}$. The first identity above shows that the horizontal subalgebra is stable under the operators $\theta(h)$. However, in general it is *not* stable under δ.

Similarly, the differential forms satisfying

$$\theta(h)\Phi = 0, \qquad h \in E,$$

form a graded subalgebra, denoted by $A(M)_{\theta=0}$. Since δ commutes with $\theta(h)$, the subalgebra $A(M)_{\theta=0}$ is stable under δ.

The intersection of the subalgebras $A(M)_{i=0}$ and $A(M)_{\theta=0}$ will be denoted by $A(M)_{i=0,\theta=0}$. This subalgebra is stable under δ. In fact, if $\theta(h)\Phi = 0$ and $i(h)\Phi = 0$, $h \in E$, it follows that

$$\theta(h)\,\delta\Phi = \delta\theta(h)\Phi = 0 \qquad \text{and} \qquad i(h)\,\delta\Phi = \theta(h)\Phi - \delta i(h)\Phi = 0, \qquad h \in E.$$

Proposition VI: $A_I(M) \subset A(M)_{\theta=0}$. If G is connected, then

$$A_I(M) = A(M)_{\theta=0}.$$

Proof: Recall from sec. 3.9 that the orbits of a fundamental vector field Z_h are given by

$$\beta_z(t) = z \cdot \exp th, \qquad z \in M, \quad t \in \mathbb{R}.$$

It follows (cf. the corollary to Proposition X, sec. 4.11, volume I) that, if $\Phi \in A(M)$, the conditions

$$\theta(h)\Phi = 0 \qquad \text{and} \qquad T^*_{\exp th}\Phi = \Phi, \qquad t \in \mathbb{R},$$

are equivalent. Thus $A_I(M) \subset A(M)_{\theta=0}$. If G is connected, $\exp E$ generates G, and so

$$A_I(M) = A(M)_{\theta=0}.$$

Q.E.D.

3.14. Equivariant maps. Suppose $\hat{T}$ is a right action of G on N, and let $\varphi\colon M \to N$ be a smooth equivariant map. Then every pair of fundamental vector fields $Z_h \in \mathscr{X}(M)$ and $\hat{Z}_h \in \mathscr{X}(N)$ are φ-related (cf. sec. 3.9). Hence (cf. Proposition III, sec. 4.4, volume I or sec. 0.13)

$$\varphi^* \circ i_N(h) = i_M(h) \circ \varphi^* \qquad \text{and} \qquad \varphi^* \circ \theta_N(h) = \theta_M(h) \circ \varphi^*, \qquad h \in E,$$

where $i_N(h)$, $\theta_N(h)$, $i_M(h)$, and $\theta_M(h)$ denote the obvious operators on $A(N)$ and $A(M)$. In particular, φ^* restricts to homomorphisms

$$\varphi^*_{i=0}\colon A(M)_{i=0} \leftarrow A(N)_{i=0}$$

$$\varphi^*_{\theta=0}\colon A(M)_{\theta=0} \leftarrow A(N)_{\theta=0}$$

and

$$\varphi^*_{i=0,\theta=0}\colon A(M)_{i=0,\theta=0} \leftarrow A(N)_{i=0,\theta=0}.$$

Finally, the relation

$$\varphi \circ T_a = \hat{T}_a \circ \varphi, \qquad a \in G,$$

implies that

$$T_a^* \circ \varphi^* = \varphi^* \circ \hat{T}_a^*, \qquad a \in G,$$

and so φ restricts to a homomorphism

$$\varphi_I^*\colon A_I(M) \leftarrow A_I(N).$$

3.15. Equivariant differential forms. Suppose P is a representation of G in a vector space W. Then each $a \in G$ determines the operator $P(a)_*$ in the space $A(M; W)$ of W-valued differential forms given by

$$(P(a)_*\Omega)(z; \zeta_1, \ldots, \zeta_p) = P(a)(\Omega(z; \zeta_1, \ldots, \zeta_p)), \qquad z \in M, \quad \zeta_i \in T_z(M).$$

We denote $P(a)_*$ simply by $P(a)$.

A left action of G in the set $A(M; W)$ is given by

$$a \cdot \Omega = (P(a) \circ T_a^*)\Omega = (T_a^* \otimes P(a))\Omega, \qquad \Omega \in A(M; W), \quad a \in G,$$

where (as in sec. 0.13) we write $A(M; W) = A(M) \otimes W$. Evidently

$$\delta(a \cdot \Omega) = a \cdot \delta\Omega.$$

A W-valued form Ω is called *equivariant* with respect to P if

$$a \cdot \Omega = \Omega, \qquad a \in G.$$

This is equivalent to the condition

$$T_a^* \Omega = P(a)^{-1} \Omega, \qquad a \in G.$$

The space of equivariant forms is denoted by $A_I(M; W)$. It is a module over the algebra $A_I(M)$, and is stable under δ.

Now consider the induced representation P' of E in W. For each $h \in E$, $P'(h)$ determines the operator $P'(h)_*$ in $A(M; W)$; it is denoted simply by $P'(h)$. The following relations are immediate from the definitions:

$$P'([h, k]) = P'(h) \circ P'(k) - P'(k) \circ P'(h), \qquad P'(h) \circ T_a^* = T_a^* \circ P'(h)$$

and

$$P'(h) \circ \delta = \delta \circ P'(h), \qquad h, k \in E, \quad a \in G.$$

Now recall that the operators $i(h)$ and $\theta(h)$ in $A(M)$ extend to operators in $A(M; W)$ (cf. sec. 0.13). The extensions will also be denoted by $i(h)$ and $\theta(h)$.

Proposition VII: An equivariant differential form Ω satisfies the relation

$$\theta(h)\Omega = -P'(h)\Omega, \qquad h \in E.$$

If G is connected, this condition is equivalent to equivariance.

Proof: Recall, from sec. 0.13, that the decomposition,

$$\tau_{M \times W^*} = \tau_M \times \tau_{W^*},$$

leads to a bigradation of $A(M \times W^*)$; $A^{p,q}(M \times W^*)$ consists of those forms which depend on p vectors tangent to M and q vectors tangent to W^*. Define a linear *injection*

$$\lambda: A^p(M; W) \to A^{p,0}(M \times W^*)$$

by setting

$$(\lambda\Omega)(z, w^*; \zeta_1, \ldots, \zeta_p) = \langle w^*, \Omega(z; \zeta_1, \ldots, \zeta_p)\rangle,$$
$$z \in M, \quad w^* \in W^*, \quad \zeta_i \in T_z(M).$$

Let $\hat{T}$ be the right action of G on $M \times W^*$ given by

$$\hat{T}_a(z, w^*) = (z \cdot a, P(a)^* w^*), \qquad a \in G, \quad z \in M, \quad w^* \in W^*,$$

and let $\hat{Z}_h$ denote the corresponding fundamental vector field generated by h ($h \in E$). A simple computation shows that

$$\lambda \circ P(a) \circ T_a^* = \hat{T}_a^* \circ \lambda \qquad \text{and} \qquad \lambda \circ (P'(h) + \theta(h)) = \theta(\hat{Z}_h) \circ \lambda.$$

Since λ is injective, the proposition follows from Proposition VI, sec. 3.13, with M replaced by $M \times W^*$.

Q.E.D.

3.16. Examples: 1. Suppose $W = \mathbb{R}$ and $P(a) = \iota$, $a \in G$. Then the equivariant forms in $A(M)$ are precisely the invariant forms (cf. sec. 3.12), and Proposition VII coincides in this case with Proposition VI.

2. Suppose $W = E$ and $P = \mathrm{Ad}$. An equivariant E-valued form Ω is a form satisfying

$$T_a^* \Omega = (\mathrm{Ad}\, a^{-1}) \Omega, \qquad a \in G.$$

If G is connected, this is equivalent to (cf. Proposition VII, sec 3.15)

$$\theta(h)\Omega = -(\mathrm{ad}\, h)\Omega, \qquad h \in E.$$

In particular, recall that each E-valued function f on M determines the vector field Z_f on M (cf. sec. 3.9). Moreover, Example 3 of sec. 3.10 states that if f is equivariant, then Z_f is invariant. Finally, recall from sec. 3.11 that if the action of G is almost free, then $f \mapsto Z_f$ is injective. Thus, in this case, Z_f is invariant if and only if f is equivariant.

3. *Scalar products*: Define bilinear maps,

$$\langle\!\langle\, , \rangle\!\rangle : A^p(M; W^*) \times A^q(M; W) \to A^{p+q}(M),$$

by

$$\langle\!\langle \Phi, \Psi \rangle\!\rangle (z; \zeta_1, \ldots, \zeta_{p+q})$$
$$= \frac{1}{p!\, q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \langle \Phi(z; \zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(p)}), \Psi(z; \zeta_{\sigma(p+1)}, \ldots, \zeta_{\sigma(p+q)}) \rangle,$$

$$\Phi \in A^p(M; W^*), \qquad \Psi \in A^q(M; W), \qquad z \in M, \quad \zeta_i \in T_z(M).$$

Thus if $\Phi_1, \Psi_1 \in A(M)$, $w \in W$, $w^* \in W^*$, then

$$\langle\!\langle \Phi_1 \otimes w^*, \Psi_1 \otimes w \rangle\!\rangle = \langle w^*, w \rangle\, \Phi_1 \wedge \Psi_1 .$$

The contragredient representation, $P^\natural$, of G in W^* determines the left action $a \mapsto P(a)^\natural \circ T_a^*$ of G in $A(M; W^*)$, denoted by $\Phi \mapsto a \cdot \Phi$. Since $P(a)^\natural = (P(a)^*)^{-1}$, it follows that

$$T_a^*\langle \Phi, \Psi \rangle = \langle a \cdot \Phi, a \cdot \Psi \rangle, \qquad a \in G, \quad \Phi \in A(M; W^*), \quad \Psi \in A(M; W).$$

In particular, if Φ and Ψ are equivariant, then $\langle \Phi, \Psi \rangle$ is an invariant differential form.

4. *Action of G on a bundle*: Let $\mathscr{B} = (M, \pi, B, F)$ be a smooth fibre bundle. Assume that right actions

$$T: M \times G \to M, \qquad \hat{T}: B \times G \to B,$$

are given such that π is equivariant. In this case, the diffeomorphisms T_a are all fibre preserving and G is said to *act on the bundle*.

Since π is equivariant the fundamental fields Z_h on M and $\hat{Z}_h$ on B are π-related. Thus (cf. sec. 3.14)

$$\pi^* \circ i(h) = i(h) \circ \pi^* \qquad \text{and} \qquad \pi^* \circ \theta(h) = \theta(h) \circ \pi^*.$$

Moreover, if $\mathscr{B}$ is oriented, then Proposition X, sec. 7.13, volume I, gives

$$\oint_F \circ\, i(h) = i(h) \circ \oint_F \qquad \text{and} \qquad \oint_F \circ\, \theta(h) = \theta(h) \circ \oint_F.$$

Now assume that G is connected. We shall show that each T_a preserves the bundle orientations, so that (cf. Proposition VIII, sec. 7.12, volume I).

$$\oint_F \circ\, T_a^* = \hat{T}_a^* \circ \oint_F, \qquad a \in G.$$

To see that G preserves the bundle orientations observe first that the components of M are stable under G (because G is connected). Thus we may assume that M is connected. In this case each T_a either preserves or reverses the bundle orientations. Since

$$T_{\exp h} = (T_{\exp(h/2)})^2, \qquad h \in E,$$

it follows that $T_{\exp h}$ preserves the orientation. But, because G is connected, $\exp E$ generates G; hence each T_a preserves the orientation.

§5. Invariant cross-sections

In this article $\xi = (N, \pi, B, F)$ denotes a fixed vector bundle.

3.17. Action of G on ξ. *A right action of G on ξ* consists of right actions

$$T: N \times G \to N, \qquad \hat{T}: B \times G \to B$$

subject to the conditions:

(1) π is equivariant
(2) The right translations T_a are bundle maps (i.e., linear in each fibre).

A *left action of G on ξ* is defined analogously.

Assume that T, $\hat{T}$ define a right action of G on ξ. Define a right action of G on Sec ξ, $(\sigma, a) \mapsto \sigma \cdot a$, by setting

$$(\sigma \cdot a)(x) = T_a(\sigma(x \cdot a^{-1})), \qquad \sigma \in \operatorname{Sec} \xi,\ a \in G,\ x \in B.$$

A cross-section σ is called *invariant* if

$$\sigma \cdot a = \sigma, \qquad a \in G.$$

Thus σ is invariant if and only if the map $\sigma: B \to E$ is equivariant. The set of invariant cross-sections forms a subspace of the vector space Sec ξ which we denote by $\operatorname{Sec}^I(\xi)$. $\operatorname{Sec}^I(\xi)$ is a module over $\mathscr{S}_I(B)$ (cf. sec. 3.12).

Example: A right action $\hat{T}$ of G on M induces a right action of G on the tangent bundle, T_M, with T: $T_M \times G \to T_M$ given by

$$T(z, a) = (d\hat{T}_a)z$$

(cf. Example 7, sec. 3.2). As usual, denote $T(z, a)$ by $z \cdot a$.

If X is a vector field on M, then $X \cdot a = (T_a)_* X$, and so the definition above coincides with that of sec. 3.10. Thus the definitions of invariant vector field and of $\mathscr{X}^I(M)$ given in sec. 3.10 agree with the definitions above.

3.18. Integration of cross-sections. Assume that G is compact. Give G a left orientation, and let $\Delta \in A_L^n(G)$ ($n = \dim G$) be the unique left invariant n-form such that $\int_G \Delta = 1$ (cf. sec. 1.15). We write (as in sec. 1.15)

$$\int_G f(a)\,da = \int_G f \cdot \Delta,$$

if f is a vector-valued function on G.

Now suppose G acts on ξ and fix $\sigma \in \operatorname{Sec} \xi$ and $x \in B$. Then a smooth F_x-valued function f_x on G is given by

$$f_x(a) = (\sigma \cdot a)(x).$$

Hence a map $\tau\colon B \to N$ is defined by

$$\tau(x) = \int_G f_x(a)\,da = \int_G (\sigma \cdot a)(x)\,da.$$

It is denoted by $\int_G \sigma$ and is called the *integral of σ over G.*

τ is a cross-section in ξ. Indeed, this follows from Proposition VII, sec. 7.11, volume I, once we observe that $\tau = \fint_G \Phi$, where

$$\Phi\colon B \times \wedge^n T_G \to N$$

is the bundle map given by $\Phi(x, a; \eta_1, \ldots, \eta_n) = \Delta(a; \eta_1, \ldots, \eta_n)(\sigma \cdot a)(x)$. (Observe that $B \times T_G$ is the vertical bundle of the trivial bundle $(B \times G, \pi_B, B, G)$.)

Proposition VIII: (1) For any $\sigma \in \operatorname{Sec} \xi$, $\int_G \sigma$ is invariant.

(2) If τ is invariant, then $\int_G \tau = \tau$.

(3) The correspondence $\sigma \mapsto \int_G \sigma$ is linear (over $\mathbb{R}$).

Proof: (1) Let $\sigma \in \operatorname{Sec} \xi$, $b \in G$. It is immediate from the definitions that

$$\left[\left(\int_G \sigma\right) \cdot b\right](x) = T_b\left(\int_G (\sigma \cdot a)(x \cdot b^{-1})\,da\right).$$

Since $T_b\colon F_{x\cdot b^{-1}} \to F_x$ is linear, it commutes with $\int_G$. Thus, by formula (1.2), sec. 1.15,

$$\left[\left(\int_G \sigma\right) \cdot b\right](x) = \int_G (\sigma \cdot ab)(x)\,da = \int_G (\sigma \cdot a)(x)\,da = \left(\int_G \sigma\right)(x).$$

This proves (1).

(2) follows from the relation,

$$\left(\int_G \tau\right)(x) = \int_G (\tau \cdot a)(x)\, da = \left(\int_G da\right) \tau(x) = \tau(x),$$

and (3) is obvious.

Q.E.D.

Examples: 1. If G is a compact Lie group that acts on a vector bundle $\xi = (N, \pi, B, F)$ via T, $\hat{T}$, then there exists a Riemannian metric in ξ with respect to which the translations $T_a: N \to N$ $(a \in G)$ are isometries.

In fact, the action T determines the (right) action of G in $\vee^2 \xi^*$ given by

$$(\Phi \cdot a)(u, v) = \Phi(u \cdot a^{-1}, v \cdot a^{-1}) \qquad \Phi \in \vee^2 F_x^*, \quad x \in B, \quad a \in G, \quad u, v \in F_{x \cdot a}.$$

Now let g be any Riemannian metric in ξ and regard g as a cross-section in the vector the bundle $\vee^2 \xi^*$. Then

$$g_0 = \int_G g$$

is a metric with the desired properties.

Suppose now that η is a subbundle of ξ which is stable under the action of G on ξ. Then there is a G-stable subbundle, ζ, of ξ such that $\eta \oplus \zeta = \xi$ (Whitney sum).

In fact, choose a Riemannian metric in ξ such that the translations by G are isometries, as above, and then let ζ be the bundle $\eta^\perp$ whose fibres are the orthogonal complements of those of η (cf. Proposition VII, sec. 2.18, volume I).

2. Suppose G acts on B and consider the induced action,

$$T: (B \times \mathbb{R}) \times G \to B \times \mathbb{R},$$

given by $T((x, t), a) = (x \cdot a, t)$. This is an action of G on the trivial bundle $\xi = (B \times \mathbb{R}, \pi, B, \mathbb{R})$.

The cross-sections of ξ are simply the smooth functions on B. If $f \in \mathscr{S}(B)$, then the integral over G of f is the invariant function f_I given by

$$f_I(x) = \int_G f(x \cdot a)\, da.$$

3.19. Remark. All of the results of this chapter have analogues if right actions are replaced by left actions, $T: G \times M \to M$. Among the notational differences in formulae, recall that $A_z(a)$ becomes $a \cdot z$ so that

$$A_{a \cdot z} = T_a \circ A_z \circ \tau_a^{-1}$$

(cf. sec. 3.1). This in turn implies that the (left) fundamental vector field Z_h generated by $h \in E$ is T_a-related to $Z_{(\mathrm{Ad}\,a)h}$ (cf. sec. 3.9). A form $\Omega \in A(M; W)$ will be called *equivariant* (cf. sec. 3.15) if

$$T_a^* \Omega = P(a)\Omega, \qquad a \in G.$$

Problems

G denotes a Lie group with Lie algebra E and M denotes a manifold.

1. Let $T: M \times G \to M$ be a right action of G on M. Show that

$$i(h) \circ T_a^* = T_a^* \circ i(\mathrm{Ad}(a^{-1})h), \qquad a \in G, \quad h \in E.$$

2. Suppose G is connected, and let G act on M. Show that a horizontal form Φ is invariant if and only if $\delta\Phi$ is horizontal.

3. Let G act on M and consider the induced action on T_M. Show how the fundamental vector fields on M determine the fundamental vector fields on T_M.

4. Construct an almost free action of S^1 on a 3-manifold such that every finite subgroup of S^1 appears as the isotropy subgroup for some point.

5. **Proper actions, I.** A left (right) action of G on M is called *proper*, if for all compact subsets $A, B \subset M$, the subset S of G given by $S = \{a \in G \mid (a \cdot A) \cap B \neq \varnothing\}$ is compact.

(i) Show that the isotropy subgroups of a proper action are all compact. Show that the orbits of a proper action are all closed submanifolds of M.

(ii) Construct an action of $\mathbb{R}$ on $S^1 \times \mathbb{R}$ subject to the following conditions: (a) $S^1 \times \mathbb{R}$ is covered by stable open subsets, each of which is equivariantly diffeomorphic to $(0, 1) \times \mathbb{R}$; (b) the action is not proper. Show, nonetheless, that the action is free and that the orbits are all closed submanifolds.

6. **Orbit space.** Let G act from the left on M. Let M/G denote the set of orbits of G, endowed with the quotient topology via the canonical projection $\pi: M \to M/G$. It is called the *orbit space* of the action.

(i) Show that π is an open map, and that M/G is second countable.

(ii) If the action is proper, show that M/G is Hausdorff and locally compact. Find examples of actions where M/G is not Hausdorff (cf. problem 5, (ii)).

(iii) Assume that the action is proper and free. Fix $z \in M$. Find a submanifold N_z of M and an open subset U_z of M such that $z \in N_z$ and the action restricts to an equivariant diffeomorphism $G \times N_z \xrightarrow{\cong} U_z$.

(iv) (Gleason) Show, if the action is proper and free, that M/G possesses a unique smooth structure for which π is a submersion. Construct a smooth bundle $(M, \pi, M/G, G)$.

7. **Bundles over a homogeneous space, I.** Let G act from the left on a bundle $\mathscr{B} = (M, \rho, G/K, Q)$, where the action of G on G/K is defined as in sec. 2.11. Identify Q with $Q_{\bar{e}}$ ($Q_{\bar{e}}$, the fibre over $\bar{e}$).

(i) Obtain an action of K on Q.

(ii) Define a right action of K on $G \times Q$ by setting

$$(a, y) \cdot b = (ab, b^{-1}y), \qquad a \in G, \quad y \in Q, \quad b \in K.$$

Show that this action is free. Use the bundle $(G, \pi, G/K, K)$ (cf. sec. 2.13) to make the orbit space $(G \times Q)/K$ into a manifold; denote this manifold by $G \times_K Q$.

(iii) Construct a smooth bundle $\xi = (G \times_K Q, p, G/K, Q)$ and an action of G on ξ. Construct a G-equivariant fibre preserving diffeomorphism $G \times_K Q \xrightarrow{\cong} M$.

(iv) Show that every K-stable submanifold Q_1 of Q leads to a bundle $G \times_K Q_1$ and a smooth fibre preserving map $G \times_K Q_1 \to M$.

(v) Let K_y denote the isotropy subgroup at $y \in Q$ for the action of K on Q. Show that every isotropy subgroup G_x for the action of G on M is conjugate to one of the K_y. Show that the inclusion $Q \to M$ induces a homeomorphism $Q/K \xrightarrow{\cong} M/G$ of orbit spaces.

(vi) Show that G acts properly on M if and only if the action of K on Q is proper. Conclude that if K is compact, then the action of G is proper.

8. **Bundles over a homogeneous space, II.** Adopt the hypotheses of problem 7.

(i) Show that the vertical subbundle V_M of τ_M is stable under the action of G. If K is compact, construct a G-stable horizontal subbundle.

(ii) Assume that K is compact. Denote the Lie algebra of K by F. Use the adjoint representation to obtain a representation of K

in $\wedge(E/F)^*$. Denote by $A_I(M)$ the algebra of G-invariant differential forms on M and by $A_I(Q; \wedge(E/F)^*)$ the algebra of K-equivariant differential forms on Q with values in $\wedge(E/F)^*$. Obtain an isomorphism

$$A_I(M) \xrightarrow{\cong} A_I(Q; \wedge(E/F)^*).$$

(iii) If K is compact, show that the inclusion map $i: Q \to M$ induces an isomorphism

$$i^*_{i=0,I}: A_I(Q)_{i=0} \xleftarrow{\cong} A_I(M)_{i=0}\,,$$

where $A_I(Q)_{i=0}$ (respectively, $A_I(M)_{i=0}$) denotes the algebra of differential forms on Q that are K-horizontal and K-invariant (respectively, the algebra of differential forms on M that are G-horizontal and G-invariant).

9. **Vector bundles over a homogeneous space.** Let $\xi = (M, \rho, G/K, F)$ be a vector bundle acted on by G so as to induce the standard action on G/K. Identify F with $F_{\dot{e}}$.

(i) Show that the induced action of K on F is a representation (cf. problem 7, (i)). Show that $G \times_K F \to G/K$ is a vector bundle. Construct a strong equivariant isomorphism $G \times_K F \xrightarrow{\cong} \xi$.

(ii) Obtain a bijection between direct decompositions of F into K-stable subspaces and decompositions of ξ as a Whitney sum of G-stable subbundles.

(iii) Construct a bijection between K-invariant Euclidean metrics in F and G-invariant Riemannian metrics in ξ.

(iv) Assume K compact and fix a G-invariant Riemannian metric in ξ. Show that the action of G restricts to actions on the unit sphere bundle and on the open disc bundle of vectors of length $< r$. Identify these bundles with $G \times_K S$ and $G \times_K F_r$, respectively, where S (respectively, F_r) denotes the unit sphere (respectively, the open disc of radius r) in F.

(v) With the hypotheses and notation of (iv), let M_r denote the open disc bundle of radius r. Construct a G-equivariant, fibre preserving diffeomorphism $M \xrightarrow{\cong} M_r$ inducing the identity map in G/K.

(vi) With the hypotheses and notation of (iv), construct a smooth G-invariant function f on M such that: (a) $0 \leqslant f(z) \leqslant 1$, $z \in M$; (b) $f(0_x) = 1$, $x \in G/K$; (c) f has fibre compact carrier.

10. **Affine sprays.** Assume G is compact, and acts on M. Recall, from the Appendix of volume I, the definition of an affine spray as a

vector field on T_M. It is called *complete*, if it generates a global 1-parameter group of transformations $\varphi_t : T_M \to T_M$ $(t \in \mathbb{R})$.

(i) Show that M admits a complete G-invariant affine spray. Show that the corresponding map exp: $T_M \to M$ is G-equivariant.

(ii) Show that the map $\exp_G\colon E \to G$ is the restriction of the exponential map of a certain affine spray.

11. Isotropy representation and stable tubular neighbourhoods. Let G act from the left on M.

(i) Use the action to define a representation of the isotropy subgroup G_x in $T_x(M)$ $(x \in M)$. This is called the *isotropy representation.*

(ii) Let $\mathrm{Ad}_x^{\perp}$ denote the representation of G_x in E/E_x (E_x, the Lie algebra of G_x). Construct a representation of G_x in a space N_x and a G_x-linear short exact sequence

$$0 \to E/E_x \to T_x(M) \to N_x \to 0.$$

(iii) Let G act on a manifold P and let $\varphi\colon P \to M$ be an equivariant immersion. Obtain an action of G on the normal bundle of P in M. In the case that φ is the inclusion map $\bar{A}_x\colon G/G_x \to M$, show that the normal bundle is the vector bundle $G \times_{G_x} N_x$ (cf. problem 9).

(iv) Suppose that G_x is compact. Use a complete G_x-invariant affine spray (cf. problem 10) to construct a G_x-equivariant smooth map $\varphi\colon N_x \to M$ satisfying $\varphi(0_x) = x$. Show that the smooth map $G \times N_x \to M$ given by $(a, y) \mapsto a \cdot \varphi(y)$ factors to yield a smooth G-equivariant map $\psi\colon G \times_{G_x} N_x \to M$ (cf. problem 7).

(v) Assume G_x is compact and let $O_x(r)$ denote the open disc of radius r in N_x with respect to a G_x-invariant Euclidean inner product. Show that, for sufficiently small r, ψ restricts to an equivariant local diffeomorphism $G \times_{G_x} O_x(r) \xrightarrow{\cong} V_x$, where V_x is a G-stable neighbourhood of the orbit $G \cdot x$.

(vi) Assume that the action of G is proper. Construct a G-equivariant diffeomorphism σ of the normal bundle of $G \cdot x$ onto a neighbourhood of $G \cdot x$, such that $\sigma(0_x) = x$ (cf. problem 5).

12. Proper actions, II. Assume that G acts properly from the left on M. Use problems 5–11 to establish the following properties:

(i) Every covering of M by G-stable open sets admits a subordinate G-invariant partition of unity.

(ii) M admits a complete G-invariant affine spray. If N is a closed G-stable submanifold of M, then there is a G-equivariant diffeomorphism from the normal bundle of N onto a neighbourhood of N.

(iii) M admits a G-invariant Riemannian metric.

(iv) If G acts *effectively* on M (i.e., $\bigcap_{x\in M} G_x = e$), the isotropy representations are all faithful.

13. Orbit types. Let G act on M. The conjugacy classes (G_x) are called *orbit types* for the action (cf. problem 24, Chap. II).

(i) If the action is proper and $x \in M$, find a neighbourhood U_x of G_x such that $(G_y) \leqslant (G_x)$, for $y \in U_x$.

(ii) If the action is proper and M/G is compact, conclude that there are only finitely many orbit types.

(iii) Assume that the action is proper. Show that there is a unique orbit type (H) such that $(H) \leqslant (G_x)$ for every $x \in M$. It is called the *principal orbit type*. Show that the set $\{x \in M \mid (G_x) = (H)\}$ is open, connected, and dense in M.

(iv) Show that the principal orbit type for the action of G on T_M is strictly contained in the principal orbit type for the action on M. Show by example that the principal orbit type of T_M need not be (e). Find the principal orbit type of the adjoint representation of a compact Lie group.

(v) Show that (G_x) is the principal orbit type if and only if the representation of G_x in N_x (cf. problem 11) is trivial. In this case show that the normal bundle is $G/G_x \times N_x$.

(vi) Fix $x_0 \in M$. Show that the union of the points $x \in M$ such that $(G_x) \leqslant (G_{x_0})$ is an open G-stable subset of M. Show that the union of the points $x \in M$ such that $(G_x) = (G_{x_0})$ is a closed submanifold of this open set.

14. Fixed point sets. Let G act properly on M and let F be the set of points $x \in M$ such that $G_x = G$.

(i) Show that each component of F is a closed submanifold of M.

(ii) Let F_0 be a component of F and suppose $\dim F_0 = p$. Construct a representation of G in $\mathbb{R}^{n-p}$ $(n = \dim M)$ and a G-equivariant coordinate representation $\psi_\alpha\colon U_\alpha \times \mathbb{R}^{n-p} \xrightarrow{\cong} \rho^{-1}(U_\alpha)$ for the normal bundle $(N_0, \rho, F_0, \mathbb{R}^{n-p})$.

(iii) If G is a torus, give the normal bundle of a component, F_λ, of F an invariant complex structure. Conclude that $\dim M - \dim F_\lambda \equiv 0$ (mod 2), and that the normal bundle is orientable.

15. Actions on vector bundles. Suppose G acts on a vector bundle $\xi = (M, \rho, B, F)$. Assume the action on B is proper.

(i) Show that the action on M is proper.

(ii) Construct a G-invariant Riemannian metric in ξ. Conclude that the fundamental fields on M are tangent to the sphere bundles.

16. Actions of Lie algebras. Suppose G is connected, and that G is its own universal covering group. Assume $\Phi: E \to \mathscr{X}(M)$ is a Lie algebra homomorphism such that each vector field $\Phi(h)$ can be integrated to produce a 1-parameter group of diffeomorphisms of M.

Prove that there is a unique smooth action of G on M such that $\Phi(h)$ is the fundamental field generated by h (cf. problem 20, Chap. I).

17. Let X_ν ($\nu = 1, \ldots, n$) be vector fields on a connected n-manifold M such that

(1) $[X_i, X_j] = \sum_k c_{ij}^k X_k$ $(c_{ij}^k \in \mathbb{R})$.

(2) For each $x \in M$, $X_\nu(x)$ ($\nu = 1, \ldots, n$) is a basis of $T_x(M)$.

(3) Every real linear combination of the X_i generates a global flow φ_t $(t \in \mathbb{R})$.

Show that M is the quotient of a Lie group by a closed discrete subgroup. If the c_{ij}^k are all zero, show that M is an abelian Lie group.

18. Let $f_1, \ldots, f_p$ be smooth functions on a manifold M. Fix real constants $c_1, \ldots, c_p$ and set $N = f_1^{-1}(c_1) \cap \cdots \cap f_p^{-1}(c_p)$. Assume that, for each $x \in N$, $(\delta f_1 \wedge \cdots \wedge \delta f_p)(x) \neq 0$.

(i) Show that N is a closed submanifold of M. Let U be a tubular neighbourhood of N; identify U with the normal bundle ν and let $\rho: U \to N$ be the projection. Show that, for suitable U,

$$x \mapsto (\rho(x), f_1(x), \ldots, f_p(x)), \qquad x \in U,$$

is a diffeomorphism from U to $N \times V$ (V, a neighbourhood of 0 in $\mathbb{R}^p$).

(ii) Assume that the dimension of M is even and that M admits a closed 2-form Φ such that $\Phi \wedge \cdots \wedge \Phi$ orients M. Show that vector fields Y_j on M are determined by the equations $i(Y_j)\Phi = \delta f_j$. Show that the Y_j restrict to vector fields X_j on N. Show that $[Y_{j_1}, Y_{j_2}] = 0$ and conclude that, if N is compact and connected, it is a torus.

(iii) Let Y be defined by $i(Y)\Phi = \delta f$, where $f \in \mathscr{S}(M)$ satisfies $Y_j(f) = 0$ $(j = 1, ..., p)$. Show that Y restricts to an invariant vector field on N.

19. Non-Euclidean geometry in the unit disc. Define a Riemannian metric in the open unit disc Ω of the complex plane by

$$g(z; \zeta_1, \zeta_2) = \frac{\operatorname{Re}(\zeta_1\bar{\zeta}_2)}{(1 - |z|^2)^2}, \qquad |z| < 1, \quad \zeta_1, \zeta_2 \in \mathbb{C}.$$

(cf. problem 16, Chap. II.) Define $\rho\colon \Omega \times \Omega \to \mathbb{R}$ by

$$\rho(z_1, z_2) = \log\frac{1 + r}{1 - r}, \qquad \text{where} \quad r = \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|.$$

Let G be the subgroup of the Möbius group consisting of the fractional linear transformations which map Ω onto itself.

(i) Show that the angles with respect to g coincide with the Euclidean angles.

(ii) Show that $\rho(z_1, z_2) = \rho(Tz_1, Tz_2)$, $T \in G$.

(iii) A *hyperbolic straight line* in Ω is a circle orthogonal to the unit circle. Show that, for any two distinct points z_1 and z_2 of Ω, there is a unique straight line joining z_1 to z_2.

(iv) The *hyperbolic* length of a smooth curve in Ω: $c\colon t \mapsto z(t)$, $0 \leqslant t \leqslant 1$ is defined by

$$l(c) = \int_0^1 \frac{|\dot{z}|}{1 - |z|^2}\, dt.$$

Show that $\rho(z_1, z_2) = l(s) \leqslant l(c)$, where s is the hyperbolic straight line segment joining z_1 to z_2 and c is any smooth curve in Ω from z_1 to z_2.

(v) Show that ρ makes Ω into a metric space with the standard topology.

(vi) Let τ_Ω be the tangent bundle of Ω and let M be the total space of the corresponding sphere bundle (with respect to g). Show that the left action of G on Ω induces an action on τ_Ω which restricts to an action on M.

(vii) Let $x \in M$. Show that the map $T \mapsto T \cdot x$ defines a diffeomorphism $G \cong M$. Conclude that G is diffeomorphic to $\Omega \times S^1$.

Conclude also that, given any $z_1, z_2 \in \Omega$, $\zeta_1, \zeta_2 \in S^1$, there is a unique $T \in G$ such that

$$T(z_1) = z_2 \qquad \text{and} \qquad T'(z_1; \zeta_1) = (z_2, \zeta_2).$$

(viii) Show that, if $z_1, w_1, z_2, w_2 \in \Omega$ are given such that $\rho(z_1, z_2) = \rho(w_1, w_2)$, there is a unique $T \in G$ such that $T(z_1) = w_1$ and $T(z_2) = w_2$.

20. Convex polygons in Ω. A subset $A \subset \Omega$ is called *hyperbolic convex* if, whenever $z_1 \in A$ and $z_2 \in A$, then the hyperbolic straight line segment between z_1 and z_2 is contained in A. A *convex polygon* is a closed convex set Δ in Ω whose boundary consists of a finite number of hyperbolic straight line segments, called its *sides*. If the boundary of Δ consists of n sides, Δ will be called *an n-polygon.*

Let Δ be an n-polygon. Show that

$$\sum_{\nu=1}^{n} \alpha_\nu + \tfrac{1}{4} \int_\Delta \Phi = (n-2)\pi,$$

where the α_ν denote the interior angles of Δ and Φ is the 2-form given by

$$\Phi(z; \zeta_1, \zeta_2) = \frac{\operatorname{Im}(\bar{\zeta}_1 \zeta_2)}{(1 - |z|^2)^2}, \qquad z \in \Omega, \qquad \zeta_1, \zeta_2 \in \mathbb{C}.$$

21. Discontinuous actions. An effective action (cf. problem 12) of a group Γ on a manifold M is called *discontinuous*, if every point $x \in M$ has a neighbourhood U such that no two distinct points of U are in the same orbit of Γ. A *fundamental domain* is an open subset F of M such that

(1) any two distinct points of F are in different orbits,

and

(2) every point $x \in M$ is in the orbit of some point of the closure $\bar{F}$.

(i) If Γ acts discontinuously on M, show that Γ is finite or countable, and that the action is free. Is the action necessarily proper?

(ii) Let $M = \Omega$ and let Γ be a group of fractional linear transformations of Ω that acts discontinuously on Ω. Set $z_\nu = T_{\gamma_\nu}(0)$, where the γ_ν are the elements of Γ. Show that the set given by

$$\{z \in \Omega \mid \rho(z, 0) < \rho(z, z_\nu), \quad \nu = 1, 2, \ldots,\}$$

is a convex fundamental domain for the action.

22. Poincaré polygons. We adopt the notation of problems 19–21. A convex $4p$-polygon Δ in Ω with consecutive sides a_1, b_1, a_1', b_1', ..., a_p, b_p, a_p', b_p' is called a *Poincaré polygon* if it satisfies the following conditions:

(a) $l(a_i) = l(a_i')$ and $l(b_i) = l(b_i')$ $(i = 1, ..., p)$.
(b) The sum of interior angles is 2π.

(i) Construct Poincaré polygons for each $p \geqslant 2$.
(ii) Show that if Δ is a Poincaré polygon, then the 2-form Φ of problem 20 satisfies

$$\tfrac{1}{4}\iint_\Delta \Phi = 4(p-1)\pi.$$

Conclude that there is no Poincaré polygon for $p = 1$.

23. Fuchsian groups. Adopt the notation of problems 19–22. Fix a Poincaré $4p$-polygon, with consecutive vertices $z_0, ..., z_{4p}$ $(z_0 = z_{4p})$. Let $\vec{a}_i$, $\vec{b}_i$, $\vec{a}_i'$, $\vec{b}_i'$ denote the sides as defined above, directed from the lower to higher vertex (e.g. $\vec{a}_i$ is the side from z_{4i} to z_{4i+1}) and let $\overleftarrow{a}_i$, $\overleftarrow{b}_i$, $\overleftarrow{a}_i'$, $\overleftarrow{b}_i'$ be the same sides with opposite orientation.

(i) Show that there are unique elements α_i, β_i, $(i = 1, ..., p)$ in G such that $\alpha_i^{-1}(\vec{a}_i) = \overleftarrow{a}_i'$ and $\beta_i(\vec{b}_i) = \overleftarrow{b}_i'$. Denote the subgroup of G generated by $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p$ by Γ. Γ is called the *Fuchsian group associated with* Δ. (cf. problem 19 for the definition of G.)

(ii) Show that $\Delta \cap \alpha_i^{-1}(\Delta) = a_i'$ and $\Delta \cap \beta_i(\Delta) = b_i'$ $(i = 1, ..., p)$

(iii) Consider the sequence

$$\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, ..., \alpha_p, \beta_p, \alpha_p^{-1}, \beta_p^{-1},$$

of elements in Γ. Denote the product (in the given order) of the first m elements by τ_m $(m = 1, ..., 4p)$. Show that for a suitable permutation σ of $(1, ..., 4p)$,

$$\tau_j(z_{\sigma(j)}) = z_0, \qquad j = 1, ..., 4p.$$

(iv) Set $\tau_j(\Delta) = \Delta_j$. Show that $\Delta_j \cap \Delta_{j+1}$ is a common side having z_0 as an endpoint. Show that

$$\Delta_j \cap \Delta_k = z_0 \qquad \text{if} \quad |k - j| > 1, \quad 1 \leqslant j, k \leqslant 4p.$$

(v) Show that $\Delta_{4p} = \Delta$ and that the polygons $\Delta_1, \ldots, \Delta_{4p}$ cover a neighbourhood of z_0. Conclude that $\tau_{4p} = \iota$; i.e., Γ has the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_p\beta_p\alpha_p^{-1}\beta_p^{-1} = \iota.$$

24. Poincaré polygons as fundamental domains. Adopt the notation of problem 23.

(i) Set $w_j = \tau_j^{-1}(z_0)$ (i.e., $w_j = z_{\sigma(j)}, j = 1, \ldots, 4p$). Define a symmetric relation in $\Gamma \times \Delta$ as follows:

(a) If z is an interior point of Δ, then $(g, z) \sim (g', z')$ if and only if $g = g'$ and $z = z'$.

(b) If z is an interior point of a_i, then

$$(g, z) \sim (g, z) \qquad \text{and} \qquad (g, z) \sim (g\alpha_i, \alpha_i^{-1}z).$$

(c) If z is an interior point of b_i, then

$$(g, z) \sim (g, z) \qquad \text{and} \qquad (g, z) \sim (g\beta_i^{-1}, \beta_i z).$$

(d) If z is a vertex ($z = w_i$), then

$$(g, w_i) \sim (g\tau_i^{-1}\tau_j, w_j), \qquad j = 1, \ldots, 4p.$$

Show that this relation is an equivalence relation, and write down the equivalence classes.

(ii) Give Γ the discrete topology. Let X be the quotient space under the equivalence relation above (quotient topology) and let $q: \Gamma \times \Delta \to X$ be the projection. Show that X is second countable, Hausdorff, and pathwise connected.

(iii) Define a map $\varphi: \Gamma \times \Delta \to \Omega$ by $\varphi(g, z) = g \cdot z$. Show that φ factors over the projection q to yield a continuous map $\psi: X \to \Omega$. Show that ψ is a local homeomorphism.

(iv) Let $t \mapsto z(t)$ be a continuous map from $[0, 1]$ into Ω. Let $x_0 \in X$ be any point such that $\psi(x_0) = z(0)$. Show that there is a unique continuous map $t \mapsto x(t)$ from $[0, 1]$ into X such that

$$x(0) = x_0 \qquad \text{and} \qquad \psi(x(t)) = z(t), \qquad 0 \leqslant t \leqslant 1.$$

(*Hint*: Cover the curve $z(t)$ by finitely many Γ-translates of Δ.) Conclude that ψ is a homeomorphism, onto Ω.

(v) Show that Γ acts discontinuously and properly on Ω and that the interior of Δ is a fundamental domain for the action. Conclude that the orbit space ($M = \Omega/\Gamma$) is a smooth compact connected orientable 2-manifold, and that $\pi: \Omega \to M$ is the universal covering projection.

(vi) Compute the cohomology algebra and Euler–Poincaré characteristic of M.

(vii) Generalize to nonconvex polygons.

25. The Möbius group. Consider the action of the Möbius group M on S^2 (cf. problem 14, Chap. II).

(i) Show that this action is transitive and determine the isotropy subgroups.

(ii) Consider the induced action on the tangent bundle τ_{S^2}. Determine the isotropy subgroups. Show that there are exactly two orbits, namely the zero cross-section and the deleted bundle. Thus obtain a smooth bundle $(M, \pi, \dot{\tau}_{S^2}, \mathbb{C})$ (cf. Example 5, sec. 3.10, volume I).

(iii) Show that M is diffeomorphic to $\dot{T}_{S^2} \times \mathbb{C}$. Conclude that M is diffeomorphic to $\mathbb{R}P^3 \times \mathbb{R}^3$. Construct an inclusion $SO(3) \to M$ of Lie groups that is a smooth strong deformation retract.

(iv) Find the fundamental fields for the action of M on S^2 and on T_{S^2}.

Chapter IV

Invariant Cohomology

In this chapter G denotes an n-dimensional Lie group with Lie algebra E.

§1. Group actions

4.1. Invariant cohomology. Consider a right action, $T: M \times G \to M$, of G on a manifold M. Recall that $\Phi \in A(M)$ is invariant if $T_a^*\Phi = \Phi$, $a \in G$, (cf. sec. 3.12) and that the invariant forms constitute a graded subalgebra, $A_I(M)$. $A_I(M)$ is stable under δ and the corresponding cohomology algebra is denoted by

$$H_I(M) = \sum_p H_I^p(M).$$

In particular, if $M = G$ and T is the group multiplication, $A_I(M)$ is denoted by $A_R(G)$, and the cohomology algebra is denoted by $H_R(G)$.

The inclusion map $i: A_I(M) \to A(M)$ induces a homomorphism

$$i_\#: H_I(M) \to H(M)$$

of graded algebras.

If $\hat{T}: N \times G \to N$ is a second right action and $\varphi: M \to N$ is equivariant, then φ^* restricts to a homomorphism

$$\varphi_I^*: A_I(M) \leftarrow A_I(N),$$

and so induces a homomorphism

$$\varphi_I^\#: H_I(M) \leftarrow H_I(N).$$

Assume that $\psi: M \to N$ is a second equivariant map, and that

$$H: \mathbb{R} \times M \to N$$

is a homotopy connecting φ and ψ and satisfying

$$H(t, x \cdot a) = H(t, x) \cdot a, \qquad t \in \mathbb{R}, \quad x \in M, \quad a \in G$$

(H is called an *equivariant homotopy*). Then the associated homotopy operator $h: A(M) \leftarrow A(N)$ (cf. sec. 0.14), satisfies

$$h \circ \hat{T}_a^* = T_a^* \circ h.$$

Hence it restricts to a linear map

$$h_I: A_I(M) \leftarrow A_I(N).$$

Now we have (cf. sec. 0.14)

$$\psi_I^* - \varphi_I^* = h_I \circ \delta + \delta \circ h_I,$$

whence $\psi_I^\# = \varphi_I^\#$.

Next, assume that $M = U \cup V$ where U and V are open sets, stable under the action of G. Then so is $U \cap V$ and, as in sec. 5.4, volume I, we can form the sequence

$$0 \longrightarrow A_I(M) \xrightarrow{\alpha} A_I(U) \oplus A_I(V) \xrightarrow{\beta} A_I(U \cap V) \longrightarrow 0,$$

where

$$\alpha(\Phi) = (\Phi|_U, \Phi|_V), \qquad \beta(\Phi, \Psi) = \Phi|_{U \cap V} - \Psi|_{U \cap V}.$$

Lemma I: If G is compact, then the above sequence is short exact.

Proof: Let (f, g) be a partition of unity for M with

$$\operatorname{carr} f \subset U \qquad \text{and} \qquad \operatorname{carr} g \subset V.$$

Let Δ be the unique left invariant n-form on G ($n = \dim G$) such that, with respect to some left orientation, $\int_G \Delta = 1$.

Define new functions $f_I, g_I \in \mathscr{S}(M)$ by

$$f_I(x) = \int_G f(x \cdot a)\, da, \qquad g_I(x) = \int_G g(x \cdot a)\, da.$$

According to Example 2, sec. 3.18, f_I and g_I are invariant. Moreover,

$$\operatorname{carr} f_I \subset (\operatorname{carr} f) \cdot G \subset U \quad \text{and} \quad \operatorname{carr} g_I \subset (\operatorname{carr} g) \cdot G \subset V.$$

Finally,

$$f_I(x) + g_I(x) = \int_G (f+g)(x \cdot a)\, da = \int_G da = 1.$$

It follows that (f_I, g_I) is again a partition of unity for M subordinate to the open cover U, V. Now mimic the proof of Lemma I, sec. 5.4, volume I, using f_I and g_I.

Q.E.D.

Corollary: There is an exact triangle

$$\begin{tikzcd} H_I(M) \arrow[rr] & & H_I(U) \oplus H_I(V) \arrow[dl] \\ & H_I(U \cap V). \arrow[ul] & \end{tikzcd}$$

4.2. Group projection. Consider again an action $T: M \times G \to M$ (with no additional hypothesis on G) and assume that M is connected. Fix a point $z \in M$ and consider the map $A_z: G \to M$ given by $A_z(a) = z \cdot a$. It induces a homomorphism

$$A_z^\#: H(M) \to H(G).$$

If $w \in M$ is a second point then a path $x(t)$ joining z to w provides a homotopy,

$$H: (t, a) \mapsto A_{x(t)}(a),$$

joining A_z and A_w. Hence $A_z^\# = A_w^\#$.

It follows that the homomorphism,

$$p: H(M) \to H(G),$$

defined by $p = A_z^\#$ is independent of the choice of $z \in M$; p is called the *group projection.*

Since A_z is equivariant (with respect to the right action of G on G), it induces a homomorphism

$$(A_z^*)_I: A_I(M) \to A_R(G).$$

Moreover, because the homotopy above is also equivariant, the homomorphism,

$$p_I\colon H_I(M) \to H_R(G),$$

defined by $p_I = (A_z)_I^\#$ is independent of the choice of z; p_I is called the *invariant group projection.*

Example: Let M be a connected Lie group and let G be a subgroup. Consider the right action of G on M given by restricting the group multiplication. Then the map $A_z\colon G \to M$ is given by

$$A_z(a) = za, \qquad z \in M, \quad a \in G.$$

In particular, $A_e = j_G$ is the inclusion map of G into M. Hence

$$p = A_e^\# = j_G^\#.$$

4.3. Compact groups. Assume that G is compact and that T: $M \times G \to G$ is an action. We shall construct a linear map

$$\rho\colon A(M) \to A_I(M)$$

homogeneous of degree zero, and satisfying (cf. sec. 4.1 for i)

$$\rho \circ i = \iota.$$

Orient G and let Δ be the unique left invariant n-form ($n = \dim G$) on G such that (with respect to some left orientation)

$$\int_G \Delta = 1.$$

Regard $(M \times G, \pi_M, M, G)$ as a trivial, oriented bundle, and let $\pi_G\colon M \times G \to G$ denote the projection. Then a linear map, homogeneous of degree zero,

$$I_\Delta\colon A(M \times G) \to A(M),$$

is defined by

$$I_\Delta \Omega = \oint_G \Omega \wedge \pi_G^* \Delta$$

(cf. sec. 0.15).

Thus we can consider the linear map $I_\Delta \circ T^*\colon A(M) \to A(M)$; it is given by

$$(I_\Delta \circ T^*)(\Phi) = \oint_G T^*\Phi \wedge \pi_G^* \Delta.$$

Lemma II: Fix $x \in M$. Then $(I_\Delta \circ T^*)(\Phi)(x) = \int_G (T_a^*\Phi)(x)\, da$.

Proof: The retrenchment of $T^*\Phi \wedge \pi_G^*\Delta$ to $x \times G$ is a $\wedge T_x(M)^*$-valued n-form on G:

$$(T^*\Phi \wedge \pi_G^*\Delta)_x \in A^n(G; \wedge T_x(M)^*).$$

A short, straightforward computation shows that

$$(T^*\Phi \wedge \pi_G^*\Delta)_x = f \cdot \Delta,$$

where $f \in \mathscr{S}(G; \wedge T_x(M)^*)$ is given by $f(a) = (T_a^*\Phi)(x)$.

It follows that

$$[(I_\Delta \circ T^*)\Phi](x) = \int_G (T^*\Phi \wedge \pi_G^*\Delta)_x = \int_G (T_a^*\Phi)(x)\, da.$$

Q.E.D.

Proposition I: (1) If $\Phi \in A(M)$, then $(I_\Delta \circ T^*)\Phi \in A_I(M)$.
(2) If $\Phi \in A_I(M)$, then $(I_\Delta \circ T^*)\Phi = \Phi$.

Proof: (1) Fix $a \in G$, $x \in M$, and write

$$\alpha_a = (\wedge dT_a)_x^* : \wedge T_x(M)^* \leftarrow \wedge T_{x\cdot a}(M)^*.$$

Set $\Psi = (I_\Delta \circ T^*)\Phi$. Then using Lemma II and the linearity of α_a we find

$$\begin{aligned}(T_a^*\Psi)(x) &= \alpha_a(\Psi(x \cdot a)) = \alpha_a\left(\int_G (T_b^*\Phi)(x \cdot a)\, db\right)\\ &= \int_G \alpha_a(T_b^*\Phi(x \cdot a))\, db\\ &= \int_G (T_{ab}^*\Phi)(x)\, db.\end{aligned}$$

Thus formula (1.2), sec. 1.15, yields

$$(T_a^*\Psi)(x) = \int_G (T_b^*\Phi)(x)\, db = \Psi(x),$$

and so Ψ is invariant.

(2) If Φ is invariant, Lemma II yields

$$[(I_\Delta \circ T^*)\Phi](x) = \Phi(x) \int_G da = \Phi(x).$$

Q.E.D.

Proposition I shows that $I_\Delta \circ T^*$ may be regarded as a linear map

$$\rho\colon A(M) \to A_I(M)$$

satisfying $\rho \circ i = \iota$.

Theorem I: Let $M \times G \to M$ be a right action of a compact Lie group. Then

$$i_\#\colon H_I(M) \to H(M)$$

is injective. If G is connected, then $i_\#$ is an isomorphism.

Proof: Recall from sec. 0.15 that $\oint_G \circ \delta = \delta \circ \oint_G$. Hence $\rho \circ \delta = \delta \circ \rho$, where ρ: $A(M) \to A_I(M)$ is the operator constructed above. Thus it induces $\rho_\# : H(M) \to H_I(M)$. Since $\rho \circ i = \iota$, clearly

$$\rho_\# \circ i_\# = \iota$$

and so $i_\#$ is injective.

Next, assume that G is connected. In Theorem II, sec. 4.4 below, we shall construct an operator,

$$h_M\colon A(M \times G) \to A(M),$$

homogeneous of degree -1, such that

$$I_\Delta - j_e^* = \delta h_M + h_M \delta$$

(j_e: $M \to M \times G$ is inclusion opposite e).

Since $T \circ j_e = \iota$, precomposing both sides of this relation with T^* yields

$$i \circ \rho - \iota = I_\Delta \circ T^* - \iota = \delta h_M T^* + h_M T^* \delta.$$

It follows that $i_\# \circ \rho_\# = \iota_{H(M)}$ and hence $i_\#$ is an isomorphism.

Q.E.D.

Remark: Theorem I applies equally well to left actions.

4.4. The operator I_Φ. Let M, N be manifolds with N connected, oriented, and of dimension n. Each $\Phi \in A_c^n(N)$ which satisfies

$$\int_N \Phi = 1$$

determines the operator I_Φ: $A(M \times N) \to A(M)$, given by

$$I_\Phi(\Omega) = \oint_N \Omega \wedge \pi_N^*\Phi.$$

I_Φ is linear and homogeneous of degree zero. It follows from Propositions IX and X, sec. 7.13, volume I (or sec. 0.15), that

$$I_\Phi \circ \delta = \delta \circ I_\Phi \qquad \text{and} \qquad I_\Phi \circ \pi_M^* = \iota.$$

Now fix $b \in N$ and let j_b: $M \to M \times N$ denote inclusion opposite b.

Theorem II: There exists a linear map $h_M : A(M \times N) \to A(M)$, homogeneous of degree -1, and such that

$$I_\Phi - j_b^* = \delta h_M + h_M \delta.$$

Proof: Let $(U, u, \mathbb{R}^n)$ be a chart on N such that $u^{-1}(0) = b$. Choose $\Psi \in A_c^n(U)$ so that

$$\int_U \Psi = \int_N \Psi = 1.$$

Then (cf. Theorem II, sec. 5.13, volume I) there is an $(n - 1)$-form $X \in A_c^{n-1}(N)$ satisfying

$$\Phi - \Psi = \delta X.$$

Fix one such X and define an operator, k_M: $A(M \times N) \to A(M)$, by

$$k_M(\Omega) = (-1)^p \oint_N \Omega \wedge \pi_N^* X, \qquad \Omega \in A^p(M \times N).$$

Then $\delta k_M + k_M \delta = I_\Phi - I_\Psi$.

Next let λ: $M \times U \to M \times N$ denote the inclusion. Since $\Psi \in A_c^n(U)$ it determines an operator,

$$I_\Psi: A(M \times U) \to A(M),$$

which we denote by $\tilde{I}_\Psi$ to avoid confusion. Evidently $\tilde{I}_\Psi \circ \lambda^* = I_\Psi$; hence

$$I_\Phi - \tilde{I}_\Psi \circ \lambda^* = \delta k_M + k_M \delta. \tag{4.1}$$

Finally, fix a homotopy H: $U \times \mathbb{R} \to U$ which connects the identity map with the constant map $U \to b$. Then $\iota \times H$ is a homotopy connecting the identity map of $M \times U$ with $\tilde{j}_b \circ \tilde{\pi}_M$ ($\tilde{\pi}_M$: $M \times U \to M$ is the projection and $\tilde{j}_b : M \to M \times U$ is the inclusion opposite b).

The induced homotopy operator $\tilde{h}_M$ satisfies

$$\tilde{\pi}_M^* \circ \tilde{j}_b^* - \iota = \delta \tilde{h}_M + \tilde{h}_M \delta.$$

It follows as above that $\tilde{I}_\Psi \circ \tilde{\pi}_M^* = \iota$; hence

$$\tilde{j}_b^* - \tilde{I}_\Psi = \delta \tilde{I}_\Psi \tilde{h}_M + \tilde{I}_\Psi \tilde{h}_M \delta.$$

Precompose both sides with λ^* to obtain

$$j_b^* - \tilde{I}_\Psi \circ \lambda^* = \delta\, \tilde{I}_\Psi \tilde{h}_M \lambda^* + \tilde{I}_\Psi \tilde{h}_M \lambda^* \,\delta. \tag{4.2}$$

Thus, setting $h_M = k_M - \tilde{I}_\Psi \tilde{h}_M \lambda^*$, we find, on subtracting (4.2) from (4.1), that

$$I_\Phi - j_b^* = \delta h_M + h_M \delta.$$

Q.E.D.

§2. Left invariant forms on a Lie group

4.5. Left invariant differential forms. Consider the *left* action of G on itself by left translations. The differential forms on G that are invariant under this action will be called *left invariant*, and the graded subalgebra of left invariant differential forms will be denoted by $A_L(G)$. Thus $\Phi \in A_L(G)$ if and only if

$$a \cdot \Phi = \lambda_a^* \Phi = \Phi, \qquad a \in G,$$

or equivalently (when $\deg \Phi = p$) if

$$\Phi(a; L_a h_1, \ldots, L_a h_p) = \Phi(e; h_1, \ldots, h_p), \qquad h_1, \ldots, h_p \in E, \quad a \in G.$$

Now let X_h be a left invariant vector field (cf. sec. 1.2). Since, for each $a \in G$,

$$(\lambda_a)_* X_h = X_h,$$

Proposition III, sec. 4.4, volume I, (or sec. 0.13) implies that the algebra $A_L(G)$ is stable under the operators $i(X_h)$ and $\theta(X_h)$. It is clearly stable under δ. The corresponding cohomology algebra $H(A_L(G), \delta)$ will be denoted by $H_L(G)$.

Proposition II: The correspondence, $\Phi \mapsto \Phi(e)$, defines an isomorphism,

$$\tau_L: A_L(G) \xrightarrow{\cong} \wedge E^*,$$

of graded algebras. In particular, the left invariant functions are constant.

Proof: According to Proposition I, sec. 1.2, a strong bundle isomorphism,

$$G \times E \xrightarrow{\cong} T_G,$$

is given by $(a, h) \mapsto X_h(a)$. It induces a strong bundle isomorphism,

$$\varphi: G \times \wedge E^* \xrightarrow{\cong} \wedge T_G^*,$$

and so we obtain an isomorphism,

$$\varphi_*: \mathscr{S}(G; \wedge E^*) \xrightarrow{\cong} A(G).$$

A simple computation (using the left invariance of the vector fields X_h) shows that the diagrams

$$\begin{array}{ccc} \mathscr{S}(G; \wedge E^*) & \xrightarrow[\cong]{\varphi_*} & A(G) \\ \lambda_a^* \downarrow & & \downarrow \lambda_a^* \\ \mathscr{S}(G; \wedge E^*) & \xrightarrow[\cong]{\varphi_*} & A(G), \end{array} \qquad a \in G,$$

commute (cf. sec. 0.13). It follows that the left invariant forms correspond under φ_* to the constant functions $G \to \wedge E^*$. The proposition follows.

Q.E.D.

Corollary: $A_L(G)$ is the exterior algebra over the vector space $A_L^1(G)$.

4.6. The differential algebra $\wedge E^*$. Since $A_L(G)$ is stable under the operators $i(X_h)$, $\theta(X_h)$ $(h \in E)$, and δ, there are uniquely determined operators $i_E(h)$, $\theta_E(h)$, and δ_E in $\wedge E^*$, such that

$$\tau_L \circ i(X_h) = i_E(h) \circ \tau_L, \qquad \tau_L \circ \theta(X_h) = \theta_E(h) \circ \tau_L, \qquad h \in E,$$

and

$$\tau_L \circ \delta = \delta_E \circ \tau_L.$$

$i_E(h)$ and δ_E are antiderivations in $\wedge E^*$, homogeneous of degrees -1 and $+1$, respectively, while $\theta_E(h)$ is a derivation, homogeneous of degree zero.

From Proposition I, sec. 4.2, and Proposition II, sec. 4.3, both of volume I (or sec. 0.13), we obtain the relations

$$i_E([h, k]) = \theta_E(h)\, i_E(k) - i_E(k)\, \theta_E(h), \qquad \theta_E([h, k]) = \theta_E(h)\, \theta_E(k) - \theta_E(k)\, \theta_E(h),$$

$$\theta_E(h) = i_E(h)\, \delta_E + \delta_E i_E(h)$$

and

$$\delta_E^2 = 0, \qquad h, k \in E.$$

The second formula shows that θ_E is a representation of E in the vector space $\wedge E^*$. Since δ_E is an antiderivation in $\wedge E^*$ whose square is zero, $(\wedge E^*, \delta_E)$ is a graded differential algebra. The corresponding cohomology algebra is called the *cohomology algebra of the Lie algebra E* and will be denoted by $H(E)$.

It follows from the definitions that

$$\tau_L\colon A_L(G) \xrightarrow{\cong} \wedge E^*$$

is an isomorphism of differential algebras. Thus it induces an isomorphism

$$(\tau_L)_\#\colon H_L(G) \xrightarrow{\cong} H(E).$$

Now we shall determine the operators $i_E(h)$, $\theta_E(h)$, and δ_E explicitly.

Proposition III: The operators $i_E(h)$, $\theta_E(h)$, and δ_E are given by

(1) $[i_E(h)\Phi](h_1, \dots, h_{p-1}) = \Phi(h, h_1, \dots, h_{p-1})$.

(2) $[\theta_E(h)\Phi](h_1, \dots, h_p) = -\sum\limits_{i=1}^{p} \Phi(h_1, \dots, [h, h_i], \dots, h_p)$.

(3) $[\delta_E\Phi](h_0, h_1, \dots, h_p) = \sum\limits_{i<j} (-1)^{i+j}\Phi([h_i, h_j], h_0, \dots, \hat{h}_i, \dots, \hat{h}_j, \dots, h_p)$,

$$\Phi \in \wedge^p E^*, \; h_i \in E.$$

Proof: (1) is immediate. To obtain (2) observe that, for $\Psi \in A_L^p(G)$,

$$\begin{aligned}[\theta_E(h)\,\tau_L\Psi](h_1, \dots, h_p) &= [\theta(X_h)\Psi](e; h_1, \dots, h_p)\\ &= X_h(\Psi(X_{h_1}, \dots, X_{h_p}))(e) - \sum_{i=1}^{p} \Psi(e; h_1, \dots, [h, h_i], \dots, h_p).\end{aligned}$$

Since Ψ is left invariant, so is the function $\Psi(X_{h_1}, \dots, X_{h_p})$. Thus this function is constant, and the first term in the above equation is zero. This proves (2). (3) follows in the same way.

Q.E.D.

Example: Let $h^* \in E^*$. Then

$$\langle\theta_E(h)\, h^*, k\rangle = -\langle h^*, [h, k]\rangle = -\langle h^*, (\operatorname{ad} h)k\rangle$$

and

$$\langle\delta_E h^*, h \wedge k\rangle = -\langle h^*, [h, k]\rangle, \qquad h, k \in E.$$

Hence the restriction of $\theta_E(h)$ to E^* is given by

$$\theta_E(h) = -(\operatorname{ad} h)^*, \qquad h \in E,$$

while the restriction of δ_E to E^* satisfies

$$\delta_E^*(h \wedge k) = -[h, k].$$

4.7. Homomorphisms. Let $\varphi: G \to K$ be a homomorphism of Lie groups. Let F be the Lie algebra of K and recall that the derivative of φ at e,

$$\varphi': E \to F,$$

is a homomorphism of Lie algebras (sec. 1.3).

For every $h \in E$, the vector fields $X_h \in \mathscr{X}_L(G)$, $X_{\varphi'(h)} \in \mathscr{X}_L(K)$ are φ-related. Thus Proposition III, sec. 4.4, volume I (or sec. 0.13) gives the relations

$$i(X_h) \circ \varphi^* = \varphi^* \circ i(X_{\varphi'(h)}),$$

$$\theta(X_h) \circ \varphi^* = \varphi^* \circ \theta(X_{\varphi'(h)}),$$

and

$$\delta \circ \varphi^* = \varphi^* \circ \delta.$$

On the other hand, the equation $\lambda_a^* \circ \varphi^* = \varphi^* \circ \lambda_{\varphi(a)}^*$, $a \in G$, shows that φ^* restricts to a homomorphism

$$\varphi_L^*: A_L(G) \leftarrow A_L(K).$$

It is immediate from the definitions that the diagram,

$$\begin{tikzcd} A_L(G) \arrow[d, "\tau_L"', "\cong"] & A_L(K) \arrow[l, "\varphi_L^*"'] \arrow[d, "\tau_L", "\cong"'] \\ \wedge E^* & \wedge F^* \arrow[l, "(\wedge\varphi')^*"] \end{tikzcd},$$

commutes.

But this yields the relations

$$i_E(h) \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ i_F(\varphi'(h)),$$

$$\theta_E(h) \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ \theta_F(\varphi'(h)),$$

and

$$\delta_E \circ (\wedge\varphi')^* = (\wedge\varphi')^* \circ \delta_F, \qquad h \in E.$$

In particular, $(\wedge\varphi')^*$ is a homomorphism of graded differential algebras. Thus it induces a homomorphism of cohomology algebras, which we denote by

$$(\varphi')^\#: H(E) \leftarrow H(F).$$

It follows from the definitions that

$$\begin{array}{ccc} H_L(G) & \xleftarrow{\varphi_L^\#} & H_L(K) \\ {\scriptstyle (\tau_L)_\#}\downarrow\cong & & \cong\downarrow{\scriptstyle (\tau_L)_\#} \\ H(E) & \xleftarrow[(\varphi')^\#]{} & H(F) \end{array}$$

commutes.

4.8. The adjoint representation in $\wedge E^*$. Consider the adjoint representation of G (cf. sec. 1.10). The contragredient representation, $\mathrm{Ad}^\natural$, of G in E^* extends to a representation, $\wedge \mathrm{Ad}^\natural = \sum_p \wedge^p \mathrm{Ad}^\natural$, of G in the graded algebra $\wedge E^*$; it will be denoted by $\mathrm{Ad}^\wedge$. Thus

$$\mathrm{Ad}^\wedge(a)(h^{*1} \wedge \cdots \wedge h^{*p}) = (\mathrm{Ad}\, a^{-1})^* h^{*1} \wedge \cdots \wedge (\mathrm{Ad}\, a^{-1})^* h^{*p}.$$

On the other hand, recall that in sec. 4.6 we defined a representation, θ_E, of E in $\wedge E^*$.

Lemma III: θ_E is the derivative of the representation $\mathrm{Ad}^\wedge$.

Proof: Since $\theta(X_h)$ is a derivation in $A_L(G)$, $\theta_E(h)$ is a derivation in $\wedge E^*$. On the other hand, if $\theta^\wedge$ denotes the derivative of $\mathrm{Ad}^\wedge$, then

$$\theta^\wedge(h)(h^{*1} \wedge \cdots \wedge h^{*p}) = \sum_{i=1}^{p} h^{*1} \wedge \cdots \wedge \theta^\wedge(h)\, h^{*i} \wedge \cdots \wedge h^{*p}$$

(cf. Example 2, sec. 1.9). It follows that $\theta^\wedge(h)$ is a derivation in $\wedge E^*$ as well. Hence we need only prove that

$$\theta^\wedge(h)\, h^* = \theta_E(h)\, h^*, \qquad h \in E, \quad h^* \in E^*.$$

But this follows from the example in sec. 4.6.

Q.E.D.

Next, fix $a \in G$. Since $\mathrm{Ad}\ a$ is the derivative of the inner automorphism τ_a, it follows that

$$\mathrm{Ad}^\wedge(a^{-1}) = \wedge(\tau_a')^*.$$

Hence $\mathrm{Ad}^\wedge(a^{-1})$ commutes with δ_E. In particular, the representation, $\mathrm{Ad}^\wedge$, of G in $\wedge E^*$ induces a representation,

$$a \mapsto \mathrm{Ad}^\#(a),$$

of G in $H(E)$.

Lemma IV: If G is connected, then the representation $\mathrm{Ad}^\#$ is trivial,

$$\mathrm{Ad}^\#(a) = \iota, \qquad a \in G.$$

Proof: It follows from Lemma III and Example 4, sec. 1.9, that the derivative of the representation $\mathrm{Ad}^\#$ is given by

$$h \mapsto \theta_E(h)^\#, \qquad h \in E.$$

But by the relations of sec. 4.6

$$\theta_E(h) = i_E(h)\,\delta_E + \delta_E\, i_E(h).$$

Hence $\theta_E(h)^\# = 0$.

Since G is connected, the lemma follows now from Proposition IX, sec. 1.8.

Q.E.D.

Proposition IV: If G is a connected Lie group, then

$$\det(\iota - \mathrm{Ad}\, a) = 0, \qquad a \in G.$$

Proof: Elementary considerations from linear algebra (cf. sec. A.2) show that

$$\det(\iota - \mathrm{Ad}\, a) = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p \mathrm{Ad}\, a = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^*.$$

We have seen above that

$$\mathrm{Ad}^\wedge(a^{-1}) = \sum_{p=0}^{n} \wedge^p (\mathrm{Ad}\, a)^*$$

is an automorphism of the graded differential algebra $\wedge E^*$. Hence the algebraic Lefschetz formula (cf. sec. 0.8) yields

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^* = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \mathrm{Ad}^{(p)}(a^{-1}),$$

where $\mathrm{Ad}^{(p)}(a^{-1})$ denotes the restriction of $\mathrm{Ad}^\#(a^{-1})$ to $H^p(E)$.

Now Lemma IV yields

$$\begin{aligned}\sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, a)^* &= \sum_{p=0}^{n} (-1)^p \operatorname{tr} \mathrm{Ad}^{(p)}(e)^* \\ &= \sum_{p=0}^{n} (-1)^p \operatorname{tr} \wedge^p (\mathrm{Ad}\, e)^*;\end{aligned}$$

i.e.,

$$\det(\iota - \operatorname{Ad} a) = \det(\iota - \operatorname{Ad} e) = \det(0) = 0.$$

Q.E.D.

Corollary: Let $a \in G$. Then of the normalizer, N_a, of a (Example 4, sec. 2.4) has at least dimension 1.

Proof: It follows from the proposition that there exists a nonzero vector $h \in E$ such that

$$(\operatorname{Ad} a)h = h.$$

Hence h is in the Lie algebra of N_a (cf. Example 4, sec. 2.4) and so $\dim N_a \geqslant 1$.

Q.E.D.

§3. Invariant cohomology of Lie groups

4.9. Invariant forms. A differential form $\Phi \in A(G)$ will be called *bi-invariant*, or simply *invariant*, if

$$\lambda_a^*\Phi = \Phi \quad \text{and} \quad \rho_a^*\Phi = \Phi, \qquad a \in G.$$

The set of invariant differential forms is a graded subalgebra of $A(G)$ which we denote by $A_I(G)$. Clearly $A_I(G)$ is stable under δ.

Proposition V: The invariant forms on G are closed.

Lemma V: If $\nu: G \to G$ is the inversion map of G, then

$$\nu^*\Phi = (-1)^p\Phi, \qquad \Phi \in A_I^p(G).$$

Proof: We have (cf. sec. 1.1)

$$(d\nu)_a = -R_{a^{-1}} \circ L_a^{-1}, \qquad a \in G.$$

Thus, for $a \in G$, $h_1, ..., h_p \in E$,

$$\begin{aligned}(\nu^*\Phi)(a; L_a h_1, ..., L_a h_p) &= \Phi(a^{-1}; -R_{a^{-1}}h_1, ..., -R_{a^{-1}}h_p)\\ &= (-1)^p(\rho_{a^{-1}}^*\Phi)(e; h_1, ..., h_p)\\ &= (-1)^p\Phi(a; L_a h_1, ..., L_a h_p).\end{aligned}$$

Q.E.D.

Proof of the proposition: Since $A_I(G)$ is stable under δ, the lemma yields

$$(-1)^{p+1}\,\delta\Phi = \nu^*\,\delta\Phi = \delta\nu^*\Phi = (-1)^p\,\delta\Phi, \qquad \Phi \in A_I^p(G),$$

whence $\delta\Phi = 0$.

Q.E.D.

It follows from Proposition V that the inclusion $A_I(G) \to A_L(G)$ induces a homomorphism of graded algebras

$$A_I(G) \to H_L(G).$$

Next we determine the subalgebra of $\wedge E^*$ corresponding to $A_I(G)$ under τ_L (cf. sec. 4.5).

Lemma VI: If $\Phi \in A_L(G)$, then $\rho_a^*\Phi \in A_L(G)$ and (cf. sec. 4.8)

$$\tau_L(\rho_a^*\Phi) = \mathrm{Ad}^\wedge(a)(\tau_L\Phi), \qquad a \in G.$$

Proof: That $A_L(G)$ is stable under ρ_a^*, $a \in G$, follows (cf. sec. 1.1) from the relation

$$\rho_a^* \circ \lambda_b^* = \lambda_b^* \circ \rho_a^*, \qquad a, b \in G.$$

Moreover, if $\Phi \in A_L(G)$, then

$$\tau_L(\rho_a^*\Phi) = (\rho_a^*\lambda_{a^{-1}}^*\Phi)(e) = (\tau_{a^{-1}}^*\Phi)(e)$$

($\tau_{a^{-1}}$ is conjugation by a^{-1}). Also $\tau_{a^{-1}}(e) = e$ and

$$\wedge(\tau'_{a^{-1}})^* = \mathrm{Ad}^\wedge(a).$$

Thus

$$\tau_L(\rho_a^*\Phi) = (\wedge\tau'_{a^{-1}})^*(\Phi(e)) = \mathrm{Ad}^\wedge(a)(\tau_L\Phi).$$

Q.E.D.

Now let $(\wedge E^*)_I$ denote the subalgebra of $\wedge E^*$ invariant with respect to the representation $\mathrm{Ad}^\wedge$. Lemma VI implies that the isomorphism τ_L restricts to an isomorphism $\tau_I : A_I(G) \xrightarrow{\cong} (\wedge E^*)_I$. Thus the diagram

$$\begin{tikzcd} A_I(G) \arrow[r] \arrow[d, "\tau_I"', "\cong"] & A_L(G) \arrow[d, "\cong"', "\tau_L"] \\ (\wedge E^*)_I \arrow[r] & \wedge E^* \end{tikzcd}$$

commutes (the horizontal maps are inclusions).

In particular the elements of $(\wedge E^*)_I$ are in the kernel of δ_E, and we have the commutative diagram

$$\begin{tikzcd} A_I(G) \arrow[r] \arrow[d, "\tau_I"', "\cong"] & H_L(G) \arrow[d, "\cong"', "(\tau_L)_*"] \\ (\wedge E^*)_I \arrow[r] & H(E). \end{tikzcd}$$

Finally, if G is connected, then $(\wedge E^*)_I = (\wedge E^*)_{\theta_E=0}$ (cf. Lemma III, sec. 4.8, and Proposition IX, sec. 1.8).

4.10. Compact connected groups. Suppose G is connected. The inclusion $A_L(G) \to A(G)$ induces a homomorphism $H_L(G) \to H(G)$. Combining this with the diagram just above yields the commutative diagram

$$\begin{tikzcd} A_I(G) \arrow[r] \arrow[d, "\tau_I"', "\cong"] & H_L(G) \arrow[r] \arrow[d, "\cong"', "(\tau_L)_*"] & H(G) \\ (\wedge E^*)_{\theta=0} \arrow[r] & H(E). & \end{tikzcd}$$

Theorem III: If G is compact and connected, all the above maps are isomorphisms (of algebras).

Proof: It is sufficient to show that the inclusions

$$A_I(G) \to A(G), \qquad A_L(G) \to A(G)$$

induce isomorphisms $A_I(G) \xrightarrow{\cong} H(G)$, $H_L(G) \xrightarrow{\cong} H(G)$.

Since $A_L(G)$ is the algebra of differential forms invariant under the left action of G on itself, Theorem I, sec. 4.3. implies that $H_L(G) \to H(G)$ is an isomorphism.

On the other hand, consider the right action, T, of the compact connected group $G \times G$ on G given by

$$T_{(a,b)}(x) = a^{-1}xb, \qquad a, b, x \in G.$$

$A_I(G)$ is the algebra of differential forms on G which are invariant under this action. Since the forms in $A_I(G)$ are closed, Theorem I, sec. 4.3, implies that $A_I(G) \to H(G)$ is an isomorphism.

Q.E.D.

Corollary: The Poincaré polynomial, $f_G(t)$, of a compact connected Lie group G is given by

$$f_G(t) = \int_G \det(\mathrm{Ad}\, a + t\iota)\, da.$$

Proof: By definition (cf. sec. 0.14)

$$f_G(t) = \sum_{p=0}^{n} b_p t^p$$

($n = \dim G$ and $b_p = \dim H^p(G)$).

It follows from Theorem III that

$$b_p = \dim(\wedge^p E^*)_I, \qquad p = 0, 1, \ldots, n.$$

Hence Corollary III to Proposition XV, sec. 1.16 (applied with $P = \mathrm{Ad}$) yields

$$\sum_{p=0}^{n} b_p t^p = \int_G \det(\mathrm{Ad}\, a + t\iota)\, da.$$

Q.E.D.

4.11. Noncompact groups. It can be shown that every connected Lie group contains a compact subgroup as deformation retract (cf. [9, p. 180]). Thus the computation of the cohomology of any Lie group is reduced to the compact case. In particular, as is shown in the example below, the group $SO(n)$ is a deformation retract of $GL^+(n; \mathbb{R})$ (the 1-component of $GL(n; \mathbb{R})$) and hence the cohomology algebras of these groups are isomorphic.

It will be shown in volume III, that the map,

$$A_I(G) \to H_L(G),$$

which is induced by the inclusion map is still an isomorphism if the Lie algebra of G is *reductive*. On the other hand, the homomorphism,

$$H_L(G) \to H(G),$$

is *not* in general an isomorphism if the group is not compact.

In fact, as will be shown in volume III, if the adjoint representation of G is semisimple, then

$$\dim H_L^n(G) = 1, \qquad n = \dim G.$$

(This holds in particular for $G = GL^+(n; \mathbb{R})$.) On the other hand, if G contains a compact subgroup K of lower dimension which is a deformation retract of G (for example $G = GL^+(n; \mathbb{R})$, $K = SO(n)$), we have

$$H^n(G) \cong H^n(K) = 0.$$

Thus $H^n(G)$ and $H_L^n(G)$ are not isomorphic.

Examples: 1. Let V be an n-dimensional Euclidean space, and denote the space of self-adjoint transformations of V by $S(V)$. Then the map,

$$\alpha: SO(V) \times S(V) \to GL^+(V),$$

given by $\alpha(\varphi, \psi) = \varphi \exp \psi$ is a diffeomorphism. In particular, $SO(V)$ is a deformation retract of $GL^+(V)$.

In fact, it was shown in Example 11, sec. 1.5, volume I, that exp maps $S(V)$ diffeomorphically onto the open subset $S^+(V) \subset S(V)$ of self-adjoint transformations with strictly positive eigenvalues. In particular, since $\exp 2\varphi = (\exp \varphi)^2$, $\varphi \in S(V)$, it follows that the map

$$\sigma \mapsto \sigma^2$$

is a diffeomorphism of $S^+(V)$. Denote its inverse by

$$\sigma \mapsto \sigma^{1/2}$$

and write $(\sigma^{-1})^{1/2} = \sigma^{-1/2}$.

Then a smooth map $\beta\colon GL^+(V) \to SO(V) \times S(V)$ is given by

$$\beta(\varphi) = (\varphi \circ (\varphi^* \circ \varphi)^{-1/2}, \exp^{-1}(\varphi^* \circ \varphi)^{1/2})$$

and β is inverse to α. Thus α is a diffeomorphism.

2. Similarly, if W is an complex n-dimensional Hermitian space, then the map

$$\alpha\colon U(W) \times S(W) \to GL(W)$$

given by

$$\alpha(\varphi, \psi) = \varphi \exp \psi$$

is a diffeomorphism. ($S(W)$ is the space of self-adjoint transformations of W.)

§4. Cohomology of compact connected Lie groups

In this article G denotes a compact connected Lie group.

4.12. The primitive space and the main theorem. Since G is compact, we have the Künneth isomorphism (cf. sec. 0.14)

$$\kappa_{\#}: H(G) \otimes H(G) \xrightarrow{\cong} H(G \times G).$$

Henceforth we shall identify $H(G \times G)$ and $H(G) \otimes H(G)$ under this isomorphism. Thus, if $\mu: G \times G \to G$ denotes the multiplication map, $\mu^{\#}$ becomes a homomorphism

$$\mu^{\#}: H(G) \otimes H(G) \leftarrow H(G).$$

Let $j_1: G \to G \times G$ and $j_2: G \to G \times G$ be the inclusion maps given by

$$j_1(a) = (a, e) \qquad \text{and} \qquad j_2(a) = (e, a).$$

In view of Example 2, sec. 5.17, volume I, if $\gamma \in H^+(G \times G)$, then

$$\gamma = j_1^{\#}\gamma \otimes 1 + \beta + 1 \otimes j_2^{\#}\gamma, \tag{4.3}$$

where $\beta \in H^+(G) \otimes H^+(G)$. Observing that $\mu \circ j_1 = \mu \circ j_2 = \iota$, we obtain

$$\mu^{\#}\alpha = \alpha \otimes 1 + \beta + 1 \otimes \alpha, \qquad \alpha \in H^+(G), \quad \beta \in H^+(G) \otimes H^+(G). \tag{4.4}$$

Definition: An element $\alpha \in H^+(G)$ is called *primitive* if

$$\mu^{\#}\alpha = \alpha \otimes 1 + 1 \otimes \alpha.$$

The primitive elements form a graded subspace, P_G, of $H(G)$ (i.e., $P_G = \sum_{p=0}^{n} P_G \cap H^p(G)$) and $P_G \cap H^p(G) = 0$, if p is even. To see the latter, assume that α is primitive, and has even degree. Then the elements $\alpha \otimes 1$ and $1 \otimes \alpha$ commute; whence

$$\mu^{\#}(\alpha^m) = (\alpha \otimes 1 + 1 \otimes \alpha)^m = \sum_{k=0}^{m} \binom{m}{k} \alpha^k \otimes \alpha^{m-k}, \qquad m \geq 0.$$

Now choose m to be the least integer such that $\alpha^m = 0$. Then

$$\sum_{k=0}^{m} \binom{m}{k} \alpha^k \otimes \alpha^{m-k} = 0.$$

It follows that

$$\alpha^k \otimes \alpha^{m-k} = 0, \qquad k = 0, \ldots, m.$$

In particular, $\alpha \otimes \alpha^{m-1} = 0$ and so $\alpha = 0$.

Since every homogeneous element of P_G has odd degree, it follows that the square of a primitive element in $H(G)$ is zero. Thus the inclusion map $P_G \to H(G)$ extends to a homomorphism

$$\lambda_G \colon \wedge P_G \to H(G)$$

of graded algebras. The purpose of this article is to establish

Theorem IV: Let G be a compact connected Lie group. Then λ_G is an isomorphism. Moreover, if r is the dimension of a maximal torus, then

$$\dim P_G = r \qquad \text{and} \qquad \dim H(G) = 2^r.$$

Definition: The number r is called the *rank of* G.

Although the actual proof of Theorem IV does not come till sec. 4.17, the key steps are established in the preceding section (Propositions VIII, IX, and X). These in turn depend on the preliminary results on power maps which are proved in sec. 4.14 and sec. 4.15.

However, before proceeding with the proof, we consider the case that our group is a torus.

4.13. Cohomology of a torus. Let T be an r-dimensional torus with Lie algebra F. Since T is abelian, the adjoint representation is trivial, and hence $(\wedge F^*)_I = \wedge F^*$. Thus Theorem III, sec. 4.10, yields an isomorphism

$$\alpha_T : \wedge F^* \xrightarrow{\cong} H(T).$$

Moreover, if S is a second torus with Lie algebra L and $\varphi \colon S \to T$ is a homomorphism, then the diagram

$$\begin{tikzcd} H(S) & \wedge L^* \arrow[l, "\alpha_S"', "\cong"] \\ H(T) \arrow[u, "\varphi^*"] & \wedge F^* \arrow[l, "\cong"', "\alpha_T"] \arrow[u, "\wedge(\varphi')^*"'] \end{tikzcd}$$

commutes.

Now, since T is abelian, the multiplication map $\mu\colon T \times T \to T$ is a homomorphism.

The derivative of μ at e is the linear map, $\mu'\colon F \oplus F \to F$, given by $\mu'(h, k) = h + k$. Hence $(\mu')^*(h^*) = h^* \otimes 1 + 1 \otimes h^*$, $h^* \in F^*$. Thus the diagram above reads

$$\begin{tikzcd} H(T) \otimes H(T) & \wedge F^* \otimes \wedge F^* \arrow[l, "\alpha_T \otimes \alpha_T"', "\cong"] \\ H(T) \arrow[u, "\mu^*"] & \wedge F^*. \arrow[l, "\cong", "\alpha_T"'] \arrow[u, "\wedge(\mu')^*"'] \end{tikzcd}$$

It follows at once that α_T restricts to a linear isomorphism from F^* onto the primitive subspace of $H(T)$. (This proves Theorem IV for tori.)

4.14. The power maps. The kth *power map* $P_k\colon G \to G$ is defined by

$$P_k(x) = x^k, \qquad P_0(x) = e, \qquad P_{-k}(x) = (x^{-1})^k, \qquad k \geqslant 1.$$

In particular, P_1 is the identity and P_{-1} is the inversion map, ν.

Example: The power maps, P_k, for an r-dimensional torus, T, are homomorphisms. Moreover P_k' is simply scalar multiplication by k. Thus it follows from sec. 4.13 that $P_k^\#$ is given by

$$P_k^\# \alpha = k^p \alpha, \qquad \alpha \in H^p(T).$$

In particular, the degree and Lefschetz number of P_k are given by

$$\deg P_k = k^r \qquad \text{and} \qquad L(P_k) = \sum_{p=0}^{r} (-1)^p \binom{r}{p} k^p = (1 - k)^r.$$

(cf. sec. 0.14, and note from sec. 4.13 that $\dim H^p(T) = \binom{r}{p}$.)

In the next sections we generalize these results to arbitrary compact connected Lie groups G.

Let $\mu\colon G \times G \to G$ and $\sigma\colon G \times G \to G \times G$ denote, respectively, the multiplication map and the interchange map $\sigma(x, y) = (y, x)$.

Proposition VI: With the notation above

(1) $\nu^\#(\alpha) = (-1)^p \alpha, \qquad \alpha \in H^p(G).$

(2) The diagram

$$\begin{tikzcd} H(G) \otimes H(G) & \\ & H(G) \arrow[ul, "\mu^*"'] \arrow[dl, "\mu^*"] \\ H(G) \otimes H(G) \arrow[uu, "\sigma^*"] & \end{tikzcd}$$

commutes.

(3) The diagrams

$$\begin{tikzcd} H(G) \otimes H(G) & H(G) \arrow[l, "\mu^*"'] \\ H(G) \otimes H(G) \arrow[u, "P_k^\# \otimes P_k^\#"] & H(G), \arrow[l, "\mu^*"] \arrow[u, "P_k^\#"'] \end{tikzcd} \qquad k \in \mathbb{Z},$$

commute.

Proof: (1) This follows from Lemma V, sec. 4.9, and Theorem III, sec. 4.10.

(2) Since the inversion map $\nu_{G \times G}$ for the Lie group $G \times G$ is given by

$$\nu_{G\times G}(a, b) = (a^{-1}, b^{-1}),$$

it follows that $\mu \circ \sigma \circ \nu_{G\times G} = \nu \circ \mu$. Hence (1) yields, for $\alpha \in H^p(G)$,

$$(-1)^p \mu^\# \alpha = \mu^\# \nu^\# \alpha = (\nu^\#_{G\times G} \circ \sigma^\# \circ \mu^\#)\alpha = (-1)^p \sigma^\# \mu^\# \alpha.$$

(3) Let

$$\mu_p: \underbrace{G \times \cdots \times G}_{(p \text{ factors})} \to G$$

be the multiplication map and let σ_i be the diffeomorphism of $G \times \cdots \times G$ that interchanges the ith and the $(i + 1)$-th component:

$$\sigma_i(x_1, \dots, x_p) = (x_1, \dots, x_{i+1}, x_i, \dots, x_p).$$

Then (2) implies that $\sigma_i^\# \circ \mu_p^\# = \mu_p^\#$. It follows that if τ is any permutation of the elements $(1, \dots, p)$ and if τ also denotes the diffeomorphism

$$\tau: (x_1, \dots, x_p) \mapsto (x_{\tau(1)}, \dots, x_{\tau(p)}),$$

then

$$\tau^\# \circ \mu_p^\# = \mu_p^\#.$$

Now fix $k \geqslant 1$ and define maps

$$\Delta_1, \Delta_2 : G \times G \to \underset{(2k \text{ factors})}{G \times \cdots \times G}$$

by

$$\Delta_1(x, y) = (x, ..., x, y, ..., y) \qquad \text{and} \qquad \Delta_2(x, y) = (x, y, ..., x, y).$$

Then

$$\mu_{2k} \circ \Delta_1 = \mu \circ (P_k \times P_k) \qquad \text{and} \qquad \mu_{2k} \circ \Delta_2 = P_k \circ \mu.$$

Since, for a suitable permutation τ, $\Delta_1 = \tau \circ \Delta_2$, and since $\tau^\# \circ \mu_{2k}^\# = \mu_{2k}^\#$, it follows that

$$\mu^\# \circ P_k^\# = \Delta_2^\# \circ \mu_{2k}^\# = \Delta_1^\# \circ \mu_{2k}^\# = (P_k \times P_k)^\# \circ \mu^\#.$$

The case $k \leqslant -1$ can be treated in the same way and the case $k = 0$ is obvious.

Q.E.D.

4.15. The Lefschetz class. In this section we assume that G is oriented. Denote its orientation class by $\omega_G \in H^n(G)$ (cf. sec. 0.14). Define the *quotient map*

$$q: G \times G \to G$$

by $q(a, b) = a^{-1}b$.

Proposition VII: The Lefschetz class, Λ_G, for G is given by

$$\Lambda_G = q^\# \omega_G$$

(cf. sec. 10.3, volume I).

Proof: Let π_L, π_R: $G \times G \to G$ be the left and right projections. It has to be shown (cf. Corollary I to Proposition I, sec. 10.3, volume I) that

$$\oint_G^\# \pi_R^\# \alpha \cdot q^\# \omega_G = \alpha, \qquad \alpha \in H(G).$$

Let φ be the diffeomorphism of $G \times G$ given by

$$\varphi(a, b) = (a, ab).$$

φ is a fibre preserving and orientation preserving map of the trivial bundle $(G \times G, \pi_L, G, G)$. Moreover, it induces the identity map in the base. Hence Proposition VIII, sec. 7.12, volume I, yields

$$\oint_G \circ \varphi^* = \oint_G,$$

whence

$$\oint_G^\# \varphi^\#(\pi_R^\# \alpha \cdot q^\# \omega_G) = \oint_G^\# \pi_R^\# \alpha \cdot q^\# \omega_G, \qquad \alpha \in H(G).$$

But $q \circ \varphi = \pi_R$ and so this relation becomes

$$\oint_G^\# \varphi^\# \pi_R^\# \alpha \cdot \pi_R^\# \omega_G = \oint_G^\# \pi_R^\# \alpha \cdot q^\# \omega_G, \qquad \alpha \in H(G).$$

It remains to prove that

$$\alpha = \oint_G^\# \varphi^\# \pi_R^\# \alpha \cdot \pi_R^\# \omega_G, \qquad \alpha \in H(G).$$

Recall that we identify $H(G) \otimes H(G)$ with $H(G \times G)$ via the Künneth isomorphism $\kappa_\#$ (cf. sec. 0.14). It follows from Example 2, sec. 5.17, volume I, that if $\gamma \in H(G \times G)$, then

$$\gamma - j_1^\# \gamma \otimes 1 \in H(G) \otimes H^+(G),$$

where $j_1 : G \to G \times G$ is given by $j_1(a) = (a, e)$. Since $\omega_G \cdot H^+(G) = 0$, this yields

$$\gamma \cdot \pi_R^\# \omega_G = j_1^\# \gamma \otimes \omega_G = \pi_L^\# j_1^\# \gamma \cdot \pi_R^\# \omega_G .$$

Now set $\gamma = \varphi^\# \pi_R^\# \alpha$. Observing that $\pi_R \circ \varphi \circ j_1 = \iota$ we find that

$$\oint_G^\# \varphi^\# \pi_R^\# \alpha \cdot \pi_R^\# \omega_G = \oint_G^\# \pi_L^\# \alpha \cdot \pi_R^\# \omega_G = \alpha \cdot \int_G^\# \omega_G = \alpha$$

(cf. Example 2, sec. 7.12, volume I).

Q.E.D.

Corollary I: Let M be a compact connected oriented manifold and let $\varphi, \psi: M \to G$ ($\dim M = \dim G = n$) be smooth maps. Then the coincidence number (cf. sec. 0.14) for φ and ψ is given by

$$L(\varphi, \psi) = \deg(\varphi^{-1} \cdot \psi),$$

where $\varphi^{-1} \cdot \psi \colon M \to G$ is given by

$$(\varphi^{-1} \cdot \psi)(x) = \varphi(x)^{-1} \cdot \psi(x), \qquad x \in M.$$

Proof: Apply Proposition VII, sec. 10.7, volume I, noting that

$$\varphi^{-1} \cdot \psi = q \circ (\varphi \times \psi) \circ \Delta_M,$$

where $\Delta_M : M \to M \times M$ is the diagonal map.

Q.E.D.

Corollary II: Let $\varphi \colon G \to G$ be a smooth map and denote by $\varphi^{(p)}$ the restriction of $\varphi^{\#}$ to $H^p(G)$. Then the Lefschetz number of φ is given by

$$L(\varphi) = \sum_{p=0}^{n} (-1)^p \operatorname{tr} \varphi^{(p)} = \deg \varphi_1 ,$$

where $\varphi_1 = \varphi^{-1} \cdot \iota$.

Corollary III: Let $k \in \mathbb{Z}$. Then the Lefschetz number of the power map P_k is given by

$$L(P_k) = \deg P_{1-k} .$$

In particular, the Euler–Poincaré characteristic of G is 0 (set $k = 1$).

4.16. The spaces $H_p(G)$. Let T be a maximal torus in G and let $r = \dim T$. Recall that a smooth map $\psi \colon G/T \times T \to G$ is given by

$$\psi(\pi a, y) = aya^{-1}, \qquad a \in G, \quad y \in T,$$

where $\pi \colon G \to G/T$ denotes the projection (cf. sec. 2.17).

Clearly, the diagrams

$$\begin{array}{ccc} G/T \times T & \xrightarrow{\psi} & G \\ {\scriptstyle \iota \times \hat{P}_k}\big\downarrow & & \big\downarrow{\scriptstyle P_k} \\ G/T \times T & \xrightarrow[\psi]{} & G, \end{array} \qquad k \in \mathbb{Z}, \tag{4.5}$$

commute, where $\hat{P}_k$ denotes the power map for T.

These yield the commutative diagrams

$$\begin{tikzcd} H(G/T) \otimes H(T) & H(G) \arrow[l, "\psi^*"'] \\ H(G/T) \otimes H(T) \arrow[u, "\iota \otimes \hat{P}_k^*"] & H(G), \arrow[l, "\psi^*"] \arrow[u, "P_k^*"'] \end{tikzcd} \qquad k \in \mathbb{Z}. \tag{4.6}$$

Proposition VIII: Let $H_p(G)$ denote the eigenspace of the linear map $P_2^\# : H(G) \to H(G)$ corresponding to the eigenvalue 2^p. Then

(1) $H(G) = \sum_{p=0}^r H_p(G)$.

(2) For every $k \neq 0$, $H_p(G)$ is an eigenspace of the linear map, $P_k^\#$, corresponding to the eigenvalue k^p.

Proof: Recall from the example of sec. 4.14 that, for $\alpha \in H^p(T)$,

$$\hat{P}_k^\#(\alpha) = k^p \cdot \alpha.$$

Thus $H(G/T) \otimes H(T)$ is the direct sum of the eigenspaces $H(G/T) \otimes H^p(T)$ of $\iota \otimes \hat{P}_k^\#$ corresponding to the eigenvalues k^p $(p = 0, \ldots, r)$.

In view of the diagram above, $\operatorname{Im} \psi^\#$ is stable under the map $\iota \otimes \hat{P}_k^\#$. This implies that

$$\operatorname{Im} \psi^\# = \sum_{p=0}^{r} \operatorname{Im} \psi^\# \cap [H(G/T) \otimes H^p(T)].$$

Next observe that, according to Proposition IV, sec. 2.18, $\deg \psi \neq 0$ and so $\psi^\#$ is injective (cf. Corollary I to Proposition III, sec. 6.5, volume I). Hence the relation above shows that

$$H(G) = \sum_{p=0}^{r} (\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$$

and that $P_k^\#$ restricts to $k^p \cdot \iota$ in $(\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$. In particular, it follows that

$$H_p(G) = (\psi^\#)^{-1}(H(G/T) \otimes H^p(T))$$

(consider the case $k = 2$), and so both parts of the proposition are obvious.

Q.E.D.

Corollary: $\mu^{\#}$ restricts to linear maps

$$\mu^{\#}: H_p(G) \to \sum_{i+j=p} H_i(G) \otimes H_j(G).$$

Proof: Apply Proposition VI (3), sec. 4.14.

Q.E.D.

Lemma VII: Each space $H_p(G)$ is graded,

$$H_p(G) = \sum_{q=0}^{n} H_p^q(G),$$

where $H_p^q(G) = H_p(G) \cap H^q(G)$. Moreover, if $p \not\equiv q \pmod 2$, then $H_p^q(G) = 0$.

Proof: The first part of the lemma is obvious. Now assume that $\alpha \in H_p^q(G)$. Then Proposition VI, (1), sec. 4.14, yields

$$(-1)^q \alpha = \nu^{\#} \alpha = P_{-1}^{\#} \alpha = (-1)^p \alpha.$$

Thus, if $p \not\equiv q \pmod 2$, $\alpha = 0$.

Q.E.D.

Proposition IX: The dimension of $H_p(G)$ is given by

$$\dim H_p(G) = \binom{r}{p}, \qquad 0 \leqslant p \leqslant r,$$

where $r = \dim T$.

Proof: First observe that, in view of the commutative diagram (4.5),

$$\deg \psi \cdot \deg(\iota \times \hat{P}_k) = \deg P_k \cdot \deg \psi$$

so that (cf. Proposition IV, sec. 2.18)

$$\deg P_k = \deg \hat{P}_k = k^r, \qquad k \in \mathbb{Z}.$$

Thus in view of Corollary III to Proposition VII, sec. 4.15, we have

$$L(P_k) = \deg P_{1-k} = (1-k)^r = \sum_{p=0}^{r} (-k)^p \binom{r}{p}.$$

On the other hand, Proposition VIII and Lemma VII give

$$L(P_k) = \sum_{p,q} (-1)^q k^p \dim H_p^q(G)$$

$$= \sum_{p,q} (-1)^p k^p \dim H_p^q(G) = \sum_{p=0}^{r} (-k)^p \dim H_p(G), \qquad k \in \mathbb{Z}.$$

These relations yield

$$\sum_{p=0}^{r} k^p \left(\dim H_p(G) - \binom{r}{p}\right) = 0, \qquad k = 1, 2, \dots .$$

Since the Vandermonde matrix (k^p) $(k = 1, \dots, r+1, p = 0, \dots, r)$ has nonzero determinant, we obtain

$$\dim H_p(G) = \binom{r}{p}.$$

Q.E.D.

Corollary: $H_0(G) = H^0(G) = \mathbb{R}$.

Proposition X: The spaces $H_1(G)$ and P_G coincide.

Proof: Let $\alpha \in P_G$. Then, if $\Delta : G \to G \times G$ is the diagonal map,

$$P_2^\#(\alpha) = \Delta^\# \mu^\# \alpha = \Delta^\#(\alpha \otimes 1 + 1 \otimes \alpha) = 2\alpha,$$

whence $\alpha \in H_1(G)$.

On the other hand, if $\alpha \in H_1(G)$, the corollary to Proposition VIII implies that

$$\mu^\# \alpha \in H_1(G) \otimes \mathbb{R} + \mathbb{R} \otimes H_1(G).$$

Hence, by formula (4.4), sec. 4.12, $\alpha \in P_G$.

Q.E.D.

4.17. Proof and consequences of Theorem IV. Lemma VIII: If $\alpha_1, \dots, \alpha_k \in P_G$ are homogeneous and linearly independent, then

$$\prod_{i=1}^{k} \alpha_i \neq 0.$$

Proof: Suppose $\deg \alpha_i = p_i$ with $p_1 \leqslant \cdots \leqslant p_k$. Then the component of $\mu^{\#}(\alpha_1 \cdot \cdots \cdot \alpha_k)$ in $H^{p_1}(G) \otimes H^{p_2+\cdots+p_k}(G)$ is given by

$$\sum_i (-1)^{i-1} \alpha_i \otimes (\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k),$$

where the sum ranges over those indices i such that $\deg \alpha_i = p_1$. We may assume by induction that $\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k \neq 0$ $(i = 1, ..., k)$. Since the α_i are linearly independent, it follows that

$$\sum_i (-1)^{i-1} \alpha_i \otimes (\alpha_1 \cdot \cdots \hat{\alpha}_i \cdots \cdot \alpha_k) \neq 0,$$

whence $\mu^{\#}(\alpha_1 \cdot \cdots \cdot \alpha_k) \neq 0$. In particular, $\alpha_1 \cdot \cdots \cdot \alpha_k \neq 0$.

Q.E.D.

Proof of Theorem IV: Lemma VIII implies that λ_G is injective. On the other hand, by Propositions IX and X, sec. 4.16,

$$\dim P_G = \dim H_1(G) = r \qquad \text{and} \qquad \dim H(G) = 2^r.$$

Thus $\dim \wedge P_G = \dim H(G)$, and so λ_G is an isomorphism.

Q.E.D.

Corollary I: The cohomology algebra of G is isomorphic to the cohomology algebra of the product of r spheres ($r = \dim T$) each of which has odd dimension.

Proof: Choose a homogeneous basis of P_G and denote by P_j the subspace generated by the jth basis vector. Then P_j is a graded one-dimensional vector space whose elements are homogeneous of degree g_j (g_j odd). Hence (cf. sec. 5.6 and sec. 5.20, volume I)

$$\begin{aligned} H(G) \cong \wedge P_G \cong \wedge P_1 \otimes \cdots \otimes \wedge P_r \cong H(S^{g_1}) \otimes \cdots \otimes H(S^{g_r}) \\ \cong H(S^{g_1} \times \cdots \times S^{g_r}). \end{aligned}$$

Q.E.D.

Corollary II: The Poincaré polynomial of G is of the form

$$f_G(t) = (1 + t^{g_1}) \cdots (1 + t^{g_r}), \qquad g_i \text{ odd}.$$

In particular

$$\sum_{q=1}^{n} (-1)^q \dim H^q(G) = 0, \quad \sum_{\mu=1}^{r} g_\mu = n, \quad \text{and} \quad n \equiv r \pmod 2 \qquad (n = \dim G).$$

Corollary III: The exponents g_i are all equal to 1 if and only if G is a torus.

Corollary IV: The isomorphism λ_G restricts to isomorphisms

$$\lambda_G^p : \wedge^p P_G \xrightarrow{\cong} H_p(G).$$

Proof: Since $P_2^\#$ is a homomorphism, it follows that

$$H_i(G) \cdot H_j(G) \subset H_{i+j}(G),$$

Thus Proposition X, sec. 4.16, implies that $\lambda_G(\wedge^p P_G) \subset H_p(G)$. But, in view of Proposition IX, sec. 4.16,

$$\dim H_p(G) = \binom{r}{p} = \dim \wedge^p P_G\,.$$

Q.E.D.

Corollary V: Let $\varphi: G \to K$ be a smooth map between compact connected Lie groups such that

$$(\varphi x)^2 = \varphi(x^2), \qquad x \in G.$$

Then $\varphi^\#$ restricts to a linear map $\varphi_P : P_G \leftarrow P_K$ and the diagram

$$\begin{tikzcd} \wedge P_G \arrow[r, "\lambda_G", "\cong"'] & H(G) \\ \wedge P_K \arrow[u, "\wedge\varphi_P"] \arrow[r, "\cong", "\lambda_K"'] & H(K) \arrow[u, "\varphi^\#"'] \end{tikzcd}$$

commutes. In particular, if $K = G$, then

$$L(\varphi) = \det(\iota - \varphi_P).$$

Proof: Observe that $P_G = H_1(G)$ is the eigenspace of the map $P_2^\#$ corresponding to the eigenvalue 2, and conclude that $\varphi^\#(P_K) \subset P_G$. The commutativity of the diagram follows immediately. In view of Lemma VII, sec. 4.16, this implies that

$$L(\varphi) = \sum_{p=0}^{r} (-1)^p \operatorname{tr} \wedge^p \varphi_P = \det(\iota - \varphi_P).$$

Q.E.D.

§5. Homogeneous spaces

In this article K denotes a closed q-dimensional subgroup of G with Lie algebra F. The left action of G on G/K is denoted by $T\colon G \times G/K \to G/K$.

4.18. The representation $\mathrm{Ad}^{\perp}$. Since K is a subgroup of G, its Lie algebra F is stable under the operators $\mathrm{Ad}\, y$ $(y \in K)$. Thus the orthogonal complement, $F^{\perp}$, of F in E^* is stable under $\mathrm{Ad}^{\natural}(y)$ $(y \in K)$. The restrictions of these operators to $F^{\perp}$ define a representation

$$\mathrm{Ad}^{\perp}\colon K \to GL(F^{\perp}).$$

It extends to a representation, $\wedge \mathrm{Ad}^{\perp}$, of K in the exterior algebra $\wedge F^{\perp}$.

Now consider the projection, $\pi\colon G \to G/K$, and recall (Corollary I, sec. 2.11) that $(d\pi)_e$ induces a linear isomorphism

$$E/F \xrightarrow{\cong} T_{\bar{e}}(G/K).$$

Hence the dual map can be regarded as a linear isomorphism

$$(d\pi)_e^*\colon T_{\bar{e}}(G/K)^* \xrightarrow{\cong} F^{\perp}$$

and $\wedge(d\pi)_e^*$ is an isomorphism

$$\wedge T_{\bar{e}}(G/K)^* \xrightarrow{\cong} \wedge F^{\perp}.$$

Since $\pi(yxy^{-1}) = \pi(yx) = y \cdot \pi(x)$, $y \in K$, $x \in G$, we have

$$(d\pi)_e \circ \mathrm{Ad}\; y = (dT_y)_{\bar{e}} \circ (d\pi)_e\,,$$

whence

$$\wedge \mathrm{Ad}^{\perp}(y^{-1}) \circ \wedge(d\pi)_e^* = \wedge(d\pi)_e^* \circ \wedge(dT_y)_{\bar{e}}^*, \qquad y \in K. \tag{4.7}$$

Next denote by $A_I(G/K)$ the algebra of differential forms on G/K invariant under the action of G. Since π is equivariant with respect to the left action of G on itself, π^* restricts to a homomorphism

$$\pi_I^*\colon A_I(G/K) \to A_L(G).$$

On the other hand, let $(\wedge F^{\perp})_I$ denote the invariant subalgebra of the representation $\wedge \mathrm{Ad}^{\perp}$. Relation (4.7) shows that if $\Phi \in A_I(G/K)$, then

$$\wedge(d\pi)_e^*(\Phi(\bar{e})) \in (\wedge F^{\perp})_I .$$

Thus a homomorphism, σ: $A_I(G/K) \to (\wedge F^{\perp})_I$, is defined by

$$\sigma(\Phi) = \wedge(d\pi)_e^*(\Phi(\bar{e})).$$

Recall the isomorphism, τ_L, of sec. 4.5.

Proposition XI: σ is an isomorphism of graded algebras which makes the diagram,

$$\begin{tikzcd} A_L(G) \arrow[r, "\tau_L", "\cong"'] & \wedge E^* \\ A_I(G/K) \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "\sigma"'] & (\wedge F^{\perp})_I \arrow[u, "i"'] \end{tikzcd} ,$$

commute (where i is the inclusion).

Proof: Evidently,

$$\sigma(\Phi) = (\pi^*\Phi)(e) = (\tau_L \pi_I^*)(\Phi), \qquad \Phi \in A_I(G/K),$$

and so the diagram commutes. Since π is a submersion, π^* is injective; it follows that σ is injective. It remains to prove that σ is surjective.

Fix $\alpha \in (\wedge F^{\perp})_I$ and let $\beta \in \wedge T_{\bar{e}}(G/K)^*$ be the unique element satisfying

$$\wedge(d\pi)_e^*(\beta) = \alpha.$$

Since α is invariant we have, for $y \in K$,

$$\wedge(dT_y)_{\bar{e}}^*(\beta) = \beta.$$

Thus a set map, Ψ: $G/K \to \wedge T_{G/K}^*$, is defined by

$$\Psi(\pi(x)) = \wedge(dT_{x^{-1}})_{\bar{e}}^*(\beta), \qquad x \in G.$$

To check that Ψ is a differential form, let $\Phi \in A_L(G)$ be the unique left invariant form such that $\Phi(e) = \alpha$. Then, for $x \in G$,

$$\Phi(x) = \wedge(d\pi)_x^* \Psi(\pi(x)).$$

Fix $\bar{x} \in G/K$ and let φ: $U \to G$ be a local cross-section, where U is a neighbourhood of $\bar{x}$ (cf. Corollary II, sec. 2.11). The relation just obtained implies that, in U, $\varphi^*\Phi = \Psi$. Hence Ψ is a differential form.

Ψ is clearly invariant and satisfies $\sigma(\Psi) = \alpha$. It follows that σ is surjective.

Q.E.D.

Corollary: Assume that K is compact and connected. Then G/K can be oriented by an invariant $(n - q)$-form.

Proof: Since K is compact and connected, $\det \mathrm{Ad}^{\perp}(y) = 1$, for $y \in K$ (cf. the example of sec. 1.13). It follows that

$$\dim A_I^{n-q}(G/K) = \dim(\wedge^{n-q}F^{\perp})_I = 1.$$

Every nonzero element of this space orients G/K.

Q.E.D.

