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Abstract

Let G be a compact Lie group with a maximal torus T. Based on a presentation of the integral cohomology ring $H^*(G/T)$ of the flag manifold G/T in [7] we have presented in [8] an explicit and unified construction of the integral cohomology rings $H^*(G)$ for the 1–connected Lie groups G. In this paper we extend this construction to all compact Lie groups.

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1 Introduction

In this paper the Lie groups under consideration are compact and connected, the coefficient for cohomology is either the ring $\mathbb{Z}$ of integers, or one of the finite fields $\mathbb{F}_p$, unless otherwise stated.

Let G be a Lie group with a maximal torus T. Based on a presentation of the integral cohomology $H^*(G/T)$ of the quotient manifold G/T we have presented in [8] a unified construction of the integral cohomology of the 1–connected Lie groups. In this paper we extend this construction to all compact Lie groups.

The paper is so arranged. In view of a Gysin type exact sequence for cyclic coverings of Lie groups we clarify in Section 2 our approach to reduce the cohomology of arbitrary Lie groups to that of the 1–connected ones. Sections 3, 4 and 5 develop theoretical results required to implement the procedure. They are applied in Section 6 to determine the cohomologies of the adjoint Lie groups $PSU(n), PSp(n), PE_6, PE_7$, see Theorems 6.6, 6.11 and 6.12.

The problem of computing the cohomology of Lie groups was raised by E. Cartan in 1929. It is a focus of algebraic topology due to the fundamental roles of Lie groups playing in geometry and topology, see [3, Chapter VI], [13]. On the other hand, the classical Schubert calculus amounts to the computation in the cohomology rings of the flag manifolds G/T [21, p.331]. The present work, together with the companion ones [8, 9], completes our project to determine the integral cohomologies of all compact Lie groups G, as well as the structure of the mod p cohomology $H^*(G;\mathbb{F}_p)$ as a module over the Steenrod algebra $\mathcal{A}_p$, in the context of Schubert calculus.

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2 Topological classification of Lie groups

All the 1–*connected simple Lie groups* G, together with their centers $\mathcal{Z}(G)$, are classified by the types Φ_G of their root systems as shown in the table bellow

G	$SU(n)$	$Sp(n)$	$Spin(2n+1)$	$Spin(2n)$	G_2	F_4	E_6	E_7	E_8
Φ_G	A_{n-1}	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$\mathcal{Z}(G)$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$, $n=2k+1$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $n=2k$	$\{e\}$	$\{e\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{e\}$

,

Table 1. The types and centers of the 1–connected simple Lie groups.

where $e \in G$ is the group unit. In general, for a Lie group G let $\mathcal{Z}_0(G)$ be the identity component of the center of G, and let G' be the commutator subgroup of G. Then the intersection $F = G' \cap \mathcal{Z}_0(G)$ is a finite abelian group. The Cartan's classification on compact Lie groups states that ([19, Theorem 5.22])

Theorem 2.1. *The isomorphism type of a compact Lie group G is*

(2.1) $G = [G' \times \mathcal{Z}_0(G)]/F$,

where F is embedded in the numerator group $G' \times \mathcal{Z}_0(G)$ as $\{(g, g^{-1}) \mid g \in F\}$.

Moreover, the commutator subgroup G' admits a canonical presentation as

(2.2) $G' = [G_1 \times \cdots \times G_k]/K$,

where each G_t, $1 \leq t \leq k$, is one of the 1–connected simple Lie groups listed in Table 1, and where K is a subgroup of the finite group $\mathcal{Z}(G_1) \times \cdots \times \mathcal{Z}(G_k)$.□

In views of (2.1) and (2.2) we call a Lie group G *semi–simple* if $\dim \mathcal{Z}_0(G) = \{e\}$; *simple* if $\mathcal{Z}_0(G) = \{e\}$ and $k = 1$. Since the commutator subgroup G' is always semi–simple we shall call it *the semi–simple part* of G. Let T^r be the r–dimensional torus group. In comparison with Theorem 2.2 we have

Theorem 2.2. *The diffeomorphism type of G with semi–simple part G' is*

(2.3) $G \cong G' \times T^r$, $r = \dim \mathcal{Z}_0(G)$.

Proof. Since G' is normal in G the quotient space $G/G' = \mathcal{Z}_0(G)/F$ has the structure of an abelian group isomorphic to the r–dimensional torus group T^r, $r = \dim \mathcal{Z}_0(G)$, and the quotient map $h : G \to G/G' = T^r$ is both a group homomorphism and a submersion with fiber G'.

Take a maximal torus T' on G'. By (2.1) a maximal torus on G is $T = [T' \times \mathcal{Z}_0(G)]/F$. Since the restriction $h \mid T : T \to T^r$ of h on T is a fiber bundle in torus groups there is a monomorphism $\sigma : T^r \to T$ so that the composition $(h \mid T) \circ \sigma$ is the identity on T^r. Via the inclusion $T \subset G$ the map σ can be regarded as a section of the bundle h. We obtain the diffeomorphism (2.3) by the fact that the fiber of h is a group.□

The unitary group $U(n)$ of order n may serve as the simplest example illustrating the subtle difference between the isomorphism and diffeomorphism types of a Lie group. As a group it is isomorphic to $\left[SU(n) \times S^1\right]/\mathbb{Z}_n$, while as a smooth manifold it is diffeomorphic to the product space $SU(n) \times S^1$.

For a semi–simple Lie group G whose center $\mathcal{Z}(G)$ contains the cyclic group $\mathbb{Z}_q$ of order q, consider the cyclic covering $c : G \to G/\mathbb{Z}_q$ of Lie groups. Since the classifying space $B\mathbb{Z}_q$ of the group $\mathbb{Z}_q$ agrees with the Elienberg–MaClane space $K(\mathbb{Z}_q;1)$ the classifying map $f_c : G/\mathbb{Z}_q \to B\mathbb{Z}_q$ of c defines a 1–dimensional cohomology class $\iota \in H^1(G/\mathbb{Z}_q;\mathbb{F}_q)$, called *the characteristic class of the covering.* We set $\omega := \beta_q(\iota) \in H^2(G/\mathbb{Z}_q)$ with β_q the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_q \to 0$ of coefficients.

On the other hand let $\mathbb{Z}_q$ acts on the circle S^1 as the anti–clockwise rotation through the angle $2\pi/q$. The obvious group homomorphism

$$(2.4)\quad C : \left[G \times S^1\right]/\mathbb{Z}_q \to G/\mathbb{Z}_q$$

is an oriented cycle bundle on the quotient group $G/\mathbb{Z}_q$ whose total space $\left[G \times S^1\right]/\mathbb{Z}_q$ is diffeomorphic to the product space $G \times S^1$ by Theorem 2.2. Set $\xi_1 := p^*(\varepsilon) \in H^1(G \times S^1)$ with $p : G \times S^1 \to S^1$ the projection onto the second factor, and with $\varepsilon \in H^1(S^1)$ the canonical generator on S^1.

Theorem 2.3. *The induced map C^* fits in the exact sequence*

$$(2.5)\quad \cdots \to H^r(G/\mathbb{Z}_q) \overset{C^*}{\to} H^r(G \times S^1) \overset{\theta}{\to} H^{r-1}(G/\mathbb{Z}_q) \overset{\omega}{\to} H^{r+1}(G/\mathbb{Z}_q) \overset{C^*}{\to} \cdots$$

where ω denotes the homomorphism of taking product with the class ω.

In addition, the homomorphism θ has the following properties

i) $\theta(\xi_1) = q \in H^0(G/\mathbb{Z}_q)$;

ii) $\theta(x \cup C^*(y)) = \theta(x) \cup y$, $x \in H^*(G \times S^1)$, $y \in H^*(G/\mathbb{Z}_q)$.

Proof. The sequence (2.5) will come as the Gysin sequence of the circle bundle C, with the total space $\left[G \times S^1\right]/\mathbb{Z}_q$ being replaced by its diffeomorphism type $G \times S^1$, as long as one shows that ω is the Euler class of C [17, P.143].

The lens space $L^{2n+1}(q) := S^{2n+1}/\mathbb{Z}_q$ acts as the $(2n+1)$–dimensional skeleton of the classifying space $B\mathbb{Z}_q$, where S^{2n+1} is the $(2n+1)$–dimensional sphere. As the group $G/\mathbb{Z}_q$ is finite dimensional the classifying map f_c goes into $L^{2n+1}(q)$ for some sufficiently large n and henceforth, induces the commutative diagrams (with f a $\mathbb{Z}_q$–equivariant map over f_c)

$$\begin{array}{ccc} G & \overset{f}{\to} & S^{2n+1} \\ c\downarrow & & \downarrow \\ G/\mathbb{Z}_q & \overset{f_c}{\to} & L^{2n+1}(q) \end{array}, \qquad \begin{array}{ccc} \left[G \times S^1\right]/\mathbb{Z}_q & \overset{[f\times id]/\mathbb{Z}_q}{\to} & \left[S^{2n+1} \times S^1\right]/\mathbb{Z}_q \\ C\downarrow & & \downarrow\gamma \\ G/\mathbb{Z}_q & \overset{f_c}{\to} & L^{2n+1}(q) \end{array},$$

The proof is done by the universal property that the Euler class of the circle bundle γ on $L^{2n+1}(q)$ is $\beta_q(x)$ with $x \in H^1(L^{2n+1}(q);\mathbb{Z}_q) = \mathbb{Z}_q$ the canonical generator. Finally, property i) is standard for the Gysin sequence of the bundle γ, while the relation ii) can be found in [15, Lemma 1].□

Granted with Theorems 2.2 and 2.3 we clarify the main idea in our approach. Firstly, Theorem 2.2 reduces the cohomology of a compact Lie group G to that of its semi–simple part G' by the Kunneth formula $H^*(G) = H^*(G') \otimes H^*(T^r)$. Next, for a semi–simple Lie group G its universal covering $c : G_0 \to G$ can be decomposed into a sequence of cyclic coverings

$$c : G_0 \xrightarrow{c_1} G_1 \xrightarrow{c_2} \cdots \xrightarrow{c_k} G_k = G.$$

With the cohomology $H^*(G_0)$ having been explicit constructed in [8, Theorem 1.9], the cohomology $H^*(G)$ in question can be accessed, in principle, from the known one $H^*(G_0)$ by repeatedly applying the sequence (2.5).

Our calculation with the sequence (2.5) actually takes place in the third page $E_3^{*,*}(G)$ of the Leray–Serre spectral sequence $\{E_r^{*,*}(G), d_r\}$ of the fibration

(2.6) $T \hookrightarrow G \xrightarrow{\pi} G/T$

whose E_2 page is the Koszul complex with

a) $E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T)$;

b) $d_2(a \otimes t) = (\tau(t) \cup a) \otimes 1$ for $a \in H^*(G/T)$, $t \in H^1(T)$

where $\tau : H^1(T) \to H^2(G/T)$ is the *transgression* in the fibration (2.6). To access the term $E_3^{*,*}(G)$ we deduce a formula for τ in Theorem 3.4, and give an account for the factor ring $H^*(G/T)$ of $E_2^{*,*}(G)$ in Theorem 4.1. With these preparations the idea to construct generators of the ring $H^*(G)$ by certain polynomials in the Schubert classes on G/T is entailed in Section 4, where a formula for the homomorphism θ in (2.5) is obtained in Theorem 4.12.

3 The transgression $\tau : H^1(T) \to H^2(G/T)$

For a Lie group G with maximal torus T consider the diagram with the top row a section of the cohomology exact sequence of the pair (G,T)

$$\begin{array}{ccccccc} \cdots \to H^1(G) & \xrightarrow{i^*} & H^1(T) & \xrightarrow{\delta} & H^2(G,T) & \xrightarrow{j^*} & H^2(G) \to \cdots \\ & & & \searrow \tau & \cong\uparrow \pi^* & & \\ & & & & H^2(G/T) & & \end{array},$$

where by the 1–connectness of the pair (G,T) the induced map π^* is an isomorphism. The *transgression* in (2.6) is the composition (see [16, p.185])

$$\tau = (\pi^*)^{-1} \circ \delta : H^1(T) \to H^2(G/T).$$

Let $Tor(A)$ be the torsion subgroup of an abelian group A.

Lemma 3.1. *The diffeomorphism type of the base space G/T of (2.6) depends only on the semi–simple part (2.2) of the group G as*

(3.1) $G/T \cong \frac{G_1}{T_1} \times \cdots \times \frac{G_k}{T_k}$,

where T_i is a maximal torus of the 1–connected simple Lie group G_i, $1 \leq i \leq k$.

The transgression τ fits into the exact sequence

(3.2) $0 \to H^1(G) \xrightarrow{j^*} H^1(T) \xrightarrow{\tau} H^2(G/T) \xrightarrow{\pi^*} TorH^2(G) \to 0.$

Proof. Let T' be a maximal torus of the semi–simple part G' of G. By (2.1) a maximal torus of G can be taken to be $T = [T' \times \mathcal{Z}_0(G)]/F$. The diffeomorphism (3.1) comes from the obvious relation $G/T = G'/T'$ and (2.2).

Since the second homotopy group of a Lie group is trivial, the homotopy exact sequence of π contains the free resolution of the group $\pi_1(G)$

$$0 \to \pi_2(G/T) \to \pi_1(T) \to \pi_1(G) \to 0.$$

Applying the co–functor $Hom(,\mathbb{Z})$, together with the Huriwicz isomorphisms

$$\pi_2(G/T) = H_2(G/T),\ \pi_1(T) = H_1(T),\ \pi_1(G) = H_1(G),$$

and the relation $Ext(H^1(G);\mathbb{Z}) = TorH^2(G)$, yields (3.2).□

By (2.3) we have $H^1(G) = H^1(T^r)$, $r = \dim \mathcal{Z}_0(G)$. According to (3.2) the transgression τ annihilates the direct summand $H^1(T^r)$ of $H^1(T)$ and hence, depends only on the semi–simple part of G. For this reason we can assume below that the Lie group G under consideration is semi–simple.

Equip the Lie algebra $L(G)$ of G with an inner product $(,)$ so that the adjoint representation acts as isometries on $L(G)$. Let $L(T) \subset L(G)$ be the *Cartan subalgebra* corresponding to the fixed maximal torus T on G, and fix a set $\Delta = \{\alpha_1, \cdots, \alpha_n\} \subset L(T)$ of simple roots of G, $n = \dim T$.

The Euclidean space $L(T)$ possesses three distinguished lattices. Firstly, the set $\{\alpha_1, \cdots, \alpha_n\}$ of simple roots generates *the root lattice* Λ_r of G. Next, the pre–image of the exponential map $\exp : L(T) \to T$ at the unit $e \in T$ gives rise to the *unit lattice* $\Lambda_e := \exp^{-1}(e)$ of G. Finally, using simple roots one defines the set $\Omega = \{\varrho_1, \cdots, \varrho_n\} \subset L(T)$ of *fundamental dominant weights* of G by the formula $2(\varrho_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{i,j}$ that spans *the weight lattice* Λ_ω of G.

Lemma 3.2 (see [5, (3.4)]). *The three lattices Λ_r, Λ_e and Λ_ω on $L(T)$ are subject to the relations $\Lambda_r \subseteq \Lambda_e \subseteq \Lambda_\omega$ in which*

i) the group G is 1–connected if and only if $\Lambda_r = \Lambda_e$;

ii) the group G is adjoint if and only if $\Lambda_e = \Lambda_\omega$;

iii) the basis Δ on Λ_r can be expressed by the basis Ω on Λ_ω by the formula

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = A \begin{pmatrix} \varrho_1 \\ \vdots \\ \varrho_n \end{pmatrix},$$

where $A = (c_{ij})_{n\times n}$ with $c_{ij} = 2(a_i, \alpha_j)/(\alpha_j, \alpha_j)$ is the Cartan matrix of G.□

For a root $\alpha \in \Delta$ let $K(\alpha) \subset G$ be the subgroup with Lie algebra $l_\alpha \oplus L_\alpha$, where $l_\alpha \subset L(T)$ is the 1–dimensional subspace spanned by α and where $L_\alpha \subset L(G)$ is the root space (viewed as an oriented real 2–plane) belonging to the root α ([11, p.35]). It is known that the group $K(\alpha)$ has the circle subgroup $S^1 = \exp(l_\alpha)$ as its maximal torus, and that the quotient space K_α/S^1 is canonically diffeomorphic to the 2–dimensional sphere S^2. Moreover, the embedding

(3.3) $s_\alpha : S^2 = K_\alpha/S^1 \to G/T$,

induced by the inclusion $(K_\alpha, S^1) \subset (G, T)$ is the *Schubert variety* associated to α [6], and the basis theorem of Chevalley [2] implies that the maps s_α with $\alpha \in \Delta$ represent a basis of the homology $H_2(G/T)$. As a result letting $\omega_i \in H^2(G/T)$ be the Kronnecker dual of the homology class represented by the maps s_{α_i}, then

Lemma 3.3. *The set $\{\omega_1, \cdots, \omega_n\}$ is a basis of the group $H^2(G/T)$.*□

On the other hand take a basis $\Theta = \{\theta_1, \cdots, \theta_n\}$ of the unit lattice Λ_e. It defines n oriented circle subgroups on the maximal torus

(3.4) $\widetilde{\theta}_i : S^1 = \mathbb{R}/\mathbb{Z} \to T$, $\widetilde{\theta}_i(t) := \exp(t\theta_i)$, $1 \le i \le n$,

that represent a basis of the first homology $H_1(T)$. Denote by $t_i \in H^1(T)$ the Kronnecker dual of the homology class represented by $\widetilde{\theta}_i$. Then

(3.5) $H^*(T) = \Lambda(t_1, \cdots, t_n)$ (the exterior ring generated by $t_1, \cdots, t_n$).

Let $C(\Theta)$ be the matrix expressing the ordered basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ of the lattice Λ_r by the ordered basis $\Theta = \{\theta_1, \cdots, \theta_n\}$ of the lattice Λ_e in view of the inclusion $\Lambda_r \subseteq \Lambda_e$, and let $C(\Theta)'$ be the transpose of the matrix $C(\Theta)$.

Theorem 3.4. *With respect to the basis of the groups $H^1(T)$ and $H^2(G/T)$ specified above, the transgression τ is given by the formula*

(3.6) $$\begin{pmatrix} \tau(t_1) \\ \vdots \\ \tau(t_n) \end{pmatrix} = C(\Theta)' \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

Proof. Firstly consider the case where the group G is 1–connected, and a basis Θ of the unit lactic $\Lambda_e = \Lambda_r$ is taken to be $\Delta = \{\alpha_1, \cdots, \alpha_n\}$. Since $C(\Theta)$ is then the identity matrix we are bound to show that $\tau(t_i) = \omega_i$, $1 \le i \le n$.

For a root α_i the inclusion $(K(\alpha_i), S^1) \subset (G, T)$ induces the bundle map

$$\begin{array}{ccccc} S^1 & \to & K(\alpha_i) & \overset{\pi_i}{\to} & K(\alpha_i)/S^1 = S^2 \\ \widetilde{\alpha}_1 \downarrow & & \downarrow & & \downarrow s_{\alpha_i} \\ T & \to & G & \overset{\pi}{\to} & G/T \end{array}$$

over the inclusion s_{α_i}. Since G is 1–connected the subgroup $K(\alpha_i)$ is isomorphic to the 3–dimensional sphere S^3, and the bundle π_i is the classical Hopf fibration over the 2–sphere S^2. This indicates that the connecting homomorphism ∂ in the homotopy exact sequence of π_i satisfies that $\partial\,[\iota_2] = [\iota_1]$, where ι_r is the identity of the r–sphere S^r. By the naturality of ∂ one gets in the homotopy exact sequence of π the relation $\partial\,[s_{\alpha_i}] = [\widetilde{\alpha}_i]$. This shows $\tau(t_i) = \omega_i$ as τ is dual to ∂ by the proof of Lemma 3.1.

