Lecture Notes
Geometric Graph Theory

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Chapter 1

Drawing planar graphs

A **graph** G consists of a finite set $V(G)$ of **vertices** (points) and a set $E(G)$ of **edges**, where every edge is a 2-element subset $\{u, v\} \subseteq V(G)$, $u \neq v$. For the sake of simplicity, an edge $\{u, v\}$ is often denoted by uv (or vu). Vertices u and v are called the **endpoints** of $uv \in E(G)$. If $uv \in E(G)$, then we say that u and v are **connected** (**joined** by an edge) in G, or that they are **adjacent**. A graph H is a **subgraph** of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a k-element set $V = \{v_1, v_2, \ldots, v_k\}$, the graphs P_k and C_k defined by

$$V(P_k) = V,\ E(P_k) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\};$$
$$V(C_k) = V,\ E(C_k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$$

are called a **path** of length $k-1$ and a **cycle** of length k respectively. Obviously $P_k \subseteq C_k$.

The most natural way of representing a graph in the plane is to assign distinct points to its vertices and connect two points by a simple curve if and only if the corresponding vertices are adjacent. A **simple curve** (also a **Jordan arc**) connecting two points $u, v \in \mathbb{R}^2$ is a continuous non-self-intersecting curve $\phi\colon [0, 1] \to \mathbb{R}^2$ with $\phi(0) = u$ and $\phi(1) = v$. If no confusion is likely to occur, we often talk about the points and curves in the representation as vertices and edges, respectively. The actual positions of the points and the curves play no role in this representation. However, we usually require that a **drawing** of a graph satisfies the following conditions:

(1) the edges pass through no vertices except their endpoints,

(2) every two edges have only a finite number of intersection points,

(3) every intersection point of two edges is either a common endpoint or a proper crossing ("touching" of the edges is not allowed), and

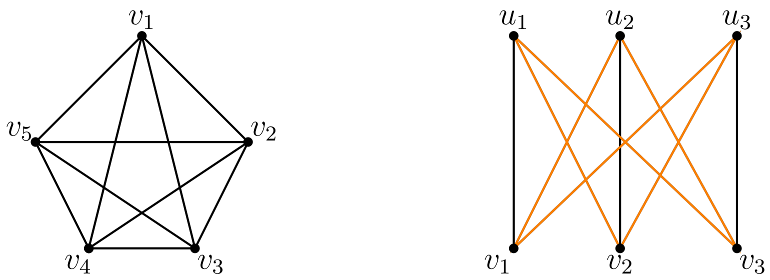


Figure 1.1: Non-planar graphs: K_5 (left), $K_{3,3}$(right)

(4) no three edges pass through the same crossing.

Some drawings of a graph are much simpler than some others, and usually we also want to produce a visually pleasing diagram. For instance we may require our arcs to be straight-line segments, we may wish to avoid or minimize crossings, maximize angles between edges etc.

Definition 1.1. A graph G that can be represented in the plane so that no two arcs meet at a point different from their endpoints is said to be **embeddable in the plane** or **planar**. A particular representation of a planar graph satisfying this property is called a **plane graph**.

Intuitively, it is easy to "see" that the graphs K_5 and $K_{3,3}$ depicted in Figure 1.1 are not planar. However, to prove this precisely, one needs at least a polygonal version of the following "intuitively obvious" fact. A **simple closed curve** (also a **Jordan curve**) is a continuous map $\varphi\colon [0,1] \to \mathbb{R}^2$ that is injective on $[0,1)$ and satisfies $\varphi(0) = \varphi(1)$.

Theorem 1.2 (Jordan curve theorem)**.** *The complement of a simple closed curve in the plane has exactly two connected components, one bounded and the other one unbounded.*

The closures of the two components of the complement of a simple closed curve φ are also called the **interior** and **exterior** of φ.

For the readers who are interested in the proof of the of the Jordan curve theorem, we recommend Thomassen's proof [64]. Quite surprisingly, his proof is based on the fact that $K_{3,3}$ is not planar.

Now suppose that $K_{3,3}$ can be embedded in the plane. Then the arcs u_1v_2, v_2u_3, u_3v_1, v_1u_2, u_2v_3, v_3u_1 would form a closed simple curve φ, and every arc u_iv_i $(i = 1, 2, 3)$ would be entirely in the interior of φ or in the exterior of φ. Assume without loss of generality that u_1v_1 and u_2v_2 lie in the interior of φ.

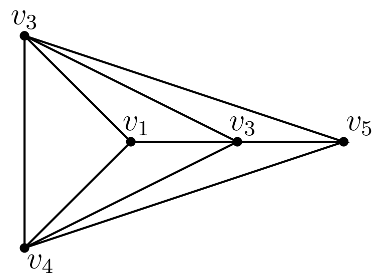


Figure 1.2: A straight-line embedding of $K_5 - v_1v_5$

Then they should cross each other, contradicting our assumption. (Notice that we used the Jordan curve theorem in two places in this argument.) Similarly, one can check that K_5 is not embeddable in the plane. In fact, a well-known theorem by Kuratowski (see Theorem 2.2) states that a graph is not planar if and only if it has a subgraph that can be obtained from K_5 or $K_{3,3}$ by replacing the edges with paths, all of whose interior vertices are distinct.

Deleting any edge (say v_1v_5) from K_5, we obtain a planar graph. Moreover, this new graph can be embedded in the plane by using only straight-line segments (see Figure 1.2).

Does every planar graph have such a representation? As we shall see, the answer to this question is in the affirmative. Moreover, we will be able to impose some further restrictions on our drawings to ensure that the resulting diagrams are relatively balanced. But first we need some preparations.

1.1 Euler's formula

Let G be a graph. The **degree** $d_G(v)$ (or simply $d(v)$) of a vertex $v \in V(G)$ is the number of vertices adjacent to v (recall that v is never adjacent to itself!). Denoting the number of vertices and edges of G by $v(G)$ and $e(G)$, respectively, we clearly have

$$\sum_{v \in V(G)} d(v) = 2e(G). \tag{1.1}$$

Consequently, G must have a vertex whose degree is at most $2e(G)/v(G)$.

A graph G is said to be **connected** if for any two vertices $v, v' \in V(G)$ there is a sequence of vertices $v_1 = v, v_2, v_3, \ldots, v_k = v'$ such that $v_iv_{i+1} \in E(G)$ for all i $(1 \leq i \leq k)$. In other words, G is connected if for any

$v, v' \in V(G)$ there is a path $P_k \subseteq G$ with $v, v' \in V(P_k)$. If G is connected and $v(G) \geq 2$, then $d(v) \geq 1$ for any $v \in V(G)$, that is, G has no isolated vertices. It is easy to see that any connected graph has at least $v(G) - 1$ edges, and equality holds if and only if G has no cycle as a subgraph.

The arcs of a plane graph partition the rest of the plane into a number of connected components, called **faces**. Exactly one of these faces is unbounded, which is called the **exterior face**. The number of faces of a plane graph G is denoted by $f(G)$.

Theorem 1.3 (Euler's formula)**.** *If G is a connected plane graph, then*

$$v(G) - e(G) + f(G) = 2.$$

Proof. By induction on f. If $f(G) = 1$, then G has no cycle, thus by the above remark $e(G) = v(G) - 1$ and the assertion is true. Assume that $f(G) = f \geq 2$, and we have already proved the theorem for all connected plane graphs having fewer than f faces. Obviously, G must contain a cycle. Delete any edge e that belongs to a cycle of G. For the resulting plane graph $G - e$, $f(G - e) = f(G) - 1$, so we can apply the induction hypothesis to obtain

$$v(G) - (e(G) - 1) + (f(G) - 1) = 2.$$

□

Let G be a plane graph. If an edge (arc) e of G belongs to the boundary of only one face of G, then e is called a **bridge**. Let $F(G)$ denote the set of faces of G. For any $f \in F(G)$, let $s(f)$ be the number of **sides** of f, that is, the number of edges belonging to the boundary of f, where all bridges are counted twice. Obviously,

$$\sum_{f \in F(G)} s(f) = 2e(G). \tag{1.2}$$

Corollary 1.4. *Let G be any plane graph with at least three vertices. Then*

(i) $e(G) \leq 3v(G) - 6$,

(ii) $f(G) \leq 2v(G) - 4$.

In both cases equality holds if and only if all faces of G have three sides.

Proof. It is sufficient to prove the statement for connected plane graphs. Clearly, $s(f) \geq 3$ for any face $f \in F(G)$. Then

$$3f(G) \leq \sum_{f \in F(G)} s(f) = 2e(G).$$

By Euler's formula, we obtain

$$v(G) - e(G) + \frac{2}{3}e(G) \geq 2,$$
$$v(G) - \frac{3}{2}f(G) + f(G) \geq 2,$$

as required. $\square$

If $s(f) = 3$ for some face $f \in F(G)$, then f is called a **triangle**. If all faces of G are triangles, then G is a **triangulation**. It is easy to show that any plane graph can be extended to a triangulation by the addition of edges (without introducing new vertices). Corollary 1.4 implies that, if G is a triangulation, then it is **maximal** in the sense that no further edges can be added to G without violating its planarity.

The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colors necessary to color the vertices of G so that no two vertices of the same color are adjacent. According to the **four-color theorem** of Appel and Haken, which settled a famous conjecture of Guthrie posed in the last century, the chromatic number of any planar graph is at most 4. (The graph in Figure 1.2 has chromatic number 4, showing that this bound cannot be improved.) The proof of Appel and Haken is quite complicated, and uses lengthy calculations by computers. However, a weaker statement can easily be deduced from Corollary 1.4.

Corollary 1.5. *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. By induction on $v(G)$. If $v(G) \leq 5$, then the statement is true, because one can assign a different color to each vertex of G. Assume that $v(G) = v \geq 6$, and that we have already established the result for all planar graphs with fewer than v vertices.

It follows from Equation (1.1) and Corollary 1.4(i) that G has a vertex u with $d(u) \leq 5$. If $d(u) \leq 4$, then we apply the induction hypothesis to the graph $G - u$ obtained from G by the removal of u (and all edges incident to u). We get that the vertices of $G - u$ can be colored by 5 colors so that no two vertices of the same color are adjacent. Clearly, we can assign a color to u, different from the (at most 4) colors used for its neighbors.

Suppose next that u is adjacent to 5 vertices u_i $(1 \leq i \leq 5)$. Since G is planar, it cannot contain K_5 as a subgraph. Thus, we can assume that, say u_1 and u_2 are not adjacent. Let G' denote the graph obtained from $G - u$ by merging u_1 and u_2. That is, $V(G') = (V(G) \setminus \{u, u_1, u_2\}) \cup \{u'\}$ and $E(G')$ consists of all edges of G, whose both endpoints belong to $V(G) \setminus \{u, u_1, u_2\}$, plus those pairs wu', for which $w \in V(G) \setminus \{u, u_1, u_2\}$ and either wu_1 or wu_2

belongs to $E(G)$. It is easy to see that G' is a planar graph, hence we can apply the induction hypothesis to obtain a proper coloring of the vertices of G' by 5 colors. If we assign the color of u' to both u_1 and u_2, then we obtain a proper coloring of $G - u$ such that the vertices u_i $(1 \le i \le 5)$ have at most 4 different colors. Therefore, we can again color u differently from its neighbors. $\square$

1.2 Straight-line drawing

In this section we are going to show that every planar graph G can be embedded in the plane so that the arcs representing the edges of G are straight-line segments that can meet only at their endpoints. An embedding with this property is called a **straight-line embedding** of G. The existence of such an embedding was discovered independently by Fáry, Tutte and Wagner, but it also follows from an ancient theorem of Steinitz.

The proof presented here is based on a simple canonical way of constructing a plane graph, which will allow us to use an inductive argument for finding a proper position of the vertices one by one.

We need the following observation.

Lemma 1.6. *Let G be a plane graph, whose exterior face is bounded by a cycle $u_1, u_2, \ldots, u_k$. Then there is a vertex u_i $(i \neq 1, k)$ not adjacent to any u_j other than u_{i-1} and u_{i+1}.*

Proof. If there are no two non-consecutive vertices along the boundary of the exterior face that are adjacent, then there is nothing to prove. Otherwise, pick an edge $u_i u_j \in E(G)$, for which $j > i+1$ and $j-i$ is minimal. Then u_{i+1} cannot be adjacent to any element of $\{u_1, \ldots, u_{i-1}, u_{j+1}, \ldots, u_k\}$ by planarity, nor can it be adjacent to any other vertex of the exterior face different from u_i and u_{i+2}, by minimality of $j-i$. $\square$

Let G be a graph (or a plane graph), and let $U \subseteq V(G)$. The subgraph of G **induced** by U is a graph (a plane graph) whose vertex set is U and whose edge set consists of all edges of $E(G)$ such that both of their endpoints belong to U.

Now we are in the position to establish the following useful theorem.

Theorem 1.7 (Canonical Construction of Triangulations)**.** *Let G be a triangulation on n vertices, with exterior face uvw. Then there is a labeling of the vertices $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ satisfying the following conditions for every $k \in \{4, \ldots, n\}$:*

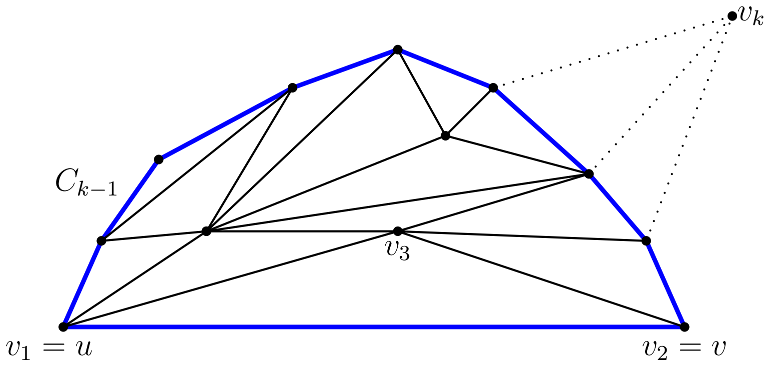


Figure 1.3: G_{k-1} and v_k in the exterior

(i) *the boundary of the exterior face of the subgraph G_{k-1} of G induced by $\{v_1, v_2, \dots, v_{k-1}\}$ is a cycle C_{k-1} containing the edge uv;*

(ii) *v_k is in the exterior face of G_{k-1}, and its neighbors in $V(G_{k-1})$ are some (at least two) consecutive elements along the path obtained from C_{k-1} by removal of the edge uv. (See Figure 1.3)*

Proof. The vertices $v_n, v_{n-1}, \dots, v_3$ will be defined by reverse induction. Set $v_n = w$, and let G_{n-1} be the graph obtained from G by the deletion of v_n. Since G is a triangulation, the neighbors of w form a cycle C_{n-1} containing uv, and this cycle is the boundary of the exterior face of G_{n-1}.

Let $k \in \{4, \dots, n\}$ be fixed and assume that $v_n, v_{n-1}, \dots, v_k$ have already been determined so that the subgraph G_{k-1} induced by $V(G) \setminus \{v_k, v_{k+1}, \dots, v_n\}$ satisfies conditions (i) and (ii). Let C_{k-1} denote the boundary of the exterior face of G_{k-1}. Applying Lemma 1.6 to G_{k-1}, we obtain that there is a vertex u' on C_{k-1}, different from u and v, which is adjacent only to two other vertices of C_{k-1} (that is, to its immediate neighbors). Letting $v_{k-1} = u'$, the subgraph $G_{k-2} \subseteq G$ induced by $V(G) \setminus \{v_{k-1}, v_k, \dots, v_n\}$ obviously meets the requirements. □

Using this theorem, we can easily prove the main result of this section.

Corollary 1.8. *Every planar graph has a straight-line embedding in the plane.*

Proof. It is sufficient to show that the statement is true for any **maximal** planar graph, that is, for any graph that can be represented by a triangulation (see Exercise 1.4 and the remark after Corollary 1.4).

Let G be any triangulation with the canonical labeling $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$, described above. We will determine the positions $f(v_k) = (x(v_k), y(v_k))$ of the vertices by induction on k.

Set $f(v_1) = (0,0)$, $f(v_2) = (2,0)$, $f(v_3) = (1,1)$. Assume that $f(v_1), f(v_2), \ldots, f(v_{k-1})$ have already been defined for some $k \geq 4$ so that by connecting the images of the adjacent vertex pairs by segments, we obtain a straight-line embedding of G_{k-1} whose exterior face is bounded by the segments corresponding to the edges of C_{k-1}. Suppose further that

$$\begin{aligned} &x(u_1) < x(u_2) < \ldots < x(u_m), \\ &y(u_i) > 0 \qquad \text{for } 1 < i < m, \end{aligned} \tag{1.3}$$

where $u_1 = u, u_2, u_3, \ldots, u_m = v$ denote the vertices of C_{k-1} listed in cyclic order. By condition (ii) of Theorem 1.7, v_k is connected to $u_p, u_{p+1}, \ldots, u_q$ for some $1 \leq p \leq q \leq m$. Let $x(v_k)$ be any number strictly between $x(u_p)$ and $x(u_q)$. If we choose $y(v_k) > 0$ to be sufficiently large and connect $f(v_k) = (x(v_k), y(v_k))$ to $f(u_p), f(u_{p+1}), \ldots, f(u_q)$ by segments, then we obtain a straight-line embedding of G_k meeting all the requirements (including the auxiliary conditions (1.3) for the vertices of C_k). □

Note that by the same method we can also establish the existence of straight-line embeddings with some special geometric properties. For example, we can require that the segments corresponding to the edges of C_{k-1} form a convex polygon for every $k \geq 4$.

Corollary 1.9. *Let G be a planar graph with n vertices and $3n-6$ edges. Then there are a labeling of the vertices $v_1, v_2, \ldots, v_n$ and a straight-line embedding of G such that for every $4 \leq k \leq n$,*

- (i) *the image of the subgraph of G induced by $\{v_1, v_2, \ldots, v_{k-1}\}$ is a triangulated convex polygon C_{k-1}, and*
- (ii) *the image of v_k lies in the exterior of C_{k-1}.*

The same technique can be used to obtain a different kind of representation of planar graphs, found by Rosenstiehl and Tarjan.

Corollary 1.10. *The vertices and the edges of any planar graph can be represented by horizontal and vertical segments, respectively, such that*

- (i) *no two segments have an interior point in common,*
- (ii) *two horizontal segments are connected by a vertical segment if and only if the corresponding vertices are adjacent.*

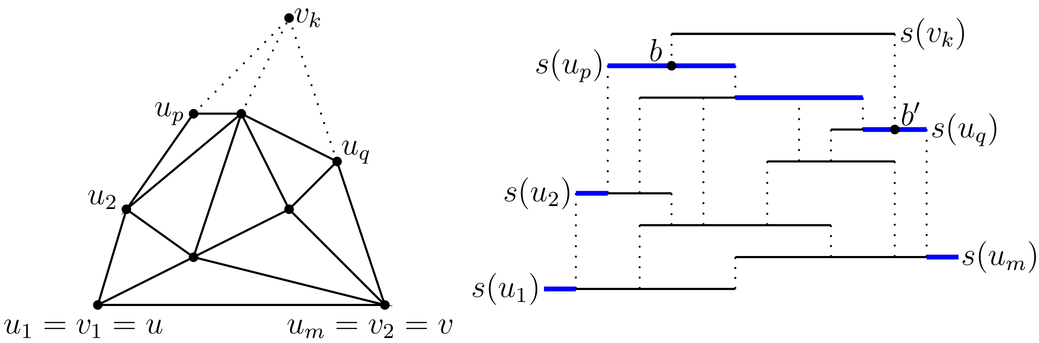


Figure 1.4: Illustration to Corollary 1.10. The thick segments are the pieces of the upper envelope of $s(v_1), s(v_2), \dots, s(v_{k-1})$.

Proof. As in the proof of Corollary 1.8, it is sufficient to establish the statement for triangulations.

Let G be any triangulation with canonical labeling $v_1 = u, v_2 = v, v_3, \dots, v_n$. To every v_k we shall assign a horizontal segment $s(v_k)$ whose endpoints are (x_k, k) and (x'_k, k). Set $x_1 = 0, x'_1 = 2, x_2 = 2, x'_2 = 4, x_3 = 1, x'_3 = 3$. Assume that $s(v_1), s(v_2), \dots, s(v_{k-1})$ have already been determined for some $k \geq 4$ so that the segments corresponding to adjacent vertex pairs can be connected by vertical segments, that is, the subgraph $G_{k-1} \subseteq G$ induced by $\{v_1, v_2, \dots, v_{k-1}\}$ has a representation satisfying conditions (i) and (ii). Let $u_1 = u, u_2, \dots, u_m = v$ denote the vertices of the exterior face of G_{k-1}, listed in cyclic order. Suppose further that the upper envelope of the segments $s(v_1), s(v_2), \dots, s(v_{k-1})$ consists of some portion of $s(u_1), s(u_2), \dots, s(u_m)$, in this order. (A point $b \in s(v_i)$ belongs to the **upper envelope** of $s(v_1)$, $s(v_2), \dots, s(v_{k-1})$, if the vertical ray starting from b and pointing upwards does not intersect any other segment $s(v_j)$, $1 \leq j \leq k-1$).