4.19. Invariant cohomology. It is an immediate consequence of Proposition XI, sec. 4.18, that $(\wedge F^{\perp})_I$ is stable under the operator δ_E defined in sec. 4.6. Thus we have the commutative diagram

$$\begin{tikzcd} H_L(G) \arrow[r, "(\tau_L)_*", "\cong"'] & H(E) \\ H_I(G/K) \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "\sigma_*"'] & H((\wedge F^{\perp})_I, \delta_E). \arrow[u, "i_*"'] \end{tikzcd}$$

Applying Theorem I, sec. 4.3, we obtain

Theorem V: Suppose that G and K are compact and connected. Then, in the commutative diagram,

$$\begin{tikzcd} H(G) & H_L(G) \arrow[l, "\cong"'] \arrow[r, "(\tau_L)_*", "\cong"'] & H(E) \\ H(G/K) \arrow[u, "\pi^*"] & H_I(G/K) \arrow[l, "\cong"'] \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "\sigma_*"'] & H((\wedge F^{\perp})_I), \arrow[u, "i_*"'] \end{tikzcd}$$

all horizontal maps are isomorphisms.

4.20. Invariant Euler-Poincaré characteristic. Suppose again that K is an arbitrary closed subgroup of the Lie group G.

Then the *invariant Euler–Poincaré characteristic* of G/K is defined by

$$\chi_I(G/K) = \sum_{p=0}^{n-q} (-1)^p \dim H_I^p(G/K) = \sum_{p=0}^{n-q} (-1)^p \dim H^p((\wedge F^\perp)_I).$$

Now assume that K is compact. Then there exists an inner product $\langle\ ,\ \rangle$ in E, invariant under the transformations $\mathrm{Ad}\, y$, $y \in K$ (cf. Proposition XVI, sec. 1.17). If we identify E^* with E under this inner product, then $F^\perp$ becomes the orthogonal complement of F in E, and we have the direct decomposition $E = F^\perp \oplus F$. Moreover, in this case $\mathrm{Ad}^\perp(y)$ is simply the restriction of $\mathrm{Ad}\, y$ to $F^\perp$.

Proposition XII: If K is compact, then

$$\chi_I(G/K) = \int_K \det(\iota - \mathrm{Ad}^\perp(y))\, dy.$$

Proof: It follows from Corollary III to Proposition XV, sec. 1.16, that

$$\int_K \det(\iota - \mathrm{Ad}^\perp(y))\, dy = \sum_{p=0}^{n-q} (-1)^p \dim(\wedge^p F^\perp)_I .$$

On the other hand, the algebraic Lefschetz formula (sec. 0.8) yields

$$\sum_{p=0}^{n-q} (-1)^p \dim(\wedge^p F^\perp)_I = \sum_{p=0}^{n-q} (-1)^p \dim H^p((\wedge F^\perp)_I) = \chi_I(G/K).$$

Q.E.D.

Corollary: If K is connected, then $\chi_I(G/K) \geqslant 0$. Equality holds if and only if, for every $y \in K$,

$$F^\perp \cap T_e(N_y) \neq 0,$$

where N_y denotes the normalizer of y.

Proof: Since K is compact and connected, $\mathrm{Ad}^\perp(y)$ is a proper rotation with respect to a suitable Euclidean inner product in $F^\perp$. Hence,

$$\det(\iota - \mathrm{Ad}^\perp(y)) \geqslant 0, \qquad y \in K.$$

Now the proposition shows that $\chi_I(G/K) \geqslant 0$ and $\chi_I(G/K) = 0$ if and only if

$$\det(\iota - \mathrm{Ad}^\perp(y)) = 0, \qquad y \in K;$$

i.e., if and only if, for every $y \in K$, there exists a nonzero vector $h \in F^\perp$ satisfying $\mathrm{Ad}^\perp(y)h = h$. But these are precisely the vectors of $F^\perp \cap T_e(N_y)$ (cf. Example 4, sec. 2.4).

Q.E.D.

4.21. Euler–Poincaré characteristic. Proposition XIII: Let K be a closed connected subgroup of a compact connected Lie group G. Then the Euler–Poincaré characteristic of G/K is given by

$$\chi_{G/K} = \int_K \det(\iota - \mathrm{Ad}^\perp(x))\, dx.$$

In particular, $\chi_{G/K} \geqslant 0$. Moreover, $\chi_{G/K} > 0$ if and only if G and K have the same rank, and in this case

$$\chi_{G/K} = |W_G|/|W_K|.$$

Proof: The first formula follows from Proposition XII and Theorem V, and shows that $\chi_{G/K} \geqslant 0$.

Let S be a maximal torus in K, and let L, F, and E denote the Lie algebras of S, K, and G. Write

$$E = F \oplus F^\perp = L \oplus (L^\perp \cap F) \oplus F^\perp.$$

Let $\mathrm{Ad}_K^\perp$, $\mathrm{Ad}^\perp$, and $\mathrm{Ad}_G^\perp$ denote the representations of S in $L^\perp \cap F$, $F^\perp$ and $(L^\perp \cap F) \oplus F^\perp$, induced by the adjoint representation of G; thus

$$\mathrm{Ad}_G^\perp(y) = \mathrm{Ad}_K^\perp(y) \oplus \mathrm{Ad}^\perp(y), \qquad y \in S.$$

The Weyl integration formula (cf. Theorem IV, sec. 2.19) yields

$$\int_K \det(\iota - \mathrm{Ad}^\perp(x))\, dx = |W_K|^{-1} \int_S \det(\iota - \mathrm{Ad}^\perp(y)) \cdot \det(\iota - \mathrm{Ad}_K^\perp(y))\, dy.$$

On the other hand, since $\mathrm{Ad}_G^\perp(y) = \mathrm{Ad}_K^\perp(y) \oplus \mathrm{Ad}^\perp(y)$, it follows that

$$\det(\iota - \mathrm{Ad}^\perp(y)) \cdot \det(\iota - \mathrm{Ad}_K^\perp(y)) = \det(\iota - \mathrm{Ad}_G^\perp(y)), \qquad y \in S.$$

Thus the first formula in the proposition applied to both G/S and G/K gives

$$\chi_{G/K} = |W_K|^{-1} \int_S \det(\iota - \mathrm{Ad}_G^\perp(y))\, dy = |W_K|^{-1} \chi_{G/S}.$$

Now assume that rank $K <$ rank G. Then S is not a maximal torus in G. Hence the corollary to Proposition XII implies that $\chi_{G/S} = 0$; it follows that $\chi_{G/K} = 0$.

On the other hand, if rank K = rank G, then S is a maximal torus in G. Now the first formula in the proposition (applied when $K = S$) together with the corollary to Theorem IV, sec. 2.20, yields $\chi_{G/S} = |W_G|$. This shows that

$$\chi_{G/K} = |W_G|/|W_K|.$$

Q.E.D.

Problems

1. Left invariant p-vector fields. A *p-vector* field on an n-manifold M is a cross-section in the vector bundle $\wedge^p \tau_M$. Denote the space of p-vector fields on M by $A_p(M)$ $(p = 0, \ldots, n)$.

(i) Given a Lie group G, use left multiplication to define a left action of G on τ_G, $\wedge^p \tau_G$ and on $A_p(G)$. A p-vector field Φ is called *left invariant*, if it is invariant under this action. The space of left invariant p-vector fields on G is denoted by $A_p^L(G)$.

(ii) Show that the map, $\tau^L\colon A_p^L(G) \to \wedge^p E$, given by evaluation at e, is an isomorphism.

(iii) Consider the space $D_p(G)$ of p-densities on G $(0 \leqslant p \leqslant n)$ and let $\partial\colon D_p(G) \to D_{p-1}(G)$ denote the divergence operator (cf. problem 8, Chap. IV, volume I). Show that an isomorphism,

$$\mu\colon A_p(G) \xrightarrow{\cong} D_p(G) \qquad (p = 0, \ldots, n),$$

is defined by $\mu(\Phi) = \Phi \otimes \Delta$, where Δ is a fixed nonzero left invariant n-form on G. Define an operator, $\partial_G : A_p(G) \to A_{p-1}(G)$, by

$$\partial_G = \mu^{-1} \circ \partial \circ \mu.$$

Show that ∂_G is independent of the choice of Δ. Show that ∂_G restricts to an operator in the space of left invariant multivector fields.

2. Let G be a Lie group with Lie algebra E.

(i) Use ∂_G (cf. problem 1) to obtain an operator ∂_E in $\wedge E$.

(ii) Show that ∂_E is explicitly given by

$$\partial_E(h_1 \wedge \cdots \wedge h_p) = \sum_{i<j} (-1)^{i+j+1} [h_i, h_j] \wedge h_1 \wedge \cdots \hat{h}_i \cdots \hat{h}_j \cdots h_p .$$

(iii) Show that the operators ∂_E and $-\delta_E$ are dual, where δ_E is the operator in $\wedge E^*$ defined in sec. 4.6.

(iv) Establish the Koszul formula

$$\delta_E = \frac{1}{2} \sum_\nu \mu(e^{*\nu}) \circ \theta_E(e_\nu),$$

where $\{e^\nu\}$, $\{e^{*\nu}\}$ is a pair of dual bases for E and E^*. *Hint*: Show that both sides are antiderivations in $\wedge E^*$.

(v) Find an analogous formula for ∂_E.

3. Define $\Phi \in A^1(GL^+(n; \mathbb{R}))$ by $\Phi(\alpha; \varphi) = \operatorname{tr}(\alpha^{-1} \circ \varphi)$, $\alpha \in GL^+(n; \mathbb{R})$, $\varphi \in L_{\mathbb{R}^n}$.

(i) Show that Φ is biinvariant.

(ii) Construct a scalar function f on $GL^+(n; \mathbb{R})$ such that $\delta f = \Phi$.

4. Let $T\colon M \times G \to M$ be a right action of a Lie group on a manifold M. Denote by $\theta(M)$ the subspace of $A(M)$ that is linearly generated by the differential forms $\theta(h)\Phi$, $h \in E$, $\Phi \in A(M)$.

(i) Prove the formula

$$\theta(h) \circ T_a^* - T_a^* \circ \theta(h) = \theta(h - \operatorname{Ad}(a)\, h) \circ T_a^*, \qquad a \in G, \quad h \in E.$$

Conclude that $\theta(M)$ is stable under T_a^*, $a \in G$.

(ii) If G is connected, show that

$$T_a^*\Phi - \Phi \in \theta(M), \qquad \Phi \in A(M).$$

(iii) Assume that G is compact and connected and let $\rho\colon A(M) \to A_I(M)$ denote the projection defined by

$$\rho(\Phi) = \int_G T_a^*\Phi \, da.$$

Prove that $\ker \rho = \theta(M)$, so that

$$A(M) = A_I(M) \oplus \theta(M) = A(M)_{\theta=0} \oplus \theta(M).$$

5. Let G_1, G_2 be Lie groups with Lie algebras E_1, E_2.

(i) Establish a canonical isomorphism $H(E_1 \oplus E_2) \cong H(E_1) \otimes H(E_2)$.

(ii) If E_1 is unimodular, show that multiplication in $H(E_1)$ determines nondegenerate scalar products, $H^p(E_1) \times H^{n-p}(E_1) \to \mathbb{R}$, where $n = \dim E_1$ (Poincaré duality).

(iii) Assume that G_1 and G_2 are connected and compact. Show that the Künneth isomorphism and the Poincaré isomorphisms correspond to the isomorphisms (i) and (ii) under the map of Theorem III.

6. Let $H \subset K \subset G$ be a sequence of compact connected Lie groups.

(i) Construct a subgroup $W_{K,G}$ of the Weyl group W_G and a surjective homomorphism $W_{K,G} \to W_K$ (cf. problem 25, Chap. II). If K has the same rank as G, show that this is an isomorphism; i.e., that W_K is a subgroup of W_G.

(ii) Show that $\chi_{G/K}$ is the index of W_K in W_G. Conclude that $\chi_{G/H} = \chi_{G/K} \cdot \chi_{K/H}$.

(iii) If L is a compact subgroup of G with 1-component L^0, show that

$$\chi_{G/L} \cdot |L/L^0| = \chi_{G/L^0}.$$

7. (i) Use the Weyl integration formula and residue calculus to show that the Poincaré polynomial of $U(n)$ is the coefficient of $(z_1 \cdot \cdots \cdot z_n)^{2n-1}$ in the polynomial, P, given by

$$n!\, P(z_1, \ldots, z_n) = \prod_{\nu,\mu=1}^{n} (tz_\nu + z_\mu) \cdot \prod_{1 \leqslant \nu \neq \mu \leqslant n} (z_\nu - z_\mu)$$

(t, a parameter). Show that the Poincaré polynomials of $U(2)$ and $U(3)$ are respectively given by

$$f(t) = (1+t)(1+t^3) \qquad \text{and} \qquad f(t) = (1+t)(1+t^3)(1+t^5).$$

(ii) Compute $H_I(SO(3))$ and verify that it coincides with $H(\mathbb{R}P^3)$.

(iii) Compute $H_L(SL(2;\mathbb{R}))$, $H_I(SL(2;\mathbb{R}))$, and $H(SL(2;\mathbb{R}))$.

8. Let E be the Lie algebra of a compact connected Lie group G.

(i) Show that $H^1(E) = Z_E^*$ and that $H(E) \cong \wedge Z_E^* \otimes H(E')$ (cf. problem 7, Chap. II). Interpret these statements in terms of $H(G)$.

(ii) Show that $H^3(G) = P_G \cap H^3(G)$, if $E = E'$.

(iii) Assume that G is not abelian and let K denote the Killing form of E (cf. problem 7, Chap. II). Define a 3-linear function Φ in E by

$$\Phi(h_1, h_2, h_3) = K(h_1, [h_2, h_3]), \qquad h_j \in E.$$

Show that Φ is skew-symmetric and depends only on the vectors in E'. Show that Φ is invariant, and conclude that it represents nonzero classes $\alpha_E \in H^3(E')$ and $\alpha_G \in H^3(G)$.

(iv) Show that the only spheres which are Lie groups are S^1 and S^3.

9. Conjugation. The set of elements in a Lie group G conjugate to a given element a is called the *conjugacy class of a*. The set of elements in the Lie algebra E of G of the form $(\mathrm{Ad}\, x)h$ (fixed h, all $x \in G$) is called the *conjugacy class of h*.

(i) Show that each conjugacy class is an embedded homogeneous space.

(ii) Show that "exp" maps the conjugacy class of h onto the conjugacy class of $\exp h$. Identify the sets of conjugacy classes in G (respectively, in E) with an orbit space of an action of G. Denote the second orbit space by E/G.

(iii) Assume G is compact and regard the elements of $(\vee E^*)_I$ as functions in E. Show that, for such a function $f, f(h)$ depends only on the conjugacy class of h.

(iv) In volume III it will be shown that, if G is compact and connected, then $(\vee E^*)_I$ is a polynomial algebra over a graded subspace Q_E with $\dim Q_E = \mathrm{rank}\, G$. Use this fact to obtain an embedding of E/G in $\mathbb{R}^r$ ($r = \mathrm{rank}\, G$).
Show that the image of the embedding contains an open set of $\mathbb{R}^r$.

(v) Assume G compact and connected. Show that an automorphism, τ, of G determines a homeomorphism $\bar{\tau}: E/G \to E/G$. Show that, for $f \in (\vee E^*)_I$, $\bar{h} \in E/G$,

$$f(\bar{\tau}\bar{h}) = ((\tau')^\vee f)(\bar{h}).$$

(vi) Let G, τ be as in (v). In volume III we shall construct a linear isomorphism $\lambda: P_G \xrightarrow{\cong} Q_E$ such that $\lambda \circ \tau^\# = (\tau')^\vee \circ \lambda$. Use this fact to conclude that $\tau^\#$ is the identity map of $H(G)$ if and only if $\bar{\tau}$ is the identity map of E/G.

10. Automorphisms. Let τ be an automorphism of a compact connected Lie group G. τ is called *inner* if, for some $a \in G$, $\tau(x) = axa^{-1}$, $x \in G$.

(i) Show that $\tau^\#(\alpha_G) = \alpha_G$, where α_G is the class defined in problem 8, (iii).

(ii) Let $Z(\tau) = \{x \in G \mid \tau(x) = x\}$. Show that $Z(\tau)$ is a compact Lie subgroup of G. If $G' \neq e$, show that $Z(\tau)$ contains a nontrivial 1-parameter subgroup of G'. If S is a maximal torus of $Z(\tau)$ conclude that its centralizer Z_S is a maximal torus of G.

(iii) Suppose $Z(\tau)$ contains a maximal torus of G. Prove that τ is inner. (*Hint*: Use problems 28 and 29, Chap. II).

(iv) Show that the following conditions are equivalent: (a) $\tau^\# = \iota$.

(b) τ is inner. (c) for each $x \in G$ there is some $a_x \in G$ such that $\tau(x) = a_x x a_x^{-1}$. (*Hint*: Use problem 9, (vi).)

11. Toral actions. Let a torus, T, with Lie algebra E act on a manifold M so that the isotropy subgroups are all different from T. Choose $h \in E$ so that the 1-parameter subgroup generated by h is dense in T.

(i) Show that the fundamental vector field Z_h has no zeros.

(ii) Give M a T-invariant Riemannian metric. Define a 1-form ω on M by

$$\omega(X) = \langle Z_h , Z_h\rangle^{-1}\langle Z_h , X\rangle, \qquad X \in \mathscr{X}(M).$$

Show that $i(h)\omega = 1$ and, for $a \in T$, $k \in E$,

$$T_a^*\omega = \omega, \qquad \theta(k)\omega = 0.$$

(iii) Set

$$A(M)_{i(h)=0} = \ker i(h), \qquad A(M)_{\theta(h)=0} = \ker \theta(h)$$

and

$$A(M)_{i(h)=0,\theta(h)=0} = A(M)_{i(h)=0} \cap A(M)_{\theta(h)=0} .$$

Show that the multiplication induces an isomorphism,

$$A(M)_{i(h)=0,\theta(h)=0} \otimes \wedge\omega \xrightarrow{\cong} A_I(M),$$

where $\wedge\omega$ denotes the exterior algebra over the one-dimensional space spanned by ω.

(iv) Show that $A(M)_{i(h)=0,\theta(h)=0}$ is stable under δ and that

$$\delta\omega \in A^2(M)_{i(h)=0,\theta(h)=0} .$$

Show that the differential operator d in the tensor product, induced by δ under the isomorphism of (iii), is given by ($p = \deg \Psi$)

$$d(\Phi \otimes 1 + \Psi \otimes \omega) = \delta\Phi \otimes 1 + \delta\Psi \otimes \omega + (-1)^p\, \delta\omega \wedge \delta\Psi \otimes 1.$$

(v) Obtain a short exact sequence of differential spaces

$$0 \longrightarrow A(M)_{i(h)=0,\theta(h)=0} \xrightarrow{\lambda} A_I(M) \xrightarrow{i(h)} A(M)_{i(h)=0,\theta(h)=0} \longrightarrow 0,$$

where λ is the inclusion map. Derive an exact triangle

$$\begin{tikzcd} H(A(M)_{i(h)=0,\theta(h)=0}) \arrow[rr, "\lambda_\#"] & & H(M) \arrow[dl, "i(h)_\#"] \\ & H(A(M)_{i(h)=0,\theta(h)=0}). \arrow[ul, "D"] & \end{tikzcd}$$

If $\tau_h \in H^2(A(M)_{i(h)=0,\,\theta(h)=0})$ is the class represented by $\delta\omega$, show that

$$D(\alpha) = \tau_h \cdot \alpha, \qquad \alpha \in H(A(M)_{i(h)=0,\theta(h)=0}).$$

(vi) Show that $H(M)$ has finite dimension if and only if $H(A(M)_{i(h)=0,\,\theta(h)=0})$ has finite dimension.

(vii) Assume that $H(M)$ has finite dimension. Show that $\chi_M = 0$ (even if M is not compact). Show that the Lefschetz number of an equivariant map is zero.

(viii) If M is compact and dim $M = 4k$, prove that M has signature zero.

(ix) Show that any toral action on $\mathbb{R}^n$ has a fixed point.

12. Action on homogeneous spaces. Let G be a compact connected Lie group and let K be a closed connected subgroup. Let T be the action of G on G/K.

(i) Show that the isotropy subgroups are all conjugate to K. Hence show that each T_a has a fixed point if and only if

$$\bigcup_{a\in G} aKa^{-1} = G.$$

(ii) Let a be a generator of a maximal torus in G. Show that the fixed point set of T_a is finite (possibly empty). Show that the set of elements $a \in G$ such that T_a has only finitely many fixed points, is dense in G.

(iii) Obtain the results of the text and problem 6 on $\chi_{G/K}$ by considering the Lefschetz number of T_a, where a is a generator of a maximal torus.

(iv) If rank $G = r$, rank $K = s$, show that a subtorus of rank $r - s$ can act almost freely on G/K. Show that this is the maximum dimension for such an action.

13. Symmetric spaces. Let τ be an automorphism of a compact connected Lie group G such that $\tau^2 = \iota$. Let K be the 1-component of the subgroup of G left pointwise fixed by τ. Then G/K is called a *symmetric space of compact type with connected fibre.* We refer to it simply as a *symmetric space.* Denote the Lie algebras of G and K by E and F.

(i) Show that a compact connected Lie group is diffeomorphic to a symmetric space.

(ii) Let G/K be a symmetric space. Show that the restriction of δ_E to $(\wedge F^{\perp})_I$ is zero and conclude that

$$H(G/K) \cong (\wedge F^{\perp})_I .$$

(iii) Assume G is compact and connected. Show that there are elements $a \in G$ such that $\tau_a \neq \iota$, $\tau_a^2 = \iota$, where τ_a is conjugation by a. Let K be the 1-component of the centralizer of a. Show that $a \in K$ and that $\tau_a' = -\iota$ in $F^{\perp}$. Conclude that $(\wedge F^{\perp})_I$ and $H(G/K)$ are evenly graded (i.e., $(\wedge^p F^{\perp})_I = 0 = H^p(G/K)$ if p is even).

14. The representation of W_G. G is a compact connected Lie group with maximal torus T.

(i) By considering the projection $G/T \to G/N_T$, construct a smooth bundle $(G/T, \pi, G/N_T, W_G)$.

(ii) Show that G/N_T is the orbit space (cf. problem 6, Chap. 3) for a suitable free action of W_G on G/T.

(iii) From the action of W_G on G/T obtain a representation of W_G in $H(G/T) \otimes \mathbb{C}$. In volume III it will be shown that $H^p(G/T) = 0$, p odd. Use this fact to determine the character of this representation (cf. problem 12, Chap. I). Conclude that it is equivalent to the left regular representation of W_G (cf. problem 14, Chap. I).

(iv) Let W_G^+ be the subgroup of W_G that acts in G/T by orientation preserving diffeomorphisms. Show that W_G^+ is a normal subgroup of index 2 in W_G. Is it the only normal subgroup of index 2?

(v) Show that $H^+(G/N_T) = 0$.

15. Let G, T be as in problem 14, and consider the map $\psi: G/T \times T \to G$ of sec. 2.17.

(i) Construct an action of W_G on T (by conjugation). Hence obtain an action of W_G on $G/T \times T$ and construct a smooth bundle $(G/T \times_{W_G} T, \rho, G/N_T, T)$ (cf. problem 7, Chap. III).

(ii) Show that ψ factors to yield the following smooth map: $\bar{\psi}: G/T \times_{W_G} T \to G$. Show that $\deg \bar{\psi} = 1$.

(iii) Show that $H(G/T \times_{W_G} T)$ is isomorphic to the subalgebra of $H(G/T) \otimes H(T)$ whose elements are invariant under the action of W_G. Conclude that $\bar{\psi}^{\#}: H(G) \to H(G/T \times_{W_G} T)$ is an isomorphism of graded algebras.

(iv) Show that the cohomology algebra of the total space of the bundle in (i) is isomorphic to the tensor product of the cohomology of fibre and base as algebras, but *not* as graded vector spaces.

16. Use the map ψ of problem 15 to obtain a smooth map

$$G/T \times S_F \to S_E$$

(S_F and S_E are the unit spheres in the Lie algebras of T and G). Compute the degree of this map.

17. Let G be a connected Lie group with Lie algebra E.

(i) Assume that G acts on M and N and that $\varphi, \psi: M \to N$ are equivariant smooth maps connected by an equivariant homotopy H. Conclude that the homomorphisms $\varphi^*_{i=0,\theta=0}$ and $\psi^*_{i=0,\theta=0}$ (respectively, $(\varphi^*_{i=0})_I$ and $(\psi^*_{i=0})_I$) are homotopic.

(ii) Let U be a suitable tubular neighbourhood of an orbit G/K of G under a proper action (cf. problem 11, Chap. III). Show that the orbit space U/G is homeomorphic to the cone over an orbit space S/K, where K acts on a sphere S by orthogonal transformations. (The *cone* over a space X is obtained from $X \times [0, 1]$ by identifying the points $(x, 1)$, $x \in X$.)

(iii) Let U be as in (ii). Construct an equivariant retraction ρ of U onto the orbit and show that $i \circ \rho$ is equivariantly homotopic to the identity map of U. Hence find isomorphisms

$$H_I(U) \cong H_I(G/K) \cong H((\wedge F^\perp)_I)$$

and

$$H((A(U)_{i=0})_I) \cong H(\text{point})$$

(F denotes the Lie algebra of K).

(iv) Establish a Mayer–Vietoris axiom and a disjoint union axiom for $H_I(M)$ and $H(A(M)_{i=0,\theta=0})$ (with respect to proper actions of a fixed Lie group).

(v) Assume that G acts properly on M and that, for all isotropy subgroups K, $H(G/K) = H_I(G/K)$. Conclude that $H(M) = H_I(M)$.

18. Čech cohomology. Let G act on M. Establish a bijection between open coverings of M/G and G-stable open coverings of M. If the action is proper, define an isomorphism

$$\check{H}(M/G) \xrightarrow{\cong} H(A_I(M)_{i=0}),$$

where $\check{H}(M/G)$ denotes the Čech cohomology of M/G (cf. problem 25, Chap. V, volume I).

19. Equivariant cohomology of sphere and vector bundles. Generalize as far as possible the results of Chaps. VIII and IX, volume I, to the equivariant case (i.e., invariant cohomology and proper actions). In particular, define equivariant Gysin and Thom classes.

20. Give an elementary example where the orbit space of an action of a compact connected Lie group on a compact connected manifold does not satisfy Poincaré duality.

21. Represent S^1 in $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{ik_1\theta}z_1, \ldots, e^{ik_n\theta}z_n), \qquad z_r \in \mathbb{C}, \quad \theta \in \mathbb{R},$$

where the k_r are integers with greatest common divisor 1. Obtain an action of S^1 on S^{2n-1}. Find the fundamental vector field and determine $H(A(S^{2n-1})_{i=0,\theta=0})$. Show that any equivariant smooth map $\varphi: S^{2n-1} \to S^{2n-1}$ has degree 1.

Chapter V

Bundles with Structure Group

§1. Principal bundles

5.1. Definition. Let G be a Lie group. A (smooth) *principal bundle with structure group* G is a pair $(\mathscr{P}, T)$, where

(i) $\mathscr{P} = (P, \pi, B, G)$ is a smooth fibre bundle.
(ii) $T: P \times G \to P$ is a right action of G on P.
(iii) $\mathscr{P}$ admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that

$$\psi_\alpha(x, ab) = \psi_\alpha(x, a) \cdot b, \qquad x \in U_\alpha, \quad a, b \in G.$$

(Note that we write $T(z, a) = z \cdot a$.)

The action T is called the *principal action* and a coordinate representation satisfying condition (iii) is called a *principal coordinate representation.*

Condition (iii) implies that

$$\pi(z \cdot a) = \pi(z), \qquad z \in P, \quad a \in G.$$

Moreover, it follows that the action T is free and that the orbit of G through a point $z \in P$ is the fibre containing z. In particular, the orbits are submanifolds of P. They will be denoted by $G_x = \pi^{-1}(x)$ $(x \in B)$, (since the action is free there is no confusion with the notation for isotropy subgroups). Note that $G_x \mapsto x$ defines a set bijection between the orbits and B.

Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with principal action $\hat{T}$. A smooth equivariant map $\varphi: P \to \hat{P}$ is called a *homomorphism of principal bundles.* Such a homomorphism is orbit preserving, and hence fibre preserving. Thus it induces a smooth map $\psi: B \to \hat{B}$ such that $\hat{\pi} \circ \varphi = \psi \circ \pi$ (cf. sec. 1.13, volume I).

Moreover, φ restricts to smooth maps $\varphi_x : G_x \to G_{\psi(x)}$ $(x \in B)$. The relations

$$\varphi_x(z \cdot a) = \varphi_x(z) \cdot a, \qquad z \in G_x, \quad a \in G,$$

imply that each φ_x is a diffeomorphism. It follows that φ is a diffeomorphism if and only if ψ is. In this case φ^{-1} is also a homomorphism of principal bundles and φ and φ^{-1} are called *isomorphisms of principal bundles*. If $B = \hat{B}$ and $\psi = \iota$, then φ is called a *strong isomorphism of principal bundles.*

Examples: 1. *The product bundle*: The trivial bundle,

$$(B \times G, \pi, B, G),$$

together with the right action

$$(x, a) \cdot b = (x, ab), \qquad x \in B, \quad a, b \in G$$

is a principal bundle. It is called the *trivial*, or *product bundle.*

2. *Homogeneous spaces:* Let K be a closed subgroup of G. Then the fibre bundle $(G, \pi, G/K, K)$ (cf. sec. 2.13), together with the action of K on G by right multiplication, is a principal bundle with structure group K.

3. *Frame bundles*: Let $\xi = (E, \rho, B, F)$ be a vector bundle, and, for $x \in B$, let G_x denote the set of linear isomorphisms from F to F_x. We shall construct a principal bundle, $(P, \pi, B, GL(F))$, where $P = \bigcup_x G_x$ and π is the projection which carries G_x to x.

In fact let $\{(U_\alpha, \psi_\alpha)\}$ be a coordinate representation for ξ. The isomorphisms $\psi_{\alpha,x} : F \xrightarrow{\cong} F_x$ determine set bijections

$$\varphi_{\alpha,x} : GL(F) \to G_x, \qquad x \in U_\alpha,$$

by

$$\varphi_{\alpha,x}(\varphi) = \psi_{\alpha,x} \circ \varphi, \qquad \varphi \in GL(F).$$

Thus set bijections $\varphi_\alpha : U_\alpha \times GL(F) \to \pi^{-1}(U_\alpha)$ are given by

$$\varphi_\alpha(x, \varphi) = \psi_{\alpha,x} \circ \varphi, \qquad x \in U_\alpha, \quad \varphi \in GL(F).$$

Evidently

$$(\varphi_\alpha^{-1} \circ \varphi_\beta)(x, \varphi) = (x, \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x} \circ \varphi),$$
$$x \in U_\alpha \cap U_\beta, \quad \varphi \in GL(F).$$

It follows that $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a diffeomorphism of $(U_\alpha \cap U_\beta) \times GL(F)$. Hence (cf. Proposition X, sec. 1.13, volume I), there is a unique smooth structure on the set P such that $(P, \pi, B, GL(F))$ becomes a smooth bundle.

Finally, define a right action of $GL(F)$ on each set G_x by setting

$$\varphi_x \cdot \varphi = \varphi_x \circ \varphi, \qquad \varphi_x \in G_x, \quad \varphi \in GL(F).$$

These actions define a right action of $GL(F)$ on the set P. Moreover,

$$\varphi_\alpha(x, \varphi) \cdot \varphi_1 = \varphi_\alpha(x, \varphi \circ \varphi_1), \qquad x \in U_\alpha, \quad \varphi, \varphi_1 \in GL(F).$$

It follows that the action of $GL(F)$ on P is smooth and that $\mathscr{P} = (P, \pi, B, GL(F))$ is a *principal* bundle.

Fix a basis $e_1, ..., e_r$ of F. Then a bijection from G_x to the set of bases (or *frames*) of F_x is given by

$$\varphi \mapsto (\varphi e_1, ..., \varphi e_r).$$

For this reason $\mathscr{P}$ is often called the *frame bundle* associated with ξ. Frame bundles are discussed again in article 5 of this chapter, and then extensively in article 7 of Chapter VIII.

5.2 Elementary properties. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle admitting a cross-section σ over an open set $U \subset B$. σ determines the homomorphism $\varphi: U \times G \to P$ of principal bundles, given by

$$\varphi(x, a) = \sigma(x) \cdot a, \qquad x \in U, \quad a \in G.$$

φ may be regarded as a strong isomorphism from the trivial bundle to the restriction of $\mathscr{P}$ to U. In particular, if $\mathscr{P}$ admits a cross-section, it is the trivial bundle.

If τ is a second cross-section over a second open set V, then there is a unique smooth map

$$g_{UV}: U \cap V \to G$$

such that $\varphi(x, g_{UV}(x)) = \tau(x)$. We have

$$\tau(x) = \sigma(x) \cdot g_{UV}(x), \qquad x \in U \cap V,$$

and this equation determines g_{UV}.

Lemma I: Let $\mathscr{P} = (P, \pi, B, G)$ be a smooth bundle. Let T be a smooth free right action of G on P, whose orbits coincide with the fibres of the bundle. Then $\mathscr{P}$ is a principal bundle with principal action T.

Proof: Let $\{U_\alpha\}$ be an open cover of B such that each U_α admits a cross-section $\sigma_\alpha : U_\alpha \to P$. Define $\psi_\alpha : U_\alpha \times G \xrightarrow{\cong} \pi^{-1}(U_\alpha)$ by setting

$$\psi_\alpha(x, a) = \sigma_\alpha(x) \cdot a.$$

Then $\{(U_\alpha, \psi_\alpha)\}$ is a coordinate representation satisfying condition (iii). Q.E.D.

Next, let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a principal bundle, and let $\psi: B \to \hat{B}$ be a smooth map. We shall construct a principal bundle (P, π, B, G) together with a homomorphism, $\varphi: P \to \hat{P}$, of principal bundles which induces ψ.

In fact, let P be the disjoint union,

$$P = \bigcup_{x \in B} (\{x\} \times G_{\psi(x)}),$$

and define π by setting $\pi(\{x\} \times G_{\psi(x)}) = x$. Define a right action, T, of G on the set P and an equivariant set map $\varphi: P \to \hat{P}$ by

$$T((x, z), a) = (x, z \cdot a) \qquad \text{and} \qquad \varphi(x, z) = z,$$
$$z \in G_{\psi(x)}, \quad x \in B, \quad a \in G.$$

Give P a smooth structure, as follows. Choose an open cover $\{V_\nu\}$ of $\hat{B}$ such that each V_ν admits a cross-section $\sigma_\nu : V_\nu \to \hat{P}$. Set $U_\nu = \psi^{-1}(V_\nu)$ and define bijections $\chi_\nu : U_\nu \times G \to \pi^{-1}(U_\nu)$ by

$$\chi_\nu(x, a) = (x, \sigma_\nu(\psi(x)) \cdot a).$$

Then for $x \in U_\nu \cap U_\mu$,

$$(\chi_\mu^{-1} \circ \chi_\nu)(x, a) = (x, g_{\mu\nu}(\psi(x))a),$$

where $g_{\mu\nu} : V_\mu \cap V_\nu \to G$ is the smooth map satisfying

$$\sigma_\nu(y) = \sigma_\mu(y) \cdot g_{\mu\nu}(y), \qquad y \in V_\mu \cap V_\nu .$$

We can thus apply Proposition X, sec. 1.13, volume I, to obtain a unique smooth structure on P such that $\mathscr{P} = (P, \pi, B, G)$ is a smooth bundle with coordinate representation $\{(U_\nu, \chi_\nu)\}$. Since the maps χ_ν are equivariant, T is a smooth action and $(\mathscr{P}, T)$ is a principal bundle. Moreover, φ is a homomorphism of principal bundles.

$\mathscr{P}$ is called the *pull-back* of $\hat{\mathscr{P}}$ to B via ψ and it is often written $\psi^*\hat{\mathscr{P}}$.

Let $\mathscr{P}_1 = (P_1, \pi_1, B, G)$ be a second principal bundle over B which admits a homomorphism $\varphi_1: P_1 \to \hat{P}$ of principal bundles inducing $\psi: B \to \hat{B}$. Then a strong isomorphism $\varphi_2: P \xrightarrow{\cong} P_1$ is defined by

$$\varphi_2(z) = ((\varphi_1)_x^{-1} \circ \varphi_x)(z), \qquad z \in \pi^{-1}(x).$$

Note that $\varphi_1 \circ \varphi_2 = \varphi$.

§2. Associated bundles

Notation convention: In this article $\mathscr{P} = (P, \pi, B, G)$ denotes a fixed principal bundle with principal action T. Moreover,

$$S\colon G \times F \to F$$

will denote a fixed left action of G on a manifold F.

5.3. Associated bundles. Consider the right action, Q, of G on the product manifold $P \times F$ given by

$$Q_a(z, y) = (z, y) \cdot a = (z \cdot a,\ a^{-1} \cdot y), \qquad z \in P, \quad y \in F, \quad a \in G.$$

Q will be called the *joint action of* G. The set of orbits for the joint action will be denoted by $P \times_G F$ and

$$q\colon P \times F \to P \times_G F$$

will denote the corresponding projection; i.e., $q(z, y)$ is the orbit through (z, y).

The map q determines a map $\rho\colon P \times_G F \to B$ via the commutative diagram

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "\pi_P"'] & P \times_G F \arrow[d, "\rho"] \\ P \arrow[r, "\pi"'] & B, \end{tikzcd} \tag{5.1}$$

where π_P is the obvious projection. Denote $\rho^{-1}(x)$ by F_x, $x \in B$.

Proposition I: There is a unique smooth structure on $P \times_G F$ such that

(1) $\xi = (P \times_G F, \rho, B, F)$ is a smooth fibre bundle.

(2) $q\colon P \times F \to P \times_G F$ is a smooth fibre preserving map, restricting to diffeomorphisms

$$q_z\colon z \times F \xrightarrow{\cong} F_{\pi(z)}, \qquad z \in P,$$

on each fibre.

(3) $(P \times F, q, P \times_G F, G)$ is a smooth principal bundle with principal action Q.

(4) π_P is a homomorphism of principal bundles.

Definition: ξ is called the *fibre bundle with fibre F and structure group G associated with* $\mathscr{P}$; q is called the *principal map*.

Proof of Proposition I: If a smooth structure satisfies 3, it makes $P \times_G F$ into a quotient manifold of $P \times F$ under q. Hence, by the corollary to Proposition V, sec. 3.9, volume I, it is uniquely determined.

Proof of (1): We construct a smooth structure on $P \times_G F$ for which ξ is a smooth bundle. Let $\{U_\alpha\}$ be an open cover of B and consider cross-sections $\sigma_\alpha\colon U_\alpha \to P$. These are related by

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta\,,$$

where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ are smooth maps. Define set maps,

$$\varphi_\alpha\colon U_\alpha \times F \to \rho^{-1}(U_\alpha),$$

by setting

$$\varphi_\alpha(x, y) = q(\sigma_\alpha(x), y), \qquad x \in U_\alpha\,, \quad y \in F.$$

Then $\rho(\varphi_\alpha(x, y)) = x$ and so φ_α restricts to set maps

$$\varphi_{\alpha,x}\colon F \to \rho^{-1}(x), \qquad x \in U_\alpha\,.$$

Moreover, to each orbit in $\rho^{-1}(x)$ there corresponds a unique $y \in F$ such that the orbit passes through $(\sigma_\alpha(x), y)$. Hence $\varphi_{\alpha,x}$ is bijective, and so φ_α is bijective.

Further, the relations $q(z \cdot a, y) = q(z, a \cdot y)$ imply that

$$\varphi_\alpha^{-1} \circ \varphi_\beta(x, y) = (x, g_{\alpha\beta}(x) \cdot y), \qquad x \in U_\alpha \cap U_\beta\,, \quad y \in F.$$

Thus Proposition X, sec. 1.13, volume I, yields a smooth structure on $P \times_G F$ for which ξ is a smooth bundle with coordinate representation $\{(U_\alpha\,, \varphi_\alpha)\}$.

Proof of (3): To show that $(P \times F, q, P \times_G F, G)$, is a smooth principal bundle with principal action Q consider the commutative diagrams,

$$\begin{array}{ccc} U_\alpha \times G \times F & \xrightarrow[\cong]{\psi_\alpha \times \iota} & \pi^{-1}(U_\alpha) \times F \\ {\scriptstyle \iota \times S}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ U_\alpha \times F & \xrightarrow[\varphi_\alpha]{\cong} & \rho^{-1}(U_\alpha)\,, \end{array} \tag{5.2}$$

where $\psi_\alpha(x, a) = \sigma_\alpha(x) \cdot a$. Set $V_\alpha = \rho^{-1}(U_\alpha)$; then

$$q^{-1}(V_\alpha) = (\pi \circ \pi_P)^{-1}(U_\alpha) = \pi^{-1}(U_\alpha) \times F.$$

Thus diffeomorphisms $\chi_\alpha \colon V_\alpha \times G \xrightarrow{\cong} q^{-1}(V_\alpha)$ are given by

$$\chi_\alpha(\varphi_\alpha(x, y), a) = (\psi_\alpha(x, a), a^{-1} \cdot y).$$

They satisfy the relations

$$(q \circ \chi_\alpha)(w, a) = w, \qquad \text{and} \qquad \chi_\alpha(w, ab) = Q(\chi_\alpha(w, a), b), \qquad w \in V_\alpha, \quad a, b \in G$$

(cf. diagram 5.2). (3) follows.

Proof of (2): The commutative diagram (5.1) shows that q is fibre preserving, while the commutative diagrams (5.2) imply that the maps

$$q_z \colon F \xrightarrow{\cong} F_{\pi(z)}$$

are diffeomorphisms.

Proof of (4): This is obvious.

Q.E.D.

5.4. Equivariant maps. Assume $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ is a second principal bundle and that $\hat{S}$ is a left action of G on a manifold $\hat{F}$. Suppose further that

$$\varphi \colon P \to \hat{P} \qquad \text{and} \qquad \alpha \colon F \to \hat{F}$$

are smooth equivariant maps.

Then the map $\varphi \times \alpha \colon P \times F \to \hat{P} \times \hat{F}$ is equivariant with respect to the joint actions of G; i.e., it is a homomorphism of principal bundles. Thus it induces a smooth map,

$$\varphi \times_G \alpha : P \times_G F \to \hat{P} \times_G \hat{F},$$

which makes the diagram,

$$\begin{array}{ccc} P \times F & \xrightarrow{\varphi \times \alpha} & \hat{P} \times \hat{F} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \hat{q}} \\ P \times_G F & \xrightarrow[\varphi \times_G \alpha]{} & \hat{P} \times_G \hat{F} \end{array},$$

commute.

Let $\psi: B \to \hat{B}$ be the smooth map induced by φ. Then the diagram,

$$\begin{tikzcd} P \times_G F \arrow[r, "\varphi \times_G \alpha"] \arrow[d, "\rho"'] & \hat{P} \times_G \hat{F} \arrow[d, "\hat{\rho}"] \\ B \arrow[r, "\psi"'] & \hat{B} \end{tikzcd},$$

commutes; i.e., $\varphi \times_G \alpha$ is a fibre preserving map between the associated bundles. The commutative diagrams

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "q_z"', "\cong"] & \hat{F} \arrow[d, "\hat{q}_{\varphi(z)}", "\cong"'] \\ F_x \arrow[r, "(\varphi \times_G \alpha)_x"'] & \hat{F}_{\psi(x)} \end{tikzcd}, \qquad x = \pi(z), \qquad z \in P,$$

show that, if α is a diffeomorphism, then so is each $(\varphi \times_G \alpha)_x$.

The case that $\mathscr{P} = \hat{\mathscr{P}}$ and $\varphi = \iota$, is of particular importance; in this case we obtain a fibre preserving map,

$$(\iota \times_G \alpha) : P \times_G F \to P \times_G \hat{F},$$

which induces the identity map in B.

5.5. Examples: 1. $F = \{point\}$. Then $P \times_G F = B$ and the principal bundle $(P \times F, q, P \times_G F, G)$ coincides with $\mathscr{P}$.

2. Assume the action of G on F is trivial. Then $\xi = (B \times F, \rho, B, F)$ is trivial. Also, if the principal bundle $\mathscr{P}$ is trivial, then so is ξ.

3. Suppose $y \in F$ is fixed under the action of G: $a \cdot y = y$, $a \in G$. Then the inclusion $j : \{y\} \to F$ is equivariant. It induces (sec. 5.4) a smooth commutative diagram

$$\begin{tikzcd} P \times_G \{y\} \arrow[r, "\sigma"] \arrow[d, "\cong"'] & P \times_G F \arrow[d, "\rho"] \\ B \arrow[r, "\iota"'] & B \end{tikzcd};$$

thus σ is a cross-section in ξ.

4. *λ-extension:* Let $\lambda: G \to K$ be a homomorphism of Lie groups. Then G acts from the left on K by

$$a \cdot y = \lambda(a)y, \qquad a \in G, \quad y \in K.$$

Thus we obtain a bundle $\mathscr{P}_\lambda = (P \times_G K, \rho, B, K)$.

On the other hand, the multiplication map of K determines a right action

$$(P \times K) \times K \to P \times K.$$

This map factors over q to yield a free right action

$$T_\lambda : (P \times_G K) \times K \to P \times_G K.$$

The orbits of T_λ are precisely the fibres of $P \times_G K$. Thus it follows from Lemma I, sec. 5.2, that $(\mathscr{P}_\lambda, T_\lambda)$ is a principal K-bundle. It is called the *λ-extension of $\mathscr{P}$.*

Next, define a smooth map φ_λ: $P \to P \times_G K$ by setting $\varphi_\lambda(z) = q(z, e)$. The diagram,

$$\begin{tikzcd} P \times G \arrow[r, "\iota\times\lambda"] \arrow[d, "T"'] & P \times K \arrow[d, "q"] \\ P \arrow[r, "\varphi_\lambda"] \arrow[dr, "\pi"'] & P \times_G K \arrow[d, "\rho"] \\ & B \end{tikzcd} \tag{5.3}$$

commutes (cf. diagram 5.1, sec. 5.3). This shows that φ_λ is a fibre preserving map from P to $P \times_G K$, inducing the identity in B.

In particular, consider the case that $G = K$ and $\lambda = \iota$; thus G acts on itself by left multiplication. In this case φ_λ is a strong isomorphism of principal bundles, and the diagram shows that $(P \times G, q, P \times_G G, G)$ is the trivial principal bundle.

5. *Reduction of structure group*: Again, let λ: $G \to K$ be a homomorphism of Lie groups. Assume that $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ is a principal bundle. A *reduction of the structure group of $\hat{\mathscr{P}}$ from K to G via λ* is a principal bundle $\mathscr{P} = (P, \pi, B, G)$ and a smooth fibre preserving map φ: $P \to \hat{P}$, inducing the identity in the base, and satisfying

$$\varphi(z \cdot a) = \varphi(z) \cdot \lambda(a), \qquad a \in G.$$

Such a reduction induces an obvious isomorphism of principal bundles from the λ-extension of $\mathscr{P}$ to $\hat{\mathscr{P}}$ (cf. Example 4). Conversely, if $\mathscr{P} = (P, \pi, B, G)$ is any principal bundle with λ-extension $\mathscr{P}_\lambda = (P \times_G K, \rho, B, K)$, then the homomorphism φ_λ of Example 4 is a reduction of the structure group of $\mathscr{P}_\lambda$ from K to G.

5.6. Associated vector bundles. Assume now that F is a finite-dimensional (real or complex) vector space and S is a representation of G in F. In this case $P \times_G F$ is a *vector bundle.*

In fact, for each $x \in B$, $z \in \pi^{-1}(x)$, the diffeomorphisms $q_z: F \xrightarrow{\cong} F_x$ are connected by

$$q_{z \cdot a} = q_z \circ S(a), \qquad a \in G.$$

Since each map $S(a)$ is a linear isomorphism, there is a unique linear structure in F_x for which the maps q_z are linear isomorphisms. The zero vector of F_x is given by $0_x = q(z, 0)$, $z \in \pi^{-1}(x)$.

Each $\varphi_{\alpha,x}$ of the coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ for ξ defined in sec. 5.3 is a linear isomorphism. Hence ξ is a vector bundle with vector bundle coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$. Since q restricts to isomorphisms in the fibres, the trivial bundle $(P \times F, \pi_P, P, F)$ is the pull-back of ξ to P via π (cf. sec. 2.5, volume I).

To the trivial representation S corresponds the trivial vector bundle.

Next, consider a representation of G in a second vector space H and let $\alpha: F \to H$ be an equivariant linear map. Then the induced map (cf. sec. 5.4),

$$\iota \times_G \alpha : P \times_G F \to P \times_G H,$$

is linear in each fibre, and so it is a (strong) bundle map.

Denote the vector bundles corresponding to F and H by ξ and η and consider the induced representations of G in the spaces

$$F \oplus H, \qquad F \otimes H, \qquad L(F; H), \qquad F^*, \qquad \wedge F.$$

The associated vector bundles corresponding to these representations are given, respectively, by

$$\xi \oplus \eta, \qquad \xi \otimes \eta, \qquad L(\xi; \eta), \qquad \xi^*, \qquad \wedge \xi.$$

The various canonical maps between these vector spaces, such as

evaluation:	$L(F; H) \otimes F$	$\to$	H,
composition:	$L_F \otimes L_F$	$\to$	L_F,
projection:	$F \oplus H$	$\to$	F,
trace:	L_F	$\to$	$\mathbb{R}$,

commute with the representations of G. Thus they induce maps between the corresponding vector bundles. For the four examples above we have (cf. sec. 2.10, volume I).

$$\begin{array}{lll} \text{evaluation:} & L(\xi;\eta)\otimes\xi \to & \eta, \\ \text{composition:} & L_\xi\otimes L_\xi \quad\to & L_\xi\,, \\ \text{projection:} & \xi\oplus\eta \quad\to & \xi, \\ \text{trace:} & L_\xi \quad\to & \mathscr{S}(B). \end{array}$$

§3. Bundles and homogeneous spaces

In this article K denotes a closed subgroup of a Lie group G. Their Lie algebras are denoted by F and E $(F \subset E)$. The corresponding principal bundle (cf. Example 2, sec. 5.1) is denoted by $\mathscr{P}_K = (G, \pi_K, G/K, K)$ and we write $\bar{e} = \pi_K(e)$ (e, the unit element of G). The left action of G on G/K is denoted by T.

5.7. Bundles with fibre a homogeneous space. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal action R. The left action of G on G/K determines an associated bundle

$$\xi = (P \times_G (G/K), \rho, B, G/K)$$

(cf. sec. 5.3). To simplify notation we shall write

$$P \times_G (G/K) = P/K.$$

Consider the commutative diagram,

$$\begin{tikzcd} P \times G/K \arrow[r, "q"] \arrow[d, "\pi_P"'] & P/K \arrow[d, "\rho"] \\ P \arrow[r, "\pi"'] & B, \end{tikzcd}$$

and define $p\colon P \to P/K$ by $p(z) = q(z, \bar{e})$.

Proposition II: With the notation above, $(P, p, P/K, K)$ is a principal bundle whose principal action is the restriction of R to K.

Proof: It is sufficient to show that each $w \in P/K$ has a neighbourhood W such that $(p^{-1}(W), p, W, K)$ is a principal bundle.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be the coordinate representation for ξ defined in sec. 5.3. Set $W = \rho^{-1}(U_\alpha)$, where α is chosen so that $w \in W$. Then

$$p^{-1}(W) = \pi^{-1}(U_\alpha).$$

Finally, let $j\colon G \to G \times G/K$ be the inclusion opposite $\bar{e}$. From diagram (5.2) of sec. 5.3 we obtain the commutative diagram

$$\begin{tikzcd} U_\alpha \times G \arrow[r, "\iota\times j"] \arrow[d, "\iota\times\pi_K"'] & U_\alpha \times G \times G/K \arrow[r, "\psi_\alpha\times\iota", "\cong"'] \arrow[d, "\iota\times T"] & \pi^{-1}U_\alpha \times G/K \arrow[d, "q"] \\ U_\alpha \times G/K \arrow[r, "\iota"'] & U_\alpha \times G/K \arrow[r, "\cong", "\varphi_\alpha"'] & \rho^{-1}(U_\alpha). \end{tikzcd}$$

It restricts to the commutative diagram,

$$\begin{tikzcd} U_\alpha \times G \arrow[r, "\psi", "\cong"'] \arrow[d, "\iota\times\pi_K"'] & p^{-1}(W) \times \{\bar{e}\} \arrow[d, "p"] \\ U_\alpha \times G/K \arrow[r, "\cong", "\varphi_\alpha"'] & W, \end{tikzcd}$$

where $\psi = (\psi_\alpha \times \iota) \circ (\iota \times j)$.

Now $\mathscr{P}_K$ is a principal K-bundle, and ψ is equivariant with respect to the given actions of K. It follows that $(p^{-1}(W), p, W, K)$ is a principal K-bundle.

Q.E.D.

Next, fix $z \in P$ and write $\pi(z) = x$. Then the fibre inclusion, $j_{G/K} : G/K \to P/K$, for the bundle ξ is given by

$$j_{G/K} = q_z : G/K \xrightarrow{\cong} \rho^{-1}(x).$$

Let $j_G\colon G \to P$ and $j_K\colon K \to P$ denote the fibre inclusions given by

$$j_G(b) = z \cdot b \qquad \text{and} \qquad j_K(a) = z \cdot a, \qquad b \in G, \quad a \in K,$$

and let $i\colon K \to G$ be the inclusion map. Then the diagram,

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, "i"'] & K \arrow[d, "j_K"] & \\ G \arrow[r, "j_G"] \arrow[d, "\pi_K"'] & P \arrow[r, "\pi"] \arrow[d, "p"] & B \arrow[d, equal] \\ G/K \arrow[r, "j_{G/K}"'] & P/K \arrow[r, "\rho"'] & B, \end{tikzcd} \tag{5.4}$$

commutes. Moreover j_G is a homomorphism of principal K-bundles.

5.8. Subgroup of a subgroup. Assume now that G is a closed subgroup of a Lie group H, and apply the results of sec. 5.7 to the principal bundle $\mathscr{P} = (H, \pi, H/G, G)$. We obtain an associated bundle,

$$\xi = (H \times_G G/K, \rho, H/G, G/K),$$

and a principal bundle

$$\hat{\mathscr{P}} = (H, p, H \times_G G/K, K).$$

The left action of H on H/K restricts to a smooth map,

$$H \times G/K \to H/K,$$

which factors to yield a diffeomorphism

$$H \times_G G/K \xrightarrow{\cong} H/K$$

(equivariant with respect to the left actions of H, cf. sec. 5.9). We identify these manifolds via this diffeomorphism and write

$$\xi = (H/K, \rho, H/G, G/K), \qquad \hat{\mathscr{P}} = (H, p, H/K, K).$$

Then $\hat{\mathscr{P}}$ is the standard principal bundle, while ρ is given by

$$\rho(aK) = aG, \qquad a \in H.$$

Moreover, diagram (5.4) reads

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\ G \arrow[r] \arrow[d, "\pi_K"'] & H \arrow[r, "\pi"] \arrow[d, "p"] & H/G \arrow[d, equal] \\ G/K \arrow[r] & H/K \arrow[r, "\rho"'] & H/G. \end{tikzcd} \tag{5.5}$$

Now suppose that K is normal in G. Then a smooth free right action of the factor group G/K on H/K is given by

$$\bar{x} \cdot \bar{a} = \overline{x \cdot a}, \qquad x \in H, \quad a \in G.$$

The orbits of G/K under this action coincide with the fibres in the bundle $\xi = (H/K, \rho, H/G, G/K)$. It follows from Lemma I, sec. 5.2, that ξ is a principal G/K-bundle.

5.9. Bundles with base a homogeneous space. Let K act from the left on a manifold N. There is a unique left action,

$$\Lambda : G \times (G \times_K N) \to G \times_K N,$$

of G that makes the diagram,

$$\begin{tikzcd} G \times G \times N \arrow[r, "\mu \times \iota"] \arrow[d, "\iota \times q"'] & G \times N \arrow[d, "q"] \\ G \times (G \times_K N) \arrow[r, "\Lambda"'] & G \times_K N, \end{tikzcd}$$

commute. Clearly Λ, together with T, is an action of G on the bundle $\xi = (G \times_K N, \rho, G/K, N)$ associated with $\mathscr{P}_K$; i.e., G acts on the total and base spaces and the projection is equivariant:

$$\rho \circ \Lambda = T \circ (\iota \times \rho).$$

Let $N_{\bar{e}} = \rho^{-1}(\bar{e})$. Since $a \cdot \bar{e} = \bar{e}$ $(a \in K)$, it follows that Λ restricts to a left action

$$K \times N_{\bar{e}} \to N_{\bar{e}}.$$

The projection q restricts to a K-equivariant diffeomorphism,

$$q_e : N \xrightarrow{\cong} N_{\bar{e}}$$

(cf. Proposition I, (2), sec. 5.3).

Conversely, assume that $\eta = (M, \rho_M, G/K, Q)$ is a smooth bundle over G/K and that Λ (with T) is a left action of G on η. Then we can construct the bundle,

$$\xi = (G \times_K Q_{\bar{e}}, \rho, G/K, Q_{\bar{e}}),$$

via the induced action of K on $Q_{\bar{e}}$.

Λ restricts to a smooth map $G \times Q_{\bar{e}} \to M$. This factors over q to yield an equivariant fibre preserving diffeomorphism,

$$\psi: G \times_K Q_{\bar{e}} \xrightarrow{\cong} M,$$

which induces the identity map in G/K.

5.10. Vector bundles. In this section we apply the results of sec. 5.9 to vector bundles. Each representation of K in a vector space N yields a vector bundle over G/K associated with $\mathscr{P}_K$ (cf. sec. 5.6) in which G acts by bundle maps. Conversely, if G acts by bundle maps in a vector bundle η over G/K so as to induce the standard action in G/K, then the action restricts to a representation of K in the fibre over $\bar{e}$.

If these two constructions are applied consecutively, starting off with a representation of K (respectively, a vector bundle over G/K acted on by G), we obtain a representation (respectively, a vector bundle acted on by G) which is equivariantly (respectively, equivariantly and strongly) isomorphic to the original.

Examples: 1. If the representation of K in N is trivial, then

$$G \times_K N = G/K \times N$$

and ξ is trivial.

2. Assume that the representation of K in N extends to a representation of G in N. Define a diffeomorphism φ of $G \times N$ by setting

$$\varphi(b, y) = (b, b^{-1} \cdot y), \qquad b \in G, \quad y \in N.$$

Then (letting Q denote the joint action of K in $G \times N$)

$$\varphi \circ (\rho_a \times \iota) = Q_a \circ \varphi, \qquad a \in K$$

(ρ_a denotes the right translation of G by a). It follows that φ induces a diffeomorphism

$$\psi: G/K \times N \xrightarrow{\cong} G \times_K N.$$

Evidently ψ is a strong vector bundle isomorphism. Moreover,

$$\psi(b \cdot z, b \cdot y) = b \cdot \psi(z, y), \qquad b \in G, \quad z \in G/K, \quad y \in N,$$

(where G acts on $G \times_K N$ as defined in sec. 5.9).

5.11. Tangent bundle of a homogeneous space. Recall that the Lie algebras of K and G are denoted by F and E. The adjoint representation of G restricts to a representation, $\mathrm{Ad}_{G,K}$, of K in E. Since the Lie algebra F is stable under the maps $\mathrm{Ad}_{G,K}(a)$, $a \in K$, we obtain a representation, $\mathrm{Ad}^{\perp}$, of K in E/F. The sequence

$$0 \to F \to E \to E/F \to 0$$

is short exact and K-equivariant with respect to the representations Ad, $\mathrm{Ad}_{G,K}$, and $\mathrm{Ad}^{\perp}$ of K.

Now form the vector bundles

$$\xi = (G \times_K (E/F), \rho_\xi, G/K, E/F) \qquad \text{and} \qquad \eta = (G \times_K F, \rho_\eta, G/K, F).$$

G acts on both ξ and η. On the other hand, the left action, T, of G on G/K induces a left action, dT, of G on the tangent bundle $\tau_{G/K}$ (cf. Example 7, sec. 3.2).

Proposition III: With the hypotheses and notation above

(1) ξ is strongly and equivariantly isomorphic to $\tau_{G/K}$.

(2) The vector bundle $\xi \oplus \eta$ is trivial.

Proof: (1) According to sec. 2.11, $(d\pi_K)_e$ induces a linear isomorphism

$$E/F \xrightarrow{\cong} T_{\bar{e}}(G/K).$$

Since $\pi_K \circ \lambda_a = T_a \circ \pi_K$ and $\pi_K \circ \rho_a = \pi_K$ $(a \in K)$, we have

$$(d\pi_K)_e \circ \mathrm{Ad}_{G,K}(a) = dT_a \circ (d\pi_K)_e, \qquad a \in K.$$

Thus this isomorphism is equivariant with respect to $\mathrm{Ad}^\perp$ and dT. Now apply sec. 5.10.

(2) Since the sequence $F \to E \to E/F$ is K-equivariant, it determines a sequence of strong bundle maps

$$\eta \xrightarrow{i} G \times_K E \xrightarrow{p} \xi.$$

For each $z \in G/K$, the restriction,

$$0 \to F_z \to E_z \to (E/F)_z \to 0,$$

is short exact.

Hence, there is a strong bundle map $\sigma: \xi \to G \times_K E$ such that $p \circ \sigma = \iota$ (cf. Lemma III, sec. 2.23, volume I). Thus a strong bundle isomorphism,

$$\varphi: \xi \oplus \eta \xrightarrow{\cong} G \times_K E,$$

is defined by

$$\varphi(u, v) = \sigma(u) + i(v), \qquad u \in (E/F)_z, \quad v \in F_z, \quad z \in G/K.$$

On the other hand, the representation $\mathrm{Ad}_{G,K}$ of K in E is the restriction of a representation of G. Hence, by Example 2 of sec. 5.10, $G \times_K E$ is a trivial bundle over G/K. Thus $\xi \oplus \eta$ is trivial.

Q.E.D.

5.12. Tori. Suppose now that G is compact and connected, and that K is a torus in G. Then the adjoint representation of K is trivial and hence, so is the bundle

$$\eta = (G \times_K F, \rho_\eta, G/K, F).$$

Thus, by Proposition III, sec. 5.11, the Whitney sum of $\tau_{G/K}$ with a trivial bundle is trivial. This implies (as will be shown in sec. 7.19) that the Whitney sum of $\tau_{G/K}$ with the trivial bundle *of rank one* is trivial,

$$\tau_{G/K} \oplus \epsilon^1 \cong \epsilon^{r+1}, \qquad r = \dim G/K. \tag{5.6}$$

Now we distinguish two cases:

Case I: K is a *maximal* torus (cf. sec. 2.15). Then the Euler–Poincaré characteristic of G/K is positive (cf. sec. 4.21). Hence Theorem II, sec. 10.1, volume I, implies that every vector field on G/K has at least one zero. In particular, the tangent bundle of G/K is nontrivial.

Case II: K is *not* maximal. Then K is properly contained in a maximal torus, T. Since T is compact and connected, the factor group T/K is again a torus (cf. Proposition XIII, sec. 1.12).

Thus according to sec. 5.8 we can form the principal T/K-bundle

$$\mathscr{P} = (G/K, \pi, G/T, T/K).$$

Write $T_{G/K} = H_{G/K} \oplus V_{G/K}$, where $V_{G/K}$ is the vertical subbundle and $H_{G/K}$ is a horizontal bundle (cf. sec. 0.15).

Since $\mathscr{P}$ is a principal bundle, the vertical subbundle is trivial (as will be shown in sec. 6.1),

$$V_{G/K} = \epsilon^m, \qquad m = \dim T/K.$$

By hypothesis, K is properly contained in T and so we have $m \geqslant 1$. On the other hand, $H_{G/K}$ is the pull-back of $\tau_{G/T}$ under π.

It follows that $\tau_{G/K}$ is the pull-back of $\tau_{G/T} \oplus \epsilon^m$. In view of relation (5.6), with K replaced by T, the bundle $\tau_{G/T} \oplus \epsilon^1$ is trivial. Hence so is $\tau_{G/K}$.

Thus if K is a nonmaximal torus, then the homogeneous space G/K has trivial tangent bundle.

§4. The Grassmannians

5.13. The Grassmann manifolds. Let Γ be one of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and consider the vector space $\Gamma^n = \Gamma \oplus \cdots \oplus \Gamma$. Introduce a positive definite inner product $\langle\ ,\ \rangle$ in Γ^n which is Euclidean, Hermitian, or quaternionic according as $\Gamma = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In the case $\Gamma = \mathbb{R}$ also choose an orientation in Γ^n.

A *k-plane* in Γ^n is a Γ-subspace of Γ-dimension k. The set of all k-planes in Γ^n is denoted by $G_\Gamma(n; k)$. An *oriented k-plane* in $\mathbb{R}^n$ is a k-plane F together with an orientation of F. The set of oriented k-planes in $\mathbb{R}^n$ will be denoted by $\tilde{G}(n; k)$ if $k < n$. Finally, we define $\tilde{G}(n; n)$ to be the set consisting of a single element, namely the oriented vector space $\mathbb{R}^n$.

This article deals with each of the four cases listed below. In each case, Γ, $I(n)$, $G(n; k)$ is to be interpreted as described below.

Case	Γ	$I(n)$	$G(n; k)$
I	$\mathbb{R}$	$O(n)$	$G_{\mathbb{R}}(n; k)$
II	$\mathbb{R}$	$SO(n)$	$\tilde{G}(n; k)$
III	$\mathbb{C}$	$U(n)$	$G_{\mathbb{C}}(n; k)$
IV	$\mathbb{H}$	$Q(n)$	$G_{\mathbb{H}}(n; k)$

Observe that in each case the Lie algebra of $I(n)$ consists of the Γ-linear transformations of Γ^n that are skew with respect to the inner product $\langle\ ,\ \rangle$. The Lie algebra of $I(n)$ is denoted by $E(n)$.

The set $G(n; k)$ is made into a manifold in the following way: First define a transitive left action of the Lie group $I(n)$ on $G(n; k)$ by setting

$$(\varphi, F) \mapsto \varphi(F), \qquad \varphi \in I(n), \qquad F \in G(n; k).$$

This yields a surjection, $\alpha: I(n) \to G(n; k)$, given by

$$\alpha(\varphi) = \varphi(\Gamma^k), \qquad \varphi \in I(n)$$

(where Γ^k is regarded as the subspace of Γ^n consisting of those vectors whose last $n - k$ components are zero).

Denote $(\Gamma^k)^\perp$ by Γ^{n-k}: $\Gamma^n = \Gamma^k \oplus \Gamma^{n-k}$. This decomposition determines an inclusion, $I(k) \times I(n-k) \to I(n)$, and clearly

$$\alpha^{-1}(\Gamma^k) = I(k) \times I(n-k).$$

Hence α induces a commutative diagram,

$$\begin{tikzcd} & I(n) \arrow[dl, "\pi"'] \arrow[dr, "\alpha"] & \\ I(n)/(I(k) \times I(n-k)) \arrow[rr, "\cong", "\beta"'] & & G(n;k), \end{tikzcd}$$

and β is an equivariant bijection. Give $G(n; k)$ the unique manifold structure such that β is a diffeomorphism. The manifold so obtained is called the *Grassmannian of k-planes in* Γ^n. Since β is equivariant the action of $I(n)$ on $G(n; k)$ defined above is smooth.

Observe that the canonical isomorphism

$$I(k) \times I(n-k) \xrightarrow{\cong} I(n-k) \times I(k)$$

induces, via β, a diffeomorphism

$$\Omega\colon G(n; k) \xrightarrow{\cong} G(n; n-k).$$

If $F \in G(n; k)$, then $\Omega(F)$ is the orthogonal complement of F in Γ^n.

5.14. Examples: 1. *The Grassmannian of k-planes in* $\mathbb{R}^n$: Assume that $0 < k < n$. Then an involution, ω, of $\tilde{G}(n; k)$ is defined as follows: If F is an oriented k-plane, then $\omega(F)$ is the same k-plane with the opposite orientation. On the other hand, a projection,

$$p\colon \tilde{G}(n; k) \to G_{\mathbb{R}}(n; k),$$

is defined by forgetting the orientations of the elements of $\tilde{G}(n; k)$. Evidently, p is a double covering and ω is the involution that interchanges the two points in each $p^{-1}(F)$.

To see that p and ω are smooth note that $SO(n)$ acts transitively on $G_{\mathbb{R}}(n; k)$, and that the isotropy subgroup at $\mathbb{R}^k$ is the group

$$K = SO(n) \cap (O(k) \times O(n-k)).$$

This group consists of two components,

$$K_0 = \{(\varphi, \psi) \mid \det \varphi = 1, \det \psi = 1\} = SO(k) \times SO(n-k),$$

and

$$K_1 = \{(\varphi, \psi) \mid \det \varphi = -1, \quad \det \psi = -1\}.$$

The commutative diagram,

$$\begin{tikzcd} SO(n)/(SO(k) \times SO(n-k)) \arrow[r, "\cong"] \arrow[d, "\pi"'] & \tilde{G}(n; k) \arrow[d, "p"] \\ SO(n)/K \arrow[r, "\cong"] & G_{\mathbb{R}}(n; k), \end{tikzcd}$$

shows that p is smooth, a local diffeomorphism and a double covering. Hence ω is also smooth.

The dimension of $G_{\mathbb{R}}(n; k)$ is given by

$$\dim G_{\mathbb{R}}(n; k) = \binom{n}{2} - \binom{k}{2} - \binom{n-k}{2} = k(n-k).$$

2. *Real projective space*: Assume that $n \geqslant 2$ and consider the manifold $\tilde{G}(n; 1)$. Its points are the oriented lines in $\mathbb{R}^n$ through the origin. Identifying each such line with its positive unit vector, we obtain an $SO(n)$-equivariant bijection between $\tilde{G}(n; 1)$ and S^{n-1}. Since $SO(n)$ acts smoothly on S^{n-1}, the commutative diagram (cf. Example 2, sec. 3.6)

$$\begin{tikzcd} & SO(n)/SO(n-1) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \tilde{G}(n; 1) \arrow[rr, "\cong"] & & S^{n-1} \end{tikzcd}$$

shows that this identification is a diffeomorphism.

Moreover, the involution, ω, in $\tilde{G}(n; 1)$ defined in Example 1 corresponds under this diffeomorphism to the antipodal involution of S^{n-1}. Thus we obtain a diffeomorphism

$$G_{\mathbb{R}}(n; 1) \xrightarrow{\cong} \mathbb{R}P^{n-1}$$

(cf. Example 2, sec. 1.4, volume I). Hence $G_{\mathbb{R}}(n; 1)$ is diffeomorphic to the real projective space of dimension $n - 1$.

3. *Complex and quaternionic projective space*: Let $n \geqslant 2$. The manifolds $G_{\mathbb{C}}(n; 1)$ (respectively, $G_{\mathbb{H}}(n; 1)$) of complex (respectively, quaternionic) lines in $\mathbb{C}^n$ (respectively, $\mathbb{H}^n$) through the origin are called *complex* (*quaternionic*) *projective space* and are denoted by $\mathbb{C}P^{n-1}$ and $\mathbb{H}P^{n-1}$ respectively.

4. *Complex and quaternionic projective lines*: We shall construct diffeomorphisms

$$\mathbb{C}P^1 \xrightarrow{\cong} S^2 \qquad \text{and} \qquad \mathbb{H}P^1 \xrightarrow{\cong} S^4.$$

Define a map $\mathbb{C} \to \mathbb{C}P^1$ by sending $z \in \mathbb{C}$ to the one-dimensional complex subspace of $\mathbb{C}^2$ generated by the pair $(1, z)$. This is a smooth embedding. Since $\dim \mathbb{C} = 2 = \dim \mathbb{C}P^1$, it is a diffeomorphism onto an open subset of $\mathbb{C}P^1$. The only point which is not in the image is the one-dimensional subspace of $\mathbb{C}^2$ generated by $(0, 1)$. Since $\mathbb{C}P^1$ is compact, it is the one-point compactification of $\mathbb{C}$; i.e., $\mathbb{C}P^1$ is diffeomorphic to S^2.

Similarly, $\mathbb{H}P^1$ is the one-point compactification of $\mathbb{H}$ and hence it is diffeomorphic to S^4.

5.15. Canonical vector bundles over $G(n; k)$. Recall that in secs. 2.1 and 2.22, volume I, we defined real and complex vector bundles. Quaternionic vector bundles are defined in a similar way, and the definition of all three may be given simultaneously as follows: A *Γ-vector bundle* is a smooth bundle $\xi = (M, \pi, B, F)$, in which F and F_x $(x \in B)$ are Γ-vector spaces, and which admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that each map,

$$\psi_{\alpha,x}: F \xrightarrow{\cong} F_x,$$

is a Γ-linear isomorphism.

We shall construct canonical Γ-vector bundles over $G(n; k)$. It will be important to distinguish between a k-plane, F, as a subspace of Γ^n, and as a point in $G(n; k)$.

Consider the disjoint union

$$M = \bigcup_{F \in G(n;k)} F.$$

Thus a point of M is a pair (F, v) with $v \in F$. Let $\rho: M \to G(n; k)$ be the projection given by

$$\rho(F, v) = F.$$

Observe that a left action of $I(n)$ on the *set* M is given by

$$\varphi(F, v) = (\varphi(F), \varphi(v)), \qquad \varphi \in I(n), \quad (F, v) \in M.$$

We shall make $\xi_k = (M, \rho, G(n; k), \Gamma^k)$ into a Γ-vector bundle so that this action becomes a smooth action.

Consider the representation of $I(k) \times I(n-k)$ in Γ^k given by

$$(\varphi, \psi)(u) = \varphi(u), \qquad \varphi \in I(k), \quad \psi \in I(n-k), \quad u \in \Gamma^k.$$

It determines a Γ-vector bundle (cf. sec. 5.10),

$$\hat{\xi}_k = (I(n) \times_{I(k) \times I(n-k)} \Gamma^k, \hat{\rho}, I(n)/(I(k) \times I(n-k)), \Gamma^k),$$

which admits a canonical left action of $I(n)$. Now define a surjective set map,

$$\Phi: I(n) \times \Gamma^k \to M,$$

by setting

$$\Phi(\varphi, v) = (\varphi(\Gamma^k), \varphi(v)), \qquad \varphi \in I(n), \quad v \in \Gamma^k.$$

Factoring through the joint action, we obtain the commutative diagram,

$$\begin{tikzcd} I(n) \times_{I(k)\times I(n-k)} \Gamma^k \arrow[r, "\Psi"] \arrow[d, "\hat{\rho}"'] & M \arrow[d, "\rho"] \\ I(n)/(I(k) \times I(n-k)) \arrow[r, "\cong", "\beta"'] & G(n; k), \end{tikzcd}$$

where β is the equivariant diffeomorphism of sec. 5.13 and Ψ is an $I(n)$-equivariant bijection restricting to linear isomorphisms on the fibres.

Give M the manifold structure for which Ψ is a diffeomorphism. Then ξ_k becomes a vector bundle acted on by $I(n)$ and

$$\Psi: \hat{\xi}_k \xrightarrow{\cong} \xi_k$$

is an equivariant isomorphism.

Similarly, we obtain a vector bundle $\xi_k^\perp = (M^\perp, \rho, G(n; n-k), \Gamma^{n-k})$ by setting

$$M^\perp = \bigcup_{F \in G(n;k)} F^\perp.$$

It admits an action of $I(n)$ and is equivariantly isomorphic to the bundle

$$(I(n) \times_{I(k)\times I(n-k)} \Gamma^{n-k}, \hat{\rho}, I(n)/(I(k) \times I(n-k)), \Gamma^{n-k}).$$

(Replace Γ^k by $\Gamma^{n-k} = (\Gamma^k)^\perp$ in the discussion above.) ξ_k and $\xi_k^\perp$ are called the *canonical k-plane and $(n-k)$-plane bundles over $G(n; k)$*. The direct decomposition,

$$\Gamma^k \oplus \Gamma^{n-k} \xrightarrow{\cong} \Gamma^n,$$

determines a strong bundle isomorphism

$$\xi_k \oplus \xi_k^\perp \xrightarrow{\cong} G(n; k) \times \Gamma^n.$$

Finally, the actions of $I(n)$ on ξ_k and $\xi_k^\perp$ defined above, together with the standard actions of $I(n)$ on $G(n; k)$ and Γ^n define actions on the bundles $\xi_k \oplus \xi_k^\perp$ and $G(n; k) \times \Gamma^n$. Moreover, the isomorphism defined above is equivariant.

5.16. The tangent bundle of $G(n; k)$. Given two Γ-vector bundles ξ and η over the same base B, we can form the (real) vector bundle $L_\Gamma(\xi; \eta)$ whose fibre at $x \in B$ consists of the Γ-linear maps between the fibres of ξ and η at x.

Proposition IV: The tangent bundle of $G(n; k)$ satisfies

$$\tau_{G(n;k)} \cong L_\Gamma(\xi_k; \xi_k^\perp).$$

Proof: Identify $G(n; k)$ with $I(n)/(I(k) \times I(n-k))$. According to sec. 5.11 its tangent bundle is obtained from the representation $\mathrm{Ad}^\perp$ of $I(k) \times I(n-k)$ in $E(n)/(E(k) \oplus E(n-k))$.

On the other hand, $L_\Gamma(\xi_k; \xi_k^\perp)$ is obtained from the representation of $I(k) \times I(n-k)$ in $L_\Gamma(\Gamma^k; \Gamma^{n-k})$ given by

$$(\sigma, \tau)(\varphi) = \tau \circ \varphi \circ \sigma^{-1}, \qquad \sigma \in I(k), \quad \tau \in I(n-k), \quad \varphi \in L_\Gamma(\Gamma^k; \Gamma^{n-k}).$$

Thus we must construct an $(I(k) \times I(n-k))$-linear isomorphism

$$L_\Gamma(\Gamma^k; \Gamma^{n-k}) \cong E(n)/(E(k) \oplus E(n-k)).$$

Recall that $E(n)$ is the real vector space of Γ-linear skew transformations of Γ^n, and that $\Gamma^{n-k} = (\Gamma^k)^\perp$. The Lie algebras $E(k)$ of $I(k)$ and $E(n-k)$ of $I(n-k)$ (considered as subalgebras of $E(n)$) are given by

$$E(k) = \{\alpha \in E(n) \mid \alpha(\Gamma^{n-k}) = 0\} \qquad \text{and} \qquad E(n-k) = \{\alpha \in E(n) \mid \alpha(\Gamma^k) = 0\}.$$

Define a subspace $L \subset E(n)$ by setting

$$L = \{\alpha \in E(n) \mid \alpha(\Gamma^k) \subset \Gamma^{n-k} \text{ and } \alpha(\Gamma^{n-k}) \subset \Gamma^k\}.$$

Then

$$E(n) = E(k) \oplus E(n-k) \oplus L.$$

Moreover, since the adjoint representation of $I(n)$ is given by

$$(\mathrm{Ad}\,\sigma)\alpha = \sigma \circ \alpha \circ \sigma^{-1}, \qquad \sigma \in I(n), \quad \alpha \in E(n),$$

it follows that L is stable under $I(k) \times I(n-k)$. In particular, there is an isomorphism of $(I(k) \times I(n-k))$-spaces

$$L \cong E(n)/(E(k) \oplus E(n-k)).$$

Finally, define a linear isomorphism,

$$\Phi\colon L_\Gamma(\Gamma^k; \Gamma^{n-k}) \xrightarrow{\cong} L,$$

by setting

$$\Phi(\alpha)(x \oplus y) = \alpha(x) - \tilde{\alpha}(y), \qquad \alpha \in L_\Gamma(\Gamma^k; \Gamma^{n-k}), \quad x \in \Gamma^k, \quad y \in \Gamma^{n-k},$$

where $\tilde{\alpha}$ denotes the adjoint of α. Since $I(k)$ and $I(n-k)$ consist of isometries, it follows easily that Φ is $(I(k) \times I(n-k))$-equivariant. Hence

$$L_\Gamma(\Gamma^k; \Gamma^{n-k}) \cong L \cong E(n)/(E(k) \oplus E(n-k)),$$

which completes the proof.

Q.E.D.

Corollary: There are isomorphisms

$$\tau_{G_{\mathbb{R}}(n;k)} \cong \xi_k \otimes_{\mathbb{R}} \xi_k^\perp, \qquad \tau_{\tilde{G}(n;k)} \cong \xi_k \otimes_{\mathbb{R}} \xi_k^\perp, \qquad \text{and} \qquad \tau_{G_{\mathbb{C}}(n;k)} \cong \xi_k^* \otimes_{\mathbb{C}} \xi_k^\perp$$

(where ξ_k is interpreted as a vector bundle over the appropriate manifold, and ξ_k^* is the complex dual of ξ_k).

§5. The Stiefel manifolds

We continue the notational conventions of article 4.

5.17. Stiefel manifolds. An *orthonormal k-frame* in Γ^n is a sequence of k vectors, $(u_1, \ldots, u_k)$, such that

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

An n-frame in the oriented space, $\mathbb{R}^n$, is called *positive*, if it represents the orientation of $\mathbb{R}^n$.

We extend the conventions of this article by letting $V(n; k)$ denote any one of the sets $\hat{V}_{\mathbb{R}}(n; k)$, $V_{\mathbb{R}}(n; k)$, $V_{\mathbb{C}}(n; k)$, and $V_{\mathbb{H}}(n; k)$ defined by:

Case I	$\hat{V}_{\mathbb{R}}(n; k)$	Orthonormal k-frames in $\mathbb{R}^n$.
Case II	$V_{\mathbb{R}}(n; k)$	Orthonormal k-frames in $\mathbb{R}^n$ if $k < n$; positive orthonormal n-frames in $\mathbb{R}^n$ if $k = n$.
Case III	$V_{\mathbb{C}}(n; k)$	Orthonormal k-frames in $\mathbb{C}^n$.
Case IV	$V_{\mathbb{H}}(n; k)$	Orthonormal k-frames in $\mathbb{H}^n$.

A transitive left action of $I(n)$ on $V(n; k)$ is given by

$$\varphi \cdot (u_1, \ldots, u_k) = (\varphi(u_1), \ldots, \varphi(u_k)),$$
$$\varphi \in I(n), \quad (u_1, \ldots, u_k) \in V(n; k).$$

In particular, write $\Gamma^n = \Gamma^k \oplus \Gamma^{n-k}$ and let $(e_1, \ldots, e_k)$ be a fixed orthonormal basis of Γ^k. Then the subgroup of $I(n)$ which fixes the k-frame $(e_1, \ldots, e_k)$ is exactly $I(n-k)$ (cf. sec. 5.13). Thus the action of $I(n)$ on $V(n; k)$ determines an equivariant bijection

$$I(n)/I(n-k) \xrightarrow{\cong} V(n; k).$$

Assign $V(n; k)$ the unique manifold structure such that this bijection is a diffeomorphism. (Then the action above is smooth.) The manifold $V(n; k)$ is called the *Stiefel manifold of orthonormal k-frames in n-space.*

5.18. The universal frame bundle over $G(n; k)$. A canonical principal bundle,

$$\mathscr{P}(n; k) = (V(n; k), \pi_k, G(n; k), I(k)),$$

is defined as follows:

If $(u_1, \ldots, u_k) \in V(n; k)$, let $\pi_k(u_1, \ldots, u_k)$ be the (oriented) k-plane with $u_1, \ldots, u_k$ as (positive) basis. Then $\pi_k: V(n; k) \to G(n; k)$ is a well defined map. Moreover, we have the smooth commutative diagram,

$$\begin{array}{ccc} I(n)/I(n-k) & \xrightarrow{\cong} & V(n; k) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ I(n)/(I(k) \times I(n-k)) & \xrightarrow{\cong} & G(n; k), \end{array}$$

where the horizontal diffeomorphisms are defined in sec. 5.17 and sec. 5.13 respectively, and $p(\sigma \cdot I(n-k)) = \sigma \cdot (I(k) \times I(n-k))$, $\sigma \in I(n)$.

We can apply sec. 5.8 to obtain a smooth principal bundle

$$(I(n)/I(n-k), p, I(n)/(I(k) \times I(n-k)), I(k)).$$

Thus the diagram above shows that π_k is the projection of a smooth principal bundle, $\mathscr{P}(n; k) = (V(n; k), \pi_k, G(n; k), I(k))$. Note that, if $F \in G(n; k)$ then $\pi_k^{-1}(F)$ consists of the (positive) orthonormal k-frames in F. For this reason $\mathscr{P}(n; k)$ is called the *universal frame bundle over* $G(n; k)$.

The inclusion maps,

$$\Gamma^n \to \Gamma^{n+1} \to \Gamma^{n+2} \to \cdots,$$

determine smooth commutative diagrams,

$$\begin{array}{ccccc} V(n; k) & \longrightarrow & V(n+1; k) & \longrightarrow & \cdots \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_k} & & \\ G(n; k) & \longrightarrow & G(n+1; k) & \longrightarrow & \cdots, \end{array}$$

which are, in fact, homomorphisms of principal $I(k)$-bundles.

The vector bundle, η_k, associated with $\mathscr{P}(n; k)$ via the action of $I(k)$ in Γ^k, is canonically isomorphic to the bundle $\xi_k = (M, \rho, G(n; k), \Gamma^k)$ of sec. 5.15. Indeed, fix a (positive) orthonormal basis $(e_1, \ldots, e_k)$ of Γ^k.

Define a map,

$$q\colon V(n; k) \times \Gamma^k \to M,$$

as follows:

$$q((u_1, \ldots, u_k), \sum_i \lambda^i e_i) = \sum_i \lambda^i u_i, \qquad \lambda^i \in \Gamma, \quad (u_1, \ldots, u_k) \in V(n; k).$$

It is easy to check that q induces an $I(n)$-equivariant, strong isomorphism

$$V(n; k) \times_{I(k)} \Gamma^k \xrightarrow{\cong} M.$$

5.19. The manifolds $I(n; k)$. Let $I(n; k)$ denote the set of isometric inclusions $\Gamma^k \to \Gamma^n$ (except in case II when $k = n$; then $I(n; n)$ will denote the set of orientation preserving isometries of $\mathbb{R}^n$). Note that $I(n)$ and $I(k)$ act, respectively, from the left and right on $I(n; k)$ via

$$\varphi \cdot \psi = \varphi \circ \psi$$

and

$$\psi \cdot \sigma = \psi \circ \sigma, \qquad \varphi \in I(n), \quad \psi \in I(n; k), \quad \sigma \in I(k).$$

Now fix a (positive) orthonormal basis, $e_1, \ldots, e_k$, of Γ^k. Then an $I(n)$-equivariant bijection,

$$I(n; k) \to V(n; k),$$

is given by $\varphi \mapsto (\varphi e_1, \ldots, \varphi e_k)$. We use this bijection to make $I(n; k)$ into a smooth manifold, and to identify it with $V(n; k)$.

In particular, we may write

$$\mathscr{P}(n; k) = (I(n; k), \pi_k, G(n; k), I(k)).$$

Then $\pi_k(\varphi) = \varphi(\Gamma^k)$, $\varphi \in I(n; k)$. Moreover the principal action of $I(k)$ is the right action given above.

Finally, observe that the isomorphism $I(n; k) \times_{I(k)} \Gamma^k \xrightarrow{\cong} \xi_k$ of sec. 5.18 is induced by the map, $q : I(n; k) \times \Gamma^k \to M$, given by

$$q(\varphi, v) = \varphi(v), \qquad \varphi \in I(n; k), \quad v \in \Gamma^k.$$

Proposition V: Let $\mathscr{P} = (P, \pi, B, I(k))$ be a principal bundle. Then, for some $n \geq k$, there is a homomorphism of principal bundles

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi"'] & I(n; k) \arrow[d, "\pi_k"] \\ B \arrow[r, "\psi"'] & G(n; k). \end{tikzcd}$$

Definition: ψ is called a *classifying map* for the principal bundle $\mathscr{P}$.

According to the proposition, $\mathscr{P}$ is the pull-back of $\mathscr{P}(n; k)$ to B via ψ.
Before proving the proposition we establish

Lemma II: Let $\xi = (P \times_{I(k)} \Gamma^k, \rho_\xi, B, \Gamma^k)$ be the vector bundle associated with $\mathscr{P}$ via the action of $I(k)$ on Γ^k. Then, for some $n \geqslant k$, there is a strong bundle map $\sigma: \xi \to B \times \Gamma^n$ restricting to Γ-linear injections on the fibres.

Proof: This lemma is proved in sec. 2.23, volume I, in case I and case II. The same argument holds in cases III and IV, using Hermitian and quaternionic inner products.

Q.E.D.

Proof of the proposition: Let σ be the bundle map constructed in Lemma II and let

$$q: P \times \Gamma^k \to P \times_{I(k)} \Gamma^k$$

be the principal map(cf. sec. 5.3). Then a smooth map $\varphi: P \to I(n; k)$ is defined by the relation

$$(\pi(z), \varphi(z)u) = \sigma(q(z, u)), \qquad z \in P, \quad u \in \Gamma^k.$$

Clearly,

$$\varphi(z \cdot \tau)u = \varphi(z)(\tau(u)) = (\varphi(z) \circ \tau)u, \qquad z \in P, \quad \tau \in I(k), \quad u \in \Gamma^k.$$

Hence $\varphi(z \cdot \tau) = \varphi(z) \circ \tau$ and so φ is equivariant; i.e., φ is a homomorphism of principal bundles.

Q.E.D.

5.20. Examples: 1. *Hopf fiberings:* A point of $V(n; 1)$ is just a unit vector in Γ^n. Thus, if $n > 1$,

$$V_{\mathbb{R}}(n; 1) = S^{n-1}, \qquad V_{\mathbb{C}}(n; 1) = S^{2n-1}, \qquad V_{\mathbb{H}}(n; 1) = S^{4n-1}.$$

Moreover, the left action (cf. sec. 5.17) of $O(n)$, $U(n)$, and $Q(n)$ on these spheres is the standard one.

Next observe that in cases I, III, and IV, $I(1)$ can be identified with the unit sphere of Γ ($\Gamma = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) as follows: For each unit vector $a \in \Gamma$, define $\mu_a \in I(1)$ by

$$\mu_a(z) = za^{-1}, \qquad z \in \Gamma.$$

Then $a \mapsto \mu_a$ is an isomorphism of Lie groups (cf. Example 2, sec. 2.6 and Example 3, sec. 2.7). Thus the universal 1-frame bundles become

$$(S^{n-1}, \pi, \mathbb{R}P^{n-1}, S^0), \qquad (S^{2n-1}, \pi, \mathbb{C}P^{n-1}, S^1), \qquad \text{and} \qquad (S^{4n-1}, \pi, \mathbb{H}P^{n-1}, S^3).$$

Notice that the first bundle is simply the double covering of Example 2, sec. 5.14. Moreover $\mathbb{C}P^1 = S^2$ and $\mathbb{H}P^1 = S^4$ (cf. Example 4, sec 5.14). Thus the bundles $\mathscr{P}_{\mathbb{C}}(2; 1)$ and $\mathscr{P}_{\mathbb{H}}(2; 1)$ can be written

$$(S^3, \pi, S^2, S^1) \qquad \text{and} \qquad (S^7, \pi, S^4, S^3).$$

Consider the right action of S^0 (respectively, S^1, S^3) on $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$, $\mathbb{H}^n$) given by

$$(z_1, ..., z_n) \cdot z = (z^{-1}z_1, ..., z^{-1}z_n), \qquad z_i \in \Gamma.$$

This action restricts to an action of S^0 (respectively, S^1, S^3) on S^{n-1} (respectively, S^{2n-1}, S^{4n-1}). We shall show that these actions are the principal actions of S^0, S^1, and S^3 on the 1-frame bundles.

In fact, let $\sigma \in I(n; 1)$ and write $\sigma(1) = (z_1, ..., z_n)$. Then $\sigma(1) \in V_\Gamma(n; 1)$ and the principal action of $I(1)$ is given by (cf. sec. 5.19)

$$\sigma(1) \cdot z = (\sigma \circ \mu_z)(1) = \sigma(z^{-1}) = (z^{-1}z_1, ..., z^{-1}z_n).$$

2. *The Stiefel manifold* $V_{\mathbb{R}}(n; 2)$: Let $\Gamma = \mathbb{R}$ and consider the Stiefel manifold $V_{\mathbb{R}}(n; 2)$. Its points are the isometries $\alpha: \mathbb{R}^2 \to \mathbb{R}^n$. An embedding $\varphi: V_{\mathbb{R}}(n; 2) \to \mathbb{R}^n \oplus \mathbb{R}^n$ is defined as follows: Choose an orthonormal basis e_1, e_2 in $\mathbb{R}^2$ and set $\varphi(\alpha) = (\alpha(e_1), \alpha(e_2))$. The image of φ consists precisely of the pairs (x, y) satisfying

$$|x| = 1, \qquad |y| = 1, \qquad \langle x, y\rangle = 0.$$

On the other hand, consider the bundle $(M, \pi, S^{n-1}, S^{n-2})$, of unit tangent vectors of S^{n-1}. Then the map,

$$\psi: z \mapsto (\pi(z), z), \qquad z \in M,$$

defines an embedding of M into $\mathbb{R}^n \oplus \mathbb{R}^n$ and the images of ψ and φ coincide. Composing φ with the inverse of ψ yields a diffeomorphism of $V_{\mathbb{R}}(n; 2)$ onto M.

§6. The cohomology of the Stiefel manifolds and the classical groups

The notation conventions of articles 4 and 5 are continued in this article. We shall frequently make the identifications

$$V_{\mathbb{C}}(n; k) = U(n)/U(n-k) \qquad \text{and} \qquad V_{\mathbb{R}}(n; k) = SO(n)/SO(n-k).$$

The tensor product of graded algebras is always the anticommutative tensor product.

5.21. Complex and quaternionic Stiefel manifolds. Theorem I: The cohomology algebras of the manifolds $V_{\mathbb{C}}(n; k)$ and $V_{\mathbb{H}}(n; k)$ are exterior algebras over oddly graded subspaces (i.e., subspaces whose homogeneous elements all have odd degree). The Poincaré polynomials are given by

$$f_{V_{\mathbb{C}}(n;\, k)} = \prod_{i=1}^{k} (1 + t^{2(n-k+i)-1})$$

and

$$f_{V_{\mathbb{H}}(n;\, k)} = \prod_{i=1}^{k} (1 + t^{4(n-k+i)-1})$$

Corollary: The Poincaré polynomials for $U(n)$ and $Q(n)$ are respectively given by (since $V_{\mathbb{C}}(n; n) = U(n)$ and $V_{\mathbb{H}}(n; n) = Q(n)$)

$$f_{U(n)} = \prod_{i=1}^{n} (1 + t^{2i-1}) \qquad \text{and} \qquad f_{Q(n)} = \prod_{i=1}^{n} (1 + t^{4i-1}).$$

Proof: We consider the complex case; the argument in the quaternionic case is identical. The proof is by induction on k (for fixed n).

Since $V_{\mathbb{C}}(n, 1) = S^{2n-1}$, the theorem is clear for $k = 1$.

Suppose it holds for some k. From sec. 5.8, we obtain a bundle

$$\xi = (U(n)/U(n-k-1), \rho, U(n)/U(n-k), U(n-k)/U(n-k-1)).$$

Since $U(n-k)/U(n-k-1) = S^{2(n-k)-1}$, ξ is a sphere bundle.

Moreover, since $U(n-k)$ acts on the sphere by orientation preserving diffeomorphisms, the bundle is orientable. Thus its Euler class,

$$\chi_\xi \in H^{2(n-k)}(U(n)/U(n-k)),$$

is defined.

On the other hand, by our induction hypothesis, the theorem holds for k, and so the formula in the theorem shows that

$$H^{2(n-k)}(U(n)/U(n-k)) = 0.$$

Thus $\chi_\xi = 0$. Now it follows from Corollary II to Proposition IV of sec. 8.4, volume I, that

$$H(U(n)/U(n-k-1)) \cong H(U(n)/U(n-k)) \otimes H(S^{2(n-k)-1})$$

(as graded algebras). This closes the induction.

Q.E.D.

5.22. The Stiefel manifolds $V_{\mathbb{R}}(n; 2)$. Proposition VI: The cohomology algebra of $V_{\mathbb{R}}(n; 2)$ (for $n \geqslant 3$) is given by

$$H(V_{\mathbb{R}}(2m; 2)) \cong H(S^{2m-1}) \otimes H(S^{2m-2}) \quad \text{and} \quad H(V_{\mathbb{R}}(2m+1; 2)) \cong H(S^{4m-1}).$$

Proof: Recall from Example 2, sec. 5.20, that the sphere bundle associated with the tangent bundle of S^{n-1} is given by

$$\xi = (V_{\mathbb{R}}(n; 2), \pi, S^{n-1}, S^{n-2}).$$

Moreover (cf. Example 1, sec. 9.10, volume I)

$$\chi_\xi = \begin{cases} 0, & n-1 \text{ odd} \\ 2\omega_{n-1}, & n-1 \text{ even,} \end{cases}$$

where ω_{n-1} denotes the orientation class of S^{n-1}.

Case A: $n = 2m$, $m > 1$. Then since $\chi_\xi = 0$ there is a class $\omega \in H^{2m-2}(V_{\mathbb{R}}(2m; 2))$ such that $\oint_S^\# \omega = 1$ (cf. sec. 8.4, volume I). Moreover, the map,

$$\alpha \otimes 1 + \beta \otimes \omega_{2m-2} \to \pi^\#\alpha + \pi^\#\beta \cdot \omega, \qquad \alpha, \beta \in H(S^{2m-1}),$$

defines a linear isomorphism

$$H(S^{2m-1}) \otimes H(S^{2m-2}) \xrightarrow{\cong} H(V_{\mathbb{R}}(2m; 2)).$$

In particular, $H^{4m-4}(V_{\mathbb{R}}(2m; 2)) = 0$, and so $\omega^2 = 0$. It follows that this isomorphism is an isomorphism of graded algebras.

Case B: $n = 2m + 1$, $m \geqslant 1$. The Gysin sequence for ξ reads

$$\begin{array}{ccccc} \downarrow & & & & \\ H^i(S^{2m}) & \xrightarrow{\pi^*} & H^i(V_{\mathbb{R}}(2m+1;2)) & \xrightarrow{\oint_S^*} & H^{i-2m+1}(S^{2m}) \\ & & & & \big\downarrow{\scriptstyle D} \\ & & & & H^{i+1}(S^{2m}) \rightarrow , \end{array}$$

(cf. sec. 8.2, volume I). This shows that, for $i \neq 0, 2m - 1, 2m, 4m - 1$,

$$H^i(V_{\mathbb{R}}(2m + 1; 2)) = 0.$$

Since $D(1) = \chi_\xi = 2\omega_{2m}$, D restricts to an isomorphism

$$H^0(S^{2m}) \xrightarrow{\cong} H^{2m}(S^{2m}).$$

Now the exactness of the Gysin sequence yields

$$H^{2m}(V_{\mathbb{R}}(2m + 1; 2)) = 0 = H^{2m-1}(V_{\mathbb{R}}(2m + 1; 2)).$$

Q.E.D.

5.23. Bundles with fibre $V_{\mathbb{R}}(2m + 1; 2)$. Let $\eta = (E, \pi, B, F)$ be an oriented bundle with $F = V_{\mathbb{R}}(2m + 1; 2)$. In view of sec. 5.22,

$$H(F) \cong H(S^{4m-1}).$$

Now the proofs of the results for sphere bundles established in article 1, Chap. VIII, volume I, depend only on the *cohomology* and compactness of the fibre; in particular, the identical results hold for η.

This implies that there is a class $\chi_\eta \in H^{4m}(B)$, depending only on η, and determined by the following condition: Let $\Phi \in A^{4m}(B)$ represent χ_η. Then, for some $\Omega \in A^{4m-1}(E)$,

$$\pi^*\Phi = \delta\Omega, \qquad \text{and} \qquad \oint_F \Omega = -1.$$

Moreover there is a long exact sequence,

$$\cdots \longrightarrow H^i(B) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\oint_F^*} H^{i-4m+1}(B) \xrightarrow{D} H^{i+1}(B) \longrightarrow \cdots,$$

where $D\alpha = \alpha \cdot \chi_\eta$. If $\chi_\eta = 0$, then there is an isomorphism of graded algebras,

$$H(E) \cong H(B) \otimes H(S^{4m-1}).$$

5.24. The real Stiefel manifolds $V_{\mathbb{R}}(n; k)$. Theorem II: The cohomology algebra of $V_{\mathbb{R}}(n; k)$ $(k < n)$ is an exterior algebra over a graded vector space. The Poincaré polynomials are the polynomials given below.

$n = 2m, \quad k = 2l + 1, \quad l \geqslant 0$	$(1 + t^{2m-1}) \prod_{i=1}^{l} (1 + t^{4m-4i-1})$
$n = 2m + 1, \quad k = 2l, \quad l \geqslant 1$	$\prod_{i=1}^{l} (1 + t^{4m-4i+3})$
$n = 2m, \quad k = 2l, \quad m > l \geqslant 1$	$(1 + t^{2m-2l})(1 + t^{2m-1}) \prod_{i=1}^{l-1} (1 + t^{4m-4i-1})$
$n = 2m + 1, k = 2l + 1, m > l \geqslant 0$	$(1 + t^{2m-2l}) \prod_{i=1}^{l-1} (1 + t^{4m-4i+3})$

Theorem III: The Poincaré polynomials for the groups $SO(n)$ are given by

$$f_{SO(2m)} = (1 + t^{2m-1}) \prod_{i=1}^{m-1} (1 + t^{4i-1})$$

and

$$f_{SO(2m+1)} = \prod_{i=1}^{m} (1 + t^{4i-1}).$$

Proof of Theorem II: Since $V_{\mathbb{R}}(n, 1) = S^{n-1}$, the theorem is correct for $k = 1$. If $2 = k < n$ the theorem is contained in Proposition VI, sec. 5.22. Now we use induction on k. Assume the theorem holds for $V_{\mathbb{R}}(n; i)$, $i < k$, and consider two cases separately.

Case A: $n - k$ is odd. Write $n - k = 2q - 1$. Consider the bundle

$$(SO(n)/SO(n-k), \rho, SO(n)/SO(n-k+2), V_{\mathbb{R}}(2q+1; 2)).$$

By induction the theorem holds for $SO(n)/SO(n-k+2)$. It follows that

$$H^{4q}(SO(n)/SO(n-k+2)) = H^{2n-2k+2}(SO(n)/SO(n-k+2)) = 0.$$

Hence it follows from sec. 5.23 that

$$H(SO(n)/SO(n-k)) \cong H(SO(n)/SO(n-k+2)) \otimes H(S^{2n-2k+1})$$

and the induction is closed.

Case B: $n - k = 2q$ and $q > 0$. Since (always) $k \geqslant 1$, we have $2q < n$. Now consider the sphere bundle

$$(SO(n)/SO(n-k), \rho, SO(n)/SO(n-k+1), S^{n-k}).$$

Since $n - k$ is even, we have a linear isomorphism,

$$H(SO(n)/SO(n-k)) \xrightarrow{\cong} H(SO(n)/SO(n-k+1)) \otimes H(S^{n-k}),$$

of graded vector spaces (cf. Corollary II to Proposition IV, sec. 8.4, volume I).

It follows from our induction hypothesis that

$$H^{2(n-k)}(SO(n)/SO(n-k+1)) = 0.$$

This, as in the proof of Proposition VI, implies that the isomorphism is an isomorphism of graded algebras.

Q.E.D.

Proof of Theorem III: Let $(v_1, \ldots, v_{n-1})$ be an orthonormal $(n-1)$-frame in $\mathbb{R}^n$. Then there is a unique vector, $v_n \in \mathbb{R}^n$, such that $(v_1, \ldots, v_n)$ is a positive orthonormal n-frame. This provides a diffeomorphism, $V_{\mathbb{R}}(n; n-1) \xrightarrow{\cong} SO(n)$. Now apply Theorem II.

Q.E.D.

Problems

1. **Free actions.** (i) Let G be a Lie group that acts freely and properly on a manifold M (cf. problem 5, Chap. III). Show that $(M, \pi, M/G, G)$ is a principal bundle (cf. problem 6, Chap. III).

(ii) Apply this when G is discrete and the action is discontinuous (problem 21, Chap. III). Show that the universal covering projection for any connected manifold is the projection of a principal bundle (problem 18, Chap. I).

2. (i) Show that the closed proper subgroups of S^1 are finite, and are in $1-1$ correspondence with the groups $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, $p = 1, 2, \ldots$.

(ii) Construct principal bundles $(S^1, \pi, S^1, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ acts by multiplication. Let $(S^1 \times_{\mathbb{Z}_p} S^1, \rho, S^1, S^1)$ be the associated bundle (same action of $\mathbb{Z}_p$). Identify it as a principal S^1-bundle, and show that it is the trivial bundle.

(iii) Construct a principal bundle $(\mathbb{R}^2, \pi, S^1 \times_{\mathbb{Z}_p} S^1, \mathbb{Z} \times \mathbb{Z})$, where $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by

$$(x, y) \cdot (m, n) = (x + pm + n, y + n), \qquad x, y \in \mathbb{R}, \quad m, n \in \mathbb{Z}.$$

(iv) Let $\mathbb{Z}_2$ act on S^1 via $e^{i\theta} \mapsto e^{-i\theta}$. Show that $S^1 \times_{\mathbb{Z}_2} S^1$ is not diffeomorphic to $S^1 \times S^1$.

3. Let $M(n, m; k)$ denote the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ of rank k.

(i) Make $M(n, m; k)$ into a smooth manifold.

(ii) Show that composition defines a smooth map

$$\rho\colon M(n, k; k) \times M(k, m; k) \to M(n, m; k).$$

(iii) Show that ρ is the projection of a principal bundle.

4. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let G act on itself by conjugation. Show that the resulting associated bundle is a bundle over B with fibre G. Construct an example in which this bundle cannot be made into a principal bundle.

5. Let (P, π, B, G) be a principal bundle. Assume that G acts on an r-manifold, Y, and let (M, ρ, B, Y) be the associated bundle. Let $\xi = (V_M, p, M, \mathbb{R}^r)$ be the vertical subbundle of the tangent bundle τ_M.

(i) Show that $(V_M, \rho \circ p, B, T_Y)$ is a smooth bundle. Identify it with the bundle $P \times_G T_Y \to B$.

(ii) Assume that $Y = G/K$, where K is a closed subgroup of G. Identify ξ with the bundle $(P \times_K E/F, p_1, P/K, E/F)$, where E and F denote the Lie algebras of G and K.

6. (i) Let $\xi = (M, \pi, B, F)$ be a Riemannian vector bundle. Construct a principal $O(n)$-bundle whose fibre at x is the set of isometries $F \to F_x$. Show that ξ is the associated vector bundle.

(ii) Make similar constructions for real vector bundles, oriented real bundles, oriented Riemannian bundles, complex bundles, Hermitian bundles, and quaternionic vector bundles.

(iii) Apply (i) and (ii) to the tangent bundle of a manifold. Show that the resulting principal bundle has trivial tangent bundle.

7. Flag manifolds. A *flag* in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$, $\mathbb{H}^n$) is a sequence of subspaces,

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{R}^n,$$

such that $\dim F_j = j$ $(1 \leqslant j \leqslant n)$.

Make the flags into a compact manifold (in each case) and identify it with a homogeneous space.

8. Grassmann manifolds. (i) Define an action of $GL(n; \mathbb{R})$ on $G_{\mathbb{R}}(n; k)$. Determine the orbits and isotropy subgroups.

(ii) Make a similar construction in the other three cases.

9. Projective vector bundles. Let $\xi = (E, \pi, B, F)$ be a real vector bundle.

(i) Define a manifold M whose points are the one-dimensional subspaces of the fibres F_x.

(ii) Show that M is the total space of a fibre bundle over B with the real projective space as typical fibre. Represent this bundle as an associated bundle.

(iii) Show that the pull-back of ξ to M can be written in the form $\eta \oplus \zeta$, where ζ is a vector bundle of rank 1. Is ζ necessarily trivial?

(iv) Establish analogous results in the complex and quaternionic cases.

10. Actions on principal bundles. A Lie group K *acts on a principal bundle* $\mathscr{P} = (P, \pi, B, G)$ if it acts from the left on P and B so that the projection π and the right translations T_a ($a \in G$) are K-equivariant.

(i) Show that an action of K on $\mathscr{P}$ induces an action of $K \times G$ on P.

(ii) Show that an action of K on $\mathscr{P}$ induces an action of K on all the associated bundles.

(iii) If K acts on $\mathscr{P}$, show that it acts on associated vector bundles by bundle maps. Obtain a geometric description of its action on the corresponding associated bundles with fibre a Grassmannian.

11. Parallelizable homogeneous spaces. Recall that a manifold M is parallelizable (respectively, stably parallelizable) if the tangent bundle τ_M is trivial (respectively, if $\tau_M \oplus \epsilon^1$ is trivial).

(i) Suppose $H \subset K \subset G$ is a sequence of closed Lie groups. If G/H is stably parallelizable, show that so is K/H.

(ii) Let G be a Lie group with Lie algebra E. Then Ad: $G \to GL(E)$. Show that $GL(E)/\mathrm{Im}\,\mathrm{Ad}$ is stably parallelizable.

(iii) Let K be a closed subgroup of a Lie group G. Assume the Lie algebra F of K satisfies

$$F = I_0 \supset I_1 \supset \cdots \supset I_k = 0,$$

where $[I_\nu, I_\nu] \subset I_{\nu+1}$. Show that G/K is stably parallelizable.

(iv) Show that the real and complex Stiefel manifolds $V(n; k)$ are parallelizable if $k \geqslant 2$. (Notice that $V_{\mathbb{R}}(n; 2)$ requires special attention.) Discuss the quaternionic case.

12. Vector fields on homogeneous spaces. Let K be a closed subgroup of a Lie group G. Let $F \subset E$ be the corresponding Lie algebras.

(i) Establish an isomorphism $f \mapsto Y_f$ between the space of K-equivariant functions $f: G \to E/F$ and vector fields on G/K. Given f_1 and f_2, describe the function f_3 satisfying $Y_{f_3} = [Y_{f_1}, Y_{f_2}]$.

(ii) Show that the zero sets $f^{-1}(0)$ and $Y_f^{-1}(0)$ are related by $f^{-1}(0) = \pi^{-1} Y_f^{-1}(0)$ ($\pi: G \to G/K$).

(iii) Show that the isomorphism of (i) determines an isomorphism from $(E/F)_I$ to the space of G-invariant vector fields on G/K. Describe the corresponding Lie product in $(E/F)_I$.

(iv) Let S be a q-dimensional torus in $U(n)$. Construct a family of

$(n-q)^2$, $U(n)$-invariant, vector fields on $U(n)/S$, linearly independent at each point.

13. Division algebras. Let E be an n-dimensional Euclidean space and let $e \in E$ be a fixed unit vector. Assume that a bilinear map $(x, y) \to xy$ is defined, subject to the following conditions: (a) $xe = ex = x$. (b) The maps, $y \mapsto xy$ and $y \mapsto yx$, are isomorphisms if $x \neq 0$. Then E is called a *real division algebra*. Assume E is a real division algebra, and let S^{2n-1} be the unit sphere in $E \oplus E$ (with respect to the induced inner product) and let S^n be the one-point compactification of E with z_∞ as compactifying point. If $y \neq 0$ define y^{-1} by $yy^{-1} = e$.

(i) Show that the map $\pi\colon S^{2n-1} \to S^n$ given by

$$\pi(x, y) = \begin{cases} xy^{-1}, & y \neq 0 \\ z_\infty, & y = 0 \end{cases}$$

is a smooth submersion.