For the general situation assume that the group G is semi–simple, and that the matrix expresses the basis Δ of Λ_r by the basis Θ of Λ_e is $C(\Theta) = (c_{i,j})_{n\times n}$. Let $d : (G_0, T_0) \to (G, T)$ be the universal covering of G with T_0 the maximal torus on G_0 corresponding to T. Then the exponential map exp of G factors through that $\exp_0$ of G_0 in the fashion

$$\exp = d \circ \exp_0 : (L(G_0), L(T_0)) \to (G_0, T_0) \to (G, T).$$

It follows that, if we let $p(\Lambda_r, \Lambda_e) : T_0 = L(T_0)/\Lambda_r \to T = L(T_0)/\Lambda_e$ be the covering map induced by the inclusion $\Lambda_r \subset \Lambda_e$ of the lattices, then

(3.8) $d \mid T_0 = p(\Lambda_r, \Lambda_e) : T_0 \to T$,

and with respect to the basis $\{[\widetilde{\alpha}_1], \cdots, [\widetilde{\alpha}_n]\}$ of $\pi_1(T_0)$ and the basis $\{[\widetilde{\theta}_1], \cdots, [\widetilde{\theta}_n]\}$ of $\pi_1(T)$, the induced map $p(\Lambda_r, \Lambda_e)_*$ on $\pi_1(T_0)$ is determined by $C(\Theta)$ as

(3.9) $p(\Lambda_r, \Lambda_e)_*[\widetilde{\alpha}_i] = c_{i,1}[\widetilde{\theta}_1] + \cdots + c_{i,n}[\widetilde{\theta}_n]$.

On the other hand the restriction $d \mid T_0$ fits in the commutative diagram

$$(3.10)\qquad \begin{array}{ccc} \pi_2(G_0/T_0) & \overset{\partial_0}{\underset{\cong}{\to}} & \pi_1(T_0) \\ \| & & \downarrow (d \mid T_0)_* \\ \pi_2(G/T) & \overset{\partial}{\to} & \pi_1(T) \end{array}$$

with ∂_0, ∂ the connecting homomorphisms of the homotopy exact sequences of the bundles $G_0 \to G_0/T_0$, $G \to G/T$, respectively, where the vertical identification on the left comes from (3.1). It follows that, for each root $\alpha_i \in \Delta$,

$$\begin{aligned} \partial\,[s_{\alpha_i}] &= (d \mid T_0)_* \circ \partial_0\,[s_{\alpha_i}] \text{ (by the diagram (3.10))} \\ &= (d \mid T_0)_*\,[\widetilde{\alpha}_i] \text{ (by the proof of the previous case)} \\ &= p(\Lambda_r, \Lambda_e)_*([\widetilde{\alpha}_i]) \text{ (by (3.8)).} \end{aligned}$$

The proof is completed by (3.9), and by the fact that τ is dual to ∂.□

4 The structure of $E_3^{*,r}(G)$, $r = 0, 1$

In term of the fiber degrees the third page of the spectral sequence of the fibration (2.6) has the ready made decomposition

$$E_3^{*,*}(G) = E_3^{*,0}(G) \oplus E_3^{*,1}(G) \oplus \cdots \oplus E_3^{*,N}(G),\ N = \dim T.$$

In this section we clarify the structure of the first two summands $E_3^{*,0}(G)$ and $E_3^{*,1}(G)$, and establish in Theorem 4.12 a formula for the map θ in (2.5).

With the d_2 action on $E_2^{*,*}(G)$ having been determined by formula (3.7) to access the third page $E_3^{*,*}(G)$ we need a concise description of the subring $H^*(G/T) \subset E_2^{*,*}(G)$. Assume therefore that the rank of the semi–simple part of G is n ($\leq N$), and let $\{\omega_1, \cdots, \omega_n\}$ be the Schubert basis on $H^2(G/T)$ given by Lemma 3.3. For a subset S of a ring A write $\langle S \rangle \subset A$ for the ideal generated by S, and denote by $A/\langle S \rangle$ the quotient ring.

The following result, whose proof is shown in [7, Theorem 1.2], is the basis of the computation and construction throughout the remaining part of this paper.

Theorem 4.1. *For a Lie group G there exist a set $\{y_1, \cdots, y_m\}$ of Schubert classes on G/T with $\deg y_i > 2$, so that the set $\{\omega_1, \cdots, \omega_n, y_1, \cdots, y_m\}$ is a minimal system of generators of the integral cohomology ring $H^*(G/T)$.*

Moreover, with respect to these generators the integral cohomology of G/T has the presentation

$$(4.1)\quad H^*(G/T) = \mathbb{Z}[\omega_1, \cdots, \omega_n, y_1, \cdots, y_m]/\left\langle h_i, f_j, g_j \right\rangle_{1 \leq i \leq k; 1 \leq j \leq m},$$

where

i) for each $1 \leq i \leq k$, $h_i \in \langle \omega_1, \cdots, \omega_n \rangle$;

ii) for each $1 \leq j \leq m$, the pair (f_j, g_j) of polynomials is related to the Schubert class y_j in the fashion

$$f_j = p_j y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j,$$

with $p_j \in \{2, 3, 5\}$ and $\alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$.□

4.1 Canonical maps $E_3^{*,0}(G)$, $E_3^{*,1}(G) \to H^*(G)$

The property $H^{2s+1}(G/T) = 0$ by Theorem 4.1 indicates the relations

a) $E_r^{2s+1,q} = 0$ for all $s, r, q \geq 0$;

b) $E_3^{4k,2} = E_4^{4k,2} = \cdots = E_\infty^{4k,2}$

that are useful to handle the extension problem from $E_3^{*,*}(G)$ to $H^*(G)$. To be precise we recall that

$$E_\infty^{p,q}(G) = F^p(H^{p+q}(G))/F^{p+1}(H^{p+q}(G)),$$

where F^p is the filtration on $H^*(G)$ defined by fibration (2.6)

$$0 = F^{r+1}(H^r(G)) \subseteq F^r(H^r(G)) \subseteq \cdots \subseteq F^0(H^r(G)) = H^r(G).$$

The routine property $d_r(E_r^{*,0}(PG)) = 0$, $r \geq 2$, yields a series of quotient maps

$$H^r(G/T) = E_2^{r,0} \to E_3^{r,0} \to \cdots \to E_\infty^{r,0} = F^r(H^r(G)) \subset H^r(G)$$

whose composition agrees with the induces map $\pi^* : H^*(G/T) \to H^*(G)$ [16, P.147]. For this reason we shall reserve π^* also for the composition

(4.2) $\pi^* : E_3^{*,0}(G) \to \cdots \to E_\infty^{*,0}(G) = F^r(H^r(G)) \subset H^*(G)$.

Next, from $F^{2k+1}(H^{2k+1}(G)) = F^{2k+2}(H^{2k+1}(G)) = 0$ by a) one finds that

$$E_\infty^{2k,1}(G) = F^{2k}(H^{2k+1}(G)) \subset H^{2k+1}(G).$$

Combining this with the property $d_r(E_r^{*,1}) = 0$, $r \geq 3$, yields the composition

(4.3) $\kappa : E_3^{*,1}(G) \to \cdots \to E_\infty^{*,1}(G) \subset H^{2t+1}(G)$

which interprets elements in $E_3^{*,1}$ as cohomology classes of the group G.

Finally, with the multiplicative structure inherited from that on $E_2^{*,*}(G)$ the page $E_3^{*,*}(G)$ is a bi-graded ring [22, P.668]. With the presence of the maps π^* and κ in (4.2) and (4.3) we show further that

Lemma 4.2. *For any $\xi \in \operatorname{Im}\kappa$ one has $\xi^2 \in \operatorname{Im}\pi^*$.*

Proof. For an element $x \in E_3^{2k,1}$ the obvious relation $x^2 = 0$ in

$$E_3^{4k,2} = E_\infty^{4k,2} = \mathcal{F}^{4k}H^{4k+2}/\mathcal{F}^{4k+1}H^{4k+2} \text{ (by b))}.$$

implies that $\kappa(x)^2 \in \mathcal{F}^{4k+1}H^{4k+2}$. From

$$\mathcal{F}^{4k+1}H^{4k+2}/\mathcal{F}^{4k+2}H^{4k+2} = E_\infty^{4k+1,1} = 0 \text{ (by a))}$$

one gets further that $\kappa(x)^2 \in \mathcal{F}^{4k+2}H^{4k+2}$. The proof is completed by

$$\mathcal{F}^{4k+2}H^{4k+2} = E_\infty^{4k+2,0} = \operatorname{Im}\pi^*.\square$$

4.2 The term $E_3^{*,0}(G)$

Recall that with respect to the presentation $E_2^{*,*}(G) = H^*(G/T) \otimes \Lambda(t_1, \cdots, t_N)$, $N = \dim T \geq n$, the differential $d_2 : E_2^{*,*}(G) \to E_2^{*,*}(G)$ is

$$d_2(x \otimes t_k) = (x \cup \tau(t_k)) \otimes 1,\ x \in H^*(G/T),\ 1 \leq k \leq N.$$

It implies that $E_3^{*,0}(G) = H^*(G/T)/\langle \operatorname{Im} \tau \rangle$. From (4.1) one gets that

Lemma 4.3. $E_3^{*,0}(G) = H^*(G/T)\mid_{\tau(t_1) = \cdots = \tau(t_N)}$.□

4.3 Constructions in $E_3^{*,1}(G)$

For a d_2–cocycle $\gamma \in E_2^{*,*}(G)$ write $[\gamma] \in E_3^{*,*}(G)$ for its cohomology class. Based on Theorem 4.1 and in view of the transgression $\tau : H^1(T) \to H^2(G/T)$, we present two ways to construct elements in $E_3^{*,1}(G)$. The first one resorts to $\ker \tau$, while the second utilizes $\operatorname{Im} \tau$.

For each $t \in \ker \tau$ the class $1 \otimes t \in E_2^{0,1}$ is clear a d_2–cocycle. The cohomology class $\iota(t) := [1 \otimes t] \in E_3^{0,1}(G)$ will be called a *primary* 1*–form* of G with base degree 0. Moreover, argument based on degrees suffices to show that

Lemma 4.4. *The map* $\iota : \ker \tau \to E_3^{0,1}(G) = H^1(G)$ *is an isomorphism.*□

We begin the second construction by taking the ring $\mathbb{Z}$ as coefficient for cohomologies. The inclusion $\{\omega_i, y_j\} \subset H^*(G/T)$ of Schubert classes by (4.1) induces the surjective ring map

$$f : \mathbb{Z}[\omega_i, y_j]_{1 \leq i \leq n, 1 \leq j \leq m} \to H^*(G/T)$$

with $\ker f = \langle h_i, f_j, g_j \rangle$. Since f is an isomorphism in degree 2 the map τ has a unique lift $\widetilde{\tau}$ into the free polynomial ring $\mathbb{Z}[\omega_i, y_j]$ that is subject to the relation $\tau = f \circ \widetilde{\tau}$. For a polynomial $P \in \langle \operatorname{Im} \widetilde{\tau} \rangle$ we can write

(4.4) $P = p_1 \cdot \widetilde{\tau}(t_1) + \cdots + p_N \cdot \widetilde{\tau}(t_N)$ with $p_i \in \mathbb{Z}[\omega_i, y_j]$,

and set $\widetilde{P} := f(p_1) \otimes t_1 + \cdots + f(p_N) \otimes t_N \in E_2^{*,1}(G)$. Observe that $P \in \langle \operatorname{Im} \widetilde{\tau} \rangle \cap \ker f$ implies that $\widetilde{P} \in \ker d_2$.

Lemma 4.5. *The map* $\varphi : \langle \operatorname{Im} \widetilde{\tau} \rangle \cap \ker f \to E_3^{*,1}(G)$ *by* $\varphi(P) = [\widetilde{P}]$ *is well defined.*

Proof. We need to show the class $[\widetilde{P}] \in E_3^{*,1}(G)$, $P \in \langle \operatorname{Im} \widetilde{\tau} \rangle \cap \ker f$, is independent of a choice of the expansion (4.4). Assume in addition to (4.4) that one has a second expansion $P = h_1 \cdot \widetilde{\tau}(t_1) + \cdots + h_N \cdot \widetilde{\tau}(t_N)$. Then the equation

$$(p_1 - h_1) \cdot \widetilde{\tau}(t_1) + \cdots + (p_N - h_N) \cdot \widetilde{\tau}(t_N) = 0$$

holds in the ring $\mathbb{Z}[\omega_i, y_j]$. We can assume further that $p_1 - h_1 \neq 0$.

Case 1. The set $\{\widetilde{\tau}(t_1), \cdots, \widetilde{\tau}(t_N)\}$ is a basis of $\operatorname{Im} \widetilde{\tau}$: Since $\{\widetilde{\tau}(t_1), \cdots, \widetilde{\tau}(t_N)\} \subset \mathbb{Z}[\omega_i, y_j]$ is algebraically independent, the above equation implies that all the differences $p_i - h_i$ with $i \neq 1$ are divisible by $\widetilde{\tau}(t_1)$. That is $p_i - h_i = q_i \cdot \widetilde{\tau}(t_1)$ for some $q_i \in \mathbb{Z}[\omega_i, y_j]$, $2 \leq i \leq N$. The proof for the present situation is shown by the calculation

$$d_2(f(q_2)\otimes t_1t_2+\cdots+f(q_N)\otimes t_1t_N)$$
$$=f(p_1-h_1)\otimes t_1+\cdots+f(p_N-h_N)\otimes t_N.$$

Case 2. The set $\{\widetilde{\tau}(t_1),\cdots,\widetilde{\tau}(t_N)\}$ is linearly dependent $\operatorname{Im}\widetilde{\tau}$: Take a subset $\{\overline{t}_1,\cdots,\overline{t}_{n'}\}\subset H^1(T;\mathbb{F}_p)$ $(n'\leq N)$ so that its $\widetilde{\tau}_p$–image $\{\widetilde{\tau}_p(\overline{t}_1),\cdots,\widetilde{\tau}_p(\overline{t}_{n'})\}$ is a basis of the group $\operatorname{Im}\widetilde{\tau}_p$, and let $B=(b_{ij})_{n'\times N}$ be the matrix expressing the elements $\tau(t_i)$ by the basis elements $\{\widetilde{\tau}_p(\overline{t}_1),\cdots,\widetilde{\tau}_p(\overline{t}_{n'})\}$. Denote by B^{τ} the transpose of B and set

$$(p_1',\cdots,p_{n'}')=(p_1,\cdots,p_N)\cdot B^{\tau},\ (h_1',\cdots,h_{n'}')=(h_1,\cdots,h_N)\cdot B^{\tau}.$$

Then, in addition to the next two expansions of P in $\mathbb{Z}[\omega_i,y_j]_{1\leq i\leq n,1\leq j\leq m}$

$$P=p_1'\cdot\widetilde{\tau}(\overline{t}_1)+\cdots+p_{n'}'\cdot\widetilde{\tau}(\overline{t}_{n'})=h_1'\cdot\widetilde{\tau}(\overline{t}_1)+\cdots+h_{n'}'\cdot\widetilde{\tau}(\overline{t}_{n'}),$$

one has the following relations in $E_2^{*,1}(G)$

$$f(p_1)\otimes t_1+\cdots+f(p_N)\otimes t_N=f(p_1')\otimes\overline{t}_1+\cdots+f(p_{n'}')\otimes\overline{t}_{n'},$$
$$f(h_1)\otimes t_1+\cdots+f(h_N)\otimes t_N=f(h_1')\otimes\overline{t}_1+\cdots+f(h_{n'}')\otimes\overline{t}_{n'}.$$

The proof is completed by the computation in $E_3^{*,1}(G)$, where the second equality has been shown in Case 1

$$[f(p_1)\otimes t_1+\cdots+f(p_N)\otimes t_N]=\left[f(p_1')\otimes\overline{t}_1+\cdots+f(p_{n'}')\otimes\overline{t}_{n'}\right]$$
$$=\left[f(h_1')\otimes\overline{t}_1+\cdots+f(h_{n'}')\otimes\overline{t}_{n'}\right]=[f(h_1)\otimes t_1+\cdots+f(h_N)\otimes t_N].\square$$

The map φ has its analogue for cohomology over a finite field $\mathbb{F}_p$. Precisely, by Theorem 4.1 one can deduce a presentation of the algebra $H^*(G/T;\mathbb{F}_p)$ in the following form ([8, Lemma 2.3])

$$(4.5)\quad H^*(G/T;\mathbb{F}_p)=\mathbb{F}_p[\omega_1,\ldots,\omega_n,y_t]/\left\langle\delta_1,\cdots,\delta_n,y_t^{k_t}+\sigma_t\right\rangle_{t\in E(G,p)},$$

with $\delta_s\in\mathbb{F}_p[\omega_1,\ldots,\omega_n]$, $\sigma_t\in\langle\omega_1,\ldots,\omega_n\rangle$, $E(G,p)=\{1\leq t\leq m;p_t=p\}$. Based on (4.5) one formulates the $\mathbb{F}_p$–analogies of the maps f, $\widetilde{\tau}$ and φ as

$$f_p:\mathbb{F}_p[\omega_1,\ldots,\omega_n,y_t]_{\ t\in E(G,p)}\to H^*(G/T;\mathbb{F}_p),$$
$$\widetilde{\tau}_p:H^1(T;\mathbb{F}_p)\to\mathbb{F}_p[\omega_1,\ldots,\omega_n,y_t]_{\ t\in E(G,p)},$$
$$\varphi_p:\langle\operatorname{Im}\widetilde{\tau}_p\rangle\cap\langle\ker f_p\rangle\to E_3^{*,1}(G;\mathbb{F}_p).$$

The proof of Lemma 4.5 is applicable to show that

Lemma 4.6. *The correspondence φ_p is well defined. In particular*

$$\varphi_p(P'P)=0 \textit{ if } P'\in\langle\operatorname{Im}\widetilde{\tau}_p\rangle \textit{ and } P\in\ker f_p.\square$$

Definition 4.7. For a polynomial $P\in\ker f\cap\langle\operatorname{Im}\widetilde{\tau}\rangle$ (resp. $P\in\ker f_p\cap\langle\operatorname{Im}\widetilde{\tau}_p\rangle$) the class $\kappa\varphi(P)\in H^*(G)$ (resp. $\kappa\varphi_p(P)\in H^*(G;\mathbb{F}_p)$) will be called *a primary 1–form* of the group G. In turn, P is called a *characteristic polynomial* of the class $\kappa\varphi(P)$ (resp. $\kappa\varphi_p(P)$).$\square$

Example 4.8. The maps κ, π^* and φ above have played a crucial role as to construct the cohomology of 1–connected Lie groups.

Indeed, with the group G being 1–connected one has $\langle\operatorname{Im}\widetilde{\tau}\rangle=\langle\omega_1,\cdots,\omega_n\rangle$ (resp. $\langle\operatorname{Im}\widetilde{\tau}_p\rangle=\langle\omega_1,\cdots,\omega_n\rangle$) by Theorem 3.4. Theorem 4.1 (resp. the presentation (4.5)) implies that the set of polynomials

$$S(G) = \{h_i, p_j\beta_j - y_j^{k_j}\alpha_j; 1 \le i \le k, 1 \le j \le m\}$$
$$(\text{resp. } S_p(G) := \{\delta_1, \cdots, \delta_n\})$$

belongs to $\ker f \cap \langle \operatorname{Im} \widetilde{\tau} \rangle$ (resp. $\ker f_p \cap \langle \operatorname{Im} \widetilde{\tau}_p \rangle$). It is shown in [8] that

i) the square free products of the primary 1–forms in the set $\{\kappa \circ \varphi(P) \in H^*(G); P \in S(G)\}$ is a basis of the free part of the integral cohomology $H^*(G)$;

ii) the set $\{\kappa \circ \varphi_p(\delta) \in H^*(G; \mathbb{F}_p); \delta \in S_p(G)\}$ of primary 1–forms, together with $\pi^*(y_t) \in H^*(G; \mathbb{F}_p)$, $t \in E(G, p)$, generates the algebra $H^*(G; \mathbb{F}_p)$.□

4.4 Steenrod operations on $H^*(G; \mathbb{F}_p)$

Let $\mathcal{A}_p$ be the mod p *Steenrod algebra* with $\mathcal{P}^k \in \mathcal{A}_p$, $k \ge 1$, the k^{th} *reduced power* (if $p = 2$ it is also customary to write Sq^{2k} instead of $\mathcal{P}^k$), and with $\delta_p = r_p \circ \beta_p \in \mathcal{A}_p$ the *Bockstein operator* [20]. Clarifying the $\mathcal{A}_p$ action on $H^*(G; \mathbb{F}_p)$ is useful in determining the ring structure on the integral cohomology $H^*(G)$, as well as the Hopf algebra structure on $H^*(G; \mathbb{F}_p)$. In this section we reduce the $\mathcal{A}_p$ actions on $H^*(G; \mathbb{F}_p)$ to computing with characteristic polynomials.