By condition (ii) of Theorem 1.7, v_k is connected to $u_p, u_{p+1}, \dots, u_q$ for some $1 \leq p \leq q \leq m$. Let b and b' be any interior points of those portions of $s(u_p)$ and $s(u_q)$, respectively, that belong to the upper envelope of $s(v_1)$, $s(v_2), \dots, s(v_{k-1})$. Letting x_k and x'_k be equal to the x-coordinates of b and b', respectively, and drawing the vertical segments connecting $s(v_k)$ with $s(v_p), \dots, s(v_q)$, we obtain a representation of G_k with the required properties (see Figure 1.4). $\square$

1.3 Drawing a planar graph on a grid

In the previous section, we have seen that any planar graph has a straight-line embedding (Corollary 1.8). However, the solution has a serious drawback: as we embed the vertices recursively in the plane, we may be forced to map a new vertex far away from all previous points, so that the size of the picture may increase exponentially with the number of vertices. To put it differently, if we want to view the resulting drawing on a computer screen, then many points will bunch together and become indistinguishable. To handle this problem, in this section we shall restrict our attention to straight-line drawings where each point is mapped into a **grid point**, that is, a point with integer coordinates. Our goal is to minimize the size of the grid needed for the embedding of any planar graph of n vertices. The set of all grid points (x, y) with $0 \leq x \leq m$, $0 \leq y \leq n$ is said to be an **m by n grid**.

Theorem 1.11. *Any planar graph with n vertices has a straight-line embedding in the $2n-4$ by $n-2$ grid.*

Proof. It suffices to prove the theorem for triangulations. Let G be a triangulation with exterior face uvw, and let $v_1 = u, v_2 = v, v_3, \dots, v_n = w$ be a canonical labeling of the vertices (see Theorem 1.7).

We are going to show by induction on k that G_k, the subgraph of G induced by $\{v_1, v_2, \dots, v_k\}$, can be straight-line embedded on the $2k-4$ by $k-2$ grid, for every $k \geq 3$. Let f_3 be the following embedding of G_3:

$$f_3(v_1) = (0, 0), f_3(v_2) = (2, 0), f_3(v_3) = (1, 1).$$

Suppose now that for some $k \geq 4$ we have already found an embedding $f_{k-1}(v_i) = (x_{k-1}(v_i), y_{k-1}(v_i))$, $1 \leq i \leq k-1$, with the following properties:

(*a*) $f_{k-1}(v_1) = (0, 0)$, $f_{k-1}(v_2) = (2k-6, 0)$;

(*b*) If $u_1 = u, u_2, \dots, u_m = v$ denote the vertices of the exterior face of G_{k-1} in cyclic order, then

$$x_{k-1}(u_1) < x_{k-1}(u_2) < \ldots < x_{k-1}(u_m);$$

(*c*) The segments $f_{k-1}(u_i)f_{k-1}(u_{i+1})$, $1 \leq i < m$, all have slope $+1$ or -1.

Note that (*c*) implies that the Manhattan (or Iowa) distance $|x_{k-1}(u_j) - x_{k-1}(u_i)| + |y_{k-1}(u_j) - y_{k-1}(u_i)|$ between the images of any two vertices u_i and u_j on the exterior face of G_{k-1} is even. Consequently, if we take a line with slope $+1$ through u_i and a line with slope -1 through u_j, then they always intersect at a grid point $P(u_i, u_j)$.

Let $u_p, u_{p+1}, \ldots, u_q$ be the neighbors of v_k in G_k $(1 \le p < q \le m)$. Clearly, $P(u_p, u_q)$ is a good candidate for $f_k(v_k)$, except that we may not be able to connect it to e.g. $f_{k-1}(u_p)$ by a segment avoiding $f_{k-1}(u_{p+1})$. To resolve this problem, we have to modify f_{k-1} before embedding v_k. We shall move the images of $u_{p+1}, u_{p+2}, \ldots, u_m$ one unit to the right, and then move the images of $u_q, u_{q+1}, \ldots, u_m$ to the right by an additional unit. That is, let

$$x_k(u_i) = \begin{cases} x_{k-1}(u_i), & \text{for } 1 \le i \le p, \\ x_{k-1}(u_i) + 1, & \text{for } p < i < q, \\ x_{k-1}(u_i) + 2, & \text{for } q \le i \le m, \end{cases}$$

$$\begin{aligned} y_k(u_i) &= y_{k-1}(u_i), && \text{for } 1 \le i \le m, \\ f_k(u_i) &= (x_k(u_i), y_k(u_i)), && \text{for } 1 \le i \le m, \end{aligned}$$

and let $f_k(v_k)$ be the point of intersection of the lines of slope $+1$ and -1 through $f_k(u_p)$ and $f_k(u_q)$, respectively. Of course, $f_k(v_k)$ is a grid point that can be connected by disjoint segments to the points $f_k(u_i)$, $p \le i \le q$, without intersecting the polygon $f_k(u_1)f_k(u_2)\ldots f_k(u_m)$. However, as we move the image of some u_i, it may be necessary to move some other points (not on the exterior face) as well, otherwise we may create crossing edges.

In order to tell exactly which set of points has to move together with the image of a given exterior vertex u_i, we define recursively a total order '$\prec$' on $\{v_1, v_2, \ldots, v_n\}$. Originally, let $v_1 \prec v_3 \prec v_2$. If the order has already been defined on $\{v_1, v_2, \ldots, v_{k-1}\}$, then insert v_k just before u_{p+1}. According to this rule, obviously

$$u_1 \prec u_2 \prec \cdots \prec u_m .$$

Now we can extend the definition of f_k to the interior vertices of G_{k-1}, as follows. For any $1 \le i \le k-1$, let

$$x_k(v_i) = \begin{cases} x_{k-1}(v_i), & \text{if } v_i \prec u_{p+1}, \\ x_{k-1}(v_i) + 1, & \text{if } u_{p+1} \preceq v_i \prec u_q, \\ x_{k-1}(v_i) + 2, & \text{if } u_q \preceq v_i, \end{cases}$$

$$\begin{aligned} y_k(v_i) &= y_{k-1}(v_i), \\ f_k(v_i) &= (x_k(v_i), y_k(v_i)). \end{aligned}$$

Evidently, f_k satisfies conditions (a), (b) and (c).

To complete the proof, it remains to verify that f_k is a straight-line embedding, that is, no two segments cross each other. A slightly stronger statement follows by straightforward induction.

Claim 1.12. *Let $f_{k-1} = (x_{k-1}, y_{k-1})$ be the straight-line embedding of G_{k-1}, defined above, and let $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$. For any $1 \leq i \leq k-1$, $1 \leq j \leq m$, let*

$$x(v_i) = x_{k-1}(v_i) + \alpha_1 + \alpha_2 + \cdots + \alpha_j \text{ if } u_j \preceq v_i \prec u_{j+1}\,,$$

$$y(v_i) = y_{k-1}(v_i)\,.$$

Then $f'_{k-1} = (x, y)$ is also a straight-line embedding of G_{k-1}.

The claim is trivial for $k = 4$. Assume that it has already been confirmed for some $k \geq 4$, and we want to prove the same statement for G_k. The vertices of the exterior face of G_k are $u_1, \dots, u_p, v_k, u_q, \dots, u_m$. Fix now any nonnegative numbers $\alpha(u_1), \dots, \alpha(u_p), \alpha(v_k), \alpha(u_q), \dots, \alpha(u_m)$. Applying the induction hypothesis to G_{k-1} with $\alpha_1 = \alpha(u_1), \dots, \alpha_p = \alpha(u_p), \alpha_{p+1} = \alpha(v_k)$, $\alpha_{p+2} = \cdots = \alpha_{q-1} = 0, \alpha_q = \alpha(u_q), \alpha_{q+1} = \alpha(u_{q+1}), \dots, \alpha_m = \alpha(u_m)$, we obtain that the restriction of f'_k to G_{k-1} is a straight-line embedding. To see that the edges of G_k incident to v_k do not create any crossing, it is enough to notice that f_k and f'_k map $\{v_k, u_{p+1}, \dots, u_{q-1}\}$ to two translations of the same set. $\square$

Chapter 2

Characterization of planar graphs

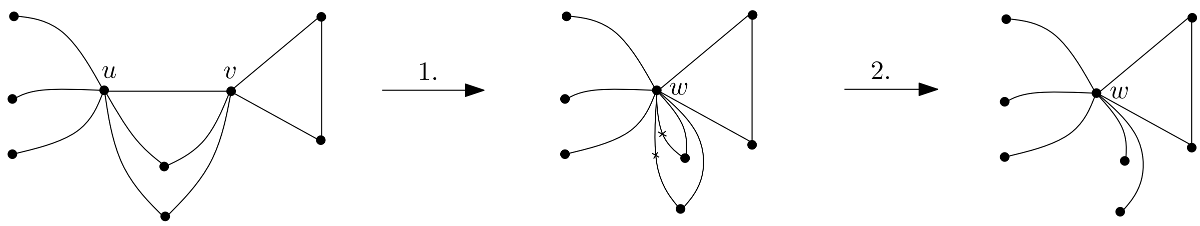
In this chapter we investigate various equivalent conditions for graphs to be planar. Then in the last section we briefly visit the third dimension.

Definition 2.1. Take a graph G and put additional vertices arbitrarily on the edges of G (but not on their crossings). This divides the original edges of G into smaller ones. Alternatively (and more precisely), we may say that we replace the edges of G by paths of length at least 1 whose internal vertices are disjoint. The resulting graph is called a **subdivision** of G.

Theorem 2.2 (Kuratowski, 1930)**.** *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Definition 2.3. A graph G contains H as a **minor** if H can be obtained from G by deleting edges and vertices and by *contracting* edges.
Contracting an edge uv consists of

1. removing the edge uv and identifying the vertices u and v, and then
2. removing all parallel edges.



Theorem 2.4 (Wagner, 1937)**.** *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

In the literature, the following terminology is also used: G contains H as a **topological minor** if G contains a subdivision of H.

It is an easy exercise to show that if G contains a subdivision of H, then G contains H as a minor. Consequently, Kuratowski's theorem implies Wagner's theorem. The other implication, that Wagner's theorem implies Kuratowski's theorem, is also not hard to show, without knowing the proof of either of them. There is just a small catch: containing $K_{3,3}$ as a minor implies containing a subdivision of $K_{3,3}$, but containing K_5 as a minor implies containing a subdivision of K_5 or a subdivision of $K_{3,3}$.

2.1 Hanani–Tutte theorems

Hanani–Tutte theorems characterize planar graphs in terms of the parity of the numbers of crossings between their edges. We start with a very basic variant and then show two slightly stronger versions.

Theorem 2.5 (A "very weak" Hanani–Tutte theorem)**.** *A graph G is planar if and only if G can be drawn in the plane so that every two edges cross an even number of times.*

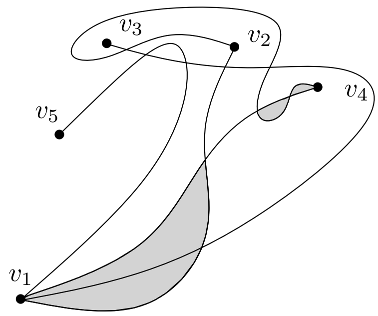
A drawing where every two edges cross an even number of times is also called an **even drawing**.

Sketch of the proof. "⇒" This direction is trivial: if G is planar then it can by definition be drawn without crossings, that is, each pair of edges cross zero times and zero is an even number.

"⇒" This direction can be proved in an easy way using Kuratowski's theorem. Namely, we only need to show that no subdivision of K_5 and $K_{3,3}$ has an even drawing.

As an example we show that K_5 has no even drawing. Suppose for contradiction that there exists an even drawing of K_5. Take a vertex v_1, and let the edges $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ leave v_1 in this clockwise order in a small neighborhood of v_1. Of course, outside this neighborhood these edges may cross one another. Consider the image of the triangle $v_1v_2v_4$. It is a closed, possibly self-intersecting curve γ. It divides the plane into several regions. It is a simple exercise to show that these regions can be colored by two colors (say, black and white) so that no two regions whose boundaries share an arc get the same color. Notice that the initial portions of the edges v_1v_3 and

v_1v_5 around v_1 belong to regions of opposite colors. Assume that the initial portion of v_1v_3 in a small neighborhood of v_1 runs in a black region.



According to our assumptions, the edge v_1v_3 must cross each of the edges v_1v_2, v_2v_4, v_4v_1 an even number of times. Therefore, the curve v_1v_3 crosses γ an even number of times, and after each crossing it switches colors. This yields that v_3 must lie in a black region. Analogously, since the initial portion of the edge v_1v_5 runs in a white region, we can conclude that v_5 lies in a white region. Since v_3 and v_5 lie in regions of opposite colors, the edge v_3v_5 crosses gamma an odd number of times, contradicting our assumption that v_3v_5 crosses every edge an even number of times. □

Hanani [32] and Tutte [68] originally proved the following stronger version of the above theorem.

Definition 2.6. Two edges $\{a, b\}$ and $\{c, d\}$ are **independent** (also **non-adjacent**) if $\{a, b\} \cap \{c, d\} = \emptyset$; that is, they do not share any vertex.

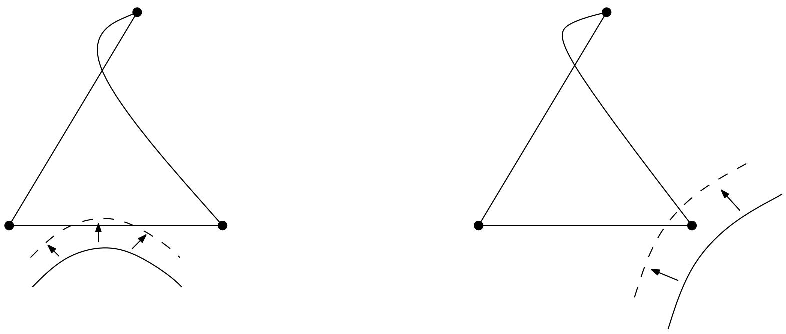
Theorem 2.7 (The strong Hanani–Tutte theorem, 1934 [32], 1970 [68])**.** *A graph G is planar if and only if G can be drawn in the plane so that any two independent edges cross an even number of times.*

A drawing where every two independent edges cross an even number of times is also called an **independently even drawing**.

Sketch of the proof. For the first direction, the same argument applies as before. For the second direction we again use Kuratowski's theorem and show as an example that K_5 has no independently even drawing. It is an easy exercise to show that this also implies that no subdivision of K_5 has an even independently even drawing.

Take an arbitrary drawing of K_5 in the plane; for example, the usual straight-line drawing with vertices on a circle, which has exactly five crossings of independent edges. We use as a fact that every drawing of K_5 in the plane can be obtained from any other by a continuous deformation of the plane and a sequence of continuous deformations of the individual edges. A proper proof of this fact would need the Jordan–Schönflies theorem [64].

We are going to prove that the parity of the total number N of crossings of all independent pairs of edges does not change during any continuous deformation of the edges. To see this, take an edge $e = v_4v_5$ of K_5 and slightly deform it. We only have to check how the intersection between this edge and the edges of the triangle $T = v_1v_2v_3$ changes. As we pull e through an edge or over a vertex vertex of T, the total number of crossings between e and T changes by two. The possible two cases are illustrated in the following figure:



Similar arguments apply for $K_{3,3}$ in place of K_5. □

An elementary proof of the strong Hanani–Tutte theorem, which does not use Kuratowski's theorem, was given by Pelsmajer, Schaefer and Štefankovič [56].

The weak Hanani–Tutte theorem was discovered later than the strong variant, by several different authors [12, 55, 56]. It does not directly from the strong variant, as the name would suggest, because it offers an additional conclusion.

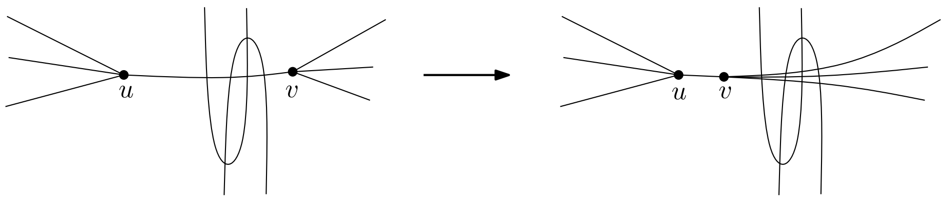
Definition 2.8. The **rotation** of a vertex v in a drawing of a graph is the clockwise cyclic order in which the edges incident to v leave the vertex v in the drawing in a small neighborhood of v. The collection of the rotations of all vertices in a drawing D is called the **rotation system** of D.

Theorem 2.9 (The weak Hanani–Tutte theorem, 2000+ [12, 55, 56])**.** *If D is a drawing of G where every two edges cross an even number of times, then G has a plane drawing with the same rotation system as D.*

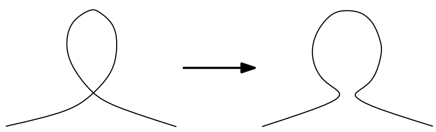
We show two different elementary proofs of Theorem 2.9, which do not need Kuratowski's theorem or any advanced topology.

Proof 1. (Pelsmajer, Schaefer and Štefankovič, 2007 [56]). We may assume that G is connected, since components may be redrawn arbitrarily far apart. Fix an even plane drawing D of G. We prove the result by induction on the number of edges in G. To make the induction possible, we prove the theorem for *multigraphs*, that is, a generalization of graphs where we allow *parallel edges* (more edges between the same pair of vertices) and *loops* (edges attached to the same vertex by both endpoints).

We begin with the inductive step: if there are at least two vertices in G, then there is an edge $e = uv$ that has two different vertices. Pull v towards u until there remains no crossing between v and u.



Since e was an even edge, the edges incident to v remain even. The pull move will introduce self-crossings in curves that intersect e and are adjacent to v. To correct this, we remove each self-crossing by a local redrawing like in this figure.



Now that the edge uv no longer has any crossings, we contract it while keeping all resulting loops or parallel edges that might arise (we may call this operation a *multigraph edge contraction*). We obtain a new multigraph G' in which the rotations of u and v are combined appropriately. By the inductive assumption, there is a planar drawing of G' respecting the rotation system.

In such a drawing, we can simply split the vertex corresponding to u and v, reintroducing the edge e between them without any intersections. We obtain a plane drawing of G respecting the rotations of all its vertices from D. Notice that the condition on the rotation system was necessary here for the induction step.

If G contains only a single vertex v, then it might have several loops attached to it. Since all the loops in G are even, it cannot happen that we find edges leaving v in the order a, b, a, b since this would force an odd number of crossings between a and b. Hence, if we consider the regions enclosed within the two loops in a small enough neighborhood of v, either they are disjoint or one region contains the other. Then it is easy to show that there must be a loop e whose ends are consecutive in the rotation of v. Removing e we obtain a smaller multigraph G' which, by inductive assumption, can be drawn without crossings while respecting the rotation system. We can then reinsert the missing loop in the right location according to the rotation of v by making it small enough.

In the base case, we simply draw a single vertex with no edges. □

Proof 2. (Fulek, Pelsmajer, Schaefer and Štefankovič, 2012 [27]). Let $G = (V, E)$ where $E = \{e_1, e_2, \ldots, e_m\}$. For every $i \in [m]$, let $E_i = \{e_1, e_2, \ldots, e_i\}$. Let $E_0 = \emptyset$. Let D_0 be the original even drawing of G in the plane. In m successive steps, we construct drawings $D_1, D_2, \ldots, D_m$ such that for every $i \in [m]$, the edges of E_i have no crossings in D_i, and D_i has the same rotation system as D_0. In particular, D_m will satisfy the theorem.

Let $i \in [m]$ and assume that we have constructed D_{i-1}. For every edge f of G that crosses e_i in D_{i-1}, we do the following operations. Since f crosses e_i an even number of times, we can match the crossings together in consecutive pairs in the order as they are encountered along e_i. We cut the edge f at each of these crossings and reconnect the severed ends of f by drawing curves between the neighborhoods of the pairs of matched crossings close to e_i, from both sides of e_i, like in Figure 2.1.

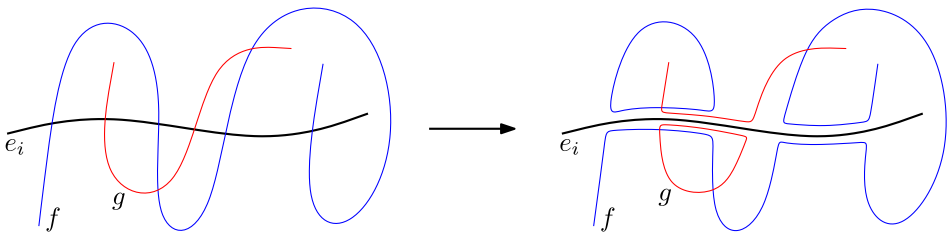


Figure 2.1: Cutting and reconnecting f along e_i.