(ii) Construct a smooth bundle $(S^{2n-1}, \pi, S^n, S^{n-1})$. (*Hint*: cf. problem 12, Chap. VII, volume I).

(iii) Show that in the cases $E = \mathbb{C}$, $\mathbb{H}$ this is a principal bundle.

(iv) Use the Cayley numbers to show that such a multiplication exists for $n = 8$ and construct a fibre bundle (S^{15}, π, S^8, S^7). Show that this is *not* a principal bundle (cf. problem 5, Chap. III, volume I).

14. Consider the principal bundles $(SO(n), \pi, S^{n-1}, SO(n-1))$, where π is given by $\pi(\tau) = \tau(e)$ (e, a fixed unit vector in $\mathbb{R}^n$). In the case $n = 4$ and $n = 8$ show that this principal bundle admits a cross-section and conclude that it is trivial. Conclude that S^3 and S^7 are parallelizable.

15. Coordinate representations. Let $\mathscr{B} = (M, \rho, B, F)$ be a smooth bundle, and assume that G is a Lie group acting on F from the left.

(i) Suppose $\mathscr{B}$ is the associated bundle of a principal G-bundle. Show that there is a coordinate representation (U_i, ψ_i) for $\mathscr{B}$ such that

$$\psi_i^{-1} \circ \psi_j(x, y) = (x, \gamma_{ij}(x) \cdot y),$$

where $\gamma_{ij} : U_i \cap U_j \to G$ are smooth maps satisfying

$$\gamma_{ij}(x)\, \gamma_{jk}(x) = \gamma_{ik}(x), \qquad x \in U_i \cap U_j \cap U_k .$$

(ii) Conversely, assume $\mathscr{B}$ has such a coordinate representation. Construct a principal bundle for which $\mathscr{B}$ is the associated bundle (via the given action of G on F).

(iii) If the action is effective show that the first equation of (i) implies the second.

(iv) Show that the constructions in problem 6 are special cases of the construction in (ii).

16. Let H_1 and H_2 be closed subgroups of a Lie group G such that $H_1 \subset H_2$. Consider the fibre bundle

$$\mathscr{B} = (G/H_1, \pi, G/H_2, H_2/H_1).$$

Define H_0 by

$$H_0 = \bigcap_{x \in H_2} xH_1x^{-1}.$$

(i) Show that H_0 is the largest subgroup of H_1 which is normal in H_2. Conclude that H_0 is a closed Lie subgroup of H_1.

(ii) Show that $(G/H_0, \pi_0, G/H_2, H_2/H_0)$ is a principal bundle, and that $\mathscr{B}$ is an associated bundle.

(iii) Show that $\mathscr{B} = (V_{\mathbb{R}}(n; j), \pi, V_{\mathbb{R}}(n; k), V_{\mathbb{R}}(n - k; j - k))$ is associated with the principal bundle $(SO(n), \pi, V_{\mathbb{R}}(n; k), SO(n - k))$ and use problem 14 to conclude that $\mathscr{B}$ is trivial if $j > 1$ and $n = 4$ or 8.

17. Vector fields on fibre bundles. Let $\mathscr{B} = (M, \pi, B, F)$ be a fibre bundle. A vector field, Y, on M is called *basic*, if there is a vector field, X, on B such that

$$Y \underset{\pi}{\sim} X.$$

A vector field, Z, on M is called *vertical*, if

$$(d\pi)_z Z(z) = 0, \qquad z \in M.$$

(i) Show that the Lie product of vertical vector fields is vertical.

(ii) Show that the Lie product of a vertical and a basic vector field is vertical.

(iii) Show that the Lie product of two basic vector fields is basic.

(iv) Show that the $\mathscr{S}(M)$-module $\mathscr{X}(M)$ is generated by the basic and vertical vector fields

(v) If $\mathscr{B}$ is a principal bundle, show that Y is basic if and only if $Y - (T_a)_* Y$ is vertical for each a in the structure group.

18. Differential forms on fibre bundles. Consider the homomorphism $\pi^*: A(M) \leftarrow A(B)$ ($\mathscr{B} = (M, \pi, B, F)$, a bundle).

(i) Show that π^* is injective.

(ii) Show that, if $\Phi \in \operatorname{Im} \pi^*$, then $i(Z)\Phi = 0$ and $\theta(Z)\Phi = 0$ for every vertical vector field. Show that if F is connected, then the converse is true.

(iii) Show that if $\mathscr{B}$ admits a cross-section, then the map, $\pi^\#: H(M) \leftarrow H(B)$, is injective.

19. Let E and F be the Lie algebras of $GL(n; \mathbb{R})$ and $U(n)$.

(i) Construct an isomorphism of graded differential algebras

$$(\wedge E^* \otimes \mathbb{C}, \delta_E \otimes \iota) \cong (\wedge F^* \otimes \mathbb{C}, \delta_F \otimes \iota).$$

(ii) Compute $H_L(GL(n; \mathbb{R}))$ and compare it with $H(SO(n))$.

(iii) Compute $H_L(O(p, q))$ (cf. problem 12, Chap. II).

20. Outer automorphisms. Construct an automorphism of $U(n)$ which is not an inner automorphism. Determine its action on $H(U(n))$. Do the same for $SO(2n)$.

Chapter VI

Principal Connections and the Weil Homomorphism

In this chapter G denotes an r-dimensional Lie group with Lie algebra E. $\mathscr{P} = (P, \pi, B, G)$ denotes a fixed principal bundle ($\dim B = n$). $T: P \times G \to P$ denotes the principal action of G on P. The fibre over $x \in B$ is denoted by G_x ; note that this is *not* an isotropy subgroup.

For every $h \in E$, Z_h denotes the fundamental vector field generated by h. The operators $i(Z_h)$, $\theta(Z_h)$ in $A(P)$ are denoted by $i(h)$ and $\theta(h)$ (cf. sec. 3.13). The Lie algebra of invariant vector fields on P is denoted by $\mathscr{X}^I(P)$.

The vertical subbundle of τ_P will be denoted by $\mathbf{V}_P$; we use the boldface notation to avoid confusion with the notation for a principal connection (cf. sec. 6.8). A cross-section of $\mathbf{V}_P$ is called a *vertical* vector field; thus a vector field, Z, on P is vertical if and only if $Z \underset{\pi}{\sim} 0$. The module of vertical vector fields is denoted by $\mathscr{X}_V(P)$.

§1. Vector fields

6.1. The vertical subbundle. Recall that the vertical subbundle is the subbundle $\mathbf{V}_P$ of the tangent bundle τ_P of P whose fibre at z is given by

$$\mathbf{V}_z(P) = \ker(d\pi)_z, \qquad z \in P,$$

(sec. 7.1, volume I).

Since G acts freely on P, we also have the fundamental bundle $F_P \subset T_P$ (cf. sec. 3.11).

Proposition I: The fundamental and vertical subbundles coincide.

Proof: Since $d\pi \circ dA_z = 0$, it follows that

$$F_P \subset \mathbf{V}_P.$$

On the other hand,

$$\operatorname{rank}(F_P) = \dim G = \operatorname{rank}(\mathbf{V}_P).$$

Hence

$$F_P = \mathbf{V}_P.$$

Q.E.D.

Corollary I: The map $P \times E \to T_P$ given by $(z, h) \mapsto Z_h(z)$ defines a strong bundle isomorphism

$$P \times E \xrightarrow{\cong} \mathbf{V}_P.$$

Proof: Apply sec. 3.9 and sec. 3.11.

Q.E.D.

Corollary II: The map $\mathscr{S}(P) \otimes E \to \mathscr{X}(P)$ given by

$$f \otimes h \mapsto f \cdot Z_h, \qquad f \in \mathscr{S}(P), \quad h \in E,$$

defines an isomorphism of $\mathscr{S}(P) \otimes E$ onto $\mathscr{X}_V(P)$. In particular, $\mathscr{X}_V(P)$ is a free $\mathscr{S}(P)$-module, generated by the fundamental vector fields.

Proof: Apply Corollary I.

Q.E.D.

Corollary III: An isomorphism $\mathscr{S}(P; E) \xrightarrow{\cong} \mathscr{X}_V(P)$ is given by $f \mapsto Z_f$, where

$$Z_f(z) = Z_{f(z)}(z).$$

Proof: This is the isomorphism of Corollary II.

Q.E.D.

Example: Suppose B is a single point and $P = G$. Then

$$V_P = T_P = T_G$$

and the isomorphism,

$$G \times E \xrightarrow{\cong} T_G,$$

is given by

$$(a, h) \mapsto X_h(a),$$

where X_h is the left invariant vector field generated by h.

6.2. Invariant vector fields. Recall from sec. 3.10 that the action of G on P determines the action $(Z, a) \mapsto Z \cdot a$ of G on $\mathscr{X}(P)$, where $Z \cdot a = (T_a)_* Z$. If $Z \cdot a = Z$, $a \in G$, then Z is called an invariant vector field and the space of invariant vector fields is denoted by $\mathscr{X}^I(P)$.

Example: Recall from Example 3, sec. 3.10, that

$$Z_f \cdot a^{-1} = Z_{a \cdot f}, \qquad f \in \mathscr{S}(P; E), \quad a \in G,$$

where $a \cdot f$ is the E-valued function defined by

$$(a \cdot f)(z) = (\operatorname{Ad} a) f(z \cdot a).$$

In particular, the vector field Z_f is invariant if and only if the function f is equivariant.

Proposition II: Let Z be an invariant vector field on P. Then there is a unique vector field X on B such that $Z \underset{\pi}{\sim} X$. The correspondence $Z \mapsto X$ is a surjective Lie algebra homomorphism

$$\pi_*: \mathscr{X}^I(P) \to \mathscr{X}(B).$$

Its kernel is given by

$$\ker \pi_* = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

Proof: Since Z is invariant,

$$Z(z \cdot a) = (dT_a)\, Z(z), \qquad a \in G, \quad z \in P.$$

It follows that

$$(d\pi)\, Z(z \cdot a) = (d\pi)\, Z(z), \qquad a \in G, \quad z \in P.$$

This shows that, for each $x \in B$, there is a unique tangent vector $X(x)$ at x satisfying

$$(d\pi)\, Z(z) = X(x), \qquad z \in G_x .$$

The correspondence $x \mapsto X(x)$ defines a set map $X: B \to T_B$. To show that X is smooth, let $\sigma: U \to P$ be a cross-section over an open set U. Then $X = (d\pi) \circ Z \circ \sigma$, and so X is smooth in U (and hence in B). Hence it is a vector field on B. Clearly, $Z \underset{\pi}{\sim} X$. Since π is surjective, X is uniquely determined by Z.

To prove the second part, consider the map $\pi_*: \mathscr{X}^I(P) \to \mathscr{X}(B)$ defined by $Z \mapsto X$. It follows directly from Proposition VIII, sec. 3.13, volume I, that π_* is a homomorphism of Lie algebras. Moreover, $\pi_* Z = 0$ if and only if $(d\pi)\, Z(z) = 0$, $z \in P$; i.e., if and only if Z is vertical. This shows that

$$\ker \pi_* = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

It remains to show that π_* is surjective. Let $X \in \mathscr{X}(B)$ and choose a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for $\mathscr{P}$. Let $\{p_\alpha\}$ be a partition of unity for B subordinate to the covering $\{U_\alpha\}$. Define vector fields, X_α, in $U_\alpha \times G$ by

$$X_\alpha(x, a) = X(x), \qquad x \in U_\alpha, \quad a \in G.$$

Then $(\psi_\alpha)_* X_\alpha \in \mathscr{X}^I(\pi^{-1}(U_\alpha))$ and so an invariant vector field Z on P is given by

$$Z = \sum_\alpha \pi^* p_\alpha \cdot (\psi_\alpha)_* X_\alpha .$$

Evidently, $\pi_* Z = X$.

Q.E.D.

Corollary: If $Z \in \mathscr{X}^I(P)$ and $Y \in \mathscr{X}_V(P)$, then

$$[Z, Y] \in \mathscr{X}_V(P).$$

Proof: Since $Z \underset{\pi}{\sim} \pi_* Z$ and $Y \underset{\pi}{\sim} 0$, it follows that

$$[Z, Y] \underset{\pi}{\sim} [\pi_* Z, 0] = 0.$$

Q.E.D.

§2. Differential forms

6.3. The homomorphism π^*. A differential form, Φ, on P is called *invariant* if it is invariant under the right action of G. The algebra of invariant forms is denoted by $A_I(P)$.

A differential form Φ on P is called *horizontal* if $i(Y)\,\Phi = 0$, $Y \in \mathscr{X}_V(P)$. Since the fundamental bundle coincides with the vertical bundle (Proposition I, sec. 6.1), Φ is horizontal if and only if it is horizontal with respect to the action of G (cf. sec. 3.13). The algebra of horizontal forms is denoted by $A(P)_{i=0}$.

Now consider the homomorphism π^*: $A(P) \leftarrow A(B)$.

Proposition III: The homomorphism π^* is injective. The image of π^* consists precisely of the differential forms which are both invariant and horizontal.

Proof: Since the maps π and $(d\pi)_z$ ($z \in P$) are surjective, π^* must be injective. Moreover, the relations,

$$T_a^* \circ \pi^* = \pi^*, \qquad a \in G,$$

and

$$(d\pi)\, Z_h(z) = 0, \qquad h \in E, \quad z \in P,$$

imply that the differential forms in $\operatorname{Im} \pi^*$ are invariant and horizontal.

Now assume $\Psi \in A(P)$ is invariant and horizontal. Choose a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for $\mathscr{P}$. Since ψ_α is equivariant, $\psi_\alpha^*\Psi \in A(U_\alpha \times G)$ is invariant and horizontal with respect to the action, $((x, a), b) \mapsto (x, ab)$, of G on $U_\alpha \times G$. It follows that there is a *unique* $\Phi_\alpha \in A(U_\alpha)$ such that

$$\Phi_\alpha \times 1 = \psi_\alpha^*\Psi.$$

This uniqueness implies that Φ_α and Φ_β agree in $U_\alpha \cap U_\beta$. Hence there is a unique differential form, $\Phi \in A(B)$, such that

$$\Phi(x) = \Phi_\alpha(x), \qquad x \in U_\alpha .$$

Clearly, $\pi^*\Phi = \Psi$.

Q.E.D.

Definition: The differential forms which are both invariant and horizontal are called *basic*. They form the *basic subalgebra*, $A_B(P)$, of $A(P)$.

Remark: Proposition III shows that π^* can be considered as an isomorphism

$$\pi^*: A(B) \xrightarrow{\cong} A_B(P).$$

Finally recall that $A(P)_{\theta=0}$ denotes the subalgebra of $A(P)$ consisting of differential forms, Φ, satisfying

$$\theta(h)\Phi = 0, \qquad h \in E.$$

Set $A(P)_{i=0} \cap A(P)_{\theta=0} = A(P)_{i=0,\theta=0}$ (cf. sec. 3.13). If G is connected, Proposition VI, sec. 3.13, shows that $A(P)_{i=0,\theta=0}$ is the basic subalgebra. Thus in this case we can write

$$\pi^*: A(B) \xrightarrow{\cong} A(P)_{i=0,\theta=0}\,.$$

6.4. Homomorphisms. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with the same group G and let $\varphi: P \to \hat{P}$ be a homomorphism of principal bundles inducing $\psi: B \to \hat{B}$. Since φ is equivariant, the fundamental vector fields on P and $\hat{P}$ generated by the same vector, $h \in E$, are φ-related,

$$Z_h \underset{\varphi}{\sim} \hat{Z}_h\,, \qquad h \in E$$

(cf. sec. 3.9). This yields the commutation relations (cf. sec. 3.14)

$$\varphi^* \circ \hat{\theta}(h) = \theta(h) \circ \varphi^*, \qquad \varphi^* \circ \hat{i}(h) = i(h) \circ \varphi^*, \qquad h \in E,$$

where $\hat{\theta}(h) = \theta(\hat{Z}_h)$ and $\hat{i}(h) = i(\hat{Z}_h)$. Moreover,

$$\varphi^* \circ \hat{T}_a^* = T_a^* \circ \varphi^*, \qquad a \in G.$$

Hence the homomorphism $\varphi^*: A(\hat{P}) \to A(P)$ restricts to homomorphisms $A_I(\hat{P}) \to A_I(P)$ and $A_B(\hat{P}) \to A_B(P)$ and we have the commutative diagram

$$\begin{tikzcd} A_B(P) & A_B(\hat{P}) \arrow[l, "\varphi^*"'] \\ A(B) \arrow[u, "\pi^*", "\cong"'] & A(\hat{B}) \arrow[l, "\psi^*"] \arrow[u, "\cong", "\hat{\pi}^*"'] \end{tikzcd}\,.$$

6.5. Integration over the fibre. An orientation of E (the Lie algebra of G) determines an orientation in the fibre bundle $\mathscr{P}$ (cf. sec. 7.4, volume I) as follows: Give the trivial vector bundle, $P \times E$, the induced orientation, and then use the bundle isomorphism,

$$P \times E \xrightarrow{\cong} \mathbf{V}_P,$$

(Corollary I to Proposition I, sec. 6.1) to orient $\mathbf{V}_P$. Finally, recall from sec. 7.4, volume I, that an orientation of $\mathbf{V}_P$ determines an orientation of $\mathscr{P}$.

Example: If B is a point, $P = G$, then $\mathbf{V}_P = T_P = T_G$ and the induced orientation of $\mathscr{P}$ is simply an orientation of G. It is the *left* invariant orientation induced by that of E (cf. sec. 1.13) as follows from the example of sec. 6.1.

More generally, if $P = B \times G$, then $\mathbf{V}_P = B \times T_G$ and the orientation of $\mathbf{V}_P$ is that obtained from the orientation of T_G. Thus the orientation of $\{x\} \times G$ induced from that of $\mathscr{P}$ is simply the orientation of G just defined.

Now, let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle, and suppose $\varphi: P \to \hat{P}$ is a homomorphism of principal bundles inducing $\psi: B \to \hat{B}$. Then (since φ is equivariant) the diagram,

$$\begin{array}{ccc} P \times E & \xrightarrow{\varphi \times \iota} & \hat{P} \times E \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{V}_P & \xrightarrow[d\varphi]{} & \mathbf{V}_{\hat{P}} \end{array},$$

commutes. It follows that φ preserves the induced bundle orientations.

In particular, if $\{(U_\alpha, \psi_\alpha)\}$ is a principal coordinate representation for $\mathscr{P}$, then

$$\psi_{\alpha,x}: (G, p, \{x\}, G) \to (P, \pi, B, G)$$

can be considered as a homomorphism of principal bundles. It follows that the maps $\psi_{\alpha,x}: G \to G_x$ are orientation preserving, where G_x is given the orientation induced from the orientation of $\mathscr{P}$.

Next, assume that G is compact and connected. Since $\mathscr{P}$ is orientable, the fibre integral (cf. sec. 7.12, volume I) is defined, depending of course, on the orientation of $\mathscr{P}$.

On the other hand let $\Delta \in A^r(G)$ be the unique invariant r-form such that $\int_G \Delta = 1$ (cf. sec. 1.15). Let $\epsilon \in \wedge^r E$ be the element satisfying

$$\langle \Delta(e), \epsilon \rangle = 1.$$

Write $\epsilon = h_1 \wedge \cdots \wedge h_r$ ($h_i \in E$). The operator,

$$i(\epsilon) = i(h_r) \circ \cdots \circ i(h_1),$$

in $A(P)$ depends only on ϵ. Moreover, since

$$i(h) \circ T_{a^{-1}}^* = T_{a^{-1}}^* \circ i((\mathrm{Ad}\ a)h), \qquad a \in G, \quad h \in E$$

(cf. diagram (3.1), sec. 3.9), it follows that

$$i(\epsilon) \circ T_{a^{-1}}^* = T_{a^{-1}}^* \circ i(\det \mathrm{Ad}\ a \cdot \epsilon) = T_{a^{-1}}^* \circ i(\epsilon), \qquad a \in G.$$

Since $h \wedge \epsilon = 0$ ($h \in E$), we also have $i(h) \circ i(\epsilon) = 0$. These relations show that $i(\epsilon)$ restricts to an operator

$$i(\epsilon): A_I(P) \to A_B(P).$$

Proposition IV: The diagram,

$$\begin{tikzcd}
A_I(P) \arrow[r, "\text{inclusion}"] \arrow[d, "\omega \circ i(\epsilon)"'] & A(P) \arrow[d, "\int_G"] \\
A_B(P) & A(B), \arrow[l, "\cong", "\pi^*"']
\end{tikzcd}$$

commutes, where ω is the involution defined by

$$\omega(\Phi) = (-1)^{pr}\Phi, \qquad \Phi \in A_B^p(P).$$

Proof: It is clearly sufficient to consider the case that $\mathscr{P}$ is the product bundle: $P = B \times G$. We must show that

$$\pi^* \int_G \Phi = \omega(i(\epsilon)\Phi), \qquad \Phi \in A_I(P).$$

Recall the bigradation of $A(P) = A(B \times G)$ (cf. sec. 3.20, volume I). Evidently, $A_I(P)$ is a bigraded subalgebra of $A(P)$,

$$A_I(P) = \sum_{p=0}^{n} \sum_{q=0}^{r} A_I^{p,q}(P).$$

Moreover, the operators $i(\epsilon)$ and $\pi^* \circ \oint_G$ are both homogeneous of bidegree $(0, -r)$. Hence it is sufficient to consider the case that that $\Phi \in A_I^{p,r}(P)$.

In this case a simple computation shows that

$$\Phi = (-1)^{pr}\, i(\epsilon)\, \Phi \wedge \pi_G^* \Delta$$

(where $\pi_G\colon B \times G \to G$ is the projection). Since $i(\epsilon)\Phi \in \operatorname{Im} \pi^*$, it follows that (cf. Example 2, sec. 7.12, volume I)

$$\pi^* \oint_G \Phi = (-1)^{pr}\, i(\epsilon)\Phi = \omega(i(\epsilon)\Phi),$$

as desired.

Q.E.D.

6.6. Vector-valued differential forms. We recall, for convenience, some facts from volume I, and from Chap. III. Let W be a finite-dimensional vector space. Then $A(P; W)$, the space of W-valued differential forms in P, is a graded left module over the graded algebra $A(P)$, and an isomorphism, $A(P) \otimes W \to A(P; W)$, is given by $\Phi \otimes w \mapsto \Phi \wedge w$, $w \in W$. (Here w also denotes the constant function $P \to w$.)

The operators $i(Z)$, $\theta(Z)$, T_a^*, and δ (where $Z \in \mathscr{X}(P)$ and $a \in G$) extend to operators $i(Z) \otimes \iota$, $\theta(Z) \otimes \iota$, $T_a^* \otimes \iota$, and $\delta \otimes \iota$ in $A(P; W)$, again denoted by $i(Z)$, $\theta(Z)$, T_a^* and δ. In particular, $i(h)$ and $\theta(h)$ ($h \in E$) are regarded as operators in $A(P; W)$.

A W-valued differential form, Ω, is called *horizontal* if $i(h)\Omega = 0$, $h \in E$ (cf. sec. 6.3). The horizontal forms are a graded subspace of $A(P; W)$, denoted by $A(P; W)_{i=0}$. The isomorphism,

$$A(P) \otimes W \xrightarrow{\cong} A(P; W),$$

restricts to an isomorphism

$$A(P)_{i=0} \otimes W \xrightarrow{\cong} A(P; W)_{i=0}\,.$$

Now suppose that R is a representation of G in W, and let R' be the derived representation of E in W. The operators $\iota \otimes R(a)$ and $\iota \otimes R'(h)$ ($a \in G$, $h \in E$) in $A(P; W)$ are denoted simply by $R(a)$ and $R'(h)$.

Thus (cf. sec. 3.15) a W-valued form Φ is equivariant if

$$T_a^*\Phi = R(a^{-1})\Phi, \qquad a \in G.$$

According to Proposition VII, sec. 3.15, if G is connected this is equivalent to

$$\theta(h)\Phi = -R'(h)\Phi, \qquad h \in E.$$

The space of equivariant forms is written $A_I(P; W)$.

On the other hand, a W-valued form, Φ, is called *invariant* if $T_a^*\Phi = \Phi$, $a \in G$. Thus Φ is invariant if and only if $\Phi \in A_I(P) \otimes W$. If the representation is trivial, then the definitions of equivariant and invariant forms coincide.

Finally, the space $A(P; W)_{i=0} \cap A_I(P; W)$ is called the space of *basic W-valued differential forms* and is denoted by $A_B(P; W)$. If $W = \mathbb{R}$ and R is the trivial representation, this reduces to the definition of sec. 6.3. A generalization of Proposition III of that section to vector-valued forms will be given in sec. 8.22.

6.7. Multilinear maps of vector-valued forms. Recall that if W_1 and W are finite-dimensional vector spaces, then a linear map $\varphi: W_1 \to W$ induces the $\mathscr{S}(P)$-linear map,

$$\varphi_* = \iota \otimes \varphi: A(P; W_1) \to A(P; W),$$

given by

$$\varphi_*\Psi(Z_1, ..., Z_p) = \varphi(\Psi(Z_1, ..., Z_p)), \qquad \Psi \in A^p(P; W_1), \quad Z_1, ..., Z_p \in \mathscr{X}(P).$$

More generally, let $\varphi: W_1 \times \cdots \times W_k \to W$ be a k-linear map of finite-dimensional vector spaces. Then φ determines the k-linear map (over $\mathscr{S}(P)$),

$$\varphi_*: A(P; W_1) \times \cdots \times A(P; W_k) \to A(P; W),$$

given by

$$\varphi_*(\Psi_1, ..., \Psi_k)(Z_1, ..., Z_m) = \frac{1}{p_1! \cdots p_k!} \sum_{\sigma \in S^m} \epsilon_\sigma \varphi(\Psi_1(Z_{\sigma(1)}, ...), ..., \Psi_k(..., Z_{\sigma(m)})),$$

where

$$\Psi_i \in A^{p_i}(P; W_i) \quad (i = 1, ..., k), \qquad Z_\nu \in \mathscr{X}(P) \quad (\nu = 1, ..., m),$$

$$\sum_{i=1}^k p_i = m.$$

If we identify $A(P; W_i)$ with $A(P) \otimes W_i$, we can write

$$\varphi_*(\Phi_1 \otimes w_1, ..., \Phi_k \otimes w_k) = (\Phi_1 \wedge \cdots \wedge \Phi_k) \otimes \varphi(w_1, ..., w_k),$$

$$\Phi_i \in A(P), \quad w_i \in W_i, \quad i = 1, 2, ..., k.$$

In particular, if R represents G in W, then a bilinear map, $E \times W \to W$, is given by

$$(h, w) \mapsto R'(h)w.$$

The corresponding map of differential forms is written

$$(\Phi, \Psi) \mapsto \Phi(\Psi), \qquad \Phi \in A(P; E), \quad \Psi \in A(P; W).$$

Thus if $h \in E$ and h also denotes the constant function $P \to h$, then

$$h(\Psi) = R'(h)\Psi.$$

As a special case suppose $W = E$ and R is the adjoint representation. In this case the original bilinear map is given by $(h, k) \mapsto [h, k]$ $(h, k \in E)$ and the corresponding map of differential forms is written

$$(\Phi, \Psi) \mapsto [\Phi, \Psi].$$

The relation, $R'([h, k]) = R'(h) \circ R'(k) - R'(k) \circ R'(h)$, leads to the formula

$$[\Phi_1, \Phi_2](\Psi) = \Phi_1(\Phi_2(\Psi)) - (-1)^{pq}\, \Phi_2(\Phi_1(\Psi)),$$
$$\Phi_1 \in A^p(P; E), \quad \Phi_2 \in A^q(P; E), \quad \Psi \in A(P; W).$$

In particular, if $\Phi \in A(P; E)$ has odd degree, then

$$[\Phi, \Phi](\Psi) = 2\Phi(\Phi(\Psi)), \qquad \Psi \in A(P; W).$$

§3. Principal connections

6.8. Connections in a principal bundle. The right action of G on P induces a right action, dT, of G in the tangent bundle T_P. It is given by

$$dT(\zeta, a) = (dT_a)\zeta, \qquad a \in G, \quad \zeta \in T_P$$

(cf. Example 7, sec. 3.2). The equations $\pi \circ T_a = \pi \quad (a \in G)$ yield

$$d\pi \circ dT_a = d\pi.$$

Thus the vertical subbundle $\mathbf{V}_P$ is stable under dT.

Definition: A *principal connection* in $\mathscr{P}$ is a strong bundle map $V: T_P \to T_P$ satisfying the conditions:

(i) $V^2 = V$.
(ii) $\operatorname{Im} V_z = \mathbf{V}_z(P)$, $z \in P$.
(iii) V is equivariant; i.e..

$$dT_a \circ V = V \circ dT_a, \qquad a \in G.$$

Remark: We remind the reader of the following notation conventions:

(1) $\mathbf{V}_P$ is the vertical bundle with fibre $\mathbf{V}_z(P)$ at $z \in P$.
(2) V is a principal connection restricting to linear projections

$$V_z: T_z(P) \to \mathbf{V}_z(P), \qquad z \in P.$$

Examples: 1. For the trivial bundle $P = B \times G$, the vertical subbundle is $B \times T_G$, and a principal connection, V, is given by

$$V(\xi, \eta) = (0, \eta), \qquad \xi \in T_x(B), \quad \eta \in T_a(G).$$

2. Let $\{U_\alpha\}$ be an open cover of B and let V_α be a principal connection in the bundle $(\pi^{-1}U_\alpha, \pi, U_\alpha, G)$. Let $\{\hat{U}_\alpha\}$ be a locally finite refinement of the open cover $\{U_\alpha\}$ and suppose that $\{p_\alpha\}$ is a family of smooth functions on B such that $\operatorname{carr} p_\alpha \subset \hat{U}_\alpha$ and $\sum_\alpha p_\alpha = 1$. (Note that

$\{p_\alpha\}$ need not satisfy $p_\alpha(x) \geqslant 0$.) Then a principal connection, V, in $\mathscr{P}$ is given by

$$V = \sum_\alpha \pi^* p_\alpha \cdot V_\alpha .$$

Remark: Examples 1 and 2 show that every principal bundle admits a principal connection.

6.9. Horizontal subbundles. Let V be a principal connection in $\mathscr{P}$. The subspaces $\ker V_z \subset T_z(P)$ $(z \in P)$ are the fibres of a subbundle, $\mathbf{H}_P$, of τ_P. Evidently

$$\tau_P = \mathbf{H}_P \oplus \mathbf{V}_P;$$

i.e., $\mathbf{H}_P$ is a horizontal bundle (cf. see 7.2, volume I). It is called *the horizontal bundle associated with the connection.* Its fibres are called the *horizontal subspaces* and are written $\mathbf{H}_z(P)$.

The bundle $\mathbf{H}_P$ is stable under the action of G. Moreover the map, $V \mapsto \mathbf{H}_P$, is a bijection between principal connections and G-stable horizontal bundles.

Examples: 1. The horizontal subbundle corresponding to the principal connection of Example 1, sec. 6.8, is given by $\mathbf{H}_P = T_B \times G$.

2. Suppose a Riemannian metric has been defined in P so that the bundle maps dT_a: $T_P \to T_P$ $(a \in G)$ are all isometries. Then $\mathbf{H}_P = \mathbf{V}_P^\perp$ is a G-stable horizontal subbundle. The corresponding principal connection is simply the orthogonal projection $T_z(P) \to \mathbf{V}_z(P)$ at each point $z \in P$.

Now let V be a fixed principal connection in $\mathscr{P}$ and let $\mathbf{H}_P$ be the corresponding horizontal subbundle. Then

$$H = \iota - V\colon T_P \to \mathbf{H}_P$$

is the projection with kernel $\mathbf{V}_P$.

Since V and H are strong bundle maps, they determine module homomorphisms,

$$V_*\colon \mathscr{X}(P) \to \mathscr{X}(P) \qquad \text{and} \qquad H_*\colon \mathscr{X}(P) \to \mathscr{X}(P),$$

given by

$$(V_*Z)(z) = V(Z(z)) \qquad \text{and} \qquad (H_*Z)(z) = H(Z(z)), \quad Z \in \mathscr{X}(P), \quad z \in P.$$

The cross-sections in $\mathbf{H}_P$ are called *horizontal vector fields*, and the module of horizontal vector fields is denoted by $\mathscr{X}_H(P)$. It is, in general, not stable under the Lie bracket. The decomposition $\tau_P = \mathbf{H}_P \oplus \mathbf{V}_P$ leads to the direct decomposition,

$$\mathscr{X}(P) = \mathscr{X}_H(P) \oplus \mathscr{X}_V(P),$$

which is given explicitly by

$$Z \mapsto (H_*Z, V_*Z).$$

Since the operator V is equivariant with respect to the action of G, so is H. It follows that H_* and V_* commute with the isomorphisms,

$$(T_a)_* : \mathscr{X}(P) \xrightarrow{\cong} \mathscr{X}(P), \qquad a \in G.$$

In particular, if Z is invariant then so are H_*Z and V_*Z. Thus the direct decomposition above restricts to a direct decomposition,

$$\mathscr{X}^I(P) = \mathscr{X}_H^I(P) \oplus \mathscr{X}_V^I(P),$$

where

$$\mathscr{X}_H^I(P) = \mathscr{X}^I(P) \cap \mathscr{X}_H(P) \qquad \text{and} \qquad \mathscr{X}_V^I(P) = \mathscr{X}^I(P) \cap \mathscr{X}_V(P).$$

Now consider the surjective homomorphism,

$$\pi_*: \mathscr{X}^I(P) \to \mathscr{X}(B),$$

of $\mathscr{S}(B)$-modules (cf. sec. 6.2). Since $\ker \pi_* = \mathscr{X}_V^I(P)$, it follows that π_* restricts to an isomorphism

$$\pi_*: \mathscr{X}_H^I(P) \xrightarrow{\cong} \mathscr{X}(B).$$

The inverse isomorphism, $\lambda: \mathscr{X}(B) \xrightarrow{\cong} \mathscr{X}_H^I(P)$, is called the *horizontal lifting isomorphism* for the principal connection V.

Proposition V: The lifting isomorphism satisfies

$$\lambda([X_1, X_2]) = H_*([\lambda X_1, \lambda X_2]), \qquad X_1, X_2 \in \mathscr{X}(B).$$

Proof: In fact,

$$\pi_*\lambda([X_1, X_2]) = [X_1, X_2] = [\pi_*\lambda X_1, \pi_*\lambda X_2] = \pi_*([\lambda X_1, \lambda X_2]),$$

whence $\pi_*(\lambda([X_1, X_2]) - [\lambda X_1, \lambda X_2]) = 0$.

Thus $\lambda([X_1, X_2]) - [\lambda X_1, \lambda X_2]$ is vertical. It follows that

$$\lambda([X_1, X_2]) = H_*\lambda([X_1, X_2]) = H_*([\lambda X_1, \lambda X_2]).$$

Q.E.D.

6.10. The connection form. Let $V\colon T_P \to T_P$ be a principal connection in $\mathscr{P}$. In sec. 6.1 we obtained a strong bundle isomorphism

$$P \times E \xrightarrow{\cong} \mathbf{V}_P\,.$$

Composing V with the inverse of this isomorphism gives a strong bundle map

$$\alpha\colon T_P \to P \times E.$$

The isomorphism, $P \times E \xrightarrow{\cong} \mathbf{V}_P$, is given by

$$(z, h) \mapsto Z_h(z) = (dA_z)_e\,(h), \qquad z \in P, \quad h \in E.$$

It follows that, for $\zeta \in T_z(P)$, $\alpha(\zeta) = (z, (dA_z)_e^{-1} V_z \zeta)$. Thus an E-valued 1-form, ω, on P is given by

$$\omega(z; \zeta) = (dA_z)_e^{-1}\,(V_z \zeta).$$

Definition: ω is called the *connection form* associated with V.

Recall from Corollary III to Proposition I, sec. 6.1, that every E-valued function f on P determines a vertical vector field Z_f. In particular, suppose $Y \in \mathscr{X}(P)$ and consider the function $\omega(Y)$. It follows from the definition of ω that

$$Z_{\omega(Y)} = V_* Y.$$

Thus, $\omega(Y) = 0$ if and only if Y is horizontal.

Proposition VI: The connection form has the following properties:

(1) $i(h)\omega = h,\ h \in E.$
(2) $T_a^*\omega = (\operatorname{Ad} a^{-1})\omega,\ a \in G.$

Conversely, if $\sigma \in A^1(P; E)$ satisfies these conditions, there is a unique principal connection in $\mathscr{P}$ for which it is the connection form.

Remark: Note that (2) asserts that ω is equivariant with respect to the adjoint representation of G.

Proof: Suppose first that $\omega \in A^1(P; E)$ is derived from a principal connection V as described above. Then

$$(i(h)\omega)(z) = \omega(z; Z_h(z)) = (dA_z)^{-1}(dA_z)h = h, \qquad z \in P, \quad h \in E,$$

whence (1).

Moreover, according to sec. 3.1, $T_a \circ A_z = A_{z\cdot a} \circ \tau_{a^{-1}}$. Hence

$$dT_a \circ (dA_z)_e = (dA_{z\cdot a})_e \circ \mathrm{Ad}\, a^{-1}, \qquad a \in G.$$

Since V is equivariant, it follows that for $a \in G$, $z \in P$, $\zeta \in T_z(P)$,

$$\omega(z \cdot a; (dT_a)\,\zeta) = (\mathrm{Ad}\, a^{-1})\,\omega(z; \zeta),$$

whence (2).

Conversely, assume that σ is an E-valued 1-form on P which satisfies (1) and (2). Thus each $\sigma(z)$ is a linear map $T_z(P) \to E$. Define $V\colon T_P \to T_P$ by setting

$$V(z) = (dA_z)_e \circ \sigma(z), \qquad z \in P.$$

Then V is the unique principal connection inducing σ.

Q.E.D.

Corollary I: The connection form satisfies the relations

$$i(h)\omega = h \qquad \text{and} \qquad \theta(h)\omega = -(\mathrm{ad}\, h)\omega, \quad h \in E.$$

Conversely, let σ be an E-valued 1-form on P which satisfies these relations. Assume that G is connected. Then σ is a connection form on P.

Proof: This is an immediate consequence of the proposition and Proposition VII, sec. 3.15.

Q.E.D.

Recall from Proposition V, sec. 3.10, that the Lie product of a fundamental field and an invariant field is zero. On the other hand, we have

Corollary II: The Lie product of a fundamental field and a horizontal field is horizontal.

Proof: We must show that $\omega([Z_h, Y]) = 0$, where Z_h is a fundamental field and Y is horizontal. Since Y is horizontal,

$$i(Y)\omega = \omega(Y) = 0.$$

Thus, by Corollary I,

$$i(Y)\,\theta(h)\omega = -i(Y)(\mathrm{ad}\,h)\omega = -(\mathrm{ad}\,h)\,i(Y)\omega = 0$$

and so

$$\omega([Z_h\,,\,Y]) = i([Z_h\,,\,Y])\omega = \theta(h)\,i(Y)\omega - i(Y)\,\theta(h)\omega = 0.$$

Q.E.D.

§4. The covariant exterior derivative

6.11. The operator H^*. Fix a principal connection, V, in $\mathscr{P}$ and set $H = \iota - V$. Consider the space $A(P; W)$, where W is a finite-dimensional vector space. The operator, $H^*: A(P; W) \to A(P; W)$, defined by

$$(H^*\Omega)(z; \zeta_1, ..., \zeta_p) = \Omega(z; H\zeta_1, ..., H\zeta_p), \quad z \in P, \quad \zeta_i \in T_z(P), \quad \Omega \in A^p(P; W),$$

is called the *horizontal projection associated with V.*

Lemma I: The operator H^* has the following properties:

(1) $H^*(\Phi \wedge \Omega) = H^*\Phi \wedge H^*\Omega, \quad \Phi \in A(P), \Omega \in A(P; W).$
(2) H^* is a projection on the subspace of horizontal forms:

$$(H^*)^2 = H^* \qquad \text{and} \qquad \operatorname{Im} H^* = A(P; W)_{i=0}.$$

(3) $H^* \circ T_a^* = T_a^* \circ H^*, \quad a \in G.$
(4) $H^* \circ \theta(h) = \theta(h) \circ H^*, \quad h \in E.$
(5) $H^*\omega = 0$ (ω, the connection form).

Proof: Property (1) is obvious. Properties (2) and (3) follow from the relations

$$H^2 = H, \qquad H \circ V = V \circ H = 0, \qquad H \circ dT_a = dT_a \circ H.$$

(4) is a consequence of (3) and Proposition X, sec. 4.11, volume I, and (5) is obvious.

Q.E.D.

6.12. Covariant exterior derivative. The *covariant exterior derivative* associated with a principal connection, V, is the linear map, $\nabla: A(P; W) \to A(P; W)$, given by

$$\nabla = H^* \circ \delta.$$

Proposition VII: The covariant exterior derivative has the following properties:

(1) $\nabla(\Phi \wedge \Omega) = \nabla\Phi \wedge H^*\Omega + (-1)^p H^*\Phi \wedge \nabla\Omega,$
$\Phi \in A^p(P), \quad \Omega \in A(P; W).$

(2) $i(h) \circ \nabla = 0, \quad h \in E.$
(3) $\nabla \circ T_a^* = T_a^* \circ \nabla, \quad a \in G.$
(4) $\nabla \circ \theta(h) = \theta(h) \circ \nabla, \quad h \in E.$
(5) $\nabla \circ \pi^* = \delta \circ \pi^*.$

Proof: (1): Apply H^* to the formula

$$\delta(\Phi \wedge \Omega) = \delta\Phi \wedge \Omega + (-1)^p \Phi \wedge \delta\Omega.$$

(2), (3), and (4) follow from Lemma I, and (5) is a consequence of the relation $H^* \circ \pi^* = \pi^*$.

Q.E.D.

Corollary: ∇ restricts to a map ∇_H: $A(P; W)_{i=0} \to A(P; W)_{i=0}$.

Remark: In general, $\nabla^2 \neq 0$.

Proposition VIII: Let φ: $W_1 \times \cdots \times W_k \to W$ be a k-linear map and let Φ_i be a W_i-valued differential form of degree p_i ($i = 1, ..., k$). Then

$$\nabla[\varphi_*(\Phi_1, ..., \Phi_k)] = \sum_{i=1}^{k} (-1)^{p_1 + \cdots + p_{i-1}} \varphi_*(H^*\Phi_1, ..., \nabla\Phi_i, ..., H^*\Phi_k).$$

Proof: It is sufficient to consider the case $\Phi_i = \Psi_i \otimes w_i$ with $\Psi_i \in A^{p_i}(P)$ and $w_i \in W_i$. Then

$$\varphi_*(\Phi_1, ..., \Phi_k) = (\Psi_1 \wedge \cdots \wedge \Psi_k \otimes) \, \varphi(w_1, ..., w_k)$$

and so the proposition follows from Proposition VII, (1).

Q.E.D.

Applying the covariant exterior derivative to functions on P we obtain an operator

$$\nabla: \mathscr{S}(P) \to A^1(P)$$

which satisfies the relations

(1) $\nabla(f \cdot g) = \nabla f \cdot g + f \cdot \nabla g, \quad f, g \in \mathscr{S}(P).$
(2) $i(h) \circ \nabla = 0, \quad h \in E.$
(3) $T_a^* \circ \nabla = \nabla \circ T_a^*, \quad a \in G.$
(4) $\nabla f = \delta f, f \in \mathscr{S}_I(P).$

Conversely, assume that an operator ∇: $\mathscr{S}(P) \to A^1(P)$ which satisfies these equations is given. Then there is a unique principal connection on $\mathscr{P}$ such that ∇ is the corresponding covariant exterior derivative. In fact, with each vector field Z on P associate the map

$$Q_Z: \mathscr{S}(P) \to \mathscr{S}(P)$$

given by

$$Q_Z(f) = i(Z)(\delta f - \nabla f).$$

In view of (1), Q_Z is a derivation in the algebra $\mathscr{S}(P)$. Hence there is a unique vector field, Y_Z, on P such that $Q_Z(f) = Y_Z(f)$. The operator $Z \mapsto Y_Z$ in $\mathscr{X}(P)$ is $\mathscr{S}(P)$-linear, and hence it determines a bundle map, $V: T_P \to T_P$, such that

$$V_*(Z) = Y_Z, \qquad Z \in \mathscr{X}(P).$$

Condition (4) implies that each Y_Z is vertical and so V maps T_P into $\mathbf{V}_P$. On the other hand, if Z is vertical, condition (2) implies that

$$Y_Z = Z, \qquad Z \in \mathscr{X}_V(P);$$

thus V restricts to the identity on $\mathbf{V}_P$. Finally, (3) shows that

$$dT_a \circ V = V \circ dT_a, \qquad a \in G.$$

Hence the bundle map, V, is a principal connection in $\mathscr{P}$.

Now set $H = \iota - V$. Then

$$\begin{aligned} H_*(Z)(f) &= Z(f) - V_*(Z)(f) \\ &= i(Z)\nabla f, \qquad Z \in \mathscr{X}(P), \quad f \in \mathscr{S}(P). \end{aligned}$$

Hence $H^*\delta f = \nabla f$. It follows that ∇ is the covariant exterior derivative of f with respect to this connection.

Finally, if V_1 is any connection on $\mathscr{P}$ such that the corresponding covariant exterior derivative coincides with ∇, then we have

$$\nabla f(z; \zeta) = \delta f(z; H_1\zeta), \qquad z \in P, \quad \zeta \in T_z(P), \quad f \in \mathscr{S}(P).$$

This relation shows that the operator H_1 (and hence the connection) is uniquely determined by ∇.

6.13. Basic forms. Let R be a representation of G in W. It follows from Proposition VII, sec. 6.12, that the space $A_B(P; W)$ of basic forms (cf. sec. 6.6) is stable under the covariant exterior derivative of a principal connection.

Proposition IX: Let ∇ and ω be the covariant exterior derivative and connection form of a principal connection. Then (cf. sec. 6.7)

$$\nabla\Phi = \delta\Phi + \omega(\Phi), \qquad \Phi \in A_B(P; W).$$

Proof: Since Φ is horizontal,

$$i(h)\,\delta\Phi = \theta(h)\Phi \qquad \text{and} \qquad i(h)\,(\omega(\Phi)) = (i(h)\omega)\,(\Phi), \quad h \in E.$$

Moreover, according to Proposition VI, sec. 6.10, $i(h)\omega$ is the constant function $P \mapsto h$. Thus (cf. sec. 6.7)

$$(i(h)\,\omega)\,(\Phi) = R'(h)\,\Phi.$$

Since Φ is equivariant, these relations yield (cf. Proposition VII, sec. 3.15)

$$i(h)(\delta\Phi + \omega(\Phi)) = \theta(h)\Phi + R'(h)\Phi = 0, \qquad h \in E,$$

and so $\delta\Phi + \omega(\Phi)$ is horizontal. It follows that

$$\delta\Phi + \omega(\Phi) = H^*(\delta\Phi + \omega(\Phi)) = \nabla\Phi + (H^*\omega)(H^*\Phi) = \nabla\Phi,$$

(because $H^*\omega = 0$).

Q.E.D.

Corollary: If $W = E$ and R is the adjoint representation, then

$$\nabla\Phi = \delta\Phi + [\omega, \Phi], \qquad \Phi \in A_B(P; E).$$

§5. Curvature

In this article V denotes a principal connection in the principal bundle $\mathscr{P}$. The corresponding connection form, horizontal projection, and covariant exterior derivative are denoted by ω, H^*, and ∇, respectively.

6.14. Curvature. The *curvature form* of the connection V is the E-valued 2-form, Ω, on P given by

$$\Omega = \nabla\omega.$$

Proposition X: The curvature form has the following properties:

(1) Ω is horizontal: $i(h)\Omega = 0$, $h \in E$.

(2) Ω is equivariant: $T_a^*\Omega = (\mathrm{Ad}\ a^{-1})\Omega$, $a \in G$. In particular, $\theta(h)\Omega = -(\mathrm{ad}\ h)\Omega$, $h \in E$.

(3) Let $Y_1, Y_2 \in \mathscr{X}_H(P)$ be horizontal vector fields. Then

$$V_*([Y_1, Y_2]) = -Z_{\Omega(Y_1, Y_2)}.$$

(Recall, from sec. 6.1, that Z_f denotes the vertical vector field generated by $f \in \mathscr{S}(P; E)$.)

Proof: (1) is obvious. (2) follows from the equivariance of ω (cf. Proposition VI, sec. 6.10). To prove (3) observe that, since Y_1 and Y_2 are horizontal, $\omega(Y_1) = \omega(Y_2) = 0$. Thus

$$\Omega(Y_1, Y_2) = \delta\omega(Y_1, Y_2) = -\omega([Y_1, Y_2]).$$

According to sec. 6.10, $V_*Y = Z_{\omega(Y)}$, $Y \in \mathscr{X}(P)$. Now (3) follows.

Q.E.D.

Recall that $\lambda: \mathscr{X}(B) \xrightarrow{\cong} \mathscr{X}_H^I(P)$ denotes the horizontal lift (cf. sec. 6.9).

Corollary I: If $X_1, X_2 \in \mathscr{X}(B)$, then the decomposition of $[\lambda X_1, \lambda X_2]$ into horizontal and vertical parts is given by

$$[\lambda X_1, \lambda X_2] = \lambda([X_1, X_2]) - Z_{\Omega(\lambda X_1, \lambda X_2)}.$$

Proof: Apply Proposition V, sec. 6.9, and part (3) of the proposition above.

Q.E.D.

Corollary II: The curvature is zero if and only if the Lie product of any two horizontal fields is horizontal.

Next, consider the real bilinear map,

$$[\,,\,]\colon A(P; E) \times A(P; E) \to A(P; E),$$

induced by the Lie multiplication in E (cf. sec. 6.7). The differential form, $[\omega, \omega] \in A^2(P; E)$, is given by

$$[\omega, \omega](z; \zeta_1, \zeta_2) = 2[\omega(z; \zeta_1), \omega(z; \zeta_2)], \qquad \zeta_1, \zeta_2 \in T_z(P).$$

Proposition XI: The curvature form satisfies the *structure equation of Maurer–Cartan*

(1) $\Omega = \delta\omega + \frac{1}{2}[\omega, \omega]$

and the *Bianchi identity*

(2) $\nabla\Omega = 0.$

Proof: To verify (1) it is sufficient to check that

$$i(h)\Omega = i(h)(\delta\omega + \tfrac{1}{2}[\omega, \omega]), \qquad h \in E, \qquad \text{and} \qquad H^*\Omega = H^*(\delta\omega + \tfrac{1}{2}[\omega, \omega]).$$

Proposition VI, sec. 6.10, implies that $i(h)\omega$ is the constant function $P \to h$. Hence

$$i(h)\,\delta\omega = \theta(h)\omega = -\mathrm{ad}(h)\omega, \qquad h \in E.$$

On the other hand,

$$i(h)[\omega, \omega] = 2[i(h)\omega, \omega] = 2[h, \omega] = 2\,(\mathrm{ad}\,h)\omega.$$

Thus $i(h)(\delta\omega + \frac{1}{2}[\omega, \omega]) = 0 = i(h)\Omega$.
Since, clearly, $H^*([\omega, \omega]) = [H^*\omega, H^*\omega] = 0$, we have

$$H^*\Omega = \Omega = H^*\delta\omega = H^*(\delta\omega + \tfrac{1}{2}[\omega, \omega])$$

and so (1) is proved.
To verify (2) apply $H^* \circ \delta$ to the structure equation just established. This gives

$$\nabla\Omega = H^*\delta\tfrac{1}{2}[\omega, \omega] = H^*[\delta\omega, \omega] = [H^*\delta\omega, H^*\omega].$$

But $H^*\omega = 0$ and so we obtain (2).

Q.E.D.

Proposition XII: If R is a representation of G in a vector space W and $\Phi \in A_B(P; W)$, then

$$\nabla^2\Phi = \Omega(\Phi).$$

Proof: In view of Proposition IX, sec. 6.13,

$$\nabla\Phi = \delta\Phi + \omega(\Phi).$$

Since $\nabla\Phi$ is again basic, the proposition can be applied a second time to yield

$$\begin{aligned}\nabla^2\Phi &= \delta(\omega(\Phi)) + \omega(\delta\Phi) + \omega(\omega(\Phi))\\ &= \delta\omega(\Phi) + \tfrac{1}{2}[\omega, \omega](\Phi) = \Omega(\Phi)\end{aligned}$$

(cf. sec. 6.7).

Q.E.D.

Corollary: If $f \in \mathscr{S}_I(P; W)$, then $\nabla^2 f = \Omega(f)$.

6.15. Induced connection. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, \hat{B}, G)$ be a second principal bundle with the same group G and let $\varphi: P \to \hat{P}$ be a homomorphism of principal bundles. Then a principal connection $\hat{V}$ in $\hat{\mathscr{P}}$ induces a principal connection in $\mathscr{P}$.

In fact, if $\hat{\omega}$ is the connection form in $\hat{\mathscr{P}}$ corresponding to $\hat{V}$, then $\varphi^*\hat{\omega}$ is a connection form in $\mathscr{P}$. The principal connection V determined by ω is called the *connection induced by* φ. It makes the diagram

$$\begin{tikzcd} T_P \arrow[r, "d\varphi"] \arrow[d, "V"'] & T_{\hat{P}} \arrow[d, "\hat{V}"] \\ T_P \arrow[r, "d\varphi"'] & T_{\hat{P}} \end{tikzcd}$$

commute. (These results follow easily from sec. 6.4 and sec. 6.10.)

The following relations are immediate:

$$H^* \circ \varphi^* = \varphi^* \circ \hat{H}^*, \qquad \nabla \circ \varphi^* = \varphi^* \circ \hat{\nabla}, \qquad \Omega = \varphi^*\hat{\Omega}.$$

§6. The Weil homomorphism

∇ continues to denote a principal connection in the principal bundle $\mathscr{P}$. Its connection and curvature forms are denoted by ω and Ω, while H^* and ∇ denote, respectively, the horizontal projection and covariant derivative.

6.16. Multilinear functions. Recall that we may regard an element $\Gamma \in \otimes^k E^*$ as the real-valued k-linear function in E given by

$$\Gamma(h_1, ..., h_k) = \langle \Gamma, h_1 \otimes \cdots \otimes h_k \rangle, \qquad h_1, ..., h_k \in E.$$

Thus Γ determines a map

$$\Gamma_*: \underbrace{A(P; E) \times \cdots \times A(P; E)}_{(k \text{ terms})} \to A(P).$$

We shall denote Γ_* simply by Γ, and write

$$\Gamma(\Psi_1, ..., \Psi_k) = \Gamma_*(\Psi_1, ..., \Psi_k), \qquad \Psi_1, ..., \Psi_k \in A(P; E).$$

As an immediate consequence of the definitions of sec. 6.7, we have

Lemma II: Let $\Gamma_1 \in \otimes^p E^*$, $\Gamma_2 \in \otimes^q E^*$ and form $\Gamma_1 \otimes \Gamma_2 \in \otimes^{p+q} E^*$. Then

$$(\Gamma_1 \otimes \Gamma_2)(\Psi_1, ..., \Psi_{p+q}) = \Gamma_1(\Psi_1, ..., \Psi_p) \wedge \Gamma_2(\Psi_{p+1}, ..., \Psi_{p+q}),$$
$$\Psi_i \in A(P; E), \quad i = 1, ..., p + q.$$

6.17. The homomorphism γ. Recall that $\vee E^*$ is the symmetric algebra over E^*. The purpose of this section is to construct a homomorphism

$$\gamma: \vee E^* \to A(P).$$

Recall that the curvature form is a 2-form on P with values in E. Define a linear map,

$$\beta: \otimes E^* \to A(P),$$

by

$$\beta(\Gamma) = \Gamma \underbrace{(\Omega, ..., \Omega)}_{(p \text{ arguments})}, \qquad \Gamma \in \otimes^p E^*.$$

Lemma III: (1) β is a homomorphism of algebras.
(2) $\beta(\otimes^p E^*) \subset A^{2p}(P)$.
(3) Let $\pi_S\colon \otimes E^* \to \vee E^*$ be the canonical projection given by

$$\pi_S(h_1^* \otimes \cdots \otimes h_p^*) = h_1^* \vee \cdots \vee h_p^* .$$

Then β factors over π_S to yield a homomorphism $\gamma\colon \vee E^* \to A(P)$ making the diagram,

$$\begin{tikzcd} \otimes E^* \arrow[dd, "\pi_S"'] \arrow[dr, "\beta"] & \\ & A(P) \\ \vee E^* \arrow[ur, "\gamma"'] & \end{tikzcd},$$

commute.

Proof: (1) follows from Lemma II (set $\Psi_1 = \cdots = \Psi_{p+q} = \Omega$). (2) is a consequence of the fact that Ω is a 2-form. To prove (3), simply observe (via (2)) that

$$\operatorname{Im}\beta \subset \sum_p A^{2p}(P)$$

and that this is a commutative algebra.

Q.E.D.

The adjoint representation of G in E determines the representation, $\mathrm{Ad}^\vee$, of G in $\vee E^*$ given by

$$\mathrm{Ad}^\vee(a)(h_1^* \vee \cdots \vee h_p^*) = (\mathrm{Ad}\, a^{-1})^* h_1^* \vee \cdots \vee (\mathrm{Ad}\, a^{-1})^* h_p^*$$
$$a \in G, \quad h_i^* \in E^*, \quad i = 1, \dots, p,$$

cf. sec. 1.9. Since G acts via homomorphisms in the graded algebra $\vee E^*$, it follows that the invariant subspace $(\vee E^*)_I$ is a graded subalgebra of $\vee E^*$; $(\vee E^*)_I = \sum_{k=0}^{\infty} (\vee^k E^*)_I$.

Proposition XIII: The homomorphism γ defined in Lemma III has the properties:

(1) $\operatorname{Im}\gamma \subset A(P)_{i=0}$.
(2) $T_a^* \circ \gamma = \gamma \circ \mathrm{Ad}^\vee(a)$, $a \in G$.
(3) $\nabla \circ \gamma = 0$.

Proof: (1) Since γ is a homomorphism of algebras and since $\vee E^*$ is generated by E^*, it is sufficient to show that

$$\gamma(h^*) \in A(P)_{i=0}, \qquad h^* \in E^*.$$

But for $h \in E$, $i(h)\,(\gamma(h^*)) = i(h)\,(h^*(\Omega)) = h^*(i(h)\Omega) = 0$ (cf. Proposition X, (1), sec. 6.14).

(2) Since both sides of (2) are algebra homomorphisms we need only verify that

$$(T_a^* \circ \gamma)(h^*) = (\gamma \circ (\operatorname{Ad} a^{-1})^*)(h^*), \qquad a \in G, \quad h^* \in E^*.$$

But since Ω is equivariant (cf. Proposition X, (2), sec. 6.14),

$$\begin{aligned}(T_a^* \circ \gamma)(h^*) &= h^*(T_a^*\Omega) = h^*(\operatorname{Ad}(a^{-1})\Omega)\\ &= (\operatorname{Ad}(a^{-1})^*\, h^*)(\Omega) = (\gamma \circ \operatorname{Ad}(a^{-1})^*)(h^*).\end{aligned}$$

(3) Every element $\Gamma \in \vee^p E^*$ can be written in the form $\pi_S(\Gamma_1)$, where $\Gamma_1 \in \otimes^p E^*$. Then

$$\nabla(\gamma(\Gamma)) = \nabla(\Gamma_1(\Omega, \ldots, \Omega)) = \sum_{i=1}^{p} \Gamma_1(\Omega, \ldots, \underset{(i\text{th position})}{\nabla\Omega}, \ldots, \Omega)$$

(cf. Proposition VIII, sec. 6.12).

The Bianchi identity (Proposition XI, sec. 6.14) states that $\nabla\Omega = 0$. Thus

$$\nabla(\gamma(\Gamma)) = 0, \qquad \Gamma \in \vee^p E^*;$$

i.e., $\nabla \circ \gamma = 0$.

Q.E.D.

Corollary: γ restricts to a homomorphism,

$$\gamma_I\colon (\vee E^*)_I \to A_B(P),$$

and the differential forms in $\operatorname{Im} \gamma_I$ are closed:

$$\delta \circ \gamma_I = 0.$$

Proof: Clearly $\gamma((\vee E^*)_I) \subset A_I(P) \cap A(P)_{i=0} = A_B(P)$ (cf. sec. 6.3). Moreover, Proposition VII, (5), sec. 6.12, shows that ∇ reduces to δ in the basic subalgebra. Thus, $\delta \circ \gamma_I = \nabla \circ \gamma_I = 0$.

Q.E.D.

6.18. Explicit formulae for β and γ. Identify $\otimes^p E^*$ with the space $T^p(E)$ of p-linear functions in E (cf. sec. 6.16). Then, if $\Gamma \in T^p(E)$ and $Z_i \in \mathscr{X}(P)$, we have

$$\beta(\Gamma)(Z_1, ..., Z_{2p}) = \frac{1}{2^p} \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Gamma(\Omega(Z_{\sigma(1)}, Z_{\sigma(2)}), ..., \Omega(Z_{\sigma(2p-1)}, Z_{\sigma(2p)})).$$

Moreover, Lemma III, sec. 6.17, shows that $\beta(\Gamma)$ depends only on the symmetric part of Γ.

Next, identify $\vee^p E^*$ with the space $S^p(E)$ of p-linear symmetric functions in E by writing

$$(h_1^* \vee \cdots \vee h_p^*)(h_1, ..., h_p) = \operatorname{perm}(\langle h_i^*, h_j \rangle).$$

Then the projection $\otimes^p E^* \xrightarrow{\pi_S} \vee^p E^*$, interpreted as a map $T^p(E) \to S^p(E)$, is given by

$$(\pi_S \Gamma)(h_1, ..., h_p) = \sum_{\sigma \in S^p} \Gamma(h_{\sigma(1)}, ..., h_{\sigma(p)}).$$

On the other hand, the inclusion i_S: $S^p(E) \to T^p(E)$, interpreted as a map $\vee^p E^* \to \otimes^p E^*$, is given by

$$i_S(h_1^* \vee \cdots \vee h_p^*) = \sum_{\sigma \in S^p} h_{\sigma(1)}^* \otimes \cdots \otimes h_{\sigma(p)}^*.$$

Hence, for $\Gamma \in \vee^p E^*$,

$$\pi_S i_S \Gamma = p!\, \Gamma.$$

It follows that, for $\Gamma \in \vee^p E^*$,

$$\begin{aligned} \gamma(\Gamma) &= \left(\frac{1}{p!}\right) \gamma(\pi_S i_S(\Gamma)) = \left(\frac{1}{p!}\right) \beta(i_S \Gamma) \\ &= \left(\frac{1}{p!}\right) (i_S \Gamma)(\Omega, ..., \Omega). \end{aligned}$$

Interpret Γ as a symmetric p-linear function; this equation then yields

$$\gamma\Gamma(Z_1, ..., Z_{2p}) = \frac{1}{p!\, 2^p} \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Gamma(\Omega(Z_{\sigma(1)}, Z_{\sigma(2)}), ..., \Omega(Z_{\sigma(2p-1)}, Z_{\sigma(2p)})), \qquad Z_i \in \mathscr{X}(P).$$

6.19. The Weil homomorphism. Recall from sec. 6.3 that π^*: $A(B) \to A(P)$ may be considered as an isomorphism

$$\pi^*: A(B) \xrightarrow{\cong} A_B(P).$$

Hence the corollary to Proposition XIII, sec. 6.17, shows that there is a unique homomorphism,

$$\gamma_B\colon (\vee E^*)_I \to A(B),$$

such that $\pi^* \circ \gamma_B = \gamma_I$. It satisfies $\delta \circ \gamma_B = 0$.

Thus, composing γ_B with the projection $Z(B) \to H(B)$, $(Z(B) = \ker \delta)$, we obtain an algebra homomorphism

$$h_{\mathscr{P}}\colon (\vee E^*)_I \to H(B).$$

Observe that $h_{\mathscr{P}}((\vee^p E^*)_I) \subset H^{2p}(B)$.

Note that we needed only the principal bundle, together with the principal connection, V, in order to define $h_{\mathscr{P}}$.

Theorem I: $h_{\mathscr{P}}$ is independent of the choice of connection. Thus it is an invariant of the bundle $\mathscr{P}$.

Proof: Assume that two principal connections are defined in $\mathscr{P}$ and let ω_0, ω_1 be the corresponding connection forms. Consider the principal bundle $\mathscr{P} \times \mathbb{R} = (P \times \mathbb{R}, \pi \times \iota, B \times \mathbb{R}, G)$. Let $f \in \mathscr{S}(\mathbb{R})$ be the function given by $f(t) = t$. Then the E-valued 1-form, ω, on $P \times \mathbb{R}$, given by

$$\omega = \omega_0 \times (1 - f) + \omega_1 \times f$$

is a connection form (cf. Example 2, sec. 6.8, and Proposition VI, sec. 6.10).

Next consider the injections,

$$j_\nu\colon P \to P \times \mathbb{R} \qquad \text{and} \qquad i_\nu\colon B \to B \times \mathbb{R} \quad (\nu = 0, 1),$$

given by

$$j_0(z) = (z, 0) \qquad j_1(z) = (z, 1) \qquad z \in P,$$

and

$$i_0(x) = (x, 0) \qquad i_1(x) = (x, 1), \qquad x \in B.$$

Then j_0 and j_1 are homomorphisms of principal bundles. Evidently,

$$j_0^*\omega = \omega_0 \qquad \text{and} \qquad j_1^*\omega = \omega_1,$$

whence (cf. sec. 6.15)

$$j_0^*\Omega = \Omega_0 \qquad \text{and} \qquad j_1^*\Omega = \Omega_1$$

(Ω, Ω_0, Ω_1 denote the curvatures corresponding to ω, ω_0, and ω_1).

Now let $(\gamma_0)_I$, $(\gamma_1)_I$, γ_I denote the homomorphisms defined via ω_0, ω_1, and ω. Clearly

$$(\gamma_0)_I = j_0^* \circ \gamma_I \qquad \text{and} \qquad (\gamma_1)_I = j_1^* \circ \gamma_I.$$

It follows that $(\gamma_0)_B = i_0^* \circ \gamma_B$ and $(\gamma_1)_B = i_1^* \circ \gamma_B$. Hence $h_0 = i_0^\# h$ and $h_1 = i_1^\# h$. But i_1 and i_0 are homotopic and so (cf. sec. 5.2, volume I or sec. 0.14) $i_0^\# = i_1^\#$. It follows that $h_0 = h_1$.

Q.E.D.

Definition: $h_{\mathscr{P}}$ is called the *Weil homomorphism* for the principal bundle $\mathscr{P}$. The subalgebra Im $h_{\mathscr{P}}$ is called the *characteristic subalgebra* of $H(B)$ and its elements are called the *characteristic classes* for $\mathscr{P}$.

Remarks: 1. Im $h_{\mathscr{P}}$ is a graded subalgebra of the commutative algebra $\sum_p H^{2p}(B)$.

2. If the bundle $\mathscr{P}$ admits a connection with curvature zero, then the Weil homomorphism is trivial and the characteristic subalgebra is zero in positive degrees. In particular, the Weil homomorphism of a product bundle is trivial (cf. Corollary II to Proposition X, sec. 6.14).

3. If G is connected, we have $(\vee E^*)_I = (\vee E^*)_{\theta=0}$, where θ is the representation of E in $\vee E^*$ given by

$$\theta(h)(h_1^* \vee \cdots \vee h_p^*) = -\sum_{i=1}^{p} h_1^* \vee \cdots \vee \operatorname{ad}(h)^* h_i^* \vee \cdots \vee h_p^*, \qquad h_1^*, \ldots, h_p^* \in E^*,$$

(cf. Example 2, sec. 1.9). Hence, in this case, $h_{\mathscr{P}}$ is a homomorphism from $(\vee E^*)_{\theta=0}$ into $H(B)$.

4. Suppose G is compact and connected. Then the cohomology algebra $H(P)$ is determined by the graded differential algebra $(A(B), \delta)$ and the Weil homomorphism $h_{\mathscr{P}}$. Moreover, given $A(B)$ and $h_{\mathscr{P}}$ it is possible to determine $H(P)$ explicitly. This will be done in volume III.

Theorem II: Let $\varphi: \mathscr{P} \to \hat{\mathscr{P}}$ be a homomorphism of principal bundles with the same group G and let $\psi: B \to \hat{B}$ be the induced map. Then the diagram,

$$\begin{tikzcd}
& H(\hat{B}) \arrow[dd, "\psi^*"] \\
(\vee E^*)_I \arrow[ur, "h_{\hat{\mathscr{P}}}"] \arrow[dr, "h_{\mathscr{P}}"'] & \\
& H(B),
\end{tikzcd}$$

commutes ($h_{\mathscr{P}}$ and $h_{\hat{\mathscr{P}}}$ denote the Weil homomorphisms for $\mathscr{P}$ and $\hat{\mathscr{P}}$).

Proof: In fact, let $\hat{\omega}$ be a connection form for $\hat{\mathscr{P}}$ and let $\omega = \varphi^*\hat{\omega}$ be the induced connection form for $\mathscr{P}$ (cf. sec. 6.15). Then $\Omega = \varphi^*\hat{\Omega}$. This relation implies that

$$\gamma = \varphi^*\hat{\gamma} \qquad \text{and} \qquad \gamma_B = \psi^*\hat{\gamma}_B ,$$

whence

$$h_{\mathscr{P}} = \psi^\# h_{\hat{\mathscr{P}}} .$$

Q.E.D.

Corollary: Let $h^+_{\mathscr{P}}$ denote the restriction of $h_{\mathscr{P}}$ to $(\vee^+ E^*)_I$. Then $\pi^\# \circ h^+_{\mathscr{P}} = 0$. ($(\vee^+ E^*)_I = \sum_{j>0} (\vee^j E^*)_I$.)

Proof: Regard the action $T\colon P \times G \to P$ as a homomorphism from the product bundle $\hat{\mathscr{P}} = (P \times G, \pi_P, P, G)$ to $\mathscr{P}$, inducing $\pi\colon P \to B$ between the base manifolds. Since $\hat{\mathscr{P}}$ is trivial, we have $h^+_{\hat{\mathscr{P}}} = 0$ (cf. Remark 2, above), whence $\pi^\# h^+_{\mathscr{P}} = h^+_{\hat{\mathscr{P}}} = 0$.

Q.E.D.

6.20. Change of connection. Let ω_0 and ω_1 be connection forms in $\mathscr{P}$ and set $\theta = \omega_1 - \omega_0$. Then

$$i(h)\theta = \omega_1(Z_h) - \omega_0(Z_h) = h - h = 0, \qquad h \in E,$$

$$T_a^*\theta = (\mathrm{Ad}\, a^{-1})\theta, \qquad a \in G,$$

and

$$\theta(h)\theta = -(\mathrm{ad}\, h)\theta, \qquad h \in E.$$

In particular, θ is a basic E-valued 1-form on P (cf. sec. 6.6).

Now adopt the notation established in the proof of Theorem I, sec. 6.19, and observe that the connection form ω in $P \times \mathbb{R}$ can be written

$$\omega = \omega_0 \times 1 + \theta \times f.$$

Theorem I implies that, for each $\Gamma \in (\vee^p E^*)_I$, there exists a $\Phi \in A^{2p-1}(B)$ such that $(\gamma_1)_B \Gamma - (\gamma_0)_B \Gamma = \delta\Phi$.

In this section we construct an explicit Φ. Use $i_S\colon \vee^p E^* \to \otimes^p E^*$ to identify $\vee^p E^*$ with the p-linear symmetric functions in E (cf, sec. 6.18). We shall use the notation

$$\langle \Gamma, \Psi_1^{k_1} \vee \cdots \vee \Psi_r^{k_r} \rangle = \Gamma(\underbrace{\Psi_1 \cdots \Psi_1}_{(k_1 \text{ arguments})}, \ldots, \underbrace{\Psi_r \cdots \Psi_r}_{(k_r \text{ arguments})}),$$

$\Gamma \in \vee^p E^*$, $\Psi_1, \ldots, \Psi_r \in A(P; E)$, cf. sec. 6.16.

Proposition XIV: With the notation and hypotheses above,

$$(\gamma_1)_B\Gamma - (\gamma_0)_B\Gamma = \delta\Phi,$$

where Φ is the $(2p-1)$-form on B determined by

$$\pi^*\Phi = \sum_{i+j+k=p-1} \frac{1}{i+2j+1} \left\langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}(\tfrac{1}{2}[\theta,\theta])^j \vee \frac{1}{k!}(\Omega_0)^k \right\rangle.$$

Proof: Since the homotopy connecting i_0 and i_1 is just the identity map of $B \times \mathbb{R}$, we have $i_1^* - i_0^* = k \circ \delta + \delta \circ k$, where

$$(k\Psi)(x; \xi_1, \ldots, \xi_{p-1}) = \int_0^1 \Psi(x, t; d/dt, \xi_1, \ldots, \xi_{p-1})\, dt,$$

$$\Psi \in A^p(B \times \mathbb{R}), \qquad \xi_i \in T_x(B),$$

(cf. sec. 0.14). It follows that

$$(\gamma_1)_B\Gamma - (\gamma_0)_B\Gamma = (i_1^* - i_0^*)\gamma_B\Gamma = \delta\Phi,$$

where

$$\Phi(x; \xi_1, \ldots, \xi_{2p-1}) = \int_0^1 (\gamma_B\Gamma)(x, t; d/dt, \xi_1, \ldots, \xi_{2p-1})\, dt.$$

Hence (cf. sec. 6.16)

$$\begin{aligned}(\pi^*\Phi)(z; \zeta_1, \ldots, \zeta_{2p-1}) &= \int_0^1 (\gamma_I\Gamma)(z, t; d/dt, \zeta_1, \ldots, \zeta_{2p-1})\, dt \\ &= \int_0^1 \frac{1}{p!}\Gamma(\Omega, \ldots, \Omega)(z, t; d/dt, \zeta_1, \ldots, \zeta_{2p-1})\, dt.\end{aligned}$$

On the other hand, the Maurer–Cartan formula (Proposition XI, sec. 6.14) applied to the relation above for ω yields

$$\Omega = \Omega_0 \times 1 + (\delta\theta + [\omega_0, \theta]) \times f + \tfrac{1}{2}[\theta, \theta] \times f^2 - \theta \times \delta f.$$

Since θ is basic, we obtain from the corollary to Proposition IX, sec. 6.13, that

$$\Omega = \Omega_0 \times 1 + \nabla_0\theta \times f + \tfrac{1}{2}[\theta, \theta] \times f^2 - \theta \times \delta f.$$

This implies that

$$\Gamma(\Omega,\dots,\Omega) = -\sum_{i+j+k=p-1} \frac{p!}{i!\,j!\,k!}\frac{1}{2^j}\langle \Gamma, \theta \vee (\nabla_0\theta)^i \vee [\theta,\theta]^j \vee \Omega_0^k\rangle \times f^{i+2j}\delta f$$

$$+ \sum_{i+j+k=p} \frac{p!}{i!\,j!\,k!}\frac{1}{2^j}\langle \Gamma, (\nabla_0\theta)^i \vee [\theta,\theta]^j \vee \Omega_0^k\rangle \times f^{i+2j}.$$

It follows that

$$\pi^*\Phi = \sum_{i+j+k=p-1} \langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}(\tfrac{1}{2}[\theta,\theta])^j \vee \frac{1}{k!}(\Omega_0)^k\rangle$$

$$\times \int_0^1 (f^{i+2j}\delta f)\left(t; \frac{d}{dt}\right) dt.$$

But

$$\int_0^1 (f^{i+2j}\delta f)\left(t; \frac{d}{dt}\right) dt = \int_0^1 t^{i+2j}\, dt = \frac{1}{i+2j+1}.$$

The proposition follows.

Q.E.D.

Corollary: Suppose $\mathscr{P}$ admits a connection form ω_0 whose curvature Ω_0 is zero. Let ω_1 be any connection form in $\mathscr{P}$ and set $\theta = \omega_1 - \omega_0$. Then, for $\Gamma \in (\vee^p E^*)_I$, $(\gamma_1)_B\Gamma = \delta\Phi$, where

$$\pi^*\Phi = \sum_{i+j=p-1} \frac{1}{p+j}\langle \Gamma, \theta \vee \frac{1}{i!}(\nabla_0\theta)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\theta,\theta]\right)^j\rangle.$$

Example: Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let ω be a connection form in $\mathscr{P}$ with curvature form Ω. Consider the trivial bundle $\hat{\mathscr{P}} = (P \times G, \pi_P, P, G)$ and let ω_0 denote the connection form on $\hat{\mathscr{P}}$ corresponding to the horizontal subbundle $T_P \times G$. Then

$$\omega_0(z, a; \zeta, \eta) = L_a^{-1}(\eta)$$

and the corresponding curvature form is zero as follows from Corollary II to Proposition X, sec. 6.14.

On the other hand, since $T\colon P \times G \to P$ is a homomorphism of principal bundles inducing $\pi\colon P \to B$, it follows that $\omega_1 = T^*\omega$ is a connection form in $\hat{\mathscr{P}}$ with curvature $\Omega_1 = T^*\Omega$.

A straightforward calculation shows that, in this case,

$$\theta(z, a) = ((\operatorname{Ad} a^{-1})\omega \times 1)(z, a), \qquad z \in P, \quad a \in G,$$

$(\theta = \omega_1 - \omega_0)$. It follows that

$$\nabla_0\theta(z, a) = ((\mathrm{Ad}\, a^{-1})(\delta\omega) \times 1)(z, a)$$

and

$$[\theta, \theta](z, a) = ((\mathrm{Ad}\, a^{-1})[\omega, \omega] \times 1)(z, a).$$

Now let $\Gamma \in (\vee^p E^*)_I$ $(p > 0)$. Then, since Γ is invariant, the corollary to Proposition XIV reads

$$\gamma_I \Gamma \times 1 = \delta\left\{\sum_{i+j=p-1} \frac{1}{p+j} \langle \Gamma, \omega \vee \frac{1}{i!}(\delta\omega)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\omega, \omega]\right)^j\rangle \times 1\right\};$$

i.e.,

$$\pi^*\gamma_B\Gamma = \gamma_I\Gamma = \delta\left\{\sum_{i+j=p-1} \frac{1}{p+j} \langle \Gamma, \omega \vee \frac{1}{i!}(\delta\omega)^i \vee \frac{1}{j!}\left(\tfrac{1}{2}[\omega, \omega]\right)^j\rangle\right\}.$$

Substitution of the relation $\Omega = \delta\omega + \frac{1}{2}[\omega, \omega]$ yields the formula

$$\begin{aligned}\pi^*(\gamma_B\Gamma) &= \gamma_I\Gamma \\ &= \frac{(p-1)!}{(2p-1)!}\,\delta\left\{\sum_{k=0}^{p-1} (-\tfrac{1}{2})^{p-1-k}\binom{2p-1}{k}\langle \Gamma, \omega \vee \Omega^k \vee [\omega, \omega]^{p-1-k}\rangle\right\}.\end{aligned}$$

(The calculation is long but elementary except for the observation that

$$\sum_{l=0}^{r}\binom{r}{l}\frac{(-1)^l}{p+l} = \int_0^1 x^{p-1}(1-x)^r\,dx = \frac{r!}{p\cdots(p+r)}.)$$

6.21. Formal power series and the Taylor homomorphism. Consider the infinite sequences

$$\Gamma = (\Gamma_0, \Gamma_1, \ldots) \qquad \text{with} \quad \Gamma_k \in \vee^k E^*.$$

Define addition and multiplication by

$$(\Gamma + \hat{\Gamma})_k = \Gamma_k + \hat{\Gamma}_k \qquad \text{and} \qquad (\Gamma \cdot \hat{\Gamma})_k = \sum_{i+j=k} \Gamma_i \vee \hat{\Gamma}_j \qquad (k = 0, 1, \ldots).$$

The associative algebra so obtained is called the *algebra of formal power series in* E^* and is denoted by $\vee^{**} E^*$.

Next, recall from sec. 1.9, volume I, that $\mathscr{S}_0(E)$ denotes the algebra of smooth function germs at 0. That is, an element of $\mathscr{S}_0(E)$ is an equivalence class of functions $f \in \mathscr{S}(E)$ under the following equivalence relation: $f \sim g$ if $f - g$ is zero in a neighbourhood of 0. If U is a neighbourhood of 0 in E and $g \in \mathscr{S}(U)$, then there is a unique germ, $[g]_0 \in \mathscr{S}_0(E)$, such that any $f \in [g]_0$ agrees with g sufficiently close to 0. We say g is a *representative* of $[g]_0$.

Now let $f \in \mathscr{S}(U)$ (U, a neighbourhood of 0 in E). Then the kth derivative of f is the smooth map $f^{(k)} \in \mathscr{S}(U;\ \vee^k E^*)$ defined inductively by

$$f^{(0)} = f$$

and

$$f^{(k)}(x; h_1, \dots, h_k) = \lim_{t\to 0} \frac{f^{(k-1)}(x + th_1; h_2, \dots, h_k) - f^{(k-1)}(x; h_2, \dots, h_k)}{t}.$$

(Note that we identify $\vee^k E^*$ with $S^k(E)$ via i_S as described in sec. 6.18.)

The Leibniz formula states that

$$(fg)^{(k)} = \sum_{i+j=k} f^{(i)} \vee g^{(j)}, \qquad f, g \in \mathscr{S}(U);$$

i.e., the map,

$$f \mapsto (f(0), f'(0), f''(0), \dots),$$

is a homomorphism of $\mathscr{S}(U)$ into $\vee^{**}E^*$. Since the derivatives of f at 0 depend only on the germ of f at 0, this homomorphism determines a homomorphism

$$\mathrm{Tay}\colon \mathscr{S}_0(E) \to \vee^{**}E^*$$

called the *Taylor homomorphism.*

Next recall that G acts on E by the automorphisms $\operatorname{Ad} a$. Thus an action of G on $\mathscr{S}_0(E)$ is defined by

$$a \cdot [f]_0 = [(\operatorname{Ad} a^{-1})^* f]_0, \qquad f \in \mathscr{S}(E), \quad a \in G.$$

The corresponding invariant subalgebra is denoted by $\mathscr{S}_0(E)_I$. On the other hand, we have an induced action of G on $\vee^{**}E^*$. Clearly, the Taylor homomorphism is equivariant with respect to these actions and hence it restricts to a homomorphism,

$$\mathrm{Tay}_I\colon \mathscr{S}_0(E)_I \to (\vee^{**}E^*)_I,$$

called the *invariant Taylor homomorphism.*

6.22. The homomorphisms $h_{\mathscr{P}}^{**}$ and $s_{\mathscr{P}}$. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle over an n-manifold B and consider the Weil homomorphism

$$h_{\mathscr{P}}\colon (\vee E^*)_I \to H(B).$$

Since $H^p(B) = 0$, $p > n$, $h_{\mathscr{P}}$ extends to a homomorphism

$$h_{\mathscr{P}}^{**}\colon (\vee^{**}E^*)_I \to H(B).$$

Clearly the image of $h_{\mathscr{P}}^{**}$ coincides with the image of $h_{\mathscr{P}}$.

On the other hand, we have the invariant Taylor homomorphism

$$(\mathrm{Tay})_I\colon \mathscr{S}_0(E)_I \to (\vee^{**}E^*)_I$$

Composing these homomorphisms we obtain a homomorphism

$$s_{\mathscr{P}}\colon \mathscr{S}_0(E)_I \to H(B).$$

Explicitly, $s_{\mathscr{P}}[f]_0 = \sum_{p=0}^{\infty} h_{\mathscr{P}}(f^{(p)}(0))$.

If $\varphi\colon P \to \hat{P}$ is a homomorphism of principal bundles inducing $\psi\colon B \to \hat{B}$, then $\psi^{\#} \circ h_{\hat{\mathscr{P}}}^{**} = h_{\mathscr{P}}^{**}$ and $\psi^{\#} \circ s_{\hat{\mathscr{P}}} = s_{\mathscr{P}}$ as follows from Theorem II, sec. 6.19, and the definitions.

Remark: The advantage of using $h_{\mathscr{P}}^{**}$ or $s_{\mathscr{P}}$ rather than $h_{\mathscr{P}}$ is the following: Let $[f]_0 \in \mathscr{S}_0(E)_I$, $\Gamma \in (\vee^{**}E^*)_I$, $\alpha \in H(B)$. These elements are invertible in their respective algebras if and only if $f(0) \neq 0$ (respectively $\Gamma_0 \neq 0$, $\alpha_0 \neq 0$, where α_0 is the component of α in $H^0(B)$). Moreover, if $f(0) \neq 0$, then

$$s_{\mathscr{P}}([f]_0^{-1}) = (s_{\mathscr{P}}([f]_0))^{-1}.$$

On the other hand, an element $\Gamma \in (\vee E^*)_I$ is only invertible if $\Gamma_0 \neq 0$ and $\Gamma_i = 0$, $i > 0$, while $h_{\mathscr{P}}(\Gamma)$ is invertible whenever $\Gamma_0 \neq 0$. Hence, if $\Gamma_0 \neq 0$, and $\Gamma_i \neq 0$ for some $i > 0$, then $(h_{\mathscr{P}}(\Gamma))^{-1}$ exists but it is expressible in the $h_{\mathscr{P}}(\Gamma_i)$ only via a complicated polynomial. To obtain simple expressions it is necessary to introduce $(\vee^{**}E^*)_I$.

§7. Special cases

6.23. Principal bundles with abelian structure group. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle whose structure group G is abelian. Let ω be a connection form in $\mathscr{P}$ with curvature form Ω. Then

$$i(h)\Omega = 0 \qquad \text{and} \qquad T_a^*\Omega = \Omega, \qquad a \in G. \tag{6.1}$$

Moreover, the Maurer–Cartan equation (Proposition XI, sec. 6.14) reduces to $\delta\omega = \Omega$. In particular, it follows that $\delta\Omega = 0$.

In view of Proposition III, sec. 6.3, relations (6.1) imply that there is a (unique) E-valued 2-form Ω_B on B such that $\Omega = \pi^*\Omega_B$. Since

$$\pi^*\,\delta\Omega_B = \delta\pi^*\Omega_B = \delta\Omega = 0,$$

it follows that $\delta\Omega_B = 0$.

Next observe that, since G is abelian, $(\vee E^*)_I = \vee E^*$ and so γ_I and γ_B become homomorphisms

$$\gamma_I \colon \vee E^* \to A_B(P) \qquad \text{and} \qquad \gamma_B \colon \vee E^* \to A(B).$$

Evidently (cf. sec. 6.18)

$$\gamma_B(\Gamma) = \frac{1}{p!}\,\Gamma(\Omega_B, \ldots, \Omega_B), \qquad \Gamma \in \vee^p E^*.$$

In particular,

$$\gamma_B(h^*) = \langle h^*, \Omega_B \rangle, \qquad h^* \in E^*. \tag{6.2}$$

Proposition XV: For every $h^* \in E^*$, let χ_{h^*} denote the 1-form on P given by

$$\chi_{h^*}(Z) = \langle h^*, \omega(Z) \rangle, \qquad Z \in \mathscr{X}(P).$$

Then

$$\delta\chi_{h^*} = \pi^*\gamma_B(h^*).$$

Proof: In fact,

$$\begin{aligned}\pi^*\gamma_B(h^*) &= \pi^*\langle h^*, \Omega_B \rangle = \langle h^*, \Omega \rangle \\ &= \langle h^*, \delta\omega \rangle = \delta\langle h^*, \omega \rangle = \delta\chi_{h^*}.\end{aligned}$$

Q.E.D.

Remark: In volume III it will be shown that Proposition XV generalizes to principal bundles with compact connected structure group.

Example: Assume that $G = S^1$. Let e^* be the basis vector of E^* which generates the invariant 1-form whose integral over S^1 equals 1. Then χ_{e^*} is a 1-form on P satisfying

$$\oint_{S^1} \chi_{e^*} = 1 \qquad \text{and} \qquad \delta\chi_{e^*} = \pi^*\gamma_B(e^*).$$

Hence, $-\gamma_B(e^*)$ represents the Euler class, $\chi_{\mathscr{P}}$, of the circle bundle $\mathscr{P}$ (cf. sec. 8.2, volume I).

This shows that $\chi_{\mathscr{P}} = -h_{\mathscr{P}}(e^*)$ and that $\chi_{\mathscr{P}}$ is represented by the 2-form

$$\Phi = -\langle e^*, \Omega_B \rangle.$$

6.24. The cohomology of $\mathbb{C}P^n$. Recall from sec. 5.20 the Hopf fibration $\mathscr{P} = (S^{2n+1}, \pi, \mathbb{C}P^n, S^1)$. The principal action of S^1 is the restriction to S^{2n+1} of the representation R of S^1 in $\mathbb{C}^{n+1}$ given by

$$R(e^{i\theta}) \cdot z = e^{i\theta} z, \qquad z \in \mathbb{C}^{n+1}.$$

Next we define a connection in $\mathscr{P}$. Identify the Lie algebra of the principal S^1-bundle $\mathscr{P}$ with $\mathbb{R}$ so that the invariant 1-form generated by 1^* has integral 1. Let Z (respectively, $\hat{Z}$) denote the fundamental fields generated by 1 on S^{2n+1} (respectively, $\mathbb{C}^{n+1}$). Then

$$Z(z) = \hat{Z}(z), \qquad z \in S^{2n+1},$$

and

$$\hat{Z}(z) = (z, 2\pi i z), \qquad z \in \mathbb{C}^{n+1}.$$

Define a 1-form θ on $\mathbb{C}^{n+1}$ by

$$\theta(z; \zeta) = -\frac{1}{2\pi} \operatorname{Im}\langle z, \zeta \rangle,$$

where $\langle \, , \rangle$ denotes the Hermitian inner product. Then θ is S^1-invariant and

$$\theta(z; \hat{Z}(z)) = \langle z, z \rangle, \qquad z \in \mathbb{C}^{n+1}.$$

Thus, if ω denotes the restriction of θ to S^{2n+1},

$$\omega(Z) = 1,$$

and so ω is a connection form in $\mathscr{P}$. Since S^1 is abelian, the corresponding curvature form is given by $\Omega = \delta\omega$, and (cf. sec. 6.23) we have

$$\delta\omega = \pi^*\Omega_B .$$

Proposition XVI: Let $\chi_{\mathscr{P}}$ denote the Euler class of the S^1-bundle $\mathscr{P}$. Then (1) the classes $1, \chi_{\mathscr{P}}, \dots, (\chi_{\mathscr{P}})^n$ form a basis for $H(\mathbb{C}P^n)$.

(2) $(\chi_{\mathscr{P}})^n$ is an orientation class for $\mathbb{C}P^n$.

Proof: (1) Recall the Gysin sequence

$$\begin{array}{ccccc} \downarrow & & & & \\ H^p(S^{2n+1}) & \xrightarrow{\oint_{S^1}^*} & H^{p-1}(\mathbb{C}P^n) & \xrightarrow{D} & H^{p+1}(\mathbb{C}P^n) \\ & & & & \downarrow{\scriptstyle \pi^*} \\ & & & & H^{p+1}(S^{2n+1}) \longrightarrow , \end{array}$$

from sec. 8.2, volume I, where D is given by

$$D\alpha = \alpha \cdot \chi_{\mathscr{P}}, \qquad \alpha \in H(\mathbb{C}P^n).$$

Observe that, if $\alpha \in H^+(\mathbb{C}P^n)$, then $\alpha \in \sum_{j=1}^{2n} H^j(\mathbb{C}P^n)$. It follows that $\pi^\#\alpha \in \sum_{j=1}^{2n} H^j(S^{2n+1})$ and so $\pi^\#\alpha = 0$. Hence the Gysin sequence yields the exact sequences,

$$0 \longrightarrow H^1(\mathbb{C}P^n) \longrightarrow 0$$

and

$$0 \longrightarrow H^p(\mathbb{C}P^n) \xrightarrow{D} H^{p+2}(\mathbb{C}P^n) \longrightarrow 0 \qquad (0 \leqslant p \leqslant 2n-2).$$

This shows that the elements $1, \chi_{\mathscr{P}}, \dots, \chi_{\mathscr{P}}^n$ form a basis for $H(\mathbb{C}P^n)$.

(2) We must show that for a suitable orientation of $\mathbb{C}P^n$,

$$\int_{\mathbb{C}P^n} \Omega_B^n = 1.$$

Orient the bundle $\mathscr{P}$ by ω and give $\mathbb{C}P^n$ the orientation such that the induced local product orientation in S^{2n+1} (cf. sec. 7.6, volume I or sec. 0.15) is the standard orientation. Then

$$\oint_{S^1} \omega = 1,$$

and so the Fubini theorem together with Stokes' theorem (cf. sec. 4.17, volume I, and sec. 7.14, volume I) imply that

$$\int_{\mathbb{C}P^n} \Omega_B^n = \int_{S^{2n+1}} (\pi^*\Omega_B)^n \wedge \omega$$

$$= \int_{S^{2n+1}} \omega \wedge (\delta\omega)^n = \int_B (\delta\theta)^{n+1},$$

where B is the unit ball in $\mathbb{R}^{2n+2}$.

Next we show that

$$(\delta\theta)^{n+1} = \frac{(n+1)!}{\pi^{n+1}} \Delta,$$

where Δ denotes the normed positive determinant function in $\mathbb{R}^{2n+2}$. In fact, fix an orthonormal basis e_ν $(\nu = 1, ..., n+1)$ in $\mathbb{C}^{n+1}$ and let X_ν, Y_ν $(\nu = 1, ..., n+1)$ be the constant vector fields corresponding to the vectors e_ν, ie_ν. Then, if a vector $z \in \mathbb{C}^{n+1}$ is written

$$z = \sum_\nu \xi^\nu e_\nu + \sum_\nu \eta^\nu(ie_\nu), \qquad \xi^\nu, \eta^\nu \in \mathbb{R},$$

we have

$$\langle\theta, X_\nu\rangle(z) = -\frac{1}{2\pi}\eta^\nu \quad \text{and} \quad \langle\theta, Y_\nu\rangle(z) = \frac{1}{2\pi}\xi^\nu.$$

These relations yield

$$\delta\theta(X_\nu, X_\mu) = 0, \qquad \delta\theta(Y_\nu, Y_\mu) = 0, \qquad \text{and} \qquad \delta\theta(X_\nu, Y_\mu) = \frac{1}{\pi}\delta_{\nu\mu}.$$

It follows that

$$(\delta\theta)^{n+1}(X_1, Y_1, ..., X_{n+1}, Y_{n+1}) = (n+1)!/\pi^{n+1},$$

whence $(\delta\theta)^{n+1} = [(n+1)!/\pi^{n+1}]\Delta$.

Finally, recall from Example 2, sec. 4.15, volume I, that

$$\int_{S^{2n+1}} i(T)\,\Delta = 2\pi^{n+1}/n!,$$

where T is the vector field in $\mathbb{C}^{n+1}$ given by $T(z) = (z, z)$. Moreover,

$$\delta\, i(T)\,\Delta = \theta(T)\,\Delta = 2(n+1)\,\Delta.$$

These relations yield

$$\int_{\mathbb{C}P^n} \Omega_B^n = \int_B (\delta\theta)^{n+1} = \frac{(n+1)!}{\pi^{n+1}} \int_B \Delta = \frac{(n+1)!}{2(n+1)\,\pi^{n+1}} \int_{S^{2n+1}} i(T)\,\Delta = 1.$$

Q.E.D.

Corollary I: The Euler class of the Hopf fibration (S^3, π, S^2, S^1) is an orientation class of S^2.

Corollary II: The inclusion maps, $i\colon \mathbb{C}P^k \to \mathbb{C}P^n$ $(k \leqslant n)$, induce linear isomorphisms

$$i^\#\colon H^p(\mathbb{C}P^k) \xleftarrow{\cong} H^p(\mathbb{C}P^n) \qquad (0 \leqslant p \leqslant 2k).$$

6.25. Reduction of structure group. Let $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ be a second principal bundle over the same base. Assume that $\sigma\colon K \to G$ is a homomorphism and that $\varphi\colon \hat{P} \to P$ is a smooth fibre preserving map inducing the identity in B and satisfying

$$\varphi(z \cdot a) = \varphi(z) \cdot \sigma(a), \qquad z \in P, \quad a \in K;$$

thus, $(\hat{P}, \varphi)$ is a reduction of structure group from G to K via σ (cf. Example 5, sec. 5.5).

Denote the Lie algebra of K by F. The derivative $\sigma'\colon F \to E$ induces a homomorphism

$$(\sigma')^\vee\colon \vee F^* \leftarrow \vee E^*.$$

Since

$$\operatorname{Ad} \sigma(a) \circ \sigma' = \sigma' \circ \operatorname{Ad} a, \qquad a \in K,$$

$(\sigma')^\vee$ restricts to a homomorphism

$$\sigma_I\colon (\vee F^*)_I \leftarrow (\vee E^*)_I .$$

Theorem III: With the notation and hypotheses above, the diagram,

$$\begin{tikzcd}
(\vee F^*)_I \arrow[dr, "h_{\hat{\mathscr{P}}}"'] & & (\vee E^*)_I \arrow[ll, "\sigma_I"'] \arrow[dl, "h_{\mathscr{P}}"] \\
& H(B) &
\end{tikzcd},$$

commutes.

Corollary: Let $\lambda\colon G \to H$ be a homomorphism from G into a Lie group H with Lie algebra L. Let $\mathscr{P}_\lambda$ be the λ-extension of $\mathscr{P}$ (cf. Example 4, sec. 5.5). Then

$$h_{\mathscr{P}} \circ \lambda_I = h_{\mathscr{P}_\lambda}.$$

The proof of Theorem III is preceded by three lemmas.

Lemma IV: There are principal coordinate representations $\{(U_\alpha, \hat{\psi}_\alpha)\}$ and $\{(U_\alpha, \psi_\alpha)\}$ for $\hat{\mathscr{P}}$ and for $\mathscr{P}$ such that the diagrams,

$$\begin{array}{ccc} U_\alpha \times K & \xrightarrow{\iota \times \sigma} & U_\alpha \times G \\ {\scriptstyle \hat{\psi}_\alpha} \Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \psi_\alpha} \\ \hat{\pi}^{-1} U_\alpha & \xrightarrow[\varphi]{} & \pi^{-1} U_\alpha \end{array},$$

commute.

Proof: Let $\{(U_\alpha, \hat{\psi}_\alpha)\}$ be any principal coordinate representation for $\hat{\mathscr{P}}$. Consider the cross-sections $U_\alpha \to P$ defined by

$$x \mapsto \varphi(\hat{\psi}_\alpha(x, e))$$

and define maps ψ_α: $U_\alpha \times G \to P$ by

$$\psi_\alpha(x, b) = \varphi(\hat{\psi}_\alpha(x, e)) \cdot b, \qquad x \in U_\alpha, \quad b \in G.$$

Then $\{(U_\alpha, \psi_\alpha)\}$ is a principal coordinate representation for $\mathscr{P}$. Moreover,

$$\begin{aligned} \varphi\hat{\psi}_\alpha(x, a) &= \varphi(\hat{\psi}_\alpha(x, e) \cdot a) \\ &= \varphi(\hat{\psi}_\alpha(x, e)) \cdot \sigma(a) = \psi_\alpha(x, \sigma(a)), \qquad a \in K, \quad x \in B, \end{aligned}$$

as desired.

Q.E.D.

Lemma V: There are principal connections $\hat{V}$ for $\hat{\mathscr{P}}$ and V for $\mathscr{P}$ such that.

$$d\varphi \circ \hat{V} = V \circ d\varphi.$$

In particular, if W is a vector space the operators $\hat{H}^*$, $\hat{\nabla}$ in $A(\hat{P}; W)$ and H^*, ∇ in $A(P; W)$ satisfy

$$\hat{H}^* \circ \varphi^* = \varphi^* \circ H^* \qquad \text{and} \qquad \hat{\nabla} \circ \varphi^* = \varphi^* \circ \nabla.$$

Proof: If the principal bundles are trivial, $\hat{P} = B \times K$, $P = B \times G$ and if φ is given by $\varphi = \iota \times \sigma$, then the connections

$$\hat{V}(\xi, \eta) = (0, \eta), \qquad \xi \in T_x(B), \quad \eta \in T_a(K),$$

and

$$V(\xi, \zeta) = (0, \zeta), \qquad \xi \in T_x(B), \quad \zeta \in T_b(G),$$

satisfy the conditions above.

In the general case let $\{U_\alpha\}$ be the covering of B used in Lemma IV. Then, in view of that lemma, there are principal connections $\hat{V}_\alpha$ and V_α in the restrictions of $\hat{\mathscr{P}}$ and $\mathscr{P}$ to U_α which satisfy

$$d\varphi \circ \hat{V}_\alpha = V_\alpha \circ d\varphi.$$

Choose a partition of unity $\{p_\alpha\}$ in B subordinate to the open covering $\{U_\alpha\}$ and set

$$\hat{V} = \sum_\alpha (\hat{\pi}^* p_\alpha)\, \hat{V}_\alpha\,, \qquad V = \sum_\alpha (\pi^* p_\alpha)\, V_\alpha\,.$$

Q.E.D.

Lemma VI: Let V, $\hat{V}$ be principal connections in $\mathscr{P}$ and $\hat{\mathscr{P}}$ satisfying the condition of Lemma V. Then the corresponding connection forms, ω and $\hat{\omega}$, and curvature forms, Ω and $\hat{\Omega}$, are related by the equations

(1) $(\sigma')_*\, \hat{\omega} = \varphi^*\omega$

and

(2) $(\sigma')_*\, \hat{\Omega} = \varphi^*\Omega.$

Proof: (1) It follows from Lemma V that both sides of (1) give zero when applied to horizontal vectors. Thus it is sufficient to check that

$$(\sigma'_*\hat{\omega})(\hat{Z}_h) = (\varphi^*\omega)(\hat{Z}_h), \qquad h \in F.$$

The equations $\varphi(z \cdot a) = \varphi(z) \cdot \sigma(a)$ ($z \in \hat{P}$, $a \in K$) imply that

$$\hat{Z}_h \underset{\varphi}{\sim} Z_{\sigma'(h)}\,.$$

Hence (for $h \in F$)

$$(\sigma'_*\hat{\omega})(\hat{Z}_h) = \sigma'(h) \;= \omega(Z_{\sigma'(h)}) = (\varphi^*\omega)(\hat{Z}_h).$$

(2) In fact,

$$\begin{aligned}(\sigma')_*\, \hat{\Omega} &= (\sigma')_*\, \hat{\nabla}\hat{\omega} = \hat{\nabla}(\sigma')_*\, \hat{\omega} \\ &= \hat{\nabla}\varphi^*\omega = \varphi^*\nabla\omega = \varphi^*\Omega.\end{aligned}$$

Q.E.D.

6.26. Proof of Theorem III: Choose V, ω, Ω and $\hat{V}$, $\hat{\omega}$, $\hat{\Omega}$ as in the lemmas above. Let

$$\beta\colon \otimes E^* \to A(P), \qquad \hat{\beta}\colon \otimes F^* \to A(\hat{P})$$

be the corresponding homomorphisms as defined in sec. 6.17. Then for $\Gamma \in \otimes^p E^*$

$$\begin{aligned}\varphi^*(\beta\Gamma) &= \varphi^*(\Gamma(\Omega, \ldots, \Omega)) = \Gamma(\varphi^*\Omega, \ldots, \varphi^*\Omega)\\ &= \Gamma((\sigma')_*\hat{\Omega}, \ldots, (\sigma')_*\hat{\Omega}) = (\otimes^p (\sigma')^* \Gamma)(\hat{\Omega}, \ldots, \hat{\Omega}).\end{aligned}$$

It follows that $\varphi^* \circ \beta = \hat{\beta} \circ \otimes (\sigma')^*$.

Thus the homomorphisms γ, γ_I, $\hat{\gamma}$ and $\hat{\gamma}_I$ (cf. sec. 6.17) are connected by the relations

$$\varphi^* \circ \gamma = \hat{\gamma} \circ (\sigma')^\vee \qquad \text{and} \qquad \varphi^* \circ \gamma_I = \hat{\gamma}_I \circ \sigma_I .$$

Since $\hat{\pi}^* = \varphi^* \circ \pi^*$, we have $\gamma_B = \hat{\gamma}_B \circ \sigma_I$ and the theorem follows.

Q.E.D.

6.27. Example. Given a principal bundle, $\mathscr{P} = (P, \pi, B, G)$, let K be a closed subgroup of G and consider the principal bundle $\mathscr{P}_1 = (P, p, P/K, K)$ (cf. sec. 5.7) and its λ-extension

$$\hat{\mathscr{P}} = (P \times_K G, \hat{\pi}, P/K, G)$$

(cf. Example 4, sec. 5.5), where $\lambda\colon K \to G$ is the inclusion. Then we have the commutative diagram,

$$\begin{tikzcd}
& P \times G \arrow[d, "q"] \arrow[dr, "T"] & \\
P \arrow[ur, "\varphi_1"] \arrow[r, "\varphi"] \arrow[d, "p"'] & P \times_K G \arrow[r, "\psi"] \arrow[d, "\hat{\pi}"] & P \arrow[d, "\pi"] \\
P/K \arrow[r, "\iota"'] & P/K \arrow[r, "\rho"'] & B,
\end{tikzcd}$$

where φ_1 is inclusion opposite e and T is the principal action. Thus φ is a reduction of structure group with respect to the inclusion map, $\lambda\colon K \to G$, and ψ is a homomorphism of principal bundles.

Let E and F be the Lie algebras of G and K and let

$$\lambda_I\colon (\vee F^*)_I \leftarrow (\vee E^*)_I$$

be the homomorphism induced by λ. Then, Theorem III, sec. 6.25, and Theorem II, sec. 6.19, yield the commutative diagrams

$$\begin{tikzcd}[column sep=small] (\vee F^*)_I \arrow[dr, "h_{\mathscr{P}_1}"'] & & (\vee E^*)_I \arrow[ll, "\lambda_I"'] \arrow[dl, "h_{\mathscr{P}}"] \\ & H(P/K) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & (\vee E^*)_I \arrow[dl, "h_{\mathscr{P}}"'] \arrow[dr, "h_{\mathscr{P}}"] & \\ H(P/K) & & H(B). \arrow[ll, "\rho^*"] \end{tikzcd}$$

Combining these we obtain the commutative diagram

$$\begin{tikzcd} (\vee F^*)_I \arrow[d, "h_{\mathscr{P}_1}"'] & (\vee E^*)_I \arrow[l, "\lambda_I"'] \arrow[d, "h_{\mathscr{P}}"] \\ H(P/K) & H(B) \arrow[l, "\rho^*"] \end{tikzcd} .$$

Remark: Assume in addition that G is compact and connected and that K is a maximal torus. Then the map ρ^* is injective, as will be shown in volume III. Moreover (cf. sec. 6.23) since in this case F is abelian, the diagram above becomes

$$\begin{tikzcd} \vee F^* \arrow[d, "h_{\mathscr{P}_1}"'] & (\vee E^*)_I \arrow[l, "\lambda_I"'] \arrow[d, "h_{\mathscr{P}}"] \\ H(P/K) & H(B) \arrow[l, "\rho^*"] \end{tikzcd} .$$

Because of the simple structure of $\vee F^*$, the following becomes an important technique: first establish properties of $h_{\mathscr{P}_1}$; then use the injectivity of $\rho^\#$ to draw conclusions about $h_{\mathscr{P}}$. This technique forms the basis of the fundamental papers [1], [2] and [3] by Borel and Hirzebruch.

6.28. Connections invariant under a group action. Suppose that $\hat{\mathscr{P}} = (\hat{P}, \hat{\pi}, B, K)$ is a principal bundle with structure group a Lie group K. Denote the corresponding principal action of K on $\hat{P}$ by $\hat{T}: \hat{P} \times K \to \hat{P}$. Assume that

$$\begin{tikzcd} G \times \hat{P} \arrow[r, "S"] \arrow[d, "\iota \times \hat{\pi}"'] & \hat{P} \arrow[d, "\hat{\pi}"] \\ G \times B \arrow[r, "\tilde{S}"'] & B \end{tikzcd}$$

is a smooth commutative diagram in which S and $\tilde{S}$ are left actions of G. Then $(S, \tilde{S})$ is called an *action of G on the principal bundle $\hat{\mathscr{P}}$* if the maps S_g and $\hat{T}_a$ commute for each $g \in G$, $a \in K$. Assume $(S, \tilde{S})$ is such an action.

A principal connection V in $\hat{\mathscr{P}}$ will be called *G-invariant* if

$$dS_g \circ V = V \circ dS_g, \qquad g \in G.$$

This holds if and only if the connection form satisfies

$$S_g^* \omega = \omega, \qquad g \in G.$$

If V is G-invariant then $H = \iota - V$ also commutes with the operators dS_g. It follows that

$$H^* \circ S_g^* = S_g^* \circ H^*, \qquad g \in G.$$

Hence the covariant derivative ∇ satisfies

$$\nabla \circ S_g^* = S_g^* \circ \nabla, \qquad g \in G.$$

In particular, the curvature form Ω is G-invariant:

$$S_g^* \Omega = \Omega, \qquad g \in G.$$

This, in turn, implies that the homomorphism $\gamma_B\colon (\vee F^*)_I \to A(B)$ (F, the Lie algebra of K) satisfies

$$\tilde{S}_g^* \circ \gamma_B = \gamma_B .$$

Thus γ_B can be considered as a homomorphism,

$$(\gamma_B)_I : (\vee F^*)_I \to A_I(B),$$

where $A_I(B)$ denotes the subalgebra of $A(B)$ invariant under the action of G. Since $\delta \circ \gamma_B = 0$, $(\gamma_B)_I$ induces a homomorphism

$$(h_{\mathscr{P}})_I : (\vee F^*)_I \to H_I(B).$$

The diagram,

$$\begin{tikzcd} & H(B) \\ (\vee F^*)_I \arrow[ur, "h_{\mathscr{P}}"] \arrow[dr, "(h_{\mathscr{P}})_I"'] & \\ & H_I(B), \arrow[uu, "i_\#"'] \end{tikzcd}$$

commutes, where i: $A_I(B) \to A(B)$ is the inclusion.

Proposition XVII: If G is compact and acts on the principal bundle, $\mathscr{P}$, then $\mathscr{P}$ admits a G-invariant principal connection.

Proof: Let V be any principal connection. Regard V as a cross-section in the vector bundle $L_{T_{\hat{P}}}$ over $\hat{P}$ (whose fibre at z is the space of linear transformations of $T_z(\hat{P})$). Using the actions (cf. sec. 3.2),

$$S: G \times \hat{P} \to \hat{P}, \qquad dS: G \times T_{\hat{P}} \to T_{\hat{P}},$$

we can integrate V over G (cf. sec. 3.18) to obtain a G-invariant cross-section V^I. We show that V^I is a (G-invariant) principal connection.

For $z \in \hat{P}$, V_z^I is the endomorphism of $T_z(\hat{P})$ given by

$$V_z^I = \int_G (dS_g)_{g^{-1}z} \circ V_{g^{-1}z} \circ (dS_{g^{-1}})_z \, dg.$$

Since the vertical spaces $V_z(\hat{P})$ are dS_g-stable (because $\hat{\pi}$ is equivariant), and because each V_z is a projection of $T_z(\hat{P})$ onto $V_z(\hat{P})$, it follows from this relation that V_z^I is also a projection of $T_z(\hat{P})$ onto $V_z(\hat{P})$.