Since the group $E_2^{*,*}(G)$ is torsion free one has for each prime p the short exact sequence of complexes

$$0 \to E_2^{*,*}(G) \overset{\cdot p}{\to} E_2^{*,*}(G) \overset{r_p}{\to} E_2^{*,*}(G; \mathbb{F}_p) \to 0.$$

With respect to the maps κ and π^* the connecting homomorphism $\widehat{\beta}_p$ in the associated cohomology exact sequence clearly fits into the commutative diagram

$$\begin{array}{ccc} E_3^{*,1}(G; \mathbb{F}_p) & \overset{\widehat{\beta}_p}{\to} & E_3^{*,0}(G) \\ \kappa \downarrow & & \pi^* \downarrow \\ H^*(G; \mathbb{F}_p) & \overset{\beta_p}{\to} & H^*(G) \end{array} .$$

In addition, in term of Lemma 4.2 the quotient map

$$H^*(G/T) \to E_3^{*,0}(G) = H^*(G/T) / \langle \operatorname{Im} \tau \rangle$$

is given by $f(P) \to f(P) \mid_{\tau(t_1) = \cdots = \tau(t_N) = 0}$, $P \in \mathbb{Z}[\omega_i, y_j]$. Granted with these conventions the diagram chasing in the cohomology exact sequence associated to above short exact sequence

$$\begin{array}{ccccc} & & \varphi(P_0) & \overset{r_p}{\to} & \varphi_p(P) \\ & & d_2 \downarrow & & d_2 \downarrow \\ \frac{1}{p} f(P_0) & \overset{\cdot p}{\longrightarrow} & f(P_0) & & 0 \end{array}$$

shows that

Lemma 4.9. *If $P_0 \in \langle \operatorname{Im} \widetilde{\tau} \rangle$ is an integral lift of a polynomial $P \in \ker f_p \cap \langle \operatorname{Im} \widetilde{\tau}_p \rangle$ (i.e. $P_0 \equiv P \bmod p$), then*

(4.6) $\beta_p(\kappa \circ \varphi_p(P)) = \pi^*(\frac{1}{p} f(P_0) \mid_{\tau(t_1) = \cdots = \tau(t_N) = 0})$.□

Let $c : (G_0, T_0) \to (G, T)$ be the universal covering of a semi–simple Lie group G, and consider the fibration induced by the inclusion $i : T_0 \to G_0$

(4.7) $G/T \overset{\psi}{\hookrightarrow} BT_0 \overset{Bi}{\to} BG_0$.

where BT_0 (resp. BG_0) is the classifying space of T_0 (resp. G_0). It is known that the maps $s_\alpha : S^2 \to G/T, \alpha \in \Delta$, in (3.3) admit a lift $\widetilde{s_\alpha} : S^2 \to BT_0$ with respect to ψ (i.e. $\psi \circ \widetilde{s_\alpha} = s_\alpha$) that represent a basis of the group $H_2(BT_0)$. As a result if we let $\{\omega_1, \ldots, \omega_n\}$ be the basis on $H^2(BT_0)$ Kronnecker dual to the ordered basis $\{\widetilde{s_{\alpha_i}}; \alpha_i \in \Delta\}$ on $H_2(BT_0)$, then by Lemma 3.3 the induced map

$$\psi^* : H^*(BT_0; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \ldots, \omega_n] \to H^*(G/T; \mathbb{F}_p).$$

agrees with the restriction $\overline{f}_p$ of f_p on the subalgebra $\mathbb{F}_p[\omega_1, \ldots, \omega_n]$. Since the lift $\widetilde{\tau}_p$ of the transgression τ takes values in $H^*(BT_0; \mathbb{F}_p)$ one has the relation

$$\ker \overline{f}_p \cap \langle \operatorname{Im} \widetilde{\tau}_p \rangle \subset H^*(BT_0; \mathbb{F}_p).$$

With respect to this inclusion the subspace $\ker \overline{f}_p \cap \langle \operatorname{Im} \widetilde{\tau}_p \rangle$ is clearly invariant with respect to the $\mathcal{A}_p$ action $H^*(BT_0; \mathbb{F}_p)$. The proof of [9, Lemma 3.2] is applicable to show the following formula, that reduces the $\mathcal{A}_p$ action on $H^*(G; \mathbb{F}_p)$ to that on the much simpler $\mathcal{A}_p$–algebra $H^*(BT_0; \mathbb{F}_p)$.

Lemma 4.10. *For $P \in \ker \overline{f}_p \cap \langle \operatorname{Im} \widetilde{\tau}_p \rangle$ the $\mathcal{A}_p$ action on $H^*(G; \mathbb{F}_p)$ satisfies*

(4.8) $\mathcal{P}^k(\kappa \circ \varphi_p(P)) = \kappa \circ \varphi_p(\mathcal{P}^k(P))$.□

4.5 The map $\theta : H^r(G \times S^1) \to H^{r-1}(G/\mathbb{Z}_q)$ in (2.5)

Returning to the situation concerned by Theorem 2.3 the circle bundle C on the quotient group $G/\mathbb{Z}_q$ fits into the commutative diagram

$$(4.9)\quad \begin{array}{ccccc} S^1 & \hookrightarrow & \left[T \times S^1\right]/\mathbb{Z}_q & \overset{C'}{\to} & T' \\ \| & & \cap & & \cap \\ S^1 & \hookrightarrow & \left[G \times S^1\right]/\mathbb{Z}_q & \overset{C}{\to} & G/\mathbb{Z}_q \\ & & \pi' \downarrow & & \pi \downarrow \\ & & G/T & = & G/T \end{array},$$

where $T \subset G$ is a fixed maximal torus on G, $T' := T/\mathbb{Z}_q$, the vertical maps π' and π are the obvious quotients by the maximal torus, and where C' denotes the restriction of C to $\left[T \times S^1\right]/\mathbb{Z}_q$. Since the maximal torus $\left[T \times S^1\right]/\mathbb{Z}_q$ of $\left[G \times S^1\right]/\mathbb{Z}_q$ has the factorization $T' \times S^1$ so that C' is the projection onto the first factor (see Theorem 2.2), one can take a basis $\Theta = \{\vartheta_1, \cdots, \vartheta_n, \vartheta\}$ ($n = \dim T'$) for the unit lattice of the group $\left[G \times S^1\right]/\mathbb{Z}_q$ so that the tangent map of C' at the group unit $e \in \left[T \times S^1\right]/\mathbb{Z}_q$ carries the subset $\{\vartheta_1, \cdots, \vartheta_n\}$ of Θ to a basis of the unit lattice of the quotient group $G/\mathbb{Z}_q$. As a result if we let $\{t_1, \cdots, t_n, t_0\}$ (resp. $\{t_1, \cdots, t_n\}$) be the basis of $H^1(\left[T \times S^1\right]/\mathbb{Z}_q)$ (resp. the basis of $H^1(T')$) corresponding to Θ (resp. corresponding to $\{\vartheta_1, \cdots, \vartheta_n\}$) in the manner of (3.5), then

a) the induced map C'^* identifies $H^*(T')$ with the subring $\Lambda^*(t_1, \cdots, t_n)$ of $H^*(\left[T \times S^1\right]/\mathbb{Z}_q) = \Lambda^*(t_1, \cdots, t_n, t_0)$,

b) the transgression τ in π is the restriction $\tau' \mid H^1(T')$ of τ' in π'.

It follows that the bundle map C from π' to π induces the short exact sequence

(4.10) $0 \to E_2^{*,*}(G/\mathbb{Z}_q) \overset{C^*}{\to} E_2^{*,*}(\left[G \times S^1\right]/\mathbb{Z}_q) \overset{Q}{\to} t_0 \cdot E_2^{*,*}(G/\mathbb{Z}_q) \to 0$

where the quotient map Q has the following description: for $x \in H^*(G/T)$, $y \in H^*(\left[T \times S^1\right]/\mathbb{Z}_q)$, write $y = y_0 + t_0 \cdot y_1$ with $y_0, y_1 \in \Lambda^*(t_1, \cdots, t_n)$. Then

(4.11) $Q(x \otimes y) = t_0 \cdot (x \otimes y_1)$.

By Lemma 4.4 the group $E_3^{0,1}(\left[G \times S^1\right]/\mathbb{Z}_q)$ has a generator that corresponds to the generator $\xi_1 \in H^1(\left[G \times S^1\right]/\mathbb{Z}_q)$ specified in Theorem 2.3, which we denote still by ξ_1.

Lemma 4.11. *To a direct summand $\mathbb{Z}_q$ in the center of a semi–simple Lie group G, there is associated an exact sequence of the form*

(4.12) $\cdots \to E_3^{*,r}(G/\mathbb{Z}_q) \overset{C^*}{\to} E_3^{*,r}(\left[G \times S^1\right]/\mathbb{Z}_q) \overset{\theta_1}{\to} E_3^{*,r-1}(G/\mathbb{Z}_q)$
$\overset{\cup\tau'(t_0)}{\to} E_3^{*,r-1}(G/\mathbb{Z}_q) \overset{C^*}{\to} E_3^{*,r-1}(\left[G \times S^1\right]/\mathbb{Z}_q) \to \cdots$

where τ' is the transgression in the fibration π', the map $\cup\tau'(t_0)$ is induced by the endomorphism $x \otimes y \to (x \cup \tau'(t_0)) \otimes y$ on $E_2^{*,*}(G/\mathbb{Z}_q)$.

In addition, the homomorphism θ_1 has the following properties

i) $\theta_1(\xi_1) = q \in E_3^{0,0}(G/\mathbb{Z}_q)$;
ii) $\theta_1(x \cup C^*(y)) = \theta_1(x) \cup y$, $x \in E_3^{*,*}(\left[G \times S^1\right]/\mathbb{Z}_q)$, $y \in E_3^{*,*}(G/\mathbb{Z}_q)$,

while the class $\tau'(t_0) \in E_3^{2,0}(G/\mathbb{Z}_q)$ satisfies, in view of (4.2), that

iii) $\pi^*(\tau'(t_0)) = \omega \in H^2(G/\mathbb{Z}_q)$.

Proof. In the long homology exact sequence

$$\cdots \to E_3^{*,r}(G/\mathbb{Z}_q) \overset{C^*}{\to} E_3^{*,r}(\left[G \times S^1\right]/\mathbb{Z}_q) \overset{Q^*}{\to} H^*(t_0 \cdot E_2^{*,*}(G/\mathbb{Z}_q)) \overset{\delta}{\to} \cdots$$

associated to the short exact sequence (4.10) one has the relations

c) $H^*(t_0 \cdot E_2^{*,*}(G/\mathbb{Z}_q)) = t_0 \cdot E_3^{*,*}(G/\mathbb{Z}_q)$; d) $\delta(t_0 \cdot 1) = \tau'(t_0) \otimes 1$.

Using c) to substitute the isomorphism group $E_3^{*,*}(G/\mathbb{Z}_q)$ in place of $H^*(t_0 \cdot E_2^{*,*}(G/\mathbb{Z}_q))$ the map Q^* gives rise to the map θ_1 with fiber degree -1. Subsequently, taking into account of the relation d) the map δ should be modified by $\cup\tau'(t_0)$. These establish the exact sequence (4.12).

In view of the obvious commutative diagram

$$\begin{array}{ccc} E_3^{0,1}(\left[G \times S^1\right]/\mathbb{Z}_q) & \overset{\theta_1}{\to} & E_3^{0,0}(G/\mathbb{Z}_q) \\ \| & & \| \\ H^1(\left[G \times S^1\right]/\mathbb{Z}_q) & \overset{\theta}{\to} & H^0(G/\mathbb{Z}_q) \end{array}$$

and by the choice of the generator $\xi_1 \in E_3^{0,1}(\left[G \times S^1\right]/\mathbb{Z}_q)$, property i) corresponds to i) of Theorem 2.3, while property ii) comes directly from the relation (4.11). Finally, the relation iii) is shown by the obvious commutative diagram induced by the map π^* in (4.2)

$$\begin{array}{ccc} E_3^{0,0}(G/\mathbb{Z}_q) & \overset{\cup\tau'(t_0)}{\to} & E_3^{2,0}(G/\mathbb{Z}_q) \\ \| & & \pi^* \quad \downarrow \\ H^0(G/\mathbb{Z}_q) & \overset{\cup\omega}{\to} & H^2(G/\mathbb{Z}_q) \end{array} .\square$$

The sequence (4.12) can be seen as an approximation to the Gysin sequence (2.5). In addition, the maps π^* and κ in (4.2) and (4.3) build up the obvious commutative diagram relating the operator θ in (2.5) with the map θ_1 in (4.12)

$$(4.13) \quad \begin{array}{ccc} E_3^{2r,1}(\left[G \times S^1\right]/\mathbb{Z}_q) & \overset{\theta_1}{\to} & E_3^{2r,0}(G/\mathbb{Z}_q) \\ \kappa \quad \downarrow & & \pi^* \downarrow \\ H^{2r+1}(G \times S^1) & \overset{\theta}{\to} & H^{2r}(G/\mathbb{Z}_q) \end{array} .$$

However, the map θ_1 (hence θ) admits a simple formula.

For a polynomial $P \in \left\langle \operatorname{Im} \widetilde{\tau'} \right\rangle$ we set $P_0 = P \mid_{\widetilde{\tau}(t_1)=\cdots=\widetilde{\tau}(t_n)=0}$. Then $P - P_0 \in \langle \operatorname{Im} \widetilde{\tau} \rangle$ and P_0 is divisible by $\widetilde{\tau'}(t_0)$. The latter enables us to define the *derivation* $\partial P/\partial t_0$ of P with respect to t_0 by the formula

$$(4.14) \quad \partial P/\partial t_0 := P_0/\widetilde{\tau'}(t_0).$$

Moreover, with $\widetilde{\tau'}(t_i) = \widetilde{\tau}(t_i)$ for $i \leq n$ by b) we have the expansion

$$P = p_1 \cdot \widetilde{\tau'}(t_1) + \cdots + p_n \cdot \widetilde{\tau'}(t_n) + \partial P/\partial t_0 \cdot \widetilde{\tau'}(t_0).$$

It implies that, if $P \in \ker f \cap \left\langle \operatorname{Im} \widetilde{\tau'} \right\rangle$ then

$$\varphi(P) = [f(p_1) \otimes t_1 + \cdots + f(p_n) \otimes t_n + f(\partial P/\partial t_0) \otimes t_0].$$

The diagram (4.13), together with the relation (4.11), implies that

Theorem 4.12. *The map θ_1 in (4.12) (resp. the map θ in (2.5)) satisfies that*

$$\theta_1(\varphi(P)) = f(\partial P/\partial t_0) \text{ (resp. } \theta(\kappa \circ \varphi(P)) = \pi^* f(\partial P/\partial t_0)),$$

where $P \in \ker f \cap \left\langle \operatorname{Im} \widetilde{\tau'} \right\rangle$.$\square$

By the exactness of the sequence (4.12), if $P \in \ker f \cap \left\langle \operatorname{Im} \widetilde{\tau'} \right\rangle$ is a polynomial with $\theta_1(\varphi(P)) = 0$ there exists a class $\eta \in E_3^{*,1}(G/\mathbb{Z}_q)$ satisfying $C^*(\eta) = \varphi(P)$. The proof of the next result indicates an algorithm to construct from P a characteristic polynomial $P' \in \ker f \cap \langle \operatorname{Im} \widetilde{\tau} \rangle$ for such a class η.

Lemma 4.13. *For a polynomial $P \in \ker f \cap \left\langle \operatorname{Im} \widetilde{\tau'} \right\rangle$ with $\theta_1(\varphi(P)) = 0$, there exists a polynomial $P' \in \ker f \cap \langle \operatorname{Im} \widetilde{\tau} \rangle$ so that $C^*\varphi(P') = \varphi(P)$.*

Proof. With $\theta_1(\varphi(P)) = 0$ we have by Theorem 4.11 that $d_2(\gamma) = f(\partial P/\partial t_0)$ for some $\gamma \in E_2^{*,1}(G/\mathbb{Z}_q)$. We can assume further that $\gamma = \widetilde{H}$ for some

$$H = h_1 \widetilde{\tau}(t_1) + \cdots + h_n \widetilde{\tau}(t_n) \in \langle \operatorname{Im} \widetilde{\tau} \rangle.$$

The desired polynomial P' is given by $P' := P + (H - \partial P/\partial t_0)\cdot \widetilde{\tau}'(t_0)$.

Indeed, the obvious relation $H - \partial P/\partial t_0 \in \ker f$ implies that

$$(H - \partial P/\partial t_0)\cdot \widetilde{\tau}'(t_0) \in \left\langle \operatorname{Im} \widetilde{\tau}' \right\rangle \cap \ker f.$$

Consequently, $P' \in \langle \operatorname{Im} \widetilde{\tau} \rangle \cap \ker f$. The calculation in $E_2^{*,1}(\left[G \times S^1\right]/\mathbb{Z}_q)$

$$\begin{aligned} & d_2(\sum_{1\le i\le n} f(h_i)\otimes t_i t_0) = -f(\partial P/\partial t_0)\otimes t_0 + \sum_{1\le i\le n} f(h_i)\widetilde{\tau}'(t_0)\otimes t_i \\ & = \widetilde{P'} - \widetilde{P} \end{aligned}$$

verifies the relation $C^*(\varphi(P')) - \varphi(P) = 0$ in $E_3^{*,1}(\left[G \times S^1\right]/\mathbb{Z}_q)$.□

5 Preliminaries in cohomology theory

In this section we develop some general results on the cohomology of Lie groups.

5.1. Let G be a semi–simple Lie group whose center $\mathcal{Z}(G)$ contains the cyclic group $\mathbb{Z}_q$. Denote by $J(\omega)$, $\langle \omega \rangle \subset H^*(G/\mathbb{Z}_q)$, respectively, the subring and the ideal generated by the Euler class $\omega \in H^2(G/\mathbb{Z}_q)$. Write $H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle}$ for the quotient ring of $H^*(G/\mathbb{Z}_q)/\langle \omega \rangle$ with quotient map g. Then, in addition to the short exact sequence

(5.1) $0 \to \langle \omega \rangle \to H^*(G/\mathbb{Z}_q) \overset{g}{\to} H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \to 0,$

one has the concise form of the exact sequence (2.5)

(5.2) $0 \to H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \overset{C^*}{\to} H^*(\left[G \times S^1\right]/\mathbb{Z}_q) \overset{\theta}{\to} H^*(G/\mathbb{Z}_q) \overset{\omega}{\to} \langle \omega \rangle \to 0.$

It implies in particular that the map C^* induces a ring isomorphism

(5.3) $H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \cong \operatorname{Im} C^* = \ker \theta.$

Theorem 5.1. *If $j : H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \to H^*(G/\mathbb{Z}_q)$ is a split homomorphism of the sequence (5.1), then the map $h : J(\omega) \otimes H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \to H^*(G/\mathbb{Z}_q)$ by $h(\omega^r \otimes x) = \omega^r \cup j(x)$ induces an isomorphism of $J(\omega)$–modules*

(5.4) $H^*(G/\mathbb{Z}_q) \cong \frac{J(\omega)\otimes H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle}}{\langle \omega \cdot h^{-1}(\operatorname{Im}\theta)\rangle}.$

Proof. As a graded group the ring $J(\omega)$ has the basis $\{1, \omega, \cdots, \omega^r\}$ for some $r \ge 1$. Granted with the map j a repeatedly application of the exact sequence (5.1) shows that every element $x \in H^*(G/\mathbb{Z}_q)$ admits an expression of the form

$$x = j(a_0) + \omega \cup j(a_1) + \cdots + \omega^r \cup j(a_r),\ a_i \in H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle},\ 0 \le i \le r,$$

showing that the map h is surjective. Given an element $y = a_0 + \omega \otimes a_1 + \cdots + \omega^r \otimes a_r \in \ker h$ one infers from $g \circ h(y) = 0$ that $a_0 = 0$. It follows then from

$$0 = h(y) = \omega \cdot (1 \cup j(a_1) + \cdots + \omega^{r-1} \cup j(a_r))$$

that $1 \cup j(a_1) + \cdots + \omega^{r-1} \cup j(a_r) \in \operatorname{Im}\theta$ by (5.2). That is $\ker h = \left\langle \omega \cdot h^{-1}(\operatorname{Im}\theta) \right\rangle$, establishing the isomorphism (5.4).□

5.2. Given a graded ring (resp. algebra) A and a set $S = \{u_1, \cdots, u_t\}$ of graded elements we write $A\{S\} := A\{u_i\}_{1 \leq \leq i \leq t}$ for the graded free A–module with basis $\{u_1, \cdots, u_t\}$. Denote by $A \otimes \Delta(S) := A \otimes \Delta(u_i)_{1 \leq \leq i \leq t}$ the graded free A–module in the simple system of generators $u_1, \cdots, u_t$. If $A = \mathbb{Z}$ (resp. $\mathbb{F}_p$) we use $\Delta(S)$ (resp. $\Delta_{\mathbb{F}_p}(S)$) instead of $A \otimes \Delta(S)$.