By this operations, we removed all crossings of e_i with f. We might have created new crossings of f with other edges, but these always come in pairs as we draw the new portions of f from both sides of e_i. Moreover, the edges participating in these new crossings with f must cross e_i, so they do not belong to E_i. In general, the edge f is now represented by a "disconnected curve" consisting of one arc-component containing both endpoints of f, and several closed components. Therefore, we next try to connect some of these components together. As long as there are two components of f in the same face of the plane graph (V, E_i) in the current drawing, we connect them by a *tunnel* consisting of a pair of arcs running close to each other, see Figure 2.2.

Again, we might have created new crossings on f, but always in pairs, one on each side of the tunnel.

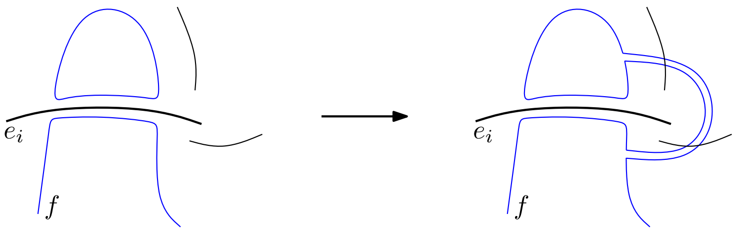


Figure 2.2: Connecting two components of f by a tunnel.

After performing these operations for all edges that crossed e_i in D_{i-1}, we have removed all crossings from e_i, and did not introduce any new crossings on the edges of E_{i-1}. It may still be the case that some edges are represented by disconnected curves, however. In this situation we just remove all the closed components. We need to verify that the resulting drawing D_i is still even. Suppose for contradiction that some two edges, f and g, cross an odd number of times in D_i. Then they are both in the same face of (V, E_i) in D_i. All closed components of f and g that we removed are thus in a different face, and cannot cross the arc-component of g and f, respectively. Since every two closed curves cross an even number of times, by removing the closed components, we changed the number of crossings between f and g by an even number. This implies that the number of crossings between f and g in D_{i-1} was odd, a contradiction. □

For the reader interested in more information about the Hanani–Tutte theorems, their history and future, other variants, and applications, we highly recommend the surveys by Schaefer [59, 60].

2.2 Intersection representations of planar graphs

One of the most important theorems about representation of planar graphs is the Koebe–Andreev–Thurston theorem, also known as the circle packing theorem.

Theorem 2.10 (The Koebe–Andreev–Thurston theorem, 1936–1970–1985)**.** *The vertices $v \in V(G)$ of any planar graph G can be represented by closed disks D_v in the plane such that D_u and D_v are tangent to each other if and only if $uv \in E(G)$, otherwise D_u and D_v are disjoint.*

Theorem 2.11 (de Fraysseix, de Mendez, Rosenstiehl, 1994 [26])**.** *The vertices $v \in V(G)$ of any planar graph G can be represented by non-overlapping triangles T_v in the plane so that T_u and T_v have a point of contact if and only if $uv \in E(G)$.*

These two theorems give rise to the following question:

Question. *Is it true that the vertices of every planar graph can be represented by (pseudo-)segments so that two of them intersect if and only if the corresponding vertices are adjacent? A collection of* ***pseudosegments*** *is a collection of simple curves such that every two of them cross at most once and do not touch.*

The following two theorems answer part of this problem.

Theorem 2.12 (Hartman, Newman, Ziv, 1991 [36]; de Fraysseix, de Mendez, Pach, 1994 [25])**.** *True for bipartite planar graphs.*

Theorem 2.13 (Castro, Cobos, Dana, Márquez, Noy, 2002 [14])**.** *True for triangle-free planar graphs.*

The problem was finally solved by Chalopin, Gonçalves and Ochem for pseudosegments.

Theorem 2.14 (Chalopin, Gonçalves, Ochem, 2010 [16])**.** *Every planar graph has an intersection representation by pseudosegments in the plane.*

Chalopin and Gonçalves then strengthened the proof to representations by segments.

Theorem 2.15 (Chalopin, Gonçalves, 2009 [15])**.** *Every planar graph has an intersection representation by segments in the plane.*

2.3 Embeddings of graphs in three dimensions

Our next subject are graphs in higher dimensions. A graph can be drawn in $\mathbb{R}^3$ in the following way: the vertices are points in $\mathbb{R}^3$ and the edges are simple curves such that they do not pass through any vertex and do not cross any other edge.

Definition 2.16. (i) Let γ_1, γ_2 be two simple closed curves in $\mathbb{R}^3$. Notice that we cannot always transform γ_1 into γ_2 just by deforming the space (such a deformation is called *ambient isotopy*), since the curves may be knotted in different ways. If we allow deformations of the curve during which the curve may cross itself, then it is possible to deform γ_1 into a circle γ which bounds a disc D. By reversing this deformation while dragging the disc D with the curve, we obtain a disc-like surface D_1, which may intersect itself, and whose boundary is γ_1. Then γ_1 and γ_2 are called **linked** if the number of times γ_2 intersects D_1 from "above" is different from the number of times it intersects D_1 from "below".

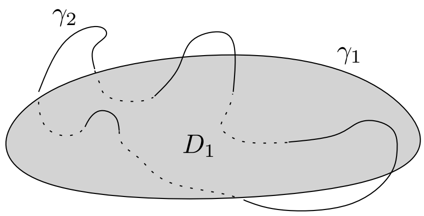


Figure 2.3: Two unlinked curves in $\mathbb{R}^3$.

(ii) Two cycles $C_1,\ C_2$ in an embedding of a graph in $\mathbb{R}^3$ are **linked** if the corresponding closed curves $\gamma_1,\ \gamma_2$ are linked.

(iii) G is a **linkless graph** if it can be drawn in $\mathbb{R}^3$ so that no two disjoint cycles are linked.

Theorem 2.17 (Robertson–Seymour–Thomas)**.** *A graph G has a linkless embedding in $\mathbb{R}^3$ if and only if G has no minor belonging to the Petersen family (shown in Figure 2.4).*

Example 2.1. *K_6 is not a linkless graph, that is, it cannot be drawn without two linked cycles.*

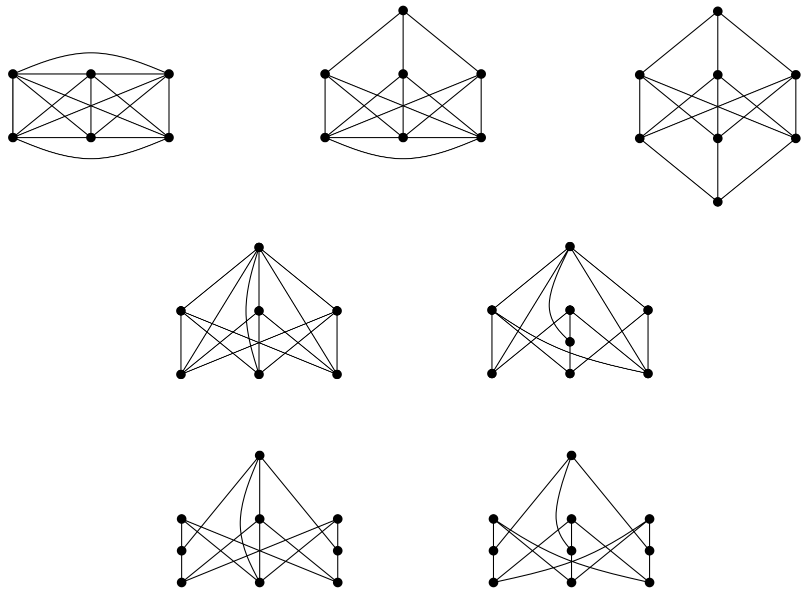


Figure 2.4: The Petersen family.

Chapter 3

Planar separator theorem

Given a planar graph with n vertices, we would like to cut it into two significantly smaller parts, by taking out not too many vertices. Lipton and Tarjan proved that it is possible to find such a cut where "significantly smaller" means of size at most cn for $c < 1$, and "not too many" is $O(\sqrt{n})$. This *planar separator theorem* found many applications mostly in computer science. The theorem allows to design "divide and conquer" algorithms for planar graphs that solve or approximate otherwise NP-hard problems in polynomial time.

In general, it is not possible to find a separator of size $o(\sqrt{n})$: for example, it can be shown that every separator of the $\sqrt{n} \times \sqrt{n}$ grid has at least $\Omega(\sqrt{n})$ vertices.

Theorem 3.1 (Planar separator theorem, Lipton–Tarjan, 1979 [42])**.** *Let G be a planar graph with n vertices. The vertex set of G can be partitioned into three sets A, B, S such that $|S| \leq 4\sqrt{n}$, $|A|, |B| \leq \frac{9}{10}n$, and there is no edge between A and B in G.*

Lipton and Tarjan [42] proved the separator theorem with better bounds than those stated in Theorem 3.1, namely, $|S| \leq \sqrt{8}\sqrt{n}$ and $|A|, |B| \leq \frac{2}{3}n$. Their proof was based on the breadth-first search, and they obtained an $O(n)$-time algorithm for finding the separator. Alon, Seymour and Thomas [6] later found a simple proof of the separator theorem using Menger's theorem. Miller and Thurston [45] derived the separator theorem from Koebe's circle packing theorem (Theorem 2.10), using a stereographic projection on the two-dimensional sphere, centerpoint theorem and probabilistic method. Har-Peled [35] proved the separator theorem using similar ideas but in a simpler way.

Proof of Theorem 3.1 (Har-Peled, 2013 [35]) Assume that $n \geq 20$, otherwise any set S with $\lfloor 4\sqrt{n} \rfloor$ vertices will do. Let $\mathcal{D}$ be the set of discs from Koebe's theorem realizing G as a touching graph. Let P be the set of the centers of these discs.

Let d be a disc (not from $\mathcal{D}$) of smallest possible radius such that $|d \cap P| \geq \frac{n}{10}$. Such a disc exists, since we need to consider only discs that are circumscribed to a triangle formed by three points of P, or those that have two points from P as their diameter. We assume without loss of generality that d is a unit disc centered in the origin; that is, $d = \{z \in \mathbb{R}^2; \|z\| \leq 1\}$. Let d_2 be the disc with radius 2 concentric with d; that is, $d_2 = \{z \in \mathbb{R}^2; \|z\| \leq 2\}$.

For $x \in (1,2)$, let C_x be the circle of radius x centered in the origin; that is, $C_x = \{z \in \mathbb{R}^2; \|z\| = x\}$. Let $S_x \subseteq \mathcal{D}$ be the set of discs from $\mathcal{D}$ that intersect C_x.

We will choose one of the sets S_x for some $x \in (1,2)$ as the separator S. If we denote by A_x the subset of discs of $\mathcal{D}$ inside C_x and by B_x the subset of discs of $\mathcal{D}$ outside C_x, then A_x, B_x and S_x form a partition of $\mathcal{D}$ such that no disc from A_x touches a disc from B_x. This is not enough to prove the theorem; we still need to show that the sets of this partition are small enough. First we show that A_x and B_x always satisfy the conditions of the theorem.

Lemma 3.2. *For every $x \in (1,2)$, there are at most $\frac{9}{10}n$ discs in A_x and at most $\frac{9}{10}n$ discs in B_x.*

Proof. By the definition of d, there are at least $\frac{n}{10}$ points from P inside C_x, thus at most $\frac{9}{10}n$ points from P outside C_x. Hence, $|B_x| \leq \frac{9}{10}n$. To bound the size of A_x, we observe that the disc d_2 can be covered by the interiors of nine unit discs. For example, take the unit discs with centers at $(-2,0)$, $(0,0), (2,0), (-1,1), (1,1), (-1,-1), (1,-1), (0,2), (0,-2)$. By the choice of d, every unit disc in the plane contains at most $\frac{n}{10}$ points of P in its interior. (If it contains exactly $\lceil \frac{n}{10} \rceil$ points, like d, than at least two of the points are on the boundary). Therefore, $|d_2 \cap P| \leq \frac{9}{10}n$. This implies that there are at most $\frac{9}{10}n$ points of P inside C_x and, consequently, that $|A_x| \leq \frac{9}{10}n$. $\square$

Clearly, not every set S_x is good, since a circle C_x may intersect many discs from $\mathcal{D}$. However, in the next lemma we show that S_x is good *in average*. More precisely, we will compute the expected value of $|S_x|$ when we pick x *uniformly at random* from the interval $(1,2)$. That is, for every interval $I \subseteq (1,2)$, the probability that we pick x from I is equal to the length of I.

Lemma 3.3. *For the expected value of $|S_x|$, we have $\mathbf{E}|S_x| \leq 4\sqrt{n}$. Therefore, there exists an $x \in (1,2)$ such that $|S_x| \leq 4\sqrt{n}$.*

Proof. The idea is to relate the probability that a given disc from $\mathcal{D}$ intersects C_x with the area of the intersection of the disc with d_2.

Let $\mathcal{D} = \{u_1, \dots, u_n\}$. For the disc u_i, let p_i be its center and r_i its radius. By the linearity of expectation, we have

$$\mathbf{E}|S_x| = \sum_{i=1}^{n} \mathbf{P}(u_i \in S_x) = \sum_{i=1}^{n} \mathbf{P}(u_i \cap C_x \neq \emptyset).$$

We now estimate the probability $\mathbf{P}(u_i \cap C_x \neq \emptyset)$ for a given disc $u_i \in \mathcal{D}$. If $u_i \subseteq d_2$, then C_x intersects u_i if and only if $x \in [\|p_i\| - r_i, \|p_i\| + r_i]$. The probability of this occurring is equal to the length of the interval $[\|p_i\| - r_i, \|p_i\| + r_i] \cap (1, 2)$, which is at most $2r_i$. Since the area of u_i satisfies $\operatorname{area}(u_i) = \pi r_i^2$, we have

$$\mathbf{P}(u_i \cap C_x \neq \emptyset) \leq 2r_i = 2\sqrt{\frac{\operatorname{area}(u_i)}{\pi}} = 2\sqrt{\frac{\operatorname{area}(u_i \cap d_2)}{\pi}}.$$

If u_i is not contained in d_2, we consider a disc v_i that has the same area as the lens-shaped intersection $u_i \cap d_2$ and touches d_2 from inside. Clearly, v_i is closer to the origin than u_i, so $\mathbf{P}(u_i \cap C_x \neq \emptyset) \leq \mathbf{P}(v_i \cap C_x \neq \emptyset)$. Similarly as before, we have

$$\mathbf{P}(v_i \cap C_x \neq \emptyset) \leq 2\sqrt{\frac{\operatorname{area}(v_i)}{\pi}} = 2\sqrt{\frac{\operatorname{area}(u_i \cap d_2)}{\pi}}.$$

Since the discs u_i are internally disjoint, their intersections with d_2 are internally disjoint as well. We thus have

$$\sum_{i=1}^{n} \operatorname{area}(u_i \cap d_2) \leq \operatorname{area}(d_2) = 4\pi.$$

Putting things together, and using the inequality between the arithmetic mean and the quadratic mean (or the Cauchy–Schwarz inequality), we have

$$\mathbf{E}|S_x| \leq \sum_{i=1}^{n} 2\sqrt{\frac{\operatorname{area}(u_i \cap d_2)}{\pi}} \leq 2\sqrt{n}\sqrt{\sum_{i=1}^{n} \frac{\operatorname{area}(u_i \cap d_2)}{\pi}} \leq 2\sqrt{n}\sqrt{\frac{4\pi}{\pi}} = 4\sqrt{n}.$$

□

The proof of Theorem 3.1 is now finished.

Observation 3.4. *If G is a planar triangulation, then the separator from Theorem 3.1 forms a cycle in G.*

Proof. If $\mathcal{D}$ is the set of discs from Koebe's theorem representing G, then every connected region of $\mathbb{R}^2 \setminus \bigcup \mathcal{D}$ is bounded by three pairwise touching discs. If we follow the circle C_x, every two consecutive discs u_i, u_j intersecting C_x are separated by an arc contained in a connected region of $\mathbb{R}^2 \setminus \bigcup \mathcal{D}$. It follows that u_i and u_j touch. □

The following weighted version of the separator theorem can be obtained by a simple modification of the proof of the separator theorem.

Theorem 3.5 (Weighted separator theorem, Lipton–Tarjan, 1979 [42])**.** *Let G be a planar graph with n vertices. Let $f : V(G) \to [0, 1)$ be a so-called* weight function *assigning a nonnegative real weight to each vertex of G. Suppose that $\sum_{v \in V(G)} f(v) = 1$; that is, the total weight of the vertices of G is 1. Then the vertex set of G can be partitioned into three sets A, B, S such that $|S| \leq 2\sqrt{n}$, each of A, B has total weight at most $\frac{2}{3}$, and there is no edge between A and B in G.*

Chapter 4

Crossings: few and many

These crossovers are like rabbits...they have a tendency to multiply at a terrifying rate.

— *Yona Friedman [10]*

4.1 Turán's brick factory problem

In 1944, Turán posed the following problem [10]. Suppose that there are m kilns and n storage yards. How can we connect every kiln to every storage yard with paths so that the number of crossings of the paths is minimum? This problem can be modeled as follows. What is the minimum number of crossings of edges in a drawing of the graph $K_{m,n}$ in the plane?

Definition 4.1. Let G be an arbitrary graph. The **crossing number** of G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings of edges over all possible drawings of G in the plane. Here it is important to assume that no three edges cross at the same point.

The brick factory problem of Turán is then to find $\mathrm{cr}(K_{m,n})$.

Suppose that the vertices of $K_{m,n}$ are partitioned into two parts A and B with $|A| = m$ and $|B| = n$ and every vertex in A is connected to every vertex in B. The following simple straight-line drawing of $K_{n,m}$ gives the best known upper bound on $\mathrm{cr}(K_{m,n})$. Namely, place the vertices of A on the y-axis to the points

$$(0, -\lfloor m/2 \rfloor), (0, -\lfloor m/2 \rfloor + 1), \ldots, (0, -1), (0, 1), (0, 2), \ldots, (0, \lceil m/2 \rceil)$$

and the vertices of B on the x-axis to the points

$$(-\lfloor n/2 \rfloor, 0), (-\lfloor n/2 \rfloor + 1, 0), \ldots, (-1, 0), (1, 0), (2, 0), \ldots, (\lceil n/2 \rceil, 0),$$

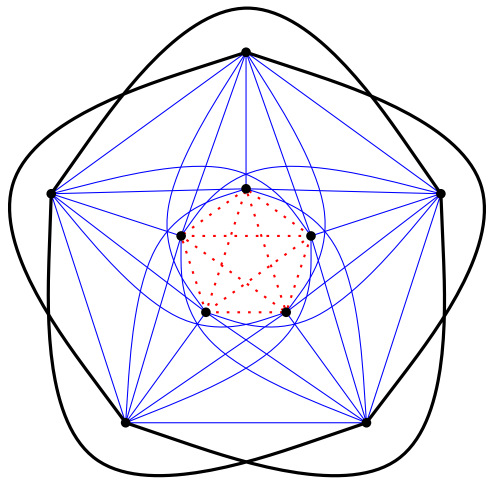


Figure 4.1: A cylindrical drawing of K_{10}.

and then join every vertex in A to every vertex in B by a straight-line segment. The number of crossings in this drawing is exactly $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

Conjecture 4.2 (Zarankiewicz [71]). *We have*

$$\operatorname{cr}(K_{n,m}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Zarankiewicz actually published his conjecture as a theorem, but later his proof was found incomplete [10]. Zarankiewicz's conjecture has been verified for $m \leq 6$ [38].

The following conjecture about the crossing number of the complete graph K_n is usually known as Hill's conjecture.

Conjecture 4.3 (Harary–Hill [33], Guy [31]). *We have*

$$\operatorname{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

There are two families of drawings of K_n that attain the number of crossings stated in Hill's conjecture: cylindrical drawings and 2-page book drawings [9, 31, 33, 34]. In the *cylindrical drawing* of K_{2n}, n vertices are put on the boundary of each circular base of a cylinder in the vertices of a regular n-gon, and the vertices are connected by shortest arcs on the surface of the cylinder. Figure 4.1 shows a "deformed" cylindrical drawing of K_{10}.