Finally, since (for $a \in K$) $d\hat{T}_a: T_z(\hat{P}) \to T_{z\cdot a}(\hat{P})$ is linear, we have

$$\begin{aligned} d\hat{T}_a \circ V_z^I &= \int_G (d\hat{T}_a \circ dS_g \circ V_{g^{-1}\cdot z} \circ dS_{g^{-1}}) \, dg \\ &= \int_G (dS_g \circ V_{g^{-1}\cdot z\cdot a} \circ dS_{g^{-1}} \circ d\hat{T}_a) \, dg \\ &= V_{z\cdot a}^I \circ d\hat{T}_a \,. \end{aligned}$$

Hence V^I is a principal connection.

Q.E.D.

§8. Homogeneous spaces

In this article, K denotes a closed subgroup of G with Lie algebra F and $\mathscr{P}_K = (G, \pi, G/K, K)$ is the principal bundle defined in Example 2, sec. 5.1.

6.29. The cohomology of G/K. The principal action of K on G is denoted by μ_K :

$$\mu_K(g, a) = ga, \qquad g \in G, \quad a \in K.$$

On the other hand, the maps

$$(g_1, g_2) \mapsto g_1 g_2 \qquad \text{and} \qquad (g_1, \pi g_2) \mapsto \pi(g_1 g_2)$$

define left actions of G on G and G/K, with respect to which π is equivariant. Thus π^* restricts to a homomorphism (cf. sec. 4.18)

$$\pi_I^*: A_I(G/K) \to A_L(G).$$

Moreover, since $(G, \pi, G/K, K)$ is a principal bundle, Proposition III, sec. 6.3, shows that

$$\pi^*: A(G/K) \xrightarrow{\cong} A_B(G)$$

is an isomorphism. (Recall that $A_B(G)$ consists of those forms which are horizontal, and invariant under the right action of K on G). Thus π^* restricts to an isomorphism,

$$\pi_I^*: A_I(G/K) \xrightarrow{\cong} A_L(G) \cap A_B(G).$$

Since the action, μ_K, of K on G is right multiplication, the corresponding fundamental vector fields are the left invariant vector fields, X_k ($k \in F$), on G. Thus the horizontal and invariant subalgebras of $A(G)$ are given by

$$\bigcap_{k\in F} \ker i(X_k) \qquad \text{and} \qquad \bigcap_{a\in K} \ker(\rho_a^* - \iota),$$

respectively. We denote them by

$$A(G)_{i_F=0} \qquad \text{and} \qquad A(G)_{K=I}.$$

The basic subalgebra, $A_B(G)$, is their intersection.

Now recall the isomorphism $\tau_L\colon A_L(G) \xrightarrow{\cong} \wedge E^*$ of sec. 4.5. It satisfies

$$\tau_L \circ i(X_h) = i_E(h) \circ \tau_L \qquad \text{and} \qquad \tau_L \circ \rho_g^* = \mathrm{Ad}^\wedge(g) \circ \tau_L\,, \qquad h \in E,\quad g \in G,$$

(cf. sec. 4.6 and sec. 4.8). Hence it restricts to isomorphisms

$$A_L(G) \cap A(G)_{i_F=0} \xrightarrow{\cong} (\wedge E^*)_{i_F=0} \qquad \text{and} \qquad A_L(G) \cap A(G)_{K=I} \xrightarrow{\cong} (\wedge E^*)_{K=I}\,.$$

(Here $(\wedge E^*)_{K=I}$ denotes the subalgebra invariant under the operators $\mathrm{Ad}^\wedge(a)$, $a \in K$ and $(\wedge E^*)_{i_F=0} = \bigcap_{k\in F} \ker i_E(k)$.) Thus

$$\tau_L\colon A_L(G) \cap A_B(G) \xrightarrow{\cong} (\wedge E^*)_{i_F=0,K=I}\,,$$

where $(\wedge E^*)_{i_F=0,K=I}$ denotes the intersection of $(\wedge E^*)_{i_F=0}$ and $(\wedge E^*)_{K=I}$.

Composing the isomorphisms τ_L and π_I^*, we obtain the commutative diagram

$$\begin{tikzcd}
A(G) & A_L(G) \arrow[l] \arrow[r, "\tau_L", "\cong"'] & \wedge E^* \\
A(G/K) \arrow[u, "\pi^*"] & A_I(G/K) \arrow[l] \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "\tau_L\circ\pi_I^*"'] & (\wedge E^*)_{i_F=0,K=I}\,. \arrow[u, "k"']
\end{tikzcd}$$

The right-hand square coincides with the diagram of Proposition XI, sec. 4.18 and k is the inclusion.

Next, assume that K is connected. Then (cf. Proposition VI, sec. 3.13) the subalgebra, $A(G)_{K=I}$, is given by

$$A(G)_{K=I} = A(G)_{\theta_F=0} = \bigcap_{k\in F} \ker \theta(X_k).$$

Set (cf. sec. 4.6)

$$(\wedge E^*)_{\theta_F=0} = \bigcap_{k\in F} \ker \theta_E(k) \qquad \text{and} \qquad (\wedge E^*)_{i_F=0,\theta_F=0} = (\wedge E^*)_{i_F=0} \cap (\wedge E^*)_{\theta_F=0}\,.$$

Then we can rewrite the diagram above in the form

$$\begin{tikzcd}
A(G) & A_L(G) \arrow[l] \arrow[r, "\tau_L", "\cong"'] & \wedge E^* \\
A(G/K) \arrow[u, "\pi^*"] & A_I(G/K) \arrow[l] \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "\tau_L\circ\pi_I^*"'] & (\wedge E^*)_{i_F=0,\theta_F=0}\,. \arrow[u, "k"']
\end{tikzcd}$$

Theorem IV: Let K be a closed connected subgroup of a compact connected Lie group G. Then, in the commutative diagram,

$$\begin{tikzcd} H(G) & H_L(G) \arrow[l, "\cong"'] \arrow[r, "(\tau_L)_*", "\cong"'] & H(E) \\ H(G/K) \arrow[u, "\pi^*"] & H_I(G/K) \arrow[l, "\cong"'] \arrow[u, "\pi_I^*"] \arrow[r, "\cong", "(\tau_L \circ \pi_I^*)_*"'] & H((\wedge E^*)_{i_F=0,\theta_F=0}) \arrow[u, "k_*"'] \end{tikzcd}$$

all the horizontal maps are algebra isomorphisms.

Proof: This is a restatement of Theorem V, sec. 4.19.

Q.E.D.

6.30. Connections in $(G, \pi, G/K, K)$. Recall that $\pi: G \to G/K$ is equivariant with respect to the left actions of G. We shall find the G-invariant principal connections for the principal bundle $(G, \pi, G/K, K)$ (cf. sec. 6.28).

Let V be a G-invariant principal connection. Since the vertical space at e is given by

$$V_e(G) = \ker(d\pi)_e = F$$

(cf. sec. 2.11), it follows that the restriction V_e of V to E is a projection

$$V_e: E \to F.$$

Moreover, since V is a G-invariant principal connection,

$$\mathrm{Ad}\, a \circ V_e = L_a \circ R_a^{-1} \circ V_e = V_e \circ \mathrm{Ad}\, a, \qquad a \in K.$$

In particular, $\ker V_e$ is stable under the operators $\mathrm{Ad}\, a$ ($a \in K$). Note that $\ker V_e$ is the horizontal subspace at e.

Proposition XVIII: The map $\alpha: V \mapsto \ker V_e$ is a bijection from the set of G-invariant principal connections to the set of K-stable subspaces of E complementing F.

Proof: If W, V are two such connections with $\ker V_e = \ker W_e$ then, since

$$\mathrm{Im}\, V_e = F = \mathrm{Im}\, W_e,$$

we have $V_e = W_e$. Now the G-invariance implies that $V = W$ and so α is injective.

On the other hand, assume $F_1 \subset E$ is a subspace stable under $\operatorname{Ad} a$ ($a \in K$) and complementary to F:

$$E = F_1 \oplus F.$$

Let $V_e: E \to F$ be the projection with kernel F_1 and define a G-invariant strong bundle map, V, in T_G by

$$V_g = L_g \circ V_e \circ L_g^{-1}, \qquad g \in G.$$

V_g is a projection onto $L_g(F)$. But since π is equivariant, $L_g(= d\lambda_g)$ maps F isomorphically to the vertical space at g; i.e., V_g is a projection onto the vertical subspace. Moreover since F_1 is stable under $\operatorname{Ad} a$ ($a \in K$), it follows that

$$\operatorname{Ad} a \circ V_e = V_e \circ \operatorname{Ad} a, \qquad a \in K.$$

Since $R_a \circ L_g = L_g \circ R_a$ (cf. sec. 1.1) this yields

$$R_a \circ V_g = V_{g \cdot a} \circ R_a, \qquad g \in G, \quad a \in K.$$

Thus V is a G-invariant principal connection. By definition, $\ker V_e = F_1$, and so α is surjective.

Q.E.D.

Corollary I: $(G, \pi, G/K, K)$ admits a G-invariant connection if and only if there is a K-stable subspace $F_1 \subset E$ such that $E = F_1 \oplus F$.

Corollary II: If K is connected, the G-invariant principal connections are in one-to-one correspondence with the subspaces $F_1 \subset E$ such that

$$(\operatorname{ad} h) F_1 \subset F_1 \quad (h \in F) \qquad \text{and} \qquad E = F_1 \oplus F.$$

Corollary III: If K is compact, the bundle, $\mathscr{P}_K$, admits a G-invariant principal connection.

Proof: Apply Proposition XVII, sec. 1.17.

Q.E.D.

6.31. Curvature and the Weil homomorphism. Assume that E admits a decomposition $E = F_1 \oplus F$, where F_1 is stable under the operators $\operatorname{Ad} a$, $a \in K$. Let $p: E \to F$ and $p_1: E \to F_1$ be the projections.

Then p and p_1 are precisely the vertical and horizontal projections in $T_e(G)$ corresponding to the induced G-invariant principal connection V.

It follows that the connection form ω is the unique left invariant 1-form in $A^1(G; F)$ which satisfies

$$\omega(e; h) = p(h), \qquad h \in E.$$

Next we compute the curvature Ω of V. Observe that if X_h, X_k are left invariant vector fields on G, then

$$\delta\omega(X_h, X_k) = -\omega([X_h, X_k]) = -\omega(e; [h, k])$$

(because the functions $\omega(X_h)$, $\omega(X_k)$ are left invariant, and so constant). Similarly,

$$\tfrac{1}{2}[\omega, \omega](X_h, X_k) = [\omega(e; h), \omega(e; k)].$$

It follows from Proposition XI, sec. 6.14 that Ω is the unique left invariant E-valued 2-form such that

$$\Omega(e; h, k) = [p(h), p(k)] - p([h, k]), \qquad h, k \in E.$$

Thus if $h, k \in F_1$, then

$$\Omega(e; h, k) = -p([h, k]).$$

Finally, consider the invariant Weil homomorphism

$$(h_{\mathscr{P}_K})_I : (\vee F^*)_I \to H_I(G/K).$$

If $\Gamma \in (\vee^k F^*)_I$, then $(h_{\mathscr{P}_K})_I(\Gamma)$ is represented by the unique left invariant differential form $\Phi \in A_I^{2k}(G/K)$ which satisfies (cf. sec. 6.18)

$$\pi^*\Phi(e; h_1, \ldots, h_{2k}) = \frac{(-1)^k}{2^k\, k!} \sum_{\sigma \in S^{2k}} \epsilon_\sigma \Gamma(p([h_{\sigma(1)}, h_{\sigma(2)}]), \ldots, p([h_{\sigma(2k-1)}, h_{\sigma(2k)}])),$$

for $h_i \in F_1$. Clearly this differential form also represents $h_{\mathscr{P}_K}(\Gamma)$ in $H(G/K)$.

6.32. Symmetric spaces. Suppose that φ is an automorphism of G such that

$$\varphi^2 = \iota \qquad \text{and} \qquad \varphi \neq \iota.$$

The elements $a \in G$ satisfying $\varphi(a) = a$ form a closed subgroup; let K be its one-component. Then the Lie algebra, F, of K is the subspace of vectors $h \in E$ satisfying $\varphi'(h) = h$.

The homogeneous space G/K is called a *symmetric space with connected fibre*. If G is compact, we say G/K has *compact type*.

Since φ is an involution, so is φ'. Hence, setting

$$E^+ = \ker(\varphi' - \iota) \qquad \text{and} \qquad E^- = \ker(\varphi' + \iota),$$

we have $E = E^+ \oplus E^-$ and $E^+ = F$. Now we show that

$$[E^+, E^-] \subset E^- \qquad \text{and} \qquad [E^-, E^-] \subset E^+$$

(where, for subspaces $A, B \subset E$, $[A, B]$ is the space spanned by vectors of the form $[h, k]$, $h \in A$, $k \in B$).

In fact, for $h \in E^+$, $k \in E^-$,

$$\varphi'([h, k]) = [\varphi'(h), \varphi'(k)] = -[h, k],$$

whence $[h, k] \in E^-$. The second relation is proved in the same way. It follows that E^- is stable under the operators ad h ($h \in F$) and so, by Corollary II to Proposition XVIII, sec. 6.30, it determines a G-invariant principal connection on $(G, \pi, G/K, K)$. It is called the *symmetric space connection.*

Examples: 1. The Grassmann manifolds (cf. sec. 5.13)

$$SO(n)/(SO(k) \times SO(n-k)), \qquad U(n)/(U(k) \times U(n-k))$$

and

$$Q(n)/(Q(k) \times Q(n-k))$$

are symmetric spaces of compact type. In fact, consider the first case. Choose a decomposition $W = W_1 \oplus W_1^\perp$ of a Euclidean space, W, with $\dim W = n$, $\dim W_1 = k$. Define a rotation $\tau\colon W \to W$ by

$$\tau(w) = \begin{cases} w, & w \in W_1 \\ -w, & w \in W_1^\perp. \end{cases}$$

Then define an involution $\varphi\colon SO(n) \to SO(n)$ by

$$\varphi(\sigma) = \tau \circ \sigma \circ \tau^{-1}.$$

Evidently $\varphi(\sigma) = \sigma$ if and only if σ stabilizes W_1 and $W_1^\perp$. Thus the one-component of the subgroup left fixed by φ is $SO(k) \times SO(n-k)$.

The other two cases are established in the same way.

2. Endow $\mathbb{R}^n$ with a Euclidean metric. Define an involution, φ, of $GL^+(n; \mathbb{R})$ by setting

$$\varphi(\sigma) = (\sigma^*)^{-1}, \qquad \sigma \in GL^+(n; \mathbb{R}),$$

where σ^* denotes the dual of σ with respect to the inner product. The subgroup left fixed by φ is $SO(n)$.

Since φ' is given by $\varphi'(\alpha) = -\alpha^*$, $\alpha \in L(n)$, we have

$$L(n)^+ = \mathrm{Sk}(n) \qquad \text{and} \qquad L(n)^- = S(n)$$

($S(n)$ is the space of symmetric transformations of $\mathbb{R}^n$).

In this case the invariant connection leads to a homomorphism

$$(\vee\, \mathrm{Sk}(n)^*)_I \xrightarrow{(h_{\mathscr{P}})_I} H_I(GL^+(n; \mathbb{R})/SO(n)).$$

This homomorphism is in general *nontrivial*, as will be shown in volume III.

On the other hand, according to Example 1 of sec. 4.11, $GL^+(n; \mathbb{R})/SO(n)$ is diffeomorphic to the vector space $S(n)$. Thus its cohomology is trivial, as is the Weil homomorphism $h_{\mathscr{P}}$.

Problems

G is a Lie group with Lie algebra E.

1. **Trivial bundles.** Let $\mathscr{P} = (B \times G, \pi, B, G)$ be a trivial principal bundle. With each connection form, ω, associate the E-valued 1-form θ on B defined by

$$\omega(x, e; \xi, h) = h + \theta(x; \xi), \qquad x \in B, \quad \xi \in T_x(B), \quad h \in E.$$

(i) Show that this correspondence defines a bijection between principal connections in $\mathscr{P}$ and elements of $A^1(B; E)$.

(ii) Fix a principal connection, V, in $\mathscr{P}$ with corresponding 1-form $\theta \in A^1(B; E)$. Show that the linear map H_z at $z = (x, y)$ is given by

$$H_z(\xi, \eta) = (\xi, -R_y\theta(x; \xi)), \qquad \xi \in T_x(B), \quad \eta \in T_y(G).$$

(iii) Consider the E-valued 2-form Φ on B given by $\Phi = \delta\theta + \frac{1}{2}[\theta, \theta]$. Show that

$$(\pi^*\Phi)(x, y) = (\operatorname{Ad} y(\Omega(x, y)), \qquad x \in B, \quad y \in G,$$

where Ω is the curvature of V.

(iv) Let $z(t) = (x(t), y(t))$ $(0 \leqslant t \leqslant 1)$ be a smooth path in $B \times G$. Show that $\dot{z}(t)$ is horizontal if and only if

$$\dot{y}(t) = -R_{y(t)}\theta(x(t); \dot{x}(t)).$$

2. **Local formulae for principal connections.** Let $\{(U_\alpha, \psi_\alpha)\}$ be a principal coordinate representation for a principal bundle $\mathscr{P} = (P, \pi, B, G)$. Fix a principal connection in $\mathscr{P}$.

(i) As in problem 1, use the connection form to define local 1-forms $\theta_\alpha \in A^1(U_\alpha; E)$.

(ii) Find the relation between the restrictions $\theta_\alpha |_{U_\alpha \cap U_\beta}$ and $\theta_\beta |_{U_\alpha \cap U_\beta}$.

(iii) Set $\Phi_\alpha = \delta\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha]$. Find the relation between $\Phi_\alpha |_{U_\alpha \cap U_\beta}$ and $\Phi_\beta |_{U_\alpha \cap U_\beta}$.

3. **Horizontal lifts.** Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with a fixed principal connection V. A *horizontal lift* of a path $x(t)$ $(0 \leqslant t \leqslant 1)$

in B is a smooth path $z(t)$ $(0 \leqslant t \leqslant 1)$ in P such that $\pi z(t) = x(t)$ and each tangent vector $\dot{z}(t)$ is horizontal.

(i) Let $x(t)$, $0 \leqslant t \leqslant 1$, be a smooth path in B. Given $z_0 \in G_{x(0)}$, show that there is a unique horizontal lift $z(t)$ of $x(t)$ such that $z(0) = z_0$. (Hint: cf. problem 1, (iv)), and problem 21, Chap. I).

(ii) Let $\psi\colon \mathbb{R}^2 \to B$ be a smooth map. Fix $z_0 \in G_{\psi(0,0)}$. Let $z(\tau)$ $(\tau \in \mathbb{R})$ be the horizontal lift of $\psi(\tau, 0)$ that satisfies $z(0) = z_0$. For fixed τ, let $z(\tau, t)$ be the horizontal lift of $\psi(\tau, t)$ that satisfies $\psi(\tau, 0) = z(\tau)$. Show that the map $\varphi\colon \mathbb{R}^2 \times G \to P$ given by

$$\varphi(\tau, t, a) = z(\tau, t) \cdot a$$

is a homomorphism of principal bundles.

(iii) Let τ and t denote the first and second coordinate functions in $\mathbb{R}^2$, with gradients $\delta\tau$ and δt. Let $\hat{V}$ be the principal connection in $\mathbb{R}^2 \times G$ induced via φ from V. Let $\theta \in A^1(\mathbb{R}^2; E)$ be the corresponding 1-form (cf. problem 1). Show that $\theta = f \cdot \delta\tau$, where $f \in \mathscr{S}(\mathbb{R}^2; E)$ satisfies $f(\tau, 0) = 0$. Conclude that, if Ω is the curvature of V, then

$$(\varphi^*\Omega)(\tau, t, e) = \frac{\partial f}{\partial t}(\tau, t)\, \delta t \wedge \delta\tau.$$

Conclude that $\Omega = 0$ implies that $\theta = 0$.

4. Homotopic paths. Let $\mathscr{P}$, V be as in problem 3. Let α and β be smooth paths in B such that

$$\alpha(0) = \beta(0) = x_0 \qquad \text{and} \qquad \alpha(1) = \beta(1) = x_1 .$$

Assume that Φ is a homotopy connecting α and β such that

$$\Phi(0, t) = x_0 \qquad \text{and} \qquad \Phi(1, t) = x_1 , \qquad t \in \mathbb{R}.$$

(i) Assume that the curvature of V is zero. Prove that if $\hat{\alpha}$ and $\hat{\beta}$ are horizontal lifts of α and β, both starting at the same point, then

$$\hat{\alpha}(1) = \hat{\beta}(1).$$

(ii) Establish the converse.

5. Holonomy groups I. Let $(\mathscr{P}, V)$ be as in problem 3. Assume that B is connected. Fix base points $x \in B$ and $z \in G_x$. Identify G with G_x via $a \mapsto z \cdot a$.

A *loop* in B, based at x, is a smooth map $\gamma\colon t \mapsto \gamma(t)$ $(0 \leqslant t \leqslant 1)$ such that $\gamma(0) = \gamma(1) = x$. Two loops based at x are called *homotopic* if they are homotopic in the sense of problem 4. A loop is called *conctractible* if it is homotopic to the constant loop. If every loop is contractible, B is called *simply connected.*

(i) Let γ be a loop based at x. Show that there is a unique element $a(\gamma) \in G$ such that for every horizontal lift, $\hat{\gamma}$, of γ

$$\hat{\gamma}(0) \cdot a(\gamma) = \hat{\gamma}(1).$$

(ii) If γ_τ is a homotopy of loops based at x, show that $\tau \mapsto a(\gamma_\tau)$ is a smooth path in G. (*Hint*: Use problem 3, (ii).)

(iii) Let $\mathscr{L}$ denote the set of loops, based at x, and let $\mathscr{L}_0$ denote the subset of contractible loops. Show that $\Gamma = \{a(\gamma) \mid \gamma \in \mathscr{L}\}$ is a subgroup of G and that $\Gamma_0 = \{a(\gamma) \mid \gamma \in \mathscr{L}_0\}$ is a normal subgroup of Γ. Show that Γ/Γ_0 is finite or countable.

(iv) Show that Γ_0 is a connected Lie subgroup of G (use problem 6, Chap. II). Conclude that Γ is a Lie subgroup of G with Γ_0 as 1-component.

Γ is called the *holonomy group* of the connection with respect to z.

6. Holonomy groups II. Adopt the hypotheses and notation of problem 5.

(i) Reduce the structure group of $\mathscr{P}$ to Γ; i.e., construct a principal bundle, $\mathscr{P}_1 = (P_1, \pi_1, B, \Gamma)$, and a Γ-equivariant inclusion map $\varphi\colon P_1 \to P$ such that $\pi \circ \varphi = \pi_1$.

Interpret P_1 as a maximal connected integral submanifold of an involutive distribution on P.

(ii) Let ω be the connection form of V. Prove that $\varphi^*\omega$ takes values in the Lie algebra of Γ_0. Conclude that $\varphi^*\omega$ is a connection form in P_1.

(iii) *Ambrose-Singer*: Assume that $\Gamma = G$ and let Ω be the curvature of V. Show that the vectors $\Omega(z; h, k)$, $z \in P$, $h, k \in T_z(P)$, span the Lie algebra E. (*Hint*: Use problem 3).

(iv) Suppose that two principal connections have the same curvature. Show that their holonomy groups have the same 1-component.

7. Zero curvature. Let $(\mathscr{P}, V)$ be as in problem 3, with B connected.

(i) Show that the following conditions are equivalent: (a) the curvature Ω is zero; (b) the holonomy group, Γ, is discrete; (c) the horizontal subbundle is an involutive distribution on P.

(ii) Assume $\Omega = 0$. Let M be a maximal connected integral manifold for the horizontal subbundle, and construct a principal covering projection (M, π, B, Γ) (cf. Problem 18, Chap. I).

(iii) Assume $\Omega = 0$ and B is simply connected. Construct an isomorphism $P = B \times G$ of principal bundles, which carries V to the standard connection in $B \times G$.

8. Principal bundles with abelian structure group. Assume that $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with abelian structure group. Let ω be a connection form and let Ω_B be the corresponding curvature form in B. Let $\varphi: D \to B$ be a smooth map of the two-dimensional disk D into B. Denote by γ the image of ∂D (∂D is the boundary of D) under φ and let $x_0 \in \partial D$ be a fixed point.

(i) Show that, for some fixed $a \in G$,

$$\hat{\gamma}(1) = \hat{\gamma}(0) \cdot a,$$

where $\hat{\gamma}$ is any horizontal lift of γ.

(ii) Show that $a = \exp_G(-\int_D \varphi^*\Omega_B)$.

9. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal connection V.

(i) Show that every horizontal vector field X on P can be written as a finite sum $\sum_i f_i \cdot X_i$, where the X_i are horizontal and invariant and $f_i \in \mathscr{S}(P)$.

(ii) Assume that G is connected. Show that a differential form Φ is in $A_B(P)$ if and only if $\theta(X)\Phi = 0$ for every vertical vector field X.

10. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let $G \times F \to F$ be a left action of G on a manifold F. Let $\xi = (M, \rho, B, F)$ be the associated bundle ($M = P \times_G F$).

(i) Let $\mathbf{H}_P$ be the horizontal subbundle associated with a principal connection in $\mathscr{P}$. Show that the vector spaces

$$\mathbf{H}_{q(z,y)}(M) = (dq)_{(z,y)}\mathbf{H}_z(P), \qquad z \in P, \quad y \in F$$

($q: P \times F \to M$ is the principal map) are the fibres of a subbundle $\mathbf{H}_M$ of τ_M. Show that $\tau_M = \mathbf{H}_M \oplus \mathbf{V}_M$. $\mathbf{H}_M$ is called the *associated horizontal subbundle for* M.

(ii) With the aid of $\mathbf{H}_M$, define the notion of horizontal lifts in the bundle ξ. Establish an analogue of problem 3, (i) for ξ.

11. Let (P, π, B, G) be a principal bundle with a principal connection and corresponding homomorphism $\gamma_B\colon (\vee E^*)_I \to A(B)$. Suppose $\Gamma \in (\vee^p E^*)_I$ is an element such that $\gamma_B(\Gamma) = 0$. Show that Γ determines a closed $(2p - 1)$-form on P; hence obtain an element of $H_I^{2p-1}(P)$.

12. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let K be a closed subgroup of G. Consider the bundles $\mathscr{P}_K = (G, \pi_K, G/K, K)$ and $\mathscr{P}_1 = (P, p, P/K, K)$.

(i) Show that a G-invariant principal connection in $\mathscr{P}_K$ and a principal connection in $\mathscr{P}$ together determine the principal connection in $\mathscr{P}_1$ given by

$$\omega_1(z; \zeta) = \omega_K(e; \omega(z; \zeta)),$$

where ω_1, ω_K, ω are the connection forms.

(ii) Describe the horizontal subbundle, the horizontal projection, and the curvature.

13. Define connections in the principal bundles of article 5, Chap. V. Obtain the corresponding curvatures.

14. Let G be a compact connected Lie group with maximal torus T. Show that the principal bundle $G \to G/T$ admits a unique G-invariant connection and determine its curvature.

15. **Bundles with compact support.** Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Let $O \subset B$ be an open set so that $B - O$ is compact and let $\sigma\colon O \to P$ be a cross-section over O. Then the pair $(\mathscr{P}, \sigma)$ is called a *principal bundle with compact support*. If $U \subset O$ is any open set such that $B - U$ is compact, then U is called an *open complement* for $(\mathscr{P}, \sigma)$.

A *homomorphism between principal bundles* $(\mathscr{P}, \sigma)$ *and* $(\hat{\mathscr{P}}, \hat{\sigma})$ *with compact support* is a homomorphism, $\varphi\colon \mathscr{P} \to \hat{\mathscr{P}}$, of principal bundles such that

(a) The induced map $\psi\colon B \to \hat{B}$ is proper.

(b) For some open complement V of $(\hat{\mathscr{P}}, \hat{\sigma})$

$$\varphi(\sigma(x)) = \hat{\sigma}(\psi(x)), \qquad x \in \psi^{-1}(V).$$

A *compact principal connection* in $(\mathscr{P}, \sigma)$ is a principal connection, V, in $\mathscr{P}$ such that for some open complement U of $(\mathscr{P}, \sigma)$

$$V_{\sigma(x)} \circ (d\sigma)_x = 0, \qquad x \in U.$$

(i) Let $(\mathscr{P}, \sigma)$ be a principal bundle with compact support. Show that a trivializing map α: $O \times G \xrightarrow{\cong} \pi^{-1}(O)$ is given by $(x, a) \mapsto \sigma(x) \cdot a$. Restate the definitions in terms of α.

(ii) Show that $(\mathscr{P}, \sigma)$ admits a compact principal connection. Show that the curvature of such a connection has support in $\pi^{-1}(K)$ for some compact subset K of B. Conclude that the induced homomorphism γ_B: $(\vee E^*)_I \to A(B)$ can be regarded as a homomorphism into $A_c(B)$.

(iii) Show that γ_B induces a homomorphism, h_σ^c: $(\vee^+ E^*)_I \to H_c(B)$, the *Weil homomorphism with compact support*. Show that h_σ^c is independent of the choice of compact connection. Show that

$$\lambda_\# \circ h_\sigma^c = h_{\mathscr{P}} ,$$

where λ: $A_c(B) \to A(B)$ is the inclusion map.

(iv) Establish a naturality property for h_σ^c.

(v) Show that a compactly supported principal bundle over $\mathbb{R}^n$ determines a principal bundle over S^n and that the diagram

$$\begin{tikzcd} & H_c^n(\mathbb{R}^n) \arrow[dd, "\cong"] \\ (\vee^+ E^*)_I \arrow[ur, "h_\sigma^c"] \arrow[dr, "h_{\mathscr{P}}"'] & \\ & H^n(S^n) \end{tikzcd}$$

commutes. Hence construct an example where $h_\sigma^c \neq 0$ but $h_{\mathscr{P}} = 0$. Conclude that h_σ^c is *not* independent of σ.

16. Odd characteristic homomorphism. Let B be a manifold and let G be a Lie group with Lie algebra E. Let f: $B \to G$ be a smooth map such that for some compact subset $A \subset B$, $f(x) = e$, $x \notin A$.

(i) Set $\mathscr{P} = (B \times \mathbb{R} \times G, \pi, B \times \mathbb{R}, G)$ and $O = B \times \mathbb{R} - A \times I$, where I denotes the closed unit interval. Define σ: $O \to B \times \mathbb{R} \times G$ by

$$\sigma(x, t) = \begin{cases} (x, t, f(x)), & t \geqslant \frac{1}{2} \\ (x, t, e), & t \leqslant \frac{1}{2}. \end{cases}$$

Show that $(\mathscr{P}, \sigma)$ is a compactly supported principal bundle.

(ii) Let p: $\mathbb{R} \to [0, 1]$ be smooth and satisfy

$$p(t) = 0, \quad t \leqslant 0, \qquad \text{and} \qquad p(t) = 1, \quad t \geqslant 1.$$

Define $\theta_f \in A^1(B \times \mathbb{R}; E)$ by

$$\theta_f(x, t; \xi, \eta) = -p(t)\, L^{-1}_{f(x)}(df)_x\xi, \qquad \xi \in T_x(B), \quad \eta \in T_t(\mathbb{R}).$$

Show that the corresponding principal connection V_f in P is compact (cf. problem 1 and problem 15). Compute its curvature.

(iii) Define a map ρ_E: $(\vee^p E^*)_I \to (\wedge^{2p-1}E^*)_I$ by

$$(\rho_E\Gamma)(h_1, \dots, h_{2p-1}) = \frac{(-1)^{p-1}(p-1)!}{2^{p-1}(2p-1)!} \times \sum_\sigma \epsilon_\sigma \Gamma(h_{\sigma(1)}, [h_{\sigma(2)}, h_{\sigma(3)}], \dots, [h_{\sigma(2p-2)}, h_{\sigma(2p-1)}]).$$

Regard $\rho_E\Gamma$ as an element of $A_I^{2p-1}(G)$. Establish the relation

$$f^*(\rho_E\Gamma) = \oint_{\mathbb{R}} \gamma_{B\times\mathbb{R}}(\Gamma),$$

where $\gamma_{B\times\mathbb{R}}$: $(\vee^+E^*)_I \to A_c(B \times \mathbb{R})$ is constructed via the connection V_f.

(iv) Obtain a map, $\bar{\rho}_E$: $(\vee^+E^*)_I \to H(G)$, from ρ_E. Show that

$$\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c : (\vee^+E^*)_I \to H_c^+(B)$$

is a canonical map, independent of the connection. Show that

$$\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c = f^\# \circ \bar{\rho}_E .$$

Conclude that $\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c$ depends only on $f^\#$.

17. Covering by two open sets. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle. Assume $B = U \cup V$ is an open covering of B such that $\mathscr{P}$ is trivial over U and V.

(i) Let (U, ψ_U) and (V, ψ_V) be a principal coordinate representation for $\mathscr{P}$. Construct a smooth map φ: $U \cap V \to G$ such that

$$(\psi_V^{-1} \circ \psi_U)(x, a) = (x, \varphi(x)a), \qquad x \in U \cap V, \quad a \in G.$$

(ii) Let $\bar{\rho}_E$: $(\vee^+E^*)_I \to H(G)$ be the linear map defined in problem 16, (iv). Let ∂: $H(U \cap V) \to H(B)$ be the connecting homomorphism of the Mayer–Vietoris sequence for (B, U, V). Prove that

$$h_{\mathscr{P}} = \partial \circ \varphi^\# \circ \bar{\rho}_E .$$

18. If $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with finite structure group, show that $\pi^\#$ maps $H(B)$ isomorphically onto $H_I(P)$.

19. The operator $i(a)$. Let M be a manifold. Define an $\mathscr{S}(M)$-linear map

$$i : \mathrm{Sec}\, \wedge\tau_M \to \mathrm{Hom}_M(A(M); A(M))$$

such that

(a) $i(\sigma \wedge \tau) = i(\tau) \circ i(\sigma)$, $\sigma, \tau \in \mathrm{Sec}\, \wedge\tau_M$,
(b) $i(X)$ is the substitution operator, $X \in \mathscr{X}(M)$ and
(c) $i(1) = \iota$.

(i) Show that i is uniquely determined by these conditions.

(ii) Let (P, π, B, G) be a principal bundle. Obtain canonical operators $i(a)$ $(a \in \wedge E)$ in $A(P)$ such that

$$i(h_1 \wedge \cdots \wedge h_p) = i(h_p) \circ \cdots \circ i(h_1).$$

Find expressions for the commutators $i(a) \circ \theta(h) - \theta(h) \circ i(a)$ and $i(a) \circ \delta - \delta \circ i(a)$.

(iii) Show that, for $a \in (\wedge^p E)_{\theta=0}$ and $\Phi \in A(P)_{\theta=0}$,

$$i(a)\, \delta\Phi = (-1)^p \delta i(a)(\Phi).$$

Hence obtain an operator, $i(a)_\#$, in $H(A(P))_{\theta=0}$.

(iv) Assume that G is compact and connected. Define

$$\varphi : H(A(P)_{\theta=0}) \to H(A(P)_{\theta=0}) \otimes (\wedge E^*)_{\theta=0}$$

by $\varphi(\alpha) = \sum_\nu \epsilon_{\nu,\alpha} i(a_\nu)_\# \alpha \otimes a^{*\nu}$, where a_ν, $a^{*\nu}$ is a pair of homogeneous bases of $(\wedge E)_{\theta=0}$ and $(\wedge E^*)_{\theta=0}$, and

$$\epsilon_{\nu,\alpha} = \deg a_\nu (\deg a_\nu + \deg \alpha).$$

Establish a commutative diagram,

$$\begin{tikzcd}
H(A(P)_{\theta=0}) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & H(A(P)_{\theta=0}) \otimes (\wedge E^*)_{\theta=0} \arrow[d, "\cong"] \\
H(P) \arrow[r, "T^\#"'] & H(P) \otimes H(G)
\end{tikzcd},$$

with vertical isomorphisms induced by inclusion maps.

(v) Extend the result of (iv) to any (i.e., not necessarily principal) action of a compact connected Lie group.

20. The operator D. Let ω be a connection form with curvature Ω in a principal bundle $\mathscr{P} = (P, \pi, B, G)$. With each representation of G in a space W, associate an operator D in $A(P; W)$ by setting

$$D\Phi = \delta\Phi + \omega(\Phi).$$

Prove the relations

(i) $D = \nabla$ in $A_B(P; E)$;

(ii) $D(\Psi(\Phi)) = D\Psi(\Phi) + (-1)^p\Psi(D\Phi), \Psi \in A^p(P; E), \Phi \in A(P; W)$;

(iii) $D\omega = \Omega + \frac{1}{2}[\omega, \omega]$;

(iv) $D^2\Psi = \Omega(\Psi), \Psi \in A(P; W)$;

(v) $D\Omega = 0$;

(vi) if $\langle,\rangle$ is a bilinear function in W, invariant under the representation, and $\boldsymbol{\langle,\rangle}: A(P; W) \times A(P; W) \to A(P)$ is the induced map, then

$$\delta\boldsymbol{\langle}\Phi, \Psi\boldsymbol{\rangle} = \boldsymbol{\langle}D\Phi, \Psi\boldsymbol{\rangle} + (-1)^p\boldsymbol{\langle}\Phi, D\Psi\boldsymbol{\rangle}, \qquad \Phi \in A^p(P; W), \qquad \Psi \in A(P; W).$$

21. Algebraic connections. An *algebraic connection* in a principal bundle, $\mathscr{P} = (P, \pi, B, G)$, is a linear map $\chi: E^* \to A^1(P)$ satisfying the conditions:

(a) $i(h)\chi(h^*) = \langle h^*, h\rangle, h \in E, h^* \in E^*$.

(b) $T_a^* \circ \chi = \chi \circ \mathrm{Ad}^\natural(a), a \in G$.

(i) Let ω be a connection form in $\mathscr{P}$. Show that an algebraic connection χ is defined by

$$\chi(h^*)(z; \zeta) = \langle h^*, \omega(z; \zeta)\rangle, \qquad z \in P, \quad \zeta \in T_z(P).$$

χ is called the *associated algebraic connection.* Show that the correspondence $\omega \mapsto \chi$ defines a bijection between principal connections and algebraic connections.

(ii) Show that an algebraic connection χ extends to the homomorphism $\chi_\wedge: \wedge E^* \to A(P)$ given by

$$(\chi_\wedge \Phi)(z; \zeta_1, ..., \zeta_p) = \langle \Phi, \omega(z; \zeta_1) \wedge \cdots \wedge \omega(z; \zeta_p)\rangle.$$

Show that $\chi_\wedge$ satisfies

$$i(h) \circ \chi_\wedge = \chi_\wedge \circ i_E(h),$$

$$T_a^* \circ \chi_\wedge = \chi_\wedge \circ \mathrm{Ad}^\wedge(a),$$

$$\theta(h) \circ \chi_\wedge = \chi_\wedge \circ \theta_E(h),$$

$$i(a) \circ \chi_\wedge = \chi_\wedge \circ i_E(a), \qquad a \in \wedge E.$$

(iii) Show that, for each $z \in P$, the map $\Phi \mapsto (\chi_\wedge \Phi)(z)$ defines an isomorphism $\wedge E^* \xrightarrow{\cong} \wedge \mathbf{V}_z(P)^*$. Conclude that an isomorphism

$$f : A(P)_{i=0} \otimes \wedge E^* \xrightarrow{\cong} A(P)$$

is given by $f(\Psi \otimes \Phi) = \Psi \wedge (\chi_\wedge \Phi)$.

(iv) Consider the linear map $\mathfrak{X}: E^* \to A^2(P)_{i=0}$ given by

$$\mathfrak{X}(h^*)(z; \zeta_1, \zeta_2) = \langle h^*, \Omega(z; \zeta_1, \zeta_2)\rangle, \qquad z \in P, \quad \zeta_1, \zeta_2 \in T_z(P),$$

where Ω is the curvature of the principal connection corresponding to the algebraic connection χ. Show that $\mathfrak{X}$ extends to a homomorphism, $\mathfrak{X}_\vee: \vee E^* \to A(P)_{i=0}$, and that

$$(\mathfrak{X}_\vee \Psi)(z; \zeta_1, \dots, \zeta_{2p}) = (1/2^p) \sum_\sigma \epsilon_\sigma \Psi(\Omega(z; \zeta_{\sigma(1)}, \zeta_{\sigma(2)}), \dots, \Omega(z; \zeta_{\sigma(2p-1)}, \zeta_{\sigma(2p)})).$$

Establish the relations ($a \in G$, $h \in E$)

$$\mathfrak{X}_\vee \circ \mathrm{Ad}^\vee(a)) = T_a^* \circ \mathfrak{X}_\vee \qquad \text{and} \qquad \mathfrak{X}_\vee \circ \theta_S(h) = \theta(h) \circ \mathfrak{X}_\vee .$$

(v) Prove that

$$\mathfrak{X}(h^*) = \delta \chi h^* - \chi_\wedge \delta_E h^*, \qquad h^* \in E^*,$$

$$\nabla \chi = \mathfrak{X} \qquad \text{and} \qquad \nabla \mathfrak{X} = 0 \quad \text{(Bianchi identity)}.$$

(vi) Show that $\mathfrak{X}_\vee$ coincides with the homomorphism γ of sec. 6.17. Thus describe the Weil homomorphism in terms of $\mathfrak{X}_\vee$.

22. Horizontal projection. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle with principal connection V and associated algebraic connection χ. Let $\{e^{*\nu}\}$, $\{e_\nu\}$ be a pair of dual bases of E^* and E and let $\mu(\Phi)$ denote left multiplication by Φ ($\Phi \in A(P)$).

(i) Define operators Y_k in $A(P)$ by

$$Y_0 = \iota$$
$$Y_k = \sum_\nu \mu(\chi e^{*\nu}) \circ Y_{k-1} \circ i(e_\nu), \qquad k \geqslant 1.$$

(ii) Show that the horizontal projection H^* is given by

$$H^* = \sum_k \frac{(-1)^k}{k!} Y_k .$$

(iii) Show that, for $\Phi \in A^p(P)$,

$$H^*\Phi = (\iota - Y_1)(\iota - \tfrac{1}{2}Y_1) \cdots (\iota - (1/p)\, Y_1)\Phi.$$

(iv) Show that Y_1 is an antiderivation, and that

$$Y_1\Phi(Z_1, \dots, Z_p) = \sum_{j=1}^p \Phi(Z_1, \dots, V_* Z_j, \dots, Z_p).$$

(v) Set $\delta_Y = Y_1 \circ \delta$. Show that, for $\Phi \in A^p(P)_{i=0}$ and $\Psi \in A(P)_{i=0}$,

$$i(h)\, \delta_Y\Phi = \theta(h)\Phi, \qquad \delta_Y\Phi = \sum_\nu (\chi e^{*\nu}) \wedge \theta(e_\nu)\Phi$$

and

$$\delta_Y(\Phi \wedge \Psi) = \delta_Y\Phi \wedge \Psi + (-1)^p\, \Phi \wedge \delta_Y\Psi.$$

23. The homomorphism g. Continue the hypotheses and notation of problem 22.

(i) Make $A(P; \wedge E^*)$ into a bigraded algebra by regarding it as the *skew* tensor product of the algebras $A(P)$ and $\wedge E^*$. Interpret the elements of $A^p(P; \wedge^q E^*)$ as functions

$$\underbrace{\mathscr{X}(P) \times \cdots \times \mathscr{X}(P)}_{(p\text{ factors})} \times \underbrace{E \times \cdots \times E}_{(q\text{ factors})} \to \mathbb{R}.$$

(ii) Define linear maps, $g\colon A^p(P; \wedge^q E^*) \to A^{p+q}(P)$, by setting

$$g\Phi(z; \zeta_1, \dots, \zeta_{p+q}) = \frac{1}{p!\, q!} \sum_\sigma \epsilon_\sigma \Phi(z; \zeta_{\sigma(1)}, \dots, \zeta_{\sigma(p)}, \omega(z; \zeta_{\sigma(p+1)}), \dots, \omega(z; \zeta_{\sigma(p+q)})).$$

Show that the resulting linear map $g\colon A(P; \wedge E^*) \to A(P)$ restricts to an isomorphism $A(P; \wedge E^*)_{i=0} \xrightarrow{\cong} A(P)$.

(iii) Show that g restricts to an isomorphism $A_B(P; \wedge E^*) \xrightarrow{\cong} A_I(P)$.

(iv) Show that the diagram

$$\begin{tikzcd} A(P; \wedge E^*)_{i=0} \arrow[d, "\cong"] \arrow[dr, "g", "\cong"'] & \\ & A(P) \\ A(P)_{i=0} \otimes \wedge E^* \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, where f is the isomorphism of problem 21, (iii).

(v) Let $\Phi \in A^m(P)$. Show that $g^{-1}\Phi = \sum_p \Psi_p$, where

$$\Psi_p \in A^p(P; \wedge^{m-p} E^*)_{i=0}$$

is given by

$$\Psi_p(Z_1, ..., Z_p, h_1, ..., h_{m-p}) = \Phi(H_* Z_1, ..., H_* Z_p, Z_{h_1}, ..., Z_{h_{m-p}}).$$

24. The decomposition of δ. Let V be a principal connection in a principal bundle $\mathscr{P} = (P, \pi, B, G)$.

(i) Show that antiderivations $D_\wedge$, $D_\times$, D_∇ are defined in $A(P)$ by the following equations ($\Phi \in A(P)$, $X_\nu \in \mathscr{X}(P)$):

$$\begin{aligned} D_\wedge \Phi(X_0, ..., X_p) &= \sum_i (-1)^i (V_* X_i)(\Phi(X_0, ..., \hat{X}_i, ..., X_p)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \Phi([X_i, X_j] - [H_* X_i, H_* X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p). \end{aligned}$$

$$\begin{aligned} D_\times \Phi(X_0, ..., X_p) &= \sum_{i<j} (-1)^{i+j} \Phi(V_*[H_* X_i, H_* X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p). \end{aligned}$$

$$\begin{aligned} D_\nabla \Phi(X_0, ..., X_p) &= \sum_i (-1)^i (H_* X_i)(\Phi(X_0, ..., \hat{X}_i, ..., X_p)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \Phi(H_*[H_* X_i, H_* X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p). \end{aligned}$$

(ii) Show that $\delta = D_\wedge + D_\times + D_\nabla$.

(iii) Let χ be the associated algebraic connection. Show that, under the isomorphism $f\colon A(P)_{i=0} \otimes \wedge E^* \xrightarrow{\cong} A(P)$, (cf. problem 21, (iii)), $D_{\mathbb{X}}$, and D_∇ correspond to the operators,

$$\sum_\nu \mu(\chi e^{*\nu})\omega \otimes i_E(e_\nu) \qquad \text{and} \qquad \nabla \otimes \iota,$$

where $\{e^{*\nu}\}$, $\{e_\nu\}$ is a pair of dual bases in E^* and E, ∇ is the covariant exterior derivative, and ω is the degree involution in $A(P)$.

(iv) Show that the covariant exterior derivative, ∇, satisfies

$$\nabla^2 \Phi = \sum_\nu \chi(e^{*\nu}) \wedge (\theta(e_\nu)\, H^*\Phi - \nabla i(e_\nu)\Phi), \qquad \Phi \in A(P).$$

(v) Establish the relations

$$D_\wedge^2 = 0, \qquad D_{\mathbb{X}}^2 = 0,$$
$$D_\wedge \circ D_{\mathbb{X}} + D_{\mathbb{X}} \circ D_\wedge = -D_\nabla^2,$$
$$D_\wedge \circ D_\nabla + D_\nabla \circ D_\wedge = 0,$$
$$D_{\mathbb{X}} \circ D_\nabla + D_\nabla \circ D_{\mathbb{X}} = 0.$$

(vi) Let $\mathscr{B} = (M, \rho, B, F)$ be any smooth bundle. Show that a decomposition $\tau_M = \mathbf{H}_M \oplus \mathbf{V}_M$ determines a bigradation of $A(M)$. Write $\delta_M = \sum_p \delta_p$, where δ_p is homogeneous of bidegree $(p, 1-p)$. Find expressions for the operators δ_p. Interpret the operators δ_p, when $\mathscr{B}$ is a principal bundle and $\mathbf{H}_M$ is the horizontal bundle of a principal connection.

25. The operators D_E and D_θ. Adopt the notation of problem 24. Let δ_E denote the operator $\omega \otimes \delta_E$ in $A(P; \wedge E^*)$ and let δ_θ be the operator in $A(P; \wedge E^*)$ given by $\delta_\theta = \sum_\nu \omega\theta(e_\nu) \otimes \mu(e^{*\nu})$.

(i) Show that δ_E and δ_θ are antiderivations with respect to which the algebra $A(P; \wedge E^*)_{i=0}$ is stable.

(ii) Use the isomorphism, g, of problem 23 to identify δ_E and δ_θ with operators D_E and D_θ in $A(P)$.

(iii) Show that $D_\wedge = D_E + D_\theta$.

(iv) Obtain a relation between D_θ and δ_Y (cf. problem 22).

26. Let $\mathscr{P} = (P, \pi, B \times \mathbb{R}, G)$ be a principal bundle. Fix $t \in \mathbb{R}$, and let $\mathscr{P}_t = (P_t, \pi_t, B \times \{t\}, G)$ be the restriction of $\mathscr{P}$.

(i) Construct an isomorphism from $\mathscr{P}$ to the principal bundle $(P_0 \times \mathbb{R}, \pi_0 \times \iota, B \times \mathbb{R}, G)$ (*Hint:* Use problem 3, (i).)

(ii) Conclude that $\mathscr{P}_0 \cong \mathscr{P}_t$.

(iii) Let $\varphi, \psi\colon M \to N$ be homotopic maps, where N is the base of a principal bundle. Prove that the pull-backs of this bundle to M via φ and ψ are strongly isomorphic.

(iv) Conclude that every principal bundle over a contractible space is trivial.

27. Principal bundles over S^n. Let $\mathscr{P} = (P, \pi, S^n, G)$ be a principal bundle over S^n.

(i) Show that $\mathscr{P}$ admits a coordinate representation consisting of only two elements.

(ii) Obtain a smooth map $\varphi\colon S^{n-1} \to G$ such that $\mathscr{P}$ is trivial if and only if φ is homotopic to the constant map.

(iii) Show that every principal bundle over S^3 is trivial (*Hint:* cf. problem 35, Chap. II.)

(iv) Apply problem 17.

28. Construct a fibre bundle over S^3 which is not the associated bundle of a principal bundle.

Hint: Proceed as follows:

(i) Construct a nontrivial bundle over S^4 with fibre S^3.

(ii) Pull this bundle back to a bundle $M \to S^3 \times S^1$ via a degree 1 map $S^3 \times S^1 \to S^4$.

(iii) Show that the induced projection $M \to S^3$ is the projection of the desired bundle.

29. Compact structure group. Suppose $\mathscr{P} = (P, \pi, B, G)$ is a principal bundle with compact group G. Let $\rho\colon A(P) \to A_I(P)$ denote the projection given by

$$\rho(\Phi) = \int_G T_a^*\Phi\, da, \qquad \Phi \in A(P).$$

Show that $\delta\rho\Phi = \rho\nabla\Phi$, $\Phi \in A(P)_{i=0}$, where ∇ is the covariant exterior derivative with respect to a principal connection.

Chapter VII

Linear Connections

In this chapter Γ denotes either $\mathbb{R}$ or $\mathbb{C}$.

$$\xi = (M, \pi, B, F), \qquad \eta = (N, \rho, B, H), \qquad \xi^i = (M^i, \pi^i, B, F^i)$$

and

$$\tilde{\xi} = (\tilde{M}, \tilde{\pi}, \tilde{B}, \tilde{F})$$

denote vector bundles over Γ, with $\dim B = n$, $\operatorname{rank} \xi = r$, and $\operatorname{rank} \eta = s$. All linear and multilinear operations (e.g., duality, sums, tensor products) are with respect to Γ unless otherwise specified. If Γ is taken as $\mathbb{C}$, then $\operatorname{Sec} \xi$, $\operatorname{Sec} \eta$,... are considered as modules over $\mathscr{S}(B; \mathbb{C})$ and multilinear operations are with respect to this ring. In particular, $\operatorname{Hom}_B(\operatorname{Sec} \xi; \operatorname{Sec} \eta)$ and $\operatorname{Sec} \xi \otimes_B \operatorname{Sec} \eta$ denote, respectively, the module of $\mathscr{S}(B; \mathbb{C})$-linear maps and the tensor product with respect to $\mathscr{S}(B; \mathbb{C})$.

§1. Bundle-valued differential forms

7.1. Real vector bundles. In this section $\Gamma = \mathbb{R}$. We shall generalize the notion of vector-valued differential forms on a manifold (cf. sec. 4.7, volume I or sec. 0.13) to differential forms with values in a vector bundle.

Consider the tangent bundle τ_B of B, and form the bundle $A^p(\tau_B\,;\,\xi)$ whose fibre at x consists of the skew-symmetric p-linear maps

$$T_x(B) \times \cdots \times T_x(B) \to F_x\,.$$

A *ξ-valued p-form, Ω, on B* is a cross-section in the bundle $A^p(\tau_B\,;\,\xi)$. It assigns (smoothly) to every point $x \in B$ a skew-symmetric p-linear map, $\Omega(x)$, of $T_x(B)$ into F_x. The ξ-valued p-forms on B form a module, $A^p(B; \xi)$, over the ring $\mathscr{S}(B)$,

$$A^p(B; \xi) = \text{Sec}\, A^p(\tau_B; \xi).$$

Thus $A^0(B; \xi) = \text{Sec}\, \xi$.

If ξ is the trivial bundle ($M = B \times F$), then a ξ-valued p-form is a p-form on B with values in the vector space F; i.e., $A^p(B; \xi) = A^p(B; F)$.

A ξ-valued p-form, Ω, on B determines the skew-symmetric p-linear map,

$$\tilde{\Omega}: \mathscr{X}(B) \times \cdots \times \mathscr{X}(B) \to \text{Sec}\, \xi,$$

given by

$$\tilde{\Omega}(X_1, ..., X_p)(x) = \Omega(x; X_1(x), ..., X_p(x)).$$

Recall from sec. 2.24, volume I, that the correspondence $\Omega \mapsto \tilde{\Omega}$ defines an isomorphism

$$A^p(B; \xi) \xrightarrow{\cong} A_B^p(\mathscr{X}(B); \text{Sec}\, \xi), \tag{7.1}$$

where $A_B^p(\mathscr{X}(B); \text{Sec}\, \xi)$ is the module of skew-symmetric p-linear (over $\mathscr{S}(B)$) maps from $\mathscr{X}(B)$ to $\text{Sec}\, \xi$. Henceforth we shall identify $\tilde{\Omega}$ with Ω via this isomorphism.

Now consider the direct sum

$$A(B; \xi) = \sum_{p=0}^{n} A^p(B; \xi).$$

$A(B; \xi)$ is a graded left module over the graded algebra $A(B)$, the module multiplication being given by

$$(\Phi \wedge \Omega)(x; h_1, ..., h_{p+q})$$
$$= \frac{1}{p!\, q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \Phi(x; h_{\sigma(1)}, ..., h_{\sigma(p)})\, \Omega(x; h_{\sigma(p+1)}, ..., h_{\sigma(p+q)})$$
$$\Phi \in A^p(B), \quad \Omega \in A^q(B; \xi), \quad x \in B, \quad h_i \in T_x(B).$$

It is immediate from sec. 2.24, volume I, that the $\mathscr{S}(B)$-bilinear map,

$$(\Phi, \sigma) \mapsto \Phi \wedge \sigma, \qquad \Phi \in A(B), \quad \sigma \in \text{Sec}\, \xi,$$

induces an isomorphism of graded $A(B)$-modules

$$A(B) \otimes_B \text{Sec}\, \xi \xrightarrow{\cong} A(B; \xi). \tag{7.2}$$

We shall identify these modules under this isomorphism and, when convenient, write $\Phi \otimes \sigma = \Phi \wedge \sigma$.

Example: *Whitney sum*: Suppose $\Omega_1 \in A^p(B; \xi)$ and $\Omega_2 \in A^p(B; \eta)$. Define $\Omega_1 \oplus \Omega_2 \in A^p(B; \xi \oplus \eta)$ by setting

$$(\Omega_1 \oplus \Omega_2)(x; h_1, ..., h_p) = \Omega_1(x; h_1, ..., h_p) \oplus \Omega_2(x; h_1, ..., h_p)$$

(recall that the fibre at x of $\xi \oplus \eta$ is the direct sum $F_x \oplus H_x$). The correspondence $(\Omega_1, \Omega_2) \mapsto \Omega_1 \oplus \Omega_2$ defines an isomorphism,

$$A(B; \xi) \oplus A(B; \eta) \xrightarrow{\cong} A(B; \xi \oplus \eta), \tag{7.3}$$

of graded $A(B)$-modules.

In particular, if $\xi \oplus \eta$ is trivial this becomes an isomorphism

$$A(B; \xi) \oplus A(B; \eta) \xrightarrow{\cong} A(B; \mathbb{R}^{r+s}).$$

Since η can always be chosen so that $\xi \oplus \eta$ is trivial (Theorem I, sec. 2.23, volume I), this shows that $A(B; \xi)$ can always be regarded as a complemented submodule of the $A(B)$-module $A(B; \mathbb{R}^m)$ for m sufficiently large.

To each vector field X on B corresponds an operator $i(X)$ in $A(B; \xi)$. It is given by

$$i(X)\sigma = 0, \qquad \sigma \in \operatorname{Sec} \xi,$$

and

$$(i(X)\Omega)(x; h_1, ..., h_{p-1}) = \Omega(x; X(x), h_1, ..., h_{p-1}), \qquad x \in B, \quad h_i \in T_x(B),$$
$$\Omega \in A^p(B; \xi), \quad p \geq 1.$$

$i(X)$ is called the *substitution operator*. It is $\mathscr{S}(B)$-linear and homogeneous of degree -1. Moreover, it satisfies the relations

$$i(X)(\Phi \wedge \Omega) = i(X)\Phi \wedge \Omega + (-1)^p\Phi \wedge i(X)\Omega,$$
$$(i(X)\Omega)(X_1, ..., X_{q-1}) = \Omega(X, X_1, ..., X_{q-1}),$$

and

$$i(X)(\Phi \otimes \sigma) = i(X)\Phi \otimes \sigma,$$

for $\Phi \in A^p(B)$, $\Omega \in A^q(B; \xi)$, $\sigma \in \operatorname{Sec} \xi$, and $X_i \in \mathscr{X}(B)$ $(i = 1, ..., p - 1)$ (cf. formulae (7.1) and (7.2)).

Finally note that if $i(X)\Omega = 0$ for a fixed $\Omega \in A^p(B; \xi)$ $(p \geq 1)$ and for every $X \in \mathscr{X}(B)$, then $\Omega = 0$.

7.2. Complex vector bundles. In this section $\Gamma = \mathbb{C}$; thus ξ and η, are complex vector bundles. Consider the vector bundle, $A^p(\tau_B\,;\,\xi)$, whose fibre at x is the space of p-linear (over $\mathbb{R}$) skew maps

$$T_x(B) \times \cdots \times T_x(B) \to F_x\,.$$

The complex structure of F_x makes this space into a complex vector space; consequently the bundle is a complex vector bundle. The cross-sections of this vector bundle are *called ξ-valued p-forms on B.* They form a complex vector space, denoted by $A^p(B;\,\xi)$, and the direct sum of these spaces is denoted by $A(B;\,\xi)$.

If ξ is a trivial bundle, then $A^p(B;\,\xi) = A^p(B;F)$; it is the space of F-valued forms on B. In particular,

$$A^p(B;\mathbb{C}) = \mathbb{C} \otimes A^p(B) = \mathrm{Sec}(\mathbb{C} \otimes \wedge^p \tau_B^*) = \mathrm{Sec}(\wedge^p(\mathbb{C} \otimes \tau_B)^*),$$

where $\wedge^p(\mathbb{C} \otimes \tau_B)^*$ denotes the pth complex exterior power of the complex dual of $\mathbb{C} \otimes \tau_B$. $A(B;\,\mathbb{C})$ is an algebra over $\mathbb{C}$, called the *algebra of complex differential forms on B.*

If ξ is any vector bundle, then $A(B;\,\xi)$ is a module over $A(B;\,\mathbb{C})$, and we adopt the same notation as in sec. 7.1. The isomorphisms (7.1), (7.2), and (7.3) have obvious analogues for complex bundles. Regarding the fibre of $A^p(\tau_B\,;\,\xi)$ as the space $A^p(T_x(B);\,F_x)$, we extend the definition of the substitution operator (cf. sec. 7.1) to complex bundles.

7.3. Bundle maps. We shall consider three types of maps between modules of bundle-valued differential forms, all of which arise from bundle maps. Fix a bundle map φ: $\xi \to \tilde{\xi}$ inducing ψ: $B \to \tilde{B}$.

Type I: The map φ^*. Let ξ^* and $\tilde{\xi}^*$ be vector bundles dual, respectively, to ξ and $\tilde{\xi}$. Let φ^*: Sec $\xi^* \leftarrow$ Sec $\tilde{\xi}^*$ be the linear map given by

$$(\varphi^*\sigma)(x) = \varphi_x^*(\sigma(\psi(x))), \qquad x \in B, \quad \sigma \in \mathrm{Sec}\,\tilde{\xi}^*,$$

(cf. sec. 2.15, volume I).

We extend φ^* to a linear map of bundle-valued forms,

$$\varphi^*\colon A(B;\,\xi^*) \leftarrow A(\tilde{B};\,\tilde{\xi}^*),$$

by setting

$$(\varphi^*\Omega)(x;\,h_1\,,\,\dots,\,h_p) = \varphi_x^*(\Omega(\psi(x);\,(d\psi)h_1\,,\,\dots,\,(d\psi)h_p)),$$
$$x \in B, \quad h_i \in T_x(B), \quad \Omega \in A^p(\tilde{B};\,\tilde{\xi}^*).$$

Then

$$\varphi^*(\Phi \wedge \Omega) = \psi^*\Phi \wedge \varphi^*\Omega, \qquad \Phi \in A(\tilde{B}), \quad \Omega \in A(\tilde{B}; \tilde{\xi}^*).$$

Type II: *The map* $\varphi^{\#}$. Suppose that φ restricts to linear isomorphisms in each fibre. Recall that a linear map $\varphi^{\#}$: Sec $\xi \leftarrow$ Sec $\tilde{\xi}$ is given by

$$(\varphi^{\#}(\sigma))(x) = \varphi_x^{-1}\sigma(\psi(x)), \qquad x \in B, \quad \sigma \in \operatorname{Sec} \tilde{\xi}$$

(cf. sec. 2.15, volume I).

We extend $\varphi^{\#}$ to a linear map $A(B; \xi) \leftarrow A(\tilde{B}; \tilde{\xi})$ by setting

$$(\varphi^{\#}\Omega)(x; h_1, ..., h_p) = \varphi_x^{-1}(\Omega(\psi(x); (d\psi)h_1, ..., (d\psi)h_p)),$$
$$x \in B, \quad h_i \in T_x(B), \quad \Omega \in A^p(\tilde{B}; \tilde{\xi}), \quad p \geq 1.$$

Then

$$\varphi^{\#}(\Phi \wedge \Omega) = \psi^*\Phi \wedge \varphi^{\#}\Omega, \qquad \Phi \in A(\tilde{B}), \quad \Omega \in A(\tilde{B}; \tilde{\xi}).$$

The map ψ^* makes $A(B)$ into a module over $\mathscr{S}(\tilde{B})$, the module multiplication being given by

$$\Phi \cdot f = \Phi \cdot \psi^* f, \qquad f \in \mathscr{S}(\tilde{B}), \quad \Phi \in A(B),$$

where the scalar has been written on the right for notational convenience. Thus we can form the tensor product $A(B) \otimes_{\tilde{B}} \operatorname{Sec} \tilde{\xi}$.

Proposition I: With the hypotheses and notation above, an isomorphism of graded $A(B)$ modules,

$$A(B) \otimes_{\tilde{B}} \operatorname{Sec} \tilde{\xi} \xrightarrow{\cong} A(B; \xi),$$

is given by

$$\Phi \otimes \sigma \mapsto \Phi \wedge \varphi^{\#}\sigma, \qquad \Phi \in A(B), \quad \sigma \in \operatorname{Sec} \tilde{\xi}.$$

Proof: In view of the isomorphism (7.2) this is immediate from the isomorphism $\mathscr{S}(B) \otimes_{\tilde{B}} \operatorname{Sec} \tilde{\xi} \cong \operatorname{Sec} \xi$ of sec. 2.26, volume I. (Use the associative law for tensor products.)

Q.E.D.

Example: Let ξ_U be the restriction of ξ to U (open in B). The inclusion, j: $\xi_U \to \xi$, induces a *restriction map*

$$j^{\#}: A(U; \xi_U) \leftarrow A(B; \xi).$$

Identifying the fibres of ξ and ξ_U, we can write

$$(j^{\#}\Omega)(x) = \Omega(x), \qquad x \in U.$$

Type III: *The map* φ_*. Finally, assume that the vector bundles ξ and $\tilde{\xi}$ have the same base B, and assume that φ is a strong bundle map ($\psi = \iota$). Then a homomorphism of $A(B)$-modules,

$$\varphi_*: A(B; \xi) \to A(B; \tilde{\xi}),$$

is given by

$$(\varphi_*\Omega)(x; h_1, \ldots, h_p) = \varphi_x(\Omega(x; h_1, \ldots, h_p)),$$
$$x \in B, \quad h_i \in T_x(B), \quad \Omega \in A^p(B; \xi), \quad p \geqslant 1,$$

and

$$(\varphi_*\sigma)(x) = \varphi_x(\sigma(x)), \qquad x \in B, \quad \sigma \in \operatorname{Sec} \xi.$$

The restriction φ_*: $\operatorname{Sec} \xi \to \operatorname{Sec} \tilde{\xi}$ of φ_* coincides with the map defined in sec. 2.15, volume I. On the other hand, if ξ and $\tilde{\xi}$ are trivial bundles, and φ is given by

$$\varphi(x, a) = (x, \alpha(a)), \qquad x \in B, \quad a \in F,$$

for some fixed linear map $\alpha: F \to \tilde{F}$, then φ_* coincides with the map

$$\alpha_*: A(B; F) \to A(B; \tilde{F})$$

of sec. 4.7, volume I. Note that, if $\varphi: \xi \xrightarrow{\cong} \tilde{\xi}$ is a strong bundle isomorphism, $\varphi^{\#} = (\varphi_*)^{-1}$.

7.4. Multilinear bundle maps. Recall that a multilinear bundle map $\alpha \in \operatorname{Hom}(\xi^1, \ldots, \xi^m; \eta)$ assigns to each $x \in B$ an m-linear map,

$$\alpha_x : F_x^1 \times \cdots \times F_x^m \to H_x,$$

in a smooth way (cf. sec. 2.4, volume I). Such an α determines a map,

$$\alpha_*: A(B; \xi^1) \times \cdots \times A(B; \xi^m) \to A(B; \eta),$$

as follows: Let $\Omega_i \in A^{p_i}(B; \xi^i)$. Then $\alpha_*(\Omega_1, \ldots, \Omega_m)$ has degree $p = p_1 + \cdots + p_m$ and is given by

$$\alpha_*(\Omega_1, \ldots, \Omega_m)(x; h_1, \ldots, h_p)$$
$$= \frac{1}{p_1! \cdots p_m!} \sum_{\sigma \in S^p} \epsilon_\sigma \alpha_x(\Omega_1(x; h_{\sigma(1)}, \ldots), \ldots, \Omega_m(x; \ldots, h_{\sigma(p)})).$$

With the identification, (7.2), α_* is given by

$$\alpha_*(\Phi_1 \otimes \sigma_1, \ldots, \Phi_m \otimes \sigma_m) = (\Phi_1 \wedge \cdots \wedge \Phi_m) \otimes \alpha_*(\sigma_1, \ldots, \sigma_m). \tag{7.4}$$

Note as well that, if $\Omega_i \in A^{p_i}(B; \xi^i)$ and $X \in \mathscr{X}(B)$, then

$$i(X)\, \alpha_*(\Omega_1, \ldots, \Omega_m) = \sum_{i=1}^{m} (-1)^{q_i}\, \alpha_*(\Omega_1, \ldots, i(X)\Omega_i, \ldots, \Omega_m), \tag{7.5}$$

where $q_i = p_1 + \cdots + p_{i-1}$. Moreover, if, in addition, $\Phi \in A^q(B)$, then

$$\alpha_*(\Omega_1, \ldots, \Phi \wedge \Omega_i, \ldots, \Omega_m) = (-1)^{q_i q}\, \Phi \wedge \alpha_*(\Omega_1, \ldots, \Omega_m). \tag{7.6}$$

§2. Examples

In this article we consider examples of multilinear maps

$$\alpha \in \operatorname{Hom}(\xi^1, \ldots, \xi^m; \eta).$$

7.5. Dual bundles. Suppose ξ^* is dual to ξ and let

$$\alpha \in \operatorname{Hom}(\xi^*, \xi; B \times \Gamma)$$

be the scalar product $\langle\ ,\ \rangle$. We write,

$$\alpha_*(\Omega^*, \Omega) = \boldsymbol{\langle} \Omega^*, \Omega \boldsymbol{\rangle}, \qquad \Omega^* \in A(B; \xi^*), \quad \Omega \in A(B; \xi);$$

thus $\boldsymbol{\langle} \Omega^*, \Omega \boldsymbol{\rangle} \in A(B; \Gamma)$. Note that $\boldsymbol{\langle}\ ,\ \boldsymbol{\rangle}$ extends the scalar product $\langle\ ,\ \rangle$ between Sec ξ^* and Sec ξ.

Lemma I: Let $\varphi: \xi \to \tilde{\xi}$ be a bundle map inducing $\psi: B \to \tilde{B}$ and restricting to isomorphisms in the fibres. Assume ξ^*, $\tilde{\xi}^*$ are dual to ξ, $\tilde{\xi}$. Then

$$\boldsymbol{\langle} \varphi^*\Omega^*, \varphi^\# \Omega \boldsymbol{\rangle} = \psi^* \boldsymbol{\langle} \Omega^*, \Omega \boldsymbol{\rangle}, \qquad \Omega^* \in A(\tilde{B}; \tilde{\xi}^*), \quad \Omega \in A(\tilde{B}; \tilde{\xi}).$$

Proof: In view of the isomorphism (7.2) it is sufficient to consider the case $\Omega^* = \Phi \wedge \sigma^*$, $\Omega = \Psi \wedge \sigma$, where $\Phi, \Psi \in A(\tilde{B})$ and $\sigma^* \in \operatorname{Sec} \tilde{\xi}^*$, $\sigma \in \operatorname{Sec} \tilde{\xi}$. But in view of the relations of sec. 7.3, and formula (7.4) the lemma follows, in this case, from the relation

$$\begin{aligned} \boldsymbol{\langle} \varphi^*\sigma^*, \varphi^\#\sigma \boldsymbol{\rangle}(x) &= \langle \varphi_x^*(\sigma^*(\psi(x))), \varphi_x^{-1}(\sigma(\psi(x)))\rangle \\ &= \langle \sigma^*(\psi(x)), \sigma(\psi(x))\rangle \\ &= (\psi^* \boldsymbol{\langle} \sigma^*, \sigma \boldsymbol{\rangle})(x), \qquad x \in B. \end{aligned}$$

Q.E.D.

A similar proof establishes

Lemma II: Let $\varphi: \xi \to \eta$ be a strong bundle map. Then

$$\boldsymbol{\langle} \varphi^*\Omega^*, \Omega \boldsymbol{\rangle} = \boldsymbol{\langle} \Omega^*, \varphi_*\Omega \boldsymbol{\rangle}, \qquad \Omega^* \in A(B; \eta^*), \quad \Omega \in A(B; \xi).$$

7.6. Bilinear maps and algebras. Consider a bilinear map

$$\alpha \in \mathrm{Hom}(\xi, \xi; \xi).$$

If α is symmetric, then

$$\alpha_*(\Omega_1, \Omega_2) = (-1)^{pq}\alpha_*(\Omega_2, \Omega_1), \qquad \Omega_1 \in A^p(B; \xi), \quad \Omega_2 \in A^q(B; \xi),$$

while if α is skew-symmetric, then

$$\alpha_*(\Omega_1, \Omega_2) = (-1)^{pq+1}\alpha_*(\Omega_2, \Omega_1).$$

Next, observe that the map $\alpha \in \mathrm{Hom}(\xi, \xi; \xi)$ makes each fibre F_x into an algebra, while α_* makes $A(B; \xi)$ into an algebra. We write

$$\alpha_*(\Omega_1, \Omega_2) = \Omega_1 \cdot \Omega_2, \quad \Omega_i \in A(B; \xi),$$

for the algebra multiplication. If the algebras F_x are all associative, so is $A(B; \xi)$,

$$\Omega_1 \cdot (\Omega_2 \cdot \Omega_3) = (\Omega_1 \cdot \Omega_2) \cdot \Omega_3 .$$

In this case we define the *pth power* of Ω ($\Omega \in A(B; \xi)$) to be the bundle-valued form $\Omega^p \in A(B; \xi)$ given by

$$\Omega^p = \underbrace{\Omega \cdot \cdots \cdot \Omega}_{(p\,\text{factors})}, \qquad p \geqslant 1,$$

and

$$\Omega^0 = 1.$$

Bundle-valued forms $\Omega_1 \in A^p(B; \xi)$ and $\Omega_2 \in A^q(B; \xi)$ will be said to *commute* (with respect to the map α_*) if, for each $x \in B$ and $h_i \in T_x(B)$, the vectors $\Omega_1(x; h_1, \ldots, h_p)$ and $\Omega_2(x; h_{p+1}, \ldots, h_{p+q})$ commute in the algebra F_x. If Ω_1 and Ω_2 commute, then

$$\Omega_1 \cdot \Omega_2 = (-1)^{pq}\Omega_2 \cdot \Omega_1 .$$

Thus if in addition, p or q is even, then $\Omega_1 \cdot \Omega_2 = \Omega_2 \cdot \Omega_1$ and so the binomial formula,

$$(\Omega_1 + \Omega_2)^k = \sum_{i+j=k} \binom{k}{i} \Omega_1^i \cdot \Omega_2^j,$$

holds.

7.7. The bundle L_ξ. Recall (sec. 2.10, volume I) that L_ξ is the vector bundle whose fibre at x is the space of linear transformations of F_x. Evaluation and composition are the bilinear maps,

$$\epsilon \in \mathrm{Hom}(L_\xi, \xi; \xi) \qquad \text{and} \qquad \circ \in \mathrm{Hom}(L_\xi, L_\xi; L_\xi),$$

given by

$$\epsilon(\alpha, v) = \alpha(v) \qquad \text{and} \qquad \circ(\alpha, \beta) = \alpha \circ \beta, \qquad \alpha, \beta \in L_{F_x}, \quad v \in F_x, \quad x \in B$$

(cf. sec. 2.10, volume I).

The induced maps of bundle-valued forms are denoted by

$$\epsilon_*(\Omega, \Phi) = \Omega(\Phi) \qquad \text{and} \qquad \circ_*(\Omega_1, \Omega_2) = \Omega_1 \circ \Omega_2,$$

Ω, Ω_1, $\Omega_2 \in A(B; L_\xi)$, $\Phi \in A(B; \xi)$. They satisfy

$$(\Omega_1 \circ \Omega_2)(\Phi) = \Omega_1(\Omega_2(\Phi)).$$

In particular, each $\Omega \in A(B; L_\xi)$ determines the $\mathscr{S}(B)$-linear map, $\hat{\Omega}$: Sec $\xi \to A(B; \xi)$, given by

$$\hat{\Omega}(\sigma) = \Omega(\sigma).$$

This correspondence defines an isomorphism

$$A(B; L_\xi) \xrightarrow{\cong} \mathrm{Hom}_B(\mathrm{Sec}\,\xi; A(B; \xi)).$$

Let φ: $\xi \to \tilde{\xi}$ induce ψ: $B \to \tilde{B}$ and restrict to isomorphisms in the fibres. Then a bundle map $\hat{\varphi}$: $L_\xi \to L_{\tilde{\xi}}$ is given by

$$\hat{\varphi}_x(\alpha) = \varphi_x \circ \alpha \circ \varphi_x^{-1}, \qquad \alpha \in L_{F_x}, \quad x \in B,$$

and $\hat{\varphi}$ restricts to isomorphisms in the fibres.

Lemma III: With the hypotheses and notation above

$$\varphi^*(\Omega(\Phi)) = (\hat{\varphi}^*\Omega)(\varphi^*\Phi), \qquad \hat{\varphi}^*(\Omega_1 \circ \Omega_2) = \hat{\varphi}^*\Omega_1 \circ \hat{\varphi}^*\Omega_2,$$

and

$$\hat{\varphi}^*(\iota_{\tilde{\xi}}) = \iota_\xi, \qquad \Omega, \Omega_1, \Omega_2 \in A(\tilde{B}; L_{\tilde{\xi}}), \quad \Phi \in A(\tilde{B}; \tilde{\xi})$$

(ι_ξ and $\iota_{\tilde{\xi}}$ are the cross-sections in L_ξ and $L_{\tilde{\xi}}$ assigning to each x and $\tilde{x}$ the identity transformation in F_x and $\tilde{F}_{\tilde{x}}$).

Finally, define tr $\in$ Sec L_ξ^* by

$$\langle \mathrm{tr}, \varphi_x \rangle = \mathrm{tr}\, \varphi_x, \qquad \varphi_x \in L_{F_x}, \quad x \in B.$$

This cross-section determines the linear map,

$$\mathbf{tr}: A(B; L_\xi) \to A(B; \Gamma),$$

given by

$$(\mathbf{tr}\Phi)(x; h_1, \ldots, h_p) = \mathrm{tr}(\Phi(x; h_1, \ldots, h_p)), \quad \Phi \in A^p(B; L_\xi), \; x \in B, \; h_i \in T_x(B).$$

7.8. Tensor products. Let $\alpha \in \mathrm{Hom}(\xi^1, \ldots, \xi^m; \xi^1 \otimes \cdots \otimes \xi^m)$ be the tensor product,

$$\alpha_x(v^1, \ldots, v^m) = v^1 \otimes \cdots \otimes v^m, \qquad v^i \in F_x^i, \quad x \in B.$$

If $\Omega_i \in A(B; \xi^i)$, then $\alpha_*(\Omega_1, \ldots, \Omega_m) \in A(B; \xi^1 \otimes \cdots \otimes \xi^m)$; we write

$$\alpha_*(\Omega_1, \ldots, \Omega_m) = \Omega_1 \circledast \cdots \circledast \Omega_m.$$

Note that this extends the tensor product between cross-sections (cf. sec. 2.24, volume I). Note also that $\circledast$ is *not* a tensor product unless the Ω_i's are of degree zero. (Use the isomorphism 7.2.)

Now recall that $\otimes^m \xi^*$ and $\otimes^m \xi$ are dual with respect to the scalar product given by

$$\langle v_1^* \otimes \cdots \otimes v_m^*, v_1 \otimes \cdots \otimes v_m \rangle = \langle v_1^*, v_1 \rangle \cdots \langle v_m^*, v_m \rangle.$$

Thus, if $\Phi \in A(B; \otimes^p \xi^*)$ and $\Omega_i \in A(B; \xi)$, $i = 1, \ldots, p$, we obtain the ordinary differential form $\boldsymbol{\langle} \Phi, \Omega_1 \circledast \cdots \circledast \Omega_p \boldsymbol{\rangle}$. This will often be denoted by $\Phi(\Omega_1, \ldots, \Omega_p)$.

Lemma IV: Let $\Phi_1 \in \mathrm{Sec} \otimes^p \xi^*$, $\Phi_2 \in \mathrm{Sec} \otimes^q \xi^*$ and let $\Omega_i \in A(B; \xi)$ $(i = 1, \ldots, p + q)$. Then

$$\begin{aligned} &\boldsymbol{\langle} \Phi_1 \otimes \Phi_2, \Omega_1 \circledast \cdots \circledast \Omega_{p+q} \boldsymbol{\rangle} \\ &\quad = \boldsymbol{\langle} \Phi_1, \Omega_1 \circledast \cdots \circledast \Omega_p \boldsymbol{\rangle} \wedge \boldsymbol{\langle} \Phi_2, \Omega_{p+1} \circledast \cdots \circledast \Omega_{p+q} \boldsymbol{\rangle}. \end{aligned}$$

Proof: The isomorphism (7.2) and formula (7.4) allow us to reduce to the case that $\Omega_i \in \mathrm{Sec}\, \xi$. But in this case the lemma is obvious.

Q.E.D.

Let $\xi^{p,q}$ denote the tensor product $\xi^* \otimes \cdots \otimes \xi^* \otimes \xi \otimes \cdots \otimes \xi$ (p factors ξ^*, q factors ξ). We write

$$A(B; \otimes \xi^*) = \textstyle\bigoplus_{p=0}^{\infty} A(B; \xi^{p,0}); \qquad A(B; \otimes \xi) = \textstyle\bigoplus_{q=0}^{\infty} A(B; \xi^{0,q})$$

and

$$A(B; (\otimes \xi^*) \otimes (\otimes \xi)) = \textstyle\bigoplus_{p,q} A(B; \xi^{p,q}).$$

These are associative algebras with multiplication $\circledast$ induced by the bilinear maps

$$\otimes : \xi^{p,q} \times \xi^{r,s} \to \xi^{p+r,q+s}.$$

The identity element is the scalar 1, and $A(B) = A(B) \otimes 1$ is a subalgebra of each. Moreover, these algebras contain (respectively) the subalgebras

$$\mathrm{Sec} \bigotimes \xi^* = \bigoplus_p \mathrm{Sec}\, \xi^{p,0}, \qquad \mathrm{Sec} \bigotimes \xi = \bigoplus_q \mathrm{Sec}\, \xi^{0,q}$$

and

$$\mathrm{Sec}((\bigotimes \xi^*) \otimes (\bigotimes \xi)) = \bigoplus_{p,q} \mathrm{Sec}\, \xi^{p,q}.$$

Next, define a strong bundle map $\theta: L_\xi \to L_{\xi^{p,q}}$ by setting, for $\alpha \in L_{F_x}$, $z_i^* \in F_x^*$, $z_j \in F_x$, $x \in B$,

$$\begin{aligned}
&\theta(\alpha)(z_1^* \otimes \cdots \otimes z_p^* \otimes z_1 \otimes \cdots \otimes z_q) \\
&\quad = -\sum_{i=1}^{p} (z_1^* \otimes \cdots \otimes \alpha^*(z_i^*) \otimes \cdots \otimes z_p^* \otimes z_1 \otimes \cdots \otimes z_q) \\
&\qquad + \sum_{j=1}^{q} (z_1^* \otimes \cdots \otimes z_p^* \otimes z_1 \otimes \cdots \otimes \alpha(z_j) \otimes \cdots \otimes z_q),
\end{aligned}$$

and $\theta(\alpha)(1) = 0$.

The map, θ, induces a map $\theta_*: A(B; L_\xi) \to A(B; L_{\xi^{p,q}})$.

7.9. The bundle $\wedge\xi$. Recall that $\wedge \xi$ is the bundle with fibre $\wedge F_x$ at x. The exterior multiplication in the fibres determines an associative multiplication in $A(B; \wedge \xi)$ (cf. sec. 7.6), which we denote by

$$(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2, \qquad \Omega_i \in A(B; \wedge \xi).$$

We write $A(B; \wedge \xi) = \sum_{p,q} A^p(B; \wedge^q \xi)$; this makes it into a bigraded algebra, the 1-element of which is the constant function 1. The isomorphism (7.2) reads

$$A(B; \wedge \xi) \cong A(B) \otimes_B \mathrm{Sec} \wedge \xi \cong A(B) \otimes_B \wedge_B (\mathrm{Sec}\, \xi)$$

and under this isomorphism the multiplication is given by

$$(\Phi \otimes \sigma) \wedge (\Psi \otimes \tau) = (\Phi \wedge \Psi) \otimes (\sigma \wedge \tau), \qquad \Phi, \Psi \in A(B), \quad \sigma, \tau \in \mathrm{Sec} \wedge \xi.$$

In particular $A(B; \wedge \xi)$ is the canonical (*not* the anticommutative) tensor product of the algebras $A(B)$ and $\mathrm{Sec} \wedge \xi$. Observe as well that

$$A(B) = A(B) \otimes 1 \qquad \text{and} \qquad \mathrm{Sec} \wedge \xi = 1 \otimes \mathrm{Sec} \wedge \xi$$

are subalgebras of $A(B; \wedge \xi)$.

Next note that the skew-symmetry of the multiplication in $\wedge\xi$ implies that

$$\Omega_1 \wedge \Omega_2 = (-1)^{p_1p_2+q_1q_2}\Omega_2 \wedge \Omega_1\,, \qquad \Omega_i \in A^{p_i}(B;\, \wedge^{q_i}\xi), \quad i = 1,2.$$

Thus, if $\Omega \in A^p(B;\, \wedge^q\xi)$ and $p + q$ is odd, then

$$\Omega \wedge \Omega = 0.$$

Moreover, since $\wedge\xi$ is associative, we can form the kth power of $\Omega \in A(B;\, \wedge\xi)$ (cf. sec. 7.6). If $\Omega \in A^1(B;\, \xi)$, then

$$\frac{1}{k!}\,\Omega^k(x;\, h_1\,, \ldots, h_k) = \Omega(x;\, h_1) \wedge \cdots \wedge \Omega(x;\, h_k).$$

Now define (as in sec. 7.8) $\theta : L_\xi \to L_{\wedge\xi}$ by setting $\theta(\alpha)(1) = 0$ and

$$\theta(\alpha)(z_1 \wedge \cdots \wedge z_p) = \sum_{i=1}^{p} z_1 \wedge \cdots \wedge \alpha(z_i) \wedge \cdots \wedge z_p\,,$$
$$\alpha \in L_{F_x}\,, \quad z_i \in F_x\,, \quad x \in B, \quad p \geqslant 1.$$

Let $\theta_*\colon A(B;\, L_\xi) \to A(B;\, L_{\wedge\xi})$ be the induced map and note that we use the same notation as in sec. 7.8.

Finally, let $\sigma^* \in \operatorname{Sec} \xi^*$ (ξ^*, a bundle dual to ξ). Define an operator $i_\xi(\sigma^*)$ in $A(B;\, \wedge\xi)$ by setting

$$(i_\xi(\sigma^*)\Omega)(x;\, h_1\,, \ldots, h_p) = i(\sigma^*(x))(\Omega(x;\, h_1\,, \ldots, h_p)),$$
$$x \in B, \quad h_i \in T_x(B), \quad \Omega \in A^p(B;\, \wedge\xi),$$

where $i(\sigma^*(x))$ is the substitution operator in $\wedge F_x$. $i_\xi(\sigma^*)$ is homogeneous of bidegree $(0, -1)$ and satisfies

$$i_\xi(\sigma^*)(\Omega_1 \wedge \Omega_2) = i_\xi(\sigma^*)\,\Omega_1 \wedge \Omega_2 + (-1)^p\,\Omega_1 \wedge i_\xi(\sigma^*)\Omega_2\,,$$
$$\Omega_1 \in A(B;\, \wedge^p\xi), \quad \Omega_2 \in A(B;\, \wedge\xi).$$

This formula yields, for $\sigma \in \operatorname{Sec} \xi$, $\Omega \in A(B;\, \wedge\xi)$,

$$i_\xi(\sigma^*)(\sigma \wedge \Omega) = \langle\sigma^*, \sigma\rangle\, \Omega - \sigma \wedge i_\xi(\sigma^*)\Omega.$$

In particular, if $\Omega \in A(B;\, \wedge^r\xi)$ ($r = \operatorname{rank} \xi$), then $\sigma \wedge \Omega = 0$, and the formula above reduces to

$$\langle\sigma^*, \sigma\rangle\Omega = \sigma \wedge i_\xi(\sigma^*)\Omega.$$

7.10. The bundles $\vee^p\xi$. Recall that $\vee^p\xi$ is the vector bundle whose fibre at x is the pth symmetric power $\vee^p F_x$, of F_x (cf. sec. 2.12, volume I). The multiplication maps,

$$\vee^p\xi \times \vee^q\xi \to \vee^{p+q}\xi,$$

determine maps

$$A(B; \vee^p\xi) \times A(B; \vee^q\xi) \to A(B; \vee^{p+q}\xi),$$

which we denote by

$$(\Omega_1, \Omega_2) \mapsto \Omega_1 \vee \Omega_2 .$$

The direct sum of the spaces $A(B; \vee^p\xi)$ is denoted by

$$A(B; \vee\xi) = \bigoplus_{p=0}^{\infty} A(B; \vee^p\xi)$$

and the maps above make it into an associative algebra. The strong bundle maps π_S: $\otimes^p\xi \to \vee^p\xi$ given by

$$\pi_S(z_1 \otimes \cdots \otimes z_p) = z_1 \vee \cdots \vee z_p, \qquad z_i \in F_x, \quad x \in B,$$

determine a homomorphism

$$(\pi_S)_*: A(B; \otimes\,\xi) \to A(B; \vee\xi),$$

of bigraded algebras.

Finally, note that

$$\Omega_1 \vee \Omega_2 = (-1)^{p_1p_2}\Omega_2 \vee \Omega_1, \qquad \Omega_i \in A^{p_i}(B; \vee\xi), \quad i = 1, 2,$$

(cf. sec. 7.8).

§3. Linear connections

7.11. Definition: A *linear connection* in the vector bundle, ξ, is a Γ-linear map,

$$\nabla\colon \operatorname{Sec}\xi \to A^1(B;\xi),$$

which satisfies the condition

$$\nabla(f\cdot\sigma) = \delta f \wedge \sigma + f\cdot\nabla\sigma, \qquad f\in\mathscr{S}(B),\quad \sigma\in\operatorname{Sec}\xi.$$

If ξ is complex, ∇ is sometimes called a *complex linear connection.*

A cross-section σ is called *parallel with respect to* ∇ if $\nabla\sigma = 0$.

Let ∇ be a linear connection in ξ. Then, for each vector field X on B, an operator, ∇_X, in $\operatorname{Sec}\xi$ is given by $\nabla_X = i(X)\circ\nabla$. These operators satisfy the relations

$$\nabla_X(f\cdot\sigma) = X(f)\cdot\sigma + f\cdot\nabla_X\sigma, \qquad \sigma\in\operatorname{Sec}\xi,\quad f\in\mathscr{S}(B),$$

and

$$\nabla_{f\cdot X}(\sigma) = f\cdot\nabla_X\sigma.$$

Similarly, if $h\in T_x(B)$, we can form the operator $\nabla_h\colon \operatorname{Sec}\xi\to F_x$ given by

$$\nabla_h\sigma = i(h)((\nabla\sigma)(x)).$$

Clearly, $(\nabla_X\sigma)(x) = \nabla_{X(x)}(\sigma)$.

Examples: 1. Suppose ξ is trivial: $M = B\times F$. Then (cf. sec. 7.1) $A(B;\xi) = A(B;F)$. In this case the *exterior derivative*,

$$\delta\colon \mathscr{S}(B;F)\to A^1(B;F),$$

is a linear connection in ξ. It is called the *standard connection.*

2. Let ∇_1 be a linear connection in ξ and suppose $\Psi\in A^1(B;L_\xi)$. Then the map, $\operatorname{Sec}\xi\to A^1(B;\xi)$, given by

$$\sigma\mapsto \nabla_1(\sigma) + \Psi(\sigma)$$

is again a linear connection in ξ. Conversely, if ∇_2 is a second linear connection in ξ, then the map,

$$\nabla_1 - \nabla_2 : \operatorname{Sec}\xi\to A^1(B;\xi),$$

is $\mathscr{S}(B)$-linear. Hence there is a unique $\Psi \in A^1(B; L_\xi)$ such that

$$\nabla_1\sigma - \nabla_2\sigma = \Psi(\sigma), \qquad \sigma \in \operatorname{Sec} \xi$$

(cf. sec. 7.7).

3. Let $\{U_\alpha\}$ be a locally finite open cover of B, and assume that $\{p_\alpha\}$ are smooth functions on B such that carr $p_\alpha \subset U_\alpha$, and $\sum_\alpha p_\alpha = 1$. Let ξ_α be the restriction of ξ to U_α.

Assume that ∇_α is a linear connection in ξ_α. Define a Γ-linear map, ∇: Sec $\xi \to A^1(B; \xi)$, by setting

$$\nabla\sigma = \sum_\alpha p_\alpha \cdot \nabla_\alpha \sigma_\alpha ,$$

where σ_α is the restriction of σ to U_α. Then ∇ is a linear connection in ξ.

4. Assume $\xi \oplus \eta = B \times \Gamma^q$. Then we have the strong bundle maps

$$i: \xi \to B \times \Gamma^q \quad \text{(inclusion)} \qquad \text{and} \qquad \rho: B \times \Gamma^q \to \xi \quad \text{(projection)}.$$

A linear connection ∇ in ξ is given by

$$\nabla\sigma = \rho_* \, \delta i_*(\sigma), \qquad \sigma \in \operatorname{Sec} \xi.$$

5. *Tangent bundle*: A *linear connection in a manifold B* is a linear connection, ∇, in the tangent bundle, τ_B. Given such a connection we define a map S: $\mathscr{X}(B) \times \mathscr{X}(B) \to \mathscr{X}(B)$ by setting

$$S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

S is a τ_B-valued 2-form: $S \in A^2(B; \tau_B)$. It is called the *torsion of* ∇.

S determines the 1-form $\Psi \in A^1(B; L_{\tau_B})$ given by

$$\Psi(X)(Y) = S(X, Y), \qquad X, Y \in \mathscr{X}(B).$$

By Example 2 above, $\tilde{\nabla} = \nabla - \Psi$ is again a linear connection in B. $\tilde{\nabla}$ is called the *conjugate connection* of ∇ and satisfies

$$\tilde{\nabla}_X Y = \nabla_X Y - S(X, Y).$$

It follows that the torsions for ∇ and $\tilde{\nabla}$ are related by $S = -\tilde{S}$.

Proposition II: Every vector bundle ξ admits a linear connection.

Proof: Example 1, above, shows that trivial bundles admit linear connections. In view of Example 3, a partition of unity argument now shows that every ξ admits a connection.

Example 4, above, provides a second construction of a linear connection.

We give a third, purely algebraic proof. Let

$$t_\xi = \sum_{i=1}^{m} \sigma_i^* \otimes \sigma_i \qquad (\sigma_i{}^* \in \operatorname{Sec} \xi^*,\ \sigma_i \in \operatorname{Sec} \xi)$$

be a representation of the unit tensor for ξ (recall, from p. 81, volume I, that $t_\xi \in \operatorname{Sec}(\xi^* \otimes \xi)$ corresponds to $\iota_\xi \in \operatorname{Sec} L_\xi$ under the isomorphism $\xi^* \otimes \xi \cong L_\xi$). Define a Γ-linear map $\nabla\colon \operatorname{Sec} \xi \to A^1(B; \xi)$ by setting

$$\nabla\sigma = \sum_{i=1}^{m} \delta\langle\sigma_i^*, \sigma\rangle \wedge \sigma_i\,.$$

(If $\Gamma = \mathbb{C}$ recall, from sec. 7.2, that $A(B; \xi)$ is a module over $A(B; \mathbb{C})$).

The relation (p. 81, volume I),

$$\sum_{i=1}^{m} \langle\sigma_i^*, \sigma\rangle\sigma_i = \sigma, \qquad \sigma \in \operatorname{Sec} \xi,$$

implies that ∇ is a linear connection. Note that ∇ depends on the particular representation of t.

Q.E.D.

7.12. Induced connections. In this section ∇ and ∇_ξ are linear connections in ξ, ∇_η is a linear connection in η and ∇^i is a linear connection in ξ^i. These connections determine linear connections in the associated linear and multilinear bundles, as described in the examples below.

Examples: 1. *Dual bundles*: Let ξ^* be dual to ξ. Then there is a unique linear connection, ∇^*, in ξ^* such that

$$\langle\nabla^*\sigma^*, \sigma\rangle + \langle\sigma^*, \nabla\sigma\rangle = \delta\langle\sigma^*, \sigma\rangle, \qquad \sigma^* \in \operatorname{Sec} \xi^*, \quad \sigma \in \operatorname{Sec} \xi.$$

∇^* will be called the *dual* of ∇.

In fact, fix $\sigma^* \in \operatorname{Sec} \xi^*$. A simple computation shows that the map $\operatorname{Sec} \xi \to A^1(B)$ given by

$$\sigma \mapsto \delta\langle\sigma^*, \sigma\rangle - \langle\sigma^*, \nabla\sigma\rangle$$

is $\mathscr{S}(B; \Gamma)$-linear. Hence there is a unique element, $\nabla^*\sigma^* \in A^1(B; \xi^*)$, such that

$$\langle \nabla^*\sigma^*, \sigma \rangle = \delta\langle \sigma^*, \sigma \rangle - \langle \sigma^*, \nabla\sigma \rangle.$$

Evidently the assignment $\sigma^* \mapsto \nabla^*\sigma^*$ defines a linear connection, ∇^*, in ξ^*, which is the only one satisfying the desired relation.

2. *The bundle L_ξ*: Fix $\alpha \in \operatorname{Sec} L_\xi$. The operator,

$$(\nabla \circ \alpha - \alpha \circ \nabla)\colon \quad \operatorname{Sec} \xi \to A^1(B; \xi),$$

is linear over $\mathscr{S}(B; \Gamma)$. Via the isomorphism,

$$A(B; L_\xi) \cong \operatorname{Hom}_B(\operatorname{Sec} \xi; A(B; \xi))$$

(cf. sec. 7.7), it determines an element $\hat{\nabla}\alpha$ in $A^1(B; L_\xi)$. Evidently $\alpha \mapsto \hat{\nabla}\alpha$ defines a linear connection, $\hat{\nabla}$, in L_ξ. $\hat{\nabla}$ is called the *connection induced by* ∇. It satisfies the relations,

$$\hat{\nabla}(\alpha \circ \beta) = \hat{\nabla}\alpha \circ \beta + \alpha \circ \hat{\nabla}\beta, \qquad \alpha, \beta \in \operatorname{Sec} L_\xi,$$

$$\hat{\nabla}\iota_\xi = 0,$$

and

$$\nabla(\alpha(\sigma)) = (\hat{\nabla}\alpha)(\sigma) + \alpha(\nabla\sigma), \qquad \alpha \in \operatorname{Sec} L_\xi, \quad \sigma \in \operatorname{Sec} \xi.$$

3. *Whitney sums*: A linear connection $\nabla_\xi \oplus \nabla_\eta$ in $\xi \oplus \eta$ is given by

$$(\nabla_\xi \oplus \nabla_\eta)(\sigma \oplus \tau) = \nabla_\xi\sigma \oplus \nabla_\eta\tau, \qquad \sigma \in \operatorname{Sec} \xi, \quad \tau \in \operatorname{Sec} \eta.$$

It is called the *direct sum* of ∇_ξ and ∇_η. If ∇_{ξ^*}, ∇_{η^*} are connections in ξ^*, η^* dual to ∇_ξ, ∇_η, then $\nabla_{\xi^*} \oplus \nabla_{\eta^*}$ is dual to $\nabla_\xi \oplus \nabla_\eta$.

4. *Tensor products*: There is a unique linear connection, $\nabla_{\xi\otimes\eta}$, in $\xi \otimes \eta$ such that

$$\nabla_{\xi\otimes\eta}(\sigma \otimes \tau) = \nabla_\xi\sigma \otimes \tau + \sigma \otimes \nabla_\eta\tau, \qquad \sigma \in \operatorname{Sec} \xi, \quad \tau \in \operatorname{Sec} \eta.$$

$\nabla_{\xi\otimes\eta}$ is called the *tensor product* of ∇_ξ and ∇_η.

In fact, consider the Γ-bilinear map φ: $\operatorname{Sec} \xi \times \operatorname{Sec} \eta \to A^1(B; \xi \otimes \eta)$ given by

$$\varphi(\sigma, \tau) = \nabla_\xi\sigma \otimes \tau + \sigma \otimes \nabla_\eta\tau.$$

The relation,

$$\varphi(f \cdot \sigma, \tau) = \delta f \wedge (\sigma \otimes \tau) + f \cdot \varphi(\sigma, \tau) = \varphi(\sigma, f \cdot \tau), \qquad f \in \mathscr{S}(B; \Gamma),$$

implies that φ induces a Γ-linear map

$$\nabla_{\xi\otimes\eta}\colon \operatorname{Sec}\xi \otimes_B \operatorname{Sec}\eta \to A^1(B; \xi\otimes\eta).$$

Identify $\operatorname{Sec}\xi \otimes_B \operatorname{Sec}\eta$ with $\operatorname{Sec}(\xi\otimes\eta)$ (cf. Proposition XIV, sec. 2.24, volume I); then the same relation shows that $\nabla_{\xi\otimes\eta}$ is a linear connection.

More generally, there is a unique linear connection, $\nabla^{\otimes}$, in $\xi^1\otimes\cdots\otimes\xi^m$ such that

$$\nabla^{\otimes}(\sigma^1\otimes\cdots\otimes\sigma^m) = \sum_{i=1}^{m} \sigma^1\otimes\cdots\otimes\nabla\sigma^i\otimes\cdots\otimes\sigma^m.$$

In particular, ∇ and ∇^* extend to unique linear connections (again denoted by ∇) in each $\xi^{p,q}$ with the property that

$$\nabla(\alpha\otimes\beta) = \nabla\alpha\otimes\beta + \alpha\otimes\nabla\beta, \qquad \alpha\in\operatorname{Sec}\xi^{p,q},\quad \beta\in\operatorname{Sec}\xi^{k,l}.$$

If ∇_{ξ^*} and ∇_{η^*} are dual to ∇_ξ and ∇_η, then the induced connections in $\xi\otimes\eta$ and $\xi^*\otimes\eta^*$ are again dual. This gives the formula

$$\begin{aligned}\delta(\Phi(\sigma_1,\dots,\sigma_p)) &= (\nabla\Phi)(\sigma_1,\dots,\sigma_p) + \sum_{i=1}^{p}\Phi(\sigma_1,\dots,\nabla\sigma_i,\dots,\sigma_p)\\ &= \langle\nabla\Phi, \sigma_1\otimes\cdots\otimes\sigma_p\rangle + \langle\Phi, \nabla(\sigma_1\otimes\cdots\otimes\sigma_p)\rangle,\\ &\qquad\qquad \Phi\in\operatorname{Sec}(\xi^{p,0}),\quad \sigma_i\in\operatorname{Sec}\xi.\end{aligned}$$

Finally, if $\omega\in\operatorname{Sec}L_\xi$ then in $\operatorname{Sec}\xi^{p,q}$ (cf. sec. 7.8)

$$\nabla\circ\theta_*(\omega) - \theta_*(\omega)\circ\nabla = \theta_*(\hat{\nabla}\omega).$$

5. *The bundles $\wedge^p\xi$ and $\vee^p\xi$:* Let ∇ be a linear connection in ξ. Then there are unique linear connections ∇ in $\wedge^p\xi$ and $\vee^p\xi$ such that

$$\nabla(\sigma_1\wedge\cdots\wedge\sigma_p) = \sum_{i=1}^{p}\sigma_1\wedge\cdots\wedge\nabla\sigma_i\wedge\cdots\wedge\sigma_p$$

and

$$\nabla(\sigma_1\vee\cdots\vee\sigma_p) = \sum_{i=1}^{p}\sigma_1\vee\cdots\vee\nabla\sigma_i\vee\cdots\vee\sigma_p, \qquad \sigma_i\in\operatorname{Sec}\xi.$$

They satisfy

$$\nabla(\alpha\wedge\beta) = \nabla\alpha\wedge\beta + \alpha\wedge\nabla\beta, \qquad \alpha,\beta\in\operatorname{Sec}\wedge\xi,$$

and

$$\nabla(\alpha\vee\beta) = \nabla\alpha\vee\beta + \alpha\vee\nabla\beta, \qquad \alpha,\beta\in\operatorname{Sec}\vee\xi.$$

Moreover, if $\omega \in \operatorname{Sec} L_\xi$, then, in $\operatorname{Sec} \wedge \xi$,

$$\nabla \circ \theta_*(\omega) - \theta_*(\omega) \circ \nabla = \theta_*(\hat{\nabla}\omega)$$

(cf. sec. 7.9).

7.13. Connection preserving bundle maps. Let ∇ and $\tilde{\nabla}$ be linear connections in ξ and $\tilde{\xi}$, with dual connections ∇^* and $\tilde{\nabla}^*$. A bundle map $\varphi: \xi \to \tilde{\xi}$ (inducing $\psi: B \to \tilde{B}$) will be called *connection preserving* if the diagram,

$$\begin{array}{ccc} \operatorname{Sec} \xi^* & \xleftarrow{\varphi^*} & \operatorname{Sec} \tilde{\xi}^* \\ {\scriptstyle \nabla^*}\downarrow & & \downarrow{\scriptstyle \tilde{\nabla}^*} \\ A^1(B; \xi^*) & \xleftarrow[\varphi^*]{} & A^1(\tilde{B}; \tilde{\xi}^*), \end{array}$$

commutes.

Proposition III: With the notation and hypotheses above, assume that $B = \tilde{B}$ and that φ is a strong bundle map ($\psi = \iota$). Then φ is connection preserving if and only if the diagram,

$$\begin{array}{ccc} \operatorname{Sec} \xi & \xrightarrow{\varphi_*} & \operatorname{Sec} \tilde{\xi} \\ {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle \tilde{\nabla}} \\ A^1(B; \xi) & \xrightarrow[\varphi_*]{} & A'(B; \tilde{\xi}), \end{array}$$

commutes.

Proof: Fix $\sigma \in \operatorname{Sec} \xi$, $\tau^* \in \operatorname{Sec} \tilde{\xi}^*$. Lemma II of sec. 7.5 yields

$$\langle \nabla^* \varphi^*(\tau^*) - \varphi^*(\tilde{\nabla}^* \tau^*), \sigma \rangle = \langle \tau^*, \tilde{\nabla} \varphi_*(\sigma) - \varphi_*(\nabla \sigma) \rangle.$$

The proposition follows.

Q.E.D.

Example: A linear connection ∇ in ξ determines linear connections ∇ and $\hat{\nabla}$ in $\xi^* \otimes \xi$ and L_ξ (cf. Examples 4 and 2, sec. 7.12). With respect to these connections the strong isomorphism f: $\xi^* \otimes \xi \xrightarrow{\cong} L_\xi$ given by

$$f(z^* \otimes w)(v) = \langle z^*, v \rangle w, \qquad z^* \in F_x^*, \quad v, w \in F_x, \quad x \in B,$$

is connection preserving.

Proposition IV: With the notation and hypotheses at the beginning of the section, assume that φ restricts to isomorphisms in each fibre. Then φ is connection preserving if and only if the diagram,

$$\begin{array}{ccc} \operatorname{Sec}\xi & \xleftarrow{\varphi^{\#}} & \operatorname{Sec}\tilde{\xi} \\ \nabla\downarrow & & \downarrow\tilde{\nabla} \\ A^1(B;\xi) & \xleftarrow[\varphi^{\#}]{} & A^1(\tilde{B};\tilde{\xi}), \end{array}$$

commutes.

Proof: Fix $\sigma \in \operatorname{Sec}\tilde{\xi}$, $\tau^* \in \operatorname{Sec}\tilde{\xi}^*$. Lemma I of sec. 7.5 yields

$$\langle \nabla^*\varphi^*(\tau^*) - \varphi^*(\tilde{\nabla}^*\tau^*), \varphi^{\#}\sigma\rangle = -\langle \varphi^*\tau^*, \nabla\varphi^{\#}(\sigma) - \varphi^{\#}(\tilde{\nabla}\sigma)\rangle.$$

Since each $\varphi_x\colon F_x \xrightarrow{\cong} F_{\psi(x)}$ is an isomorphism, the proposition follows.

Q.E.D.

Proposition V: Let $\varphi\colon \xi \to \tilde{\xi}$ be a bundle map restricting to isomorphisms in the fibres, and inducing $\psi\colon B \to \tilde{B}$. Then to each linear connection $\tilde{\nabla}$ in $\tilde{\xi}$ corresponds a unique connection ∇ in ξ such that φ is connection preserving.

Proof: Recall the isomorphism of Proposition I, sec. 7.3,

$$A(B) \otimes_{\tilde{B}} \operatorname{Sec}\tilde{\xi} \cong A(B;\xi).$$

It shows that $\operatorname{Sec}\xi$ is generated as an $\mathscr{S}(B)$-module by sections of the form $\varphi^{\#}\sigma$ ($\sigma \in \operatorname{Sec}\tilde{\xi}$). In view of Proposition IV it follows that ∇ is uniquely determined by $\tilde{\nabla}$.

To construct ∇ define an $\mathbb{R}$-bilinear map,

$$\beta\colon \mathscr{S}(B) \times \operatorname{Sec}\tilde{\xi} \to A^1(B;\xi),$$

by setting

$$\beta(f,\sigma) = \delta f \wedge \varphi^{\#}\sigma + f\cdot\varphi^{\#}(\tilde{\nabla}\sigma).$$

Then

$$\beta(f\cdot\tilde{g},\sigma) = \beta(f,\tilde{g}\cdot\sigma), \qquad \tilde{g} \in \mathscr{S}(\tilde{B}),$$

where the dots denote the $\mathscr{S}(\tilde{B})$-module multiplications (cf. sec. 7.3). Thus β induces an $\mathbb{R}$-linear map

$$\nabla\colon \mathscr{S}(B) \otimes_{\tilde{B}} \operatorname{Sec}\tilde{\xi} \to A^1(B;\xi).$$

If $\Gamma = \mathbb{C}$, it is clear that ∇ is complex-linear.

Identify $\mathscr{S}(B) \otimes_{\tilde{B}} \operatorname{Sec} \tilde{\xi}$ with $\operatorname{Sec} \xi$ via the isomorphism above; then ∇ becomes the desired connection.

Q.E.D.

Definition: ∇ is called the *pull-back* of $\tilde{\nabla}$ via φ.

Examples: 1. If $\psi: B \to \tilde{B}$ is a local diffeomorphism, then $d\psi: \tau_B \to \tau_{\tilde{B}}$ restricts to isomorphisms in the fibres. Thus in this case each linear connection in $\tilde{B}$ pulls back to a linear connection in B.

2. If $\psi: B \to \tilde{B}$ is a constant map ($\psi(B) = a$), then the pull-back of every vector bundle is trivial, and the pull-back of every linear connection is the standard connection δ (since the pull-back of every cross-section is constant).

3. If ξ is the pull-back of $\tilde{\xi}$ via $\psi: B \to \tilde{B}$, then the identity maps, $\iota_x: F_x = F_{\psi(x)} \to U_{\psi(x)}$ $(x \in B)$, define a bundle map, $\xi \to \tilde{\xi}$, which restricts to isomorphisms in the fibres and hence may be used to pull back a connection from $\tilde{\xi}$ to ξ.

§4. Curvature

In this article ∇ denotes a linear connection in the vector bundle $\xi = (M, \pi, B, F)$.