In addition, if $V = V^0 \oplus V^1 \oplus V^2 \oplus \cdots$ is a graded vector space (resp. a graded ring), define its subspace (resp. subring) V^+ by $V^1 \oplus V^2 \oplus \cdots$. For instance if $\mathbb{Z}[x_1, \cdots, x_n]$ is the ring of integral polynomials in $x_1, \cdots, x_n$ with $\deg x_i > 0$, then $\mathbb{Z}[x_1, \cdots, x_n]^+$ is the subring of $\mathbb{Z}[x_1, \cdots, x_n]$ consisting of the polynomials without constant terms.

Definition 5.2. Let A be a graded truncated polynomial algebra of the form

(5.5) $A = \mathbb{F}_p[x_1, \cdots, x_r] / \left\langle x_1^{k_1}, \cdots, x_r^{k_r} \right\rangle$, $\deg x_t \equiv 0 \bmod 2$, $1 \leq t \leq r$.

The *Koszul complex* associated to A is the cochain complex $\{\mathcal{C}, \delta\}$ defined by

i) $\mathcal{C} = A \otimes \Delta_{\mathbb{F}_p}(\theta_1, \cdots, \theta_r)$, $\deg \theta_t = \deg x_t - 1$;

ii) $\delta(\theta_t) = -x_t$, $\delta(x_t) = 0$.□

The cohomology $H^*(\{\mathcal{C}, \delta\})$ of the complex $\{\mathcal{C}, \delta\}$ can be explicitly presented. For a multi–index $I \subseteq \{1, \cdots, r\}$ introduce in the A–module $\mathcal{C}$ the next elements

$$\theta_I = \prod_{t \in I} \theta_t; \;\; f_I = \sum_{t \in I} -x_t \theta_{I_t} (= \delta(\theta_I)); \quad g_I = (\prod_{t \in I} x_t^{k_t - 1}) \theta_I,$$

where I_t denotes the complement of t in I. We shall also put

(5.6) $c_I = \delta\theta_I, \quad D_I = \sum_{t \in I} -x_t c_{I_t}, \quad R_I = (\prod_{t \in I} x_t^{k_t - 1}) c_I.$

Theorem 5.3. *For the complex $\{\mathcal{C}, \delta\}$ we have*

i) $H^*(\{\mathcal{C}, \delta\}) = \Delta_{\mathbb{F}_p}(x_1^{k_1 - 1}\theta_1, \cdots, x_r^{k_r - 1}\theta_r)$;

ii) $\operatorname{Im}\delta = \frac{A\{1, c_I\}^+}{\langle D_J, R_K \rangle}$, *$I, J, K \subseteq \{1, \cdots, r\}$ with $|I|, |J|, |K| \geq 2$*

where $\langle D_J, R_K \rangle \subset A\{1, c_I\}^+$ is the sub A–module spanned by D_J and R_K.

Proof. Since $\mathcal{C} = \underset{1 \leq t \leq r}{\otimes} \mathcal{C}_t$ with each factor $\mathcal{C}_t = (\mathbb{F}_p[x_t] / \left\langle x_t^{k_t} \right\rangle) \otimes \Delta_{\mathbb{F}_p}(\theta_t)$ a closed subspace of δ, one has $H^*(\{\mathcal{C}, \delta\}) = \underset{1 \leq t \leq r}{\otimes} H^*(\mathcal{C}_t, \delta)$ by the Künneth formula. Assertion i) comes from the obvious fact that the cohomology $H^*(\mathcal{C}_t, \delta)$ has a basis represented by the subset $\left\{1, x_t^{k_t - 1}\theta_t\right\} \subset \mathcal{C}_t$, $1 \leq t \leq r$.

For ii) one notes by the exact sequence $0 \to \ker\delta \to \mathcal{C} \xrightarrow{\delta} \operatorname{Im}\delta \to 0$ that δ induces an isomorphism of A–modules

a) $\overline{\delta} : \mathcal{C} / \ker\delta \overset{\cong}{\to} \operatorname{Im}\delta$

Moreover, in the quotient space $\mathcal{C}/\ker\delta$ the numerator $\mathcal{C}$ has the A–basis $\{1, \theta_I \mid I \subseteq \{1, \cdots, r\}\}$, while the denominator $\ker\delta$ has the decomposition $H^*(\{\mathcal{C}, \delta\}) \oplus \operatorname{Im}\delta$ in which the first summand $H^*(\{\mathcal{C}, \delta\})$ has the $\mathbb{F}_p$–basis $\{1, g_K; K \subseteq \{1, \cdots, r\}\}$ by assertion i), while the second summand $\operatorname{Im}\delta$ is spanned over A by the subset $\{f_J = \delta(\theta_J); J \subseteq \{1, \cdots, r\}\}$. That is

b) $\mathcal{C}/\ker\delta = \frac{A\{1,\theta_I\}^+}{\langle f_J, g_K\rangle}, I, J, K \subseteq \{1, \cdots, r\}$.

The presentation ii) is shown by a), b), the obvious relations $\overline{\delta}(\theta_I) = c_I$, $\overline{\delta}(f_J) = D_J$, $\overline{\delta}(g_K) = R_K$, together with the facts that if $I = \{t\}$ is a singleton, then

$$c_I = -x_t; \quad D_I = 0; \quad R_I = -x_t^{k_t} = 0.\square$$

5.3. Given a topological space X and a prime p the *Bockstein operator* δ_p on the algebra $H^*(X; \mathbb{F}_p)$ is the differential with degree 1

$$\delta_p := r_p \circ \beta_p : H^r(X; \mathbb{F}_p) \overset{\beta_p}{\to} H^{r+1}(X) \overset{r_p}{\to} H^{r+1}(X; \mathbb{F}_p).$$

The pair $\{H^*(X; \mathbb{F}_p); \delta_p\}$ is a cochain complex whose cohomology, denoted by $\overline{H}^*(X; \mathbb{F}_p)$, is called the $\operatorname{mod} p$ *Bockstein cohomology of* X.

Recall that any finitely generated abelian group A admits the canonical decomposition

(5.7) $A = \mathcal{F}(A) \underset{q}{\oplus} \sigma_q(A)$,

where $\mathcal{F}(A)$ is the *free part of* A, the sum is over all primes q, and where $\sigma_q(A) := \{x \in A; q^r x = 0, r \geq 1\}$ is the *q–primary component* of the group A. If A is the integral cohomology $H^*(X)$ of a space X we use $\mathcal{F}(X)$ in place of $\mathcal{F}(H^*(X))$, $\sigma_q(X)$ instead of $\sigma_q(H^*(X))$.

Theorem 5.4. *For a topological space X and a prime p one has*

(5.8) $\dim \mathcal{F}(X) \otimes \mathbb{Q} \leq \dim \overline{H}^*(X; \mathbb{F}_p)$,

while the equality implies that

i) $\sigma_p(X) = \operatorname{Im}\beta_p$;

ii) r_p *restricts to an isomorphism* $\sigma_p(X) \overset{\cong}{\to} \operatorname{Im}\delta_p$ *of algebras.*

Proof. Let r_p^1 be the restriction of r_p on the subgroup $\operatorname{Im}\beta_p \subset H^*(X)$, and consider the exact ladder induced by the decomposition $\delta_p = r_p \circ \beta_p$

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Im} r_p & \to & H^*(X; \mathbb{F}_p) & \overset{\beta_p}{\to} & \operatorname{Im}\beta_p & \to 0 \\ & i \downarrow & & \| & & r_p^1 \downarrow & \\ 0 \to & \ker\delta_p & \to & H^*(X; \mathbb{F}_p) & \overset{\delta_p}{\to} & \operatorname{Im}\delta_p & \to 0 \end{array}.$$

Since the vertical map i on the left is injective while the vertical map r_p^1 on the right is surjective, one has $\ker r_p^1 \cong co\ker i$ by [22, Exercise 5, p.726]. In addition, with respect to the obvious decompositions

$$\operatorname{Im} r_p \cong \mathcal{F}(X)/p \cdot \mathcal{F}(X) \oplus \sigma_p(X)/p \cdot \sigma_p(X),$$
$$\ker \delta_p = \overline{H}^*(X; \mathbb{F}_p) \oplus \operatorname{Im} \delta_p$$

the monomorphism i satisfies the relations

a) $i(\mathcal{F}(X)/p \cdot \mathcal{F}(X)) \subseteq \overline{H}^*(X; \mathbb{F}_p)$;
b) $\operatorname{Im} \delta_p \subseteq i(\sigma_p(X)/p \cdot \sigma_p(X))$ (since $\operatorname{Im} \beta_p \subseteq \sigma_p(X)$)

With $\dim_{\mathbb{F}_p} \mathcal{F}(X)/p \cdot \mathcal{F}(X) = \dim \mathcal{F}(X) \otimes \mathbb{Q}$ the relation (5.8) is shown by a).

In addition, if the equality in (5.8) (i.e. in a)) holds then i is surjective by a) and b), and hence r_p^1 is an isomorphism. We get i) and ii) from $\operatorname{Im} \beta_p \subseteq \sigma_p(X)$, as well as the fact that $\ker r_p^1 = 0$ implies $\operatorname{Im} \beta_p = \sigma_p(X)$.□

5.4. For an integer $n \geq 2$ let $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ be the binomial coefficients and set

$$b_{n,r} = g.c.d.\{\tbinom{n}{1}, \cdots, \tbinom{n}{r}\},\ 1 \leq r \leq n.$$

In view of the standard property that $b_{k+1} \mid b_k$ with $b_{n,1} = n, b_{n,n} = 1$, we define $a_{n,k} := \frac{b_{n,k-1}}{b_{n,k}}$, $k \geq 2$.

By *the prime factorization of an integer* $n \geq 2$ we mean the unique expression $n = p_1^{r_1} \cdots p_t^{r_t}$ with $1 < p_1 < \cdots < p_t$ the set of all prime factors of n. The next result, whose proof will appear in [4], express the sequences $(b_{n,1}, \cdots, b_{n,n})$ and $(a_{n,2}, \cdots, a_{n,n})$ as explicit functions in the given integer n.

Theorem 5.5. *Let $n > 2$ be with the prime factorization $n = p_1^{r_1} \cdots p_t^{r_t}$. Then*

i) $a_{n,k} = \begin{cases} p_s \text{ if } k = p_s^r \text{ for some } 1 \leq s \leq t,\ 1 \leq r \leq r_s, \\ 1 \text{ otherwise;} \end{cases}$

ii) $b_{n,k}$ has the prime factorization $\underset{1 \leq s \leq t}{\Pi} p_s^{r_s - l(s,k)}$,

where $l(s,k) = r_s$ if $k \geq p_s^{r_s}$, and satisfies $p_s^{l(s,k)} \leq k < p_s^{l(s,k)+1}$ if $k < p_s^{r_s}$.□

Example 5.6. For an integer n the graded ring $J_n(\omega) = \mathbb{Z}[\omega]/\left\langle \binom{n}{k}\omega^k \right\rangle_{1 \leq k \leq n}$, $\deg \omega = 2$, in the single generator ω is clearly isomorphic to $\mathbb{Z}[\omega]/\left\langle b_{n,k}\omega^k \right\rangle_{1 \leq k \leq n}$. Assume that the prime decomposition of the given integer n is $p_1^{r_1} \cdots p_t^{r_t}$. Then ii) of Theorem 5.5 implies the canonical decomposition (5.7) of $J_n(\omega)$

$$(5.10)\quad J_n(\omega) = \mathbb{Z} \bigoplus_{s \in \{1, \cdots, t\}} \mathbb{Z}[\omega]^+ / \left\langle p_s^{r_s}\omega, p_s^{r_s - 1}\omega^{p_s}, p_s^{r_s - 2}\omega^{p_s^2}, \cdots, \omega^{p_s^{r_s}} \right\rangle.\square$$

6 The cohomology of adjoint Lie groups

A Lie group G is called *adjoint* if its center subgroup $\mathcal{Z}(G)$ is trivial. In particular, for any Lie group G the quotient group $PG := G/\mathcal{Z}(G)$ is adjoint. As applications of the cohomology theory of Lie groups developed through Section 2 to Section 5 we compute the cohomologies of the adjoint Lie groups PG with $G = SU(n), Sp(n), E_6, E_7$. In these cases the quotient maps $c : G \to PG$ are cyclic. The corresponding circle bundle over PG is denoted by

$C : \left[G \times S^1\right]/\mathbb{Z}_q \to PG,\ q = |\mathcal{Z}(G)|.$

Briefly, the computation is carried out by three steps. Granted with the presentations of the rings $H^*(G/T)$ in Theorem 7.1, as well as the constructions in Section 4, we obtain firstly the subgroups $\operatorname{Im}\pi^*$ and $\operatorname{Im}\kappa$. The exact sequence (2.5) is then applied to formulate the additive cohomology $H^*(PG)$ by $\operatorname{Im}\pi^*$ and $\operatorname{Im}\kappa$. Finally, the ring (resp. algebra) structure on $H^*(PG)$ is decided by expressing the squares x^2 with $x \in \operatorname{Im}\kappa$ as elements of $\operatorname{Im}\pi^*$, see Lemma 4.2.

As the groups G are 1–connected the algebras $H^*(G;\mathbb{F}_p)$ are known [8]. We exclude the algebras $H^*(PG;\mathbb{F}_p)$ with $(p,q) = 1$ from further consideration.

Theorem 6.1. *If $(p,q) = 1$ the map C induces an isomorphism of algebras*

$$C^* : H^*(PG;\mathbb{F}_p) \cong H^*(G;\mathbb{F}_p).$$

Proof. With p co–prime to q one has $\omega \equiv 0 \bmod p$. Therefore the Gysin sequence (2.5) becomes the short exact sequence of algebras

$$0 \to H^*(PG;\mathbb{F}_p) \overset{C^*}{\to} H^*(G\times S^1;\mathbb{F}_p) \overset{\theta}{\to} H^*(PG;\mathbb{F}_p) \to 0.$$

Moreover, the relation $\theta(\xi_1) = 1$ by i) of Theorem 2.3 implies that, with respect to the decomposition $H^*(G\times S^1;\mathbb{F}_p) = H^*(G;\mathbb{F}_p) \oplus \xi_1 \cdot H^*(G;\mathbb{F}_p)$ the map C^* carries $H^*(PG;\mathbb{F}_p)$ injectively into the first summand $H^*(G;\mathbb{F}_p)$, while by ii) of Theorem 2.3 the map θ maps the second summand $\xi_1 \cdot H^*(G;\mathbb{F}_p)$ surjectively onto $H^*(PG;\mathbb{F}_p)$. This establishes the isomorphism.$\square$

6.1 The map $C^* : E_3^{*,0}(G/\mathbb{Z}_q) \to E_3^{*,0}(\left[G\times S^1\right]/\mathbb{Z}_q)$ in (4.12)

To begin with we take the ring $\mathbb{Z}$ as coefficients for cohomology. As the groups G are 1–connected $TorH^2(G\times S^1) = 0$ by Theorem 2.2. Therefore, the transgression τ' in π' is surjective by (3.2). In view of the rings $H^*(G/T)$ given by Theorem 7.1 the relation $E_3^{*,0}(\left[G\times S^1\right]/\mathbb{Z}_q) = H^*(G/T)\mid_{\omega_1=\cdots=\omega_n=0}$by Theorem 4.2 yields that

$$(6.1)\quad \begin{aligned} &E_3^{*,0}(\left[SU(n)\times S^1\right]/\mathbb{Z}_n) = E_3^{*,0}(\left[Sp(n)\times S^1\right]/\mathbb{Z}_2) = \mathbb{Z};\\ &E_3^{*,0}(\left[E_6\times S^1\right]/\mathbb{Z}_3) = \frac{\mathbb{Z}[x_6,x_8]}{\langle 2x_6,3x_8,x_6^2,x_8^3\rangle};\\ &E_3^{*,0}(\left[E_7\times S^1\right]/\mathbb{Z}_2) = \frac{\mathbb{Z}[x_6,x_8,x_{10},x_{18}]}{\langle 2x_6,3x_8,2x_{10},2x_{18},x_6^2,x_8^3,x_{10}^2,x_{18}^2\rangle}\end{aligned}$$

where x_i's are the Schubert classes on E_n/T, $n = 6,7$, specified in Section 7.1.

For the groups PG we take a set $\Omega = \{\varrho_1,\cdots,\varrho_n\}$ of fundamental dominant weights as a basis for the unit lattice Λ_e, and let $\{t_1,\cdots,t_n\}$ be the corresponding basis on the group $H^1(T)$ (see (3.6)). By iii) of Lemma 3.2 the transition matrix $C(\Omega)$ from the basis Δ of root lattice Λ_r to Ω is the Cartan matrix of G. Granted with the presentation of the rings $H^*(G/T)$ in Theorem 7.1 and the results of Lemma 7.2 one get from Lemma 4.2 that

$$(6.2)\quad \begin{aligned} &E_3^{*,0}(PSU(n)) = \frac{\mathbb{Z}[\omega_1]}{\langle b_{n,r}\omega_1^r,\ 1\le r\le n\rangle} \text{ (see Example 5.6)}\\ &E_3^{*,0}(PSp(n)) = \frac{\mathbb{Z}[\omega_1]}{\langle 2\omega_1,\omega_1^{2^{r+1}}\rangle},\ n = 2^r(2s+1);\\ &E_3^{*,0}(PE_6) = \frac{\mathbb{Z}[\omega_1,x_3',x_4]}{\langle 3\omega_1,2x_3',3x_4,x_3'^2,\omega_1^9,x_4^3\rangle},\ x_3' = x_3+\omega_1^3;\\ &E_3^{*,0}(PE_7) = \frac{\mathbb{Z}[\omega_2,x_3,x_4,x_6,x_9]}{\langle 2\omega_2,\omega_2^2,2x_3,3x_4,2x_5,2x_9,x_3^2,x_4^3,x_5^2,x_9^2\rangle}.\end{aligned}$$

Inputting the presentations (6.1) and (6.2) into the section

$$E_3^{*,0}(PG) \overset{\tau'(t_0)}{\to} E_3^{*,0}(PG) \overset{C^*}{\to} E_3^{*,0}(\left[G \times S^1\right]/\mathbb{Z}_q) \to 0$$

of the exact sequence (4.12) shows that

Lemma 6.2. *Over the ring $\mathbb{Z}$ of integers we have*

(6.3) $\tau'(t_0) = \omega_1$ *for* $SU(n), Sp(n), E_6$; $\quad \omega_2$ *for* E_7.