In the "optimal" 2-*page book* drawing of K_n, there is a cycle of length n without crossings, forming the "spine" of the book, which can be drawn as

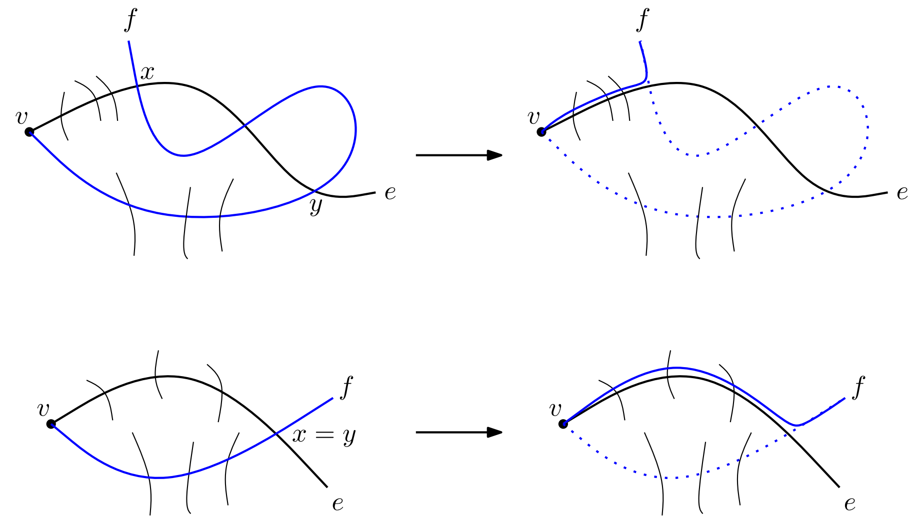


Figure 4.2: Reducing the number of crossings in the case when two adjacent edges cross.

a regular n-gon, for example. Then half of the other edges are drawn inside the cycle and the other half outside the cycle. Roughly speaking, the edges that are drawn inside are those whose slope is between $-45°$ and $45°$.

It is not hard to show that Zarankiewicz's conjecture implies an asymptotic version of Hill's conjecture [58]. We will prove this in Lemma 4.6.

Now we show that $\mathrm{cr}(K_{n,n})$ and $\mathrm{cr}(K_n)$ are of the order n^4. First we observe the following property of optimal drawings.

Lemma 4.4. *Let D be a drawing of a graph G with exactly $\mathrm{cr}(G)$ crossings. Then every two edges have at most one point in common (either an endpoint or a crossing).*

Proof. Suppose that e, f are two adjacent edges with a common vertex v that cross at least once. Let x be a crossing of e and f that is closest to v along e, and let y be a crossing of e and f that is closest to v along f. See Figure 4.2. The crossings x and y might be the same or different. Let e_{vx} be the portion of e between v and x, and let f_{vy} be the portion of f between v and y. Let a be the number of crossings of e_{vx} with other edges of G, and let b be the number of crossings of f_{vy} with other edges of G. Without loss of generality, assume that $b \geq a$. Replace a portion of f slightly longer than the part between v and x by a curve f' drawn along e_{vx}, from an appropriate side. In this way, we get rid of the crossing x (and perhaps some other crossings as well), and we exchange b old crossings on f for a new crossings on f'.

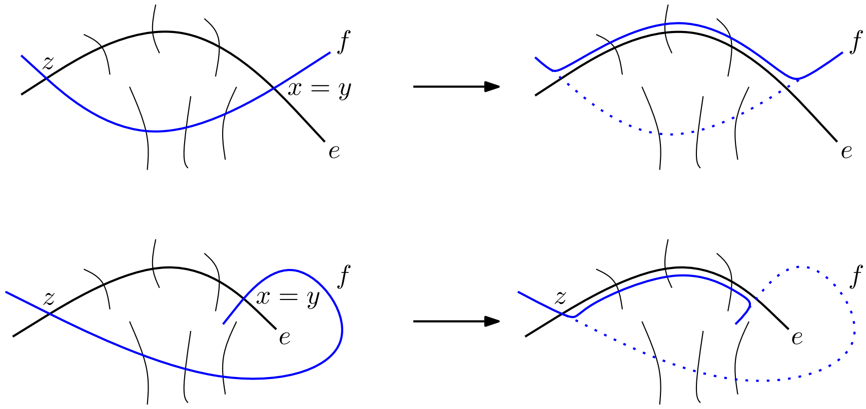


Figure 4.3: Reducing the number of crossings in the case when two edges cross more than once.

Now suppose that e, f are two independent edges with at least two crossings. Let z, z' be arbitrary two crossings of e with f. Let x be a crossing of e and f that is closest to z along e in the direction of z', and similarly, let y be a crossing of e and f that is closest to z along f in the direction of z'. The redrawing step is now analogous to the previous case where we substitute z for v. See Figure 4.3, where only the case $x = y$ is illustrated. Note that we cannot always get rid of both crossings x and z by this redrawing operation. $\square$

There are alternative ways of proving Lemma 4.4. For example, we could first take any pair of crossings z, z' (or a vertex v and a crossing x) between the two edges, and redraw a portion of e or f between z and z' (between v and x). In this way, we could introduce self-crossings, but those may be removed rather easily.

Observe that if two edges e, f cross at least four times, it is not always possible to find two crossings, x and y, so that the portion of e between x and y contains no other crossings with f, and the portion of f between x and y contains no other crossings with e.

Theorem 4.5. *The limits* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}$ *and* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ *exist and both are positive numbers.*

Proof. We will prove the theorem only for the graph K_n. The proof for the graph $K_{n,n}$ is similar and is left as an exercise. By Lemma 4.4, for every drawing of K_n with $\mathrm{cr}(K_n)$ crossings and for every four vertices in it, there are at most three possible crossings among the edges between these four vertices (in fact, there is at most one). This observation shows that $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ never exceeds 3. Now, in order to show that the limit exists, it is sufficient to show that the sequence $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$, $n = 1, 2, 3, \ldots$, is an increasing sequence. The theorem will follow from the fact that every increasing upper bounded sequence whose first term is positive has a positive limit.

To complete the proof, we need to show that for every positive integer n we have $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \leq \frac{\mathrm{cr}(K_{n+1})}{\binom{n+1}{4}}$. Expanding the binomial coefficients in both sides and ignoring the common factors in both sides, we observe that this inequality is equivalent to the inequality

$$\frac{\mathrm{cr}(K_n)}{n} \geq \frac{\mathrm{cr}(K_{n-1})}{n-4} \quad \text{or equivalently} \quad (n-4)\,\mathrm{cr}(K_n) \geq n\,\mathrm{cr}(K_{n-1}).$$

Fix a drawing D of K_n with exactly $\mathrm{cr}(K_n)$ crossings. Removing each vertex in D yields a copy of K_{n-1}, which has at least $\mathrm{cr}(K_{n-1})$ crossings in D. In total, this gives at least $n \cdot \mathrm{cr}(K_{n-1})$ crossings. But notice that every crossing in D is counted precisely $n-4$ times. Therefore, the number of the crossings in D is at least $\frac{n}{n-4} \cdot \mathrm{cr}(K_{n-1})$. This shows that $\mathrm{cr}(K_n) \geq \frac{n}{n-4} \cdot \mathrm{cr}(K_{n-1})$. $\square$

Observe that the above proof tells us more than just the existence of a limit. It says that the sequence $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ is an increasing sequence. Therefore, every term of this sequence is a lower bound for $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$.

Note that Zarankiewicz conjecture would imply that $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2} = 1/4$ and Hill's conjecture would imply that $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} = 3/8$.

Lemma 4.6 (Richter and Thomassen [58])**.** *If* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2} = 1/4$ *then* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} = 3/8$.

Proof. Let n be given and let D be a drawing of the graph K_{2n} with $\mathrm{cr}(K_{2n})$ crossings. If we color n vertices red and the remaining n vertices blue, the color classes induce a drawing of the bipartite graph $K_{n,n}$, which has at least $\mathrm{cr}(K_{n,n})$ crossings. There are exactly $\binom{2n}{n}$ such colorings. A crossing of edges uv and xy in D is counted if and only if u and v get different color and x and y get different color. The number of such colorings is exactly $4 \cdot \binom{2n-4}{n-2}$. Therefore, we get

$$4 \cdot \binom{2n-4}{n-2} \cdot \mathrm{cr}(K_{2n}) \geq \binom{2n}{n} \cdot \mathrm{cr}(K_{n,n}).$$

After simplifying, this gives

$$\frac{\mathrm{cr}(K_{2n})}{\binom{2n}{4}} \geq \frac{3}{2} \cdot \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}.$$

Since for every n, there are drawings of K_n attaining the number of crossings in Hill's conjecture, we have $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \leq 3/8$ and the lemma follows. $\square$

4.2 Conway's thrackle conjecture

A **thrackle** is a graph drawn in the plane so that the edges are represented by simple curves, any pair of which either meet at a common vertex or cross precisely once. A graph is **thrackleable** if it can be drawn as a thrackle.

Conjecture 4.7. *In every thrackle, the number of edges is at most equal to the number of vertices.*

Conway's thrackle conjecture is analogous to the following combinatorial theorem known as nonuniform Fisher's inequality [8], which generalizes Fisher's inequality [24], and was originally proved by de Bruijn and Erdős [11].

Theorem 4.8 (a nonuniform Fisher's inequality, 1948 [11])**.** *If $A_1, A_2, \ldots, A_m$ are distinct subsets of a finite set X such that every two of the subsets have precisely one element in common, then $m \leq |X|$.*

Proof. Let $n = |X| \geq 1$. If some of the sets A_i is empty then $m \leq 1$. If some of the elements $x \in X$ is contained in all the sets A_i, then the sets $A_i \setminus \{x\}$ are pairwise disjoint, and thus we can select a unique point from each of the sets A_i, which implies that $m \leq |X|$. If some of the sets A_i is equal to X, then every other set A_j has only one element, and again, $m \leq |X|$. For the rest of the proof assume that $1 \leq |A_i| \leq n-1$ for every i and that $\bigcap_{i=1}^m A_i = \emptyset$.

For every $x \in X$, let $\deg(x)$ be the number of sets A_i containing x. Observe that if $x \notin A_i$, then $|A_i| \geq \deg(x)$: indeed, every two sets containing x must intersect A_i and these intersections must be disjoint.

Draw a rectangular table (a matrix) with rows indexed by the elements of X and the columns indexed by the sets A_i (or by the numbers $1, 2, \ldots, m$). Write a '1' at the position (x, A_i) if $x \in A_i$ and '0' otherwise. By our assumption, every column and every row has at least one 0-entry. Obviously, the total number of entries in the table is mn. We will now count the number of entries in the table in two other ways, while "stepping" only on the 0-entries. First, we will count according to the columns. We have $n - |A_i|$ 0-entries in the ith column, thus

$$mn = \sum_{i=1}^{m} \sum_{x \in X; x \notin A_i} \frac{n}{n - |A_i|} = \sum_{x \notin A_i} \frac{n}{n - |A_i|}. \tag{4.1}$$

Now we count according to the rows. We have $m - \deg(x)$ 0-entries in row x, thus

$$mn = \sum_{x \in X} \sum_{i \in \{1,\ldots,m\}; x \notin A_i} \frac{m}{m - \deg(x)} = \sum_{x \notin A_i} \frac{m}{m - \deg(x)}. \tag{4.2}$$

Suppose that $m > n$. We observed that if $x \notin A_i$, then $|A_i| \geq \deg(x)$. Since $|A_i| \geq 1$, this further implies the following inequalities:

$$\begin{aligned} m|A_i| &> n \cdot \deg(x), \\ mn - m|A_i| &< mn - n \cdot \deg(x), \\ \frac{n - |A_i|}{n} &< \frac{m - \deg(x)}{m}, \\ \frac{n}{n - |A_i|} &> \frac{m}{m - \deg(x)}. \end{aligned}$$

Summing the last inequality over all $x \notin A_i$, we get

$$\sum_{x \notin A_i} \frac{n}{n - |A_i|} > \sum_{x \notin A_i} \frac{m}{m - \deg(x)},$$

which contradicts equations (4.1) and (4.2). □

An example of a thrackleable graph is the cycle C_5. This can be easily seen from the star-like drawing of C_5 (Figure 4.4). We now show that C_4 cannot be drawn as a thrackle. If the vertices of C_4 are a, b, c, d and each vertex is joined to the next vertex in this order, then in every thrackle drawing of C_4, there is only one possible configuration for the path $abcd$ shown in Figure 4.5. These three edges create a triangle whose one side is the edge bc. The edge da must cross the edge bc, so it has to get inside the triangle and when it goes out of the triangle it either crosses the edge bc for the second time or it must cross one of the other edges. None of these situations is allowed in a thrackle.

Clearly, every subgraph of a thrackle is also a thrackle. This together with the previous observation shows that if G is thrackleable then G has no C_4 as a subgraph.

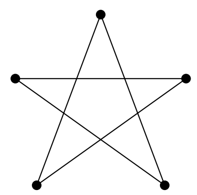


Figure 4.4: A thrackle drawing of C_5.

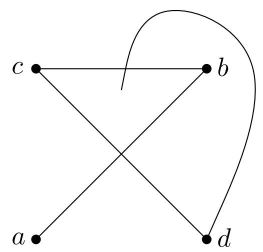


Figure 4.5: An unsuccessful attempt of drawing C_4 as a thrackle.

Theorem 4.9 (Erdős, Kővári–Sós–Turán, 1954 [39]). *Any graph G with n vertices with no C_4 as a subgraph has at most $n^{3/2}$ edges.*

Proof. Suppose that G is a graph with n vertices with no C_4 as its subgraph. We count the number of paths of length 2 in G in two ways. Since G has no C_4, every pair of its vertices have at most one common neighbor and therefore the number of 2-paths in G is at most $\binom{n}{2}$. Now we count the number of 2-paths as follows. Let v be a vertex of G of degree d. Every pair of the neighbors of v form a path of length 2 and conversely every such path is obtained in this way precisely once (just consider the middle point of the 2-path). So, the number of 2-paths in G is equal to

$$\sum_{i=1}^{n} \binom{d_i}{2}$$

where d_i's are the degrees of the vertices of G. So we have $\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2}$. Since the function $f(x) = \binom{x}{2} = x(x-1)/2$ is a convex function, we can use Jensen's inequality to conclude that

$$\binom{n}{2} \geq \sum_{i=1}^{n} \binom{d_i}{2} \geq n \cdot \binom{\frac{\sum_{i=1}^{n} d_i}{n}}{2} = n \cdot \binom{\frac{2|E(G)|}{n}}{2}.$$

Thus,

$$n \geq n-1 \geq \frac{2|E(G)|}{n} \cdot \left(\frac{2|E(G)|}{n} - 1\right) \geq \left(\frac{2|E(G)|}{n} - 1\right)^2$$

and therefore $|E(G)| \leq \frac{1}{2}(n^{3/2} + n) \leq n^{3/2}$. □

Corollary 4.10. *If G is a thrackle with n vertices then $|E(G)| \leq n^{3/2}$.*

Proof. Since no thrackle has C_4 as a subgraph, the assertion is true by Theorem 4.9. □

Notice that the previous corollary is still very far from Conway's thrackle conjecture. Now we try to obtain a better upper bound on the number of the edges of a thrackle. We need the following useful lemmas to obtain an $O(n)$ upper bound on the number of edges of a thrackle.

Lemma 4.11. *Let C_1 and C_2 be two closed curves (possibly self-intersecting) that may cross but do not touch each other. The number of crossings of C_1 and C_2 is even.*

Proof. The closed curve C_1 divides the plane into regions and each of these regions can be colored black or white so that every two adjacent regions have different colors. Now, a point traveling along C_2 observes a change of color every time it crosses C_1. Therefore, after returning to its initial position, the color must have changed an even number of times. □

Lemma 4.12. *Every graph G has a bipartite subgraph H such that $|E(H)| \geq |E(G)|/2$.*

Proof. Let H be a bipartite subgraph of G with the maximum number of edges. Without loss of generality we can assume that H has all the vertices of G. Let A and B be a bipartition of the vertices of H. Let v be an arbitrary vertex of G. Assume that $v \in A$. The bipartite subgraph of G induced by the bipartition $A \setminus \{v\}, B \cup \{v\}$ cannot have more edges than the graph H because of the way we have chosen H. This means that in the graph G, v has at least as many neighbors in B as in A. So, the degree of v in H is at least half the degree of v in G. This argument is valid for every vertex v. Therefore H has the property that each of its vertices has degree at least half the degree in the graph G. Therefore $|E(H)| \geq |E(G)|/2$. □

Theorem 4.13. *Every bipartite thrackleable graph is planar.*

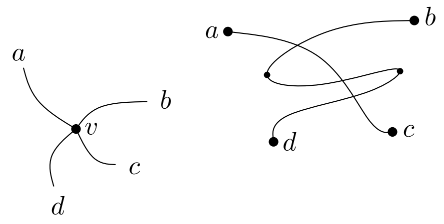


Figure 4.6: A neighborhood of v and the paths $a \dots c$ and $b \dots d$ in $D(H)$.

Proof (sketch). First we prove that if G is bipartite and thrackleable then G contains no subdivision of K_5. Let $D(G)$ be a thrackle drawing of G. Suppose for contradiction that there is a subdivision H of K_5 in G. Clearly, H is also bipartite and the drawing $D(H)$ of H in $D(G)$ is a thrackle. Let v, a, b, c, d be the vertices of $D(H)$ of degree 4 (notice that $D(H)$ has five vertices of degree 4 and the other vertices are of degree 2). Suppose that the neighborhood of v looks like in Figure 4.6. Let C_1 be the closed curve formed by the paths $v \dots a$, $a \dots c$ and $c \dots v$. Let C_2 be the closed curve formed by the paths $v \dots b$, $b \dots d$ and $d \dots v$. Since H is bipartite, each of the two closed curves is formed by an even number of edges. Since $D(H)$ is a thrackle, every edge of C_1 must cross every edge of C_2. On the other hand, Lemma 4.11 ensures that C_1 and C_2 will cross an even number of times. This is a contradiction since C_1 and C_2 intersect an even number of times at interior points of their edges and one more time at the point v. Similarly, it can also be shown that no subdivision of $K_{3,3}$ can be both bipartite and thrackleable and therefore G has no subdivision of K_5 or $K_{3,3}$. Thus G is a planar graph. $\square$

A graph drawn in the plane is called an **odd thrackle** (also a **generalized thrackle**) if every two independent edges cross an odd number of times and every two adjacent edges cross an even number of times (that is, they have an odd number of intersecting points, including the vertex they share). Theorem 4.13 is still true if we replace "thrackle" with "odd thrackle". In fact, we have the reverse implication as well.

Theorem 4.14. *A bipartite graph G is planar if and only if it can be drawn as an odd thrackle.*

Proof. Let D be a plane drawing of G. Deform the plane so that the two vertex classes of G are separated by the x-axis. It is an interesting exercise to show that the deformation can be chosen in such a way that every edge crosses the x-axis exactly once. However, we do not need this stronger observation as we use only the fact that every edge crosses the x-axis an odd number of times. Now cut the plane along the x-axis, move the lower part of the drawing one unit down, and reflect this lower part over the y-axis. Then in the empty strip between the lines $y = 0$ and $y = -1$, reconnect the severed edges by straight-line segments (the ith end from the left on the x-axis with the ith end from the right on the line $y = -1$), and remove all self-crossings that were created. By this, we introduced an odd number of crossings between every pair of edges, since every two segments in the strip cross. Finally, in a small neighborhood of every vertex v, do a similar trick: deform the neighborhood of v so that all the edges are directed in the lower halfplane, cut the edges by a horizontal line, move the part containing v and reflect it over a vertical line passing through v, and reconnect the edges. This operation introduces one crossing on every pair of edges incident with v. The resulting drawing is an odd thrackle.

If D is a drawing of G that is an odd thrackle, we perform the same procedure, and in the end we obtain a drawing of G where every two edges cross an even number of times. By the weak (or strong) Hanani–Tutte theorem (Theorem 2.9 or 2.7), G is planar. □

Now, we are able to prove the following upper bound for the number of edges of a thrackle.

Corollary 4.15. *If G is a thrackle or an odd thrackle then $|E(G)| \leq 4|V(G)|$.*

Proof. Suppose that G is a thrackle with n vertices. By Lemma 4.12, there is a bipartite subgraph H of G with at least $\frac{|E(G)|}{2}$ edges. Since H is a bipartite thrackle, it is a drawing of a planar graph by Theorem 4.14. Thus, by Euler's formula, we have $\frac{|E(G)|}{2} \leq |E(H)| \leq 2n - 4$. □

For thrackles, one can obtain a better upper bound, $|E(G)| \leq 3|V(G)|$, using the fact that there are no cycles of length four. This is left as an exercise.

The upper bound has been further improved several times during recent years.

Theorem 4.16 (Lovász, Pach and Szegedy, 1997 [44])**.** *If G is a thrackle then $|E(G)| \leq 2|V(G)|$.*

Theorem 4.17 (Cairns and Nikolayevsky, 2000 [12])**.** *If G is a thrackle then $|E(G)| \leq 1.5|V(G)|$.*

Theorem 4.18 (Fulek and Pach, 2011 [28])**.** *If G is a thrackle then $|E(G)| \leq \frac{167}{117}|V(G)| < 1.428|V(G)|$.*

Theorem 4.19 (Xu, 2014 [70])**.** *If G is a thrackle then $|E(G)| \leq 1.4|V(G)|$.*

4.3 The crossing lemma

We recall that the **crossing number** of a graph G, denoted by $\mathrm{cr}(G)$, is the smallest possible number of crossing in a drawing of G in the plane. Here we consider drawings with not necessarily straight-line edges, and such that no three edges cross at the same point.