7.14. Covariant exterior derivative. Recall the operators, ∇_X, of sec. 7.11. Operators,

$$\nabla : A^p(B; \xi) \to A^{p+1}(B; \xi),$$

are defined by

$$\nabla\Psi(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j \nabla_{X_j}(\Psi(X_0, \ldots, \hat{X}_j, \ldots, X_p))$$

$$+ \sum_{i<j} (-1)^{i+j} \Psi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),$$
$$\Psi \in A^p(B; \xi), \quad X_i \in \mathscr{X}(B).$$

In particular, if $\Psi \in A^1(B; \xi)$,

$$\nabla\Psi(X, Y) = \nabla_X(\Psi(Y)) - \nabla_Y(\Psi(X)) - \Psi([X, Y]). \tag{7.7}$$

Note that

$$\nabla(\Phi \wedge \Psi) = \delta\Phi \wedge \Psi + (-1)^p \Phi \wedge \nabla\Psi, \qquad \Phi \in A^p(B), \quad \Psi \in A^q(B; \xi).$$

Together these operators consitute a single operator, ∇, in $A(B; \xi)$; it is called the *covariant exterior derivative* with respect to the linear connection. The formulae of sec. 7.12 extend in an obvious way to relations involving covariant exterior derivatives; we do not repeat them here. Observe that, if ∇ is the standard connection in a trivial bundle, then the covariant exterior derivative reduces to the ordinary exterior derivative for vector-valued differential forms.

7.15. Curvature. Consider the map $\mathscr{X}(B) \times \mathscr{X}(B) \times \operatorname{Sec} \xi \to \operatorname{Sec} \xi$, given by

$$(X, Y, \sigma) \mapsto (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})\sigma.$$

It is easy to check that this map is trilinear over $\mathscr{S}(B)$. Hence, there is a unique 2-form $R \in A^2(B; L_\xi)$ such that

$$R(X, Y)(\sigma) = (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})\sigma.$$

It is called the *curvature* of the linear connection ∇.

Proposition VI: The curvature satisfies the relations

(1) $\nabla^2\Psi = R(\Psi)$, $\Psi \in A(B; \xi)$, and

(2) $\hat{\nabla} R = 0$ (*Bianchi identity*), where $\hat{\nabla}$ denotes the induced connection in the vector bundle L_ξ (cf. Example 2, sec. 7.12).

Proof: It follows, from formula 7.7, that $\nabla^2\sigma = R(\sigma)$, $\sigma \in \operatorname{Sec} \xi$. Now let $\Phi \in A^p(B)$, $\sigma \in \operatorname{Sec} \xi$. Then (cf. formula 7.6, sec. 7.4, for the last equality)

$$\begin{aligned}\nabla^2(\Phi \wedge \sigma) &= \delta^2\Phi \wedge \sigma + (-1)^p\, \delta\Phi \wedge \nabla\sigma + (-1)^{p+1}\, \delta\Phi \wedge \nabla\sigma + \Phi \wedge \nabla^2\sigma \\ &= \Phi \wedge R(\sigma) = R(\Phi \wedge \sigma).\end{aligned}$$

To prove the Bianchi identity, let $\sigma \in \operatorname{Sec} \xi$. Then

$$(\hat{\nabla} R)(\sigma) = \nabla(R(\sigma)) - R(\nabla\sigma) = \nabla^3\sigma - \nabla^3\sigma = 0,$$

whence $\hat{\nabla} R = 0$.

Q.E.D.

Suppose, next, that $\varphi: \xi \to \tilde{\xi}$ is a bundle map restricting to isomorphisms in the fibres and inducing $\psi: B \to \tilde{B}$. Assume further that ∇ and $\tilde{\nabla}$ are linear connections in ξ and $\tilde{\xi}$ and that φ is connection preserving. Then (cf. Proposition IV, sec. 7.13) the covariant exterior derivatives are related by

$$\varphi^\# \circ \tilde{\nabla} = \nabla \circ \varphi^\#.$$

In particular, it follows that

$$\varphi^\#(\tilde{R}(\sigma)) = R(\varphi^\#\sigma), \qquad \sigma \in \operatorname{Sec} \tilde{\xi},$$

where R and $\tilde{R}$ are the curvatures of ∇ and $\tilde{\nabla}$. Thus, letting $\hat{\varphi}: L_\xi \to L_{\tilde{\xi}}$ be the induced map (cf. Lemma III of sec. 7.7), we have

$$\hat{\varphi}^\#(\tilde{R}) = R.$$

7.16. Examples: **1.** If ξ is trivial, then $\nabla = \delta + \Psi$, where $\Psi \in A^1(B; L_F)$ (cf. Example 2, sec. 7.11). Moreover the curvature is given by

$$R = \delta\Psi + \Psi \circ \Psi.$$

2. *Dual bundles*: A strong bundle map $L_\xi \to L_{\xi^*}$ is given by $\alpha \mapsto \alpha^*$ ($\alpha \in L_{F_x}$, $x \in B$). Denote the induced map $A(B; L_\xi) \to A(B; L_{\xi^*})$ by

$$\Omega \mapsto \Omega^\star.$$

Then if ∇^* is the connection in ξ^* dual to ∇, its curvature is given by

$$R_{\xi^*} = -(R_\xi)^\star.$$

In fact, for $\tau^* \in \operatorname{Sec} \xi^*$, $\sigma \in \operatorname{Sec} \xi$,

$$\begin{aligned}
\langle (\nabla^*)^2\tau^*, \sigma\rangle &= \delta\langle \nabla^*\tau^*, \sigma\rangle + \langle \nabla^*\tau^*, \nabla\sigma\rangle \\
&= \delta^2\langle \tau^*, \sigma\rangle - \delta\langle \tau^*, \nabla\sigma\rangle + \langle \nabla^*\tau^*, \nabla\sigma\rangle \\
&= -\langle \tau^*, \nabla^2\sigma\rangle
\end{aligned}$$

(cf. Example 1, sec. 7.12).

3. *Whitney sums and tensor products*: Let ∇_ξ and ∇_η be linear connections in ξ and η with curvatures R_ξ and R_η. Then the curvatures of the induced connections in $\xi \oplus \eta$ and $\xi \otimes \eta$ are given by

$$R_{\xi\oplus\eta} = R_\xi \oplus R_\eta \qquad \text{and} \qquad R_{\xi\otimes\eta} = R_\xi \otimes \iota_\eta + \iota_\xi \otimes R_\eta$$

(cf. Examples 3 and 4, sec. 7.12).

4. *The bundles $\xi^{p,q}$, $\wedge\xi$*: Let R be the curvature of ∇ in ξ. Then the curvatures of the induced connections in $\xi^{p,q}$ and $\wedge\xi$ are given, respectively, by

$$R_{\xi^{p,q}} = \theta_*(R) \qquad \text{and} \qquad R_{\wedge\xi} = \theta_*(R)$$

(cf. sec. 7.8 and sec. 7.9 for the definitions of $\theta_*(R)$).

5. *Torsion*: Let τ_M be the tangent bundle of a manifold M and let ω denote the 1-form on M with values in τ_M given by

$$\omega(x; h) = h, \qquad x \in M, \quad h \in T_x(M).$$

Then $\omega(X) = X$, $X \in \mathscr{X}(M)$. Hence, the covariant exterior derivative of ω is given by

$$\begin{aligned}(\nabla\omega)(X, Y) &= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) \\ &= \nabla_X Y - \nabla_Y X - [X, Y] = S(X, Y),\end{aligned}$$

where S denotes the torsion of the linear connection (cf. Example 5, sec. 7.11). Thus we have the formula

$$\nabla\omega = S.$$

Applying ∇ again and using Proposition VI, sec. 7.14, we obtain

$$\nabla S = R(\omega).$$

Remark: Observe that ω can also be interpreted as the cross-section ι_{τ_M} in the bundle L_{τ_M}. Thus, if $\hat{\nabla}$ denotes the induced connection in this bundle, we have

$$\hat{\nabla}\omega = \hat{\nabla}\,\iota_{\tau_M} = 0,$$

(cf. Example 2, sec. 7.12).

6. *Trivial bundles.* The standard connection in a trivial bundle (cf. Example 1, sec. 7.11) has zero curvature (because $\delta^2 = 0$). Moreover, if U is an open subset of a real vector space and τ_U is given the standard trivialization (cf. Example 1, sec. 3.5, volume I), then the corresponding standard connection has zero curvature and zero torsion.

In fact, if X and Y are constant vector fields, then $[X, Y] = 0$ and so

$$S(X, Y) = \delta_X Y - \delta_Y X - [X, Y] = 0.$$

§5. Parallel translation

In this article ∇ denotes a fixed linear connection in the vector bundle ξ.

7.17. Parallel translation. Proposition VII: Let $\psi\colon \mathbb{R} \to B$ be a smooth path and let $\psi(0) = b$. Then there is a unique bundle map

$$\varphi\colon \mathbb{R} \times F_b \to \xi,$$

which induces $\psi\colon \mathbb{R} \to B$, restricts to isomorphisms in the fibres, and satisfies

$$\varphi^{\#} \circ \nabla = \delta \circ \varphi^{\#} \qquad \text{and} \qquad \varphi_0 = \iota\colon\ F_b \to F_b$$

(δ is the standard connection in the bundle $\mathbb{R} \times F_b$ over $\mathbb{R}$).

Proof: We proceed in three steps.

Step I: *Assume* $B = \mathbb{R}$, $b = 0$, $\psi = \iota$, *and* ξ is *trivial* ($M = \mathbb{R} \times F$). In this case φ will be a strong bundle map, and so we must construct φ to satisfy

$$\varphi_* \circ \delta = \nabla \circ \varphi_*$$

(cf. Propositions III and IV of sec. 7.13). Let T be the standard vector field d/dt on $\mathbb{R}$; since $T(t)$ is a basis of each $T_t(\mathbb{R})$, this condition is equivalent to

$$\varphi_* \circ i(T)\delta = \nabla_T \circ \varphi_* .$$

Moreover, since $M = \mathbb{R} \times F$, we may write (cf. Example 2, sec. 7.11) $\nabla = \delta + \Psi$, where $\Psi \in A^1(\mathbb{R}; L_F)$. Define a path $\psi\colon \mathbb{R} \to L_F$ by

$$\psi_t = \Psi(t; T(t)), \qquad t \in \mathbb{R}.$$

Furthermore, again because $M = \mathbb{R} \times F$, we may regard strong bundle maps, $\mathbb{R} \times F \to \xi$, as smooth maps, $\mathbb{R} \to L_F$. Thus we must find a smooth map $\varphi\colon \mathbb{R} \to L_F$ such that for each smooth path σ_t in F

$$\varphi_t(\dot{\sigma}_t) = \dot{\varphi}_t(\sigma_t) + \psi_t(\varphi_t(\sigma_t)), \qquad t \in \mathbb{R},$$

and $\varphi_0 = \iota$.

Equivalently, φ must satisfy the *linear* ordinary differential equation

$$\dot{\varphi}_t + \psi_t(\varphi_t) = 0, \qquad t \in \mathbb{R},$$

and the initial condition, $\varphi_0 = \iota$. The existence of a unique solution of such an equation in all of $\mathbb{R}$ is standard [6, Theorem 5.1 and Theorem 5.2, p. 20]; moreover each φ_t is an isomorphism.

Step II: *Assume* $B = \mathbb{R}$, $\psi = \iota$. Choose points $t_i \in \mathbb{R}$ $(i \in \mathbb{Z})$ so that $t_0 = 0$, $t_i < t_{i+1}$ and so that $\{(t_i, t_{i+2}) \mid i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ by trivializing neighbourhoods for ξ. By step I the proposition holds in each (t_i, t_{i+2}); now an obvious step by step procedure establishes it for ξ.

Step III: *General case.* Let ζ and ∇_ζ be the pull-back of ξ and ∇ to $\mathbb{R}$ via ψ (cf. Example 3, sec. 7.13). The fibre at t of ζ is $F_{\psi(t)}$ and the identification maps $F_t = F_{\psi(t)}$ define a connection preserving bundle map

$$\chi_1 \colon \zeta \to \xi.$$

On the other hand, we can apply step II to (ζ, ∇_ζ) to obtain a bundle map,

$$\chi_2 \colon \mathbb{R} \times F_b \to \zeta,$$

which is connection preserving with respect to δ and ∇_ζ. Define

$$\varphi \colon \mathbb{R} \times F_b \to \xi$$

by

$$\varphi = \chi_1 \circ \chi_2 .$$

Q.E.D.

Corollary I: Every vector bundle over $\mathbb{R}$ is trivial.

Proof: Apply step II.

Q.E.D.

Corollary II: Let ξ be a vector bundle over $\mathbb{R}$ with a linear connection ∇. Then for each $z_0 \in F_0$, there is a unique parallel cross-section $\sigma \in \operatorname{Sec} \xi$ such that $\sigma(0) = z_0$ (cf. sec. 7.11).

Proof: Set $\sigma(t) = \varphi(t, z_0)$. Then $\varphi^{\#}(\sigma)$ is constant and so

$$\varphi^{\#}(\nabla\sigma) = \delta\varphi^{\#}(\sigma) = 0.$$

It follows that $\nabla\sigma = 0$.

Q.E.D.

Now, consider again the situation in the proposition.

Definition: The linear isomorphism $\varphi_t: F_b \xrightarrow{\cong} F_{\psi(t)}$ is called *parallel translation along the path* ψ *from* $\psi(0)$ $(= b)$ *to* $\psi(t)$. If $\sigma \in \mathrm{Sec}\,\xi$ satisfies

$$\varphi_t(\sigma(b)) = \sigma(\psi(t)), \qquad t \in \mathbb{R},$$

then σ is called *parallel along* ψ.

Proposition VIII: Let σ be a cross-section in ξ which is parallel with respect to a linear connection ∇. Then σ is parallel along every smooth path ψ.

Proof: Let $\varphi: \mathbb{R} \times F_b \to \xi$ be the bundle map of Proposition VII. Since φ is connection preserving,

$$\delta(\varphi^{\#}\sigma) = \varphi^{\#}\nabla\sigma = 0.$$

Thus $\varphi^{\#}\sigma: \mathbb{R} \to F_b$ is constant, whence

$$\begin{aligned}\sigma(\psi(t)) &= \varphi_t(\varphi^{\#}\sigma(t)) \\ &= \varphi_t(\varphi^{\#}\sigma(0)) = \varphi_t(\sigma(b)).\end{aligned}$$

Q.E.D.

Corollary: Assume that B is connected. Let $\sigma_1, \ldots, \sigma_m$ be parallel cross-sections in ξ such that, for some $a \in B$, $\sigma_1(a), \ldots, \sigma_m(a)$ are linearly independent. Then $\sigma_1(x), \ldots, \sigma_m(x)$ are linearly independent for each $x \in B$.

7.18. Bundles over $B \times \mathbb{R}$. Let $\zeta = (M_\zeta, \pi_\zeta, B \times \mathbb{R}, F)$ be a vector bundle. Denote by i_0 the inclusion map $B \to B \times \mathbb{R}$ opposite 0, and let $\zeta_0 = (M_0, \pi_0, B, F)$ be the pull-back of ζ to B via i_0. Then we can form the bundle

$$\zeta_0 \times \mathbb{R} = (M_0 \times \mathbb{R}, \pi_0 \times \iota, B \times \mathbb{R}, F).$$

(the Cartesian product of ζ_0 with $(\mathbb{R}, \iota, \mathbb{R}, 0)$, cf. Example 3, sec. 2.3, volume I.)

Theorem I: With the hypotheses and notation above, there is a strong bundle isomorphism

$$\varphi: \zeta_0 \times \mathbb{R} \xrightarrow{\cong} \zeta.$$

Proof: Give ζ a linear connection and let

$$\varphi_{x,t}: F_{(x,0)} \times \{t\} \xrightarrow{\cong} F_{(x,t)}$$

be the parallel translation along the path $t \mapsto (x, t)$ in $B \times \mathbb{R}$.

Q.E.D.

Corollary I: Let $\psi_0: \hat{B} \to B$ and $\psi_1: \hat{B} \to B$ be smooth homotopic maps. Then the pull-backs $\psi_0^*\xi$ and $\psi_1^*\xi$ of the vector bundle ξ to $\hat{B}$ are strongly isomorphic.

Proof: Let $h: \hat{B} \times \mathbb{R} \to B$ be a homotopy connecting ψ_0 and ψ_1 and let $i_0: \hat{B} \to \hat{B} \times \mathbb{R}$ and $i_1: \hat{B} \to \hat{B} \times \mathbb{R}$ be the inclusions opposite 0 and 1, respectively. Then

$$h \circ i_0 = \psi_0 \qquad \text{and} \qquad h \circ i_1 = \psi_1,$$

whence

$$\psi_0^*\xi = i_0^*(h^*\xi) \qquad \text{and} \qquad \psi_1^*\xi = i_1^*(h^*\xi).$$

On the other hand, the theorem shows that $h^*\xi$ is isomorphic to $i_0^*(h^*\xi) \times \mathbb{R}$, whence

$$i_0^*(h^*\xi) \cong i_1^*(h^*\xi).$$

Q.E.D.

Corollary II: Every vector bundle, ξ, over a contractible (cf. Example 1, sec. 5.5, volume I) base is trivial.

Proof: Since B is contractible the identity map of B is homotopic to the constant map $\varphi: B \to a$ $(a \in B)$. It follows that (cf. p. 325)

$$\xi \cong \varphi^*\xi = B \times F_a.$$

Q.E.D.

7.19. Theorem II: Let ξ and η be real vector bundles over B with the same rank r and assume that $r > \dim B$. Let ϵ^m denote the trivial bundle over B with rank m. Assume that the bundles $\xi \oplus \epsilon^m$ and $\eta \oplus \epsilon^m$ are strongly isomorphic. Then ξ and η are strongly isomorphic.

Proof: It is sufficient to consider the case $m = 1$. Then there is a vector bundle, $\zeta = (M_\zeta, \pi_\zeta, B, F \oplus \mathbb{R})$, together with strong isomorphisms

$$\varphi_0: \xi \oplus \epsilon \xrightarrow{\cong} \zeta \qquad \text{and} \qquad \varphi_1: \eta \oplus \epsilon \xrightarrow{\cong} \zeta.$$

The fibre at x of $\xi \oplus \epsilon$ is $F_x \oplus \mathbb{R}$ and hence $x \mapsto (0_x, 1)$ defines a cross-section in $\xi \oplus \epsilon$. Let $\sigma_0 \in \operatorname{Sec} \zeta$ correspond to this cross-section under φ_0 and define σ_1 analogously via φ_1.

Now consider the vector bundle,

$$\zeta \times \mathbb{R} = (M_\zeta \times \mathbb{R}, \pi_\zeta \times \iota, B \times \mathbb{R}, F \oplus \mathbb{R}),$$

and regard σ_0 and σ_1 as cross-sections

$$\sigma_0 : B \times \{0\} \to M_\zeta \times \mathbb{R} \qquad \text{and} \qquad \sigma_1 : B \times \{1\} \to M_\zeta \times \mathbb{R}.$$

Because $\zeta \times \mathbb{R}$ has rank $r + 1 > \dim(B \times \mathbb{R})$, there is a cross-section σ in $\zeta \times \mathbb{R}$ that extends σ_0 and σ_1 and which has no zeros (cf. Lemma V below). This cross-section determines a subbundle of $\zeta \times \mathbb{R}$ whose fibre at (x, t) is the one-dimensional space spanned by $\sigma(x, t)$.

Let $\hat{\zeta} = (\hat{M}, \hat{\pi}, B \times \mathbb{R}, F)$ be a complementary subbundle (cf. sec. 2.18, volume I). Let i_0, i_1: $B \to B \times \mathbb{R}$ be the inclusions opposite 0 and 1. Then

$$i_0^*\hat{\zeta} \cong \xi \qquad \text{and} \qquad i_1^*\hat{\zeta} \cong \eta.$$

Now Corollary I of Theorem I yields

$$\xi \cong i_0^*\hat{\zeta} \cong i_1^*\hat{\zeta} \cong \eta.$$

Q.E.D.

Corollary: Suppose $\tau_B \oplus \epsilon^m$ is trivial for some $m \geqslant 1$. Then so is $\tau_B \oplus \epsilon$.

Lemma V: Let $\zeta_1 = (M_1, \pi_1, B_1, F_1)$ be a real vector bundle with rank $\zeta_1 > \dim B_1$. Let $A \subset U \subset B_1$ (A closed, U open) and suppose that σ: $U \to M_1$ is a cross-section with no zeros. Then there is a cross-section τ in ζ_1 with no zeros and such that $\tau|_A = \sigma|_A$.

Proof: Consider $\zeta_1 \times \mathbb{R}$. Then $\sigma \times \iota$: $U \times \mathbb{R} \to M_1 \times \mathbb{R}$ is a cross-section without zeros. Since rank $(\zeta_1 \times \mathbb{R}) \geqslant \dim(B_1 \times \mathbb{R})$, it follows from Proposition VIII, sec. 8.13, volume I, that there exists a cross-section $\hat{\tau}$ in $\zeta_1 \times \mathbb{R}$ with countably many zeros and such that $\hat{\tau}|_{A\times\mathbb{R}} = \sigma \times \iota|_{A\times\mathbb{R}}$. Thus, for some $t_0 \in \mathbb{R}$, $\hat{\tau}(x, t_0) \neq 0$ ($x \in B$). Set $\tau(x) = \hat{\tau}(x, t_0)$.

Q.E.D.

§6. Horizontal subbundles

In this article, $\Gamma = \mathbb{R}$. Thus $\xi = (M, \pi, B, F)$ is a real vector bundle.

7.20. The horizontal map for a connection. Recall that the vertical subbundle, V_M, of τ_M is the vector bundle whose fibre at $z \in M$ is given by

$$V_z(M) = \ker(d\pi)_z$$

(cf. sec. 7.1, volume I). On the other hand, let $\pi^*(\tau_B)$ denote the pull-back of τ_B to M via π. Then the linear maps $(d\pi)_z\colon T_z(M) \to T_{\pi z}(B)$ define a strong bundle map,

$$\widetilde{d\pi}\colon \tau_M \to \pi^*(\tau_B),$$

and the sequence of strong bundle maps,

$$0 \longrightarrow V_M \longrightarrow \tau_M \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \longrightarrow 0,$$

restricts to a short exact sequence at each point $z \in M$.

In particular, a strong bundle map $\gamma\colon \pi^*(\tau_B) \to \tau_M$ is a family of linear maps,

$$\gamma_z\colon T_{\pi z}(B) \to T_z(M), \qquad z \in M,$$

which depends smoothly on z.

A strong bundle map $\gamma\colon \pi^*(\tau_B) \to \tau_M$ is called a *horizontal map for* ξ if

$$\widetilde{d\pi} \circ \gamma = \iota.$$

If γ is a horizontal map for ξ, then

$$T_z(M) = \operatorname{Im}\gamma_z \oplus V_z(M), \qquad z \in M.$$

Thus the subspaces $\operatorname{Im}\gamma_z$ are the fibres of a horizontal subbundle, H_M, of τ_M (cf. sec. 7.2, volume I). Moreover, it is easy to see that the correspondence $\gamma \mapsto H_M$ defines a bijection between the set of horizontal maps for ξ and the set of horizontal subbundles of τ_M.

Now we shall show that a linear connection, ∇, in ξ determines a horizontal map.

Observe that each $\sigma \in \operatorname{Sec} \xi$ determines linear maps,

$$(d\sigma)_x: T_x(B) \to T_{\sigma(x)}(M), \qquad x \in B.$$

Moreover, recall from sec. 7.7, volume I, that if $j_x: F_x \to M$ denotes the inclusion then we identify $F_{\pi z}$ with $T_z(F_{\pi z})$ and regard $(dj_{\pi z})_z$ as a linear isomorphism

$$\omega_z: F_{\pi z} \xrightarrow{\cong} V_z(M).$$

Thus, if ∇ is a linear connection in ξ, each $\sigma \in \operatorname{Sec} \xi$ determines the linear maps,

$$(\omega \circ \nabla\sigma)_x: T_x(B) \to V_{\sigma(x)}(M)$$

given by

$$(\omega \circ \nabla\sigma)_x(h) = \omega_{\sigma(x)}(\nabla\sigma(x; h)).$$

Proposition IX: Let ∇ be a linear connection in ξ. Then there is a unique horizontal map, γ, for ξ such that

$$\gamma_{\sigma(x)} = (d\sigma)_x - (\omega \circ \nabla\sigma)_x, \qquad x \in B, \quad \sigma \in \operatorname{Sec} \xi.$$

Proof: The uniqueness is clear. Thus, to prove existence, we may assume that ξ is trivial: $M = B \times F$. In this case

$$T_{(x,y)}(M) = T_x(B) \oplus T_y(F) = T_x(B) \oplus F$$

and, under the identification $F = T_y(F)$, $\omega_{(x,y)}$ becomes the identity map of F.

Now write $\nabla = \delta + \Psi$, $\Psi \in A^1(B; L_F)$. Then a horizontal map, $\gamma: \pi^*(\tau_B) \to \tau_M$, for ξ is given by

$$\gamma_{(x,y)}(h) = h - \Psi(x; h)(y), \qquad x \in B, \quad y \in F, \quad h \in T_x(B).$$

Each $\sigma \in \operatorname{Sec} \xi$ has the form $\sigma(x) = (x, \sigma_F(x))$ ($\sigma_F \in \mathscr{S}(B; F)$). Thus the proposition will be proved once we have established the relation

$$(d\sigma)_x(h) - \nabla\sigma(x; h) = h - \Psi(x; h)(\sigma_F(x)), \qquad x \in B, \quad h \in T_x(B), \quad \sigma \in \operatorname{Sec} \xi.$$

But this follows from the formula

$$(d\sigma)_x(h) = h + (\delta\sigma_F)(x; h).$$

Q.E.D.

Definition: γ is called the *horizontal map* associated with ∇. The corresponding horizontal bundle, H_M, is called the *horizontal subbundle associated with the linear connection* ∇.

The decomposition, $T_M = H_M \oplus V_M$, determines a projection

$$\kappa\colon T_M \to V_M$$

which is called the *vertical projection* associated with ∇. It restricts to linear projections $\kappa_z\colon T_z(M) \to V_z(M)$. Proposition IX shows that, for $\sigma \in \operatorname{Sec} \xi$,

$$[\omega_{\sigma(x)}^{-1} \circ \kappa_{\sigma(x)} \circ (d\sigma)_x](h) = \nabla\sigma(x; h), \qquad x \in B, \quad h \in T_x(B). \tag{7.8}$$

7.21. General connections. Again consider the vector bundle $\xi = (M, \pi, B, F)$. Let $H_M \subset T_M$ be *any* horizontal subbundle, and let $\kappa_M\colon T_M \to V_M$ be the corresponding vertical projection. Define a set map, $D\colon \operatorname{Sec} \xi \to A^1(B; \xi)$, by setting

$$(D\sigma)(x; h) = [\omega_{\sigma(x)}^{-1} \circ (\kappa_M)_{\sigma(x)} \circ (d\sigma)_x](h), \qquad \sigma \in \operatorname{Sec} \xi, \quad x \in B, \quad h \in T_x(B).$$

D will be called the *general connection* in ξ corresponding to H_M. If H_M is the horizontal bundle associated with a linear connection, ∇, then $D = \nabla$ (cf. formula 7.8). In the next section we shall characterize those horizontal bundles H_M for which D is a linear connection (and which, consequently, are the horizontal bundles associated with D).

Examples: **1.** *Whitney sums*: Consider the Whitney sum $\xi \oplus \eta = (M \oplus N, \pi_{\xi\oplus\eta}, B, F \oplus H)$. The identity maps

$$F_x \oplus H_x = F_x \times H_x, \qquad x \in B,$$

define a bundle map into the Cartesian product (cf. Example 3, sec. 2.3, volume 1),

$$i\colon \xi \oplus \eta \to \xi \times \eta,$$

restricting to isomorphisms in the fibres and inducing the diagonal map $\Delta\colon B \to B \times B$.

A typical point of $M \oplus N$ is a vector $z \oplus w$ ($z \in F_x$, $w \in H_x$) and the derivative of i restricts to linear injections,

$$di\colon T_{z\oplus w}(M \oplus N) \to T_z(M) \times T_w(N).$$

These restrict further to isomorphisms,

$$(di)_V\colon V_{z\oplus w}(M \oplus N) \xrightarrow{\cong} V_z(M) \times V_w(N).$$

Now assume H_M and H_N are horizontal subbundles for ξ and η with corresponding vertical projections κ_M and κ_N. Then

$$H_M \times H_N \subset T_M \times T_N$$

is a horizontal subbundle for $\xi \times \eta$; the corresponding vertical projection is $\kappa_M \times \kappa_N$. Let $\kappa_{M\oplus N}\colon T_{M\oplus N} \to V_{M\oplus N}$ be the unique strong bundle map that satisfies

$$(di)_V \circ \kappa_{M\oplus N} = (\kappa_M \times \kappa_N) \circ di.$$

Then $\kappa_{M\oplus N}$ is the vertical projection corresponding to the horizontal subbundle $H_{M\oplus N}$ of $T_{M\oplus N}$ given by

$$H_{z\oplus w}(M \oplus N) = \ker(\kappa_{M\oplus N})_{z\oplus w}.$$

$H_{M\oplus N}$ and $\kappa_{M\oplus N}$ are called the *horizontal subbundle* and *vertical projection induced in $\xi \oplus \eta$ by H_M and H_N*. The corresponding general connection, $D_{M\oplus N}$, is given by

$$D_{M\oplus N}(\sigma \oplus \tau) = D_M\sigma \oplus D_N\tau, \qquad \sigma \in \operatorname{Sec} \xi, \quad \tau \in \operatorname{Sec} \eta,$$

where D_M and D_N are the general connections corresponding to H_M and H_N.

2. *Bundle maps*: Again let H_M and H_N be horizontal subbundles for ξ and η, with corresponding vertical projections and general connections κ_M, κ_N and D_M, D_N. Assume $\varphi\colon \xi \to \eta$ is a strong bundle map. Then $d\varphi$ restricts to a bundle map

$$(d\varphi)_V\colon V_M \to V_N.$$

Moreover, it is easy to see that the following conditions are equivalent:

(1) $d\varphi$ restricts to a bundle map $(d\varphi)_H\colon H_M \to H_N$,
(2) $(d\varphi)_V \circ \kappa_M = \kappa_N \circ d\varphi$, and
(3) $\varphi_* \circ D_M = D_N \circ \varphi_*$.

If they hold, φ will be called *connection preserving*. (Note that if D_M and D_N are *linear* connections, this coincides with the definition of sec. 7.13 as follows from Proposition III, sec. 7.13).

7.22. Let H_M be an arbitrary horizontal subbundle for ξ and let κ_M and D_M be the corresponding vertical projection and general connection. Give $\xi \oplus \xi = (M \oplus M, \pi_{M\oplus M}, B, F \oplus F)$ the induced horizontal subbundle $H_{M\oplus M}$, vertical projection $\kappa_{M\oplus M}$ and general connection $D_{M\oplus M}$ (cf. Example 1 above).

A strong bundle map $\alpha: \xi \oplus \xi \to \xi$ is given by

$$\alpha(z \oplus w) = z + w, \qquad z, w \in F_x, \quad x \in B.$$

Moreover, to each $f \in \mathscr{S}(B)$ corresponds the strong bundle map, $\mu_f: \xi \to \xi$, given by

$$(\mu_f)(z) = f(\pi z)z, \qquad z \in M.$$

Then $d\mu_f$ is a bundle map in τ_M. If f is the constant function $B \to t$, we denote μ_f by μ_t.

A second bundle map $\theta_f: \tau_M \to \tau_M$ is determined by f as follows: Fix $k \in T_z(M)$ $(z \in F_x)$. Set

$$\theta_f(k) = \delta f(x; (d\pi)k) \cdot \omega_{f(x)z}(z).$$

Lemma VI: The following conditions are equivalent:

(1) $D_M(f \cdot \sigma) = \delta f \wedge \sigma + f \cdot D_M\sigma, \qquad f \in \mathscr{S}(B), \quad \sigma \in \operatorname{Sec} \xi.$

(2) $\kappa_M \circ d\mu_f - d\mu_f \circ \kappa_M = \theta_f, \qquad f \in \mathscr{S}(B).$

(3) $\kappa_M \circ d\mu_t = d\mu_t \circ \kappa_M, \qquad t \in \mathbb{R}.$

Proof: A straightforward computation shows that

$$(D_M(f \cdot \sigma) - f \cdot D_M\sigma)(x; h) = [\omega_{f(x)\sigma(x)}^{-1} \circ (\kappa_M \circ d\mu_f - d\mu_f \circ \kappa_M) \circ (d\sigma)_x](h),$$
$$x \in B, \quad h \in T_x(B),$$

and

$$(\delta f \wedge \sigma)(x; h) = [\omega_{f(x)\sigma(x)}^{-1} \circ \theta_f \circ (d\sigma)_x](h).$$

Since, if $k \in V_z(M)$,

$$(\kappa_M \circ d\mu_f - d\mu_f \circ \kappa_M)(k) = 0 = \theta_f(k),$$

it follows that (1) is equivalent to (2).

Condition (3) is simply condition (2) in the case $f = t$. Thus (2) implies (3). To prove that (3) implies (2) observe that if $k \in T_z(M)$ and $x = \pi(z)$, then

$$(\kappa_M \circ d\mu_f - d\mu_f \circ \kappa_M - \theta_f)(k) = (\kappa_M \circ d\mu_{f(x)} - d\mu_{f(x)} \circ \kappa_M)(k).$$

Q.E.D.

Proposition X: Let D_M be the general connection in ξ corresponding to a horizontal bundle H_M. Then, with the hypotheses and notation above, D_M is a linear connection if and only if

(1) $\kappa_M \circ d\alpha = d\alpha \circ \kappa_{M \oplus N}$ and
(2) $\kappa_M \circ d\mu_t = d\mu_t \circ \kappa_M$, $t \in \mathbb{R}$.

Proof: α_* is given by $\alpha_*(\sigma \oplus \tau) = \sigma + \tau$, σ, $\tau \in \operatorname{Sec} \xi$. Thus, in view of Examples 1 and 2 above, condition (1) is equivalent to

$$D_M(\sigma + \tau) = D_M\sigma + D_M\tau, \qquad \sigma, \tau \in \operatorname{Sec} \xi.$$

Now the proposition follows from Lemma VI.

Q.E.D.

§7. Riemannian connections

In this article $\Gamma = \mathbb{R}$. The vector bundle ξ is equipped with a fixed Riemannian metric $\langle\ ,\ \rangle$. The Riemannian metric can be considered as a cross-section in $\vee^2\xi^*$, and will then be denoted by g.

7.23. Riemannian connections. Let ∇ be a linear connection in ξ, and let ∇ denote as well all the induced connections. Then

$$\nabla g \in A^1(B;\ \vee^2\xi^*)$$

and (cf. Example 4, sec. 7.12)

$$\delta(g(\sigma_1, \sigma_2)) = (\nabla g)(\sigma_1, \sigma_2) + g(\nabla\sigma_1, \sigma_2) + g(\sigma_1, \nabla\sigma_2), \qquad \sigma_i \in \operatorname{Sec} \xi.$$

Definition: ∇ is called a *Riemannian connection* if $\nabla g = 0$, or, equivalently, if

$$\delta\langle\sigma_1, \sigma_2\rangle = \langle\nabla\sigma_1, \sigma_2\rangle + \langle\sigma_1, \nabla\sigma_2\rangle, \qquad \sigma_i \in \operatorname{Sec} \xi.$$

Proposition XI: Every Riemannian vector bundle admits a Riemannian connection.

Proof: Let ∇_0 be any linear connection in the Riemannian bundle $(\xi, \langle\ ,\ \rangle)$. Define $\Psi \in A^1(B; L_\xi)$ by

$$\langle\Psi(\sigma), \tau\rangle = (\nabla_0 g)(\sigma, \tau), \qquad \sigma, \tau \in \operatorname{Sec} \xi.$$

Then $\nabla_0 + \frac{1}{2}\Psi$ is a Riemannian connection.

Q.E.D.

Remarks: 1. Recall from sec. 2.21, volume I, that Sk_ξ denotes the subbundle of L_ξ whose fibre at x is the space of skew transformations of F_x. If ∇_1 and ∇_2 are Riemannian connections in ξ, then $\nabla_2 - \nabla_1$ is a 1-form on B taking values in Sk_ξ. Conversely, if $\Psi \in A^1(B; \mathrm{Sk}_\xi)$ and ∇_1 is a Riemannian connection in ξ, then so is $\nabla_1 + \Psi$.

2. Let φ: $\tilde{\xi} \to \xi$ be an isometry of Riemannian bundles. Then the pull-back, via φ, of a Riemannian connection in ξ is a Riemannian connection in $\tilde{\xi}$.

7.24. Basic properties. Fix a Riemannian connection, ∇, in ξ. Recall that a strong bundle isomorphism,

$$f: \xi \xrightarrow{\cong} \xi^*,$$

is defined by the equation,

$$\langle f_x(z), w\rangle = \langle z, w\rangle, \qquad z, w \in F_x, \quad x \in B$$

(cf. sec. 2.17, volume I). It is immediate from the relation above, and the defining relation for the dual connection ∇^* (cf. Example 1, sec. 7.12) that f is connection preserving,

$$f_* \circ \nabla = \nabla^* \circ f_* .$$

Henceforth we identify ξ^* and ξ under this isomorphism.

Next recall that $\langle\ ,\ \rangle$ induces Riemannian metrics in the bundles $\otimes^p\xi$, $\wedge^p\xi$, $\vee^p\xi$; they are given by

$$\langle z_1 \otimes \cdots \otimes z_p , w_1 \otimes \cdots \otimes w_p\rangle = \langle z_1 , w_1\rangle \cdots \langle z_p , w_p\rangle,$$
$$\langle z_1 \wedge \cdots \wedge z_p , w_1 \wedge \cdots \wedge w_p\rangle = \det(\langle z_i , w_j\rangle),$$

and

$$\langle z_1 \vee \cdots \vee z_p , w_1 \vee \cdots \vee w_p\rangle = \operatorname{perm}(\langle z_i , w_j\rangle), \qquad z_i , w_j \in F_x , \quad x \in B.$$

On the other hand, ∇ determines linear connections in $\otimes^p\xi$, $\wedge^p\xi$, $\vee^p\xi$ (cf. sec. 7.12). Evidently these connections are Riemannian with respect to the induced metrics.

Lemma VII: Suppose $\sigma \in \operatorname{Sec} \xi$ and $\langle\!\langle\sigma, \sigma\rangle\!\rangle = 1$. Then

$$\langle\!\langle\nabla\sigma, \sigma\rangle\!\rangle = 0.$$

In particular, if ξ has rank 1, then $\nabla\sigma = 0$.

Proof: Since $\delta\langle\!\langle\sigma, \sigma\rangle\!\rangle = 0$, we have

$$\langle\!\langle\nabla\sigma, \sigma\rangle\!\rangle + \langle\!\langle\sigma, \nabla\sigma\rangle\!\rangle = 0.$$

Because $\langle\ ,\ \rangle$ is symmetric it follows that

$$\langle\!\langle\nabla\sigma, \sigma\rangle\!\rangle = 0.$$

Q.E.D.

Example: Suppose ξ is oriented and of rank r. Then, since $\wedge^r\xi$ has rank 1, the positive normed determinant function $\Delta_\xi \in \operatorname{Sec} \wedge^r\xi$ is parallel:

$$\nabla(\Delta_\xi) = 0.$$

7.25. Riemannian curvature. Let ∇ be a Riemannian connection in ξ. Then, for $\sigma, \tau \in \operatorname{Sec} \xi$,

$$0 = \delta^2\langle\sigma, \tau\rangle = \delta(\langle\nabla\sigma, \tau\rangle + \langle\sigma, \nabla\tau\rangle)$$
$$= \langle\nabla^2\sigma, \tau\rangle + \langle\sigma, \nabla^2\tau\rangle.$$

Since the curvature, R, of ∇ satisfies$R(\ \sigma) = \nabla^2\sigma$ (cf. Proposition VI, (1), sec. 7.15), this shows that for each $h_1, h_2 \in T_x(B)$, $R(x; h_1, h_2)$ is a skew transformation of F_x. It follows that R takes values in Sk_ξ,

$$R \in A^2(B; \mathrm{Sk}_\xi).$$

On the other hand, a strong bundle isomorphism $\beta\colon \wedge^2\xi \xrightarrow{\cong} \mathrm{Sk}_\xi$ is defined by

$$\beta_x(z \wedge w)(v) = \langle z, v\rangle w - \langle w, v\rangle z, \qquad v, w, z \in F_x, \quad x \in B$$

(cf. sec. 2.21, volume I). Applying $\beta^\#$ to R we obtain a 2-form

$$R^\bullet = \beta^\# R \in A^2(B; \wedge^2\xi).$$

Definition: $R^\bullet$ is called the *Riemannian curvature* of ∇.

The Riemannian curvature $R^\bullet$ satisfies the relation

$$i_\xi(\sigma)R^\bullet = R(\sigma), \qquad \sigma \in \operatorname{Sec} \xi,$$

where $i_\xi(\sigma)$ is the operator defined in sec. 7.9 (recall that ξ and ξ^* are identified via the Riemannian metric). Moreover, β is clearly connection preserving. Thus the Bianchi identity (Proposition VI, sec. 7.15) gives

$$\nabla R^\bullet = 0.$$

Example: Assume $\xi = \tau_B$. Then, for each $x \in B$, $R^\bullet(x)$ is a linear map

$$R^\bullet(x)\colon \wedge^2 T_x(B) \to \wedge^2 T_x(B).$$

If for some fixed scalar, λ,

$$R^\bullet(x) = \lambda \cdot \iota, \qquad x \in B$$

(ι, the identity map of $\wedge^2 T_x(B)$), then the Riemannian curvature is called *constant*.

7.26. Riemannian connections in a manifold. Recall, from Example 5, sec. 7.11, the definition of the torsion in the tangent bundle.

Proposition XII (*Ricci lemma*): Let $\langle\ ,\ \rangle$ be a Riemannian metric in the tangent bundle τ_B of B. Let $S \in A^2(B; \tau_B)$ be any 2-form. Then there exists precisely one Riemannian connection in τ_B with torsion S.

Proof: Let ∇_1 be any Riemannian connection in τ_B and denote by S_1 the torsion of ∇_1. Then every Riemannian connection in τ_B is of the form

$$\nabla_2 = \nabla_1 + \Psi, \qquad \Psi \in A^1(B; \operatorname{Sk} \tau_B),$$

and conversely (cf. Remark 1, sec. 7.23).

An easy computation shows that the torsion, S_2, of ∇_2 is given by

$$S_2(X, Y) = S_1(X, Y) + \Psi_X(Y) - \Psi_Y(X), \qquad X, Y \in \mathscr{X}(B),$$

where $\Psi_X = i(X)\Psi$ ($\Psi_X \in \operatorname{Sec}(\operatorname{Sk} \tau_B)$). Now Lemma VIII below, applied with $N = \mathscr{X}(B)$, $\Lambda = \mathscr{S}(B)$, shows that there is precisely one $\Psi \in A^1(B; \operatorname{Sk} \tau_B)$ such that

$$S(X, Y) = S_1(X, Y) + \Psi_X(Y) - \Psi_Y(X).$$

Q.E.D.

Definition: The unique Riemannian connection in τ_B with torsion equal to zero is called the *Levi-Civita connection.*

Lemma VIII: Let Λ be a commutative ring such that the map $\lambda \mapsto 2\lambda$ ($\lambda \in \Lambda$) is bijective and let N be a Λ-module. Let

$$f: N \to \operatorname{Hom}_\Lambda(N; \Lambda)$$

be an isomorphism of Λ-modules and assume that the map,

$$\langle\ ,\ \rangle : N \times N \to \Lambda,$$

given by

$$\langle x, y \rangle = f(x)(y),$$

is symmetric.

Then each skew-symmetric Λ-bilinear map $\Omega\colon N \times N \to N$ determines a unique Λ-linear map,

$$\Psi\colon N \to \operatorname{Hom}_\Lambda(N; N),$$

such that

$$\langle \Psi(x)(y), z\rangle + \langle y, \Psi(x)(z)\rangle = 0, \qquad x, y, z \in N,$$

and

$$\Omega(x, y) = \Psi(x)(y) - \Psi(y)(x).$$

Proof: Define $\Omega_x\colon N \to N$ by setting,

$$\Omega_x(y) = \Omega(x, y),$$

and let $\tilde{\Omega}_x$ be the module endomorphism satisfying

$$\langle \tilde{\Omega}_x(y), z\rangle = \langle y, \Omega_x(z)\rangle, \qquad y, z \in N.$$

Define Ψ by

$$\Psi(x)(y) = \tfrac{1}{2}\{\Omega(x, y) - \tilde{\Omega}_x(y) - \tilde{\Omega}_y(x)\}.$$

Then

$$\begin{aligned}\langle \Psi(x)(y), y\rangle &= \tfrac{1}{2}\langle \Omega(x, y), y\rangle - \tfrac{1}{2}\langle y, \Omega(x, y)\rangle - \tfrac{1}{2}\langle x, \Omega(y, y)\rangle \\ &= 0.\end{aligned}$$

On the other hand,

$$2\Psi(x)(y) = \Omega(x, y) - \tilde{\Omega}_x(y) - \tilde{\Omega}_y(x)$$

and

$$2\Psi(y)(x) = \Omega(y, x) - \tilde{\Omega}_y(x) - \tilde{\Omega}_x(y), \qquad x, y \in N.$$

Subtracting these equations and dividing by 2, we obtain

$$\Psi(x)(y) - \Psi(y)(x) = \Omega(x, y), \qquad x, y \in N.$$

To prove uniqueness it is sufficient to show that, if $\Omega = 0$, then Ψ must be zero as well. If $\Omega = 0$, we have

$$\Psi(x)(y) = \Psi(y)(x), \qquad x, y \in N,$$

whence

$$\langle \Psi(x)(y), z\rangle = \langle \Psi(y)(x), z\rangle, \qquad x, y, z \in N.$$

It follows that

$$\begin{aligned}\langle \Psi(x)(y), z\rangle &= -\langle y, \Psi(x)(z)\rangle = -\langle y, \Psi(z)(x)\rangle \\ &= \langle \Psi(z)(y), x\rangle = \langle \Psi(y)(z), x\rangle \\ &= -\langle z, \Psi(y)(x)\rangle = -\langle \Psi(x)(y), z\rangle\end{aligned}$$

and so $\Psi = 0$.

Q.E.D.

§8. Sphere maps

7.27. Degree of a sphere map. Let B be a connected, compact, oriented n-manifold, and let

$$\psi: B \to S^n$$

be a smooth map. Regard S^n as an oriented submanifold of the oriented Euclidean space $\mathbb{R}^{n+1}$; then ψ becomes an $\mathbb{R}^{n+1}$-valued function on B. In particular, we can form the exterior derivative $\delta\psi \in A^1(B; \mathbb{R}^{n+1})$.

From the equation $\langle \psi(x), \psi(x) \rangle = 1$ $(x \in B)$, we obtain

$$\delta\psi(x; h) \in \psi(x)^{\perp} = T_{\psi(x)}(S^n), \qquad x \in B, \quad h \in T_x(B).$$

Moreover, the linear map $T_x(B) \to T_{\psi(x)}(S^n)$ given by

$$h \mapsto \delta\psi(x; h),$$

coincides with $(d\psi)_x$.

Now let Δ be the positive normed determinant function in $\mathbb{R}^{n+1}$ and regard Δ as a constant function $B \to \wedge^{n+1}(\mathbb{R}^{n+1})^*$. Thus we can form

$$\psi \wedge \delta\psi \wedge \cdots \wedge \delta\psi = \psi \wedge (\delta\psi)^n \in A^n(B; \wedge^{n+1}\mathbb{R}^{n+1})$$

and

$$\langle \Delta, \psi \wedge (\delta\psi)^n \rangle \in A^n(B).$$

Proposition XIII: With the notation and hypotheses above,

$$\deg \psi = \frac{1}{n!\,\kappa_n} \int_B \langle \Delta, \psi \wedge (\delta\psi)^n \rangle,$$

where κ_n is the volume of S^n (cf. Example 2, sec. 4.15, volume I or sec. 0.13); i.e.,

$$\kappa_{2m} = \frac{2^{m+1}}{1 \cdot 3 \cdot 5 \cdots (2m-1)} \pi^m \qquad \text{and} \qquad \kappa_{2m+1} = \frac{2}{m!} \pi^{m+1}.$$

Proof: Define $\Omega \in A^n(S^n)$ by

$$\Omega(y; k_1, \ldots, k_n) = \Delta(y, k_1, \ldots, k_n), \qquad y \in S^n, \quad k_i \in T_y(S^n).$$

Then (cf. Example 2, sec. 4.15, volume I), $\int_{S^n} \Omega = \kappa_n$. Hence

$$\deg \psi = (1/\kappa_n) \int_B \psi^*\Omega.$$

But (cf. sec. 7.9)

$$\begin{aligned}\psi^*\Omega(x; h_1, ..., h_n) &= \Delta(\psi(x), \delta\psi(x; h_1), ..., \delta\psi(x; h_n)) \\ &= \langle \Delta, \psi(x) \wedge \delta\psi(x; h_1) \wedge \cdots \wedge \delta\psi(x; h_n)\rangle \\ &= n! \, \langle\!\langle \Delta, \psi \wedge (\delta\psi)^n \rangle\!\rangle(x; h_1, ..., h_n).\end{aligned}$$

Q.E.D.

Corollary: Suppose $B = S^n$ and $\psi: S^n \to S^n$ is the identity map. Then

$$\frac{1}{n!\,\kappa_n} \int_{S^n} \langle\!\langle \Delta, \psi \wedge (\delta\psi)^n \rangle\!\rangle = 1.$$

7.28. Stably trivial bundles. Let $\xi = (M, \pi, B, F)$ be a real vector bundle. Assume that rank $\xi = \dim B = n$ and that there is a strong isomorphism

$$\xi \oplus \epsilon \xrightarrow{\cong} \epsilon^{n+1}$$

($\epsilon^p = B \times \mathbb{R}^p$). Fix a Riemannian metric in ξ and give $\xi \oplus \epsilon$ the induced metric ($\xi \perp \epsilon$). Give ϵ^{n+1} the metric defined by a fixed inner product in $\mathbb{R}^{n+1}$. Then there is a strong isometry between $\xi \oplus \epsilon$ and ϵ^{n+1} (cf. Proposition VI, sec. 2.17, volume I).

We identify $\xi \oplus \epsilon$ and ϵ^{n+1} via such an isometry; this bundle will be denoted by η. The exterior derivative, δ, is a Riemannian connection in η. Moreover, since ϵ is trivial of rank 1, there is a cross-section, $\sigma \in \operatorname{Sec} \epsilon \subset \operatorname{Sec} \eta$, of constant length 1. Thus σ is given by

$$\sigma(x) = (x, \psi(x)), \qquad x \in B,$$

where $\psi: B \to S^n$ is a smooth map.

Moreover, our identification restricts to isometries,

$$\varphi_x: F_x \xrightarrow{\cong} \psi(x)^\perp = T_{\psi(x)}(S^n),$$

and so it defines an isometry,

$$\varphi: \xi \to \tau_{S^n},$$

of Riemannian bundles.

Now we construct a Riemannian connection in ξ. σ determines an operator, $i_\eta(\sigma)$, in $A(B;\wedge\eta)$ (cf. sec. 7.9). Thus an $\mathbb{R}$-linear map, θ: Sec $\eta \to A^1(B;\eta)$, is given by

$$\theta: \tau \mapsto i_\eta(\sigma)(\sigma \wedge \delta\tau), \qquad \tau \in \operatorname{Sec} \eta.$$

Lemma IX: The linear map θ defined above restricts to a Riemannian connection in ξ. The corresponding covariant exterior derivative is given by

$$\nabla\Omega = i_\eta(\sigma)(\sigma \wedge \delta\Omega), \qquad \Omega \in A(B;\xi),$$

where Ω, $\delta\Omega$ are considered as elements of $A(B;\eta) \cong A(B;\mathbb{R}^{n+1})$.

Proof: Since $\langle \sigma, \theta(\tau) \rangle = i_\eta(\sigma)^2(\sigma \wedge \delta\tau) = 0$, $\tau \in \operatorname{Sec}\eta$, it follows that $\theta(\tau) \in A^1(B;\xi)$. Thus θ restricts to a map ∇: Sec $\xi \to A^1(B;\xi)$.

Moreover for $\tau \in \operatorname{Sec}\xi$, $f \in \mathscr{S}(B)$,

$$\nabla(f\cdot\tau) = \delta f \wedge i_\eta(\sigma)(\sigma \wedge \tau) + f\cdot\nabla\tau.$$

Since $\tau \in \operatorname{Sec}\xi$, $i_\eta(\sigma)(\tau) = \langle \sigma, \tau \rangle = 0$. It follows that

$$i_\eta(\sigma)(\sigma \wedge \tau) = \tau.$$

Hence ∇ is a linear connection.

To show that ∇ is Riemannian, observe that, for $\tau_1, \tau_2 \in \operatorname{Sec}\xi$,

$$\langle \nabla\tau_1, \tau_2 \rangle = \langle \sigma \wedge \delta\tau_1, \sigma \wedge \tau_2 \rangle = \langle \delta\tau_1, \tau_2 \rangle,$$

whence

$$\langle \nabla\tau_1, \tau_2 \rangle + \langle \tau_1, \nabla\tau_2 \rangle = \delta\langle \tau_1, \tau_2 \rangle.$$

The expression for the covariant exterior derivative is obvious (consider the case $\Omega = \Phi \wedge \tau$, $\Phi \in A^p(B)$, $\tau \in \operatorname{Sec}\xi$).

Q.E.D.

Recall that, if $\tau \in \operatorname{Sec}\eta$, then $\delta\tau$ is defined by writing

$$\tau(x) = (x, \alpha(x)),$$

where $\alpha \in \mathscr{S}(B;\mathbb{R}^{n+1})$, and setting $\delta\tau = \delta\alpha$. In particular $\delta\sigma = \delta\psi$.

Proposition XIV: The 1-form $\delta\sigma$ takes values in ξ. Moreover,

(1) $\nabla(\delta\sigma) = 0$ and
(2) $R^\bullet = -\frac{1}{2}\delta\sigma \wedge \delta\sigma$,

where ∇ and $R^\bullet$ are, respectively, the covariant exterior derivative and Riemannian curvature of the linear connection defined in Lemma IX.

Proof: Since σ has constant length, $\langle\sigma, \delta\sigma\rangle = 0$. Hence

$$\delta\sigma \in A^1(B; \xi).$$

(1) is immediate from the lemma, while (2) follows from the relation

$$\begin{aligned} -\tfrac{1}{2}i_\eta(\tau)(\delta\sigma \wedge \delta\sigma) &= \delta\sigma \wedge \langle\delta\sigma, \tau\rangle = -i_\eta(\sigma)(\sigma \wedge \delta\sigma \wedge i_\eta(\sigma)\,\delta\tau) \\ &= \nabla^2\tau = R(\tau) \\ &= i_\eta(\tau)R^\bullet, \qquad \tau \in \operatorname{Sec}\xi \end{aligned}$$

(cf. sec. 7.25).

Q.E.D.

Now let Δ be the positive normed determinant function in $\mathbb{R}^{n+1}$. Set $\Delta_\xi = i_\eta(\sigma)\,\Delta$; then Δ_ξ orients ξ, and, with respect to this orientation, φ is orientation preserving.

Proposition XV: If B is compact, connected, oriented, and $n = 2m$, the degree of ψ is given by

$$\deg\psi = \frac{(-1)^m 2^m}{n!\,\kappa_n}\int_B \langle\Delta_\xi, (R^\bullet)^m\rangle.$$

Proof: Apply Proposition XIII, sec. 7.27, and Proposition XIV.

Q.E.D.

Corollary: The Euler class χ_S of the associated sphere bundle of ξ is given by

$$\int_B^\# \chi_S = \frac{(-1)^m 2^{m+1}}{n!\,\kappa_n}\int_B \langle\Delta_\xi, (R^\bullet)^m\rangle.$$

Proof: Apply Example 3, sec. 9.10, volume I, to obtain $2\deg\psi = \int_B^\# \chi_S$.

Q.E.D.

Remark: In Chapter 10 it will be shown that the formula in the corollary holds for all oriented Riemannian vector bundles η satisfying $\operatorname{rank}\eta = \dim B = 2m$ and all Riemannian connections (Gauss–Bonnet–Chern theorem).

7.29. Tangent bundle of the sphere. Consider the special case of sec. 7.28 when $B = S^n$, $\xi = \tau_{S^n}$ and $\psi = \iota$. Then Lemma IX shows that a Riemannian connection, ∇, is defined in τ_{S^n} by

$$\nabla X = i_\eta(\sigma)(\sigma \wedge \delta X), \qquad X \in \mathscr{X}(S^n).$$

Proposition XVI: The connection ∇ has zero torsion and constant curvature.

Proof: Consider δ as a linear connection in the tangent bundle $(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \pi, \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ of $\mathbb{R}^{n+1}$. δ has zero torsion (cf. Example 6, sec. 7.16):

$$\delta_X Y - \delta_Y X = [X, Y], \qquad X, Y \in \mathscr{X}(\mathbb{R}^{n+1}).$$

In particular, if X and Y restrict to vector fields on S^n, this equation continues to hold when δ_X, δ_Y are considered as maps $\operatorname{Sec} \eta \to \operatorname{Sec} \eta$.

It follows that

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= i(X)\,\nabla Y - i(Y)\,\nabla X \\ &= i_\eta(\sigma)[\sigma \wedge (\delta_X Y - \delta_Y X)] \\ &= i_\eta(\sigma)(\sigma \wedge [X, Y]) \\ &= [X, Y], \qquad X, Y \in \mathscr{X}(S^n), \end{aligned}$$

and so ∇ has zero torsion.

To show that ∇ has constant curvature, recall from Proposition XIV, sec. 7.28, that

$$R^\bullet = -\tfrac{1}{2}\delta\sigma \wedge \delta\sigma.$$

It follows that

$$\begin{aligned} R^\bullet(x; h, k) &= -\delta\sigma(x; h) \wedge \delta\sigma(x; k) \\ &= -(d\psi)_x(h) \wedge (d\psi)_x(k). \end{aligned}$$

Since ψ is the identity map, so is each map $(d\psi)_x$. Thus

$$R^\bullet(x; h, k) = -h \wedge k.$$

Q.E.D.

Problems

1. **The operators ∇_X.** Let ∇ be a linear connection in

$$\xi = (M, \pi, B, F).$$

(i) Show that ∇ is uniquely determined by the operators ∇_X, $X \in \mathscr{X}(B)$.

(ii) Show that $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$ and

$$\nabla_X(f\sigma + g\tau) = X(f)\sigma + X(g)\tau + f\nabla_X\sigma + g\nabla_X\tau, \quad f, g \in \mathscr{S}(B), \quad \sigma, \tau \in \operatorname{Sec}\xi.$$

(iii) Given operators θ_X $(X \in \mathscr{X}(B))$ in $\operatorname{Sec}\xi$ which satisfy the conditions of (ii) construct a linear connection, $\hat{\nabla}$, such that $\hat{\nabla}_X = \theta_X$.

2. **Covariant Lie derivative.** Let ∇ be the covariant exterior derivative of a linear connection in $\xi = (M, \pi, B, F)$. Set

$$\theta_\nabla(X) = i(X)\nabla + \nabla i(X), \qquad X \in \mathscr{X}(B);$$

it is called the *covariant Lie derivative*. Establish the formulae:

(i) $$[\theta_\nabla(X)(\Omega)](X_1, \dots, X_p) = \nabla_X(\Omega(X_1, \dots, X_p)) - \sum_{j=1}^{p} \Omega(X_1, \dots, [X, X_j], \dots, X_p).$$

(ii) $\theta_\nabla(X)(\Phi \wedge \Omega) = \theta(X)\Phi \wedge \Omega + (-1)^q \Phi \wedge \theta_\nabla(X)\Omega$.

(iii) $i([X, Y]) = \theta_\nabla(X) \circ i(Y) - i(Y) \circ \theta_\nabla(X)$.

(iv) $(\theta_\nabla(X) \circ \nabla - \nabla \circ \theta_\nabla(X))\Omega = (i(X)R)(\Omega)$.

$$X, Y, X_1, \dots, X_p \in \mathscr{X}(B), \quad \Omega \in A^p(B; \xi), \quad \Phi \in A^q(B).$$

3. **Framings.** Let $\xi = (M, \pi, B, F)$ be a vector bundle. A *framing* of ξ over an open set $U \subset B$ is a system of r cross-sections $e_1, \dots, e_r$ over U such that the vectors $e_1(x), \dots, e_r(x)$ are a basis for F_x, $x \in U$.

Let $\{e_i\}$ be a framing of ξ over U with dual framing, $\{e^{*i}\}$, in ξ^*.

(i) Show that 1-forms $\omega_i^j \in A^1(U)$ and 2-forms $R_i^j \in A^2(U)$ are defined by

$$\nabla e_i = \sum_{j=1}^{r} \omega_i^j \wedge e_j \qquad \text{and} \qquad R_i^j = \langle e^{*j}, R(e_i)\rangle.$$

Prove that

$$\delta\omega_i^j = -\sum_{k=1}^{r} \omega_k^j \wedge \omega_i^k + R_i^j .$$

(ii) Express the Bianchi identity in the form

$$\delta R_i^j = -\sum_{k=1}^{r} \omega_k^j \wedge R_i^k + \sum_{k=1}^{r} \omega_i^k \wedge R_k^j .$$

4. Local coordinate representation. Let ∇ be a linear connection in a vector bundle $\xi = (M, \pi, B, F)$. Let $(x^1, ..., x^n)$ be a system of coordinate functions in an open subset $U \subset B$ (i.e., $x^\nu \in \mathscr{S}(U)$ and $x \mapsto (x^1(x), ..., x^n(x))$ is a diffeomorphism of U onto an open subset of $\mathbb{R}^n$). Assume that $e_1, ..., e_r$ is a framing of ξ over U.

(i) Define the components $\Phi^j_{\alpha_1,...,\alpha_p} \in \mathscr{S}(U)$ of $\Phi \in A^p(B; \xi)$, by

$$\Phi\left(x; \frac{\partial}{\partial x^{\alpha_1}}, ..., \frac{\partial}{\partial x^{\alpha_p}}\right) = \sum_{j=1}^{r} \Phi^j_{\alpha_1,...,\alpha_p} e_j .$$

Find their transformation law with respect to change of coordinate functions and change of framing.

(ii) Show that ∇ determines unique functions $\Gamma^i_{j\alpha} \in \mathscr{S}(U)$ such that

$$(\nabla\sigma)^i_\alpha = \frac{\partial\sigma^i}{\partial x^\alpha} + \sum_{j=1}^{r} \Gamma^i_{j\alpha}\sigma^j, \qquad \sigma \in \operatorname{Sec} \xi.$$

They are called the *connection parameters* for ∇. Find the transformation law of the connection parameters. Deduce that they are not the components of an element of $A^1(B; L_\xi)$. Find the components of $\nabla\Phi$ in terms of the partial derivatives of the components of Φ and the $\Gamma^i_{j\alpha}$.

(iii) Show that a framing e^i_j in L_ξ over U is given by $e^i_j(e_k) = \delta^i_k e_j$. Find the induced framings for the multilinear bundles associated with ξ. Find the parameters of the induced connections.

(iv) Show that the components, $R^i_{j\alpha\beta}$, of the curvature, R, of ∇ with respect to the induced framing e^i_j of L_ξ are given by

$$R^i_{j\alpha\beta} = \frac{\partial\Gamma^i_{j\beta}}{\partial x^\alpha} - \frac{\partial\Gamma^i_{j\alpha}}{\partial x^\beta} + \sum_{k=1}^{r} \Gamma^i_{k\alpha}\Gamma^k_{j\beta} - \sum_{k=1}^{r} \Gamma^i_{k\beta}\Gamma^k_{j\alpha} .$$

Express the Bianchi identity in terms of components.

5. Structure equations. Let M be an n-manifold. Let $\omega \in A^1(M; \tau_M)$ be the 1-form defined by $\omega(x; h) = h$. Let ∇ be a connection in τ_M with torsion S and curvature R.

(i) Establish a bijection between framings (over U) of τ_M and 1-forms $\Phi \in A^1(U; \mathbb{R}^n)$ for which each $\Phi(x) : T_x(U) \to \mathbb{R}^n$ is an isomorphism. Show that the isomorphism $\varphi: U \times \mathbb{R}^n \xrightarrow{\cong} \tau_U$, induced by Φ, identifies Φ with ω.

(ii) Fix a trivialization, φ, with corresponding 1-form Φ. Use φ to write $\nabla = \delta + \Psi$, $\Psi \in A^1(M; L_{\mathbb{R}^n})$. Let $S_\varphi \in A^2(U; \mathbb{R}^n)$ and $R_\varphi \in A^2(U; L_{\mathbb{R}^n})$ correspond to S and R under φ. Establish the structure equations

$$\delta\Phi = S_\varphi - \Psi(\Phi), \qquad \delta\Psi = R_\varphi - \Psi \circ \Psi,$$

and

$$\delta S_\varphi = R_\varphi(\Phi) - \Psi(S_\varphi).$$

(iii) Express the equations of (ii) in components with respect to the framing of τ_U corresponding to Φ (cf. problem 3).

6. Components in the tangent bundle. Let $(x^1, ..., x^n)$ be a system of coordinate functions in an open subset U of a manifold M. Let ∇ be a linear connection in τ_M.

(i) Show that $\partial/\partial x^1, ..., \partial/\partial x^n$ is a local framing of τ_M.

(ii) Show that an element $\Phi \in A^p(M; \tau_M)$ is a tensor field of type $(p, 1)$ on M. Show that the components of Φ with respect to the framing (i) (as defined in problem 4) coincide with the components of the tensor field, Φ, with respect to the given local coordinate system.

(iii) Let Γ^i_{jk} be the connection parameters for ∇. Show that the components, S^i_{jk}, of the torsion are given by $S^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}$.

(iv) Let g be a Riemannian metric in M. Regard g as a tensor field on M and let g_{ij} be the components of g. Obtain, from g, a Riemannian metric in τ_M^* with components g^{ij}. Show that $\sum_{j=1}^n g_{ij}g^{jk} = \delta_i^k$.

(v) Let ∇ be the unique Riemannian connection in M with no torsion. Show that

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^n g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

Hence express the components of the curvature in terms of the g_{ij}.

7. Symmetric part of a connection. Let ∇ be a linear connection in τ_M with torsion S. Show that a new linear connection, $\mathring{\nabla}$, is given by

$$\mathring{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2} S(X, Y).$$

Show that $\mathring{\nabla}$ has torsion zero. It is called the *symmetric part of* ∇.

Suppose g is a Riemannian metric in M with $\mathring{\nabla} g = 0$. Does it follow that $\nabla g = 0$?

8. Covariant derivative. Let $\xi = (M, \pi, B, F)$ be a vector bundle. A *covariant tensor field of degree p with values in ξ* is a cross-section in $\otimes^p \tau_B^* \otimes \xi$. Denote the space of these tensor fields by $\mathscr{X}^p(B; \xi)$.

(i) Show that linear connections ∇_B in τ_B and ∇_ξ in ξ determine a connection in $\otimes^p \tau_B^* \otimes \xi$. Hence obtain an $\mathbb{R}$-linear operator $D: \mathscr{X}^p(B; \xi) \to \mathscr{X}^{p+1}(B; \xi)$. It is called the *covariant derivative* induced by ∇_B and ∇_ξ.

(ii) If $\xi = B \times \mathbb{R}$, and $\nabla_\xi = \delta$, then $\mathscr{X}^p(B; \xi)$ is written $\mathscr{X}^p(B)$ and D is called the *covariant derivative for* B. If $\xi = \otimes^q \tau_B^*$ and ∇_ξ is induced by ∇_B, show that $D: \mathscr{X}^p(B; \xi) \to \mathscr{X}^{p+1}(B; \xi)$ may be identified with the covariant derivative for B.

(iii) Assume $\xi = B \times \mathbb{R}$, $\nabla_\xi = \delta$. Regard D as a map from $A^1(B)$ to $\mathscr{X}^2(B)$. Show that

$$(D\omega)(X, Y) - (D\omega)(Y, X) - (\delta\omega)(X, Y) = -\omega(S(X, Y))$$

and

$$(D\omega)(X, Y) - (\theta(X)\omega)(Y) = \omega((\nabla_B)_Y X) - \omega(S(X, Y)),$$
$$\omega \in A^1(B), \quad X, Y \in \mathscr{X}(B)$$

(S is the torsion of ∇_B).

(iv) Regard $A^p(B; \xi)$ as the subspace of $\mathscr{X}^p(B; \xi)$ consisting of skew symmetric tensor fields. Show that an $\mathbb{R}$-linear operator,

$$D_A: \mathscr{X}^p(B; \xi) \to A^{p+1}(B; \xi),$$

is defined by

$$(D_A\Phi)(X_0, \dots, X_p) = \frac{1}{p!} \sum_{\sigma \in S^{p+1}} \epsilon_\sigma (D\Phi)(X_{\sigma(0)}, \dots, X_{\sigma(p)}).$$

(v) Show that an $\mathscr{S}(B)$-linear operator, $I_S: \mathscr{X}^p(B; \xi) \to A^{p+1}(B; \xi)$, is defined by

$$(I_S\Phi)(X_0, \dots, X_p) = \frac{1}{2 \cdot p!} \sum_{\sigma \in S^{p+1}} \epsilon_\sigma \sum_{i=0}^{p} \Phi(X_{\sigma(0)}, \dots, S(X_{\sigma(i)}, X_{\sigma(i+1)}), \dots, X_{\sigma(p)}).$$

Show that the restriction of D_A to $A^1(B;\,\xi)$ is given by

$$D_A = \nabla - I_S\,,$$

where ∇ is the covariant exterior derivative of ∇_ξ.

(vi) Let R be the curvature of ∇_ξ. Show that

$$(D^2\sigma)(X, Y) - (D^2\sigma)(Y, X) = R(X, Y)(\sigma) - (\nabla_\xi)_{S(X,Y)}(\sigma),$$
$$\sigma \in \operatorname{Sec}\xi, \quad X, Y \in \mathscr{X}(B).$$

Compute D_A^2.

9. Curvature identities. Let ∇ be a linear connection in τ_B with zero torsion and curvature R. Let D be the induced covariant derivative in $\mathscr{X}(B; L\tau_B)$.

(i) Establish the identities:

$$R(X, Y) = -R(Y, X).$$
$$R(X, Y)(Z) + R(Z, X)(Y) + R(Y, Z)(X) = 0.$$
$$(DR)(X, Y, Z) + (DR)(Z, X, Y) + (DR)(Y, Z, X) = 0.$$

Show that the third identity is equivalent to the Bianchi identity, $\nabla R = 0$, of the text.

(ii) Assume ∇ is the Levi–Civita connection for a Riemannian metric, $\langle\ ,\ \rangle$. Show that

$$\langle R(X, Y)(Z), W\rangle + \langle R(X, Y)(W), Z\rangle = 0$$

and

$$\langle R(X, Y)(Z), W\rangle = \langle R(Z, W)(X), Y\rangle.$$

Hence interpret $R^{\boldsymbol{\cdot}}(x)$ as a self-adjoint operator in $\wedge^2 T_x(B)$, $x \in B$.

10. Let $\xi = (M, \pi, B, F)$ be a vector bundle. Let ∇_ξ and ∇ be linear connections in ξ and τ_B.

(i) Use ∇_ξ and ∇ to determine a linear connection, $\hat{\nabla}$, in $\wedge^p\tau_B^* \otimes \xi$. For $X \in \mathscr{X}(B)$, interpret $\hat{\nabla}_X$ as a linear map in $A^p(B;\,\xi)$.

(ii) Let $(\nabla_\xi)_X$ be the covariant Lie derivative determined by ∇_ξ (cf. problem 2) and set $\Xi_X = \hat{\nabla}_X - (\nabla_\xi)_X$. Show that Ξ_X is an $\mathscr{S}(B)$-linear endomorphism of $A^p(B;\,\xi)$.

(iii) Show that there is a unique operator Φ_X: $A^p(B) \to A^p(B)$ such that

$$\Xi_X = \Phi_X \otimes \iota.$$

Conclude that Ξ_X is independent of ξ and ∇_ξ.

(iv) Prove the formula

$$(\Phi_X \Omega)(X_1, \ldots, X_p) = -\sum_{i=1}^{p} \Omega(X_1, \ldots, \nabla_{X_i} X, \ldots, X_p)$$

$$- \sum_{i=1}^{p} \Omega(X_1, \ldots, S(X, X_i), \ldots, X_p)$$

(where S denotes the torsion for ∇).

(v) Specialize to the case $p = 2$, $\xi = \tau_B$ and $\nabla_\xi = \nabla$. In particular, compute $\Xi_X(S)$.

11. Covariant derivative along a path. Let ∇ be a linear connection in a vector bundle $\xi = (M, \pi, B, F)$. Let $\gamma: t \mapsto x(t)$ $(0 \leqslant t \leqslant 1)$ be a smooth path on B. A *cross-section along the path* γ is a smooth path $t \mapsto \sigma(t)$ on M such that $\sigma(t) \in F_{x(t)}$ $(0 \leqslant t \leqslant 1)$. Set $I = [0,1]$.

(i) Make the cross-sections along γ into an $\mathscr{S}(I)$ module, $\mathrm{Sec}_\gamma\, \xi$. Show that there is a unique linear operator, ∇_γ, in $\mathrm{Sec}_\gamma\, \xi$ such that: (a) if $\sigma \in \mathrm{Sec}_\gamma\, \xi$ and $\tau \in \mathrm{Sec}\, \xi$ are related by $\sigma(t) = \tau(x(t))$, then $(\nabla_\gamma \sigma)(t) = (\nabla \tau)(x(t); \dot{x}(t))$, and (b) if $\sigma \in \mathrm{Sec}_\gamma\, \xi$ and $f \in \mathscr{S}(I)$, then $\nabla_\gamma(f \cdot \sigma) = f' \cdot \sigma + f \cdot \nabla_\gamma \sigma$. $\nabla_\gamma \sigma$ is called the *covariant derivative along* γ.

(ii) Show that in local coordinates (cf. problem 4)

$$(\nabla_\gamma \sigma)^i = \dot{\sigma}^i + \Gamma^i_{j\alpha} \sigma^j \dot{x}^\alpha.$$

(iii) Show that $\sigma \in \mathrm{Sec}_\gamma\, \xi$ is obtained from $\sigma(0)$ by parallel translation along γ if and only if $\nabla_\gamma \sigma = 0$.

(iv) Show that ∇ is a Riemannian connection (with respect to a Riemannian metric $\langle\ ,\ \rangle$) if and only if parallel translation along each path γ is an isometry from $F_{x(0)}$ to $F_{x(1)}$.

(v) If ∇ is a Riemannian connection (with respect to a Riemannian metric $\langle\ ,\ \rangle$), show that for $\sigma, \tau \in \mathrm{Sec}_\gamma\, \xi$

$$\frac{d}{dt} \langle \sigma(t), \tau(t) \rangle = \langle \nabla_\gamma \sigma, \tau \rangle(t) + \langle \sigma, \nabla_\gamma \tau \rangle(t).$$

Conclude that

$$\frac{d}{dt}\langle\sigma,\sigma\rangle = 2\langle\nabla_\gamma\sigma,\sigma\rangle \qquad \text{and} \qquad |\sigma|\frac{d}{dt}|\sigma| = \langle\nabla_\gamma\sigma,\sigma\rangle.$$

12. Autoparallels. Fix a linear connection, ∇, in the tangent bundle of a manifold M.

(i) Let $\gamma: t \mapsto x(t)$ be a path in M. Show that $\dot{x}(t)$ is a vector field along γ. γ is called *autoparallel*, if $\nabla_\gamma\dot{x} = 0$.

(ii) Show that in local coordinates the equations for an autoparallel are

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$

Conclude that the autoparallels depend only on the symmetric part of the connection (cf. problem 7).

(iii) Show that given a point $a \in M$ and a vector $h \in T_a(M)$ there is a unique autoparallel $\gamma: t \mapsto x(t)$ $(0 \leqslant t < \delta)$ such that $x(0) = a$ and $\dot{x}(0) = h$.

13. Geometric interpretation of the curvature. Let ∇ be a linear connection in a vector bundle $\xi = (M, \pi, B, F)$. Fix a point $x \in B$ and tangent vectors $h, k \in T_x(B)$. Set $\Delta = \{(s,t) \in \mathbb{R}^2 \mid 0 \leqslant s, 0 \leqslant t, s + t \leqslant 1\}$ and suppose that $\sigma: \Delta \to B$ is a smooth map such that $\sigma(0) = x$, $(d\sigma)_0(1,0) = h$, $(d\sigma)_0(0,1) = k$. Define $\sigma_\tau: \Delta \to B$ by $\sigma_\tau(u) = \sigma(\tau u)$ $(0 \leqslant \tau \leqslant 1)$ and let Φ_τ be the automorphism of F_x obtained by parallel translation along the boundary of σ_τ. Show that

$$\lim_{\tau\to 0}\frac{\Phi_\tau - \iota}{\tau^2} = R(x; h, k).$$

14. Geometric interpretation of torsion. Let ∇ be a linear connection on a manifold M with torsion S. Let $\varphi(t,\tau)$ be a smooth map from U (an open subset of $\mathbb{R}^2$) to M. Set $\dot{\varphi} = \partial\varphi/\partial t$ and $\varphi' = \partial\varphi/\partial\tau$. For fixed t, let φ_t, $\dot{\varphi}_t$, φ'_t denote the paths $\tau \mapsto \varphi(t,\tau)$, $\tau \mapsto \dot{\varphi}(t,\tau)$, and $\tau \mapsto \varphi'(t,\tau)$. Define φ_τ, $\dot{\varphi}_\tau$, φ'_τ in a similar way. Show that

$$\nabla_{\varphi_t}(\dot{\varphi}_t) - \nabla_{\varphi_\tau}(\varphi'_\tau) = S(\varphi(t,\tau); \varphi'(t,\tau), \dot{\varphi}(t,\tau)).$$

15. Arc length. Fix a Riemannian metric, $\langle\ ,\ \rangle$, in M. The *arc length* of a path, $\gamma: t \mapsto x(t)$ $(0 \leqslant t \leqslant a)$ is defined by

$$l_\gamma = \int_0^a |\dot{x}(t)|\, dt.$$

(i) Show that l_γ is independent of the parametrization.

(ii) If $t \mapsto x(t)$ is an immersion, show that the function,

$$t \mapsto \int_0^t |\dot{x}(\tau)|\, d\tau,$$

has a smooth inverse. Hence obtain a new map $\gamma_1\colon s \mapsto x_1(s)$ with the same image and the same endpoints and such that $|\dot{x}_1(s)| = 1$. γ_1 is said to be *parametrized by arc length.*

(iii) Let γ be a smooth map from the unit square in $\mathbb{R}^2$ to M, and let γ_τ be the path $t \mapsto \gamma(t, \tau)$. Assume $\gamma_\tau(0) = x_0$ and $\gamma_\tau(1) = x_1$ (independent of τ). Suppose $|\dot{\gamma}_0(t)|$ is constant, for $0 \leqslant t \leqslant 1$. Show that (cf. problem 11)

$$\left[\frac{d}{d\tau} l_{\gamma_\tau}\right]_{\tau=0} = -\int_0^1 \frac{1}{|\dot{\gamma}_0|} \left\langle \frac{\partial \gamma}{\partial \tau}(t, 0), \nabla_{\gamma_0}(\dot{\gamma}_0) \right\rangle dt,$$

where ∇ is the Levi–Civita connection in M. Conclude that if $l\gamma_0 \leqslant l\hat{\gamma}$ for all paths $\hat{\gamma}$ joining x_0 to x_1, then γ_0 is an autoparallel.

(iv) Assume M is connected, and fix $x_0 \in M$. Show that M admits a metric with the following property: For each $\lambda > 0$ there is a compact subset $A_\lambda \subset M$ such that all paths γ starting at x_0 with $l_\gamma \leqslant \lambda$ are contained in A_λ. Show that this property continues to hold if x_0 is replaced by any other point x_1. (Such a metric is called *complete.*)

16. Differential equations. Let M and N be connected Riemannian manifolds and fix $x_0 \in M$ and $z_0 \in N$. Denote by ξ and η the vector bundles over $M \times N$ whose fibres at (x, z) are the spaces $L(T_x(M); T_z(N))$ and $L(\wedge^2 T_x(M); T_z(N))$. A cross-section, Φ, in ξ determines the differential equation,

(1) $(d\varphi)_x = \Phi(x, \varphi(x)), \ x \in M,$

with initial condition $\varphi(x_0) = z_0$ for a map $\varphi\colon M \to N$. Moreover, if $\gamma\colon t \mapsto x(t)$, $0 \leqslant t \leqslant 1$, is a path on M, then

(2) $\dot{z}(t) = \Phi(x(t), z(t))\, \dot{x}(t), \qquad z(0) = z_0,$

is an ordinary differential equation for a path on N.

(i) Show that there is a constant $\delta > 0$ such that the differential equation (2) has a solution in the full interval $0 \leqslant t \leqslant 1$ whenever $l_\gamma < \delta$ (l_γ denotes the length of γ, cf. problem 15).

(ii) Assign a Riemannian metric to M and a complete Riemannian metric to N (cf. problem 15). Assume that the cross-section Φ satisfies the condition,

$$|\Phi(x, z)h| \leqslant B(x)\,|h|, \qquad x \in M, \quad z \in N, \quad h \in T_x(M),$$

where B is a continuous function on M. Prove that then the differential equation (2) has a solution in the full interval $0 \leqslant t \leqslant 1$.

(iii) The *curvature operator*, R_Φ, for Φ is the cross-section in η given as follows: Let $z \in N$, $x \in M$, $h \in T_x(M)$. Define $\Phi_z \in A^1(M; T_z(N))$ and $\Phi_{x,h} \in \mathscr{X}(N)$ by

$$\Phi_z(x; h) = \Phi(x, z)h = \Phi_{x,h}(z).$$

Then

$$R_\Phi(x, z; h, k) = (\delta\Phi_z)(x; h, k) + [\Phi_{x,h}, \Phi_{x,k}](z).$$

Let γ_τ: $t \mapsto x(t, \tau)$ be a family of paths on M all starting at x_0, and denote by $t \mapsto z(t, \tau)$ the solution of equation (2) along the path γ_τ. Set

$$y(t, \tau) = \frac{\partial}{\partial\tau} z(t, \tau) - \Phi(x(t, \tau), z(t, \tau))\left(\frac{\partial}{\partial\tau} x(t, \tau)\right).$$

Use charts in M and N to interpret y as a smooth map from the unit square into $\mathbb{R}^n$ ($n = \dim N$). Establish the relations

$$\frac{\partial}{\partial t} y(t, \tau) = R_\Phi\left(x, z; \frac{\partial x}{\partial\tau}, \frac{\partial x}{\partial t}\right) + \Phi'_{x,\partial x/\partial t}(z; y)$$

and $y(0, \tau) = 0$. (Interpret $\Phi_{x,h}$ as a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$; then $\Phi'_{x,h}$ is its derivative.)

(iv) Show that if $R_\Phi = 0$ near (x_0, z_0), then (1) has a solution in some neighbourhood of x_0.

(v) Assume that $R_\Phi = 0$ and that Φ satisfies the condition of (ii). Assume M is simply connected (cf. problem 5, Chap. VI). Show that (1) has a global solution. Show that in both cases (local and global) the solution is unique.

17. Distributions. (i) With the notation of problem 16 show that the vectors of the form,

$$Z(x, z) = (h, \Phi(x, z)h), \qquad h \in T_x(M),$$

determine a distribution on $M \times N$ (i.e., a subbundle of $\tau_{M\times N}$) whose rank is the dimension of M.

(ii) Show that the distribution is involutive (i.e., cross-sections are closed under the Lie product) if and only if $R_\Phi = 0$. In this case show that $\varphi(x)$ is the unique point of N such that $(x, \varphi(x))$ is on the same integral manifold of the distribution as (x_0, z_0).

(iii) Given the curvature, R, of a linear connection in a vector bundle (M, π, B, F), obtain a curvature operator in $M \times F$. Show that in this case (ii) implies: $R = 0$ if and only if the horizontal subbundle H_M of T_M is involutive.

18. Flat connections. A linear connection in a vector bundle, ξ, is called *flat*, if its curvature is zero.

(i) Show that a linear connection is flat if and only if there is a coordinate representation, $\{(U_\alpha, \psi_\alpha)\}$, such that the cross-sections $\sigma_{\alpha,y}$ determined by

$$\sigma_{\alpha,y}(x) = \psi_\alpha(x, y), \qquad x \in U_\alpha,$$

(where $y \in F$ is fixed) satisfy $\nabla\sigma_{\alpha,y} = 0$. (*Hint*: cf. problem 7, Chap. VI, or problems 16 and 17.)

(ii) Show that ξ admits a flat connection if and only if there is a coordinate representation with constant transition functions

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to GL(F).$$

19. Parallelisms. Recall from problem 14, Chap. IV, volume I, that a *parallelism* on an n-manifold, M, is a trivialization of the tangent bundle, $M \times \mathbb{R}^n \xrightarrow{\cong} T_M$. The standard connection on $M \times \mathbb{R}^n$ determines (via the parallelism) a linear connection, ∇, in T_M. It is called the *connection of the parallelism.*

(i) Show that a vector field X on M is parallel with respect to the parallelism (respectively, parallel along a curve γ) if and only if $\nabla X = 0$ (respectively, $\nabla_\gamma X = 0$).

(ii) Regard the parallelism as a family of isomorphisms,

$$P(x, y)\colon T_x(M) \xrightarrow{\cong} T_y(M),$$

and hence as a cross-section, P, in the vector bundle, η, over $M \times M$ whose fibre at (x, y) is the space $L(T_x(M); T_y(M))$. Show that any linear connection in τ_M determines a connection in η. Show that $\hat{\nabla}P = 0$, where $\hat{\nabla}$ denotes the connection in η induced from the connection of the parallelism.

(iii) Show that the connection of a parallelism is flat. Conversely, show that a flat connection in a simply connected manifold determines a unique parallelism for which it is the connection. (*Hint*: cf. problems 5 and 7, Chap. VI, or problem 16.)

(iv) Let ∇ be a flat connection in τ_B, with torsion S. Let $e_1, \dots, e_n$ be a local parallel framing for τ_B, and let e^{*i} be the dual framing. Write $S(e_j, e_k) = \sum_{i=1}^n S_{jk}^i e_i$. Show that the e^{*i} are local 1-forms satisfying

$$\delta e^{*i} = \sum_{j<k} S_{jk}^i e^{*j} \wedge e^{*k}.$$

Compute $\nabla S(e_i, e_j, e_k)$ in terms of the components of S (∇, the covariant exterior derivative). Conclude that $\nabla S = 0$ if and only if the bracket

$$\left[\sum_{i=1}^n f_i e_i, \sum_{j=1}^n g_j e_j\right] = \sum_{i,j} f_i g_j S_{ij}^k e_k$$

satisfies the Jacobi identity.

(v) (*Moscow parallelism*). Consider an oriented Euclidean plane, E, and let σ be the rotation of E through the angle $\pi/2$. Define a frame in the deleted plane $E - \{0\}$:

$$e_1(x) = \frac{-x}{|x|}, \qquad e_2(x) = \frac{\sigma(x)}{|x|}, \qquad x \neq 0.$$

Compute the torsion of this parallelism.

20. Horizontal maps. Let γ be the horizontal map associated with a general connection, D_M, in $\xi = (M, \pi, B, F)$.

(i) Find necessary and sufficient conditions on γ for D_M to be a linear connection.

(ii) Show that D_M is a linear connection if and only if it is an $\mathbb{R}$-linear map.

(iii) Assume D_M is a linear connection. Show that $\sigma \in \operatorname{Sec} \xi$ is parallel if and only if the spaces $\operatorname{Im}(d\sigma)_x$ are horizontal for each x. Establish an analogous result for cross-sections along a path (cf. problem 11).

(iv) Assume $\xi = \tau_B$ and D_M is a linear connection. Express the torsion of D_M in terms of γ. (*Hint*: cf. problem 9, Chap. III, volume I).

21. Let ∇ be a linear connection in a Riemannian vector bundle (M, π, B, F). Show that ∇ is a Riemannian connection if and only if the corresponding horizontal subspaces $H_z(M)$ are tangent to the associated sphere bundle of radius $|z|$ $(z \in M, |z| \neq 0)$.

22. Linear connections in L_ξ. Let ξ be a vector bundle and let ∇ be a linear connection in the bundle L_ξ which satisfies

$$\nabla(\sigma \circ \tau) = \nabla\sigma \circ \tau + \sigma \circ \nabla\tau, \qquad \sigma, \tau \in \operatorname{Sec} L_\xi .$$

Show that there is a linear connection, ∇_ξ, in ξ such that ∇ is the connection induced in L_ξ.

23. Let (M, π, B, F) be a fibre bundle and assume that $\dim H(F) < \infty$. Construct an induced vector bundle $\xi = (H, \pi_M, B, H(F))$ whose fibre at x is the space $H(F_x)$. Show that ξ admits a connection with curvature zero. Show that if B is simply connected (cf. problem 5, Chap. VI), then ξ is the product bundle $B \times H(F)$.

24. Isometries. Let M be a Riemannian manifold and let $\varphi: M \to M$ be a local diffeomorphism. Consider $d\varphi$ as a 1-form on M with values in the vector bundle $\varphi^*(\tau_M)$. Let ∇ be a Riemannian connection in M.

(i) Show that if φ is an isometry, then $\nabla(d\varphi) = 0$, where ∇ also denotes the induced connection in $\varphi^*(\tau_M)$.

(ii) Show that if $\nabla(d\varphi) = 0$ and, for a fixed point $a \in M$, the linear map $(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(M)$ is an isometry, then φ is an isometry.

25. Sprays. Let $\tau_M = (T_M, \pi, M, \mathbb{R}^n)$ and $\tau_M^2 = (T_M^2, \pi^2, T_M, \mathbb{R}^{2n})$ be respectively the tangent bundle of a manifold, M, and the tangent bundle of T_M. Recall that a *spray* for M is a vector field, Y, on T_M such that $d\pi \circ Y = \iota$ (cf. Appendix A, volume I).

(i) Let $\alpha(t)$ be an orbit of a spray, Y, and set $\beta(t) = \pi\alpha(t)$. Show that $\dot\beta(t) = \alpha(t)$ and $\ddot\beta(t) = Y(\dot\beta(t))$.

(ii) Let H be a horizontal subbundle of τ_M^2. Show that (with respect to H) there is a unique horizontal spray.

(iii) Recall that a spray, Y, is called *affine* if $(d\mu_t)(Y(z)) = (1/t)\, Y(tz)$ ($\mu_t: T_M \to T_M$ is multiplication by t.) Show that the spray determined by a horizontal bundle is affine if the corresponding general connection satisfies $D(tX) = tDX$, $X \in \mathscr{X}(M)$, $t \in \mathbb{R}$. Show that in this case $D(fX) = \delta f \wedge X + fD(X), f \in \mathscr{S}(M)$. Conclude that a linear connection determines an affine spray.

(iv) Let Y be an affine spray for M. Let W be a radial neighbourhood of $\{0\} \times T_M$ in $\mathbb{R} \times T_M$, and let ψ: $W \to T_M$ be a local flow for Y. Fix $(s, h) \in W$ and $\lambda \in \mathbb{R}$. Show that $t \mapsto \lambda\psi(\lambda t, h)$ ($| t | \leqslant s/| \lambda |$) is an orbit of Y. Show that, for t sufficiently small,

$$\psi(t, \lambda h) = \lambda\psi(\lambda t, h).$$

Conclude that $\psi(t, 0_x) = 0_x$, $x \in M$.

(v) Use (iv) to show that W may be chosen to contain $I \times O$, where $I = \{t \in \mathbb{R} \mid | t | \leqslant 1\}$ and $O \subset T_M$ is an open set satisfying $tO \subset O$, if $| t | \leqslant 1$. Show that the equation $\psi(t, \lambda h) = \lambda\psi(\lambda t, h)$ holds for $t, \lambda \in I$ and $h \in O$.

(vi) Assume W chosen as in (v). Set $\varphi = \pi \circ \psi$: $W \to M$, and write $\varphi_h(t) = \varphi(t, h)$. Show that $\varphi_h(0) = \pi(h)$, $\dot{\varphi}_h(0) = h$ and $\ddot{\varphi}_h(t) = Y(\dot{\varphi}_h(t))$. Show that $\varphi_h(\lambda t) = \varphi_{\lambda h}(t)$.

26. Exponential map. Let O, W, φ, ψ be as in problem 25, (vi). Define exp: $O \to M$ by $\exp h = \varphi(1, h)$. It is called the *exponential map generated by the affine spray* Y. Let $\exp_x$ denote the restriction of exp to $O \cap T_x(M)$.

(i) Show that $\exp_x(0_x) = x$ and $(d \exp_x)_{0_x} = \iota$. Conclude that there is an open subset $U \subset O$ such that $tU \subset U$ if $| t | \leqslant 1$ and such that $h \mapsto (\pi(h), \exp h)$ is a diffeomorphism from U onto an open subset, V, of $M \times M$.

(ii) Suppose Y is the affine spray determined by a linear connection, ∇ (cf. problem 25, (iii)). Then exp is called the *exponential map for* ∇. Show that in this case γ: $t \mapsto \exp th$ ($h \in T_M$, $| t | \leqslant \delta$) is the unique autoparallel such that $\gamma(0) = \pi(h)$ and $\dot{\gamma}(0) = h$. Conclude that if $(x, y) \in V$, then x and y can be joined by an autoparallel.

(iii) Suppose exp is the exponential map of a linear connection. Fix a Euclidean inner product, $\langle\ ,\ \rangle$, in $T_a(M)$ and let $T_{a,\epsilon}(M)$ be the open ball of radius ϵ centered at 0_a. Set $U_\epsilon = \exp(T_{a,\epsilon}(M))$. Show that, for ϵ sufficiently small, any two points $x, y \in U_\epsilon$ can be joined by a unique autoparallel, γ, such that $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma(t) \in U_\epsilon$, $0 \leqslant t \leqslant 1$. (*Hint*: Define $f \in \mathscr{S}(U_\epsilon)$ by $f(z) = \langle \exp_a^{-1}(z), \exp_a^{-1}(z)\rangle$. Show that $(d^2/dt^2) f(\exp_a(th)) > 0$.)

(iv) Show that M admits a locally finite open cover $\{U_\alpha\}$ such that the nonvoid intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ are all contractible.

27. Polar trivialization. Let ∇ be a linear connection in a vector bundle $\xi = (M, \pi, U, F)$, where U is the unit ball in a Euclidean space $\mathbb{R}^n$. A *polar trivialization* for ξ is an isomorphism $\varphi: U \times F \xrightarrow{\cong} M$ under which the constant cross-sections in ξ correspond to cross-sections parallel along the radial paths $x(t) = ta$ $(0 \leqslant t \leqslant 1,\ a \in U)$. Let Z denote the radial vector field in U: $Z(x) = x$, $x \in U$.

(i) Show that an isomorphism $F \xrightarrow{\cong} F_0$ (F_0, the fibre over $x = 0$) extends to a unique polar trivialization.

(ii) Let $\psi: U \times F \xrightarrow{\cong} M$ be any trivialization of ξ and write the induced connection in $U \times F$ in the form $\nabla = \delta + \Psi$, $\Psi \in A^1(U; L_F)$. Show that ψ is polar if and only if $i(Z)\Psi = 0$.

(iii) Use a polar trivialization, φ, to write the curvature as a 2-form, R_φ, with values in L_F. Write $\nabla = \delta + \Psi$ and show that

$$\Psi(x; h) = \int_0^1 R_\varphi(tx; x, th)\, dt, \qquad x \in U, \quad h \in \mathbb{R}^n.$$

Hint: If $\mu_t: U \to U$ is the multiplication by t, prove the operator identity

$$\theta(Z) \circ \mu_t^* = \mu_t^* \circ \theta(Z) = t \frac{d}{dt} \mu_t^*.$$

Conclude that ∇ is determined by the polar trivialization and the curvature.

28. Let F be a real finite dimensional vector space and let Z be the radial vector field in F: $Z(x) = x$, $x \in F$. Consider the linear map $i(Z): A(F) \to A(F)$.

(i) Show that $i(Z)^2 = 0$.

(ii) Show that $\Phi \in \operatorname{Im} i(Z)$ if and only if $i(Z)\Phi = 0$.

Hint: If $\Phi \in A^p(F)$ and $i(Z)\Phi = 0$, show that $\Phi = i(Z)\Psi$, where

$$\Psi(x) = \int_0^1 (\delta\Phi)(tx) t^p\, dt.$$

29. Let F be a real finite dimensional vector space and let Z be the radial vector field in F. Establish the following theorem.

Theorem: Let ∇ be a linear connection in F with curvature R and torsion S. Define $\Omega \in A^1(F; L_{\tau_F})$ and $\Xi \in A^1(F; \tau_F)$ by

$$\Omega = i(Z)R \qquad \text{and} \qquad \Xi = i(Z)S.$$

Then

$$i(Z)\Omega = 0 \qquad \text{and} \qquad i(Z)\Xi = 0.$$

Conversely, let $\Omega \in A^1(F; L_{\tau_F})$ and $\Xi \in A^1(F; \tau_F)$ satisfy $i(Z)\Omega = 0$ and $i(Z)\Xi = 0$. Then there is a neighbourhood V of the origin and a unique linear connection, ∇, in V such that

(a) The curves $x(t) = ta$ $(0 \leqslant t \leqslant 1)$ are autoparallel.

(b) The curvature, R, of ∇ satisfies $i(Z)R = \Omega$.

(c) The torsion, S, of ∇, satisfies $i(Z)S = \Xi$.

Hint: Let U be an open ball about 0 of radius r $(0 < r \leqslant \infty)$:

(i) Suppose ∇ is a linear connection in U which satisfies conditions (a), (b), and (c). Let $\lambda: U \times F \xrightarrow{\cong} T_U$ be the standard trivialization of τ_U given by

$$\lambda(x, h)(f) = f'(x; h), \qquad x \in U, \quad h \in F, \quad f \in \mathscr{S}(U),$$

and let $\sigma: U \times F \to T_U$ be the polar trivialization. For notational convenience use λ to identify $U \times F$ with T_U so that λ becomes the identity map.

Define $\Phi \in A^1(U; L_F)$, $\varphi \in \mathscr{S}(U; L_F)$, and $\Psi \in A^1(U; L_F)$ by

$$\Phi(x; h) = \sigma_x^{-1}(h), \qquad \varphi(x) = \sigma_x^{-1}, \qquad \text{and} \qquad \delta + \Psi = \hat{\nabla},$$

where $\hat{\nabla}$ is the pull-back of ∇ to $U \times F$ via σ.

Use the structure equations of problem 5 to derive the ordinary differential equations (for fixed $x \in U$, $h \in F$)

$$\frac{d}{dt}\Phi(tx; th) = \Phi(tx; S(tx; x, th)) + \Psi(tx; th)(x) + h$$

and

$$\frac{d}{dt}\Psi(tx; th) = \varphi(tx) \circ R(tx; x, th) \circ \varphi(tx)^{-1}.$$

(ii) *Uniqueness.* Fix $x \in U$. Define smooth functions

$$r, s: [-1, 1] \to L_F$$

by

$$r(t)(h) = R(tx; x, h)(x) \qquad \text{and} \qquad s(t)(h) = S(tx; x, h).$$

Show that r and s are determined by $i(Z)R$ and $i(Z)S$.

Define $\alpha: [-1, 1] \to L_F$ by

$$\alpha(t) = t\varphi(tx).$$

Show that α satisfies the differential equation,

$$\ddot{\alpha} - \dot{\alpha} \circ s - \alpha \circ (\dot{s} + r) = 0,$$

and the initial conditions

$$\alpha(0) = 0, \qquad \dot{\alpha}(0) = \iota.$$

Conclude that φ is uniquely determined by conditions (a), (b), and (c). Hence (cf. problem 27) establish uniqueness.

(iii) *Existence*: Assume that Ω and Ξ are given subject to the stated conditions. Use problem 28 to construct 2-forms $\hat{R} \in A^2(F; L\tau_F)$ and $\hat{S} \in A^2(F; \tau_F)$ satisfying

$$i(Z)\hat{R} = \Omega \qquad \text{and} \qquad i(Z)\hat{S} = \Xi.$$

Use $\hat{S}$ and $\hat{R}$ to define $\hat{r}$ and $\hat{s}$ as in (ii). Construct $\hat{\varphi}$: $\mathscr{S}(F; L_F)$ so that $\hat{\alpha}(t) = t\hat{\varphi}(tx)$ satisfies the corresponding differential equation of (ii). Show that in some neighbourhood, V, of 0, $\hat{\varphi}(x)$ is an isomorphism if $x \in V$.

Finally solve the second differential equation of (i) (with φ and R replaced by $\hat{\varphi}$ and $\hat{R}$) in V to obtain $\hat{\Psi} \in A^1(V; L_F)$. Set $\hat{\sigma}_x = \hat{\varphi}(x)^{-1}$ and let ∇ be the unique linear connection in V which pulls back under $\hat{\sigma}$ to $\delta + \hat{\Psi}$. Show that ∇ has the desired properties.

30. (i) Use problems 26 and 29 to conclude that a linear connection, ∇, in τ_M is completely determined in the neighbourhood of a point a by its torsion, S, its curvature, R, and the map $\exp_a$.

(ii) Suppose $R = 0$. Obtain a first order linear differential equation for the polar trivialization.

(iii) Suppose $S = 0$. Show (in the notation of problem 29) that

$$\ddot{\alpha} = \alpha \circ r.$$

Conclude that the function β given by $\beta(t) = t\alpha^{-1} \circ \dot{\alpha}$ satisfies

$$\dot{\beta} + \beta^2 - t\beta = t^2 r.$$

(iv) Suppose two linear connections ∇_1 and ∇_2 in τ_M have the same curvature and the same torsion. Is it necessarily true that $\nabla_1 = \nabla_2$?

31. Conjugate connections. Let ∇ and $\bar{\nabla}$ be conjugate connections in τ_B. Let S be the torsion of ∇ and let R and $\bar{R}$ denote the respective curvatures. Let D denote the covariant derivative of ∇ in $\mathscr{X}(B; \tau_B)$ (cf. problem 8). The torsion, S, is called *parallel* if $DS = 0$.

(i) Show that

$$\bar{R}(X, Y)(Z) + R(Z, X)(Y) + R(Y, Z)(X) = DS(Z, X, Y).$$

(ii) Assume $R = 0$. Conclude that $\bar{R} = 0$ if and only if the torsion is parallel.

(iii) Assume the torsion is parallel. Show that the Lie bracket of ∇-parallel vector fields X and Y is given by $[X, Y] = -S(X, Y)$. Conclude that $[X, Y]$ is parallel.

(iv) Assume $R = 0$ and $\bar{R} = 0$. Show that the ∇-parallel vector fields (respectively, the $\bar{\nabla}$-parallel vector fields) form a Lie algebra, E, of dimension n ($n = \dim B$). Show that if $\nabla X = 0$ and $\bar{\nabla} Y = 0$ then $[X, Y] = 0$.

(v) Assume $R = 0$ and $\bar{R} = 0$, and M is simply connected (cf. problem 5, Chap. VI). Obtain a trivialization $M \times E \xrightarrow{\cong} T_M$ such that $[X_h, X_k] = X_{[h,k]}$, where X_h is the vector field corresponding to the constant function $M \to h$. Show that, with respect to this trivialization, S becomes a constant function from M to $\wedge^2 E^* \otimes E$. Show that this function defines the Lie multiplication in E.

32. Local Lie groups. Let E be a finite dimensional real Lie algebra. Define $\omega \in A^1(E; E)$ by

$$\omega(x; h) = \int_0^1 \exp(-t \operatorname{ad} x) h \, dt, \qquad x, h \in E.$$

(i) Show that ω is the unique 1-form in E which satisfies

$$\delta\omega = -\tfrac{1}{2}[\omega, \omega]$$

and

$$\omega(x; x) = x, \qquad x \in E.$$

Show that for some neighbourhood, U, of the origin, ω defines a parallelism on U with parallel torsion. Show that the induced Lie algebra structure in the parallel vector fields coincides with the given Lie product in E (cf. problem 31, (v). (*Hint*: Use problems 26, 27, and 29.)

(ii) For $h \in E$ let $X_h \in \mathscr{X}(U)$ be the parallel vector field which satisfies $X_h(0) = h$. Let $Y_h \in \mathscr{X}(U)$ be parallel with respect to the conjugate parallelism and satisfy $Y_h(0) = h$ (cf. problem 31). Define a distribution on $U \times U \times U$ whose vectors have the form

$$Z_{h,k}(x, y, z) = (Y_h(x), X_k(y), X_k(z) + Y_h(z)), \qquad h, k \in E.$$

Show that this distribution is involutive.

(iii) Find a neighbourhood, V, of 0 in E and a smooth map $\mu\colon V \times V \to U$ induced by the distribution of (ii) such that $\mu(0, 0) = 0$. Show that this map is locally associative. Show that 0 is a local identity, and that elements near zero have inverses (cf. problem 17). μ is said to make V into a *local Lie group*.

33. Lie groups. Let G be a Lie group. Use right and left translations on G to define two parallelisms on G (cf. problem 19).

(i) Show that the corresponding torsions satisfy

$$S_L = -S_R$$

and conclude that the corresponding linear connections are conjugate (cf. problem 31).

(ii) Show that the torsion fields are parallel.

(iii) Let ∇_L and ∇_R denote the linear connections in the tangent bundle τ_G which are determined by these parallelisms. Show that the linear connection $\nabla_0 = \frac{1}{2}(\nabla_L + \nabla_R)$ has torsion zero and that its curvature is given by

$$R(X_h, X_k) = \tfrac{1}{4}\operatorname{ad}([h, k]), \qquad h, k \in E.$$

(iv) Show that for a left invariant Riemannian metric, g_L, $\nabla_L g_L = 0$ and that for a biinvariant Riemannian metric g_0, $\nabla_0 g_0 = 0$.

(v) Show that exp is an isomorphism with respect to the local multiplication of E defined in problem 32. Conclude that G admits an analytic structure with respect to which multiplication is analytic.

34. Immersions. Let $\varphi\colon M \to N$ be an immersion, and suppose N has a fixed Riemannian metric.

(i) Assign M the unique Riemannian metric such that each $(d\varphi)_x$ is an isometry *into*. This metric is called the *first fundamental tensor* of the immersion. Write $\varphi^*\tau_N = \tau_M \oplus \tau_M^\perp$, and identify $\tau_M^\perp$ with the normal bundle, ν, of M in N (cf. problem 19, Chap. III, volume I).

(ii) Let $i\colon \tau_M \to \varphi^*\tau_N$ and $\rho\colon \varphi^*\tau_N \to \tau_M$ be the induced inclusion and projection. If ∇_N is a Riemannian connection in N, and ∇_N pulls back to $\hat{\nabla}$ in $\varphi^*\tau_N$, show that $\nabla = \rho_* \hat{\nabla} i_*$ is a Riemannian connection in M.

(iii) Show that the torsions S_M of ∇ and S_N of ∇_N are related by

$$S(x; h, k) = \rho_x(S_N(\varphi(x); (d\varphi)h, (d\varphi)k)).$$

Conclude that if ∇_N is the Levi–Civita connection for N, then ∇ is the Levi–Civita connection for M.

(iv) Assume ∇_N is the Levi–Civita connection for N. Show that a covariant field $\Phi \in \mathscr{X}^2(M; \nu)$ is defined by

$$\hat{\nabla}_X(Y) = \nabla_X(Y) + \Phi(X, Y), \qquad X, Y \in \mathscr{X}(M).$$

Show that $\Phi(X, Y) = \Phi(Y, X)$. Φ is called the *second fundamental tensor* of the immersion.

35. Immersions in $\mathbb{R}^{n+1}$. Suppose $\varphi\colon M \to \mathbb{R}^{n+1}$ is an immersion of an oriented n-manifold in an oriented Euclidean $(n + 1)$-space. Give M the induced Riemannian metric.

(i) Show that there is a unique smooth map $Z\colon M \to S^n$ such that

$$(h, t) \mapsto (d\varphi)_x(h) + tZ(x), \qquad h \in T_x(M), \quad t \in \mathbb{R},$$

defines an orientation preserving isometry from $\tau_M \oplus \epsilon$ to $M \times \mathbb{R}^{n+1}$.

(ii) Define $B \in \mathscr{X}^2(M)$ by $B(X, Y) = \langle \Phi(X, Y), Z \rangle$, where Φ is the second fundamental tensor for the immersion. Show that B can be interpreted as a cross-section, Ξ, in L_{τ_M} or as a 1-form, $\Psi \in A^1(M; \tau_M)$, via the relations

$$B(x; h, k) = \langle \Xi(x)(h), k \rangle = \langle \Psi(x; h), k \rangle.$$

Show that each $\Xi(x)$ is symmetric.

(iii) Establish the formulae

(a) $\delta Z = -\varphi_*(\Psi)$ (Weingarten)

(b) $R^{\boldsymbol{\cdot}} = -\frac{1}{2}\Psi \wedge \Psi$ (Gauss)

(c) $\nabla \Psi = 0$ (Codazzi–Mainardi).

(Here $R^{\boldsymbol{\cdot}}$ is the Riemannian curvature for M.)

(iv) The *Gaussian curvature* of the immersed manifold (M, φ) is the function, K, on M given by

$$K(x) = \det \Xi(x), \qquad x \in M.$$

Assume that dim M is even. Show that the Gaussian curvature depends only on the Riemannian metric of M and hence is invariant under isometries of M. (Theorema egregium.)

(v) Show that the even characteristic coefficients of Ξ can be expressed in terms of $R^{\bullet}$.

36. Local diffeomorphisms into $\mathbb{R}^n$. (i) Let $\varphi: M \to \mathbb{R}^n$ be a local diffeomorphism from an n-manifold to $\mathbb{R}^n$. Denote by ∇ the connection in M induced from the standard connection in $\mathbb{R}^n$ via φ. Show that $R = 0$ and $S = 0$.

(ii) Assume that M is a simply connected n-manifold and that ∇ is a linear connection in τ_M such that $R = 0$ and $S = 0$. Construct a local diffeomorphism $\varphi: M \to \mathbb{R}^n$ which is connection preserving with respect to ∇ and the standard connection in $\mathbb{R}^n$.

37. The Bonnet immersion theorem. Let (M, g) be a simply connected Riemannian n-manifold (cf. problem 5, Chap. VI). Let $B \in \mathscr{T}^2(M)$ be a symmetric tensor field, and define $\Psi \in A^1(M; \tau_M)$ as in problem 35. Assume Ψ satisfies the Gauss and Codazzi–Mainardi equations.

(i) Construct an immersion $\varphi: M \to \mathbb{R}^{n+1}$ such that g and B are the corresponding fundamental tensors. (*Hint*: Use problem 36).

(ii) If φ and ψ are two such immersions show that

$$\psi = \tau \circ \varphi,$$

where τ is an isometry of $\mathbb{R}^{n+1}$.

38. Flat connections on $\mathbb{R}P^3$. Assume that ∇ is a linear connection in $\mathbb{R}P^3$ with zero curvature. Show that there exists a parallelism on $\mathbb{R}P^3$ which induces ∇. Do the same for $\mathbb{R}P^7$.

Chapter VIII

Characteristic Homomorphism for Σ-bundles

In this chapter $\Gamma = \mathbb{R}$ or $\mathbb{C}$ and vector spaces and vector bundles are defined over Γ. We continue the linear and multilinear conventions announced at the beginning of Chapter VII.

§1. Σ-bundles

8.1. Definition. Let F be a vector space and write

$$F^{p,q} = \bigotimes^p F^* \otimes \bigotimes^q F.$$

A linear isomorphism $\alpha: F \xrightarrow{\cong} H$ induces the isomorphism

$$\bigotimes^p (\alpha^*)^{-1} \otimes \bigotimes^q \alpha : F^{p,q} \xrightarrow{\cong} H^{p,q} ;$$

it will be denoted by $\alpha^{p,q}$, or simply by α.