Setting $\varpi := \tau'(t_0)$ amd letting $J(\varpi) \subset E_3^{*,0}(PG)$ be the subring generated by ϖ, then in the order of $G = SU(n), Sp(n)$ with $n = 2^r(2s+1), E_6$ and E_7,

i) $J(\varpi) = \frac{\mathbb{Z}[\varpi]}{\langle b_{n,r}\varpi^r,\ 1\leq r\leq n\rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 2\varpi, \varpi^{2^{r+1}}\rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 3\varpi, \varpi^9\rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 2\varpi, \varpi^2\rangle}$;

ii) the map $C^* : E_3^{*,0}(PG) \to E_3^{*,0}(\left[G \times S^1\right]/\mathbb{Z}_q)$ *is given by*

$$C^*(x_s) = x_s,\ C^*(x_3') = x_3;\ C^*(\varpi) = 0.\square$$

For a prime p let $J_p(\varpi) \subset E_3^{*,0}(PG;\mathbb{F}_p)$ be the subalgebra generated by ϖ and set $E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle} = E_3^{*,0}(PG;\mathbb{F}_p)\mid_{\varpi=0}$. Since the ring $H^*(G/T)$ is torsion free we have $E_3^{*,0}(X;\mathbb{F}_p) = E_3^{*,0}(X)\otimes\mathbb{F}_p$ for both $X = G/\mathbb{Z}_q$ and $\left[G \times S^1\right]/\mathbb{Z}_q$. Lemma 6.2 implies that

Lemma 6.3. *If $(p,q) = p$ we have the decomposition*

(6.4) $E_3^{*,0}(PG;\mathbb{F}_p) = J_p(\varpi) \otimes E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle}$,

where, in the order of $(G,p) = (SU(n),p)$ with $n = p^r n'$ and $(p,n\prime) = 1$, $(Sp(n),2)$ with $n = 2^r(2d+1)$, $(E_6,3)$ and $(E_7,2)$

i) $J_p(\varpi) = \frac{\mathbb{F}_p[\varpi]}{\langle\varpi^{h(G)}\rangle}$ *with* $h(G) = p^r, 2^{r+1}, 9, 2$;

ii) $E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle} = \mathbb{F}_p, \mathbb{F}_2, \frac{\mathbb{F}_3[x_4]}{\langle x_4^3\rangle}, \frac{\mathbb{F}_2[x_3,x_5,x_9]}{\langle x_3^2,x_5^2,x_9^2\rangle}$;

iii) the map C^* annihilates ϖ and carries the second factor $1 \otimes E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle}$ in (6.4) isomorphically onto $E_3^{*,0}(\left[G \times S^1\right]/\mathbb{Z}_q;\mathbb{F}_p)$.$\square$

6.2 The algebra $H^*(PG;\mathbb{F}_p)$

By Theorem 6.1 we can assume $(G,p) = (SU(n),p)$ with $p \mid n$; $(Sp(n),2)$; $(E_6,3)$; $(E_7,2)$. Since G is 1–connected a unified method constructing the algebra $H^*(G\times S^1;\mathbb{F}_p)$ is available in [8]. Precisely, in view of the set $S_p(G)$ of polynomials given in (7.1) (see also Example 4.8) let $D(G,p)$ be the degree set of the elements in $S_p(G)$, arranged in the increasing order in the table below

(G,p)	$D(G,p)$
$(SU(n),p)$	$\{2,3,\cdots,n\}$
$(Spin(n),2)$	$\{2,4,6,\cdots,2n\}$
$(E_6,3)$	$\{2,4,5,6,8,9\}$
$(E_7,2)$	$\{2,3,5,8,9,12,14\}$

.

For each $s \in D(G,p)$ let $\xi_{2s-1} := \kappa \circ \varphi_p(P) \in H^*(G \times S^1, \mathbb{F}_p)$ be with $P \in S_p(G)$, $\deg P = s$. By [9, Theorem 5.4] the map $\pi^{'}$ in (4.9) induces an isomorphism

(6.4) $\pi'^* : E_3^{*,0}(G \times S^1, \mathbb{F}_p) \cong \operatorname{Im} \pi'^* \subset H^*(G \times S^1, \mathbb{F}_p)$ (see (4.2))

and the inclusions $\{\xi_{2s-1}\} \subset H^*(G \times S^1, \mathbb{F}_p)$ induces an isomorphism

(6.5) $H^*(G \times S^1, \mathbb{F}_p) = \operatorname{Im} \pi'^* \otimes \Delta(\xi_1, \xi_{2s-1})_{s \in D(G,p)}$,

of $\operatorname{Im} \pi'^*$–modules, where the class ξ_1 has been specified in Theorem 2.3. Let $J_p(\omega)$ be the subalgebra generated by the Euler class $\omega \in H^*(PG, \mathbb{F}_p)$.

Lemma 6.4. *With respect to (6.5) the map θ in (2.5) satisfies that*

i) $\theta(\xi_{2s-1}) = 0$ if $s \neq h(G)$, $\omega^{h(G)-1}$ if $s = h(G)$ (see Lemma 6.3).

The map π^* in (4.2) restricts an isomorphism

ii) $J_p(\varpi) \cong J_p(\omega)$.

Proof. With $p \mid q$ one has $\theta(\xi_1) = 0$ by i) of Theorem 2.3. In term of the set $S_p(G)$ of characteristic polynomials given in (7.1) one computes the derivations $\partial P / \partial t_0$ with $P \in S_p(G)$ as that presented in table (7.2). Results on $\theta(\xi_{2s-1})$ with $s \in D(G,p)$ are verified by Theorem 4.12 and the contents of table (7.2).

With p a prime we can assume that $J_p(\omega) = \mathbb{F}_p[\omega] / \langle \omega^r \rangle$ for some $r \geq 2$. The exact sequence (2.5) implies that $\omega^{r-1} \in \operatorname{Im} \theta$, hence $r = h(G)$ by i). The isomorphism ii) is verified by iii) of Lemma 4.11 and i) of Lemma 6.3.□

Let $D(PG,p)$ be the complement of $h(G)$ in $D(G,p)$. By i) of Lemma 6.4 $\theta(\xi_{2s-1}) = 0$, $s \in D(PG,p)$. In view of the exact sequence (2.5)

$$H^*(PG; \mathbb{F}_p) \xrightarrow{C^*} H^*(G \times S^1; \mathbb{F}_p) \xrightarrow{\theta} H^*(PG; \mathbb{F}_p)$$

and by Lemma 4.13 there is a polynomial $P' \in \langle \operatorname{Im} \widetilde{\tau} \rangle_p \cap \ker f_p$ with $\deg P' = s$ so that 1–form $\zeta_{2s-1} = \kappa \circ \varphi_p(P') \in H^*(PG; \mathbb{F}_p)$ satisfies the relation $C^*(\zeta_{2s-1}) = \xi_{2s-1}$. A set of such polynomials P' obtained, denoted by $S_p(PG)$, are presented in table (7.3). As in Section 2 let $\iota \in H^1(PG; F_p)$ be the characteristic class of the covering c.

Lemma 6.5. *The inclusion $\iota, \zeta_{2s-1} \in H^*(PG; \mathbb{F}_p)$ induces an isomorphism of $\operatorname{Im} \pi^*$–modules, where $\operatorname{Im} \pi^* \cong E_3^{*,0}(PG; \mathbb{F}_p)$ via the map π^* in (4.2).*

$$H^*(PG; \mathbb{F}_p) \cong \operatorname{Im} \pi^* \otimes \Delta(\iota, \zeta_{2r-1})_{r \in D(PG,p)}.$$

Proof. With the field $\mathbb{F}_p$ as coefficients the exact sequence (5.1) is split. Therefore, formula (5.4) is applicable to express $H^*(PG; \mathbb{F}_p)$. With the subalgebra $J_p(\omega)$ having been determined in Lemma 6.4 we are bound to decide the subspaces $H^*(PG; \mathbb{F}_p)_{\langle \omega \rangle}$ and $\langle \omega \cdot h^{-1}(\operatorname{Im} \theta) \rangle$ requested by the formula (5.4).

Since $\theta(\xi_1) = 0$ by Lemma 6.4, one has $C^*(\iota) = \xi_1$ by the degree reason. According to (6.5) the algebra $H^*(G \times S^1; \mathbb{F}_p)$ has the decomposition

$$H^*(G \times S^1; \mathbb{F}_p) = B \oplus \xi_{2h(G)-1} \cdot B,$$

with $B = \operatorname{Im}\pi'^* \otimes \Delta(\xi_1, \xi_{2s-1})_{s\in D(PG,p)}$. On the other hand, by $C^*(\zeta_{2s-1}) = \xi_{2s-1}$, $s \in D(PG,p)$, $C^*(\iota) = \xi_1$, and by Lemma 6.3, the inclusions $\{\iota, \zeta_{2s-1}\}$, $\operatorname{Im}\pi^* \subset H^*(PG;\mathbb{F}_p)$ induces a linear map

$$j : E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle} \otimes \Delta(\iota, \zeta_{2s-1})_{s\in D(PG,p)} \to H^*(PG;\mathbb{F}_p)$$

such that $C^* \circ j$ is an isomorphism onto the subspace B. These imply that

$$\dim H^*(G \times S^1;\mathbb{F}_p) = 2\dim B \text{ and } B \subseteq \operatorname{Im} C^*.$$

Combining these with the relation $\dim H^*(G \times S^1;\mathbb{F}_p) = 2\dim H^*(PG;\mathbb{F}_p)_{\langle\omega\rangle}$ by (5.2) yields that $\operatorname{Im} C^* = B$. As a result

(6.6) $H^*(PG;\mathbb{F}_p)_{\langle\omega\rangle} \cong E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle} \otimes \Delta(\iota, \zeta_{2s-1})_{s\in D(PG,p)}$.

With $\operatorname{Im} C^* = B$ the map θ in (5.2) restricts to an isomorphism

$$\xi_{2h(G)-1} \cdot B \cong \omega^{h(G)-1} \cdot H^*(PG;\mathbb{F}_p)_{\langle\omega\rangle}$$

(by ii) of Theorem 2.5 and i) of Lemma 6.4). It implies the relation $\left\langle \omega \cdot h^{-1}(\operatorname{Im}\theta)\right\rangle = \left\langle \omega^{h(G)}\right\rangle$ that has already been recorded in the presentation of the algebra $J_p(\omega)$. The isomorphism in the lemma is shown by the calculation

$$\begin{aligned} &H^*(PG;\mathbb{F}_p) = J_p(\omega) \otimes H^*(PG;\mathbb{F}_p)_{\langle\omega\rangle} \text{ (by (5.4))} \\ &= J_p(\omega) \otimes E_3^{*,0}(PG;\mathbb{F}_p)_{\langle\varpi\rangle} \otimes \Delta(\iota, \zeta_{2s-1})_{s\in D(PG,p)} \text{ (by (6.6))} \\ &= E_3^{*,0}(PG;\mathbb{F}_p) \otimes \Delta(\iota, \zeta_{2s-1})_{s\in D(PG,p)} \text{ (by ii) of Lemma 6.4)}, \end{aligned}$$

which implies also that the map π^* in (4.2) is an isomorphism onto $\operatorname{Im}\pi^*$.□

By Lemma 6.5 the maps π^* and κ are monomorphisms. For this reason we can reserve $x \in E_3^{*,0}(PG;\mathbb{F}_p)$ for $\pi^*(x) \in H^*(PG;\mathbb{F}_p)$.

Theorem 6.6. *One has the isomorphisms of algebras*

i) $H^*(PSU(n);\mathbb{F}_2) = \frac{\mathbb{F}_2[\omega]}{\langle\omega^{2^r}\rangle} \otimes \Delta(\iota) \otimes \Lambda_{\mathbb{F}_2}(\zeta_3, \zeta_5, \cdots, \widehat{\zeta}_{2^{r+1}-1}, \cdots, \zeta_{2n-1})$, *where* $n = 2^r(2b+1)$, $\iota^2 = \omega$ *or* 0 *in accordance to* $r = 1$ *or* $r > 1$;

ii) $H^*(PSU(n);\mathbb{F}_p) = \frac{\mathbb{F}_p[\omega]}{\langle\omega^{p^r}\rangle} \otimes \Lambda_{\mathbb{F}_p}(\iota, \zeta_3, \cdots, \widehat{\zeta}_{2p^r-1}, \cdots, \zeta_{2n-1})$, *where* $p \neq 2$, $n = p^r n'$ *with* $(n', p) = 1$;

iii) $H^*(PSp(n);\mathbb{F}_2) = \frac{\mathbb{F}_2[\omega]}{\left\langle\omega^{2^{r+1}}\right\rangle} \otimes \Delta(\iota) \otimes \Lambda_{\mathbb{F}_2}(\zeta_3, \zeta_7, \cdots, \widehat{\zeta}_{2^{r+2}-1}, \cdots, \zeta_{4n-1})$, *where* $\iota^2 = \omega$, $n = 2^r(2b+1)$;

iv) $H^*(PE_6;\mathbb{F}_3) = \frac{\mathbb{F}_3[\omega, x_4]}{\left\langle\omega^9, x_4^3\right\rangle} \otimes \Lambda_{\mathbb{F}_3}(\iota, \zeta_3, \zeta_7, \zeta_9, \zeta_{11}, \zeta_{15})$;

v) $H^*(PE_7;\mathbb{F}_2) = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9]}{\left\langle\omega^2, x_3^2, x_5^2, x_9^2\right\rangle} \otimes \Delta(\iota, \zeta_5, \zeta_9) \otimes \Lambda_{\mathbb{F}_2}(\zeta_{15}, \zeta_{17}, \zeta_{23}, \zeta_{27})$, *where* $\iota^2 = \omega, \zeta_5^2 = x_5, \zeta_9^2 = x_9$.

Proof. Through the presentations i) to v) the first factors is $\operatorname{Im}\pi^* = E_3^{*,0}(PG;\mathbb{F}_p)$ by Lemma 6.5, whose presentations are given by Lemma 6.3. It remains to verify the expressions of the squares ι^2, ζ_{2r-1}^2 as that indicated in the theorem.

The cases ii) and iv) are trivial, as in a characteristic $p \neq 2$ the square of any odd degree cohomology class is zero. In the remaining cases i), iii) and v) we have $p = 2$. The relations on ι^2 follows from the definition $\omega = \beta_q(\iota) \in H^2(PG)$ of ω as an integral class (see in Section 2), where $q = n, 2, 2$ in accordance to $G = SU(n), Sp(n), E_7$. To evaluate the squares ζ_{2s-1}^2 with $s \in D(PG,p)$ we make use of the Steenrod operators Sq^{2k}, $k \geq 1$, by which

$$\zeta_{2s-1}^2 = \delta_2 \circ Sq^{2s-2}(\zeta_{2s-1}) \text{ (see [20])}.$$

Computation in Lemma 7.5 shows that $\zeta_{2s-1}^2 = 0$ with the only exceptions that $\zeta_5^2 = x_5$, $\zeta_9^2 = x_9$ when $(G,p) = (E_7, 2)$. This completes the proof.□

6.3 The Bockstein cohomologies $\overline{H}^*(PG;\mathbb{F}_p)$

In preparation to compute the torsion ideals $\sigma_p(PG)$ of the integral cohomology $H^*(PG)$ we determine in this section the Bockstein cohomologies $\overline{H}^*(PG;\mathbb{F}_p)$ for $(G,p) = (Sp(n),2)$ with $n = 2^r(2b+1)$, $(E_6,3)$ and $(E_7,2)$. To begin with we note that the cochain complexes $\{H^*(PG;\mathbb{F}_p);\delta_p\}$ have already been decided. Indeed, combining Theorem 6.6 with Lemma 7.5 we have

(6.7a) $H^*(PSp(n);\mathbb{F}_2) = [\operatorname{Im}\pi^* \otimes \Delta_{\mathbb{F}_2}(\iota)] \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_3, \cdots, \widehat{\widetilde{\zeta}}_{2^{r+2}-1}, \cdots, \widetilde{\zeta}_{4n-1})$ with

a) $\operatorname{Im}\pi^* = \frac{\mathbb{F}_2[\omega]}{\langle \omega^{2^{r+1}} \rangle}, \delta_2(\iota) = \omega;$

b) $\delta_2(\widetilde{\zeta}_{2s-1}) = 0$ for $s \in D(PSp(n),2)$.

where in view of iii) of Theorem 6.6 and for $1 \leq s \leq n$

$\widetilde{\zeta}_{4s-1} := \zeta_{4s-1}$, $s \neq 2^{r-1}$; $\widetilde{\zeta}_{2^{r+1}-1} = \zeta_{2^{r+1}-1} + \iota\omega^{2^{r+1}-2}$;

(6.7b) $H^*(PE_6;\mathbb{F}_3) = [\operatorname{Im}\pi^* \otimes \Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_1, \widetilde{\zeta}_7)] \otimes \Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_{2s-1})_{s\in\{2,5,6,8\}}$ with

a) $\operatorname{Im}\pi^* = \frac{\mathbb{F}_3[\omega, x_4]}{\langle \omega^9, x_4^3 \rangle}, \delta_3(\widetilde{\zeta}_{2s-1}) = x_s$ for $s = 1, 4$;

b) $\delta_2(\widetilde{\zeta}_{2s-1}) = 0$ for $s = 2, 5, 6, 8$.

where in view of iv) of Theorem 6.6 and in the order of $s = 1, 2, 4, 5, 6, 8$

$\widetilde{\zeta}_{2s-1} := \iota, \zeta_3, \zeta_7, \zeta_9, \zeta_{11}, \zeta_{15} - x_4\zeta_7;$

(6.7c) $H^*(PE_7;\mathbb{F}_2) = [\operatorname{Im}\pi^* \otimes \Delta_{\mathbb{F}_2}(\widetilde{\zeta}_{2s-1})_{s\in\{1,3,5,9\}}] \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{2s-1})_{s\in\{8,12,14\}}$ with

a) $\operatorname{Im}\pi^* = \frac{\mathbb{F}_2[x_1, x_3, x_5, x_9]}{\langle x_1^2, x_3^2, x_5^2, x_9^2 \rangle}$, $\delta_3(\widetilde{\zeta}_{2s-1}) = x_s$ for $s = 1, 3, 5, 9$;

b) $\delta_3(\widetilde{\zeta}_{2s-1})) = 0$ for $s = 8, 12, 14$.

where in view of v) of Theorem 6.6 and in the order of $s = 1, 3, 5, 8, 9, 12, 14$

$\widetilde{\zeta}_{2s-1} := \iota, \zeta_5, \zeta_9, \zeta_{15} + x_3\zeta_9, \zeta_{17}, \zeta_{23} + x_3\zeta_{17}, \zeta_{27} + x_5\zeta_{17}$

In what follows we put $c_{\{1,4\}} = \delta_3(\widetilde{\zeta}_1\widetilde{\zeta}_7) \in H^9(PE_6;\mathbb{F}_3)$. For each multi–index $I \subseteq \{1,3,5,9\}$ define in $H^*(PE_7;\mathbb{F}_2)$ the elements

$$\widetilde{\zeta}_I = \prod_{s\in I}\widetilde{\zeta}_{2s-1},\ \ c_I = \delta_2(\widetilde{\zeta}_I).$$

Lemma 6.7. *The cohomology $\overline{H}^*(PG;\mathbb{F}_p)$, together with $\operatorname{Im}\delta_p$, are given by*

i) $\overline{H}^*(PSp(n);\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\omega^{2^{r+1}-1}\iota, \widetilde{\zeta}_3, \widetilde{\zeta}_7, \cdots, \widehat{\widetilde{\zeta}}_{2^{r+2}-1}, \cdots, \widetilde{\zeta}_{4n-1})$;

$\operatorname{Im}\delta_2 = \frac{\mathbb{F}_2[\omega]^+}{\langle \omega^{2^{r+1}}\rangle} \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_3, \widetilde{\zeta}_7, \cdots, \widehat{\widetilde{\zeta}}_{2^{r+2}-1}, \cdots, \widetilde{\zeta}_{4n-1})$

ii) $\overline{H}^*(PE_6;\mathbb{F}_3) \cong \Lambda_{\mathbb{F}_3}(\omega^8\widetilde{\zeta}_1, x_4^2\widetilde{\zeta}_7, \widetilde{\zeta}_3, \widetilde{\zeta}_9, \widetilde{\zeta}_{11}, \widetilde{\zeta}_{15})$

$\operatorname{Im}\delta_3 = \frac{\mathbb{F}_3[\omega, x_4, c_{\{1,4\}}]^+}{\langle \omega^9, x_4^3, c_{\{1,4\}}^2, x_1^8x_4^2c_{\{1,4\}}\rangle} \otimes \Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_3, \widetilde{\zeta}_9, \widetilde{\zeta}_{11}, \widetilde{\zeta}_{15})$

iii) $\overline{H}^*(PE_7;\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\omega\widetilde{\zeta}_1, x_3\widetilde{\zeta}_5, x_5\widetilde{\zeta}_9, x_9\widetilde{\zeta}_{17}, \widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27})$

$\operatorname{Im}\delta_2 = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9, c_I]^+}{\langle \omega^2, x_3^2, x_5^2, x_9^2, D_I, R_I, S_{I,J}\rangle} \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27})$

where $I, J \subset \{1,3,5,9\}$ with $|I|, |J| \geq 2$, the relations $D_I, R_I, S_{I,J}$ in the presentation of $\operatorname{Im}\delta_2$ are, respectively,

$$(6.8)\quad \sum_{t\in I} x_t c_{I_t} = 0, \quad (\prod_{t\in I} x_t)c_I = 0,\ c_Ic_J + \sum_{t\in I} x_t \prod_{s\in I_t\cap J} \widetilde{\zeta}^2_{2s-1} c_{\langle I_t, J\rangle} = 0,$$

and where $(\widetilde{\zeta}^2_1, \widetilde{\zeta}^2_5, \widetilde{\zeta}^2_9, \widetilde{\zeta}^2_{17}) = (\omega, x_5, x_9, 0)$, I_t is the complement of $t \in I$, $\langle I, J\rangle$ denotes the complement of the intersection $I \cap J$ in the union $I \cup J$.