Lemma 4.20. *If G is a graph with $n \geq 3$ vertices, then*

$$\mathrm{cr}(G) \geq e(G) - (3n-6).$$

Proof. Let D be a drawing of G with $k = \mathrm{cr}(G)$ crossings. By removing one edge from a pair of edges that cross, we decrease the number of crossings. Therefore, by removing at most k edges from D, we obtain a plane drawing of a graph with at least $e(G)-k$ edges. By Corollary 1.4, we have $e(G)-k \leq 3n-6$, which proves the lemma. □

The following lower bound on the crossing number of a graph is known under different names, including the crossing lemma, the crossing number theorem, or the crossing number inequality.

Theorem 4.21 (The crossing lemma)**.** *If G is a graph with n vertices and $e \geq 4n$ edges, then*

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{e^3}{n^2}.$$

The crossing lemma was proved independently by Leighton [41, Theorem 7.6] with constant 1/375 and by Ajtai, Chvátal, Newborn and Szemerédi [5] with constant 1/100. The constant was later improved by Pach and Tóth [54] to 1/33.75 (if $e \geq 7.5n$), by Pach, Radoičić, Tardos and Tóth [49] to $1024/31827 \approx 1/31.08$ (if $e \geq 6.44n$), and by Ackerman [2] to 1/29 (if $e \geq 6.95n$).

Proof. The idea of the proof is to use the weak bound from Lemma 4.20 and amplify it using a probabilistic trick. We do not apply the weaker bound directly to G, but to sufficiently sparse induced subgraphs, containing, in average, cn^2/e vertices and $c'n^2/e$ edges.

Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. We choose a random subset $V' \subset V(G)$ by including each vertex independently with probability p (which we choose later). Let G' be the subgraph induced by V' and D' the corresponding subdrawing of D. Let x be the number of crossings in D'. We have

$$\mathbf{E}[|V'|] = np, \qquad \mathbf{E}[e(G')] = ep^2, \qquad \mathbf{E}[x] = \mathrm{cr}(G)p^4.$$

By Lemma 4.20, we have $x \geq e(G') - 3|V'|$, hence $\mathrm{cr}(G)p^4 \geq ep^2 - 3np$. Setting $p = 4n/e$ (which is at most 1) we get

$$\mathrm{cr}(G) \geq \frac{e^3}{64n^2}.$$

□

The order of magnitude of the lower bound in Theorem 4.21 cannot be improved. To see this, take a graph G consisting of $n^2/(2e)$ disjoint complete graphs as equal in size as possible. Then in each component of G there are $\Theta(e/n)$ vertices, $\Theta(e^2/n^2)$ edges, and it can be drawn with $O(e^4/n^4)$ crossings. Therefore, G can be drawn with $O(e^3/n^2)$ crossings, which matches asymptotically the lower bound from the crossing lemma.

The following construction by Pach and Tóth [54] shows that the constant from the crossing lemma is not far from optimal. Suppose that $n \ll e \ll n^2$. Take for the vertex set the vertices of the $\sqrt{n} \times \sqrt{n}$ grid and connect two vertices by an edge if and only if their distance is at most $\sqrt{2e/\pi n}$. Then

$$\mathrm{cr}(G) \leq \left(\frac{16}{27\pi}\right)\frac{e^3}{n^2} \approx \frac{1}{16.65}\frac{e^3}{n^2}.$$

The following lemma can be used to improve the lower bound from Theorem 4.21.

Lemma 4.22. *The maximum number of edges in a graph with $n \geq 3$ vertices that can be drawn in the plane so that every edge crosses at most one other edge is $4n - 8$.*

Proof (sketch). Let G' be a maximal plane subgraph of G. The edges in $E(G) - E(G')$ are all split into two by some edge of G'. We call these pars *half-edges*. It is easy to prove by induction that in each face f of G' with $s(f)$ sides we can have at most $s(f) - 2$ half-edges. Now we only use Euler's formula and its corollaries:

$$\begin{aligned} e(G) &= e(G') + (e(G) - e(G')) \\ &\leq e(G') + \frac{1}{2}\sum_f (s(f) - 2) = 2e(G') - f(G') \\ &= e(G') + (e(G') - f(G')) \leq 3n - 6 + (n - 2) = 4n - 8. \end{aligned}$$

For an optimal construction take a planar graph whose all faces are quadrilaterals and then add two diagonals of each quadrilateral. □

Denote by $e_k(n)$ the maximum number of edges in a graph on n vertices that can be drawn in the plane so that every edge crosses at most k other edges. Lemma 4.22 says that $e_1(n) = 4n - 8$. It can be also proved that $e_2(n) = 5n - 10$, $e_3(n) = 6n - 12$, $e_4(n) = 7n - 14$. It can be conjectured that $e_k(n) = (k + 3)(n - 2)$. As a consequence we have:

$$\mathrm{cr}(G) > (e - 3n) + (e - 4n) + (e - 5n) + (e - 6n) + (e - 7n) = 5e - 25n.$$

4.4 Incidences and unit distances

Let P be a set of n points and L a set of m lines in the plane. An **incidence** between P and L is a pair (p, ℓ) such that $p \in P, \ell \in L$ and $p \in \ell$.

Theorem 4.23 (Szemerédi–Trotter, 1983 [63])**.** *The maximum number of incidences between n points and m lines in the plane is $O(n^{2/3}m^{2/3}+m+n)$.*

Proof. (Székely [62]) Let P be the given set of points and L the given set of lines. We may assume without loss of generality that every line is incident to at least one point and that every point is incident to at least one line. Define a graph G drawn in the plane as follows. The vertex set of G is P, and two vertices are joined by an edge drawn as a straight line segment if the two vertices are consecutive points of P on one of the lines from L. This drawing shows that $\mathrm{cr}(G) \le \binom{m}{2}$. The number of points on any of the lines of L is one greater than the number of edges drawn along that line. Therefore, the number of incidences among the points and the lines is at most $e(G) + m$. Theorem 4.21 finishes the proof: either $e(G) \le 4n$, in which case the number of incidences is at most $4n+m$, or $\binom{m}{2} \ge \mathrm{cr}(G) \ge \mathrm{cr}(G)^3/n^2$, in which case $e(G) \le O(n^{2/3}m^{2/3})$ and the number of incidences is thus at most $O(n^{2/3}m^{2/3}) + m$. $\square$

Theorem 4.24 (Spencer, Szemerédi and Trotter, 1984 [61])**.** *The maximum number of unit distances determined by n points in the plane is $O(n^{4/3})$.*

Proof. (Székely [62]) Draw a multigraph G in the plane in the following way. The vertex set of G is the set of n given points. Draw a unit circle around each point; in this way, consecutive points on the unit circles are connected by circular arcs. These arcs form the edges of the multigraph G. The number of edges of G is equal to the number of point-circle incidences, and this is equal to twice the number of unit distances. Discard the circles that contain at most two points. By this, we delete at most $2n$ edges from G and obtain a multigraph G'. In G', there are no loops, and every two vertices are connected by at most two edges, each of them coming from a different circle, since at most two unit circles can pass through two given points. Then, for every two vertices joined by two edges in G', delete one of the edges. The resulting drawing is a drawing of a graph G''. For the number of edges of G'', we have $e(G'') \geq e(G')/2 \geq (e(G) - 2n)/2 = e(G)/2 - n$. The number of crossings of G'' in this drawing is at most n^2, since any pair of circles intersect in at most two points. By Theorem 4.21, $e(G'')^3/n^2 = O(n^2)$ and so $e(G'') = O(n^{4/3})$. This implies that the number of unit distances is at most $e(G)/2 \leq e(G'') + n \leq O(n^{4/3})$. $\square$

Chapter 5

Turán-type problems

5.1 Disjoint edges in geometric graphs

Theorem 5.1 (Hopf–Pannwitz, 1934)**.** *For every $n \geq 3$, the maximum number of times the diameter (the largest distance) can occur among n points in the plane is n.*

Proof. Let S be the given set of n points. Let G be a graph with vertex set S such that two vertices form an edge if and only if the corresponding two points form a diameter of S. If the degree of every vertex in G is at most 2, then we are done. If G has a vertex v whose degree is at least 3, then let u be one of its neighbors that is not the leftmost nor the rightmost one. A simple geometric observation shows that the degree of u is 1; see Figure 5.1. Indeed, all vertices of S must lie in the region that is an intersection of the unit discs centered in u, v, and the leftmost and the rightmost neighbor of v. The only point in the intersection region that has distance 1 from u is the point v. By induction, $G - u$ has at most $n - 1$ edges, thus G has at most n edges.

On the other hand, the vertices of a unit triangle and a set of $n-3$ points on a unit circle centered in one of its vertices show that n diameters can be achieved. $\square$

Now comes the definition that gave this course its name.

Definition 5.2. A **geometric graph** is a graph whose vertices are represented by distinct points in general position in the plane and whose edges are drawn as straight-line segments, possibly with crossings.

Using the triangle inequality, it is an easy exercise to show that in the geometric graph formed by diameters of a finite point set in the plane there are no two disjoint edges. Theorem 5.1 can be generalized as follows.

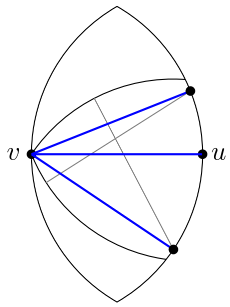


Figure 5.1: A vertex of degree at least 3 in the diameter graph has a neighbor of degree 1.

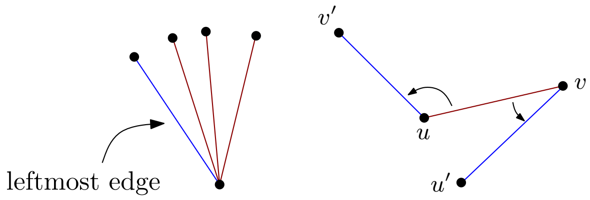


Figure 5.2: Perles' argument

Theorem 5.3. *Let G be a geometric graph with no two disjoint edges. Then* $|E(G)| \leq |V(G)|$.

Proof. (Perles) We call a vertex $v \in V(G)$ *pointed* if there is a line ℓ passing through v such that all edges incident to v lie in a halfplane bounded by ℓ. At each pointed vertex v, a chicken lays an egg on the "leftmost" edge incident to v, that is, the first edge in the clockwise order of edges around v, starting from the line ℓ. Now we observe that every edge of G has an egg on it, which proves the theorem. Indeed, suppose that there is no egg on an edge $uv \in E(G)$. Thus, originally G contained two edges, uv' and $u'v$, with clockwise angles $v'uv$ and $u'vu$ smaller than $180°$. These two edges must lie on opposite sides of the line uv; see Figure 5.2. Hence, they are disjoint, contradicting the assumption. □

In other words, Theorem 5.3 says that the number of edges in a straight-line thrackle is at most n.

Problem 5.4 (Avital, Hanani, 1966; Kupitz, 1979; Perles, Erdős)**.** *Fix a $k \geq 2$. What is the maximum number $f_k(n)$ of edges that a geometric graph on n vertices can have without containing k pairwise disjoint edges?*

The following table summarizes the current knowledge.

k	$f_k(n)$	
2	$= n$	Hopf–Pannwitz, 1934
3	$= 2.5n + O(1)$	Černý, 2005 [17]
4	$\leq 10n$	Goddard, Katchalski and Kleitman, 1996 [30]
> 4	$\leq 2^9(k-1)^2 n$	Tóth, 2000 [65]

Definition 5.5. A geometric graph is **convex** if its vertices are in convex position; that is, they form the vertex set of a convex polygon.

Proposition 5.6 (Kupitz, 1982)**.** *For any $n \geq 2k+1$, the maximum number of edges that a convex geometric graph with n vertices can have without containing $k+1$ pairwise disjoint edges is kn.*

Proof. Let G be a convex geometric graph with n vertices. Without loss of generality we can assume that the vertex set of G is the set of vertices of a regular n-gon. Partition the set of edges of G into n classes so that two segments belong to the same class if and only if they are parallel. If G has no $k+1$ pairwise disjoint edges, then each class contains at most k elements of $E(G)$. Thus, $|E(G)| \leq kn$.

To show that the bound can be attained, take a graph G with vertices $V = \{x_0, \dots, x_{n-1}\}$ that appear clockwise in this order and with the edges $x_i x_{i+\lfloor n/2 \rfloor + j}$, $0 \leq i \leq n-1$, $1 \leq j \leq k$, where the index $i + \lfloor n/2 \rfloor + j$ is computed modulo n. $\square$

5.2 Partial orders and Dilworth's theorem

Definition 5.7. A binary relation $\preceq$ on a set X is a **partial order** on X if $\preceq$ is reflexive, antisymmetric and transitive. That is, for every $x, y, z \in X$, we have $x \preceq x$, $(x \preceq y) \wedge (y \preceq x) \Rightarrow (x = y)$, and $(x \preceq y) \wedge (y \preceq z) \Rightarrow (y \preceq z)$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. Two elements of X are **comparable** by $\preceq$ if $x \preceq y$ or $y \preceq x$, otherwise they are **incomparable**. A partial order $\preceq$ on X is a **total order** if every two elements of X are comparable. A pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order on X is called a **partially ordered set** or also a **poset**. The **comparability graph** $G(P)$ of a partially ordered set $P = (X, \preceq)$ is the graph with vertex set X such that for every two distinct elements $x, y \in X$, xy is an edge of $G(P)$ if and only if x and y are comparable by $\preceq$. A **chain** in a poset is a totally ordered subset, that is, a subset whose elements are pairwise comparable. An **antichain** in a poset is a subset of elements that are pairwise incomparable. See Figure 5.3 for an illustration of the power set of a four-element set ordered by inclusion.

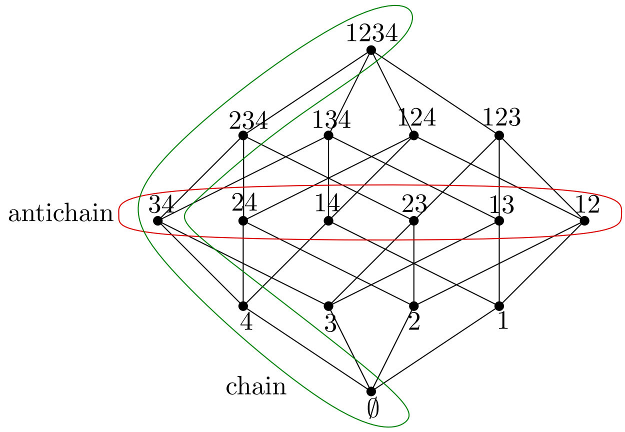


Figure 5.3: A *Hasse diagram* of the set of subsets of $\{1, 2, 3, 4\}$ ordered by inclusion. The comparability graph is obtained by joining all pairs of vertices connected by a vertically monotone path in the diagram. A chain and an antichain are highlighted.

Dilworth's theorem and Mirsky's theorem are important results about partially ordered sets. Notice they are "dual" to each other.

Theorem 5.8 (Dilworth, 1950 [20]). *If the maximum size of an antichain in a partially ordered set P is k, then P is a union of k chains.*

See e.g. [48, Theorem 14.9] for the proof. Dilworth's theorem is closely related (in fact, easily shown to be equivalent) to Hall's marriage theorem, König's theorem about vertex covers in bipartite graphs, and can be also derived from the max-flow min-cut theorem.

Theorem 5.9 (Mirsky, 1971 [46]). *If the maximum length of a chain in a partially ordered set P is k, then P is a union of k antichains.*

Although Mirsky's theorem was published later than Dilworth's theorem, its proof is significantly easier. We warn the reader that some authors include Mirsky's theorem as a part of Dilworth's theorem.

Proof. Let $P = (X, \preceq)$. For $x \in P$, let $r(x)$ denote the maximum length of an increasing chain starting at x. By the assumption, we have $1 \leq r(x) \leq k$ for every x. Let $X_i = \{x \in X : r(x) = i\}$. Since $X = \bigcup_i X_i$, it suffices to prove that X_i is an antichain for every i. Take $x, y \in X_i$. Suppose for contradiction that $x \prec y$. We have

$$x \prec y = y_1 \prec y_2 \prec \cdots \prec y_i$$

for certain $y_2, \ldots, y_i \in X$, which implies that $x \notin X_i$; a contradiction. □

Both Dilworth's theorem and Mirsky's theorem (separately) imply the following interesting corollaries.

Corollary 5.10. *Every partially ordered set of size at least $kl+1$ has either a chain of length $k+1$ or an antichain of size $l+1$.*

In particular, every partially ordered set of size n has either a chain or an antichain of size at least $\sqrt{n}$. Notice that by a straightforward application of Ramsey's theorem to the corresponding comparability graph we get only a chain or an antichain of size $\log n$.

Corollary 5.11 (Gallai, Hajós)**.** *Every system of at least $kl+1$ intervals on a line has either $k+1$ disjoint members or $l+1$ intersecting members.*

Corollary 5.12 (Erdős–Szekeres lemma)**.** *Every sequence of at least $kl+1$ distinct real numbers has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $l+1$.*

We now use Mirsky's theorem to obtain a generalization of Theorem 5.3 to geometric graphs with at most k pairwise disjoint edges.

Theorem 5.13 (Pach–Törőcsik, 1994)**.** *If the maximum number of pairwise disjoint edges in a geometric graph G is k, then $|E(G)| \leq k^4|V(G)|$.*

Proof. We start by defining four strict partial orders $\prec_1, \prec_2, \prec_3, \prec_4$ on the family of segments in the plane. Analogous partial orders can also be defined for compact convex sets [40]. Hopefully the reader will be satisfied with a pictorial definition (Figure 5.4). For each of the orders, only disjoint segments are comparable, and every pair of disjoint segments is comparable by at least one of the four orders. The four orders are distinguished by the relative ordering of the x-coordinates of the endpoints of the segments; there are exactly six such possible orderings. If two segments e, f comparable by $\prec_i$ intersect a common vertical line ℓ, then $e \prec_i f$ if e intersects ℓ below f. We invite the reader to verify that each of the four orders is indeed a strict partial order. Note that if e lies completely to the left of f, then $e \prec_1 f$ and $f \prec_2 e$ simultaneously.

Interpret $E(G)$ as the set of closed segments representing the edges of G. Since G has no $k+1$ disjoint edges, there is no chain of length $k+1$ in either of the posets $(E_G, \prec_i)$. By Mirsky's theorem, we can divide $E(G)$ into k subsets $E_1 \cup \cdots \cup E_k$ that are antichains $(E_G, \prec_1)$. Now pick the largest of the subsets E_i, which is of size at least $\frac{|E(G)|}{k}$, and further divide it into

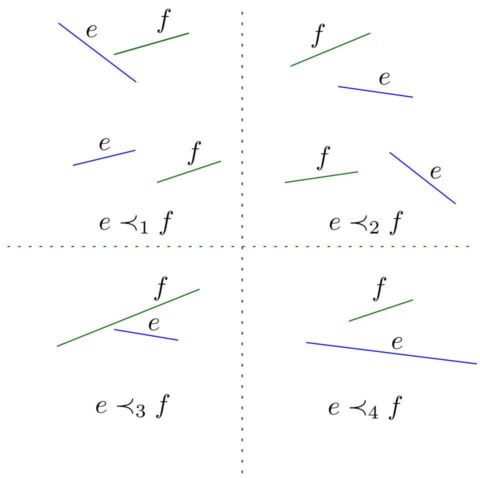


Figure 5.4: A schematic definition of the four partial orders of segments.

k subsets that are antichains in $(E_G, \prec_2)$. One of the parts will have size at least $\frac{|E(G)|}{k^2}$. Continue dividing in this fashion four times in total. In the end we obtain a set $H \subset E(G)$ such that $|H| \geq \frac{|E(G)|}{k^4}$, and H is an antichain in each of the four posets $(E_G, \prec_i)$. This implies that every two segments in H intersect. By Theorem 5.3, we have $|H| \leq n$. Therefore, $|E(G)| \leq k^4 n$. ☐

Definition 5.14. Let $f(n)$ denote the largest number such that any family of n convex sets in the plane has $f(n)$ disjoint or $f(n)$ pairwise intersecting members.

From the proof of Theorem 5.13 we have $f(n) \geq n^{1/5}$ [40].

The rest of this section was not presented during the lectures in 2014/2015.

We now show a trivial construction for an upper bound. Consider $\sqrt{n}$ groups of $\sqrt{n}$ segments, so that segments in each group are mutually intersecting, while the groups are pairwise disjoint. With this trivial construction we have a simple upper bound: $f(n) \leq \sqrt{n}$.