If $\xi = (M, \pi, B, F)$ is a vector bundle, we shall write

$$\xi^{p,q} = (\bigotimes^p \xi^*) \otimes (\bigotimes^q \xi)$$

(cf. sec. 7.8). The fibre of this bundle at $x \in B$ is the space $F_x^{p,q}$. A bundle map $\varphi: \xi \to \eta$ that restricts to isomorphisms in the fibres induces the bundle maps $\varphi^{p,q}: \xi^{p,q} \to \eta^{p,q}$ defined by

$$\varphi_x^{p,q} = (\varphi_x)^{p,q}, \qquad x \in B.$$

We shall frequently denote $\varphi^{p,q}$ simply by φ. Thus, if σ is a cross-section in $\eta^{p,q}$, then $\varphi^\#\sigma$ is the pulled-back cross-section in $\xi^{p,q}$.

In particular, if $\{(U_\alpha, \psi_\alpha)\}$ is a coordinate representation for ξ, then $\{(U_\alpha, \psi_\alpha^{p,q})\}$ is a coordinate representation for $\xi^{p,q}$.

Definition: A *Σ-bundle* is a pair (ξ, Σ_ξ) where:

(i) $\xi = (M, \pi, B, F)$ is a smooth vector bundle.

(ii) $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$ is a finite ordered set of cross-sections $\sigma_i \in \operatorname{Sec} \xi^{p_i, q_i}$, subject to the following condition:

(iii) There is a finite ordered system $\Sigma_F = (v_1, ..., v_m)$ of tensors $v_i \in F^{p_i, q_i}$ and there is a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ of ξ such that

$$\psi_\alpha(x, v_i) = \sigma_i(x), \qquad x \in U_\alpha, \quad i = 1, ..., m.$$

A coordinate representation for ξ that satisfies condition (iii) will be called a *Σ-coordinate representation* for the Σ-bundle (ξ, Σ_ξ). Thus $\{(U_\alpha, \varphi_\alpha)\}$ is a Σ-coordinate representation if and only if for each i and α, $\varphi_\alpha^\# \sigma_i$ (regarded as an F^{p_i, q_i}-valued function in U_α) is the constant function $U_\alpha \to v_i$.

Remark: A Σ-structure in a vector bundle can be regarded as a reduction of the structure group from $GL(F)$ to an *algebraic* subgroup (cf. article 7).

Now suppose (ξ, Σ_ξ) and (η, Σ_η) are Σ-bundles, where $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$ and $\Sigma_\eta = (\tau_1, ..., \tau_m)$. Then a *$\Sigma$-homomorphism* between these Σ-bundles is a bundle map, $\varphi: \xi \to \eta$, restricting to isomorphisms in the fibres and satisfying

$$\varphi^\#(\tau_i) = \sigma_i, \qquad i = 1, ..., m.$$

8.2. 0-deformable cross-sections. Let $\xi = (M, \pi, B, F)$ be a vector bundle, and let $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$ be some ordered set of cross-sections: $\sigma_i \in \operatorname{Sec} \xi^{p_i, q_i}$, $i = 1, ..., m$.

Definition: The set Σ_ξ is called *0-deformable* if, for each pair $x, y \in B$, there is a linear isomorphism,

$$\alpha_{x,y}: F_x \xrightarrow{\cong} F_y,$$

such that

$$\alpha_{x,y}(\sigma_i(x)) = \sigma_i(y), \qquad i = 1, ..., m.$$

Theorem I: Let $\xi = (M, \pi, B, F)$ be a vector bundle, and let $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$ $(\sigma_i \in \operatorname{Sec} \xi^{p_i, q_i})$. Then (ξ, Σ_ξ) is a Σ-bundle if and only if Σ_ξ is 0-deformable.

Proof: If (ξ, Σ_ξ) is a Σ-bundle, then Σ_ξ is obviously 0-deformable.

Conversely, assume Σ_ξ is 0-deformable. We must show that (ξ, Σ_ξ) admits a Σ-coordinate representation. Since Σ_ξ is 0-deformable, we may choose a fixed set of tensors $v_i \in F^{p_i, q_i}$ and linear isomorphisms,

$$\alpha_x : F \xrightarrow{\cong} F_x, \qquad x \in B,$$

satisfying $\alpha_x(v_i) = \sigma_i(x)$, $i = 1, \ldots, m$.

To construct a Σ-coordinate representation, we lose no generality in assuming $M = B \times F$. Thus the σ_i are smooth maps $\sigma_i : B \to F^{p_i, q_i}$. Now fix $a \in B$ and use α_a to arrange that

$$\sigma_i(a) = v_i, \qquad i = 1, \ldots, m.$$

Then consider the vector space,

$$H = F^{p_1, q_1} \oplus \cdots \oplus F^{p_m, q_m},$$

and let $\sigma: B \to H$ be the smooth map given by

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_m(x)), \qquad x \in B.$$

Set $\sigma(a) = v$ and let K be the isotropy subgroup of $GL(F)$ at v (with respect to the obvious representation of $GL(F)$ in H). Then (cf. sec. 3.5) the smooth map $A_v : GL(F) \to H$ given by $A_v(\varphi) = \varphi(v)$ induces an embedding

$$\bar{A}_v : GL(F)/K \to H.$$

Moreover, the image of $\bar{A}_v$ is the orbit of $GL(F)$ through v.

Now, since Σ_ξ is 0-deformable, each vector $\sigma(x)$ lies on the orbit of $GL(F)$ through v; i.e.,

$$\operatorname{Im} \sigma \subset \operatorname{Im} \bar{A}_v .$$

Thus we can apply Theorem I, sec. 3.7, to obtain a smooth map, $\tau: B \to GL(F)/K$, such that the diagram,

$$\begin{tikzcd} & GL(F)/K \arrow[dd, "\bar{A}_v"] \\ B \arrow[ur, "\tau"] \arrow[dr, "\sigma"'] & \\ & H \end{tikzcd}$$

commutes.

Let $\pi_K: GL(F) \to GL(F)/K$ be the projection. Since π_K is a bundle projection, (cf. sec. 2.13) there is a neighbourhood U_a of a in B and a smooth map, $\omega_a: U_a \to GL(F)$, such that

$$\pi_K \circ \omega_a = \tau.$$

This implies that $(\omega_a(x))v = \sigma(x)$, $x \in U_a$.

Hence a Σ-coordinate representation $\{(U_a, \varphi_a)\}_{a \in B}$ for ξ is given by

$$\varphi_a(x, y) = (x, \omega_a(x)y), \qquad x \in U_a, \quad y \in F.$$

Q.E.D.

8.3. Examples: 1. $\Sigma_F = \varnothing$. In this case the Σ-bundles are just vector bundles.

2. *Riemannian bundles*: A Riemannian bundle (ξ, g) is a Σ-bundle. (This follows from Theorem I, sec. 8.2, which in this case coincides with Proposition V, sec. 2.17, volume I). We may choose $\Sigma_F = (\langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is an inner product in F; then a Σ-coordinate representation is a Riemannian coordinate representation.

3. *Oriented Riemannian bundles*: Let ξ be an oriented Riemannian vector bundle with Riemannian metric g_ξ. Let Δ_ξ be the unique positive normed determinant function in ξ (cf. sec. 2.19, volume I). Set $\Sigma_\xi = (g_\xi, \Delta_\xi)$. Then (ξ, Σ_ξ) is a Σ-bundle (apply Theorem I, which in this case coincides with Proposition VIII, sec. 2.19, volume I).

4. *Complex bundles as real Σ-bundles*: Let $\xi_{\mathbb{R}}$ be the underlying real bundle of a complex vector bundle ξ (cf. sec. 2.22, volume I). Let $i_\xi \in \operatorname{Sec} L_{\xi_{\mathbb{R}}}$ ($= \operatorname{Sec} \xi_{\mathbb{R}}^{1,1}$) be defined by

$$i_\xi(x)(z) = iz, \qquad z \in F_x.$$

Set $\Sigma_{\xi_{\mathbb{R}}} = (i_\xi)$. Then $(\xi_{\mathbb{R}}, \Sigma_{\xi_{\mathbb{R}}})$ is a Σ-bundle. Its Σ-coordinate representations coincide with the coordinate representations of the *complex* bundle ξ.

5. *Whitney sums*: Let ξ, η be vector bundles over the same base. The projections,

$$\rho_\xi: \xi \oplus \eta \to \xi, \qquad \rho_\eta: \xi \oplus \eta \to \eta,$$

may be regarded as strong bundle maps of the bundle $\xi \oplus \eta$. Thus we may write

$$\rho_\xi, \rho_\eta \in \operatorname{Sec} L_{\xi \oplus \eta} = \operatorname{Sec}(\xi \oplus \eta)^{1,1}.$$

It follows from Theorem I (or sec. 2.8, volume I) that, with $\Sigma_{\xi\oplus\eta} = (\rho_\xi, \rho_\eta)$, the pair $(\xi \oplus \eta, \Sigma_{\xi\oplus\eta})$ is a Σ-bundle.

6. *Cross-sections in* L_ξ : Let $\xi = (M, \pi, B, F)$ be a vector bundle and let $\sigma \in \operatorname{Sec} L_\xi$. Then the set $\Sigma_\xi = (\sigma)$ is 0-deformable if and only if for each $x, y \in B$, there is a linear isomorphism $\alpha\colon F_x \xrightarrow{\cong} F_y$ such that

$$\alpha \circ \sigma(x) \circ \alpha^{-1} = \sigma(y).$$

This is equivalent to the following two conditions: For all $x, y \in B$,

(1) $\sigma(x)$ and $\sigma(y)$ have the same characteristic polynomial $p(t)$.

(2) If $p(t) = p_1(t)^{r_1} \cdots p_m(t)^{r_m}$ is the decomposition of p into prime factors, then

$$\operatorname{rank}[p_i(\sigma(x))]^j = \operatorname{rank}[p_i(\sigma(y))]^j, \qquad j = 1, \dots, r_i, \quad i = 1, \dots, m.$$

Moreover, these two conditions are equivalent to the conditions:

(1′) $\sigma(x)$ and $\sigma(y)$ have the same minimal polynomial $\mu(t)$.

(2′) If $\mu(t) = \mu_1(t)^{l_1} \cdots \mu_m(t)^{l_m}$ is the decomposition of μ into prime factors, then

$$\operatorname{rank}[\mu_i(\sigma(x))]^j = \operatorname{rank}[\mu_i(\sigma(y))]^j, \qquad j = 1, \dots, l_i, \quad i = 1, \dots, m.$$

Thus we can apply Theorem I, sec. 8.2, to obtain

Proposition I: Let $\sigma \in \operatorname{Sec} L_\xi$ and set $\Sigma_\xi = (\sigma)$. Then the following are equivalent:

(i) σ satisfies (1) and (2).

(ii) σ satisfies (1′) and (2′).

(iii) (ξ, Σ_ξ) is a Σ-bundle.

Corollary: Assume that B is connected. Suppose $\tau \in \operatorname{Sec} L_\xi$ and assume $f(t)$ is a polynomial with no repeated roots such that

$$f(\tau(x)) = 0, \qquad x \in B.$$

Then $(\xi, (\tau))$ is a Σ-bundle.

Proof: Write $f(t) = f_1(t) \cdots f_m(t)$, where the f_i are irreducible and relatively prime, and set $\tau_i(x) = f_i(\tau(x))$ $(i = 1, \dots, m)$. Then

$$\operatorname{rank}(\tau_i(x)^j) = \operatorname{rank} \tau_i(x), \qquad i = 1, \dots, m, \quad j = 1, 2, \dots,$$

and

$$\sum_{i=1}^{m} \operatorname{rank} \tau_i(x) = (m-1) \dim F_x, \qquad x \in B.$$

Now fix $a \in B$. Then there is a neighbourhood U of a such that

$$\operatorname{rank} \tau_i(x) \geqslant \operatorname{rank} \tau_i(a), \qquad x \in U, \quad i = 1, \ldots, m.$$

Since the sum of these ranks is constant, we obtain

$$\operatorname{rank} \tau_i(x) = \operatorname{rank} \tau_i(a), \qquad x \in U, \quad i = 1, \ldots, m.$$

Thus these ranks are locally constant, and since B is connected, they are constant. It follows easily that τ satisfies (1′) and (2′).

Q.E.D.

Examples: 7. *Algebra bundles*: A vector bundle ξ is called an *algebra bundle* if each fibre F_x, and the typical fibre F are algebras (not necessarily associative), and if ξ admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that each map

$$\psi_{\alpha,x}: F \xrightarrow{\cong} F_x$$

is an isomorphism of algebras.

In particular, if η is any vector bundle then $\wedge\eta$ and L_η are algebra bundles.

Algebra bundles may be regarded as Σ-bundles as follows. Regard the multiplication in F_x,

$$\mu_x: F_x \otimes F_x \to F_x,$$

as an element of $F_x^{2,1}$. Then the hypothesis above implies that $x \mapsto \mu_x$ is a cross-section in $\xi^{2,1}$ and that (ξ, Σ_ξ) is a Σ-bundle where $\Sigma_\xi = (\mu)$.

Let ξ be a vector bundle and let $\mu \in \operatorname{Sec} \xi^{2,1}$. Thus μ makes each fibre F_x into an algebra. Theorem I, sec. 8.2, implies that (ξ, μ) is an algebra bundle if and only if the algebras F_x $(x \in B)$ are all isomorphic.

8. *Lie algebra bundles*: An algebra bundle is called a *Lie algebra bundle* if the fibres F and F_x are Lie algebras. As an example, consider the bundle L_ξ with Lie product given by

$$[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha, \qquad \alpha, \beta \in L_{F_x}, \qquad x \in B.$$

8.4. The associated Lie algebra bundle. Let $\xi = (M, \pi, B, F)$, $\Sigma_\xi = (\sigma_1, \ldots, \sigma_m)$ be a Σ-bundle. If $x \in B$, let $G_{(x)}$ be the closed subgroup of $GL(F_x)$ consisting of those φ which satisfy

$$\varphi(\sigma_i(x)) = \sigma_i(x), \qquad i = 1, \ldots, m.$$

Its Lie algebra, $E_{(x)}$, is the subalgebra of L_{F_x} consisting of the linear transformations ψ which satisfy

$$\theta(\psi)(\sigma_i(x)) = 0, \qquad i = 1, \ldots, m,$$

where $\theta(\psi)$ is the linear transformation of $F_x^{p,q}$ defined as in sec. 7.8 (cf. sec. 1.9).

Next, fix a set of tensors $v_i \in F^{p_i,q_i}$ such that the set $\Sigma_F = (v_1, \ldots, v_m)$ corresponds to Σ_ξ under a Σ-coordinate representation. The subgroup G of $GL(F)$ consisting of the automorphisms that fix each v_i is called the *structure group* of the Σ-bundle (ξ, Σ_ξ). Its Lie algebra is denoted by E.

Now we shall construct a Lie algebra bundle (cf. Example 8, sec. 8.3),

$$\xi_E = (M_E, \pi_E, B, E),$$

whose fibre at x is the Lie algebra $E_{(x)}$. It will be called the *Lie algebra bundle associated with the Σ-bundle* (ξ, Σ_ξ).

In fact, let $\{(U_\alpha, \varphi_\alpha)\}$ be a Σ-coordinate representation for (ξ, Σ_ξ). It determines the coordinate representation $\{(U_\alpha, \hat{\varphi}_\alpha)\}$ for L_ξ given by

$$\hat{\varphi}_{\alpha,x}(\varphi) = \varphi_{\alpha,x} \circ \varphi \circ \varphi_{\alpha,x}^{-1}, \qquad \varphi \in L_F.$$

Since $\varphi_{\alpha,x}(v_i) = \sigma_i(x)$ (for each i, α), it follows that $\hat{\varphi}_{\alpha,x}$ restricts to a Lie algebra isomorphism

$$\hat{\varphi}_{\alpha,x}\colon E \xrightarrow{\cong} E_{(x)}.$$

Thus the $E_{(x)}$ are the fibres of a subbundle, ξ_E, of L_ξ with coordinate representation $\{(U_\alpha, \hat{\varphi}_\alpha)\}$. ξ_E is the desired Lie algebra bundle.

A Σ-homomorphism $\varphi\colon (\xi, \Sigma_\xi) \to (\eta, \Sigma_\eta)$ induces the bundle map $\hat{\varphi}\colon L_\xi \to L_\eta$ which corresponds to $\varphi^{1,1}$ under the isomorphisms $L_\xi \cong \xi^* \otimes \xi$, $L_\eta \cong \eta^* \otimes \eta$. Thus $\hat{\varphi}$ restricts to a bundle map,

$$\varphi_E\colon \xi_E \to \eta_E,$$

and each $(\varphi_E)_x$ is a Lie algebra isomorphism.

In particular, suppose (ξ, Σ_ξ) is a complex Σ-bundle ($\Gamma = \mathbb{C}$). Then L_ξ is a complex vector bundle. Since $E_{(x)}$ consists of the elements $\psi \in L_{F_x}$ satisfying

$$\theta(\psi)(\sigma_i(x)) = 0, \qquad i = 1, ..., m,$$

it follows that $E_{(x)}$ is a *complex* subalgebra of L_{F_x}. Thus ξ_E is a complex subbundle of L_ξ. Finally, if $\varphi: \xi \to \eta$ is a Σ-homomorphism, then each map $(\varphi_E)_x$ is complex linear. Hence φ_E is a homomorphism of complex vector bundles.

§2. Σ-connections

8.5. Definition. Let ξ be a vector bundle and let

$$\Sigma_\xi = (\sigma_1, ..., \sigma_m), \qquad \sigma_i \in \operatorname{Sec} \xi^{p_i, q_i}, \quad i = 1, ..., m,$$

be an ordered set of cross-sections. A *Σ-connection* in (ξ, Σ_ξ) is a linear connection, ∇, satisfying

$$\nabla \sigma_i = 0, \qquad i = 1, ..., m.$$

Theorem II: Let ξ be a vector bundle over a connected base B and let $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$ be an ordered set of cross-sections.

Then (ξ, Σ_ξ) is a Σ-bundle if and only if it admits a Σ-connection.

Proof: Assume (ξ, Σ_ξ) is a Σ-bundle. Choose a Σ-coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ for ξ and let $\{p_\alpha\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$.

Further, let ∇_α be the linear connection in the restriction of ξ to U_α corresponding to δ under φ_α. Then (since $\varphi_\alpha^\# \sigma_i$ is constant)

$$\nabla_\alpha \sigma_i = 0, \qquad \text{for} \quad \text{each } i, \alpha.$$

Thus setting $\nabla = \sum_\alpha p_\alpha \cdot \nabla_\alpha$ (Example 3, sec. 7.11) we find

$$\nabla \sigma_i = 0, \qquad i = 1, ..., m.$$

Conversely, let ∇ be a Σ-connection in ξ. Since B is connected, Proposition VIII of sec. 7.17, shows that Σ_ξ is 0-deformable. Now Theorem I, sec. 8.2, implies that (ξ, Σ_ξ) is a Σ-bundle.

Q.E.D.

If $\varphi: \xi \to \eta$ is a homomorphism of Σ-bundles and if ∇_η is a Σ-connection in η, then the pull-back of ∇_η is a Σ-connection in ξ.

8.6. Examples: 1. $\Sigma_\xi = \varnothing$; any linear connection is a Σ-connection.

2. *Riemannian connections*: Suppose $\Sigma_\xi = (\langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is a Riemannian metric. Then the Σ-connections are precisely the Riemannian connections (cf. sec. 7.23).

If in addition ξ is orientable, let Δ_ξ be the positive normed determinant function. Then (cf. sec. 7.24) a Riemannian connection ∇ satisfies $\nabla(\Delta_\xi) = 0$. It follows that the Riemannian connections are the Σ-connections in the Σ-bundle (ξ, Σ_ξ) with $\Sigma_\xi = (\langle\ ,\ \rangle, \Delta_\xi)$.

3. *Complex bundles*: The linear connections in a complex bundle ξ coincide with the Σ-connections in the corresponding Σ-bundle $(\xi_{\mathbb{R}}, \Sigma_{\xi_{\mathbb{R}}})$ of Example 4, sec. 8.3.

4. *Algebra bundles*: Let $\Sigma_\xi = (\mu)$, where $\mu \in \operatorname{Sec} \xi^{2,1}$ makes ξ into an algebra bundle (cf. Example 7, sec. 8.3). Denote the induced multiplication in $A(B; \xi)$ by $\cdot$. Then a linear connection ∇ in ξ is a Σ-connection if and only if

$$\nabla(\sigma \cdot \tau) = \nabla\sigma \cdot \tau + \sigma \cdot \nabla\tau, \qquad \sigma, \tau \in \operatorname{Sec} \xi.$$

5. *Trivial Σ-bundles*: Consider the trivial Σ-bundle $(B \times F, \Sigma_F)$, where $\Sigma_F = (v_1, \ldots, v_m)$ is a set of tensors in F, regarded as constant cross-sections. Every linear connection in $B \times F$ is of the form

$$\nabla = \delta + \Psi, \qquad \Psi \in A^1(B; L_F).$$

Let E be the Lie algebra of the subgroup $G \subset GL(F)$ which fixes the v_i. Then E is a subalgebra of L_F, and ∇ is a Σ-connection if and only if $\Psi \in A^1(B; E)$.

8.7. The bundle ξ_E. Proposition II: Let ∇ be a Σ-connection in a Σ-bundle (ξ, Σ_ξ), where $\Sigma_\xi = (\sigma_1, \ldots, \sigma_m)$. Then

(1) The connection, $\hat{\nabla}$, induced in L_ξ restricts to a linear connection in the associated Lie algebra bundle ξ_E.

(2) The curvature, R, of ∇ takes values in ξ_E; $R \in A^2(B; \xi_E)$.

Proof: (1) Let $\tau \in \operatorname{Sec} \xi_E$; we must show that

$$\hat{\nabla}\tau \in A^1(B; \xi_E).$$

This is equivalent to (cf. sec. 7.8)

$$(\theta_*(\hat{\nabla}\tau))(\sigma_i) = 0, \qquad i = 1, \ldots, m.$$

But (cf. Example 4, sec. 7.12)

$$\theta_*(\hat{\nabla}\tau) = \nabla \circ \theta_*(\tau) - \theta_*(\tau) \circ \nabla$$

and

$$(\theta_*(\tau))(\sigma_i) = 0 = \nabla\sigma_i\,, \qquad i = 1, 2, ..., m.$$

(2) Recall from Example 4, sec. 7.16, that $\theta_*(R)$ is the curvature of the connection induced in ξ^{p_i, q_i}. It follows that

$$\theta_*(R)(\sigma_i) = \nabla^2\sigma_i = 0, \qquad i = 1, ..., m,$$

and so R takes values in ξ_E.

Q.E.D.

§3. Invariant subbundles

In this article (ξ, Σ_ξ) denotes a Σ-bundle with $\xi = (M, \pi, B, F)$ and $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$: σ_i is a cross-section in ξ^{p_i,q_i}. $\Sigma_F = (v_1, ..., v_m)$ is a set of tensors ($v_i \in F^{p_i,q_i}$) corresponding to the σ_i under a Σ-coordinate representation. ∇ denotes a Σ-connection in (ξ, Σ_ξ) with curvature R. G is the structure group of (ξ, Σ_ξ) (cf. sec. 8.4); its Lie algebra is denoted by E. If the underlying coefficient field Γ is $\mathbb{C}$, then E is a complex Lie algebra. In this case E^* is the complex dual of E. The adjoint representation of G in E is given by

$$(\mathrm{Ad}\,\varphi)(\psi) = \varphi \circ \psi \circ \varphi^{-1}, \qquad \varphi \in G, \quad \psi \in E;$$

the contragredient representation, $\mathrm{Ad}^\natural$, of G in E^* is defined by

$$\mathrm{Ad}^\natural(\varphi) = (\mathrm{Ad}\,\varphi^{-1})^*.$$

Note that the contragredient representation of Ad was defined in sec. 1.9 to be a representation in the space $\mathrm{Hom}(E; \mathbb{R})$. However, if E is complex, a complex structure is induced in $\mathrm{Hom}(E; \mathbb{R})$. In this case, there is a natural identification of the complex spaces E^* and $\mathrm{Hom}(E; \mathbb{R})$, and under this identification the two definitions of $\mathrm{Ad}^\natural$ coincide.

We continue to use the notation established in sec. 8.4, and note that, if $\Gamma = \mathbb{C}$, then the associated Lie algebra bundle, ξ_E, is a complex vector bundle. We remind the reader that in this case all linear and multilinear operations with respect to ξ, ξ_E, and E are taken over $\mathbb{C}$.

8.8. Invariant subbundles. The representation $\mathrm{Ad}^\natural$ of G extends to representations of G in $\otimes^p E^*$ and $\vee^p E^*$. In particular, we have the invariant subspaces $(\otimes^p E^*)_I$ and $(\vee^p E^*)_I$. Similarly, for $x \in B$, representations of $G_{(x)}$ in $\otimes^p E^*_{(x)}$ and $\vee^p E^*_{(x)}$ are defined; their invariant subspaces are denoted by $(\otimes^p E^*_{(x)})_I$ and $(\vee^p E^*_{(x)})_I$.

Now let $\{(U_\alpha, \varphi_\alpha)\}$ be a Σ-coordinate representation for (ξ, Σ_ξ). Recall (sec. 8.4) that a coordinate representation $\{(U_\alpha, \hat{\varphi}_\alpha)\}$ for L_ξ is given by

$$\hat{\varphi}_{\alpha,x}(\sigma) = \varphi_{\alpha,x} \circ \sigma \circ \varphi_{\alpha,x}^{-1}, \qquad \sigma \in L_F, \quad x \in U_\alpha,$$

and that it restricts to a coordinate representation for ξ_E. Further, $\hat{\varphi}_{\alpha,x}$ restricts to an isomorphism of Lie groups $G \xrightarrow{\cong} G_{(x)}$.

Next, let $\{(U_\alpha, \hat{\varphi}_\alpha)\}$ also denote the induced coordinate representations for $\otimes^p \xi_E^*$ and $\vee^p \xi_E^*$. Then, for $x \in U_\alpha$, $\hat{\varphi}_{\alpha,x}$ restricts to isomorphisms

$$\psi_{\alpha,x}: (\otimes^p E^*)_I \xrightarrow{\cong} (\otimes^p E^*_{(x)})_I, \qquad \chi_{\alpha,x}: (\vee^p E^*)_I \xrightarrow{\cong} (\vee^p E^*_{(x)})_I.$$

It follows that the spaces $(\otimes^p E^*_{(x)})_I$ and $(\vee^p E^*_{(x)})_I$ are the fibres of subbundles,

$$(\otimes^p \xi_E^*)_I \subset \otimes^p \xi_E^* \qquad \text{and} \qquad (\vee^p \xi_E^*)_I \subset \vee^p \xi_E^*,$$

with coordinate representations $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\alpha, \chi_\alpha)\}$, respectively.

Definition: $(\otimes^p \xi_E^*)_I$ and $(\vee^p \xi_E^*)_I$ are called the *pth invariant tensor bundle*, and the *pth invariant symmetric bundle* associated with (ξ, Σ_ξ). A cross-section of $(\otimes^p \xi_E^*)_I$ (respectively, $(\vee^p \xi_E^*)_I$) is called *invariant.*

Proposition III: There are unique strong bundle isomorphisms,

$$\psi: B \times (\otimes^p E^*)_I \xrightarrow{\cong} (\otimes^p \xi_E^*)_I, \qquad \chi: B \times (\vee^p E^*)_I \xrightarrow{\cong} (\vee^p \xi_E^*)_I,$$

with the following property: If $\{(U_\alpha, \varphi_\alpha)\}$ is any Σ-coordinate representation for ξ, and $\psi_{\alpha,x}$ and $\chi_{\alpha,x}$ are the maps defined above, then

$$\psi_x = \psi_{\alpha,x} \qquad \text{and} \qquad \chi_x = \chi_{\alpha,x}, \qquad x \in U_\alpha.$$

Proof: It is sufficient to show that $\psi_{\alpha,x}$ and $\chi_{\alpha,x}$ are independent of the choice of U_α and of the choice of coordinate representation. Since the union of two Σ-coordinate representations is again a Σ-coordinate representation, it is sufficient to show that

$$\psi_{\alpha,x} = \psi_{\beta,x} \qquad \text{and} \qquad \chi_{\alpha,x} = \chi_{\beta,x}, \qquad x \in U_\alpha \cap U_\beta.$$

Write $\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x} = \tau$. Then $\tau \in G$ and

$$\hat{\varphi}_{\alpha,x}^{-1} \circ \hat{\varphi}_{\beta,x} = \operatorname{Ad} \tau: E \xrightarrow{\cong} E.$$

It follows that the induced isomorphisms $\hat{\varphi}_{\alpha,x}, \hat{\varphi}_{\beta,x}: \otimes^p E^* \xrightarrow{\cong} \otimes^p E^*_{(x)}$ satisfy

$$\hat{\varphi}_{\alpha,x}^{-1} \circ \hat{\varphi}_{\beta,x} = \otimes^p \operatorname{Ad}(\tau)^\natural.$$

Restricting this equation to $(\otimes^p E^*)_I$ yields

$$\psi_{\alpha,x}^{-1} \circ \psi_{\beta,x} = \iota, \qquad x \in U_\alpha \cap U_\beta.$$

Similarly, $\chi_{\alpha,x} = \chi_{\beta,x}$. Q.E.D.

Identify $(\otimes^p E^*)_I$ and $(\vee^p E^*)_I$ with the constant cross-sections of the trivial bundles. As in secs. 7.8 and 7.10, write

$$(\otimes E^*)_I = \sum_{p=0}^{\infty} (\otimes^p E^*)_I, \qquad (\vee E^*)_I = \sum_{p=0}^{\infty} (\vee^p E^*)_I$$

and

$$\mathrm{Sec}(\otimes \xi_E^*)_I = \bigoplus_{p=0}^{\infty} \mathrm{Sec}(\otimes^p \xi_E^*)_I, \qquad \mathrm{Sec}(\vee \xi_E^*)_I = \bigoplus_{p=0}^{\infty} \mathrm{Sec}(\vee^p \xi_E^*)_I.$$

These spaces are all graded associative algebras. Moreover, ψ and χ induce canonical homomorphisms

$$\psi_*: (\otimes E^*)_I \to \mathrm{Sec} \otimes \xi_E^* \qquad \text{and} \qquad \chi_*: (\vee E^*)_I \to \mathrm{Sec} \vee \xi_E^*.$$

Remark: The constructions of this article depend only on the isomorphism class of the pair (F, Σ_F). In fact, suppose (F_1, Σ_{F_1}) is a second pair, in the same isomorphism class, with corresponding Lie algebra $E_1 \subset L_{F_1}$. Then there is an isomorphism $\sigma: F \xrightarrow{\cong} F_1$ carrying Σ_F to Σ_{F_1}. It induces isomorphisms

$$\otimes E^* \xrightarrow{\cong} \otimes E_1^* \qquad \text{and} \qquad \vee E^* \xrightarrow{\cong} \vee E_1^*.$$

These restrict to isomorphisms,

$$(\otimes E^*)_I \xrightarrow{\cong} (\otimes E_1^*)_I \qquad \text{and} \qquad (\vee E^*)_I \xrightarrow{\cong} (\vee E_1^*)_I,$$

which are independent of the choice of σ. Moreover, the diagrams,

$$\begin{tikzcd}
(\otimes E^*)_I \arrow[d, "\cong"'] \arrow[dr, "\psi_*"] & \\
& \mathrm{Sec} \otimes \xi_E^* \\
(\otimes E_1^*)_I \arrow[ur, "\psi_*"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
(\vee E^*)_I \arrow[d, "\cong"'] \arrow[dr, "\chi_*"] & \\
& \mathrm{Sec} \vee \xi_E^* \\
(\vee E_1^*)_I \arrow[ur, "\chi_*"'] &
\end{tikzcd},$$

commute; this shows that, *up to canonical isomorphism*, ψ_* *and* χ_* *are independent of the choice of* F *and* Σ_F.

8.9. Σ-connections. Recall that the connection, $\hat{\nabla}$, in L_ξ induced by ∇ restricts to a linear connection in ξ_E (cf. Proposition II, (1), sec. 8.7). Hence it determines a linear connection, ∇, in the bundles $\otimes^p \xi_E^*$ and $\vee^p \xi_E^*$.

Proposition IV: The inclusions,

$$\psi: B \times (\otimes^p E^*)_I \to \otimes^p \xi_E^* \qquad \text{and} \qquad \chi: B \times (\vee^p E^*)_I \to \vee^p \xi_E^*,$$

are connection preserving with respect to the standard connection, δ, and ∇.

Proof: It is sufficient to consider the case that

$$\xi = (B \times F, \pi, B, F) \qquad \text{and} \qquad \sigma_i(x) = (x, v_i), \qquad x \in B, \quad i = 1, ..., m.$$

In this case $\nabla = \delta + \Psi$, where $\Psi \in A^1(B; E)$ (cf. Example 5, sec. 8.6). Moreover the total space of ξ_E is $B \times E$ and the induced connection in ξ_E is given by

$$\hat{\nabla}\tau = \delta\tau + [\Psi, \tau], \qquad \tau \in \mathscr{S}(B; E).$$

Thus the induced connection in $\otimes^p \xi_E^*$ is given by $\nabla = \delta + \Psi^p$, where Ψ^p is the $L_{\otimes^p E^*}$-valued 1-form defined by

$$\Psi^p(x; h)(z_1 \otimes \cdots \otimes z_p) = -\sum_{i=1}^{p} z_1 \otimes \cdots \otimes \operatorname{ad}^*(\Psi(x; h))\, z_i \otimes \cdots \otimes z_p,$$
$$x \in B, \quad h \in T_x(B), \quad z_i \in E^*.$$

In view of Proposition IX, sec. 1.8, it follows that

$$\Psi^p(x; h)(v) = 0, \qquad v \in (\otimes^p E^*)_I.$$

This proves the proposition for ψ; the proof for χ is identical.

Q.E.D.

Corollary: The inclusions,

$$\psi_*: (\otimes E^*)_I \to \text{Sec} \otimes \xi_E^* \qquad \text{and} \qquad \chi_*: (\vee E^*)_I \to \text{Sec} \vee \xi_E^*,$$

of sec. 8.8 are isomorphisms into the graded subalgebras of invariant, ∇-parallel cross-sections. If B is connected they are surjective.

8.10. Homomorphisms. Let $\varphi: (\xi, \Sigma_\xi) \to (\eta, \Sigma_\eta)$ be a homomorphism of Σ-bundles (cf. sec. 8.1). It induces a bundle map, $\varphi_E: \xi_E \to \eta_E$, whose fibre maps $(\varphi_E)_x$ are isomorphisms. Thus the linear isomorphisms,

$$\otimes^p((\varphi_E)_x^*)^{-1} \qquad \text{and} \qquad \vee^p((\varphi_E)_x^*)^{-1},$$

define bundle maps $\otimes^p \xi_E^* \to \otimes^p \eta_E^*$ and $\vee^p \xi_E^* \to \vee^p \eta_E^*$. These restrict to bundle maps,

$$(\otimes^p \xi_E^*)_I \to (\otimes^p \eta_E^*)_I \qquad \text{and} \qquad (\vee^p \xi_E^*)_I \to (\vee^p \eta_E^*)_I ,$$

and all these bundle maps will be denoted by φ_E.

Moreover, all the φ_E induce the same map $\varphi_B \colon B \to \tilde{B}$ ($\tilde{B}$, the base of η). The same argument as that given in Proposition III, sec. 8.8, shows that the diagrams,

$$\begin{tikzcd} B \times (\otimes^p E^*)_I \arrow[r, "\psi", "\cong"'] \arrow[d, "\varphi_B \times \iota"'] & (\otimes^p \xi_E^*)_I \arrow[d, "\varphi_E"] \\ \tilde{B} \times (\otimes^p E^*)_I \arrow[r, "\cong", "\psi"'] & (\otimes^p \eta_E^*)_I \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} B \times (\vee^p E^*)_I \arrow[r, "\chi", "\cong"'] \arrow[d, "\varphi_B \times \iota"'] & (\vee^p \xi_E^*)_I \arrow[d, "\varphi_E"] \\ \tilde{B} \times (\vee^p E^*)_I \arrow[r, "\cong", "\chi"'] & (\vee^p \eta_E^*)_I , \end{tikzcd}$$

commute. Thus the diagrams,

$$\begin{tikzcd} & \operatorname{Sec} \otimes \xi_E^* \\ (\otimes E^*)_I \arrow[ur, "\psi_*"] \arrow[dr, "\psi_*"'] & \\ & \operatorname{Sec} \otimes \eta_E^* \arrow[uu, "\varphi_E^*"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & \operatorname{Sec} \vee \xi_E^* \\ (\vee E^*)_I \arrow[ur, "\chi_*"] \arrow[dr, "\chi_*"'] & \\ & \operatorname{Sec} \vee \eta_E^* \arrow[uu, "\varphi_E^*"'] \end{tikzcd} ,$$

also commute.

§4. Characteristic homomorphism

In this article we continue the notation conventions of article 3.

8.11. The homomorphisms β_ξ and γ_ξ. Define a linear map,

$$\beta_\xi\colon \mathrm{Sec} \bigotimes \xi_E^* \to A(B; \Gamma),$$

by setting

$$\beta_\xi(\Lambda) = \langle \Lambda, \underbrace{R \otimes \cdots \otimes R}_{(p \text{ factors})} \rangle, \qquad \Lambda \in \mathrm{Sec} \bigotimes^p \xi_E^*$$

(recall that $R \in A^2(B; \xi_E)$ is the curvature of the Σ-connection ∇).

Lemma I: β_ξ is an algebra homomorphism. It factors over the canonical projection $(\pi_S)_*\colon \mathrm{Sec} \bigotimes \xi_E^* \to \mathrm{Sec} \vee \xi_E^*$ to yield a commutative diagram,

$$\begin{tikzcd} \mathrm{Sec} \bigotimes \xi_E^* \arrow[d, "(\pi_S)_*"'] \arrow[dr, "\beta_\xi"] & \\ \mathrm{Sec} \vee \xi_E^* \arrow[r, "\gamma_\xi"'] & A(B; \Gamma) \end{tikzcd},$$

of algebra homomorphisms.

Proof: Let $\Lambda_i \in \mathrm{Sec} \bigotimes^{p_i} \xi_E^*$ $(i = 1, 2)$. Then (cf. Lemma IV, sec. 7.8)

$$\begin{aligned} \beta_\xi(\Lambda_1 \otimes \Lambda_2) &= \langle \Lambda_1 \otimes \Lambda_2, R \otimes \cdots \otimes R \rangle \\ &= \langle \Lambda_1, R \otimes \cdots \otimes R \rangle \wedge \langle \Lambda_2, R \otimes \cdots \otimes R \rangle \\ &= \beta_\xi(\Lambda_1) \wedge \beta_\xi(\Lambda_2). \end{aligned}$$

Thus β_ξ is a homomorphism.

Since R is a 2-form, $\operatorname{Im} \beta_\xi$ is a graded subalgebra of the commutative algebra $\sum_p A^{2p}(B; \Gamma)$; in particular,

$$\beta_\xi(\Lambda_1) \wedge \beta_\xi(\Lambda_2) = \beta_\xi(\Lambda_2) \wedge \beta_\xi(\Lambda_1), \qquad \Lambda_i \in \operatorname{Sec} \bigotimes \xi_E^*.$$

Hence β_ξ factors as desired.

Q.E.D.

Extend β_ξ and γ_ξ to homomorphisms,

$$\beta_\xi: A(B; \bigotimes \xi_E^*) \to A(B; \Gamma) \qquad \text{and} \qquad \gamma_\xi: A(B; \vee \xi_E^*) \to A(B; \Gamma),$$

by setting

$$\beta_\xi(\Phi \wedge \Lambda) = \Phi \wedge \beta_\xi(\Lambda) \qquad \text{and} \qquad \gamma_\xi(\Phi \wedge \Xi) = \Phi \wedge \gamma_\xi(\Xi),$$
$$\Lambda \in \operatorname{Sec} \bigotimes \xi_E^*, \quad \Xi \in \operatorname{Sec} \vee \xi_E^*, \quad \Phi \in A(B).$$

The analogue of Lemma I holds and

$$\beta_\xi(\Omega) = \langle \Omega, R \otimes \cdots \otimes R \rangle, \qquad \Omega \in A(B; \bigotimes^p \xi_E^*).$$

Lemma II: The maps β_ξ and γ_ξ satisfy

$$\beta_\xi \circ \nabla = \delta \circ \beta_\xi \qquad \text{and} \qquad \gamma_\xi \circ \nabla = \delta \circ \gamma_\xi .$$

Proof: The second relation follows trivially from the first. To prove the first, fix $\Lambda \in \operatorname{Sec} \bigotimes^p \xi_E^*$. According to the Bianchi identity (cf. Proposition VI, sec. 7.15), $\nabla R = 0$. Thus it follows from Example 4, sec. 7.12, that

$$\begin{aligned} \beta_\xi(\nabla \Lambda) &= \langle \nabla \Lambda, R \otimes \cdots \otimes R \rangle \\ &= \delta \langle \Lambda, R \otimes \cdots \otimes R \rangle \\ &= \delta \beta_\xi(\Lambda). \end{aligned}$$

Q.E.D.

Finally, set

$$\beta_\xi^I = \beta_\xi \circ \psi_* : (\bigotimes E^*)_I \to A(B; \Gamma)$$

and

$$\gamma_\xi^I = \gamma_\xi \circ \chi_* : (\vee E^*)_I \to A(B; \Gamma).$$

Proposition V: The linear maps β_ξ^I and γ_ξ^I are algebra homomorphisms. They make the diagram,

$$\begin{tikzcd}
(\bigotimes E^*)_I \arrow[dd, "\pi_S"'] \arrow[dr, "\beta_\xi^I"] & \\
& A(B;\Gamma) \\
(\vee E^*)_I \arrow[ur, "\gamma_\xi^I"'] &
\end{tikzcd},$$

commute, and satisfy

$$\delta \circ \beta_\xi^I = 0, \qquad \delta \circ \gamma_\xi^I = 0.$$

Proof: The first part of the proposition follows from Lemma I, together with the commutative diagram

$$\begin{tikzcd}
(\bigotimes E^*)_I \arrow[r, "\psi_*"] \arrow[d, "\pi_S"'] & \mathrm{Sec} \bigotimes \xi_E^* \arrow[d, "(\pi_S)_*"] \\
(\vee E^*)_I \arrow[r, "\chi_*"'] & \mathrm{Sec} \vee \xi_E^*
\end{tikzcd}.$$

The last relation is a consequence of Lemma II, and the corollary to Proposition IV, sec. 8.9.

Q.E.D.

8.12. Explicit formulae for β_ξ^I and γ_ξ^I. As in sec. 6.18, we identify $T^p(E)$ with $\otimes^p E^*$, and $S^p(E)$ with $\vee^p E^*$. Then β_ξ^I and γ_ξ^I satisfy

$$\beta_\xi^I(\Lambda)(X_1, \ldots, X_{2p}) = \left(\frac{1}{2^p}\right) \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Lambda\{R(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, R(X_{\sigma(2p-1)}, X_{\sigma(2p)})\},$$

$$X_i \in \mathscr{X}(B), \quad \Lambda \in (\bigotimes^p E^*)_I,$$

and

$$\gamma_\xi^I(\Lambda) = \left(\frac{1}{p!}\right) \langle \Lambda, R \vee \cdots \vee R \rangle, \qquad \Lambda \in (\vee^p E^*)_I.$$

Thus

$$\gamma_\xi^I(\Lambda)(X_1, \ldots, X_{2p}) = \left(\frac{1}{p!\, 2^p}\right) \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Lambda\{R(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, R(X_{\sigma(2p-1)}, X_{\sigma(2p)})\}. \tag{8.1}$$

Note that γ_ξ^I is *not* the restriction of β_ξ^I to $S^p(E)_I$.

8.13. Characteristic homomorphism. Proposition V of sec. 8.11 shows that the differential forms in $\operatorname{Im}\gamma_\xi^I$ are closed. Thus we can compose γ_ξ^I with the projection $Z(B; \Gamma) \to H(B; \Gamma)$ ($Z(B; \Gamma) = \ker \delta$) to obtain a homomorphism

$$h_\xi \colon (\vee E^*)_I \to H(B; \Gamma).$$

It is called the *characteristic homomorphism* of the Σ-bundle (ξ, Σ_ξ). Its image is called the *characteristic subalgebra*. The restriction of h_ξ to $(\vee^p E^*)_I$ will be denoted by h_ξ^p,

$$h_\xi^p \colon (\vee^p E^*)_I \to H^{2p}(B; \Gamma).$$

Clearly, $h_\xi^p = 0$ if $2p > \dim B$.

Theorem III: The homomorphism h_ξ is independent of the choice of connection, and hence an invariant of the Σ-bundle (ξ, Σ_ξ).

Theorem IV: Let $\varphi\colon (\xi, \Sigma_\xi) \to (\eta, \Sigma_\eta)$ be a homomorphism of Σ-bundles inducing $\varphi_B : B \to \tilde{B}$ ($\tilde{B}$, the base of η). Then the diagram,

$$\begin{tikzcd} & H(\tilde{B}; \Gamma) \arrow[dd, "\varphi_B^*"] \\ (\vee E^*)_I \arrow[ur, "h_\eta"] \arrow[dr, "h_\xi"'] & \\ & H(B; \Gamma) \end{tikzcd},$$

commutes.

The proofs of both theorems depend on

Lemma III: Let $\varphi\colon \xi \to \eta$ be a connection preserving Σ-homomorphism with respect to Σ-connections ∇ and $\tilde{\nabla}$. Then the diagram,

$$\begin{tikzcd} & A(\tilde{B}; \Gamma) \arrow[dd, "\varphi_B^*"] \\ (\vee E^*)_I \arrow[ur, "\gamma_\eta^I"] \arrow[dr, "\gamma_\xi^I"'] & \\ & A(B; \Gamma) \end{tikzcd},$$

commutes, where γ_η^I and γ_ξ^I are defined as in sec. 8.11 via $\tilde{\nabla}$ and ∇.

Proof: In view of sec. 7.15, the curvatures of ∇ and $\tilde{\nabla}$ satisfy

$$R = \varphi_E^{\#}\tilde{R}$$

(where $\varphi_E : \xi_E \to \eta_E$ is the map induced by φ between the Lie algebra bundles). Thus we can apply Lemma I, sec. 7.5, and the commutative diagram at the end of sec. 8.10 to obtain

$$\begin{aligned}
\varphi_B^*\beta_\eta^I(\Lambda) &= \varphi_B^*\langle\psi_*\Lambda, \tilde{R} \otimes \cdots \otimes \tilde{R}\rangle \\
&= \langle\varphi_E^{\#}\psi_*\Lambda, \varphi_E^{\#}\tilde{R} \otimes \cdots \otimes \varphi_E^{\#}\tilde{R}\rangle \\
&= \langle\psi_*\Lambda, R \otimes \cdots \otimes R\rangle \\
&= \beta_\xi^I(\Lambda), \qquad \Lambda \in (\textstyle\bigotimes^p E^*)_I .
\end{aligned}$$

The lemma follows.

Q.E.D.

Proof of Theorem III: Consider the Σ-bundle,

$$\xi \times \mathbb{R} = (M \times \mathbb{R}, \pi \times \iota, B \times \mathbb{R}, F),$$

with $\Sigma_{\xi\times\mathbb{R}} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_m)$ given by

$$\tilde{\sigma}_i(x, t) = (\sigma_i(x), t), \qquad x \in B, \quad t \in \mathbb{R}, \quad i = 1, ..., m.$$

Σ-homomorphisms ρ: $\xi \times \mathbb{R} \to \xi$ and $j_t : \xi \to \xi \times \mathbb{R}$ are defined by

$$\rho(z, t) = z \qquad \text{and} \qquad j_t(z) = (z, t), \qquad z \in M, \quad t \in \mathbb{R}.$$

The induced maps $\rho_B : B \times \mathbb{R} \to B$ and $i_t : B \to B \times \mathbb{R}$ are given by

$$\rho_B(x, t) = x \qquad \text{and} \qquad i_t(x) = (x, t), \qquad x \in B, \quad t \in \mathbb{R}.$$

Now let ∇_0 and ∇_1 be any two Σ-connections in ξ. Let $\tilde{\nabla}_0$ and $\tilde{\nabla}_1$ be the pull-backs (via ρ) of these connections to $\xi \times \mathbb{R}$. Define a Σ-connection, ∇, in $\xi \times \mathbb{R}$ by setting

$$\nabla = t\tilde{\nabla}_1 + (1 - t)\tilde{\nabla}_0 .$$

Because $\rho \circ j_0 = \iota$ and $\rho \circ j_1 = \iota$, it follows that j_0 (respectively, j_1) is connection preserving with respect to ∇_0 and ∇ (respectively, with respect to ∇_1 and ∇).

Now let h_ξ^0, h_ξ^1, and $h_{\xi\times\mathbb{R}}$ denote the characteristic homomorphisms defined via ∇_0, ∇_1, and ∇. Since $i_0^{\#} = i_1^{\#}$, Lemma III gives

$$h_\xi^0 = i_0^{\#} \circ h_{\xi\times\mathbb{R}} = i_1^{\#} \circ h_{\xi\times\mathbb{R}} = h_\xi^1 .$$

Q.E.D.

Proof of Theorem IV: Choose any Σ-connection, ∇, in η and give ξ the pull-back of ∇ via φ. According to Theorem III, h_η and h_ξ can be defined via these connections; hence Lemma III implies that

$$h_\xi = \varphi_B^\# \circ h_\eta .$$

Q.E.D.

8.14. Smooth functions. In this section ξ denotes a real vector bundle ($\Gamma = \mathbb{R}$). Exactly as in sec. 6.22 we extend h_ξ to a homomorphism

$$h_\xi^{**}: (\vee^{**}E^*)_I \to H(B).$$

Precomposing this map with the Taylor homomorphism,

$$\mathscr{S}_0(E)_I \to (\vee^{**}E^*)_I ,$$

(cf. sec. 6.21) yields the homomorphism

$$s_\xi: \mathscr{S}_0(E)_I \to H(B).$$

Examples: 1. Let $f \in \mathscr{S}(E)_I$ be given by

$$f(y) = \frac{1}{p!}\, \Psi(y, ..., y), \qquad y \in E,$$

where $\Psi \in (\vee^p E^*)_I$. Then $s_\xi(f) = h_\xi(\Psi)$.

2. Let ξ be a Σ-bundle with $\Sigma = \varnothing$; thus $E = L_F$. Define $f \in \mathscr{S}(L_F)_I$ by

$$f(\varphi) = \operatorname{tr} \exp \varphi, \qquad \varphi \in L_F .$$

Then $s_\xi(\operatorname{tr} \circ \exp) \in H(B)$; it is (in general) a non-homomogeneous class.

§5. Examples

In this article we continue the notation conventions of article 3.

8.15. Dual Σ-bundles. Canonical isomorphisms,

$$*: F_x^{p,q} \xrightarrow{\cong} (F_x^*)^{q,p}, \qquad x \in B,$$

are defined by

$$*: w_1^* \otimes \cdots \otimes w_p^* \otimes w_1 \otimes \cdots \otimes w_q \mapsto w_1 \otimes \cdots \otimes w_q \otimes w_1^* \otimes \cdots \otimes w_p^*,$$
$$w_i^* \in F_x^*, \quad w_j \in F_x.$$

Thus they induce isomorphisms

$$*: \mathrm{Sec}\, \xi^{p,q} \xrightarrow{\cong} \mathrm{Sec}(\xi^*)^{q,p} \qquad \text{and} \qquad *: A(B; \xi^{p,q}) \xrightarrow{\cong} A(B; (\xi^*)^{q,p}).$$

Now define a Σ-bundle, (ξ^*, Σ_{ξ^*}), as follows: $\xi^* = (M^*, \pi, B, F^*)$ is a dual bundle for ξ, and $\Sigma_{\xi^*} = (*\sigma_1, ..., *\sigma_m)$. This Σ-bundle is called *the dual Σ-bundle for* (ξ, Σ_ξ).

Next, let G and G_* be the structure groups of ξ and ξ^*. The isomorphism $GL(F) \xrightarrow{\cong} GL(F^*)$ given by $\varphi \mapsto (\varphi^*)^{-1}$ restricts to an isomorphism $G \xrightarrow{\cong} G_*$. Its derivative is the Lie algebra isomorphism $\varphi \mapsto -\varphi^*$ between the Lie algebras E and E_* of G and G_*. In the same way, a canonical isomorphism, $\xi_E \cong (\xi^*)_{E_*}$, of the Lie algebra bundles is defined.

Now let ∇ be a Σ-connection in ξ. Then the dual connection ∇^* is a Σ-connection in ξ^*, as follows from the equation

$$\nabla^*(*\sigma) = *(\nabla\sigma), \qquad \sigma \in \mathrm{Sec}\, \xi^{p,q}.$$

On the other hand, Example 2, sec. 7.16, shows that the curvatures of ∇^* and ∇ are related by

$$R_{\xi^*}(x; h, k) = -(R_\xi(x; h, k))^*.$$

Thus the canonical isomorphism $\xi_E \xrightarrow{\cong} (\xi^*)_{E_*}$ maps R_ξ to R_{ξ^*}.

Finally, notice that the canonical isomorphisms, $E \xrightarrow{\cong} E_*$, $\xi_E \xrightarrow{\cong} (\xi^*)_{E_*}$, induce isomorphisms

$$(\vee E^*)_I \xrightarrow{\cong} (\vee(E_*)^*)_I \qquad \text{and} \qquad \mathrm{Sec}\, \vee\xi_E^* \xrightarrow{\cong} \mathrm{Sec}\, \vee(\xi_{E_*}^*)^*.$$

Since R_ξ is carried to R_{ξ^*}, we obtain the commutative diagram

$$\begin{tikzcd} (\vee E^*)_I \arrow[dd, "\cong"'] \arrow[dr, "h_\xi"] & \\ & H(B;\Gamma) \\ (\vee(E_*)^*)_I \arrow[ur, "h_{\xi^*}"'] & \end{tikzcd}$$

8.16. Whitney sums. Consider a second Σ-bundle (η, Σ_η) $(\eta = (N, \tilde{\pi}, B, H))$ over the same base B, where $\Sigma_\eta = \{\tau_1, \ldots, \tau_l\}$. Let $K \subset GL(H)$ be the corresponding structure group, with Lie algebra L.

The cross-sections σ_i of Σ_ξ and the cross-sections τ_j may be regarded as homogeneous elements of $\mathrm{Sec}[\otimes(\xi \oplus \eta) \otimes (\xi \oplus \eta)^*]$. Moreover, the projection operators,

$$\rho_\xi \colon \xi \oplus \eta \to \xi \qquad \text{and} \qquad \rho_\eta \colon \xi \oplus \eta \to \eta,$$

may be regarded as elements of $\mathrm{Sec}(\xi \oplus \eta)^{1,1}$. Set

$$\Sigma_{\xi\oplus\eta} = (\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_l, \rho_\xi, \rho_\eta);$$

then $(\xi \oplus \eta, \Sigma_{\xi\oplus\eta})$ is a Σ-bundle. It is called the *Whitney sum* of (ξ, Σ_ξ) and (η, Σ_η).

Its structure group is $G \times K \subset GL(F \oplus H)$ and the corresponding Lie algebra is $E \oplus L \subset L_{F\oplus H}$. The canonical isomorphism

$$\vee E^* \otimes \vee L^* \cong \vee(E \oplus L)^*$$

restricts to an isomorphism

$$(\vee E^*)_I \otimes (\vee L^*)_I \xrightarrow{\cong} (\vee(E \oplus L)^*)_I.$$

Proposition VI: The characteristic homomorphisms h_ξ, h_η, and $h_{\xi\oplus\eta}$ of the Σ-bundles ξ, η, and $\xi \oplus \eta$ are related by the commutative diagram

$$\begin{tikzcd} (\vee E^*)_I \otimes (\vee L^*)_I \arrow[r, "\cong"] \arrow[d, "h_\xi \otimes h_\eta"'] & (\vee(E \oplus L)^*)_I \arrow[d, "h_{\xi\oplus\eta}"] \\ H(B;\Gamma) \otimes H(B;\Gamma) \arrow[r, "\text{(multiplication)}"'] & H(B, \Gamma) \end{tikzcd}$$

Proof: Observe first that the Lie algebra bundle associated with the Σ-bundle $\xi \oplus \eta$ is given by

$$(\xi \oplus \eta)_{E\oplus L} = \xi_E \oplus \eta_L \,.$$

This induces an isomorphism

$$\mathrm{Sec}(\vee \xi_E^*) \otimes_B \mathrm{Sec}(\vee \eta_L^*) \xrightarrow{\cong} \mathrm{Sec}\, \vee(\xi \oplus \eta)_{E\oplus L}^*$$

(where $\otimes_B$ denotes the tensor product over $\mathscr{S}(B; \Gamma)$). It follows directly from the definitions that the diagram,

$$\begin{tikzcd}
(\vee E^*)_I \otimes (\vee L^*)_I \arrow[r, "\cong"] \arrow[d, "\chi_* \otimes \chi_*"'] & \vee(E \oplus L)_I^* \arrow[d, "\chi_*"] \\
\mathrm{Sec}(\vee \xi_E^*) \otimes_B \mathrm{Sec}(\vee \eta_L^*) \arrow[r, "\cong"'] & \mathrm{Sec}\, \vee(\xi \oplus \eta)_{E\oplus L}^*
\end{tikzcd} ,$$

commutes.

Fix Σ-connections ∇_ξ in ξ and ∇_η in η. Then $\nabla_\xi \oplus \nabla_\eta$ is a Σ-connection in $\xi \oplus \eta$; its curvature is given by (cf. Example 3, sec. 7.16)

$$R_{\xi\oplus\eta} = R_\xi \oplus R_\eta \,.$$

It is now easy to check that the diagram,

$$\begin{tikzcd}
\mathrm{Sec}(\vee \xi_E^*) \otimes_B \mathrm{Sec}(\vee \eta_L^*) \arrow[r] \arrow[d, "\gamma_\xi \otimes \gamma_\eta"'] & \mathrm{Sec}\, \vee(\xi \oplus \eta)_{E\oplus L}^* \arrow[d, "\gamma_{\xi\oplus\eta}"] \\
A(B; \Gamma) \otimes_B A(B; \Gamma) \arrow[r, "\text{(multiplication)}"'] & A(B; \Gamma)
\end{tikzcd} ,$$

commutes. (In fact, since all maps are homomorphisms, it is sufficient to check commutativity on functions, on $\mathrm{Sec}\, \xi_E^* \otimes 1$ and on $1 \otimes \mathrm{Sec}\, \eta_L^*$.

Now combine the two diagrams above and pass to cohomology, to complete the proof.

Q.E.D.

Corollary: Suppose ξ and η are real Σ-bundles. If $f \in \mathscr{S}_0(E)_I$ and $g \in \mathscr{S}_0(L)_I$, then $f \times g \in \mathscr{S}_0(E \oplus L)_I$, and

$$s_{\xi\oplus\eta}(f \times g) = s_\xi(f) \cdot s_\eta(g).$$

8.17. Σ-substructures. Assume that two Σ-structures, Σ and $\hat{\Sigma}$ are given in a vector bundle ξ. Then $\hat{\Sigma}$ is called a *Σ-substructure of Σ* if $\hat{\Sigma} \subset \Sigma$. If $\hat{\Sigma}$ is a Σ-substructure of Σ, then the corresponding Lie groups

$G \subset GL(F)$ and $\hat{G} \subset GL(F)$ satisfy $G \subset \hat{G}$. It follows that $E \subset \hat{E}$ (where E and $\hat{E}$ are the corresponding Lie algebras). The inclusion map $j\colon E \to \hat{E}$ induces a homomorphism

$$j_I^\vee : (\vee E^*)_I \leftarrow (\vee \hat{E}^*)_I .$$

Moreover, there is an obvious inclusion map $j_E\colon \xi_E \to \xi_{\hat{E}}$ between the associated Lie algebra bundles.

Proposition VII: Let $\hat{\Sigma}$ be a Σ-substructure of Σ and let

$$h_\xi\colon (\vee E^*)_I \to H(B;\Gamma) \qquad \text{and} \qquad \hat{h}_\xi\colon (\vee \hat{E}^*)_I \to H(B;\Gamma)$$

denote the corresponding characteristic homomorphisms. Then the diagram,

$$\begin{tikzcd}
(\vee E^*)_I \arrow[dr, "h_\xi"'] & & (\vee \hat{E}^*)_I \arrow[ll, "j_I^\vee"'] \arrow[dl, "\hat{h}_\xi"] \\
& H(B;\Gamma) &
\end{tikzcd},$$

commutes.

Proof: Choose a Σ-connection, ∇, for (ξ, Σ_ξ). Then ∇ is also a Σ-connection for $(\xi, \hat{\Sigma}_\xi)$. The corresponding curvatures R and $\hat{R}$ satisfy

$$(j_E)_* R = \hat{R}.$$

This gives the commutative diagram,

$$\begin{tikzcd}
(\vee \hat{E}^*)_I \arrow[r, "\hat{\chi}_*"] \arrow[dd, "j_I^\vee"'] & \mathrm{Sec}(\vee \xi_{\hat{E}}^*) \arrow[dd, "j_E^*"'] \arrow[dr, "\hat{\gamma}_\xi"] & \\
& & A(B;\Gamma) \\
(\vee E^*)_I \arrow[r, "\chi_*"'] & \mathrm{Sec}(\vee \xi_E^*) \arrow[ur, "\gamma_\xi"'] &
\end{tikzcd},$$

and the proposition follows.

Q.E.D.

Corollary I: Assume that ξ is a real bundle. The homomorphisms,

$$s_\xi\colon \mathscr{S}_0(E)_I \to H(B) \qquad \text{and} \qquad \hat{s}_\xi\colon \mathscr{S}_0(\hat{E})_I \to H(B),$$

are related by $\hat{s}_\xi(f) = s_\xi(j^*f), f \in \mathscr{S}_0(\hat{E})_I$.

Corollary II: The characteristic subalgebra for (ξ, Σ_ξ) contains the characteristic subalgebra for $(\xi, \hat{\Sigma}_\xi)$. In particular, the characteristic subalgebra for $(\xi, \varnothing)$ (ξ considered as a $GL(F)$-bundle) is contained in the characteristic algebra for (ξ, Σ_ξ), where Σ_ξ is an arbitrary Σ-structure in ξ.

Example: Let $\xi = (M_\xi, \pi_\xi, B, F)$ and $\eta = (M_\eta, \pi_\eta, B, H)$ be vector bundles and regard these bundles as Σ-bundles with $\Sigma_\xi = \varnothing$ and $\Sigma_\eta = \varnothing$. Then the Whitney sum is a Σ-bundle with $\Sigma_{\xi\oplus\eta} = (\rho_\xi, \rho_\eta)$ (cf. Example 5, sec. 8.3). Denote its characteristic homomorphism by $\hat{h}_{\xi\oplus\eta}$.

On the other hand, $\xi \oplus \eta$ may be considered as a Σ-bundle with $\Sigma_{\xi\oplus\eta} = \varnothing$. The corresponding characteristic homomorphism will be denoted by $h_{\xi\oplus\eta} : (\vee(L^*_{F\oplus H}))_I \to H(B; \Gamma)$. Applying Proposition VII (with $E = L_F, L = L_H$) we obtain the commutative diagram,

$$\begin{tikzcd}
(\vee(L_F \oplus L_H)^*)_I \arrow[d, "\hat{h}_{\xi\oplus\eta}"'] & (\vee L^*_{F\oplus H})_I \arrow[l, "j_I^\vee"'] \arrow[dl, "h_{\xi\oplus\eta}"] \\
H(B;\Gamma) &
\end{tikzcd},$$

where $j\colon L_F \oplus L_H \to L_{F\oplus H}$ denotes the inclusion map.

Combining this with the commutative diagram of Proposition VI, sec. 8.16, we obtain the commutative diagram

$$\begin{tikzcd}
(\vee L_F^*)_I \otimes (\vee L_H^*)_I \arrow[r, "\cong"] \arrow[d, "h_\xi\otimes h_\eta"'] & (\vee(L_F \oplus L_H)^*)_I \arrow[d, "\hat{h}_{\xi\oplus\eta}"] & (\vee L^*_{F\oplus H})_I \arrow[l, "j_I^\vee"'] \arrow[dl, "h_{\xi\oplus\eta}"] \\
H(B;\Gamma) \otimes H(B;\Gamma) \arrow[r, "\text{multiplication}"'] & H(B;\Gamma) &
\end{tikzcd}$$

§6. Σ-bundles with compact carrier

8.18. Bundles with compact carrier. A *vector bundle with compact carrier* (or *compact support*) is a pair (ξ, α), where

(1) $\xi = (M, \pi, B, F)$ is a smooth vector bundle.

(2) $\alpha: O \times F \xrightarrow{\cong} \xi|_O$ is a trivializing map, and

(3) O is an open subset of B such that $B - O$ is compact.

Any open subset $U \subset O$ such that $B - U$ is compact will be called a *complement* for (ξ, α).

Note that a vector bundle with compact carrier is a vector bundle which is trivial off some compact set, *together with an explicit trivialization.*

Suppose (η, β) is a second compactly supported bundle with base $\tilde{B}$ and typical fibre H. A *homomorphism* $\varphi: (\xi, \alpha) \to (\eta, \beta)$ of *compactly supported bundles* is a bundle map $\varphi: \xi \to \eta$ with the following properties:

(1) The induced map $\psi: B \to \tilde{B}$ is proper.

(2) There is a commutative diagram,

$$\begin{array}{ccc} U \times F & \xrightarrow[\cong]{\alpha} & \xi|_U \\ {\scriptstyle \psi\times\gamma}\downarrow & & \downarrow{\scriptstyle \varphi} \\ V \times H & \xrightarrow[\beta]{\cong} & \eta|_V, \end{array}$$

where U and V are complements for (ξ, α) and (η, β), and $\gamma: F \to H$ is a linear isomorphism.

If φ is a bundle isomorphism (respectively, a strong bundle isomorphism), we say the pairs (ξ, α) and (η, β) are *isomorphic* (respectively, *strongly isomorphic*).

More generally, a *Σ-bundle with compact carrier* is a triple, $(\xi, \Sigma_\xi, \alpha)$, where

(1) (ξ, Σ_ξ) is a Σ-bundle and (ξ, α) is a bundle with compact support and

(2) the cross-sections $\alpha^{\#}\sigma_i$ $(\sigma_i \in \Sigma_\xi)$ are constant in some complement, U, for (ξ, α).

A *homomorphism of compactly supported Σ-bundles* is a bundle map which is simultaneously a homomorphism of Σ-bundles and a homomorphism of compactly supported bundles.

Henceforth $(\xi, \Sigma_\xi, \alpha)$ is a fixed compactly supported Σ-bundle, where (ξ, Σ_ξ) is the Σ-bundle described at the start of article 3.

A *compact Σ-connection* in $(\xi, \Sigma_\xi, \alpha)$ is a Σ-connection, ∇, such that, in some complement, U, for $(\xi, \Sigma_\xi, \alpha)$,

$$\alpha^{\#} \circ \nabla = \delta \circ \alpha^{\#},$$

where δ is the standard connection in $A(U; F)$.

The following proposition is obvious.

Proposition VIII: (1) The curvature, R, of a compact Σ-connection has compact support,

$$R \in A_c^2(B; \xi_E).$$

(2) If $\tilde{\nabla}$ is a compact Σ-connection in $(\eta, \Sigma_\eta, \beta)$, and, if

$$\varphi: (\xi, \Sigma_\xi, \alpha) \to (\eta, \Sigma_\eta, \beta)$$

is a homomorphism of compactly supported Σ-bundles, then the pull-back of $\tilde{\nabla}$ to ξ via φ is a compact Σ-connection.

8.19. Compact characteristic homomorphism. Recall from sec. 8.11 that each Σ-connection in (ξ, Σ_ξ) determines a homomorphism,

$$\gamma_\xi^I: (\vee E^*)_I \to A(B; \Gamma),$$

satisfying $\delta \circ \gamma_\xi^I = 0$. It is immediate from Proposition VIII that, if ∇ is a compact Σ-connection in $(\xi, \Sigma_\xi, \alpha)$, then

$$\gamma_\xi^I(\vee^+ E^*)_I \subset A_c(B; \Gamma).$$

Hence γ_ξ^I determines a homomorphism

$$h_\xi^c: (\vee^+ E^*)_I \to H_c(B; \Gamma).$$

Moreover, the diagram,

$$\begin{tikzcd} & H_c(B;\Gamma) \arrow[dd, "\lambda_*"] \\ (\vee^+ E^*)_I \arrow[ur, "h_\xi^c"] \arrow[dr, "h_\xi"'] & \\ & H(B;\Gamma) \end{tikzcd} \qquad (8.2)$$

,

commutes, where $\lambda_\#$ is induced by the inclusion λ: $A_c(B) \to A(B)$. h_ξ^c is called the *compact characteristic homomorphism for* $(\xi, \Sigma_\xi, \alpha)$.

In analogy with Theorems III and IV of sec. 8.13, we have

Theorem V: The homomorphism h_ξ^c is independent of the choice of compact Σ-connection. Thus it is an invariant of the compactly supported Σ-bundle $(\xi, \Sigma_\xi, \alpha)$.

Theorem VI: Let φ: $(\xi, \Sigma_\xi, \alpha) \to (\eta, \Sigma_\eta, \beta)$ be a homomorphism of compactly supported Σ-bundles, inducing $\varphi_B : B \to \tilde{B}$ between the base manifolds. Then the diagram,

$$\begin{tikzcd} & H_c(\tilde{B};\Gamma) \arrow[dd, "(\varphi_B)_c^*"] \\ (\vee^+ E^*)_I \arrow[ur, "h_\eta^c"] \arrow[dr, "h_\xi^c"'] & \\ & H_c(B;\Gamma) \end{tikzcd} ,$$

commutes.

Proof of Theorem V: A compactly supported Σ-bundle, $(\xi \times S^1, \Sigma_{\xi\times S^1}, \alpha \times \iota)$, is given by

$$\xi \times S^1 = (M \times S^1, \pi_\xi \times \iota, B \times S^1, F),$$

with $\Sigma_{\xi\times S^1} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_k)$. Here

$$\tilde{\sigma}_i(x, z) = (\sigma_i(x), z), \qquad x \in B, \quad z \in S^1,$$

while, since $\xi\,|_O \times S^1 = (\xi \times S^1)|_{O\times S^1}$,

$$\alpha \times \iota : O \times S^1 \xrightarrow{\cong} (\xi \times S^1)|_{O\times S^1}.$$

Choose compact Σ-connections ∇_0 and ∇_1 in $(\xi, \Sigma_\xi, \alpha)$. The projection ρ: $\xi \times S^1 \to \xi$ is a homomorphism of compactly supported Σ-bundles; hence ∇_0 and ∇_1 pull back to compact Σ-connections $\tilde{\nabla}_0$ and $\tilde{\nabla}_1$ in $\xi \times S^1$.

Fix two points $a, b \in S^1$ and let $f \in \mathscr{S}(S^1)$ satisfy $f(a) = 1$ and $f(b) = 0$. Then

$$\tilde{\nabla} = f \cdot \tilde{\nabla}_0 + (1 - f) \cdot \tilde{\nabla}_1$$

is a compact Σ-connection in $(\xi \times S^1, \Sigma_{\xi\times S^1}, \alpha \times \iota)$. Moreover, the inclusions,

$$j_a, j_b : \xi \to \xi \times S^1,$$

are homomorphisms of compactly supported Σ-bundles, and satisfy

$$j_a^\# \circ \tilde{\nabla} = \nabla_0 \circ j_a^\# \quad \text{and} \quad j_b^\# \circ \tilde{\nabla} = \nabla_1 \circ j_b^\#.$$

In particular if γ_0^I, γ_1^I, γ^I are the homomorphisms corresponding to ∇_0, ∇_1, and $\tilde{\nabla}$, we have the commutative diagrams,

$$\begin{tikzcd} & A_c(B \times S^1; \Gamma) \arrow[d, "(i_a)_c^*"] \\ (\vee^+ E^*)_I \arrow[ur, "\gamma^I"] \arrow[r, "\gamma_0^I"'] & A_c(B; \Gamma) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A_c(B \times S^1; \Gamma) \arrow[d, "(i_b)_c^*"] \\ (\vee^+ E^*)_I \arrow[ur, "\gamma^I"] \arrow[r, "\gamma_1^I"'] & A_c(B; \Gamma) \end{tikzcd},$$

as follows from Lemma III, sec. 8.13 (i_a, i_b are the inclusions $B \to B \times S^1$ opposite a and b).

Finally, since i_a and i_b are properly homotopic, $(i_a)_c^\# = (i_b)_c^\#$ (cf. sec. 5.10, volume I). It follows that the compact characteristic homomorphisms h_0^c, h_1^c, and $h_{\xi\times S^1}^c$, defined via ∇_0, ∇_1, and $\tilde{\nabla}$, satisfy

$$h_0^c = (i_a)_c^\# h_{\xi\times S^1}^c = (i_b)_c^\# h_{\xi\times S^1}^c = h_1^c.$$

Q.E.D.

Proof of Theorem VI: This is an obvious consequence of Lemma III, sec. 8.13, and Proposition VIII, sec. 8.18.

Q.E.D.

§7. Associated principal bundles

In this article $\Gamma = \mathbb{R}$. The notation conventions of article 3 are continued; in particular (ξ, Σ_ξ) is a Σ-bundle with structure group G whose Lie algebra is denoted by E. Moreover, $\xi = (M, \pi_\xi, B, F)$, $\Sigma_\xi = (\sigma_1, \ldots, \sigma_m)$, and $\Sigma_F = (v_1, \ldots, v_m)$ is a fixed set of tensors over F corresponding to the σ_i under a Σ-coordinate representation.

8.20. ***The associated principal bundle.*** The purpose of this and the next section is to construct a principal bundle $\mathscr{P}_\xi = (P, \pi, B, G)$ such that ξ is the associated vector bundle (with respect to the natural representation of G in F, cf. sec. 5.6).

A linear isomorphism $\varphi: F \xrightarrow{\cong} F_x$ will be called *admissible* if

$$\varphi(v_i) = \sigma_i(x), \qquad i = 1, \ldots, m.$$

The set of admissible isomorphisms is denoted by G_x. Define a right action of G on the set G_x by

$$\varphi_x \cdot \psi = \varphi_x \circ \psi, \qquad \varphi_x \in G_x, \quad \psi \in G.$$

Now consider the disjoint union $P = \bigcup_{x \in B} G_x$ and let $\pi: P \to B$ be the natural projection. The right actions of G on each G_x define a right action, T, of G on the set P.

Finally, let $\{(U_\alpha, \psi_\alpha)\}$ be a Σ-coordinate representation for (ξ, Σ_ξ). Then G-equivariant bijections $\varphi_\alpha : U_\alpha \times G \xrightarrow{\cong} \pi^{-1}(U_\alpha)$ are given by

$$\varphi_\alpha(x, \tau) = \psi_{\alpha,x} \circ \tau, \qquad x \in U_\alpha, \quad \tau \in G.$$

They satisfy

$$(\varphi_\alpha^{-1} \circ \varphi_\beta)(x, \tau) = (x, \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x} \circ \tau), \qquad x \in U_\alpha \cap U_\beta, \quad \tau \in G.$$

We show now that $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a diffeomorphism.

In fact, for $v \in F$, $x \in U_\alpha \cap U_\beta$, we have

$$(x, (\psi_{\alpha,x}^{-1} \circ \psi_{\beta,x})v) = (\psi_\alpha^{-1} \circ \psi_\beta)(x, v).$$

It follows that the map $x \mapsto (\psi_{\alpha,x}^{-1} \circ \psi_{\beta,x})(v)$ is a smooth map from $U_\alpha \cap U_\beta$ to F. This implies that the correspondence,

$$x \mapsto \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x},$$

defines a smooth map of $U_\alpha \cap U_\beta$ into $GL(F)$.

Since $\psi_{\alpha,x}^{-1} \circ \psi_{\beta,x} \in G$ and G is a submanifold of $GL(F)$, this map may be regarded as a smooth map into G (cf. Proposition VI, sec. 3.10, volume I). It follows that $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a diffeomorphism.

Now Proposition X, sec. 1.13, volume I, applies and shows that $\mathscr{P}_\xi = (P, \pi, B, G)$ is a smooth fibre bundle with coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$. Since the maps φ_α are equivariant, $\mathscr{P}_\xi$ is a smooth principal bundle with principal action, T, and principal coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$. It is called the *principal bundle associated with the Σ-bundle* (ξ, Σ_ξ).

Next fix a basis e_j $(j = 1, \ldots, r)$ in F. Then the admissible maps are in 1-1 correspondence with the r-tuples,

$$(\varphi_x(e_1), \ldots, \varphi_x(e_r)), \qquad \varphi_x \in G_x,$$

of vectors in F_x. These r-tuples are called *frames* in F_x and so G_x may be identified with a set of frames in F_x. For this reason, $\mathscr{P}_\xi$ is sometimes called the *frame bundle of the* Σ-bundle (ξ, Σ_ξ).

Examples: 1. If $\Sigma = \varnothing$, then $G = GL(F)$, G_x is the set of all frames in F_x and $\mathscr{P}_\xi$ is the frame bundle of ξ (cf. Example 3, sec. 5.1).

2. If $\Sigma_\xi = (g)$, g a Riemannian metric, then $G = O(F)$ and G_x may be identified with the set of orthonormal frames of F_x via an orthonormal basis of F. $\mathscr{P}_\xi$ is called the *orthonormal frame bundle*.

3. $\Sigma_\xi = (g, \Delta)$, where g is a Riemannian metric and Δ is an orienting determinant function in ξ. In this case $G = SO(F)$ and G_x consists of the positive orthonormal frames in F_x.

8.21. The principal map. Let $\mathscr{P}_\xi = (P, \pi, B, G)$ be the associated principal bundle of the Σ-bundle (ξ, Σ_ξ), and recall that $\xi = (M, \pi_\xi, B, F)$. Consider the trivial Σ-bundle (η, Σ_η) given by $\eta = (P \times F, \pi_P, P, F)$, $\Sigma_\eta = (v_1, \ldots, v_m)$. Since the fibre, G_x, of $\mathscr{P}_\xi$ consists of linear maps $F \to F_x$, a set map $q\colon P \times F \to M$ is given by

$$q(\tau_x, y) = \tau_x(y), \qquad x \in B, \quad \tau_x \in G_x, \quad y \in F.$$

It makes the diagram,

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "\pi_P"'] & M \arrow[d, "\pi_\xi"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd},$$

commute.

Use the local coordinate representations of sec. 8.20 to show that q is smooth, and hence a bundle map. It restricts to isomorphisms in the fibres and the induced maps, $q\colon P \times \otimes^{p,q} F \to \xi^{p,q}$, satisfy

$$q(z, v_i) = \sigma_i(\pi z), \qquad z \in P, \quad i = 1, \dots, m.$$

Thus $q\colon (\eta, \Sigma_\eta) \to (\xi, \Sigma_\xi)$ is a homomorphism of Σ-bundles. It follows that the pull-back of (ξ, Σ_ξ) to P, via π, is trivial.

The bundle map q factors over the projection $P \times F \to P \times_G F$ to yield a strong isomorphism from the associated bundle $(P \times_G F, \rho, B, F)$ to ξ. These bundles will be identified via this isomorphism; in particular q is then identified with the principal map (cf. sec. 5.3).

Examples: 1. *Tangent bundle*: Assume the tangent bundle of B is made into a Σ-bundle (τ_B, Σ_B) (possibly by setting $\Sigma_B = \varnothing$). Let (P, π, B, G) be the associated principal bundle. We shall show that the manifold P is *parallelizable*; i.e., the tangent bundle τ_P is trivial.

In fact, $\tau_P = H_P \oplus V_P$, where H_P is some horizontal subbundle. Since H_P is the pull-back of τ_B to P, and P is the principal bundle associated with τ_B, it follows that H_P is trivial. On the other hand, Corollary I to Proposition I, sec. 6.1, shows that the vertical subbundle V_P is trivial. Hence so is τ_P.

2. *The associated Lie algebra bundle*: Since the principal map is a homomorphism of Σ-bundles it determines a bundle map,

$$q_E\colon P \times E \to \xi_E,$$

between the associated Lie algebra bundles.

On the other hand, using the adjoint action of G in E we obtain a vector bundle $P \times_G E$ over B (with fibre E) and a projection

$$\hat{q}\colon P \times E \to P \times_G E.$$

q_E factors over $\hat{q}$ to yield a strong isomorphism $P \times_G E \xrightarrow{\cong} \xi_E$ of bundles; this isomorphism identifies q_E with the principal map $\hat{q}$.

8.22. Associated connections. Let $\mathscr{P}_\xi = (P, \pi, B, G)$ be the principal bundle associated with (ξ, Σ_ξ). Recall that the space of F-valued forms on P which are both horizontal and equivariant is called the space of basic forms, and is denoted by $A_B(P; F)$ (cf. sec. 6.6).

On the other hand, the principal map q induces a linear map

$$q^{\#}: A(B; \xi) \to A(P; F)$$

(cf. sec. 7.3). It follows from the relations,

$$q^{\#}(\Phi \wedge \sigma) = \pi^*\Phi \wedge q^{\#}\sigma, \qquad \Phi \in A(B), \quad \sigma \in \operatorname{Sec} \xi,$$

and

$$q(z \cdot \tau, y) = q(z, \tau(y)), \qquad z \in P, \quad \tau \in G, \quad y \in F,$$

that $\operatorname{Im} q^{\#} \subset A_B(P; F)$.

Exactly the same argument as that given for Proposition III, sec. 6.3, establishes

Proposition IX: With the hypotheses and notation above, $q^{\#}$ is a linear isomorphism of $A(B; \xi)$ onto $A_B(P; F)$,

$$q^{\#}: A(B; \xi) \xrightarrow{\cong} A_B(P; F).$$

Using the isomorphism $q^{\#}$, we shall now construct a canonical bijection between Σ-connections in (ξ, Σ_ξ) and principal connections in $\mathscr{P}_\xi$.

Recall from sec. 6.12 that a principal connection, V, in $\mathscr{P}_\xi$ determines a covariant exterior derivative,

$$\nabla_{\mathscr{P}} = H^* \circ \delta$$

in $A(P; F)$. Since $\nabla_{\mathscr{P}}$ is equivariant and $H^* \circ \nabla_{\mathscr{P}} = \nabla_{\mathscr{P}}$, the covariant exterior derivative restricts to an operator in $A_B(P; F)$.

Define an operator, $\nabla_\xi : \operatorname{Sec} \xi \to A^1(B; \xi)$, by

$$\nabla_\xi = (q^{\#})^{-1} \circ \nabla_{\mathscr{P}} \circ q^{\#}.$$

Lemma IV: ∇_ξ is a Σ-connection in ξ.

Proof: The relation

$$\nabla_{\mathscr{P}}(f \cdot \Phi) = \delta f \wedge \Phi + f \cdot \nabla_{\mathscr{P}}\Phi, \qquad f \in \mathscr{S}_I(P; F), \quad \Phi \in A_B(P; F)$$

(cf. Proposition VII, sec. 6.12) implies that ∇_ξ is a linear connection. To show that ∇_ξ is a Σ-connection, consider the induced map,

$$q: P \times F^{p,q} \to \xi^{p,q},$$

of tensor bundles. The covariant exterior derivative, $\nabla_{\mathscr{P}}$, in $A(P; F^{p,q})$ and the induced connection, ∇_ξ, in $\xi^{p,q}$ satisfy

$$q^\# \circ \nabla_\xi = \nabla_{\mathscr{P}} \circ q^\#.$$

Since $q^\#(\sigma_i)$ is the constant cross-section, $x \mapsto v_i$, it follows that

$$q^\#(\nabla_\xi \sigma_i) = H^*(\delta v_i) = 0.$$

Hence ∇_ξ is a Σ-connection.

Q.E.D.

Proposition X: The set map,

$$\lambda: \{\text{principal connections in } \mathscr{P}\} \to \{\Sigma\text{-connections in } \xi\},$$

defined by $\lambda: V \mapsto \nabla_\xi$, is a bijection.

Proof: We construct the inverse map. Let $\tilde{\nabla}$ be the pull-back, via π, of a Σ-connection, ∇_ξ, in ξ to the trivial Σ-bundle

$$((P \times F, \pi_P, P, F), \Sigma_F).$$

According to Example 5, sec. 8.6, we can write

$$\tilde{\nabla} = \delta + \omega, \qquad \omega \in A^1(P; E)$$

(recall that $E \subset L_F$ is the Lie algebra of G).

We show now that ω is a connection form for $\mathscr{P}$. Let Z_h be the fundamental field on P generated by $h \in E$. Then, for $\sigma \in \operatorname{Sec} \xi$,

$$0 = i(Z_h)\, q^\#(\nabla\sigma) = i(Z_h)\, \tilde{\nabla}(q^\#\sigma) = Z_h(q^\#\sigma) + (i(Z_h)\omega)(q^\#\sigma).$$

Since $q^\#\sigma$ is equivariant. Proposition VII, sec. 3.15, yields

$$Z_h(q^\#\sigma) = -h(q^\#\sigma).$$

These relations show that $i(h)\omega = h$. Similarly, ω is equivariant and thus it is a connection form.

Finally, let V be the unique principal connection in $\mathscr{P}$ with connection form ω (cf. Proposition VI, sec. 6.10). The correspondence $\nabla_\xi \mapsto V$ defines a set map

$$\mu: \{\Sigma\text{-connections in } \xi\} \to \{\text{principal connections in } \mathscr{P}\},$$

and it has to be shown that λ and μ are inverse.

First, fix a Σ-connection ∇_ξ in ξ with pull-back $\tilde{\nabla}$ to $P \times F$. Let H^* and ω be the horizontal projection and connection form associated with $\mu(\nabla_\xi)$. Then $H^*\omega = 0$, and so

$$\begin{aligned}(H^*\delta) \circ q^\# &= H^* \circ (\delta + \omega) \circ q^\# \\ &= H^* \circ \tilde{\nabla} \circ q^\# = H^* \circ q^\# \circ \nabla_\xi = q^\# \circ \nabla_\xi .\end{aligned}$$

It follows that $\lambda\mu(\nabla_\xi) = \nabla_\xi$.

On the other hand, fix a principal connection V in $\mathscr{P}$ with horizontal projection, H^*, and connection form, ω. Let $\nabla_\xi = \lambda(V)$, and let ω_1 be the connection form of $\mu(\nabla_\xi)$. Then

$$(\delta + \omega_1) \circ q^\# = q^\# \circ \nabla_\xi = H^* \delta \circ q^\#.$$

Applying H^* to both sides, we find that

$$H^*\omega_1 \circ q^\# = 0,$$

whence $H^*\omega_1 = 0$. This implies that $\omega_1 = \omega$ and so $\mu\lambda(V) = V$.

Q.E.D.

Definition: A principal connection, V, and a Σ-connection, ∇_ξ, are called *associated* if $\lambda(V) = \nabla_\xi$ (or, equivalently, if $\mu(\nabla_\xi) = V$).

8.23. Curvature. Recall from Example 2, sec. 8.21, that the principal map for $\xi_E = P \times_G E$ is the bundle map q_E which is the restriction to $P \times E$ of $\hat{q}\colon P \times L_F \to L_\xi$ (cf. sec. 8.4). Thus (cf. Lemma III, sec. 7.7)

$$(q_E^\#\Psi)(q^\#\Phi) = q^\#(\Psi(\Phi)), \qquad \Psi \in A(B; \xi_E), \quad \Phi \in A(B; \xi).$$

Now let V be the principal connection in $\mathscr{P}$ associated with a Σ-connection, ∇. Their curvatures,

$$\Omega \in A^2(P; E) \qquad \text{and} \qquad R \in A^2(B; \xi_E),$$

are, respectively, E-valued and ξ_E-valued 2-forms (cf. sec. 6.14 and sec. 8.7).

Proposition XI: The curvatures of associated connections are related by

$$\Omega = q_E^\# R.$$

Proof: Let $\nabla_{\mathscr{P}}$ be the covariant exterior derivative with respect to the connection V. If $\tau \in \mathscr{S}(P; E)$ is equivariant, then

$$\Omega(\tau) = \nabla_{\mathscr{P}}^2(\tau)$$

(cf. the corollary to Proposition XII, sec. 6.14). Hence, for $\sigma \in \operatorname{Sec} \xi$,

$$\begin{aligned} \Omega(q^{\#}\sigma) &= \nabla_{\mathscr{P}}^2(q^{\#}\sigma) = q^{\#}\nabla^2\sigma \\ &= q^{\#}(R(\sigma)) = q_E^{\#}R(q^{\#}\sigma), \end{aligned}$$

and so $\Omega = q_E^{\#}R$. Q.E.D.

8.24. Weil and characteristic homomorphisms. Theorem VII: The characteristic homomorphism for ξ and the Weil homomorphism for $\mathscr{P}_\xi$ coincide.

Proof: q_E induces bundle maps,

$$q_E\colon P \times \vee^p E^* \to \vee^p \xi_E^*,$$

and, since q was a homomorphism of Σ-bundles, the diagram,

$$\begin{tikzcd} & \mathscr{S}_I(P; \vee E^*) \\ (\vee E^*)_I \arrow[ur, "\chi_*"] \arrow[dr, "\chi_*"'] & \\ & \operatorname{Sec} \vee \xi_E^* \arrow[uu, "\cong"', "q_E^*"] \end{tikzcd},$$

commutes (cf. sec. 8.10). Moreover $\chi_* : (\vee^p E^*)_I \to \mathscr{S}_I(P; \vee^p E^*)$ simply identifies $(\vee^p E^*)_I$ with the constant functions.

Next, consider a principal connection V in $\mathscr{P}$. It induces a homomorphism,

$$\gamma_{\mathscr{P}}\colon \vee E^* \to A(P)$$

(cf. sec. 6.17), defined via the curvature Ω of V. Extend $\gamma_{\mathscr{P}}$ to a homomorphism, $\mathscr{S}(P; \vee E^*) \to A(P)$, in the obvious way, and then restrict this to a homomorphism

$$\gamma_{\mathscr{P}}\colon \mathscr{S}_B(P; \vee E^*) \to A_B(P)$$

(basic means with respect to the representation of G in $\vee E^*$).

On the other hand, in sec. 8.11 we defined (for a Σ-connection ∇_ξ) a homomorphism

$$\gamma_\xi\colon \operatorname{Sec} \vee\xi_E^* \to A(B).$$

If V and ∇_ξ are associated, then it follows at once from Proposition XI, sec. 8.23, that the diagram (note that $\mathscr{S}_I(P;\ \vee E^*) = \mathscr{S}_B(P;\ \vee E^*)$),

$$\begin{tikzcd}
\mathscr{S}_I(P;\ \vee E^*) \arrow[r, "\gamma_{\mathscr{P}}"] & A_B(P) \\
\operatorname{Sec} \vee\xi_E^* \arrow[u, "q_E^*", "\cong"'] \arrow[r, "\gamma_\xi"'] & A(B) \arrow[u, "\pi^*"', "\cong"]
\end{tikzcd} ,$$

commutes.

Combining this with the previous diagram yields the commutative diagram,

$$\begin{tikzcd}
 & A_B(P) \\
(\vee E^*)_I \arrow[ur, "\gamma_I"] \arrow[dr, "\gamma_\xi^I"'] & \\
 & A(B) \arrow[uu, "\pi^*"', "\cong"]
\end{tikzcd} ,$$

where γ_I and γ_ξ^I are respectively the homomorphisms defined in sec. 6.17 and sec. 8.11. The theorem follows.

Q.E.D.

§8. Characteristic homomorphism for associated vector bundles

8.25. Representations. In this article $\Gamma = \mathbb{R}$. $\mathscr{P} = (P, \pi, B, G)$ denotes a principal bundle, and E is the Lie algebra of the Lie group G. Further, $\Phi: G \to GL(W)$ denotes a representation of G and $(v_1, ..., v_m)$ denotes a set of tensors over W such that $v_i \in W^{p_i, q_i}$, and such that each v_i is left fixed by G.

We let K denote the subgroup of $GL(W)$ consisting of those transformations which fix each v_i. The Lie algebra of K is denoted by F. Thus Φ is a homomorphism from G to K and $\Phi': E \to F$ is its derivative. Φ' induces the homomorphisms (cf. sec. 6.25)

$$(\Phi')^\vee: \vee E^* \leftarrow \vee F^* \qquad \text{and} \qquad (\Phi')^\vee_I: (\vee E^*)_I \leftarrow (\vee F^*)_I .$$

On the other hand, recall from sec. 5.6 that Φ determines an associated vector bundle,

$$\xi = (P \times_G W, \rho_\xi, B, W),$$

and a principal map $q: P \times W \to P \times_G W$. Denote $P \times_G W$ simply by M.

Since the v_i are G-invariant, there are unique cross-sections $\sigma_i \in \operatorname{Sec} \xi^{p_i, q_i}$ such that $q^\# \sigma_i = v_i$ $(i = 1, ..., m)$. It is evident from the construction that (ξ, Σ_ξ) is a Σ-bundle, with $\Sigma_\xi = (\sigma_1, ..., \sigma_m)$.

Now, in view of sec. 8.20, (ξ, Σ_ξ) determines an associated principal bundle,

$$\mathscr{P}_\xi = (P_\xi, \pi_\xi, B, K),$$

whose fibre, K_x, at $x \in B$ consists of the admissible isomorphisms from W to W_x. But by definition the linear maps,

$$q_z: W \xrightarrow{\cong} W_{\pi(z)}, \qquad z \in P,$$

carry v_i to $\sigma_i(\pi(z))$. Thus each q_z is admissible. It follows that maps,

$$\varphi_x: G_x \to K_x, \qquad x \in B,$$

are defined by $\varphi_x(z) = q_z$, $z \in G_x$.

A simple calculation using appropriate coordinate representations shows that the maps φ_x together define a smooth fibre preserving map

$$\varphi: P \to P_\xi,$$

which satisfies

$$\varphi(z \cdot a) = \varphi(z) \cdot \Phi(a), \qquad z \in P, \quad a \in G,$$

and which induces the identity map in B. Thus φ is a reduction of the structure group of $\mathscr{P}_\xi$ from K to G (cf. Example 5, sec. 5.5).

Moreover, if $q_\Sigma : P_\xi \times W \to M$ is the principal map, then the diagram,

$$\begin{tikzcd} P \times W \arrow[rr, "\varphi \times \iota"] \arrow[dr, "q"'] & & P_\xi \times W \arrow[dl, "q_\Sigma"] \\ & M & \end{tikzcd},$$

commutes.

Now, exactly as in sec. 8.22, we show that a principal connection, V_P, in $\mathscr{P}$ determines a Σ-connection, ∇_ξ, in (ξ, Σ_ξ). In fact, the argument of Proposition III, sec. 6.3, shows that $q^\#$ is an isomorphism from $A(B; \xi)$ to $A_B(P; W)$. On the other hand (cf. Proposition VII, sec. 6.12), the covariant exterior derivative for V_P restricts to an operator, ∇, in $A_B(P; W)$. Thus we set

$$\nabla_\xi = (q^\#)^{-1} \circ \nabla \circ q^\#$$

and argue as in Lemma IV, sec. 8.22, that ∇_ξ is in fact a Σ-connection.

Next, let V_{P_ξ} be the principal connection in $\mathscr{P}_\xi$ determined by ∇_ξ (cf. sec. 8.22). Then it is a straightforward consequence of the definitions that

$$d\varphi \circ V_P = V_{P_\xi} \circ d\varphi.$$

Thus Lemma VI, sec. 6.25, applies, and shows that the curvatures Ω_P of V_P and Ω_{P_ξ} of V_{P_ξ} are related by

$$(\Phi')_*(\Omega_P) = \varphi^*(\Omega_{P_\xi}).$$

Finally, use the commutative diagram above to obtain the commutative diagram,

$$\begin{tikzcd} P \times F \arrow[rr, "\varphi \times \iota"] \arrow[dr, "q_F"'] & & P_\xi \times F \arrow[dl, "(q_\Sigma)_F"] \\ & \xi_F & \end{tikzcd},$$

where ξ_F is the associated Lie algebra bundle for (ξ, Σ_ξ). According to Proposition XI, sec. 8.23, the curvature, R, of ∇_ξ satisfies

$$(q_\Sigma)_F^\# (R) = \Omega_{P_\xi}.$$

It follows that Ω_P and R are related by the equation

$$(\Phi')_* (\Omega_P) = (q_F)^{\#} (R). \tag{8.3}$$

Now a simple calculation (as in the proofs of Theorem III, sec. 6.25, and Theorem VII, sec. 8.24) gives the commutative diagram,

$$\begin{tikzcd} (\vee E^*)_I \arrow[r, "\gamma_I"] & A_B(P) \\ (\vee F^*)_I \arrow[u, "(\Phi')_I^\vee"] \arrow[r, "\gamma_\xi^I"'] & A(B), \arrow[u, "\pi^*"', "\cong"] \end{tikzcd}$$

where γ_I and γ_ξ^I are the homomorphisms determined respectively by Ω and R as described in secs. 6.17 and 8.11. Passing to cohomology we obtain the formula

$$h_\xi = h_{\mathscr{P}} \circ (\Phi')_I^\vee . \tag{8.4}$$

It, in turn, yields the relation

$$s_\xi = s_{\mathscr{P}} \circ (\Phi')^*.$$

8.26. Examples: 1. *Exterior algebra bundles*: Let ζ be any real vector bundle with associated principal bundle $\mathscr{P} = (P, \pi, B, GL(W))$. Let Φ denote the natural representation of $GL(W)$ in $\wedge W$. Then

$$\Phi'(\varphi) = \theta(\varphi), \qquad \varphi \in L_W ,$$

where $\theta(\varphi)$ is the unique derivation in $\wedge W$ that extends φ.

Since the bundle $\xi = (P \times_{GL(W)} \wedge W, \rho, B, \wedge W)$ is simply the exterior algebra bundle $\wedge\zeta$ and since $h_{\mathscr{P}} = h_\zeta$, formula (8.4) gives the commutative diagram

$$\begin{tikzcd} (\vee L_{\wedge W}^*)_I \arrow[dd, "\theta_I^\vee"'] \arrow[dr, "h_{\wedge\zeta}"] & \\ & H(B) \\ (\vee L_W^*)_I \arrow[ur, "h_\zeta"'] & \end{tikzcd}$$

It follows that $s_{\wedge\zeta} = s_\zeta \circ \theta^*$.

2. *Homogeneous spaces*: Let H be a closed subgroup of a Lie group G and let Φ be a representation of H in a vector space W. Consider the principal bundle $\mathscr{P} = (G, \pi, G/H, H)$ and let

$$\xi = (G \times_H W, \rho, G/H, W)$$

be the associated vector bundle (with respect to Φ).

Let E and F denote the Lie algebras of G and H. Assume that there is a stable decomposition $E = F \oplus F_1$ under the action of H. This decomposition determines a G-invariant principal connection in $\mathscr{P}$ (cf. sec. 6.30) whose curvature, Ω, satisfies

$$\Omega(e; h, k) = -p([h, k]), \qquad h, k \in F_1,$$

where $p: E \to F$ is the projection with kernel F_1 (cf. sec. 6.31).

In view of sec. 8.25 this principal connection determines a linear connection, ∇_ξ, in ξ. Moreover, because the principal connection was G-invariant, it follows that the natural action of G on $G \times_H W$ is by connection preserving bundle maps.

Now consider the curvature, R, of ∇_ξ. It takes values in the bundle $G \times_H L_W$, and is invariant under the action of G. Thus R is completely determined by $R(\bar{e})$, and $R(\bar{e}) \in \wedge^2 T_{\bar{e}}(G/H)^* \otimes L_W$. Moreover, formula (8.3) of sec. 8.25 shows that

$$R(\bar{e}; (d\pi)h, (d\pi)k) = -\Phi'(p([h, k])), \qquad h, k \in F_1.$$

It follows from this that the homomorphism h_ξ can be described as follows: Let $\Gamma \in (\vee^p L_W^*)_I$. Then $h_\xi(\Gamma)$ is represented by the (uniquely determined) differential form $\Psi \in A_I^{2p}(G/H)$ that satisfies

$$(\pi^*\Psi)(e; h_1, \ldots, h_{2p})$$
$$= \frac{(-1)^p}{2^p p!} \sum_{\sigma \in S^{2p}} \epsilon_\sigma \Gamma(\Phi'(p[h_{\sigma(1)}, h_{\sigma(2)}], \ldots, \Phi'(p[h_{\sigma(2p-1)}, h_{\sigma(2p)}])), \qquad h_i \in F_1.$$

Now suppose that G and H are compact and connected. Then G/H is orientable. Assume that $\dim G/H = 2p$. Then we can form the integral

$$\int_{G/H}^{\#} h_\xi(\Gamma) = \int_{G/H} \Psi.$$

Let $\Delta_{G/H}$ be the unique G-invariant $2p$-form on G/H which satisfies

$$\int_{G/H} \Delta_{G/H} = 1$$

(cf. sec. 2.14). Since Ψ is also G-invariant, it follows that

$$\Psi = \left(\int_{G/H}^{\#} h_\xi(\Gamma)\right) \Delta_{G/H}\,.$$

Thus to compute the integral it is sufficient to have an explicit formula for $(\pi^*\Delta_{G/H})(e)$.

In sec. 9.15 such an explicit formula will be established in the case rank $H =$ rank G. By contrast, in volume III it will be shown that if rank $H <$ rank G, then $\int_{G/H}^{\#} h_\xi(\Gamma) = 0$, for every $\Gamma \in (\vee^p L_W^*)_I$.

Problems

1. **Tensor product.** (i) Let F and W be finite dimensional vector spaces. Show that $(\varphi, \psi) \mapsto \varphi \otimes \psi$ is a representation of $GL(F) \times GL(W)$ in $F \otimes W$. Construct a finite set, Σ, of tensors over $F \otimes W$ such that the subgroup of $GL(F \otimes W)$ leaving Σ fixed is $GL(F) \times GL(W)$.

(ii) Represent the tensor product of two vector bundles ξ and η as a Σ-bundle.

(iii) Find an analogue of the results in sec. 8.16 and the example of sec. 8.17 for tensor products. In particular, if (ξ, Σ_ξ) and (η, Σ_η) are Σ-bundles, define a Σ-bundle $(\xi \otimes \eta, \Sigma_{\xi\otimes\eta})$.

2. Let ξ be a complex vector bundle of rank r. Show that $\wedge^r\xi$ is trivial if and only if the structure group of ξ can be reduced to $SU(r)$. Interpret this in terms of Σ-bundles.

3. Extend the notion of Σ-bundles to sets Σ_ξ with infinitely many elements. Show that the structure group for such a bundle is the same as the structure group for some finite substructure.

4. Let (ξ, Σ) be a Σ-bundle with structure group G and let $\hat{G} \supset G$ denote the group which fixes the set Σ (but not necessarily pointwise). Show that $\hat{G}^0 = G^0$.

Construct a Σ-bundle $(\xi, \hat{\Sigma})$ such that $\hat{G}$ is the structure group of $(\xi, \hat{\Sigma})$.

5. **Compact carrier.** Let $(\xi, \Sigma_\xi, \alpha)$ be a Σ-bundle with compact carrier and compact Σ-connection.

Show that (ξ, Σ_ξ) determines an associated principal bundle, $\mathscr{P}$, with compact carrier. Extend the results of article 7 to this case. In particular, show that $h_{\mathscr{P}}^c = h_\xi^c$ (cf. problem 15, Chap. VI).

6. Let (M, π, B, F) be a smooth bundle. Define the notion of a Σ-bundle over M with fibre-compact carrier. Show that such a bundle determines a canonical homomorphism $h_\xi^F\colon (\vee^+E^*)_I \to H_F(M)$.

7. **Manifold algebras.** A real *manifold algebra* is an algebra, A, over $\mathbb{R}$ such that the derivations in A form a finitely generated projective A-module, Der A.

(i) Define substitution operator, Lie derivative and exterior derivative in $\wedge_A(\text{Der } A)^*$. Set $H(\wedge_A(\text{Der } A)^*, \delta) = H(A)$.

(ii) Let M be a finitely generated projective A-module. Define a *linear connection* in M as a map,

$$\nabla: M \to (\text{Der } A)^* \otimes_A M,$$

which satisfies $\nabla(x + y) = \nabla x + \nabla y$ and $\nabla(f \cdot x) = \delta f \otimes x + f \cdot \nabla x$. Show that M always admits a linear connection.

(iii) Define the curvature of a linear connection. Establish the Bianchi identity.

(iv) Show that ∇ induces linear connections in the modules $(\otimes^p M^*) \otimes (\otimes^q M)$, $\wedge M$, $\vee^p M$ and obtain their curvatures.

(v) Let $\sigma_i \in (\otimes M^*) \otimes (\otimes M)$ $(i = 1, \ldots, m)$ satisfy $\nabla \sigma_i = 0$. Set $\Sigma = (\sigma_1, \ldots, \sigma_m)$. Let L_M act by derivations in this algebra and let E be the submodule of endomorphisms φ such that $\varphi(\sigma_i) = 0$ $(i = 1, \ldots, m)$. Show that

$$\hat{\nabla}: L_M \to (\text{Der } A)^* \otimes_A L_M$$

restricts to a map $\hat{\nabla}: E \to (\text{Der } A)^* \otimes_A E$.

(vi) Construct a characteristic homomorphism $(\vee_A E^*)_{\theta=0,\nabla=0} \to H(A)$ and prove it is independent of the connection.

(vii) If the homomorphism in (vi) is nonzero (with $\Sigma = \varnothing$), conclude that M is not free.

8. (i) Define the odd characteristic homomorphism for Σ-bundles of compact carrier over $B \times \mathbb{R}$ which are trivial as Σ-bundles (cf. problem 16, Chap. VI). Show that it coincides with the map $\oint_{\mathbb{R}}^{\#} \circ h_\sigma^c$ of problem 16, Chap. VI.

(ii) Convert problem 17, Chap. VI, to a theorem on Σ-bundles.

9. The ring $V_\Sigma(B)$. The *isomorphism class* of a Σ-bundle (ξ, Σ_ξ) over B is the collection of all vector bundles over B which are strongly isomorphic to (ξ, Σ_ξ). Denote the set of isomorphism classes of Σ-bundles by $\text{Vect}_\Sigma(B)$. Let $\mathscr{F}_\Sigma(B)$ be the free abelian group with the elements of $\text{Vect}_\Sigma(B)$ as basis. Consider the factor group generated by elements of the form $[\xi, \Sigma_\xi] + [\eta, \Sigma_\eta] - [\xi \oplus \eta, \Sigma_{\xi\oplus\eta}]$ and denote this factor group by $V_\Sigma(B)$. (Note that we get two separate groups, depending on whether the field Γ is $\mathbb{R}$ or $\mathbb{C}$.)

(i) Show that every element of $V_\Sigma(B)$ can be represented in the form $[\xi, \Sigma_\xi] - [\eta, \Sigma_\eta]$.

(ii) Show that the tensor product of problem 1, (iii) makes $V_\Sigma(B)$ into a commutative ring with identity.

(iii) Show that a smooth map $\varphi: B_1 \to B_2$ induces a ring homomorphism $\varphi^*: V_\Sigma(B_1) \leftarrow V_\Sigma(B_2)$ which depends only on the homotopy class of φ.

(iv) Define the rings $V_\Sigma^c(B)$ corresponding to Σ-bundles with compact support. Obtain an analogue of (iii) for proper maps and proper homotopies. If B is compact, prove that $V_\Sigma^c(B) = V_\Sigma(B)$.

(v) Let (M, π, B, F) be a smooth bundle. Define the ring $V_\Sigma^F(M)$ corresponding to Σ-bundles with fibre-compact carrier.

10. The ring $V(B)$. (i) Repeat the construction of problem 9 for ordinary vector bundles to obtain a ring $V(B)$ in which $\bar{\xi} + \bar{\eta} = \overline{\xi \oplus \eta}$ and $\bar{\xi} \cdot \bar{\eta} = \overline{\xi \otimes \eta}$, where ξ and η are vector bundles representing $\bar{\xi}$ and $\bar{\eta}$.

(ii) Similarly obtain rings $V^c(B)$ and $V^F(M)$ as in problem 9, (iv) and problem 9, (v). If the field is $\mathbb{C}$ then $V^c(B)$ is denoted by $K(B)$.

(iii) Show that the map $(\xi, \Sigma_\xi) \mapsto \xi$ defines surjective ring homomorphisms $V_\Sigma(B) \to V(B)$, $V_\Sigma^c(B) \to V^c(B)$ and $V_\Sigma^F(M) \to V^F(M)$.

(iv) Show that $\xi \mapsto \operatorname{rank} \xi$ determines surjective ring homomorphisms $V(B) \to \mathbb{Z}$, $V^c(B) \to \mathbb{Z}$, and $V^F(M) \to \mathbb{Z}$. Construct left inverses for these homomorphisms. Denote their kernels respectively by $\tilde{V}(B)$, $\tilde{V}^c(B)$, and $\tilde{V}^F(M)$.

11. Represent quaternionic vector bundles as real Σ-bundles.

12. Let (ξ, Σ_ξ) be a Σ-bundle with total space M.

(i) Find necessary and sufficient conditions on a horizontal subbundle, H_M, for M so that the corresponding general connection is a Σ-connection.

(ii) Consider the principal map $q: P_\xi \times F \to M$. Assume that V_{P_ξ} is a principal connection for P_ξ with horizontal bundle H_{P_ξ}. Show that the spaces $(dq)_{(z,y)}(H_z(P_\xi))$, $z \in P_\xi$, $y \in F$, are the fibres of a horizontal bundle for ξ. Show that the corresponding general connection is the Σ-connection in ξ associated with V_{P_ξ}.

13. Construct Σ-bundles (ξ, Σ_ξ) and $(\xi, \hat{\Sigma}_\xi)$ with the same underlying vector bundle, ξ, but such that $(\xi, \Sigma_\xi \cup \hat{\Sigma}_\xi)$ is not a Σ-bundle.

14. Let Φ be a tensor field on a connected manifold M whose covariant derivative (with respect to some linear connection) is zero (cf. problem 8, Chap. VII). Interpret $(\tau_M, \{\Phi\})$ as a Σ-bundle.

15. Let ξ be a vector bundle over B and suppose $d \in \operatorname{Sec} L_\xi$ satisfies $d_x^2 = 0$, $x \in B$. Show that the following are equivalent:

(i) $(\xi, \{d\})$ is a Σ-bundle.

(ii) The spaces $H(F_x, d_x)$ all have the same dimension.

(iii) The spaces $\ker d_x$ are the fibres of a subbundle, ζ, of ξ, and the projections, $\ker d_x \to H(F_x)$, define a strong bundle map of ζ onto a vector bundle κ.

16. Let ξ be a vector bundle and assume $\langle\ ,\ \rangle$ is a skew-symmetric nondegenerate bilinear form in ξ (i.e., $\langle\ ,\ \rangle$ restricts to scalar products in each fibre). Prove that $(\xi, \{\langle\ ,\ \rangle\})$ is a Σ-bundle.

17. Extend the results of Example 2, sec. 8.26, to Σ-bundles over G/H.

18. Let ∇_1 and ∇_2 be two Σ-connections in a Σ-bundle (ξ, Σ_ξ), with Lie algebra E. Let $\Gamma \in (\vee^p E^*)_I$, and write (as in sec. 6.20) $((\gamma_\xi^I)^1 - (\gamma_\xi^I)^2)(\Gamma)$ as an explicit coboundary, giving a "vector bundle" proof.