Proof. In (6.7a) (resp. (6.7b)) the first factor $[\operatorname{Im}\pi^* \otimes \Delta_{\mathbb{F}_2}(\iota)]$ (resp. $[\operatorname{Im}\pi^* \otimes \Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_1, \widetilde{\zeta}_7)]$) is the Koszul complex associated to the truncated polynomial algebra $\operatorname{Im}\pi^*$ by a), while the differential δ_2 (resp. δ_3) acts trivially on the second factor $\Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{4s-1})_{s\in D(PSp(n),2)}$ (resp. $\Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_{2s-1})_{s\in\{2,5,6,8\}}$) by b). One gets i) (resp. ii)) from Theorem 5.3, together with the Künneth formula.

Similarly, one gets from (6.7c), Theorem 5.3 and the Künneth formula that

a) $\overline{H}^*(PE_7;\mathbb{F}_2) = \Delta_{\mathbb{F}_2}(x_1\widetilde{\zeta}_1, x_3\widetilde{\zeta}_5, x_5\widetilde{\zeta}_9, x_9\widetilde{\zeta}_{17}) \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27})$,

b) $\operatorname{Im}\delta_2 = \frac{\operatorname{Im}\pi^*\{1, c_I\}^+}{\langle D_J, R_K\rangle} \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27})$,

where $I, J, K \subseteq \{1,3,5,9\}$ with $|I|, |J|, |K| \geq 2$. Taking into account of the algebra structure on $H^*(PE_7;\mathbb{F}_2)$ given by v) of Theorem 6.6 we find that a) is identical to iii). To convert the additive presentation of $\operatorname{Im}\delta_2$ in b) into its algebra presentation in iii) one needs to clarify the multiplicative rule among the classes c_I's. This brings us the relations of the type $S_{I,J}$ which are established by the following calculation. Let $I, J \subseteq \{1,3,5,9\}$ be with $|I|, |J| \geq 2$. Then

$$\begin{aligned}
c_Ic_J &= \delta_2(\widetilde{\zeta}_I)\delta_2(\widetilde{\zeta}_J) = \delta_2(\delta_2(\widetilde{\zeta}_I)\widetilde{\zeta}_J) \text{ (since } \delta_2^2 = 0)\\
&= \delta_2(\sum_{t\in I} x_t\widetilde{\zeta}_{I_t}\widetilde{\zeta}_J) \text{ (since } \delta_2(\widetilde{\zeta}_I) = \sum_{t\in I} -x_t\widetilde{\zeta}_{I_t})\\
&= \delta_2(\sum_{t\in I} x_t \prod_{s\in I_t\cap J} \zeta^2_{2s-1}\zeta_{\langle I_t, J\rangle}) \text{ (with } \prod_{s\in I\cap J}\widetilde{\zeta}^2_{2s-1} = 1 \text{ if } I\cap J = \emptyset)\\
&= \sum_{t\in I} x_t \prod_{s\in I_t\cap J} \widetilde{\zeta}^2_{2s-1} c_{\langle I_t, J\rangle} \text{ (since } \delta_2(x_t) = 0,\ \delta_2(\widetilde{\zeta}^2_{2s-1}) = 0).\square
\end{aligned}$$

Corollary 6.8. *For $(G,p) = (Sp(n),2)$ with $n = 2^r(2b+1)$, $(E_6,3)$ and $(E_7,2)$*

i) the reduction r_p restricts to an isomorphism $\sigma_p(PG) \cong \operatorname{Im}\delta_p$;

ii) the subspace $\operatorname{Im}\kappa \cap \overline{H}^*(PG;\mathbb{F}_p) \subset H^*(PG;\mathbb{F}_p)$ has the basis

$B(PSp(n),2) = \{\omega^{2^{r+1}-1}\iota, \widetilde{\zeta}_{2s-1}\}_{s\in D(PSp(n),2)}$;

$B(PE_6,3) = \{\omega^8\widetilde{\zeta}_1, x_4^2\widetilde{\zeta}_7, \widetilde{\zeta}_3, \widetilde{\zeta}_9, \widetilde{\zeta}_{11}, \widetilde{\zeta}_{15}\}$,

$B(PE_7,2) = \{\omega\widetilde{\zeta}_1, x_3\widetilde{\zeta}_5, x_5\widetilde{\zeta}_9, x_9\widetilde{\zeta}_{17}, \widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27}\}$.

Proof. By the presentations of the spaces $\overline{H}^*(PG;\mathbb{F}_p)$ we have $\dim\overline{H}^*(PG;\mathbb{F}_p) = 2^n, 2^6$ and 2^7 in the order of $G = Sp(n), E_6$ and E_7. This shows i) by Theorem 5.4. Noticing that $\operatorname{Im}\kappa$ is a module over $\operatorname{Im}\pi^*$ property ii) is transparent in view of the presentations of $\overline{H}^*(PG;\mathbb{F}_p)$ in Lemma 6.7.□

6.4 The integral cohomology $H^*(PG)$

In view of the set $S(G)$ of characteristic polynomials (see Example 4.8) given in (7.5) let $D(G)$ be the degree set of elements in $S(G)$, arranged in the increasing order as shown in the table below

G	$D(G)$
$SU(n)$	$\{2,3,\cdots,n\}$
$Spin(n)$	$\{2,4,6,\cdots,2n\}$
E_6	$\{2,5,6,8,9,12\}$
E_7	$\{2,6,8,10,12,14,18\}$

.

For each $s \in D(G)$ we set $\gamma_{2s-1} = \kappa\circ\varphi(P) \in H^*(G)$, where $P \in S(G)$ with $\deg P = s$. For $G = E_7$ and in view of the presentation (6.4) of the algebra $H^*(E_7;\mathbb{F}_2)$, introduce the classes

$$C_I := \beta_2(\prod_{t\in I}\xi_{2t-1}) \in \sigma_2(E_7),\ I \subseteq \{3,5,9\}.$$

By [8, Theorem 1.9] the integral cohomology $H^*(G)$ has the presentations

(6.9a) $H^*(SU(n)) = \Lambda(\gamma_3,\gamma_5\cdots,\gamma_{2n-1})$;

(6.9b) $H^*(Sp(n)) = \Lambda(\gamma_3,\gamma_7,\cdots,\gamma_{4n-1})$;

(6.9c) $H^*(E_6) = \Delta(\gamma_3)\otimes\Lambda(\gamma_9,\gamma_{11},\gamma_{15},\gamma_{17},\gamma_{23}) \oplus \sigma_2(E_6) \oplus \sigma_3(E_6)$ with

$\sigma_2(E_6) = \mathbb{F}_2[x_3]^+/\langle x_3^2\rangle \otimes \Delta(\gamma_3)\otimes\Lambda(\gamma_9,\gamma_{15},\gamma_{17},\gamma_{23})$,

$\sigma_3(E_6) = \mathbb{F}_3[x_4]^+/\langle x_4^3\rangle \otimes \Lambda(\gamma_3,\gamma_9,\gamma_{11},\gamma_{15},\gamma_{17})$,

where $\gamma_3^2 = x_3$, $x_3\gamma_{11} = 0$, $x_4\gamma_{23} = 0$.

(6.9d) $H^*(E_7) = \Delta(\gamma_3)\otimes\Lambda_{\mathbb{Z}}(\gamma_{11},\gamma_{15},\gamma_{19},\gamma_{23},\gamma_{27},\gamma_{35}) \underset{p=2,3}{\oplus} \sigma_p(E_7)$ with

$$\sigma_2(E_7) = \frac{\mathbb{F}_2[x_3,x_5,x_9,\mathcal{C}_I]^+}{\langle x_3^2,x_5^2,x_9^2,\mathcal{D}_I,\mathcal{R}_I,\mathcal{S}_{I,J},\mathcal{H}_{r,I}\rangle}\otimes\Delta(\gamma_3)\otimes\Lambda(\gamma_{15},\gamma_{23},\gamma_{27})$$

$$\sigma_3(E_7) = \frac{\mathbb{F}_3[x_4]^+}{\langle x_4^3\rangle}\otimes\Lambda(\gamma_3,\gamma_{11},\gamma_{15},\gamma_{19},\gamma_{27},\gamma_{35}),$$

where $\gamma_3^2 = x_3$, $x_4\gamma_{23} = 0$, $I,J \subseteq \{3,5,9\}$ with $|I|,|J| \geq 2$, $r \in \{11,19,35\}$.

In addition, we remark that in (6.9c) and (6.9d) the classes x_i's are the Schubert classes on E_n/T, $n = 6, 7$, specified in Section 7.1, and that the relations $\mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_{I,J}, \mathcal{H}_{r,I}$ appearing in the presentation of $\sigma_2(E_7)$, slightly delicate to be stated here, will not be referred to in the sequel.

Lemma 6.9. *With respect to (6.9a)–(6.9d), the map θ in (2.5) satisfies that*

i) $\theta(\xi_1) = n, 2, 3$ and 2, in the order $G = SU(n), Sp(n), E_6$ and E_7;

ii) $\theta(\gamma_{2s-1}) = 0$ with the following exceptions

$$\theta(\gamma_{2s-1}) = \tbinom{n}{s}\omega^{s-1},\ \omega^{2^{r+1}-1},\ \omega^8,\ \omega$$

in the order of $G = SU(n)$ with $2 \leq s \leq n$; $G = Sp(n)$ with $n = 2^r(2b+1)$ and $s = 2^{r+1}$; $G = E_6$ and $s = 9$; $G = E_7$ and $s = 2$.

In addition, letting $J(\omega)$ be the subring generated by $\omega \in H^*(PG)$,

iii) the map π^* in (4.2) restricts an isomorphism $J(\varpi) \cong J(\omega)$.

Proof. With the ring $\mathbb{Z}$ as coefficients i) follows from i) of Theorem 2.3.

With the set $S(G)$ of characteristic polynomials of the 1–forms γ_{2s-1}'s being presented in (7.5) one computes the derivations $\partial P/\partial t_0$, $P \in S(G)$, as that tabulated in (7.6). Results in ii) are verified by Theorem 4.12, together with the contents of table (7.6).

As a graded group the ring $J(\omega)$ has the basis $\{1, \omega, \cdots, \omega^r\}$ for some $r \geq 1$. Let c_t be the order of the basis element ω^t, $1 \leq t \leq r$. Then, in addition to the presentation $J(\omega) = \mathbb{Z}[\omega]/\left\langle c_t\omega^t\right\rangle_{1\leq t\leq r}$ one has $c_t\omega^{t-1} \in \operatorname{Im}\theta$ by the exactness of the sequence (2.5). Noticing that $\operatorname{Im}\theta \subset H^*(PG)$ is an ideal the relations i) and ii) imply, in the order of $G = SU(n)$, $G = Sp(n)$ with $n = 2^r(2b+1)$, $G = E_6$ and $G = E_7$, that

$$\operatorname{Im}\theta = \left\langle \tbinom{n}{s}\omega^{s-1}, 1 \leq s \leq n\right\rangle, \left\langle 2, \omega^{2^{r+1}-1}\right\rangle, \left\langle 3, \omega^8\right\rangle, \left\langle 2, \omega\right\rangle$$

and henceforth

$$J(\omega) = \tfrac{\mathbb{Z}[\omega]}{\langle b_{n,s}\omega^s\rangle_{1\leq s\leq n}}, \tfrac{\mathbb{Z}[\omega]}{\left\langle 2\omega, \omega^{2^{r+1}}\right\rangle}, \tfrac{\mathbb{Z}[\omega]}{\langle 3\omega, \omega^9\rangle}, \tfrac{\mathbb{Z}[\omega]}{\langle 2\omega, \omega^2\rangle}$$

The isomorphism iii) is shown by iii) of Lemma 4.11 and i) of Lemma 6.2.$\square$

Let a_s be the order of the class $\theta(\gamma_{2s-1}) \in H^*(PG)$. By Lemma 6.9 we have

(6.10)

G	$SU(n)$	$Sp(n)$, $n = 2^r(2b+1)$	E_6	E_7
a_s, $s \in D(G)$	$a_{n,s}$	2 if $s = 2^{r+1}$ 1 if $s \neq 2^{r+1}$	3 if $s = 9$ 1 if $s \neq 9$	2 if $s = 2$ 1 if $s \neq 2$

,

see Theorem 5.5 for the numbers $a_{n,s}$. By the section of exact sequence (2.5)

$$H^*(PG) \overset{C^*}{\to} H^*(\left[G \times S^1\right]/\mathbb{Z}_q) \overset{\theta}{\to} H^*(PG)$$

and by Lemma 4.13, for each $s \in D(G)$ there exists a polynomial $P' \in \langle \operatorname{Im}\widetilde{\tau}\rangle \cap \ker f$ with $\deg P' = s$ so that the primary 1–form $\rho_{2s-1} := \kappa \circ \varphi(P') \in H^*(PG)$ satisfies the relation $C^*(\rho_{2s-1}) = a_s \cdot \gamma_{2s-1}$.

For a multi–index $I \subseteq D(G)$ we put $a_I = \prod\limits_{s\in I} a_s \in \mathbb{Z}$ and let

$$\gamma_I = \prod_{s\in I} \gamma_{2s-1} \in H^*(G), \quad \rho_I = \prod_{s\in I} \rho_{2s-1} \in H^*(PG).$$

Lemma 6.10. *The inclusions $\gamma_I \in H^*(G)$, $\rho_I \in H^*(PG)$ induce, respectively, the isomorphisms of free abelian groups*

i) $\mathcal{F}(G) \cong \Delta(\gamma_{2s-1})_{s\in D(G)}$; $\mathcal{F}(PG) \cong \Delta(\rho_{2s-1})_{s\in D(G)}$.

For $(G,p) = (Sp(n),2)$, $(E_6,3)$, $(E_7,2)$ and subject to the constraint (6.10), the classes ρ_{2s-1} can be modified within $\operatorname{Im}\kappa$ so that

ii) $B(PG,p) = \{r_p(\rho_{2s-1}), s \in D(G)\}$ (compare with Corollary 6.8).

Moreover, for $G = Sp(n), E_6, E_7$ one has

iii) $\rho_{2s-1}^2 = 0$ with the only exception $\rho_3^2 = x_3$ when $G = E_6$.

Proof. The first isomorphism in i) is clear by (6.9a)–(6.9d). For the second we note that the mapping degree of the covering $c : G \to PG$ is $a_{D(G)} = n, 2, 3, 2$ in the order $G = SU(n), Sp(n), E_6, E_7$. On the other hand, setting $m = \dim PG$ then the monomial $\gamma_{D(G)}$ is a generator of the top degree cohomology $H^m(G) = \mathbb{Z}$ by (6.9), while the relation $C^*(\rho_{D(G)}) = a_{D(G)} \cdot \gamma_{D(G)}$ by (6.10) indicates that the class $\rho_{D(G)}$ is a generator of $H^m(PG) = \mathbb{Z}$. By [8, Lemma 4.2] the set $\{1, \rho_I, I \subseteq D(G)\}$ spans a direct summand of $\mathcal{F}(PG)$ with rank $2^{|D(G)|}$. The second isomorphism is verified by the relation

$$\dim(\mathcal{F}(PG) \otimes \mathbb{Q}) = \dim(\mathcal{F}(G) \otimes \mathbb{Q}) = 2^{|D(G)|}.$$

Turning to ii) we put $B(PG,p)' = \{r_p(\rho_{2s-1}), s \in D(G)\}$ with $(G,p) = (Sp(n),2)$, $(E_6,3)$, $(E_7,2)$. Since $B(PG,p)' \subset \operatorname{Im}\kappa$, and since r_p induces an isomorphism $\mathcal{F}(PG)/p \cdot \mathcal{F}(PG) \cong \overline{H}^*(PG;\mathbb{F}_p)$ by Corollary 6.8, the set $B(PG,p)'$ is also a basis of the space $\operatorname{Im}\kappa \cap \overline{H}^*(PG;\mathbb{F}_p)$. By the degree reason for each $s \in D(G)$ there exists a unique $x_s \in B(PG,p)$ so that $r_p(\rho_{2s-1}) = \varepsilon_s x_s + y_s$ with $y_s \in \operatorname{Im}\delta_p$, $\varepsilon_s = 1$ if $p = 2$, ± 1 if $p = 3$. Changing the sign of the basis elements x_s whenever is necessary we can assume that $\varepsilon_s = 1$. Also with $y_s \in \operatorname{Im}\delta_p$ one finds $z_s \in H^*(PG;\mathbb{F}_p)$ so that $y_s = \delta_p(z_s)$. These imply that if we use $\rho'_{2s-1} = \rho_{2s-1} - \beta_p(z_s)$ instead of ρ_{2s-1}, then relation ii) is satisfied by the set $\{\rho'_{2s-1}, s \in D(G)\}$ of classes.

To show iii) recall that $2\rho^2 = 0$ for any odd degree cohomology class ρ of a space X. If $G = Sp(n)$ or E_7 we have by property ii) that $r_2(\rho_{2s-1}^2) = x_s^2 = 0$, $s \in D(G)$, where $x_s \in B(PG,2)$ is the unique element with $\deg x_s = 2s-1$. One obtains $\rho_{2s-1}^2 = 0$ from $\rho_{2s-1}^2 \in \sigma_2(PG)$, as well as i) of Corollary 6.8.

For $G = E_6$ we shall show in the proof of Theorem 6.11 that the map C^* in (2.5) restricts to a monomorphism $\sigma_2(PE_6) \to \sigma_2(E_6 \times S^1)$. Property iii) is shown by

$$C^*(\rho_{2s-1}) \equiv \gamma_{2s-1} \bmod 2, \ C^*(x_3) \equiv x_3 \bmod 2 \text{ (see ii) of Lemma 6.2)},$$

as well as the relations $\gamma_3^2 = x_6, \gamma_{2s-1}^2 = 0$, $s \geq 3$, on $H^*(E_6)$ by (6.9c).□

In term of the classes $\widetilde{\zeta}_1, \widetilde{\zeta}_7 \in H^*(PE_6;\mathbb{F}_3)$ in (6.7b) we set $\mathcal{C}_{\{1,4\}} = \beta_3(\widetilde{\zeta}_1\widetilde{\zeta}_7) \in \sigma_3(PE_6)$. In view of the algebra $H^*(PE_7;\mathbb{F}_2)$ given in (6.7c) introduce the elements

$$\mathcal{C}_K = \beta_2(\prod_{t\in K} \widetilde{\zeta}_{2t-1}) \in \sigma_2(E_7),\ K \subseteq \{1,3,5,9\}.$$

For $I, J \subset \{1,3,5,9\}$ with $|I|, |J| \geq 2$ let $\mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_{I,J} \in \sigma_2(E_7)$ be respectively the elements obtained by substituting in the polynomials $D_I, R_I, S_{I,J} \in \mathbb{F}_2[\omega, x_3, x_5, x_9, c_I]^+$ in (6.8) the classes c_I by C_I.