A less trivial construction tightens the upper bound to approximately $n^{0.431}$, where $0.431 \geq \frac{\log 2}{\log 5}$. The construction method is commonly used in combinatorics and graph theory in general: we find a small configuration that is good and we iterate it. We start with a pentagon formed by five segments. This configuration shows that $f(5) \leq 2 < \sqrt{5}$. Now we replace each segment with what would look like a very thin squeezed pentagon like

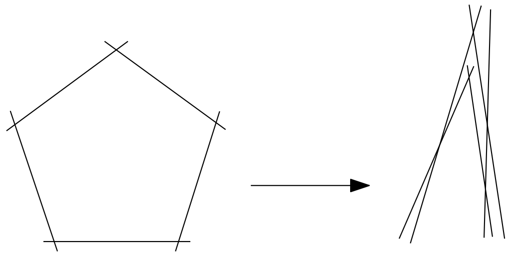


Figure 5.5: An iterative construction for an upper bound on $f(n)$.

in Figure 5.5. Now we have 5^2 segments with the maximum number of intersecting (or disjoint) segments being 2^2. Iterating in this fashion we get $n = 5^k$ segments, with at most 2^k pairwise disjoint or intersecting segments. Since $2^k = n^{\log 2/\log 5} \leq n^{0.431}$, we have

$$n^{1/5} \leq f(n) \leq n^{0.431}.$$

As one can see, the bounds are not quite tight. Further small refinements can be made, but this is more difficult.

Definition 5.15. Let $F_k(n)$ be the largest number such that any graph G that is a union of k comparability graphs $G_1, \ldots, G_k$ contains $F_k(n)$ vertices that form a complete subgraph or $F_k(n)$ vertices that are independent.

Similarly as in Theorem 5.13, we get that

$$F_k(n) \geq n^{\frac{1}{k+1}},$$

as follows. Let $l = n^{1/(k+1)}$. Color the edges of G in k different colors corresponding to the k partial orders. If there is no complete subgraph of size l in the first color, say, red, we find a subset of at least n/l vertices with no red edge. We repeat this step for each of the colors. After k steps we will either find a complete subgraph with all edges of the same color or a set of at least $n/l^k = l$ vertices with no edges of any color, which means that it is an independent set.

An upper bound was constructed by Dumitrescu and Tóth, in a purely combinatorial way.

Theorem 5.16 (Dumitrescu and Tóth, 2002 [21])**.**

$$n^{1/(k+1)} \leq F_k(n) \leq n^{(1+\log k)/k}.$$

5.3 Crossing edges and bisection width

A graph that has a drawing in the plane in which there are no k pairwise crossing edges is called **k-quasiplanar**. Clearly, a graph is 2-quasiplanar if and only if it is planar, thus 2-quasiplanar grphs with $n \geq 3$ vertices have at most $3n - 6$ edges. The following table summarizes current known upper bounds on the number of edges of k-quasiplanar graphs with n vertices.

$k = 2$	$3n - 6$	Euler
$k = 3$	$O(n)$	Pach, Radoičić and Tóth, 2006 [50]
$k = 3$	$8n - 20$	Ackerman and Tardos, 2007 [3]
$k = 4$	$72(n - 2)$	Ackerman, 2009 [1]
$k > 4$	$O(n \log^{4k-12} n)$	Pach, Radoičić and Tóth, 2006 [50]
$k > 4$	$O(n \log^{4k-16} n)$	Ackerman, 2009 [1]

A linear upper bound is conjectured.

Conjecture 5.17. *For every k there is a constant c_k such that every k-quasiplanar graph with n vertices has at most $c_k n$ edges.*

For geometric graphs with no k pairwise crossing edges, slightly better upper bounds are known.

$k = 3$	$O(n)$	Agarwal et al., 1997 [4]
$k = 3$	$6.5n - O(1)$	Ackerman and Tardos, 2007 [3]
$k \geq 2$	$O(n \log^{2k-4} n)$	Pach, Shahrokhi and Szegedy, 1996 [51]
$k \geq 4$	$O(n \log^{2k-6} n)$	Agarwal et al., 1997 [4]
$k \geq 4$	$O(n \log n)$	Valtr, 1998 [69]

For convex geometric graphs, a linear upper bound is known.

Theorem 5.18 (Capoyleas–Pach, 1992 [13])**.** *For any $n \geq 2k - 1$, the maximum number of edges in a convex geometric graph with n vertices and no k pairwise crossing edges is*

$$2(k-1)n - \binom{2k-1}{2}.$$

The upper bound in Theorem 5.18 is tight due to the following construction. Let $x_1, \ldots, x_n$ be the vertices of a convex n-gon in clockwise order. Connect x_i and x_j by an edge if and only if they are separated by fewer than k vertices along the boundary of the polygon, or $1 \leq i \leq k - 1$.

Now we prove a weaker upper bound on the number of edges in geometric graphs with no three pairwise crossing edges.

Theorem 5.19. *The maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges is $O(n^{3/2})$.*

Proof. Let G be a geometric graph with n vertices, m edges and no three pairwise crossing edges. By the crossing lemma, G has at least $\frac{1}{64} \cdot \frac{m^3}{n^2}$ crossings, so there is an edge e with at least $\frac{1}{32} \cdot \frac{m^2}{n^2}$ crossings. By our assumption, the edges that cross e do not cross each other, in other words, they form a plane graph, and thus there are at most $3n$ of them. In fact, since they also form a bipartite graph, there are at most $2n$ of them. Therefore, we have $\frac{1}{32} \cdot \frac{m^2}{n^2} \leq 2n$, which implies that $m \leq 8n^{3/2}$. □

Next we prove a general upper bound for geometric graphs with no k pairwise crossing edges.

Definition 5.20. Let G be a graph with n vertices. The **bisection width** of G, denoted by $b(G)$, is the minimum number of edges one has to remove from G so that the vertex set of the resulting graph G' can be divided into two parts, A and B, such that there is no edge between A and B in G' and $|A|, |B| \leq 2n/3$. Instead of the last inequality, we can equivalently require that $|A|, |B| \geq n/3$.

Clearly, $b(G) \leq 2n^2/9$ for every graph G with n vertices. This is of course tight if G is the complete graph. The bisection width of a planar graph with n vertices can be as large as $2n/3$; this contrasts with the separator theorem and shows that removing vertices might be much more powerful in disconnecting the graph. It is a simple exercise to show that the bisection width of an m times m grid is at least $m/3$ and at most m.

The following theorem may be considered as a variant of the crossing lemma, which gives a lower bound on the crossing number in terms of the bisection width.

Theorem 5.21 (Leighton [41]; Pach, Sharokhi and Szegedy [51]). *Let G be a graph with n vertices and degree sequence $d_1, d_2, \ldots, d_n$. For the bisection with of G, we have*

$$b(G) \leq 1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}.$$

It is an easy exercise to see that if G is a random graph with n vertices where every edge is taken independently with probability $1/2$, then with probability more than 0.99, G has at least $n^2/10$ edges, and thus the crossing number of G is at least cn^4 for some constant c, by the crossing lemma. It is also an easy exercise to show that with probability at least $1/2$, the bisection width of G is at least $n^2/100$.

For the proof of Theorem 5.21, we use the following variant of the weighted separator theorem, where we remove edges instead of vertices.

Theorem 5.22 (Gazit–Miller, 1990 [29]). *Let G be a planar graph with n vertices. Let $f : V(G) \to [0, 2/3]$ be a weight function assigning a nonnegative real weight to each vertex of G. Suppose that $\sum_{v \in V(G)} f(v) = 1$. Then the vertex set of G can be partitioned into two sets A, B such that each of A, B has total weight at most $\frac{2}{3}$, and the number of edges between A and B in G is at most $1.58\sqrt{\sum_{i=1}^{n} d_i^2}$.*

Proof of Theorem 5.21. Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. Construct a drawing D' of a graph G' by replacing each crossing in D by a *new* vertex of degree 4, subdividing the two edges participating in the crossing. The vertices of G are called the *old* vertices in G'. The new graph G' is planar and satisfies $|V(G')| = |V(G)| + \mathrm{cr}(G)$. Note that when the number of edges of G grows asymptotically faster than the number of vertices, the crossing lemma implies that the new vertices significantly outnumber the old vertices in G'.

We assign weight 0 to every new vertex in G', and weight $1/n$ to every old vertex. Now we apply Theorem 5.22 to G'. We get a set S' of at most $1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}$ edges separating G' into two parts, each containing at most $2n/3$ old vertices. From S', we create a corresponding set S of edges in G by copying every edge of S' between two old vertices, and for every edge $e' \in S'$ incident to a new vertex, we take the edge e of G that extends e'. Now S separates G into two parts of size at most $2n/3$, and contains at most $1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}$ edges. This gives an upper bound on the bisection width of G. $\square$

Sketch of proof of Theorem 5.22. (by Pach, Spencer and Tóth [52]). We consider only the case when G has two types of vertices, one type with weight 0 and the other type with weight $1/m$. Let G'' be a graph obtained from G by replacing each vertex v_i of weight $1/m$ of degree d_i by a $d_i \times d_i$ grid, and by connecting the edges that were incident to v_i to the vertices on one side of the grid (called *special* vertices), so that each vertex in the grid has degree at most 4. See Figure 5.6. Then all vertices of G'' have degree at most 4. The number of vertices in G'' is $|V(G'')| = \sum_{i=1}^{n} d_i^2$.

For every i, we assign weight $1/(md_i)$ to every special vertex in the $d_i \times d_i$ grid constructed from v_i, and weight 0 to all other vertices.

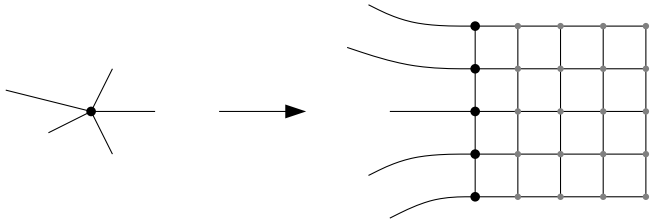


Figure 5.6: Replacing a vertex by a grid. Special vertices form the left column of the grid.

By the weighted separator theorem (Theorem 3.5), we can split G'' into two parts A, B with total weight at most $2/3$ by removing a set S of at most $2\sqrt{|V(G'')|} = 2\sqrt{\sum_{i=1}^n d_i^2}$ vertices of G''.

Using the partition A, B, S of G'', we want to define a partition of G into two parts, separated by few edges. The idea is to put the vertices v_i such that the corresponding $d_i \times d_i$ grid has many points in A to one part, the vertices v_i such that the corresponding $d_i \times d_i$ grid has many points in B to the second part, and distribute the remaining vertices so that the sizes of the two parts are as equal as possible. We omit the details. $\square$

Using the inequality between the bisection width and the crossing number, we improve the upper bound from Theorem 5.19 as follows.

Theorem 5.23. *The maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges is $O(n \log^2 n)$.*

Proof. Let G be a geometric graph with n vertices and degree sequence $d_1, d_2, \ldots, d_n$. By Theorem 5.21, we have $b(G) \le 1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^n d_i^2}$. For every edge e, the edges crossing e in G form a planar subgraph. Hence, $\mathrm{cr}(G) \le |E(G)| \cdot 3n$. By the estimate $d_i \le n$, we have $\sum_{i=1}^n d_i^2 \le n \cdot \sum_{i=1}^n d_i = 2n|E(G)|$. Therefore, $b(G) \le 1.58\sqrt{50n|E(G)|} \le 12\sqrt{n|E(G)|}$.

By this inequality, there is a set of at most $12\sqrt{n|E(G)|}$ edges that separates the graph G into two parts G_1, G_2 with n_1 and n_2 vertices, respectively, so that $n/3 \le n_1, n_2 \le 2n/3$.

Let $f_3(n)$ be the maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges. By induction on n we prove that $f_3(n) \le cn \log^2 n$ for some constant c and $n \ge 2$. For $n = 2$ this is true with $c \ge 1/(4 \log 2)$. Let $n \ge 3$. Let G be a geometric graph with n vertices, no three pairwise crossing edges and with $f_3(n)$ edges. Consider the partition from the previous paragraph. By induction hypothesis, we have

$$f_3(n) \le f_3(n_1) + f_3(n_2) + b(G) \le cn_1 \log^2 n_1 + cn_2 \log^2 n_2 + 12\sqrt{nf_3(n)}.$$

Since $x \to x \log^2 x$ is a convex function on $(0, \infty)$, we have

$$f_3(n) \leq c \cdot \lfloor 2n/3 \rfloor \cdot \log^2(\lfloor 2n/3 \rfloor) + c \cdot \lceil n/3 \rceil \cdot \log^2(\lceil n/3 \rceil) + 12\sqrt{n f_3(n)}.$$

The rest of the computation is left as an exercise. □

5.4 Crossing lemma revisited

We start with a corollary of Theorem 5.21.

Corollary 5.24. *Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$, and let $G_1, G_2, \ldots, G_j$ be edge-disjoint subgraphs of G. Then the sum of their bisection widths satisfies*

$$\sum_{i=1}^{j} b(G_i) \leq 2\sqrt{j} \cdot \sqrt{16\mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2}.$$

Proof. By the inequality between the arithmetic mean and the quadratic mean (or the Cauchy–Schwarz inequality), we have

$$\sum_{i=1}^{j} b(G_i) \leq \sqrt{j} \cdot \sqrt{\sum_{i=1}^{j} (b(G_i))^2}.$$

For $i = 1, 2, \ldots, j$, let $d_{1,i}, d_{2,i}, \ldots, d_{n,i}$ be the degree sequence of G_i. By Theorem 5.21 applied to G_i, we have

$$(b(G_i))^2 \leq 4 \left(16\mathrm{cr}(G_i) + \sum_{k=1}^{n} d_{k,i}^2 \right).$$

This implies that

$$\sum_{i=1}^{j} (b(G_i))^2 \leq 4 \left(16 \sum_{i=1}^{j} \mathrm{cr}(G_i) + \sum_{k=1}^{n} \sum_{i=1}^{j} d_{k,i}^2 \right).$$

Since the graphs G_i are edge-disjoint, we have

$$\sum_{i=1}^{j} \mathrm{cr}(G_i) \leq \mathrm{cr}(G) \quad \text{and}$$

$$\sum_{i=1}^{j} d_{k,i}^2 \leq \left(\sum_{i=1}^{j} d_{k,i} \right)^2 = d_k^2.$$

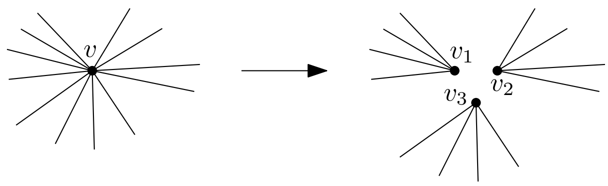


Figure 5.7: Splitting a vertex of large degree.

Therefore,

$$\sum_{i=1}^{j}(b(G_i))^2 \leq 4\left(16\mathrm{cr}(G)+\sum_{k=1}^{n} d_k^2\right)$$

and the corollary follows. □

The following theorem strengthens the crossing lemma for C_4-free graphs.

Theorem 5.25 (Pach, Spencer and Tóth, 2000 [52]). *Let G be a graph with n vertices, e edges, and with no C_4 as a subgraph. If $e \geq 1000n$, then*

$$\mathrm{cr}(G) \geq c \cdot \frac{e^4}{n^3}$$

where c is a positive constant.

In the original paper [52], the theorem is proved with $c = 1/10^8$. We prove it with $c = 1/10^7$.

Proof. The idea of the proof is the following. We recursively cut G into smaller parts by removing few edges. When a part has fewer than s vertices (where the s will be chosen later), we stop the recursion. The number of edges in all resulting parts will be small, at most $3e/4$. If the crossing numbers of the parts are small, we would delete less than $e/4$ edges, which would be a contradiction. This will imply that the crossing number of G is large. The same idea can be also used to prove the crossing lemma for general graphs.

First we modify the graph G so that all degrees are at most $\Delta = \lfloor 4e/n \rfloor$. Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. For every vertex v with degree more than Δ, do the following. Split the neighbors of v into $k = \lceil d(v)/\Delta \rceil$ sets $A_1, A_2, \ldots, A_k$ of size at most Δ, so that each set forms an interval of consecutive vertices in the rotation at v in D. Then remove the vertex v from G and replace it by k vertices $v_1, v_2, \ldots, v_k$ placed close to the original location of v in D, on a small circle, and connect v_i to all the vertices in A_i by an edge so that these new edges do not cross. See Figure 5.7. Let G' be the resulting graph. Clearly, $e(G') = e$ and G' has no C_4 as a subgraph.

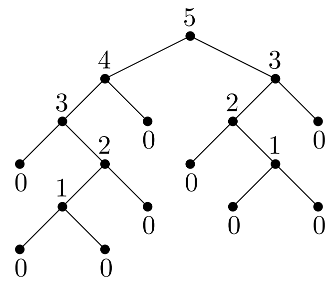


Figure 5.8: The heights of the nodes in the tree T.

Since we did not create new crossings, we have $\mathrm{cr}(G') \leq \mathrm{cr}(G)$. A vertex v of degree $d(v)$ was replaced by $\lceil d(v)/\Delta \rceil \leq d(v)/\Delta + 1$ new vertices. This implies that

$$v(G') \leq \sum_{v \in V(G)} \left(\frac{d(v)}{\Delta} + 1 \right) = n + \frac{2e}{\Delta} = n + \frac{2e}{\lfloor 4e/n \rfloor} \leq n + \frac{2e}{4e/n - 1}$$
$$= n + \frac{2en}{4e - n} = n + \frac{2en - 1/2}{4e - n} + \frac{1}{8e - 2n} \leq \frac{3n}{2} + \frac{1}{2} \leq 2n.$$

Thus, if we prove that $\mathrm{cr}(G') \geq ce^4/v(G')^3$, this will imply that $\mathrm{cr}(G) \geq (c/8) \cdot e^4/n^3$. Hence, for the rest of the proof we assume that all degrees in G are at most $4e/n$.

Now we describe the recursive decomposition of G in detail. Let V be the vertex set of G. In step i, we will have a decomposition of V into several subsets, and these subsets will be arranged in a rooted tree T_i whose root is V, each node has either two children or is a leaf, and each node that is not a leaf is the union of its two children. The leaves of T_i are exactly the sets of the decomposition. In the beginning, we have a single set V in the decomposition, and the corresponding tree T_0 has just one vertex—the root.

We set $s = e^2/(16n^2)$. Let $i \geq 0$. If $i \geq 1$ and all the leaves in T_i have at most s vertices, we stop. Otherwise, if $i \geq 1$, let W be a leaf of T_i with more than s vertices. If $i = 0$, let $W = V$. We cut $G[W]$ into two parts W_1, W_2 with at most $2|W|/3$ vertices each, by removing $b(G[W])$ edges from $G[W]$. Note that here we used just the definition of the bisection width. We attach W_1 and W_2 as children of W to T_i and obtain a tree T_{i+1}.

Let T be the tree obtained by the decomposition algorithm. For a node W of T, the *height* of W is the length of the longest path from W to a leaf in its subtree. That is, the leaves of T are exactly the nodes of height 0, the nodes whose both children are leaves are the nodes of height 1 and so on. See Figure 5.8. For $i \geq 0$, let $\mathcal{A}_i$ be the set of nodes of T of height i. Observe that each $\mathcal{A}_i$ is a decomposition of V. Let h be the height of the root of T.

Since G has no C_4 as a subgraph, Theorem 4.9 implies that $e \le n^{3/2} \;\Rightarrow\; e^2 \le n^3 \;\Rightarrow\; n \ge e^2/n^2 = 16s$. It follows that T has at least 16 leaves, in particular, $h \ge 2$. It also follows that every set of $\mathcal{A}_0$ has at least $s/3$ (and less than s) vertices, and every set of $\mathcal{A}_1$ has at least s vertices. By induction, for every $i = 1, 2, \ldots, h$, every set of $\mathcal{A}_i$ has at least $(3/2)^{i-1} \cdot s$ vertices. This implies that $|\mathcal{A}_0| \le 3n/s$ and $|\mathcal{A}_i| \le (n/s) \cdot (2/3)^{i-1}$ for every $i = 1, 2, \ldots, h$.

By Theorem 4.9, the total number of edges in $\bigcup_{W \in \mathcal{A}_0} G[W]$ is at most $|\mathcal{A}_0| \cdot s^{3/2} \le (3n/s) \cdot s^{3/2} = 3ns^{1/2} = 3e/4$. Therefore, we have deleted at least $e/4$ edges during the decomposition.