19. Let (ξ, Σ_ξ) be a Σ-bundle which admits a Σ-coordinate representation with p elements. Prove that the product of any $p + 1$ elements in $h_\xi(\vee^+ E^*)_I$ is zero.

20. Let ξ be a vector bundle, and assume $\varphi \in \operatorname{Sec} L_\xi$. Suppose $f(t)$ is a polynomial with constant coefficients such that $f(\varphi_x) = 0$ for all x.

(i) Prove that there are unique cross-sections $\varphi_S, \varphi_N \in \operatorname{Sec} L_\xi$ such that (a) each $(\varphi_S)_x$ is semisimple, (b) each $(\varphi_N)_x$ is nilpotent, (c) $\varphi = \varphi_S + \varphi_N$, and (d) $\varphi_S \circ \varphi_N = \varphi_N \circ \varphi_S$. Prove that φ_S is 0-deformable.

(ii) Establish a converse to (i).

Chapter IX

Pontrjagin, Pfaffian, and Chern Classes

§1. The modified characteristic homomorphism for real Σ-bundles

9.1. Definition. Let (ξ, Σ_ξ) be a real Σ-bundle with Σ-connection ∇ and curvature R. The *modified curvature* of ∇ is the element $\tilde{R} \in \mathbb{C} \otimes A^2(B; \xi_E)$ given by

$$\tilde{R} = \frac{-1}{2\pi i} R = \left(\frac{-1}{2\pi i}\right) \otimes R.$$

In analogy with secs. 8.11, 8.12, and 8.13 we define the *modified characteristic homomorphism,*

$$\tilde{h}_\xi \colon (\vee E^*)_I \to H(B; \mathbb{C}),$$

as follows: Regard $\Psi \in (\vee^p E^*)_I$ as a parallel cross-section in $\vee^p \xi_E^*$. Choose $\Phi \in (\otimes^p E^*)_I$ so that $\pi_S \Phi = \Psi$ and define $\tilde{h}_\xi(\Psi)$ to be the cohomology class represented by

$$\Phi\left(\frac{-1}{2\pi i} R, \cdots, \frac{-1}{2\pi i} R\right).$$

The maps h_ξ and $\tilde{h}_\xi$ are connected by the relation

$$\tilde{h}_\xi(\Psi) = \left(\frac{-1}{2\pi i}\right)^p h_\xi(\Psi), \qquad \Psi \in (\vee^p E^*)_I .$$

This shows that $\tilde{h}_\xi$ is indeed a homomorphism (of *real* algebras) and has the same properties as h_ξ (naturality, independence of connection). Moreover, $\tilde{h}_\xi(\Psi)$ is a real (respectively, purely imaginary) class if Ψ is homogeneous of even (respectively, odd) degree.

Finally, we extend $\tilde{h}_\xi$ to the algebra $(\vee^{**} E^*)_I$ and precompose with the Taylor homomorphism to obtain a homomorphism

$$\tilde{s}_\xi \colon \mathscr{S}_0(E)_I \to H(B; \mathbb{C})$$

(cf. sec. 8.14).

Remark: The reasons for introducing both h_ξ and $\tilde{h}_\xi$ are as follows: On the one hand, h_ξ has an essentially formal, algebraic definition. For this reason, we shall work almost exclusively with this homomorphism in volume III.

On the other hand, the cohomology classes to be discussed in this chapter can be constructed as integral classes in a purely topological fashion. In the approach presented here, they arise out of classical invariants of linear transformations, and, in order to obtain the correct signs and dimension factors it is more convenient to use $\tilde{h}_\xi$.

§2. Real bundles: Pontrjagin and trace classes

In this article $\xi = (N, \pi, B, F)$ denotes a real vector bundle of rank r, and

$$\tilde{h}_\xi\colon (\vee L_F^*)_I \to H(B; \mathbb{C})$$

denotes the corresponding modified characteristic homomorphism. ($(\vee L_F^*)_I$ is the invariant subalgebra of $\vee L_F^*$ for the action of $GL(F)$.)

9.2. Proposition I: Every nonzero homogeneous characteristic class of ξ has degree $\equiv 0 \pmod 4$.

Proof: Give ξ a Riemannian metric, $\langle\ ,\ \rangle$. From Proposition VII, sec. 8.17, we obtain the commutative diagram,

$$\begin{tikzcd}
(\vee L_F^*)_I \arrow[d, "j_I^\vee"'] \arrow[dr, "\tilde{h}_\xi"] & \\
& H(B; \mathbb{C}) \\
(\vee \mathrm{Sk}_F^*)_I \arrow[ur, "\tilde{k}_\xi"'] &
\end{tikzcd},$$

where $j\colon \mathrm{Sk}_F \to L_F$ is the inclusion, $(\vee \mathrm{Sk}_F^*)_I$ is the invariant subalgebra for the *orthogonal group* $O(F)$ and $\tilde{k}_\xi$ is the modified characteristic homomorphism for the Σ-bundle $(\xi, \langle\ ,\ \rangle)$. Thus it is sufficient to show that if $\Phi \in (\vee^p L_F^*)_I$, p odd, then

$$j_I^\vee(\Phi) = 0.$$

But, for every $\varphi \in L_F$, there exists an $\alpha \in GL(F)$ such that

$$\alpha \circ \varphi \circ \alpha^{-1} = \varphi^*,$$

where φ^* denotes the adjoint of φ with respect to the inner product. Hence, if $\varphi \in \mathrm{Sk}_F$,

$$\alpha \circ \varphi \circ \alpha^{-1} = -\varphi.$$

It follows that, for $\Phi \in (\vee^p L_F^*)_I$,

$$\Phi(\varphi,..., \varphi) = (-1)^p \Phi(\varphi,..., \varphi), \qquad \varphi \in \mathrm{Sk}_F ;$$

i.e.,

$$j_I^\vee \Phi = (-1)^p j_I^\vee \Phi.$$

In particular, $j_I^\vee \Phi = 0$, if p is odd.

Q.E.D.

Corollary: Every class in the image of $\tilde{h}_\xi$ is real, and so $\tilde{h}_\xi$ may be regarded as a homomorphism into $H(B)$.

9.3. Examples: 1. *Involutive distributions*: An *involutive distribution* on a manifold M is a subbundle η of the tangent bundle τ_M such that for all X, $Y \in \mathrm{Sec}\,\eta$, $[X, Y] \in \mathrm{Sec}\,\eta$.

Assume that τ_M is decomposed in the form

$$\tau_M = \xi \oplus \eta,$$

where η is an involutive distribution. Let $\rho_\xi\colon \tau_B \to \xi$ be the corresponding projection. Let ∇_ξ be any linear connection in ξ. Then there is a unique linear connection, ∇, in ξ that satisfies (cf. sec. 7.11)

$$\nabla_X = (\nabla_\xi)_X, \qquad X \in \mathrm{Sec}\,\xi,$$

and

$$\nabla_Y X = (\rho_\xi)_*[Y, X], \qquad X \in \mathrm{Sec}\,\xi, \quad Y \in \mathrm{Sec}\,\eta.$$

We shall show that the curvature, R, of ∇ satisfies

$$R(Y_1, Y_2) = 0, \qquad Y_i \in \mathrm{Sec}\,\eta.$$

In fact, fix $X \in \mathrm{Sec}\,\xi$ and define $Y \in \mathrm{Sec}\,\eta$ by

$$\nabla_{Y_2} X = [Y_2, X] + Y.$$

Since η is involutive, $[Y_1, Y] \in \mathrm{Sec}\,\eta$. Hence $(\rho_\xi)_* [Y_1, Y] = 0$, and so

$$\nabla_{Y_1}(\nabla_{Y_2} X) = (\rho_\xi)_*([Y_1, [Y_2, X]]).$$

It follows from this and the Jacobi identity that

$$\begin{aligned} R(Y_1, Y_2)(X) &= \nabla_{Y_1}(\nabla_{Y_2} X) - \nabla_{Y_2}(\nabla_{Y_1} X) - \nabla_{[Y_1, Y_2]}(X) \\ &= (\rho_\xi)_*([Y_1, [Y_2, X]] - [Y_2, [Y_1, X]] - [[Y_1, Y_2], X]) \\ &= 0; \end{aligned}$$

i.e., $R(Y_1, Y_2) = 0$.

Thus, if $\Phi \in (\vee^p L_F^*)_I$ (F, the typical fibre of ξ) and $p >$ rank ξ, the differential form,

$$\Psi = \frac{1}{p!}\,\Phi\Big(\frac{-1}{2\pi i}\,R, \ldots, \frac{-1}{2\pi i}\,R\Big),$$

is zero. Since Ψ represents $\tilde{h}_\xi(\Phi)$, it follows that $\tilde{h}_\xi(\Phi) = 0$; i.e.,

$$\tilde{h}_\xi^p = 0, \qquad p > \operatorname{rank} \xi.$$

2. *Group actions*: Let $T\colon G \times M \to M$ be an action of a compact connected Lie group on a manifold M. Recall that every point $x \in M$ determines the smooth map $A_x\colon G \to M$ given by

$$A_x(a) = a \cdot x, \qquad a \in G, \quad x \in M.$$

Assume that the subspaces $\operatorname{Im}(dA_x)_e \subset T_x(M)$ all have the same dimension (equivalently, all the orbits of G have the same dimension). Then these spaces are the fibres of a distribution, η, on M. Moreover, the module Sec η is generated by the fundamental vector fields. It follows that η is involutive.

Give τ_M a G-invariant Riemannian metric (cf. Example I, sec. 3.18), and let ξ be the orthogonal complement of η. Then the fundamental vector fields Z_h ($h \in E$) satisfy

$$Z_h(\langle X, Y\rangle) = \langle [Z_h, X], Y\rangle + \langle X, [Z_h, Y]\rangle, \qquad X, Y \in \mathscr{X}(M).$$

It follows that

$$[Z_h, X] \in \operatorname{Sec} \xi \qquad \text{if} \quad X \in \operatorname{Sec} \xi.$$

Next, construct a G-invariant connection, ∇_I, in ξ as follows: Let ∇_ξ be any linear connection and set

$$(\nabla_I X)(x; Z(x)) = \int_G a^{-1} \cdot (\nabla_\xi(a \cdot X)(a \cdot x; a \cdot Z(x)))\, da.$$

Then

$$(\nabla_I)_{a \cdot Z}(a \cdot X) = a \cdot ((\nabla_I)_Z X), \qquad Z \in \mathscr{X}(M), \quad X \in \operatorname{Sec} \xi, \quad a \in G.$$

It follows that (set $a = \exp th$ and differentiate with respect to t)

$$\begin{aligned}&(\nabla_I)_{[Z_h, Z]}(X) + (\nabla_I)_Z([Z_h, X])\\ &\qquad = [Z_h, (\nabla_I)_Z(X)], \qquad h \in E,\ Z \in \mathscr{X}(M), \quad X \in \operatorname{Sec} \xi\end{aligned}$$

(E, the Lie algebra of G).

Finally, define a linear connection, ∇, in ξ by setting

$$\nabla_Y X = (\nabla_I)_Y X, \qquad X, Y \in \operatorname{Sec} \xi$$

and

$$\nabla_Y X = (\rho_\xi)_*[Y, X], \qquad X \in \operatorname{Sec} \xi, \quad Y \in \operatorname{Sec} \eta.$$

Then the corresponding curvature satisfies

$$i(Y)R = 0, \qquad Y \in \operatorname{Sec} \eta. \tag{9.1}$$

In fact, according to Example 1, $R(Y_1, Y_2) = 0$, $Y_1, Y_2 \in \operatorname{Sec} \eta$. Thus we have only to show that

$$R(Y, X) = 0, \qquad Y \in \operatorname{Sec} \eta, \quad X \in \operatorname{Sec} \xi.$$

Since the fundamental fields span the fibres of η, it is sufficient to show that

$$R(Z_h, X) = 0, \qquad X \in \operatorname{Sec} \xi, \quad h \in E.$$

But for $X' \in \operatorname{Sec} \xi$,

$$\begin{aligned} R(Z_h, X)(X') &= [Z_h, (\nabla_I)_X(X')] - (\nabla_I)_X([Z_h, X']) - (\nabla_I)_{[Z_h, X]}(X') \\ &= 0, \end{aligned}$$

as follows from the relation above.

Next observe that

$$\nabla_{a\cdot Z}(a \cdot X) = a \cdot \nabla_Z X, \qquad Z \in \mathscr{X}(M), \quad X \in \operatorname{Sec} \xi, \quad a \in G.$$

It follows that

$$\begin{aligned} &R(a \cdot Z_1, a \cdot Z_2)(a \cdot X) \\ &\quad = a \cdot R(Z_1, Z_2)(X), \qquad a \in G, \quad Z_i \in \mathscr{X}(M), \quad X \in \operatorname{Sec} \xi. \end{aligned} \tag{9.2}$$

Let $\Phi \in (\vee^p L_F^*)_I$ (F, the typical fibre of ξ). Relations (9.1) and (9.2) imply that the differential form,

$$\Psi = \frac{1}{p!}\, \Phi\Big(\frac{-1}{2\pi i} R, \ldots, \frac{-1}{2\pi i} R\Big),$$

is both horizontal and invariant with respect to the action of G. Thus

$$i(Z_h)\Psi = 0 \qquad \text{and} \qquad T_a^*\Psi = \Psi, \qquad h \in E, \quad a \in G.$$

Hence

$$\theta(Z_h)\Psi = 0, \qquad h \in E.$$

It follows that

$$i(Y)\Psi = 0 = \theta(Y)\Psi, \qquad Y \in \operatorname{Sec} \eta,$$

and so $\Psi = 0$ if $2p > \operatorname{rank} \xi$.

Let $A_I(M)_{i=0}$ denote the subalgebra of horizontal invariant forms. The remarks above show that the homomorphism $\gamma_\xi^I\colon (\vee L_F^*)_I \to A(M)$ determined by R is in fact a homomorphism into $A_I(M)_{i=0}$. Thus it determines a modified homomorphism $(\vee L_F^*)_I \to H(\mathbb{C} \otimes A_I(M)_{i=0})$, and the diagram,

$$\begin{tikzcd}
 & H(M;\mathbb{C}) \\
(\vee L_F^*)_I \arrow[ur, "\tilde{h}_\xi"] \arrow[dr] & \\
 & H(\mathbb{C} \otimes A_I(M)_{i=0}), \arrow[uu]
\end{tikzcd}$$

commutes. In particular, if $\Phi \in (\vee^p L_F^*)_I$ and $2p > \operatorname{rank} \xi$, then

$$\tilde{h}_\xi(\Phi) = 0.$$

9.4. Pontrjagin classes. In sec. A.2 of Appendix A the characteristic coefficients $C_k^F \in (\vee^k L_F^*)_I$ are defined. They satisfy

$$\det(\varphi + \lambda\iota) = \sum_{k=0}^{r} C_k^F(\varphi)\,\lambda^{r-k}, \qquad \varphi \in L_F, \qquad r = \dim F,$$

where

$$C_0^F(\varphi) = 1, \qquad C_k^F(\varphi) = \frac{1}{k!}\, C_k^F(\varphi, \ldots, \varphi), \qquad k = 1, \ldots, r.$$

The cohomology classes $p_k(\xi)$ given by

$$p_k(\xi) = \tilde{h}_\xi(C_{2k}^F), \qquad 0 \leqslant 2k \leqslant r,$$

are called the *Pontrjagin classes of* ξ. According to the corollary of Proposition I, sec. 9.2, these classes are real; $p_k(\xi) \in H^{4k}(B)$. In volume III it will be shown that the Pontrjagin classes generate the characteristic algebra of ξ.

The class $p_k(\xi)$ is represented by the differential form, P_k, given by

$$P_k = \frac{(-1)^k}{(2\pi)^{2k}(2k)!} C^F_{2k}(R, \dots, R) = \frac{(-1)^k}{(2\pi)^{2k}(2k)!} \operatorname{tr}(R \,\square\, \cdots \,\square\, R) \qquad (9.3)$$

(cf. sec. A.2) where R is the curvature of a linear connection in ξ.

The nonhomogeneous class $p(\xi) = \sum_{k=0}^{r} p_k(\xi)$ is called the *total Pontrjagin class of* ξ. Since $p_0(\xi) = 1$, $p(\xi)$ is invertible in the algebra $H(B)$. It can be written in the form,

$$p(\xi) = \tilde{s}_\xi(\det(\iota + \varphi)),$$

where $\det(\iota + \varphi)$ is the invariant function in L_F given by $\varphi \mapsto \det(\iota + \varphi)$.

Note that $p(\epsilon^q) = 1$ because ϵ^q (the trivial bundle of rank q) admits a connection with curvature zero. (cf. Example 1, sec. 7.16.)

Remark: The definition of the Pontrjagin classes, p_k, differs from the definition given in [10, p. 78] by the sign factor $(-1)^k$.

Proposition II: The Pontrjagin classes have the following properties:

(1) *Naturality.* Let $\varphi: \xi \to \zeta$ be a bundle map restricting to isomorphisms in the fibres and inducing a smooth map ψ between the base manifolds. Then

$$p(\xi) = \psi^\#(p(\zeta)).$$

(2) *Whitney sums.* Let η be a second vector bundle over B with typical fibre H. Then

$$p(\xi \oplus \eta) = p(\xi) \cdot p(\eta).$$

Proof: (1) This follows from Theorem IV, sec. 8.13.

(2) This follows from the example of sec. 8.17 and the relation

$$C_k^{F+H}(\varphi \oplus \psi) = \sum_{i+j=k} C_i^F(\varphi)\, C_j^H(\psi), \qquad \varphi \in L_F, \quad \psi \in L_H$$

(cf. Proposition I, sec. A.2).

Q.E.D.

Corollary I: Assume $\xi \oplus \epsilon^q = \eta \oplus \epsilon^p$. Then

$$p(\xi) = p(\eta).$$

Corollary II: Assume $\xi \oplus \cdots \oplus \xi \oplus \epsilon^p = \epsilon^{rq+p}$. Then

$$p(\xi) = 1.$$

Proof: Clearly $p(\xi)^q = 1$. Now an easy degree argument shows that $p(\xi) = 1$.

Q.E.D.

Corollary III: Assume $\xi \oplus \eta$ is trivial. Then

$$p(\xi) = p(\eta)^{-1}.$$

9.5. Pontrjagin numbers. Assume that B is compact, connected, oriented, and of dimension $4k$. Let m be the largest integer such that $2m \leqslant r$. Consider the m-tuples $(j_1, \ldots, j_m)$ of nonnegative integers that satisfy

$$\sum_{\mu=1}^{m} \mu \cdot j_\mu = k.$$

With each such m-tuple we associate the cohomology class,

$$p_1(\xi)^{j_1} \cdot \cdots \cdot p_m(\xi)^{j_m} \in H^{4k}(B),$$

and the real numbers $\lambda_{j_1 \cdots j_m}(\xi)$ given by

$$\lambda_{j_1 \cdots j_m}(\xi) = \int_B^{\#} p_1(\xi)^{j_1} \cdots p_m(\xi)^{j_m}.$$

They are called the *Pontrjagin numbers of* ξ. If $\xi = \tau_B$, these are called the *Pontrjagin numbers of* B and are written $\lambda_{j_1 \cdots j_m}(B)$.

Evidently $H^{4k}(B) \subset \operatorname{Im} \tilde{h}_\xi$ if and only if some Pontrjagin number of ξ is nonzero.

As an immediate consequence of Proposition II, (1), above we have

Proposition III: Let $\hat{\xi} = (\hat{M}, \hat{\pi}, \hat{B}, F)$ be a second vector bundle of rank r over a compact, connected oriented $4k$-dimensional base. Let $\varphi: \xi \to \hat{\xi}$ be a bundle map which restricts to linear isomorphisms in the fibres and let $\psi: B \to \hat{B}$ be the induced map. Then the Pontrjagin numbers of ξ are given by

$$\lambda_{j_1 \cdots j_m}(\xi) = \deg \psi \cdot \lambda_{j_1 \cdots j_m}(\hat{\xi}).$$

Corollary I: If some Pontrjagin number of ξ is different from zero, then $\deg \psi \neq 0$.

Corollary II: If $\hat{\xi} = \xi$ and at least one Pontrjagin number is different from zero, then $\deg \psi = 1$.

Corollary III: Suppose that at least one Pontrjagin number of B is nonzero. Then every local diffeomorphism of B is an orientation preserving diffeomorphism. In particular, B is an irreversible manifold.

Proof: Apply Corollary II to the bundle map $d\psi\colon T_B \to T_B$ to obtain $\deg \psi = 1$. Now the corollary to Theorem I, sec. 6.3, volume I, implies that ψ is a diffeomorphism.

Q.E.D.

9.6. Trace classes and the Pontrjagin character. The *trace coefficients* of an r-dimensional vector space, F, are the elements $\mathrm{Tr}_p^F \in (\vee^p L_F^*)_I$ given by

$$\mathrm{Tr}_p^F(\varphi_1, \ldots, \varphi_p) = \sum_{\sigma \in S^p} \mathrm{tr}(\varphi_{\sigma(1)} \circ \cdots \circ \varphi_{\sigma(p)}), \qquad p \geq 1, \quad \varphi_i \in L_F,$$

and

$$\mathrm{Tr}_0^F = \dim F.$$

The cohomology classes, $\mathrm{tr}_p(\xi)$, given by

$$\mathrm{tr}_p(\xi) = \tilde{h}_\xi(\mathrm{Tr}_{2p}^F), \qquad p = 0, 1, \ldots,$$

are called the *trace classes of* ξ. They are represented by the differential forms,

$$T_p = \frac{(-1)^p}{(2\pi)^{2p}} \,\mathrm{tr}(R \circ \cdots \circ R), \tag{9.4}$$

where R is the curvature of a linear connection in ξ (cf. sec. 7.15).

Proposition III, sec. A.3, implies that the trace and Pontrjagin classes of ξ are related by

$$p_k(\xi) = \frac{-1}{2k} \sum_{j=0}^{k-1} p_j(\xi)\, \mathrm{tr}_{k-j}(\xi), \qquad k \geq 1. \tag{9.5}$$

The *Pontrjagin character of* ξ is the nonhomogeneous class $\mathrm{tr}(\xi)$ given by

$$\mathrm{tr}(\xi) = \sum_{p=0}^{\infty} \frac{1}{(2p)!}\, \mathrm{tr}_p(\xi).$$

(Observe that this is a finite sum.) It can also be written in the form

$$\operatorname{tr}(\xi) = \tilde{s}_\xi(\operatorname{tr} \circ \exp),$$

where $\operatorname{tr} \circ \exp$ denotes the smooth invariant function in L_F given by $\varphi \mapsto \operatorname{tr} \exp \varphi$.

The trace classes (Pontrjagin classes) of the tangent bundle of a manifold M are called the *trace classes* (*Pontrjagin classes*) *of* M and are denoted by $\operatorname{tr}(M)$ and $(p_k(M))$.

Proposition IV: Let ξ and η be vector bundles over the same base. Then

(1) $\operatorname{tr}(\xi \oplus \eta) = \operatorname{tr}(\xi) + \operatorname{tr}(\eta)$.
(2) $\operatorname{tr}(\xi \otimes \eta) = \operatorname{tr}(\xi) \cdot \operatorname{tr}(\eta)$.
(3) $\operatorname{tr}(\wedge^p \xi) = \tilde{s}_\xi(\operatorname{tr} \wedge^p \exp)$.
(4) $\operatorname{tr}(\wedge \xi) = \tilde{s}_\xi \det(\iota + \exp)$.

($\det(\iota + \exp)$ denotes the function $\varphi \mapsto \det(\iota + \exp \varphi)$, $\varphi \in L_F$).

Proof: Let R_ξ and R_η be the curvatures of linear connections in ξ and η.

(1) The induced connection in $\xi \oplus \eta$ has curvature $R_\xi \oplus R_\eta$ (cf. Example 3, sec. 7.16). Thus (1) follows from the relation

$$(R_\xi \oplus R_\eta) \circ \cdots \circ (R_\xi \oplus R_\eta) = (R_\xi \circ \cdots \circ R_\xi) \oplus (R_\eta \circ \cdots \circ R_\eta).$$

(2) The induced connection in $\xi \otimes \eta$ has curvature $R_\xi \otimes \iota + \iota \otimes R_\eta$. Since the $L_{\xi \otimes \eta}$-valued differential forms $R_\xi \otimes \iota$ and $\iota \otimes R_\eta$ commute, we have

$$(R_\xi \otimes \iota + \iota \otimes R_\eta)^k = \sum_{i=0}^{k} \binom{k}{i} R_\xi^i \otimes R_\eta^{k-i}.$$

Thus (2) follows from the relation (cf. sec. 7.7)

$$\operatorname{tr}(\Phi \otimes \Psi) = \operatorname{tr}(\Phi) \wedge \operatorname{tr}(\Psi), \qquad \Phi \in A(B; L_\xi), \quad \Psi \in A(B; L_\eta).$$

(3) The induced connection in $\wedge^p \xi$ has curvature $\theta_*(R)$ (cf. Example 4, sec. 7.16), where $\theta: L_F \to L_{\wedge^p F}$ is the map given by

$$\theta(\varphi)(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} x_1 \wedge \cdots \wedge \varphi x_i \wedge \cdots \wedge x_p, \qquad x_i \in F, \quad \varphi \in L_F.$$

Hence the diagram,

$$\begin{tikzcd}
(\vee L^*_{\wedge^p F})_I \arrow[d, "\theta_I^\vee"'] \arrow[dr, "\tilde{h}_{\wedge^p \xi}"] & \\
& H(B; \mathbb{C}) \\
(\vee L_F^*)_I \arrow[ur, "\tilde{h}_\xi"'] &
\end{tikzcd},$$

commutes. This in turn yields

$$\tilde{s}_{\wedge^p \xi}(f) = \tilde{s}_\xi(\theta^* f), \qquad f \in \mathscr{S}_0(L_{\wedge^p F})_I.$$

In particular, since

$$(\theta^* \operatorname{tr} \exp)(\varphi) = \operatorname{tr} \exp \theta(\varphi) = (\operatorname{tr} \wedge^p \exp)(\varphi),$$

it follows that

$$\operatorname{tr}(\wedge^p \xi) = \tilde{s}_\xi(\operatorname{tr} \wedge^p \exp).$$

(4) follows from (3) and the formula

$$\sum_{p=0}^{r} \operatorname{tr}(\wedge^p \exp \varphi) = \det(\iota + \exp \varphi), \qquad \varphi \in L_F.$$

Q.E.D.

9.7. Characters. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let Φ be a representation of G in the real vector space F. The *character of* Φ is the smooth function χ_Φ on G given by

$$\chi_\Phi(a) = \operatorname{tr} \Phi(a), \qquad a \in G.$$

If E denotes the Lie algebra of G, then $\chi_\Phi \circ \exp_G$ is a smooth invariant function in E (cf. Proposition VII, sec. 1.7).

As in sec. 9.1 modify the characteristic homomorphism of $\mathscr{P}$ to a homomorphism,

$$\tilde{h}_{\mathscr{P}}: (\vee E^*)_I \to H(B; \mathbb{C}),$$

by replacing the curvature form, Ω, by $(-1/2\pi i)\Omega$. Let

$$\tilde{s}_{\mathscr{P}}: \mathscr{S}_0(E)_I \to H(B; \mathbb{C})$$

be the resulting homomorphism. Then we can form the cohomology class

$$\tilde{s}_{\mathscr{P}}(\chi_{\Phi} \circ \exp_G) \in H(B; \mathbb{C}).$$

Proposition V: Let ξ be the vector bundle over B associated with $\mathscr{P}$ via Φ. (cf. sec. 5.6.) Then

$$\operatorname{tr}(\xi) = \tilde{s}_{\mathscr{P}}(\chi_{\Phi} \circ \exp_G).$$

Proof: It follows from sec. 8.25 that

$$\operatorname{tr}(\xi) = \tilde{s}_{\xi}(\operatorname{tr} \circ \exp) = \tilde{s}_{\mathscr{P}}((\Phi')^*(\operatorname{tr} \circ \exp)).$$

But

$$\begin{aligned} ((\Phi')^*(\operatorname{tr} \circ \exp))(h) &= \operatorname{tr}(\exp \Phi'(h)) \\ &= \operatorname{tr}(\Phi(\exp_G h)) = (\chi_{\Phi} \circ \exp_G)(h), \qquad h \in E, \end{aligned}$$

and so

$$(\Phi')^*(\operatorname{tr} \circ \exp) = \chi_{\Phi} \circ \exp_G .$$

The proposition follows.

Q.E.D.

9.8. Decomposable curvature Let R be the curvature of a linear connection in ξ. Then, for each $x \in B$, $R(x)$ can be regarded as a linear map

$$R(x)\colon \wedge^2 T_x(B) \to F_x^* \otimes F_x .$$

Definition: R is called *decomposable*, if, for each $x \in B$, there are linear maps,

$$\Phi(x)\colon T_x(B) \to F_x^* \qquad \text{and} \qquad \Psi(x)\colon T_x(B) \to F_x ,$$

such that

$$R(x; h, k) = \Phi(x; h) \otimes \Psi(x; k) - \Phi(x; k) \otimes \Psi(x; h), \qquad h, k \in T_x(B).$$

Proposition VI: If ξ admits a linear connection with decomposable curvature, then

$$\underbrace{R \circ \cdots \circ R}_{(k \text{ factors})} = (-1)^{k-1}(\operatorname{tr} R)^{k-1} \wedge R.$$

In particular, $\operatorname{tr}_k(\xi) = 0 = p_k(\xi)$, $k > 0$.

Proof: Fix $x \in B$ and let $\Phi(x)$, $\Psi(x)$ satisfy the condition above. Then

$$\begin{aligned}(R \circ R)(x; h_1, h_2, h_3, h_4) &= \sum_{\sigma \in S^4} \epsilon_\sigma \langle \Phi(x; h_{\sigma(1)}), \Psi(x; h_{\sigma(4)}\rangle\, \Phi(x; h_{\sigma(3)}) \otimes \Psi(x; h_{\sigma(2)}) \\ &= -((\mathbf{tr}\, R) \wedge R)(x; h_1, h_2, h_3, h_4).\end{aligned}$$

It follows by induction that

$$\underset{(k \text{ factors})}{(R \circ \cdots \circ R)} = (-1)^{k-1}(\mathbf{tr}\, R)^{k-1} \wedge R.$$

This formula implies that $\mathrm{tr}_k(\xi)$ is represented by a scalar multiple of the differential form $(\mathbf{tr}\, R)^k$. On the other hand, $\mathbf{tr}\, R$ represents $h_\xi(\mathrm{tr})$, which, in view of Proposition I, sec. 9.2, is zero. Thus $\mathbf{tr}\, R$ is exact and hence $(\mathbf{tr}\, R)^k$ is also exact.

This shows that $\mathrm{tr}_k(\xi) = 0$, $k \geqslant 1$. Now formula (9.5), sec. 9.6, implies that $p_k(\xi) = 0$, $k \geqslant 1$.

Q.E.D.

Corollary: If $\mathbf{tr}\, R = 0$, then $R \circ R = 0$ and the differential forms representing $\mathrm{tr}_k(\xi)$ and $p_k(\xi)$ are zero for $k \geqslant 1$.

Examples: 1. Assume that ξ admits a Riemannian connection such that the Riemannian curvature, $R^\bullet$, (cf. sec. 7.25) is of the form

$$R^\bullet = f \cdot (\Psi \wedge \Psi), \qquad f \in \mathscr{S}(B), \quad \Psi \in A^1(B; \xi),$$

where ξ^* is identified with ξ via the Riemannian metric. Then the curvature R is decomposable. Moreover, since each linear map $R(x; h, k): F_x \to F_x$ is skew, we have

$$\mathbf{tr}\, R = 0.$$

2. *Constant curvature*: Let M be a Riemannian manifold and consider the tangent bundle τ_M. The identity maps $T_x(M) \to T_x(M)$ determine a 1-form $\Omega \in A^1(M; \tau_M)$. The condition,

$$R^\bullet = \lambda(\Omega \wedge \Omega), \qquad \lambda \in \mathbb{R},$$

is precisely equivalent to the condition that the curvature be constant (cf. sec. 7.25). Thus manifolds with constant curvature have decomposable curvature. In particular, the spheres have decomposable curvature.

§3. Pseudo-Riemannian bundles: Pontrjagin classes and Pfaffian class

In this article, $\xi = (N, \pi, B, F)$ denotes a real vector bundle over a *connected* base B. $\langle\ ,\ \rangle$ denotes a pseudo-Riemannian metric in ξ; thus, for each $x \in B$, $\langle\ ,\ \rangle_x$ is a symmetric and nondegenerate (but not necessarily positive definite) inner product in F_x. We use the inner product to identify ξ with ξ^*.

9.9. The decomposition of ξ. Let $(\ ,\ )$ be a Riemannian metric in ξ. Then a strong bundle isomorphism, λ, of ξ is determined by

$$\langle u, v\rangle = (\lambda_x u, v), \qquad u, v \in F_x, \quad x \in B.$$

It is called the *associated isomorphism* for $(\ ,\ )$.

Proposition VII: ξ admits a Riemannian metric $(\ ,\ )$ with the following property: There is a direct decomposition of ξ into subbundles ξ^+ and ξ^- such that

(1) ξ^+ is orthogonal to ξ^- with respect to both $\langle\ ,\ \rangle$ and $(\ ,\ )$.
(2) In ξ^+ the metrics $\langle\ ,\ \rangle$ and $(\ ,\ )$ coincide.
(3) In ξ^- the metric $\langle\ ,\ \rangle$ is the negative of the metric $(\ ,\ )$.

Proof: Let $(\ ,\ )_1$ be any Riemannian metric in ξ. Let $S(F_x)$ denote the space of linear transformations of F_x that are self-adjoint with respect to $(\ ,\ )_1$ and let $S^+(F_x)$ be the open subset of $S(F_x)$ consisting of the transformations with strictly positive eigenvalues (cf. Example 11, sec. 1.5, volume I).

If λ denotes the associated isomorphism for $(\ ,\ )_1$, then $\lambda_x \in S(F_x)$ and so $\lambda_x^2 \in S^+(F_x)$. Thus, for each $x \in B$, there is a unique transformation $\psi_x \in S^+(F_x)$ such that $\psi_x^2 = \lambda_x^2$. Moreover, $\psi_x \circ \lambda_x = \lambda_x \circ \psi_x$.

With the aid of a Riemannian coordinate representation for ξ and the fact that the correspondence $\psi \mapsto \psi^2$ defines a diffeomorphism of $S^+(F)$ (cf. Example 12, sec. 1.5, volume I), it is easy to see that the maps ψ_x depend smoothly on x. Hence, they define a strong bundle isomorphism of ξ.

Next define a new Riemannian metric $(\ ,\ )$ in ξ by setting

$$(u, v) = (\psi_x u, v)_1\,, \qquad u, v \in F_x\,.$$

The associated isomorphism, φ, for (,) is given by

$$\varphi_x = \psi_x^{-1} \circ \lambda_x , \qquad x \in B.$$

Since ψ_x and λ_x commute, it follows that $\varphi_x^2 = \iota$, $x \in B$. Hence

$$F_x = F_x^+ \oplus F_x^- ,$$

where $F_x^+ = \ker(\varphi_x - \iota)$ and $F_x^- = \ker(\varphi_x + \iota)$.

A straightforward argument, using the fact that B is connected, shows that F_x^+ and F_x^- are the fibres of subbundles ξ^+ and ξ^- of ξ. That ξ^+ and ξ^- satisfy conditions (1), (2), and (3), is immediate from the definitions.

Q.E.D.

Corollary I: $(\xi, \langle\ ,\ \rangle, (\ ,\ ))$ is a Σ-bundle.

Corollary II: $(\xi, \langle\ ,\ \rangle)$ is a Σ-bundle.

9.10. The bundle Sk_ξ. Since $(\xi, \langle\ ,\ \rangle)$ is a Σ-bundle, it admits a coordinate representation $\{(U_\alpha , \psi_\alpha)\}$ such that the maps $\psi_{\alpha,x}: F \xrightarrow{\cong} F_x$ are isometries (with respect to a fixed indefinite inner product $\langle\ ,\ \rangle_F$ in F). The structure group of this Σ-bundle is the group of isometries of F (with respect to $\langle\ ,\ \rangle_F$) and the corresponding Lie algebra is the Lie algebra of skew transformations (with respect to $\langle\ ,\ \rangle_F$).

The Lie subalgebras, Sk_{F_x}, of L_{F_x}, are the fibres of the associated Lie algebra bundle Sk_ξ. Exactly as in the Riemannian case (cf. sec. 7.25), a strong bundle isomorphism β: $\wedge^2\xi \xrightarrow{\cong} \mathrm{Sk}_\xi$ is given by

$$(\beta(u \wedge v))(z) = \langle u, z\rangle v - \langle v, z\rangle u, \qquad u, v, z \in F_x , \quad x \in B.$$

9.11. Pontrjagin classes. A Σ-connection, ∇, in the Σ-bundle $(\xi, \langle\ ,\ \rangle)$ will be called a *pseudo-Riemannian connection*. It satisfies

$$\nabla(\langle\ ,\ \rangle) = 0.$$

The corresponding curvature R takes values in the bundle Sk_ξ; hence it determines a 2-form, $R^\bullet$, with values in $\wedge^2\xi$. It is called the *pseudo-Riemannian curvature of* ∇ (cf. sec. 7.25).

The *kth pseudo-Riemannian curvature of* ∇ is the $2k$-form with values in $\wedge^{2k}\xi$ given by

$$(R^\bullet)^k = R^\bullet \wedge \cdots \wedge R^\bullet.$$

If $\langle\ ,\ \rangle$ is a Riemannian metric, $(R^\bullet)^k$ will be called the *kth Riemannian curvature.*

Recall that each $\wedge^k\xi$ is a pseudo-Riemannian bundle with inner product given by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k\rangle = \det(\langle u_i, v_j\rangle), \qquad u_i \in F_x, \quad v_j \in F_x, \quad x \in B.$$

Hence we can construct the $4k$-forms

$$\Phi_k = \langle\!\langle (R^\bullet)^k, (R^\bullet)^k\rangle\!\rangle.$$

Proposition VIII: With the hypotheses and notation above the differential form,

$$\frac{(-1)^k}{(k!)^2(2\pi)^{2k}} \langle\!\langle (R^\bullet)^k, (R^\bullet)^k\rangle\!\rangle,$$

is closed and represents the kth Pontrjagin class of ξ.

Proof: This is an immediate consequence of Proposition V, sec. A.5, and formula (9.3), sec. 9.4.

Q.E.D.

Proposition VIII is due to A. Stehney.

9.12. Pfaffian class of an oriented pseudo-Riemannian bundle. In the rest of this article we shall assume that ξ is oriented. Then there is a unique cross-section, Δ_ξ, in $\wedge^r\xi$ ($r = \operatorname{rank} \xi$) such that

(i) Δ_ξ is positive.
(ii) Δ_ξ is normed; i.e., $\langle\Delta_\xi, \Delta_\xi\rangle = \pm 1$.

Δ_ξ will be called the *positive normed determinant function* in ξ.

Clearly, $(\xi, \langle\ ,\ \rangle, \Delta_\xi)$ is a Σ-bundle. The corresponding group is $SO(F)$ (the group of proper rotations of F) and the Lie algebra consists of the skew linear transformations of F (with respect to $\langle\ ,\ \rangle_F$).

The modified characteristic homomorphism for this Σ-bundle will be denoted by $\tilde{k}_\xi$ rather than $\tilde{h}_\xi$; thus $\tilde{k}_\xi$ is a homomorphism,

$$\tilde{k}_\xi\colon (\vee \mathrm{Sk}_F^*)_I \to H(B; \mathbb{C}),$$

where $(\vee\mathrm{Sk}_F^*)_I$ is the subalgebra of $\vee\mathrm{Sk}_F^*$ invariant under the action of $SO(F)$.

Assume now that the rank of ξ is even, $r = 2m$. Then the Pfaffian, Pf^F, of the oriented inner product space, F, is defined (cf. Example 1, sec. A.7). It is an element of $(\vee^m\mathrm{Sk}_F^*)_I$. The corresponding characteristic class $\tilde{k}_\xi(\mathrm{Pf}^F)$ is real if m is even and purely imaginary if m is odd.

Hence a real cohomology class $\mathrm{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) \in H^r(B)$ is given by

$$\mathrm{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) = i^m\tilde{k}_\xi(\mathrm{pf}^F).$$

It is called the *Pfaffian class* of the oriented pseudo-Riemannian bundle ξ. It is convenient to extend the definition of the Pfaffian to vector bundles with odd rank by setting it equal to zero in this case.

The Pfaffian class of ξ is represented by the r-form,

$$P = \left(\frac{-1}{2\pi}\right)^m \frac{1}{m!} \langle \Delta_\xi , \underbrace{R^\bullet \wedge \cdots \wedge R^\bullet}_{(m \text{ factors})} \rangle, \tag{9.6}$$

where $R^\bullet$ is the pseudo-Riemannian curvature of a pseudo-Riemannian connection.

Now recall that $\langle\ ,\ \rangle_F$ denotes an inner product in F corresponding isometrically to the inner products in F_x (cf. sec. 9.10). Let $F = F^+ \oplus F^-$ be a fixed orthogonal decomposition of F such that the restriction of $\langle\ ,\ \rangle_F$ to F^+ (respectively, to F^-) is positive (respectively, negative) definite. Then the dimension of F^- is called the *index* of $\langle\ ,\ \rangle_F$.

Let $(\ ,\ )_F$ be the positive definite inner product in F defined by

$$(x_+ + x_- , y_+ + y_-)_F = \langle x_+ , y_+\rangle_F - \langle x_- , y_-\rangle_F, \qquad x_+ , y_+ \in F^+, \quad x_- , y_- \in F^-.$$

Lemma I: Let $(\ ,\ )$ be the Riemannian metric constructed in Proposition VII, sec. 9.9. If the index of $\langle\ ,\ \rangle_F$ is odd, then

$$\operatorname{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) = 0 = \operatorname{pf}(\xi, (\ ,\ ), \Delta_\xi).$$

If the index of $\langle\ ,\ \rangle$ is $2q$, then

$$\operatorname{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) = (-1)^q \operatorname{pf}(\xi, (\ ,\ ), \Delta_\xi).$$

Proof: Recall, from sec. 9.9, that $(\xi, \langle\ ,\ \rangle, (\ ,\ ))$ is a Σ-bundle and that $\xi = \xi^+ \oplus \xi^-$. The corresponding Lie algebra is the subalgebra of L_F consisting of the transformations which are skew with respect to both $\langle\ ,\ \rangle_F$ and $(\ ,\ )_F$; this is exactly $\mathrm{Sk}_{F^+} \oplus \mathrm{Sk}_{F^-}$. Thus the associated Lie algebra bundle is $\mathrm{Sk}_{\xi^+} \oplus \mathrm{Sk}_{\xi^-}$.

Since $(\xi, \langle\ ,\ \rangle, (\ ,\ ))$ is a Σ-bundle, it admits a Σ-connection ∇. ∇ is both a Riemannian and a pseudo-Riemannian connection; thus its curvature, R, which takes values in $\mathrm{Sk}_{\xi^+} \oplus \mathrm{Sk}_{\xi^-}$, can be used to obtain both $\operatorname{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi)$ and $\operatorname{pf}(\xi, (\ ,\ ), \Delta_\xi)$. Now the lemma follows from Proposition IX, sec. A.7.

Q.E.D.

Proposition IX: The Pfaffian class has the following properties:

(1) Suppose $\varphi: \xi \to \eta$ is a bundle map between oriented pseudo-Riemannian bundles which restricts to orientation preserving (respectively, orientation reversing) isometries in the fibres. Then

$$\operatorname{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) = \epsilon\psi^{\#} \operatorname{pf}(\eta, \langle\ ,\ \rangle, \Delta_\eta),$$

where ψ denotes the induced map between the base manifolds and $\epsilon = +1$ (respectively, $\epsilon = -1$).

(2) $\mathrm{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi)$ depends only on ξ, the orientation of ξ and the index of $\langle\ ,\ \rangle_F$.

(3) $(\mathrm{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi))^2 = (-1)^m p_m(\xi)$ $(2m = \operatorname{rank} \xi)$.

(4) Suppose ξ is the orthogonal direct sum of oriented pseudo-Riemannian subbundles η and ζ, and assume ξ has the orientation induced from the orientations of η and ζ, and the decomposition $\xi = \eta \oplus \zeta$. Then

$$\mathrm{pf}(\xi, \langle\ ,\ \rangle, \Delta_\xi) = \mathrm{pf}(\eta, \langle\ ,\ \rangle, \Delta_\eta) \cdot \mathrm{pf}(\zeta, \langle\ ,\ \rangle, \Delta_\zeta).$$

Proof: (1) This follows from Theorem IV, sec. 8.13.

(2) In view of Lemma I, it is sufficient to prove that any two Riemannian metrics $(\ ,\ )_1$ and $(\ ,\ )_2$ in ξ determine the same Pfaffian class. According to Proposition VI, sec. 2.17, volume I, there is a strong bundle isometry from $(\xi, (\ ,\ )_1)$ to $(\xi, (\ ,\ )_2)$. It is evident from the construction that this isometry preserves the orientation of the fibres. Thus (2) follows from (1).

(3) Proposition VII, sec. 8.17, implies that the diagram,

$$\begin{tikzcd} (\vee L_F^*)_I \arrow[d, "j_I^\vee"'] \arrow[dr, "\tilde{h}_\xi"] & \\ (\vee \mathrm{Sk}_F^*)_I \arrow[r, "\tilde{k}_\xi"'] & H(B; \mathbb{C}) \end{tikzcd},$$

commutes. Moreover, according to Proposition VI, sec. A.6,

$$j_I^\vee(C_{2m}^F) = \mathrm{Pf}^F \vee \mathrm{Pf}^F.$$

Now (3) follows.

(4) Choose pseudo-Riemannian connections in η and ζ and give ξ the induced pseudo-Riemannian connection. Its curvature takes values in $\mathrm{Sk}_\eta \oplus \mathrm{Sk}_\zeta$. Thus (4) follows from the corollary to Proposition VIII, sec. A.6.

Q.E.D.

9.13. The Pfaffian class of a Riemannian vector bundle. In view of Proposition IX, sec. 9.12, the Pfaffian class of an oriented Riemannian vector bundle, ξ, is independent of the choice of the Riemannian metric; this class will be called the *Pfaffian of* ξ and will be denoted simply by $\mathrm{pf}(\xi)$. In Chapter X it will be shown that the Pfaffian class coincides with the Euler class of the associated sphere bundle (Gauss–Bonnet–Chern theorem).

The Pfaffian class of the tangent bundle of an oriented Riemannian manifold M will be written $\mathrm{pf}(M)$ and called the *Pfaffian class of M*.

Let Δ_M be the positive normed determinant function in τ_M; it is called the *volume form* for M and $\int_M \Delta_M$ is called the *volume* of M. Moreover if $R^{\cdot}$ is the Riemannian curvature of a Riemannian connection in M, then

$$\frac{(-1)^m}{m!(2\pi)^m}\langle \Delta_M, (R^{\cdot})^m\rangle$$

represents $\mathrm{pf}(M)$ $(2m = \dim M)$.

Now let $K \in \mathscr{S}(M)$ be the function determined by the equation

$$\frac{(-1)^m \operatorname{vol} S^{2m}}{(2\pi)^m m!}\langle \Delta_M, (R^{\cdot})^m\rangle = 2K \cdot \Delta_M .$$

This function is called the *Gaussian curvature* of the connection. Evidently $\mathrm{pf}(M)$ is represented by $(2/\operatorname{vol} S^{2m})\, K\Delta_M$.

Proposition IX specializes to

Proposition X: The Pfaffian class of an oriented Riemannian vector bundle, ξ, has the following properties:

(1) It changes sign if the orientation is reversed.

(2) If $\varphi: \xi \to \eta$ is a bundle map restricting to orientation preserving isomorphisms in the fibres and inducing ψ between the base manifolds, then

$$\psi^{\#}\,\mathrm{pf}(\eta) = \mathrm{pf}(\xi).$$

(3) Suppose $\xi = \eta \oplus \zeta$, where η and ζ are oriented subbundles and ξ is given the induced orientation. Then

$$\mathrm{pf}(\xi) = \mathrm{pf}(\eta) \cdot \mathrm{pf}(\zeta).$$

In particular, if ξ admits an oriented subbundle of odd rank, then $\mathrm{pf}(\xi) = 0$.

9.14. Decomposable curvature. Suppose ξ is an oriented Riemannian vector bundle that admits a Riemannian connection whose Riemannian curvature is of the form

$$R^{\boldsymbol{\cdot}} = f \cdot (\Psi \wedge \Psi), \qquad f \in \mathscr{S}(B), \quad \Psi \in A^1(B; \xi)$$

(cf. Example I, sec. 9.8). Assume that ξ has even rank $r = 2m$. Then

$$\langle \Delta_\xi , (R^{\boldsymbol{\cdot}})^m \rangle = f^m \cdot \langle \Delta_\xi , \Psi^{2m} \rangle.$$

It follows that pf(ξ) is represented by the differential form

$$P = \left(\frac{-1}{2\pi}\right)^m \frac{1}{m!} f^m \cdot \langle \Delta_\xi , \Psi^{2m} \rangle.$$

Next observe that Ψ determines the strong bundle map $\psi: \tau_B \to \xi$ given by

$$\psi_x(h) = \Psi(x; h), \qquad x \in B, \quad h \in T_x(B).$$

The bundle map $\wedge\psi: \wedge\tau_B \to \wedge\xi$ yields a homomorphism

$$(\wedge\psi)^*: A(B) \leftarrow \operatorname{Sec} \wedge\xi^*.$$

Now the differential form P can be written

$$P = \frac{(-1)^m (2m)!}{(2\pi)^m m!} f^m \cdot (\wedge\psi)^* (\Delta_\xi) .$$

In particular, assume that B is compact and that $\xi = \tau_B$. Write $\Delta_\xi = \Delta_B$. Then Δ_B is the volume form *of* B.

In this case we have

$$((\wedge\psi)^*\Delta_B)(x) = \det \psi_x \cdot \Delta_B(x), \qquad x \in B.$$

Moreover,

$$\operatorname{Vol}(S^{2m}) = \frac{(2\pi)^m 2^{m+1}}{(2m)!} m!$$

(cf. sec. 0.13).

Thus

$$\int_B^\# \operatorname{pf}(B) = \frac{2}{\operatorname{Vol}(S^{2m})} \int_B (-2f)^m \cdot \det \psi \cdot \Delta_B .$$

In particular, if B has *constant* curvature, then $\psi(x) = \iota$ and f is constant. Set $f = -\frac{1}{2}\kappa$. In this case the formula above yields

$$\int_B^{\#} \mathrm{pf}(B) = 2\kappa^m \operatorname{Vol}(B)/\operatorname{Vol}(S^{2m}).$$

If $B = S^{2m}$, then $\kappa = 1$ (cf. sec. 7.29) and this formula reduces to the formula

$$\int_{S^{2m}}^{\#} \mathrm{pf}(S^{2m}) = 2.$$

(cf. the corollary to Proposition XV, sec. 7.28.)

9.15. Homogeneous spaces. Let G be a compact connected Lie group and let H be a closed subgroup. We shall determine the Pontrjagin classes of the homogeneous space G/H.

Denote the Lie algebras of G and H by E and F and assign E a G-invariant positive definite inner product. Then we have vector bundles,

$$\xi = G \times_H F^{\perp} \qquad \text{and} \qquad \eta = G \times_H F,$$

over G/H. (cf. sec. 5.6).

Recall from Proposition III, sec. 5.11, that

$$\xi = \tau_{G/H} \qquad \text{and} \qquad \xi \oplus \eta = \epsilon^n, \qquad (n = \dim G).$$

It follows that $p(G/H) = p(\xi) = p(\eta)^{-1}$. (cf. sec. 9.4.)

Denote by ad and $\mathrm{ad}^{\perp}$ the induced representations of F in F and $F^{\perp}$. Let p: $E \to F$ be the projection. Since

$$p_k(G/H) = \tilde{h}_\xi(C_{2k}) = \frac{(-1)^k}{(2\pi)^{2k}} h_\xi(C_{2k}),$$

we may apply Example 2, sec. 8.26, to obtain an explicit representative Φ_k for $p_k(G/H)$. In fact, Φ_k is the unique closed invariant $4k$-form on G/H satisfying

$$(\pi^*\Phi_k)(e; h_1, \dots, h_{4k})$$
$$= \frac{(-1)^k}{(2\pi)^{2k}2^{2k}(2k)!} \sum_{\sigma \in S^{4k}} \epsilon_\sigma C_{2k}(\mathrm{ad}^{\perp} p[h_{\sigma(1)}, h_{\sigma(2)}], \dots, \mathrm{ad}^{\perp} p[h_{\sigma(4k-1)}, h_{\sigma(4k)}]),$$
$$h_i \in F^{\perp}.$$

Similarly $p_k(\eta)$ is represented by the closed invariant $4k$-form, Ψ_k, which satisfies

$$(\pi^*\Psi_k)(e; h_1, ..., h_{4k})$$
$$= \frac{(-1)^k}{(2\pi)^{2k}2^{2k}(2k)!} \sum_\sigma \epsilon_\sigma C_{2k}(\operatorname{ad} p[h_{\sigma(1)}, h_{\sigma(2)}], ..., \operatorname{ad} p[h_{\sigma(4k-1)}, h_{\sigma(4k)}]),$$
$$h_i \in F^\perp.$$

(Here $\pi: G \to G/H$ denotes the projection.) Thus the differential forms,

$$\Phi = \sum_k \Phi_k \qquad \text{and} \qquad \Psi = \sum_k \Psi_k,$$

represent $p(G/H)$ and $p(G/H)^{-1}$.

Next assume that G/H is oriented and that G/H has even dimension $2m$. Then the Pfaffian class for G/H is represented by the unique invariant $2m$-form Ξ which satisfies

$$\pi^*\Xi(e; h_1, ..., h_{2m}) = \frac{1}{(2\pi)^m 2^m m!} \sum_\sigma \epsilon_\sigma \operatorname{Pf}(\operatorname{ad}^\perp p[h_{\sigma(1)}, h_{\sigma(2)}], ..., \operatorname{ad}^\perp p[h_{\sigma(2m-1)}, h_{\sigma(2m)}].$$

In Chapter X (Theorem II) it will be shown that

$$\int_{G/H}^{\#} \operatorname{pf}(G/H) = \chi_{G/H}.$$

Hence $\int_{G/H} \Xi = \chi_{G/H}$. If H has the same rank as G, then Proposition XIII, sec. 4.21, shows that $\chi_{G/H} \neq 0$. Thus, in this case, the unique invariant $2m$-form, $\Delta_{G/H}$ on G/H, which satisfies $\int_{G/H}\Delta_{G/H} = 1$ is given by

$$\Delta_{G/H} = \frac{1}{\chi_{G/H}} \Xi.$$

Now we can apply the formula of Example 2, sec. 8.26, to find that for any vector bundle $\zeta = (G \times_H W, \rho, G/H, W)$ which is associated with a representation, Φ, of H and, for any $\Gamma \in (\vee^m L_W^*)_I$,

$$\int_{G/H}^{\#} h_\zeta(\Gamma)$$
$$= (-2\pi)^m \chi_{G/H} \frac{\Sigma_\sigma \epsilon_\sigma \Gamma(\Phi'(p[h_{\sigma(1)}, h_{\sigma(2)}]), ..., \Phi'(p[h_{\sigma(2m-1)}, h_{\sigma(2m)}]))}{\Sigma_\sigma \epsilon_\sigma \operatorname{Pf}(\operatorname{ad}^\perp(p[h_{\sigma(1)}, h_{\sigma(2)}]), ..., \operatorname{ad}^\perp(p[h_{\sigma(2m-1)}, h_{\sigma(2m)}]))},$$

where $h_1, ..., h_{2m}$ is a basis of $F^\perp$.

§4. Complex vector bundles

In this article $\xi = (M, \pi, B, F)$ denotes a complex vector bundle of complex rank r. Thus F is a complex r-dimensional vector space.

9.16. The modified characteristic homomorphisms $\tilde{l}_\xi$ and $\tilde{m}_\xi$. The characteristic homomorphism for ξ will be denoted by l_ξ. Thus l_ξ is the complex linear homomorphism (cf. sec. 8.13),

$$l_\xi \colon (\vee L_F^*)_I \to H(B; \mathbb{C}),$$

defined as follows: If $\Phi \in (\vee^p L_F^*)_I$ then $l_\xi(\Phi)$ is represented by $(1/p!)$ $\Phi(R, \ldots, R)$, where Φ is identified with the corresponding constant invariant cross-section and R denotes the curvature of a complex linear connection. Note that all linear and multilinear operations are with respect to $\mathbb{C}$ and $(\vee L_F^*)_I$ is the subalgebra of $\vee L_F^*$ whose elements are invariant under $GL(F)$.

In particular, we can modify l_ξ to a homomorphism,

$$\tilde{l}_\xi \colon (\vee L_F^*)_I \to H(B; \mathbb{C}),$$

by setting

$$\tilde{l}_\xi(\Phi) = \left(\frac{-1}{2\pi i}\right)^p l_\xi(\Phi), \qquad \Phi \in (\vee^p L_F^*)_I .$$

$\tilde{l}_\xi$ is called the *modified characteristic homomorphism for* ξ.

Next, let $\langle\ ,\ \rangle$ be a Hermitian metric in ξ. Then a Riemannian metric, $\langle\ ,\ \rangle_{\mathbb{R}}$, is defined in the underlying real vector bundle,

$$\xi_{\mathbb{R}} = (M, \pi, B, F_{\mathbb{R}}),$$

by

$$\langle u, v\rangle_{\mathbb{R}} = \operatorname{Re}\langle u, v\rangle, \qquad u, v \in F_x .$$

Then $(\xi_{\mathbb{R}}, i_\xi, \langle\ ,\ \rangle_{\mathbb{R}})$ is a Σ-bundle, where i_ξ denotes the cross-section in $\xi_{\mathbb{R}}^{1,1}$ corresponding to multiplication by i (cf. Example 4, sec. 8.3).

The structure group of this Σ-bundle is the group $U(F)$ of unitary transformations of F, and the corresponding Lie algebra is the (real) Lie algebra, Sk_F, of skew Hermitian, complex linear transformations of F. The modified characteristic homomorphism of this Σ-bundle is a real linear homomorphism

$$\tilde{m}_\xi \colon (\vee \mathrm{Sk}_F^*)_I \to H(B; \mathbb{C}).$$

It determines the homomorphism (also written $\tilde{m}_\xi$)

$$\tilde{m}_\xi \colon \mathbb{C} \otimes (\vee \mathrm{Sk}_F^*)_I \to H(B; \mathbb{C})$$

given by

$$\tilde{m}_\xi(\lambda \otimes \Phi) = \lambda \cdot \tilde{m}_\xi(\Phi), \qquad \lambda \in \mathbb{C}, \quad \Phi \in (\vee \mathrm{Sk}_F^*)_I .$$

To establish the relation between $\tilde{l}_\xi$ and $\tilde{m}_\xi$ make $\mathbb{C} \otimes \mathrm{Sk}_F$ into a complex Lie algebra by setting

$$[\lambda \otimes \varphi, \mu \otimes \psi] = \lambda\mu \otimes [\varphi, \psi], \qquad \lambda, \mu \in \mathbb{C}, \quad \varphi, \psi \in \mathrm{Sk}_F .$$

Then the correspondence $\lambda \otimes \varphi \mapsto \lambda \cdot \varphi$ defines an isomorphism of complex Lie algebras

$$\mathbb{C} \otimes \mathrm{Sk}_F \xrightarrow{\cong} L_F .$$

This in turn induces isomorphisms,

$$\vee(\mathbb{C} \otimes \mathrm{Sk}_F)^* \xleftarrow{\cong} \vee L_F^* ,$$

and

$$[\vee(\mathbb{C} \otimes \mathrm{Sk}_F)^*]_{\theta=0} \xleftarrow{\cong} (\vee L_F^*)_{\theta=0} . \tag{9.7}$$

(All multilinear operations are over $\mathbb{C}$ and θ denotes the obvious representations of $\mathbb{C} \otimes \mathrm{Sk}_F$ and of L_F .)

On the other hand, consider the isomorphism of complex algebras,

$$\mathbb{C} \otimes \vee \mathrm{Sk}_F^* \xrightarrow{\cong} \vee(\mathbb{C} \otimes \mathrm{Sk}_F)^*,$$

given by

$$\lambda \otimes (h_1^* \vee \cdots \vee h_p^*) \mapsto$$
$$(\lambda \otimes h_1^*) \vee (1 \otimes h_2^*) \vee \cdots \vee (1 \otimes h_p^*), \qquad \lambda \in \mathbb{C}, \ h_j^* \in \mathrm{Sk}_F^* , \ j = 1, \ldots, p.$$

Since the representation of $\mathbb{C} \otimes \mathrm{Sk}_F$ in $\mathbb{C} \otimes \vee \mathrm{Sk}_F^*$ is given by

$$\theta(\lambda \otimes \varphi) = \lambda \otimes \theta(\varphi), \qquad \lambda \in \mathbb{C}, \quad \varphi \in \mathrm{Sk}_F ,$$

this isomorphism restricts to an isomorphism

$$\mathbb{C} \otimes (\vee \mathrm{Sk}_F^*)_{\theta=0} \xrightarrow{\cong} [\vee(\mathbb{C} \otimes \mathrm{Sk}_F^*)]_{\theta=0} \,. \tag{9.8}$$

Combining (9.7) with the inverse of (9.8) we obtain an isomorphism

$$\mathbb{C} \otimes (\vee \mathrm{Sk}_F^*)_{\theta=0} \xleftarrow{\cong} (\vee L_F^*)_{\theta=0} \,.$$

Since both $U(F)$ and $GL(F)$ are connected, this is an isomorphism

$$\kappa : \mathbb{C} \otimes (\vee \mathrm{Sk}_F^*)_I \xleftarrow{\cong} (\vee L_F^*)_I$$

(cf. Proposition IX, sec. 1.8).

Proposition XI: The diagram,

$$\begin{tikzcd}
(\vee L_F^*)_I \arrow[d, "\kappa", "\cong"'] \arrow[dr, "l_\xi"] & \\
& H(B; \mathbb{C}) \\
\mathbb{C} \otimes (\vee \mathrm{Sk}_F^*)_I \arrow[ur, "\tilde{m}_\xi"'] &
\end{tikzcd} ,$$

commutes.

Proof: A Σ-connection in $(\xi_{\mathbb{R}}, i_\xi, \langle\,,\,\rangle_{\mathbb{R}})$ is, in particular, a complex linear connection in the complex vector bundle ξ. Thus its curvature can be used to define both l_ξ and $\tilde{m}_\xi$. In view of this, the proposition is a straightforward consequence of the definitions.

Q.E.D.

Corollary: $\tilde{m}_\xi$ is independent of the Hermitian metric.

Finally, in analogy with sec. 8.14 we extend $\tilde{m}_\xi$ to a homomorphism

$$\tilde{m}_\xi^{**} : \mathbb{C} \otimes (\vee^{**}\mathrm{Sk}_F^*)_I \to H(B; \mathbb{C}).$$

Precomposing with the Taylor homomorphism,

$$[\mathscr{S}_0(\mathrm{Sk}_F\,; \mathbb{C})]_I \to \mathbb{C} \otimes (\vee^{**}\mathrm{Sk}_F^*)_I \,,$$

yields a homomorphism

$$\tilde{s}_\xi\colon \mathscr{S}_0(\mathrm{Sk}_F\,;\,\mathbb{C})_I \to H(B;\,\mathbb{C}).$$

Suppose $f \in \mathscr{S}(\mathrm{Sk}_F;\,\mathbb{C})_I$. Define functions $f_1 \in \mathscr{S}(\mathrm{Sk}_F;\,\mathbb{C})_I$ and $f_2 \in \mathscr{S}(\mathrm{Sk}_F;\,\mathbb{C})_I$ by

$$f_1(\varphi) = \tfrac{1}{2}(f(\varphi) + \overline{f(-\varphi)}) \qquad \text{and} \qquad f_2(\varphi) = \frac{1}{2i}(f(\varphi) - \overline{f(-\varphi)}).$$

(Here $\overline{f(-\varphi)}$ denotes the complex conjugate of $f(-\varphi)$.) Then $f = f_1 + if_2$. Moreover, a simple calculation shows that, if $\varphi_1, \ldots, \varphi_k \in \mathrm{Sk}_F$, then

$$f_\nu^{(k)}(0;\varphi_1, \ldots, \varphi_k) \in \begin{cases} \mathbb{R}, & k \text{ even}, \\ i\mathbb{R}, & k \text{ odd} \end{cases} \qquad \nu = 1, 2.$$

It follows that $\tilde{s}_\xi(f_1) \in H(B)$ and $\tilde{s}_\xi(f_2) \in H(B)$. Thus

$$\tilde{s}_\xi(f) = 1 \otimes \tilde{s}_\xi(f_1) + i \otimes \tilde{s}_\xi(f_2)$$

is the decomposition of $\tilde{s}_\xi(f)$ into its real and imaginary parts. In particular if $f(\varphi) = \overline{f(-\varphi)}$, $\varphi \in \mathrm{Sk}_F$, then $\tilde{s}_\xi(f)$ is a real class; i.e., $\tilde{s}_\xi(f) \in H(B)$.

9.17. Examples: **1.** *Complexification*: Let $\eta = (W, \rho, B, H)$ be a (real) Riemannian vector bundle and let $\xi = (M, \pi, B, \mathbb{C} \otimes H)$ be the complex bundle whose fibre at x is the complex space $\mathbb{C} \otimes H_x$; ξ is called the *complexification* of η and we write $\xi = \mathbb{C} \otimes \eta$. The Riemannian metric $\langle\ ,\ \rangle$ in η determines the Hermitian metric $\langle\ ,\ \rangle$ in ξ given by

$$\langle \lambda_1 \otimes y_1, \lambda_2 \otimes y_2 \rangle = \lambda_1 \bar{\lambda}_2 \langle y_1, y_2 \rangle, \qquad \lambda_i \in \mathbb{C}, \quad y_i \in H_x, \quad x \in B.$$

Thus we have four modified characteristic homomorphisms:

$$\tilde{h}_\eta\colon \mathbb{C} \otimes (\vee L_H^*)_I \to H(B;\,\mathbb{C}), \qquad \tilde{k}_\eta\colon \mathbb{C} \otimes (\vee \mathrm{Sk}_H^*)_I \to H(B;\,\mathbb{C})$$

and

$$\tilde{l}_\xi\colon (\vee L_{\mathbb{C}\otimes H}^*)_I \to H(B;\,\mathbb{C}), \qquad \tilde{m}_\xi\colon \mathbb{C} \otimes (\vee \mathrm{Sk}_{\mathbb{C}\otimes H}^*)_I \to H(B;\,\mathbb{C}),$$

where $\tilde{h}_\eta$, $\tilde{k}_\eta$, and $\tilde{m}_\xi$ are extended in the obvious way (cf. sec. 9.2 for $\tilde{h}_\eta$ and $\tilde{k}_\eta$).

Now regard $\mathbb{C} \otimes L_H$ as a complex Lie algebra, and observe that an isomorphism,

$$\mathbb{C} \otimes L_H \xrightarrow{\cong} L_{\mathbb{C}\otimes H},$$

of complex Lie algebras is given by

$$\lambda \otimes \varphi \mapsto \lambda(\iota \otimes \varphi), \qquad \lambda \in \mathbb{C}, \quad \varphi \in L_H.$$

This isomorphism induces an isomorphism,

$$\vee(\mathbb{C} \otimes L_H)^* \xleftarrow{\cong} \vee L^*_{\mathbb{C}\otimes H},$$

of complex symmetric algebras; we identify $\vee(\mathbb{C} \otimes L_H)^*$ with $\mathbb{C} \otimes \vee L_H^*$ and write

$$\mathbb{C} \otimes \vee L_H^* \xleftarrow{\cong} \vee L^*_{\mathbb{C}\otimes H}.$$

This isomorphism induces the commutative diagram,

$$\begin{tikzcd}
\mathbb{C} \otimes (\vee L_H^*)_I \arrow[d, "\cong"'] & (\vee L^*_{\mathbb{C}\otimes H})_I \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
\mathbb{C} \otimes (\vee L_H^*)_{\theta=0} & (\vee L^*_{\mathbb{C}\otimes H})_{\theta=0} \arrow[l, "\cong"],
\end{tikzcd}$$

where the notation I (respectively, $\theta = 0$) is with respect to $GL(H)$ (respectively, L_H) on the left, and with respect to $GL(\mathbb{C} \otimes H)$ (respectively, $L_{\mathbb{C}\otimes H}$) on the right. In fact, the right-hand vertical arrow is an isomorphism because $GL(\mathbb{C} \otimes H)$ is connected (cf. Proposition IX, sec. 1.8) and the lower horizontal arrow is an isomorphism because of the isomorphism $\mathbb{C} \otimes L_H \cong L_{\mathbb{C}\otimes H}$. Now simple diagram chasing shows that the other arrows are isomorphisms.

It follows (by an argument analogous to that of Proposition XI, sec. 9.16) that the diagram,

$$\begin{tikzcd}
(\vee L^*_{\mathbb{C}\otimes H})_I \arrow[rr, "\cong"] \arrow[dr, "\tilde{l}_\xi"'] & & \mathbb{C} \otimes (\vee L_H^*)_I \arrow[dl, "\tilde{h}_\eta"] \\
& H(B; \mathbb{C}) &
\end{tikzcd},$$

commutes. Combining this with the diagrams of sec. 9.16 and sec. 9.2 yields the commutative diagram

$$\begin{tikzcd}
(\vee L^*_{\mathbb{C}\otimes H})_I \arrow[rr, "\cong"] \arrow[dr, "\tilde{l}_\xi"'] \arrow[dd, "\cong"'] & & \mathbb{C} \otimes (\vee L_H^*)_I \arrow[dl, "\tilde{h}_\eta"] \arrow[dd] \\
& H(B; \mathbb{C}) & \\
\mathbb{C} \otimes (\vee \mathrm{Sk}^*_{\mathbb{C}\otimes H})_I \arrow[rr] \arrow[ur, "\tilde{m}_\xi"] & & \mathbb{C} \otimes (\vee \mathrm{Sk}^*_H)_I \arrow[ul, "\tilde{k}_\eta"']
\end{tikzcd}.$$

2. *Realification*: Consider a Hermitian metric, $\langle\ ,\ \rangle$, in the complex bundle ξ and let $(\xi_{\mathbb{R}}, i_\xi, \langle\ ,\ \rangle_{\mathbb{R}})$ be the underlying real Σ-bundle defined in sec. 9.16. Then $i_\xi \in \operatorname{Sec} \mathrm{Sk}_{\xi_{\mathbb{R}}}$ and the cross-section,

$$i_\xi \wedge \cdots \wedge i_\xi \in \operatorname{Sec} \wedge^{2r} \xi_{\mathbb{R}},$$

is nowhere zero; i.e., it orients $\xi_{\mathbb{R}}$. This orientation is called the *orientation induced from the complex structure.*

Observe that the orientation in $\xi_{\mathbb{R}}$ is uniquely determined by the condition

$$\mathrm{Pf}(i_\xi(x)) > 0, \qquad x \in B.$$

Denote $i_\xi \wedge \cdots \wedge i_\xi$ by Δ; then Δ is the positive normed determinant function in $\xi_{\mathbb{R}}$. Thus $(\xi_{\mathbb{R}}, \langle\ ,\ \rangle_{\mathbb{R}}, \Delta)$ is a Σ-substructure of $(\xi_{\mathbb{R}}, i_\xi, \langle\ ,\ \rangle_{\mathbb{R}})$. In view of Proposition VII, sec. 8.17, this leads to the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \otimes (\vee\, \mathrm{Sk}^*_{F_{\mathbb{R}}})_I \arrow[d, "\iota\otimes j_I^\vee"'] \arrow[dr, "\tilde{k}_{\xi_{\mathbb{R}}}"] & \\
\mathbb{C} \otimes (\vee\, \mathrm{Sk}^*_F)_I \arrow[r, "\tilde{m}_\xi"] \arrow[d, "\cong"'] & H(B; \mathbb{C}) \\
(\vee L^*_F)_I \arrow[ur, "l_\xi"'] &
\end{tikzcd}$$

Here $(\vee \mathrm{Sk}^*_{F_{\mathbb{R}}})_I$ is the subalgebra invariant under $SO(F_{\mathbb{R}})$, and $j\colon \mathrm{Sk}_F \to \mathrm{Sk}_{F_{\mathbb{R}}}$ is the inclusion.

3. *Dual bundles*: Let ξ^* be the complex dual of ξ. Section 8.15 yields a commutative diagram,

$$\begin{tikzcd}
(\vee L^*_F)_I \arrow[dd, "\lambda_I"', "\cong"] \arrow[dr, "l_\xi"] & \\
& H(B; \mathbb{C}) \\
(\vee L^*_{F^*})_I \arrow[ur, "l_{\xi^*}"'] &
\end{tikzcd}$$

where λ_I is the isomorphism induced by the Lie algebra isomorphism $\varphi \mapsto -\varphi^*$.

On the other hand, F and F^* are only determined up to a complex isomorphism; if F is replaced by an isomorphic complex space F_1, then the identification $(\vee L_F^*)_I \cong (\vee L_{F_1}^*)_I$ is the canonical isomorphism induced by an isomorphism $F \xrightarrow{\cong} F_1$ (cf. sec. 8.15). In particular, we may identify F with F^* and $(\vee L_F^*)_I$ with $(\vee L_{F^*}^*)_I$. Then λ_I becomes an automorphism of $(\vee L_F^*)_I$.

Lemma II: The automorphism λ_I: $(\vee L_F^*)_I \xrightarrow{\cong} (\vee L_F^*)_I$ is given by

$$\lambda_I \Phi = (-1)^p \Phi, \qquad \Phi \in (\vee^p L_F^*)_I .$$

Proof: Fix $\varphi \in L_F$ and choose a linear isomorphism $\sigma: F \xrightarrow{\cong} F$ so that

$$\varphi^* = \sigma \circ \varphi \circ \sigma^{-1}.$$

Then, for $\Phi \in (\vee^p L_F^*)_I$,

$$\begin{aligned}(\lambda_I \Phi)(\varphi, \ldots, \varphi) &= \Phi(-\sigma \circ \varphi \circ \sigma^{-1}, \ldots, -\sigma \circ \varphi \circ \sigma^{-1}) \\ &= (-1)^p \Phi(\varphi, \ldots, \varphi).\end{aligned}$$

Q.E.D.

The lemma shows that $\tilde{l}_\xi$ and $\tilde{l}_{\xi^*}$ are related by the formula

$$\tilde{l}_\xi(\Phi) = (-1)^p \tilde{l}_{\xi^*}(\Phi), \qquad \Phi \in (\vee^p L_F^*)_I .$$

Moreover if $f \in \mathscr{S}_0(\mathrm{Sk}_F)_I$, define $\hat{f} \in \mathscr{S}_0(\mathrm{Sk}_F)_I$ by $\hat{f}(\varphi) = f(-\varphi)$. Then the lemma implies that

$$\tilde{s}_{\xi^*}(f) = \tilde{s}_\xi(\hat{f}).$$

4. *Homomorphisms*: Let $\hat{\xi} = (\hat{M}, \hat{\pi}, \hat{B}, \hat{F})$ be a second complex vector bundle. A smooth fibre preserving map φ: $M \to \hat{M}$ is called a *complex linear* (respectively, *conjugate complex linear*) bundle map if it restricts to complex (respectively, conjugate complex) linear maps in the fibres.

Proposition XII: Assume that φ: $\xi \to \hat{\xi}$ is a fibre preserving map inducing ψ: $B \to \hat{B}$ and restricting to isomorphisms in the fibres.

(1) If φ is a complex linear bundle map, then

$$\psi^* \tilde{l}_{\hat{\xi}}(\Phi) = \tilde{l}_\xi(\Phi), \qquad \Phi \in (\vee L_F^*)_I .$$

(2) If φ is a conjugate complex linear bundle map, then

$$\psi^{\#}\tilde{l}_{\xi}(\Phi) = (-1)^p \tilde{l}_{\xi}(\Phi), \qquad \Phi \in (\vee^p L_F^*)_I .$$

Proof: (1) is a direct consequence of Theorem IV, sec. 8.13.

(2) A Hermitian metric in ξ determines a strong complex conjugate bundle isomorphism $\sigma\colon \xi^* \xrightarrow{\cong} \xi$. Thus $\varphi \circ \sigma$ is complex linear, and so

$$\psi^{\#}\tilde{l}_{\xi}(\Phi) = \tilde{l}_{\xi^*}(\Phi) = (-1)^p \tilde{l}_{\xi}(\Phi), \qquad \Phi \in (\vee^p L_F^*)_I$$

(cf. Example 3 above).

Q.E.D.

Corollary: If $\tilde{l}_{\xi}(\Phi) \neq 0$, for some $\Phi \in (\vee^p L_F^*)_I$ with p odd, then the complex bundles ξ and ξ^* are *not* strongly isomorphic.

Remark: The situation in the corollary above (cf. Example 7, sec. 9.20) is in contrast with the situation for a real bundle, which is always strongly isomorphic to its dual.

§5. Chern classes

In this article $\xi = (M, \pi, B, F)$ denotes a complex vector bundle of complex rank r. The notation established in sec. 9.16 remains in force.

9.18. Chern classes. In sec. A.2 the characteristic coefficients $C_k^F \in (\vee^k L_F^*)_I$ are defined. The cohomology class,

$$c_k(\xi) = \bar{l}_\xi(C_k^F), \qquad k = 0, 1, \ldots, r,$$

is called the *kth Chern class of* ξ. In volume III it will be shown that the Chern classes generate the characteristic algebra of ξ.

The Chern classes are *real* classes:

$$c_k(\xi) \in H^{2k}(B).$$

In fact, introduce a Hermitian metric in ξ and observe that, for $\varphi_j \in \mathrm{Sk}_F$,

$$C_k^F\left(\frac{1}{i}\,\varphi_1, \ldots, \frac{1}{i}\,\varphi_k\right) \in \mathbb{R}.$$

Thus the commutative diagram in Proposition XI, sec. 9.16, implies that $\bar{l}_\xi(C_k^F)$ is a real class.

The nonhomogeneous class

$$c(\xi) = \sum_{k=0}^{r} c_k(\xi)$$

is called the *total Chern class of* ξ. It is given by

$$c(\xi) = \tilde{s}_\xi \det(\iota + \varphi)$$

(cf. sec. 9.16 and sec. 9.4). In particular, $c_r(\xi) = \tilde{s}_\xi(\det)$.

It follows from Example 3, sec. 9.17, that the Chern classes for the dual bundle are given by

$$c_k(\xi^*) = (-1)^k c_k(\xi).$$

Thus

$$c(\xi^*) = \tilde{s}_\xi \det(\iota - \varphi) = \tilde{s}_\xi \det(\iota - \varphi^*).$$

Moreover, the Chern class of the Whitney sum of two complex bundles is given by

$$c(\xi \oplus \eta) = c(\xi)c(\eta);$$

this follows in exactly the same way as the analogous result for Pontrjagin classes (Proposition II, (2), sec. 9.4).

In particular, if the Whitney sum of a number of copies of ξ and a trivial complex vector bundle is a trivial complex bundle, then $c(\xi) = 1$.

9.19. Chern character. The cohomology classes corresponding to the trace coefficients $\mathrm{Tr}_p \in (\vee^p L_F^*)_I$ (cf. sec. A.3) are denoted by $\mathrm{tr}_p(\xi)$,

$$\mathrm{tr}_p(\xi) = \tilde{l}_\xi(\mathrm{Tr}_p).$$

These classes are real (by the same argument as in sec. 9.18). In particular,

$$\mathrm{tr}_0(\xi) = r \qquad \text{and} \qquad \mathrm{tr}_1(\xi) = c_1(\xi).$$

Proposition III, sec. A.3, shows that the trace classes and the Chern classes are related by

$$c_k(\xi) = -\frac{1}{k}\sum_{j=0}^{k-1}(-1)^{k-j}c_j(\xi)\,\mathrm{tr}_{k-j}(\xi).$$

The *Chern character* of ξ is the nonhomogeneous class, $\mathrm{ch}(\xi)$, defined by

$$\mathrm{ch}(\xi) = \sum_{p=0}^{\infty}\frac{1}{p!}\,\mathrm{tr}_p(\xi).$$

(Observe that this is a finite sum.) It is given by

$$\mathrm{ch}(\xi) = \tilde{s}_\xi(\mathrm{tr}\,\mathrm{exp}).$$

In exactly the same way as the analogous formulae for the Pontrjagin character are established in sec. 9.6, it follows that

$$\begin{aligned} \mathrm{ch}(\xi \oplus \eta) &= \mathrm{ch}(\xi) + \mathrm{ch}(\eta), \\ \mathrm{ch}(\xi \otimes \eta) &= \mathrm{ch}(\xi)\cdot\mathrm{ch}(\eta), \qquad (9.9) \\ \mathrm{ch}(\wedge^p\xi) &= \tilde{s}_\xi(\mathrm{tr}\,\wedge^p\mathrm{exp}) \end{aligned}$$

and

$$\mathrm{ch}(\wedge\xi) = \tilde{s}_\xi\,\det(\iota + \mathrm{exp}),$$

where η is a second complex bundle over B.

Next, let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let $\Phi: G \to U(F)$ be a representation of G in the Hermitian space F. Let χ_Φ be the smooth complex-valued function in G given by

$$\chi_\Phi(a) = \operatorname{tr} \Phi(a), \qquad a \in G$$

(χ_Φ is called the *character of* Φ).

If ξ is the associated complex vector bundle $P \times_G F$, then

$$\operatorname{ch}(\xi) = \tilde{s}_{\mathscr{P}}(\chi_\Phi \circ \exp_G).$$

(The proof coincides with the proof of Proposition V, sec. 9.7.)

9.20. Examples: 1. *Todd class*: Consider the smooth invariant complex valued function, f, in Sk_F given by

$$f(\varphi) = \det \sum_{p=0}^{\infty} \frac{(-1)^p}{(p+1)!} \varphi^p, \qquad \varphi \in \mathrm{Sk}_F .$$

It is the unique smooth function which satisfies

$$\det \varphi \cdot f(\varphi) = \det(\iota - \exp(-\varphi)), \qquad \varphi \in \mathrm{Sk}_F .$$

Since $f(0) = 1$, a smooth invariant function, Td, is defined near the origin of Sk_F by

$$\mathrm{Td}(\varphi) = \frac{1}{f(\varphi)} .$$

It is uniquely determined by the formula

$$\det \varphi = \mathrm{Td}(\varphi) \cdot \det(\iota - \exp(-\varphi)),$$

whence

$$\det(\tfrac{1}{2}\varphi) \cdot \det(\exp \tfrac{1}{2}\varphi) = \mathrm{Td}(\varphi) \cdot \det(\sinh \tfrac{1}{2}\varphi), \qquad \varphi \in \mathrm{Sk}_F .$$

Evidently $\mathrm{Td}(0) = 1$.

The cohomology class $\mathrm{td}(\xi) = \tilde{s}_\xi(\mathrm{Td})$ is called the *Todd class of* ξ. It follows from the definition of $f(\varphi)$ that $\mathrm{Td}(-\varphi) = \overline{\mathrm{Td}(\varphi)}$, $\varphi \in \mathrm{Sk}_F$. Hence (cf. sec. 9.16) $\mathrm{td}(\xi)$ is a real class; i.e., $\mathrm{td}(\xi) \in H(B)$. Since $\mathrm{Td}(0) = 1$, $\mathrm{td}(\xi)$ is an invertible element of $H(B)$.

The Todd class of the dual bundle is given by

$$\mathrm{td}(\xi^*) = \tilde{s}_\xi(\mathrm{Td}(-\varphi)),$$

(cf. Example 3, sec. 9.17). Since

$$\mathrm{Td}(-\varphi) = \det \exp(-\varphi)\,\mathrm{Td}(\varphi) = \frac{\mathrm{Td}(\varphi)}{\exp \mathrm{tr}(\varphi)}, \qquad \varphi \in \mathrm{Sk}_F,$$

it follows that $\mathrm{td}(\xi^*) = \mathrm{td}(\xi)(\exp c_1(\xi))^{-1}$. Moreover, the relation,

$$\begin{aligned}\det \varphi &= \mathrm{Td}(-\varphi)\det(\exp \varphi - \iota)\\ &= \mathrm{Td}(-\varphi)\sum_{p=0}^{r}(-1)^{r-p}\,\mathrm{tr}\,\wedge^p \exp \varphi, \qquad \varphi \in \mathrm{Sk}_F,\end{aligned}$$

(cf. formula A.1, sec. A.2) implies that (cf. secs. 9.19 and 9.18)

$$\sum_{p=0}^{r}(-1)^p\,\mathrm{ch}(\wedge^p\xi) = \frac{(-1)^r c_r(\xi)}{\mathrm{td}(\xi^*)} = \frac{(-1)^r c_r(\xi)}{\mathrm{td}(\xi)}\exp c_1(\xi).$$

2. *Complexification*: If η is a real vector bundle, then

$$p(\eta) = c(\mathbb{C}\otimes\eta) \qquad \text{and} \qquad \mathrm{tr}(\eta) = \mathrm{ch}(\mathbb{C}\otimes\eta)$$

(cf. Example 1, sec. 9.17).

3. *Realification*: Let $\xi_{\mathbb{R}}$ be the real vector bundle underlying a complex vector bundle ξ. Then there is an isomorphism of complex bundles:

$$\mathbb{C}\otimes\xi_{\mathbb{R}} \cong \xi\oplus\xi^*.$$

In fact, fix a Hermitian metric in ξ and let $\sigma\colon \xi \xrightarrow{\cong} \xi^*$ be the induced complex conjugate isomorphism. Define $\varphi\colon \mathbb{C}\otimes\xi_{\mathbb{R}} \xrightarrow{\cong} \xi\oplus\xi^*$ by

$$\varphi(\lambda\otimes z) = \lambda(z\oplus\sigma(z)), \qquad \lambda\in\mathbb{C}, \quad z\in(F_x)_{\mathbb{R}}.$$

This isomorphism implies that

$$p(\xi_{\mathbb{R}}) = c(\xi)\cdot c(\xi^*) \qquad \text{and} \qquad \mathrm{tr}(\xi_{\mathbb{R}}) = \mathrm{ch}(\xi) + \mathrm{ch}(\xi^*).$$

4. *Pfaffian*: Fix a Hermitian metric in the complex vector bundle ξ and consider the underlying real vector bundle $\xi_{\mathbb{R}}$. Multiplication by i orients each fibre (cf. Example 2, sec. 9.17). Thus $\xi_{\mathbb{R}}$ becomes an oriented Riemannian vector bundle. The Pfaffian class of $\xi_{\mathbb{R}}$ (cf. sec. 9.13) is given by

$$\mathrm{pf}(\xi_{\mathbb{R}}) = c_r(\xi). \tag{9.10}$$

Indeed, if $\varphi \in \mathrm{Sk}_F$ and $j\colon \mathrm{Sk}_F \to \mathrm{Sk}_{F_{\mathbb{R}}}$ is the inclusion, then

$$C_r^F(\varphi) = i^r\, \mathrm{Pf}^{F_{\mathbb{R}}}(j(\varphi))$$

(cf. Example 3, sec. A.7).
Thus formula (9.10) follows from the relations

$$c_r(\xi) = \tilde{l}_\xi(C_r^F), \qquad \mathrm{pf}(\xi_{\mathbb{R}}) = i^r \tilde{k}_{\xi_{\mathbb{R}}}(\mathrm{Pf}^{F_{\mathbb{R}}})$$

and the commutative diagram of Example 2, sec. 9.17.

5. *Complex line bundles*: A complex vector bundle of rank 1 is called a ***complex line bundle.*** Each Hermitian line bundle determines an oriented Riemannian bundle of rank 2. Conversely, if η is an oriented Riemannian vector bundle of rank 2, then a complex structure is determined in each fibre as follows: The linear transformation i_x is the unique skew transformation that satisfies

$$i_x^2 + \iota = 0 \qquad \text{and} \qquad \mathrm{Pf}(i_x) = 1.$$

These constructions are inverse to each other. Hence $\xi \mapsto \xi_{\mathbb{R}}$ defines a bijection between Hermitian line bundles and oriented Riemannian bundles of rank 2.

On the other hand, a Hermitian line bundle ξ determines an associated principal circle bundle $\mathscr{P}_\xi$ and ξ is the bundle associated with $\mathscr{P}_\xi$ via the standard representation of S^1 in $\mathbb{C}$. Moreover, it is easy to see that $\mathscr{P}_\xi$ is the associated circle bundle, $(\xi_{\mathbb{R}})_S$, for the Riemannian bundle $\xi_{\mathbb{R}}$. Thus 1-1 correspondences,

$$\begin{tikzcd}[column sep=tiny] \text{Hermitian line bundles} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{oriented rank-2 Riemannian bundles} \arrow[dl, leftrightarrow] \\ & \text{principal circle bundles} & \end{tikzcd},$$

are defined by

$$\begin{tikzcd} \xi \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \xi_{\mathbb{R}} \arrow[dl, leftrightarrow] \\ & \mathscr{P}_\xi = (\xi_{\mathbb{R}})_S . & \end{tikzcd}$$

Proposition XIII: Let ξ be a Hermitian line bundle. Then

$$c_1(\xi) = \mathrm{pf}(\xi_{\mathbb{R}}) = \chi_{(\xi_{\mathbb{R}})_S},$$

where $\chi_{(\xi_{\mathbb{R}})_S}$ is the Euler class of $(\xi_{\mathbb{R}})_S$ (cf. sec. 0.16).

Remark: The second equality is a special case of the Gauss–Bonnet–Chern theorem of Chapter X.

Proof: The first equality is established in Example 4. To establish the second, observe that $\xi_{\mathbb{R}}$ is the real vector bundle associated with $\mathscr{P}_\xi$ via the standard representation of S^1 in $\mathbb{R}^2$. Hence (cf. (8.4), sec. 8.25),

$$\mathrm{pf}(\xi_{\mathbb{R}}) = -\frac{1}{2\pi} h_{\mathscr{P}_\xi}(\mathrm{Pf}).$$

Next observe that the standard representation of S^1 in $\mathbb{R}^2$ identifies S^1 with $SO(2)$. Thus in this case Pf becomes a linear function in the Lie algebra of S^1. Let $\omega \in A^1(S^1)$ be the unique invariant 1-form such that $\omega(e) = \mathrm{Pf}$. Then a simple computation shows that

$$\int_{S^1} \omega = 2\pi.$$

Hence, $(-1/2\pi)\,\mathrm{Pf}$ determines the invariant 1-form on S^1 whose integral is -1. Now the example of sec. 6.23 shows that

$$\chi_{(\xi_{\mathbb{R}})_S} = \chi_{\mathscr{P}_\xi} = h_{\mathscr{P}_\xi}\left(\frac{-1}{2\pi}\,\mathrm{Pf}\right),$$

whence $\chi_{(\xi_{\mathbb{R}})_S} = \mathrm{pf}(\xi_{\mathbb{R}})$.

Q.E.D.

6. *The canonical complex line bundle*: Consider the Hopf fibration $(S^{2n+1}, \pi, \mathbb{C}P^n, S^1)$ (cf. sec. 5.20). The standard action of S^1 in $\mathbb{C}$ determines an associated complex line bundle

$$\xi_1^n = (S^{2n+1} \times_{S^1} \mathbb{C}, \rho, \mathbb{C}P^n, \mathbb{C}).$$

Recall from sec. 5.14 that a point x in $\mathbb{C}P^n$ is a one-dimensional complex subspace of $\mathbb{C}^{n+1}$; this subspace, regarded as a complex space, is the fibre over x in ξ_1^n (cf. sec. 5.15).

The orientation of $(\xi_1^n)_{\mathbb{R}}$ determines an orientation in the Hopf fibration. Give $\mathbb{C}P^n$ the orientation so that the corresponding local product orientation in S^{2n+1} is the standard orientation. Then it follows from Example 5, above, and sec. 6.24, that $(c_1(\xi_1^n))^n$ is the orientation class of $\mathbb{C}P^n$.

In particular consider the case $n = 1$ and write $\xi_1^1 = \xi_1$. Then $\mathbb{C}P^1 = S^2$ (cf. Example 4, sec. 5.14) and the Hopf fibration is a bundle (S^3, π, S^2, S^1). The corresponding line bundle ξ_1 is called the *canonical complex line bundle*, and $c_1(\xi_1)$ is the orientation class for S^2.

7. *The tangent bundle for S^2*: Give τ_{S^2} the standard metric and orientation; let ξ denote the resulting complex bundle. Then (cf. sec. 9.14 and Example 5, above)

$$\int_{S^2}^{\#} c_1(\xi) = \int_{S^2}^{\#} \mathrm{pf}(\tau_{S^2}) = 2.$$

This shows that $c_1(\xi) \neq 0$.

Hence, by the corollary to Proposition XII, sec. 9.17, ξ is not complex isomorphic to ξ^*. Moreover, if ϵ^q is the trivial complex bundle over S^2 of rank q, then $\xi \oplus \epsilon^q$ is not trivial. On the other hand, the real vector bundle $(\xi \oplus \epsilon)_{\mathbb{R}}$ is the Whitney sum of τ_{S^2} with the trivial real bundle of rank 2; hence it is trivial. (Consider the embedding of S^2 in $\mathbb{R}^3$.)

9.21. The axioms for Chern classes. Theorem I: The Chern classes of a complex vector bundle satisfy the following axioms:

1. *Dimension*: If ξ has rank r, then $c(\xi) \in \sum_{j=0}^{r} H^{2j}(B)$.

2. *Naturality*: If $\varphi: \xi \to \eta$ is a homomorphism of complex vector bundles which restricts to linear isomorphisms on the fibres, then

$$\psi^* c(\eta) = c(\xi),$$

where ψ denotes the induced map between the base manifolds.

3. *Whitney duality*: If ξ and η are vector bundles over the same base, then

$$c(\xi \oplus \eta) = c(\xi) \cdot c(\eta).$$

4. *Normalization*: Let ξ_1 be the canonical complex line bundle over S^2. Then

$$c(\xi_1) = 1 + \omega_{S^2},$$

where ω is the orientation class of S^2 for the orientation defined in Example 6, sec. 9.20.

Conversely, let $\xi \mapsto d(\xi)$ be a rule that associates with each complex vector bundle a cohomology class in the base such that the above conditions hold. Then

$$d(\xi) = c(\xi).$$

Proof: Axiom (1) follows from the definition. Axiom (2) follows from Proposition XII, sec. 9.17. Axiom (3) is observed in sec. 9.18, and Axiom (4) is established in Example 6, sec. 9.20.

Conversely, assume that $\xi \mapsto d(\xi)$ satisfies axioms (1)–(4). Then certainly $d(\xi_1) = c(\xi_1)$.

Next, consider the complex line bundles ξ_1^n (cf. Example 6, sec. 9.20). The inclusions $\lambda_n\colon \mathbb{C}P^1 \to \mathbb{C}P^n$ extend to homomorphisms of complex vector bundles. Thus, by naturality,

$$\lambda_n^\# d_1(\xi_1^n) = d_1(\xi_1) = c_1(\xi_1) = \lambda_n^\# c_1(\xi_1^n).$$

Since $\lambda_n^\#\colon H^2(\mathbb{C}P^n) \to H^2(\mathbb{C}P^1)$ is injective (cf. sec. 6.24), it follows that

$$d(\xi_1^n) = 1 + d_1(\xi_1^n) = 1 + c_1(\xi_1^n) = c(\xi_1^n).$$

Now suppose $\xi = (M, \pi, B, \mathbb{C})$ is any complex vector bundle of rank 1. Then there is a bundle map $\varphi\colon \xi \to \xi_1^n$ restricting to isomorphisms in the fibres and inducing a smooth map $\psi\colon B \to \mathbb{C}P^n$. (This follows from Proposition V, sec. 5.19, with $k = 1$.) Hence,

$$d(\xi) = \psi^\# d(\xi_1^n) = \psi^\# c(\xi_1^n) = c(\xi).$$

Finally, let $\xi = (M, \pi_\xi, B, F)$ be any complex vector bundle of rank r. Introduce a Hermitian metric and let $\mathscr{P} = (P, \pi, B, U(r))$ be the associated principal bundle. Let T be the maximal torus in $U(r)$ that consists of the diagonal isometries (with respect to a fixed orthonormal basis of F). Then F decomposes into one-dimensional T-stable subspaces

$$F = F_1 \oplus \cdots \oplus F_r .$$

Thus the complex vector bundle $\eta = P \times_T F$ over P/T is the Whitney sum of the vector bundles $\eta_i = P \times_T F_i$ (cf. sec. 5.6).

Now consider the commutative diagram of complex bundle maps:

$$\begin{array}{ccccc}
P \times_T F & \longrightarrow & P \times_{U(r)} F & \xrightarrow{\cong} & M \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_\xi} \\
P/T & \xrightarrow[\rho]{} & P/U(r) & \xrightarrow[\cong]{} & B .
\end{array}$$

Using axioms (2) and (3) we obtain

$$\rho^\# d(\xi) = d(\eta) = \prod_{i=1}^r d(\eta_i) = \prod_{i=1}^r c(\eta_i) = \rho^\# c(\xi).$$

Finally, consider the bundle $(P/T, \rho, B, U(r)/T)$ and let $V_{P/T}$ be the corresponding vertical subbundle (over P/T) (cf. sec. 0.15). Let $\chi \in H(P/T)$ be its Euler class. Then the fibre integral, $\oint^\#_{U(r)/T} \chi$, is a scalar. Moreover, the restriction of $V_{P/T}$ to the fibre over z ($z \in P/T$) is the tangent bundle of the fibre, and the restriction, χ_z, of χ to that fibre is the corresponding Euler class. It follows from Theorem I, sec. 10.1, volume I, that

$$\left(\oint^{\#}_{U(r)/T} \chi\right)(z) = \int^{\#} \chi_z = \chi_{U(r)/T}, \qquad z \in P/T,$$

where $\chi_{U(r)/T}$ is the Euler Poincaré characteristic of $U(r)/T$.

Now Proposition XIII, sec. 4.21, together with the example of sec. 2.16, yield

$$\oint^{\#}_{U(r)/T} \chi = r!.$$

It follows that

$$d(\xi) = \frac{1}{r!} \oint^{\#}_{U(r)/T} \rho^{\#}(d(\xi)) \cdot \chi = \frac{1}{r!} \oint^{\#}_{U(r)/T} \rho^{\#}(c(\xi)) \cdot \chi = c(\xi).$$

Q.E.D.

Problems

1. Trace series and characteristic polynomial. Let F be an n-dimensional vector space. Write

$$\mathrm{Tr}(t) = \mathrm{Tr}^F(t) = \sum_{p=0}^{\infty} \frac{1}{p!} \mathrm{Tr}_p^F t^p \qquad \text{and} \qquad C(t) = C^F(t) = \sum_{p=0}^{\infty} C_p^F t^p.$$

(i) Verify the differential equation $tC'(t) = C(t)(n - \mathrm{Tr}(-t))$.

(ii) Evaluate $\mathrm{Tr}(t)$ and $C(t)$ for the linear transformation φ given by $\varphi e_\nu = \lambda_\nu e_\nu$, $\lambda_\nu \in \mathbb{R}$, where $e_1, \ldots, e_n$ is a basis for F.

(iii) Use the differential equation in (i) to prove that

$$C^{F \oplus H}(t) = C^F(t) \cdot C^H(t).$$

(iv) Prove that $C(t) = \exp \int_0^t (1/\tau)(n - \mathrm{Tr}(-\tau))\, d\tau$.

2. Elementary symmetric functions. Define polynomials σ_i, s_i in n indeterminates $\lambda_1, \ldots, \lambda_n$ by

$$\sigma_0 = 1, \qquad \sigma_i = \sum_{1 \leqslant \nu_1 < \cdots < \nu_i \leqslant n} \lambda_{\nu_1} \cdots \lambda_{\nu_i}$$

and

$$s_0 = n, \qquad s_i = \sum_{\nu=1}^{n} (\lambda_\nu)^i, \qquad i = 1, \ldots, n.$$

(i) Find polynomials P_i and Q_i in i indeterminates such that

$$s_i = P_i(\sigma_1, \ldots, \sigma_i) \qquad \text{and} \qquad \sigma_i = Q_i(s_1, \ldots, s_i), \qquad i = 1, \ldots, n$$

(ii) Show that P_i and Q_i are independent of n and that

$$\mathrm{Tr}_i = P_i(C_1, \ldots, C_i) \qquad \text{and} \qquad C_i = Q_i(\mathrm{Tr}_1, \ldots, \mathrm{Tr}_i).$$

(iii) Show that $\sigma_p = -(1/p) \sum_{k=1}^{p} (-1)^k \sigma_{p-k} s_k$.

3. Let A denote the subalgebra of $(\vee L_F^*)_I$ generated by the trace coefficients (or equivalently, the subalgebra generated by the characteristic coefficients, cf. problem 2, (ii)).

(i) Show that A is the symmetric algebra over the subspace spanned by the elements $\mathrm{Tr}_p^F(p = 1,\dots, n)$ where $n = \dim F$.

(ii) Show that A is the symmetric algebra over the subspace spanned by the elements $C_p^F(p = 1,\dots, n)$.

4. Suppose σ and τ are, respectively, a skew and a self-adjoint transformation of an oriented even-dimensional Euclidean space. Assume that $\sigma \circ \tau = \tau \circ \sigma$ and that $\det \tau > 0$. Show that $\sigma \circ \tau$ is skew and that

$$\mathrm{Pf}(\sigma \circ \tau) = \mathrm{Pf}(\sigma) \cdot (\det \tau)^{1/2}.$$

5. Double covers and Pfaffian class. Let ξ and η be Riemannian vector bundles of ranks r and s over a connected manifold B. Let $\rho_\xi\colon \tilde{B}_\xi \to B$ and $\rho_\eta\colon \tilde{B}_\eta \to B$ be the unit sphere bundles of $\wedge^r\xi$ and $\wedge^s\eta$.

(i) Show that ρ_ξ is a double cover. Let ω_ξ interchange the two points of $\rho_\xi^{-1}(x)$ $(x \in B)$. Show that $A(\tilde{B}_\xi) = A_+(\tilde{B}_\xi) \oplus A_-(\tilde{B}_\xi)$, where

$$A_+(\tilde{B}_\xi) = \{\Phi \mid \omega_\xi^*\Phi = \Phi\} \qquad \text{and} \qquad A_-(\tilde{B}_\xi) = \{\Phi \mid \omega_\xi^*\Phi = -\Phi\}.$$

Establish a similar result for cohomology.

(ii) Show that $\wedge^{r+s}(\xi \oplus \eta) = \wedge^r\xi \otimes \wedge^s\eta$. Fix $z \in \rho_\xi^{-1}(x)$ and $w \in \rho_\eta^{-1}(x)$. Use ρ_ξ and ρ_η to obtain isomorphisms of $\wedge T_z^*(\tilde{B}_\xi)$, $\wedge T_w^*(\tilde{B}_\eta)$, and $\wedge T_{z\otimes w}(\tilde{B}_{\xi\oplus\eta})$ onto $\wedge T_x^*(B)$. Hence obtain a bilinear map

$$\wedge T_z^*(\tilde{B}_\xi) \times \wedge T_w^*(\tilde{B}_\eta) \to \wedge T_{z\otimes w}^*(\tilde{B}_{\xi\oplus\eta}).$$

(iii) Construct a bilinear map,

$$A_-(\tilde{B}_\xi) \times A_-(\tilde{B}_\eta) \to A_-(\tilde{B}_{\xi\oplus\eta}),$$

written $(\Phi, \Psi) \mapsto \Phi \cdot \Psi$ such that $(\Phi \cdot \Psi)(z \otimes w) = \Phi(z) \cdot \Psi(w)$. Show that

$$\delta(\Phi \cdot \Psi) = \delta\Phi \cdot \Psi + (-1)^p\, \Phi \cdot \delta\Psi \qquad (p = \deg \Phi).$$

Thus obtain an induced bilinear map $H_-(\tilde{B}_\xi) \times H_-(\tilde{B}_\eta) \to H_-(\tilde{B}_{\xi\oplus\eta})$.

(iv) Let $\tilde{\xi}$ be the pull-back of ξ to $\tilde{B}_\xi$. Show that $\tilde{\xi}$ has a *canonical* orientation. Show that ω_ξ extends to an orientation-reversing isomorphism of $\tilde{\xi}$ and conclude that $\mathrm{pf}(\tilde{\xi}) \in H_-(\tilde{B}_\xi)$.

(v) Prove that $\mathrm{pf}(\widetilde{\xi \oplus \eta}) = \mathrm{pf}(\tilde{\xi}) \cdot \mathrm{pf}(\tilde{\eta})$.

(vi) Show that an orientation in ξ determines a linear isomorphism $H(B) \xrightarrow{\cong} H_-(\tilde{B}_\xi)$ which carries $\mathrm{pf}(\xi)$ to $\mathrm{pf}(\tilde{\xi})$.

(vii) Show that if ξ is an oriented vector bundle containing a subbundle of odd rank, then $\mathrm{pf}(\xi) = 0$.

6. Conjugate bundle. (i) Let F be a complex vector space. Show that multiplication by $-i$ defines a second complex structure in $F_{\mathbb{R}}$; the resulting complex space, $\bar{F}$, is called the *conjugate* of F.

(ii) Let ξ be a complex vector bundle. Replace the fibres of ξ by their conjugates to obtain a new complex vector bundle $\bar{\xi}$ called the *conjugate bundle*.

(iii) Show that $\bar{\xi} \cong \xi^*$. Conclude that

$$\bar{l}_{\bar{\xi}}(\Gamma) = (-1)^p\, \bar{l}_{\xi}(\Gamma), \qquad \Gamma \in \vee^p L_F^* \quad (F\text{, the fibre of } \xi).$$

7. Todd class. Let ξ and η be complex vector bundles of ranks r and s and let ζ be a real vector bundle (all over the same manifold).

(i) Show that $\mathrm{td}(\xi \oplus \eta) = \mathrm{td}(\xi) \cdot \mathrm{td}(\eta)$.

(ii) Mimic the definitions in the text to define the Todd class for a real vector bundle. Show that $\mathrm{td}(\zeta) = \mathrm{td}(\mathbb{C} \otimes \zeta)$.

(iii) Prove the relations

$$\mathrm{td}(\xi_{\mathbb{R}}) = \mathrm{td}(\xi) \cdot \mathrm{td}(\xi^*)$$

and

$$\mathrm{td}(\xi_{\mathbb{R}})\, \tilde{s}_{\xi}(\det(\iota - \cosh)) = 2^{-r}\mathrm{pf}(\xi_{\mathbb{R}}).$$

8. Let $\mathscr{P} = (P, \pi, B, G)$ be a principal bundle and let R be a representation of G in a real vector space F. Let $\xi = (P \times_G F, \rho, B, F)$ be the associated vector bundle. Show that

$$\mathrm{tr}\, \xi = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!\,(2\pi)^p}\, h_{\mathscr{P}}((R')^{\vee}\, \mathrm{Tr}_{2p}^F).$$

9. Let $\xi = (E, \pi, B, F)$ be a complex vector bundle of rank r. Show that

$$\mathrm{ch}(\wedge^r \xi) = \sum_{p=0}^{\infty} \frac{1}{p!}\, c_1(\xi)^p \qquad \text{and} \qquad c(\wedge^r \xi) = 1 + c_1(\xi).$$

Conclude that if $c_1(\xi) \neq 0$, then the structure group of ξ can not be reduced to $SU(F)$.

10. Projective bundles I. For every complex vector space F denote by $P(F)$ the corresponding complex projective space (i.e., the points of

$P(F)$ are the complex lines in F). Let $\xi = (E, \pi_\xi, B, F)$ be a complex vector bundle of rank r.

(i) Construct a fibre bundle, $\xi_P = (M, \pi, B, \mathbb{C}P^{r-1})$, whose fibre at $x \in B$ is the complex projective space $P(F_x)$.

(ii) Construct a complex line bundle, $\eta = (E_\eta, \pi_\eta, M, \mathbb{C})$ over M whose restriction to $P(F_x)$ $(x \in B)$ is the canonical line bundle over $P(F_x)$. Show that the pull-back of ξ to M contains η as a subbundle.

(iii) Introduce a Hermitian metric in ξ and let $(P, \pi_P, B, U(r))$ be the principal bundle associated with ξ. Show that

$$M \cong P \times_{U(r)} \mathbb{C}P^{r-1} \cong P/(U(1) \times U(r-1)).$$

(iv) Show that $E_\eta \cong P \times_{U(r)} N$, where η is the vector bundle of (ii) and N is the canonical line bundle over $\mathbb{C}P^{r-1}$.

(v) Let $(E_S, \pi_S, B, S^{2r-1})$ be the associated sphere bundle of $\xi_{\mathbb{R}}$. Show that

$$E_S = P \times_{U(r)} S^{2r-1} \cong P/U(r-1).$$

Obtain a principal bundle (E_S, p, M, S^1) and establish the commutative diagram

$$\begin{tikzcd} E_S \arrow[rr, "p"] \arrow[dr, "\pi_S"'] & & M \arrow[dl, "\pi"] \\ & B & \end{tikzcd}.$$

(vi) Show that the principal bundle of (v) is the circle bundle associated with the line bundle η.

11. Projective bundles II. Retain the notation of problem 10. Let $\theta \in H^2(\mathbb{C}P^{r-1})$ be the first Chern class of the canonical line bundle and let $\omega \in A^2(M)$ represent $c_1(\eta)$.

(i) Define a linear map, $\varphi: A(B) \otimes H(\mathbb{C}P^{r-1}) \to A(M)$, by setting

$$\varphi\left(\sum_\nu \Phi_\nu \otimes \theta^\nu\right) = \sum_\nu \pi^*\Phi_\nu \wedge \omega^\nu, \qquad \Phi_\nu \in A(B).$$

Show that φ induces a linear isomorphism

$$\varphi_\#: H(B) \otimes H(\mathbb{C}P^{r-1}) \xrightarrow{\cong} H(M).$$

Find an explicit expression for $\varphi_\#$. (*Hint*: Use Mayer–Vietoris.)

(ii) Conclude that $\pi^*\colon H(B) \to H(M)$ is injective. Thus prove that the homomorphism, $\rho^\#$, of sec. 9.21 is injective.

(iii) Show that there are unique classes $\alpha_i \in H^{2i}(B)$ ($i = 0,\dots, r$) such that $\alpha_0 = 1$ and

$$\sum_{i=0}^{r} \pi^\#(\alpha_i)(-c_1(\eta))^{r-i} = 0.$$

Let $H(B)[t]$ denote the algebra of polynomials with coefficients in $H(B)$, and let $\mathscr{I}$ denote the ideal generated by the polynomial $\sum_{i=0}^{r} \alpha_i t^{r-i}$. Construct an algebra isomorphism $H(B)[t]/\mathscr{I} \cong H(M)$.

(iv) Prove the relation

$$\sum_{i=0}^{r} \pi^\# c_i(\xi)(-c_1(\eta))^{r-i} = 0.$$

(use problem 10, (ii)). Conclude that $\alpha_i = c_i(\xi)$. Conclude that (iii) provides an alternative definition for the Chern classes.

(v) Let h be a nonzero Lie vector of S^1. Interpret the results on $H(M)$ as theorems on $H(A(M)_{i(h)=0,\theta(h)=0}, \delta)$.