In the next two theorems we present of the integral cohomology rings $H^*(PG)$ by the set $\{\rho_{2s-1}\}_{s\in D(PG)}$ of primary 1–forms, the subring $\operatorname{Im}\pi^* \subset H^*(PG)$ indicated by (6.2), as well as the torsion classes $\mathcal{C}_K$ from the Bocksteins.

Theorem 6.11. *The cohomologies $H^*(PG)$ with $G = Sp(n)$, E_6 and E_7 are*

i) $H^*(PSp(n)) = \Lambda(\rho_{4s-1})_{s\in\{1,\cdots,n\}} \oplus \sigma_2(PSp(n))$ *with* $n = 2^r(2b+1)$,

$$\sigma_2(PSp(n)) = \mathbb{F}_2[\omega]^+ / \left\langle \omega^{2^{r+1}} \right\rangle \otimes \Lambda(\rho_3, \rho_7, \cdots, \widehat{\rho_{2^{r+2}-1}}, \cdots \rho_{4n-1}),$$

that are subject the relation $\omega \cdot \rho_{2^{r+2}-1} = 0$.

ii) $H^*(PE_6) = \Delta(\rho_3) \otimes \Lambda(\rho_9, \rho_{11}, \rho_{15}, \rho_{17}, \rho_{23}) \underset{p=2,3}{\oplus} \sigma_p(PE_6)$ *with*

$$\sigma_2(PE_6) = \mathbb{F}_2[x_3]^+ / \left\langle x_3^2 \right\rangle \otimes \Delta(\rho_3) \otimes \Lambda(\rho_9, \rho_{15}, \rho_{17}, \rho_{23}),$$

$$\sigma_3(PE_6) = \frac{\mathbb{F}_3[\omega, x_4, \mathcal{C}_{\{1,4\}}]^+ \otimes \Lambda(\rho_3, \rho_9, \rho_{11}, \rho_{15}, \rho_{17})}{\left\langle \omega^9, x_4^3, \omega\cdot\rho_{17}, \mathcal{C}_{\{1,4\}}^2, \omega^8 x_4^2 \mathcal{C}_{\{1,4\}} \right\rangle},$$

that are subject the relations

$$\rho_3^2 = x_3,\ x_3\rho_{11} = 0,\ x_4\rho_{23} = 0,\ \omega\rho_{23} = x_4^2\mathcal{C}_{\{1,4\}},\ \mathcal{C}_{\{1,4\}}\rho_{23} = 0.$$

iii) $H^*(PE_7) = \Lambda(\rho_3, \rho_{11}, \rho_{15}, \rho_{19}, \rho_{23}, \rho_{27}, \rho_{35}) \underset{p=2,3}{\oplus} \sigma_p(PE_7)$ *with*

$$\sigma_2(PE_7) = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9, \mathcal{C}_I]^+}{\left\langle x_1^2, x_3^2, x_5^2, x_9^2, \mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_{I,J} \right\rangle} \otimes \Lambda(\rho_{15}, \rho_{23}, \rho_{27});$$

$$\sigma_3(PE_7) = \frac{\mathbb{F}_3[x_4]^+}{\left\langle x_4^3 \right\rangle} \otimes \Lambda(\rho_3, \rho_{11}, \rho_{15}, \rho_{19}, \rho_{27}, \rho_{35}),$$

that are subject to the relations, where $K \subseteq \{1,3,5,9\}$, $s \in \{2,6,10,18\}$,

$$x_4\varrho_{23} = 0,\ \rho_{2s-1}C_K = 0 \text{ if } s \in K,\ \rho_{2s-1}C_K = x_{\frac{s}{2}} C_{K\cup\{s\}} \text{ if } s \notin K.$$

Proof. For $G = Sp(n)$, E_6 or E_7 assume as in (5.7) that

$$(6.11)\quad H^*(PG) = \mathcal{F}(PG) \underset{p}{\oplus} \sigma_p(PG).$$

Note that the presentations of the free parts $\mathcal{F}(PG)$ in the theorem have been shown by i) and iii) of Lemma 6.10. It remains for us to justify the presentations of the torsion ideals $\sigma_p(PG)$, as well as the relations that characterizes the actions $\mathcal{F}(PG) \times \sigma_p(PG) \to \sigma_p(PG)$ of the free part $\mathcal{F}(PG)$ on $\sigma_p(PG)$.

For $G = Sp(n)$ with $\langle\omega\rangle \in \sigma_2(PSp(n))$ by $2\omega = 0$ one gets by (6.11) that

$$H^*(PSp(n))_{\langle\omega\rangle} = \mathcal{F}(PSp(n)) \underset{p\neq 2}{\oplus} \sigma_p(PSp(n)) \oplus \sigma_2(PSp(n)) / \langle\omega\rangle.$$

Therefore, if $p \neq 2$ the injection C^* on $H^*(PSp(n))_{\langle\omega\rangle}$ in (5.2) restricts to the monomorphisms $\sigma_p(PSp(n)) \to \sigma_p(Sp(n) \times S^1)(= 0$ by (6.9b)) which shows that $\sigma_p(PSp(n)) = 0$. When $p = 2$ we have by Corollary 6.8 that r_2 restricts to an isomorphism

$$\sigma_2(PSp(n)) \overset{\cong}{\to} \operatorname{Im}\delta_2(= \tfrac{\mathbb{F}_2[\omega]^+}{\langle \omega^{2^{r+1}}\rangle} \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_3, \widetilde{\zeta}_7, \cdots, \widehat{\widetilde{\zeta}}_{2^{r+2}-1}, \cdots, \widetilde{\zeta}_{4n-1})).$$

In view of this isomorphism and by the relations

$$r_2(\rho_{2s-1}) = \widetilde{\zeta}_{2s-1} \text{ for } s \neq 2^{r+1}-1,\ r_2(\rho_{2^{r+2}-1}) = \omega^{2^{r+1}-1}\iota,$$

obtained in ii) of Lemma 6.10, one translates $\operatorname{Im}\delta_2$ into the presentation of the ideal $\sigma_2(PSp(n))$ in i), together with the relation $\omega\rho_{2^{r+2}-1} = 0$. This completes the proof of i).

The same arguments applies equally well to $G = E_6$ or E_7. With $\langle\omega\rangle \in \sigma_3(PE_6)$ by $3\omega = 0$ (resp. $\langle\omega\rangle \in \sigma_2(PE_7)$ by $2\omega = 0$) we get by (6.11) that

$$H^*(PE_6)_{\langle\omega\rangle} = \mathcal{F}(PE_6) \underset{p\neq 3}{\oplus} \sigma_p(PE_6) \oplus \sigma_3(PE_6)/\langle\omega\rangle$$
$$(\text{resp. } H^*(PE_7)_{\langle\omega\rangle} = \mathcal{F}(PE_7) \underset{p\neq 2}{\oplus} \sigma_p(PE_7) \oplus \sigma_2(PE_6)/\langle\omega\rangle).$$

As result the C^* in (5.2) restricts to the monomorphism $\sigma_p(PE_6) \to \sigma_p(E_6 \times S^1)$, $p \neq 3$ (resp. $\sigma_p(PE_7) \to \sigma_p(E_7 \times S^1)$, $p \neq 2$) which implies that

a) $\sigma_p(PG) = 0$, $p \neq 2, 3$ (since $\sigma_p(G \times S^1) = 0$, $p \neq 2, 3$ by (6.9)),

b) $\sigma_2(PE_6) \cong \sigma_2(E_6)$ (resp. $\sigma_3(PE_7) \cong \sigma_3(E_7)$) under C^*.

From b) one obtains the presentation of $\sigma_2(PE_6)$ (resp. of $\sigma_3(PE_7)$), together with the relation $x_3\rho_{11} = 0$ (resp. $x_4\varrho_{23} = 0$) from (6.9c) (resp. by (6.9d)).

By Corollary 6.8 the reduction r_p restricts to the isomorphism

$$\text{c) } \sigma_3(PE_6) \cong \operatorname{Im}\delta_3(= \frac{\mathbb{F}_3[x_1, x_4, c_{\{1,4\}}]^+}{\langle x_1^9, x_4^3, c_{\{1,4\}}^2, x_1^8 x_4^2 c_{\{1,4\}}\rangle} \otimes \Lambda_{\mathbb{F}_3}(\widetilde{\zeta}_3, \widetilde{\zeta}_9, \widetilde{\zeta}_{11}, \widetilde{\zeta}_{15})),$$
$$(\text{resp. } \sigma_2(PE_7) \cong \operatorname{Im}\delta_2\ (= \frac{\mathbb{F}_2[x_1, x_3, x_5, x_9, c_I]^+}{\langle x_1^2, x_3^2, x_5^2, x_9^2, D_I, R_I, S_{I,J}\rangle} \otimes \Lambda_{\mathbb{F}_2}(\widetilde{\zeta}_{15}, \widetilde{\zeta}_{23}, \widetilde{\zeta}_{27})).$$

With $r_p(\rho_{2s-1}) = \widetilde{\zeta}_{2s-1}$ for $s \in \{2, 5, 6, 8\}$ (resp. for $s \in \{8, 12, 14\}$) by ii) of Lemma 6.10, and with $r_p(\mathcal{C}_{\{1,4\}}) = c_{\{1,4\}}$ (resp. $r_p(\mathcal{C}_I) = c_I$) by the definition of the class $\mathcal{C}_{\{1,4\}}$ (resp. the classes $\mathcal{C}_I$), one translates the presentation of $\operatorname{Im}\delta_3$ (resp. $\operatorname{Im}\delta_2$) available in Theorem 6.7 into the presentation of $\tau_3(PE_6)$ (resp. $\tau_2(PE_7)$) stated in the Theorem.

Finally, granted with Corollary 6.8, Theorem 6.6 and ii) of Lemma 6.10, one can justify the relations

d) $x_4\rho_{23} = 0$, $\omega\rho_{23} = x_4^2\mathcal{C}_{\{1,4\}}$, $\mathcal{C}_{\{1,4\}}\rho_{23} = 0$;

e) $x_4\varrho_{23} = 0$, $\rho_{2s-1}C_K = 0$ if $s \in K$, $= x_{\frac{s}{2}}C_{I\cup\{s\}}$ if $s \notin K$

that describe the actions $\mathcal{F}(PG) \times \sigma_p(PG) \to \sigma_p(PG)$, respectively for $(G, p) = (E_6, 3)$ and $(E_7, 2)$. As examples, granted with i) of Corollary 6.8 and iv) of Theorem 6.6, the relations in d) are shown by the following calculations in the known algebra $H^*(PE_6; \mathbb{F}_3)$ (note that $r_3(\rho_{23}) = x_4^2\zeta_7$ by ii) of Lemma 6.10)

$$r_3(\omega\rho_{23}) = \omega x_4^2\zeta_7 = x_4^2(\omega\zeta_7 - \iota x_4) = x_4^2 c_{\{1,4\}};$$
$$r_3(x_4\rho_{23}) = x_4^3\zeta_7 = 0;$$
$$r_3(\mathcal{C}_{\{1,4\}}\rho_{23}) = (\omega\zeta_7 - \iota x_4)x_4^2\zeta_7 = 0.\square$$

For the remaining group $PSU(n)$ the exact sequence (5.2) reads

$$(6.12)\quad 0 \to H^*(PSU(n))_{\langle\omega\rangle} \overset{C^*}{\to} H^*(SU(n)) \overset{\theta}{\to} H^*(PSU(n)) \overset{\omega}{\to} \langle\omega\rangle \to 0,$$

where $H^*(PSU(n))_{\langle\omega\rangle} = \Delta(\rho_{2s-1})_{s\in\{2,\cdots,n-1\}}$ by the proof of i) of Lemma 6.10. Since the group $H^*(PSU(n))_{\langle\omega\rangle}$ is free the exact sequence (5.1) is split. Therefore, by (5.4) the inclusion $\omega, \rho_{2s-1} \in H^*(PSU(n))$ induces a surjective homomophism of the graded $J_n(\omega)$–modules

$$(6.13)\quad h : J_n(\omega) \otimes \Delta(\rho_3, \rho_5 \cdots, \rho_{2n-1}) \to H^*(PSU(n)).$$

It follows that, if the prime factorization of the integer n is $p_1^{r_1} \cdots p_t^{r_t}$, then $\sigma_p(PSU(n)) = 0$ for any prime $p \notin \{p_1, \cdots, p_t\}$, and that h restricts to the surjections by (5.10)

$$(6.14)\quad h_s : \frac{\mathbb{Z}[\omega]^+ \otimes \Delta(\rho_3, \rho_5 \cdots, \rho_{2n-1})}{\langle p_s^{r_s}\omega, p_s^{r_s-1}\omega^{p_s}, \cdots, \omega^{p_s^{r_s}}\rangle} \to \sigma_{p_s}(PSU(n)),\ 1 \le s \le t.$$

Consequently, the problem of determining the ring $H^*(PSU(n))$ amounts to decide all the squares ρ_{2s-1}^2, and to obtain an account for the ideals $\ker h_s$.

For each prime factor p_s of $n = p_1^{r_1} \cdots p_t^{r_t}$ introduce the subset

$$I(n; p_s) = \{p_s^k;\ 0 \le k \le r_s\} \subset \{1, 2, \cdots, n-1\}.$$

For each multi–index $K \subseteq I(n; p_s)$ we set

$$g_K = \omega\theta(\prod_{s\in K} \rho_{2s-1}) \in \mathbb{Z}[\omega]^+ \otimes \Delta(\rho_3, \rho_5 \cdots, \rho_{2n-1}).$$

Theorem 6.12. *For an integer $n \ge 2$ with prime factorization $p_1^{r_1} \cdots p_t^{r_t}$, the integral cohomology ring of the adjoint group $PSU(n)$ has the presentation*

$$H^*(PSU(n)) = \Lambda(\rho_3, \rho_5 \cdots, \rho_{2n-1}) \underset{1\le s\le t}{\oplus} \sigma_{p_s}(PSU(n))$$

where

$$(6.15)\quad \sigma_{p_s}(PSU(n)) = \frac{\mathbb{Z}[\omega]^+ \otimes \Lambda(\rho_3, \rho_5 \cdots, \rho_{2n-1})}{\langle p_s^{r_s}\omega, p_s^{r_s-1}\omega^{p_s}, \cdots, \omega^{p_s^{r_s}}, g_K\rangle},\ K \subseteq I(n; p_s),\ |K| \ge 2,$$

Proof. A proof will appear in the subsequent version of this paper.

6.5 Remarks

The algebras $H^*(PG; \mathbb{F}_p)$ concerned in Theorem 6.6 have been computed by Baum and Browder [1], Toda, Kono and Ishitoya [12, 14]. Our approach applies uniformly to the groups PG, while the calculation is extendable to obtain the integral cohomology ring $H^*(PG)$.

By Grothendieck [10] the subring $\mathrm{Im}\,\pi^* \subset H^*(PG)$ is the *Chow ring* $A^*(PG^c)$ of the reductive algebraic group PG^c corresponding to PG. In this regard formulae (6.2) presents the rings $A^*(PG^c)$ with $G = SU(n)$, $Sp(n)$, E_6, E_7 by explicit Schubert classes on G/T.

Let G be a semi-simple Lie group. In [13, formula (4)] Kač stated a formula for the differential d_2 on $E_2^{*,*}(G; \mathbb{F}_p)$, which implies that the transgression τ in the fibration (2.6) is an isomorphism in any characteristic p. This convinces us that the formula (3.6) for τ is not known.

In [18, Theorem A] Ruiz stated a presentation of the integral cohomology ring of the projective complex Stiefel manifold $Y_{n,n-m}$. It implies when taking $m = 0$ that the map h in (6.12) is an isomorphism, That is, the relations g_K with $K \subseteq I(n; p_s)$ required to present $\sigma_{p_s}(PSU(n))$ in (6.15) are absent. As a comparison, taking the case $n = 6$ as example, Theorem 6.12 yields two additional relations

$$3\omega\rho_3 = 0,\ 2\omega\rho_5 = 0$$

that are missing in Ruiz's result [18, Theorem A].

7 Appendix: Schubert calculus

This appendix is set to generate and record the intermediate data facilitating the computation in Section 6. It serves also the purpose to illustrate how the construction and computation with the cohomology of Lie groups G can bee reduced to certain calculation with polynomials in the Schubert classes on G/T.

7.1 Schubert presentation of the ring $H^*(G/T)$

By the Schubert basis $\{\omega_1, \cdots, \omega_n\}$ of the group $H^2(G/T)$ and in the order of $G = SU(n), Sp(n), E_6, E_7$, we define a subset $\Omega(G) \subset H^2(G/T)$ by

$$\Omega(SU(n)) = \{\omega_1, \omega_k - \omega_{k-1}, -\omega_{n-1} \mid 2 \leq k \leq n-1\};$$
$$\Omega(Sp(n)) = \{\pm\omega_1, \pm(\omega_k - \omega_{k-1}) \mid 2 \leq k \leq n\};$$
$$\Omega(E_6) = \{\omega_6, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\};$$
$$\Omega(E_7) = \{\omega_7, \omega_6 - \omega_7, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\}.$$

Let $c_r(G) \in H^{2r}(G/T)$ be the r^{th} elementary symmetric polynomials on the set $\Omega(G)$. Using the *Weyl coordinates* of Schubert classes ([8, Definition 2]) introduce the *special Schubert classes* x_r on E_n/T in the following table

x_i	$E_n/T,\ n = 6, 7$
x_3	$\sigma_{[5,4,2]},\ n = 6, 7$
x_4	$\sigma_{[6,5,4,2]},\ n = 6, 7$
x_5	$\sigma_{[7,6,5,4,2]},\ n = 7$
x_9	$\sigma_{[1,5,4,3,7,6,5,4,2]},\ n = 7$

.

With these notation we have by [7] the presentations of the cohomologies $H^*(G/T)$ by explicit generators and relations, where in comparison with Theorem 4.1 we use $x_{\deg y_i}$ in place of y_i, $R_{\deg h_i}$, $R_{\deg f_j}$, $R_{\deg g_j}$) instead of h_i, f_j, g_j.