During the decomposition algorithm, we deleted $b(G[W])$ edges from every $G[W]$ such that $W \in \mathcal{A}_h \cup \mathcal{A}_{h-1} \cup \cdots \cup \mathcal{A}_1$. By Corollary 5.24 applied to each decomposition $\mathcal{A}_i$ with $i \in \{1, 2, \ldots, h\}$, we have

$$\frac{e}{4} \le \sum_{i=1}^{h} \sum_{W \in \mathcal{A}_i} b(G[W]) \le \sum_{i=1}^{h} 2\sqrt{|\mathcal{A}_i|} \cdot \sqrt{16\mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2}$$

where $d_1, d_2, \ldots, d_k$ is the degree sequence of G. Since $d_i \le \Delta \le 4e/n$, we have $\sum_{k=1}^{n} d_k^2 \le 16e^2/n$. Further we have

$$\sum_{i=1}^{h} \sqrt{|\mathcal{A}_i|} \le \sqrt{\frac{n}{s}} \cdot \sum_{i=0}^{h-1} \left(\sqrt{\frac{2}{3}}\right)^{i} \le \frac{4n^{3/2}}{e} \cdot \frac{1}{1 - \sqrt{2/3}} \le \frac{22n^{3/2}}{e}.$$

Putting this together, we get

$$\frac{e}{4} \le \frac{44n^{3/2}}{e} \cdot \sqrt{16\mathrm{cr}(G) + 16e^2/n},$$

which implies that

$$\mathrm{cr}(G) \ge \frac{e^4}{16 \cdot (4 \cdot 44)^2 \cdot n^3} - \frac{e^2}{n} \ge \frac{2e^4}{10^6 \cdot n^3} - \frac{e^2}{n}.$$

By our assumption, $e \ge 1000n$, so $e^2/n \le e^4/(10^6 n^3)$. Therefore, we have

$$\mathrm{cr}(G) \ge \frac{e^4}{10^6 \cdot n^3}$$

and we are finished. □

The proof of Theorem 5.25 can be also adapted to give an alternative proof of the crossing lemma: we only choose a different threshold $s = e/(2n)$. Similarly, Theorem 5.25 can be generalized to give an improved lower bound on the crossing number for graphs with no $K_{s,t}$ as a subgraph.

Chapter 6

Halving segments

Definition 6.1. Given a set P of n points in the plane in general position, a segment s connecting two points of P is called a **halving segment** or a **halving edge** if each open halfplane determined by s contains $\lfloor (n-2)/2 \rfloor$ or $\lceil (n-2)/2 \rceil$ points of P. That is, the number of points of P on the left side of s is the same as the number of points of P on the right side of s if n is even, or the two numbers differ by 1 if n is odd. The line extending a halving segment is called a **halving line**.

Observe that in a set of n points in convex position in the plane, there are exactly $n/2$ halving segments if n is even, and n halving segments if n is odd.

6.1 Upper bounds

What is the maximum possible number of halving segments of a set of n points in the plane? Lovász [43] obtained the upper bound $O(n^{3/2})$, which was later improved by Pach, Steiger and Szemerédi [53] to $O(n^{3/2}/\log^* n)$. The best known bound is $O(n^{4/3})$, first proved by Dey [19]. First we present Lovász' approach.

Theorem 6.2 (Lovász, 1971 [43])**.** *For n even, the maximum number of halving segments in a set of n points in the plane in general position is $O(n^{3/2})$.*

Proof. It is an easy exercise to show that the geometric graph G formed by halving segments has the following property: for every vertex v and every pair of halving segments s_1, s_2 (edges of G) incident to v, the cone opposite to the convex cone determined by s_1 and s_2 contains another halving segment incident with v. By the same property, the cone contains, in fact, exactly one halving segment incident with v. Also, every vertex has at least one halving segment incident to it. If we assume that no two vertices of G lie on a vertical line, it follows that the degree of every vertex is odd and the number of neighbors of v that are to the left of v differs from the number of right neighbors by 1.

The crucial observation is that every vertical line intersects at most $n/2$ halving segments. To show this, start with a vertical line p that crosses k halving segments. Assume without loss of generality that p has at most $n/2$ vertices on its left. Start translating p to the left. The number of halving segments intersected by p changes only when p passes through a vertex, and then it changes by exactly 1. After at most $n/2$ such changes all the vertices will be to the right of p, so p will not intersect any halving segment. Therefore, $k \leq n/2$.

To finish the proof, draw vertical lines $p_1, p_2, \ldots, p_{\lceil\sqrt{n}\rceil}$ so that in every region between p_i and p_{i+1}, to the left of p_1, and to the right of $p_{\lceil\sqrt{n}\rceil}$, there are at most $\sqrt{n}$ vertices. The number of halving segments crossing at least one of the lines p_i is $O(n\sqrt{n})$, since every vertical line intersects $O(n)$ halving segments. The number of halving segments that are disjoint with all the lines p_i is $O(n\sqrt{n})$, since each of the $\lceil\sqrt{n}\,\rceil + 1$ regions contains only $O(n)$ pairs of vertices. □

The following remarkable identity combined with the crossing lemma gives an improvement on the Lovász' bound.

Theorem 6.3 (Andrzejak et al., 1998 [7]). *Let n be an even positive integer. Let G be the geometric graph determined by the halving segments of n points in the plane in general position. Let k be the number of crossing in G. Then we have*

$$k + \sum_{v \in V(G)} \binom{(d(v)+1)/2}{2} = \binom{n/2}{2}.$$

Idea of the proof. Start with n points in convex position and move them continuously one by one to the vertices of G. In convex position there are $n/2$ halving segments, every two of them cross, and each of the n points is incident with exactly one halving segment, so the equality holds. During the continuous motion of the vertices, the elementary changes to the graph of halving segments do not affect the validity of the identity. □

Theorem 6.4. *For n even, the maximum number of halving segments in a set of n points in general position in the plane is $O(n^{4/3})$.*

Proof. Let G be the graph of halving segments. Theorem 6.3 implies that $\mathrm{cr}(G) = O(n^2)$. However, by the crossing lemma either $e(G) = O(n)$ or $cr(G) = \Omega(e^3/n^2)$. In any case, $e(G) = O(n^{4/3})$. □

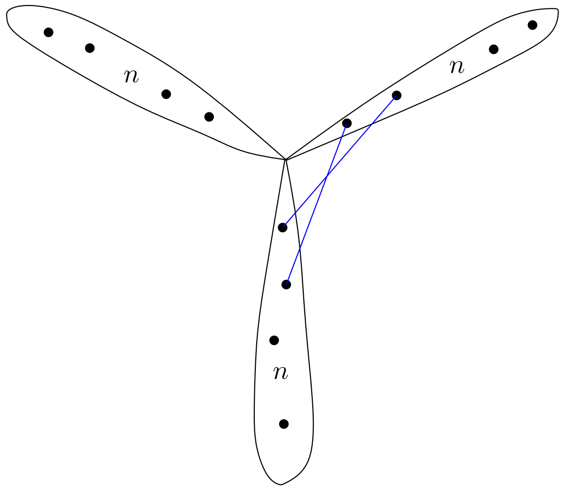


Figure 6.1: Inductive construction for halving segments. Two of the "new" $3n/2$ halving segments are drawn.

6.2 Lower bounds

Theorem 6.5 (Lovász, 1971 [43]). *For n even, there are sets of n points in general position in the plane with $\Omega(n \log n)$ halving segments.*

Proof. The construction is done by induction. Suppose we have a construction with n points and $h(n)$ halving segments. Then we can build a configuration with $3n$ points and $h(3n) \geq 3h(n) + cn$ halving segments, for a certain constant $c > 0$, as follows. We squash the configuration of n points to make it look almost like a segment, we take three copies of it and arrange them in three directions separated by 120° as in Figure 6.1. We have $h(3n) \geq 3h(n) + 3n/2$, which gives $h(n) = \Omega(n \log n)$. when h is nondecreasing. It is a simple exercise to show that from a configuration of n points with s halving segments one can create a configuration of $n + 2$ points with s halving segments. $\square$

The lower bound has been significantly improved: Tóth [66] constructed a set of n points with $ne^{\Omega(\sqrt{\log n})}$ halving segments. Nivasch [47] simplified the construction and improved the lower bound to $\Omega(ne^{\sqrt{\ln 4}\sqrt{\ln n}}/\ln n)$. (We use "log" for the binary logarithm and "ln" for the natural logarithm.)

Theorem 6.6 (Nivasch, 2008 [47]). *For n even, there are sets of n points in general position in the plane with $\Omega(ne^{\sqrt{\ln 4}\sqrt{\ln n}}/\ln n)$ halving segments.*

Instead of a set of points, we construct a *dual* set of lines. We use the following notion of duality.

Definition 6.7. Given a point $p = (c, d) \in \mathbb{R}^2$, the **dual** (or the **dual line**) of p is the (non-vertical) line $p^* = \{(x, y) \in \mathbb{R}^2; y = cx - d\}$. Given a non-vertical line $\ell = \{(x, y) \in \mathbb{R}^2; y = ax - b\}$, the **dual** (or the **dual point**) of ℓ is the point $\ell^* = (a, b) \in \mathbb{R}^2$.

The following observation is left as an exercise.

Observation 6.8. *Suppose that p is a point in $\mathbb{R}^2$ and ℓ is a non-vertical line in $\mathbb{R}^2$. Then*

a) $(p^*)^* = p$ *and* $(\ell^*)^* = \ell$,

b) $p \in \ell$ *if and only if* $\ell^* \in p^*$,

c) p *lies above* ℓ *if and only if* p^* *lies below* ℓ^*.

Definition 6.9. A finite set L of lines in the plane forms an **arrangement of lines** $\mathcal{A}(L)$, which is the decomposition of the plane into **vertices**, **edges** and **cells**, where the vertices are the intersection points of the lines, the edges are the open segments or rays of the lines that remain after removing the vertices, and the cells are the 2-dimensional open regions that are the connected components of $\mathbb{R}^2 \setminus (\bigcup L)$. The vertices, edges and cells are also called 0-, 1- and 2-dimensional **faces** of the arrangement, respectively.

The **level** of a point p with respect to a set of lines L (or with respect to an arrangement $\mathcal{A}(L)$) is the number of lines of L that lie strictly below p. The **level** of a face of $\mathcal{A}(L)$ is the level of an arbitrary point of the face with respect to $\mathcal{A}(L)$.

The following observation follows from Observation 6.8.

Observation 6.10. *Let n be even and let P be a set of n points in the plane in general position. Let L be the set of n dual lines of the points of P, and suppose that L is in general position, that is, no two lines of L are parallel, no three lines of L pass through the same point. Moreover, suppose that no line of L is vertical. Then the dual of a halving line of P is a vertex of $\mathcal{A}(L)$ of level $n/2 - 1$.*

We will call the vertices of $\mathcal{A}(L)$ of level $n/2 - 1$ the **middle-level vertices**. Similarly, the cells of $\mathcal{A}(L)$ of level $n/2$ will be called the **middle-level cells**. By Observation 6.10, constructing a point set with many halving lines is equivalent to constructing an arrangement of lines with many middle-level vertices.

Proof of Theorem 6.6

First we describe the construction, then we verify its correctness, and finally we count the middle-level vertices.

The construction

We construct an infinite sequence $L_0, L_1, L_2, \dots$ of sets of non-vertical lines in the plane in general position. Every line in every L_m, $m \geq 0$, is of one of two types: **plain** or **bold**. For every L_m, we construct a set V_m of middle-level vertices of $\mathcal{A}(L_m)$ such that each of them lies in the intersection of a plain line and a bold line. The set V_m does not necessarily contain all the middle-level vertices with this property. The construction depends on free parameters $a_0, a_1, a_2, \dots$, which we choose as $a_0 = 0$ and $a_m = 2^m$ for $m \geq 1$.

The base case, L_0, consists of a plain line ℓ_0, a bold line b_0, and a vertex v_0 in their intersection. We set $V_0 = \{v_0\}$.

Now we describe the inductive step, which is the heart of the construction. Let $m \geq 0$ and suppose that L_m and V_m have been constructed. We construct L_{m+1} and V_{m+1} as follows.

Each plain line $\ell \in L_m$ is replaced by a *bundle* of a_{m+1} plain lines parallel and close to ℓ separated by a very small distance $\varepsilon_m > 0$. The new lines are then slightly perturbed into general position, but only so little that within a square containing V_m, their displacement is almost imperceptible and they still appear *almost parallel*. Each bold line $b \in L_m$ is replaced by a bundle of $a_{m+1} + 1$ plain lines parallel and close to b separated by a very small distance $\delta_m > 0$ that is much smaller than ε_m. Again, the new lines are then slightly perturbed into general position. We will call this step a *uniform replacement*.

For every vertex $v \in V_m$, the uniform replacement creates an $a_{m+1} \times (a_{m+1} + 1)$ grid G_v in place of v; see Figure 6.2. We then draw a new bold line b'_v along the diagonal of the grid, so that its crossings with the lines of the two bundles alternate. We add these $2a_{m+1} + 1$ vertices of L_{m+1} to the set V_{m+1}. We assume that δ_m is so small compared to ε_m, that b'_v is very close to the original bold line in L_m that contained v.

The correctness

We need to verify that all the vertices in V_{m+1} are, indeed, middle-level vertices of $\mathcal{A}(L_{m+1})$. We will show a stronger property, which is needed to show this by induction. We say that a point v is **strongly balanced** in a subset L of L_m if the number of plain lines in L above v is equal to the number of plain lines in L below v, and the number of bold lines in L above v is equal to the number of bold lines in L below v.

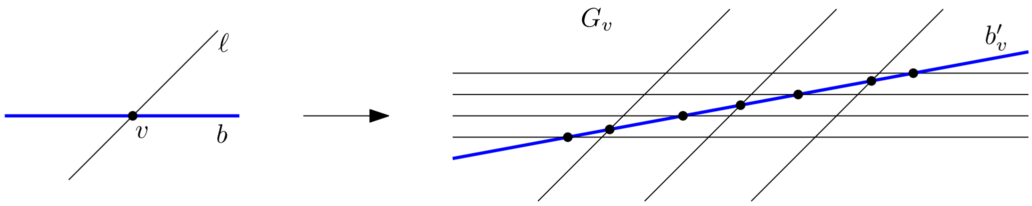


Figure 6.2: A uniform replacement creating the grid G_v from v, and a new bold line b'_v. The vertex v of V_m is replaced by $2a_m + 1$ vertices in V_{m+1}. Here $a_m = 3$.

Lemma 6.11. *For every $m \geq 0$, all vertices in V_m are strongly balanced in L_m.*

Proof. We proceed by induction on m. For $m = 0$, the vertex v_0 in L_0 is strongly balanced in L_0, since there are no lines of L_0 above or below it. For the induction step, let $m \geq 0$ and suppose that the lemma is true for all vertices of V_m. We will prove it for the vertices of V_{m+1}.

Each bold line b in L_m contains $2a_m + 1$ vertices of V_m. Moreover, the slopes of the plain lines passing through these vertices alternate between larger and smaller than the slope of b, since these plain lines form a grid in the construction. Let $v \in V_m$. Let b be the bold line and ℓ the plain line containing v. Let b'_v be the bold line of L_{m+1} through the grid G_v. Let $w \in V_{m+1} \cap b'_v$. Let $v_1, v_2, \ldots, v_{2a_m}$ be the vertices of $V_m \cap b$ other than v, and let $b'_1, b'_2, \ldots, b'_{2a_m}$ be the corresponding bold lines in L_{m+1}.

Partition L_{m+1} into three sets, S_1, S_2 and S_3, as follows. Let S_1 be the set of all lines of L_{m+1} created from lines other than ℓ or b (the set S_1 contains both plain and bold lines). By induction, v is strongly balanced in L_m. This implies that after the uniform replacement, w is strongly balanced in S_1.

Let S_2 be the set of lines of the grid G_v and the line b'_v; see Figure 6.2. The location of b'_v along the diagonal of the grid implies that w is strongly balanced in S_2.

Let $S_3 = \{b'_1, b'_2, \ldots, b'_{2a_m}\}$. These lines, together with b'_v, are created from b, and also satisfy the property that their slopes alternate between larger and smaller than the slope of b. If there are an even number of the points $v_1, v_2, \ldots, v_{2a_m}$ to the right of v (and even number to the left of v), half of the corresponding lines b'_i is above v (and thus above w) and the other half below v (and thus below w). If there are an odd number of points $v_1, v_2, \ldots, v_{2a_m}$ to the right of v (and odd number to the left of v), the two lines b'_i and b'_j corresponding to the points v_i and v_j closest to v from left and right, respectively, have both larger or both smaller slope than b. But this means that one of the lines b'_i, b'_j is above v (and w) and the other below v (and w). The rest follows from the previous even case. This shows that w is strongly balanced in S_3. Altogether, w is strongly balanced in $S_1 \cup S_2 \cup S_3 = L_{m+1}$. $\square$

Computations

For $m \geq 0$, let $n_m = |L_m|$ and $f_m = |V_m|$. Recall that $a_0 = 0$ and $a_m = 2^m$ for $m \geq 1$. From the construction of L_0 we have $n_0 = 2$ and $f_0 = 1$. By the construction of V_{m+1}, we have

$$f_{m+1} = (2a_{m+1} + 1) \cdot f_m.$$

Now we count the number of lines in L_{m+1}. For every $i \geq 1$, the number of bold lines in L_i is equal to the number of vertices in V_{i-1}, which is equal to f_{i-1}. The number of plain lines in L_i is thus $n_i - f_{i-1}$. By the uniform replacement, it follows that the number of plain lines in L_{m+1} is $a_{m+1} \cdot (n_m - f_{m-1}) + (a_{m+1} + 1) \cdot f_{m-1} = a_{m+1} n_m + f_{m-1}$. The number of bold lines in L_{m+1} is f_m. Together, the number of lines in L_{m+1} is

$$n_{m+1} = a_{m+1} n_m + f_m + f_{m-1}.$$

Now by a straightforward induction, we have

$$f_m = f_0 \cdot (2a_1 + 1)(2a_2 + 1) \cdots (2a_m + 1) = (2^2 + 1)(2^3 + 1) \cdots (2^{m+1} + 1).$$

It is an easy exercise to show that $f_m = \Theta\big(2^{(m^2+3m)/2}\big)$. Plugging this into the recursion for n_m, we get

$$n_m = 2^m \cdot n_{m-1} + \Theta(f_{m-1}) = 2^m \cdot n_{m-1} + \Theta\big(2^{(m^2+m)/2}\big).$$

Let $n'_0, n'_1, n'_2, \ldots$ be a sequence satisfying the recursion $n'_m = 2^m \cdot n'_{m-1} + k \cdot 2^{(m^2+m)/2}$, where k is a constant. It is a straightforward exercise to verify that $n'_m = 2^{(m^2+m)/2} \cdot (n'_0 + km)$. Since the sequence n_m is "sandwiched" between two sequences of the type n'_m just with a different constant k, we conclude that $n_m = \Theta\big(m \cdot 2^{(m^2+m)/2}\big)$. This further implies that

$$\begin{aligned}
\log n_m &= \log m + \frac{m^2 + m}{2} + \Theta(1) \\
\Rightarrow\ 2 \log n_m &= m^2 + m + 2\log m + \Theta(1) = \big(m + \Theta(1)\big)^2 \\
\Rightarrow\ m &= \sqrt{2 \log n_m} - \Theta(1)
\end{aligned}$$

and also that

$$\frac{f_m}{n_m} = \Theta(2^m/m).$$

Combining the last two expressions we get

$$\begin{aligned} f_m &= \Theta\left(n_m \cdot 2^{\sqrt{2\log n_m}-\Theta(1)}/\left(\sqrt{2\log n_m}-\Theta(1)\right)\right) \\ &= \Theta\left(n_m \cdot 2^{\sqrt{2\log n_m}}/\sqrt{\log n_m}\right) \\ &= \Theta\left(n_m \cdot e^{\ln 2\sqrt{2}\sqrt{\ln n_m}/\sqrt{\ln 2}}/\sqrt{\ln n_m}\right) \\ &= \Theta\left(n_m \cdot e^{\sqrt{\ln 4}\sqrt{\ln n_m}}/\sqrt{\ln n_m}\right). \end{aligned}$$

We have finished the proof of Theorem 6.6 for $n = n_m$. To prove it for all even n, we need to "fill the gaps" between consecutive members of the sequence n_m [66]. For this, we need to observe that if a set of n points in the plane in general position has s halving segments, then we can add two points so that the resulting set still has at least s halving segments (thus the maximum number of halving segments is a nondecreasing function for even n). But this is not enough, since the sequence n_m grows very fast. So we use a second observation, stating that if a set of n points in the plane has s halving segments and a is a positive integer, then there is a set of an points in the plane with at least as halving segments. This will be sufficient to interpolate the lower bound on the number of halving segments for even values of n and finish the proof of the theorem. This, including the two observations, is left as an exercise.

Chapter 7

Ramsey theory

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $R(5, 5)$. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number $R(6, 6)$, however, we would have no choice but to launch a preemptive attack.

— *Paul Erdős*

7.1 Ramsey's theorem

Among six people at a party, one can always find three who mutually know each other or three that mutually do not know each other. In a mathematical translation, this can be stated as follows: given a 2-coloring of the edges of K_6 with, say, red and blue color, there exists a red triangle or a blue triangle (or both).

Proof. Let v be a vertex of K_6. This vertex has degree 5. Given a 2-coloring of the edges, there must be at least three edges containing v of the same color, say, red. See Figure 7.1. If some pair of the other vertices of the red edges is joined with a red edge, they create a red triangle. Otherwise, the three neighbors of v are pairwise joined by blue edges, so they form a blue triangle. $\square$

The following more general statement is known as Ramsey's theorem.