12. Group actions. Suppose a Lie group G (with Lie algebra E) acts from the left on a real vector bundle, $\xi = (M, \pi, B, F)$. Let $\hat{Z}_h$ and Z_h denote the fundamental vector fields on M and B generated by $h \in E$, and set $\hat{X}_h = -\hat{Z}_h$, $X_h = -Z_h$.

(i) Write $(a \cdot \sigma)(x) = a \cdot (\sigma(a^{-1} \cdot x))$, $a \in G$, $x \in B$, $\sigma \in \operatorname{Sec} \xi$. Write

$$(\theta(h)\sigma)(x) = \frac{d}{dt}((\exp th) \cdot \sigma)(x)|_{t=0}.$$

Show that $\theta(h)$ belongs to $\operatorname{Sec} L_\xi$ and satisfies

$$\theta([h, k]) = [\theta(h), \theta(k)]$$

and

$$\theta(h)(f \cdot \sigma) = X_h(f) \cdot \sigma + f \cdot \theta(h)\sigma, \qquad h, k \in E, \quad f \in \mathscr{S}(B), \quad \sigma \in \operatorname{Sec} \xi.$$

(ii) If $\xi = \tau_B$ (respectively, $\wedge \tau_B^*$) show that $\theta(h)X = [X_h, X]$ (respectively, $\theta(h)\Omega = \theta(X_h)\Omega$).

(iii) Show that $X_{[h,k]} = [X_h, X_k]$.

(iv) If ξ is a Riemannian vector bundle and G acts by isometries show that

$$\langle\theta(h)\sigma, \tau\rangle + \langle\sigma, \theta(h)\tau\rangle = X_h(\langle\sigma, \tau\rangle), \qquad \sigma, \tau \in \operatorname{Sec} \xi.$$

Establish a converse if G is connected.

(v) If ∇ is a linear connection in ξ and G acts by connection preserving maps show that

$$\theta(h) \circ \nabla_X - \nabla_X \circ \theta(h) = \nabla_{[X_h, X]}, \qquad h \in E, \quad X \in \mathscr{X}(B).$$

Establish a converse if G is connected.

(vi) Assume that G acts by connection preserving maps. Show that a linear map $S: E \to \operatorname{Sec} L_\xi$ is given by $S(h) = \theta(h) - \nabla_{X_h}$. Show that $\nabla(S(h)) = i(X_h)R$, where R is the curvature of ∇.

(vii) Assume that the action of G on B is trivial. Show that the action of G on M restricts to representations, Φ_x, of G in the fibres F_x. Prove that

$$\Phi_x'(h)(\sigma(x)) = (\theta(h)\sigma)(x), \qquad x \in B, \quad h \in E.$$

If, in addition, G preserves a linear connection in ξ show that $\theta(h)$ is parallel. Thus, if B is connected, construct a representation of G in F and a G-equivariant coordinate representation for ξ.

(viii) Suppose that $h \in E$ is a vector such that X_h has no zeros and assume that G is compact. Construct a linear connection in ξ which is invariant under the toral subgroup generated by $\exp th$ and whose curvature satisfies $i(X_h)R = 0$. Conclude that the Pontrjagin numbers of ξ are all zero.

(ix) Assume a torus T acts on a compact oriented manifold B so that the isotropy subgroups are proper. Prove that the Pontrjagin numbers of B are all zero.

13. Parallel cross-sections. Let $\eta = (M, \pi, B, F)$ be a Riemannian vector bundle with a Riemannian connection ∇. Assume that B is connected, and that σ is a parallel cross-section in $\operatorname{Sk} \eta$.

(i) Obtain distinct, strictly positive numbers $a_1, \dots, a_r$ and a unique decomposition $\eta = \bigoplus_0^r \eta_\nu$ such that: (a) $\sigma = \bigoplus_0^r \sigma_\nu$, $\sigma_\nu \in \operatorname{Sec}(\operatorname{Sk} \eta_\nu)$, (b) $\sigma_0 = 0$, and (c) $\sigma_\nu^2 + 4\pi a_\nu^2\, \iota = 0$, $\nu = 1, \dots, r$.

(ii) Show that the decomposition of η in (i) is orthogonal. Show that $\nabla = \bigoplus_0^r \nabla_\nu$, where ∇_ν is a Riemannian connection in η_ν. Show that each σ_ν is parallel.

(iii) Construct unique complex vector bundles ξ_ν ($\nu = 1, \ldots, r$), such that $(\xi_\nu)_{\mathbb{R}} = \eta_\nu$ and $\sigma_\nu = (2\pi i a_\nu)\iota$. Show that each ∇_ν is complex linear.

(iv) Let $\Gamma \in (\vee^k \mathrm{Sk}_F^*)_I$, where I denotes the invariance under $SO(F)$ if η is orientable, and with respect to $O(F)$ otherwise. Set

$$\Omega = \frac{1}{k!}\,\Gamma\Big(\frac{-1}{2\pi i}(R+\sigma), \ldots, \frac{-1}{2\pi i}(R+\sigma)\Big)$$

and

$$\Omega_j = \frac{1}{j!(k-j)!}\,\Gamma\Big(\underbrace{\frac{-1}{2\pi i}R, \ldots, \frac{-1}{2\pi i}R}_{(j \text{ factors})},\ \underbrace{\frac{-1}{2\pi i}\sigma, \ldots, \frac{-1}{2\pi i}\sigma}_{(k-j \text{ factors})}\Big).$$

Show that Ω_j is the component of degree $2j$ of Ω. Show that the correspondence $\Gamma \mapsto \Omega$ is a homomorphism. Prove that Ω is closed, and pass to cohomology to obtain a homomorphism $\tilde{h}_\sigma \colon (\vee\, \mathrm{Sk}_F^*)_I \to H(B; \mathbb{C})$. Show that $\tilde{h}_\sigma$ is independent of the choice of ∇.

(v) Define $\alpha_{\nu\mu} \in H(B)$ ($\nu, \mu = 1, \ldots, r$) by

$$\alpha_{\nu\mu} = \sum_{\substack{0 \leqslant t \leqslant j \leqslant m_\nu \\ 0 \leqslant s \leqslant \mu - j \leqslant m_\nu}} (-1)^j \binom{m_\nu - t}{j - t}\binom{m_\nu - s}{\mu - j - s}\, c_t(\xi_\nu)\, c_s(\xi_\nu)(-a_\nu)^{\mu - t - s},$$

where m_ν denotes the rank of ξ_ν. Prove that

$$\tilde{h}_\sigma(C_{2q}) = \sum_{q_0 + \cdots + q_r = q} p_{q_0}(\eta_0)\, \alpha_{1,2q_1} \cdot \cdots \cdot \alpha_{r,2q_r}.$$

Conclude that $\tilde{h}_\sigma(C_{2q})$ is a polynomial in the Pontrjagin classes of η_0, the Chern classes of ξ_ν, and the a_ν.

(vi) Assume that η is orientable and of rank $2m$. Use the complex structure of ξ_ν to orient η_ν. Conclude that η_0 is orientable and of rank $2m_0$. Orient η_0 and give η the induced orientation. Let $c(\xi_\nu, t)$ be the polynomial $\Sigma_p c_p(\xi_\nu)\, t^{m_\nu - p}$. Show that

$$i^m \tilde{h}_\sigma(\mathrm{Pf}) = \mathrm{pf}(\eta_0) \cdot \prod_{\nu=1}^{r} c(\xi_\nu, -a_\nu).$$

Conclude that $i^m \tilde{h}_\sigma(\mathrm{Pf})$ is a real class, invertible if and only if $\eta_0 = 0$.

(vii) Interpret (iv), (v) and (vi) in the special case that B is a point.

14. Toral actions I. Let T be a torus with Lie algebra E and let $\eta = (M, \pi_\eta, B, F)$ (B connected) and $\zeta = (N, \pi_\zeta, M, H)$ be real vector bundles. Assume that T acts from the left on η and ζ so that: (a) the action on B is trivial, (b) the two actions on M coincide, and (c) the action on M restricts to faithful representations of T in the fibres.

Let $\xi = (Q, \pi_\xi, B, H)$ denote the restriction of ζ to B via the zero cross-section.

(i) Construct a T-invariant Riemannian metric and a T-invariant Riemannian connection ∇ in ξ. Do the same for η. For $h \in E$ construct a parallel cross-section $\theta(h)$, in Sk_ξ (cf. problem 12, (vii)).

(ii) Find a T-equivariant strong isomorphism $\zeta \xrightarrow{\cong} \pi_\eta^* \xi$. Hence obtain from ξ a T-invariant Riemannian metric and a T-invariant Riemannian connection ∇_ζ in ζ. Show that the curvature R_ζ of ∇_ζ satisfies $i(Z)\, R_\zeta = 0$ if Z is a vertical vector field on M.

(iii) Let Z_h be the fundamental vector field on M generated by $h \in E$ and set $X_h = -Z_h$. Show that X_h is vertical, and that $X_h(z)$ is tangent to the sphere in $F_{\pi(z)}$ through z. Conclude that $i(X_h)\, R_\zeta = 0$.

(iv) Apply problem 12, (i) to obtain operators $\theta_\zeta(h)$ $(h \in E)$, in Sec ζ. Set

$$S(h) = \theta_\zeta(h) - (\nabla_\zeta)_{X_h}$$

and show that $S(h)$ is a T-invariant, parallel cross-section in Sk_ζ. Show that $S(h)$ is the pull-back of $\theta(h)$.

(v) Assume that $\exp th$ generates T. Let $\dot{\eta} = (\dot{M}, \dot{\pi}, B, \dot{F})$ be the deleted bundle. Construct a T-invariant 1-form, ω, on $\dot{M}$ so that $i(X_h)\omega = 1$. Show that $\delta\omega$ is T-invariant and satisfies $i(X_h)\, \delta\omega = 0$.

(vi) Let $f \in \mathscr{S}(\mathbb{R})$ satisfy $f(t) = 1,\ t \geqslant \frac{1}{2}$ and $f(t) = 0,\ t \leqslant \frac{1}{3}$. Define $p \in \mathscr{S}(M)$ by $p(z) = f(|\, z\,|)$. Show that f is T-invariant. Prove that $\hat{\nabla} = \nabla_\zeta + (p \cdot \omega) \otimes S(h)$ is a T-invariant Riemannian connection in ζ whose curvature is given by $\hat{R} = R_\zeta + \delta(p \cdot \omega) \otimes S(h)$. Conclude that $i(X_h)\hat{R} = -\delta p \otimes S(h)$.

(vii) Justify the notation $[1 - \delta(p\omega)]^{-1} = \sum_{k=0}^{\infty} [\delta(p\omega)]^k$. Let Ω_j denote the jth component of a nonhomogeneous form Ω. Establish the formula,

$$\Gamma\Big(\frac{-\hat{R}}{2\pi i}, \dots, \frac{-\hat{R}}{2\pi i}\Big)(1 - \delta(p\omega))^{-1}$$
$$= \sum_{j \geqslant 2k} \Big[\pi_\eta^*\Big(\Gamma\Big(\frac{R + \theta(h)}{-2\pi i}, \dots, \frac{R + \theta(h)}{-2\pi i}\Big)\Big)(1 - \delta(p\omega))^{-1}\Big]_j,$$

for $\Gamma \in (\vee^k \mathrm{Sk}_H^*)_I$.

15. Toral actions II. Consider the special case of problem 14 in which $\xi = \eta$ and $\zeta = \pi_\eta^*\eta$. Fix $h \in E$ so that exp th generates T. Denote the resulting parallel cross-section in Sk_η by $L(h)$ (corresponding to the $\theta(h)$ in problem 14).

(i) Apply problem 13 to $L(h)$ to obtain a decomposition $\eta = \eta_1 \oplus \cdots \oplus \eta_r$, positive numbers a_ν and complex vector bundles ξ_ν such that $(\xi_\nu)_{\mathbb{R}} = \eta_\nu$ and $L(h)$ restricts to $(2\pi i a_\nu)\iota$ in ξ_ν. Let $m_\nu = \operatorname{rank} \xi_\nu$ and $m = \sum_{\nu=1}^r m_\nu$. Orient the η_ν and η (compatibly).

(ii) Give the associated sphere bundle, $\eta_S = (M_S, \pi_S, B, S^{2m-1})$, the induced orientation. If $\Phi \in A(M_S)$ satisfies $i(X_h)\Phi = 0$ prove that $\oint_S \Phi = 0$.

(iii) Choose a T-invariant 1-form, ω, on $\dot{M}$ so that $i(X_h)\omega = 1$. Prove that

$$\oint_S \omega \wedge (\delta\omega)^{m-1} = (-1)^m a_1^{-m_1} \cdots a_r^{-m_r}$$

(Show first that the integral is independent of the choice of ω).

(iv) Let ∇_η be a T-invariant Riemannian connection in η with curvature R_η. Let $\hat{\nabla}$ be the corresponding connection in $\pi^*\eta$ (with curvature $\hat{R}$), constructed as in problem 14, (vi)). Let j: $M_S \to M$ be the inclusion map. Then

$$j^*\mathrm{Pf}(\hat{R}, \dots, \hat{R}) = \delta\Phi$$

for some $\Phi \in A^{2m-1}(M_S)$ which is T-invariant and satisfies $i(X_h)\Phi = 0$ (cf. problem 9, Chap. X). Use this fact (and problem 14, (vii)) to showt hat there exists a $\Psi \in A(B)$ such that

$$\oint_S \omega \wedge (1 - \delta\omega)^{-1} = \left[\mathrm{Pf}\left(\frac{R_\eta + L(h)}{-2\pi}, \dots, \frac{R_\eta + L(h)}{-2\pi}\right)\right]^{-1} + \delta\Psi.$$

(v) Let $A(M_S)_{i(h)=0,\theta(h)=0}$ be the graded differential algebra of forms, Φ, which satisfy $i(X_h)\Phi = 0$ and $\theta(X_h)\Phi = 0$. Show that for any form Ψ on M, $\theta(X_h)\Psi = 0$ if and only if Ψ is T-invariant. Obtain from π_S^* a homomorphism ρ: $A(B) \to A(M_S)_{i(h)=0,\theta(h)=0}$.

(vi) Define a map φ: $H(B) \otimes \mathbb{R}[t] \to H(A(M_S)_{i(h)=0,\theta(h)=0})$ by $\varphi(\alpha \otimes t^k) = (\rho_\# \alpha)\, \theta^k$, where θ is the class represented by $\delta\omega$. Show that φ restricts to a linear isomorphism

$$H(B) \otimes (1 \oplus t \oplus \cdots \oplus t^{m-1}) \xrightarrow{\cong} H(A(M_S)_{i(h)=0,\theta(h)=0}).$$

(vii) Show that φ induces an algebra isomorphism

$$(H(B) \otimes \mathbb{R}[t])/\mathscr{I} \xrightarrow{\cong} H(A(M_S)_{i(h)=0,\theta(h)=0}),$$

where $\mathscr{I}$ is the ideal generated by

$$\prod_\nu c(\xi_\nu, -a_\nu t) \qquad \Big(c(\xi_\nu, \lambda) = \sum_p c_p(\xi_\nu) \otimes \lambda^{m_\nu - p}\Big).$$

(*Hint*: Use part (iv), and problem 13, (vi).)

(viii) Show that parts (vi) and (vii) generalize problem 11.

16. Toral actions III. Consider the situation of problem 14. Assume that $\dim B = 2n$ and let $\operatorname{rank} \eta = 2m$. Assume B is compact and oriented.

(i) Use problem 13 to obtain homomorphisms,

$$\tilde{h}_{\theta(h)}\colon (\vee \mathrm{Sk}_H^*)_I \to H(B; \mathbb{C}) \qquad \text{and} \qquad \tilde{h}_{L(h)}\colon (\vee \mathrm{Sk}_F^*)_I \to H(B; \mathbb{C}).$$

(ii) Let $\Gamma \in (\vee^{n+m} \mathrm{Sk}_H^*)_I$. Show that $\Gamma(\hat{R}/-2\pi i, \dots, \hat{R}/-2\pi i)$ has compact support, where $\hat{R}$ is the curvature of problem 14, (vi). Prove that

$$\frac{1}{(n+m)!} \int_M \Gamma\left(\frac{\hat{R}}{-2\pi i}, \dots, \frac{\hat{R}}{-2\pi i}\right) = \int_B^\# \tilde{h}_{\theta(h)}(\Gamma)(i^m \tilde{h}_{L(h)}(\mathrm{Pf}))^{-1}$$

(*Hint*: Use problem 14, (vi) and problem 15, (iv)).

17. Toral actions IV. Let a torus T act on a real vector bundle $\zeta = (N, \rho, M, H)$, where M is a compact connected $2q$-manifold. Let $F = \bigcup_\alpha F_\alpha$ be the fixed point set (cf. problem 14, Chap. III) and let η_α be the normal bundle of F_α. Fix a Lie vector h so that $\exp th$ generates T.

(i) Obtain homomorphisms,

$$\tilde{h}_{\theta_\alpha(h)}\colon (\vee \mathrm{Sk}_H^*)_I \to H(F_\alpha; \mathbb{C}) \qquad \text{and} \qquad \tilde{h}_{L_\alpha(h)}\colon (\vee \mathrm{Sk}_{H_\alpha}^*)_I \to H(F_\alpha; \mathbb{C}),$$

where H_α is the typical fibre of η_α (as in problem 16, (i)).

(ii) Show that, for $\Gamma \in (\vee^q \mathrm{Sk}_H^*)_I$,

$$\int_M^\# \tilde{h}_\xi(\Gamma) = \sum_\alpha i^{m_\alpha} \int_{F_\alpha}^\# \tilde{h}_{\theta_\alpha(h)}(\Gamma)(\tilde{h}_{L_\alpha(h)}(\mathrm{Pf}))^{-1},$$

where $\operatorname{rank} \eta_\alpha = 2m_\alpha$.

(*Hint*: Find a T-invariant Riemannian connection in ζ which agrees, in a tubular neighbourhood of the F_α, with the connection $\hat{\nabla}$ constructed in problem 14, (vi) and whose curvature $\hat{R}$ satisfies $i(X_h)\hat{R} = 0$ outside smaller neighbourhoods of the F_α.)

(iii) Show that, for $\Gamma \in (\vee^j \mathrm{Sk}_H^*)_I$, $j \leqslant q - 1$,

$$\sum_\alpha \int_{F_\alpha}^{\#} \tilde{h}_{\theta_\alpha(h)}(\Gamma)(\tilde{h}_{L_\alpha(h)}(\mathrm{Pf}))^{-1} = 0.$$

18. Let ξ be a real vector bundle of rank 4 over a compact oriented 4-manifold, B. Assume that ξ is trivial over $B - \{b\}$ $(b \in B)$.

(i) Identify S^3 with a small sphere in B around b. Use ξ to obtain a smooth map

$$\varphi\colon S^3 \to SO(3).$$

(ii) Show that any smooth map $S^3 \to SO(3)$ has even degree.

(iii) Show that for suitable orientations

$$\tfrac{1}{2} \deg \varphi = \int_B^{\#} p_1(\xi).$$

Conclude that $\int_B^{\#} p_1(\xi)$ is an integer.

(iv) Show that $p_1(\xi) = 0$ if and only if ξ is trivial.

(v) Given B and an integer k construct a real rank 4 vector bundle, η, over B such that $\int_B^{\#} p_1(\eta) = k$.

19. Bundles over the Grassmannians. (i) Express the Pontrjagin character of the manifold $G_{\mathbb{R}}(n; k)$ in terms of the Pontrjagin character of the canonical vector bundle of rank k over $G_{\mathbb{R}}(n; k)$.

(ii) Express the Pontrjagin character of the manifold $G_{\mathbb{C}}(n; k)$ in terms of the Chern character of the canonical vector bundle of rank k over $G_{\mathbb{C}}(n; k)$.

(iii) Find the Pontrjagin and trace classes of $\mathbb{C}P^n$.

20. K-theory. Let M be a compact manifold.

(i) Use the Chern character to define a ring homomorphism (cf. problem 10, Chap. VIII)

$$\mathrm{ch}\colon K(M) \otimes \mathbb{C} \to H(M; \mathbb{C}).$$

(ii) If ξ is a complex vector bundle of rank r over M, show that

$$\mathrm{ch}\left(\sum_{p=0}^{r} (-1)^p \wedge^p \xi\right) = \mathrm{pf}(\xi_{\mathbb{R}}) \cdot \mathrm{td}(\xi).$$

21. Weyl tensor. Let g be a pseudo-Riemannian metric in a smooth n-manifold M.

(i) Show that there is a unique pseudo-Riemannian connection ∇ in τ_M with torsion zero (Levi–Civita connection). Let R be its curvature.

(ii) Suppose R has components $R_{ij}{}^k{}_l$ with respect to a local coordinate system. Show that the expressions

$$(\mathrm{Ric})_{ij} = \sum_\alpha R_{\alpha i}{}^\alpha{}_j, \quad (\widehat{\mathrm{Ric}})^i_j = \sum_\alpha g^{\alpha i}(\mathrm{Ric})_{\alpha j},$$

define tensor fields Ric and $\widehat{\mathrm{Ric}}$ on M; they are called the *Ricci tensors.* Show that Ric is symmetric, and find intrinsic expressions for Ric and $\widehat{\mathrm{Ric}}$. Define $\Phi \in \mathscr{S}(M)$ by $\Phi = \Sigma_\alpha(\widehat{\mathrm{Ric}})^\alpha_\alpha$. Φ is called the *Ricci scalar curvature*; express it intrinsically.

(iii) Assume $n \geqslant 3$. Show that a 2-form $C \in A^2(M; \mathrm{Sk}_{\tau_M})$ is defined by

$$C_{lm}{}^h{}_k = R_{lm}{}^h{}_k - \frac{1}{n-2}\{\delta^h_l(\mathrm{Ric})_{mk} - \delta^h_m(\mathrm{Ric})_{lk} + g_{km}(\widehat{\mathrm{Ric}})^h_l - g_{kl}(\widehat{\mathrm{Ric}})^h_m\}$$
$$- \frac{\Phi}{(n-1)(n-2)}\{\delta^h_m g_{kl} - \delta^h_l g_{km}\}.$$

C is called the *Weyl conformal curvature tensor*. Find an intrinsic expression for C.

(iv) Two metrics g_1 and g_2 are called *conformally equivalent* if $g_1 = f \cdot g_2$ with $f \in \mathscr{S}(M)$ and $f(x) > 0$, $x \in M$. Show that the Weyl tensors of two conformally equivalent metrics coincide.

(v) Show that $\mathbf{tr}(R^{2k}) = \mathbf{tr}(C^{2k})$, $k = 1, 2, \ldots$. Conclude that the Pontrjagin forms for τ_M (determined by a Levi–Civita connection) are conformal invariants. (*Hint*: Prove that $R^2 = C^2 + \Psi$, where $\Psi^2 = 0$, $R \circ \Psi \circ R = 0$ and $\mathbf{tr}(\Psi) = 0$.)

22. Bundles over symmetric spaces. Let ω be an involution of a compact connected Lie group G. Let $G_1 = \{a \in G \mid \omega(a) = a\}$ and let K be a subgroup such that $G_1^0 \subset K \subset G_1$. Assume $\xi = (G \times_K F, \pi_\xi, G/K, F)$ and $\eta = (G \times_K H, \pi_\eta, G/K, H)$ are vector bundles.

(i) Show that ξ admits a G-invariant linear connection.

(ii) Show that the homomorphism $\gamma^I_\xi\colon (\vee L_F^*)_I \to A(G/K)$ determined by a G-invariant connection is independent of the choice of connection.

(iii) Establish a similar result for Σ-bundles.

(iv) Suppose that there is a G-equivariant isomorphism

$$(G \times_K F) \oplus (G \times_K H) \cong G/K \times \mathbb{R}^m.$$

Let P_ξ and P_η be the *representatives* of the total Pontrjagin classes for ξ and η determined by G-invariant connections in ξ and η. Show that

$$P_\xi \wedge P_\eta = 1.$$

(v) Apply these results to the canonical bundles over the Grassmann manifolds.

23. Let $\eta = (M, \pi_\eta, B, F)$ be a Riemannian vector bundle of rank n and assume $i: \eta \to B \times \mathbb{R}^m$ is a strong bundle map which restricts to isometries $i_x: F_x \to i_x(F_x)$ (with respect to a constant inner product in $\mathbb{R}^m$). Let $\rho: B \times \mathbb{R}^m \to \eta$ be the bundle map determined by $\rho_x \circ i_x = \iota$ and $\ker \rho_x \perp \operatorname{Im} i_x$.

(i) Show that $\rho_* \circ \delta \circ i_*$ is a Riemannian connection in η.

(ii) Consider the case that $B = SO(m)/(SO(n) \times SO(m-n))$, $\eta = \xi$, where ξ is the canonical vector bundle of rank n, and i is the inclusion induced by the isomorphism of Proposition V, sec. 5.19. Show that the resulting connection in ξ is $SO(m)$-equivariant.

(iii) With B as in (ii), use the symmetric space structure to obtain a principal connection in $(SO(m), \pi, B, SO(n) \times SO(m-n))$. Show that this induces a Riemannian connection in ξ, and that this connection coincides with the connection in (ii).

(iv) Show that the inclusion $i: \eta \to B \times \mathbb{R}^m$ determines a bundle map $\varphi: \eta \to \xi$, restricting to isometries in the fibres. Show that φ is connection preserving with respect to the connections defined in (i) and (ii).

(v) Let P_η be the representative of the total Pontrjagin class of η determined by the connection in (i). Show that P_η is invertible, and that the kth component of P_η^{-1} is zero for $k > 4(n-m)$. (*Hint*: cf. problem 22.)

(vi) Let M be a Riemannian n-manifold, and assume $\psi: M \to \mathbb{R}^m$ is an immersion such that the induced metric in M is conformally equivalent to the given one (ψ is called a *conformal immersion*). Show that the representative P_M of the total Pontrjagin class of M (determined by the Levi-Civita connection) satisfies: the kth component of P_M^{-1} is zero for $k > 4(n-m)$. (*Hint*: cf. problem 21.)

24. Let ξ and η be real vector bundles of the same rank. Let $\varphi: E_\xi \to E_\eta$ be a smooth fibre preserving map inducing $\psi: B_\xi \to B_\eta$ and restricting to diffeomorphisms on the fibres. Show that $p(\xi) = \psi^\# p(\eta)$.

25. Finitely generated projective modules. (i) Extend the definition of characteristic elements to finitely generated projective modules over a commutative ring.

(ii) Let R be the curvature of a linear connection in a vector bundle ξ over a manifold B. Show that $\bigoplus_p A^{2p}(B; \xi)$ is a finitely generated projective module over the ring $\bigoplus_p A^{2p}(B)$. Show that R is an endomorphism of this module and that $1/p!\, C_p(R, \ldots, R)$ is the pth characteristic coefficient.

26. Resultant. Let $f(t) = \sum_0^n a_i t^{n-i}$ and $g(t) = \sum_0^m b_j t^{m-j}$ be polynomials over a commutative ring A. Set

$$Q(a_0, \ldots, a_n; b_0, \ldots, b_m) = \begin{vmatrix} a_0 & a_1 & \cdots & & a_n & & & \\ & a_0 & \cdots & & & a_n & & \\ & & & & & & & \\ & & & & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & & & \\ & b_0 & \cdots & & b_m & & & \\ & & & & & & & \\ & & & & b_0 & b_1 & \cdots & b_m \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix} b \\ b \\ b \\ b \end{matrix}}\right\} n \end{array};$$

Q is called the *resultant* of f and g. Define a polynomial, $f * g$, by

$$(f * g)(t) = Q(f_0(t), \ldots, f_n(t), b_0, \ldots, b_m),$$

where

$$f_p(t) = (-1)^p \sum_{i=0}^{p} \binom{n-i}{p-i} a_i t^{p-i}.$$

(i) If $f(t) = \prod_{i=1}^n (t + \alpha_i)$ and $g(t) = \prod_{j=1}^m (t + \beta_j)$, show that

$$Q(a_0, \ldots, a_n; b_0, \ldots, b_m) = \prod_{i,j} (\beta_j - \alpha_i) \qquad \text{and} \qquad (f * g)(t) = \prod_{i,j} (t + \alpha_i + \beta_j).$$

Conclude that $(f * g)(0) = Q(a_0, -a_1, a_2, \ldots, (-1)^n a_n; b_0, \ldots, b_m)$.

(ii) Establish the relations

$$f * g = g * f, \qquad (f * g) * h = f * (g * h), \qquad f * (gh) = (f * g)(f * h),$$
$$(f * g)^{pq} = f^p * g^q \qquad \text{and} \qquad f * 1 = f,$$

for polynomials f, g, and h with leading coefficient 1.

(iii) For $\varphi \in L_F$ set $f_\varphi(t) = \det(\varphi + t\iota)$. If $\varphi \in L_F$, $\psi \in L_H$ show that $f_{\varphi\otimes\iota+\iota\otimes\psi} = f_\varphi * f_\psi$.

27. Pontrjagin and Chern polynomials. Let ξ be a real vector bundle of rank n. Its *Pontrjagin polynomial* is the polynomial $p(\xi, t) = \sum_j p_j(\xi)\, t^{n-2j}$. Let η be a second real vector bundle.

(i) Show that

$$p(\xi \oplus \eta, t) = p(\xi, t) \cdot p(\eta, t) \qquad \text{and} \qquad p(\xi \otimes \eta, t) = p(\xi, t) * p(\eta, t).$$

(ii) If $\check{h}_\xi(\mathrm{Det}^F) = 0$, $\check{h}_\eta(\mathrm{Det}^H) = 0$, prove that $\check{h}_{\xi\otimes\eta}(\mathrm{Det}^{F\otimes H}) = 0$.

(iii) Establish the recurrence relation

$$p(\wedge^m \xi, t)^{m!} = \prod_{k=0}^{m-1} \left\{ p(\wedge^k \xi, t)^{k!} * \left[(m-k)^n\, p\left(\xi, \frac{t}{m-k}\right)\right]^{(m-1)!/k!} \right\}^{(-1)^{m-1-k}}$$

Solve this relation for $m = 2, 3$.

(iv) Establish analogous results for Chern classes of complex bundles.

28. Quaternionic bundles. Let $\xi = (M, \pi, B, F)$ be a quaternionic bundle.

(i) By regarding $\mathbb{C}$ as a subalgebra of $\mathbb{H}$ make F_x, $x \in B$ (respectively, F) into complex spaces $(F_x)_\mathbb{C}$ (respectively, $F_\mathbb{C}$). Hence obtain an underlying complex bundle $\xi_\mathbb{C} = (M, \pi, B, F_\mathbb{C})$.

(ii) Show that $\xi_\mathbb{C} \cong \xi_\mathbb{C}^*$. Conclude that $c_p(\xi_\mathbb{C}) = 0$ if p is odd.

(iii) Let $\xi_\mathbb{R} = (M, \pi, B, F_\mathbb{R})$ be the underlying real bundle (so that $\xi_\mathbb{R} = (\xi_\mathbb{C})_\mathbb{R}$. Show that $p(\xi_\mathbb{R}) = c(\xi_\mathbb{C})^2$.

29. Hopf fibrations. Consider the Hopf fibrations

$$\mathscr{P} = (S^{2n+1}, \pi, \mathbb{C}P^n, S^1) \qquad \text{and} \qquad \mathscr{P}_1 = (S^{4n+3}, p, \mathbb{H}P^n, S^3),$$

and let ξ and η be the corresponding canonical complex and quaternionic line bundles.

(i) Use the orientations of $\mathbb{C}^{n+1}$ and $\mathbb{C}$ to orient S^{2n+1} and S^1. Obtain an induced orientation in $\mathbb{C}P^n$. Show that the orientation class is $[-c_1(\xi)]^n$.

(ii) Obtain a bundle $(\mathbb{C}P^{2n+1}, \rho, \mathbb{H}P^n, S^2)$ such that $p = \rho \circ \pi$. Show that $\rho^*(\eta_\mathbb{C}) = \xi \oplus \bar{\xi}$. Conclude that $\rho^\#(c(\eta_\mathbb{C})) = 1 - c_1(\xi)^2$.

(iii) Show that $c_2(\eta_{\mathbb{C}})$ is the Euler class for the Hopf fibration above, where S^3 is given the orientation induced from the complex structure of $\mathbb{H}$. Conclude that $\mathrm{pf}(\eta_{\mathbb{R}})$ is the Euler class for $\mathscr{P}_1$. Show that $1, c_2(\eta_{\mathbb{C}}), \ldots, c_2(\eta_{\mathbb{C}})^n$, is a basis for $H(\mathbb{H}P^n)$, and that $c_2(\eta_{\mathbb{C}})^n$ is an orientation class for $\mathbb{H}P^n$. Conclude that $\rho^\#$ (as in (ii)) is injective. Show that the bundle of (ii) does not admit a cross-section.

(iv) Let E be the Lie algebra of S^3 (i.e., the Lie algebra of pure quaternions). Define $\Gamma \in (\vee^2 E^*)_I$ by $\Gamma(q_1, q_2) = \mathrm{Re}(q_1 \bar{q}_2)$. Show that $\chi_{\mathscr{P}_1} = -h_{\mathscr{P}_1}(\Gamma/2\pi^2)$.

(v) Establish analogues of problems 10 and 11 for quaternionic vector bundles.

30. Let $\mathscr{P} = (P, \pi, B, S^3)$ be a principal bundle. Consider the real vector bundle $\xi = (M, \rho, B, \mathbb{R}^4)$ (where $M = P \times_{S^3} \mathbb{H}$ and S^3 acts in $\mathbb{H}$ by conjugation) and the quaternionic line bundle $\eta = (P \times_{S^3} \mathbb{H}, \hat{\rho}, B, \mathbb{H})$, (where S^3 acts on $\mathbb{H}$ by right multiplication).

(i) Show that $\xi = \epsilon \oplus \zeta$, where rank $\zeta = 3$. Conclude that $p_2(\xi) = 0$.

(ii) Show that $p(\xi) = 1 - h_{\mathscr{P}}(2\Gamma/\pi^2)$, where Γ is defined in problem 29, (iv).

(iii) Show that $c(\eta_{\mathbb{C}}) = 1 - h_{\mathscr{P}}(\Gamma/2\pi^2)$.

31. Tangent bundle of $\mathbb{H}P^n$. Let η be the canonical quaternionic line bundle over $\mathbb{H}P^n$ and let $\eta_{\mathbb{C}}$ be the underlying complex bundle.

(i) Show that

$$p(\mathbb{H}P^n) = [1 + c_2(\eta_{\mathbb{C}})]^{2n+2}\left[\sum_{k=0}^{n} (-4c_2(\eta_{\mathbb{C}}))^k\right].$$

(ii) Conclude that $p_1(\mathbb{H}P^n) = (2n - 2)\, c_2(\eta_{\mathbb{C}})$. Hence show that $\mathbb{H}P^n$ is irreversible for $n \geqslant 2$ (cf. problem 29).

(iii) Reverse $\mathbb{H}P^1$.

32. Kodaira class. Let $\mathscr{P} = (P, \pi, B, T)$ be a principal torus bundle, and let ξ be the complex vector bundle associated with a representation, Φ, of T in a complex space F. Let ξ have a linear connection ∇ induced from a principal connection in $\mathscr{P}$.

(i) Interpret the curvature, R, of ∇ as a 2-form with values in E: $R \in A^2(B; E)$, where E is the Lie algebra of T. Show that $\delta R = 0$.

(ii) Let $\mathscr{U} = \{U_\alpha\}$ be a simple cover of B, and let $\{(U_\alpha, \psi_\alpha)\}$ be a principal coordinate representation for $\mathscr{P}$. Let $g_{\alpha\beta}(x) = \psi_{\alpha,x}^{-1} \circ \psi_{\beta,x}$, $x \in U_{\alpha\beta}$. Show that there are smooth functions $f_{\alpha\beta}: U_{\alpha\beta} \to E$ such that $\exp \circ f_{\alpha\beta} = g_{\alpha\beta}$.

(iii) Show that $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}$ is constant in $U_{\alpha\beta\gamma}$. If $h_{\alpha\beta\gamma}$ is the constant value of this function, prove that the eigenvalues of $\Phi'(h_{\alpha\beta\gamma})$ are integral multiples of $2\pi i$. Interpret the correspondence $U_{\alpha\beta\gamma} \mapsto h_{\alpha\beta\gamma}$ as a simplicial 2-cocycle, Ω, on the nerve $\mathscr{N}$, of $\mathscr{U}$, with values in E.

(iv) Assume f is a function which assigns to each non-void intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ an element of $A^p(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}; E)$ (p and q fixed). Then f is a q-cochain of E-valued p-forms. Make these functions into a space $C^{p,q}$ and introduce an operator, $\nabla = \delta + D$, in the direct sum, $C = \bigoplus C^{p,q}$, exactly as in article 7, Chap. V, volume I. Regard $A(B; E)$ and $C(\mathscr{N}; E)$ as subspaces of C.

(v) Show that $R - \Omega = \nabla(\Psi)$ for some $\Psi \in C$.

(vi) Consider the case that $\dim F = 1$. Show that an integral simplicial 2-cocycle, θ, on $\mathscr{N}$ is defined by $\Phi'(h_{\alpha\beta\gamma}) = -2\pi i\theta(\alpha\beta\gamma)\iota$. Prove that the class represented by θ in $H(\mathscr{N})$(the *Kodaira class*) corresponds to $c_1(\xi)$ under the isomorphism $H(\mathscr{N}) \cong H(B)$.

(vii) For general F write $F = F_1 \oplus \cdots \oplus F_r$, where the F_i are 1-dimensional T-stable subspaces. Let $\theta_i \in C^2(\mathscr{N})$ be the cocycle corresponding to F_i. Set $\theta = \prod_{i=1}^r (1 + \theta_i)$. Show that θ is an integral cocycle in $C(\mathscr{N})$ whose class corresponds to $c(\xi)$ under the isomorphism $H(\mathscr{N}) \cong H(B)$.

33. Vector bundles over S^4. Let $\mathbb{R}^4$ be the space of quaternions and let S^4 be the one-point compactification of $\mathbb{R}^4$. Give S^4 the orientation induced by $\mathbb{R}^4$. Let M be the 8-manifold obtained from two copies of $S^4 \times \mathbb{R}^4$ via the identification, $\psi: \dot{\mathbb{R}}^4 \times \mathbb{R}^4 \xrightarrow{\cong} \dot{\mathbb{R}}^4 \times \mathbb{R}^4$, given by

$$\psi(x, y) = \left(\frac{x}{|x|^2}, \frac{x^p y x^q}{|x|}\right), \qquad x \in \dot{\mathbb{R}}^4, \quad y \in \mathbb{R}^4,$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$. Construct a vector bundle, $\xi = (M, \pi, S^4, \mathbb{R}^4)$, and show that

$$p_1(\xi) = 2(p - q)\omega,$$

where ω denotes the orientation class of S^4. *Hint*: Use problem 17, Chap. VI.

Chapter X

The Gauss–Bonnet–Chern Theorem

10.1. In this section $\xi = (M, \pi, B, F)$ denotes a fixed Riemannian vector bundle of rank $r = 2m$. Thus, if ξ is oriented, the Pfaffian class $\mathrm{pf}(\xi)$ is defined. On the other hand, the Euler class χ_ξ of the associated sphere bundle (via a Riemannian metric) is also defined. This chapter is centered around the following theorem:

Theorem I (Gauss–Bonnet–Chern): Let $\xi = (M, \pi, B, F)$ be an oriented Riemannian vector bundle of rank $r = 2m$. Then the Pfaffian class of ξ coincides with the Euler class of the associated sphere bundle:

$$\mathrm{pf}(\xi) = \chi_\xi .$$

As an immediate consequence of Theorem III, sec. 9.9, volume I, and Theorem I, sec. 10.1, volume I, and formula 9.6, sec. 9.12, we obtain

Theorem II: Let ξ be an oriented Riemannian vector bundle of rank $2m$ over a compact connected $2m$-manifold B. Let σ be a cross-section with finitely many zeros and let ∇ be a Riemannian connection in ξ with Riemannian curvature $R^\bullet$. Then the index sum of σ is given by

$$j(\sigma) = \frac{(-1)^m}{m!(2\pi)^m} \int_B \langle \Delta_\xi , R^{\bullet m} \rangle = \int_B^{\#} \chi_\xi .$$

Moreover, if ξ is the tangent bundle of B, then these numbers coincide with the Euler–Poincaré characteristic of B:

$$j(\sigma) = \frac{(-1)^m}{m!(2\pi)^m} \int_B \langle \Delta_\xi , R^{\bullet m} \rangle = \sum_{p=0}^{2m} (-1)^p \dim H^p(B) = \int_B^{\#} \chi_\xi .$$

Remarks: 1. In the theorem above, $(R^\bullet)^m$ denotes the mth power of $R^\bullet$ in the algebra $A(B; \wedge\xi)$ and Δ_ξ is the positive normed determinant function in ξ.

2. In view of Lemma I, sec. 9.12, there are almost identical analogues of Theorems I and II for pseudo-Riemannian bundles.

Corollary I: The Euler class of the Whitney sum of two oriented Riemannian vector bundles is given by

$$\chi_{\xi\oplus\eta} = \chi_\xi \cdot \chi_\eta .$$

Proof: Apply Proposition X, (3), sec. 9.13. (Recall from sec. 8.2, volume I, that $\chi_\xi = 0$ if ξ has odd rank).

Q.E.D.

Corollary II: An oriented Riemannian vector bundle with nonzero Euler class contains no oriented subbundle of odd rank.

Proof: Again apply Proposition X, (3), sec. 9.13.

Q.E.D.

Corollary III: The tangent bundles of the even dimensional spheres contain no proper nonzero orientable subbundles.

Corollary IV: Assume that B is a compact oriented n-manifold ($n = 2m$) whose tangent bundle admits a Riemannian connection with decomposable curvature $R^\bullet = f \cdot (\Psi \wedge \Psi)$ (cf. sec. 9.14). Then the Euler–Poincaré characteristic of B is given by

$$\chi_B = (-1)^m \frac{2^{m+1}}{\operatorname{vol} S^n} \int_B f^m \det \Psi \cdot \Delta_B ,$$

where $\psi_x(h) = \Psi(x; h)$, $h \in T_x(B)$.

Proof: Apply the Gauss–Bonnet theorem and sec. 9.14.

Q.E.D.

10.2. Normed cross-sections. Let $\xi = (M, \pi, B, F)$ be an oriented Riemannian vector bundle of rank r, equipped with a Riemannian connection ∇. Let $R^\bullet$ denote the corresponding Riemannian curvature and let Δ_ξ be the unique positive normed determinant function in ξ.

Assume that ξ admits a normed cross-section σ; i.e.,

$$\langle \sigma(x), \sigma(x) \rangle = 1, \qquad x \in B.$$

(Observe that this is equivalent to assuming that ξ admits a cross-section without zeros.)

Define differential forms $\Phi_k \in A^{r-1}(B)$ by

$$\Phi_k = \langle \Delta_\xi, \sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^\bullet)^{m-k-1} \rangle, \qquad 0 \leqslant k \leqslant m-1,$$

and set

$$\Phi = -\sum_{k=0}^{m-1} a_k \Phi_k,$$

where a_k is given by

$$a_k = (-1)^k \frac{m(m-1)\cdots(m-k)}{1 \cdot 3 \cdots (2k+1)}, \qquad k = 0,\dots, m-1.$$

Proposition I: The differential form Φ satisfies

$$\delta\Phi + \langle \Delta_\xi, (R^\bullet)^m \rangle = 0.$$

The proof of the proposition depends on two lemmas. Recall that we identify ξ with ξ^* via the canonical isomorphism induced by the Riemannian metric. Thus σ determines an operator $i_\xi(\sigma)$ in $A(B; \wedge\xi)$ (cf. sec. 7.9).

Lemma I:

(1) $\sigma \wedge i_\xi(\sigma)\Omega = \Omega, \qquad \Omega \in A(B; \wedge^r\xi).$

(2) $i_\xi(\sigma)(\nabla\sigma)^k = 0, \qquad k = 1, 2, \dots .$

(3) $(\nabla\sigma)^{2m} = 0.$

Proof: (1) This is established in sec. 7.9. (2) This follows from the relation (cf. Lemma VII, sec. 7.24)

$$i_\xi(\sigma)\,\nabla\sigma = \langle \nabla\sigma, \sigma \rangle = 0.$$

(3) In view of (1) and (2) we have

$$(\nabla\sigma)^{2m} = \sigma \wedge i_\xi(\sigma)(\nabla\sigma)^{2m} = 0$$

Q.E.D.

Lemma II:

$$\nabla(\sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^\bullet)^{m-k-1}) = (\nabla\sigma)^{2k+2} \wedge (R^\bullet)^{m-k-1} + \frac{2k+1}{m-k}(\nabla\sigma)^{2k} \wedge (R^\bullet)^{m-k}.$$

Proof: In view of Lemma I, (2), we can apply Lemma I, (1), with

$$\Omega = (\nabla\sigma)^{2k} \wedge (R^\bullet)^{m-k}$$

to obtain

$$(\nabla\sigma)^{2k} \wedge (R^{\bullet})^{m-k} = (m-k)\sigma \wedge (\nabla\sigma)^{2k} \wedge i_{\xi}(\sigma)\, R^{\bullet} \wedge (R^{\bullet})^{m-k-1}$$
$$= (m-k)\sigma \wedge i_{\xi}(\sigma)\, R^{\bullet} \wedge (\nabla\sigma)^{2k} \wedge (R^{\bullet})^{m-k-1}.$$

Lemma II is an immediate consequence of this, together with the relations

$$\nabla(\Phi \wedge \Psi) = \nabla\Phi \wedge \Psi + (-1)^{p}\Phi \wedge \nabla\Phi, \qquad \Phi \in A^{p}(B;\xi), \quad \Psi \in A(B;\xi),$$
$$\nabla^{2}\sigma = R(\sigma) = i_{\xi}(\sigma)\, R^{\bullet}$$

and

$$\nabla R^{\bullet} = 0.$$

Q.E.D.

10.3. Proof of Proposition I: It follows from Lemma II that (with $a_{-1} = -1$)

$$\nabla(a_k\sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^{\bullet})^{m-k-1})$$
$$= a_k(\nabla\sigma)^{2k+2} \wedge (R^{\bullet})^{m-k-1} - a_{k-1}(\nabla\sigma)^{2k} \wedge (R^{\bullet})^{m-k}, \qquad k = 0,\dots, m-1.$$

Adding these relations, we obtain

$$a_{m-1}(\nabla\sigma)^{2m} + (R^{\bullet})^{m} = \nabla\left(\sum_{k=0}^{m-1} a_k\sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^{\bullet})^{m-k-1}\right).$$

In view of Lemma I, (3), this reads

$$(R^{\bullet})^{m} = \nabla\left(\sum_{k=0}^{m-1} a_k\sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^{\bullet})^{m-k-1}\right).$$

On the other hand, since $\nabla(\Delta_{\xi}) = 0$ (cf. sec. 7.24), it follows that

$$\delta\langle \Delta_{\xi}, \Omega\rangle = \langle \Delta_{\xi}, \nabla\Omega\rangle, \qquad \Omega \in A(B; \wedge^{r}\xi).$$

Now set $\Omega = \sum_{k=0}^{m-1} a_k\sigma \wedge (\nabla\sigma)^{2k+1} \wedge (R^{\bullet})^{m-1}$ to obtain

$$\langle \Delta_{\xi}, (R^{\bullet})^{m}\rangle = \delta \sum_{k=0}^{m-1} a_k\Phi_k = -\delta\Phi.$$

Q.E.D.

10.4. The bundle η. Consider the oriented Riemannian bundle $\xi = (M, \pi, B, F)$ of sec. 10.1, and let $\xi_S = (M_S, \pi_S, B, S)$ be the associated sphere bundle (cf Example 6, sec. 3.10, volume I). According to sec. 3.10, M_S is a submanifold of M. Hence we can pull back the vertical subbundle $(V_M, \rho_M, M, \mathbb{R}^r)$ of τ_M to M_S (cf. sec. 0.15).

This yields a vector bundle $\eta = (V_S, \rho_S, M_S, \mathbb{R}^r)$. The inclusion map, $i_V: V_S \to V_M$, is a bundle map,

$$\begin{tikzcd} V_S \arrow[r, "i_V"] \arrow[d, "\rho_S"'] & V_M \arrow[d, "\rho_M"] \\ M_S \arrow[r, "i_M"'] & M \end{tikzcd},$$

which restricts to linear isomorphisms on the fibres.

Next consider the injections $j_x: F_x \to M$. For $z \in F_x$, $(dj_x)_z$ is a linear isomorphism from F_x to $V_z(M)$. Now recall from sec. 7.7, volume I, (cf. also sec. 7.20) that a bundle map,

$$\begin{tikzcd} V_M \arrow[r, "\alpha"] \arrow[d, "\rho_M"'] & M \arrow[d, "\pi"] \\ M \arrow[r, "\pi"'] & B \end{tikzcd},$$

is defined by

$$\alpha_z = (dj_{\pi(z)})_z^{-1}, \qquad z \in M.$$

Combining the diagrams above we obtain the commutative diagram,

$$\begin{tikzcd} V_S \arrow[r, "i_V"] \arrow[d, "\rho_S"'] & V_M \arrow[r, "\alpha"] \arrow[d, "\rho_M"'] & M \arrow[d, "\pi"] \\ M_S \arrow[r, "i_M"'] & M \arrow[r, "\pi"'] & B \end{tikzcd},$$

which defines a bundle map, $\gamma = \alpha \circ i_V : \eta \to \xi$, inducing linear isomorphisms in the fibres.

In particular there is a unique Riemannian metric and a unique orientation in η such that γ is isometric and orientation preserving. It follows that the positive normed determinant function Δ_η in the Riemannian bundle η is given by

$$\Delta_\eta = (\wedge\gamma)^{\#} \Delta_\xi ,$$

where $\wedge\gamma: \wedge\eta \to \wedge\xi$ is the induced bundle map.

Now recall from sec. 7.7, volume I, that the vertical radial vector field Z is the cross-section in V_M given by

$$Z(z) = \alpha_z^{-1}(z), \qquad z \in M.$$

Since i_V restricts to linear isomorphisms on the fibres, Z determines a cross-section,

$$\sigma = i_V^{\#} Z,$$

in η. For $z \in M_S$, we have $\gamma_z(\sigma(z)) = z$. Since γ is an isometry, it follows that

$$\langle \sigma(z), \sigma(z) \rangle = 1, \qquad z \in M_S;$$

thus σ is a *normed* cross-section.

Proposition II: Let ∇_ξ be a Riemannian connection in ξ, and let ∇_η denote its pull-back to η via γ. Then the corresponding Riemannian curvatures are related by

$$\langle\!\langle \Delta_\eta, (R_\eta^\bullet)^m \rangle\!\rangle = \pi_S^* \langle\!\langle \Delta_\xi, (R_\xi^\bullet)^m \rangle\!\rangle.$$

Proof: Observe that, since γ is connection preserving,

$$R_\eta^\bullet = (\wedge\gamma)^{\#} R_\xi^\bullet.$$

Since γ induces $\pi_S \colon M_S \to B$ between the base manifolds, we have

$$\begin{aligned} \langle\!\langle \Delta_\eta, (R_\eta^\bullet)^m \rangle\!\rangle &= \langle\!\langle (\wedge\gamma)^{\#} \Delta_\xi, (\wedge\gamma)^{\#} (R_\xi^\bullet)^m \rangle\!\rangle \\ &= \pi_S^* \langle\!\langle \Delta_\xi, (R_\xi^\bullet)^m \rangle\!\rangle. \end{aligned}$$

Q.E.D.

10.5. Proof of the main theorem. Theorem I: Let $\xi = (M, \pi, B, F)$ be an oriented vector bundle of rank $r = 2m$. Then the Pfaffian class of ξ coincides with the Euler class of an associated sphere bundle.

Proof: Fix a Riemannian metric in ξ. Let ∇_ξ be a Riemannian connection in ξ with Riemannian curvature $R_\xi^\bullet$. According to formula 9.6 of sec. 9.12 the differential form,

$$\frac{(-1)^m}{(2\pi)^m m!} \langle\!\langle \Delta_\xi, (R_\xi^\bullet)^m \rangle\!\rangle,$$

is closed and represents the class $\mathrm{pf}(\xi)$. Thus we must construct a differential form, $\Phi \in A^{r-1}(M_S)$, such that

$$\delta\Phi = -\pi_S^*\langle \Delta_\xi, (R_\xi^\bullet)^m\rangle$$

and

$$\oint_S \Phi = (-1)^m\, m!(2\pi)^m.$$

Recall that in sec. 10.4 we used the vertical radial vector field to obtain a normed cross-section, σ, in the oriented Riemannian vector bundle $\eta = (V_S, \rho_S, M_S, \mathbb{R}^r)$. Let ∇_η be the pull-back, via γ, of ∇_ξ, and let $R_\eta^\bullet$ be the Riemannian curvature of ∇_η, as in that section.

Define differential forms $\Phi_k \in A^{r-1}(M_S)$ $(k = 0, 1, \ldots, m-1)$ by

$$\Phi_k = \langle \Delta_\eta, \sigma \wedge (\nabla_\eta \sigma)^{2k+1} \wedge (R_\eta^\bullet)^{m-k-1}\rangle$$

and set

$$\Phi = -\sum_{k=0}^{m-1} a_k \Phi_k,$$

where

$$a_k = (-1)^k \frac{m(m-1)\cdots(m-k)}{1\cdot 3\cdot 5\cdots(2k+1)}, \qquad k = 0, \ldots, m-1.$$

Then Proposition I of sec. 10.2, applied to η, implies that

$$\delta\Phi = -\langle \Delta_\eta, (R_\eta^\bullet)^m\rangle.$$

Combining this with Proposition II of sec. 10.4 yields

$$\delta\Phi = -\pi_S^*\langle \Delta_\xi, (R_\xi^\bullet)^m\rangle.$$

It remains to be proved that

$$\oint_S \Phi = (-1)^m\, m!(2\pi)^m.$$

Fix a point $a \in B$ and let $i_a\colon S_a \to M_S$ be the inclusion map, where S_a denotes the fibre of ξ_S at a. Regard $S_a \times F_a$ as a trivial oriented Riemannian vector bundle over S_a. Then a bundle map

$$\begin{tikzcd} S_a \times F_a \arrow[r, "\varphi_a"] \arrow[d] & V_S \arrow[d, "\rho_S"] \\ S_a \arrow[r, "i_a"'] & M_S \end{tikzcd}$$

is defined by

$$\varphi_a(z, y) = \gamma_z^{-1}(y), \qquad z \in S_a, \quad y \in F_a.$$

Clearly, φ_a induces orientation preserving isometries in the fibres. Moreover, the diagram,

$$\begin{tikzcd} S_a \times F_a \arrow[r, "\varphi_a"] \arrow[d] & V_S \arrow[r, "\gamma"] \arrow[d, "\rho_S"'] & M \arrow[d, "\pi"] \\ S_a \arrow[r, "i_a"'] & M_S \arrow[r, "\pi_S"'] & B \end{tikzcd},$$

commutes.

Now the map of base manifolds induced by $\gamma \circ \varphi_a$ is the *constant* map: $S_a \to a$. It follows (cf. Example 2, sec. 7.13) that the pull-back of any linear connection in ξ to the trivial bundle $S_a \times F_a$ via $\gamma \circ \varphi_a$ is simply the standard connection δ. In particular,

$$(\wedge\varphi_a)^\# R_\eta^\bullet = \wedge(\gamma \circ \varphi_a)^\# R_\xi^\bullet = 0.$$

On the other hand, φ_a is an orientation preserving isometry of vector bundles, and so

$$(\wedge\varphi_a)^\# \Delta_\eta = \Delta_a,$$

where Δ_a denotes the positive normed determinant function in F_a. Thus, with $\sigma_a = \varphi_a^\# \sigma$, we have

$$i_a^* \Phi_k = \langle \Delta_a, \sigma_a \wedge (\delta\sigma_a)^{2k+1} \wedge (\wedge\varphi_a)^\#(R_\eta^\bullet)^{m-k-1} \rangle = 0, \qquad k = 0, \dots, m-2,$$

and

$$i_a^* \Phi_{m-1} = \langle \Delta_a, \sigma_a \wedge (\delta\sigma_a)^{r-1} \rangle.$$

It follows that

$$i_a^* \Phi = -a_{m-1} i_a^* \Phi_{m-1} = \frac{(-1)^m\, m!}{1 \cdot 3 \cdot 5 \cdots (2m-1)} \langle \Delta_a, \sigma_a \wedge (\delta\sigma_a)^{2m-1} \rangle.$$

Finally, observe that the cross-section, σ_a, is given by $\sigma_a(z) = (z, z)$, $z \in S_a$. Hence, the corollary to Proposition XIII, sec. 7.27, yields

$$\int_S \langle \Delta_a, \sigma_a \wedge (\delta\sigma_a)^{r-1} \rangle = (2m-1)! \operatorname{vol} S^{2m-1}.$$

It follows that

$$\left(\oint_S \Phi\right)(a) = \int_S i_a^* \Phi = (-1)^m\, m!(2\pi)^m, \qquad a \in B,$$

(cf. sec. 0.13 for vol S^{2m-1}). **Q.E.D.**

Problems

1. Angle function. Let M be an oriented Riemannian 2-manifold with metric tensor g and normed determinant function Δ.

Let $\alpha: [0, 1] \to M$ be a path on M. *A vector field along* α is a cross-section, X, in the pull-back, $\alpha^*\tau_M$, of τ_M under α (equivalently, X is a smooth map from the unit interval to T_M such that $X(t) \in T_{\alpha(t)}(M)$, $t \in [0, 1]$). Let ∇ be the Levi–Civita connection in M and let ∇_T be the induced connection in $\alpha^*\tau_M$ in the direction of the vector field $T = d/dt$. The *covariant derivative of a vector field along* α, denoted by ∇_α, is defined by $\nabla_\alpha X = \nabla_T X$.

(i) Let $X \in \mathscr{X}(M)$ and set $X(t) = X(\alpha(t))$. Establish the *chain rule*

$$(\nabla_\alpha X)(t) = \nabla(X(\alpha(t); \dot{\alpha}(t)).$$

(ii) Let X and Y be vector fields along α such that $|X(t)| = 1$ and $|Y(t)| = 1$. An *angle function* is a smooth function θ of t $(0 \leqslant t \leqslant 1)$ satisfying

$$\cos\theta(t) = g(\alpha(t); X(t), Y(t)),$$

$$\sin\theta(t) = \Delta(\alpha(t); X(t), Y(t)).$$

Construct an angle function for X and Y. Show that if θ_1 and θ_2 are angle functions then, for some $k \in \mathbb{Z}$, $\theta_2 - \theta_1 = 2k\pi$. Conclude that the difference $\theta(1) - \theta(0)$ does not depend on the choice of the angle function.

(iii) Show that

$$\theta(1) - \theta(0) = \int_0^1 [-\Delta(\alpha(t); X(t), \nabla_\alpha X(t)) + \Delta(\alpha(t); Y(t), \nabla_\alpha Y(t))]\, dt.$$

Hint: Use the identity

$$g(x; h, h_1) \cdot \Delta(x; h_2, h_3) + g(x; h, h_2) \cdot \Delta(x; h_3, h_1)$$
$$+ g(x; h, h_3) \cdot \Delta(x; h_1, h_2) = 0.$$

(iv) Show that θ changes sign if the orientation of M is reversed.

(v) Let X_i ($i = 1, 2, 3$) be vector fields along α and denote the corresponding angle functions by θ_{ij}. Show that, for some $k \in \mathbb{Z}$,

$$\theta_{13}(t) = \theta_{12}(t) + \theta_{23}(t) + 2\pi k.$$

Conclude that

$$\theta_{12}(t) + \theta_{21}(t) = 2\pi k.$$

(vi) Assume that α is a closed path homotopic to the constant path $\alpha_0: t \mapsto \alpha(0)$. Let $\varphi: Q \to M$ be a homotopy from α to α_0, where Q is the unit square $0 \leqslant t \leqslant 1$, $0 \leqslant \tau \leqslant 1$. Construct a cross-section Y in the bundle $\varphi^*\tau_M$ such that $|Y(t, \tau)| = 1$ $(0 \leqslant t \leqslant 1,\ 0 \leqslant \tau \leqslant 1)$. If X is a vector field along α show that $\theta(1) - \theta(0)$ is independent of the choice of Y. Write

$$\sphericalangle(X(0), X(1)) = \theta(1) - \theta(0).$$

(vii) Let X be a vector field along $\partial\sigma$ (σ a smooth 2-simplex) obtained by parallel translation of a vector $X(0)$. Show that

$$\sphericalangle(X(0), X(1)) = -\int_\sigma \mathrm{Pf}(R),$$

where R is the curvature of the Levi–Civita connection.

2. Geodesic curvature. We retain the notation and hypothesis of problem 1. The *geodesic curvature* of a path α on M is the function given by

$$\gamma(t) = \frac{1}{|\dot\alpha(t)|^3}\,\Delta(\alpha(t); \dot\alpha(t), \nabla_\alpha\dot\alpha(t)).$$

(i) If σ is a smooth 2-simplex on M, show that

$$\int_{\partial\sigma}\gamma - \int_\sigma \mathrm{Pf}(R) = \theta_1 + \theta_2 + \theta_3 - \pi$$

where θ_1, θ_2, θ_3 are the interior angles of σ (local Gauss–Bonnet formula for 2-manifolds).

(ii) Conclude that the area of a geodesic triangle on the unit sphere S^2 is given by

$$A = \theta_1 + \theta_2 + \theta_3 - \pi.$$

3. The Gauss–Bonnet formula for 2-manifolds. Let M be an oriented Riemannian 2-manifold with positive normed determinant function Δ. Assume that X is a unit vector field on M. Denote by $X^\perp$ the unique unit vector field which satisfies $\Delta(X, X^\perp) = 1$.

(i) Show that the 1-form Φ given by $\Phi = \langle \nabla X, X^{\perp} \rangle$ satisfies

$$\delta\Phi = \langle R(X), X^{\perp} \rangle.$$

(ii) If M is compact and X is a unit vector field on M with finitely many singularities, use (i) to prove the formula,

$$\int_M \mathrm{Pf}(R) = -2\pi j(X),$$

where $j(X)$ denotes the index sum of X.

4. Torus in $\mathbb{R}^3$. Let $\varphi\colon T^2 \to \mathbb{R}^3$ be the immersion of the 2-torus in $\mathbb{R}^3$ given by

$$\varphi(t, \tau) = (b + a \cos t) \cos \tau \cdot e_1 + (b + a \cos t) \sin \tau \cdot e_2 + a \sin t \cdot e_3$$

$(0 < a < b)$. Give T^2 the Riemannian metric induced from $\mathbb{R}^3$ via the immersion. Verify the formula

$$\int_T \mathrm{Pf}(R) = 0.$$

5. Poincaré polygons. Define a Riemannian metric in the open unit disk of $\mathbb{C}$ by setting

$$g(z; \zeta_1, \zeta_2) = \frac{4}{(1 - |z|^2)^2} \operatorname{Re}(\bar{\zeta}_1 \zeta_2).$$

(i) Show that the corresponding Gaussian curvature is given by $K = -1$.

(ii) Consider the compact manifold M constructed in problem 24, (v), Chap. III, from a $4p$-Poincare polygon. Use the Gauss–Bonnet formula to show that

$$\chi_M = 2(1 - p), \qquad p \geqslant 2.$$

Conclude that the first Betti number of M is p.

6. The Hodge $\ast$-operator. Let M be an oriented Riemannian n-manifold with volume form Δ_M. Identify τ_M with τ_M^* under the Riemannian metric. The *Hodge $\ast$-operator* is the bundle map,

$$\operatorname{Sec} \wedge^p \tau_M \to \operatorname{Sec} \wedge^{n-p} \tau_M,$$

defined by

$$\ast(X_1 \wedge \cdots \wedge X_p) = i(X_p) \cdots i(X_1) \Delta_M, \qquad X_j \in \mathscr{X}(M).$$

Let $R^\bullet$ be the Riemannian curvature of M and regard $(R^\bullet)^p$ as a cross-section in the vector bundle $L_{\wedge^{2p}\tau_M}$.

(i) Show that $(R^\bullet)^p$ is self-adjoint.

(ii) Assume that $n = 2m$. Show that the Gaussian curvature of M is given by each of the formulae

$$K = \frac{(-2)^m}{(2m)!}\operatorname{tr}(R^{\bullet p} * R^{\bullet m-p} *), \qquad 0 \leqslant p \leqslant m.$$

7. Sectional curvature. Let M be a Riemannian n-manifold and assume $h, k \in T_x(M)$ are linearly independent. The *sectional curvature* of M at x corresponding to h and k is the number given by

$$S(x; h, k) = \frac{\langle R(x; h, k)(k), h\rangle}{\langle h, h\rangle\langle k, k\rangle - \langle h, k\rangle^2}.$$

M is said to have *constant sectional curvature* at x, if $S(x; h, k)$ is independent of h and k; M is said to have *positive sectional curvature at* x, if $S(x; h, k) > 0$ for every independent pair h, k.

(i) Show that $S(x; h, k)$ depends only on the plane spanned by h and k.

(ii) If M has constant sectional curvature $S(x)$ at all points and $n \geqslant 3$, show that the function S is constant (Schur's Lemma).

(iii) Assume that $n = 4$ and that M has positive sectional curvature at x. Show that $K(x) > 0$. (*Hint*: Use problem 6.)

(iv) Let $\varphi\colon M \to \mathbb{R}^{n+1}$ be an immersion of an n-manifold ($n = 2m$). Give M the Riemannian metric induced by φ. Express the sectional curvature in terms of the two fundamental forms. Show that if n is even and M has positive sectional curvature at a point x, then $K(x) > 0$.

8. Homogeneous spaces. Let G be a compact connected Lie group and let K be a closed connected subgroup. Denote the Lie algebras by E and F.

(i) Equip E with a G-invariant inner product. Use the corresponding inner product in $F^\perp$ to obtain a G-invariant Riemannian metric in G/K. Show that the curvature of the corresponding Levi-Civita connection is G-invariant. Express the sectional curvature in terms of the Lie product.

(ii) Use the orthogonal projection $E \to F$ to obtain a principal connection in the bundle $(G, \pi, G/K, K)$. Use the isomorphism $\tau_{G/K} \cong G \times_K F^\perp$ to obtain a G-invariant Riemannian connection ∇ in $\tau_{G/K}$. Compare its curvature with the curvature obtained in (i). Compute the torsion of ∇.

(iii) Let $\Delta_{G/K}$ be the unique G-invariant $(n - r)$-form ($n = \dim G$, $r = \dim K$) on G/K such that $\int_{G/K} \Delta_{G/K} = 1$. Assume that $n - r = 2m$. Show that if $R^\bullet$ is the Riemannian curvature of any G-invariant Riemannian connection in $\tau_{G/K}$, then

$$\langle \Delta_{G/K}, R^{\bullet m} \rangle = (-1)^m m!(2\pi)^m \chi_{G/K} \Delta_{G/K}.$$

(iv) Let Δ be the volume form on G/K associated with the G-invariant Riemannian metric. Show that

$$\Delta = \mathrm{vol}(G/K)\, \Delta_{G/K}.$$

Hence obtain an explicit formula for $\mathrm{vol}(G/K)$ in terms of the roots of G, the roots of K, and the inner product in E. In particular, evaluate $\mathrm{vol}(G)$ and so obtain an explicit expression for the n-form Δ_G which satisfies $\int_G \Delta_G = 1$. (*Hint*: Consider a maximal torus in K, cf. problem 28, Chap. II.)

9. Establish the unproved statement in problem 15, (iv), Chap. IX.

10. Manifolds-with-boundary. (i) Let $(M, \partial M)$ be an oriented Riemannian $2m$-manifold-with-boundary and let $R^\bullet$ denote the Riemannian curvature. Let Y denote the normed outward-pointing normal vector field on ∂M. Set

$$\Omega = \sum_{k=0}^{m-1} a_k \langle \Delta_M, Y \wedge \nabla Y \wedge R^{\bullet m-k-1} \rangle,$$

where the a_k are the numbers defined in sec. 10.2. Prove that

$$\frac{(-1)^m}{(2\pi)^m m!} \left(\int_M \langle \Delta_M, R^{\bullet m} \rangle - \int_{\partial M} \Omega \right) = \chi_M .$$

Hint: Use problem 15, (vii), Chap. VIII, and problem 9, (iv), Chap. X, volume I.

(ii) If M is an oriented Riemannian 2-manifold with boundary ∂M, conclude that

$$\int_M K \Delta_M + \int_{\partial M} \gamma = 2\pi \chi_M ,$$

where K is the Gaussian curvature of M and γ is the geodesic curvature of ∂M.

11. (i) Let ξ and η be oriented vector bundles of rank $2m$ over a compact $2m$-manifold B. Assume that $\xi \oplus \epsilon$ is strongly isomorphic to $\eta \oplus \epsilon$. Show that $\int_B^\# (\mathrm{pf}(\xi) - \mathrm{pf}(\eta))$ is an even integer.

Hint: Prove that the restrictions of ξ and η to $B - \{a\}$ are isomorphic (where a is any point of B).

(ii) Conclude that if $(M, \partial M)$ is a compact oriented $(2m+1)$-manifold-with-boundary, then $\chi_{\partial M}$ is even.

12. Let ξ be an oriented Riemannian vector bundle of rank $2m$ which admits a normed cross-section σ. Let ϵ be the rank 1 bundle spanned by σ and let η be the bundle spanned by the orthogonal complement of ϵ.

(i) Show that a Riemannian connection ∇_ξ in ξ determines a Riemannian connection in η. Hence obtain a second Riemannian connection ∇ in ξ such that $\nabla\sigma = 0$.

(ii) Let $R_\xi^\bullet$ and $R^\bullet$ denote the Riemannian curvatures of ∇_ξ and ∇. Show that

$$\mathrm{Pf}(R^\bullet, \dots, R^\bullet) = 0.$$

Use a technique analogous to that of sec. 6.20 to construct a differential form such that

$$\mathrm{Pf}(R_\xi^\bullet, \dots, R_\xi^\bullet) - \mathrm{Pf}(R^\bullet, \dots, R^\bullet) = \delta\Phi.$$

Show that this provides an alternative proof of Proposition I, sec. 10.2.

13. Let $\xi = (M, \pi, B, F)$ be an oriented Riemannian vector bundle of rank $2m+1$ with associated sphere bundle $\xi_S = (M_S, \pi_S, B, S^{2m})$. Let $(V_S, \rho, M_S, \mathbb{R}^{2m})$ be the vertical subbundle of the tangent bundle of M_S.

(i) Show that the Euler class χ_{V_S} of V_S is the unique class in $H^{2m}(M_S)$ which satisfies

$$\oint_S^\# \chi_{V_S} = 2 \qquad \text{and} \qquad (\chi_{V_S})^2 \in \pi_S^\# H(B).$$

(ii) Let $((V_S)_S, \rho_S, M_S, S^{2m-1})$ be the associated sphere bundle of V_S. Regard $(V_S)_S$ as a bundle over B with fibre the Stiefel manifold

$V(2m+1, 2)$, and consider the corresponding exact cohomology triangle (cf. sec. 5.23). Prove that

$$D(1) = (-1)^m p_m(\xi),$$

where D is the connecting homomorphism.

14. Prove or disprove the following *Chern conjecture*: An even-dimensional Riemannian manifold with positive sectional curvature has positive Gaussian curvature.

Appendix A

Characteristic Coefficients and the Pfaffian

In this chapter all vector spaces are defined over a field Γ of characteristic zero. The notation conventions of sec. 0.5 are extended to vector spaces over Γ.

A.0. The algebra of homogeneous functions. Given a vector space F a function $f: F \to \Gamma$ is called *homogeneous of degree p* if

$$f(\lambda x) = \lambda^p f(x), \qquad x \in F, \quad \lambda \in \Gamma.$$

The functions homogeneous of degree p form a vector space, $\mathscr{H}^p(F)$. Multiplication of functions makes the direct sum,

$$\mathscr{H}(F) = \sum_{p=0}^{\infty} \mathscr{H}^p(F),$$

into a graded commutative algebra.

Consider the inclusion map $\alpha: F^* \to \mathscr{H}(F)$. Since $\mathscr{H}(F)$ is a commutative algebra, α extends to a homomorphism,

$$\alpha: \vee F^* \to \mathscr{H}(F),$$

of graded algebras. For simplicity, we usually denote $\alpha(\Psi)(x)$ by $\Psi(x)$.

On the other hand, a homomorphism of graded algebras $\beta: \otimes F^* \to \mathscr{H}(F)$ is given by

$$\beta(\Phi)(x) = \Phi(x, \dots, x), \qquad \Phi \in \otimes^p F^*, \quad x \in F.$$

Let $\pi_S: \otimes F^* \to \vee F^*$ be the projection (cf. sec. 6.17); then

$$(\alpha \circ \pi_S)(x^*) = x^* = \beta(x^*) \qquad x^* \in F^*.$$

It follows that (cf. sec. 6.18)

$$\alpha \circ \pi_S = \beta.$$

This shows that

$$\Psi(x) = \frac{1}{p!} \Psi(x, \dots, x), \qquad \Psi \in \vee^p F^*, \quad x \in F.$$

In particular, α is injective.

§1. Characteristic and trace coefficients

A.1. The characteristic algebra of a vector space. Let F be an n-dimensional vector space. Define bilinear maps,

$$\square : L_{\wedge^p F} \times L_{\wedge^q F} \to L_{\wedge^{p+q} F},$$

by setting

$$\begin{aligned}(\Phi \square \Psi)&(x_1 \wedge \cdots \wedge x_{p+q}) \\ &= \frac{1}{p!q!} \sum_{\sigma \in S^{p+q}} \epsilon_\sigma \Phi(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) \wedge \Psi(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}), \\ &\qquad\qquad \Phi \in L_{\wedge^p F}, \quad \Psi \in L_{\wedge^q F}, \quad x_i \in F.\end{aligned}$$

These bilinear maps make the space $\sum_{p=0}^{n} L_{\wedge^p F}$ into a graded algebra, $C(F)$. It is called the *characteristic algebra for F*.

On the other hand, make the direct sum $\Delta(F) = \sum_{p=0}^{n}(\wedge^p F^* \otimes \wedge^p F)$ into a commutative and associative algebra by setting

$$(u^* \otimes u) \cdot (v^* \otimes v) = (u^* \wedge v^*) \otimes (u \wedge v), \qquad u^*, v^* \in \wedge F^*, \quad u, v \in \wedge F.$$

Then the canonical linear isomorphisms $\wedge^p F^* \otimes \wedge^p F \xrightarrow{\cong} L_{\wedge^p F}$ define an algebra isomorphism

$$\Delta(F) \xrightarrow{\cong} C(F).$$

In particular, it follows that $C(F)$ is commutative and associative. Henceforth we shall identify the algebras $\Delta(F)$ and $C(F)$ under the isomorphism above.

The pth power of an element $\Phi \in C(F)$ will be denoted by $\Phi^{\boxed{p}}$,

$$\Phi^{\boxed{p}} = \underbrace{\Phi \square \cdots \square \Phi}_{(p \text{ factors})}.$$

In particular,

$$\varphi^{\boxed{p}} = p! \wedge^p \varphi, \qquad \varphi \in L_F.$$

More particularly, if ι denotes the identity map of F and ι_p denotes the identity map of $\wedge^p F$, this formula becomes

$$\iota^{\boxed{p}} = p! \iota_p.$$

It follows that

$$\iota_p \,\Box\, \iota_q = \frac{(p+q)!}{p!q!}\,\iota_{p+q}\,.$$

Next, recall the substitution operators $i(x)\colon \wedge F^* \to \wedge F^*$ and $i(x^*)\colon \wedge F \to \wedge F$ determined by vectors $x \in F$ and $x^* \in F^*$. They are the unique antiderivations that satisfy

$$i(x)y^* = \langle y^*, x\rangle \qquad \text{and} \qquad i(x^*)y = \langle x^*, y\rangle, \qquad y^* \in F^*, \quad y \in F.$$

An algebra homomorphism $i\colon \Delta(F) \to L_{\Delta(F)}$ is defined by

$$i(x^{*1} \wedge \cdots \wedge x^{*p} \otimes x_1 \wedge \cdots \wedge x_p) = i(x_p) \circ \cdots \circ i(x_1) \otimes i(x^{*p}) \circ \cdots \circ i(x^{*1}).$$

With the aid of the identification above we may regard i as a homomorphism

$$i\colon C(F) \to L_{C(F)}\,.$$

Finally, note that the spaces $L_{\wedge^p F}$ are self-dual with respect to the inner product given by

$$\langle \Phi, \Psi\rangle = \operatorname{tr}(\Phi \circ \Psi) = \langle \iota_p\,, \Phi \circ \Psi\rangle = i(\Phi)\Psi.$$

It satisfies

$$\langle u^* \otimes u, v^* \otimes v\rangle = \langle u^*, v\rangle\langle v^*, u\rangle, \qquad u^*, v^* \in \wedge^p F^*, \quad u, v \in \wedge^p F.$$

Moreover $i(\Phi)$ is dual to multiplication by Φ, $\Phi \in L_{\wedge^p F}$.

A.2. Characteristic coefficients. The pth *characteristic coefficient* for an n-dimensional vector space F is the element $C_p^F \in \vee^p L_F^*$ given by $C_0^F = 1$ and

$$C_p^F(\varphi_1, \ldots, \varphi_p) = \operatorname{tr}(\varphi_1 \,\Box \cdots \Box\, \varphi_p) = \langle \iota_p\,, \varphi_1 \,\Box \cdots \Box\, \varphi_p\rangle, \qquad p \geq 1, \ \varphi_i \in L_F\,.$$

Note that $C_p^F = 0$ if $p > n$. C_n^F will be denoted by Det^F.

The homogeneous functions, C_p^F, corresponding to C_p^F are given by

$$C_p^F(\varphi) = \operatorname{tr} \wedge^p \varphi, \qquad \varphi \in L_F$$

(cf. sec. A.0). We shall show that

$$\det(\varphi + \lambda\iota) = \sum_{p=0}^{n} C_p^F(\varphi)\,\lambda^{n-p}, \qquad \lambda \in \Gamma, \quad \varphi \in L_F\,. \tag{A.1}$$

In particular,

$$\det \varphi = \frac{1}{n!} \operatorname{Det}^F(\varphi, \dots, \varphi).$$

To prove formula (A.1) we argue as follows. Let $e_1, \dots, e_n$ be a basis of F. Then

$$\det(\varphi + \lambda\iota)\, e_1 \wedge \cdots \wedge e_n = (\varphi + \lambda\iota)\, e_1 \wedge \cdots \wedge (\varphi + \lambda\iota)\, e_n$$

$$= \sum_{p=0}^{n} \lambda^{n-p} \sum_{i_1 < \cdots < i_p} e_1 \wedge \cdots \wedge \varphi e_{i_1} \wedge \cdots \wedge \varphi e_{i_p} \wedge \cdots \wedge e_n$$

The elements $e_{i_1} \wedge \cdots \wedge e_{i_p}$ $(i_1 < \cdots < i_p)$ are a basis for $\wedge^p F$. Moreover, writing

$$\wedge^p \varphi(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j_1 < \cdots < j_p} \lambda_{i_1 \cdots i_p}^{j_1 \cdots j_p} e_{j_1} \wedge \cdots \wedge e_{j_p},$$

we see that $e_1 \wedge \cdots \wedge \varphi e_{i_1} \wedge \cdots \wedge \varphi e_{i_p} \wedge \cdots \wedge e_n = \lambda_{i_1 \cdots i_p}^{i_1 \cdots i_p} e_1 \wedge \cdots \wedge e_n$.
It follows that

$$\det(\varphi + \lambda\iota)\, e_1 \wedge \cdots \wedge e_n = \sum_{p=0}^{n} \lambda^{n-p} \sum_{i_1 < \cdots < i_p} \lambda_{i_1 \cdots i_p}^{i_1 \cdots i_p} e_1 \wedge \cdots \wedge e_n$$

$$= \left(\sum_{p=0}^{n} \operatorname{tr} \wedge^p \varphi \cdot \lambda^{n-p} \right) e_1 \wedge \cdots \wedge e_n.$$

Relation (A.1) is now established.

Relation (A.1) implies that $C_p^F \in (\vee^p L_F^*)_I$; i.e.,

$$C_p^F(\sigma \circ \varphi_1 \circ \sigma^{-1}, \dots, \sigma \circ \varphi_p \circ \sigma^{-1}) = C_p^F(\varphi_1, \dots, \varphi_p),$$

or, equivalently

$$C_p^F(\sigma \circ \varphi_1, \dots, \sigma \circ \varphi_p) = C_p^F(\varphi_1 \circ \sigma, \dots, \varphi_p \circ \sigma), \qquad \varphi_i \in L_F, \quad \sigma \in GL(F).$$

Setting $\sigma = \psi + \lambda\iota$ ($\psi \in L_F$, $-\lambda$ not an eigenvalue of ψ) and comparing the coefficients of λ^{p-1} we obtain

$$\sum_{i=1}^{p} C_p^F(\varphi_1, \dots, [\psi, \varphi_i], \dots, \varphi_p) = 0, \qquad \varphi_i, \psi \in L_F.$$

The nonhomogeneous element $C^F \in (\vee L_F^*)_I$, given by

$$C^F = \sum_{p=0}^{n} C_p^F,$$

is called the *characteristic element for F*.

Next, let H be a second finite-dimensional vector space. The inclusion map $j: L_F \oplus L_H \to L_{F\oplus H}$, given by $j(\varphi \oplus \psi) = \varphi \oplus \psi$, induces a homomorphism

$$j^\vee: \vee L_F^* \otimes \vee L_H^* \leftarrow \vee L_{F\oplus H}^* .$$

(Recall that multiplication induces a canonical isomorphism

$$\vee L_F^* \otimes \vee L_H^* \xrightarrow{\cong} \vee(L_F^* \oplus L_H^*) \;.)$$

Proposition I: The characteristic elements of F, H, and $F \oplus H$ are connected by the relation

$$j^\vee(C^{F\oplus H}) = C^F \otimes C^H.$$

Proof: The equation

$$\det(\varphi \oplus \psi + \lambda\iota_{F\oplus H}) = \det(\varphi + \lambda\iota) \cdot \det(\psi + \lambda\iota), \qquad \varphi \in L_F, \quad \psi \in L_H,$$

shows that

$$C_r^{F\oplus H}(\varphi \oplus \psi) = \sum_{p=0}^{r} C_p^F(\varphi)\, C_{r-p}^H(\psi).$$

Let α: $\vee(L_F^* \oplus L_H^*) \to \mathscr{H}(L_F \oplus L_H)$ be the homomorphism of sec. A.0.

The relation above yields

$$\alpha(j^\vee(C^{F\oplus H})) = \alpha(C^F \otimes C^H).$$

Since α is injective, the proposition follows.

Q.E.D.

A.3. Trace coefficients. Let F be a finite-dimensional vector space. The *trace coefficients* of F are the elements $\mathrm{Tr}_p^F \in \vee^p L_F^*$, given by

$$\mathrm{Tr}_0^F = \dim F$$

and

$$\mathrm{Tr}_p^F(\varphi_1, \dots, \varphi_p) = \sum_{\sigma\in S^p} \mathrm{tr}(\varphi_{\sigma(1)} \circ \cdots \circ \varphi_{\sigma(p)}), \qquad \varphi_\nu \in L_F, \quad p \geqslant 1.$$

Evidently,

$$\mathrm{Tr}_p^F(\sigma\circ\varphi_1\circ\sigma^{-1},\dots,\sigma\circ\varphi_p\circ\sigma^{-1}) = \mathrm{Tr}_p^F(\varphi_1,\dots,\varphi_p), \qquad \sigma\in GL(F),$$

and

$$\sum_{j=1}^{p}\mathrm{Tr}_p^F(\varphi_1,\dots,[\psi,\varphi_j],\dots,\varphi_p) = 0, \qquad \psi\in L_F.$$

Let H be a second finite-dimensional vector space. Consider the inclusion maps,

$$j: L_F\oplus L_H\to L_{F\oplus H} \qquad \text{and} \qquad i: L_F\oplus L_H\to L_{F\otimes H},$$

given by

$$j(\varphi\oplus\psi) = \varphi\oplus\psi \qquad \text{and} \qquad i(\varphi\oplus\psi) = \varphi\otimes\iota + \iota\otimes\psi.$$

A straightforward computation establishes (cf. sec. A.2)

Proposition II: The trace coefficients of F, H, $F\oplus H$, $F\otimes H$ are connected by the relations

$$j^\vee(\mathrm{Tr}_p^{F\oplus H}) = \mathrm{Tr}_p^F\otimes 1 + 1\otimes\mathrm{Tr}_p^H$$

and

$$i^\vee(\mathrm{Tr}_p^{F\otimes H}) = \sum_{i+j=p}\binom{p}{i}\mathrm{Tr}_i^F\otimes\mathrm{Tr}_j^H.$$

Next, consider the commutative algebra,

$$\vee^{**}L_F^* = \prod_{p=0}^{\infty}(\vee^p L_F^*),$$

whose elements are the infinite sequences

$$\Phi = (\Phi_0,\Phi_1,\dots,\Phi_p,\dots), \qquad \Phi_p\in\vee^p L_F^*.$$

Addition is defined componentwise, while the product is given by

$$(\Phi\cdot\Psi)_k = \sum_{i+j=p}\Phi_i\vee\Psi_j$$

(cf. sec. 6.21). Clearly $\vee^{**}L_F^*$ contains $\vee L_F^*$ as a subalgebra.

The *trace series* of F is the element $\mathrm{Tr}^F \in \vee^{**}L_F^*$, given by

$$\mathrm{Tr}^F = \left(\mathrm{Tr}_0^F, \dots, \frac{1}{p!}\mathrm{Tr}_p^F, \dots\right).$$

Proposition II implies that

$$j^\vee(\mathrm{Tr}^{F\oplus H}) = \mathrm{Tr}^F \otimes 1 + 1 \otimes \mathrm{Tr}^H$$

and

$$i^\vee(\mathrm{Tr}^{F\otimes H}) = \mathrm{Tr}^F \otimes \mathrm{Tr}^H.$$

Proposition III: The trace and characteristic coefficients of a finite-dimensional vector space F are related by

$$C_p^F = -\frac{1}{p}\sum_{j=0}^{p-1}(-1)^{p-j}C_j^F \vee \mathrm{Tr}_{p-j}^F, \qquad p \geqslant 1.$$

Lemma I: The operator, d, in $\vee L_F^*$ given by

$$(d\Phi)(\varphi_0, \dots, \varphi_p)$$
$$= \sum_{i<j}\Phi(\varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i, \varphi_0, \dots, \hat{\varphi}_i, \dots, \hat{\varphi}_j, \dots, \varphi_p), \quad \Phi \in \vee^p L_F^*, \; \varphi_i \in L_F,$$

is a derivation, homogeneous of degree 1. It satisfies

$$d\,\mathrm{Tr}_p^F = p\,\mathrm{Tr}_{p+1}^F$$

and

$$d\,C_p^F = -(p+1)C_{p+1}^F + C_p^F \vee \mathrm{Tr}_1^F, \qquad p \geqslant 0.$$

Proof: A simple calculation yields the formula

$$d(\Phi \vee \Psi)(\varphi, \dots, \varphi) = (d\Phi \vee \Psi + \Phi \vee d\Psi)(\varphi, \dots, \varphi), \qquad \varphi \in L_F.$$

This implies that d is a derivation. Clearly d is homogeneous of degree 1.

The first formula follows at once from the definition of d. To establish the second formula note that (cf. sec. A.1)

$$\begin{aligned}(p+1)\,C_{p+1}^F(\varphi_0, \dots, \varphi_p) &= \langle \iota \,\square\, \iota_p, \varphi_0 \,\square \cdots \square\, \varphi_p\rangle \\ &= \langle \iota_p, i(\iota)(\varphi_0 \,\square \cdots \square\, \varphi_p)\rangle\end{aligned}$$

and

$$i(\iota)(\varphi_0 \square \cdots \square \varphi_p)$$

$$= \sum_{j=0}^{p} \langle \iota, \varphi_j \rangle \varphi_0 \square \cdots \hat{\varphi}_j \cdots \square \varphi_p$$

$$- \sum_{j<k} (\varphi_j \circ \varphi_k + \varphi_k \circ \varphi_j) \square \varphi_1 \square \cdots \hat{\varphi}_j \cdots \square \hat{\varphi}_k \cdots \square \varphi_p .$$

Combining these relations yields the second formula.

Q.E.D.

Proof of the proposition: The proposition is trivial for $p = 1$. In the general case, it follows by induction via the formulae in the lemma and the derivation property of d.

Q.E.D.

Corollary I:

$$\sum_{j=0}^{p-1} (-1)^{p-j} C_j^F \vee \mathrm{Tr}_{p-j}^F = 0, \qquad p > n.$$

Proof: Apply the proposition and observe that $C_p^F = 0$, $p > n$.

Q.E.D.

Corollary II: The subalgebras of $\vee L_F^*$ generated respectively, by $C_0^F, \ldots, C_n^F$ and by $\mathrm{Tr}_0^F, \ldots, \mathrm{Tr}_n^F$, coincide and contain all the trace coefficients and characteristic coefficients.

Q.E.D.

§2. Inner product spaces

In this article F denotes an n-dimensional vector space and $\langle\ ,\ \rangle$ denotes an inner product in F. It induces a linear isomorphism $F \xrightarrow{\cong} F^*$ which we use to identify F with F^*. Further, $\langle\ ,\ \rangle$ extends to an inner product in each space $\wedge^p F$.

Sk_F denotes the Lie subalgebra of L_F consisting of the linear transformations which are skew with respect to $\langle\ ,\ \rangle$.

A.4. Multiplications in $\wedge F \otimes \wedge F$. In the vector space $\wedge F \otimes \wedge F$ we introduce *two* algebra structures: the first is the *canonical* tensor product of the algebras $\wedge F$ and $\wedge F$; the second is the *anticommutative* tensor product of $\wedge F$ and $\wedge F$.

The first algebra contains $\Delta(F)$ as a subalgebra (cf. sec. A.1) and so its multiplication is denoted by $\square$:

$$(u \otimes v) \square (u_1 \otimes v_1) = (u \wedge u_1) \otimes (v \wedge v_1).$$

The second algebra is canonically isomorphic to $\wedge(F \oplus F)$, and so multiplication is denoted by $\wedge$:

$$(u \otimes v) \wedge (u_1 \otimes v_1) = (-1)^{qr}(u \wedge u_1) \otimes v \wedge v_1\,, \qquad v \in \wedge^q F, \quad u_1 \in \wedge^r F.$$

The two products are connected by the relation

$$\Phi \square \Psi = (-1)^{qr}\Phi \wedge \Psi, \qquad \Phi \in \wedge F \otimes \wedge^q F, \quad \Psi \in \wedge^r F \otimes \wedge F.$$

This implies that

$$\varphi_1 \square \cdots \square \varphi_p = (-1)^{p(p-1)/2}\varphi_1 \wedge \cdots \wedge \varphi_p\,, \qquad \varphi_j \in F \otimes F. \qquad \text{(A.2)}$$

In particular,

$$\iota_p = \frac{1}{p!}\,\iota \square \cdots \square \iota = \frac{1}{p!}(-1)^{p(p-1)/2}\,\iota \wedge \cdots \wedge \iota,$$

where ι_p is regarded as an element of $\wedge^p F \otimes \wedge^p F$.

Now define an inner product in $F \oplus F$ by

$$\langle x \oplus y, x_1 \oplus y_1 \rangle = \langle x, y_1 \rangle + \langle y, x_1 \rangle.$$

(This is *not* the usual inner product!) Extend it to an inner product in $\wedge(F \oplus F)$. The induced inner product in $\wedge F \otimes \wedge F$ (via the standard algebra isomorphism $(\wedge F \otimes \wedge F, \wedge) \cong \wedge(F \oplus F)$ is given by

$$\langle \wedge^p F \otimes \wedge^q F, \wedge^r F \otimes \wedge^s F \rangle = 0, \qquad \text{unless} \quad p = s \quad \text{and} \quad q = r,$$

and

$$\langle a \otimes b, u \otimes v \rangle = (-1)^{pq} \langle a, v \rangle \langle b, u \rangle, \qquad a, v \in \wedge^p F, \quad b, u \in \wedge^q F.$$

Remark: Up to sign, this inner product agrees with the inner product in $C(F)$ defined in sec. A.1.

Next, identify $F \oplus F$ with $(F \oplus F)^*$ under the above inner product. Let $\tau: F \oplus F \to F \oplus F$ be the linear isomorphism given by

$$\tau(x, y) = (x + y, x - y), \qquad x, y \in F.$$

Its dual, τ^*, is given by

$$\tau^*(x, y) = (y - x, y + x), \qquad x, y \in F.$$

τ and τ^* extend to algebra automorphisms $\tau_\wedge$ and $\tau^\wedge$ of $(\wedge F \otimes \wedge F, \wedge)$ which are dual with respect to the inner product defined above.

Observe that

$$\tau_\wedge(x \wedge y \otimes 1) = (x \wedge y) \otimes 1 + x \otimes y - y \otimes x + 1 \otimes (x \wedge y),$$

$$\tau_\wedge(x \otimes y) = (x \wedge y) \otimes 1 - x \otimes y - y \otimes x - 1 \otimes (x \wedge y)$$

and

$$\tau_\wedge(1 \otimes x \wedge y) = (x \wedge y) \otimes 1 - x \otimes y + y \otimes x + 1 \otimes (x \wedge y).$$

Lemma II: τ has the following properties:

(1) $\tau_\wedge(x \otimes y - y \otimes x) = 2((x \wedge y) \otimes 1 - 1 \otimes (x \wedge y))$.

(2) $\tau^\wedge(\iota_p) = 2^p \iota_p$.

Proof: (1) is immediate from the formula above as is (2) in the case $p = 1$. To obtain (2) in general observe that

$$\begin{aligned}\tau^\wedge(\iota_p) &= (-1)^{p(p-1)/2} \frac{1}{p!} \tau^\wedge(\iota \wedge \cdots \wedge \iota) \\ &= (-1)^{p(p-1)/2} \frac{1}{p!} (\tau^\wedge \iota \wedge \cdots \wedge \tau^\wedge \iota) = 2^p \iota_p .\end{aligned}$$

Q.E.D.

A.5. Characteristic coefficients for F. Let $\beta\colon \wedge^2 F \xrightarrow{\cong} \mathrm{Sk}_F$ be the canonical isomorphism given by

$$\beta(x \wedge y)(z) = \langle x, z\rangle y - \langle y, z\rangle x.$$

Proposition IV: Let $\varphi \in \mathrm{Sk}_F$. Then the characteristic coefficients $C_p^F(\varphi)$ are given by

$$C_p^F(\varphi) = 0, \qquad p \text{ odd},$$

and

$$C_{2k}^F(\varphi) = \frac{1}{(k!)^2}\langle \underbrace{\beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)}_{(k\,\text{factors})}, \underbrace{\beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)}_{(k\,\text{factors})}\rangle.$$

Proof: Let $\varphi \in \mathrm{Sk}_F$. Then

$$\det(\varphi + \lambda\iota) = \det(\varphi^* + \lambda\iota) = \det(-\varphi + \lambda\iota).$$

It follows that $C_{2k+1}^F(\varphi) = -C_{2k+1}^F(\varphi)$, whence $C_{2k+1}^F(\varphi) = 0$.

To establish the second formula, regard the inclusion $j\colon \mathrm{Sk}_F \to L_F$ as a linear map from Sk_F into $F \otimes F$. Then

$$j\beta(x \wedge y) = x \otimes y - y \otimes x.$$

Thus Lemma II, (1) shows that

$$\tau_\wedge j(\varphi) = 2(\beta^{-1}(\varphi) \otimes 1 - 1 \otimes \beta^{-1}(\varphi)), \qquad \varphi \in \mathrm{Sk}_F.$$

Now let $\langle\ ,\ \rangle$ be the inner product in $\wedge F \otimes \wedge F$ defined in sec. A.4. Then, for $\varphi \in \mathrm{Sk}_F$ (cf. Lemma II and formula (A.2), sec. A.4),

$$\begin{aligned} C_{2k}^F(\varphi) &= \frac{1}{(2k)!}\langle \iota_{2k}, j(\varphi) \,\square \cdots \square\, j(\varphi)\rangle \\ &= \frac{(-1)^k}{(2k)!\,2^{2k}}\langle \tau^\wedge(\iota_{2k}), j(\varphi) \wedge \cdots \wedge j(\varphi)\rangle \\ &= \frac{1}{(k!)^2}\langle \iota_{2k}, \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi) \otimes \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)\rangle \\ &= \frac{1}{(k!)^2}\langle \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi), \beta^{-1}(\varphi) \wedge \cdots \wedge \beta^{-1}(\varphi)\rangle. \end{aligned}$$

Q.E.D.

Next, define elements $B_k \in \vee^{2k}\mathrm{Sk}_F^*$ by

$$B_k(\varphi_1, \dots, \varphi_{2k})$$
$$= \frac{1}{(k!)^2} \sum_{\sigma \in S^{2k}} \langle \beta^{-1}(\varphi_{\sigma(1)}) \wedge \cdots \wedge \beta^{-1}(\varphi_{\sigma(k)}), \beta^{-1}(\varphi_{\sigma(k+1)}) \wedge \cdots \wedge \beta^{-1}(\varphi_{\sigma(2k)})\rangle.$$

Then, as an immediate consequence of Proposition IV, we have

Proposition V: Let $j\colon \mathrm{Sk}_F \to L_F$ be the inclusion. Then

$$j^\vee(C^F_{2k+1}) = 0 \quad \text{and} \quad j^\vee(C^F_{2k}) = B_k.$$

A.6. Pfaffian. Suppose F has even dimension $n = 2m$ and let $a \in \wedge^n F$. Then the *Pfaffian of the pair* (F, a) is the element, $\mathrm{Pf}^F_a \in \vee^m \mathrm{Sk}_F^*$, given by

$$\mathrm{Pf}^F_a(\varphi_1, \dots, \varphi_m) = \langle a, \beta^{-1}(\varphi_1) \wedge \cdots \wedge \beta^{-1}(\varphi_m)\rangle, \qquad \varphi_\mu \in \mathrm{Sk}_F.$$

It determines the homogeneous function Pf^F_a given by

$$\mathrm{Pf}^F_a(\varphi) = \frac{1}{m!}\mathrm{Pf}^F_a(\varphi, \dots, \varphi), \qquad \varphi \in \mathrm{Sk}_F.$$

The scalar $\mathrm{Pf}^F_a(\varphi)$ is called the *Pfaffian* of φ with respect to a.

We extend the definition to odd-dimensional spaces by setting the Pfaffian equal to zero in this case.

Proposition VI: Let $a \in \wedge^n F$ and $b \in \wedge^n F$. Then

$$\mathrm{Pf}^F_a \vee \mathrm{Pf}^F_b = \langle a, b\rangle\, j^\vee(\mathrm{Det}),$$

where $j\colon \mathrm{Sk}_F \to L_F$ denotes the inclusion. In particular,

$$(\mathrm{Pf}^F_a(\varphi))^2 = \langle a, a\rangle \det \varphi, \qquad \varphi \in \mathrm{Sk}_F.$$

Proof: In fact,

$$(\mathrm{Pf}^F_a \vee \mathrm{Pf}^F_b)(\varphi_1, \dots, \varphi_{2m})$$
$$= \frac{1}{(m!)^2} \sum_\sigma \langle a, \beta^{-1}\varphi_{\sigma(1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(m)}\rangle\langle b, \beta^{-1}\varphi_{\sigma(m+1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(2m)}\rangle$$
$$= \frac{1}{(m!)^2} \langle a, b\rangle\langle \beta^{-1}\varphi_{\sigma(1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(m)}, \beta^{-1}\varphi_{\sigma(m+1)} \wedge \cdots \wedge \beta^{-1}\varphi_{\sigma(2m)}\rangle$$

(since $a \in \wedge^n F$ and $b \in \wedge^n F$). This shows that

$$\mathrm{Pf}_a^F \vee \mathrm{Pf}_b^F = \langle a, b\rangle\, B_m\,.$$

Now apply Proposition V, with $k = m$.

Q.E.D.

Next, let $\tau: F \to F$ be an isometry; i.e.,

$$\langle \tau x, \tau y\rangle = \langle x, y\rangle, \qquad x, y \in F.$$

Then $\det \tau = \pm 1$. If $\det \tau = 1$, τ is called *proper*.

Proposition VII: (1) If τ is an isometry of F, then

$$\mathrm{Pf}_a^F(\tau \circ \varphi_1 \circ \tau^{-1}, \dots, \tau \circ \varphi_m \circ \tau^{-1}) = \det \tau\, \mathrm{Pf}_a^F(\varphi_1, \dots, \varphi_m), \qquad \varphi_i \in \mathrm{Sk}_F\,.$$

(2) If $\psi \in \mathrm{Sk}_F$, then

$$\sum_{i=1}^m \mathrm{Pf}_a^F(\varphi_1, \dots, [\psi, \varphi_i], \dots, \varphi_m) = 0, \qquad \varphi_i \in \mathrm{Sk}_F\,.$$

Proof: In fact, since

$$\beta(\tau x \wedge \tau y) = \tau \circ \beta(x \wedge y) \circ \tau^{-1}, \qquad x, y \in F,$$

it follows that

$$\mathrm{Pf}_a^F(\tau \circ \varphi_1 \circ \tau^{-1}, \dots, \tau \circ \varphi_m \circ \tau^{-1}) = \det \tau\, \mathrm{Pf}_a^F(\varphi_1, \dots, \varphi_m),$$

which establishes (1).

Similarly, for $\psi \in \mathrm{Sk}_F$

$$\beta(\psi x \wedge y + x \wedge \psi y) = [\psi, \beta(x \wedge y)],$$

whence

$$\sum_{i=1}^m \mathrm{Pf}_a^F(\varphi_1, \dots, [\psi, \varphi_i], \dots, \varphi_m) = \mathrm{tr}\, \psi \cdot \mathrm{Pf}_a^F(\varphi_1, \dots, \varphi_m) = 0.$$

Q.E.D.

Let H be a second inner product space and give $F \oplus H$ the induced inner product; i.e.,

$$\langle x \oplus y, x_1 \oplus y_1\rangle = \langle x, x_1\rangle + \langle y, y_1\rangle.$$

The inclusion map $j\colon \mathrm{Sk}_F \oplus \mathrm{Sk}_H \to \mathrm{Sk}_{F\oplus H}$ induces a homomorphism

$$j^\vee\colon \vee\, \mathrm{Sk}_F^* \otimes \vee\, \mathrm{Sk}_H^* \leftarrow \vee\, \mathrm{Sk}_{F\oplus H}^* .$$

Moreover, multiplication defines a canonical algebra isomorphism,

$$\wedge F \otimes \wedge H \xrightarrow{\cong} \wedge(F \oplus H),$$

which preserves the inner products. We shall identify the algebras $\wedge F \otimes \wedge H$ and $\wedge(F \oplus H)$ under this isomorphism.

Proposition VIII: Let $a \in \wedge^n F$ and $b \in \wedge^r H$, where $n = \dim F$ and $r = \dim H$. Then, with the identification above,

$$j^\vee(\mathrm{Pf}_{a\otimes b}^{F\oplus H}) = \mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H .$$

Proof: If $n + r$ is odd both sides are zero. Now assume that $n + r = 2k$. Then we have, for $\varphi \in \mathrm{Sk}_F$ and $\psi \in \mathrm{Sk}_H$,

$$\begin{aligned}(j^\vee \mathrm{Pf}_{a\otimes b}^{F\oplus H})(\varphi \oplus \psi, \dots, \varphi \oplus \psi) &= \langle a \otimes b, (\varphi \otimes \iota + \iota \otimes \psi)^k\rangle \\ &= \sum_{i+j=k} \binom{k}{i} \langle a, \varphi^i\rangle\langle b, \psi^j\rangle.\end{aligned}$$

If n and r are odd, it follows that

$$j^\vee \mathrm{Pf}_{a\otimes b}^{F\oplus H} = 0 = \mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H .$$

If $n = 2m$ and $r = 2s$, we obtain

$$\begin{aligned}(j^\vee \mathrm{Pf}_{a\otimes b}^{F\oplus H})(\varphi \oplus \psi, \dots, \varphi \oplus \psi) &= \binom{k}{m} \mathrm{Pf}_a^F(\varphi)\, \mathrm{Pf}_b^H(\psi) \\ &= (\mathrm{Pf}_a^F \otimes \mathrm{Pf}_b^H)(\varphi \oplus \psi, \dots, \varphi \oplus \psi).\end{aligned}$$

Q.E.D.

Corollary: $\mathrm{Pf}_{a\otimes b}^{F\oplus H}(\varphi \oplus \psi) = \mathrm{Pf}_a^F(\varphi)\, \mathrm{Pf}_b^H(\psi), \qquad \varphi \in \mathrm{Sk}_F,\ \psi \in \mathrm{Sk}_H .$

A.7. Examples: 1. *Oriented inner product spaces*: Let F be a real inner product space of dimension $n = 2m$ (note that we do not require the inner product to be positive definite). Let $e \in \wedge^n F$ be the unique element which represents the orientation and satisfies $|\langle e, e\rangle| = 1$.

Then Pf_e^F is called the *Pfaffian of the oriented inner product space* F, and is denoted by Pf^F. Reversing the orientation changes the sign of the Pfaffian. Proposition VI implies that

$$\det \varphi = \langle e, e\rangle(\mathrm{Pf}^F\varphi)^2, \qquad \varphi \in \mathrm{Sk}_F .$$

Next let $F = F^+ \oplus F^-$ be an orthogonal decomposition of F such that the restriction of the inner product to F^+ (respectively, F^-) is positive (respectively, negative) definite. Define a positive definite inner product $(\,,\,)$ in F by setting

$$(x^+ + x^-, y^+ + y^-) = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle, \qquad x^+, y^+ \in F^+, \quad x^-, y^- \in F^-.$$

Let φ be a skew linear transformation of F that stabilizes F^+ and F^-,

$$\varphi = \varphi^+ \oplus \varphi^-, \qquad \varphi^+ : F^+ \to F^+, \qquad \varphi^- : F^- \to F^-.$$

Then φ is skew with respect to both of the inner products $\langle \,,\, \rangle$ and $(\,,\,)$ and so that Pfaffians $\mathrm{Pf}^F_{\langle,\rangle}(\varphi)$ and $\mathrm{Pf}^F_{(,)}(\varphi)$ are defined.

Proposition IX: Suppose φ satisfies the conditions above. Then:

(1) If $\dim F^-$ is odd,

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = 0, \qquad \mathrm{Pf}^F_{(,)}(\varphi) = 0.$$

(2) If $\dim F^- = 2q$. Then

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = (-1)^q\, \mathrm{Pf}^F_{(,)}(\varphi).$$

Proof: The corollary to Proposition VIII, sec. A.6, shows that, for suitable orientations of F^+ and F^-,

$$\mathrm{Pf}^F_{\langle,\rangle}(\varphi) = \mathrm{Pf}^{F^+}_{\langle,\rangle}(\varphi^+) \cdot \mathrm{Pf}^{F^-}_{\langle,\rangle}(\varphi^-)$$

and

$$\mathrm{Pf}^F_{(,)}(\varphi) = \mathrm{Pf}^{F^+}_{(,)}(\varphi^+) \cdot \mathrm{Pf}^{F^-}_{(,)}(\varphi^-).$$

Since $\langle \,,\, \rangle$ and $(\,,\,)$ coincide in F^+, it follows that

$$\mathrm{Pf}^{F^+}_{\langle,\rangle}(\varphi^+) = \mathrm{Pf}^{F^+}_{(,)}(\varphi^+)$$

We are thus reduced to the case that $\langle \,,\, \rangle$ is negative definite; i.e., $F = F^-$ and $\varphi = \varphi^-$.

In this case, $\langle \,,\, \rangle = -(\,,\,)$ and so the linear isomorphisms $\beta_{\langle,\rangle}$ and $\beta_{(,)}$ are related by

$$\beta_{\langle,\rangle} = -\beta_{(,)} .$$

If $\dim F$ is odd, then, by definition

$$\mathrm{Pf}^F_{\langle,\rangle} = \mathrm{Pf}^F_{(,)} = 0.$$

On the other hand, if $\dim F = 2q$, then

$$\begin{aligned}\mathrm{Pf}^F_{\langle\,,\,\rangle}(\varphi) &= \langle e, \beta^{-1}_{\langle\,,\,\rangle}(\varphi) \wedge \cdots \wedge \beta^{-1}_{\langle\,,\,\rangle}(\varphi)\rangle \\ &= (-1)^q \langle e, \beta^{-1}_{(\,,\,)}(\varphi) \wedge \cdots \wedge \beta^{-1}_{(\,,\,)}(\varphi)\rangle = (-1)^q \mathrm{Pf}^F_{(\,,\,)}(\varphi).\end{aligned}$$

Q.E.D.

2. *Oriented Euclidean spaces*: Let F be an oriented $2m$-dimensional Euclidean space. Fix $\varphi \in \mathrm{Sk}_F$ and choose a positive orthonormal basis $x_1, \ldots, x_{2m}$ of F so that

$$\varphi(x_{2i-1}) = \lambda_i x_{2i},$$

and

$$\varphi(x_{2i}) = -\lambda_i x_{2i-1}, \qquad i = 1, \ldots, m.$$

Then $\mathrm{Pf}^F(\varphi) = \lambda_1 \cdots \lambda_m$.

On the other hand, the characteristic coefficients of φ are given by

$$C_p^F(\varphi) = \sum_{i_1 < \cdots < i_p} \lambda_{i_1}^2 \cdots \lambda_{i_p}^2.$$

3. *Complex spaces*: Let F be an m-dimensional complex space with a Hermitian inner product. Orient the underlying real vector space $F_{\mathbb{R}}$ as described in Example 2, sec. 9.17, and define a positive definite inner product in $F_{\mathbb{R}}$ by

$$\langle\,,\,\rangle_{\mathbb{R}} = \mathrm{Re}\langle\,,\,\rangle.$$

Then a skew Hermitian linear transformation φ of F may be considered as a skew linear transformation $\varphi_{\mathbb{R}}$ of $F_{\mathbb{R}}$. We shall show that

$$i^m \,\mathrm{Pf}^{F_{\mathbb{R}}}(\varphi_{\mathbb{R}}) = \det \varphi, \qquad \varphi \in \mathrm{Sk}_F.$$

In fact, let $z_1, \ldots, z_m$ be an orthonormal basis of F and let $\lambda_\mu \in \mathbb{R}$ be scalars, such that

$$\varphi z_\mu = i\lambda_\mu z_\mu, \qquad \mu = 1, \ldots, m.$$

Then $\det \varphi = i^m \lambda_1 \cdots \lambda_m$.

On the other hand, the vectors $z_1, iz_1, \ldots, z_m, iz_m$ form a positive orthonormal basis of $F_{\mathbb{R}}$. Moreover,

$$\varphi_{\mathbb{R}}(z_\mu) = \lambda_\mu(iz_\mu) \qquad \text{and} \qquad \varphi_{\mathbb{R}}(iz_\mu) = -\lambda_\mu(z_\mu), \qquad \mu = 1, \ldots, m.$$

It follows that (cf. Example 2)

$$i^m \,\mathrm{Pf}^{F_{\mathbb{R}}}(\varphi_{\mathbb{R}}) = i^m \lambda_1 \cdots \lambda_m = \det \varphi.$$

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Notation Index

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