Theorem 7.1. *The ring $H^*(G/T)$ has the following presentations*

i) $H^*(SU(n)/T) = \mathbb{Z}[\omega_1, \cdots, \omega_{n-1}] / \langle c_2, \cdots, c_n \rangle$, $c_r = c_r(SU(n))$,

ii) $H^*(Sp(n)/T) = \mathbb{Z}[\omega_1, \cdots, \omega_n] / \langle c_2, c_4, \cdots, c_{2n} \rangle$, $c_{2r} = c_{2r}(Sp(n))$.

iii) $H^*(E_6/T) = \mathbb{Z}[\omega_1, \cdots, \omega_6, x_3, x_4]/\langle R_2, R_3, R_4, R_5, R_6, R_8, R_9, R_{12}\rangle$, *where*

$R_2 = 4\omega_2^2 - c_2;$

$R_3 = 2x_3 + 2\omega_2^3 - c_3;$

$R_4 = 3x_4 + \omega_2^4 - c_4;$

$R_5 = 2\omega_2^2 x_3 - \omega_2 c_4 + c_5;$

$R_6 = x_3^2 - \omega_2 c_5 + 2c_6;$

$R_8 = x_4(c_4 - \omega_2^4) - 2c_5 x_3 - \omega_2^2 c_6 + \omega_2^3 c_5;$

$R_9 = 2x_3 c_6 - \omega_2^3 c_6;$

$R_{12} = x_4^3 - c_6^2.$

iv) $H^*(E_7/T) = \mathbb{Z}[\omega_1, \cdots, \omega_7, x_3, x_4, x_5, x_9]/\langle R_t\rangle$, *where* $t \in \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18\}$, *and where*

$R_2 = 4\omega_2^2 - c_2;$

$R_3 = 2x_3 + 2\omega_2^3 - c_3;$

$R_4 = 3x_4 + \omega_2^4 - c_4;$

$R_5 = 2x_5 - 2\omega_2^2 x_3 + \omega_2 c_4 - c_5;$

$R_6 = x_3^2 - \omega_2 c_5 + 2c_6;$

$R_8 = 3x_4^2 - x_5(2\omega_2^3 - c_3) - 2x_3 c_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5;$

$R_9 = 2x_9 + x_4(2\omega_2^2 x_3 - \omega_2 c_4 + c_5) - 2x_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6;$

$R_{10} = x_5^2 - 2x_3 c_7 + \omega_2^3 c_7;$

$R_{12} = x_4^3 - 4x_5 c_7 - c_6^2 + (2\omega_2^3 - c_3)(x_9 + x_4 x_5) + 2\omega_2 x_5 c_6 + 3\omega_2 x_4 c_7 + c_5 c_7;$

$R_{14} = c_7^2 - (2\omega_2^2 x_3 - \omega_2 c_4 + c_5)x_9 + 2x_3 x_4 c_7 - \omega_2^3 x_4 c_7;$

$R_{18} = x_9^2 + 2x_5 c_6 c_7 - x_4 c_7^2 - (2\omega_2^2 x_3 - \omega_2 c_4 + c_5)x_4 x_9 - (2\omega_2^3 - c_3)x_5^3$
$\quad - 5\omega_2 x_5^2 c_7.\square$

As in Section 6.1, for the groups PG with $G = SU(n), Sp(n), E_6, E_7$ we take a set $\Omega = \{\varrho_1, \cdots, \varrho_n\}$ of fundamental dominant weights as a basis for the unit lattice Λ_e, and let $\{t_1, \cdots, t_n\}$ be the corresponding basis on $H^1(T)$. Let τ be the transgression in the fibration π in (4.9). By the Cartan matrices of simple Lie groups given in [11, p.59] one gets that

Theorem 7.2. *In the quotient group $H^*(G/T)/\operatorname{Im}\tau$ and in the order of $G = SU(n), Sp(n), E_6, E_7$ we have*

i) $\omega_k = k\omega_1$, $1 \le k \le n-1$, $n\omega_1 = 0$;

ii) $\omega_k = k\omega_1$, $1 \le k \le n$, $2\omega_1 = 0$;

iii) $\omega_2 = \omega_4 = 0$, $\omega_1 = \omega_5 = 2\omega_3 = 2\omega_6$, $3\omega_1 = 0$;

iv) $\omega_1 = \omega_3 = \omega_4 = \omega_6 = 0$, $\omega_5 = \omega_7 = \omega_2$, $2\omega_2 = 0$,

In particular,

a) $c_r(SU(n)) \mid_{\tau(t_1)=\cdots=\tau(t_{n-1})=0} = \binom{n}{r}\omega_1^r$, $2 \le r \le n$;

b) $c_{2r}(Sp(n)) \mid_{\tau(t_1)=\cdots=\tau(t_n)=0} = \binom{n}{r}\omega_1^{2r}$, $1 \le r \le n$;

c) $c_r(E_6) \mid_{\tau(t_1)=\cdots=\tau(t_6)=0} = (-1)^r\binom{6}{r}\omega_1^r$, $1 \le r \le 6$;

d) $c_r(E_7) \mid_{\tau(t_1)=\cdots=\tau(t_7)=0} = \binom{7}{r}\omega_2^r$, $1 \le r \le 7.\square$

7.2 Computing with characteristic polynomials over $\mathbb{F}_p$

Let $(G,p) = (SU(n),p)$, $(Sp(n),2)$, $(E_6,3)$ and $(E_7,2)$. From Theorem 7.1 one deduces a sets $S_p(G) := \{\delta_1, \cdots, \delta_n\}$ of polynomials satisfying (4.6) that are presented in the following table (see DZ)

(7.1)

(G,p)	$S_p(G) = \{\delta_1, \cdots, \delta_n\}$
$(SU(n),p)$	$(c_k)_p\,, 2 \le k \le n$
$(Sp(n),2)$	$(c_{2k})_2\,, 1 \le k \le n$
$(E_6,3)$	$(\omega_2^2 - c_2)_3, (c_2^2 - c_4)_3, (c_5 + c_2c_3)_3, \left(c_6 - c_2c_4 - c_3^2\right)_3$ $(-c_3c_5 - c_2c_6)_3, (c_6c_3)_3$
$(E_7,2)$	$(c_2)_2\,, (c_3)_2\,, (c_5 + \omega_2c_4)_2\,, \left(c_4^2 + \omega_2^2c_6 + \omega_2^3c_5 + \omega_2^8\right)_2,$ $\left(\omega_2^2c_7 + \omega_2^3c_6\right)_2, \left(c_6^2 + c_4^3\right)_2, \left(c_7^2 + c_4^2c_6 + \omega_2^2c_6^2\right)_2$

where $(H)_p \in \mathbb{F}_p[\omega_1, \cdots, \omega_n]$ denotes $H \bmod p$, $H \in \mathbb{Z}[\omega_1, \cdots, \omega_n]$.

Let τ' be the transgression in the fibration π' in the diagram (4.9). Granted with the class $\tau'(t_0) \in H^2(G/T;\mathbb{F}_p)$ determined by (6.3), as well as Lemma 7.2, one computes the derivation $\partial P/\partial t_0$ of $P \in S_p(G)$ with respect to t_0 by the formula (4.14), as that tabulated below

(7.2)

(G,p)	$\{\partial P/\partial t_0 \mid P \in S_p(G)\}$
$(SU(n),p)$	$\left(\binom{n}{k}\omega_1^{k-1}\right)_p$, $2 \le k \le n$
$(Sp(n),2)$	$\left(\binom{n}{k}\omega_1^{2k-1}\right)_2$, $1 \le k \le n$
$(E_6,3)$	$0, 0, 0, 0, 0, \left(\omega_1^8\right)_3$
$(E_7,2)$	$(\omega_2)_2\,, \left(\omega_2^2\right)_2\,, 0, 0, 0, 0, \left(\omega_2^{13}\right)_2$

.

Results in (7.2) determine the values of $\theta_1(\varphi_p(P)) \in E_3^{*,0}(PG;\mathbb{F}_p)$, $P \in S_p(G)$, as that stated in Lemma 6.4.

By the algorithm given in the proof of Lemma 4.13, for $P \in S_p(G)$ with $\theta_1(\varphi_p(P)) = 0$ one obtains a polynomial $P' \in \langle \operatorname{Im} \tau_p \rangle \cap \ker f_p$ satisfying the relation $C^*(\varphi_p(P')) = \varphi_p(P)$. Explicitly, a set of polynomials P' so obtained, denoted $S_p(PG)$, is given in the following table

(7.3)

(G,p)	$S_p(PG) \subset \langle \operatorname{Im} \widetilde{\tau}_p \rangle \cap \ker f_p$
$(SU(n),p)$	$(c_k)_p$ for $2 \le k < p^r$, $(c_k - t_{n,k}c_{p^r}\omega_1^{k-p^r})_p$ for $k \ge p^r$ where $n = p^r n'$ with $(n', p) = 1$;
$(Sp(n),2)$	$(c_{2k})_2$ for $2 \le k < 2^r$, $(c_{2k} - t_{n,2}c_{2^{r+1}}\omega_1^{2(k-2^r)})_2$ for $k \ge 2^r$ where $n = 2^r(2b+1)$;
$(E_6,3)$	$\left(\omega_2^2 - c_2\right)_3, \left(c_2^2 - c_4\right)_3, (c_5 + c_2c_3)_3\,,$ $\left(c_6 - c_2c_4 - c_3^2\right)_3, (-c_3c_5 - c_2c_6)_3;$
$(E_7,2)$	$(c_3 - c_2\omega_2)_2\,, (c_5 + \omega_2c_4)_2\,, \left(c_4^2 + \omega_2^2c_6 + \omega_2^3c_5 + \omega_2^8\right)_2$ $\left(\omega_2^2c_7 + \omega_2^3c_6\right)_2, \left(c_6^2 + c_4^3\right)_2, \left(c_7^2 + c_4^2c_6 + \omega_2^2c_6^2 - c_2\omega_2^{12}\right)_2$

where $t_{n,k} > 0$ is the smallest integer such that $t_{n,k}\binom{n}{p^r} \equiv \binom{n}{k} \bmod p$.

It is shown in Lemma 6.5 that the set $\{\zeta_{2\deg P-1} = \kappa \circ \varphi_p(P);\, P \in S_p(PG)\}$ of primary 1–forms, together with $\operatorname{Im} \pi^*$, generates the algebra $H^*(PG;\mathbb{F}_p)$. In the statement and the proof of the next result, we note that, with $\zeta_{2\deg P-1} \in \operatorname{Im} \kappa$, one has by the proof of Lemma 4.4 that

$$\beta_p(\zeta_{2\deg P-1}) \in \pi^*(E_3^{*,0}(PG)) \subset H^*(PG),$$

and that the ring $E_3^{*,0}(PG)$ has been determined in (6.2).

Lemma 7.4. *The Bockstein $\beta_p : H^*(PG;\mathbb{F}_p) \to H^*(PG)$ satisfies that*

i) for $(G,p) = (SU(n),p)$ with $n = p^r n'$, $(n\prime, p) = 1$

$\beta_p(\zeta_{2s-1}) = -p^{r-t-1}\omega^{p^t}$ *if* $s = p^t$ *with* $t < r$, 0 *otherwise;*

ii) for $(G,p) = (Sp(n),2)$ with $n = 2^r(2b+1)$:

$\beta_2(\zeta_{4s-1}) = \omega^{2^r}$ *if* $s = 2^{r-1}$, 0 *if* $s \neq 2^{r-1}$*;*

iii) for $(G,p) = (E_6,3)$ and in the order of $s = 2,4,5,6,8$

$\beta_3(\zeta_{2s-1}) = 0, -x_4, 0, 0, -x_4^2$*;*

iv) for $(G,p) = (E_7,2)$ and in the order of $s = 3,5,8,9,12,14$

$\beta_2(\zeta_{2s-1}) = x_3, x_5, x_3x_5, x_9, x_3x_9, x_5x_9$.

Proof. For $(SU(n),p)$ with $n = p^r n'$, $(n\prime,p) = 1$ (resp. $(Sp(n),2)$ with $n = 2^r(2b+1)$) one has by (6.2) that

$$\beta_p(\zeta_{2s-1}) \in \mathbb{Z}[\omega]^+/\left\langle p^r\omega,\cdots,\omega^{p^r}\right\rangle \text{ (resp. } \beta_2(\zeta_{4s-1}) \in \mathbb{Z}[\omega]/\left\langle 2\omega,\omega^{2^{r+1}}\right\rangle),$$

where $\omega = \omega_1$. For the degree reason

$$\beta_p(\zeta_{2s-1}) = 0 \text{ if } s \geq p^r \text{ (resp. } \beta_2(\zeta_{4s-1}) = 0 \text{ if } s \geq 2^r).$$

In the remaining cases $2 \leq s < p^r$ (resp. $1 \leq s < 2^r$) an integral lift of the characteristic polynomial $(c_s)_p$ (resp. $(c_{2s})_2$) of the class ζ_{2s-1} (resp. ζ_{4s-1}) is easily seen to be

$$c_s - \tbinom{n}{s}\omega_1^s \in \langle \operatorname{Im}\widetilde{\tau}\rangle \text{ (resp. } c_{2s} - \tbinom{n}{k}\omega_1^{2s} \in \langle \operatorname{Im}\widetilde{\tau}\rangle).$$

The formula (4.6) yields that

$$\beta_p(\zeta_{2s-1}) = \pi^*\tfrac{1}{p}f(c_s - \tbinom{n}{s}\omega_1^s) = \pi^*(-\tfrac{1}{p}\tbinom{n}{s}\omega_1^s) = -\tfrac{1}{p}\tbinom{n}{s}\omega^s$$
$$\text{(resp. } \beta_2(\zeta_{4s-1}) = \pi^*(\tfrac{1}{2}f(c_{2s} - \tbinom{n}{s}\omega_1^{2s}) = \pi^*(\tfrac{1}{2}\tbinom{n}{s}\omega_1^{2s}) = \tfrac{1}{2}\tbinom{n}{s}\omega^{2s})$$

where the second equality follows from $f(c_s) = 0$ (resp. $f(c_{2s}) = 0$) by i) (resp. ii)) of Theorem 7.1, while the last one is due to $\pi^*(\omega_1) = \omega$ by ii) of Lemma 6.4. Combining this with the relations

$$p^r\omega, p^{r-1}\omega^p, p^{r-2}\omega^{p^2},\cdots,\omega^{p^r} = 0 \text{ (resp. } 2\omega, \omega^{2^{r+1}} = 0)$$

on the ring $E_3^{*,0}(PSU(n))$ (resp. $E_3^{*,0}(PSp(n))$), together with formula ii) of Theorem 5.5, verifies the relation i) (resp. ii)) of the present lemma.

Turning to $(G,p) = (E_6,3)$ or $(E_7,2)$ for each $P = (H)_p \in S_p(PG)$ the enclosed polynomial H satisfies $H \in \langle \operatorname{Im}\widetilde{\tau}\rangle$ by (7.3). Therefore

$$\beta_p(\zeta_{2\deg P-1}) = \pi^*(\tfrac{1}{p}f(H)\mid_{\tau(t_1)=\cdots=\tau(t_n)=0}) \text{ (by (4.6))}.$$

This formula, together with the relations R_i's in the presentations iii) and iv) of Theorem 7.1, suffices to show the results in iii) and iv). As examples, when $(G,p)=(E_6,3)$ it gives rise to

$$\beta_3(\zeta_3)=\pi^*\tfrac{\omega_2^2-c_2}{3}=\pi^*\omega_2^2=0;$$
$$\beta_3(\zeta_7)=\pi^*\tfrac{c_2^2-c_4}{3}=\pi^*(5\omega_2^2-x_4)=-x_4;$$
$$\beta_3(\zeta_9)=\pi^*\tfrac{c_5+c_2c_3}{3}=\pi^*(\omega_2^5+\omega_2x_4+\omega_2^2c_3)=0;$$
$$\beta_3(\zeta_{11})=\pi^*\tfrac{c_2c_4+c_3^2-c_6}{3}=\pi^*(5\omega_2^2x_4+\omega_2^6+\omega_2c_5+\omega_2^3c_3-3c_6)=0;$$
$$\beta_3(\zeta_{15})=\pi^*(-\tfrac{c_3c_5+c_2c_6}{3})=\pi^*(-4x_4^2+c_3c_5+4\omega_2^3c_5)=-x_4^2,$$

where, in each of the above equations, the first equality comes from (4.6), the second is deduced from the relations R_i's in iii) of Theorem 7.1, and the last one is obtained from properties iii) and c) of Lemma 7.2.□

Lemma 7.5. *With respect to the presentation of the cohomology $H^*(PG;\mathbb{F}_2)$ in Lemma 6.5, the Steenrod operators δ_2 and Sq^{2r} satisfy the following relations*

i) $G=SU(n)$ with $n=2^r(2b+1)$:

$$\delta_2\zeta_{2s-1}=\omega_1^{2^{r-1}} \text{ if } s=2^{r-1},\ 0 \text{ if } s\neq 2^{r-1};$$
$$Sq^{2s-2}\zeta_{2s-1}=\zeta_{4s-3} \text{ for } 2s-1\leq 2^{r-1};$$

ii) $G=Sp(n)$ with $n=2^r(2b+1)$:

$$\delta_2\zeta_{4s-1}=\omega_1^{2^r} \text{ if } s=2^{r-1},\ 0 \text{ if } s\neq 2^{r-1},$$
$$Sq^{4s-2}\zeta_{4s-1}=\zeta_{8s-3} \text{ for } 4s-1\leq 2^r;$$

iii) $G=E_7$ and in accordance to $s=3,5,8,9,12,14$:

$$\delta_2\zeta_{2s-1}=y_3,y_5,y_3y_5,y_9,y_3y_9,y_5y_9;\ Sq^{2s-2}\zeta_{2s-1}=\zeta_9,\zeta_{17},0,0,0,0.$$

Proof. The results on $\delta_2\zeta_{2s-1}$ comes directly from $\delta_2=r_2\circ\beta_2$, where the action of β_2 on ζ_{2s-1} has been decided in Lemma 7.4, and the reduction r_2 is transparent by the presentations of $E_3^{*,0}(PG)$ and $E_3^{*,0}(PG;\mathbb{F}_2)$ in Section 6.1.

Assume that $G=SU(n)$ (resp. $Sp(n)$) with $n=2^r(2s+1)$. Since the class $1+c_1+c_2+\cdots$ is the total Chen class of a complex bundle on BT_0 (is the total total Pontrijagin class of a symplectic bundle on BT_0) by [9, Lemma 5.4], we have by the Wu–formula [17, p.94] the equality on $H^*(BT_0;\mathbb{F}_2)$

(7.4) $Sq^{2s-2}c_s=c_{s-1}c_s+c_{s-2}c_{s+1}+\cdots+c_{2s-1}$

(resp. $Sq^{4s-2}c_{2s}=c_{2s-2}c_{2s}+c_{2s-4}c_{2s+2}+\cdots+c_{4s-2}$).

Suppose that $2s-1<2^{r-1}$ (resp. $4s-1<2^r$). Then by Lemma 4.10

$$Sq^{2s-2}\zeta_{2s-1}=\kappa\varphi_2(Sq^{2s-2}c_s)=\kappa\varphi_2(c_{2s-1})=\zeta_{4s-3}$$
$$(\text{resp. } Sq^{4s-2}\zeta_{4s-1}=\varphi_2(Sq^{4s-2}c_{2s})=\varphi_2(c_{4s-2})=\zeta_{4s-3}),$$

where the second equality comes from $\varphi_2(c_{s-1-t}c_{s+t})=0$ for all $0\leq t\leq s-2$ by Lemma 4.6, and where the last equality follows from the fact that, with $2s-1<2^{r-1}$, c_{2s-1} is a characteristic polynomial of the class ζ_{4s-3}, see in table (7.3). This completes the proof of i) (resp. ii)).

Finally, for $G=E_7$ granted with the set $S_2(PE_7)$ of explicitly polynomials given in table (7.3), the Wu–formula (7.4), and by the method entailed above, it is straightforward to show the formulae for $Sq^{2s-2}\zeta_{2s-1}$ in iii) (alternatively, see [9, Section 5.2]). We may therefore omit the details.□

7.3 Computing with integral characteristic polynomials

From the presentation of $H^*(G/T)$ Theorem 7.1 one formulates the set

$$S(G) = \{h_i, p_j\beta_j - y_j^{k_j}\alpha_j; 1 \le i \le k, 1 \le j \le m\}$$

of polynomials (see Example 4.6) in accordance to $G = SU(n), Sp(n), E_6, E_7$ as that tabulated below

(7.5)

G	$S(G)$
$SU(n)$	$c_k, 2 \le k \le n$
$Sp(n)$	$c_{2k}, 1 \le k \le n$
E_6	$R_2, R_5, 2R_6 - x_3R_3, R_8, R_9, 3R_{12} - x_4^2R_4$
E_7	$R_2, 2R_6 - x_3R_3, R_8, 2R_{10} - x_5R_5, 3R_{12} - x_4^2R_4, R_{14}, 2R_{18} - x_9R_9$

Through (6.9a) to (6.9d) the ring $H^*(G)$ is presented in term of the primary 1–forms $\gamma_{2\deg P-1} := \kappa\varphi(P) \in E_3^{*,1}(G)$ with characteristic polynomials $P \in S(G)$.

Let τ' is the transgression in the fibration π' in (4.9). With the class $\tau'(t_0)$ determined in (6.3) one computes the derivations $\partial P/\partial t_0$, $P \in S(G)$, using the formula (4.14), as well as the results a)–d) of Theorem 7.2. The set $\{\partial P/\partial t_0; P \in S(G)\}$ of polynomials so obtained are tabulated below

(7.6)

G	$\{\partial P/\partial t_0; P \in S(G)\}$
$SU(n)$	$\binom{n}{s}\omega_1^{s-1}$, $2 \le s \le n$
$Sp(n)$	$\binom{n}{s}\omega_1^{2s-1}$, $1 \le s \le n$
E_6	$0, 0, 0, 0, \omega_1^8, 0$
E_7	$\omega_2, \omega_2^2y_3, \omega_2^2y_5, 0, \omega_2^2(y_9 + y_4y_5 + y_4\omega_2^5), \omega_2^9(y_4 + \omega_2^4), 0$

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