Theorem 7.1 (Ramsey, 1930 [57]). *For any $k \in \mathbb{N}$, there exists a smallest number $n = R(k)$ such that in any coloring of the edges of K_n by two colors, there is always a monochromatic copy of K_k (that is, a copy of K_k with all*

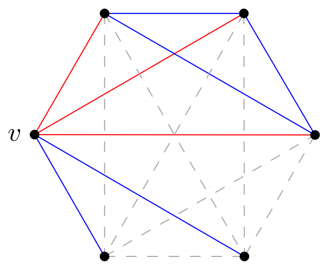


Figure 7.1: Sketching the proof of the sociological statement.

edges of the same color). In other words, every graph with n vertices has either a complete or an empty subgraph with k vertices.

The function $R(k)$ has yet to be determined. Classical results of Erdős [22] and Erdős and Szekeres [23] give the exponential bounds

$$2^{k/2} \leq R(k) \leq 2^{2k}.$$

Despite many improvements during the last sixty years (see [18] for example), the constant factors in the exponents remain the same.

Theorem 7.2 (Ramsey's theorem; Erdős–Szekeres, 1935 [23])**.** *Every graph with n vertices has either a complete or an empty subgraph with $\frac{1}{2}\log_2 n$ vertices.*

A more refined form of Ramsey's theorem is the following.

Theorem 7.3 (Ramsey's theorem, off-diagonal version)**.** *For every $r, s \in \mathbb{N}$, there exists a smallest number $R(r, s)$ such that any complete graph on $R(r, s)$ vertices with edges colored red and blue has a complete subgraph with r vertices with all edges red or a complete subgraph with s vertices with all edges blue.*

Proof. By induction on $r + s$, we show that $R(r, s) \leq \binom{r+s-2}{r-1}$. By definition, for all n, we have $R(n, 1) = R(1, n) = 1$. This proves the base case. Suppose that the claim is true for $R(r-1, s)$ and $R(r, s-1)$; we show that it is also true for $R(r, s)$.

We now claim that $R(r, s) \leq R(r-1, s) + R(r, s-1)$. This will imply that $R(r, s) \leq \binom{r+s-2}{r-1}$ since $\binom{r+s-2}{r-1} = \binom{r+s-3}{r-2} + \binom{r+s-3}{r-1}$. Let G be a complete graph with $R(r-1, s) + R(r, s-1)$ vertices, with edges colored red and blue. Pick a vertex v from G and partition the remaining vertices into two sets A, B so that the vertices in A are joined to v by a red edge and the vertices from B are joined to v by a blue edge. Since G has $R(r-1, s) + R(r, s-1) =$

$|A|+|B|+1$ vertices, it follows that either $|A| \geq R(r-1,s)$ or $|B| \geq R(r,s-1)$. Suppose that $|A| \geq R(r-1,s)$. By induction, $G[A]$ contains a blue copy of K_s, in which case we are finished, or a red copy of K_{r-1}, which together with v forms a red copy of K_r, so we are finished as well. The case that $|B| \geq R(r,s-1)$ is symmetric. □

Erdős's lower bound for the Ramsey numbers was one of the first applications of the probabilistic method in combinatorics.

Theorem 7.4 (Lower bound on Ramsey numbers; Erdős, 1947 [22])**.** *For $k \geq 2$, the Ramsey numbers $R(k)$ satisfy*

$$R(k) \geq 2^{k/2}.$$

Proof. Let k, n be fixed positive integers and let $G = K_n$. Color every edge of G independently red or blue, each color with probability 1/2. Since the number of edges of K_n is $\binom{n}{2}$, there are $2^{\binom{n}{2}}$ colorings and each of them has probability $2^{-\binom{n}{2}}$. For every subset $S \subseteq V(G)$ of k vertices, the probability that all the edges of $G[S]$ have the same color is $2 \cdot 2^{-\binom{k}{2}}$ (there are exactly two such colorings). By the union bound, the probability that there exists a subset $S \subseteq V(G)$ of k vertices such that all the edges of $G[S]$ have the same color, is at most

$$\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} \leq \left(\frac{en}{k}\right)^k \cdot \frac{2}{2^{\binom{k}{2}}} \leq 2^{k\log n + k\log e - k\log k + 1 - \binom{k}{2}}.$$

If this is less than 1, then there exists a 2-coloring of G with no monochromatic copy of K_k (but we do not know how to construct it!). If we choose $n < 2^{k/2}$, then

$$\begin{aligned} &k\log n + k\log e - k\log k + 1 - \binom{k}{2} < \frac{k^2}{2} + k\log e - k\log k - \frac{k^2}{2} + \frac{k}{2} \\ &= k(\log e + 1/2 - \log k) < k(1.95 - \log k). \end{aligned}$$

If $k \geq 4$, then $1.95 - \log k \leq 1.95 - 2 < 0$, so $R(k) \geq 2^{k/2}$. Clearly, $R(2) \geq 2 = 2^{2/2}$. Finally, the graph C_5 shows that $R(3) \geq 5 > 2^{3/2}$. This finishes the proof. □

Ramsey proved a more general form of his theorem: for more than two colors, and for colorings of r-tuples instead of just pairs.

Theorem 7.5 (Ramsey, 1930 [57])**.** *For every $k, r, c \in \mathbb{N}$, there exists a smallest number $n = R_r(k;c)$ such that in any coloring of the r-tuples of elements of $\{1, 2, \ldots, n\}$ by c colors, there is always a k-element subset $A \subseteq \{1, 2, \ldots, n\}$ such that all r-tuples of elements from A have the same color.*

The proof is analogous to the proof of the graph version. For $r \geq 3$, the gaps between the current best lower and upper bounds on $R_r(k;c)$ are enormous.

Erdős and Szekeres proved the following geometric analogue of Ramsey's theorem.

Theorem 7.6 (Erdős–Szekeres theorem, 1935 [23]). *For every $k \in \mathbb{N}$, there exists a smallest number $s(k)$ such that in any set of $s(k)$ points in the plane in general position, there is always a set of k points forming the vertices of a convex k-gon.*

Proof. Let $n = R_3(k;2)$. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n points in the plane in general position. Define the coloring of triples of S as follows: for $i < j < l$, a triple (p_i, p_j, p_l) is red if the triangle $p_ip_jp_l$ is oriented clockwise, and blue if the triangle $p_ip_jp_l$ is oriented counter-clockwise. By Theorem 7.5, there is a subset $X \subseteq P$ of k points such that all the triples in X have the same color. It is an easy exercise to show that X is the set of vertices of a convex k-gon. This shows that $s(k) \leq R_3(k;2)$. ☐

Erdős and Szekeres [23] gave two proofs of Theorem 7.6: the first was a direct application of Ramsey's theorem, but to 4-tuples instead of triples. The second is a direct proof by induction, which gives a more reasonable upper bound $s(k) \leq \binom{2k-4}{k-2} + 1$, by induction similar to that in the proof of Theorem 7.3. The best current upper bound, $s(k) \leq \binom{2k-5}{k-2} + 1$, is due to Tóth and Valtr [67].

7.2 Ramsey-type theorems in geometric graphs

Theorem 7.7 (Károlyi, Pach and Tóth, 1997 [37]). *Every complete geometric graph whose edges are colored by two colors has a noncrossing monochromatic spanning tree.*

Proof. Let G be a complete geometric graph with n vertices whose edges are colored by two colors, red and blue. We proceed by induction on n. For $n \leq 3$ the theorem is trivial. Let $k \geq 4$ and suppose that theorem has been proved for $n \leq k - 1$.

If there are two consecutive edges of the convex hull of G of different colors, then we remove their common vertex, and in the remaining graph with $k - 1$ vertices we find a monochromatic noncrossing spanning tree by induction. By adding one of the two edges of the convex hull, we obtain a monochromatic noncrossing spanning tree of G.

It remains to solve the case where all the edges of the convex hull of G are of the same color, say, blue. Denote the vertices of G by $v_1, \ldots, v_n$, sorted from left to right according to their x-coordinates. We assume without loss of generality that no two of the x-coordinates are equal (we can rotate the drawing otherwise). By induction, for every $i \in \{2, 3, \ldots, k-1\}$ there is a *left* monochromatic noncrossing tree L_i in $G[\{v_1, v_2, \ldots, v_i\}]$ and a *right* monochromatic noncrossing tree R_i in $G[\{v_i, v_{i+1}, \ldots, v_k\}]$. We can assume that for every i the two trees are of different colors, otherwise we can join them to a noncrossing monochromatic spanning tree of G. Since v_1v_2 and $v_{k-1}v_k$ are edges of the convex hull of G, the trees L_2 and R_{k-1} are both blue, and thus L_{k-1} is red. Now it follows that there is an i such that L_i is blue and L_{i+1} is red, hence R_{i+1} is blue. Finally, we connect the two blue trees L_i and R_{i+1} by one of the two available blue edges from the convex hull of G. $\square$

Chapter 8

Geometric extremal graph theory

This chapter is not part of the course in 2014/2015.

Definition 8.1. By $ex(n, G)$ we denote the maximum number of edges that a graph with n vertices can have without containing a subgraph isomorphic to G. Analogously, for geometric and convex geometric graphs, we introduce $\overline{example}(n, \overline{G}) \geq \overline{example}_c(n, \overline{G})$ (here $\overline{G}$ denotes a geometric graph).

One of the most famous result in extremal graph theory is:

Theorem 8.2 (Turán)**.**

$$ex(n, K_t) = \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right) + O(n)$$

The following theorem is a useful generalization of Turan's theorem.

Theorem 8.3 (Erdős-Stone, Simonovits)**.** *For any nonempty graph H*

$$ex(n, H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1}\right) + o(n^2)\,,$$

where $\chi(H)$ denotes the chromatic number of H.

Theorem 8.4. $\overline{example}(n, C_k) = ex(n, K_k)$*, where C_k denotes a non-selfintersecting geometric k-cycle.*

Proof. Let G be a geometric graph with n vertices and more than $ex(n, K_k)$ edges. Then G contains a complete geometric graph on k points. Every set of k points in the plane is the vertex set of a non-selfintersecting polygon. Therefore, there is an isomorphic copy of C_k in G.

On the other hand, it is easy to construct a convex geometric graph G with n vertices and $ex(n, K_k)$ edges that does not contain any simple closed polygon of length k. $\square$

Definition 8.5. A geometric graph is called *outerplanar* if it can be obtained by adding some noncrossing internal diagonals to a simple closed polygon in the plane. Two geometric graphs are said to be *isomorphic* if they are topologically equivalent, that is, if there exists a one-to-one continuous mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that carries one into the other.

We can generalize the previous theorem to arbitrary outerplanar graphs.

Theorem 8.6 (Pach, Perles)**.** *Let $\overline{G}$ be an outerplanar graph, that is, a triangulated C_k. Then $\overline{example}(n, \overline{G}) = ex(n, K_t)$*

All we need for the proof is the following lemma (the rest is the same as above).

Lemma 8.7 (Gritzmann-Mohar-Pach-Pollack)**.** *Let G be an outerplanar geometric graph with k vertices, and let P be any set of k points in the plane, no three of which are collinear. Then there exists a geometric graph G' isomorphic to G, with vertex set $V(G') = P$.*

Proof. We can assume that G is triangulated. We will prove a stronger statement: if x_1 and x_2 are two consecutive vertices of the closed polygon bounding G and p_1 and p_2 are two consecutive vertices of the convex hull of P, then we can find an assignment such that $x_1 \to p_1$ and $x_2 \to p_2$. The proof is by induction on k. For $k = 3$ it is easy. Let $k > 3$ and let $x_1, \ldots, x_k$ be the vertices of G listed clockwise. Let x_i be the common neighbor of x_1 and x_2 in G. The following statement follows easily from a continuity argument: there exists a point $p \in P - \{p_1, p_2\}$ such that:

1. there are no points of P inside the triangle p_1p_2p;

2. there is a line l passing through p, separating p_1 from p_2 and not containing other points from P, such that there are exactly $i - 2$ elements of P in the open halfplane bounded by l which contains p_2.

Having this claim, we can proceed by induction. $\square$

Proposition 8.8. *If $\overline{G}$ has a non-crossing cycle then $\overline{example}_c(n, \overline{G}) \geq const \cdot n^2$.*

Proposition 8.9. *If $\overline{G}$ has a non-crossing Y then $\overline{example_c}(n,\overline{G}) \geq const \cdot n^2$.*

Theorem 8.10 (Perles)**.** *Let $\overline{G}$ be a non-crossing caterpillar with k vertices. Then*

$$\overline{example_c}(n,\overline{G}) = \left\lfloor \frac{n(k-2)}{2} \right\rfloor$$

Definition 8.11. The convex chromatic number $\chi_c(\overline{G})$ of a convex geometric graph $\overline{G}$ is the minimum number of colors in a good coloring such that the color classes consist of consecutive vertices.

Theorem 8.12 (Perles, Brass)**.** *If the convex chromatic number $\chi_c(\overline{G}) = t$, then*

$$\overline{example}(n,\overline{G}) = \overline{example}(n,K_t) + o(n^2)$$

Theorem 8.13 (Károlyi-Pach-Tóth 1997)**.** *For any 2-coloring of the edges of a complete geometric graph, one can find a non-crossing spanning tree all of whose edges are of the same color.*

Proof. By induction on n. If there are 2 consecutive edges of the convex hull of different colors, then we are done. So, we can assume that all edges along the convex hull are, say, blue. Denote the vertices by $v_1,\dots,v_n$ with increasing x-coordinates. By induction for every point v_i we have a left monochromatic tree L_i and a right monochromatic tree R_i. We can assume that for every i the trees are of different colors, otherwise, we are done. Similarly, we can assume that R_2 and L_{n-1} are red. Now it follows that there is an i such that L_i and R_{i+1} are both blue. We can connect these trees by using an edge from the convex hull. □

Theorem 8.14 (Károlyi-Pach-Tóth 1997)**.** *For any 2-coloring of the edges of a complete geometric graph of $3n-1$ vertices, one can find n disjoint edges of the same color.*

Theorem 8.15 (Károlyi-Pach-Tóth-Valtr 1998)**.** *For any 2-coloring of the edges of a complete geometric graph of n vertices, one can find a monochromatic non-crossing path of length $\geq cn^{\frac{2}{3}}$.*

Question. *Does there also exist such a path of length $\geq cn$?*

Question. *What is the largest number $f(n)$ such that every complete geometric graph with n vertices has $f(n)$ pairwise crossing edges?*

Theorem 8.16 (Aronov et al. 1994)**.** $f(n) \geq \sqrt{n/12}$

Chapter 9

Davenport–Schinzel sequences

This chapter is not part of the course in 2014/2015.

With the development of computers, and especially computer graphics, the following problem arose: Given a camera angle on a field (usually 3D) with objects. Which portions of the objects can be seen directly from the cameras angle of view. This problem is core to fast and efficient rendering/processing of 3D computer graphics, and was originally thought to be simple. As it turned out, this problem, even in 2D was particularly difficult to solve. Davenport and Shinzel attempted the problem in 2D first and the problem goes as follows: given n segments in the plane, what is the maximum number of subsegments comprising the upper envelope of them? We denote this number by $f(n)$.

It was initially conjectured that $f(n) = O(n)$ (Atallah, 1965, D.-Sch. 1985). The lower bound can be set trivially to n. The upper bound has been gradually reduced: $f(n) = O(n \log n)$, and then $f(n) = O(n \log^* n)$, where $\log^* n$ is the iterative logaritheorem function. However, the final answer was surprising (see the theorem below).

The underlying idea is the following: let us number the segments 1 through n and write down the sequence of the labels along the upper envelope from left to right. We obtain a sequence with the following properties:

1. no two consecutive numbers are equal;

2. there is no alternating subsequence of length 5, that is, there is no subsequence of the form $a \dots b \dots a \dots b \dots a$ (where $a \neq b$).

The first property is trivial by the construction of Davenport-Schinzel sequences. The second can be easily shown by using the fact that two straight lines can intersect at most once. A sequence satisfying the above properties is called a *Davenport-Schinzel sequence* of order n. The question now is

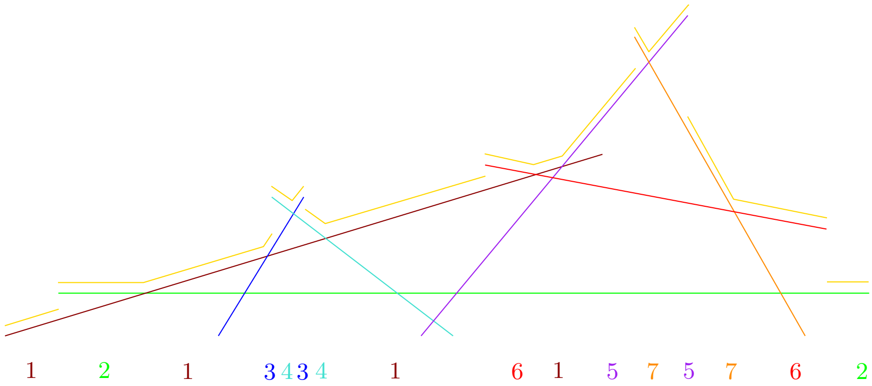


Figure 9.1: Upper envelope of segments in the plane. If we label all segments and scan from left to right which segments the upper envelope is comprised of, we obtain a Davenport-Schinzel sequence.

equivalent to: what is the largest length of a Davenport-Schinzel sequence of order n?

Theorem 9.1 (Hart-Sharir, 1984)**.** *The maximum length of a Davenport-Schinzel sequence of order n is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of Ackermann's function. The bound is tight.*

(The inverse Ackermann's function (even the iterative logaritheorem) has such a slow growth rate that, from a practical point of view, it can be considered as a constant. So, for all practical purposes, the upper bound can be seen as linear.)

Remark. The maximum length of a Davenport-Schinzel sequence with no alternating subsequence of length 4 is $2n-1$.

Klazar and Valtr considered a natural extension of Davenport-Schinzel sequences by relaxing the second property:

Theorem 9.2 (Klazar-Valtr, 1994)**.** *Consider all sequences formed by n symbols such that any l consecutive elements are pairwise distinct, and there is no subsequence of type $a_1a_2\dots a_{l-1}a_la_{l-1}\dots a_2a_1a_2\dots a_{l-1}a_l$. For every $l \geq 2$, the maximum length of such a sequence of $O(n)$.*

Valtr used this theorem to derive the following result on geometric graphs.

Theorem 9.3 (Valtr, 1997)**.** *The maximum number of edges that a geometric graph of n vertices can have without containing k pairwise crossing edges is $O(n \log n)$.*

Proof. We can assume that the x-coordinates of the vertices are pairwise different and that the y-axis partitions $V(G)$ into two parts which are of equal size (more precisely, as equal as possible). Suppose we are able to show that the number of edges that intersect the y axis is $O(n)$. For the maximum number of edges $f_k(n)$ we now have that

$$f_k(n) \leq 2 \times f_k(n/2) + O(n) ,$$

which implies $f_k(n) = O(n \log n)$.

It remains to prove that $|E'| = O(n)$, where E' is the set of edges that cross the y-axis. Let $e_1, \ldots, e_r$ be the edges in E' ordered in the order in which they cross the y-axis. Write on the left side of each crossing the label of the left endpoint of the corresponding edge and do the same thing on the right. In this way we get two sequences L and R of length $r = |E'|$ formed by n symbols. By Klazar-Valtr theorem the following two claims suffice to finish the proof.

Claim 1: For each $l \geq 1$ at least one of the sequences L and R contains a subsequence of length at least r/l^2 in which any l consecutive symbols are pairwise distinct.

Claim 2: Neither of sequences L and R contains a subsequence of type up-down-up($R(k)^2$), where $R(k)$ is the kth Ramsey number.

The first claim is proved as follows. We build two new sequences L' and R'. We start with empty sequences. Now we process the edges $e_1, \ldots, e_r$ one by one, at the ith step we add the left endpoint of e_i to L' if it does not violate regularity and the right endpoint of e_i to R' similarly. It cannot happen that we do not add anything in l^2 consecutive steps. Therefore, the longer list will have at least r/l^2 elements at the end.

As for the second claim, we proceed by contradiction. We start with a long up-down-up subsequence and by Erdős–Szekeres we extract from it a long subsequence in which the x-coordinates of the corresponding vertices in the second "down" part are monotone. Let the sequence of vertices be $v_1, \ldots, v_{R(k)}$ and the corresponding edges $e_1, \ldots, e_{R(k)}$. Consider the following 2-colored complete graph: the vertices are $v_1, \ldots, v_{R(k)}$ and the edge $v_i v_j$ with $x(v_i) < x(v_j)$ is blue iff v_j lies below the edge e_i, otherwise it is red. By Ramsey's theorem there is a monochromatic complete subgraph on k vertices. However, this implies that one can find k pairwise crossing edges in G. Contradiction. □

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